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Pure Math Grade 10 Workbook

- Polynomial Operations
- Factoring Polynomial Expressions
- Real Numbers and Radicals
- Exponents
- Relations and Functions
- Line Segments
- Linear Functions and Equations
- Trigonometry
- Sequences and Data Tables
- Rational Expressions

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About the Pure Math Grade 10 Workbook

The Pure Math Grade 10 Workbook is a complete resource for the Alberta and British Columbia Curriculum. Each curricular topic is subdivided into individual lessons. Most lessons can be covered in one hour (plus homework time), but some may require more time to complete. Lessons are composed of four parts:

- ***Warm-Ups*** - which could be review, preview or investigative work.
- ***Class Examples*** - which are intended to be teacher led.
- ***Assignments*** - short response, extended response, multiple choice and numeric response questions are provided for student practice.
- ***Answer Key*** - answers to the assignment questions.

The **Teacher Solution Manual** is a copy of the workbook with detailed solutions to all the Warm-ups, Class Examples, and Assignments.

The **Student Solution Manual** contains detailed solutions to all the Warm-ups, Class Examples, and Assignments without the questions.

The material has been piloted in several schools and adjustments have been made based on student and teacher feedback.

Acknowledgments

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- our students for their suggestions, opinions, and encouragement.

- Tony Audia for his support.

Most of all, we would like to thank our families, especially our respective wives, Susan, Linda, and Rose, for their patience and understanding.

Advantages for Students

- Students write **in** the workbook so that the math theory, worked examples, and assignments are all in one place for easy review.
- Students can write on the diagrams and graphs.
- Provides class examples and assignments so that students can use their time more efficiently by focusing on solving problems and making their own notes.
- For independent learners the workbook plus solution manual fosters self-paced learning.
- Encourages group learning and peer tutoring.
- The design of the workbook ensures that students are fully aware of the course expectations.
- We hope you enjoy using this workbook and that with the help of your teacher you realize the success that thousands of students each year are achieving using the workbook series.

Advantages for Teachers

- Written by teachers experienced in preparing students for success in high school and diploma examinations.
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- Lessons have been thoroughly piloted in the classroom and modified based on student and teacher feedback.
- Reduces school photocopying costs and time.
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Teacher, student, and parent responses to the workbook series have been very positive. We welcome your feedback. It enables us to produce a high quality resource meeting our goal of success for both teachers and students.

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Polynomial Operations Lesson #1:
Review

Warm-Up

In algebra, a letter that represents one or more numbers is called a **variable**.

Expressions like $2a - b + 4$ or $\frac{5}{x} + 3$ are called **algebraic expressions**.

Certain algebraic expressions are called polynomials as explained below.

A **monomial** is a number or variable or the product of numbers and variables. (Note that the exponent of any variable must be a positive integer in the numerator of the monomial).

eg. 6, x, $6x$, $-\frac{1}{2}xy$, $5x^3$, abc, $2p^4q^2$ etc., are all monomials.

The number part of a monomial is called the **numerical coefficient**.

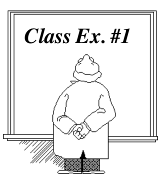
A **polynomial** is a monomial or a sum or difference of monomials.

eg. 6, x, $6 + x$, $2y + 7z$, $x^2 - 5x - 9$ etc., are all polynomials

whereas $\frac{5}{x} + 3$ is <u>not</u> a polynomial.

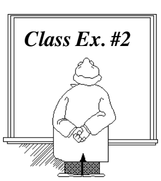
A polynomial consists of one or more **terms** (which are separated by + or - signs).

A polynomial with 1 term is a **monomial**, a polynomial with 2 terms is a **binomial** and a polynomial with 3 terms is a **trinomial**. A polynomial with 4 or more terms is simply called a polynomial.



State whether or not the following are polynomial expressions. If they are not polynomial expressions explain why not.

a) $\frac{1}{4}xy - 10$ **b)** $3pq^{\frac{1}{2}}$ **c)** $\sqrt{7}\,x^4 - x^3$ **d)** $3x^2 + 9x - 4x^{0.2}$ **e)** $\frac{7}{a}$



Complete the following table.

Polynomial Expression	# Variables	# Terms	Name of Polynomial
$4x + 3yz$			
$2a - 4b + 7c$			
$x^2 + 3x + 4$			
$\sqrt{2}\,x$			
$2x^3 + 3x^2y + 3y^2 - 8$			

Classifying Polynomials by Degree

Polynomials can also be classified according to degree.

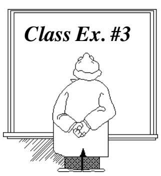
The **degree of a monomial** is the sum of the exponents of its variable(s).

eg. $2x^5$ has degree ____ $\qquad -\frac{2}{3}ab^3c^2$ has degree ____

The **degree of a polynomial** is the degree of the term with highest degree.

eg. $3x^2y^2 - 2x^4 + xy^4 - 2$ has degree ____

If a polynomial has a term of degree zero (ie there is no variable present) this term is called a **constant term**. In the polynomial $3x^2y^2 - 2x^4 + xy^4 - 2$, the constant term is ____.



Give an example of

a) a binomial of degree 1 in one variable.

b) a trinomial in two variables with a constant term.

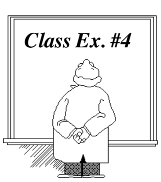
c) a monomial of degree 6 with a (numerical) coefficient of 9.

d) a binomial of degree 8 with each term containing two variables.

Polynomials in a Single Variable

Polynomials in a single variable are usually arranged in **ascending** or **descending** order of the powers of the variable.

The **leading coefficient** of a polynomial in a single variable is the coefficient of the term with highest power of the variable.



Consider the polynomial expression $2x - 4x^3 - 7 + 6x^2$.

a) Write the polynomial in descending powers of x.

b) Write the polynomial in ascending powers of x.

c) State the leading coefficient and the constant term.

Complete Assignment Questions #1 - #7

Addition and Subtraction of Polynomial Expressions

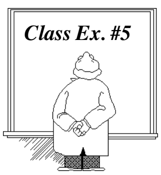
Like terms are terms with the same variable raised to the same exponent.

eg. $3a$, $7a$ and a are like terms. $2x^3$, $\frac{1}{5}x^3$ and $-4x^3$ are like terms.

Unlike terms have different variables or the same variable raised to different exponents.

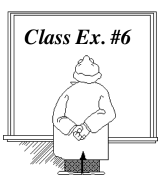
eg. $2x^3$, $\frac{1}{5}x^2$ and $-4x$ are unlike terms. $4x$ and $4y$ are unlike terms.

Like terms can be added or subtracted to produce a single term.



Simplify

a) $4x - 2x^2 + 3 - 6x^2 + 5 - x$ **b)** $a^2b - ab^2 + 4a^3b - 7ab^2 + 5a^2b$



Simplify the following polynomials expressions by collecting like terms.

a) $(3a - 4b + c) + (3b - 5c - 3a)$ **b)** $\begin{array}{r} (4x^2 - 9x + 6) \\ -\ \underline{(2x^2 - 3x - 1)} \end{array}$

c) $3[2(x - 7) + 8] - 2[7 - (x + 5)]$

Complete Assignment Questions #8 - #14

Assignment

1. State whether or not the following are polynomial expressions. If they are not polynomial expressions explain why not.

a) $\frac{1}{2}x^2 - 3x$ **b)** $8m^{-2}$ **c)** $\sqrt{6}$ **d)** $(x^4 + 2)^{1.5}$

2. Complete the following table.

Polynomial expression	*# variables*	*# terms*	*name of polynomial*	*degree*
$2y^3 + y^4 - y + 13$				
$9ab - 4x + 11c$				
25				
$\frac{3}{5}x^3yz^5 + 3x^2yz^4$				

3. Complete the following table.

Polynomial expression	*leading coefficient*	*constant term*	*degree*
$y^4 - y + 13$			
$0.2t^3 - 0.3t^2 + 0.4t - 0.5$			
$\sqrt{5} - x^6$			
$\pi x^2 - 7 - 3x$			
$-\frac{1}{10}c^2$			

4. Give an example of

a) a trinomial of degree 2 in one variable.

b) a binomial in four variables with a constant term of 6.

c) a monomial of degree 3 in two variables with a negative numerical coefficient.

d) a monomial with a degree of 0.

5. Arrange the following in descending powers of the variable.

a) $6w^2 - 9w + 5 + 2w^3$

b) $\frac{1}{4}a^2 - \frac{2}{3}a^3 - 1 - \mathrm{a}$

c) $z - 3 - 4z^6 + z^3$

6. Arrange the following in ascending powers of the variable.

a) $6w^2 - 9w + 5 - 2w^3$

b) $3x^2 - 4x^5 - 2x^4 - 4x^3 + 9x - 7$

c) $8x^3 - 8x + 8$

7. State which of the following are true and which are false.

a) -54 is a polynomial.

b) The degree of the polynomial $3x^3y^3$ is 9.

c) The numerical coefficient of $\dfrac{6x}{5}$ is 6.

d) A polynomial may have 1000 terms.

e) $\dfrac{2}{a^3} - 1$ is a binomial.

f) The degree of the polynomial 0 is 0.

g) The polynomial $x^3 + 2x^2 + 3x + 4$ is written in ascending powers of x.

8. Simplify

a) $6p - 7q - 3q - 2p$ **b)** $5x - 3x^2 + 2x - 8x^2$ **c)** $\frac{1}{2}x - 3 + \frac{3}{2}x + 18$

d) $4a^3 + 7a - 2a^2 - 6a - 4a^3 - a^2$ **e)** $3 - 2x + 7y + 4y - 2x + 8z - 9$

9. Simplify the following polynomials expressions by collecting like terms.

a) $(5a - 9b - 2c) + (c - 7b - 3a)$ **b)** $(3 - a - 2a^2) + (9 - 4a + 5a^2)$

c) $(2x^2 + 5x - 1) + (3x - 6 - 6x^2) + (4 - 5x + x^2)$ **d)** $(4a - 6b) - (5a - 2b)$

e) $\begin{array}{r} (4x^2 - 9x + 6) \\ -\ \underline{(2x^2 - 3x - 1)} \end{array}$ **f)** $\begin{array}{r} (7x^2 + 2x - 1) \\ -\ \underline{(-5x^2 - 3x - 1)} \end{array}$ **g)** $\begin{array}{r} (-4x^2 + 2x - 6) \\ -\ \underline{(3x + 6 - 2x^2)} \end{array}$

h) $5[4 - 2(z + 7)]$ **i)** $6[2(x - 5) + 1] - 3[1 - 4(3 - 2x)]$

10. a) Subtract $3x^2 - 2x + 7$ from $6x^2 - 5x - 2$.

b) Subtract the sum of $2x^3 - 7x^2 - 6x + 1$ and $8 - 3x + 5x^2 - 4x^3$ from $2x^3 - 7x + 9$.

11. A triangle has a perimeter of $(17m + 5n)$ cm. One side measures $(3m - 2n)$ cm and another side measures $(5n + m)$ cm. Write and simplify an expression for the length of the third side of the triangle.

Multiple Choice

12. Which of the following is a polynomial expression of degree 4?

A. $4x^4 - 4x^7$
B. $5x^4 - 3x^3 + 2x^{-2} + x - 1$
C. $\dfrac{4x^4 - 3x}{x}$
D. $9 + 3x - \frac{1}{3}x^2 - x^3 + \frac{2}{5}x^4$

13. Which of the following polynomial expressions, when simplified, is equal to $5x$?

A. $(3x^2 - 3x) - (2x + 3x^2)$
B. $5x - (2x^2 - 2x) + (2x^2 + 2x)$
C. $8 + (4 - 2x) - (12 - 7x)$
D. $(2x^2 - 2x + 6) - (2x^2 - 2x) + (9x - 6)$

Numerical Response

14. If the polynomial $4 - 7x + 2x^2 - 5x^3$ has degree a, leading coefficient b and constant term c, then the value of $3a - 2b - c$ is ______.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** yes **b)** no, negative exponent **c)** yes **d)** no, rational exponent of a polynomial

2.

Polynomial expression	*# variables*	*# terms*	*name of polynomial*	*degree*
$2y^3 + y^4 - y + 13$	1	4	polynomial	4
$9ab - 4x + 11c$	4	3	trinomial	2
25	0	1	monomial	0
$\frac{3}{5}x^3yz^5 + 3x^2yz^4$	3	2	binomial	9

3.

Polynomial expression	*leading coefficient*	*constant term*	*degree*
$y^4 - y + 13$	1	13	4
$0.2t^3 - 0.3t^2 + 0.4t - 0.5$	0.2	-0.5	3
$\sqrt{5} - x^6$	-1	$\sqrt{5}$	6
$\pi x^2 - 7 - 3x$	π	-7	2
$-\frac{1}{10}c^2$	$-\frac{1}{10}$	0	2

4. answers may vary **a)** $x^2 - x + 30$ **b)** $abcd + 6$ **c)** $-2xy^2$ **d)** 10

5. **a)** $2w^3 + 6w^2 - 9w + 5$ **b)** $-\frac{2}{3}a^3 + \frac{1}{4}a^2 - a - 1$ **c)** $-4z^6 + z^3 + z - 3$

6. **a)** $5 - 9w + 6w^2 - 2w^3$ **b)** $-7 + 9x + 3x^2 - 4x^3 - 2x^4 - 4x^5$ **c)** $8 - 8x + 8x^3$

7. **a)** true **b)** false **c)** false **d)** true **e)** false **f)** true **g)** false

8. **a)** $4p - 10q$ **b)** $-11x^2 + 7x$ **c)** $2x + 15$ **d)** $-3a^2 + a$ **e)** $-4x + 11y + 8z - 6$

9. **a)** $2a - 16b - c$ **b)** $3a^2 - 5a + 12$ **c)** $-3x^2 + 3x - 3$ **d)** $-a - 4b$ **e)** $2x^2 - 6x + 7$
f) $12x^2 + 5x$ **g)** $-2x^2 - x - 12$ **h)** $-10z - 50$ **i)** $-12x - 21$

10. **a)** $3x^2 - 3x - 9$ **b)** $4x^3 + 2x^2 + 2x$

11. $(13m + 2n)$ cm **12.** D **13.** C **14.**

1	5		

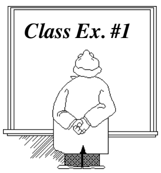
Polynomial Operations Lesson #2:
Multiplication of Polynomials - Part One

Warm-Up

In previous math courses we learned how to multiply
i) two monomials, **ii)** a monomial and a binomial or trinomial, and **iii)** two binomials.

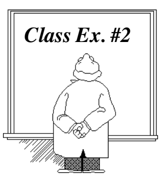
In each case we wrote a product of polynomials as a sum or difference of terms.

In this process we **expanded** the polynomial expressions by using the distributive property, $a(b + c) = ab + ac$ and the exponent rule $x^a \times x^b = x^{a+b}$.



Expand and simplify.

a) $4(2x - 3) - 2(x - 6)$ **b)** $5x(3x^2 - 7x + 1) - (4x + 3x^2)$



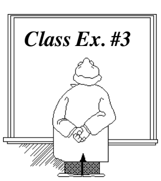
A rectangle has length $(a + 3)$ cm. and width $(a + 2)$ cm.
Determine the area of the rectangle (in cm^2) by completing each of the following solutions.

Area = length × width = ( )( )

(i) *use a diagram* (ii) *use the distributive property* (iii) *use FOIL*

(ii)
$(a + 3)(a + 2)$
$= a(a + 2) + 3(a + 2)$
$=$
$=$

(iii)
$(a + 3)(a + 2)$
$=$



Expand and simplify $4(2x - 7)(3x + 2) - (x - 1)(3x - 1)$

Complete Assignment Questions #1 - #2

Three Important Products

Complete the following

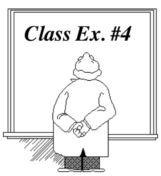
i) $(a+b)^2 = (a+b)(a+b)$ **ii)** $(a-b)^2 = (a-b)(a-b)$ **iii)** $(a-b)(a+b)$

$=$ $=$ $=$

$=$ $=$ $=$

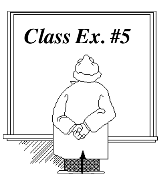
1. **The square of a binomial can be found by squaring the first term, doubling the product of the two terms and squaring the last term.**

2. **The product of the sum and difference of the same two monomials results in the difference of the squares of the monomials.**
   This important result will be considered in more detail in the next unit on factoring.



Expand each of the following .

a) $(x+7)^2$ **b)** $(3x-1)^2$ **c)** $(2m-3n)^2$ **d)** $(5a-3b)(5a+3b)$



The hypotenuse of a right triangle is $(5x+5)$ cm long and the lengths of the other two sides are $(4x+8)$ cm and $(3x-5)$ cm. Form an equation and solve it to determine the lengths of the three sides of the triangle.

Complete Assignment Questions #3 - #15

Assignment

1. Expand and simplify where possible.

a) $6(7x - 3)$ **b)** $-4(4x + 9)$ **c)** $4x(2y + 8z)$ **d)** $-x(x - 5y)$

e) $5(8x - 3y) + 2(4y + x)$ **f)** $3a(2a^2b - ab + b^2) - 6b(a^3 + 3ab - 5b^2)$

g) $3x(x - 3) - 2x(x - 1) + x(2x - 2)$ **h)** $(p^2 - 3p)(4p) - (3 + 5p)(-2p^2)$

i) $a(b - c) + b(c - a) + c(a - b)$ **j)** $20x^3y^3 - 4x^3y^2(3x + 5y - xy)$

2. Expand and simplify where possible.

a) $(7x - 2)(3x + 5)$ **b)** $(2h - 3)(2h - 1)$ **c)** $(3z + 4)(3z + 5)$

d) $2(4x - 3)(3x - 4)$ **e)** $5(8x - 3y)(2x + y)$ **f)** $-4(a + 3b)(2a - 5b)$

g) $(3x - 1)(x - 3) - 2x(x - 1)$ **h)** $(4x + 1)(2x + 3) - (3x - 7)(2x - 5)$

i) $9 - 2(x - 1)(x + 7) + (2x - 5)(x - 3)$ **j)** $3(1 + 3y)(4 - y) - (3y - 2)(3y - 5)$

3. Expand and simplify where possible.

a) $(x-8)^2$ **b)** $(2p+7)^2$ **c)** $(3x-y)^2$ **d)** $4(5x+2y)^2$

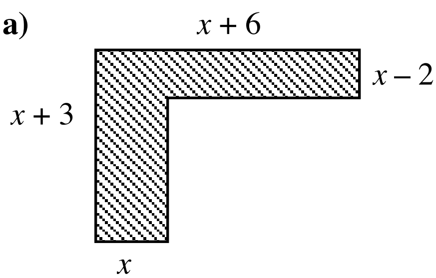
e) $(x+4)^2+(x+2)^2$ **f)** $(3a-b)^2-(2a+5b)^2$ **g)** $3(y-1)^2-2(2y-1)^2$

h) $(x-9)(x+9)$ **i)** $(3x-2)(3x+2)$ **j)** $(4m+3n)(4m-3n)$

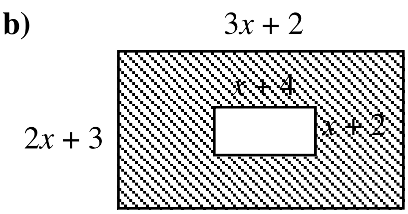
4. In each of the following **i)** write an expression for the shaded area

ii) expand and write the expression in simplest form.

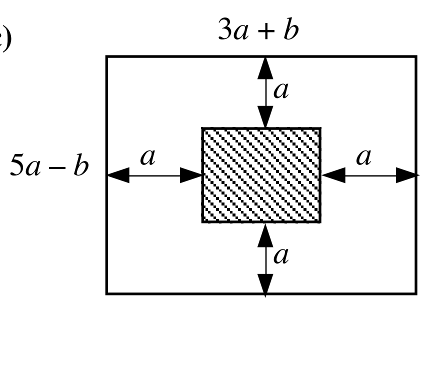
a)



b)



c)



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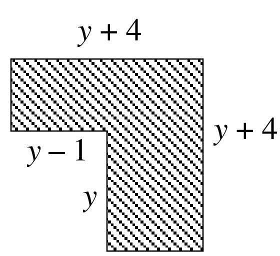


5. Solve the following equations where the variable is on the set of real numbers.

a) $(3x - 1)(x - 1) = 3x(x + 1)$

b) $(y + 2)^2 = y^2 + 2$

c) $t^2 - (t - 9)^2 = 9$

d) $2a^2 - (a - 3)^2 = (a + 2)(a - 1)$

6. A square garden with a side length of $(3x + 1)$ m contains two square flower beds each with a side length of $(x + 1)$ m. The remainder of the garden is grass.
Write and simplify an expression for the area of grass in the garden.

7. Consider a series of rectangles with sides $(4a - 3)$ cm and $(2a + 7)$ cm.

a) Write and simplify an expression for the area of the rectangles.

b) If one of these rectangles has a perimeter of 50 cm determine the length and width of this rectangle.

c) If another of these rectangles is a square determine the length of each side.

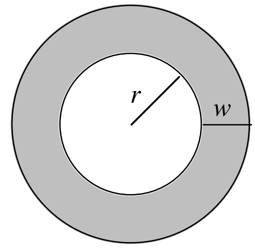
8. A square metal plate of side 25 cm is heated so that each side increases in length by x cm.

a) Write and simplify an expression for the area of the heated plate.

b) Write and simplify an expression for the increase in area of the plate.

c) If $x = 0.2$, calculate the increase in area.

9. A metal washer has internal radius r mm and width w mm as shown.



a) Write an expression for the outer radius of the washer.

b) Show that the area of the washer, A mm^2, is given by $A = 2\pi rw + \pi w^2$.

Multiple Choice

10. For all x and y, $(2x + 6y)^2$ equals

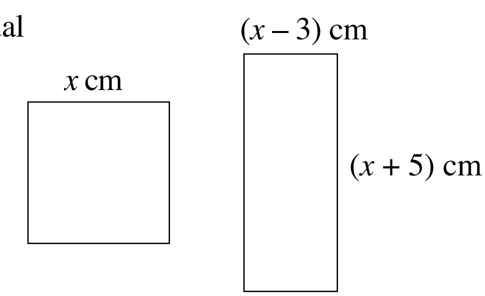
A. $4x^2 + 36y^2$
B. $4x^2 + 12xy + 36y^2$
C. $2x^2 + 6xy + 6y^2$
D. $4x^2 + 24xy + 36y^2$

Numerical Response

11. The expansion of $(3x - c)(x - 3)$, where c is a whole number, results in a polynomial in x with a leading coefficient of 3 and a constant term of 12. The value of c is ______.

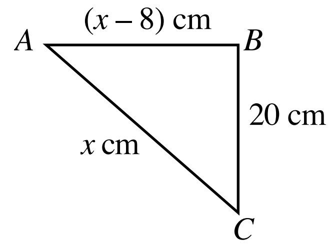
(Record your answer in the numerical response box from left to right)

12. The square and the rectangle in the diagram are equal in area.
The value of x, to the nearest tenth, is ______.



(Record your answer in the numerical response box from left to right)

13. The diagram shows the lengths of the sides of right triangle ABC.



The perimeter (to the nearest tenth of a cm) of triangle ABC is ______.

(Record your answer in the numerical response box from left to right)

14. The expression $4(3x + 7)(2x - 9) - 3(x - 4)(5x - 2)$ can be written in the form $ax^2 + bx + c$. The value of b is ______ .

(Record your answer in the numerical response box from left to right)

15. When $3(5x - 2)(5x + 2) - (5x - 1)^2$ is written in polynomial form, the coefficient of the term in x is ______ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $42x - 18$ **b)** $-16x - 36$ **c)** $8xy + 32xz$ **d)** $-x^2 + 5xy$ **e)** $42x - 7y$
f) $-3a^2b - 15ab^2 + 30b^3$ **g)** $3x^2 - 9x$ **h)** $14p^3 - 6p^2$ **i)** 0 **j)** $-12x^4y^2 + 4x^4y^3$

2. **a)** $21x^2 + 29x - 10$ **b)** $4h^2 - 8h + 3$ **c)** $9z^2 + 27z + 20$ **d)** $24x^2 - 50x + 24$
e) $80x^2 + 10xy - 15y^2$ **f)** $-8a^2 - 4ab + 60b^2$ **g)** $x^2 - 8x + 3$ **h)** $2x^2 + 43x - 32$
i) $-23x + 38$ **j)** $-18y^2 + 54y + 2$

3. **a)** $x^2 - 16x + 64$ **b)** $4p^2 + 28p + 49$ **c)** $9x^2 - 6xy + y^2$ **d)** $100x^2 + 80xy + 16y^2$
e) $2x^2 + 12x + 20$ **f)** $5a^2 - 26ab - 24b^2$ **g)** $-5y^2 + 2y + 1$ **h)** $x^2 - 81$
i) $9x^2 - 4$ **j)** $16m^2 - 9n^2$

4. answers to part i) may vary
a) **i)** $(x + 6)(x - 2) + 5x$ **ii)** $x^2 + 9x - 12$
b) **i)** $(3x + 2)(2x + 3) - (x + 4)(x + 2)$ **ii)** $5x^2 + 7x - 2$
c) **i)** $(3a - b)(a + b)$ **ii)** $3a^2 + 2ab - b^2$
d) **i)** $(y + 4)^2 - y(y - 1)$ **ii)** $9y + 16$

5. **a)** $\frac{1}{7}$ **b)** $-\frac{1}{2}$ **c)** 5 **d)** $\frac{7}{5}$

6. $7x^2 + 2x - 1$ m^2

7. **a)** $(4a - 3)(2a + 7) = 8a^2 + 22a - 21$ cm^2 **b)** 11 cm by 14 cm **c)** 17 cm

8. **a)** $625 + 50x + x^2$ cm^2 **b)** $50x + x^2$ cm^2 **c)** 10.04 cm^2

9. **a)** $r + w$ **10.** D

11.

4			

12.

7	.	5	

13.

7	0	.	0

14.

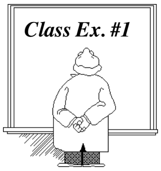
1	4		

15.

1	0		

Polynomial Operations Lesson #3:
Multiplication of Polynomials - Part Two

Product of a Binomial and a Trinomial



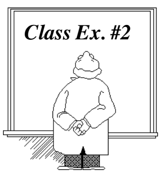
A rectangle has length $(5x + 2)$ cm. and width $(x^2 + 2x + 1)$ cm.
Determine the area of the rectangle (in cm^2) by completing each of the following solutions.

Area = length × width = (   )(   )

i) *use a diagram*

ii) *use the distributive property*

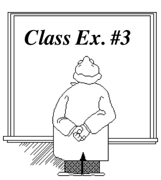
$$(5x + 2)(x^2 + 2x + 1)$$
$$= 5x(x^2 + 2x + 1) + 2(x^2 + 2x + 1)$$
$$=$$
$$=$$



Expand and simplify

a) $(x^2 - 4)(2x^3 + x - 5)$

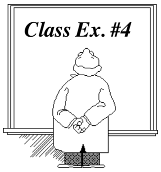
b) $(2y^2 - 3y - 7)(y - 6)$



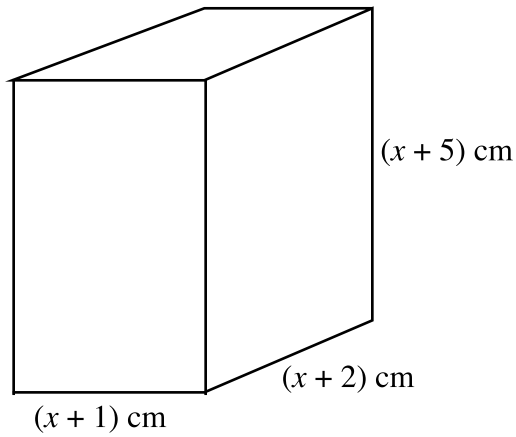
Extension. Expand and simplify $\quad 4(a - 4)(a^2 - 3a - 6) - (4a - 3)(4a + 3)$

Product of Three Binomials

In this lesson we extend the multiplication of binomials to consider three factors. This leads to applications involving the volume of a rectangular prism.

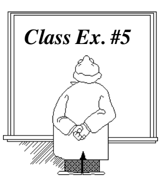


Consider the rectangular prism shown.



a) Write an expression which represents the volume of the prism in cm^3.

b) Expand the expression in a) to write the volume in the form $V = ax^3 + bx^2 + cx + d$.



Expand and simplify

a) $(x - 3)(x + 4)(2x - 1)$

b) $(2x - 1)(x - 3)(x + 4)$

c) Comment on the results to a) and b).

Complete Assignment Questions #1 - #11

Assignment

1. Expand and simplify

a) $(y-5)(y^2+2y+4)$

b) $(3m+7)(m^2-3m+6)$

c) $(x^2-7)(2x^3+4x-1)$

d) $(-m^2-m+1)(m+1)$

e) $(a-3b)\,(4a^2-3ab-2b^2)$

f) $2(5x+2)(3x^2+x-4)$

2. Expand and simplify.

a) $(x+1)(x+2)(3x+5)$

b) $(h-4)(2h-3)(3h-1)$

c) $(a+3b)(2a-5b)(2a+5b)$

d) $(3x+7y)(4x-3y)(x-4y)$

e) $(x-3)(2x+1)^2$

f) $(2z+3)^3$

3. A rectangle has length $(x^2 + 4x - 1)$ cm and width $(3x - 2)$ cm.

a) Write and simplify an expression for the area of the rectangle in cm^2.

b) If $x = 2.5$, calculate the area of the rectangle.

4. Dice for a children's board game are cubes with an edge length of $(3x - 2)$ mm.

a) Write and simplify an expression for the volume of a die in mm^3.

b) The manufacturer packages dice in cubic containers containing 64 dice.
Determine the volume of the container in cm^3 if $x = 4$.

5. A rectangular prism has length $(5x - 2)$ cm, width $(3x - 1)$ cm and height $(3x + 1)$ cm.

a) Write and simplify an expression for the volume of the rectangular prism in cm^3.

b) Write and simplify an expression for the surface area of the rectangular prism in cm^2.

c) If $x = 4$, calculate the volume and surface area of the rectangular prism.

Multiple Choice

6. A box is in the shape of a rectangular prism. The length of the box is y cm. The width is 2 cm less than the length and the height is 2 cm more than the length. If the volume of the box can be written in the form $V = ay^3 + by^2 + cy + d$ where a, b, c and d are integers then how many of the parameters a, b, c and d are equal to zero?

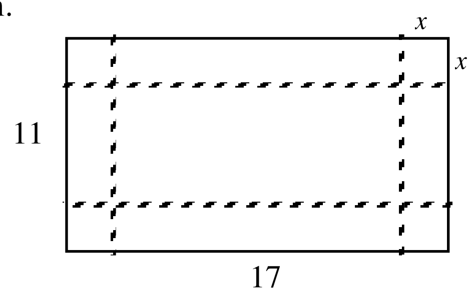
A. 0
B. 1
C. 2
D. 3

7. When the expression $(5 - x)^2(x - 2)$ is simplified the value of the constant term is

A. -50
B. 50
C. -20
D. -10

Use the following information to answer questions #8 and #9

A sheet of paper measures 17 cm by 11 cm. Squares of side x cm are cut out from each corner as shown. The paper is folded along the dotted lines to form a rectangular prism.



8. The length and width (in cm) of the rectangular prism are respectively

A. $17 - x$ and $11 - x$
B. $17 - 2x$ and $11 - 2x$
C. $17 + x$ and $11 + x$
D. $17 + 2x$ and $11 + 2x$

Numerical Response

9. The volume of the rectangular prism can be written as the polynomial expression $ax^3 + bx^2 + cx$ where a, b and c are integers.
The value of $a + c$, to the nearest whole number, is _____.
(Record your answer in the numerical response box from left to right)

10. Subtracting the product of $(3x - 1)$ and $(2x^2 - 4x + 3)$ from the sum of $(2x^3 - 7x^2 - 6)$ and $(x^2 + 6x - 3)$ results in a polynomial of the form $ax^3 + bx^2 + cx + d$.
The value of $b - 2c$ is _____ .

(Record your answer in the numerical response box from left to right)

Extension

11. Simplify

a) $-3(a^2 + 2)(3a^2 - a - 1)$

b) $(-2x^2 - 3x + 1)(x^2 - x - 3)$

c) $2(4x - 1)^2 - (3x - 2)^3$

Answer Key

1. a) $y^3 - 3y^2 - 6y - 20$ **b)** $3m^3 - 2m^2 - 3m + 42$ **c)** $2x^5 - 10x^3 - x^2 - 28x + 7$
d) $-m^3 - 2m^2 + 1$ **e)** $4a^3 - 15a^2b + 7ab^2 + 6b^3$ **f)** $30x^3 + 22x^2 - 36x - 16$

2. a) $3x^3 + 14x^2 + 21x + 10$ **b)** $6h^3 - 35h^2 + 47h - 12$ **c)** $4a^3 + 12a^2b - 25ab^2 - 75b^3$
d) $12x^3 - 29x^2y - 97xy^2 + 84y^3$ **e)** $4x^3 - 8x^2 - 11x - 3$ **f)** $8z^3 + 36z^2 + 54z + 27$

3. a) $(x^2 + 4x - 1)(3x - 2) = 3x^3 + 10x^2 - 11x + 2$ cm^2 **b)** 83.875 cm^2

4. a) $(3x - 2)^3 = 27x^3 - 54x^2 + 36x - 8$ mm^3 **b)** 64 cm^3

5. a) $(5x - 2)(3x - 1)(3x + 1) = 45x^3 - 18x^2 - 5x + 2$ cm^3
b) $2(3x - 1)(3x + 1) + 2(3x - 1)(5x - 2) + 2(3x + 1)(5x - 2) = 78x^2 - 24x - 2$ cm^2
c) volume = 2574 cm^3, surface area = 1150 cm^2

6. C **7.** A **8.** B **9.**

1	9	1	

10.

2	2		

11. a) $-9a^4 + 3a^3 - 15a^2 + 6a + 6$ **b)** $-2x^4 - x^3 + 10x^2 + 8x - 3$
c) $-27x^3 + 86x^2 - 52x + 10$

Polynomial Operations Lesson #4:
Dividing Polynomials - Part One

Warm-Up #1 *Review - Dividing a Monomial by a Monomial*

We use the exponent rule $x^a \div x^b = x^{a-b}$.

Complete the following divisions:

a) $x^8 \div x^4 =$ **b)** $\dfrac{x^{12}}{x^3} =$ **c)** $\dfrac{6a^4}{2a^3} =$ **d)** $\dfrac{-20x^8y^6}{15xy^3} =$

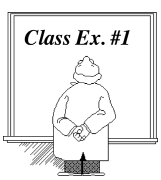
Warm-Up #2 *Review - Dividing a Polynomial by a Monomial*

Complete the following division:

$$\frac{15x^3 - 10x^2 + 5x}{5x} = \frac{15x^3}{5x} - \frac{10x^2}{5x} + \frac{}{5x} =$$

Notice that in this example the division would not be valid if $x = 0$ since division by zero is not defined. There is a **restriction** on the variable x.
In this case the value 0 is a **nonpermissible value** for x.

In this lesson we will be dividing polynomials by expressions which contain variables so there will be nonpermissible values of the variable in each question. This topic will be covered in more depth in a later unit on rational expressions. For this lesson we will not state the nonpermissible values although we should be aware that they exist.



Divide.

a) $\dfrac{9a^2b + 27ab^2 - 12a^3b^2}{3ab}$ **b)** $(20x^3y - 45x^4y^2 + 5x^2y) \div (5x^2y)$

c) $(16x^4y^2 - 28x^3y^2 - 20x^3y^4) \div (-4x^3y^2)$

Complete Assignment Questions #1 - #3

Dividing a Polynomial by a Binomial of the Form x − b

In all the previous questions the divisor divided the polynomial exactly.
This is not always the case.

Consider the division of 20 by 3. The **dividend** is 20, the **divisor** is 3, the **quotient** is ___ and the **remainder** is ___.

The result of the division can be written in two ways.

i) $20 = 3 \times 6 + 2$ or **ii)** $\dfrac{20}{3} = 6 + \dfrac{2}{3}$

In other words,

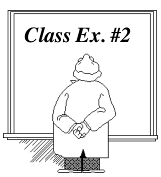
i) dividend = divisor × quotient + remainder.

ii) $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$.

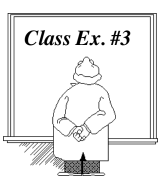
In the case where the dividend is a polynomial, we have the following:

i) **polynomial = divisor × quotient + remainder.** $\qquad \boldsymbol{P = DQ + R}$

ii) $\dfrac{\textbf{polynomial}}{\textbf{divisor}} = \textbf{quotient} + \dfrac{\textbf{remainder}}{\textbf{divisor}}$. $\qquad \boldsymbol{\dfrac{P}{D} = Q + \dfrac{R}{D}}$



Use long division to divide 739 by 35 and write the result of the division in two ways.



Use long division to divide $x^2 + 3x + 8$ by x and write the result of the division in two ways.

Class Ex. #4

Use long division to divide $2x^2 - x - 18$ by $x + 2$ and write the result in the form $P = DQ + R$.

Class Ex. #5

Divide $y^3 - 6y + 20$ by $y - 3$. Express the answer in the form $\frac{P}{D} = Q + \frac{R}{D}$.

Class Ex. #6

Consider the polynomial $10 + 4x^3 - 2x^2 + x^4$.

a) Write the polynomial in descending order of the powers of x.

b) Divide the polynomial by $x - 1$ and write the result in the form $P = DQ + R$.

Complete Assignment Questions #4 - #10

Assignment

1. Divide.

a) $\dfrac{9 - 6x + 3x^2}{3}$

b) $\dfrac{15x^3 - 25x^2 - 15x}{5x}$

c) $\dfrac{8a^2b^3 + 4ab^2 - 12a^3b^2}{4ab^2}$

d) $\dfrac{21x^3y - 35x^2y^2 + 77x^2y^5}{-7x^2y}$

e) $(36x^4 - 54x^3 - 24x^2) \div (-6x^2)$

f) $(24x^4y^2 + 9x^2y^2 - 18x^2y^4) \div (-3x^2y^2)$

g) $(3mn^2 + 6m^2n^3 - 9m^3n^4) \div (3mn)$

h) $(4a^3 - 8a^2 + 16a) \div (16a)$

2. A rectangular tennis court has an area of $(8p^3 - 4p^2 + 6p)$ m^2. If the width of the tennis court is $2p$ metres, determine an expression for the length of the court.

3. A right angled triangle has an area of $(6x^2 - 4x)$ cm^2 and a base length of $4x$ cm.

a) Write and simplify an expression for the height of the triangle.

b) If the hypotenuse has a length of $(5x - 1)$ cm, determine the value of x and the lengths of the three sides.

4. Divide the following polynomials. In each case express the answer in the form $P = DQ + R$ and in the form $\dfrac{P}{D} = Q + \dfrac{R}{D}$.

a) $\dfrac{x^2 + 5x + 4}{x + 2}$ **b)** $\dfrac{2x^2 - 5x + 2}{x - 3}$ **c)** $\dfrac{6x^2 - 5x - 3}{x - 1}$

5. Divide the following polynomials. Express the answer in the form $\dfrac{P}{D} = Q + \dfrac{R}{D}$.

a) $\dfrac{a^3 - a^2 - 4a + 12}{a - 2}$

b) $\dfrac{3x^3 - x^2 + 2x + 4}{x + 4}$

c) $(x^3 + 125) \div (x + 5)$

d) $\dfrac{x^2 - 2x^3 + x^4 + x^5 + x - 6}{x + 1}$

6. Divide the following polynomials. Express the answer in the form $P = DQ + R$.

a) $(t^3 + 6t) \div (t - 3)$

b) $(y^5 - 1) \div (y - 1)$

7. When a third degree polynomial is divided by $x + 5$ the quotient is $x^2 - 2x - 1$ and the remainder is 7. Express the polynomial in the form $ax^3 + bx^2 + cx + d$.

Multiple Choice

8. Which division simplifies to $-3xy^2 - 1$?

A. $\dfrac{5 - 15xy^2}{5}$

B. $\dfrac{-6x^2y^2 + 2x}{2x}$

C. $\dfrac{2xy + 6x^2y^3}{-2xy}$

D. $\dfrac{21x^3y^3 - 7x^2y}{-7x^2y}$

9. When $(3z^4 + 6z^3 - 18z)$ is divided by $(z + 3)$ the remainder is

A. -45
B. 45
C. -135
D. 135

Numerical Response

10. The binomial $x - 3$ divides exactly the polynomial $x^3 - 2x^2 + cx - 12$ where c is a whole number. The value of c is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $3 - 2x + x^2$ **b)** $3x^2 - 5x - 3$ **c)** $2ab + 1 - 3a^2$ **d)** $-3x + 5y - 11y^4$
e) $-6x^2 + 9x + 4$ **f)** $-8x^2 - 3 + 6y^2$ **g)** $n + 2mn^2 - 3m^2n^3$ **h)** $\frac{1}{4}a^2 - \frac{1}{2}a + 1$

2. $4p^2 - 2p + 3$ m. **3.** **a)** $\dfrac{6x^2 - 4x}{2x} = 3x - 2$ cm. **b)** $x = 1.5$ Sides are 2.5 cm, 6 cm, 6.5 cm.

4. **a)** $x^2 + 5x + 4 = (x + 2)(x + 3) - 2$ $\qquad \dfrac{x^2 + 5x + 4}{x + 2} = x + 3 - \dfrac{2}{x + 2}$

b) $2x^2 - 5x + 2 = (x - 3)(2x + 1) + 5$ $\qquad \dfrac{2x^2 - 5x + 2}{x - 3} = 2x + 1 + \dfrac{5}{x - 3}$

c) $6x^2 - 5x - 3 = (x - 1)(6x + 1) - 2$ $\qquad \dfrac{6x^2 - 5x - 3}{x - 1} = 6x + 1 - \dfrac{2}{x - 1}$

5. **a)** $\dfrac{a^3 - a^2 - 4a + 12}{a - 2} = a^2 + a - 2 + \dfrac{8}{a - 2}$ **b)** $\dfrac{3x^3 - x^2 + 2x + 4}{x + 4} = 3x^2 - 13x + 54 - \dfrac{212}{x + 4}$

c) $\dfrac{x^3 + 125}{x + 5} = x^2 - 5x + 25$ **d)** $\dfrac{x^5 + x^4 - 2x^3 + x^2 + x - 6}{x + 1} = x^4 - 2x^2 + 3x - 2 - \dfrac{4}{x + 1}$

6. **a)** $t^3 + 6t = (t - 3)(t^2 + 3t + 15) + 45$ **b)** $y^5 - 1 = (y - 1)(y^4 + y^3 + y^2 + y + 1)$

7. $x^3 + 3x^2 - 11x + 2$ **8.** C **9.** D **10.**

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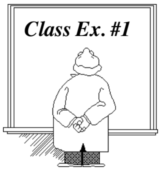
Polynomial Operations Lesson #5:
Dividing Polynomials - Part Two

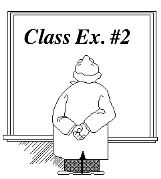
Dividing Polynomials by Binomials of the Form $ax - b$ or $ax^2 + c$



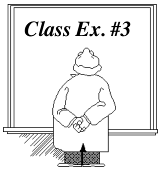
Consider the polynomial $8 + 6x^3 - 6x^2 + 4x^4$.

a) Write the polynomial in descending order of the powers of x.

b) Divide the polynomial by $2x - 3$ and write the result in the form $P = DQ + R$.



Divide $p^4 + 3p^3 - 5p^2 + p - 2$ by $p^2 + 2$ and express the answer in the form $\dfrac{P}{D} = Q + \dfrac{R}{D}$.



A rectangular table top has an area of $6a^5 - 23a^3 + 3a^2 + 20a - 4$ cm^2 and a length of $3a^2 - 4$ cm. Write and simplify an expression for the width of the table.

When dividing polynomials, the degree of the remainder must be less than the degree of the divisor.

Complete Assignment Questions #1 - #9

Assignment

1. Divide the following polynomials. In each case express the answer in the form $P = DQ + R$ and in the form $\dfrac{P}{D} = Q + \dfrac{R}{D}$.

a) $\dfrac{2x^2 + 5x + 9}{2x + 1}$ **b)** $\dfrac{6x^2 - 5x + 7}{2x - 3}$ **c)** $\dfrac{4x^2 + x - 1}{4x - 7}$

2. Divide the following polynomials. Express the answer in the form $P = DQ + R$.

a) $\dfrac{9x^2 - 9}{3x + 1}$ **b)** $\dfrac{6x^2 - 5x}{3x + 2}$ **c)** $\dfrac{12x^3 - 5x^2 + x}{4x - 3}$

3. Divide the following polynomials. Express the answer in the form $\frac{P}{D} = Q + \frac{R}{D}$.

a) $\dfrac{x^3 + 2x^2 + 6x - 7}{x^2 + 4}$

b) $\dfrac{2x^4 + 8x^3 + x^2 - 3}{x^2 - 2}$

c) $\dfrac{x^5 + 2x^3 + x^2 - 3x + 6}{x^2 + 3}$

d) $\dfrac{6x^4 + 4x^3 - 10x + 2}{2x^2 + 4}$

e) $(2a^4 - 6a^3 + a^2 + 6a - 2) \div (2a^2 - 1)$

f) $(10b^4 - 11b^2 - 2) \div (5b^2 + 2)$

4. The area of a triangle is represented by the expression $t^5 - t^3 - 6t^2 - 2t + 12$ cm^2. The base length of the triangle is $t^2 - 2$ cm. Write and simplify an expression for the height of the triangle.

5. The division shows a polynomial expression in x, written as $P(x)$, being divided by a binomial.

$$
\begin{array}{r|l}
 & x^2 + 4 \\
\hline
2x - 3 & P(x) \\
 & \\
\hline
 & \quad\quad -2
\end{array}
$$

a) Write the division in the form $P = DQ + R$

b) Write the polynomial $P(x)$ in the form $ax^3 + bx^2 + cx + d$.

Multiple Choice

6. The number of thumb tacks produced by a company in one day is given by the expression $22 + 40m + 5m^2 - 24m^3 + 6m^4$. The thumb tacks are packaged into boxes each containing $2m^2 - 3$ tacks. After as many boxes as possible have been filled there are some thumb tacks left over. The number of thumb tacks left over is

A. $4m + 43$
B. $3m^2 - 12m + 7$
C. 43
D. 7

Numerical Response

7. When the polynomial $2x^3 - 5x^2 + 4x + d$ is divided by $2x + 3$ the remainder is -6.
The value of d is _____.

(Record your answer in the numerical response box from left to right)

8. A rectangle has an area of $8x^2 - 14x - 15$ cm^2 and a length of $4x + 3$ cm.
The perimeter of the rectangle can be written in the form $ax + b$ cm.
The value of $a + b$ is _____.

(Record your answer in the numerical response box from left to right)

9. When the polynomial $ax^3 + bx^2 + cx + d$ is divided by $3x - 2$, the quotient is $2x^2 + 2x + 3$ and the remainder is 7.

Record the value of a in the first box.
Record the value of b in the second box.
Record the value of c in the third box.
Record the value of d in the fourth box.

Answer Key

1. a) $2x^2 + 5x + 9 = (2x + 1)(x + 2) + 7$ $\qquad \dfrac{2x^2 + 5x + 9}{2x + 1} = x + 2 + \dfrac{7}{2x + 1}$

b) $6x^2 - 5x + 7 = (2x - 3)(3x + 2) + 13$ $\qquad \dfrac{6x^2 - 5x + 7}{2x - 3} = 3x + 2 + \dfrac{13}{2x - 3}$

c) $4x^2 + x - 1 = (4x - 7)(x + 2) + 13$ $\qquad \dfrac{4x^2 + x - 1}{4x - 7} = x + 2 + \dfrac{13}{4x - 7}$

2. a) $9x^2 - 9 = (3x + 1)(3x - 1) - 8$ $\qquad$ **b)** $6x^2 - 5x = (3x + 2)(2x - 3) + 6$

c) $12x^3 - 5x^2 + x = (4x - 3)(3x^2 + x + 1) + 3$

3. a) $\dfrac{x^3 + 2x^2 + 6x - 7}{x^2 + 4} = x + 2 + \dfrac{2x - 15}{x^2 + 4}$

b) $\dfrac{2x^4 + 8x^3 + x^2 - 3}{x^2 - 2} = 2x^2 + 8x + 5 + \dfrac{16x + 7}{x^2 - 2}$

c) $\dfrac{x^5 + 2x^3 + x^2 - 3x + 6}{x^2 + 3} = x^3 - x + 1 + \dfrac{3}{x^2 + 3}$

d) $\dfrac{6x^4 + 4x^3 - 10x + 2}{2x^2 + 4} = 3x^2 + 2x - 6 + \dfrac{-18x + 26}{2x^2 + 4}$

e) $\dfrac{2a^4 - 6a^3 + a^2 + 6a - 2}{2a^2 - 1} = a^2 - 3a + 1 + \dfrac{3a - 1}{2a^2 - 1}$

f) $\dfrac{10b^4 - 11b^2 - 2}{5b^2 + 2} = 2b^2 - 3 + \dfrac{4}{5b^2 + 2}$

4. $h = \dfrac{2t^5 - 2t^3 - 12t^2 - 4t + 24}{t^2 - 2} = 2t^3 + 2t - 12$ cm.

5. a) $P = (2x - 3)(x^2 + 4) - 2$ $\qquad$ **b)** $P = 2x^3 - 3x^2 + 8x - 14$ $\qquad$ **6.** A

7.

1	8		

8.

8			

9.

6	2	5	1

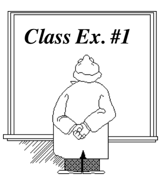
Polynomial Operations Lesson #6
Enrichment Lesson - Synthetic Division

Warm-Up

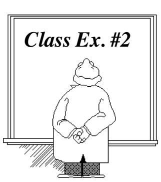
Use long division to divide the polynomial $3x^3 - 7x - 9$ by $x - 2$.
State the quotient, divisor and remainder and express the division in the form $P = DQ + R$.

Synthetic Division

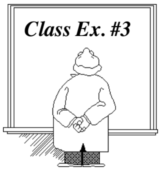
Synthetic Division is an alternative method of dividing a polynomial by a linear binomial.



Use synthetic division to divide $3x^3 - 7x - 9$ by $x - 2$.
Express the answer in the form $P = DQ + R$.



Use synthetic division to divide $2x^3 - 3x^2 - 8x + 15$ by $x - 1$ and express the answer in the form $\dfrac{P}{D} = Q + \dfrac{R}{D}$.

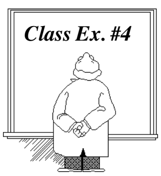


Consider the polynomial $5x^5 - 6x^4 + 3x^2 - 2x + 1$.

a) Use synthetic division to find the quotient and remainder when $5x^5 - 6x^4 + 3x^2 - 2x + 1$ is divided by $x + 2$.

b) Find the value of the polynomial when x is replaced by -2.

c) Comment on your answers in **a)** and **b)**.



If $x + 3$ is the divisor in the following synthetic division, calculate the values of m and p.

$$\begin{array}{|cccc} 2 & 2 & -m & 16 \\ & & & n \\ \hline 2 & & 2m & p \end{array}$$

Complete Assignment Questions #1 - #8

Synthetic Division by ax − b

Use synthetic division to determine the quotient and remainder when the polynomial $2x^3 + x^2 + 5x - 1$ is divided by $2x - 1$.

Note that $2x - 1 = 2\left(x - \frac{1}{2}\right)$.

We divide first by $x - \frac{1}{2}$.

$$\begin{array}{|cccc} 2 & 1 & 5 & -1 \\ & & & \\ \hline \end{array}$$

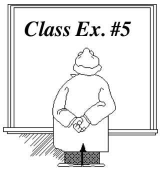
$P = DQ + R$.

so $P = \left(x - \frac{1}{2}\right)(\qquad\qquad) +$

$= (2x - 1)(\qquad\qquad) +$

Quotient is __________

Remainder is ________



Divide $6x^3 - 8x^2 - 5x + 5$ by $3x + 2$ using synthetic division and write the division in the form $P = DQ + R$.

Complete Assignment Questions #9 - #10

Assignment

1. In the synthetic division below, a polynomial is divided by $x - 2$.

1	-2	6	3
1	0	6	15

a) State the polynomial. **b)** State the quotient. **c)** State the remainder.

d) Write the synthetic division in the form $P = DQ + R$.

2. Find $p, q,$ and r in the synthetic division below in which the divisor is $x - 1$.

2	3	q	1
2	p	7	r

3. Find m and n in the synthetic division below in which the divisor is $x + 2$.

2	m	-3	n
	-4		m

4. For each pair of polynomials, divide the first expression by the second. Then express each in the form $P = DQ + R$.

a) $x^3 + 4x^2 + 9x + 6; \quad x - 1$

b) $2x^3 - 6x^2 - 5x + 3; \quad x + 3$

c) $x^4 - 8x^2 + 7; \quad x + 1$

5. Find the remainder on dividing $x^3 - 6x^2 + 3x + 4$ by $x - 2$.
Compare this with the value of $x^3 - 6x^2 + 3x + 4$ when x is replaced by 2.

6. Find the remainder on dividing $11 - x + 3x^2 + x^3$ by $x + 3$.
Compare this with the value of $11 - x + 3x^2 + x^3$ when x is replaced by -3.

Multiple Choice

7. When the polynomial $2a^3 - 7a + 6$ is divided by $a - 4$ the remainder is

A. -94
B. 10
C. 66
D. 106

Numerical Response

8. When the polynomial $3y^3 - 4y^2 + by + 6$ is divided by $y + 2$, the remainder is -40.

The value of b is _______.

(Record your answer in the numerical response box from left to right)

9. Divide $9x^3 + 18x^2 - 13x + 5$ by $3x - 1$ using synthetic division and write the division in the form $P = DQ + R$.

10. Divide $4x^3 + 11x^2 - 14x - 9$ by $4x + 3$ using synthetic division and write the division in the form $P = DQ + R$.

Answer Key

1. **a)** $x^3 - 2x^2 + 6x + 3$ **b)** $x^2 + 6$ **c)** 15 **d)** $x^3 - 2x^2 + 6x + 3 = (x - 2)(x^2 + 6) + 15$

2. $p = 5, q = 2, r = 8$ **3.** $m = 0, n = 10$

4. **a)** $x^3 + 4x^2 + 9x + 6 = (x - 1)(x^2 + 5x + 14) + 20$
b) $2x^3 - 6x^2 - 5x + 3 = (x + 3)(2x^2 - 12x + 31) - 90$
c) $x^4 - 8x^2 + 7 = (x + 1)(x^3 - x^2 - 7x + 7)$

5. both answers are -6 **6.** both answers are 14 **7.** D **8.**

3			

9. $9x^3 + 18x^2 - 13x + 5 = (3x - 1)(3x^2 + 7x - 2) + 3$

10. $4x^3 + 11x^2 - 14x - 9 = (4x + 3)(x^2 + 2x - 5) + 6$

Factoring Polynomial Expressions Lesson #1:
Common Factors

Warm-Up ***Expanding and Factoring***

In the previous unit we were concerned with multiplying polynomial expressions.
In particular we multiplied

i) a monomial by a polynomial eg. $2x(x + 5) =$ ____________

ii) a binomial by a binomial to form a binomial eg. $(x - 5)(x + 5) =$ ____________

iii) a binomial by a binomial to form a trinomial eg. $(x + 1)(x + 3) =$ ____________

$(2x + 3)(x + 4) =$ ____________

In these examples we have **expanded** a product of polynomials to form a sum or difference of monomials.

In this unit we are concerned with the opposite process. We want to write a sum or difference of monomials as a product of polynomials. This process is called **factoring**.

We will be concerned with three major types of factoring, some of which you have met in previous math courses.

i) factoring by removing a common factor. eg. $2x^2 + 10x =$ ____________

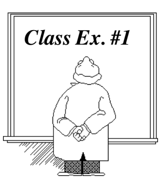
ii) factoring a difference of squares eg. $x^2 - 25 =$ ____________

iii) factoring a trinomial eg. $x^2 + 4x + 3 =$ ____________

$2x^2 + 11x + 12 =$ ____________

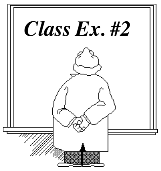
In this unit we will review and extend what you have already learned and introduce a method for factoring trinomials of the form $ax^2 + bx + c$, where $a \neq 1$.

Monomial Common Factors ***Review***

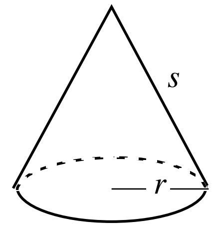


Factor the following polynomials by removing the greatest common factor.

a) $12x - 8y + 16z$ **b)** $9pq + 6pr - 15p$ **c)** $24x^3 - 60x^2$ **d)** $10a^3b^2 + 8ab^3 + 2ab^4$



The surface area of a cone is given by the formula $A = \pi r^2 + \pi rs$ where r is the radius of the base of the cone and s is the slant height.



i) Determine the surface area of a cone, to the nearest 0.01 cm^2, which has slant height 7.4 cm and base radius 2.6 cm.

ii) Write the formula for A in factored form.

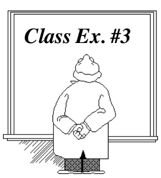
iii) Calculate the surface area of the cone, to the nearest 0.01 cm^2, using the factored form of A.

Monomial Common Factors involving Fractions

In polynomials involving fractional coefficients, it is useful to include a fraction as part of the monomial common factor so that the remaining factor is an integral polynomial.

eg. $\frac{1}{2}x^2 - 3x = \frac{1}{2}x(x - 6)$

Such a technique will prove useful in future math courses.



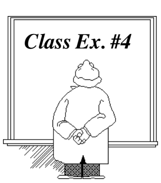
An object is thrown down from a high building. The distance, s metres, travelled by the object in t seconds is given by the formula $s = vt + \frac{1}{2}at^2$.

a) Express s in factored form.

b) Find the distance travelled in 0.1 seconds if $v = 5$ and $a = 9.8$

Binomial Common Factors

In certain circumstances the greatest common factor may be a binomial rather than a monomial. This particular type of factoring will be part of a process for factoring trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, in a later lesson.



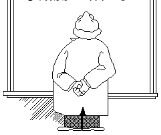
Factor the following polynomials by removing the greatest common factor.

a) $(x + 7)(4x) - (x + 7)(3)$ **b)** $(3y + 2)(5y + 1) + (3y + 2)(4y)$ **c)** $3a(a - 6) - 9(a - 6)$

Factoring by Grouping

Sometimes polynomials in four terms can be factored by removing the greatest common factor from a pair of terms followed by a binomial common factor. This method is called factoring by grouping. The method of grouping is a component of the method used to factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, in a later lesson.

Class Ex. #5



Factor the following polynomials by grouping.

a) $x^2 + 3x + 6x + 18$ **b)** $8x^2 - 2x + 12x - 3$ **c)** $8a^2 - 4a - 10a + 5$

d) $6a^2 - 9a - 2a + 3$ **e)** $pq + pr - sq - sr$ **f)** $5x^2 + 18y^2 - 15xy^2 - 6x$

Complete Assignment Questions #1 - #13

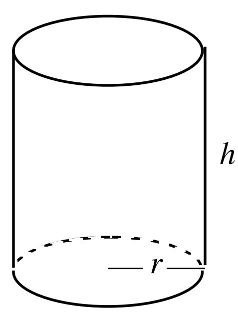
Assignment

1. Factor the following polynomials by removing the greatest common factor.

a) $6m + 9n$ **b)** $xy + y$ **c)** $9ab - 12ac$ **d)** $15x^2 - 10xy - 20xz$

e) $t^3 + t^2$ **f)** $4abc - 2abd + 8abe$ **g)** $14a^2b^2 + 21a^3b^2 - 35a^2b^3$

2. The surface area of a cylinder is given by the formula $A = 2\pi r^2 + 2\pi rh$, where r is the radius of the base and h is the height of the cylinder.



a) Calculate the surface area, to the nearest 0.01 cm^2, of a cylinder which has vertical height 14.5 cm and base diameter 11 cm.

b) Write the formula for A in factored form.

c) Calculate the surface area of the cylinder, to the nearest 0.01 cm^2, using the factored form of A.

3. A square of side $2r$ cm has semicircles drawn externally on each of two opposite sides. Find expressions in factored form for

a) the perimeter of the shape

b) the area of the shape

4. Factor the following polynomials.

a) $\frac{1}{2} + \frac{1}{2}x$

b) $\frac{1}{2}m - \frac{3}{2}$

c) $\frac{3}{4}ah - \frac{1}{2}bh$

d) $\frac{4}{3}\pi r^3 - 4\pi r^2$

5. Factor the following polynomials by removing the greatest common factor.

a) $(x + 5)(3x) - (x + 5)(7)$

b) $(x + 4)(7y) + (x + 4)(2)$

c) $(x - 1)(x) - (x - 1)$

d) $(a + b)(x) - (a + b)(y)$

e) $(a + b)(4x) - (a + b)(8y)$

f) $2a(a + 9) - 4(a + 9)$

g) $7a(b+c) + 21d(b+c)$

h) $5(x + y)(m - n)^2 - 15(x + y)^2(m - n)$

6. Factor the following polynomials by grouping.

a) $x^2 + 2x + 6x + 12$

b) $x^2 + 3x + 15x + 45$

c) $m^2 - 5m + 2m - 10$

d) $a^2 - 9a - 5a + 45$

e) $x^2 - 15x - 4x + 60$

f) $t^2 + 7t - 3t - 21$

7. Factor the following polynomials by grouping.

a) $2x^2 + 2x + 3x + 3$ **b)** $3x^2 + x + 6x + 2$ **c)** $3m^2 + 9m + 5m + 15$

d) $6b^2 - 9b - 4b + 6$ **e)** $2a^2 - 6a - a + 3$ **f)** $5x^2 + 2x - 25x - 10$

g) $16 + 4p + 4p + p^2$ **h)** $15 - 3y - 5y + y^2$ **i)** $a^2 + ax + ay + xy$

8. Factor the following polynomials by grouping.

a) $ab + x^2 - ax - bx$ **b)** $4b^2 + 3a - 12b - ab$ **c)** $4x^2 + 15y^2 - 12xy^2 - 5x$

Multiple Choice

9. $\pi r^3 + 3\pi r$ is equivalent to

A. $3\pi^2 r^4$
B. $3\pi(r^2 + r)$
C. $\pi r(2r + 3)$
D. $\pi r(r^2 + 3)$

10. One factor of $xy - 4xz - 12tz + 3ty$ is

A. $(4t + x)$
B. $(3t - x)$
C. $(y - 4z)$
D. $(3x + t)$

11. How many of the following three statements are true?

i) $p^2 + pq + p = p(p + q)$ **ii)** $\frac{2}{3}\pi r^3 + \pi r^2 h = \frac{2}{3}\pi r^2(r + 2h)$

iii) $t^2 + t^4 + t^6 = t^2(1 + t^2 + t^3)$

A. 0
B. 1
C. 2
D. 3

12. $x(x - 2) + y(2 - x)$ is equivalent to

A. $(x - 2)(x + y)$
B. $(x - 2)(x - y)$
C. $(2 - x)(x - y)$
D. $(2 - x)(x + y)$

Numerical Response

13. The polynomial $2x^2 + 3x + 8x + p$ has a factor of $(2x + 3)$. The value of p is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $3(2m + 3n)$ **b)** $y(x + 1)$ **c)** $3a(3b - 4c)$ **d)** $5x(3x - 2y - 4z)$
e) $t^2(t + 1)$ **f)** $2ab(2c - d + 4e)$ **g)** $7a^2b^2(2 + 3a - 5b)$

2. a) $691.15\ \text{cm}^2$ **b)** $A = 2\pi r(r + h)$ **c)** $691.15\ \text{cm}^2$

3. a) $2r(2 + \pi)$ cm **b)** $r^2(4 + \pi)\ \text{cm}^2$

4. a) $\frac{1}{2}(1 + x)$ **b)** $\frac{1}{2}(m - 3)$ **c)** $\frac{1}{4}h(3a - 2b)$ **d)** $\frac{4}{3}\pi r^2(r - 3)$

5. a) $(x + 5)(3x - 7)$ **b)** $(x + 4)(7y + 2)$ **c)** $(x - 1)^2$ **d)** $(a + b)(x - y)$
e) $4(a + b)(x - 2y)$ **f)** $2(a + 9)(a - 2)$ **g)** $7(b + c)(a + 3d)$ **h)** $5(x + y)(m - n)(m - n - 3x - 3y)$

6. a) $(x + 2)(x + 6)$ **b)** $(x + 3)(x + 15)$ **c)** $(m - 5)(m + 2)$ **d)** $(a - 9)(a - 5)$
e) $(x - 15)(x - 4)$ **f)** $(t + 7)(t - 3)$

7. a) $(x + 1)(2x + 3)$ **b)** $(x + 2)(3x + 1)$ **c)** $(m + 3)(3m + 5)$ **d)** $(2b - 3)(3b - 2)$
e) $(a - 3)(2a - 1)$ **f)** $(x - 5)(5x + 2)$ **g)** $(4 + p)^2$ **h)** $(5 - y)(3 - y)$ **i)** $(a + x)(a + y)$

8. a) $(b - x)(a - x)$ **b)** $(b - 3)(4b - a)$ **c)** $(x - 3y^2)(4x - 5)$

9. D **10.** C **11.** A **12.** B **13.**

1	2		

Factoring Polynomial Expressions Lesson #2:
Difference of Squares

Warm-Up *Review*

In the previous unit we multiplied a binomial by a binomial to form a binomial.

eg. $(x - 5)(x + 5)$ = _____________ $(2x - 3)(2x + 3)$ = _____________

In these examples we have expanded a product of binomials to form a **difference of squares.**

In general $(a - b)(a + b)$ = _____________

In this lesson we are concerned with the opposite process. We want to factor a difference of squares.

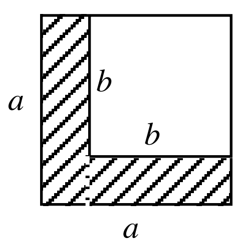
From the above examples we see that

$x^2 - 25$ = ______________ $4x^2 - 9$ = _____________

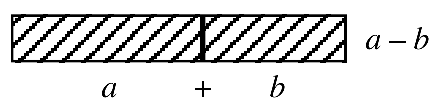
In general $a^2 - b^2$ = ______________

In each of the above cases the left side is a difference of squares and the right side is a product of two factors.

The identity $a^2 - b^2 = (a - b)(a + b)$ can be illustrated in the following diagram.

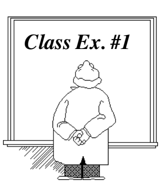


Shaded area = $a^2 - b^2$



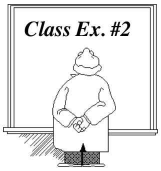
Shaded area = $(a - b)(a + b)$

The shaded area on the left is cut along the dotted line and rearranged to form the diagram on the right.
The shaded area on the left is represented by $a^2 - b^2$ and the shaded area on the right is represented by $(a - b)(a + b)$.



Factor the following polynomials using the method of difference of squares.

a) $a^2 - 100$ **b)** $16t^2 - 49s^2$ **c)** $81 - 25y^2$ **d)** $a^2x^2 - b^2y^2$



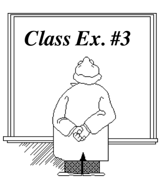
The floor of an international doubles squash court is rectangular with an area of $25a^2 - b^2$ square feet.

a) Write expressions for the length and width of the floor.

b) The perimeter of the floor is 140 feet. Determine the length and width of the floor if the length is 1.8 times the width.

Difference of Squares involving a Common Factor

The first step in factoring any polynomial expression should be to determine if we can remove a common factor.



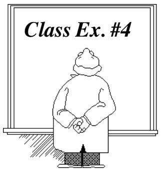
Factor the following polynomials by first removing the greatest common factor.

a) $2a^2 - 50$ **b)** $3x^2 - 12y^2$ **c)** $144p^2q^2 - 4$ **d)** $3x^3 - 27x$

In this example there were two steps in the factoring process - a common factor followed by a difference of squares. If we are asked to factor a polynomial expression it is understood this means to continue factoring until no further factoring is possible. This is sometimes written as “factor completely ...”. The operation “factor” means “factor completely”.

Complete Assignment Questions #1 - #6

More Complex Differences of Squares



Factor completely.

a) $k^4 - 1$ **b)** $80 - 5x^4$ **c)** $2p^5q^4 - 162pt^4$

Extension

Factor.

a) $(a + b)^2 - c^2$ **b)** $x^2 - (y - z)^2$ **c)** $9(p - q)^2 - 4(p + q)^2$

Complete Assignment Questions #7 - #14

Assignment

1. Factor the following polynomials all of which contain a difference of squares.

a) $m^2 - n^2$ **b)** $c^2 - 7^2$ **c)** $1 - k^2$ **d)** $g^2 - 64h^2$

e) $25x^2 - 144$ **f)** $16a^2 - 9b^2$ **g)** $4x^2 - z^2$ **h)** $121a^2 - 36b^2$

2. Factor.

a) $a^2b^2 - 9$ **b)** $c^2 - d^2e^2$ **c)** $100x^2 - y^2z^2$ **d)** $p^2q^2 - r^2s^2$

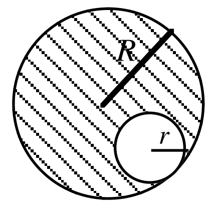
e) $25x^2y^2 - 1$ **f)** $c^2d^2 - 4f^2$ **g)** $4x^2a^2 - 49z^2t^2$ **h)** $16a^2c^2 - 225b^2d^2$

3. Factor completely.

a) $8x^2 - 32$ **b)** $4a^2 - 100y^2$ **c)** $3t^2 - 27s^2$ **d)** $7x^2 - 7y^2$

e) $9a^2b^2 - 36$ **f)** $8 - 50p^2q^2$ **g)** $xy^2 - x^3$ **h)** $20a^2b^2 - 5a^4b^4$

4. The diagram shows a circle of radius R with a circle of radius r removed.



a) Write an expression for the shaded area.

b) Write the expression in a) in factored form.

c) Determine the shaded area (as a multiple of π) if $R = 8.5$ and $r = 1.5$
Do not use a calculator.

5. The expression $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$ occurs in physics.

a) Write the expression in factored form.

b) Determine the value of the expression when $m = 10$, $v = 75$, and $u = 25$.
Do not use a calculator.

6. p and q are integers such that $p^2 - q^2 = 72$. Use the ideas in this lesson to find three different sets of values for p and q which satisfy the given equation.

7. Factor

a) $x^4 - y^4$ **b)** $a^4 - 256b^4$ **c)** $2z^4 - 162$

d) $48x^4 - 3y^4$ **e)** $9a^4b^4 - 144c^4d^4$ **f)** $z^8 - 1$

8. Consider the following in which each letter represents a whole number.

$64x^2 - y^2 = (Hx - y)(Hx + y)$

$16x^4 - y^4 = (Ix - y)(Ix + y)(Cx^2 + y^2)$

$7x^2 - 252y^2 = P(x - Ey)(x + Ey)$

$Lx^2 - Ny^2 = (3x - 5y)(Sx + Ay)$

Determine the value of each letter and hence name the country represented by the following code.

(4) (8) (2) (9) (6)

_ _ _ _ _

Multiple Choice

9. One factor of $16 - 4m^2$ is

A. $4 - m$
B. $8 - 2m$
C. $4 + m$
D. $2 + m$

10. One factor of $y^4 - 81$ is

A. $y + 9$
B. $y + 3$
C. $y^2 - 3$
D. $y^2 + 3$

11. Given that $x^2 - y^2 = 45$ and $x + y = 9$, the value of x is

A. 2
B. 5
C. 7
D. impossible to determine

Extension Questions.

12. Factor

a) $(m + n)^2 - p^2$

b) $x^2 - (y + z)^2$

c) $4a^2 - (p - q)^2$

d) $9(m + n)^2 - 4n^2$

e) $1 - (y - z)^2$

f) $81a^4 - 16(a - b)^4$

g) $(a - b)^2 - (x + y)^2$

h) $(x + 7)^2 - (x + 4)^2$

i) $(3c - 2d)^2 - (c - 4d)^2$

Multiple Choice

13. One factor of $(9a - 3b)^2 - (4a + 3b)^2$ is

A. $4a$
B. $5a$
C. $9a$
D. $13a$

Numerical Response

14. When factored, the expression $(2a + 3)^2 - (2a - 3)^2$ can be written in the form ka where k is a whole number. The value of k is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $(m - n)(m + n)$ **b)** $(c - 7)(c + 7)$ **c)** $(1 - k)(1 + k)$
d) $(g - 8h)(g + 8h)$ **e)** $(5x - 12)(5x + 12)$ **f)** $(4a - 3b)(4a + 3b)$
g) $(2x - z)(2x + z)$ **h)** $(11a - 6b)(11a + 6b)$

2. **a)** $(ab - 3)(ab + 3)$ **b)** $(c - de)(c + de)$ **c)** $(10x - yz)(10x + yz)$
d) $(pq - rs)(pq + rs)$ **e)** $(5xy - 1)(5xy + 1)$ **f)** $(cd - 2f)(cd + 2f)$
g) $(2xa - 7zt)(2xa + 7zt)$ **h)** $(4ac - 15bd)(4ac + 15bd)$

3. **a)** $8(x - 2)(x + 2)$ **b)** $4(a - 5y)(a + 5y)$ **c)** $3(t - 3s)(t + 3s)$
d) $7(x - y)(x + y)$ **e)** $9(ab - 2)(ab + 2)$ **f)** $2(2 - 5pq)(2 + 5pq)$
g) $x(y - x)(y + x)$ **h)** $5a^2b^2(2 - ab)(2 + ab)$

4. **a)** $A = \pi R^2 - \pi r^2$ **b)** $\pi(R - r)(R + r)$ **c)** 70π

5. **a)** $\frac{1}{2}m(v - u)(v + u)$ **b)** 25 000

6. $p = 19, q = 17$ $p = 11, q = 7$ $p = 9, q = 3$

7. **a)** $(x - y)(x + y)(x^2 + y^2)$ **b)** $(a - 4b)(a + 4b)(a^2 + 16b^2)$
c) $2(z - 3)(z + 3)(z^2 + 9)$ **d)** $3(2x - y)(2x + y)(4x^2 + y^2)$
e) $9(ab - 2cd)(ab + 2cd)(a^2b^2 + 4c^2d^2)$ **f)** $(z - 1)(z + 1)(z^2 + 1)(z^4 + 1)$

8. CHILE **9.** D **10.** B **11.** C

12. **a)** $(m + n - p)(m + n + p)$ **b)** $(x - y - z)(x + y + z)$ **c)** $(2a - p + q)(2a + p - q)$
d) $(3m + n)(3m + 5n)$ **e)** $(1 - y + z)(1 + y - z)$ **f)** $(a + 2b)(5a - 2b)[9a^2 + 4(a - b)^2]$
g) $(a - b - x - y)(a - b + x + y)$ **h)** $3(2x + 11)$ **i)** $4(c + d)(2c - 3d)$

13. D **14.**

2	4		

Factoring Polynomial Expressions Lesson #3:
Factoring Trinomials of the form $x^2 + bx + c$

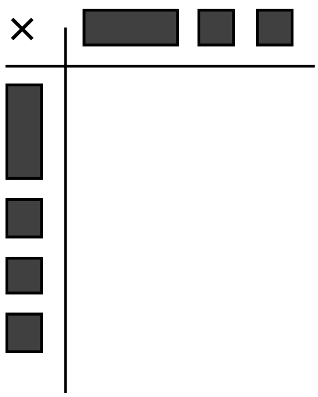
Warm-Up *Review - Algebra Tiles*

Algebra tiles can be used to represent polynomial expressions.
In this workbook we will adopt the following convention.

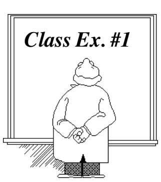
x^2 $\quad x \quad 1 \quad -x^2 \quad -x \quad -1$

Algebra tiles can be used to represent the multiplication of two binomials.

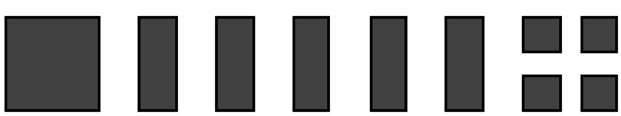
Complete the algebra tile diagram to show the product $(x + 2)(x + 3)$



Factoring Trinomials using Algebra Tiles



a) Write a polynomial expression for the group of algebra tiles shown.



b) Arrange the algebra tiles into a rectangle.

c) State the length and width of the rectangle.

d) Use the results above to express the polynomial in factored form.

Factoring Trinomials by Inspection

Factor the following from the previous page.

$x^2 + 5x + 6 =$ ____________________

$x^2 + 5x + 4 =$ ____________________

State the connection between the numbers in the trinomial and the numbers in the binomials.

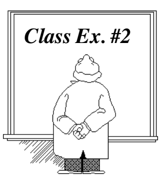
The connection can be explained by considering the product $(x + m)(x + n)$. Using FOIL, $(x + m)(x + n) = x^2 + nx + mx + mn$ which can be written in the form $x^2 + (m + n)x + mn$.

In order to factor $x^2 + bx + c$, we need to find two integers which have a <u>product equal to c</u> and a <u>sum equal to b</u>. If no two such integers exist then the polynomial cannot be factored.

If the value of c is positive, then the two integers have the same sign
- i) if b is positive, the two integers are positive
- ii) if b is negative, the two integers are negative

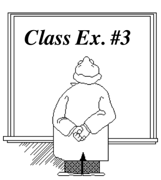
If the value of c is negative, then the two integers have opposite signs.

This process is called the **method of inspection**.



Factor where possible.

a) $x^2 + 7x + 12$ **b)** $x^2 - 11x + 10$ **c)** $x^2 - x - 30$ **d)** $x^2 + 3x + 4$



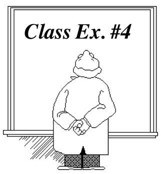
Factor the polynomial expressions by first removing a common factor.

a) $4x^2 - 32x + 48$ **b)** $3x^3 + 15x^2 - 18x$

Complete Assignment Questions #1 - #6

Factoring Trinomials of the form $f^2 + bf + c$ *where* f **is** *a monomial.*

The method of inspection can be extended to factor polynomial expressions of the form $f^2 + bf + c$ where f itself is a polynomial. In this section we will restrict f to be a monomial and in an extension to the lesson we will consider f to be a binomial.



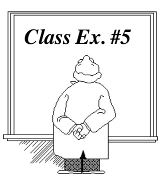
Factor completely.

a) $a^4 - 5a^2 - 14$ **b)** $x^4 + 4x^2 - 5$ **c)** $x^6 - 9x^3 + 14$

Complete Assignment Questions #7 - #13

Extension

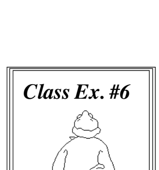
Factoring Trinomials of the form $f^2 + bf + c$ *where* f **is** *a binomial.*



Factor.

a) $(x + 3)^2 - 2(x + 3) - 24$ **b)** $(3x - 1)^2 - 12(3x - 1) + 36$

Factoring Trinomials of the form $x^2 + bxy + cy^2$



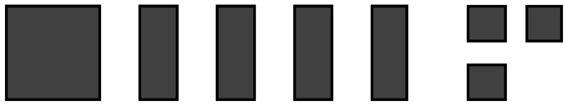
Factor.

a) $x^2 + 13xy + 30y^2$ **b)** $x^2 + 71xy - 72y^2$

Complete Assignment Questions #14 - #16

Assignment

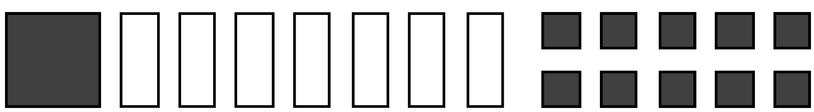
1. a) Write a polynomial expression for the group of algebra tiles shown.



b) Arrange the algebra tiles into a rectangle.

c) State the length and width of the rectangle and hence express the polynomial in factored form.

2. a) Write a polynomial expression for the group of algebra tiles shown.



b) Arrange the algebra tiles into a rectangle.

c) State the length and width of the rectangle and hence express the polynomial in factored form.

3. Factor where possible.

a) $x^2 + 10x + 16$ **b)** $x^2 - 11x + 18$ **c)** $x^2 - 2x - 8$ **d)** $x^2 + 3x - 18$

e) $x^2 - 4x + 12$ **f)** $x^2 - 4x - 12$ **g)** $x^2 - 10x + 25$ **h)** $x^2 + x - 20$

i) $m^2 + 21m + 38$ **j)** $a^2 - 17a + 42$ **k)** $p^2 - 10p - 9$ **l)** $p^2 - 9p - 10$

4. Factor.

a) $2x^2 + 6x + 4$ **b)** $4x^2 - 28x - 32$ **c)** $-2a^2 + 2a + 220$

d) $5x^2 - 20x + 15$ **e)** $ax^2 - 4ax - 45a$ **f)** $2x^3 + 2x^2 - 40x$

5. Consider the following in which the each letter represents a whole number.

$x^2 + 4x - 5 = (x + A)(x - O)$ $x^2 - 3x - 54 = (x - E)(x + I)$

$x^3 + 2x^2 - 8x = x(x - Y)(x + P)$ $3x^2 - 48x + 192 = T(x - R)^2$

$-5x^2 + 20x + 105 = -5(x + T)(x - H)$

Determine the value of each letter and hence name the fictional character represented by the following code.

(7) (5) (8) (8) (2) (4) (1) (3) (3) (9) (8)

_ _ _ _ _ _ _ _ _ _ _

6. A volleyball court has an area of $x^2 + 15x + 36$ square metres.

a) Factor $x^2 + 15x + 36$ to find binomials that represent the length and width of the court.

b) If $x = 3$, determine the length and width of the court.

7. Factor completely.

a) $x^4 + 9x^2 + 20$

b) $x^4 - 9x^2 + 20$

c) $a^4 - 17a^2 + 16$

d) $t^6 - 4t^3 - 21$

e) $3x^4 + 9x^2 - 30$

f) $2x^5 - 16x^3 + 32x$

Multiple Choice

8. One factor of $-m^3 - m^2 + 6m$ is

A. $m - 2$
B. $m + 2$
C. $m - 3$
D. $m - 6$

9. One factor of $x^4 - 16x^2 + 15$ is

A. $x + 1$
B. $x^2 + 15$
C. $x + 15$
D. $x - 15$

10. The expression $t^2 + kt + 12$ **cannot** be factored if k has the value

A. -13
B. -8
C. 7
D. 11

11. The expression $x^2 - 4x + c$ **cannot** be factored if c has the value

A. -5
B. 0
C. 4
D. 5

Numerical Response

12. The largest value of b for which $x^2 + bx + 32$ can be factored over the integers is _____

(Record your answer in the numerical response box from left to right)

13. The polynomial expressions $a^3 - 2a^2 - 15a$ and $3a^2 - 27$ have a binomial factor in common. If this factor is $a + k$, where $k \in W$, then k is equal to _____.

(Record your answer in the numerical response box from left to right)

Extension Questions.

14. Factor

a) $(5x + 1)^2 - 8(5x + 1) + 15$

b) $(m - 3)^2 - 4(m - 3) - 21$

c) $(a - 6)^2 - 20(a - 6) + 100$

d) $(2x - 3)^2 - 6(2x - 3) - 27$

15. Factor

a) $x^2 + 18xy + 45y^2$ **b)** $x^2 + 10xy - 24y^2$ **c)** $a^2 - 12ab + 36b^2$

d) $p^2 - 12pq + 11q^2$ **e)** $x^2 + xy - 72y^2$ **f)** $x^2 + 71xy - 72y^2$

Numerical Response

16. The polynomial expression $(x^2 - x)^2 - (x + 8)^2$ can be factored to the form $(x - A)(x + B)(x^2 + C)$ where $A, B, C \in W$. The value of the product ABC is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $x^2 + 4x + 3$ **c)** $x + 3$, $x + 1$, $x^2 + 4x + 3 = (x + 3)(x + 1)$

2. a) $x^2 - 7x + 10$ **c)** $x - 2, x - 5$, $x^2 - 7x + 10 = (x - 2)(x - 5)$

3. a) $(x + 8)(x + 2)$ **b)** $(x - 9)(x - 2)$ **c)** $(x + 2)(x - 4)$ **d)** $(x + 6)(x - 3)$
e) not possible **f)** $(x - 6)(x + 2)$ **g)** $(x - 5)^2$ **h)** $(x + 5)(x - 4)$
i) $(m + 2)(m + 19)$ **j)** $(a - 14)(a - 3)$ **k)** not possible **l)** $(p - 10)(p + 1)$

4. a) $2(x + 2)(x + 1)$ **b)** $4(x - 8)(x + 1)$ **c)** $-2(a - 11)(a + 10)$ **d)** $5(x - 3)(x - 1)$
e) $a(x - 9)(x + 5)$ **f)** $2x(x + 5)(x - 4)$

5. HARRY POTTER **6. a)** $(x + 12)(x + 3)$ **b)** 15m, 6m

7. a) $(x^2 + 5)(x^2 + 4)$ **b)** $(x - 2)(x + 2)(x^2 - 5)$ **c)** $(a - 1)(a + 1)(a - 4)(a + 4)$
d) $(t^3 - 7)(t^3 + 3)$ **e)** $3(x^2 + 5)(x^2 - 2)$ **f)** $2x(x - 2)^2(x + 2)^2$

8. A **9.** A **10.** D **11.** D **12.**

3	3		

13.

3			

14. a) $(5x - 4)(5x - 2)$ **b)** $m(m - 10)$ **c)** $(a - 16)^2$ **d)** $4x(x - 6)$

15. a) $(x + 15y)(x + 3y)$ **b)** $(x - 2y)(x + 12y)$ **c)** $(a - 6b)^2$ **d)** $(p - q)(p - 11q)$
e) $(x - 8y)(x + 9y)$ **f)** $(x + 72y)(x - y)$

16.

6	4		

Factoring Polynomial Expressions Lesson #4:
Factoring $ax^2 + bx + c$ (where $a \neq 1$)

Warm-Up #1

In this lesson we learn a method for factoring polynomials of the form $ax^2 + bx + c$ where $a \neq 1$. This is one of the most important processes in this course and has a large number of applications in almost every math course you will meet in high school or further education.

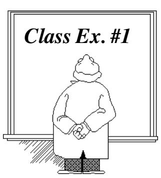
Complete the following

$(2x + 1)(3x + 4)$ = ________________ so ________________ factors to $(2x + 1)(3x + 4)$

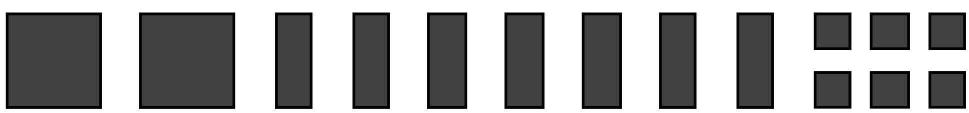
$(3x - 2)(4x + 3)$ = ________________ so ________________ factors to $(3x - 2)(4x + 3)$

Consider the following problem: "What are the factors of $2x^2 + 7x + 6$?"
We need to find two binomials whose product is $2x^2 + 7x + 6$. The first method we will consider is to use algebra tiles.

Factoring $ax^2 + bx + c$ using Algebra Tiles

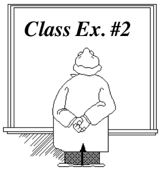


a) Write a polynomial expression for the group of algebra tiles shown.



b) Arrange the algebra tiles into a rectangle and state the length and width of the rectangle.

c) Use the algebra tile diagram to express the polynomial in factored form.



Factor $5x^2 + 7x + 2$ using algebra tiles.

Complete Assignment Questions #1 - #2

Factoring using algebra tiles will work for all trinomials of the form $ax^2 + bx + c$ which have binomial factors. However it can get rather tedious if the values of a, b, c are large.

The second method we will consider uses as part of its process factoring by grouping and we will review this concept first.

Warm-Up #2

Review - Factoring by grouping

Factor the following polynomials by grouping.

i) $6x^2 + 3x + 8x + 4$ **ii)** $12x^2 + 9x - 8x - 6$

Factoring $ax^2 + bx + c$ using the Method of Decomposition

In Warm-Up #2i) we factored

$6x^2 + 3x + 8x + 4$ or $6x^2 + 11x + 4$ to get $(2x + 1)(3x + 4)$.

In order to factor $6x^2 + 11x + 4$ we must first split $11x$ into $3x$ and $8x$ and then group.

But how do we know to split $11x$ into $3x$ and $8x$ and not $2x$ and $9x$ or $5x$ and $6x$ etc.? We will provide the answer to this on the next page.

In Warm-Up #2ii) we factored

$12x^2 + 9x - 8x - 6$ or $12x^2 + x - 6$ to get $(4x + 3)(3x - 2)$.

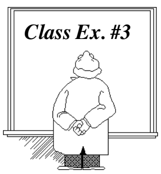
In order to factor $12x^2 + x - 6$ we must first split $1x$ into $9x$ and $-8x$ and then group.

But how do we know to split $1x$ into $9x$ and $-8x$ and not $5x$ and $-4x$ or $-3x$ and $2x$?

In 2i) how are the numbers 8 and 3 connected to the value of a (ie 6), the value of b (ie 11) and the value of c (ie 4)?

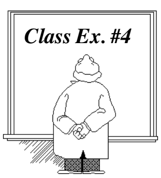
In 2ii) how are the numbers 9 and -8 connected to the value of a (ie 12), the value of b (ie 1) and the value of c (ie -6)?

The method of factoring $ax^2 + bx + c$ by splitting the value of b into two integers whose product is ac and whose sum is b is called the **method of decomposition**.



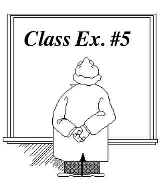
Factor using the method of decomposition and compare the answers with Class Examples #1 and #2.

a) $2x^2 + 7x + 6$ **b)** $5x^2 + 7x + 2$



Factor.

a) $6x^2 + 17x - 3$ **b)** $3n^2 - 2n - 8$ **c)** $12x^2 - 8x + 1$



Factor.

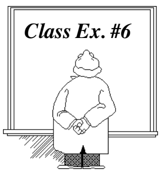
a) $15 - 7y - 2y^2$ **b)** $15k^2 + 5k - 10$

Complete Assignment Questions #3 - #12

Extension

Factoring Trinomials of the form $ax^2 + bxy + cy^2$

The method of decomposition can be applied to trinomials of the form $ax^2 + bxy + cy^2$.



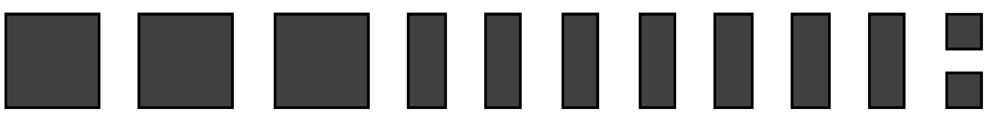
Factor.

a) $2x^2 - 5xy + 2y^2$ **b)** $2n^2 - 7nm - 15m^2$

Complete Assignment Question #13

Assignment

1. a) Write a polynomial expression for the group of algebra tiles shown.



b) Arrange the algebra tiles into a rectangle and state the length and width of the rectangle.

c) Use the algebra tile diagram to express the polynomial in factored form.

2. Factor the following expressions using algebra tiles.

a) $2x^2 + 5x + 3$

b) $2x^2 + 7x + 3$

c) $6x^2 + 7x + 2$

d) $4x^2 + 13x + 3$

3. Factor the following expressions.

a) $10x^2 + 17x + 3$

b) $9x^2 + 6x + 1$

c) $3x^2 + 14x + 15$

d) $3a^2 - 23a - 8$

e) $3a^2 + a - 2$

f) $5x^2 - 23x - 10$

g) $2p^2 - 19p + 9$

h) $6x^2 - 13x + 6$

4. Factor.

a) $6x^2 + 5x - 6$

b) $2x^2 + x - 1$

c) $3x^2 - 2x - 1$

d) $8y^2 + 2y - 3$

e) $9t^2 - 24t + 16$

f) $12m^2 - 11m - 5$

g) $12p^2 + 13p - 4$

h) $9x^2 - x - 10$

5. A rectangular garden has an area of $2a^2 + 3a - 5$ m^2.

a) Write the area as the product of two binomials.

b) The garden is to be completely enclosed by a path 1m wide.
Find and simplify an expression for the area of the path.

c) The path is poured concrete to a depth of 12 cm.
Calculate the volume (in m^3) of concrete used if $a = 7$.

6. Factor the following expressions.

a) $12 + 8x + x^2$

b) $6 - 7x - 20x^2$

c) $3 + a - 10a^2$

d) $10a^2 + 25a - 15$

e) $12z^2 + 66z + 30$

f) $4x^3 - 7x^2 - 2x$

7. Consider the following in which each letter represents a whole number.

$10x^2 + 13x - 3 = (Ax + M)(Cx - P)$

$7x^2 + 64x + 9 = (x + S)(Ox + P)$

$24x^2 - 90x + 54 = B(x - M)(Kx - M)$

$64x^2 - 1 = (Lx - 1)(Lx + 1)$

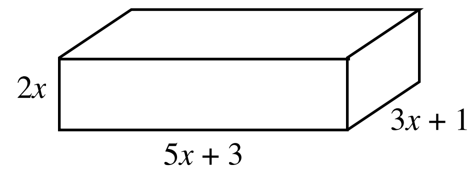
Determine the value of each letter and hence name the place in Canada represented by the following code.

(4)	(2)	(3)	(8)	(7)	(7)	(1)	(9)	(6)	(5)
_	_	_	_	_	_	_	_	_	_

8. A tank in the shape of a rectangular prism has dimensions $2x$, $3x + 1$, and $5x + 3$ metres.

A wooden block is dropped into the tank and the tank is then filled with water.

The volume of water is $22x^3 + 18x^2 + 3x$ cubic metres.



a) Determine a polynomial expression for the volume of the wooden block.

b) Factor the expression in a) to determine the dimensions of the wooden block.

9. Given that $(\sin x)^2$ is written as $\sin^2 x$ and $(\cos x)^2$ is written as $\cos^2 x$ factor

a) $6 \sin^2 x + \sin x - 2$

b) $4 \cos^2 x - 7 \cos x + 3$

Multiple Choice

10. One factor of $12x^2 + 10x - 8$ is

A. $3x + 4$
B. $3x - 4$
C. $2x + 1$
D. $6x - 1$

11. The polynomials $4x^2 + 8x - 5$ and $12x^2 - 3$ have in common a factor of

A. $4x + 1$
B. $4x - 1$
C. $2x + 1$
D. $2x - 1$

Numerical Response

12. The factored form of $3x^2 + 19x - 14$ is $(x + a)(bx + c)$ where a, b and c are integers. The value of b^c, to the nearest hundredth, is ________ .

(Record your answer in the numerical response box from left to right)

Extension Question.

13. Factor

a) $8x^2 + 22xy + 5y^2$

b) $6x^2 + 11xy - 7y^2$

c) $4a^2 - 9ab - 9b^2$

d) $2m^2 - 19mn + 9n^2$

e) $9x^2 + xy - 10y^2$

f) $8x^2 + 7xy - 15y^2$

Answer Key

1. **a)** $3x^2 + 7x + 2$ **b)** $3x + 1, x + 2$ **c)** $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

2. **a)** $(2x + 3)(x + 1)$ **b)** $(2x + 1)(x + 3)$ **c)** $(3x + 2)(2x + 1)$ **d)** $(4x + 1)(x + 3)$

3. **a)** $(5x + 1)(2x + 3)$ **b)** $(3x + 1)^2$ **c)** $(3x + 5)(x + 3)$ **d)** $(3a + 1)(a - 8)$
e) $(3a - 2)(a + 1)$ **f)** $(5x + 2)(x - 5)$ **g)** $(2p - 1)(p - 9)$ **h)** $(3x - 2)(2x - 3)$

4. **a)** $(3x - 2)(2x + 3)$ **b)** $(2x - 1)(x + 1)$ **c)** $(3x + 1)(x - 1)$ **d)** $(2y - 1)(4y + 3)$
e) $(3t - 4)^2$ **f)** $(3m + 1)(4m - 5)$ **g)** $(4p - 1)(3p + 4)$ **h)** $(9x - 10)(x + 1)$

5. **a)** $(2a + 5)(a - 1)$ **b)** $6a + 12$ m^2 **c)** 6.48 m^3

6. **a)** $(6 + x)(2 + x)$ **b)** $(3 + 4x)(2 - 5x)$ **c)** $(3 - 5a)(1 + 2a)$
d) $5(2a - 1)(a + 3)$ **e)** $6(2z + 1)(z + 5)$ **f)** $x(4x + 1)(x - 2)$

7. KAMLOOPS BC **8.** **a)** $8x^3 + 10x^2 + 3x$ m^3 **b)** $4x + 3$, $2x + 1$, and x m

9. **a)** $(3 \sin x + 2)(2 \sin x - 1)$ **b)** $(4 \cos x - 3)(\cos x - 1)$ **10.** A **11.** D

12.

0	.	1	1

13. **a)** $(2x + 5y)(4x + y)$ **b)** $(2x - y)(3x + 7y)$ **c)** $(4a + 3b)(a - 3b)$
d) $(2m - n)(m - 9n)$ **e)** $(9x + 10y)(x - y)$ **f)** $(8x + 15y)(x - y)$

Factoring Polynomial Expressions Lesson #5:
Further Factoring

Perfect Square Trinomials

A **perfect square trinomial** is formed from the product of two identical binomials. Perfect square trinomials can be factored by considering the pattern displayed when squaring binomials.

Complete the following : $(p+q)^2 =$ ________________ $(p-q)^2 =$ _______________

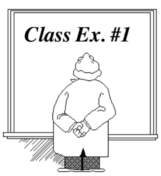
From the above we can see that :

. The first term in the trinomial is the square of the __________ term in the binomial.

. The last term in the trinomial is the square of the ___________term in the binomial.

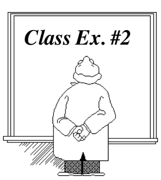
. The middle term in the trinomial is ___________the _______________of the first and last terms in the binomial.

In a perfect square trinomial, eg $x^2 + 10x + 25$, the **first and last terms must be perfect squares** and the **middle term must be twice the product of the square roots of the first and last terms.**



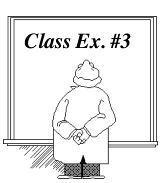
Which of the following are perfect square trinomials?

a) $a^2 + 4a + 4$ **b)** $x^2 - 9x + 6$ **c)** $4x^2 - 36x + 81$ **d)** $y^2 + 8y - 16$



Fill in the blank so that each of the following is a perfect square trinomial

a) $x^2 +$ ___ $+100$ **b)** $x^2 -$ ___ $+ 100$ **c)** $25x^2 +$ ___ $+ 36$ **d)** $9m^2 + 24m +$ ____



Factor

a) $49x^2 - 14x + 1$ **b)** $16 + 40x + 25x^2$ **c)** $\frac{1}{9}a^2 - 2ab + 9b^2$

Complete Assignment Questions #1 - #3

Factoring Trinomials of the form $af^2 + bf + c$ where f is a monomial

The method of decomposition can be extended to factor polynomial expressions of the form $af^2 + bf + c$ where f itself is a polynomial. In this section we will restrict f to be a monomial and in an extension to the lesson we will consider f to be a binomial.

Class Ex. #4

Factor.

a) $2a^2b^2 - 9ab + 9$ **b)** $4x^4 - 5x^2 - 6$ **c)** $9y^{10} - 12y^5 + 4$

Class Ex. #5

Factor completely $8x^4 + 10x^2 - 3$.

Further Applications of Decomposition

A polynomial expression of degree 3 is known as a cubic polynomial. When it is factored, the factors may be linear (degree 1, eg. $2x - 5$) or quadratic (degree 2, eg. $x^2 - 3x + 1$)

Class Ex. #6

Consider the polynomial expression $6x^3 + 23x^2 - 10x - 75$.

a) Show that $6x^3 + 23x^2 - 10x - 75$ is divisible by $x + 3$.

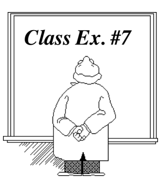
b) Use the result in a) write $6x^3 + 23x^2 - 10x - 75$ as the product of one linear and one quadratic factor.

c) Write $6x^3 + 23x^2 - 10x - 75$ as the product of three linear factors.

Complete Assignment Questions #4 - #12

Extension

Factoring Trinomials of the form $f^2 + bf + c$ where f is a binomial



Factor.

a) $7(x - 3)^2 - 4(x - 3) - 3$

b) $9(a + 4)^2 + (a + 4) - 10$

Complete Assignment Question #13

Assignment

1. Which of the following are perfect square trinomials?

a) $a^2 + 12a + 36$ **b)** $x^2 - 25x + 50$ **c)** $4x^2 - 4x + 1$ **d)** $16y^2 + 32y + 16$

e) $a^2 + 9a + 9$ **f)** $25x^2 - 90x + 81$ **g)** $1 - 16x + 64x^2$ **h)** $y^2 + 20y - 100$

2. Fill in the blank so that each of the following is a perfect square trinomial.

a) $x^2 +$ ___ $+ 49$ **b)** $x^2 -$ ___ $+ 144$ **c)** $9x^2 +$ ___ $+ 36$ **d)** $4m^2 + 24m +$ ___

e) $\frac{1}{4}a^2 +$ ___ $+1$ **f)** $225x^2 -$ ___ $+ 16$ **g)** $100x^2 +$ ___ $+ y^2$ **h)** ___ $- 30y + 9y^2$

3. Factor.

a) $16x^2 - 8x + 1$ **b)** $36 + 60x + 25x^2$ **c)** $4a^2 - 12ab + 9b^2$

d) $4x^2 - 44x + 121$ **e)** $5x^2 + 10x + 5$ **f)** $\frac{4}{9}x^2 + \frac{2}{9}x + \frac{1}{36}$

4. Factor completely.

a) $4x^2y^2 - xy - 14$ **b)** $4\pi^2r^2 - 9\pi r - 9$

c) $6x^4 + 11x^2 + 5$ **d)** $2a^4 - 5a^2 + 2$

e) $5p^6 - 8p^3 - 4$

f) $16x^4 + 8x^2 - 3$

g) $4 - 9t^2 - 9t^4$

h) $4x^5 - 50x^3 + 126x$

5. Factor the polynomial expression $16a^8 - 65a^4 + 4$.

6. a) Show that $6x^3 - 17x^2 - 31x + 12$ is divisible by $x - 4$.

b) Use the result in a) write $6x^3 - 17x^2 - 31x + 12$ as the product of one linear and one quadratic factor.

c) Write $6x^3 - 17x^2 - 31x + 12$ as the product of three linear factors.

7. The volume of a rectangular prism is $12x^3 - 49x^2 + 3x + 4$ cubic metres and the height is $4x + 1$ metres. Determine binomial expressions for the length and width of the rectangular prism.

Multiple Choice

8. When factored completely, the polynomial $k^4 + 16 - 17k^2$ is equal to

A. $(k^2 - 1)(k^2 - 16)$
B. $(k^2 + 1)(k^2 + 16)$
C. $k^2(k + 1)(k + 16)$
D. $(k + 1)(k - 1)(k + 4)(k - 4)$

9. From the expressions below, the one which does **not** represent a perfect square trinomial is

A. $x^2 - 14x + 49$
B. $144 + 24x + x^2$
C. $4x^2 - 12x + 36$
D. $9x^4 + 30x^2 + 25$

10. The volume of a rectangular box is $27a^3 + 63a^2 + 42a + 8$ cm^3. The length is greater than the width which is greater than the height. If the height is $3a + 1$ cm, then the length, in cm, is

A. $3a + 2$
B. $3a + 4$
C. $3a + 6$
D. $3a + 8$

Numerical Response

11. The polynomial expression $\frac{1}{16}x^2 + \frac{1}{3}x + \frac{4}{9}$ can be written in the form $(Ax + B)^2$. The value of the product AB, to the nearest one hundredth, is _____.

(Record your answer in the numerical response box from left to right)

12. Triangle PQR is right angled at P. The area of the triangle is $\frac{3}{2}x^2 + 10x + 16$ cm^2, where x is a positive integer. Given that the length of PQ is 12 cm more than the length of PR, the length of QR, to the nearest tenth of a cm, is _____.

(Record your answer in the numerical response box from left to right)

Extension Questions.

13. Factor.

a) $4(3x + 1)^2 - 5(3x + 1) + 1$

b) $6(x - 4)^2 - (x - 4) - 2$

c) $4(a-b)^2 - 40(a-b) + 100$

d) $5(2-3x)^2 - 28(2-3x) + 15$

Answer Key

1. **a)**, **c)**, **d)**, **f)**, **g)** are all perfect square trinomials.

2. **a)** $14x$ **b)** $24x$ **c)** $36x$ **d)** 36 **e)** a **f)** $120x$ **g)** $20xy$ **h)** 25

3. **a)** $(4x-1)^2$ **b)** $(6+5x)^2$ **c)** $(2a-3b)^2$ **d)** $(2x-11)^2$ **e)** $5(x+1)^2$ **f)** $\left(\frac{2}{3}x + \frac{1}{6}\right)^2$

4. **a)** $(4xy+7)(xy-2)$ **b)** $(4\pi r+3)(\pi r-3)$ **c)** $(6x^2+5)(x^2+1)$
d) $(2a^2-1)(a^2-2)$ **e)** $(5p^3+2)(p^3-2)$ **f)** $(2x-1)(2x+1)(4x^2+3)$
g) $(4+3t^2)(1-3t^2)$ **h)** $2x(x-3)(x+3)(2x^2-7)$

5. $(2a-1)(2a+1)(a^2-2)(a^2+2)(4a^2+1)$

6. **b)** $(x-4)(6x^2+7x-3)$ **c)** $(x-4)(3x-1)(2x+3)$

7. $3x-1,\ x-4$ **8.** D **9.** C **10.** B

11.

0	.	1	7

12.

2	1	.	5

13. **a)** $9x(4x+1)$ **b)** $(2x-7)(3x-14)$
c) $4(a-b-5)^2$ **d)** $-3(7-15x)(1+x)$ or $3(15x-7)(x+1)$

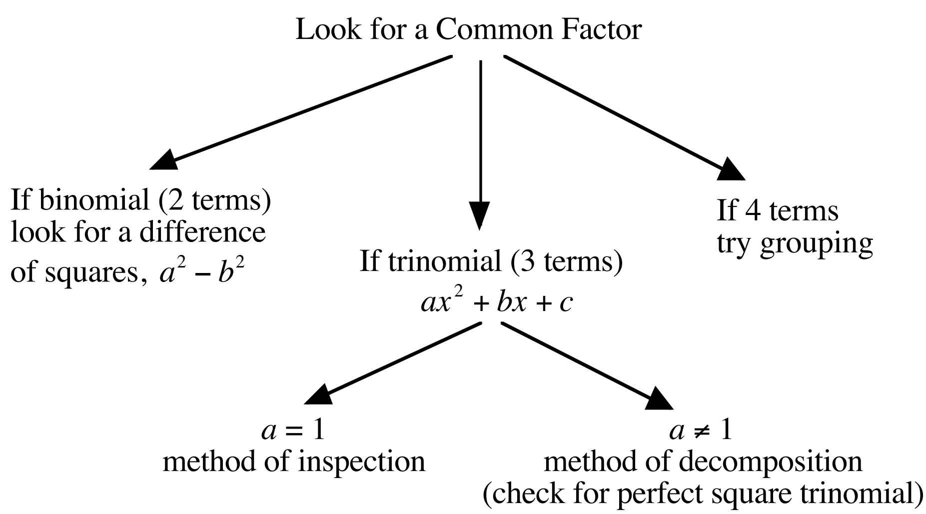
Factoring Polynomial Expressions Lesson #6:
Factoring Review

Guidelines for Factoring a Polynomial Expression

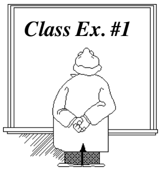
If we are asked to factor a polynomial expression, the following guidelines should help us to determine the best method.

1. Look for a common factor. If there is one, take out the common factor and look for further factoring.
2. If there is a binomial expression, look for a difference of squares.
3. If there is a trinomial expression of the form $x^2 + bx + c$, look for factoring by inspection.
4. If there is a trinomial expression of the form $ax^2 + bx + c$, look for factoring by decomposition. Watch out for perfect square trinomials.
5. If there is a polynomial with four terms, look for factoring by grouping.
6. After factoring, check to see if further factoring is possible.

The guidelines can be shown in a flowchart.



Note
Always check to see if further factoring is possible.



Factor the following

a) $9x^2 - 36$ **b)** $x^2 - 16x - 36$ **c)** $-8x^2 + 26x + 15$

d) $24x^2 - 30x + 36x - 45$ **e)** $3 - 3x^2 - 36x^4$

Complete Assignment Questions #1 - #8

Assignment

1. Factor.

a) $x^2 - 49$ **b)** $x^2 - 8x + 15$ **c)** $8x^2 + 32$

d) $a^3 + a^2 + a + 1$ **e)** $2p^2 - 5p - 7$ **f)** $v^2 + 7v + 10$

g) $a^3 - a^2 - a + 1$ **h)** $4 - 25t^2$ **i)** $x^4 - 16$

2. Factor.

a) $7x^2 - 19x - 6$ **b)** $3 + x - 2x^2$ **c)** $a^2 - 64b^2$

d) $108 - 3z^2$ **e)** $x^4 + 5x^2 + 4$ **f)** $8v^2 - 32v - 96$

g) $625p^4 - 1$ **h)** $2y^4 - y^2 - 3$ **i)** $36 - 3x - 3x^2$

3. Factor.

a) $b^2 - 16 - 6b + 24$ **b)** $t^6 - t^3 - 6$ **c)** $36a^2 + 60a + 25$

d) $5 + 17g + 6g^2$ **e)** $x^5 - 81x$ **f)** $-256 + t^4$

g) $x^2 + y - x - xy$ **h)** $2x^4 - 15x^2 - 27$ **i)** $12a^2 + 32a - 12$

Multiple Choice

In questions #4 -#5 one or more of the four responses may be correct. Answer

A. if only 1 and 2 are correct
B. if only 1, 2, and 3 are correct
C. if only 3 and 4 are correct
D. if some other response or combination of responses is correct

4. The set of factors of $5x^2 - 10x - 15$ contains

(1) $x - 1$ (2) $x + 3$ (3) $x + 1$ (4) $x - 3$

5. $x + 4$ is a factor of
(1) $3x^2+7x - 20$ (2) $2x^3 + 5x^2 - 13x - 4$ (3) x^2+16 (4) $3x^2+ 12x$

6. If $9a^2 + ka + 16$ is a perfect square trinomial, then the value of k must be _______ .

(Record your answer in the numerical response box from left to right)

Extension Questions.

7. Factor.
a) $9(a - b)^2 - c^2$ **b)** $x^2 - 8xy - 33y^2$ **c)** $4x^2 - 25(x - y)^2$

d) $6a^2 + 19ab + 10b^2$ **e)** $15t^6 + t^3p^2 - 2p^4$ **f)** $36(x - y)^2 - (a - b)^2$

g) $36(x - y)^2 - 12(x - y)$ **h)** $7(x + 3)^2 - 19(x + 3) - 6$ **i)** $(x - 2)^8 - 256$

8. Factor $2(3a - 4)^2 - (3a - 4)(a + 2) - 6(a + 2)^2$

Answer Key

1. **a)** $(x-7)(x+7)$ **b)** $(x-5)(x-3)$ **c)** $8(x^2+4)$
d) $(a+1)(a^2+1)$ **e)** $(p+1)(2p-7)$ **f)** $(v+5)(v+2)$
g) $(a+1)(a-1)^2$ **h)** $(2-5t)(2+5t)$ **i)** $(x-2)(x+2)(x^2+4)$

2. **a)** $(7x+2)(x-3)$ **b)** $(3-2x)(1+x)$ **c)** $(a-8b)(a+8b)$
d) $3(6-z)(6+z)$ **e)** $(x^2+4)(x^2+1)$ **f)** $8(v+2)(v-6)$
g) $(5p-1)(5p+1)(25p^2+1)$ **h)** $(y^2+1)(2y^2-3)$ **i)** $3(4+x)(3-x)$

3. **a)** $(b-2)(b-4)$ **b)** $(t^3-3)(t^3+2)$
c) $(6a+5)^2$ **d)** $(5+2g)(1+3g)$
e) $x(x-3)(x+3)(x^2+9)$ **f)** $-(4-t)(4+t)(16+t^2)$ or $(t-4)(t+4)(t^2+16)$
g) $(x-1)(x-y)$ **h)** $(x-3)(x+3)(2x^2+3)$ **i)** $4(3a-1)(a+3)$

4. C **5.** D **6.**

2	4		

7. **a)** $(3a-3b-c)(3a-3b+c)$ **b)** $(x-11y)(x+3y)$
c) $(5y-3x)(7x-5y)$ **d)** $(2a+5b)(3a+2b)$
e) $(5t^3+2p^2)(3t^3-p^2)$ **f)** $(6x-6y-a+b)(6x-6y+a-b)$
g) $12(x-y)(3x-3y-1)$ **h)** $x(7x+23)$
i) $x(x-4)[(x-2)^2+4][(x-2)^4+16]$

8. $(a-8)(9a-2)$

Factoring Polynomial Expressions Lesson #7
Enrichment Lesson - Solving Polynomial Equations

Warm-Up

Complete the following:

The statement $x - 3 = 0$ is true only if $x =$ ______.

The statement $x + 1 = 0$ is true only if $x =$ ______.

The statement $(x - 3)(x + 1) = 0$ is true if $x =$ ______ or if $x =$ _____.

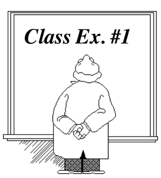
The statement $4(x - 3)(x + 1) = 0$ is true if ______________________.

All the above statements are polynomial equations in which the left side is a polynomial expression and the right side equals zero.

The **solution** to a polynomial equation is given by stating the value(s) of the variable which make(s) the left side and the right side equal. These values are said to **satisfy** the equation.

Solving Polynomial Equations

Consider the equation $x^2 - 2x - 3 = 0$. Factoring the left side leads to $(x - 3)(x + 1) = 0$. This is true if $x = 3$ or if $x = -1$. Since the equation is satisfied by both $x = 3$ and $x = -1$ the solutions to the equation are $x = 3$ **and** $x = -1$, sometimes written as $x = -1, 3$.



Complete the solution to the equation $x^2 - 9x + 20 = 0$.

$x^2 - 9x + 20 = 0$

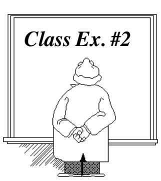
$(x - \quad)(x - \quad) = 0$

$x - \quad = 0$ or $x - \quad = 0$

$x =$ _____ or $x =$ ______

The solutions are $x =$ _____ and $x =$ ______

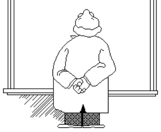
or $x =$ ____ , ____



Solve the equation.

a) $x^2 - 81 = 0$ **b**) $4x^2 - 9 = 0$ **c**) $10x^2 - 90x = 0$ **d**) $10x^2 - 90 = 0$.

Class Ex. #3



Solve the equation.

a) $3x^2 - 13x - 10 = 0$

b) $5x^2 + 30x = -25$

Complete Assignment Questions #1 - #4

Problem Solving with Polynomial Equations

Some problems in mathematics can be solved by the following procedure.

i) introduce a variable to represent an unknown value.

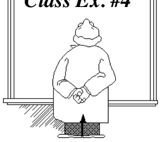
ii) form a polynomial equation from the given information.

iii) solve the polynomial equation using the methods in this lesson.

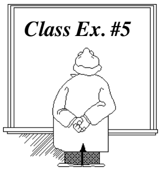
iv) state the solution to the problem.

In this section we will consider fairly routine problems. This topic will be extended in a higher level math course.

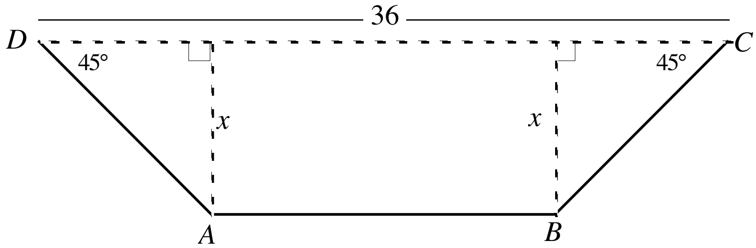
Class Ex. #4



The area of a rectangular sheet of paper is 300 cm^2. The length is 5 cm more than the width. Form a polynomial equation and solve it to determine the perimeter of the rectangular sheet.



The diagram shows the cross-section of a water trough whose sloping sides *AD* and *BC* make an angle of 45° with the horizontal. The length $DC = 36$ cm.



a) Show that the area of the cross-section is $x(36 - x)$ cm^2.

b) If the area of the cross-section is 260 cm^2, determine the value of x.

Complete Assignment Questions #5 - #10

Assignment

1. Solve the equation.

a) $(x - 2)(x + 7) = 0$ **b**) $(3x - 2)(2x + 5) = 0$ **c**) $5x(10 - x) = 0$

d) $x^2 + 2x = 0$ **e**) $x^2 - 121 = 0$ **f**) $9x^2 - 100 = 0$

g) $36x^2 = 25$ **h**) $9x - 4x^2 = 0$ **i**) $4(49 - x^2) = 0$

2. Solve the equation.

a) $x^2 - 3x + 2 = 0$ **b)** $x^2 + 13x + 30 = 0$ **c)** $x^2 + 2x - 15 = 0$

d) $3x^2 - 10x + 3 = 0$ **e)** $2x^2 + 3x - 35 = 0$ **f)** $15 - 2x - x^2 = 0$

3. Solve the equation.

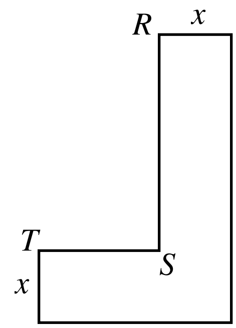
a) $2x^2 + 5x = 7$ **b)** $6x^2 = 7x + 3$ **c)** $x(x + 4) = 32$

d) $(x - 3)(2x + 3) = 5$ **e)** $(2x - 3)^2 = 1$ **f)** $(x + 1)(x - 1) = 5(x + 1)$

4. Solve the equation.

a) $6a^2 - 7 - 19a = 0$ **b)** $21 - 8k - 2k^2 = 2k^2$ **c)** $(y + 1)^2 + 6(y + 1) + 8 = 0$

5. The diagram shows a piece of wood of uniform width x cm.
$RS = 10$ cm and $ST = 7$ cm.



a) Find the area of the piece of wood in terms of x.

b) Find the value of x if the area is 60 cm^2.

6. The sum of the first n even numbers, starting with 0, is given by the formula $S = n(n - 1)$.

a) Determine the sum of the first 25 even numbers, starting with 0.

b) How many consecutive even numbers, starting with 0, add up to 870?

7. The height of a triangle is 8 mm more than the base. The area is 172.5 mm^2.

a) Write a polynomial equation to model this information.

b) Determine the height of the triangle.

Multiple Choice

8. The complete solution to the equation $x(x - 1) = 2$ is

A. $x = 0$ and $x = 1$
B. $x = 2$ and $x = 3$
C. $x = -1$ and $x = 2$
D. $x = -2$ and $x = 1$

Numerical Response

9. The equation $24x^2 + 2x = 15$ has solutions $x = a$ and $x = -b$, where a and b are positive rational numbers. The value of b, to the nearest hundredth, is _____.

(Record your answer in the numerical response box from left to right)

10. The sum of the first n natural numbers is given by the formula $S = \frac{1}{2}n(n+1)$. If the first k natural numbers have a sum of 496, the value of k is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $2, -7$ **b)** $\frac{2}{3}, -\frac{5}{2}$ **c)** $0, 10$ **d)** $0, -2$
e) ± 11 **f)** $\pm\frac{10}{3}$ **g)** $\pm\frac{5}{6}$ **h)** $0, \frac{9}{4}$ **i)** ± 7

2. a) $1, 2$ **b)** $-10, -3$ **c)** $-5, 3$ **d)** $\frac{1}{3}, 3$ **e)** $-5, \frac{7}{2}$ **f)** $-5, 3$

3. a) $-\frac{7}{2}, 1$ **b)** $-\frac{1}{3}, \frac{3}{2}$ **c)** $-8, 4$ **d)** $-2, \frac{7}{2}$ **e)** $1, 2$ **f)** $-1, 6$

4. a) $-\frac{1}{3}, \frac{7}{2}$ **b)** $-\frac{7}{2}, \frac{3}{2}$ **c)** $-5, -3$ **5. a)** $x^2 + 17x$ cm^2 **b)** 3

6. a) 600 **b)** 30 **7. a)** $x^2 + 8x - 345 = 0$ **b)** 23 mm

8. C **9.**

0	.	8	3

10.

3	1		

Real Numbers and Radicals Lesson #1:
Rational and Irrational Numbers

Warm-Up *Review*

a) Consider the number $\frac{28}{11}$.

* Convert $\frac{28}{11}$ to a decimal.

* Circle the correct alternatives in the following statement:
  The decimal representing $\frac{28}{11}$ has a (repeating / non-repeating) pattern and (terminates / does not terminate).

b) Consider the number $\frac{28}{8}$.

* Convert $\frac{28}{8}$ to a decimal.

* Circle the correct alternatives in the following.statement:
  The decimal representing $\frac{28}{8}$ has a (repeating / non-repeating) pattern and (terminates / does not terminate).

c) Consider the number $\sqrt{2}$.

* Convert $\sqrt{2}$ to a decimal.

* Circle the correct alternatives in the following statement:
  The decimal representing $\sqrt{2}$ has a (repeating / non-repeating) pattern and (terminates / does not terminate).

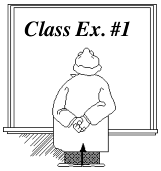
Terminology

Repeating Decimals - decimals which have a recurring pattern of digits.

Non-Repeating Decimals - decimals which have no recurring pattern of digits.

Terminating Decimals - decimals with a finite number of digits

Non-Terminating Decimals - decimals with an infinite number of digits.



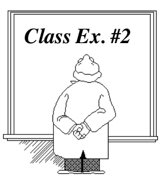
State whether the decimal equivalent of each number is
i) repeating or non-repeating and
ii) terminating or non-terminating.

a) $\frac{1}{8}$ **b)** $\frac{2}{11}$ **c)** $\frac{9}{7}$

d) $0.9\overline{4}$ **e)** $\sqrt{8}$ **f)** $\sqrt{0.16}$

Rational and Irrational Numbers

- Decimal numbers which repeat or terminate can be converted into fractions. These numbers are called **rational numbers** since they can be written as the ratio of two integers.
- Decimal numbers which are both non-repeating and non-terminating cannot be converted into fractions and are called **irrational numbers**.



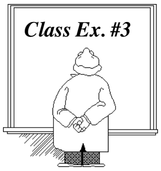
Identify each of the following numbers as rational or irrational. Give a reason in each case. Use a calculator to convert the rational numbers to improper fractions in simplest form.

a) 1.493 **b)** $\sqrt{5}$ **c)** 2.347 347 347 347. . .

d) $-\sqrt{81}$ **e)** $\sqrt{4.41}$ **f)** –8.11221112221111 . . .

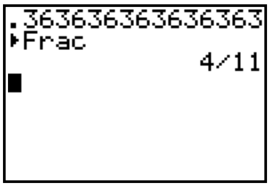
Complete Assignment Questions #1 - #4

Converting a Repeating Decimal to a Fraction using a Graphing Calculator


Class Ex. #3

Use a graphing calculator to convert each of the following to an improper fraction in simplest form.

a) $0.\overline{36}$



b) $3.9\overline{5}$

Converting a Repeating Decimal to a Fraction Algebraically

The following method can be used to convert a repeating decimal to a fraction algebraically.

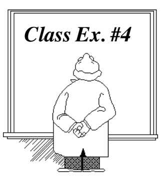
Step 1. Let x be the original number.

Step 2. Identify the digit(s) which repeat.

Step 3. Multiply both sides in step 1 by a power of 10 which moves the repeating part to the **left** of the decimal point.

Step 4. If necessary multiply both sides in step 1 by a power of 10 which moves the repeating part immediately to the **right** of the decimal point.

Step 5. Subtract the equation in step 4 from the equation in step 3 and solve for x. If there is no step 4, subtract the equation in step 1 from the equation in step 3 and solve for x.


Class Ex. #4

A student is using the above procedure to convert $2.0\overline{51}$ to a fraction.
Follow the procedure and complete the last step.

Step 1. Let $x = 2.0\overline{51}$

Step 2. The repeating digits are 51

Step 3. To move the 51 to the left of the decimal point multiply both sides in step 1 by 10^3 or 1000.

$$1000x = 2051.\overline{51}$$

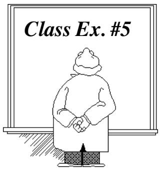
Step 4. To move the 51 to the immediate right of the decimal point multiply both sides in step 1 by 10.

$$10x = 20.\overline{51}$$

Step 5.

$$1000x = 2051.\overline{51}$$
$$10x = 20.\overline{51}$$

subtract



Use an algebraic procedure to convert each of the following to an improper fraction in simplest form.

a) $0.\overline{36}$ **b)** $3.9\overline{5}$

Complete Assignment Question #5 - #11

Assignment

1. State whether the decimal equivalent of each number is

i) repeating or non-repeating and

ii) terminating or non-terminating.

a) $\frac{1}{2}$ **b)** 0.123 123 123 ... **c)** $-\frac{3}{7}$ **d)** $\sqrt{\frac{49}{81}}$

e) $-\sqrt{17}$ **f)** $\sqrt{0.64}$ **g)** $-4\frac{3}{8}$ **h)** π

2. Classify the following statements as true or false.

a) Decimals which are terminating can be converted to fraction form.

b) Repeating decimals cannot be converted to fraction form.

c) Only decimal numbers which terminate can be converted to fraction form.

d) Rational numbers are either repeating decimals or terminating decimals.

e) A decimal number can be repeating and non-repeating at the same time.

f) π is an irrational number.

3. Determine whether the following numbers are rational or irrational.
Explain your reasoning.

a) $-\frac{7}{5}$ **b)** $0.\overline{234}$ **c)** $\sqrt{144}$ **d)** 0.23223222322223222223...

4. Identify each of the following numbers as rational or irrational.
Use a calculator to convert the rational numbers to fractions in simplest form.

a) 0.8 **b)** $\sqrt{\frac{1}{9}}$ **c)** $\sqrt{0.0064}$

d) $-\sqrt{79}$ **e)** -0.555 **f)** $-\sqrt{1\frac{9}{16}}$

g) 4.102 102 102 ... **h)** 5.724 734 744 ... **i)** $\sqrt{\sqrt{\frac{81}{256}}}$

5. Use a graphing calculator to convert each of the following to an improper fraction in simplest form.

a) $0.\overline{6}$ **b)** $0.\overline{21}$ **c)** $1.0\overline{8}$ **d)** $0.\overline{123}$ **e)** $-3.24\overline{70}$

6. Use an algebraic procedure to convert each of the following to an improper fraction in simplest form.

a) $0.\overline{2}$ **b)** $0.\overline{61}$ **c)** $0.9\overline{8}$

7. Use an algebraic procedure to convert each of the following to an improper fraction in simplest form.

a) $2.0\overline{05}$ **b)** $-1.\overline{234}$ **c)** $4.47\overline{3}$

Multiple Choice

8. The decimal number representing $\frac{4}{9}$ is

A. terminating and repeating
B. terminating and non-repeating
C. non-terminating and repeating
D. non-terminating and non-repeating

9. Which of the following is an irrational number?

A. $\sqrt{169}$
B. $\sqrt{0.025}$
C. $\frac{5}{6}$
D. 3.14

10. $9.\overline{9}$ is equal to

A. $\frac{99}{10}$
B. $\frac{999}{100}$
C. 9.9
D. 10

Numerical Response

11. When the repeating decimal $0.4\overline{76}$ is converted to a rational number of the form $\frac{a}{b}$ the value of $b - a$ is ______.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** non-repeating, terminating **b)** repeating, non-terminating **c)** repeating, non-terminating

d) repeating, non-terminating **e)** non-repeating, non-terminating **f)** non-repeating, terminating

g) non-repeating, terminating **h)** non-repeating, non-terminating

2. **a)** true **b)** false **c)** false **d)** true **e)** false **f)** true

3. **a)** rational since it is a terminating decimal **b)** rational since it is a repeating decimal

c) rational since it is a terminating decimal

d) irrational since it is a non-repeating and non-terminating decimal

4. **a)** rational, $\frac{4}{5}$ **b)** rational, $\frac{1}{3}$ **c)** rational, $\frac{2}{25}$ **d)** irrational **e)** rational, $-\frac{111}{200}$

f) rational, $-\frac{5}{4}$ **g)** rational, $\frac{1366}{333}$ **h)** irrational **i)** rational, $\frac{3}{4}$

5. **a)** $\frac{2}{3}$ **b)** $\frac{7}{33}$ **c)** $\frac{49}{45}$ **d)** $\frac{41}{333}$ **e)** $-\frac{16073}{4950}$

6. **a)** $\frac{2}{9}$ **b)** $\frac{61}{99}$ **c)** $\frac{89}{90}$

7. **a)** $\frac{397}{198}$ **b)** $-\frac{137}{111}$ **c)** $\frac{671}{150}$

8. C **9.** B **10.** D **11.**

2	5	9	

Real Numbers and Radicals Lesson #2:
Number Systems

The Real Number System

Recall from Lesson 1 the definitions of rational numbers and irrational numbers.

- Decimal numbers which repeat or terminate can be converted into fractions and are called **rational numbers** since they can be written as the ratio of two integers.

- Decimal numbers which are both non-repeating and non-terminating cannot be converted into fractions and are called **irrational numbers**.

The set of all rational numbers and the set of all irrational numbers when combined form the set of **real numbers**, numbers which can be represented on a number line.

The following sets of numbers are within the real number system:

Natural Numbers

$N = \{\ 1, 2, 3, \ldots\ \}$

Whole Numbers

$W = \{\ 0, 1, 2, 3, \ldots\ \}$

Integers

$I = \{\ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ \}$

Rational Numbers

$Q = \left\{ \dfrac{a}{b},\ \text{where } a, b \in I,\ b \neq 0 \right\}$

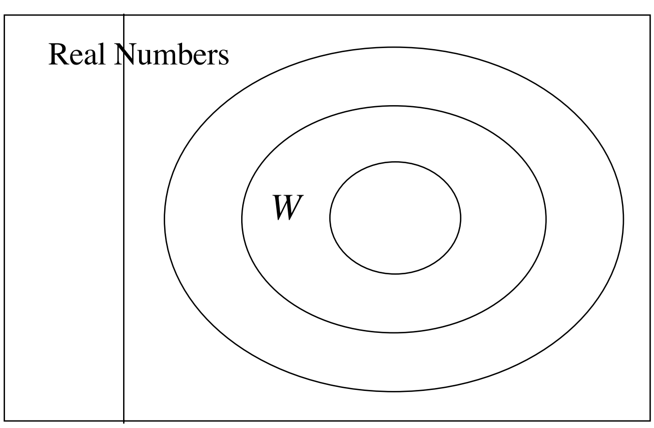
Irrational Numbers

$\overline{Q} = \{\text{non-terminating and non-repeating decimals}\}$

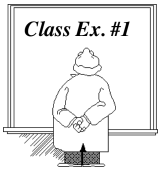
Real Numbers

$\Re = \{Q \text{ and } \overline{Q}\}$

The interrelationship between the sets can be shown in the following diagram which illustrates how the number sets are nested within the real number system. Complete the diagram.



Note that the area of each region bears no relation to the number of members in each set.



a) For each of the following write all the sets of numbers to which the given number belongs. Write the answers from the largest set to the smallest set.

i) 9 **ii)** $\frac{11}{7}$ **iii)** $\sqrt{5}$ **iv)** -7

b) Explain why 9 belongs to five number sets but -9 belongs to only three number sets.

Complete Assignment Questions #1 - #7

Square Roots

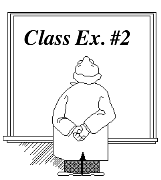
All positive numbers have two square roots, one a positive number and the other a negative number. The positive square root is called the **principal square root** and is denoted by the symbol $\sqrt{\ }$.

The square roots of a perfect square are rational numbers.
eg the square roots of 16 are 4 and -4.

The square roots of a non-perfect square are irrational numbers.
eg the square roots of 17 are $\sqrt{17}$ and $-\sqrt{17}$.

The ability to estimate mentally the square root of a non-perfect square is important when checking a calculator calculation for possible error.
A knowledge of some common perfect squares enables us to make such estimates to the nearest whole number.

1^2	$= 1$
2^2	$= 4$
3^2	$= 9$
4^2	$= 16$
5^2	$= 25$
6^2	$= 36$
7^2	$= 49$
8^2	$= 64$
9^2	$= 81$
10^2	$= 100$
11^2	$= 121$
12^2	$= 144$
13^2	$= 169$
14^2	$= 196$
15^2	$= 225$

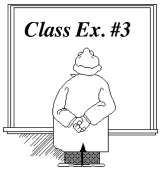


In each of the following
i) Estimate the value mentally. (Use whole numbers)
ii) Use a calculator to find the decimal approximation to the nearest tenth. Decide if the estimate in i) is reasonable.

a) $\sqrt{46}$ **b)** $2\sqrt{18} + 5\sqrt{37}$

c) $-\sqrt{5}$ **d)** $\sqrt{\sqrt{95}}$

To estimate square roots of large numbers or of positive numbers less than 1, divide the number into groups of 2 digits starting from the decimal point.



Estimate to one significant digit.

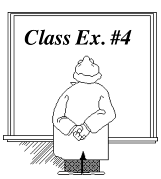
a) $\sqrt{6210}$ **b)** $\sqrt{82147.3}$ **c)** $\sqrt{0.00243}$ **d)** $\sqrt{0.0243}$

Cube Roots

All numbers (positive and negative) have one cube root, denoted by the symbol $\sqrt[3]{\ }$.

The cube root of a perfect cube is a rational number.
eg the cube root of 1000 is 10, the cube root of -27 is -3 etc.

The cube root of a non-perfect cube is an irrational number.
eg the cube root of 9 is $\sqrt[3]{9}$, which is irrational.



In each of the following

i) Estimate the value mentally. (Use whole numbers)

ii) Use a calculator to find the decimal approximation to the nearest tenth. Decide if the estimate in i) is reasonable.

a) $\sqrt[3]{11}$ **b)** $\sqrt[3]{120}$

c) $4\sqrt{70} - 4\sqrt[3]{70}$

1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

Complete Assignment Questions #8 - #17

Extension - Absolute Value

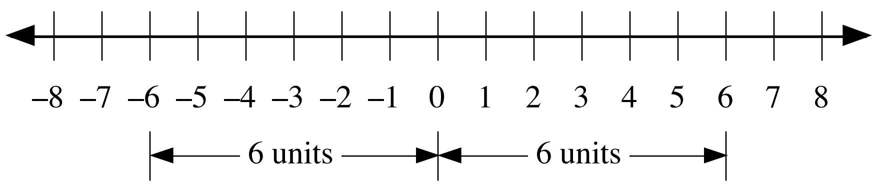
The **absolute value** of a real number can be defined as the principal square root of the square of the number.

eg. the absolute value of $6 = \sqrt{(6)^2} = \sqrt{36} = 6$

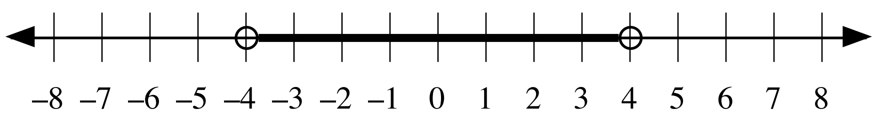
the absolute value of $-6 = \sqrt{(-6)^2} = \sqrt{36} = 6$

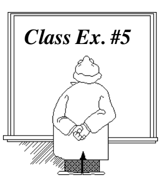
For a real number, a, the absolute value of a is written $|a|$. eg $|6| = 6$ and $|-6| = 6$.

The absolute value of a real number can be regarded as the distance of the number from zero on a number line.



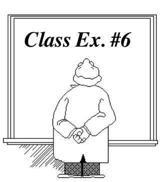
To graph the inequality $|a| < 4,\ a \in R$ we know that the distance from 0 must be less than 4 units.





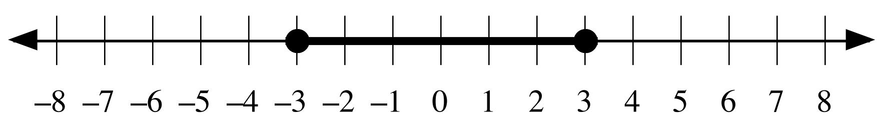
Evaluate

a) $|-7| + |7|$ **b)** $|1 - 5|$ **c)** $-|-\sqrt{81}\,|$

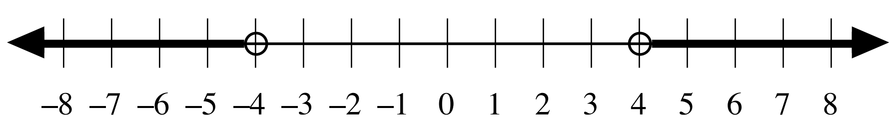


In each case state the absolute value inequality represented by the graph.

a)



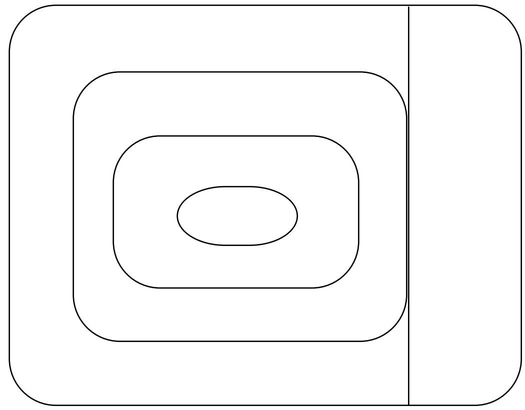
b)



Complete Assignment Questions #18 - #20

Assignment

1. Complete the Venn Diagram to show the interrelationship between the sets of numbers in the real number system.



2. For each of the following write all the sets of numbers to which the given number belongs. Write the answers from the largest set to the smallest set.

a) -2 **b)** $\sqrt{36}$ **c)** 3.14159265 **d)** $-\frac{7}{5}$

e) 0 **f)** $\sqrt{7}$ **g)** $-2.1345218...$ **h)** π

3. Explain why -7 belongs to more number sets than $-\frac{7}{2}$.

4. Use check marks to indicate all the sets to which each number belongs.

	N	W	I	Q	$\overline{Q}$	$\Re$
$1/3$						
123 983						
–2.0						
$7.53\overline{4}$						
9.5						
$\sqrt{75}$						
$-\pi$						
$-355/113$						
$-\sqrt{49}$						
0.000005						
2.232425...						
$\sqrt{0.16}$						

5. Find one number which satisfies each condition.

a) An integer but not a whole number.

b) A rational number but not an integer.

c) A real number but not a rational number

d) A whole number but not a natural number.

6. Complete the following statements using *always*, *sometimes*, or *never*.

a) A whole number is ____________ a natural number.

b) The quotient of two integers is ____________ an integer

c) A whole number is ____________ a rational number.

d) The difference between two integers is ____________ an integer.

e) The square root of a number is ____________ in set $\overline{Q}$.

f) A negative number is ____________ in set W.

g) A number in set N is ____________ a number in set $\Re$.

7. Determine whether each statement is true or false. Answer in the blank space provided.

a) ___ All natural numbers are integers.

b) ___ Real numbers consist of rational numbers and irrational numbers.

c) ___ The set of integers is nested within the set of rational numbers.

d) ___ All integers are rational numbers.

e) ___ All irrational numbers are real.

f) ___ The set $\Re$ is nested within the set N.

g) ___ The set Q is nested within the set W.

h) ___ There is only one number in set W which is not also in set N.

8. Determine whether each statement is true or false.

a) ___ Every positive number has two square roots but only one cube root.

b) ___ Every negative number has one cube root but no square roots.

9. **a)** Use estimates to explain why $\sqrt{8} + \sqrt{17}$ is not equal to $\sqrt{25}$.

b) Use estimates to explain why $\sqrt{2} + \sqrt{3} + \sqrt{4}$ is not equal to $\sqrt{9}$.

10. Determine whether each statement is true or false.

a) ___ $\sqrt{9} + \sqrt{4}$ is equal to $\sqrt{9 + 4}$

b) ___ $\sqrt{9} - \sqrt{4}$ is equal to $\sqrt{9 - 4}$

c) ___ $\sqrt{9} \times \sqrt{4}$ is equal to $\sqrt{9 \times 4}$

d) ___ $\sqrt{9} \div \sqrt{4}$ is equal to $\sqrt{9 \div 4}$

11. In each of the following

i) Estimate the value mentally. (Use whole numbers)

ii) Use a calculator to find the decimal approximation to the nearest tenth. Decide if the estimate in i) is reasonable.

a) $\sqrt{19}$ **b)** $\sqrt{26.4}$ **c)** $4\sqrt{50} - 3\sqrt{60}$ **d)** $\frac{3}{4}\sqrt{13.9} + \frac{1}{2}\sqrt{3}$

e) $\sqrt{119}$ **f)** $\sqrt{\sqrt{80}}$ **g)** $\sqrt{\sqrt{10} + \sqrt{23.9}}$ **h)** $\sqrt{\sqrt{2501}}$

12. Estimate to one significant digit.

a) $\sqrt{507.1}$ **b)** $\sqrt{7991}$ **c)** $\sqrt{10389}$ **d)** $\sqrt{823\ 775}$

e) $\sqrt{0.501}$ **f)** $\sqrt{0.0501}$ **g)** $\sqrt{0.0876}$ **h)** $\sqrt{0.0003972}$

13. In each of the following

i) Estimate the value mentally. (Use whole numbers)

ii) Use a calculator to find the decimal approximation to the nearest tenth. Decide if the answer to i) is reasonable.

a) $\sqrt[3]{25}$ **b)** $\sqrt[3]{2}$ **c)** $\sqrt[3]{202}$ **d)** $\sqrt[3]{999.9}$

e) $2\sqrt[3]{58.7} - 3\sqrt[3]{7.62}$ **f)** $\frac{2}{3}\sqrt{40} - \frac{1}{2}\sqrt[3]{60}$ **g)** $\sqrt[3]{3\sqrt{10}}$

Multiple Choice

14. Consider the following statements.

i) the set of irrational numbers is nested within the set of rational numbers
ii) the set of integers contains the set of rational numbers
iii) the set of whole numbers is nested within the set of natural numbers
iv) the set of real numbers contains the set of irrational numbers

Which of the above statements is false?

A. i), ii) and iii) only
B. i), ii) and iv) only
C. ii), iii) and iv) only
D. all are false

15. How many of the numbers $-\sqrt{6}$, $\sqrt{-6}$, $-\sqrt[3]{6}$, $\sqrt[3]{-6}$ do not belong to the real number system?

A. 0
B. 1
C. 2
D. 3

16. How many of the numbers $\sqrt{49}$, $\sqrt{4.9}$, $\sqrt{0.49}$, $\sqrt{\frac{4}{9}}$ can be expressed in the form $\frac{a}{b}$ where $a, b \in N$?

A. 1
B. 2
C. 3
D. 4

Numerical Response

17. To the nearest hundredth, the value of $5\sqrt[3]{7}$ is ______.

(Record your answer in the numerical response box from left to right)

Extension Questions.

18. Evaluate

a) $|-4|$ **b)** $|13|$ **c)** $|3-9|$ **d)** $|3|-|9|$

e) $||3|-|9||$ **f)** $-|\sqrt[3]{27}|$ **g)** $|-\sqrt[3]{27}|$ **h)** $|\sqrt[3]{-27}|$

19. Decide whether each statement is true or false

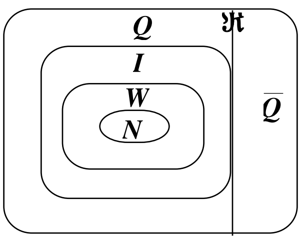
a) $|x| = x$ if $x > 0$ **b)** $|x| = -x$ if $x < 0$.

20. In each case draw a graph to represent the absolute value inequality. The variables are defined on the set of real numbers.

a) $|x| < 5$ **b)** $|a| \geq 3$

Answer Key

1.



2.a) real numbers, rational numbers, integers
b) real numbers, rational numbers, integers, whole numbers, natural numbers
c) real numbers, rational numbers
d) real numbers, rational numbers
e) real numbers, rational numbers, integers, whole numbers
f) real numbers, irrational numbers
g) real numbers, irrational numbers
h) real numbers, irrational numbers

3. –7 is a real number, rational number and integer whereas $-\frac{7}{2}$ is a real number and a rational number but not an integer.

4.

	N	W	I	Q	$\overline{Q}$	$\Re$
$1/3$				3		3
123 983	3	3	3	3		3
–2.0			3	3		3
$7.53\overline{4}$				3		3
9.5				3		3
$\sqrt{75}$					3	3
$-\pi$					3	3
$-355/113$				3		3
$-\sqrt{49}$			3	3		3
0.000005				3		3
2.232425...					3	3
$\sqrt{0.16}$				3		3

5. **a)** –3 **b)** $\frac{4}{5}$ **c)** $\sqrt{3}$ **d)** 0

6. **a)** sometimes **b)** sometimes **c)** always
d) always **e)** sometimes **f)** never **g)** always

7. **a)** true **b)** true **c)** true **d)** true
e) true **f)** false **g)** false **h)** true

8. **a)** true **b)** true **9.** **a)** $3 + 4 \neq 5$ **b)** $1 + 2 + 2 \neq 3$

10.a) false **b)** false **c)** true **d)** true

11.a)i) 4 **ii)** 4.4 **b)i)** 5 **ii)** 5.1 **c)i)** 4 **ii)** 5.0 **d)i)** 4 **ii)** 3.7
e)i) 11 **ii)** 10.9 **f)i)** 3 **ii)** 3.0 **g)i)** 3 **ii)** 2.8 **h)i)** 7 **ii)** 7.1

12.a) 20 **b)** 90 **c)** 100 **d)** 900 **e)** 0.7 **f)** 0.2 **g)** 0.3 **h)** 0.02

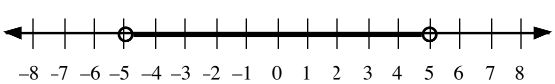
13.a)i) 3 **ii)** 2.9 **b)i)** 1 **ii)** 1.3 **c)i)** 6 **ii)** 5.9 **d)i)** 10 **ii)** 10.0
e)i) 2 **ii)** 1.9 **f)i)** 2 **ii)** 2.3 **g)i)** 2 **ii)** 2.1

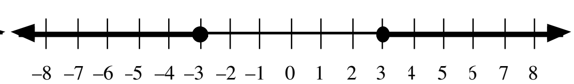
14. A **15.** B **16.** C **17.**

9	.	5	6

18.a) 4 **b)** 13 **c)** 6 **d)** –6 **19.a)** true **b)** true
e) 6 **f)** –3 **g)** 3 **h)** 3

20.a) **b)**





Real Numbers and Radicals Lesson #3:
Entire Radicals and Mixed Radicals - Part One

Warm-Up #1

Recall the following notes from Lesson #2.

Square Roots : All positive numbers have two square roots, one a positive number and the other a negative number. The positive square root is called the **principal square root** and is denoted by the symbol $\sqrt{}$.

The square roots of a perfect square are rational numbers.
eg the square roots of 16 are 4 and –4.

The square roots of a non-perfect square are irrational numbers.
eg the square roots of 17 are $\sqrt{17}$ and $-\sqrt{17}$.

Cube Roots: All numbers (positive and negative) have one cube root, denoted by the symbol $\sqrt[3]{}$.

The cube root of a perfect cube is a rational number.
eg the cube root of 1000 is 10, the cube root of –27 is –3 etc.

The cube root of a non-perfect cube is an irrational number.
eg the cube root of 49 is $\sqrt[3]{49}$, which is irrational.

Radicals

Numbers like $\sqrt{17}$, $\sqrt[3]{64}$, $\sqrt[4]{40}$, $\sqrt{25}$ etc. are examples of **radicals**.

In fact, any expression of the form $\sqrt[n]{x}$, where $n \in N$, is called a radical.

n is called the **index**. In a number like $\sqrt{17}$ the index is 2.
If the index in a radical is even, then only the positive root is taken.
x is called the **radicand** and $\sqrt{}$ is called the **radical sign**.

Index

$$\sqrt[3]{29}$$

Radicand

Radical sign

$$\sqrt[3]{29}$$

Radical

- When the index is not written in the radical, as in square root, it is assumed to be 2.
- Radicals with an index of 3 or more will be discussed in the next lesson.

Warm-Up #2

Recall the following results from Lesson #2, question 10.

$\sqrt{9} \times \sqrt{4}$ is equal to $\sqrt{9 \times 4}$ $\qquad$ $\sqrt{9} + \sqrt{4}$ is **not** equal to $\sqrt{9 + 4}$

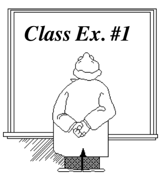
$\sqrt{9} \div \sqrt{4}$ is equal to $\sqrt{9 \div 4}$ $\qquad$ $\sqrt{9} - \sqrt{4}$ is **not** equal to $\sqrt{9 - 4}$

The calculations above are examples of some general rules involving radicals.

i) The product(quotient) of the roots of two numbers is equal to the root of the product(quotient) of the two numbers.

ii) The sum(difference) of the roots of two numbers is NOT equal to the root of the sum(difference) of the two numbers.

In general $\sqrt{\boldsymbol{a}} \times \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{ab}}$ where $a, b \geq 0$ and $\dfrac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}} = \sqrt{\dfrac{\boldsymbol{a}}{\boldsymbol{b}}}$ where $a \geq 0, b > 0$



Determine whether each statement is true or false.

a) $\sqrt{3} \times \sqrt{6} = \sqrt{18}$ **b)** $\dfrac{\sqrt{20}}{\sqrt{10}} = \sqrt{2}$ **c)** $\sqrt{6 + x} = \sqrt{6} + \sqrt{x}$ **d)** $\sqrt{9b} = 3\sqrt{b}$

Entire Radicals and Mixed Radicals

Use a calculator to approximate the value of each radical to 5 decimal places.

i) $\sqrt{80}$ = ____________ **ii)** $2\sqrt{20}$ = ____________ **iii)** $4\sqrt{5}$ = ____________

What do you notice about the answers? ______________________

Complete the following to explain why the three radicals are equivalent.

$\sqrt{80} = \sqrt{4 \times 20} = \sqrt{} \times \sqrt{}$ = _____

$\sqrt{80} = \sqrt{16 \times 5} = \sqrt{} \times \sqrt{}$ = _____

$\sqrt{80}$ is an example of an **entire radical** - the number is entirely under the root symbol.

$2\sqrt{20}$ and $4\sqrt{5}$ are examples of **mixed radicals**.

Entire/Pure Radicals

- Radicals expressed in the form $\sqrt[n]{b}$ are called entire (or pure) radicals.
- For example, $\sqrt{25}$, $\sqrt{80}$, $\sqrt[3]{17}$

Mixed Radicals

- Radicals expressed in the form $a\sqrt[n]{b}$ are called mixed radicals.
- For example $\frac{2}{3}\sqrt{5}$, $8\sqrt{7}$, $-9\sqrt[3]{17}$

Every mixed radical can be expressed as an entire radical.

To determine if an entire radical can be expressed as a mixed radical, we need to check if the number has a factor which is a perfect square.

Converting Entire Radicals (with an index of 2) to Mixed Radicals

An entire radical of index 2 may be expressed as a mixed radical when the highest perfect square has been factored out of the entire radical.

1^2	= 1
2^2	= 4
3^2	= 9
4^2	= 16
5^2	= 25
6^2	= 36
7^2	= 49
8^2	= 64
9^2	= 81
10^2	= 100
11^2	= 121
12^2	= 144
13^2	= 169
14^2	= 196
15^2	= 225

Complete the following to convert $\sqrt{108}$ to a mixed radical.

Entire Radical $\Rightarrow$ *Mixed Radical*

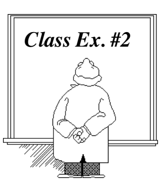
$$\sqrt{108} = \sqrt{\underline{\qquad} \times 3}$$

$$= \sqrt{\underline{\qquad}} \times \sqrt{3}$$

$$= 6 \times \sqrt{3}$$

$$\sqrt{108} =$$

If a perfect square is factored out which is not the highest perfect square, then the process will require more than one step to obtain the mixed radical in simplest form. When converting an entire radical to a mixed radical it is asssumed that the answer will be in simplest form.



Convert the following radicals to mixed radicals in simplest form.

a) $\sqrt{75}$ **b)** $2\sqrt{192}$ **c)** $\frac{3}{4}\sqrt{160}$ **d)** $\sqrt{\frac{7}{9}}$

Converting Mixed Radicals (with an index of 2) to Entire Radicals

A mixed radical of index 2 may be expressed as an entire radical by converting the number outside the radical symbol into a radical and then multiplying it by the radicand.

The number outside the radical symbol can be converted into a radical by raising it to the power of 2.

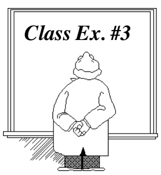
Complete the following to convert $3\sqrt{14}$ to an entire radical.

Mixed Radical $\Rightarrow$ *Entire Radical*

$$3\sqrt{14} = \sqrt{\_\_} \times \sqrt{14}$$

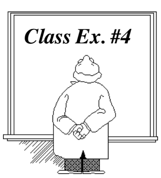
$$= \sqrt{\_\_\_\_ \times 14}$$

$$3\sqrt{14} =$$



Convert the following mixed radicals to entire radicals.

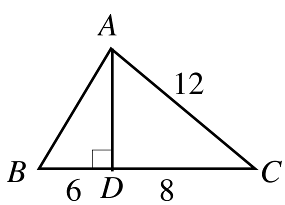
a) $2\sqrt{5}$ **b)** $\frac{3}{2}\sqrt{8}$ **c)** $0.4\sqrt{50}$ **d)** $-5\sqrt{7}$



Use the Pythagorean Theorem to determine the exact length of AB. Express the answer as:

a) an exact value

b) as a decimal to the nearest hundredth.



Complete Assignment Questions #1 - #18

Assignment

1. Determine whether each statement is true or false.

a) $\sqrt{30} = \sqrt{5}\sqrt{6}$ **b)** $\sqrt{6-4} = \sqrt{6} - \sqrt{4}$ **c)** $\sqrt{9x^2} = 3x$

d) $\sqrt{3} = \dfrac{\sqrt{45}}{\sqrt{15}}$ **e)** $\dfrac{\sqrt{20}}{\sqrt{10}} = \sqrt{10}$ **f)** $\sqrt{x^2 - 9} = x - 3$

g) $\sqrt{2} + \sqrt{2} = \sqrt{4}$ **h)** $\sqrt{\dfrac{1}{2} \times 30} = \sqrt{15}$ **i)** $\dfrac{1}{2}\sqrt{30} = \sqrt{15}$

2. Write the following in the form $\sqrt{x}$.

a) $\sqrt{5} \times \sqrt{7}$ **b)** $\sqrt{14 \times 2}$ **c)** $\sqrt{3} \cdot \sqrt{8}$ **d)** $\sqrt{6 \cdot 11}$

e) $\dfrac{\sqrt{20}}{\sqrt{10}}$ **f)** $\dfrac{\sqrt{25}}{\sqrt{5}}$ **g)** $\dfrac{\sqrt{\sqrt{81}}}{\sqrt{\sqrt{9}}}$ **h)** $\dfrac{\sqrt{10}\sqrt{6}}{\sqrt{2}}$

3. Express as a product of radicals.

a) $\sqrt{35}$ **b)** $\sqrt{33}$ **c)** $\sqrt{65}$ **d)** $\sqrt{49}$

4. Identify whether each radical is written as a mixed radical or an entire radical.

a) $\sqrt{35}$ **b)** $2\sqrt{7}$ **c)** $\sqrt{81}$ **d)** $0.3\sqrt{6}$

5. Convert the following radicals to mixed radicals in simplest form.

a) $\sqrt{8}$ **b)** $\sqrt{20}$ **c)** $\sqrt{75}$ **d)** $\sqrt{98}$

e) $3\sqrt{32}$ **f)** $-5\sqrt{45}$ **g)** $2\sqrt{54}$ **h)** $-4\sqrt{48}$

6. Convert the following radicals to mixed radicals in simplest form.
There are two which cannot be converted. Identify them and explain why they cannot be converted to mixed radicals.

a) $\sqrt{96}$ **b)** $\sqrt{242}$ **c)** $-\frac{2}{3}\sqrt{180}$ **d)** $\frac{1}{8}\sqrt{320}$

e) $\sqrt{245}$ **f)** $4\sqrt{338}$ **g)** $\sqrt{1250}$ **h)** $\sqrt{66}$

i) $-\frac{5}{6}\sqrt{304}$ **j)** $\sqrt{980}$ **k)** $4\sqrt{272}$ **l)** $-3\sqrt{288}$

m) $2\sqrt{369}$ **n)** $\sqrt{364}$ **o)** $\frac{2}{5}\sqrt{450}$ **p)** $\frac{7}{11}\sqrt{341}$

7. Convert the following radicals to mixed radicals where the radicand is a whole number.

a) $\sqrt{\frac{2}{9}}$ **b)** $\sqrt{\frac{5}{4}}$ **c)** $\sqrt{\frac{18}{25}}$ **d)** $7\sqrt{\frac{20}{49}}$

8. Convert the following to entire radical form.

a) $2\sqrt{6}$ **b)** $3\sqrt{7}$ **c)** $5\sqrt{15}$ **d)** $12\sqrt{2}$

e) $3\sqrt{25}$ **f)** $-8\sqrt{3}$ **g)** $9\sqrt{10}$ **h)** $-4\sqrt{5}$

9. Convert the following to entire radical form.

a) $\frac{1}{3}\sqrt{27}$ **b)** 15 **c)** $\frac{3}{2}\sqrt{8}$ **d)** $3^2\sqrt{21}$

Do not use a calculator to answer question #10 or #11.

10. Given that $\sqrt{6}$ is approximately equal to 2.45 and $\sqrt{60}$ is approximately equal to 7.75 find the approximate square roots of

a) $\sqrt{600}$ **b)** $\sqrt{6000}$ **c)** $\sqrt{600\,000}$ **d)** $\sqrt{0.06}$

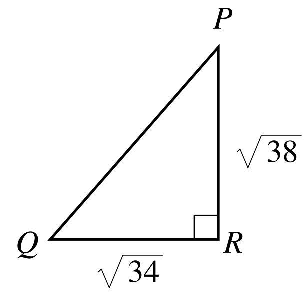
e) $\sqrt{0.6}$ **f)** $\sqrt{24}$ **g)** $\sqrt{540}$ **h)** $\sqrt{\frac{6}{25}}$

11. Arrange the following radicals in order from greatest to least.
$3\sqrt{7}$, $5\sqrt{3}$, $\sqrt{60}$, $2\sqrt{11}$, $\frac{1}{2}\sqrt{200}$

12. Consider triangle *PQR* as shown. Students are trying to determine the length of *PQ* using the Pythagorean Formula.

Louis expresses each side to the nearest hundredth and calculates the length of *PQ* to the nearest hundredth.

Asia uses the entire radical form for each side and expresses her answer to the nearest hundredth.



a) Complete each student's work.

<u>Louis</u>

$(PQ)^2 = (QR)^2 + (PR)^2$

$= 5.83^2 +$

$=$

<u>Asia</u>

$(PQ)^2 = (QR)^2 + (PR)^2$

$= (\sqrt{34})^2 +$

$= 34 +$

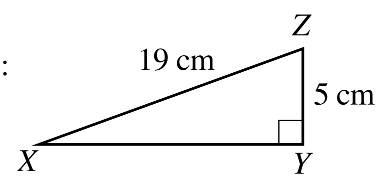
b) Which student's answer is more precise? Explain.

c) State the **exact** answer as a mixed radical.

13. Use the Pythagorean formula, $c^2 = a^2 + b^2$, in the given triangle to calculate the length of *XY*. Express the answer as:

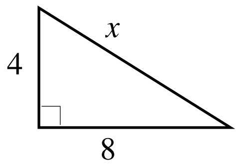
i) an entire radical **ii)** a mixed radical

iii) a decimal to the nearest hundredth

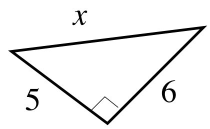


14. Find the lengths of the missing sides.
Express the answers in simplest mixed radical form.

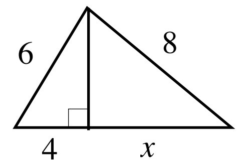
a)



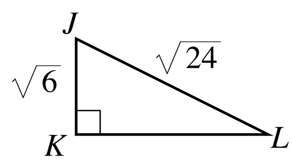
b)



c)



Multiple Choice **15.** The length of *KL* can be represented by which of the following?



A. $\sqrt{540}$
B. $3\sqrt{2}$
C. $\sqrt{30}$
D. $9\sqrt{2}$

Use the following information to answer question 16.

Devon read in his physics book that the distance, *d* kilometres, a person can see to the horizon on a clear day can be approximated by the formula $d = \sqrt{13h}$, where *h* is the person's eye level distance above the ground in metres. When standing on the ground, Devon's eye level distance is 1.8 m above the ground.

Numerical Response **16.** The distance Devon can see to the horizon when he is standing on the roof of a building 698.2 m high can be represented in simplest form by the expression $a\sqrt{b}$.

The value of $a + b$ is _____ .
(Record your answer in the numerical response box from left to right)

Use the following information to answer question #17.

In Ancient Greece a formula was developed which could be used to calculate the area of a triangle. The formula, known as Heron's Formula, is shown below.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

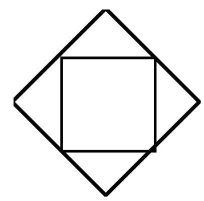
where a, b, and c are the lengths of the sides of a triangle,

and $s = \dfrac{a+b+c}{2}$

17. The area of a triangle whose sides measure 14, 15, and 25 can be written in simplest form as $a\sqrt{26}$, where $a \in N$. The value of a is _____.

(Record your answer in the numerical response box from left to right)

18. The smaller square has side length 8 cm. The side length of the larger square can be written in simplest form as $p\sqrt{q}$, where $p, q \in N$. The value of pq is _____.



(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** true **b)** false **c)** true **d)** true **e)** false **f)** false **g)** false **h)** true **i)** false

2. **a)** $\sqrt{35}$ **b)** $\sqrt{28}$ **c)** $\sqrt{24}$ **d)** $\sqrt{66}$ **e)** $\sqrt{2}$ **f)** $\sqrt{5}$ **g)** $\sqrt{3}$ **h)** $\sqrt{30}$

3. **a)** $\sqrt{5}\ \sqrt{7}$ **b)** $\sqrt{3}\ \sqrt{11}$ **c)** $\sqrt{5}\ \sqrt{13}$ **d)** $\sqrt{7}\ \sqrt{7}$ **4.** **a)** entire **b)** mixed **c)** entire **d)** mixed

5. **a)** $2\sqrt{2}$ **b)** $2\sqrt{5}$ **c)** $5\sqrt{3}$ **d)** $7\sqrt{2}$ **e)** $12\sqrt{2}$ **f)** $-15\sqrt{5}$ **g)** $6\sqrt{6}$ **h)** $-16\sqrt{3}$

6. **a)** $4\sqrt{6}$ **b)** $11\sqrt{2}$ **c)** $-4\sqrt{5}$ **d)** $\sqrt{5}$ **e)** $7\sqrt{5}$ **f)** $52\sqrt{2}$ **g)** $25\sqrt{2}$

h) cannot be converted because 66 does not have a factor which is a perfect square.

i) $-\frac{10}{3}\sqrt{19}$ **j)** $14\sqrt{5}$ **k)** $16\sqrt{17}$ **l)** $-36\sqrt{2}$ **m)** $6\sqrt{41}$ **n)** $2\sqrt{91}$ **o)** $6\sqrt{2}$

p) cannot be converted because 341 does not have a factor which is a perfect square.

7. **a)** $\frac{1}{3}\sqrt{2}$ **b)** $\frac{1}{2}\sqrt{5}$ **c)** $\frac{3}{5}\sqrt{2}$ **d)** $2\sqrt{5}$

8. **a)** $\sqrt{24}$ **b)** $\sqrt{63}$ **c)** $\sqrt{375}$ **d)** $\sqrt{288}$ **e)** $\sqrt{225}$ **f)** $-\sqrt{192}$ **g)** $\sqrt{810}$ **h)** $-\sqrt{80}$

9. **a)** $\sqrt{3}$ **b)** $\sqrt{225}$ **c)** $\sqrt{18}$ **d)** $\sqrt{1701}$

10. **a)** 24.5 **b)** 77.5 **c)** 775 **d)** 0.245 **e)** 0.775 **f)** 4.9 **g)** 23.25 **h)** 0.49

11. $5\sqrt{3}$, $3\sqrt{7}$, $\sqrt{60}$, $\frac{1}{2}\sqrt{200}$, $2\sqrt{11}$

12. **a)** Louis 8.48, Asia 8.49

b) Asia because she used exact values rather than rounded values in her calculation. **c)** $6\sqrt{2}$

13. **i)** $\sqrt{336}$ cm **ii)** $4\sqrt{21}$ cm **iii)** 18.33 cm **14.** **a)** $4\sqrt{5}$ **b)** $\sqrt{61}$ **c)** $2\sqrt{11}$

15. B **16.** 1 0 1 **17.** 1 8 **18.** 1 6

Real Numbers and Radicals Lesson #4:
Entire Radicals and Mixed Radicals - Part Two

Converting Entire Radicals (with an index of 3 or greater) to Mixed Radicals

An entire radical of index 3 may be expressed as a mixed radical when the highest perfect cube has been factored out of the entire radical.

1^3	=	1
2^3	=	8
3^3	=	27
4^3	=	64
5^3	=	125
6^3	=	216
7^3	=	343
8^3	=	512
9^3	=	729
10^3	=	1000

Complete the following to convert $\sqrt[3]{54}$ to a mixed radical.

Entire Radical $\Rightarrow$ *Mixed Radical*

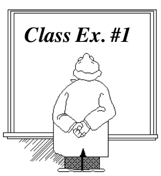
$$\sqrt[3]{54} = \sqrt[3]{\underline{\qquad} \times 2}$$

$$= \sqrt[3]{\underline{\qquad}} \times \sqrt[3]{2}$$

$$= 3 \times \sqrt[3]{2}$$

$$\sqrt[3]{54} =$$

A similar process is involved for indices greater than 3.
Questions at this level will involve simple perfect cubes, etc.



Convert the following radicals to mixed radicals in simplest form.

a) $\sqrt[3]{6000}$ **b)** $\sqrt[5]{320}$ **c)** $\sqrt[3]{-16}$

Converting Mixed Radicals (with an index of 3 or greater) to Entire Radicals

A mixed radical of index 3 may be expressed as an entire radical by converting the number outside the radical symbol into a radical and then multiplying it by the radicand.

The number outside the radical symbol can be converted into a radical by raising it to the power of 3.

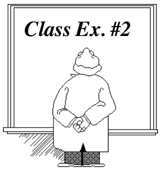
Complete the following to convert $\frac{1}{2}\sqrt[3]{80}$ to an entire radical.

Mixed Radical $\Rightarrow$ *Entire Radical*

$$\frac{1}{2}\sqrt[3]{80} = \sqrt[3]{\qquad} \times \sqrt[3]{80}$$

$$= \sqrt[3]{\qquad \times 80}$$

$$\frac{1}{2}\sqrt[3]{80} =$$



Convert the following mixed radicals to entire radicals.

a) $2\sqrt[4]{3}$ **b)** $-4\sqrt[3]{7}$ **c)** $\frac{2}{5}\sqrt[3]{100}$ **d)** $-3\sqrt[4]{2}$

Complete Assignment Questions #1 - #3

Radicals involving Variables

Since $x^3 \times x^3 = x^6$ then $\sqrt{x^6}$ = ___ . Also, since $x^5 \times x^5 \times x^5 = x^{15}$ then $\sqrt[3]{x^{15}}$ = ___

So $\sqrt{x^4}$ = ____. $\sqrt{y^{10}}$ = ____. $\sqrt{a^8b^6}$ = _____. $\sqrt[3]{x^{24}}$ = ____. $\sqrt[3]{y^6}$ = ____.

Complete the following to convert $\sqrt{x^5}$ to a mixed radical.

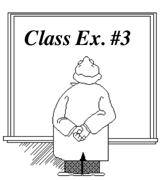
Entire Radical $\Rightarrow$ *Mixed Radical*

$\sqrt{x^5} = \sqrt{x^4 \times x}$

$= \sqrt{\quad\quad} \times \sqrt{x}$

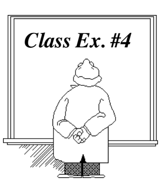
$=$ ____ $\times \sqrt{x}$

$\sqrt{x^5} =$



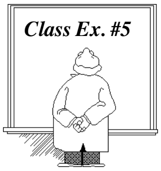
Convert the following entire radicals to mixed radicals in simplest form.

a) $\sqrt{a^7}$ **b)** $\sqrt{t^9}$ **c)** $\sqrt[3]{x^5}$ **d)** $\sqrt[3]{x^7}$



Convert the following entire radicals to mixed radicals in simplest form.

a) $\sqrt{x^6y^5}$ **b)** $\sqrt{18x^3}$ **c)** $\sqrt{32y^7z^8}$ **d)** $\sqrt[3]{40x^4y^9}$



Convert the following mixed radicals to entire radicals.

a) $2\sqrt{x^3}$ **b)** $a^2\sqrt{a}$ **c)** $x^5\sqrt{xy}$ **d)** $3xy^3\sqrt[3]{2z^4}$

Complete Assignment Questions #4 - #9

Assignment

1. Convert the following radicals to mixed radicals in simplest form.

a) $\sqrt[3]{48}$ **b)** $\sqrt[3]{128}$ **c)** $\sqrt[3]{2000}$ **d)** $5\sqrt[3]{-81}$

e) $\frac{5}{6}\sqrt[3]{108}$ **f)** $5\sqrt[4]{162}$ **g)** $\sqrt[5]{-192}$ **h)** $-2\sqrt[3]{625}$

2. Convert the following mixed radicals to entire radicals.

a) $2\sqrt[4]{2}$ **b)** $3\sqrt[3]{4}$ **c)** $-3\sqrt[4]{3}$ **d)** $-10\sqrt[3]{5}$

e) $2\sqrt[5]{6}$ **f)** $\frac{1}{2}\sqrt[3]{16}$ **g)** $\frac{3}{10}\sqrt[4]{100000}$ **h)** $-5\sqrt[3]{9}$

Do not use a calculator to answer question #3.

3. Arrange the following radicals in order from least to greatest.

$7\sqrt[6]{1}$, $-3\sqrt[3]{-27}$, $\frac{5}{2}\sqrt[4]{16}$, $3\sqrt{\sqrt[3]{64}}$

4. Express as an entire radical.

a) $6\sqrt{y}$ **b)** $8\sqrt{c^2}$ **c)** $10\sqrt{2yz^3}$ **d)** $-3\sqrt[3]{x^2}$

e) $c\sqrt{c}$ **f)** $x\sqrt{3y^3}$ **g)** $11c^2\sqrt{c^2d}$ **h)** $5a^3b\sqrt{3a^2b}$

i) $4\sqrt{3}\ a^2b$ **j)** $2p^2q\sqrt[3]{5pq^2}$ **k)** $7p^8q^9\sqrt{p^2r}$ **l)** $2xy^3\sqrt[4]{9x}$

5. Express each as a mixed radical in simplest form.

a) $\sqrt{a^5}$ **b)** $\sqrt{t^3}$ **c)** $\sqrt{x^{11}}$ **d)** $\sqrt[3]{x^4}$ **e)** $\sqrt[3]{b^8}$ **f)** $\sqrt[4]{x^6}$

6. Express each as a mixed radical in simplest form.

a) $\sqrt{8y^2}$ **b)** $\sqrt{16p^3}$ **c)** $\sqrt{75y^3z^4}$

d) $\sqrt{300a^9w^7}$ **e)** $5\sqrt{28c^4d^3}$ **f)** $-6\sqrt{29a^4b^8}$

g) $7p^3q^2\sqrt{27p^5q^6}$ **h)** $\frac{2}{3}c\sqrt{81c^3d^{12}}$ **i)** $11x^7y^{15}\sqrt{242x^9y^{10}}$

j) $\sqrt[3]{2000x^7}$ **k)** $4\sqrt[3]{250b^{13}}$ **l)** $\sqrt[4]{32x^9}$

Multiple Choice

7. $\sqrt{3x}\ \sqrt{2x}$ is equivalent to

A. $\sqrt{6x}$
B. $\sqrt{36x^2}$
C. $6\sqrt{x}$
D. $x\sqrt{6}$

8. $\sqrt[3]{240}$ is equivalent to

A. $2\sqrt[3]{40}$
B. $4\sqrt[3]{15}$
C. $2\sqrt[3]{30}$
D. $8\sqrt[3]{30}$

Numerical Response

9. If $\sqrt{\sqrt[3]{64x^n}} = 2x^2\sqrt{x}$, where n is a whole number, then n is equal to _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $2\sqrt[3]{6}$ **b)** $4\sqrt[3]{2}$ **c)** $10\sqrt[3]{2}$ **d)** $-15\sqrt[3]{3}$ **e)** $\frac{5}{2}\sqrt[3]{4}$ **f)** $15\sqrt[4]{2}$ **g)** $-2\sqrt[5]{6}$ **h)** $-10\sqrt[3]{5}$

2. **a)** $\sqrt[4]{32}$ **b)** $\sqrt[3]{108}$ **c)** $-\sqrt[4]{243}$ **d)** $-\sqrt[3]{5000}$ or $\sqrt[3]{-5000}$ **e)** $\sqrt[5]{192}$ **f)** $\sqrt[3]{2}$ **g)** $\sqrt[4]{810}$
h) $-\sqrt[3]{1125}$ or $\sqrt[3]{-1125}$

3. $\frac{5}{2}\sqrt[4]{16}$, $3\sqrt{\sqrt[3]{64}}$, $7\sqrt[6]{1}$, $-3\sqrt[3]{-27}$

4. **a)** $\sqrt{36y}$ **b)** $\sqrt{64c^2}$ **c)** $\sqrt{200yz^3}$ **d)** $-\sqrt[3]{27x^2}$
e) $\sqrt{c^3}$ **f)** $\sqrt{3x^2y^3}$ **g)** $\sqrt{121c^6d}$ **h)** $\sqrt{75a^8b^3}$
i) $\sqrt{48a^4b^2}$ **j)** $\sqrt[3]{40p^7q^5}$ **k)** $\sqrt{49p^{18}q^{18}r}$ **l)** $\sqrt[4]{144x^5y^{12}}$

5. **a)** $a^2\sqrt{a}$ **b)** $t\sqrt{t}$ **c)** $x^5\sqrt{x}$ **d)** $x\sqrt[3]{x}$ **e)** $b^2\sqrt[3]{b^2}$ **f)** $x\sqrt[4]{x^2}$

6. **a)** $2y\sqrt{2}$ **b)** $4p\sqrt{p}$ **c)** $5yz^2\sqrt{3y}$` **d)** $10a^4w^3\sqrt{3aw}$
e) $10c^2d\sqrt{7d}$ **f)** $-6a^2b^4\sqrt{29}$ **g)** $21p^5q^5\sqrt{3p}$ **h)** $6c^2d^6\sqrt{c}$
i) $121x^{11}y^{20}\sqrt{2x}$ **j)** $10x^2\sqrt[3]{2x}$` **k)** $20b^4\sqrt[3]{2b}$ **l)** $2x^2\sqrt[4]{2x}$

7. D **8.** C **9.**

1	5		

Real Numbers and Radicals Lesson #5:
Adding and Subtracting Radicals

Warm-Up #1

In Radicals Lesson #2 we discovered that $\sqrt{9} + \sqrt{4}$ is not equal to $\sqrt{9+4}$ and that $\sqrt{9} - \sqrt{4}$ is not equal to $\sqrt{9-4}$. So adding and subtracting radicals cannot be accomplished by simply adding or subtracting radicands.

In order to develop a rule for adding and subtracting radicals, complete the work below.

a) Use a calculator to investigate which of the following radical statements are true. Circle the statements which are true and place a single line through the expressions which are false.

i) $\sqrt{2} + 5\sqrt{2} = 6\sqrt{2}$

ii) $4\sqrt[3]{5} - 7\sqrt[3]{5} = -3\sqrt[3]{5}$

iii) $5\sqrt{8} - 2\sqrt{8} + 7\sqrt{8} = 10\sqrt{8}$

iv) $7\sqrt{5} + 7\sqrt[3]{5} = 14\sqrt[5]{5}$

v) $\sqrt[3]{3} + \sqrt[3]{2} = \sqrt[3]{5}$

b) Use the results in a) to suggest a rule for adding and subtracting radicals

c) Simplify the following. Express the answer as a mixed radical.

i) $8\sqrt{7} - 3\sqrt{7} + 15\sqrt{7}$

ii) $18\sqrt[5]{10} + 12\sqrt[5]{10} - 30\sqrt[5]{10} - 7\sqrt[5]{10}$

Warm-Up #2

a) Use a calculator to verify that the following statements are true.

i) $\sqrt{2} + \sqrt{8} = 3\sqrt{2}$

ii) $5\sqrt{12} + 6\sqrt{48} = 34\sqrt{3}$

b) Does this appear to contradict the rule you wrote in Warm-Up #1 b)?

c) Complete the following by writing each radical in simplest mixed form to show that the rule can be modified.

i) $\sqrt{2} + \sqrt{8}$

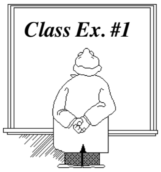
$= \sqrt{2} +$

ii) $5\sqrt{12} + 6\sqrt{48}$

$=$

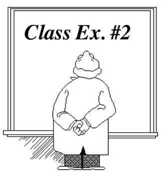
Adding and Subtracting Radicals

In order to add and subtract radicals, they must be able to be expressed as **like radicals**, ie. radicals with the SAME radicand **and** the SAME index.



Write each expression in terms of a single radical.

a) $\sqrt{80} - \sqrt{20}$ **b)** $\sqrt[3]{80} + \sqrt[3]{270}$ **c)** $7\sqrt{27} - 3\sqrt{75} + 2\sqrt{147}$



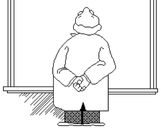
Simplify by combining like radicals.

a) $-5\sqrt{108} + \frac{3}{4}\sqrt{8} - \frac{5}{4}\sqrt{48} + \frac{1}{2}\sqrt{50}$

b) $\frac{\sqrt[3]{64}}{8} + 2\sqrt[3]{375} - \frac{2\sqrt[3]{54}}{3} - \frac{5\sqrt[3]{24}}{2}$

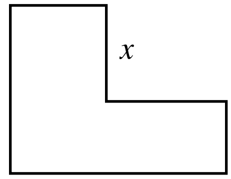
Complete Assignment Questions #1 - #5

Class Ex. #3



Find the length of x

a) as an exact value
b) as a decimal to the nearest tenth.



Complete Assignment Questions #6 - #12

Assignment

1. Simplify.

a) $5\sqrt{7} - 2\sqrt{7}$ **b)** $\sqrt{3} + 4\sqrt{3}$ **c)** $4\sqrt{11} - 9\sqrt{11} + \sqrt{11}$

d) $4\sqrt{5} - 2\sqrt{2} + 8\sqrt{2}$ **e)** $-3\sqrt{2} + 6\sqrt{3} - 9\sqrt{3} + 4\sqrt{2}$

2. Write each expression in terms of a single radical.

a) $\sqrt{125} - \sqrt{5}$ **b)** $\sqrt{27} + \sqrt{12}$ **c)** $\sqrt{24} - \sqrt{54} + 2\sqrt{6}$

d) $\sqrt{150} + \sqrt{216}$ **e)** $\sqrt[3]{16} + \sqrt[3]{128}$ **f)** $-3\sqrt{175} + 8\sqrt{28} - \sqrt{63}$

g) $\sqrt[4]{16} + \sqrt[4]{162}$ **h)** $2\sqrt{700} - 6\sqrt{63}$ **i)** $-7\sqrt[3]{54} - 2\sqrt[3]{250}$

3. Simplify by combining like radicals.

a) $\sqrt{20} + \sqrt{72} - \sqrt{45}$

b) $\sqrt{27} + \sqrt{12} - \sqrt{32} - \sqrt{8}$

c) $\sqrt{98} - \sqrt{20} + \sqrt{18}$

d) $2\sqrt{252} - \sqrt{726} - 5\sqrt{63}$

e) $-3\sqrt{810} - 6\sqrt{360} + 3\sqrt{1440}$

f) $12\sqrt{150} - 5\sqrt{54} + 3\sqrt{24}$

g) $2\sqrt[3]{108} + \sqrt[3]{32} + 3\sqrt[3]{256}$

h) $8\sqrt{45} + 7\sqrt{243} + \sqrt{507} - \sqrt{169}$

4. Write in simplest radical form.

$$\sqrt[3]{128} + 3\sqrt[3]{375} - 7\sqrt[3]{27} - 2\sqrt[3]{250} - 5\sqrt[3]{432} + 8\sqrt[3]{2000}$$

5. Simplify.

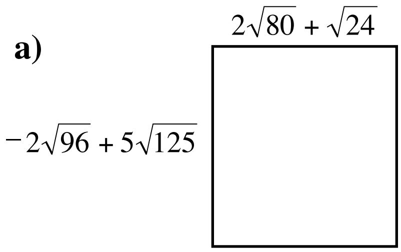
a) $\frac{1}{3}\sqrt{63} + \frac{2}{5}\sqrt{700} - \frac{2}{3}\sqrt{112} + \frac{3}{2}\sqrt{28}$

b) $\frac{3\sqrt{200}}{5} + 5\sqrt{20} - \frac{4\sqrt{500}}{5} + \frac{3\sqrt{363}}{11}$

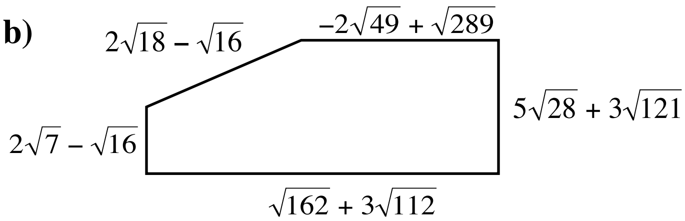
c) $\frac{7\sqrt[3]{1024}}{2} + \frac{5\sqrt[3]{2000}}{12} - 3\sqrt[3]{686} + \frac{1}{8}\sqrt[3]{128}$

6. Find the perimeter of the following figures.

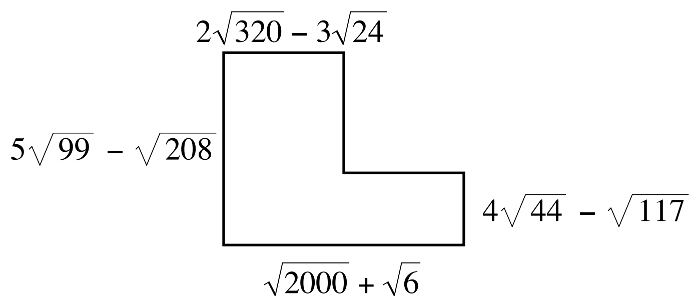
a)



b)



7. Find the length of the missing sides.



Multiple Choice

8. $\sqrt{75} + \sqrt{3}$ equals

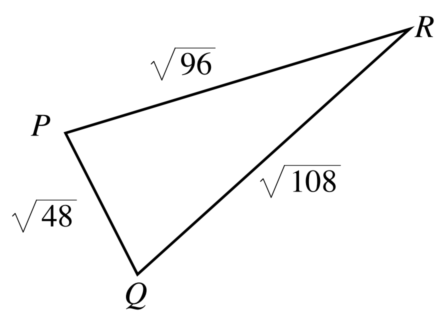
A. $6\sqrt{3}$
B. $26\sqrt{3}$
C. $\sqrt{78}$
D. $3\sqrt{5} + \sqrt{3}$

9. Given that $x - 2\sqrt{5} = \sqrt{45}$, then $\sqrt{5} + x$ is equal to

A. $2\sqrt{5}$
B. $3\sqrt{5}$
C. $4\sqrt{5}$
D. $6\sqrt{5}$

10. In simplest radical form the perimeter of $\Delta\, PQR$ is

A. $\sqrt{252}$
B. $6\sqrt{7}$
C. $10\sqrt{3} + 4\sqrt{6}$
D. $52\sqrt{3} + 16\sqrt{6}$



Numerical Response

11. When simplified, the expression $\sqrt{52} + \sqrt{208} - \sqrt{13} + \sqrt{169}$ can be written in the form $p\sqrt{13} + q$. The value of pq is _____.

(Record your answer in the numerical response box from left to right)

12. When simplified, the expression $\frac{9}{2}\sqrt[3]{48} + \frac{3}{4}\sqrt[3]{162} - \frac{3}{5}\sqrt[3]{750}$ can be written in the form $a\sqrt[3]{b}$. The value of a, to the nearest hundredth, is ______.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $3\sqrt{7}$ **b)** $5\sqrt{3}$ **c)** $-4\sqrt{11}$ **d)** $4\sqrt{5} + 6\sqrt{2}$ **e)** $\sqrt{2} - 3\sqrt{3}$

2. **a)** $4\sqrt{5}$ **b)** $5\sqrt{3}$ **c)** $\sqrt{6}$ **d)** $11\sqrt{6}$ **e)** $6\sqrt[3]{2}$ **f)** $-2\sqrt{7}$ **g)** $2 + 3\sqrt[4]{2}$

h) $2\sqrt{7}$ **i)** $-31\sqrt[3]{2}$

3. **a)** $6\sqrt{2} - \sqrt{5}$ **b)** $5\sqrt{3} - 6\sqrt{2}$ **c)** $10\sqrt{2} - 2\sqrt{5}$ **d)** $-3\sqrt{7} - 11\sqrt{6}$

e) $-27\sqrt{10}$ **f)** $51\sqrt{6}$ **g)** $20\sqrt[3]{4}$ **h)** $24\sqrt{5} + 76\sqrt{3} - 13$

4. $15\sqrt[3]{3} + 44\sqrt[3]{2} - 21$

5. **a)** $\frac{16}{3}\sqrt{7}$ **b)** $6\sqrt{2} + 2\sqrt{5} + 3\sqrt{3}$ **c)** $\frac{35}{3}\sqrt[3]{2}$

6. **a)** $66\sqrt{5} - 12\sqrt{6}$ **b)** $24\sqrt{7} + 15\sqrt{2} + 28$

7. $7\sqrt{11} - \sqrt{13}$, $4\sqrt{5} + 7\sqrt{6}$

8. A **9.** D **10.** C

11

6	5		

12.

8	.	2	5

Real Numbers and Radicals Lesson #6:
Multiplying Radicals

Warm-Up #1 *Investigating Multiplication Properties of Radicals*

Use a calculator to determine whether the following expressions are true or false.

a) $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ **b)** $\left(2\sqrt{5}\right)\left(-4\sqrt{3}\right) = -8\sqrt{15}$ **c)** $\sqrt{2} \cdot \sqrt[3]{4} = \sqrt[3]{8}$

d) $2\sqrt[3]{10} \times 3\sqrt[3]{7} = 6\sqrt[3]{70}$ **e)** $\left(4\sqrt[3]{5}\right)\left(7\sqrt{6}\right) = 28\sqrt[6]{30}$

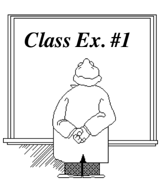
Write a rule which describes the process of multiplying radicals based on the results from a) - e).

Multiplying Radicals

To multiply radicals the index must be the same in each radical.

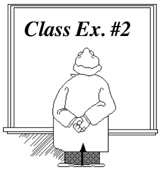
- multiply numerical coefficients by numerical coefficients
- multiply radicand by radicand
- simplify into mixed radical form if possible.

not*e* It is usually easier to convert each radical to simplest mixed form before multiplying



Multiply and simplify where possible.

a) $\sqrt{8} \cdot \sqrt{8}$ **b)** $\left(4\sqrt{5}\right)\left(3\sqrt{6}\right)$ **c)** $-2\sqrt{8} \times 5\sqrt{12}$



Expand and simplify

a) $\sqrt{5}\left(2\sqrt{10} - \sqrt{5}\right)$

b) $2\sqrt{5}\left(3\sqrt{45} - 8\sqrt{5} + 3\sqrt{20}\right)$

c) $\left(4 + \sqrt{6}\right)\left(\sqrt{6} - \sqrt{12}\right)$

d) $\left(2\sqrt{18} - \sqrt{27}\right)^2$

e) $2\left(\sqrt{3} - \sqrt{5}\right) - \sqrt{2}\left(\sqrt{6} + \sqrt{10}\right)$

Complete Assignment Questions #1 - #10

Multiplying Conjugate Binomials

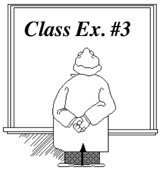
Expand the following expressions:

i) $\left(\sqrt{5} - \sqrt{2}\right)\left(\sqrt{5} + \sqrt{2}\right)$

ii) $\left(-2\sqrt{7} + 8\right)\left(-2\sqrt{7} - 8\right)$

The pairs of binomials above are called **conjugates** of each other.
What do you notice about the product of two conjugate binomials?

- Conjugate binomials are pairs of binomials in the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$.
- The product of conjugate binomials is always a rational number of the form $a^2b - c^2d$.



Write the conjugate of each. Then multiply each pair.

a) $4\sqrt{6} + 3$

b) $-3\sqrt{11} + \sqrt{2}$

Complete Assignment Questions #11 - #19

Assignment

1. Multiply and simplify where possible. Do not use a calculator.

a) $\left(\sqrt{7}\right)\left(\sqrt{3}\right)$

b) $4\sqrt{3} \times 2\sqrt{5}$

c) $8\sqrt{11} \cdot 5\sqrt{2}$

d) $\left(\sqrt{15}\right)\left(\sqrt{3}\right)$

e) $10\sqrt{5} \times 9\sqrt{5}$

f) $3\sqrt{6} \cdot 5\sqrt{10}$

g) $\left(\sqrt{18}\right)\left(\sqrt{50}\right)$

h) $-3\sqrt{5} \times 2\sqrt{2}$

i) $7\sqrt{54} \cdot 2\sqrt{6}$

j) $\left(\sqrt{32}\right)\left(\sqrt{6}\right)$

k) $\sqrt{15} \times 3\sqrt{27}$

l) $3\sqrt{20} \times 4\sqrt{45}$

2. Write each radical as the product of two mixed radicals

a) $15\sqrt{18}$

b) $35\sqrt{6}$

3. Express in simplest form. Do not use a calculator.

a) $\left(\sqrt{3}\right)^2$ **b)** $\left(4\sqrt{2}\right)^2$ **c)** $\left(-3\sqrt{5}\right)^2$ **d)** $-\left(\sqrt{12}\right)^2$ **e)** $\left(\sqrt{5}\right)^3$

4. Express in simplest form.

a) $\sqrt{5} \times 2\sqrt{3} \times 3\sqrt{2}$ **b)** $2\sqrt{6} \times 2\sqrt{3} \times 3\sqrt{2}$ **c)** $\left(-2\sqrt{6}\right)\left(2\sqrt{3}\right)\left(-3\sqrt{5}\right)$

d) $\left(\frac{2}{3}\sqrt{27}\right)\left(\sqrt{6}\right)$ **e)** $2\sqrt{\frac{8}{25}} \times 5\sqrt{2}$ **f)** $3\sqrt[3]{16} \times 2\sqrt[3]{4} \times 2\sqrt[3]{2}$

5. Consider the product $6\sqrt{5} \times 3\sqrt{8}$.

a) Use a **two decimal place approximation** for each radical to determine a two decimal place approximation for the product.

b) Determine the **exact value** of the product as a mixed radical in simplest form.

c) Determine a **two decimal place approximation** to the answer in b).

d) Which of the two decimal place approximations is more precise? Explain.

6. Expand and simplify where possible.

a) $\sqrt{6}\left(2\sqrt{6} - \sqrt{5}\right)$ **b)** $\sqrt{2}\left(1 - \sqrt{2}\right)$ **c)** $2\sqrt{3}\left(2\sqrt{7} - 4\sqrt{5}\right)$

7. Expand and simplify.

a) $\sqrt{3}\left(2\sqrt{6} - \sqrt{12}\right)$ **b)** $\sqrt{8}\left(\sqrt{6} - \sqrt{2}\right)$ **c)** $2\sqrt{10}\left(\sqrt{6} + 4\sqrt{5}\right)$

d) $2\sqrt{11}\left(3\sqrt{2} - \sqrt{50} + 3\sqrt{32}\right)$ **e)** $\sqrt{5}\left(3\sqrt{5} - \sqrt{75} + 3\sqrt{3}\right)$

f) $\sqrt{2}\left(\sqrt{5} - 12\sqrt{3}\right) - \sqrt{3}\left(\sqrt{8} - 2\sqrt{30}\right)$ **g)** $2\sqrt{3}\left(\sqrt{243} - 2\right) - \sqrt{2}\left(5 + 7\sqrt{2}\right)$

8. Simplify.

a) $\left(4+\sqrt{27}\right)\left(1-\sqrt{12}\right)$

b) $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{3}+\sqrt{2}\right)$

c) $\left(5\sqrt{8}-2\right)\left(3\sqrt{8}+4\right)$

d) $\left(2\sqrt{3}-\sqrt{10}\right)\left(\sqrt{6}-7\sqrt{20}\right)$

9. Expand and simplify.

a) $\left(5\sqrt{3}-2\right)^2$

b) $\left(4\sqrt{6}-\sqrt{2}\right)^2$

c) $\left(2\sqrt{12}+\sqrt{24}\right)^2$

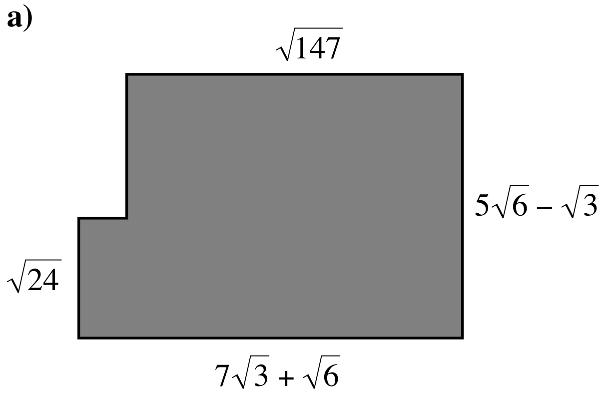
d) $\left(3\sqrt{208}-8\right)^2$

e) $2\left(\sqrt{15}-3\sqrt{5}\right)^2$

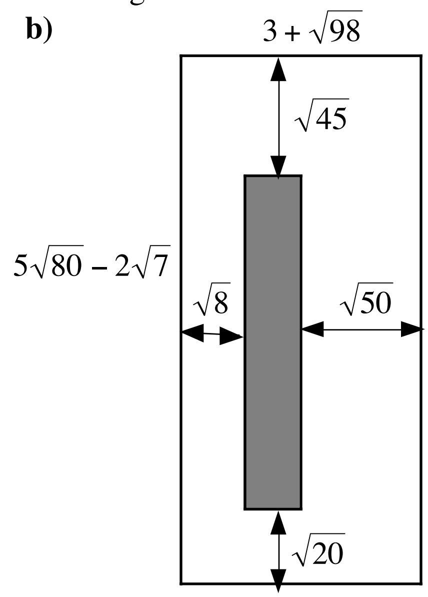
f) $\left(\sqrt{5}-3\sqrt{2}+\sqrt{10}\right)^2$

10. Find the area and perimeter of the shaded region in each figure.

a)



b)



11. Expand and simplify.

a) $\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)$

b) $\left(\sqrt{8}+\sqrt{7}\right)\left(\sqrt{8}-\sqrt{7}\right)$

c) $\left(2\sqrt{6}-\sqrt{2}\right)\left(2\sqrt{6}+\sqrt{2}\right)$

12. Write the conjugate of each.

a) $\sqrt{2} - \sqrt{5}$ **b)** $4 + \sqrt{7}$ **c)** $-3\sqrt{8} - 15$

13. Write the conjugate of each. Then multiply each pair.

a) $\sqrt{3} - 1$ **b)** $2 + \sqrt{5}$ **c)** $2\sqrt{6} - \sqrt{3}$

d) $2\sqrt{8} + \sqrt{27}$ **e)** $\sqrt{32} - \sqrt{3}$ **f)** $-3\sqrt{40} + 2\sqrt{10}$

Multiple Choice

14. For all values of a and b, $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$ is equal to

A. $\sqrt{(a - b)(a + b)}$
B. $a - b$
C. $a + b$
D. $a^2 - b^2$

15. $\left(\sqrt{2}\right)^5$ is equal to

A. $\sqrt{10}$
B. $5\sqrt{2}$
C. $4\sqrt{2}$
D. 32

16. If $2\sqrt[3]{4}\left(2\sqrt[3]{54} - \sqrt[3]{x}\right)$ is equal to 16, then x is equal to

A. 2
B. 4
C. 16
D. 64

17. The expression $\sqrt{5}\left(\sqrt{10} + 12\sqrt{5}\right) - \sqrt{7}\left(\sqrt{7} - 2\sqrt{14}\right)$ can be simplified to the form $a + b\sqrt{c}$ where a, b and c are integers. The value of $a + b + c$ is _____.

(Record your answer in the numerical response box from left to right)

18. If $p \oplus q$ means "$(p - q)$ multiplied by q" then the value of $\sqrt{6} \oplus \sqrt{3}$ can be simplified to the form $a + b\sqrt{c}$ where a, b and c are integers. The value of c is _____.

(Record your answer in the numerical response box from left to right)

19. If $m * n$ means "$(m + n)$ multiplied by m" then the value of $\sqrt{10} * \left(\sqrt{5} * \sqrt{2}\right)$ can be written as the sum of a rational number and an irrational number. The rational number is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $\sqrt{21}$ **b)** $8\sqrt{15}$ **c)** $40\sqrt{22}$ **d)** $3\sqrt{5}$ **e)** 450 **f)** $30\sqrt{15}$ **g)** 30

h) $-6\sqrt{10}$ **i)** 252 **j)** $8\sqrt{3}$ **k)** $27\sqrt{5}$ **l)** 360

2. Answers may vary **a)** $(3\sqrt{3})(5\sqrt{6})$ **b)** $(5\sqrt{2})(7\sqrt{3})$

3. **a)** 3 **b)** 32 **c)** 45 **d)** -12 **e)** $5\sqrt{5}$

4. **a)** $6\sqrt{30}$ **b)** 72 **c)** $36\sqrt{10}$ **d)** $6\sqrt{2}$ **e)** 8 **f)** $48\sqrt[3]{2}$

5. **a)** 113.94 **b)** $36\sqrt{10}$ **c)** 113.84 **d)** c) because rounding is not done until the last step.

6. **a)** $12 - \sqrt{30}$ **b)** $\sqrt{2} - 2$ **c)** $4\sqrt{21} - 8\sqrt{15}$

7. **a)** $6\sqrt{2} - 6$ **b)** $4\sqrt{3} - 4$ **c)** $4\sqrt{15} + 40\sqrt{2}$ **d)** $20\sqrt{22}$

e) $15 - 2\sqrt{15}$ **f)** $7\sqrt{10} - 14\sqrt{6}$ **g)** $40 - 4\sqrt{3} - 5\sqrt{2}$

8. **a)** $-14 - 5\sqrt{3}$ **b)** $5 + 2\sqrt{6}$ **c)** $112 + 28\sqrt{2}$ **d)** $76\sqrt{2} - 30\sqrt{15}$

9. **a)** $79 - 20\sqrt{3}$ **b)** $98 - 16\sqrt{3}$ **c)** $72 + 48\sqrt{2}$ **d)** $1936 - 192\sqrt{13}$

e) $120 - 60\sqrt{3}$ **f)** $33 - 6\sqrt{10} + 10\sqrt{2} - 12\sqrt{5}$

10. **a)** Area = $105\sqrt{2} - 9$, Perimeter = $12\sqrt{3} + 12\sqrt{6}$
b) Area = $45\sqrt{5} - 6\sqrt{7}$, Perimeter = $30\sqrt{5} - 4\sqrt{7} + 6$

11. **a)** 4 **b)** 1 **c)** 22

12. **a)** $\sqrt{2} + \sqrt{5}$ **b)** $4 - \sqrt{7}$ **c)** $-3\sqrt{8} + 15$

13. **a)** $\sqrt{3} + 1$, 2 **b)** $2 - \sqrt{5}$, -1 **c)** $2\sqrt{6} + \sqrt{3}$, 21

d) $2\sqrt{8} - \sqrt{27}$, 5 **e)** $\sqrt{32} + \sqrt{3}$, 29 **f)** $-3\sqrt{40} - 2\sqrt{10}$, 320

14. B **15.** C **16.** C

17.

7	4		

18.

2			

19.

2	0		

Real Numbers and Radicals Lesson #7:
Dividing Radicals - Part One

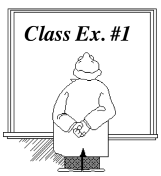
Dividing Radicals

In lesson 3 we discovered that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$, $a \geq 0,\ b > 0$, and $a, b \in \Re$.

We can use this rule to divide radicals of the form $\dfrac{m\sqrt{a}}{n\sqrt{b}}$.

To divide radicals the index must be the same in each radical.

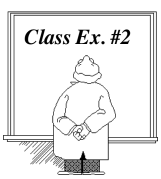
- divide numerical coefficients by numerical coefficients
- divide radicand by radicand
- simplify into mixed radical form if possible.



Divide and simplify where possible.

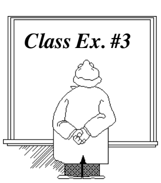
a) $\dfrac{\sqrt{30}}{\sqrt{6}}$ **b)** $\dfrac{8\sqrt{21}}{2\sqrt{3}}$ **c)** $\dfrac{15\sqrt{48}}{10\sqrt{6}}$

In some cases converting a radical into simplest mixed radical form before dividing will make the calculation easier.



Simplify numerator and denominator, then divide.

a) $\dfrac{4\sqrt{54}}{3\sqrt{8}}$ **b)** $\dfrac{8\sqrt{126}}{\sqrt{112}}$



Divide each term in the numerator by the denominator and simplify.

$$\frac{\sqrt{24} + \sqrt{48} - \sqrt{108}}{\sqrt{6}}$$

Complete Assignment Questions #1 - #4

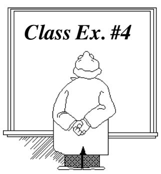
Rationalizing the Denominator

Usually answers are written in **simplest form**, eg $\frac{1}{6} + \frac{1}{3} = \frac{3}{6}$ which simplifies to $\frac{1}{2}$.

In the division of radicals in this unit, regard simplest form as the form in which

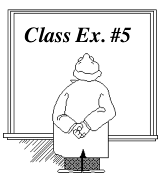
i) the denominator of the fraction is a rational number, ie it does not contain a radical.
ii) the radicand cannot contain a fraction and is expressed in simplest mixed form.

The process of eliminating the radical from the denominator (i.e. converting the denominator from an irrational number to a rational number) is called **rationalizing the denominator**. The denominators in this lesson are all of monomial form. Denominators in binomial form will be discussed in the next lesson.



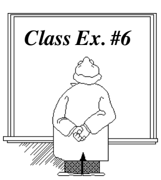
Simplify by rationalizing the denominator.

a) $\frac{1}{\sqrt{13}}$ **b)** $\frac{\sqrt{5}}{\sqrt{2}}$ **c)** $\frac{\sqrt{2}}{-\sqrt{6}}$ **d)** $\frac{\sqrt{20}}{\sqrt{3}}$



Simplify.

a) $\frac{7}{3\sqrt{7}}$ **b)** $\sqrt{\frac{18}{5}}$ **c)** $\frac{3\sqrt{12}}{\sqrt{72}}$



Simplify the radical expression $\frac{3\sqrt{18} - \sqrt{12}}{\sqrt{2}}$ by:

a) rationalising the denominator first **b)** dividing numerator and denominator by $\sqrt{2}$

Complete Assignment Questions #5 - #18

Assignment

1. Simplify.

a) $\dfrac{\sqrt{50}}{\sqrt{5}}$ **b)** $\dfrac{\sqrt{35}}{\sqrt{7}}$ **c)** $\dfrac{\sqrt{39}}{\sqrt{3}}$ **d)** $\dfrac{\sqrt{28}}{\sqrt{7}}$

e) $\dfrac{8\sqrt{42}}{2\sqrt{6}}$ **f)** $\dfrac{25\sqrt{88}}{5\sqrt{8}}$ **g)** $\dfrac{12\sqrt{51}}{-6\sqrt{17}}$ **h)** $\dfrac{4\sqrt{50}}{8\sqrt{10}}$

2. Simplify.

a) $\dfrac{\sqrt{270}}{\sqrt{10}}$ **b)** $\dfrac{\sqrt{90}}{\sqrt{5}}$ **c)** $\dfrac{\sqrt{96}}{4\sqrt{3}}$ **d)** $\dfrac{3\sqrt{200}}{2\sqrt{5}}$

3. Simplify.

a) $\dfrac{2\sqrt{150}}{\sqrt{8}}$ **b)** $\dfrac{4\sqrt{90}}{\sqrt{72}}$ **c)** $\dfrac{3\sqrt{240}}{\sqrt{108}}$ **d)** $\dfrac{18\sqrt{24}}{\sqrt{162}}$

4. Simplify.

a) $\dfrac{\sqrt{35} - \sqrt{21}}{\sqrt{7}}$ **b)** $\dfrac{9\sqrt{20} - 3\sqrt{10}}{3\sqrt{2}}$ **c)** $\dfrac{8\sqrt{42} + 12\sqrt{75}}{4\sqrt{3}}$

d) $\dfrac{8\sqrt{20} + 10\sqrt{125}}{2\sqrt{5}}$ **e)** $\dfrac{\sqrt{75} + \sqrt{48} - \sqrt{27}}{\sqrt{3}}$ **f)** $\dfrac{\sqrt{90} + 2\sqrt{40} - \sqrt{160}}{\sqrt{5}}$

5. Simplify by rationalizing the denominator.

a) $\dfrac{1}{\sqrt{2}}$ **b)** $\dfrac{6}{\sqrt{6}}$ **c)** $\dfrac{\sqrt{5}}{\sqrt{3}}$ **d)** $\dfrac{\sqrt{3}}{-\sqrt{2}}$

e) $\dfrac{\sqrt{10}}{\sqrt{7}}$ **f)** $\dfrac{\sqrt{12}}{\sqrt{5}}$ **g)** $\dfrac{2}{5\sqrt{6}}$ **h)** $\dfrac{\sqrt{32}}{\sqrt{18}}$

i) $\dfrac{5}{\sqrt{50}}$ **j)** $\dfrac{14}{\sqrt{98}}$ **k)** $\dfrac{-2}{\sqrt{88}}$ **l)** $\dfrac{3\sqrt{500}}{-\sqrt{27}}$

6. Simplify.

a) $\sqrt{\dfrac{27}{10}}$ **b)** $\dfrac{5\sqrt{14}}{\sqrt{70}}$ **c)** $\sqrt{\dfrac{243}{2}}$ **d)** $\dfrac{20\sqrt{12}}{12\sqrt{20}}$

7. Express the following with rational denominators.

a) $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{2}}$ **b)** $\dfrac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}$ **c)** $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{6}}$

8. a) Students are asked to simplify the radical expression $\dfrac{6\sqrt{40}-8\sqrt{20}}{2\sqrt{5}}$.

Erica decides to simplify the expression by rationalizing the denominator whereas Jaclyn divides each term in the numerator by the denominator. Determine the simplification by each method and state which method you prefer.

b) Without doing the simplification**, explain** why Jaclyn's method would be more difficult if the radical expression was $\dfrac{6\sqrt{40}-8\sqrt{20}}{2\sqrt{7}}$.

9. Simplify and express in lowest terms.

a) $\dfrac{10\sqrt{18} - 5\sqrt{24}}{\sqrt{5}}$

b) $\dfrac{15\sqrt{18} - 3\sqrt{242}}{-3\sqrt{8}}$

10. Simplify

a) $\dfrac{6\sqrt{18} + 5\sqrt{20} - 3\sqrt{72} - 6\sqrt{125}}{\sqrt{5}}$

b) $\dfrac{7\sqrt{50} + 8\sqrt{48} - 12\sqrt{75} - 8\sqrt{18}}{2\sqrt{6}}$

11. A rectangular garden has length $3\sqrt{6}$ metres and area $\left(9\sqrt{2}-6\sqrt{3}\right)$ square meters.

a) Write and simplify an expression for the width of the garden.

b) Determine the perimeter of the garden to the nearest tenth of a metre.

12. A triangle has an area of $\left(3\sqrt{288}-2\sqrt{12}\right)$ square metres with a base of $3\sqrt{2}$ metres. Express the height of the triangle

a) as an exact value in simplest form **b)** as a decimal to the nearest 0.01 m.

Do not use a calculator to answer question #13.

Multiple Choice

13. Which of the following expressions is **not** equivalent to the others?

A. $\dfrac{36}{\sqrt{48}}$

B. $\left(\sqrt{3}\right)^3$

C. $\sqrt{192} - \sqrt{75}$

D. $\dfrac{\sqrt{54}}{\sqrt{3}}$

14. $\dfrac{2 + \sqrt{8}}{2}$ can be simplified to

A. $1 + \sqrt{8}$
B. $1 + \sqrt{6}$
C. $1 + \sqrt{4}$
D. $1 + \sqrt{2}$

15. If $\dfrac{\sqrt{10} \times \sqrt{12}}{\sqrt{6}} = 2\sqrt{t}$, then t is equal to

A. $\sqrt{5}$
B. $\sqrt{10}$
C. 5
D. 10

Numerical Response

16. The expression $\dfrac{1}{\sqrt{27}} - \dfrac{5\sqrt{3}}{4\sqrt{24}}$ can be written in the form $a\sqrt{3} - b\sqrt{2}$, $a, b > 0$.
To the nearest hundredth, the value of b is _____.
(Record your answer in the numerical response box from left to right)

17. The expression $\dfrac{20\sqrt{5}}{\sqrt{10}} - \dfrac{16}{\sqrt{8}}$ can be expressed in the form $k\sqrt{2}$, where $k \in W$. The value of k is _____.

(Record your answer in the numerical response box from left to right)

18. When the equation $\sqrt{2} + a\sqrt{5} = \sqrt{72}$ is solved for a, the solution is $a = \sqrt{t}$, where $t \in W$. The value of t is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $\sqrt{10}$ **b)** $\sqrt{5}$ **c)** $\sqrt{13}$ **d)** 2 **e)** $4\sqrt{7}$ **f)** $5\sqrt{11}$ **g)** $-2\sqrt{3}$ **h)** $\frac{1}{2}\sqrt{5}$

2. **a)** $3\sqrt{3}$ **b)** $3\sqrt{2}$ **c)** $\sqrt{2}$ **d)** $3\sqrt{10}$

3. **a)** $5\sqrt{3}$ **b)** $2\sqrt{5}$ **c)** $2\sqrt{5}$ **d)** $4\sqrt{3}$

4. **a)** $\sqrt{5} - \sqrt{3}$ **b)** $3\sqrt{10} - \sqrt{5}$ **c)** $2\sqrt{14} + 15$ **d)** 33 **e)** 6 **f)** $3\sqrt{2}$

5. **a)** $\frac{1}{2}\sqrt{2}$ **b)** $\sqrt{6}$ **c)** $\frac{1}{3}\sqrt{15}$ **d)** $-\frac{1}{2}\sqrt{6}$ **e)** $\frac{1}{7}\sqrt{70}$ **f)** $\frac{2}{5}\sqrt{15}$ **g)** $\frac{1}{15}\sqrt{6}$
h) $\frac{4}{3}$ **i)** $\frac{1}{2}\sqrt{2}$ **j)** $\sqrt{2}$ **k)** $-\frac{1}{22}\sqrt{22}$ **l)** $-\frac{10}{3}\sqrt{15}$

6. **a)** $\frac{3}{10}\sqrt{30}$ **b)** $\sqrt{5}$ **c)** $\frac{9}{2}\sqrt{6}$ **d)** $\frac{1}{3}\sqrt{15}$

7. **a)** $\dfrac{\sqrt{14} - 2}{2}$ **b)** $\dfrac{3 + 2\sqrt{6}}{6}$ **c)** $\dfrac{\sqrt{30} + 2\sqrt{3}}{6}$

8. **a)** $6\sqrt{2} - 8$ probably Jaclyn's method **b)** 40 and 20 do not divide exactly by 7

9. **a)** $6\sqrt{10} - 2\sqrt{30}$ **b)** -2

10. **a)** -20 **b)** $\dfrac{11\sqrt{3} - 42\sqrt{2}}{6}$

11. **a)** $\sqrt{3} - \sqrt{2}$ meters **b)** 15.3 metres

12. **a)** $\dfrac{72 - 4\sqrt{6}}{3}$ meters **b)** 20.73 metres

13. D **14.** D **15.** C

16.

0	.	3	1

17.

6			

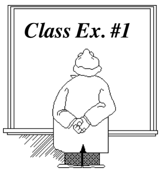
18.

1	0		

Real Numbers and Radicals Lesson #8:
Dividing Radicals - Part Two

Rationalizing a Denominator in Binomial Form

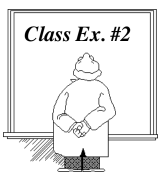
When the original denominator of the fraction is of binomial form, the process of rationalizing the denominator involves multiplying both numerator and denominator of the fraction by the **conjugate** of the binomial denominator.



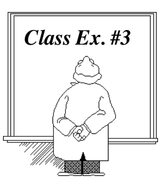
Simplify by rationalizing the denominator

a) $\dfrac{2}{\sqrt{5} - \sqrt{3}}$

b) $\dfrac{\sqrt{6} - 2}{\sqrt{6} + 2}$



Express $\dfrac{\sqrt{8} - \sqrt{3}}{4\sqrt{3} - \sqrt{2}}$ with a whole number in the denominator.



The area of a trapezoid is given by the formula $A = \frac{1}{2}h(a + b)$ where a and b are the lengths of the parallel sides and h is the shortest distance between the sides. If the area of a trapezoid is 20 cm^2 and the parallel sides are of lengths $\sqrt{6}$ cm and $\sqrt{5}$ cm determine the exact value of the distance between the parallel sides. Answer with a rational denominator.

Complete Assignment Questions #1 - #13

Assignment

1. Simplify by rationalizing the denominator.

a) $\dfrac{4}{\sqrt{5}-1}$ **b)** $\dfrac{1}{\sqrt{6}+2}$ **c)** $\dfrac{3}{3-\sqrt{3}}$

d) $\dfrac{\sqrt{7}}{\sqrt{7}-2}$ **e)** $\dfrac{3}{\sqrt{2}-\sqrt{3}}$ **f)** $\dfrac{\sqrt{2}}{\sqrt{6}+\sqrt{2}}$

2. Simplify by rationalizing the denominator.

a) $\dfrac{2\sqrt{3}}{3\sqrt{2}+\sqrt{3}}$ **b)** $\dfrac{3\sqrt{11}}{3\sqrt{11}+10}$

c) $\dfrac{\sqrt{2}}{\sqrt{12}-\sqrt{8}}$ **d)** $\dfrac{\sqrt{7}}{4-\sqrt{14}}$

3. Simplify leaving an integer in the denominator.

a) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

b) $\dfrac{\sqrt{5}-2}{\sqrt{5}-1}$

c) $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$

d) $\dfrac{5-\sqrt{10}}{3+\sqrt{10}}$

e) $\dfrac{\sqrt{11}+5\sqrt{2}}{\sqrt{11}-2\sqrt{2}}$

f) $\dfrac{2\sqrt{6}-\sqrt{3}}{3\sqrt{3}+\sqrt{6}}$

g) $\dfrac{\sqrt{30}+3\sqrt{3}}{\sqrt{30}-3\sqrt{3}}$

h) $\dfrac{3\sqrt{5}-2\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$

4. Simplify leaving a whole number in the denominator.

a) $\dfrac{5\sqrt{8} - 6\sqrt{5}}{4\sqrt{5} - 3\sqrt{2}}$

b) $\dfrac{2\sqrt{2} - 4\sqrt{3}}{-\sqrt{50} + \sqrt{75}}$

5. The area of a rectangle is 5 m^2 and the length is $3 + \sqrt{3}$ m. Calculate the width of the rectangle expressing the answer

i) as an exact value with a whole number in the denominator.
ii) as a decimal to the nearest hundredth

6. A triangle has area $\left(2\sqrt{15} - 3\sqrt{6}\right)$ square units and base $(\sqrt{15} + \sqrt{6})$ units. Determine the exact value of the height of the triangle giving the answer with a rational denominator.

Multiple Choice

7. The fraction $\dfrac{2}{\sqrt{5}-\sqrt{3}}$ expressed with a rational denominator is

A. $\dfrac{\sqrt{5}+\sqrt{3}}{4}$ **B.** $\dfrac{\sqrt{5}+\sqrt{3}}{8}$

C. $\sqrt{5}+\sqrt{3}$ **D.** $\dfrac{2\sqrt{5}+\sqrt{3}}{2}$

8. When $\dfrac{1}{2(2+\sqrt{3})}$ is expressed with a rational denominator the result is

A. $\dfrac{2-\sqrt{3}}{2}$ **B.** $\dfrac{2-\sqrt{3}}{-1}$

C. $\dfrac{2-\sqrt{3}}{14}$ **D.** $\dfrac{2-\sqrt{3}}{-10}$

9. When $\dfrac{3\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$ is expressed with a rational denominator in simplest form the result is

A. $\dfrac{33+5\sqrt{15}}{23}$

B. $\dfrac{33+5\sqrt{15}}{17}$

C. $\dfrac{27-\sqrt{15}}{23}$

D. $\dfrac{27-\sqrt{15}}{17}$

Numerical Response

10. When the denominator is rationalized, $\dfrac{\sqrt{10}-\sqrt{2}}{\sqrt{10}+\sqrt{2}}$ can be expressed in the form $a-b\sqrt{c}$, where $a, b, c \in Q$. The value of $a+b$, to the nearest tenth, is _____.

(Record your answer in the numerical response box from left to right)

11. The value, to the nearest hundredth, of the expression $\dfrac{8\sqrt{2}-\sqrt{5}}{6\sqrt{3}-\sqrt{8}}$ is _____ .

(Record your answer in the numerical response box from left to right)

Extension Questions.

12. Simplify by rationalizing the denominator.

a) $\dfrac{3}{2\sqrt{x}+3}$ **b)** $\dfrac{x+\sqrt{10}}{x-\sqrt{10}}$ **c)** $\dfrac{\sqrt{k}+\sqrt{2}}{\sqrt{k}-\sqrt{2}}$

Multiple Choice

13. $\dfrac{p}{q-\sqrt{r}}$, expressed with a rational denominator, may be written as

A. $\dfrac{p}{q^2-r}$ **B.** $\dfrac{p\left(q+\sqrt{r}\right)}{q^2-r^2}$

C. $\dfrac{p\left(q+\sqrt{r}\right)}{q^2-r}$ **D.** $\dfrac{p\left(q-\sqrt{r}\right)}{q^2+r}$

Answer Key

1. **a)** $\sqrt{5}+1$ **b)** $\dfrac{\sqrt{6}-2}{2}$ **c)** $\dfrac{3+\sqrt{3}}{2}$ **d)** $\dfrac{7+2\sqrt{7}}{3}$ **e)** $-3\sqrt{2}-3\sqrt{3}$ **f)** $\dfrac{\sqrt{3}-1}{2}$

2. **a)** $\dfrac{2\sqrt{6}-2}{5}$ **b)** $30\sqrt{11}-99$ **c)** $\dfrac{\sqrt{6}+2}{2}$ **d)** $\dfrac{4\sqrt{7}+7\sqrt{2}}{2}$

3. **a)** $2-\sqrt{3}$ **b)** $\dfrac{3-\sqrt{5}}{4}$ **c)** $2+\sqrt{3}$ **d)** $8\sqrt{10}-25$ **e)** $\dfrac{31+7\sqrt{22}}{3}$

f) $\sqrt{2}-1$ **g)** $19+6\sqrt{10}$ **h)** $\dfrac{19-4\sqrt{15}}{11}$

4. **a)** $\dfrac{11\sqrt{10}-30}{31}$ **b)** $\dfrac{-8-2\sqrt{6}}{5}$ **5.** **i)** $\dfrac{15-5\sqrt{3}}{6}$ m. **ii)** 1.06 m.

6. $\dfrac{32-10\sqrt{10}}{3}$ units. **7.** C **8.** A **9.** D

10.

2	.	0	

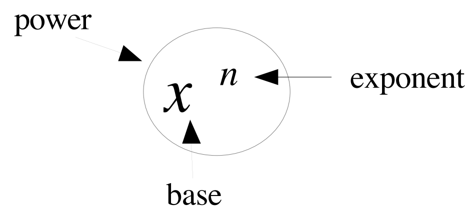
11.

1	.	2	0

12. **a)** $\dfrac{6\sqrt{x}-9}{4x-9}$ **b)** $\dfrac{x^2+2x\sqrt{10}+10}{x^2-10}$ **c)** $\dfrac{k+2\sqrt{2k}+2}{k-2}$ **13.** C

Exponents Lesson #1:
Review of Exponent Laws

Warm-Up *Review*



The exponent laws with integral exponents were covered in previous math courses.Complete the chart below as a review of the exponent laws. Negative exponents and the zero exponent will be covered in the next lesson.

Bases as Numbers	Bases as Variables	Exponent Laws
$8^3 \times 8^2 = (8 \cdot 8 \cdot 8)(8 \cdot 8)$ $= 8^5$ or $8^{\ \ +\ \ }$	$x^3 \times x^2 = (x \cdot x \cdot x)(x \cdot x)$ $= x^{\ \ +\ \ }$ $= x$	**Product Law** $x^m x^n =$
$8^3 \div 8^2 = \dfrac{8 \cdot 8 \cdot 8}{8 \cdot 8}$ $= 8^1$ or $8^{\ \ -\ \ }$	$x^3 \div x^2 = \dfrac{x \cdot x \cdot x}{x \cdot x}$ $= x$ or $x^{\ \ -\ \ }$	**Quotient Law** $x^m \div x^n = \dfrac{x^m}{x^n} =$
$(8 \cdot 7)^3 = (8 \cdot 7)(8 \cdot 7)(8 \cdot 7)$ $= (8 \cdot 8 \cdot 8)(\qquad)$ $= 8^3 \cdot 7^3$	$(x \cdot y)^3 = (x \cdot y)(x \cdot y)(x \cdot y)$ $= (x \cdot x \cdot x)(\qquad)$ $= x\ \ y$	**Power of a Product Law** $(xy)^m =$
$\left(\dfrac{8}{7}\right)^3 = \left(\dfrac{\ }{\ }\right)\left(\dfrac{\ }{\ }\right)\left(\dfrac{\ }{\ }\right)$ $= \dfrac{8^3}{7^3}$	$\left(\dfrac{x}{y}\right)^3 = \left(\dfrac{\ }{\ }\right)\left(\dfrac{\ }{\ }\right)\left(\dfrac{\ }{\ }\right)$ $= \dfrac{x}{y}$	**Power of a Quotient Law** $\left(\dfrac{x}{y}\right)^m =$
$(8^3)^2 = (8^3)(8^3)$ $= (\qquad)(\qquad)$ $= 8^6$ or $8^{\ \ \times\ \ }$	$(x^3)^2 = (x^3)(x^3)$ $= (\qquad)(\qquad)$ $= x$ or $x^{\ \ \times\ \ }$	**Power of a Power Law** $(x^m)^n =$

Class Ex. #1

Use the exponent laws to simplify.

a) $4x^5 \times 2x^3$ **b)** $(2a^4)^3$ **c)** $\dfrac{20y^{20}}{5y^5}$ **d)** $\left(\dfrac{3}{t^2}\right)^4$

Class Ex. #2

Evaluate.

a) -10^2 **b)** $(-10)^2$ **c)** -10^3 **d)** $(-10)^3$

e) $(-10)^6 \div (-10)^4$ **f)** $-10^6 \div (-10)^4$ **g)** $-10^7 \div (-10)^3$

Class Ex. #3

Simplify using the exponent laws and evaluate.

a) $\dfrac{2^5 \times 2^3}{2}$ **b)** $\left(\dfrac{9^{20}}{9^{17}}\right)^2$

Class Ex. #4

Simplify the following.

a) $(-2a^2b^3)^3(4a^5b^7)$ **b)** $\dfrac{72x^4y^{10}(-z^2)^4}{6(2xy^2)^3z^8y^3}$ **c)** $\dfrac{b^{4x+y}}{b^{x-2y}}$

Complete Assignment Questions #1 - #17

Assignment

1. Evaluate.

a) 3^8 **b)** -5^2 **c)** $(-5)^2$

d) $(-5)^3$ **e)** -5^3 **f)** $\left(\frac{3}{5}\right)^3$

2. Write in a simpler form and evaluate

a) $4^3 \cdot 4^4 \cdot 4^2$ **b)** $(7^2)^3$ **c)** $\frac{8^{15}}{8^{13}}$ **d)** $\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^3$

e) $\frac{6^6 \times 6}{6^4}$ **f)** $(-3^3)^2$ **g)** $\left(\frac{2^{10}}{2^5}\right)^3$ **h)** $\frac{(0.7)^8}{(0.7)^4 \times (0.7)^2}$

i) $-5^6 \times 5^2$ **j)** $(-5)^6 \times (-5)^2$ **k)** $-10^{10} \div (-10)^8$ **l)** $\frac{-10^{10}}{-10^8}$

3. Simplify.

a) $(a^3)(a^4)(a^5)$ **b)** $\frac{x^{12}}{x^3}$ **c)** $(xy)^7$ **d)** $(t^3)^3$

e) $(-a^2b^3)^4$ **f)** $(-a^2b^3)^5$ **g)** $\left(\frac{b^4}{a^3}\right)^3$ **h)** $\frac{c^5 \times c^2}{c^4 \times c}$

4. Simplify the following

a) $(3a^3)(5a^3)$ **b)** $(2p^3)(4p^7)(-2p)$ **c)** $(-2xy)(x^2y^3)(-3xy)$

d) $(8b^6) \div (2b^3)$ **e)** $(-10t^8) \div (-2t^7)$ **f)** $(16x^5z^7) \div (-4xz^6)$

g) $(3ab^2)^4$ **h)** $(-4a^5c^2)^4$ **i)** $(-2m^3n^4)^5(m^2n^3)$

5. Write each expression without brackets.

a) $(3a^2b^3)(5a^4b^8)$ **b)** $(-4x^2y^3)^3(8xy^8)$ **c)** $(a^3b^4c^5)(3abc^2)^3$

d) $\left(\dfrac{2d^5 \times d^4}{4d^3}\right)^3$ **e)** $\left(\dfrac{-16a^5b^3 \cdot 2a^2b^6}{8ab^7}\right)^3$ **f)** $\left(\dfrac{-5k^3 \cdot k^2}{k}\right)^2\left(\dfrac{(-k)^5 \cdot k^2}{5k^2}\right)$

6. Simplify and evaluate for $x = -2$ and $y = 3$.

a) $\frac{1}{36}(2x^3)^2(-3yx^2)$

b) $\frac{6(x^3y^5)^2}{(3xy)^4}$

7. Simplify each expression.

a) $a^{x+y}a^{2x+3y}$

b) $\frac{m^{x+9}}{m^3}$

c) $\frac{a^{3m+2}}{a^{m-3}}$

d) $\frac{x^{2y+7} \cdot x^{3y+2}}{x^{y+8}}$

Multiple Choice

8. Consider the following three statements.

1. $(x^y)^2 = x^{2y}$ **2.** $(x+y)^3 = x^3 + y^3$ **3.** $(xy)^6 = (-xy)^6$

Which statements are true?

A. **1** and **2** only
B. **1** and **3** only
C. **2** and **3** only
D. some other combination of **1, 2,** and **3**

9. The simplified form of $x^{\sqrt{40}} \div x^{\sqrt{10}}$ is

A. $x^{\sqrt{30}}$
B. $x^{\sqrt{10}}$
C. x^4
D. x^2

Numerical Response

10. The value of the expression $(-3m^2n^3)^2(-2m^3n)^3\left(\dfrac{1}{72m^2n^3}\right)$ for $m = -1$ and $n = 2$ is _____ .

(Record your answer in the numerical response box from left to right)

Match each item in column 1 on the left with the equivalent item in column 2 on the right. Each item in column 2 may be used once, more than once, or not at all.

Column 1	Column 2
11. $(-a^2)^3$	**A.** a^4
12. $(-a^3)^2$	**B.** a^5
13. $a^3 \times a^2$	**C.** a^6
14. $a^8 \div a^2$	**D.** a^{24}
15. $a^{30} \div a^6$	**E.** $-a^5$
16. $(a^2 \times a)^2$	**F.** $-a^6$
17. $-a(a^2)^2$	

Answer Key

1. **a)** 6561 **b)** -25 **c)** 25 **d)** -125 **e)** -125 **f)** $\dfrac{27}{125}$

2. **a)** $4^9 = 262144$ **b)** $7^6 = 117649$ **c)** $8^2 = 64$ **d)** $\left(\dfrac{2}{3}\right)^4 = \dfrac{16}{81}$ **e)** $6^3 = 216$
f) $3^6 = 729$ **g)** $2^{15} = 32768$ **h)** $(0.7)^2 = 0.49$ **i)** $-5^8 = -390625$
j) $(-5)^8 = 390625$ **k)** $-10^2 = -100$ **l)** $10^2 = 100$

3. **a)** a^{12} **b)** x^9 **c)** x^7y^7 **d)** t^9 **e)** a^8b^{12} **f)** $-a^{10}b^{15}$ **g)** $\dfrac{b^{12}}{a^9}$ **h)** c^2

4. **a)** $15a^6$ **b)** $-16p^{11}$ **c)** $6x^4y^5$ **d)** $4b^3$ **e)** $5t$
f) $-4x^4z$ **g)** $81a^4b^8$ **h)** $256a^{20}c^8$ **i)** $-32m^{17}n^{23}$

5. **a)** $15a^6b^{11}$ **b)** $-512x^7y^{17}$ **c)** $27a^6b^7c^{11}$ **d)** $\dfrac{d^{18}}{8}$ **e)** $-64a^{18}b^6$ **f)** $-5k^{13}$

6. **a)** -256 **b)** 216 **7.** **a)** a^{3x+4y} **b)** m^{x+6} **c)** a^{2m+5} **d)** x^{4y+1}

8. B **9.** B **10.**

6	4		

11. F **12.** C **13.** B **14.** C **15.** D **16.** C **17.** E

Exponents Lesson #2:
Integral Exponents

Warm-Up #1 ***Review of The Zero Exponent***

Consider the expression $5^4 \div 5^4$.

a) Evaluate the expression using a calculator or using the result of dividing a number by itself.

b) Complete the following to evaluate the expression.

$$5^4 \div 5^4 = \frac{5^4}{5^4} = \frac{5 \cdot 5 \cdot 5 \cdot 5}{} =$$

c) Use the quotient law to complete the following

$$5^4 \div 5^4 = 5^{\square - \square} = 5^{\square}$$

d) The results in a) to c) are examples of a general rule when a base is raised to the exponent zero. Complete: $a^0 =$ _____

e) Evaluate the following: **i)** 7^0 **ii)** $(-7)^0$ **iii)** -5^0 **iv)** $3(4^2)^0$

Warm-Up #2 ***Review of the Negative Exponent***

Consider the expression $5^4 \div 5^7$.

a) Evaluate the expression as an exact value using a calculator.

b) Complete the following to evaluate the expression.

$$5^4 \div 5^7 = \frac{5 \cdot 5 \cdot 5 \cdot 5}{} = \frac{1}{5^{\square}} =$$

c) Use the quotient law to complete the following

$$5^4 \div 5^7 = 5^{\square - \square} = 5^{\square}$$

d) The results in a) to c) are examples of a general rule when a base is raised to a negative exponent. Complete: $a^{-p} = \frac{1}{}$

e) Write the following with positive exponents and evaluate.
i) 2^{-1} **ii)** 3^{-2} **iii)** 4^{-3}

Zero Exponent Law

A base (not including zero) raised to the exponent zero is equal to one, i.e.

$$a^0 = 1,\ a \neq 0.$$

Negative Exponent Law

A base (not including zero) raised to a negative exponent has the following properties:

$$a^{-n} = \frac{1}{a^n}\ , a \neq 0 \qquad \text{and} \qquad \frac{1}{a^{-n}} = a^n\ , a \neq 0$$

Class Ex. #1

Simplify, express with positive exponents and evaluate without using a calculator.

a) $4^5 \times 4^{-3}$ **b)** $3^2 \times 3^{-5}$ **c)** $\frac{1}{2^{-5}}$ **d)** $\frac{6^{-7}}{6^{-5}}$ **e)** $(2^3)^{-1}$

Class Ex. #2

Identify the following as true or false.

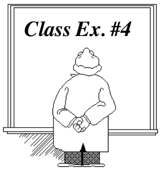
a) $\frac{8^3}{8^{-1}} = 8^4$ **b)** $\frac{8^3}{4^{-1}} = 2^4$ **c)** $a^{-3} = \frac{1}{a^3}$ **d)** $9a^{-3} = \frac{1}{9a^3}$

Class Ex. #3

Simplify and write the answer with positive exponents.

a) $a^{-4} \times a^{-3}$ **b)** $6x^2 \div 2x^7$ **c)** $\frac{y^6}{2y^{-5}}$

d) $(-2x)^{-3}$ **e)** $\frac{8a^{-5}}{4b^{-3}}$ **f)** $\frac{(5p)^{-2}}{5q^4}$



Simplify and write the answer with positive exponents.

$$5x^3y^{-8}z^{-2} \div \frac{15x^8y^3z^{-1}}{x^5y^{-3}z^2}$$

Simplifying a Fractional Base with a Negative Exponent

Consider the expression $\left(\frac{2}{3}\right)^{-4}$.

a) Complete the following $\left(\frac{2}{3}\right)^{-4} = \dfrac{1}{\left(\frac{2}{3}\right)^{\square}} = \dfrac{1}{} = 1 \times =$

b) Evaluate $\left(\frac{3}{2}\right)^{4}$.

c) Classify the following statement as true or false. $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^{4}$

d) Suggest a quick method for evaluating $\left(\frac{5}{2}\right)^{-3}$ without using a calculator.

In general, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n} \quad a, b \neq 0.$

Complete Assignment Questions #1 - #17

Assignment

1. Explain why -8^0 and $(-8)^0$ have different values.

2. Without using a calculator show that $\dfrac{3}{5^{-2}} = 75$.

3. Simplify, express with positive exponents and evaluate without using a calculator.

a) $4^3 \times 4^{-4}$ **b)** $3^0 \times 3^{-3}$ **c)** $\dfrac{1}{7^{-2}}$ **d)** $\dfrac{10^{-3}}{10}$ **e)** $(3^2)^{-2}$

4. Express with positive exponents.

a) n^2m^{-5} **b)** $c^{-2}x^{-5}$ **c)** $16h^{-1}$ **d)** $\dfrac{2}{3}b^{-8}$ **e)** $(y^{-4})^{-2}$

f) $\dfrac{t^{-5}}{4}$ **g)** $\dfrac{1}{4x^{-9}}$ **h)** $\dfrac{4}{x^{-9}}$ **i)** $\dfrac{a^2}{b^{-7}}$ **j)** $\dfrac{a^{-2}}{b^7}$

5. Evaluate the following without using a calculator.

a) -3^{-2} **b)** $(-3)^{-2}$ **c)** $-7^2 \cdot 8^{-2}$ **d)** $(-8.3)^0$ **e)** $[-(3.9)^0]^{-2}$

6. Use a calculator to find the exact value of the following.

a) -4^{-4} **b)** $(-7)^{-3}$ **c)** $(0.75)^{-3}$ **d)** $(-0.025)^{-2}$ **e)** $\left(\dfrac{4}{7}\right)^{-3}$

7. Simplify $x^3(x + x^{-3})$ and calculate the value when $x = \frac{1}{2}$.

8. Simplify and write the answer with positive exponents.

a) $x^{10} \cdot x^{-5}$ **b)** $m^5 \div m^8$ **c)** $b^{-1} \cdot b^{-3}$ **d)** $-w^0 \div w^5$

9. Simplify and write the answer with positive exponents.

a) $a^8 \times a^{-10}$ **b)** $10x^2 \div 2x^{-1}$ **c)** $\dfrac{6y^{-6}}{2y^{-4}}$ **d)** $\dfrac{2a^{-5}}{4b^6}$

e) $-7x^{-2}$ **f)** $-(7x)^{-2}$ **g)** $(-7x)^{-2}$ **h)** $\dfrac{(-7x)^{-2}}{-7x^{-2}}$

10. Simplify each expression, writing the answer with positive exponents.

a) $a^{-3}a^{-3}$ **b)** $(5b^8b^{-12})(-10b^3b^{-12})$ **c)** $(-7x^3x^{-5})(x^2x^{-3})$

d) $(-2a^3)^{-3} \cdot 3a^{12}$ **e)** $\dfrac{16a^6b^{-3}}{-4a^6b^3}$ **f)** $(-3a^5b^{-3}c^0)^{-2}$

11. Simplify. Write the final answer with positive exponents.

a) $\dfrac{32a^2b^{-4}}{4a^{-8}b^{-2}} \times \dfrac{-8a^{-2}}{-3b^{-3}}$

b) $\dfrac{10(p^3q^2r^0)^{-3}}{(8p^{-3}q^5r^3)^{-2}}$

c) $(-2x^5y^3z^8)^{-2}(-2x^2y^{-8}z^{12})^3$

d) $(5a^3b^2)(-2a^{-2}b)^{-3} \div (-5a^8b^{-9})^{-2}$

12. Evaluate the following without using a calculator.

a) $\left(\dfrac{2}{3}\right)^{-3}$ **b)** $\left(\dfrac{1}{5}\right)^{-2}$ **c)** $\left(\dfrac{8}{5}\right)^{-1}$ **d)** $\left(\dfrac{3}{2}\right)^{-4}$

13. Simplify. Write the final answers with positive exponents.

a) $\left(\dfrac{c}{d}\right)^{-3}$ **b)** $\left(\dfrac{x}{4}\right)^{-3}$ **c)** $\left(\dfrac{p^2}{r^4}\right)^{-3}$ **d)** $\left(\dfrac{a^{-2}}{b^{-5}}\right)^{-3}$

e) $\left(\dfrac{-12x^{-3}}{6y^{-8}}\right)^{-1}$

f) $\left(\dfrac{12x^3y^{-1}}{-8x^{-1}y^5}\right)^{-2}$

14. Simplify. Write the final answers with positive exponents.

a) $\left(\dfrac{-x^3}{y}\right)^{-2} \div \left(\dfrac{y^3}{x^5}\right)^2$

b) $49\left(\dfrac{7w^3x^{-5}z^4}{w^{-3}z}\right)^{-2} \times \dfrac{14(x^4z^8)^0}{x^{-8}z^8}$

Multiple Choice

15. The value of $\dfrac{1^{-3}+3^0}{2^{-1}}$ is

A. 1
B. 4
C. 8
D. 12

16. $p = 6 \times 10^{-7}$ and $q = 2 \times 10^8$. If $r = 3 \times 10^n$ and $\dfrac{pq}{r} = 4 \times 10^6$ then n is equal to

A. –5
B. –4
C. 5
D. 4

17. Which of the following statements are true?

i) $3a^{-3} = \dfrac{1}{3a^3}$ ii) $8x^4 \div 4x^7 = \dfrac{1}{2x^3}$ iii) $\dfrac{1}{2a^{-4}} = 2a^4$

A. i) only
B. ii) only
C. iii) only
D. none of the statements are true

Answer Key

1. $-8^0 = -1$ since the exponent applies only to the base 8.
$(-8)^0 = 1$ since the exponet applies to the exponent -8

2. $\frac{3}{5^{-2}} = 3 \times 5^2 = 3 \times 25 = 75$

3. **a)** $\frac{1}{4^1} = \frac{1}{4}$ **b)** $\frac{1}{3^3} = \frac{1}{27}$ **c)** $7^2 = 49$ **d)** $\frac{1}{10^4} = \frac{1}{10\,000}$ **e)** $\frac{1}{3^4} = \frac{1}{81}$

4. **a)** $\frac{n^2}{m^5}$ **b)** $\frac{1}{c^2x^5}$ **c)** $\frac{16}{h}$ **d)** $\frac{2}{3b^8}$ **e)** y^8
f) $\frac{1}{4t^5}$ **g)** $\frac{x^9}{4}$ **h)** $4x^9$ **i)** a^2b^7 **j)** $\frac{1}{a^2b^7}$

5. **a)** $-\frac{1}{9}$ **b)** $\frac{1}{9}$ **c)** $-\frac{49}{64}$ **d)** 1 **e)** 1

6. **a)** $-\frac{1}{256}$ **b)** $-\frac{1}{343}$ **c)** $\frac{64}{27}$ **d)** 1600 **e)** $\frac{343}{64}$

7. $x^4 + 1 = \frac{17}{16}$

8. **a)** x^5 **b)** $\frac{1}{m^3}$ **c)** $\frac{1}{b^4}$ **d)** $-\frac{1}{w^5}$

9. **a)** $\frac{1}{a^2}$ **b)** $5x^3$ **c)** $\frac{3}{y^2}$ **d)** $\frac{1}{2a^5b^6}$
e) $-\frac{7}{x^2}$ **f)** $-\frac{1}{49x^2}$ **g)** $\frac{1}{49x^2}$ **h)** $-\frac{1}{343}$

10. **a)** $\frac{1}{a^6}$ **b)** $-\frac{50}{b^{13}}$ **c)** $-\frac{7}{x^3}$ **d)** $-\frac{3}{8}a^3$ **e)** $-\frac{4}{b^6}$ **f)** $\frac{b^6}{9a^{10}}$

11. **a)** $\frac{64}{3}a^8b$ **b)** $\frac{640q^4r^6}{p^{15}}$ **c)** $-\frac{2z^{20}}{x^4y^{30}}$ **d)** $-\frac{125a^{25}}{8b^{19}}$

12. **a)** $\frac{27}{8}$ **b)** 25 **c)** $\frac{5}{8}$ **d)** $\frac{16}{81}$

13. **a)** $\frac{d^3}{c^3}$ **b)** $\frac{64}{x^3}$ **c)** $\frac{r^{12}}{p^6}$ **d)** $\frac{a^6}{b^{15}}$ **e)** $-\frac{x^3}{2y^8}$ **f)** $\frac{4y^{12}}{9x^8}$

14. **a)** $\frac{x^4}{y^4}$ **b)** $\frac{14x^{18}}{w^{12}z^{14}}$

15. B **16.** A **17.** D

Exponents Lesson #3:
Rational Exponents - Part One

Warm-Up #1 *The Meaning of* $x^{\frac{1}{n}}$

a) Complete and evaluate the following.

i) $\sqrt{5} \cdot \sqrt{5} = \sqrt{} =$ **ii)** $5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\square + \square} = 5^{\square} =$

Deduce a meaning for $5^{\frac{1}{2}}$.

b) Complete and evaluate the following.

i) $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{} =$ **ii)** $2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2^{\square + \square + \square} = 2^{\square}$

Deduce a meaning for $2^{\frac{1}{3}}$.

c) Write the following in radical form and evaluate manually. Verify with a calculator.

i) $25^{\frac{1}{2}} =$ **ii)** $64^{\frac{1}{3}} =$ **iii)** $81^{\frac{1}{4}} =$

d) Write the following in radical form

i) $a^{\frac{1}{2}} =$ **ii)** $b^{\frac{1}{3}} =$ **iii)** $p^{\frac{1}{10}} =$ **iv)** $x^{\frac{1}{n}} =$

Warm-Up #2 *The Meaning of* $x^{\frac{m}{n}}$

a) Complete and evaluate the following.

i) $\sqrt{5^3} \cdot \sqrt{5^3} = =$ **ii)** $5^{\frac{3}{2}} \cdot 5^{\frac{3}{2}} = 5^{\square + \square} = 5^{\square} =$

Deduce a meaning for $5^{\frac{3}{2}}$.

b) Complete and evaluate the following.

i) $\sqrt[3]{2^2} \cdot \sqrt[3]{2^2} \cdot \sqrt[3]{2^2} = =$ **ii)** $2^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} = 2^{\square + \square + \square} = 2^{\square}$

Deduce a meaning for $2^{\frac{2}{3}}$.

c) Write the following in radical form

i) $a^{\frac{5}{3}} =$ **ii)** $b^{\frac{4}{5}} =$ **iii)** $p^{\frac{5}{2}} =$ **iv)** $x^{\frac{m}{n}} =$

Warm-Up #3

a) Evaluate i) $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = \left(\sqrt[3]{8}\right)^2 =$

ii) $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{(8^2)} =$.

b) Which of the calculations above is the easier method for evaluating $8^{\frac{2}{3}}$?

c) Write the following in radical form and evaluate manually. Verify with a calculator.

i) $64^{\frac{3}{2}} =$ **ii)** $4^{\frac{5}{2}}$ **iii)** $81^{\frac{3}{4}}$

Warm-Up #4

a) Use exponent laws to simplify $8^{\frac{2}{3}} \times 8^{-\frac{2}{3}}$.

b) Use the result in c) to write $8^{-\frac{2}{3}}$ in a form with a positive exponent.
Evaluate $8^{-\frac{2}{3}}$ without using a calculator.

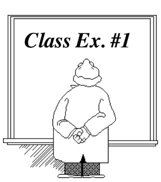
Rational Exponents

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad \text{or} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}, \; m \in I, n \in N, a \neq 0 \text{ when } m \text{ is } 0.$$

Note that $a \geq 0$ if n is even

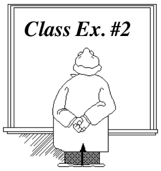
$$a^{-\frac{m}{n}} = \frac{1}{\left(\sqrt[n]{a}\right)^m} \quad \text{or} \quad a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}}, \; m \in I, n \in N, a \neq 0 \text{ when } m \text{ is } 0.$$

Note that $a > 0$ if n is even



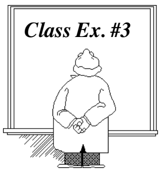
Write the following in radical form and evaluate without using a calculator.
Verify with a calculator.

a) $25^{\frac{3}{2}}$ **b)** $1000^{\frac{4}{3}}$ **c)** $27^{-\frac{2}{3}}$ **d)** $16^{-\frac{3}{4}}$



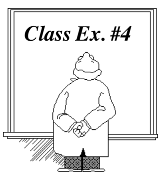
Write the following in radical form and evaluate without using a calculator. Verify with a calculator.

a) $(-8)^{\frac{2}{3}}$ **b)** $-8^{\frac{2}{3}}$ **c)** $(3^2 + 4^2)^{\frac{1}{2}}$



Write the following in radical form and evaluate without using a calculator. Verify with a calculator.

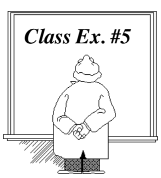
a) $\left(\frac{9}{4}\right)^{\frac{3}{2}}$ **b)** $\left(\frac{9}{4}\right)^{-\frac{3}{2}}$



Consider the following powers.

A $64^{\frac{2}{3}}$ *B* $(-64)^{\frac{2}{3}}$ *C* $64^{\frac{3}{2}}$ *D* $(-64)^{\frac{3}{2}}$

Explain why three of the above powers can be calculated but the other has no meaning.

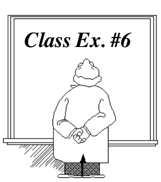


A cube has a volume of 60 m^3.

a) Write a power which represents the edge length of the cube.

b) Write a power which represents the surface area of the cube.

c) Use a calculator to calculate the edge length and surface area to the nearest tenth.



Write the number 10 in the following forms.

a) As a power with an exponent of $\frac{1}{2}$. **b)** As a power with an exponent of $\frac{1}{3}$

Complete Assignment Questions #1 - #13

Assignment

1. Evaluate without the use of a calculator.

a) $4^{\frac{1}{2}}$ **b)** $100^{\frac{1}{2}}$ **c)** $64^{\frac{1}{3}}$ **d)** $9^{\frac{3}{2}}$ **e)** $49^{\frac{3}{2}}$

f) $16^{\frac{3}{4}}$ **g)** $8^{\frac{2}{3}}$ **h)** $125^{\frac{1}{3}}$ **i)** $(6^2 + 8^2)^{\frac{3}{2}}$ **j)** $(0.04)^{0.5}$

2. Determine the exact value without using a calculator.

a) $9^{-\frac{1}{2}}$ **b)** $4^{-\frac{7}{2}}$ **c)** $25^{-\frac{3}{2}}$ **d)** $1000^{-\frac{2}{3}}$ **e)** $64^{-\frac{5}{6}}$

f) $8^{-\frac{4}{3}}$ **g)** $49^{-\frac{1}{2}}$ **h)** $32^{-\frac{2}{5}}$ **i)** $(5^2 - 3^2)^{-\frac{5}{4}}$ **j)** $(0.09)^{-\frac{3}{2}}$

3. Determine the exact value without using a calculator.

a) $\left(\frac{1}{25}\right)^{\frac{1}{2}}$ **b)** $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$ **c)** $\left(\frac{1}{8}\right)^{\frac{4}{3}}$ **d)** $\left(\frac{9}{4}\right)^{-\frac{3}{2}}$ **e)** $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

4. Determine the exact value without using a calculator.

a) $(-8)^{\frac{1}{3}}$ **b)** $(-27)^{\frac{2}{3}}$ **c)** $-25^{-\frac{1}{2}}$ **d)** $-(-32)^{-\frac{4}{5}}$ **e)** $(-0.008)^{\frac{2}{3}}$

5. Use a calculator to evaluate the following to the nearest hundredth.

a) $4^{\frac{2}{3}}$ **b)** $7^{\frac{3}{4}}$ **c)** $(-5)^{\frac{6}{5}}$ **d)** $6^{-\frac{1}{4}}$ **e)** $-(-0.8)^{\frac{2}{3}}$

6. Write an equivalent expression using radicals.

a) $a^{\frac{1}{4}} =$ **b)** $b^{\frac{1}{2}} =$ **c)** $c^{\frac{1}{5}} =$ **d)** $d^{-\frac{1}{2}} =$

e) $e^{-\frac{1}{10}} =$ **f)** $f^{\frac{2}{3}} =$ **g)** $g^{\frac{4}{3}} =$ **h)** $h^{\frac{5}{2}} =$

i) $i^{-\frac{3}{2}} =$ **j)** $j^{-\frac{4}{5}} =$ **k)** $k^{-\frac{3}{4}} =$ **l)** $l^{\frac{m}{n}} =$

7. Assuming that x represents a positive integer, state which of the following expressions has no meaning.

a) $(-x)^{\frac{7}{3}}$ **b)** $(-x)^{\frac{3}{2}}$ **c)** $-(-x)^{\frac{1}{9}}$ **d)** $-(-x)^{\frac{5}{6}}$

8. A cube has a volume of 216 cm^3.

a) Write a power which represents the edge length of the cube.

b) Write a power which represents the surface area of the cube.

c) Calculate the exact edge length and surface area of the cube.

9. A cube has a volume of V cm^3.

a) Write a power and a radical which represents the edge length of the cube.

b) Write a power and a radical for the area of one of the faces of the cube.

10. In each case write the given number as a power with the given exponent.

a) 5 as a power with an exponent of $\frac{1}{2}$. **b)** 8 as a power with an exponent of $\frac{1}{3}$

c) -3 as a power with an exponent of $\frac{1}{3}$. **d)** $\frac{1}{4}$ as a power with an exponent of $-\frac{1}{2}$

e) 6 as a power with an exponent of $-\frac{1}{2}$. **f)** 100 as a power with an exponent of $\frac{2}{3}$

Multiple Choice

11. $\left(\frac{9}{16}\right)^{-0.5}$ is equal to

A. $\frac{256}{81}$

B. $\frac{4}{3}$

C. $-\frac{4}{3}$

D. $\frac{81}{256}$

12. $\left(-\frac{1}{4}\right)^{-1.5}$ is equal to

A. -8

B. 8

C. 6

D. has no meaning

Numerical Response

13. Evaluate the following and arrange the answers from greatest to least.

Calculation 1. $-(27)^{-\frac{2}{3}}$

Calculation 2. $\left(\frac{1}{27}\right)^{\frac{1}{3}}$

Calculation 3. $(-27)^{\frac{2}{3}}$

Calculation 4. $\left(-\frac{1}{27}\right)^{-\frac{1}{3}}$

Place the calculation # with the greatest answer in the first box.
Place the calculation # with the second greatest answer in the second box.
Place the calculation # with the third greatest answer in the third box.
Place the calculation # with the smallest answer in the fourth box.

(Record your answer in the numerical response box from left to right)

Answer Key

1.a) 2 **b)** 10 **c)** 4 **d)** 27 **e)** 343 **f)** 8 **g)** 4 **h)** 5 **i)** 1000 **j)** 0.2

2.a) $\frac{1}{3}$ **b)** $\frac{1}{128}$ **c)** $\frac{1}{125}$ **d)** $\frac{1}{100}$ **e)** $\frac{1}{32}$ **f)** $\frac{1}{16}$ **g)** $\frac{1}{7}$ **h)** $\frac{1}{4}$ **i)** $\frac{1}{32}$ **j)** $\frac{1000}{27}$

3.a) $\frac{1}{5}$ **b)** 2 **c)** $\frac{1}{16}$ **d)** $\frac{8}{27}$ **e)** $\frac{27}{8}$

4.a) -2 **b)** 9 **c)** $-\frac{1}{5}$ **d)** $-\frac{1}{16}$ **e)** 0.04 **5.a)** 2.52 **b)** 4.30 **c)** 6.90 **d)** 0.64 **e)** -0.86

6.a) $\sqrt[4]{a}$ **b)** $\sqrt{b}$ **c)** $\sqrt[5]{c}$ **d)** $\frac{1}{\sqrt{d}}$ **e)** $\frac{1}{\sqrt[10]{e}}$ **f)** $\left(\sqrt[3]{f}\right)^2$ **g)** $\left(\sqrt[3]{g}\right)^4$ **h)** $\left(\sqrt{h}\right)^5$

i) $\frac{1}{\left(\sqrt{i}\right)^3}$ **j)** $\frac{1}{\left(\sqrt[5]{j}\right)^4}$ **k)** $\frac{1}{\left(\sqrt[4]{k}\right)^3}$ **l)** $\left(\sqrt[n]{l}\right)^m$ **7. b)** and **d)** have no meaning

8.a) $(216)^{\frac{1}{3}}$ **b)** $6(216)^{\frac{2}{3}}$ **c)** edge length = 6 cm, surface area = 216 cm^2

9.a) edge length = $\sqrt[3]{V} = V^{\frac{1}{3}}$ **b)** area = $\left(\sqrt[3]{V}\right)^2 = V^{\frac{2}{3}}$

10.a) $5 = 25^{\frac{1}{2}}$ **b)** $8 = 512^{\frac{1}{3}}$ **c)** $-3 = (-27)^{\frac{1}{3}}$ **d)** $\frac{1}{4} = 16^{-\frac{1}{2}}$ **e)** $6 = \left(\frac{1}{36}\right)^{-\frac{1}{2}}$ **f)** $100 = 1000^{\frac{2}{3}}$

11. B **12.** D **13.**

3	2	1	4

Exponents Lesson #4:
Rational Exponent - Part Two

Review Complete the following as a review.

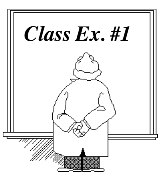
Product Law $x^m x^n =$ ***Quotient Law*** $x^m \div x^n =$

Power of a Power $(x^m)^n =$ ***Power of a Product*** $(xy)^m =$

Power of a Quotient $\left(\dfrac{x}{y}\right)^m =$ $\qquad$ $, y \neq 0$

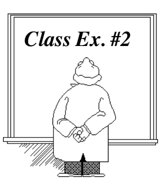
Integral Exponent Rule $x^{-m} =$ $\qquad$, where $x \neq 0$

Rational Exponents $x^{\frac{m}{n}} =$ $\qquad$ or ($\qquad$ $)^m$



Simplify the following. Write each expression as a power and as a radical.

a) $x^{\frac{3}{2}} \times x$ **b)** $y^{\frac{5}{7}} \div y^{\frac{2}{7}}$ **c)** $(a^{\frac{1}{2}})^{\frac{2}{3}}$ **d)** $\left(\dfrac{x^2}{y}\right)^{-\frac{1}{2}}$



Simplify the following. Write each expression as a power and as a radical.

a) $4x^{\frac{3}{4}} \times 3x^{-\frac{1}{2}}$ **b)** $\dfrac{5x^{\frac{3}{5}}}{25x^{-\frac{3}{5}}}$ **c)** $(8a^{\frac{1}{2}})^{\frac{4}{3}}$ **d)** $\dfrac{8^{-\frac{1}{3}}(a^2)^{-\frac{2}{5}}b^{-\frac{1}{5}}}{2(ab)^{-\frac{1}{5}}}$

Writing Radicals as Powers

We can use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ to write radicals as powers.

Class Ex. #3

Write each radical as a power.

a) $\sqrt[3]{x^5}$ **b)** $\sqrt[5]{a^2}$ **c)** $\sqrt{t^9}$ **d)** $\dfrac{1}{\sqrt{p^7}}$

Class Ex. #4

Write as a power and evaluate.

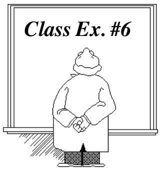
a) $\sqrt{\sqrt{1296}}$ **b)** $\dfrac{1}{\sqrt{169}}$ **c)** $\sqrt[3]{\sqrt{64}}$

Class Ex. #5

Simplify. Write each expression in simplest radical form and as a power.

a) $\sqrt[3]{8x^5}$ **b)** $\sqrt[4]{32a^3}$ **c)** $\sqrt{(-900y)}$

d) $\left(\sqrt[3]{x^5}\right)\left(\sqrt[3]{x}\right)$ **e)** $\sqrt{a} \times \sqrt[3]{a}$ **f)** $\left(\sqrt[4]{x^5y^3}\right)^{\frac{3}{2}}$



Write an equivalent expression using exponents.

a) $\sqrt{\sqrt{a^3}}$ **b)** $\sqrt{\sqrt[3]{64x^6}}$ **c)** $\dfrac{1}{\sqrt[3]{\sqrt{5y^{12}}}}$

Complete Assignment Questions #1 - #18

Assignment

1. Write each power as a radical.

a) $a^{\frac{4}{5}}$ **b)** $b^{\frac{3}{2}}$ **c)** $c^{\frac{1}{4}}$ **d)** $x^{-\frac{2}{5}}$ **e)** $y^{-\frac{1}{3}}$

f) $5h^{\frac{2}{3}}$ **g)** $(5h)^{\frac{2}{3}}$ **h)** $-r^{\frac{5}{4}}$ **i)** $(-r)^{\frac{5}{4}}$ **j)** $2x^{-\frac{1}{2}}$

2. Write each radical as a power.

a) $\sqrt[5]{x^3}$ **b)** $\sqrt[5]{a^4}$ **c)** $\sqrt{d^5}$ **d)** $\dfrac{1}{\sqrt[4]{y}}$ **e)** $\dfrac{1}{\sqrt[4]{h^5}}$

f) $\left(\sqrt[4]{2y-3}\right)^{-3}$ **g)** $-\sqrt[3]{x^2}$ **h)** $\sqrt[3]{-x^2}$ **i)** $\sqrt[3]{(-x)^2}$ **j)** $\sqrt[3]{(-x)^{-2}}$

3. Simplify the following. Write each expression as a power and as a radical.

a) $x^{\frac{7}{2}} \times x$ **b)** $y^{\frac{6}{5}} \div y^{\frac{4}{5}}$ **c)** $(a^{\frac{2}{5}})^{\frac{3}{4}}$ **d)** $(e^3 f)^{\frac{3}{2}}$

e) $x^{\frac{1}{2}} \times x^{-1}$ **f)** $y^{\frac{2}{7}} \div y^{\frac{5}{7}}$ **g)** $\left(\dfrac{x}{y^4}\right)^{\frac{1}{2}}$ **h)** $\left(\dfrac{x^2}{y}\right)^{-\frac{3}{2}}$

4. Write as a power and evaluate.

a) $\sqrt{\sqrt[3]{64}}$ **b)** $\dfrac{1}{\sqrt[4]{625}}$ **c)** $\sqrt{\sqrt{2401}}$

5. Simplify the following. Write each expression as a power and as a radical.

a) $2x^{\frac{3}{8}} \times 5x^{-\frac{1}{8}}$ **b)** $64(a^{\frac{1}{2}})^{\frac{1}{3}}$ **c)** $((64a)^{\frac{1}{3}})^{\frac{1}{2}}$ **d)** $(64a^{\frac{1}{3}})^{\frac{1}{2}}$

e) $\dfrac{y^{\frac{2}{3}} y^{\frac{1}{2}}}{y^{\frac{1}{4}}}$ **f)** $\dfrac{a^3 b^{\frac{1}{2}}}{b^3 (a^{\frac{3}{2}})^2}$ **g)** $\dfrac{10x^{\frac{1}{5}}}{5x^{-\frac{3}{5}}}$ **h)** $\dfrac{(a^4)^{\frac{1}{3}}}{9} \div \dfrac{a}{81^{3/4}}$

6. Simplify. Write each expression in simplest radical form and as a power.

a) $\sqrt[3]{27x^7}$ **b)** $\sqrt[4]{81a^3}$ **c)** $\sqrt[3]{(-270y)}$

d) $\left(\sqrt[4]{x^3}\right)\left(\sqrt{x}\right)$ **e)** $\sqrt[3]{a} \times \sqrt[3]{a}$ **f)** $\left(\sqrt[4]{x^4y^3}\right)^{\frac{3}{2}}$

7. Write an equivalent expression using exponents.

a) $\sqrt{\sqrt{x^5}}$ **b)** $\sqrt[3]{\sqrt{a^8}}$ **c)** $\sqrt[3]{\sqrt{16y^{12}}}$

d) $\sqrt[3]{\sqrt[4]{x^{\frac{2}{3}}}}$ **e)** $\left(\dfrac{25\sqrt[3]{x^5}}{5x^{\frac{1}{3}}}\right)^2$ **f)** $\left(\sqrt[4]{\sqrt[5]{y^{\frac{1}{3}}}}\right)^5$

8. Simplify and express each as a power with positive exponents.

a) $\dfrac{6x^{-\frac{1}{4}} \cdot 2x^{\frac{5}{2}}}{-3x^{-\frac{3}{4}}}$ **b)** $\left(\dfrac{768b^{-1}}{3c^{-1}}\right)^{-\frac{3}{8}}$ **c)** $\dfrac{a^{\frac{1}{3}}(a^{\frac{2}{5}})^{-\frac{5}{3}}}{a^{\frac{4}{3}}}$

Matching Match each item in column 1 with the equivalent item in column 2. Each item in column 2 may be used once, more than once, or not at all.

Column 1	Column 2
9. $\left(\frac{p}{q}\right)^{\frac{4}{3}}$	**A.** $\sqrt[4]{\frac{q^3}{p^3}}$
10. $\left(\frac{p}{q}\right)^{\frac{3}{4}}$	**B.** $\sqrt[4]{\frac{p^3}{q^3}}$
11. $\left(\frac{q}{p}\right)^{-\frac{4}{3}}$	**C.** $-\sqrt[4]{\frac{p^3}{q^3}}$
12. $\left(\frac{p}{q}\right)^{-\frac{3}{4}}$	**D.** $\sqrt[3]{\frac{p^4}{q^4}}$
13. $\left(\frac{q}{p}\right)^{\frac{3}{4}}$	**E.** $\sqrt[3]{\frac{q^4}{p^4}}$
14. $\left(\frac{p}{q}\right)^{-\frac{4}{3}}$	**F.** $-\sqrt[3]{\frac{q^4}{p^4}}$

Multiple Choice **15.** Which of the following is equivalent to $(-x^3)^{-\frac{5}{3}}$?

A. x^5

B. $-x^{\frac{1}{5}}$

C. $\frac{1}{x^5}$

D. $-\frac{1}{x^5}$

16. Which expression is not equivalent to the others?

A. $a^{-\frac{4}{3}}$

B. $\left(\frac{1}{a^4}\right)^3$

C. $\left(\sqrt[3]{a}\right)^{-4}$

D. $\frac{1}{\sqrt[3]{a^4}}$

17. For all positive integers a and b, which of the following is not equivalent to $a^3\sqrt{b}$?

A. $a^3 b^{\frac{1}{2}}$

B. $(a^6 b)^{\frac{1}{2}}$

C. $\sqrt{a^6 b}$

D. all of the expressions are equivalent to $a^3\sqrt{b}$

Numerical Response

18. The value, to the nearest tenth, of the expression $\left(\sqrt[3]{x^{\frac{4}{5}} - y^{\frac{1}{2}} + \sqrt[3]{z}}\right)^2$ when $x = 32$, $y = 36$, and $z = 125$ is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

Unless otherwise indicated in the question, radicals can be given in the form $\sqrt[n]{x^m}$ or $\left(\sqrt[n]{x}\right)^m$ and powers can be given in the form x^{-n} or $\frac{1}{x^n}$. Equivalent versions of some answers are possible.

1. a) $\sqrt[5]{a^4}$ b) $\sqrt{b^3}$ c) $\sqrt[4]{c}$ d) $\frac{1}{\sqrt[5]{x^2}}$ e) $\frac{1}{\sqrt[3]{y}}$ f) $5\sqrt[3]{h^2}$

g) $\sqrt[3]{(5h)^2}$ h) $-\sqrt[4]{r^5}$ i) $\sqrt[4]{(-r)^5}$ j) $\frac{2}{\sqrt{x}}$

2. a) $x^{\frac{3}{5}}$ b) $a^{\frac{4}{5}}$ c) $d^{\frac{5}{2}}$ d) $y^{-\frac{1}{4}}$ e) $h^{-\frac{5}{4}}$ f) $(2y-3)^{-\frac{3}{4}}$ g) $-x^{\frac{2}{3}}$ h) $-x^{\frac{2}{3}}$ i) $(-x)^{\frac{2}{3}}$

j) $(-x)^{-\frac{2}{3}}$

3. a) $x^{\frac{9}{2}} = \sqrt{x^9}$ b) $y^{\frac{2}{5}} = \sqrt[5]{y^2}$ c) $a^{\frac{3}{10}} = \sqrt[10]{a^3}$ d) $e^{\frac{9}{2}}f^{\frac{3}{2}} = e^4f\sqrt{ef}$

e) $\frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}$ f) $\frac{1}{y^{\frac{3}{7}}} = \frac{1}{\sqrt[7]{y^3}}$ g) $\frac{x^{\frac{1}{2}}}{y^2} = \frac{\sqrt{x}}{y^2}$ h) $\frac{y^{\frac{3}{2}}}{x^3} = \frac{\sqrt{y^3}}{x^3}$

4. a) $64^{\frac{1}{6}} = 2$ b) $625^{-\frac{1}{4}} = \frac{1}{5}$ c) $2401^{\frac{1}{4}} = 7$

5. a) $10x^{\frac{1}{4}} = 10\sqrt[4]{x}$ b) $64a^{\frac{1}{6}} = 64\sqrt[6]{a}$ c) $2a^{\frac{1}{6}} = 2\sqrt[6]{a}$ d) $8a^{\frac{1}{6}} = 8\sqrt[6]{a}$

e) $y^{\frac{11}{12}} = \sqrt[12]{y^{11}}$ f) $\frac{1}{b^{\frac{5}{2}}} = \frac{1}{\sqrt{b^5}}$ g) $2x^{\frac{4}{5}} = 2\sqrt[5]{x^4}$ h) $3a^{\frac{1}{3}} = 3\sqrt[3]{a}$

6. a) $3x^2\sqrt[3]{x} = 3x^{\frac{7}{3}}$ b) $3\sqrt[4]{a^3} = 3a^{\frac{3}{4}}$ c) $-3\sqrt[3]{10y} = -3(10y)^{\frac{1}{3}}$

d) $\sqrt[4]{x^5} = x^{\frac{5}{4}}$ e) $\sqrt[3]{a^2} = a^{\frac{2}{3}}$ f) $\sqrt[8]{x^{12}y^9} = x^{\frac{3}{2}}y^{\frac{9}{8}}$

7. a) $x^{\frac{5}{4}}$ b) $a^{\frac{4}{3}}$ c) $4^{\frac{1}{3}}y^2$ d) $x^{\frac{1}{18}}$ e) $25x^{\frac{8}{3}}$ f) $y^{\frac{1}{12}}$

8. a) $-4x^3$ b) $\frac{b^{\frac{3}{8}}}{8c^{\frac{3}{8}}}$ c) $\frac{1}{a^{\frac{5}{3}}}$

9. D **10.** B **11.** D **12.** A **13.** A

14. E **15.** D **16.** B **17.** D **18.**

6	.	1	

Exponents Lesson #5:
Surface Area and Volume

This lesson provides examples where students are required to "communicate a set of instructions used to solve an arithmetic problem"

In mathematics there are many formulas which involve exponents. In this lesson we review the concepts of perimeter, area, surface area and volume for one, two and three dimensional shapes which were covered in previous math courses and we extend our knowledge to include the volume and surface area of a sphere.

Perimeter

Perimeter is the total distance around a figure. Examples where the concept of perimeter may occur are;

- a rancher needs to know how much fencing is needed to coral his livestock, or,
- a landscaper wants to know how much fencing is needed around a yard.

For a circle, the term **circumference** is used instead of perimeter to express the distance around it.

Area

The area of a figure is a value expressed in square units which is needed to cover a surface. Examples where the concept of area may occur are;

- a tailor needs to know how much material is required to make a suit, or,
- a farmer needs to know the area of a field to seed it with grain.

Surface Area

The surface area of a solid is the total area needed to cover all the surfaces (or faces) of the solid. Examples of where the concept of surface area may occur are;

- a manufacturer needs to know how much material required for packaging a product, or,
- a painter needs to know how much paint to buy to paint a room.

Volume

The volume of a solid is the amount of space in a solid and is expressed in cubic units. Examples where the concept of volume may occur are;

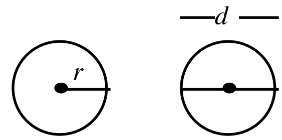
- a worker needs to know how much concrete is required to fill a driveway, or,
- a worker needs to know how much earth needs to be removed from an excavation site.

The formulas on the next page include the formulas for the volume and the surface area of a sphere. The formula for the volume of a sphere is developed on the third page of the lesson.

Formulas for Perimeter, Area, Surface Area, and Volume

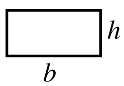
Perimeter (one dimensional)

- General → Add the distance around the figure.
- For a **circle**: $C = \pi d$ or $C = 2\pi r$

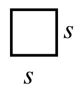


Area (two dimensional)

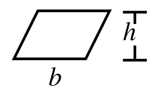
- $A_{Rectangle} = bh$



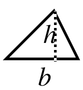
- $A_{Square} = s^2$



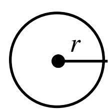
- $A_{Parallelogram} = bh$



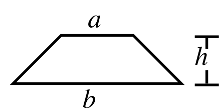
- $A_{Triangle} = \frac{1}{2}bh$ or $\frac{bh}{2}$



- $A_{Circle} = \pi r^2$

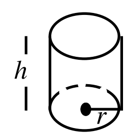


- $A_{Trapezoid} = \frac{1}{2}h(a + b)$

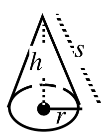


Surface Area (three dimensional)

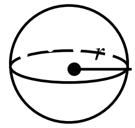
- General → Add all the areas of the surfaces of the solid.
- Cylinder → $SA_{Cylinder} = 2 \times$ area of a circle + area of a rectangle
  $= 2\pi r^2 + 2\pi rh$



- $SA_{Cone} = \pi r^2 + \pi rs$



- $SA_{Sphere} = 4\pi r^2$

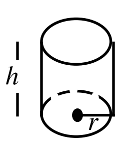


Volume

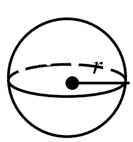
$V_{Prism} = A_{Base} \times h$

$V_{Pyramid} = \frac{1}{3}A_{Base} \times h$

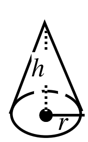
- $V_{Cylinder} = \pi r^2 h$



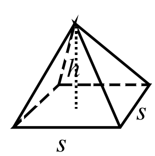
- $V_{Sphere} = \frac{4}{3}\pi r^3$



- $V_{Cone} = \frac{1}{3}\pi r^2 h$



- $V_{Square\text{-}based\ Pyramid} = \frac{1}{3}s^2 h$



Investigating the Formula for the Volume of a Sphere

A student used the method of displacement to determine the volume of a sphere. First she completely filled a cylinder with 375 ml of water and placed it on an overflow container. Then she placed a ball into the cylinder and allowed the water to overflow. Finally she measured the volume of water in the overflow container and the volume of water left in the cylinder. After three trials she determined that the volume of the water in the overflow container was 250 ml and the volume of water left in the cylinder was 125 ml .

Note: 1 ml = 1 cm^3

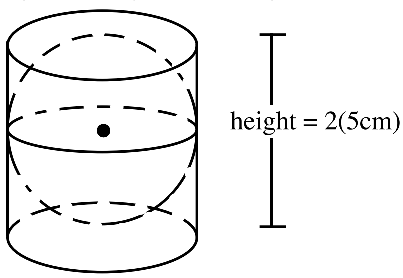
She concluded the following:

- The volume of water in the overflow container is the same as the volume of the ball.
- The volume of the water in the overflow container is $\frac{2}{3}$ of the volume of the cylinder.
- The volume of the ball (sphere) is $\frac{2}{3}$ of the volume of the cylinder.
- The volume of water left in the cylinder is $\frac{1}{3}$ of the original volume.

She then tried to determine a formula for the volume of a sphere in terms of its radius *r*.

a) First she considered a sphere and a cylinder both with a radius of 5 cm.

i) Mark the radius of the sphere and cylinder on the diagram and explain why the height of the cylinder is 10 cm.



ii) She determined the volume of the sphere using the formula for the volume of a cylinder. Complete her work.

$$V_{Sphere} = \frac{2}{3}V_{Cylinder}$$

$$= \frac{2}{3}\pi r^2 h$$

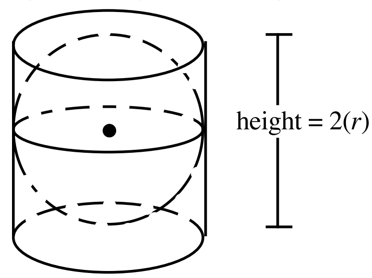
and $h = 2(5) = 10$

$$= \frac{2}{3}\pi(\quad)^2(10)$$

$$= \frac{4}{3}\pi(\quad)^3 =$$

b) Secondly, she considered a sphere and a cylinder both with a radius of *r*.

i) Mark the radius of the sphere and cylinder on the diagram and explain why the height of the cylinder is 2*r*.



ii) Using the formula for the volume of a cylinder, complete her work to find the volume of a sphere.

$$V_{Sphere} = \frac{2}{3}V_{Cylinder}$$

$$= \frac{2}{3}\pi r^2 h$$

and $h = 2(r)$

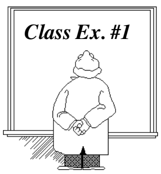
$$= \frac{2}{3}\pi(\quad)^2(2r)$$

$$=$$

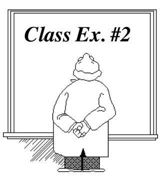
Formulas for Volume and Surface Area of a Sphere

- $SA_{Sphere} = 4\pi r^2$
- $V_{Sphere} = \frac{4}{3}\pi r^3$

Whereas the term **semi-circle** is used to describe half of a circle, the term **hemisphere** is used to describe half of a sphere.

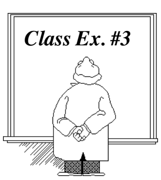


Create a set of keystroke instructions for using a calculator to find the volume of a ball of radius 12 cm. Calculate the volume to the nearest cubic centimetre.



Travis uses a rope to wrap the outer edge of a beach ball exactly once. He then measured the distance the rope wrapped around the ball by stretching it out straight and using a ruler to measure its length. He determined the length to be 88.2 cm

Calculate, to the nearest tenth, the surface area of the beach ball?



A solid hemisphere has a diameter of 8 cm. Calculate, in terms of π:

a) its volume

b) its total surface area.

Complete Assignment Questions #1 - #18

Assignment

1. Calculate the following to the nearest tenth:

a) The surface area of a sphere with a radius of 12 mm.

b) The volume of a sphere with a diameter of 18 cm.

2. Create a set of keystroke instructions for using a calculator to find the surface area of a soccer ball of diameter 21 cm. Calculate the surface area to the nearest square cm.

3. The shape of the planet Mars is approximately that of a sphere with a diameter of about 6786 km.

a) Calculate the surface area of the planet. Express the answer in scientific notation ($a \times 10^n$ where a is calculated to the nearest tenth and n is an integer).

b) Calculate the volume of the planet. Express the answer in scientific notation ($a \times 10^n$ where a is calculated to the nearest tenth and n is an integer).

4. a) Solve the equation $V = \frac{4}{3}\pi r^3$ for r. Write the answer as a radical and as a power.

b) The volume of a beach ball is 50 965 cm^3. Create a set of calculator keystroke instructions which can be used to determine the radius of the ball.

c) Calculate the radius to the nearest tenth of a cm.

5. A student gave the following instructions to calculate the volume of a solid.

- square the radius
- multiply the result by the height
- multiply the result by π
- divide the result by 3

a) Which type of solid is the student describing?

b) Write the calculator keystrokes for the above instructions to determine the volume when the radius is 4 cm and the height is 10 cm.

c) Calculate the volume in cm^3 to the nearest tenth.

6. The surface area of a cylinder is given by the formula $SA = 2\pi r^2 + 2\pi rh$.

a) Explain why the formula can be written in the form $SA = 2\pi r(r + h)$.

b) Write a set of instructions (as in the stem of question #5) to communicate to another student how to calculate the surface area in #6a).

c) Write a set of calculator keystrokes for the above instructions to calculate the surface area of a cylinder with height 9.2 cm and radius 6.8 cm.

d) Calculate the surface area in cm^2 to the nearest tenth.

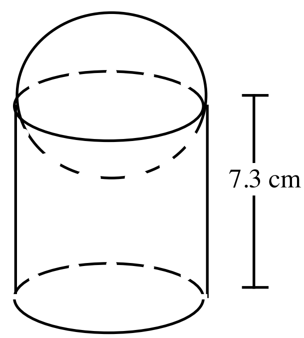
7. During a rainfall, a student leaves a 250 ml beaker outside to collect rainwater. If the diameter of each spherical raindrop is 5 mm, determine, as an exact multiple of π;

a) the surface area and volume of a raindrop.

b) the number of raindrops that will be required to fill the 250 ml beaker.

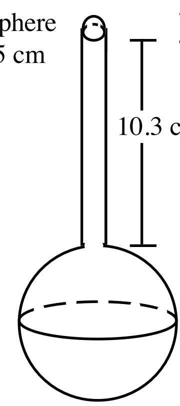
8. The diagram is a sketch of a roll-on deodorant. It consists of a cylindrical container into which a ball is placed as shown. Liquid deodorant completely fills the space below the ball.

Calculate, to the nearest ml, the volume of liquid deodorant in the container.



9. A model of a mercury thermometer consists of a spherical base and a cylindrical portion with a hemispherical top.

Use the dimensions shown in the diagram to answer the following questions to the nearest hundredth.
The diagram is not drawn to scale.



a) Determine the volume of the space inside the thermometer.

b) The thermometer is made of glass. Determine the surface area of glass required to make the thermometer. Assume that the missing glass in the sphere is equal in surface area to the base of the cylinder.

10. Carlos is to receive an award for being the top goal scorer in a local soccer league. The award is a silver miniature soccer ball. The silver ball is packaged in a cubic container which is just large enough to contain the ball. The volume of the container is 1728 cm^3.

a) Calculate the surface area of the miniature silver ball, to the nearest cm^2.

b) Once the ball is placed inside the cube, bubble wrap is added to the container to prevent damage in transit. The bubble wrap occupies 80% of the remaining space inside the cube. Determine the volume of bubble wrap used to the nearest cm^3.

Multiple Choice

11. The surface area of a sphere is 255 m^2. Its diameter to the nearest tenth of a metre is

A. 28.3
B. 14.2
C. 9.0
D. 4.5

12. A sphere has a radius of 15 mm. The volume of the sphere in mm^3 is

A. 225π
B. 900π
C. 4500π
D. 14137π

Numerical Response

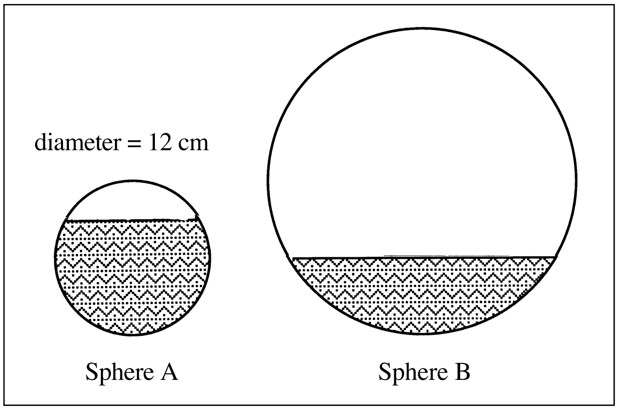
13. A ball has a volume of 890 cm^3. The surface area of the ball, to the nearest square centimetre, is _____ .

(Record your answer in the numerical response box from left to right)

14. A cylindrical jar has diameter 7 cm and contains water to the depth of 5 cm. A sphere of diameter 3 cm is dropped into the jar and sinks to the bottom. To the nearest tenth of a cm, the water depth is now _____.

(Record your answer in the numerical response box from left to right)

Use the following information to answer question #15.



15. Sphere A is 80% full of water and sphere B is 20% full of water. If both spheres contain an equal volume of water, the diameter of Sphere B, to the nearest tenth of a cm, is _____ .

(Record your answer in the numerical response box from left to right)

16. A boiler is in the shape of a cylinder with hemispherical ends. Its total length is 14 m and its diameter is 6 m. Its cubic content, to the nearest m^3, is _____ .

(Record your answer in the numerical response box from left to right)

Use the following information to answer questions #17 and #18.

A solid gold earring has the shape of a hemisphere topped by a cone.
The diameter of the hemisphere and the height of the cone are each 6.2 mm.
The mass of 1 cubic centimetre of gold is 19.3 grams.

17. The mass of the earring, to the nearest tenth of a gram, is _____ .
(Record your answer in the numerical response box from left to right)

18. The surface area of the earring, to the nearest mm^2, is _____ .
(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** 1809.6 mm^2 **b)** 3053.6 cm^3
2. answers may vary (21 ÷ 2) ^ 2 x 2nd ^ x 4 Enter, 1385 cm^2
3. **a)** 1.4×10^8 km^2 **b)** 1.6×10^{11} km^3
4. **a)** $r = \sqrt[3]{\dfrac{3V}{4\pi}} = \left(\dfrac{3V}{4\pi}\right)^{\frac{1}{3}}$
b) answers may vary ((3 x 50965) ÷ (4 x 2nd ^)) ^ (1 ÷ 3) Enter **c)** 23.0 cm

5. **a)** cone **b)** answers may vary 4 ^ 2 x 10 x 2nd ^ ÷ 3 Enter **c)** 167.6 cm^3
6. **a)** factor $2\pi r^2 + 2\pi rh$ by removing a common factor of $2\pi r$ gives $2\pi r(r + h)$
b) add the radius and the height, multiply the result by the radius, multiply the result by 2π
c) answers may vary (6.8 + 9.2) x 6.8 x 2 x 2nd ^ Enter **d)** 683.6 cm^2

7. **a)** surface area = 25π mm^2, volume = $\dfrac{125}{6}\pi$ mm^3 **b)** 3820
8. 158 ml

9. **a)** 0.76 cm^3 **b)** 10.16 cm^2

10. **a)** 452 cm^2 **b)** 659 cm^3

11. C **12.** C

13.

4	4	7	

14.

5	.	4	

15.

1	9	.	0

16.

3	3	9	

17.

2	.	4	

18.

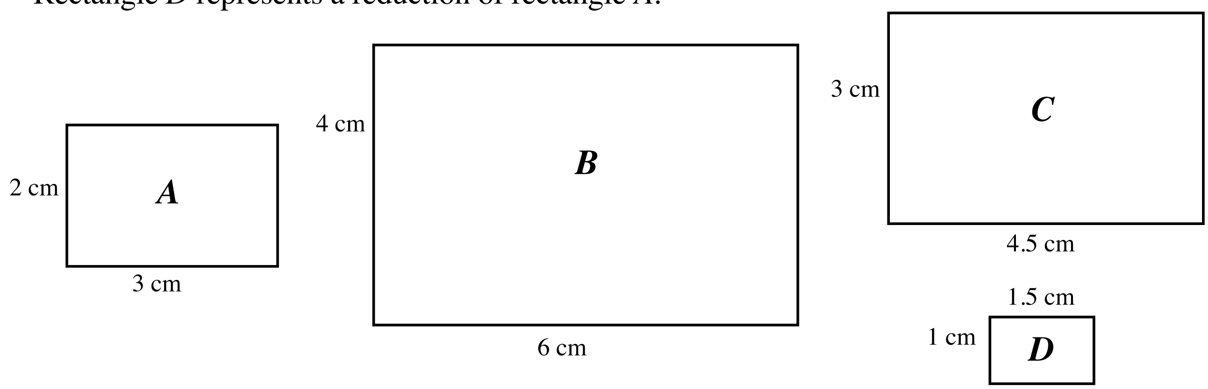
1	2	8	

Exponents Lesson #6:
Scale Factors

This lesson is enrichment for students following the Alberta curriculum.
In earlier math courses we studied the concept of **similar** objects - objects which have the same shape but not the same size.
One way of describing the enlargement or reduction of an object is by **scale factors**.

Linear Scale Factor Investigation

Rectangles *B* and *C* represent enlargements of rectangle *A*.
Rectangle *D* represents a reduction of rectangle *A*.



Complete the following.
- Give each answer as a ratio in simplest form and as an exact decimal.

a) i) $\dfrac{\text{Length of } \square B}{\text{Length of } \square A} = \dfrac{6}{3} = \dfrac{2}{1} = 2$ **ii)** $\dfrac{\text{Width of } \square B}{\text{Width of } \square A} = \dfrac{\quad}{\quad} = \dfrac{\quad}{\quad} =$

b) i) $\dfrac{\text{Length of } \square C}{\text{Length of } \square A} = \dfrac{\quad}{\quad} = \dfrac{\quad}{\quad} =$ **ii)** $\dfrac{\text{Width of } \square C}{\text{Width of } \square A} = \dfrac{\quad}{\quad} =$

c) i) $\dfrac{\text{Length of } \square D}{\text{Length of } \square A} = \dfrac{\quad}{\quad} = \dfrac{\quad}{\quad} =$ **ii)** $\dfrac{\text{Width of } \square D}{\text{Width of } \square A} = \dfrac{\quad}{\quad} =$

Linear Scale Factor

A linear scale factor describes the enlargement or reduction of length. It is described as a ratio in the form $a : b$ or in the form $\dfrac{a}{b} : 1$ in which case the 1 is usually omitted.

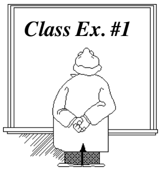
- The **linear scale factor** of □*B* to □*A* is 2 : 1 or 2
- The **linear scale factor** of □*C* to □*A* is 3 : 2 or 1.5
- The **linear scale factor** of □*D* to □*A* is 1 : 2 or 0.5 .

A <u>scale factor</u> **greater than 1** describes an **enlargement**.
A <u>scale factor</u> **between 0 and 1** describes a **reduction**.

Perimeter Scale Factor

Since perimeter is determined by adding a series of lengths, perimeter scale factor is equal to linear scale factor

For example; the **perimeter** of □*B* is 2 times the perimeter of □*A*. Verify.
the **perimeter** of □*C* is ___ times the perimeter of □*A*.
the **perimeter** of □*D* is ___ times the perimeter of □*A*.



Sara increased the length and width of a rectangular 8″ x 10″ photograph by a factor of 5 : 2.

a) Calculate the dimensions of the enlargement.

b) How does the perimeter of the enlargement compare to the perimeter of the photograph?

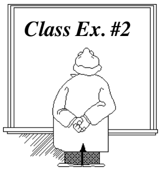
Relationship Between Linear and Area Scale Factors

a) Complete the following to investigate the relationship between linear scale factor and area scale factor.

	Original Dimensions of Rectangle (cm)	**Original Area (cm^2)**	**<u>Linear Scale Factor</u> Applied to Rectangle**	**New Dimensions of Rectangle (cm)**	**New Area (cm^2)**	**<u>Area Scale Factor</u> New Area : Original Area**
A	3 x 5	15	2:1	6 x 10	60	$\frac{\text{New Area}}{\text{Original Area}} = \frac{60}{15} \Rightarrow 4:1$
B	9 x 6		1:3			$\frac{\text{New Area}}{\text{Original Area}} = \frac{\quad}{\quad} \Rightarrow \ \ :$
C	4 x 8		3:2			$\frac{\text{New Area}}{\text{Original Area}} = \frac{\quad}{\quad} \Rightarrow \ \ :$
D	3 x 12		2:3			$\frac{\text{New Area}}{\text{Original Area}} = \frac{\quad}{\quad} \Rightarrow \ \ :$

b) Compare the <u>linear scale factors</u> to the <u>area scale factors</u> on the chart in a).

Complete: area scale factor = (linear scale factor)

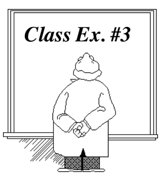


Maggie has scanned an 8″ x 10″ photograph to her computer.

a) Maggie would like to increase the size (area) by 40%.
Determine the area scale factor and the linear scale factor of the enlargement

b) Determine the dimensions of the enlargement to the nearest tenth.

c) Maggie must also produce a print which will a reduction in area of 25% of the original photograph. Calculate, to the nearest tenth, the dimensions of the print.



Marco, owner of Map-It Inc. has produced a map of Canada.

a) The area of the province of Alberta is approximately 661 850 km^2.
On Marco's map, the area of Alberta is represented as 264.74 cm^2.
The scale of the map (linear scale factor) can be written in the form 1 : x.
Calculate the value of x to the nearest whole number.

b) If the area of the province of British Columbia is represented on the map by 376.84 cm^2, determine the approximate area of British Columbia.

Relationship Between Linear and Surface Area Scale Factors

a) Complete the following to investigate the relationship between the linear scale factor and surface area scale factor of rectangular prisms.

	Original Dimensions (cm)	**Original Surface Area** (cm^2)	**Linear Scale Factor**	**New Dimensions** (cm)	**New Surface Area** (cm^2)	**Surface Area Scale Factor** New Surface Area : Original Surface Area
A	3 x 5 x 1	46	2:1	6 x 10 x 2	184	$\frac{\text{New Surface Area}}{\text{Original Surface Area}} = \frac{184}{46} \Rightarrow 4:1$
B	9 x 6 x 12		1:6			$\frac{\text{New SurfaceArea}}{\text{Original Surface Area}} = \frac{\quad}{\quad} \Rightarrow \quad :$
C	4 x 8 x 5		3:2			$\frac{\text{New SurfaceArea}}{\text{Original Surface Area}} = \frac{\quad}{\quad} \Rightarrow \quad :$
D	3 x 12 x 6		2:3			$\frac{\text{New SurfaceArea}}{\text{Original Surface Area}} = \frac{\quad}{\quad} \Rightarrow \quad :$

b) Compare the linear scale factors to the surface area scale factors on the chart in a).

Complete: surface area scale factor = (linear scale factor)

Relationship Between Linear and Volume Scale Factors

a) Complete the following.

	Original Dimensions (cm)	**Original Volume** (cm^3)	**Linear Scale Factor**	**New Dimensions** (cm)	**New Volume** (cm^3)	**Volume Scale Factor** New Volume : Original Volume
A	3 x 5 x 1	15	2:1	6 x 10 x 2	120	$\frac{\text{New Volume}}{\text{Original Volume}} = \frac{120}{15} \Rightarrow 8:1$
B	9 x 6 x 12		1:6			$\frac{\text{New Volume}}{\text{Original Volume}} = \frac{\quad}{\quad} \Rightarrow \quad :$
C	4 x 8 x 5		3:2			$\frac{\text{New Volume}}{\text{Original Volume}} = \frac{\quad}{\quad} \Rightarrow \quad :$
D	3 x 12 x 6		2:3			$\frac{\text{New Volume}}{\text{Original Volume}} = \frac{\quad}{\quad} \Rightarrow \quad :$

b) Compare the linear scale factors to the volume scale factors on the chart in a).

Complete: volume scale factor = (linear scale factor)

Scale Factor Summary

Linear scale factor is a ratio in the form $a : b$ or $\frac{a}{b} : 1$ which describes an enlargement or reduction in one dimension.

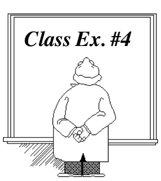
Area scale factor is a ratio in the form $c : d$ or $\frac{c}{d} : 1$ which describes how many times to enlarge or reduce the area of a two dimensional figure or the surface area of a three dimensional solid.

Volume scale factor is a ratio in the form $e : f$ or $\frac{e}{f} : 1$ which describes how many times to enlarge or reduce the volume of a three dimensional solid.

The Scale Factor Relationships

Perimeter	Perimeter Scale Factor = Linear Scale Factor
Area	Area Scale Factor = $(\text{Linear Scale Factor})^2$
Surface Area	Surface Area Scale Factor = $(\text{Linear Scale Factor})^2$
Volume	Volume Scale Factor = $(\text{Linear Scale Factor})^3$

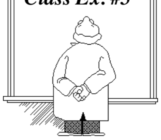
If a question asks for the **scale factor** of an enlargement or reduction it is implied that the **linear scale factor** is required.



Complete the table below writing the answers as powers and radicals (where appropriate).

Linear Scale Factor	Area Scale Factor	Volume Scale Factor
$a : b$		
	$c : d$	
		$e : f$

Class Ex. #5



Raj sells three different sizes of jars, all of which are similar in shape. The smallest jar has a volume of 50 cm^3 and the largest jar has a volume of 1.35 litres.

a) The surface area of glass in the smallest jar is 90 cm^2. Calculate the surface area of glass in the largest jar.

b) The medium jar has dimensions twice those of the smaller jar. Calculate the surface area and volume of the medium jar.

Complete Assignment Questions #1 - #12

Assignment

1. Kylie from Absolute Value Renovations has designed a plan for the renovation of a candy store. The scale drawing she has made uses a scale of 1 : 50.

a) What are the actual dimensions of the storage room if the scale drawing dimensions are 8 cm by 12 cm?

b) If the chocolate section of the store is to be expanded to 3.5 m by 4.75 m, determine the dimensions of this section on Kylie's plan.

2. Three marbles with surface areas A_1, A_2 and A_3 and volumes V_1, V_2 and V_3 respectively have radii 1 cm, 2 cm and 3 cm. Without calculating their areas or volumes give values of the following ratios.

a) $A_1 : A_2$ **b)** $A_2 : A_3$ **c)** $A_1 : A_2 : A_3$ **d)** $V_3 : V_1$ **e)** $V_1 : V_2 : V_3$

3. City Council is to vote on a proposal for a major expansion to a rectangular park with an area of 158 525 m^2.

a) If the proposal is to triple the current dimensions of the park, by what factor would the area increase?

b) Calculate the area of the proposed expanded park in square kilometres to two decimal places.

c) A scale model of the proposed park is 3.73 m by 4.25 m. Calculate the dimensions of the proposed park.

4. Complete the table.

Shape	**Surface Area**	**Volume**	**Linear Scale Factor appled**	**New Surface Area**	**New Volume**
Cube	150 m^2	125 m^3	1 : 5		
Rectangular Prism	340 mm^2	400 mm^3	5 : 4		
Sphere	804 m^2	2144 m^3	0.2		

5. Complete the table below using exact values.

Linear Scale Factor	Area Scale Factor	Volume Scale Factor
3 : 1		
	1 : 4	
		1 : 125
	16 : 1	
5 : 2		
		27 : 8
	2 : 1	
	2 : 5	
		3 : 1
0.3		
	25	
		$\frac{1}{1000}$

6. Value Pizza sells 10″, 12″ and 14″ pizzas - the measurement being the diameter of the pizza. All the pizzas have the same thickness.

a) If a 10″ Hawaiian costs \$10 what should be the price of a 14″ Hawaiian if the cost is in proportion to the amount of pizza?

b) Value Pizza decide to create a lunchtime special 7″ Hawaiian at a reduction of 20% of the proportional cost. Calculate the selling price of the 7″ Hawaiian.

Multiple Choice

7. The ratio of the volume of two spheres is 1: 64. The ratio of the surface areas is

A. 1 : 8
B. 1 : 16
C. 1 : 512
D. 3 : 128

8. Students are making scale models of a solid which has surface area 2500 cm^2 and volume 1000 cm^3. After completing their model each student measured the surface area and volume of their model. Which of the students A, B, C or D below can we be certain has not made a correct scale model?

A.	surface area = 10 000 cm^2	volume = 8000 cm^3
B.	surface area = 625 cm^2	volume =125 cm^3
C.	surface area = 250 cm^2	volume =100 cm^3
D.	surface area = 25 cm^2	volume =1 cm^3

9. An architect has designed a three dimensional model of a hotel. The swimming pool in the model holds 24 cm^3 of water and the actual pool will have a volume of 81 000 litres. The scale factor of the model is

A. 1 : 15
B. 1 : 150
C. 1 : 3375
D. 1 : 3375000

10. The circumference of a circle is reduced from a metres to b metres.
The area of the original circle is c m^2 and the area of the reduced circle is d m^2.
Which statement is true?
A. $a^2d = b^2c$
B. $a^2b = c^2d$
C. $ad^2 = bc^2$
D. $ac^2 = bd^2$

Numerical Response

11. The surface area of a sphere is increased by a factor of 3. To the nearest tenth, the volume of the sphere is increased by a factor of ________ .
(Record your answer in the numerical response box from left to right)

12. In designing a new model of a rectangular building, each face of the building is decreased in area by 20%. To the nearest percent, the decrease in volume is ________ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) 4m × 6m **b)** 7cm × 9.5cm

2. a) 1:4 **b)** 4:9 **c)** 1:4:9 **d)** 27:1 **e)** 1:8:27

3. a) 9 **b)** 1.43 km^2 **c)** 1119m × 1275m

4. see table

Shape	Surface Area	Volume	Linear Scale Factor appled	New Surface Area	New Volume
Cube	150 m^2	125 m^3	1 : 5	$150 \times \frac{1}{25} = 6\,m^2$	$125 \times \frac{1}{125} = 1\,m^3$
Rectangular Prism	340 mm^2	400 mm^3	5 : 4	$340 \times \frac{25}{16} = 531.25\,mm^2$	$400 \times \frac{125}{64} = 781.25\,mm^3$
Sphere	804 m^2	2144 m^3	0.2	$804 \times (0.2)^2 = 32.16\,m^2$	$2144 \times (0.2)^3 = 17.152\,m^3$

5. see table below **6.a)** \$19.60 **b)** \$3.92 **7.** B **8.** C **9.** B **10.** A

Linear Scale Factor	Area Scale Factor	Volume Scale Factor
3 : 1	9 : 1	27 : 1
1 : 2	1 : 4	1 : 8
1 : 5	1 : 25	1 : 125
4 : 1	16 : 1	64 : 1
5 : 2	25 : 4	125 : 8
3 : 2	9 : 4	27 : 8
$\sqrt{2} : 1$	2 : 1	$\sqrt{2^3} : 1$
$\sqrt{2} : \sqrt{5}$	2 : 5	$(\sqrt{2})^3 : (\sqrt{5})^3$
$\sqrt[3]{3} : 1$	$(\sqrt[3]{3})^2 : 1$	3 : 1
0.3	0.09	0.0027
5	25	125
$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

or $2\sqrt{2} : 1$, $2^{\frac{3}{2}} : 1$, $\sqrt{8} : 1$

$2\sqrt{2} : 5\sqrt{5}$ etc

or equivalent

11.

5	.	2	

12.

2	8		

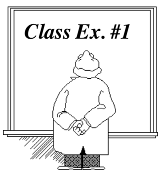
Exponents Lesson #7:
Enrichment: Solving Exponential Equations

Warm-Up *Converting Bases*

Convert each number shown to the base indicated.

a) 64 to base 8 **b)** 64 to base 4 **c)** $\frac{1}{9}$ to base $\frac{1}{3}$ **d)** $\frac{36}{25}$ to base $\frac{6}{5}$

Changing Bases



Use the laws of exponents to convert each of the following expressions to the base indicated.

a) 16^x to base 2 **b)** $125^{\frac{2}{3}x+2}$ to base 5

c) $\left(\frac{9}{64}\right)^{\frac{3}{2}x-5}$ to base $\frac{3}{8}$ **d)** $\left(\frac{25}{49}\right)^{2x}$ to base $\frac{7}{5}$

Complete Assignment Question #1 - #2

Exponential Equations

An exponential equation is an equation where the variable is in the exponent.

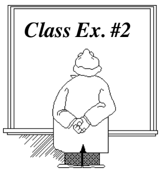
For example, $3^x = 243$, $512 = 2^{2x+1}$, $9^{x+2} = 27^x$

are all examples of exponential equations because the variable is in the exponent.

Solving Exponential Equations with a Common Base

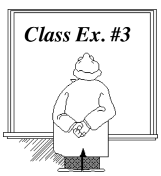
Exponential equations with a common base may be solved using the following procedure:

- Write both sides of the equation in the same base.
- Equate the exponents on both sides of the equation.
- Determine the value of the variable.



Solve for x in each of the following.

a) $2^{x+1} = 2^5$ **b)** $2^{x+1} = 128$ **c)** $7^{2x-1} = 343^{x-3}$



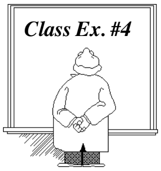
Solve.

a) $9^{x+2} = 27^x$ **b)** $81^{2x+3} = \dfrac{1}{27^{x-1}}$ **c)** $\left(\dfrac{216}{125}\right)^{2x} = \left(\dfrac{5}{6}\right)^{x-5}$

Solving Equations with Rational Exponents

Use the following method to solve equations with rational exponents.

- If necessary, rewrite the equation so that the variable has a coefficient of 1.
- Raise both sides to the reciprocal power of the exponent.
- Simplify and solve for the variable.



Solve for x.

a) $x^{\frac{1}{2}} = 16$ **a)** $x^{-\frac{1}{2}} = 5$ **b)** $2x^{\frac{3}{2}} = 16$ **c)** $(2x + 8)^{\frac{2}{3}} = 64$

Complete Assignment Questions #3 - #5

Assignment

1. Convert each number shown to the base indicated.

a) 36 to base 6 **b)** 81 to base 9 **c)** 81 to base 3

d) $\frac{81}{25}$ to base $\frac{9}{5}$ **e)** $\frac{81}{25}$ to base $\frac{5}{9}$ **f)** $\frac{343}{216}$ to base $\frac{6}{7}$

2. Use the laws of exponents to convert each of the following to the base indicated.

a) 64^{3x} to base 8 **b)** $64^{\frac{x}{3} - 3}$ to base 4 **c)** $25^{\frac{3}{2}x + 1}$ to base 5

d) $\left(\frac{27}{8}\right)^{3x - 2}$ to base $\frac{3}{2}$ **e)** $\left(\frac{512}{343}\right)^{3x}$ to base $\frac{7}{8}$

3. Identify the equations which are exponential.

a) $3x^2 = 5$ **b)** $\frac{2}{5^x} = 7$ **c)** $8^{3x} = 10$ **d)** $2x^{\frac{1}{2}} = 8$.

4. Solve for x.

a) $2^{2x+3} = 2^2$ **b)** $2^{x+1} = 8192$ **c)** $8^{3x-2} = 4096^{3x}$ **d)** $9^{4x+1} = 27^{3x}$

e) $125^{5x-2} = 25^{3x-2}$ **f)** $27^{3x} = \dfrac{1}{81^{x-2}}$ **g)** $\left(\dfrac{4}{7}\right)^{2x} = \left(\dfrac{343}{64}\right)^{x-1}$

5. Solve.

a) $x^{\frac{1}{2}} = 36$ **b)** $x^{-\frac{1}{3}} = 2$ **c)** $3x^{\frac{3}{2}} = 81$ **d)** $2x^{-\frac{5}{2}} = 64$ **e)** $4x^{\frac{5}{3}} = 128$

f) $(3x + 26)^{\frac{4}{5}} = 625$ **g)** $2(5x + 96)^{\frac{3}{4}} = 1024$ **h)** $\dfrac{2}{3}(2x - 1)^{\frac{2}{3}} = 150$

Answer Key

1. a) 6^2 **b)** 9^2 **c)** 3^4 **d)** $\left(\dfrac{9}{5}\right)^2$ **e)** $\left(\dfrac{5}{9}\right)^{-2}$ **f)** $\left(\dfrac{6}{7}\right)^{-3}$

2. a) 8^{6x} **b)** 4^{x-9} **c)** 5^{3x+2} **d)** $\left(\dfrac{3}{2}\right)^{9x-6}$ **e)** $\left(\dfrac{7}{8}\right)^{-9x}$ **3. a)** no **b)** yes **c)** yes **d)** no

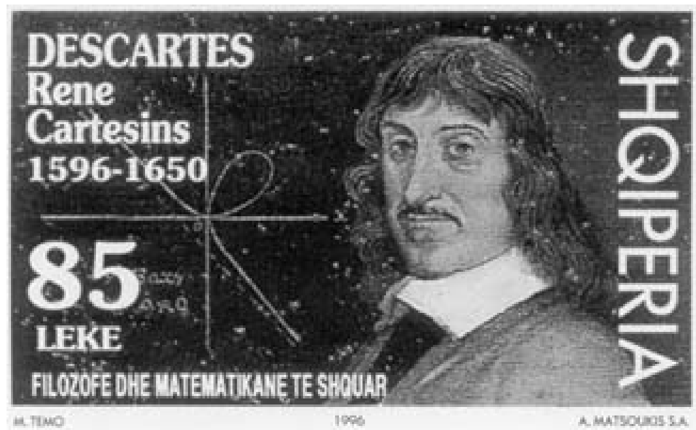
4. a) $-\dfrac{1}{2}$ **b)** 12 **c)** $-\dfrac{2}{9}$ **d)** 2 **e)** $\dfrac{2}{9}$ **f)** $\dfrac{8}{13}$ **g)** $\dfrac{3}{5}$

5. a) 1296 **b)** $\dfrac{1}{8}$ **c)** 9 **d)** $\dfrac{1}{4}$ **e)** 8 **f)** 1033 **g)** 800 **h)** 1688

Relations and Functions Lesson #1:
Review & Preview

This lesson review some important concepts that will appear through the course of the next three units. The workbook has been designed in such a way that the order of teaching this unit and the next unit (Line Segments) can be interchanged.
Omit this page if you have studied Line Segments before Relations and Functions.

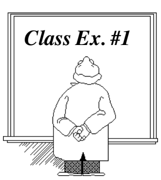
The Cartesian Coordinate System



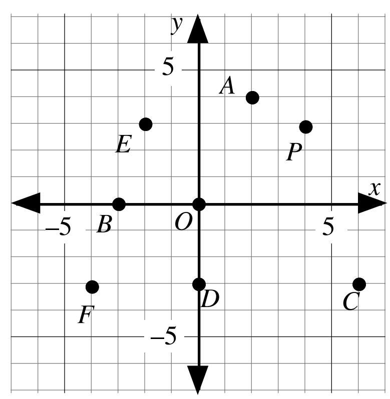
In mathematics "Cartesian" means relating to the French mathematician **René Descartes**. In 1637 he introduced the new idea of specifying the position of a point on a surface using two intersecting axes as measuring guides.

The modern Cartesian coordinate system in two dimensions is defined by two axes at right angles to each other forming a plane (called the *xy* plane). The horizontal axis is labelled *x* and the vertical axis is labelled *y*. All the points in a Cartesian coordinate system taken together form a **Cartesian plane**.

The point of intersection of the two axes is called the **origin**, usually labelled O.
On each axis a unit length is chosen and units are marked off to form a grid. To specify a particular point on the grid we use a unique **ordered pair** of numbers called **coordinates**. The first number in the ordered pair, called the ***x*-coordinate**, identifies the position with regard to the *x*-axis, while the second number, called the ***y*-coordinate**, identifies the position with regard to the *y*-axis. The point $P(4, 3)$ is shown on the grid below.
The intersection of the *x*-axis and *y*-axis creates four **quadrants**, numbered counterclockwise starting from the north-east quadrant.



a) Complete the following by writing the coordinates of the points represented by the letters on the grid.



A(*B*(*C*(
D(*O*(

b) Write the coordinates of the point in the second quadrant.

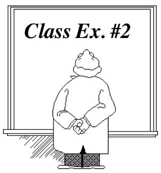
c) Write the coordinates of the point in quadrant III.

d) Complete the following table using "positive" or "negative".

Quadrant	*x*-coordinate	*y*-coordinate
I		
II		
III		
IV		

Substitution

When replacing variables with numbers **always use brackets.**



Find the value of $9x^2 - y^3 + y^2$ if $x = 2$, and $y = -3$.

Substitution by Graphing Calculator

The substitution in Class Ex. #2 can be done on a graphing calculator by accessing the $\boxed{\text{STO}\rightarrow}$ key.

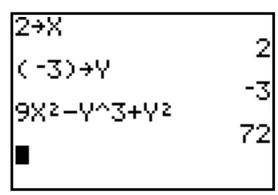
Follow the instructions below.

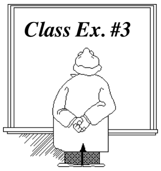
"Find the value of $9x^2 - y^3 + y^2$ if $x = 2$, and $y = -3$."

1. Press $\boxed{2}$ $\boxed{\text{STO}\rightarrow}$ $\boxed{\text{x,T,}\theta\text{,n}}$ $\boxed{\text{ENTER}}$ to enter $x = 2$.

2. Press $\boxed{(}$ $\boxed{(-)}$ $\boxed{3}$ $\boxed{)}$ $\boxed{\text{STO}\rightarrow}$ $\boxed{\text{ALPHA}}$ $\boxed{1}$ $\boxed{\text{ENTER}}$ to enter $y = -3$.

3. Enter in the equation $9x^2 - y^3 + y^2$ and press $\boxed{\text{ENTER}}$.





Determine the value of y given the value of x.

a) $y = 2x - 5, \quad x = 3$

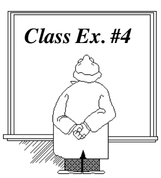
b) $-3y = 4x - 5, \quad x = 7$

c) $2y - 4x = 7, \quad x = 0.25$

d) $3x - 5y = 20, \quad x = 15$

The Relation Rule

Sometimes it is necessary to describe a relation in terms of a rule - in word or equation form.



a) Write an equation in two variables for the following statement.

"Alicia is two years older then her brother Spencer."

b) Determine an equation for the following relation.

x	y
1	1
2	4
3	9
4	16

Complete Assignment Questions #1 - #11

Assignment

Omit questions #1 - 3 if you studied Line Segments before Relations and Functions.

1. Complete the following statements.

a) The first coordinate of an ordered pair is called the _____ coordinate and the second coordinate is called the _____ coordinate.

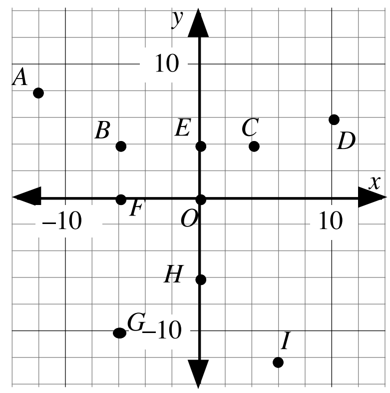
b) _____________ _____________ are used to locate or plot points on a Cartesian Plane.

c) The numbers of an ordered pair are called the ______________ of a point on the grid.

d) The x-axis and the y-axis intersect at the ____________ .

e) The Cartesian plane is divided into four ____________ .

2. The following questions refer to the grid on the right.



a) Name the points represented by the following coordinates.

i) $(-6, 4)$ **ii)** $(6, -12)$ **iii)** $(0, 4)$

b) List the coordinates of each point.

i) C **ii)** G **iii)** F

iv) O **v)** H

c) Which points have the same x-coordinate?
What visually check can be used?

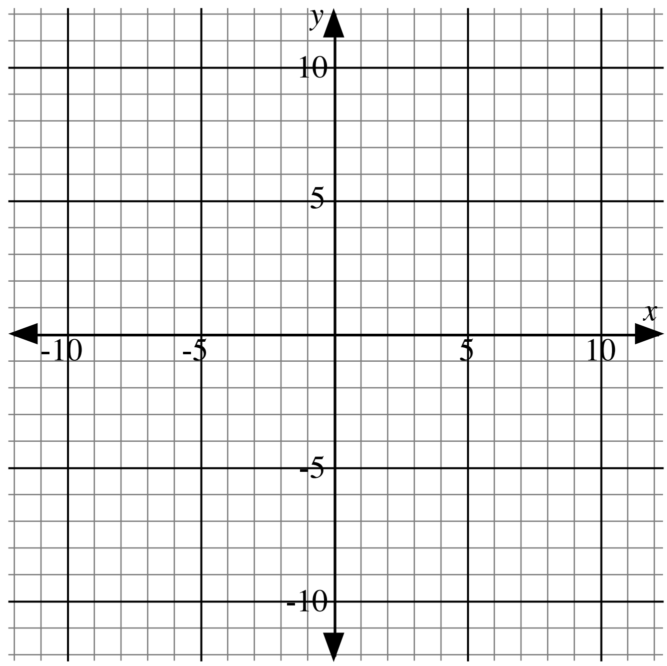
d) Which points have the same y-coordinate?
What visually check can be used?

e) Which points are in;

i) quadrant 1 **ii)** quadrant 2 **iii)** quadrant 3 **iv)** quadrant 4

f) Which points are in between quadrants?

3. The following is a scrambled message using ordered pairs. Plot the ordered pairs on the grid provided, unscramble the letters and find the message. The symbol • represents the beginning of a new letter.



- Join $(5, -9)$ to $(5, -5)$. Join $(7, -9)$ to $(5, -7)$ to $(7, -5)$.
- Join $(-11, -2)$ to $(-10, 0)$ to $(-9, 2)$ to $(-8, 0)$ to $(-7, -2)$. Join $(-8, 0)$ to $(-10, 0)$.
- Join $(0, -10)$ to $(0, -6)$ to $(2, -6)$ to $(2, -8)$ to $(0, -8)$ to $(2, -10)$.
- Join $(2, 0)$ to $(0, 0)$ to $(0, -2)$ to $(2, -2)$ to $(2, -4)$ to $(0, -4)$.
- Join $(8, 5)$ to $(8, 9)$. Join $(10, 5)$ to $(10, 9)$. Join $(8, 7)$ to $(10, 7)$.
- Join $(-3, 2)$ to $(-2, 6)$ to $(-1, 4)$ to $(0, 6)$ to $(1, 2)$.
- Join $(3, 8)$ to $(5, 8)$. Join $(4, 8)$ to $(4, 4)$.
- Join $(6, 1)$ to $(4, 1)$ to $(4, -3)$ to $(6, -3)$.
- Join $(-6, -6)$ to $(-6, -10)$ to $(-4, -10)$ to $(-4, -6)$ to $(-6, -6)$.

4. Evaluate the following expressions for the given replacements for the variables. Show the replacement step.

a) $3x - 5 - 6y \quad x = -5, y = 0$

b) $5x + 2y \quad x = -3, y = \frac{1}{2}$

c) $3x^2 + 2y^3 \quad x = -3, y = -1.2$

d) $10x^2 + 3y^2 - 25xy, \quad x = -2, y = -3$

5. Use the $\boxed{\text{STO}\rightarrow}$ key feature of a graphing calculator to verify your answers to #4b) and d).

6. Solve for y given the value of x. Round off to the nearest tenth where necessary.

a) $2y = 7x - 8, \quad x = -6$

b) $-y = 4x - 3, \quad x = 5$

c) $4y - 2x = 9, \quad x = -8$

d) $2x - 5y = 15, \quad x = 10$

e) $-8x + 0.4y = 3, \quad x = 1.2$

f) $-3y = 7x - 8, \quad x = -25$

7. Write an equation in two variables for the following statements.

a) The sum of two numbers is 15.

b) A number exceeds another by 5.

c) The width of a rectangle is 3 m less than the length.

d) Daniel is 13 years older than Bobby.

e) One number decreased by one half of another number equals 7.

f) A collection of dimes and pennies has a value of $1.97.

g) The value of some quarters exceeds the value of some nickels by $3.50.

8. Adam the paperboy is given \$10 plus 4¢ a flyer to deliver flyers to houses in his neighborhood. Write the relation rule in equation form, for his wage, W dollars, in terms of the number of flyers, f that he delivers.

9. A car rental agency charges \$50 plus 15¢/km. Write the relation rule in equation form using C for the cost in dollars and k for km travelled.

10. Write an equation for each relation.

a)

x	y
1	5
2	10
3	15
4	20

b)

x	y
1	3
2	5
3	7
4	9

11. Will's weekly salary is \$500 with a 10% commission on all his sales. The relation rule in equation form to represent his scenario is $S = mc + w$, where S dollars is the weekly salary, m is the commission, c dollars is the value of the sales made in the week, and w dollars is the fixed wage paid weekly.
The value of w in the relation rule is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** x y **b)** ordered pairs **c)** coordinates **d)** origin **e)** quadrants

2. **a)** **i)** B **ii)** I **iii)** E
b) **i)** (4, 4) **ii)** (–6, –10) **iii)** (–6, 0) **iv)** (0, 0) **v)** (0, –6)
c) $B, F, G,$ E, O, H They lie on the same vertical line.
d) B, E, C $F, O,$ They lie on the same horizontal line.
e) **i)** C, D **ii)** A, B **iii)** G **iv)** I **f)** E, F, H

3. MATH ROCKS

4. **a)** –20 **b)** –14 **c)** 23.544 **d)** –83

6. **a)** –25 **b)** –17 **c)** –1.8 **d)** 1 **e)** 31.5 **f)** 61

7. **a)** $x + y = 15$ **b)** $p - q = 5$ **c)** $w = l - 3$ **d)** $d = b + 13$
e) $x - \frac{1}{2}y = 7$ **f)** $10d + p = 197$ **g)** $25q - 5n = 350$

8. $W = 10 + 0.04f$ **9.** $C = 50 + 0.15k$

10. **a)** $y = 5x$ **b)** $y = 2x + 1$

11. 500

Relations and Functions Lesson #2:
Relationships between Two Quantities

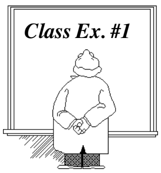
Relations

Much of mathematics involves the search for patterns and relationships between sets of data. Many real life applications of mathematics investigate the relationship between two quantities.

For example:

- the value of a computer is related to its age
- the price of a watermelon is related to its weight
- the time taken for a person to walk to school is related to the distance to be walked.

In mathematics, a comparison between two sets of elements is called a **relation**.



List one more example of a relation.

Representing the Relationship Between Two Quantities

In this unit we will consider seven ways in which the relationship between two quantities can be represented. Some of these ways are already familiar to us.

- in words
- a table of values
- a set of ordered pairs
- a mapping (or arrow) diagram
- an equation
- a graph
- function notation (some relations can be represented in this way (see Lessons 8, 9, & 10)

We will use the relation below as an example.

Investigating a Relation

Consider the following relation:

" The cost, *C* (cents per km), of driving a car, is related to the speed, *s* (km per hour), at which it is driven."

We will use this relation to introduce some ideas which will be developed throughout the course of this unit. Our task is to represent this relation in some form.

The example illustrates a relationship between two **variables**, *C* and *s*.

In the statement of the relation, the cost depends on the speed.

C is called the **dependent variable** and *s* is called the **independent variable**.

When representing a relation, we often regard the values of the independent variable as the **input** and the values of the dependent variable as the **output**.

Before considering how to represent this relation we need some data - we need input values and output values.

The input values make up the **domain** of the relation and the output values make up the **range** of the relation. These concepts will be discussed in more detail later.

Obviously we would not attempt to collect data for <u>every</u> possible input value (ie for every possible speed at which the car can be driven). Suppose that we choose as input values speeds of 20, 30, 40, ...120 km/h. and that the output values are as given in the diagrams below.

The diagrams show how the information collected can be represented as ordered pairs, in a table of values and as a mapping diagram. The ordered pairs can also be represented graphically.

Table of Values

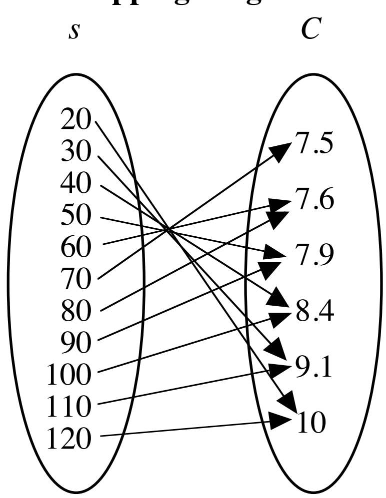
Input (s)	Output (C)
20	10
30	9.1
40	8.4
50	7.9
60	7.6
70	7.5
80	7.6
90	7.9
100	8.4
110	9.1
120	10

Ordered Pairs

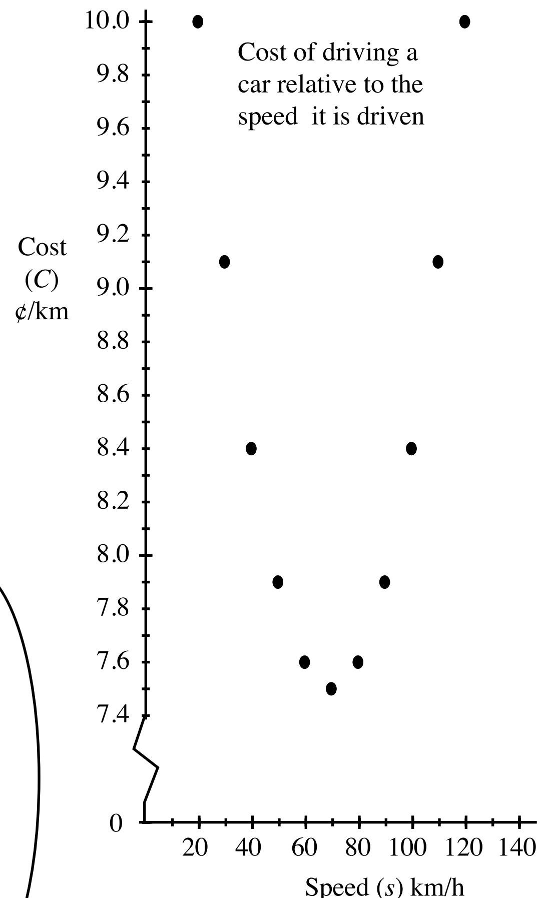
(s, C)

(20, 10)
(30, 9.1)
(40, 8.4)
(50, 7.9)
(60, 7.6)
(70, 7.5)
(80, 7.6)
(90, 7.9)
(100, 8.4)
(110, 9.1)
(120, 10)

Mapping Diagram



Graph



We have only chosen some of the possible input values but it is obvious that an output value could have been determined for any input value greater than zero and up to the maximum speed of the car. It makes sense then to connect the points on the graph in some way.

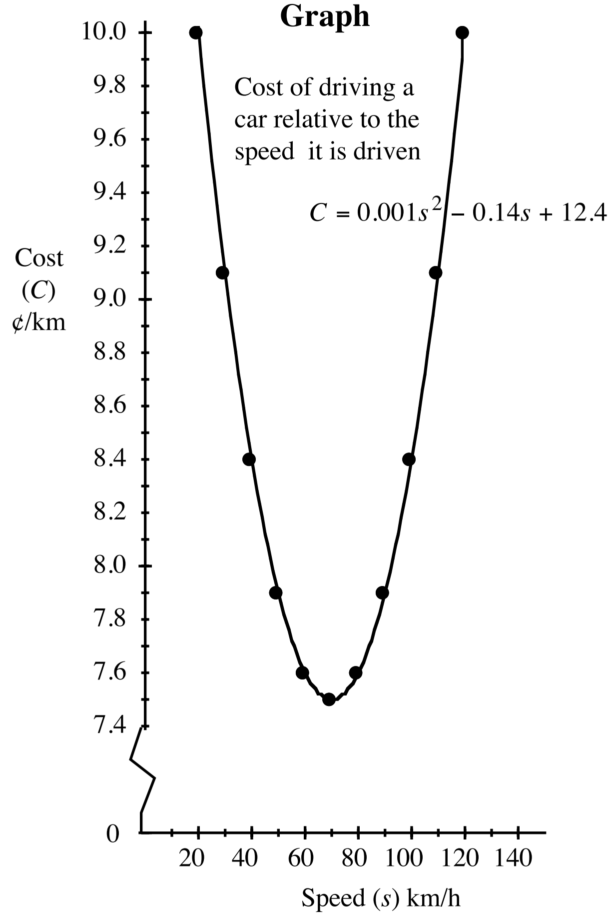
We will learn later how this can be done and how an equation can be determined that best represents the data.

The graph and equation for the relation are given below. Note that the equation is only valid for certain input values which make up the domain of the relation. For example, the equation would not be valid for $s = 5000$!

Equation

$$C = 0.001s^2 - 0.14s + 12.4$$

Graph

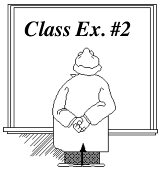


Discrete and Continous Variables

In the above example, the variables s and c are examples of **continuous variables** since they can take on every value within a particular interval. For example, s can take on any value between zero and the maximum speed of the car.

Other variables such as the number n of tickets bought for a concert, can only take on limited (in this case whole number values) and are therefore not continuous variables. Such variables are called **discrete variables**.

A graph relating two discrete variables consists of a series of unconnected points, whereas in the graph of two continuous variables the points would be connected.



Classify each of the following variables as discrete or continuous.

a) time taken to complete a 100 m sprint

b) number of students who pass Pure Math 10

c) height of students

d) shoe size

Independent and Dependent Variables in a Relation

The values of the independent variable represent the inputs and the corresponding values of the dependent variable are the outputs.

- In an ordered pair, the values of the first coordinate are those of the independent variable and the values of the second coordinate are values of the dependent variable.
- In a table of values, the independent variable is usually given first - either to the left or above the values of the dependent variable.
- In a mapping diagram, the arrows go from the independent variable to the dependent variable.
- On a graph, the independent variable is on the horizontal axis, often the x-axis, and the dependent variable is on the vertical axis, often the y-axis. The figure below uses the equation $y = 3x - 5$ as an example to illustrate the independent and dependent variables of an equation.

In an equation, we usually try to isolate the dependent variable to the left side.

$$y = 3x - 5$$

- The **dependent variable** (or responding variable).
- Values of the dependent variable represent the <u>outputs</u> of the relation.
- Values of the dependent variable are represented by the **second** coordinate of an ordered pair and are on the **vertical axis**.

- The **independent variable** (or manipulated variable).
- Values of the independent variable represent the <u>inputs</u> of the relation.
- Values of the independent variable are represented by the **first** coordinate of an ordered pair and are on the **horizontal axis** .

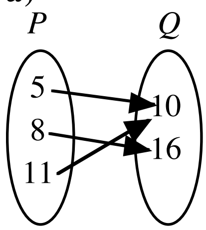
Class Ex. #3

The diagrams show relations expressed in different ways. In each case

i) state the independent variable **ii)** state the dependent variable

iii) list the inputs **iv)** list the outputs

a)



b)

V	A
4	15
10	12
25	15

c)

(B, c): $(3, 7)$, $(4, 11)$, $(5, 15)$, $(6, 19)$

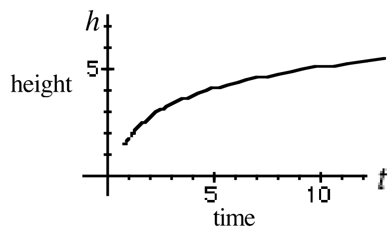
Class Ex. #4

The diagrams show relations expressed in different ways. In each case

i) state the independent variable **ii)** state the dependent variable

a) $C = 2\pi r$

b)



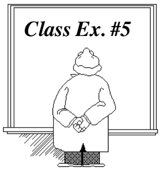
c) The amount of sap, s, obtained from maple trees is dependent on the time, t, the container is left attached to the maple tree.

Complete Assignment Question #1 - #4

Investigating Relationships by Plotting Ordered Pairs

In this section we will consider relations defined by an equation and sketch a graph by plotting ordered pairs.

- Make a table of inputs by choosing replacements for the independent variable.
- For each of the input values calculate the corresponding value (the output) of the dependent variable.
- Plot the ordered pairs on a Cartesian plane.



Consider the relation described by the equation $y = 2x - 5$.

a) Complete the first five rows of the following table of values which shows some of the possible input values.

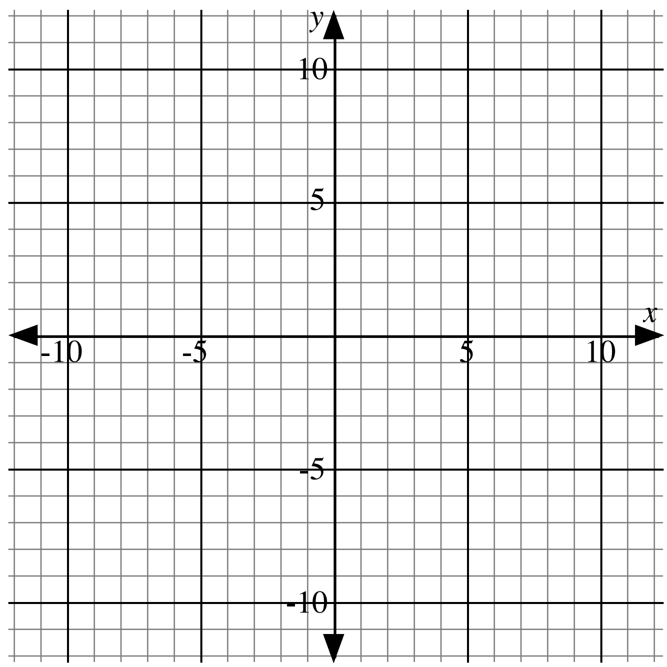
Input (x)	Output (y)	Ordered pair (x, y)
–2		
–1		
0		
1		
2		

b) Plot the ordered pairs in a) on the grid provided.

c) Connect the points on the grid and extend the line in both directions with arrows at both ends.

d) Use the graph to determine the value of y when $x = 6$.

e) Use the equation to determine the value of y when $x = 6$ and verify the answer in d).

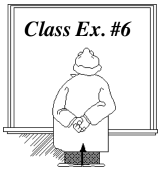


f) Write the value of y when $x = 6$ in the table of values using the first blank space in a).

g) Use the graph to determine the value of x when $y = 3$. Put this information in the last row in a).

h) Complete the following statement:

This relation is called a ______________ relation because the graph of the relation is a straight line.



Consider the relation described by the equation $y = x^2 - 6$.

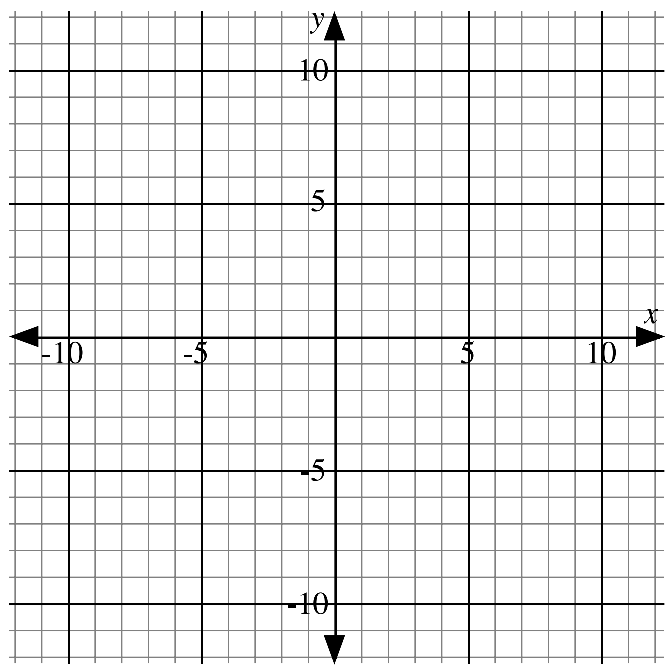
a) Complete the table of values to the right which show some of the possible input values.

Input (x)	Output (y)	Ordered pair (x, y)
4		
3		
2		
1		
0		
–1		
–2		
–3		

b) Plot the ordered pairs in a) on the grid provided.

c) Use the symmetry of the graph or table to predict the value of y when $x = –4$.

d) Use the equation to determine the value of y when $x = –4$ and verify the answer in c).



e) Write the value of y when $x = –4$ in the table of values using the first blank space in a).

f) Connect the points on the grid with a smooth curve.

g) Why do you think this type of relation is called a **nonlinear relation**?

Complete Assignment Question #5 - #9

Assignment

1. Complete the following.

a) The mathematical relationship between two quantities is called a ____________ .

b) The variable used for inputs in a relation is known as the manipulated variable or _____________ variable.

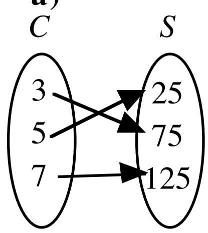
c) The variable used for outputs in a relation is known as the responding variable or _____________ variable.

d) In the equation $A = \pi r^2$, the independent variable is _____ and the dependent variable is ____.

2. The diagrams show relations expressed in different ways. In each case:

i) state the independent and dependent variables **ii)** list the inputs and outputs.

a)



b)

C	n
8	22
20	19
50	35

c)

(f, e): $(2, 3)$, $(-2, 19)$, $(8, 17)$, $(0, 2)$

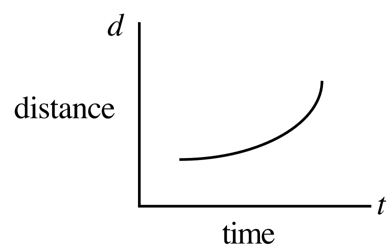
3. For each of the following relations, state;

i) the independent variable **ii)** the dependent variable

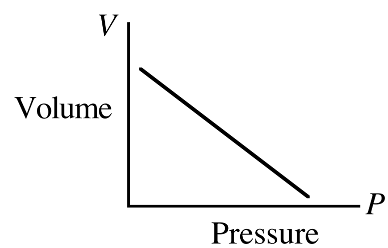
a) $V = \frac{4}{3}\pi r^3$

b) $C = \frac{5}{9}(F - 32)$

c)



d)



e) A truck's value, v, depends on its age, a.

f) The cost, C, of producing business cards is dependent on the number of cards, n, produced.

4. List the different ways a relation may be represented.

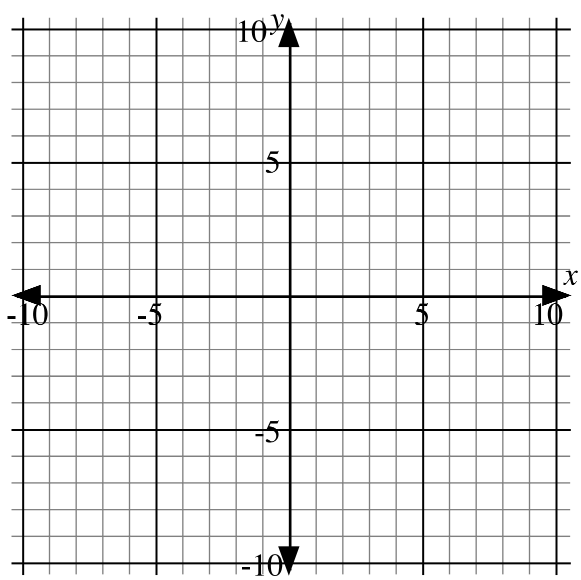
5. Consider the relation described by the equation $y = -x - 2$.

a) Identify the independent and dependent variables.

b) Complete the following table of values.

Input (x)	Output (y)	Ordered pair (x, y)
–3		
–1		
0		
1		

c) Plot the ordered pairs in b) on the grid provided.



d) Connect the points on the grid and then extend the line in both directions with arrows at both ends.

e) Use the graph to determine the value of y when $x = 5$.

f) Use the equation to determine the value of y when $x = 5$ and verify the answer in e).

g) Write the value of y when $x = 5$ in the table of values in b).

h) Use the graph to determine the value of x when $y = 0$. Include this ordered pair in the table of values.

i) Use the graph to determine the value of x when $y = 4$. Include this ordered pair in the table of values.

j) Verify the answer in i) using the equation.

k) Is this a linear or a nonlinear relation?

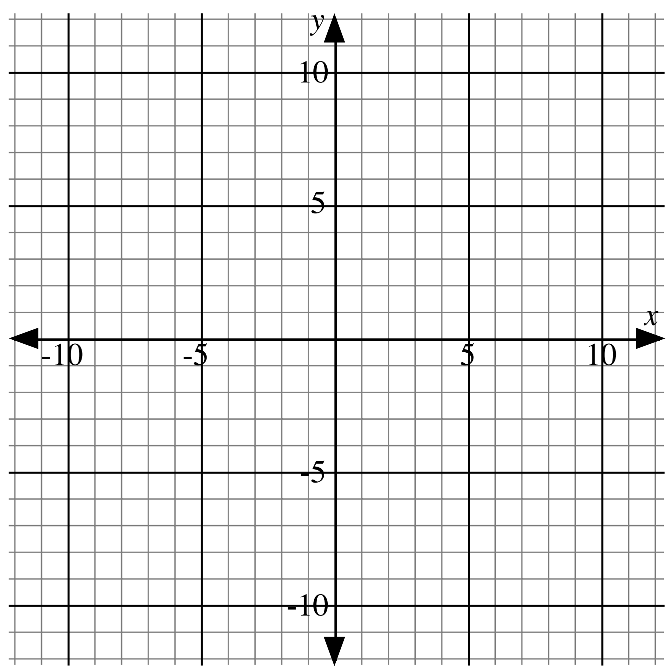
6. Consider the relation described by the equation $y = -0.5x^2 + 8$.

a) Identify the independent and dependent variable.

b) Complete the following table of values.

Input (x)	Output (y)	Ordered pair (x, y)
–6		
–4		
–2		
0		
2		
4		

c) Plot the ordered pairs in b) on the grid provided.



d) Use the plotted points and table to predict the value of y when $x = 6$. Plot this point on the grid.

e) Use the equation to determine the value of y when $x = 6$ and verify the answer in d).

f) Connect the points on the grid with a smooth curve.

g) Is this a linear or a nonlinear relation?

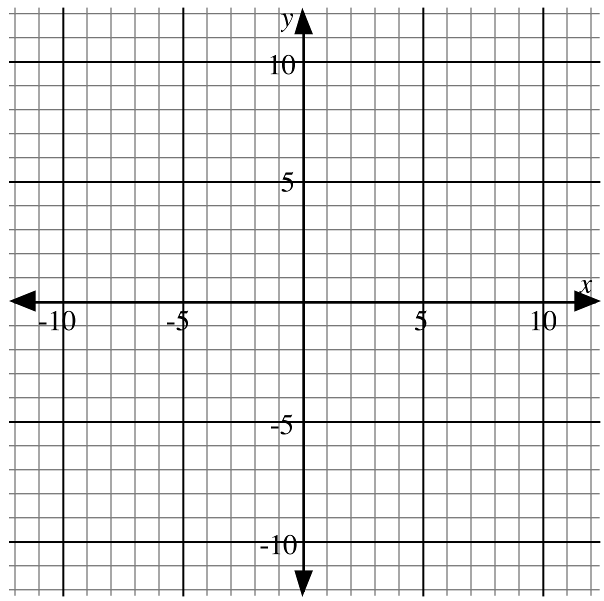
h) Use the graph to determine the values of x when $y = 3.5$

7. For the following relations:

i) Complete the table of values choosing your own input values where necessary.
ii) Plot the ordered pairs on the grid and sketch the graph of the relation.
iii) State whether the relation is linear or nonlinear.

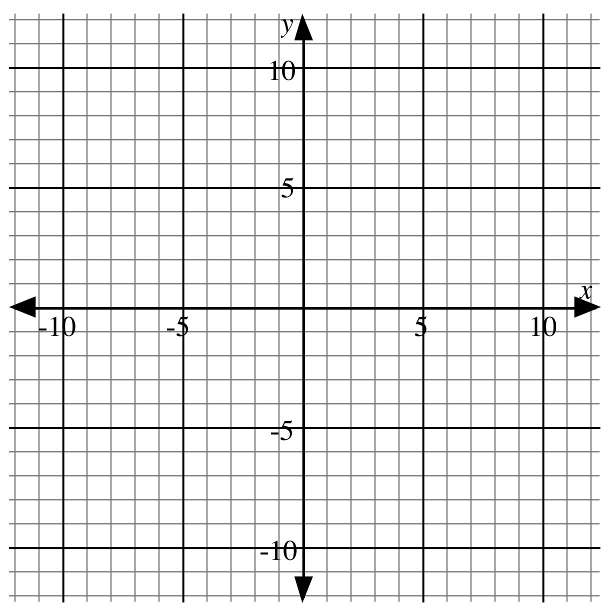
a) $y = -2x + 3$

Input (x)	Output (y)	Ordered pair (x, y)
0		
3		



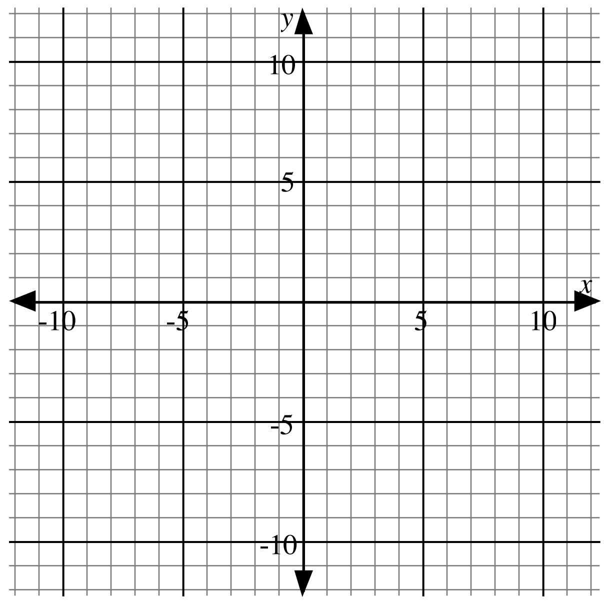
b) $y = 0.5x - 8$

Input (x)	Output (y)	Ordered pair (x, y)
−10		
0		
6		



c) $y = -x^2 + 5$

Input (x)	Output (y)	Ordered pair (x, y)
4		
3		
2		
1		
0		
–1		
–2		
–3		
–4		



Multiple Choice

8. Which of the following variables is discrete?

A. temperature
B. weight
C. altitude
D. number of goals

Numerical Response

9. Consider the relation described by the equation $y = 1.5^{x-2}$. If the input is 4, then the output is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** relation **b)** independent **c)** dependent **d)** r, A

2. **a)** **i)** independent - C
dependent - S
ii) input - 3, 5, 7
output - 25, 75, 125

b) **i)** independent - C
dependent - n
ii) input - 8, 20, 50
output - 22, 19, 35

c) **i)** independent - f
dependent - e
ii) input - 2, –2, 8, 0
output - 3, 19, 17, 2

3. **a)** **i)** independent - r
ii) dependent - V

b) **i)** independent - F
ii) dependent - C

c) **i)** independent - time
ii) dependent - distance

d) **i)** independent - pressure
ii) dependent - volume

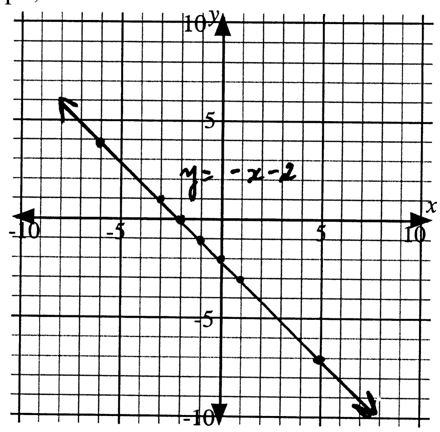
e) **i)** independent - a
ii) dependent - V

f) **i)** independent - n
ii) dependent - C

4. words, table of values, set of ordered pairs, mapping, equation, graph, function notation

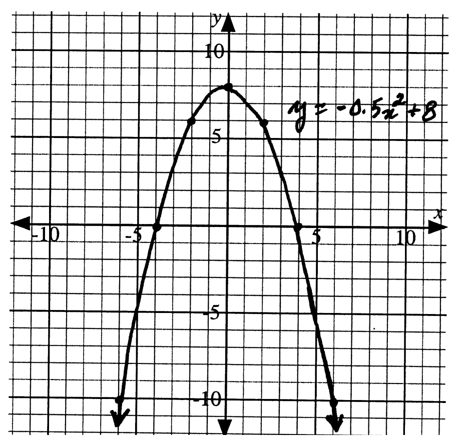
5. **a)** independent - x
dependent - y
b) see table
c) see grid →
d) see grid →
e) –7
f) –7
g) –7, and see table
h) –2, and see table
i) –6, and see table
k) linear

	Input (x)	Output (y)	Ordered pair (x, y)
	–3	1	(-3,1)
	–1	-1	(-1,-1)
	0	-2	(0,-2)
	1	-3	(1,-3)
g)	5	-7	(5,-7)
h)	-2	0	(-2,0)
i)	-6	4	(-6,4)



6. **a)** independent - x
dependent - y
b) see table
c) see grid →
d) –10
e) –10
f) see grid →
g) non-linear
h) ± 3

Input (x)	Output (y)	Ordered pair (x, y)
–6	-10	(-6,-10)
–4	0	(-4,0)
–2	6	(-2,6)
0	8	(0,8)
2	6	(2,6)
4	0	(4,0)



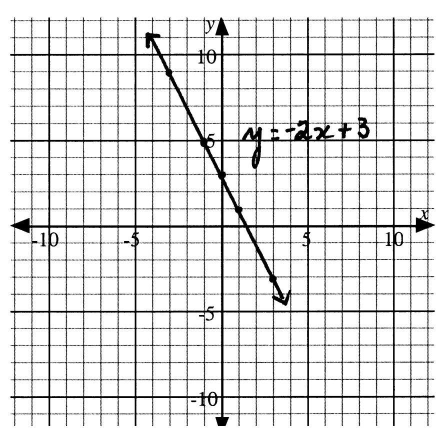
7. **a)**

i) answers may vary
see table

ii) see grid

iii) linear

Input (x)	Output (y)	Ordered pair (x, y)
-3	9	(-3,9)
-1	5	(-1,5)
0	3	(0,3)
1	1	(1,1)
3	-3	(3,-3)



7. (cont.)

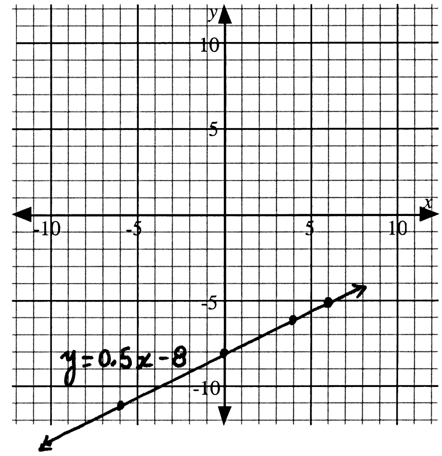
b)

i) answers may vary
see table

ii) see grid

iii) linear

Input (x)	Output (y)	Ordered pair (x, y)
–10	-13	(-10, -13)
-6	-11	(-6, -11)
0	-8	(0, -8)
4	-6	(4, -6)
6	-5	(6, -5)



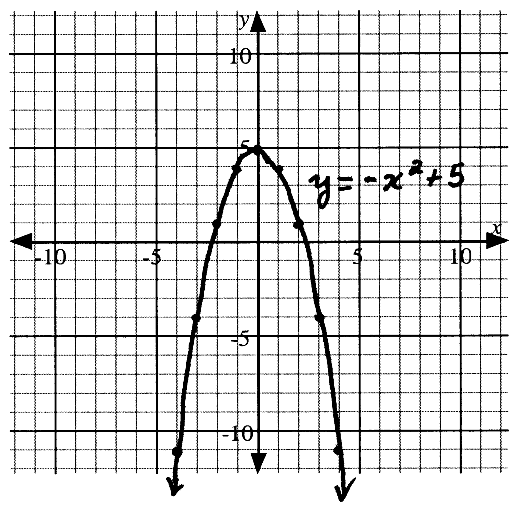
c)

i) answers may vary
see table

ii) see grid

iii) non-linear

Input (x)	Output (y)	Ordered pair (x, y)
4	-11	(4, -11)
3	-4	(3, -4)
2	1	(2, 1)
1	4	(1, 4)
0	5	(0, 5)
–1	4	(-1, 4)
–2	1	(-2, 1)
–3	-4	(-3, -4)
–4	-11	(-4, -11)



8. D

9.

2	.	2	5

Relations and Functions Lesson #3:
x- and y-intercepts and Interpreting Relations

Review

a) A **relation** is a connection between two quantities. A relation can be represented graphically by a set of ______________ __________ .

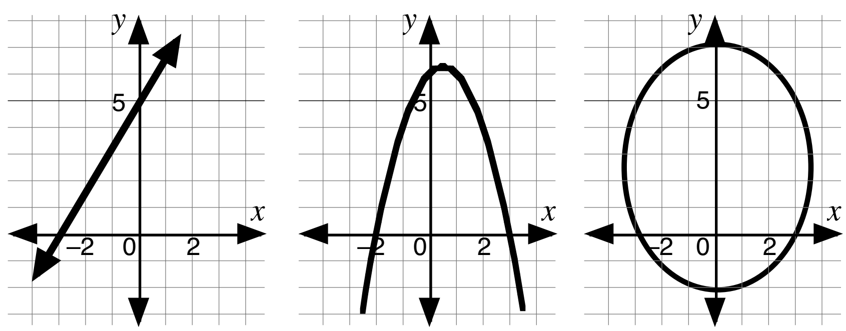
b) The first component of a set of ordered pairs is the ___ coordinate, also known as the input. Values of the input are values of the________________ variable.

c) The second component of a set of ordered pairs is the ___ coordinate, also known as the output. Values of the output are values of the________________ variable.

Exploring x- and y-intercepts

Consider the following graphs.

Graph 1 **Graph 2** **Graph 3**



a) List the coordinates of the point(s) where each graph crosses the x-axis.

- Graph 1 crosses the x-axis at (,).
- Graph 2 crosses the x-axis at (,) and (,).
- Graph 3 crosses the x-axis at (,) and (,).

b) What do all the points in a) have in common?

c) List the coordinates of the point(s) where each graph crosses the y-axis.

- Graph 1 crosses the y-axis at (,).
- Graph 2 crosses the y-axis at (,).
- Graph 3 crosses the y-axis at (,) and (,).

d) What do all the points in c) have in common?

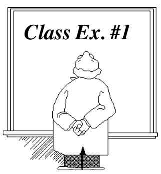
x- and y- Intercepts of a Graph

The ***x*-intercept** of a graph is the x-coordinate of the ordered pair where the graph intersects the x-axis. An x-intercept occurs at a point on the graph where the y-coordinate is zero.

The ***y*-intercept** of a graph is the y-coordinate of the ordered pair where the graph intersects the y-axis. A y-intercept occurs at a point on the graph where the x-coordinate is zero.

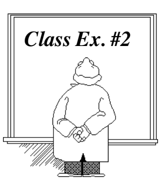
1. Given the equation of the graph of a relation:
- to determine the x-intercept, set $y = 0$ and solve for x.
- to determine the y-intercept, set $x = 0$ and solve for y.

2. The equation of a graph can be written in different forms, all of which are equivalent. The equation of Graph 1 on the previous page is $y = \frac{5}{3}x + 5$, which can be written as $3y = 5x + 15$ or $5x - 3y + 15 = 0$. Equivalent forms of an equation will be studied in detail, in a later unit. For the time being, use the instruction in note 1 to find the x- and y-intercepts of the graph of an equation given in any form.



The equation of Graph 1 on the previous page is $3y = 5x + 15$.
Algebraically determine the x-intercept and y-intercept of Graph 1.

x-intercept	y-intercept

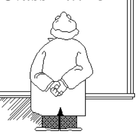


Calculate the x-intercept and the y-intercept of the graph of $x^2 + y^2 = 36$.

x-intercept	y-intercept

Complete Assignment Question #1 - #3

Class Ex. #3

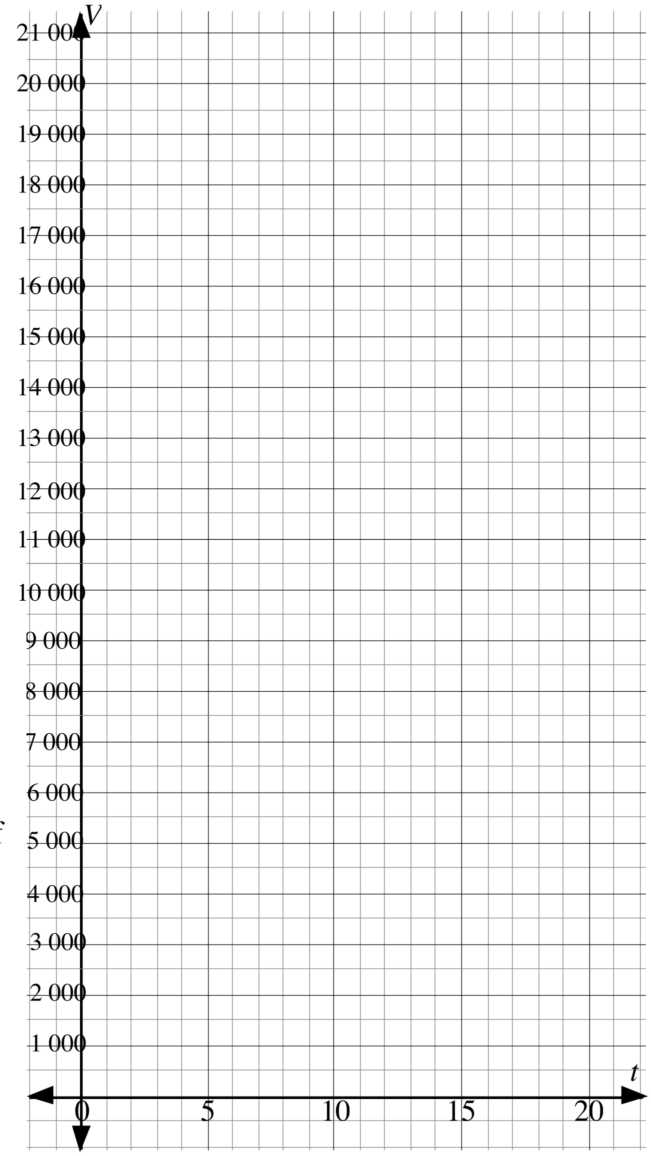


Lysa purchases a new car for \$20 000. The value of the car can be represented by the formula $V = 20\,000 - 1250t$, where V is the value of the car in dollars and t is the age of the car in years.

a) • Complete the table of values and plot the ordered pairs on the grid.

Input (t)	Output (V)	Ordered pair (t, V)
0		
2		
4		
6		

• Connect the points with a straight line and extend the line.



b) What does the ordered pair (0, 20 000) represent?

c) Use the graph to determine the t-intercept. What does the t-intercept represent?

d) Use the graph to determine the value of the car after;
i) 3 years **ii)** 10 years **iii)** 14 years.

e) Use the formula to verify d) ii).

f) Use the graph to determine when the car will be worth:
i) \$5 000 **ii)** half of the purchase price.

g) Use the formula to verify f) ii).

h) Complete the following statement to describe the relation:

The original value of the car is __________ . It depreciates in value by ________ per year and has no value after ___________ years.

In this lesson, using algebra determines the exact values for intercepts, etc. whereas using graphs gives an estimate for intercepts, etc. In lesson 5 we use the features of a graphing calculator to determine more accurate results from a graph.

In part d)i) we were asked to use the graph to find values lying between given points. This process is called **interpolation**. Extending the graph to predict values outwith the plotted points is called **extrapolation**. d)ii) and d)iii) are examples of extrapolation.

Complete Assignment Questions #4 - #9

Assignment

1. Determine the y-intercept of the graph of each equation.

a) $y = x - 5$ **b)** $y = 3x - 15$ **c)** $2y + 3x - 12 = 0$

d) $0.5x - 2.4y + 0.8 = 0$ **e)** $2y = x^2 - 60$ **f)** $y = 0.001x^2 - 0.001x + 12.44$

2. Determine the x-intercept(s) of the graph of each equation.

a) $y = x - 2$ **b)** $y = 2x - 8$ **c)** $3y + 2x - 12 = 0$

d) $0.6x - 2y + 0.5 = 0$ **e)** $y = x^2 - 9$ **f)** $y = 12 - 3x$

3. Find the x- and y-intercepts of each equation.

a) $y = 4x + 7$

b) $y = 15 - 6x$

c) $4x - 2y + 16 = 0$

d) $y = \frac{x^2}{2} - 18$

e) $x^2 + y^2 = 25$

f) $y = 3x$

g) $y = x^2 + 4$

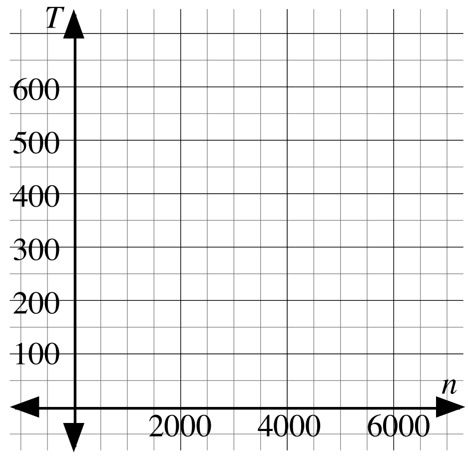
h) $9x^2 + y^2 = 81$

i) $9x^2 - y^2 = 81$

4. Triple A Car Rental charges \$100 per rental plus 10¢ per km. The total cost, T, in dollars of renting the car can be represented by the formula, $T = 100 + 0.10n$, where n is the number of km travelled.

a) • Complete the table of values and plot the ordered pairs on the grid provided.

Number of km (n)	Total Rental Cost (T) dollars
0	
1000	
3500	
5000	



• Connect the points with a straight line and extend the line in both directions.

b) What does the ordered pair (0, 100) represent?

c) Determine the n-intercept of the graph. Explain why it is not applicable to this problem.

d) Interpolate from the graph to determine the cost for a journey of:
i) 2000 km **ii)** 4500 km

e) Use the formula to verify the answers in d).

f) If the total cost of rental is \$650, use the graph to determine the number of km travelled.

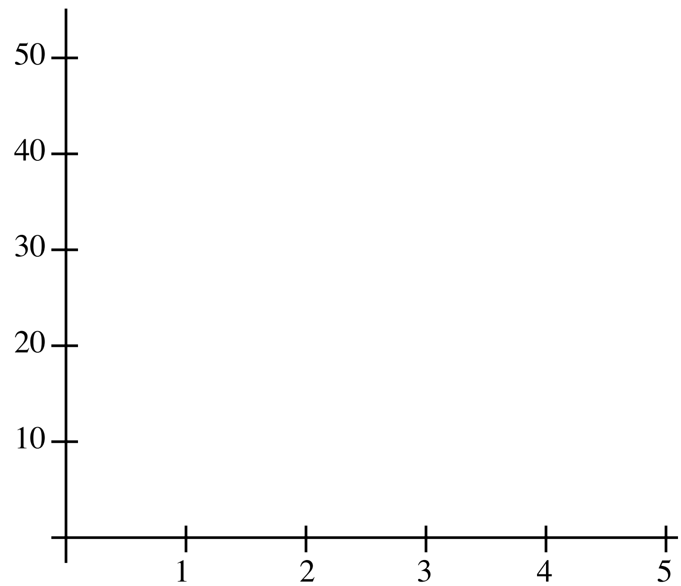
g) Verify the answer in f) using the formula.

5. An arrow is shot vertically into the air using a bow. The height, h metres, above the ground after t seconds, where $t \geq 0$ is approximated by the equation $h = -5t^2 + 20t + 25$.

a) The maximum height of the arrow is reached after 2 seconds. Calculate the maximum height.

b) Complete the table of values and plot the points on the grid.
Join the points with a smooth curve and label the graph.

time (seconds)	height (metres)
0	
1	
2	
3	
4	
5	



c) Is this a linear or nonlinear relation?

d) For how many seconds is the arrow in the air?

e) What does the h-intercept represent in the context of the question?

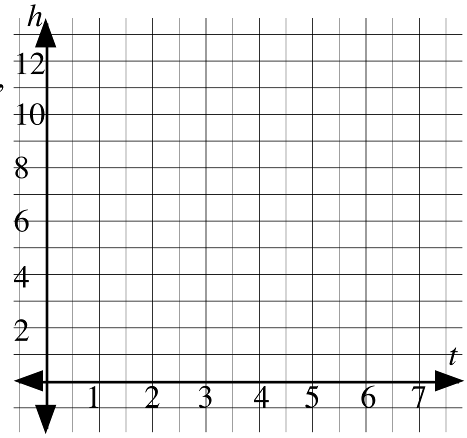
f) What does the t-intercept represent in the context of the question?

g) i) Use the graph to estimate the height of the arrow after 1.5 seconds.

ii) Use the equation to calculate the exact height of the arrow after 1.5 seconds.

h) Does it make sense to extend the graph of the relation $h = -5t^2 + 20t + 25$ further in a downward direction to the left or right? Explain.

6. A candle manufacturer determined that their "Long-Last" candles melted according to the formula $h = -2t + 12$, where h is the height of the candle, in cm, after t hours.



a) Make a table of values and use this to construct the graph of $h = -2t + 12$.

t					
h					

Use your graph to answer b – e.

b) How high is the candle before it begins to melt?

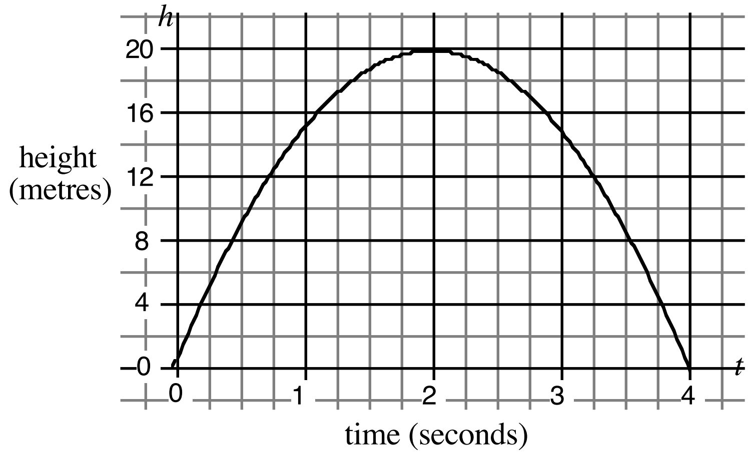
c) How many hours will the candle last before it will completely burn out?

d) How high will the candle be after burning for 5 hours?

e) How long will it take for the candle to burn down to a height of 7 cm?

f) Verify the answers from b) - e) using the formula.

7. A football is kicked by a student. The graph of the relation between the height of the football above the ground and time is shown. The formula that represents the relation is given by $h = -4.9t^2 + 19.4t + 0.6$, where h is the height in metres above the ground and t is the time in seconds the football is in the air.



Use the graph to answer a – c:

a) Estimate, to the nearest metre, the maximum height of the football above the ground.

b) Estimate how long it takes for the football to reach the ground.

c) What is the height, to the nearest metre, of the football when it is in the air for 3 seconds?

d) Use the formula to calculate the exact answer to c).

e) Calculate the h-intercept and describe what it represents in the context of the question.

Multiple Choice

8. In which of the following relations does the graph of the relation have equal x- and y-intercepts?

A. $y = x + 8$
B. $2x + 2y = 7$
C. $2x - 3y + 4 = 0$
D. none of the above

Numerical Response

9. The graph of the relation $4x^2 + 9y^2 - 36 = 0$ has x-intercepts a and b, and y-intercepts c and d. The value of the product $abcd$ is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** -5 **b)** -15 **c)** 6 **d)** $\frac{1}{3}$ **e)** -30 **f)** 12.44

2. **a)** 2 **b)** 4 **c)** 6 **d)** $-\frac{5}{6}$ **e)** ± 3 **f)** 4

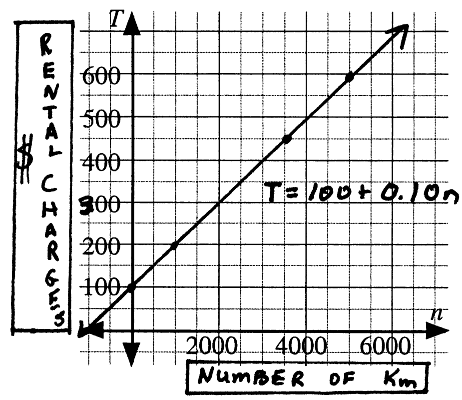
3. **a)** x-int $= -\frac{7}{4}$, y-int $= 7$ **b)** x-int $= \frac{5}{2}$, y-int $= 15$ **c)** x-int $= -4$, y-int $= 8$

d) x-int $= \pm 6$, y-int $= -18$ **e)** x-int $= \pm 5$, y-int $= \pm 5$ **f)** x-int $= 0$, y-int $= 0$

g) no x-int, y-int $= 4$ **h)** x-int $= \pm 3$, y-int $= \pm 9$ **i)** x-int $= \pm 3$, no y-int

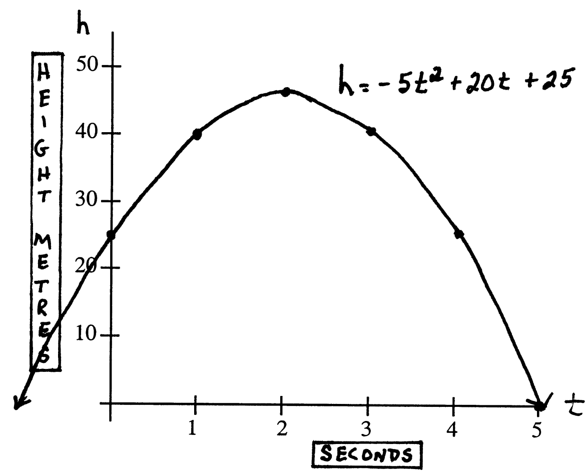
4. **a)** see table and graph
b) Triple A Car Rental charges a flat rate of $100
c) n–int = –1000, distance in this scenario cannot be represented by a negative value
d) **i)** $300 **ii)** $550
f) 5500 km

Number of km (n)	Total Rental Cost (T) dollars
0	100
1000	200
3500	450
5000	600



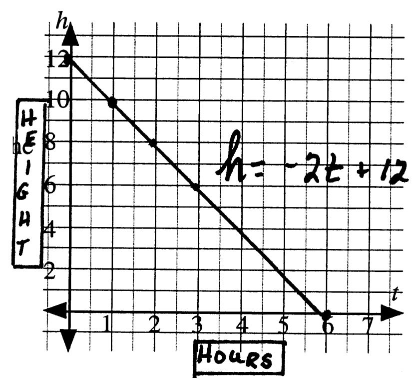
5. **a)** 45 m
b) see table and graph
c) non-linear
d) 5
e) The arrow was fired from a height of 25m above the ground
f) The number of seconds it takes to strike the ground.
g) **i)** approximately 44m
ii) 43.75
h) No to the left because time cannot be negative.
No to the right because the ground stops the arrow from going further.

time (seconds)	height (metres)
0	25
1	40
2	45
3	40
4	25
5	0



6. **a)** see table and graph answers may vary
b) 12 cm
c) 6 hours
d) 2 cm
e) 2.5 hours

t	0	1	2	3	6
h	12	10	8	6	0



7. **a)** approx 20 m
b) approx 4 seconds
c) approx 15 m
d) 14.7 m
e) h–int is 0.6 m.
The football was punted 0.6 m above the ground.

8. B **9.**

3	6		

Relations and Functions Lesson #4:
Domain and Range

Review

Complete the following statements.

a) The *x*-intercept of a graph of a relation has a _____ coordinate with a value of zero.

b) The *y*-intercept of a graph of a relation has a _____ coordinate with a value of zero.

Domain and Range

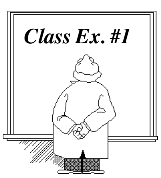
The **domain** of a relation is the set of all possible values which can be used for the **input** of the **independent variable** (x).

The **range** of a relation is the set of all possible values of the **output** of the **dependent variable** (y).

In lesson 2 we described the relation in each of the following forms

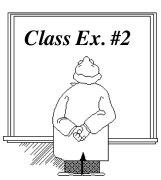
- in words
- a table of values
- a set of ordered pairs
- a mapping (or arrow) diagram
- an equation
- a graph

In this lesson we will study the domain and range given in any of these forms.

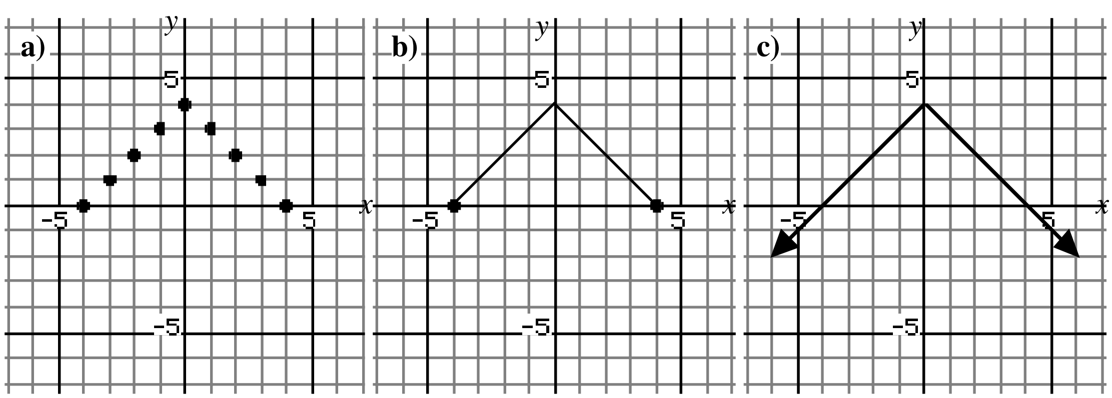


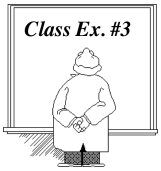
List the domain and range of the following set of ordered pairs.

a) (1, 2), (0, 5), (3, 8), (5, 9), (–3, 2)   **b)** (3, 3), (0, 3), (–3, 3), (2, 9), (–8, 3)



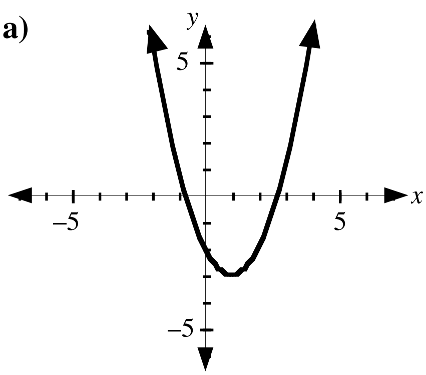
In each case, state the domain and range of the relation represented by the graph.



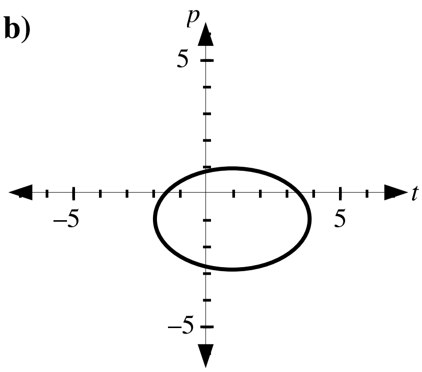


State the domain and range of the following relations.

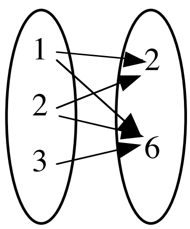
a)



b)

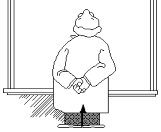


c)



d) A circle with centre (–2, 3) and a radius of 3.

Class Ex. #4



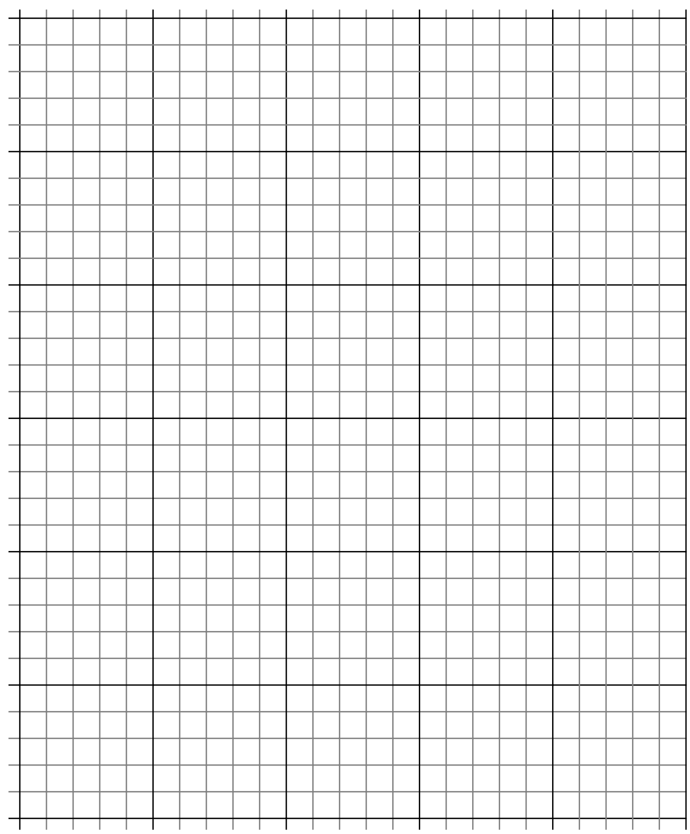
A high school football team is hosting a banquet to celebrate winning the championship. The caterer charges a set up fee of \$500 plus \$20 per person.
The equation $C = 500 + 20n$ represents the cost of hosting the banquet for n people.

a) Make a table of values with 8 entries for a minimum of 100 and a maximum of 500 people.

n	C

b) Plot the eight ordered pairs from a) on the grid.

c) If all possible ordered pairs from b) were plotted on the grid, state the domain and range of the relation and explain why there are restrictions on both.



Complete Assignment Questions #1 - #9

Assignment

1. State the domain and range of each relation.

a) (2, 3), (0, 2), (4, 8), (–1, 8), (–3, 1)

b) (–3, 3), (0, –5), (–3, 3), (5, –2), (–8, 1)

c)

Input (x)	Output (y)
0	3
2	4
4	5
6	3

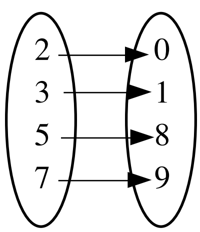
d)

Input (x)	Output (y)
2	3
0	4
–3	5
2	6

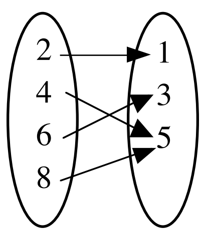
e)

Input (x)	Output (y)
1	5
–1	5
3	5
7	5

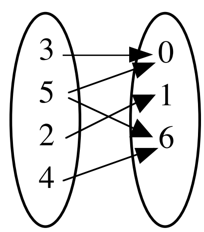
f)



g)

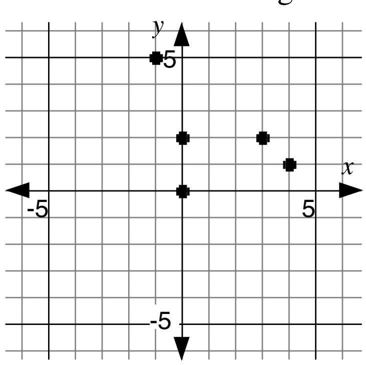


h)

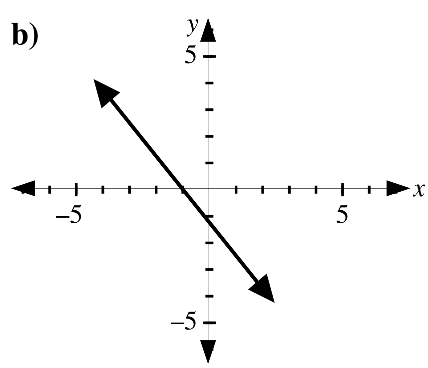


2. State the domain and range for each relation.

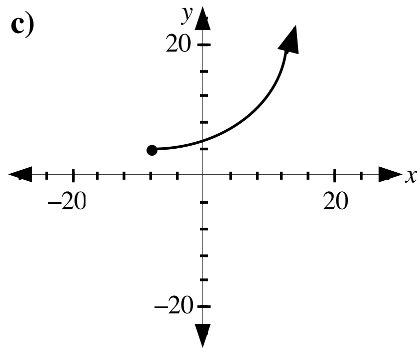
a)



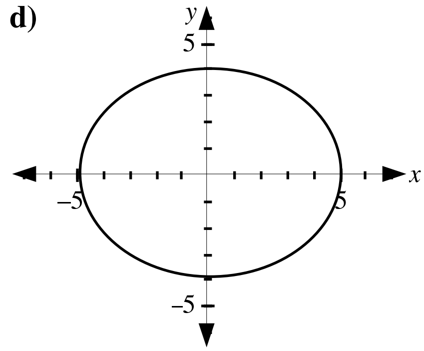
b)



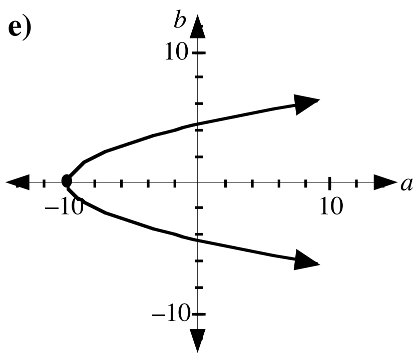
c)



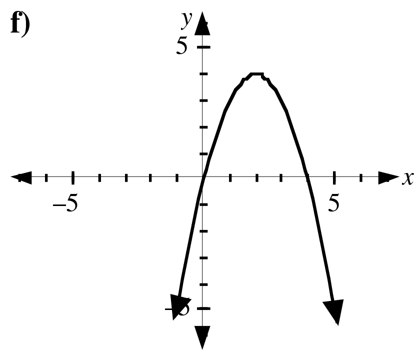
d)



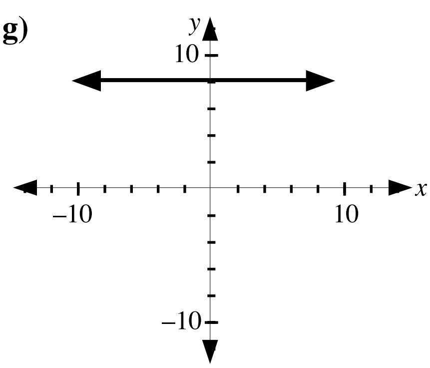
e)



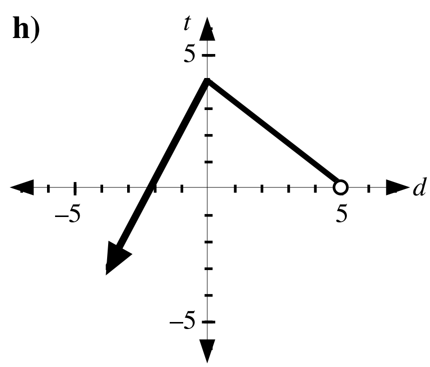
f)



g)



h)



3. In each case a relation is graphed on a grid. State the domain and range of the relation if the graph is;

a) a circle whose centre is located at $(-1, 12)$ and has a radius of 5 units.

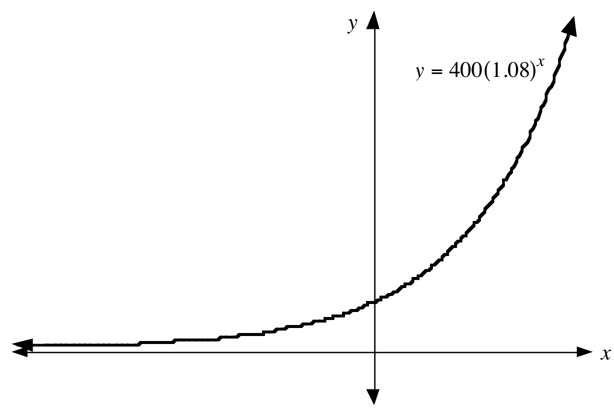
b) a circle with centre $(-3, -5)$ and diameter 40 units.

c) a rectangle with vertices $A(-8, 10)$, $B(-8, -2)$, $C(7, -2)$, and $D(7, 10)$.

d) a triangle with vertices $T(-50, -75)$, $U(-35, -25)$, and $V(-65, -25)$.

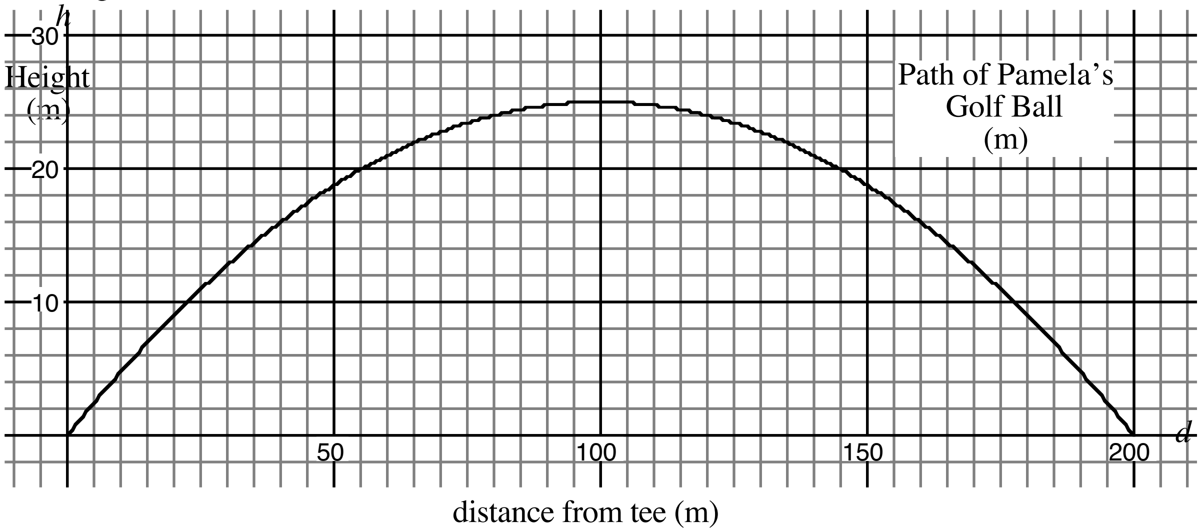
4. The graph of the relation $y = 400(1.08)^x$ is shown on the grid.

a) State the domain, range, and y-intercept of the relation.



b) The relation $A = 400(1.08)^t$ represents the amount of money when an original investment of \$400 is compounded annually at 8% for a period of t years. State the domain and range of this relation and explain why the answer is different from a).

5. The graph shows the flight of Pamela's golf ball from the tee to a sand trap at the edge of the green.



a) State the *h*-intercept and the *d*-intercepts of the graph and explain their significance in relation to the question.

b) State maximum height of the golf ball and explain its relevance to the domain or range of the relation.

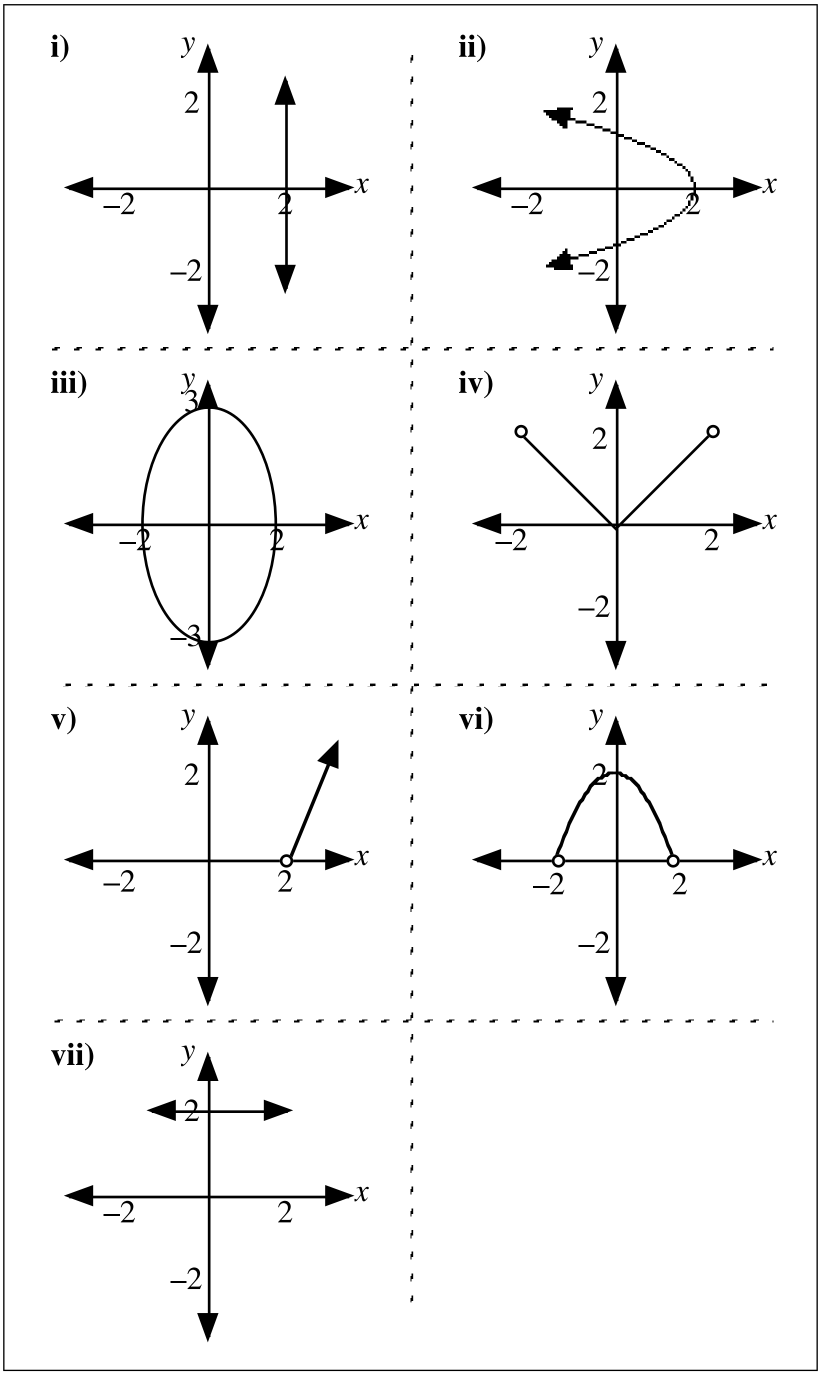
c) State the domain and range of the relation.

d) Estimate from the graph the horizontal distance the ball has travelled when it is 20 m in the air. Explain why there are two answers.

e) Estimate from the graph the height of the golf ball when the horizontal distance from the tee is 80 m.

f) Give a brief description of the relationship between the height of the golf ball and the horizontal distance from the tee.

6. Match each graph with the domain from A to F. Each domain may be used once, more than once, or not at all.



A. $x = 2$

B. $x < 2$

C. $x \leq 2$

D. $x > 2$

E. $x \geq 2$

F. $-2 < x < 2$

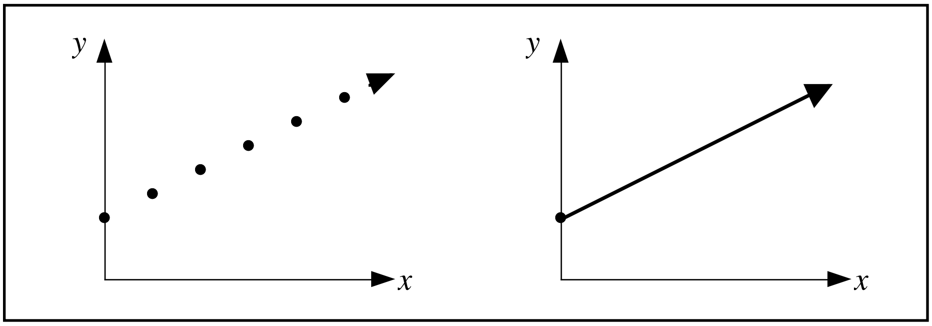
G. $-2 \leq x \leq 2$

H. $2 < x < -2$

I. $x \in \Re$

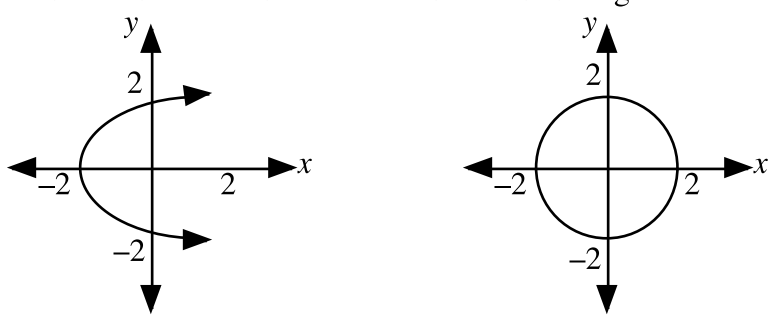
Multiple Choice

7. The graphs of two relations are shown. Which of the following statements is true?



A. The domains are the same but the ranges are different.
B. The ranges are the same but the domains are different.
C. The domains are the same and the ranges are the same.
D. The domains are different and the ranges are different.

8. The graphs of two relations are shown. Which of the following statements is true?



A. The range of each relation is $-2 \le y \le 2$.
B. The range of each relation is $y \in \Re$.
C. The domain of each relation is $-2 \le x \le 2$.
D. None of the above.

Numerical Response

9. The relation between the distance travelled, d km, and the cost, C dollars, of renting a truck is given by the formula $C = 60 + 0.27d$. The domain of the relation can be expressed in the form $d \ge x$ and the range can be expressed in the form $C \ge y$. Write the value of y in the first two boxes and the value of x in the last two boxes.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $D = \{2, 0, 4, -1, -3\}$ $R = \{3, 2, 8, 1\}$
b) $D = \{-3, 0, 5, -8\}$ $R = \{3, -5, -2, 1\}$
c) $D = \{0, 2, 4, 6\}$ $R = \{3, 4, 5\}$
d) $D = \{2, 0, -3\}$ $R = \{3, 4, 5, 6\}$

e) $D = \{1, -1, 3, 7\}$ $R = \{5\}$
f) $D = \{2, 3, 5, 7\}$ $R = \{0, 1, 8, 9\}$
g) $D = \{2, 4, 6, 8\}$ $R = \{1, 3, 5\}$
h) $D = \{3, 5, 2, 4\}$ $R = \{0, 1, 6\}$

2. **a)** $D = \{-1, 0, 3, 4\}$ $R = \{5, 0, 2, 1\}$
b) $D = \{x \in \Re\}$ $R = \{y \in \Re\}$
c) $D = \{x \geq -8, x \in \Re\}$ $R = \{y \geq 4, y \in \Re\}$
d) $D = \{-5 \leq x \leq 5, x \in \Re\}$ $R = \{-4 \leq y \leq 4, y \in \Re\}$

e) $D = \{a \geq -10, a \in \Re\}$ $R = \{b \in \Re\}$
f) $D = \{x \in \Re\}$ $R = \{y \leq 4, y \in \Re\}$
g) $D = \{x \in \Re\}$ $R = \{8\}$
h) $D = \{d < 5, d \in \Re\}$ $R = \{t \leq 4, t \in \Re\}$

3. **a)** $D = \{-6 \leq x \leq 4, x \in \Re\}$ $R = \{7 \leq y \leq 17, y \in \Re\}$
b) $D = \{-23 \leq x \leq 17, x \in \Re\}$ $R = \{-25 \leq y \leq 15, y \in \Re\}$

c) $D = \{-8 \leq x \leq 7, x \in \Re\}$ $R = \{-2 \leq y \leq 10, y \in \Re\}$
d) $D = \{-65 \leq x \leq -35, x \in \Re\}$ $R = \{-75 \leq y \leq -25, y \in \Re\}$

4. **a)** $D = \{x \in \Re\}$ $R = \{y > 0, y \in \Re\}$, y–ints 0 and 400

b) $D = \{t \geq 0, t \in \Re\}$ different from a) because time is never a negative value.
$R = \{a \geq 400, a \in \Re\}$different from a) because the amount of money can never be less than $400.

5. **a)** h–int = 0, d–int = 0 and 200. On the tee the ball is on the ground. It returns to ground level 200 m from the tee.
b) max height = 25 m. The maximum height is the upper limit of the range.
c) $D = \{0 \leq d \leq 200, d \in \Re\}$ $R = \{0 \leq h \leq 25, h \in \Re\}$
d) 55 m from the tee when the ball is rising and 145m from the tee when the ball is descending.
e) 24 m
f) Starting from a height of 0 m at the tee, the golf ball increases in height to a maximum height of 25m, 100 m from the tee. Then the golf ball starts decreasing in height until it hits the ground 200 m from the tee.

6. **i)** A **ii)** C **iii)** G **iv)** F **v)** D **vi)** F **vii)** I

7. D **8.** D **9.**

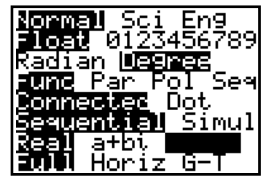
6	0	0	0

Relations and Functions Lesson #5:
Relations and the Graphing Calculator

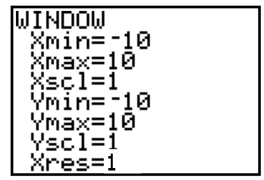
- Although features of different types of graphing calculators may be similar to each other, the instructions in this unit refer to a TI-83 Plus graphing calculator.
- This lesson takes more than one class to complete.

Preparing the Calculator to Graph

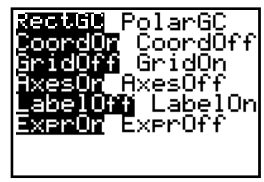
Confirm that the calculator is in "Function" mode as shown.
Press `MODE` and use the cursor keys to navigate to the proper settings and then press `ENTER`.



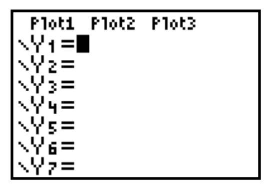
Set the display window of the calculator as shown.
Press `WINDOW` and use the cursor keys to navigate to the proper settings and then press `ENTER` , or press `ZOOM` `6`.



Set the format of the calculator as shown.
Press `2nd` `ZOOM` and use the cursor keys to navigate to the proper settings and then press `ENTER`.



Press `Y=` to confirm that the "Y= editor" window is as shown.
If Plot1, Plot2, or Plot3, are highlighted in black, then you must turn them off by scrolling to Plot1, Plot2, or Plot3, and pressing `ENTER`.



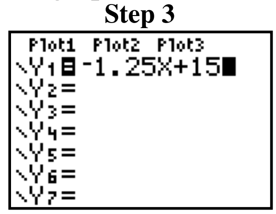
The relation with equation $y = -1.25x + 15$ is used throughout most of this lesson to illustrate some features of the graphing calculator.

Entering and Graphing a Relation using a Graphing Calculator

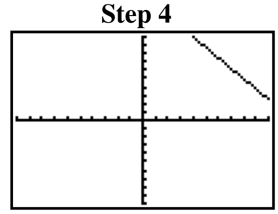
The following procedure may be used to display the graph of $y = -1.25x + 15$.

1. Access the "Y= editor" by pressing the `Y=` key.
2. Enter the equation in Y_1. (Write the equation in terms of the dependent variable if necessary).
3. Press the `GRAPH` key to display the graph.

Step 3



Step 4



Notice that the window we have chosen in this example does not allow us to see much of the graph. We need to alter the window setting to see the main features of the graph.

Using the Window and Zoom Features

Sometimes the standard display setting shown below does not allow us to see the behaviour of specific portions of a graph. For instance, if we wanted to see where the graph of $y = -1.25x + 15$ crosses the x and y-axis we need to adjust the window settings of the calculator.

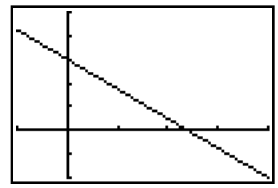
WINDOW	
Xmin=-10	**Minimum value** seen on the screen for x
Xmax=10	**Maximum value** seen on the screen for x
Xscl=1	**Scale for x-axis** - The increments of the tic marks for x
Ymin=-10	**Minimum value** seen on the screen for y
Ymax=10	**Maximum value** seen on the screen for y
Yscl=1	**Scale for y-axis** - The increments of the tic marks for y
Xres=1	**Resolution:** 1 is best screen resolution. Leave at 1

The following procedure may be used to adjust the window settings.

1. Press the WINDOW key.

2. Use the cursor to enter in appropriate values to adjust the Cartesian plane and press the GRAPH key.

```
WINDOW
 Xmin=-5
 Xmax=20
 Xscl=5
 Ymin=-10
 Ymax=25
 Yscl=5
 Xres=1
```



- An alternative method to "fit" the graph of an equation on the display screen is to use the Zoom Out feature of a graphing calculator. Although it may not give the exact window settings we would like, it can give a good first approximation.

  - Press the ZOOM 3 (or scroll down to Zoom Out and press GRAPH). Keep doing this until you see the graph.

  - Press WINDOW to see the new settings. Adjust the window.

- Another method to "fit" the graph of an equation on the display screen is to use the ZoomFit feature of a graphing calculator. Although it may not give the exact window settings we would like, it can also give a good first approximation.
- Press ZOOM 0 (or scroll down to ZoomFit and press ENTER).
- The standard display settings (as shown in the window at the top of this page) of a graphing calculator can be quickly entered by pressing ZOOM 6 (or scroll down to ZStandard and press ENTER).

- When displaying a graph on the calculator, it is generally a good idea to have the x- and y- intercepts of the graph visible.

- The following notation is used to write a graphing calculator window.

  $x:[x_{\min}, x_{\max}, x_{\text{scl}}]$ $\qquad$ $y:[y_{\min}, y_{\max}, y_{\text{scl}}]$

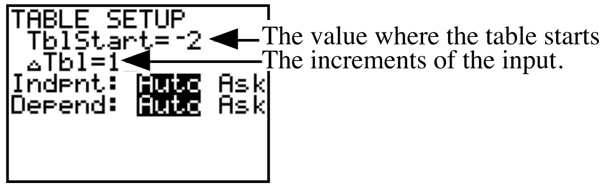
  For example, the window settings used above in step 1 can be written:

  $x:[-5, 20, 5]$ $\qquad$ $y:[-10, 25, 5]$

Displaying the Table of Values for an Equation

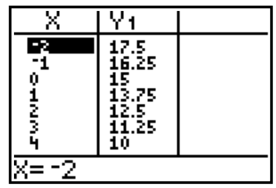
The table of values feature may be used to display the ordered pairs of a relation defined by an equation Before using this feature, we will need to set up the table using the table setup feature.

<u>Accessing the Table Set Up Feature</u>



- Press [2nd] [WINDOW] keys to access TBLSET. The illustration describes the features.
- This will lead us to a table of values starting with $x = -2$ and increasing in units of 1.

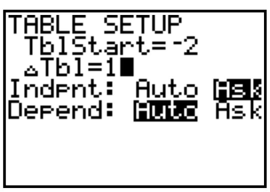
<u>Accessing the Table of Values</u>

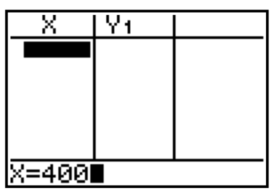


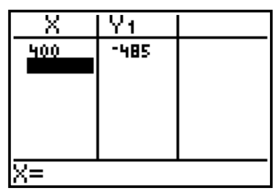
- Press [2nd] [GRAPH] keys to access the table feature of the calculator.
  We can use the cursor keys to scroll up and down the table to determine the value of y for any given value of x.

If a value of x which would require a lot of cursoring is used, for example $x = 400$, a quicker method of calculating the value of y is to use the following procedure.

- Access TBLSET feature and change Indpnt to Ask,
- Return to the table of values, type in 400 ,and press [ENTER].

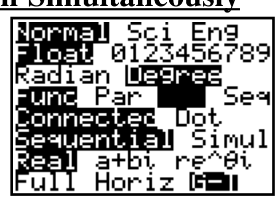


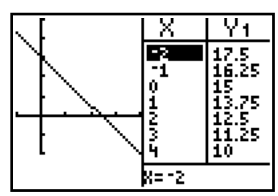




<u>Accessing the Table of Values and the Graph Simultaneously</u>

Return to Table Set Up as on the top of this page.





1. Press the [MODE] key, scroll down and across to "G-T", and then press [ENTER] and [GRAPH].

2. Switching between the graph and table of values can be done by pressing the [GRAPH] key and the [2nd] [GRAPH] keys.

3. Using the [TRACE] key in the G-T mode allows us to scroll along the graph and up and down the table of values at the same time (use the left and right arrow keys).

Complete Assignment Question #1

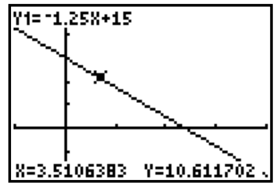
Finding the y- coordinate of a Point on the Graph of a Relation

To determine the output for a given input of a relation, the trace feature of a graphing calculator may be used.

Using the Trace Feature to Move Along the Graph

Return to Full screen.

The trace feature may be used to find the coordinates of points on the graph. The following procedures may be used:

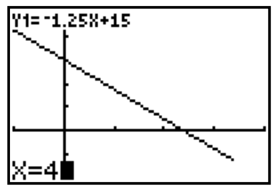


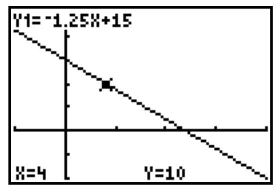
1. Set the window to x:[–5, 20, 5] y:[–10, 25, 5]

2. Press the TRACE key and use the scroll keys to move along the graph.

Using the Trace Feature to Determine the Value of *y* for a Given Value of *x*

- Press the TRACE key, enter a value for the input, and press ENTER. For example, press TRACE 4 ENTER will calculate the value y when $x = 4$.
  The answer `y = 10` will be displayed on the bottom right.





Finding the x- coordinate of a Point on the Graph of a Relation

To determine the input for a given output of a relation, the intersect feature of a graphing calculator may be used.

Using the Intersect Feature

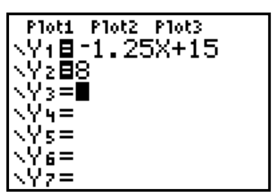
- The intersect feature can be used to determine the intersection point of the graphs of two relations. An application of this is to find the y-coordinate of a point on a graph given the x-coordinate.

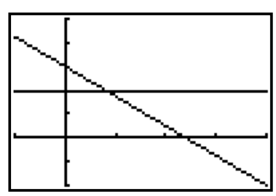
- As a working example we will find the x-coordinate of $y = -1.25x + 15$ when $y = 8$.

1. Enter the equation into Y_1.

2. Enter the given value of y into Y_2,

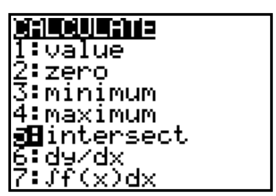
press GRAPH, and adjust the window if necessary.





3. Access the CALC menu by entering 2nd then TRACE.

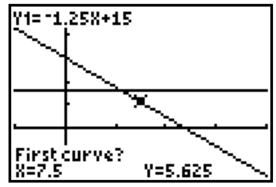
4. Select the "intersect" feature.



5. On the bottom left hand side of the screen the calculator will ask for the `First curve?`.

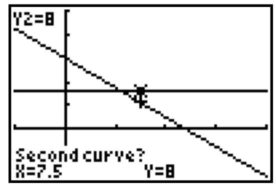
Scroll close to the intersection point.and press $\boxed{\text{ENTER}}$.

The values of x and y will vary depending on how close we scroll to the intersection point.



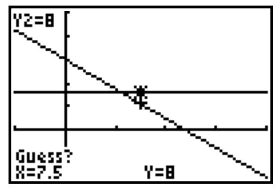
6. On the bottom left hand side of the screen the calculator will ask for a `Second curve?`.

Scroll close to the intersection point.and press $\boxed{\text{ENTER}}$.



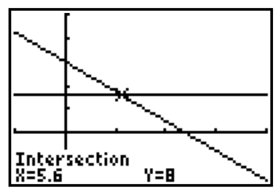
7. On the bottom left hand side of the screen the calculator will ask to `Guess?`.

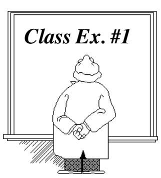
Press $\boxed{\text{ENTER}}$.



8. The x and y values of the intersection point will be displayed. In this example the intersection point is (5.6, 8).

The answer to the question is $x = 5.6$

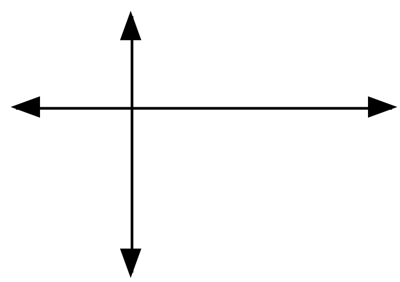




Consider the equation $y = 2x - 25$.

a) Use a graphing calculator to sketch the graph on the grid provided. Write a suitable window.

x:[, ,] y:[, ,]



b) Use a graphing calculator to determine the value. of y when $x = 4.5$.

c) Algebraically verify the answer in b).

d) Use a graphing calculator to determine the value of x when $y = -6.4$.

e) Algebraically verify the answer in d).

Complete Assignment Questions #2 - #5

Using a Graphing Calculator to Determine x-intercepts

Finding the *x*-intercept using the Intersect Feature

The intersect feature may be used to find the *x*-intercept by using the equation $y = 0$ for Y_2, and determining the intersection point of the graph and *x*-axis.

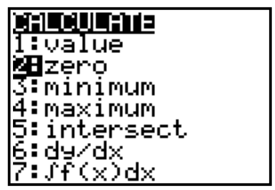
Finding the *x*-intercept using the Zero Feature

The *x*-intercept(s) on the graph of a relation can be determined using the **zero** feature on the calculator.

The following procedure may be used to determine the *x*-intercept on the graph .

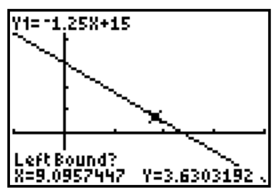
1. Enter the equation in Y_1 ($y = -1.25x + 15$, x:[–5, 20, 5], y:[–10, 25, 5]).

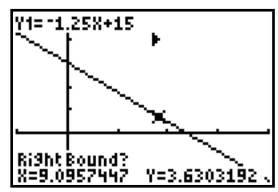
2. Press `2nd` `TRACE` keys to access the `CALCULATE` (`CALC`) menu.



3. Choose `2:zero` and press `ENTER`.

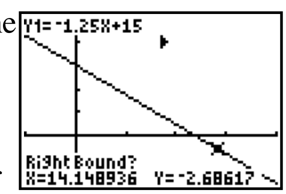
4. On the bottom left hand side of the screen the calculator will ask for a `LeftBound?`. The calculator is asking us to put the cursor on the **left side of the *x*-intercept**.

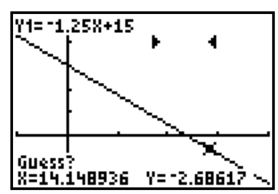




Place the flashing cursor on the <u>left</u> side of the *x*-intercept using the left and right cursor keys. Press `ENTER`.
Note the arrow to the <u>left</u> and above the graph on the screen.

5. On the bottom left hand side of the screen the calculator will ask for a `RightBound?`. The calculator is asking us to put the cursor on the **right side of the *x*-intercept**.



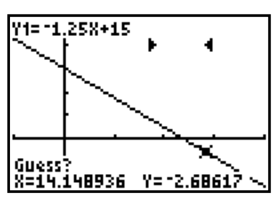


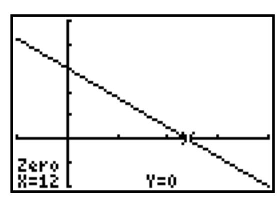
Place the flashing cursor on the <u>right</u> side of the *x*-intercept using the left and right cursor keys. Press `ENTER`.
Note the second arrow to the <u>right</u> and above the graph on the screen.

6. On the bottom left hand side of the screen the calculator will ask to `Guess?`.

Press `ENTER`.

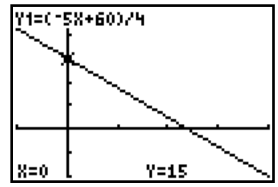
The *x* value will be the *x*-intercept.





Using a Graphing Calculator to Determine y-intercepts

Finding the *y*-intercept using the Trace Feature



1. Enter the equation in Y_1 and adjust the window to appropriate settings to display the *x*- and *y*-intercepts.

2. Press the TRACE key, 0 for the input, and then press ENTER for the output.

Finding the *y*-intercept using the Value Feature

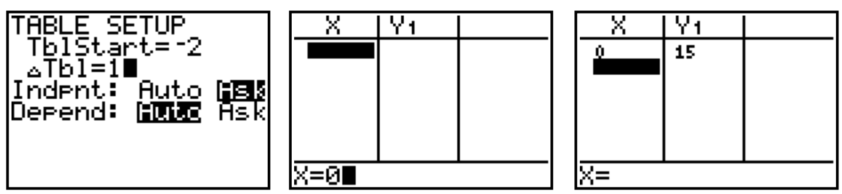
1. Enter the equation in Y_1 and adjust the window to appropriate settings to display the *x*- and *y*-intercepts.

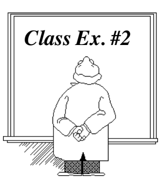
2. Press 2nd TRACE keys to access the CALCULATE (CALC) menu.

3. Choose 1:value and press ENTER 0 ENTER.

Finding the *y*-intercept using the Table Feature

Use procedure described earlier to find *y* when *x* = 0.





In each case use the features of a graphing calculator to calculate the *x*- and *y*-intercepts of the graphs of the relation.

a) $y = \dfrac{5x + 15}{3}$

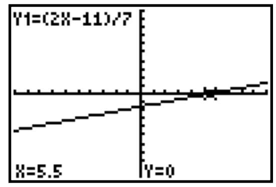
b) $y = -x^2 + x + 6$

Converting a Decimal to a Fraction from a Graph Display

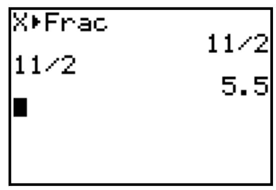
If the x- or y-intercept is not an integer, the graphing display will give the answer in decimal form. The following procedure may be used to write the decimal as an improper fraction (if applicable).

To demonstrate the procedure we will determine the x- and y-intercepts of the graph of the equation $y = \dfrac{(2x-11)}{7}$.

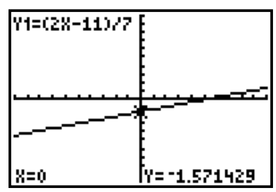
<u>Converting the *x*-value to a Fraction</u>



1. Enter the equation in Y_1, press `GRAPH`, and adjust the window to the appropriate settings.
2. Find the x-intercept using the zero feature.
3. Press `2nd` `QUIT` to exit the graph display.
4. Press `X,T,θ,n` `MATH` `ENTER` `ENTER`.
   The decimal value may be verified by dividing the fraction.

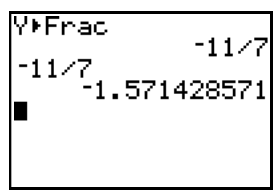


<u>Converting the *y*-value to a Fraction</u>



1. Enter the equation in Y_1, press `GRAPH`, and adjust the window to the appropriate settings.
2. Find the y-intercept using the trace or value feature.
3. Press `2nd` `QUIT` to exit the graph display.
4. Press `ALPHA` `1` to access the letter "y" and then press `MATH` `ENTER` `ENTER`.

   The decimal value may be verified by dividing the fraction.

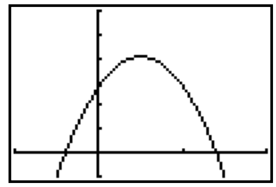


Complete Assignment Questions #6 - #7

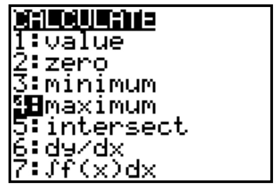
Extension:Using a Graphing Calculator to Determine a Maximum Value

- The graphing calculator can be used to calculate the coordinates of the maximum or minimum points on a graph.
- The equation $y = -x^2 + 5x + 14$ will be used to illustrate the maximum features.

1. Enter the equation into $\boxed{Y_1}$, adjust the window to x: [–5, 10, 5] y: [–5, 30, 5] and press $\boxed{\text{GRAPH}}$.

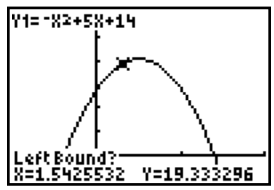


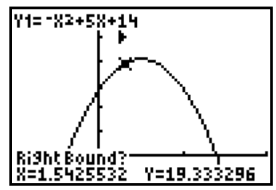
2. Access the `CALC` menu by entering $\boxed{\text{2nd}}$ then $\boxed{\text{TRACE}}$.



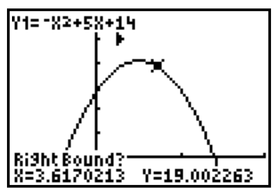
3. Select "`maximum`".

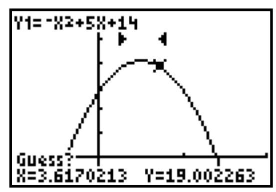
4. On the bottom left hand side of the screen the calculator will ask us for a `LeftBound?`. Select a value by scrolling to the left side of the maximum point and press $\boxed{\text{ENTER}}$.



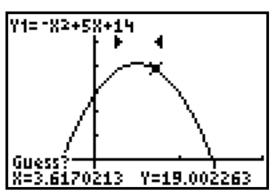


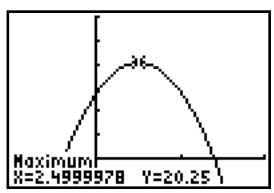
5. On the bottom left hand side of the screen the calculator will ask us for a `RightBound?`. Select a value by scrolling to the right side of the maximum point and press $\boxed{\text{ENTER}}$.

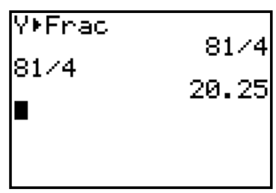




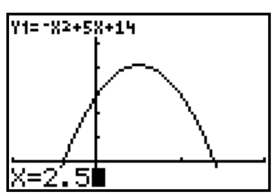
6. On the bottom left hand side of the screen the calculator will ask to `Guess?`. Press $\boxed{\text{ENTER}}$.

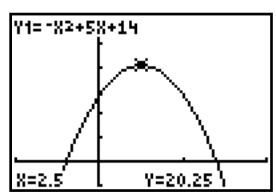






7. Due to the limitations of the calculator, the x value on the display may vary slightly from calculator to calculator. The answer obtained may depend on the window setting used and the bounds chosen. All the answers however should round to the exact answer, in this case 2.5. Test the point using the trace feature and entering $x = 2.5$.





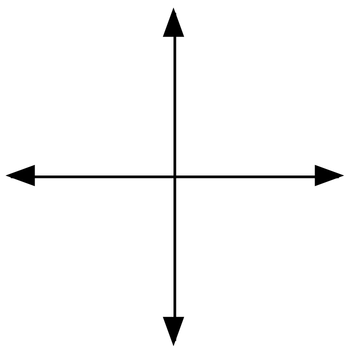
In this example the graph of $y = -x^2 + 5x + 14$ has a **maximum value** of 20.25 or $\frac{81}{4}$, and the **maximum point** has coordinates (2.5, 20.25).

Complete Assignment Questions #8 - #9

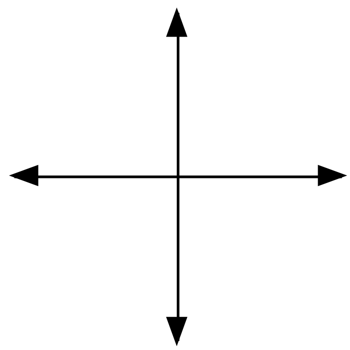
Assignment

1. For each relation represented by an equation:
- Use a graphing calculator to graph the relation.
- Adjust the window to an appropriate setting so that the x- and y- intercepts (where applicable) are visible.
- Sketch the graph on the grid provided.
- Write an appropriate window setting.
- Reset the window to `ZStandard` before beginning a new graph.

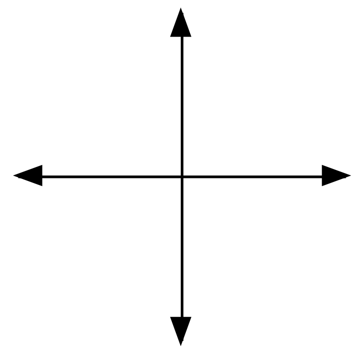
a) $y = 0.5x - 8$



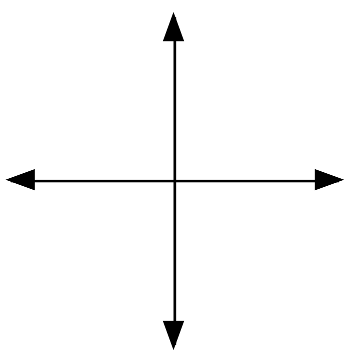
b) $y = 2x - 15$



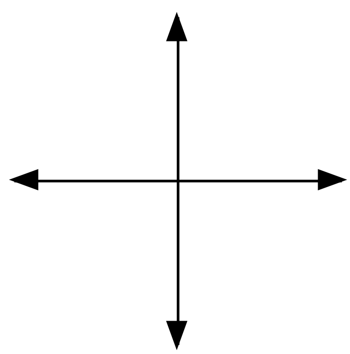
c) $y = -x + 25$



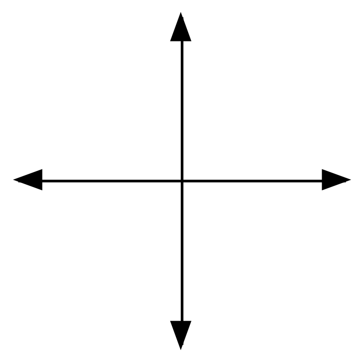
d) $y = -3x - 40$



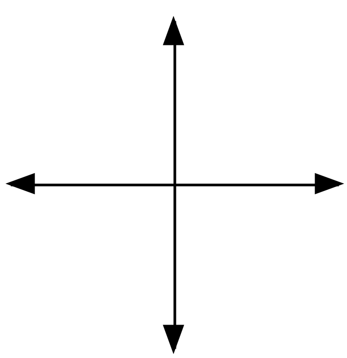
e) $y = -2x + 128$



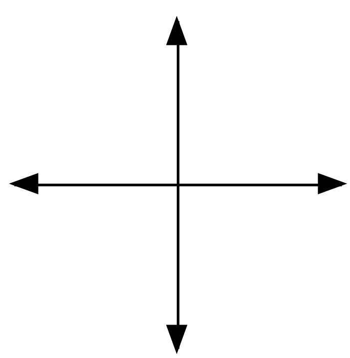
f) $y = x^2 - 15$



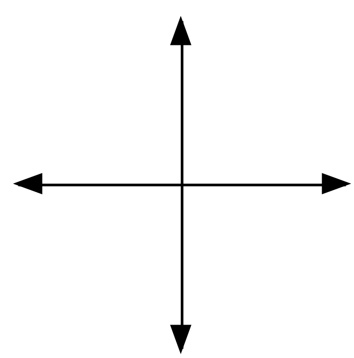
g) $y = x^2 + 20$



h) $y = 30$

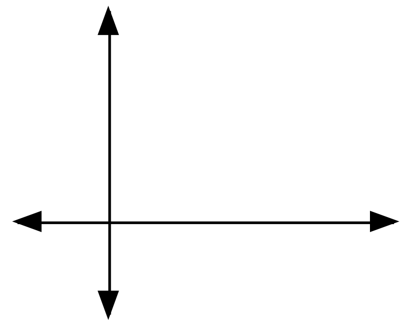


i) $y = 2^x$



2. Consider the equation $y = -3x + 36$.

a) Use a graphing calculator to sketch the graph on the grid provided. Make sure the x- and y- intercepts are visible.



b) Use a graphing calculator to determine the value of y when $x = 5$.

c) Algebraically verify the answer in b).

d) Use a graphing calculator to determine the value of x when $y = 12$.

e) Algebraically verify the answer in d).

3. Given the x-coordinate of each relation, determine the corresponding y-coordinate using the graphing features of a calculator.

a) $y = 8x - 5,\ x = 7,\ y =$

b) $y = 0.75x + 8,\quad x = 15,\quad y =$

c) $y = -2x + 12,\ x = -17,\quad y =$

d) $y = \frac{5}{3}x - 1,\ x = 0,\ y =$

e) $y = x^2 + 3x - 12,\ x = -5,\ y =$

f) $y = 2x^2 - 5x - 2,\ x = 3.5,\quad y =$

g) $y = 2x^3 + 15,\ x = 2,\ y =$

h) $y = |x| - 7,\ x = -15,\quad y =$

4. Given the y-coordinate of each relation, determine the corresponding x-coordinate using the graphing features of a calculator.
Write your answer to the nearest hundredth where necessary.

a) $y = 7x - 3,\ x = \qquad,\ y = 8$

b) $y = 5x + 8,\quad x = \quad,\ y = -3$

c) $y = -2x + 12,\ x = \qquad,\quad y = 15$

d) $y = \frac{5}{3}x - 1,\ x = \quad,\ y = -12$

e) $y = x^2 + 2x - 15,\ x = \qquad$ and $\qquad,\ y = 3$

f) $y = 2x^3 + 15,\ x = \quad,\quad y = 10$

g) $y = -x^2 - 10x - 21,\ x = \qquad$ and $\qquad,\ y = -15$

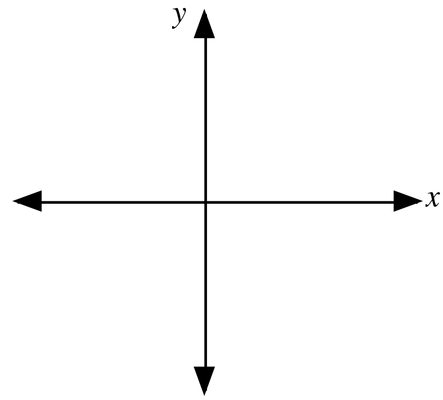
h) $y = |x| - 7,\ x = \quad,\quad y = 20$

5. Consider the relation given by the equation $y = -0.5x^2 + 8$.

a) Use the table set and table features of the calculator to complete the table of values below.

Input (x)	Output (y)
–6	
–4	
–2	
0	
2	
4	

b) Sketch the graph of the relation using an appropriate window.



c) State the window setting x:[, ,] y:[, ,].

d) Use the appropriate features of a calculator to determine:

i) the value of y when $x = 10$

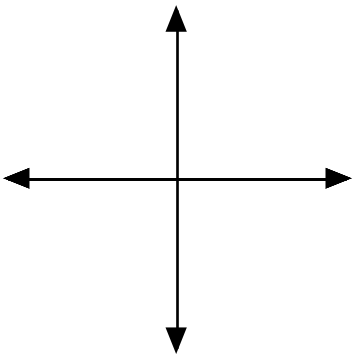
ii) the values of x when $y = 3.5$

e) Compare the answers to a), b), and d ii) with those from Lesson #2, assignment question #6.

6. For each equation use a graphing calculator to:

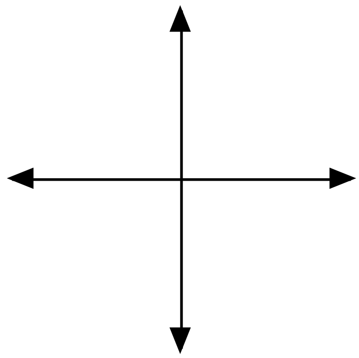
- sketch the graph of the relation on the grid provided,
- **label** any x- and y-intercepts on the graph to the nearest tenth where necessary,
- **list** in the chart provided any x- and y-intercepts as an **exact value**.

a) $y = -3 - x$



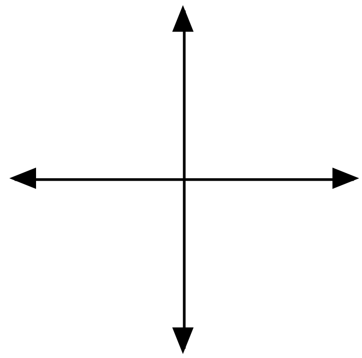
x-intercept	
y-intercept	

b) $y = -3x - 1$



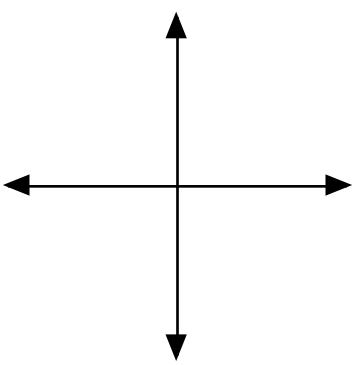
x-intercept	
y-intercept	

c) $y = 2x - 5$



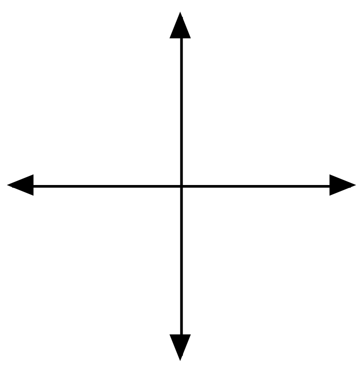
x-intercept	
y-intercept	

d) $y = -6x - 10$



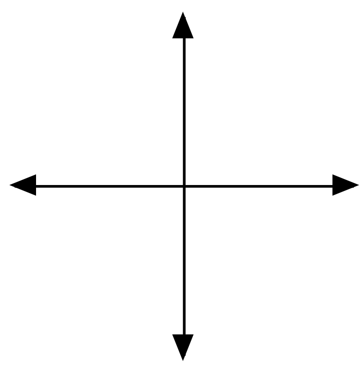
x-intercept	
y-intercept	

e) $y = \dfrac{-4x + 6}{3}$



x-intercept	
y-intercept	

f) $y = 5$

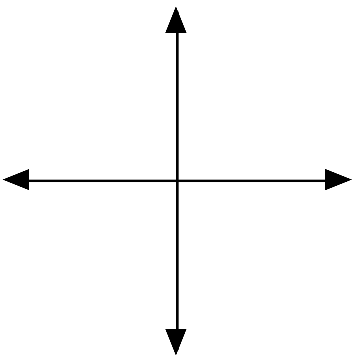


x-intercept	
y-intercept	

7. For each equation use a graphing calculator to:

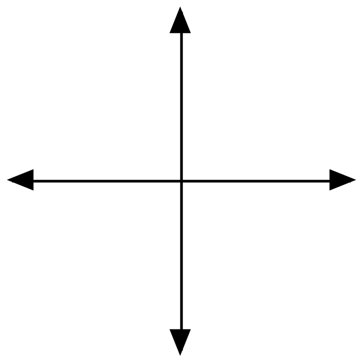
- sketch the graph of the relation on the grid provided,
- **label** any x- and y-intercepts on the graph to the nearest tenth where necessary
- **list** in the chart provided any x- and y-intercepts as an **exact value**. If an exact value cannot be determined, answer to the nearest tenth.

a) $y = x^2 - 5x - 6$



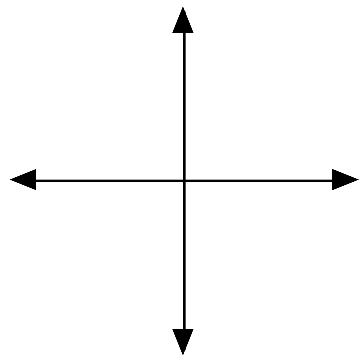
x-intercept(s)	
y-intercept	

b) $y = x^2 - 16$



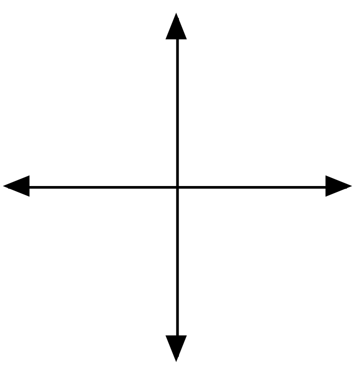
x-intercept(s)	
y-intercept	

c) $y = 2x^2 - 7x + 8$



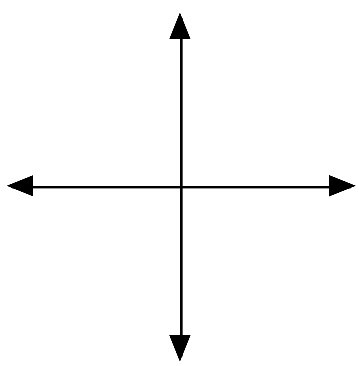
x-intercept(s)	
y-intercept	

d) $y = -x^2 + 3x + 18$



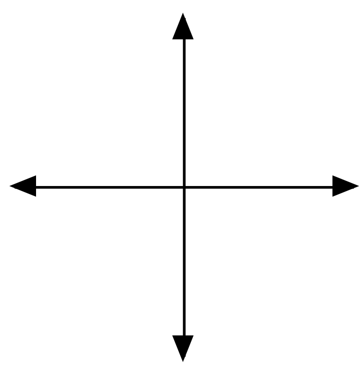
x-intercept(s)	
y-intercept	

e) $y = -2x^2 - x - 12$



x-intercept(s)	
y-intercept	

f) $y = 2x^3 + x^2 - 16x - 15$



x-intercept(s)	
y-intercept	

8. Determine the coordinates of the maximum point of each graph. Answer using exact values.

a) $y = -0.3x^2 + 5x - 10$ **b)** $y = -0.1x^2 + 5x + 7$ **c)** $y = -9x^2 - 60x + 100$

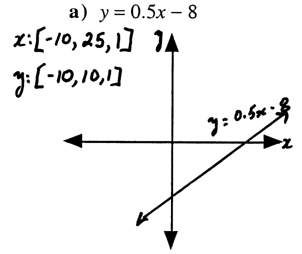
9. Determine the coordinates of the minimum point of each graph. Answer using exact values.

a) $y = x^2 + 2x + 35$ **b)** $y = 0.15x^2 - 2x - 8$ **c)** $y = 0.5x^2 + 2x - 16$

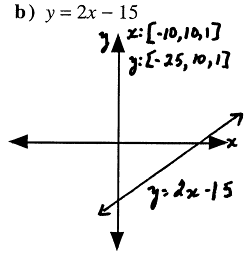
Answer Key

1. see below (windows may vary)

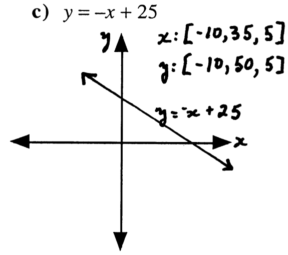
a) $y = 0.5x - 8$



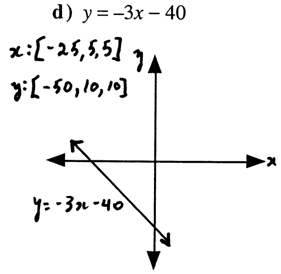
b) $y = 2x - 15$



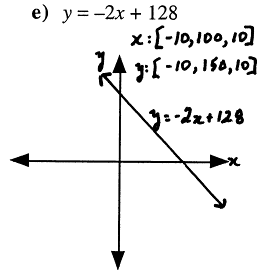
c) $y = -x + 25$



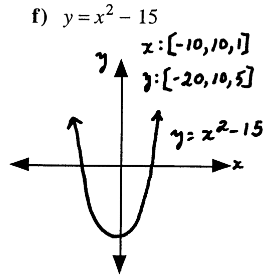
d) $y = -3x - 40$



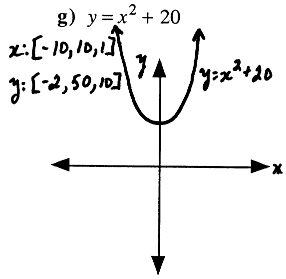
e) $y = -2x + 128$



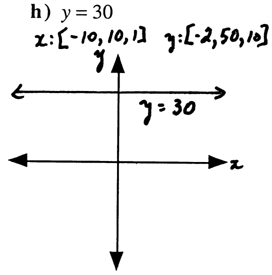
f) $y = x^2 - 15$



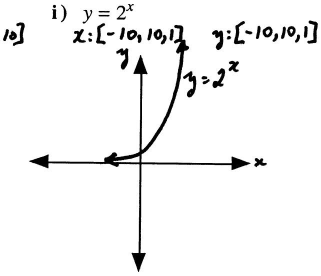
g) $y = x^2 + 20$



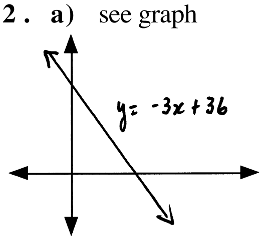
h) $y = 30$



i) $y = 2^x$



2. a) see graph **b)** 21 **c)** 21 **d)** 8 **e)** 8

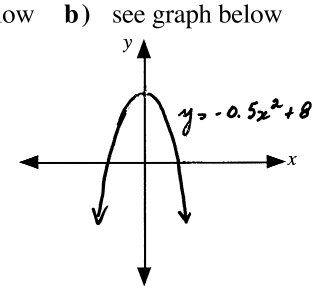


3. a) 51 **b)** 19.25 **c)** 46 **d)** –1
e) –2 **f)** 5 **g)** 31 **h)** 8

4. a) 1.57 **b)** –2.2 **c)** –1.5 **d)** –6.6
e) –5.36 and 3.36 **f)** –1.36 **g)** –9.36 and –0.64 **h)** ±27

5 a) see table below **b)** see graph below **c)** x:[–10, 10, 2] y:[–10, 10, 2]

Input (x)	Output (y)
–6	-10
–4	0
–2	6
0	8
2	6
4	0

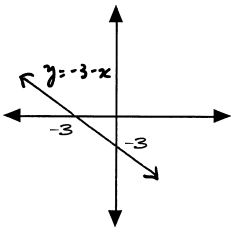


d) i) –42 **ii)** –3 and 3

e) exactly the same

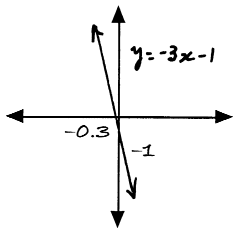
6. See graphs and tables below.

a) $y = -3 - x$



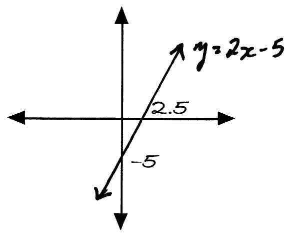
x-intercept	-3
y-intercept	-3

b) $y = -3x - 1$



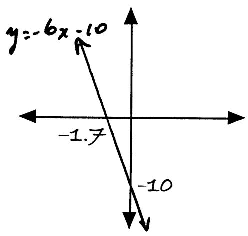
x-intercept	$-\frac{1}{3}$
y-intercept	-1

c) $y = 2x - 5$



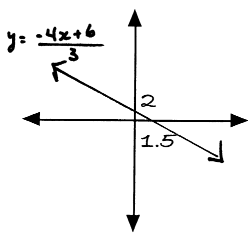
x-intercept	$\frac{5}{2}$
y-intercept	-5

d) $y = -6x - 10$



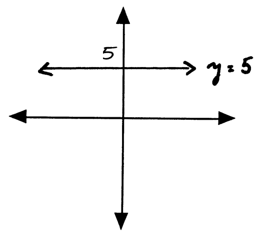
x-intercept	$-\frac{5}{3}$
y-intercept	-10

e) $y = \dfrac{-4x + 6}{3}$



x-intercept	$\frac{3}{2}$
y-intercept	2

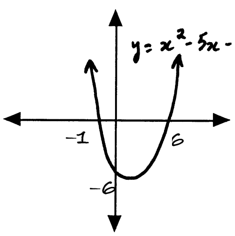
f) $y = 5$



x-intercept	none
y-intercept	5

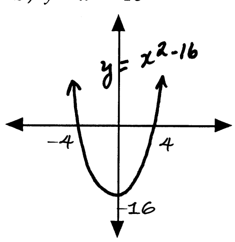
7. See graphs and tables below.

a) $y = x^2 - 5x - 6$



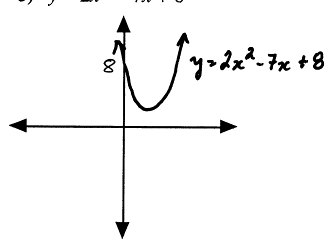
x-intercept(s)	-1 and 6
y-intercept	-6

b) $y = x^2 - 16$



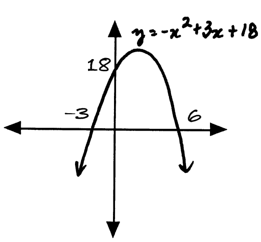
x-intercept(s)	-4 and 4
y-intercept	-16

c) $y = 2x^2 - 7x + 8$



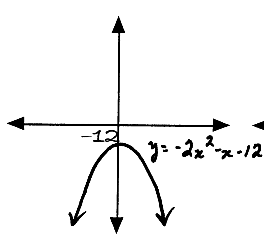
x-intercept(s)	none
y-intercept	8

d) $y = -x^2 + 3x + 18$



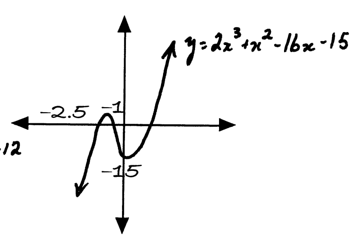
x-intercept(s)	-3 and 6
y-intercept	18

e) $y = -2x^2 - x - 12$



x-intercept(s)	none
y-intercept	-12

f) $y = 2x^3 + x^2 - 16x - 15$



x-intercept(s)	$-\frac{5}{2}, -1, 3$
y-intercept	-15

8. a) $\left(\dfrac{25}{3}, \dfrac{65}{6}\right)$

b) $\left(25, \dfrac{139}{2}\right)$

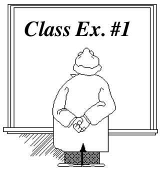
c) $\left(-\dfrac{10}{3}, 200\right)$

9. a) $(-1, 34)$

b) $\left(\dfrac{20}{3}, -\dfrac{44}{3}\right)$

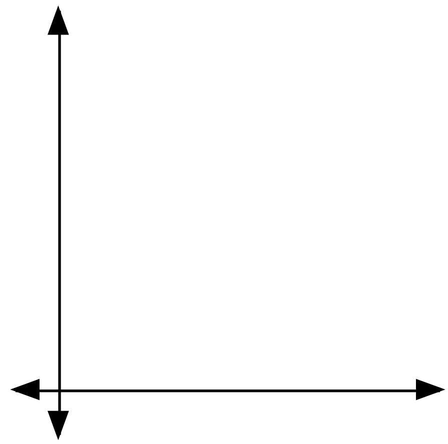
c) $(-2, -18)$

Relations and Functions Lesson #6:
Interpreting Relations Using a Graphing Calculator



Consider the scenario from Class Ex. #3 from Lesson #3 of this unit.

"Lysa purchases a new car for \$20 000.
The value of the car can be represented by the formula $V = 20\,000 - 1250t$, where V is the value of the car in dollars and t is the age of the car in years."



a) • Graph $V = 20\,000 - 1250t$ using a graphing calculator.

• Use the ZoomFit feature as a guide to adjust the graphing window. Then use the **Window** key to align the graph appropriately for this scenario. Write the window setting below.

• Sketch the graph on the grid provided.

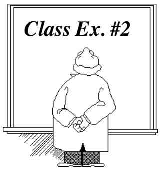
b) Calculate the t-intercept of the graph using a graphing calculator and label it on the sketch. Describe what this value represents in the context of the question.

c) Calculate the V-intercept of the graph using a graphing calculator and label it on the sketch. Describe what this value represents in the context of the question.

d) Use the trace feature of a graphing calculator to determine what the car will be worth in 5 years.

e) Use the intersect feature of a graphing calculator to determine when the car be worth half of the purchase price. Illustrate this on your sketch.

f) Write an appropriate domain and range for the function which describes the value of the car over time.



The height of a human cannon ball, "Cano", can be described by the formula $h = 12 + 6t - t^2$, where h is the height in metres above ground level and t is the time in seconds. Cano is projected out of a cannon from the top of a building and lands on a soft mat. The mat is placed in a hole in the ground so that the top of the mat is level with the ground.

a) Display the graph of $h = 12 + 6t - t^2$ on a graphing calculator.

b) Write down a window setting which would be appropriate for this situation.

c) Sketch $h = 12 + 6t - t^2$ on the grid provided.

d) What is the height of the cannon above the ground?

e) What is the maximum height Cano reaches?

f) How many seconds does it take Cano to reach the highest point on the path he is travelling?

g) To the nearest hundredth of a second, how long does it take Cano to land on the mat?

h) How high is Cano one second after he is launched?

i) When will Cano be at the height in h) again?

j) In words, describe the relation connecting height and time.

k) Write an appropriate domain and range for the relation described in j)

Complete Assignment Questions #1 - #7

Assignment

In this assignment round answers to 2 decimal places unless otherwise stated.

1. Students at a senior high school produce an art literary magazine. The cost for this magazine can be modelled by the formula $C = 2n + 30$, where C is the total cost of the magazine and n is the number of magazines produced.

a) Sketch the graph on the grid provided using a graphing calculator.

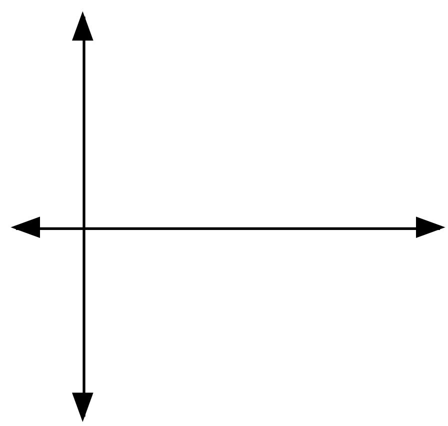
b) Is the n-intercept relevant to the graph of this relation?

c) What would be the cost for 30 magazines?

d) How many magazines are produced if the total cost is $126?

e) Describe the significance of the C-intercept.

2. For a science experiment, a pot of water is heated to a certain temperature and then put in a freezer and allowed to freeze. The rate of freezing can be estimated by the formula $T = -0.4t + 50$, where the temperature, T in degrees Celsius is recorded for each time, t in minutes.



a) Sketch the graph on the grid using a graphing calculator.

b) What was the temperature of the water at 12 minutes?

c) How long did it take for the pot of water to cool down to 40°C?

d) Describe the significance of the t-intercept in this question.

e) Describe the relation in words.

3. During a high school football game, the height of a punt can be modelled by the relation

$$h = -5t^2 + 14t + 1$$

where
t is the number of seconds which have elapsed since the football was punted, and,
h is the number of metres above the ground after t seconds

a) Sketch the graph on the grid.

b) What was the height of the football above the ground as the punter makes contact with the football?

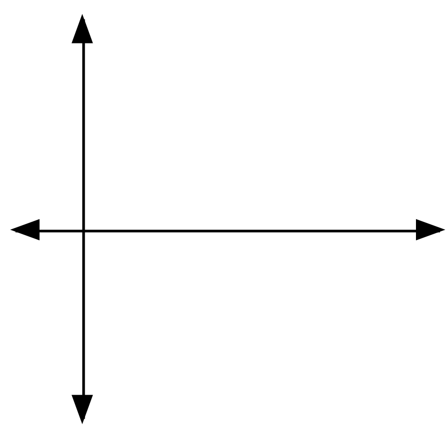
c) What was the height of the football above the ground 1 second after contact?

d) What is the maximum height of the football above the ground?

e) How many seconds had elapsed when the football reached its maximum height?

f) The punt is not caught by the opposing team and the football hits the ground. How many seconds did it take for the football to hit the ground?

4. Student Council is planning a school dance and sells tickets at \$5 per student. The expenses for the dance are \$800. The treasurer of the council generates a formula to determine the potential profit for the dance to be $P = 5t - 800$, where $\$P$ is the profit made on the dance and t is the number of tickets sold.
The maximum number of tickets which can be sold is 450.



a) Sketch the graph on the grid using a graphing calculator.

b) Determine how many tickets need to be sold to break even.

c) How many tickets need to be sold to make a \$1250 profit?

d) How much profit is made if 200 tickets are sold?

e) Write the domain and range for this relation.

5. The height of a soccer ball after a free kick for the Hawks is given by the equation $h = 0.03(-d^2 + 40d)$, where h is the height in metres and d is the horizontal distance in metres the ball travels.

a) Sketch the graph on the grid.

b) What is the height that the soccer ball is kicked from?

c) What is the maximum height the soccer ball reaches?

d) How far down the field is the ball when it reaches its maximum height?

e) Before the ball strikes the ground, a defender heads the ball after it has travelled 38 m. What is the height of the ball above the ground when the defender heads it.

f) How high is the ball when it is 3m down the field?

6. In a soccer game the ball is passed back to the goal keeper and she kicks the ball from gound level up the field. The height, h metres, of the ball above the ground can be modelled by the equation $h = -0.033d^2 + 1.6d - 10$, where d is the distance in metres from the goal line.

a) Sketch the graph on the grid.

b) What was the distance between the goal line and the point where the ball was kicked?

c) What is the maximum height the soccer ball reaches?

d) How far from the goal line is the ball when it reaches its maximum height?

e) What horizontal distance has the ball travelled when it hits the ground for the first time?

f) Write a domain and range for this relation.

g) At what distance from the goal line is the ball 5 m in the air?

7. During an airshow, the path of a stunt dive of a jet can be modelled by the equation $h = t^2 - 9t + 81$, where h is the height in metres after t seconds. The end of the stunt occurs when the plane achieves its starting height.

a) Sketch the graph on the grid.

b) How high is the jet at the start of the dive?

c) How high is the jet above ground level at its lowest point?

d) When does the jet reach its lowest point in the dive?

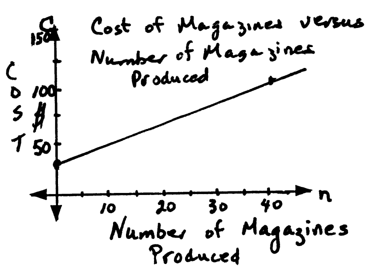
e) How high is the jet at two seconds into its dive?

f) After how many seconds does this height occur again within the jet's stunt dive?

g) Write an appropriate domain and range.

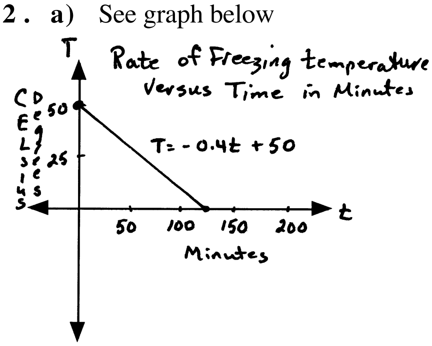
Answer Key

1. a) See graph below



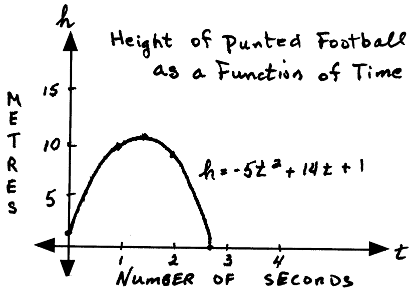
b) No, since the number of magazines cannot be negative.
c) \$90
d) 48
e) The C-intercept of 30 represents a fixed \$30 fee charged by the printing company.

2. a) See graph below



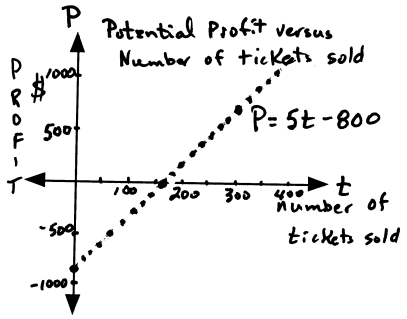
b) 45.2°C
c) 25 min
d) The t- intercept of 125 represents the number of minutes it takes the water to freeze.
e) A pot of water is heated to 50°C and placed in a freezer and allowed to freeze. It cools at a constant rate and after 125 min the water is frozen.

3 . a) See graph below



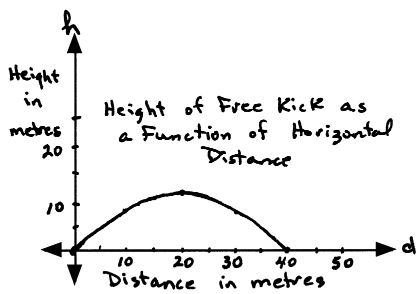
b) 1 m
c) 10 m
d) 10.8 m
e) 1.4 sec
f) 2.87 sec

4 . a) See graph below



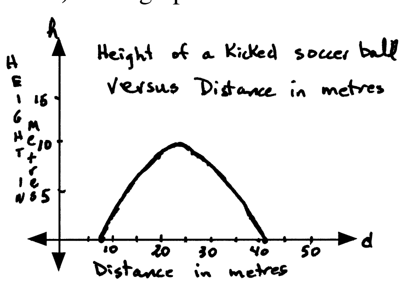
b) 160
c) 410
d) $200
e) domain: 0, 1, 2, 3 450
range: –800, –795, –790, –785 1450
(increasing in steps of 5)

5 . a) See graph below



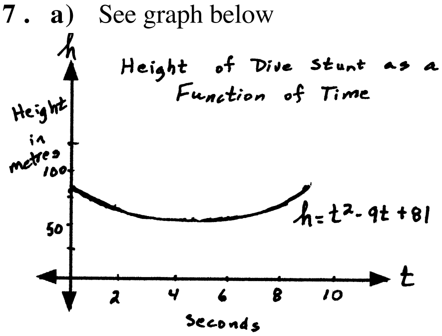
b) 0 m
c) 12 m
d) 20 m
e) 2.28 m
f) 3.33 m

6 . a) See graph below



b) 7.37 m
c) 9.39 m
d) 24.24 m
e) 33.74 m
f) domain: $\{7.37 \le d \le 41.11, d \in R\}$
range: $\{0 \le h \le 9.39, h \in R\}$
g) 12.7 m and 35.78 m

7 . a) See graph below



b) 81 m
c) 60.75 m
d) 4.5 sec
e) 67 m
f) 7 sec
g) domain: $\{0 \le t \le 9, t \in R\}$
range: $\{60.75 \le h \le 81, h \in R\}$

Relations and Functions Lesson #7:
Functions

Review

We have considered six ways in which the relationship between two quantities can be represented.

- in words
- a table of values
- a set of ordered pairs
- a mapping (or arrow) diagram
- an equation
- a graph

In a **relation** each element of the **domain** (the **input**) is related to an element or elements of the **range** (the **output**).

In this lesson we will study a special type of relation called a **function**.

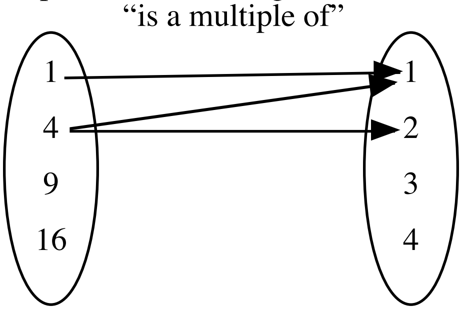
Exploration

To illustrate the concept of function we will look at two relations described in words with domain $D = \{1, 4, 9, 16\}$ and range $R = \{1, 2, 3, 4\}$.

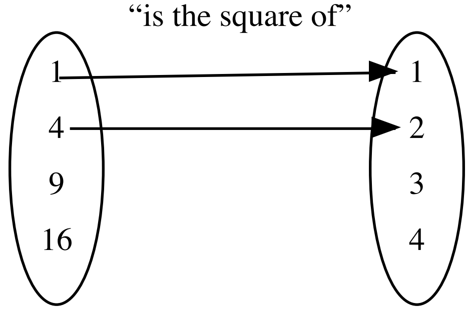
i) "is a multiple of "　　　　**ii)** "is the square of"

a) Complete the arrow diagrams.

i) "is a multiple of"



ii) "is the square of"



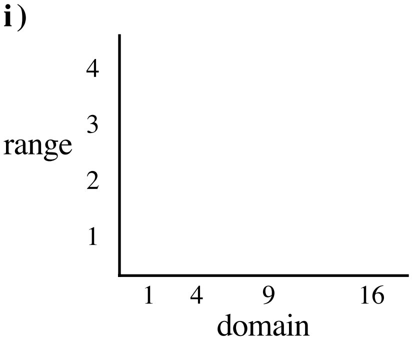
b) Complete the set of ordered pairs.

i) (1, 1), (4, 1), (

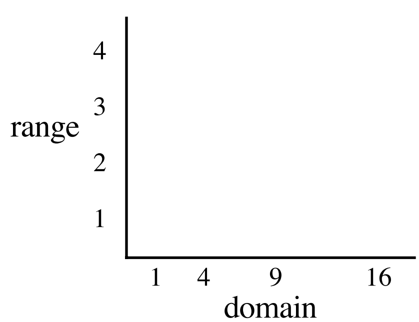
ii)

c) Plot the ordered pairs on the grid.

i)

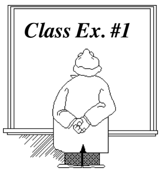


ii)



Function

A functional relation, or **function**, is a special type of relation in which each element of the domain is related to exactly one element of the range. If any element of the domain is related to more than one element of the range then the relation is not a function.



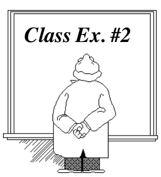
In the exploration on the previous page, one of the relations is a function and the other relation is not a function.

Explain how we can determine which relation is a function by looking at

a) the arrow diagram

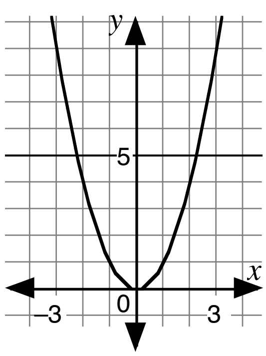
b) the ordered pairs

c) the graphs

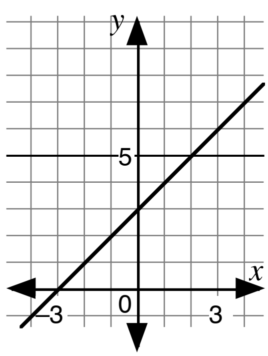


Each of the following is the graph of a relation.

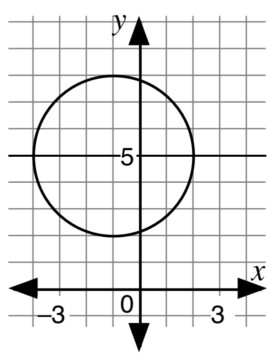
Graph A



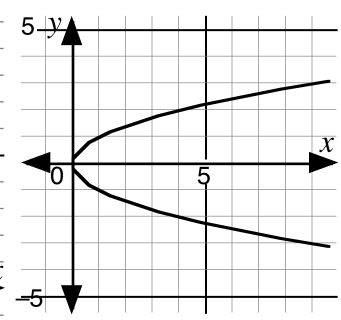
Graph B



Graph C



Graph D



a) Classify the following statements as true (T) or false (F).

- For each input value there is only one output.
- For each output value there is only one input.
- The relation is a function.

A	B	C	D

b) From graph C, write two ordered pairs which show that the relation is not a function. Draw a line joining these points.

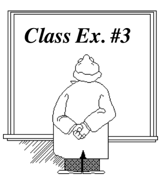
c) From graph D, write two ordered pairs which show that the relation is not a function. Draw a line joining these points.

d) On graphs A and B draw a series of vertical lines. Do any of these lines intersect the graph of the relation in more than one point?

Vertical Line Test

The vertical line test can be used on the graph of a relation to determine whether the relation is a function or not.

- If every vertical line, draw on the domain of the relation, intersects the graph exactly once, then the relation is a function.
- If any vertical line intersects the graph more than once, then it is **not** a function.

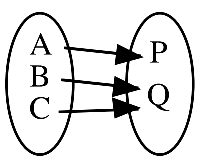


Determine which of the following are functions. Explain your answers.

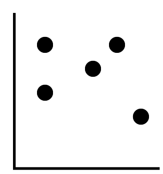
a) (5, 8), (6, 7), (–5, 3), (2, 3), (6, 8)

b) (3, 3), (2, 3), (4, 5), (–3, 2)

c)

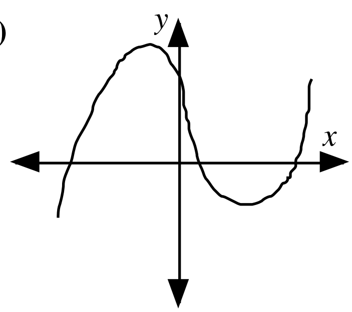


d)

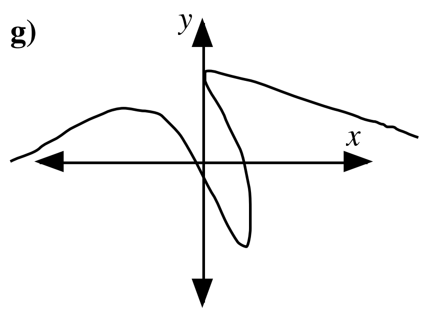


e) the relation connecting the provinces and territories of Canada with their capital cities.

f)



g)



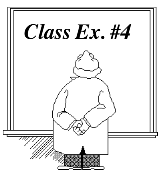
A Function as a Mapping

A function from a set D, the domain, to a set R, the range, is a relation in which each element of D is related to exactly one element of R.

If the function f maps an element x in the domain to an element y in the range we write $f: x \rightarrow y$.

Complete the following for the function "is the square of" on the first page of this lesson.

$1 \rightarrow$ $\qquad$ $4 \rightarrow$ $\qquad$ $9 \rightarrow$ $\qquad$ $\rightarrow$



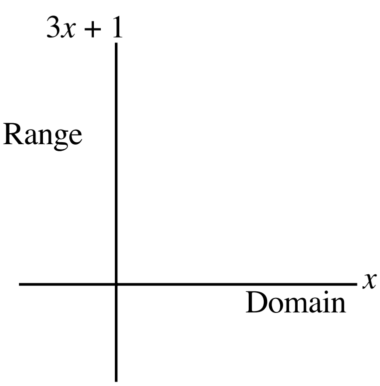
Consider the function $f: x \rightarrow 3x + 1$, for domain $\{-1, 0, 1, 2\}$.

a) Complete $\quad -1 \rightarrow \qquad 0 \rightarrow \qquad \rightarrow \qquad \rightarrow$

b) List the elements of the range of the function.

c) Show the function as

i) an arrow diagram $\qquad$ **ii)** a set of ordered pairs $\qquad$ **iii)** a Cartesian graph.



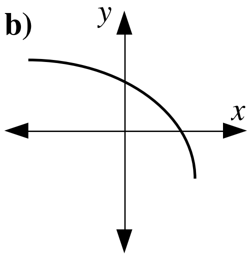
Complete Assignment Questions #1 - #12

Assignment

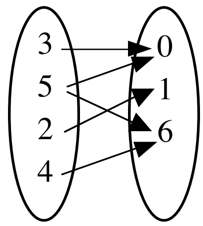
1. Determine which of the following relations are functions. Give reasons for your answers.

a) $(-1, 3)$, $(-2, 1)$, $(5, 2)$, $(7, 3)$

b)



c)



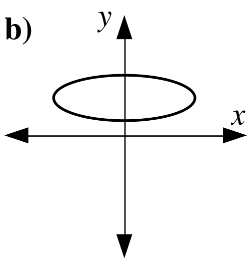
d)

Input (x)	Output (y)
2	3
0	4
–3	5
2	6

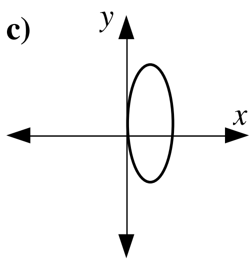
2. State which of the following relations are functions.

a) $(0, 0)$, $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 3)$

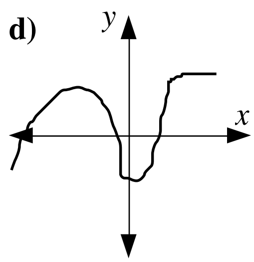
b)



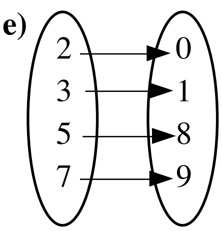
c)



d)



e)

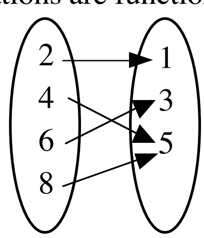


3. State which of the following relations are functions.

a)

Input (x)	Output (y)
0	3
2	4
4	5
6	3

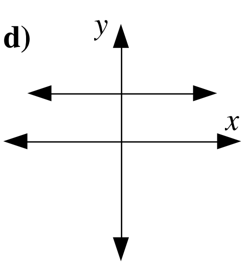
b)



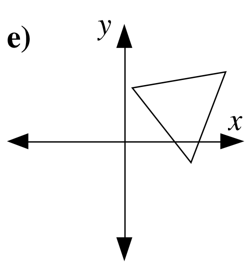
c)

Input (x)	Output (y)
1	5
–1	5
3	5
7	5

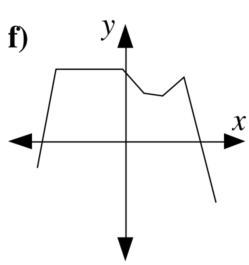
d)



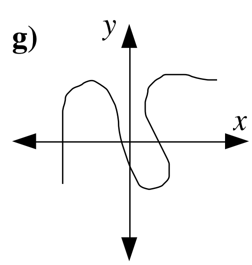
e)



f)



g)



4. Mr. A has a son Jim and a daughter Kristen. Mr. B has three daughters Lauren, Melanie and Noreen.

a) Draw an arrow diagram to illustrate the relation "is the father of" from the set of fathers to the set of children. Is the relation "is the father of" a function?

b) Draw an arrow diagram to illustrate the relation "is the child of" from the set of children to the set of fathers. Is the relation "is the child of" a function?

5. The function $f\colon x \to 2x + 5$, has domain $\{0, 1, 2, 3\}$.

a) List the elements of the range of the function.

b) Show the function f in a Cartesian graph

6. The function $g\colon x \to x^3$, has domain $\{-2, -1, 0, 1, 2\}$.

a) List the elements of the range of the function.

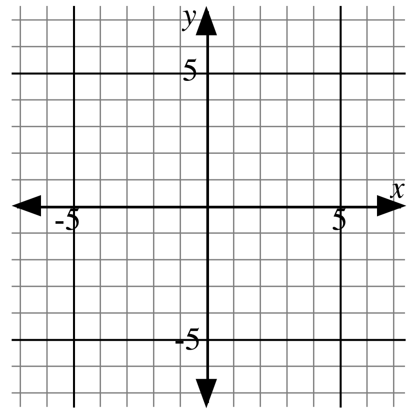
b) Show the function g in a Cartesian graph

7. Consider the function $f: x \rightarrow x^2 - 4$.
a) Complete the following table of values.

Elements of Domain	3	2	1	–1	–2	–3
Elements of Range						

b) Plot the ordered pairs on a Cartesian graph.

c) Draw a smooth curve through the points to illustrate the function $f: x \rightarrow x^2 - 4,\ x \in R$.



8. The domain of the function $h: x \rightarrow 6$, is $\{0, 10, 20\}$.

a) List the ordered pairs of the graph of the function.

b) Show the function h in a Cartesian graph.

Multiple Choice

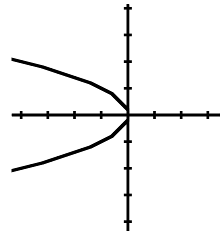
9. The function $f: x \rightarrow 6 - 2x$, has domain $\{0, 2, 4, 6, 8\}$.
Which of the following is **not** an element of the range of the function?

A. –10
B. 2
C. 4
D. –6

10. Which of the following statements is not always true for a function?

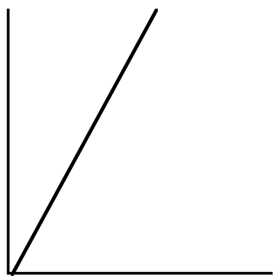
A. A set of ordered pairs (x, y) in which for every x there is only one y.
B. A vertical line must not intersect the graph of a function in more than one point.
C. For every output there is only one input.
D. For every element in the domain there is only one element in the range

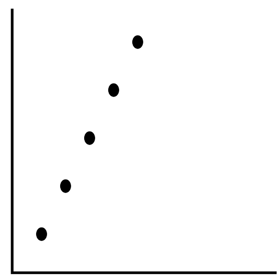
11. Which of the following represents a function?

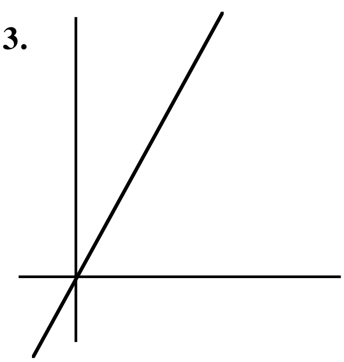
1	"multiply the number by 3 and add 5."	2	$y = -x^2$
3	$(9, 3), (4, 2), (1, 1), (0, 0). (1, -1), (4, -2), (9, -3)$	4	

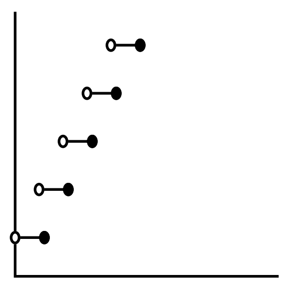
A. 1 only
B. 1 and 2 only
C. 1 and 3 only
D. some other combination of 1 - 4

Numerical Response **12.** Partial graphs of four functions are shown.

1.


2.


3.


4.


The functions are described as follows:

A: Coffee costs $8 per jar. Graph cost as a function of the number of jars purchased.

B: Distance cycled at a constant speed of 8 km/h. Graph distance as a function of time.

C: Parking costs $8 per hour (or part of an hour). Graph cost as a function of time.

D: Set of ordered pairs which satisfy the equation $y = 8x, x \in R$.
Graph y as a function of x.

Place the graph number for function A in the first box.
Place the graph number for function B in the second box.
Place the graph number for function C in the third box.
Place the graph number for function D in the last box.

(Record your answer in the numerical response box from left to right)

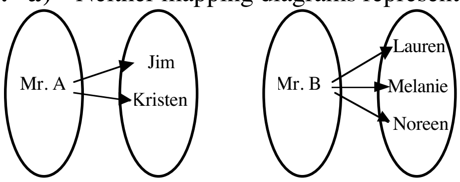
Answer Key

1. a) function: each first coordinate has only one second coordinate
b) function: vertical lines intersects the graph exactly once
c) not a function: the input 5 has two outputs
d) not a function: the input 2 has two outputs

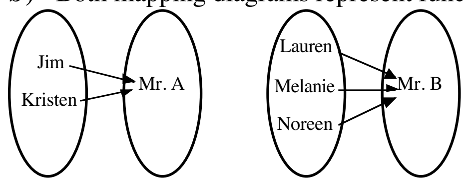
2. a) function **b)** not a function **c)** not a function **d)** not a function **e)** function

3. a) function **b)** function **c)** function **d)** function **e)** not a function
f) function **g)** not a function

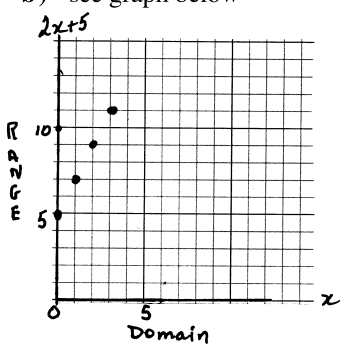
4. a) Neither mapping diagrams represents a function



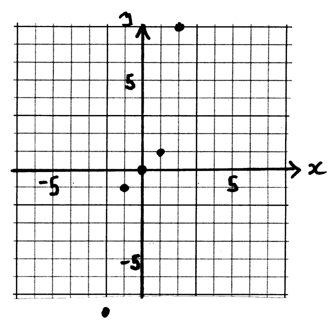
b) Both mapping diagrams represent functions



5. a) {5, 7, 9, 11}
b) see graph below

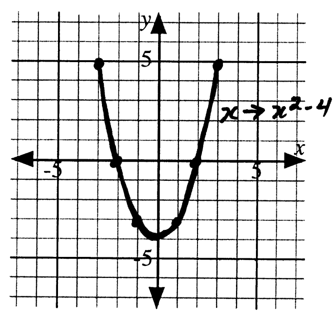


6. a) {–8, –1, 0, 1, 8}
b) see graph below

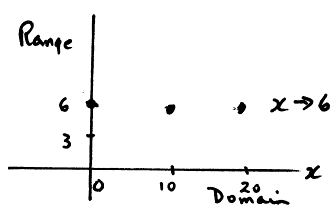


7. a) see table **b)** see grid **c)** see grid

Elements of Domain	3	2	1	–1	–2	–3
Elements of Range	5	0	-3	-3	0	5



8. a) {(0, 6), (10, 6), (20, 6)} **b)** see Cartesian graph →



9. C **10.** C **11.** B **12.**

2	1	4	3

Relations and Functions Lesson #8:
Function Notation

Mapping Notation

In the last lesson we discovered some ways in which functions can be represented:

- Graph
- Table of Values
- Arrow Diagram
- Ordered pairs
- Mapping

In the previous lesson a function was defined in mapping notation as follows.

"A function from a set D, the domain, to a set R, the range, is a relation in which each element of D is related to exactly one element of R.
If the function f maps an element x in the domain to an element y in the range we write $f: x \to y$."

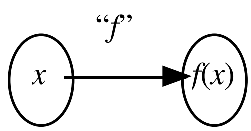
Consider the function $f: x \to 2x+3$ defined on the set of real numbers.

Under this function we know that $5 \to 2(5)+3$ ie $5 \to 13$.
We say that under the function f the **image** of 5 is 13.
We also say that the **value of the function** is 13 when $x = 5$.

Function Notation

In most math courses function notation is used to replace the mapping notation $f: x \to 2x+3$.

Under a function f, the image of an element x in the domain is denoted by $f(x)$, which is read "f of x"



In the example above the function f can be defined by the formula $f(x) = 2x + 3$.

The notation $f(x) = 2x + 3$ is called **function notation**.

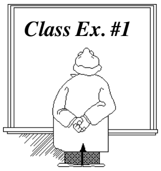
We showed above that, under the function f, the image of 5 is 13. We write $f(5) = 13$.

<u>mapping notation</u>	<u>function notation</u>	<u>equation of graph of function</u>
$f: x \to 2x+3$	$f(x) = 2x + 3$	$y = 2x + 3$
$f: 5 \to 2(5)+3$	$f(5) = 2(5) + 3$	$y = 2(5) + 3$
$f: 5 \to 13$	$f(5) = 13$	$y = 13$

The symbol $f(x)$ is read as "f at x" or "f of x".
$f(x)$ provides a formula for the function f and also represents the value of the function for a given value of x.

In function notation;

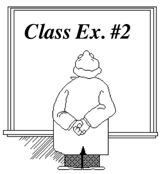
- $f(x)$ does <u>not</u> mean f times x.
- the "name" of the function is f.
- Values of the independent variable are represented by the **input** of the function and are shown on the **horizontal axis**.
- Values of the dependent variable are represented by the **output** of the function and are shown on the **vertical axis**.



Consider the function f defined by $f(x) = 2x^3 + 1, x \in R$. Find

a) the value of f when $x = 2$. **b)** the image of 5 under f **c)** $f(-3)$ **d)** $f(0)$.

e) an expression for $f(a)$. **f)** an expression for $f(5x)$.



Consider the functions $f(x) = x^2 - 9$ and $g(x) = 2x^2 - 5x$ defined on the set of real numbers.

a) Evaluate:

i) $f(1)$ **ii)** $g(-4)$ **iii)** $f\left(3\sqrt{5}\right)$

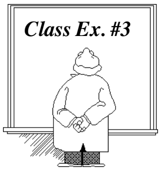
b) Find expressions for

i) $f(x + 3)$ **ii)** $g(8t)$

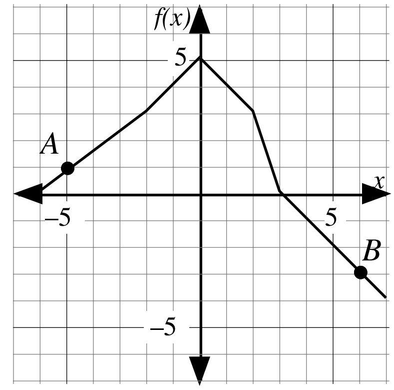
c) Solve the equation

i) $f(x) = 16$ **ii)** $g(a) = g(a + 1)$

Class Ex. #3



The graph of a function is shown.



a) Complete:
i) $f(5) =$ **ii)** $f(-2) =$ **iii)** $f(4) =$

b) Write the ordered pairs associated with i), ii), and iii).
i) **ii)** **iii)**

c) State the value(s) of x if:
i) $f(x) = -1$ **ii)** $f(x) = 3$ **iii)** $f(x) = -4$

d) Use the notation in a) to make a statement about the points A and B on the graph.

e) Write the intercepts of the graph using function notation.

f) Compete the following statements.
- The domain of f is _____ $\le x \le$ _____ , $x \in R$.
- The range of f is _____ $\le f(x) \le$ _____, $f(x) \in R$.

Complete Assignment Questions #1 - #23

Assignment

1. Function g defined by $g(x) = 3 - x^2, x \in R$. Evaluate:
a) $g(0)$ **b)** $g(2)$ **c)** $g(-3)$ **d)** $g\left(\frac{1}{2}\right)$ **e)** $g\left(\sqrt{3}\right)$

2. A function f is defined by the formula $f(x) = x^3, x \in R$. Find:
a) the image of 2 under f **b)** the value of f at -7. **c)** the number a, given that $f(a) = -8$

3. Consider the function f defined by $f(x) = 4x - 12, x \in R$. Find:

a) the value of f when $x = 1$. **b)** the value of f at -7. **c)** the image of 3 under f

d) $f(-2)$ **e)** $f(0)$ **f)** an expression for $f(t)$. **g)** an expression for $f(t + 3)$.

h) the solution to the equation $f(x) = 20$

4. The function $g(x) = 3x^2 - 4$ has a domain $\{-2, -1, 0, 1, 2\}$.

a) Draw an arrow diagram to represent the function. **b)** State the range of g.

c) Solve the equation $g(x) = -1$

5. If $g(x) = -3x^2 + 5$, determine:

a) the value of:

i) $g(6)$ **ii)** $g(-3)$ **iii)** $g\left(\sqrt{5}\right)$

b) an expression for:

i) $g(-z)$ **ii)** $g(x - 2)$ **iii)** $g(2a + 5)$

6. If $C(n) = 2n^2 - 25$, determine:

a) the value of:

i) $C(25)$ **ii)** $C\left(\frac{3}{2}\right)$ **iii)** $C\left(1+\sqrt{3}\right)$

b) an expression for:

i) $C(n - 5)$ **ii)** $C(3n^2)$

7. **a)** If $f(x) = 5x - 6$, determine the value of x if $f(x) = 44$.

b) If $g(t) = 55 - 3t$, determine the value of t if $g(t) = 10$.

c) If $f(n) = 55 - 3n^2$, determine the value of n if $f(n) = -20$.

8. A function g is defined by the formula $g(t) = t + 9$ where $t \in R$.

a) Calculate the value of $g(6) + g(-12)$.

b) If $g(a^2) = 25$, find the possible values of a.

9. A function c is defined by $c(x) = \sqrt{x}$ where $x \geq 0$.

a) Evaluate **i)** $c(9)$ **ii)** $c\left(\frac{1}{9}\right)$

b) If $c(x) = 16$, find x.

c) Without using a calculator, find the exact value of $\dfrac{c(200)}{c(2)}$.

10. Given that $f(x) = 7 + 2x$.

a) Evaluate $f(-5)$

b) Find the value of $f(z) + f(-z)$

c) Calculate the x-intercept on the graph of f.

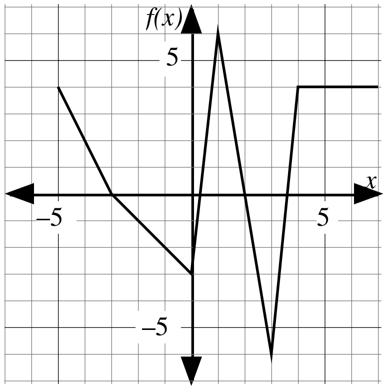
11. Consider the function $f(x) = 1 - x^2$, where x is an integer.

a) Evaluate $f(2) - f(-1)$

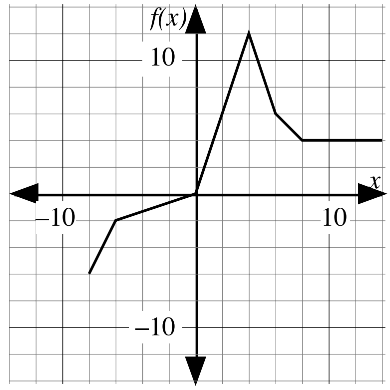
b) Given that $f\left(a^{\frac{1}{4}}\right) = -8$, calculate the value of a.

12. Complete the tables for the following graphs.

a)



b)



$f(x)$	**Ordered Pair**
$f(1) =$	
$f(0) =$	
$f(2) =$	
$f(-1) =$	
$f(-3) =$	
$f(3) =$	

x	$f(x)$	**Ordered Pair**
		(2,)
	0	
–6		
8		
	–6	
10		

c) Explain why, in part b), the solution to the equation $f(x) = 4$ has an infinite number of solutions.

13. Consider the function $f(x) = x^3 - 2x^2 - x - 5$.

a) Find the exact value of $f\left(\sqrt{12}\right)$ in simplest radical form.

b) Find expressions for:

i) $f(a - 2)$ **ii)** $f(4x^2)$

Multiple Choice

14. If $f(x) = 3x - 1$ and $f(t) = 8$, then $t =$

A. $\frac{7}{3}$
B. 3
C. $\frac{11}{3}$
D. 23

15. The graph of the function $f(x) = 4^x, x \in R$, intersects the y-axis at

A. $(0, 0)$
B. $(0, 1)$
C. $(0, 4)$
D. no point

16. Given a function g defined by $g(x) = px + q$ with $g(0) = 2$ and $g(1) = 3$ then

A. $p = 1, q = 0$
B. $p = 3, q = 2$
C. $p = 1, q = 2$
D. $p = 3, q = 0$

17. If $f(x) = 3^x$ and $f(-a) = \frac{1}{81}$, then $a =$

A. -4
B. 4
C. $-\frac{1}{4}$
D. $\frac{1}{4}$

18. Given that $f(x) = x^3 + px^2 + qx - 1$, and $f(1) = 3p$, the ratio $p : q$ equals

A. $1 : 2$
B. $1 : 3$
C. $2 : 1$
D. $3 : 1$

19. Consider the following functions:

1. $p(x) = x^2 - 4x - 2$ **2.** $p(x) = \frac{1}{3}x + 14$ **3.** $p(x) = 3x^2 + x$ **4.** $p(x) = 7 - 5x$

For each function evaluate $p(-3)$ and put the expressions in order from greatest to least. The order is

A. 4312
B. 3412
C. 3124
D. none of the above

Numerical Response

20. A function f is defined by the formula $f(x) = 5x^3, x \in R$. The value of $f(\sqrt{2})$ can be written in the form $k\sqrt{2}$. The value of k is _____ .

(Record your answer in the numerical response box from left to right)

21. If $f(x) = 1 - 2x - 5x^2$ and if $f(x + 2)$ is written in the form $ax^2 + bx + c$, the value of $a - b - c$ is _____ .

(Record your answer in the numerical response box from left to right)

22. $f(a) = \dfrac{a}{a + 4}$. The exact value of $f(5) - f(5^{-1})$ written as a rational number in simplest form is $\dfrac{p}{q}$. The value of p is _____ .

(Record your answer in the numerical response box from left to right)

Extension Question.

23. Consider the function $g(x) = 2x^2 - 5x$.

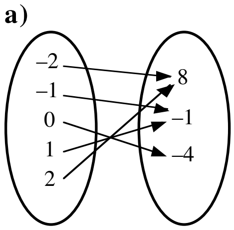
a) Solve the equation $g(a) = 12$. **b)** Solve the equation $g(2y) = 42$

Answer Key

1. a) 3 **b)** –1 **c)** –6 **d)** $\frac{11}{4}$ **e)** 0 **2. a)** 8 **b)** –343 **c)** –2

3. a) –8 **b)** –40 **c)** 0 **d)** –20 **e)** –12 **f)** $4t - 12$ **g)** $4t$ **h)** 8

4. a)



b) $\{-4, -1, 8\}$ **c)** ±1

5. a) i) –103 **ii)** –22 **iii)** –10 **b) i)** $-3z^2 + 5$ **ii)** $-3x^2 + 12x - 7$ **iii)** $-12a^2 - 60a - 70$

6. a) i) 1225 **ii)** $-\frac{41}{2}$ **iii)** $4\sqrt{3} - 17$ **b) i)** $2n^2 - 20n + 25$ **ii)** $18n^4 - 25$

7. a) 10 **b)** 15 **c)** ±5 **8. a)** 12 **b)** ±4

9. a) i) 3 **ii)** $\frac{1}{3}$ **b)** 256 **c)** 10 **10. a)** –3 **b)** 14 **c)** $-\frac{7}{2}$

11. a) –3 **b)** 81

12. a) see table below **b)** see table below

$f(x)$	Ordered Pair
$f(1) =$ 6	(1, 6)
$f(0) =$ -3	(0, -3)
$f(2) =$ 0	(2, 0)
$f(-1) =$ -2	(-1, -2)
$f(-3) =$ 0	(-3, 0)
$f(3) =$ -6	(3, -6)

x	$f(x)$	Ordered Pair
2	6	(2, 6)
0	0	(0, 0)
–6	-2	(-6, -2)
8	4	(8, 4)
-8	–6	(-8, -6)
10	4	(10, 4)

c) horizontal line has an infinite number of points

13. a) $22\sqrt{3} - 29$
b) i) $a^3 - 8a^2 + 19a - 19$
ii) $64x^6 - 32x^4 - 4x^2 - 5$

14. B **15.** B **16.** C

17. B **18.** A **19.** B

20.

1	0		

21.

4	0		

22.

3	2		

23. a) $-\frac{3}{2}$, 4 **b)** $-\frac{7}{4}$, 3

Relations and Functions Lesson #9:
Function Notation and Problem Solving

Using Function Notation

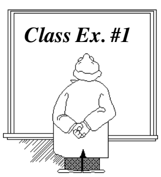
In Lesson 2 and 3 we solved problems about relations defined by an equation. In this lesson we solve problems where function notation is used to define the relation.

In lesson 2 we had the following scenario.
" A candle manufacturer found that their "Long-Last" candles melted according to the formula $h = -2t + 12$, where h is the height of the candle, in cm, after t hours."

The relation between height and time is described by an **equation.**

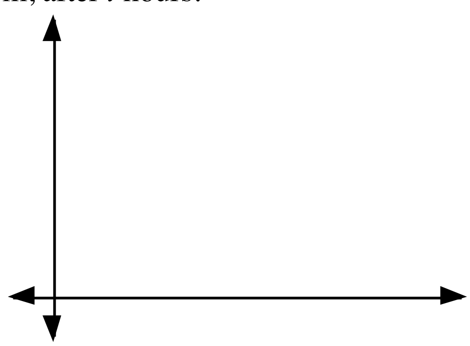
The relation is a function since for each input there is only one output and so it can be described using the **function notation** below.
" A candle manufacturer found that their "Long-Last" candles melted according to the formula $h(t) = -2t + 12$, where h is the height of the candle, in cm, after t hours."

In this example, the notation $h(4)$ is a simplified way of saying
"What is the height of the candle after four hours?"



A candle manufacturer found that their "Long-Last" candles melted according to the formula $h(t) = -2t + 12$, where h is the height of the candle, in cm, after t hours.

a) Use a graphing calculator to sketch the graph of the function and show the graph on the grid



b) Determine the value of $h(5)$.

c) Write in words the meaning of $h(5)$

d) Evaluate the following and explain their significance

i) $h(0)$ **ii)** $h(6)$ **iii)** $h(8)$

e) How long will it take for the candle to burn down to a height of 7 cm?

f) Suggest an appropriate domain and range for the function.

Complete Assignment Questions #1 - #5

Assignment

1. Ivory the botanist treated a 2 cm plant with a special growth fertilizer. With this fertilizer, the plant grew at a rate modelled by the function $H(t) = \frac{5}{3}t + 2$, where $H(t)$ represent the height of the plant in cm after t days.

a) Use a graphing calculator to sketch the graph of the function and show the graph on the grid.

b) Determine the value of $H(3)$.

c) Write in words the meaning of $H(3)$.

d) Evaluate the following.

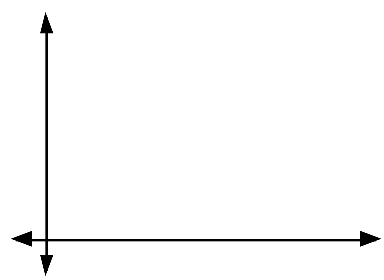
i) $H(0)$ **ii)** $H(6)$ **iii)** $H(21)$

e) How long will it take for the plant to reach a height of 21 cm?

f) It takes 27 days for the plant to mature (to reach maximum height). State the domain and range of the function $H(t)$.

2. The cost to Inner Technology of producing IT graphing calculators can be modelled by the function $C(n) = 11750 + 32n$, where $C(n)$ represents the cost in dollars of producing n calculators.

a) Sketch the graph of the function for a maximum of 4000 calculators.



b) Determine the value of $C(30)$.

c) Write in words the meaning of $C(30)$

d) Evaluate $C(0)$ and explain its significance.

e) How many calculators can be produced for \$31 270?

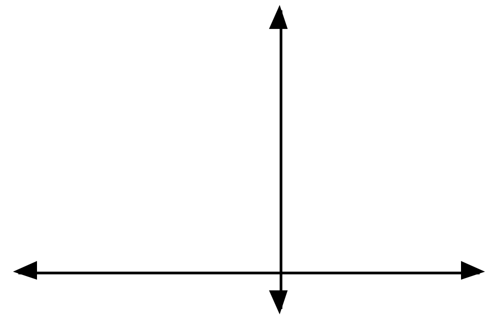
f) Last month IT produced 2600 calculators and spent \$14000 on advertising. If there are other fixed monthly costs of \$24 500 and each calculator sells for \$165 how much profit would be made if all the calculators are sold?

3. Over the last 10 years, data was recorded for the number of cups of hot chocolate sold at BGB Senior High School. It was found from the data that the warmer the weather, the less cups of hot chocolate were sold. The data can be modelled by the formula $N(t) = 150 - 10t$, where $N(t)$ is the daily number of cups of hot chocolates sold when the average daily temperature is t°C.

a) Sketch the graph of the function on the grid provided.

b) Determine the value of $N(-5)$.

c) Write in words the meaning of $N(-5)$



d) What was the average temperature if 190 cups of hot chocolate were sold?

e) Explain how to estimate the lower limit of the domain of the relation.

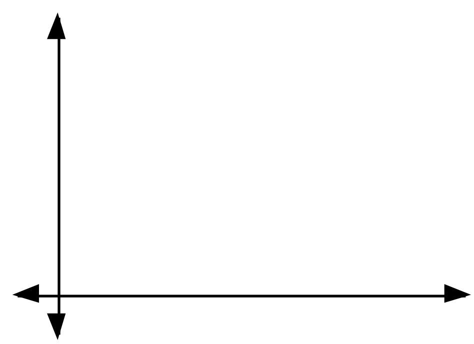
f) Suggest an appropriate domain and range for the function $N(t)$ if BGB High School is located in southern Alberta.

4. A special type of weather balloon follows a path which can be represented by the formula $h(t) = -9t^2 + 900t$, where $h(t)$ is the height in cm after t minutes.

a) Sketch the graph of the function on the grid.

b) Determine the value of $h(30)$ and $h(70)$.

c) Does $h(30) = h(70)$? Do they mean the same thing? Explain.



d) Evaluate the following and explain their significance in the context of the question.

i) $h(0)$ **ii)** $h(100)$ **iii)** $h(110)$

e) What is the highest point the balloon will reach?

f) When will the balloon land?

g) Suggest an appropriate domain and range for the function $h(t)$?

5. As part of an experiment, a 50 kg steel ball is dropped from a Canadian Air Force jet. The height of the steel ball above the ground can be described by part of the graph of the function $h(t) = 2575 - 4.9t^2$, where $h(t)$ is the height, in metres, of the steel ball after t seconds.

a) Sketch the graph of the function on the grid.

b) Determine the value of $h(30)$.

c) Why does $h(30)$ not represent the height of the ball after 30 seconds?

d) At what height is the ball dropped from the jet?

e) How long (to the nearest second) will it take the ball to make contact with the ground?

f) Suggest an appropriate domain and range for the function $h(t)$.

Answer Key

1. **b)** 7 **c)** After 3 days the height is 7 cm. **d)** i) 2 ii) 12 iii) 37 **e)** 11.4 days
f) domain $\{t \mid 0 \le t \le 27, t \in R\}$ range $\{H \mid 2 \le H \le 47, H \in R\}$

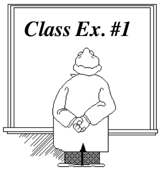
2. **b)** 12710 **c)** It costs \$12 710 to produce 30 calculators
d) $C(0) = 11750$, there are fixed costs of \$11750 before any calculators are produced
e) 610 **f)** \$295 550

3. **b)** 200 **c)** 200 cups are sold when the average temperature is –5°C **d)** –4°C
e) estimate the minimum average daily temperature
f) answers may vary domain $\{t \mid -35 \le t \le 15, t \in R\}$ range $\{N \mid 0 \le N \le 500, N \in W\}$

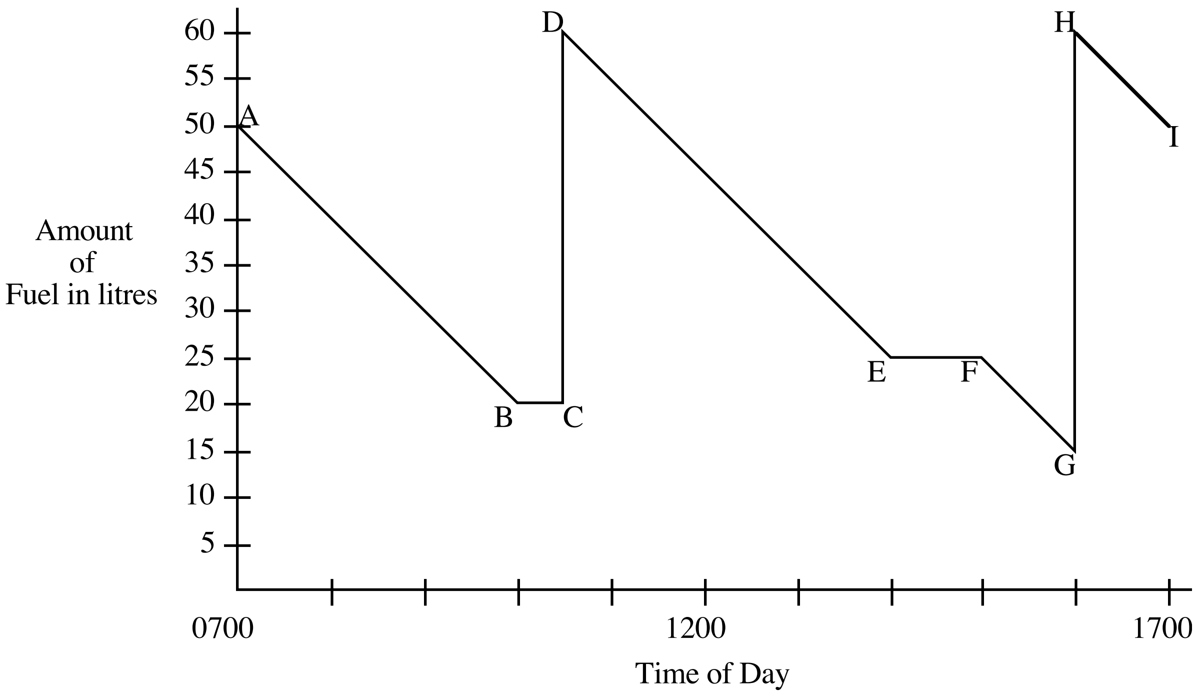
4. **b)** both = 18 900
c) they are equal but do not represent the same thing. $h(30)$ is the height after 30 minutes and $h(70)$ is the height after 70 minutes
d) i) 0 Initial height = 0 m ii) 0 After 100 min the balloon has landed on the ground
iii) –9900 this has no meaning since the balloon has already landed
e) 22 500 cm = 225 m **f)** after 100 min
g) domain $\{t \mid 0 \le t \le 100, t \in R\}$ range $\{h \mid 0 \le h \le 22\,500, h \in R\}$

5. **b)** –1835
c) the ball has already hit the ground, so the function no longer represents the height of the ball
d) 2575 m **e)** 23 seconds **f)** domain $\{t \mid 0 \le t \le 23, t \in R\}$ range $\{h \mid 0 \le h \le 2575, h \in R\}$

Relations and Functions Lesson #10:
Interpreting Graphs of Functions



The Carter Family are driving to Yukon for a family vacation. The graph represents the amount of fuel (in litres) in the gas tank of their car on the first day of their journey.



The graph of the journey is divided into eight line segments.

a) With reference to the journey, explain what is happening between:

i) A and B

ii) B and C

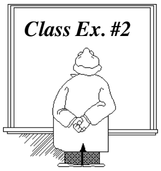
iii) C and D

b) What is the rate of fuel consumption (in litres per hour) between D and E?

c) Which line segment represents the car being refueled for the second time?

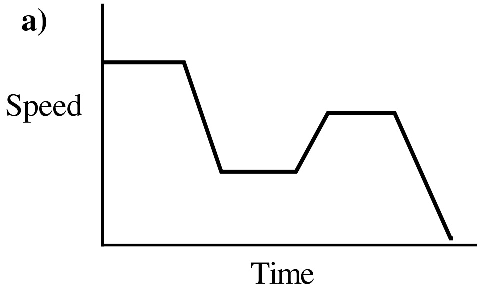
d) Calculate the total time when the car is being driven.

e) If fuel costs 85¢ per litre, calculate the cost of the fuel used for the first day of the journey?

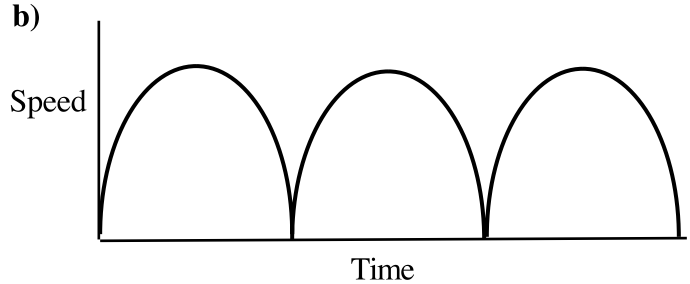


Suggest a possible scenario for each of the following graphs.

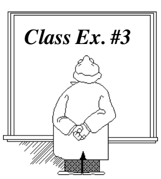
a)



b)



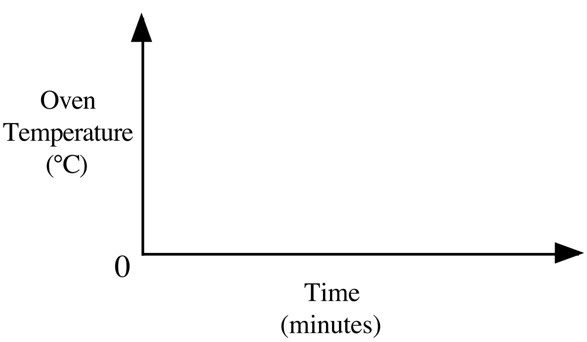
Sketching a Graph

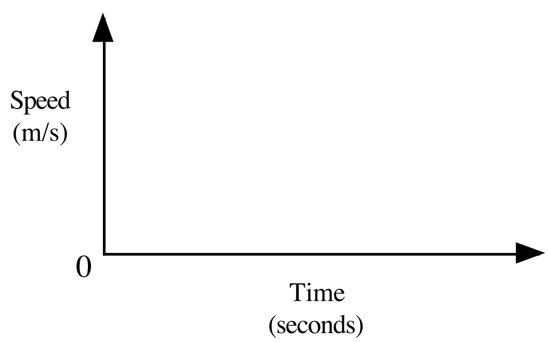


Sketch a graph with no scale for each of the following:

a) the oven temperature when baking a pie.

b) Ben taking part in a 100 m sprint.

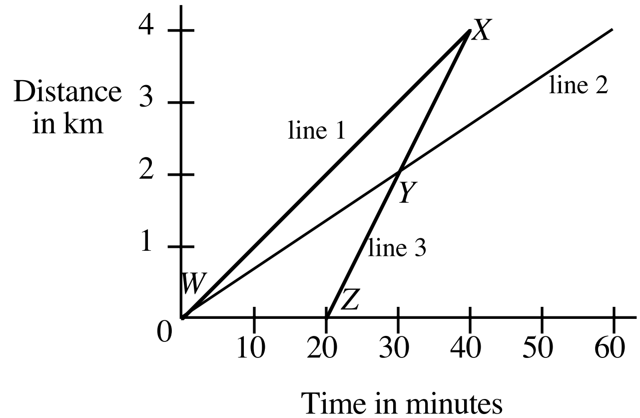




Complete Assignment Questions #1 - #14

Assignment

1. Three sisters, Amanda, Brittany, and Chelsea each follow the same route to school. One morning Amanda cycles to school, Brittany walks to school, and Chelsea runs to school.
Lines 1, 2, and 3 on the graph represent the three routes.



a) Complete the table below.

	Line 1	**Line 2**	**Line 3**
Distance (km)	4		
Time (hrs)	$\frac{2}{3}$		
Rate (km/hr)	6		
Student			

b) Explain what is happening at the following points.

i) *W*

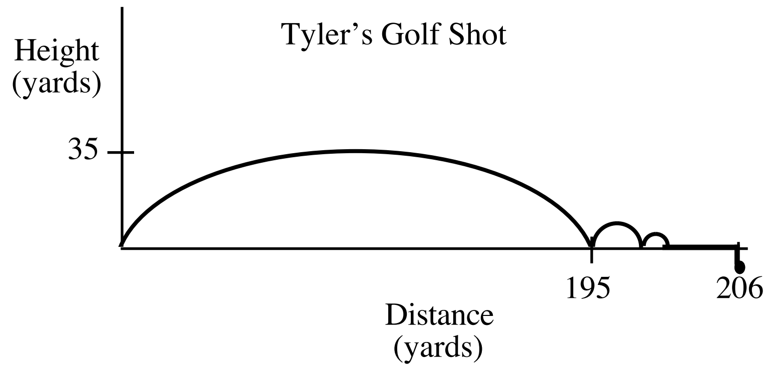
ii) *X*

iii) *Y*

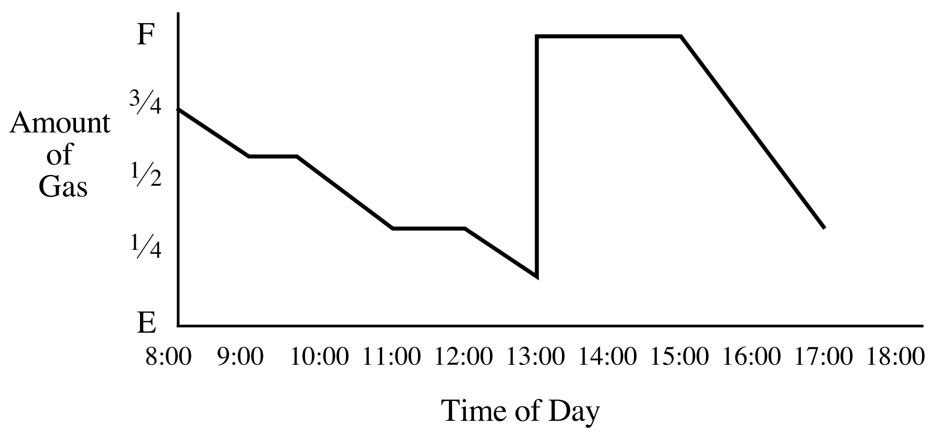
iv) *Z*

c) How can you tell from the steepness of the lines which line represents the route of each student?

2. Tyler, a member of St. Andrews High School golf team, hits a golf ball. The graph shows the path of the ball. Describe Tyler's golf shot.

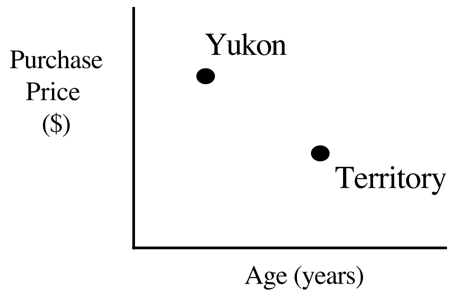


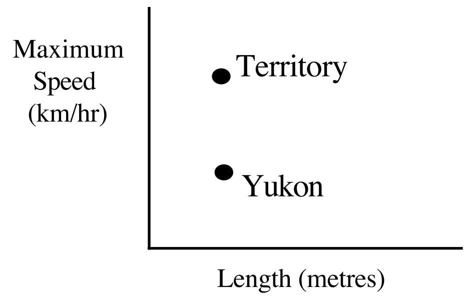
3. Dar sell medical supplies. The graph shows the amount of gasoline in his car during a particular day.



Describe how Dar may have spent the day.

4. The two graphs shown compare two yachts, the Yukon and the Territory. The first graph compares the yachts by age and cost. The second graph compares the boats by speed and length. Describe the comparison between the two yachts.





5. A super ball is dropped from a 10 m building and bounces 80% of its previous height. Create a graph of height as a function of time.

6. Patrick leaves home and travels at a steady speed of 100 km/hr for two hours. He stops for 30 minutes and sets off at a steady speed of 50 km/hr for one hour. He has a business meeting for two hours, and then drives home at a steady speed of 100 km/hr.

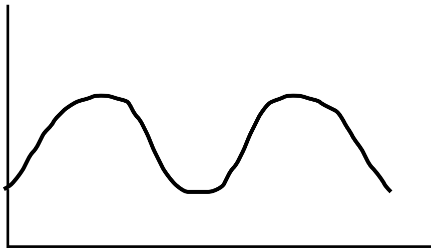
a) Draw a sketch of the distance travelled as a function of time.

b) Draw a sketch of distance from home as a function of time.

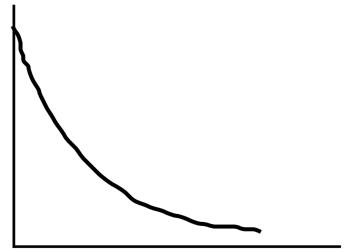
c) If Patrick left home at 9:30 a.m., at what time did he return home?

7. Suggest a possible scenario for each of the following graphs.

a)

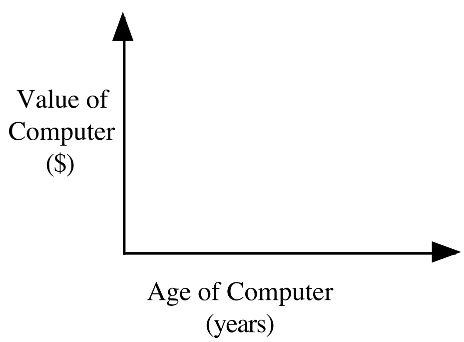


b)

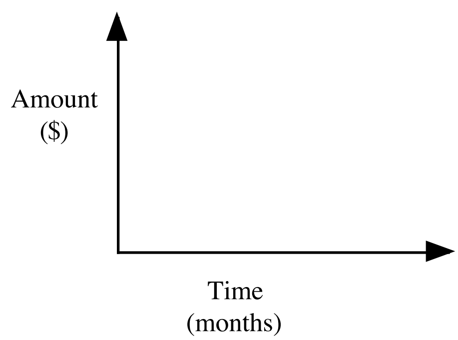


8. Sketch a graph with no scale to represent each of the following.

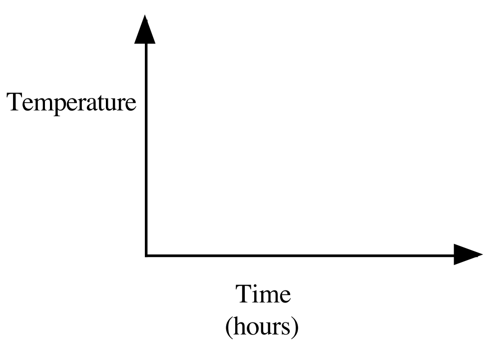
a) A computers value compared to its age.



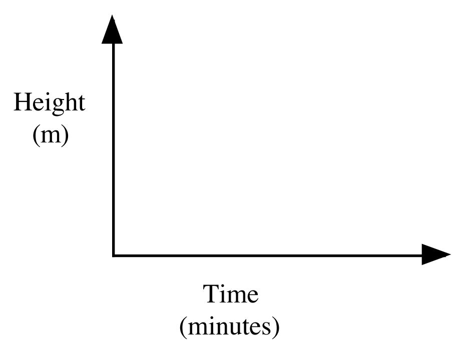
b) The amount of your savings if you save $10 every month for a period of six months.



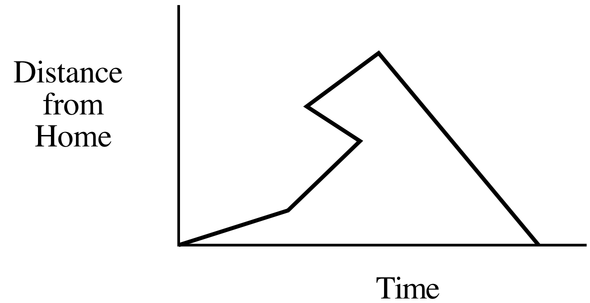
c) The air temperature during a spring day from 6:00 a.m. to 6:00 p.m.



d) You are sitting in the bottom chair of a ferris wheel. Graph your height above the ground during two rotations of the wheel.

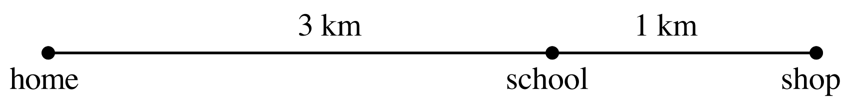


9. A student drew the following graph to represent a journey. Explain why the graph must be incorrect.



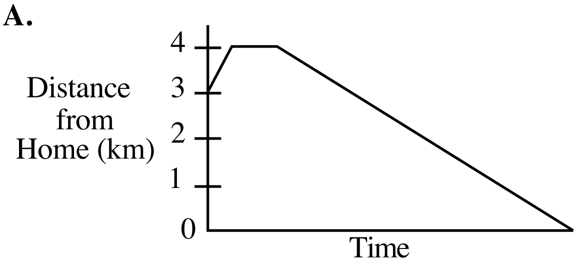
Use the following information to answer the next question.

Melanie leaves school and runs to the shop. She spends some time in the shop and walks home.

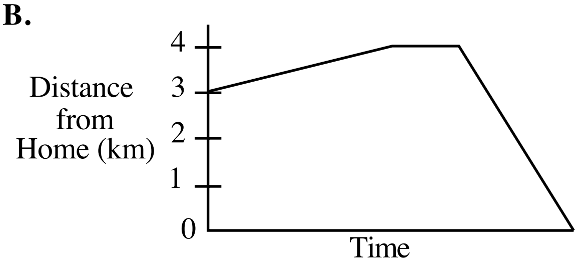


Multiple Choice **10.** Which graph best describes Melanie's distance from home starting from when she left school?

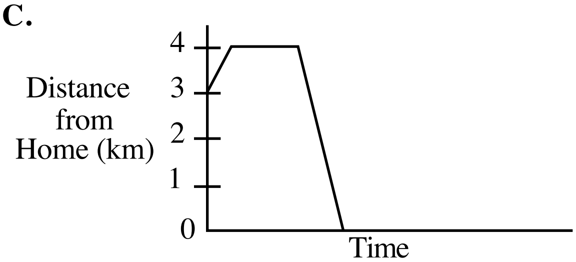
A.



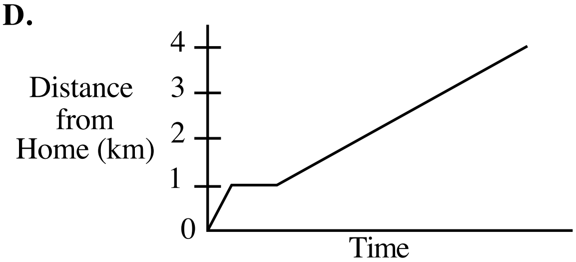
B.



C.



D.



Matching Match each description on the left with the best graph on the right. Each graph may be used once, more than once, or not at all.

<u>Description</u>

11. Sketch a graph of a person's height as a function of their age.

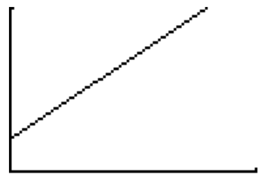
12. The number of hours of daylight in a given town in Alberta depends on the day of the year. Sketch a graph of number of hours of daylight as a function of day of the year.

13. Sketch a graph of the area of a circle as a function of its radius.

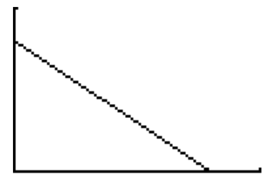
14. You start driving at a constant speed with a full tank of gas. Sketch a graph of litres of gas in the tank as a function of distance travelled.

<u>Graph</u>

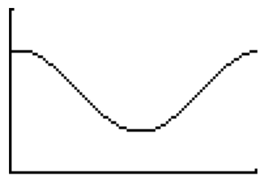
A.



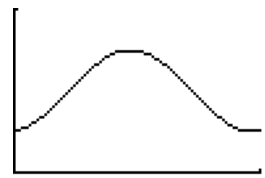
B.



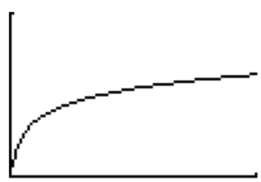
C.



D.



E.



F.

Answer Key (Answers may vary)

1. a) see table below

	Line 1	Line 2	Line 3
Distance (km)	4	4	4
Time (hrs)	$\frac{2}{3}$	1	$\frac{1}{3}$
Rate (km/hr)	6	4	12
Student	Chelsea	Brittany	Amanda

b) i) Chelsea and Brittany leave home at the same time.

ii) Chelsea and Amanda arrive at school.

iii) Amanda overtakes Brittany.

iv) Amanda leaves home 20 minutes after Brittany and Chelsea.

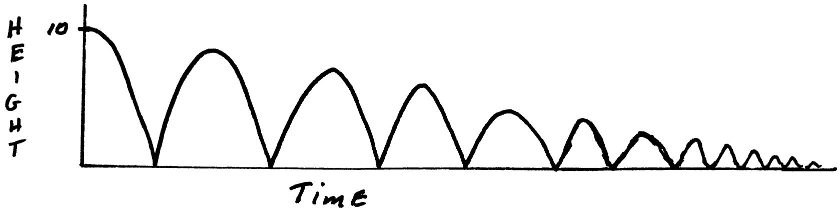
c) The steeper the graph, the less time is taken to travel to school. The steepest slope represents the cyclist Amanda, the next steepest slope represents the runner Chelsea, and the remaining line represents the walker Brittany.

2. Tyler hits the ball 195 yards in the air. It bounces twice and rolls into the hole. The golf shot travelled a total of 206 yards, and had a maximum height of 35 yards.

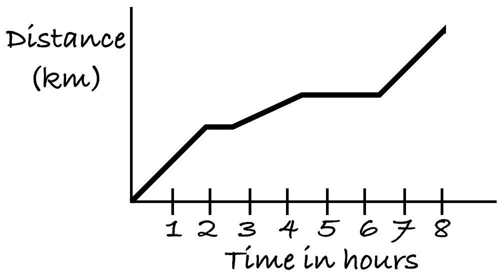
3. Dar started with $\frac{3}{4}$ tank of gas in his car. He drove for about one hour, had a meeting for about $\frac{1}{2}$ hour, drove for about $1\frac{1}{2}$ hours, had a second meeting for one hour, and drove for about one hour. He refuelled at 1 pm and had a lunch meeting for about 2 hours. He then drove home and arrived about 6 p.m. with a quarter tank of gas left.

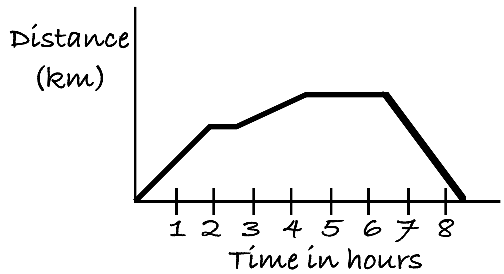
4. The Territory is older than the Yukon, and its purchase price was less. Both the Territory and the Yukon are the same length, but the Territory can achieve a greater maximum speed.

5. see graph below



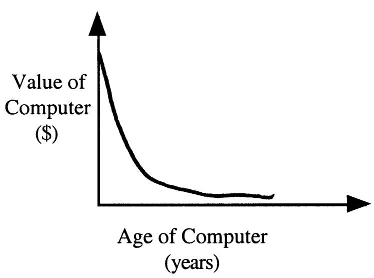
6. a) see graph below **b)** see graph below **c)** 5:30 pm



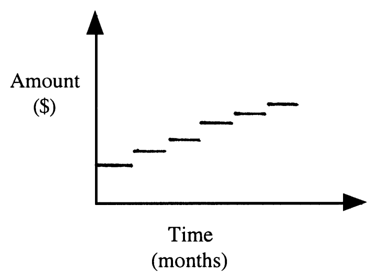


7. a) The number of hours of daylight per day over a period of two years for a location in the northern hemisphere.
b) The value of a car depreciating over time.

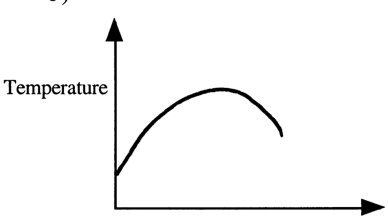
8. a)



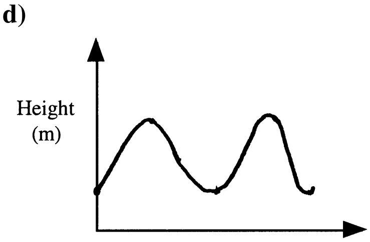
b)



c)



d)



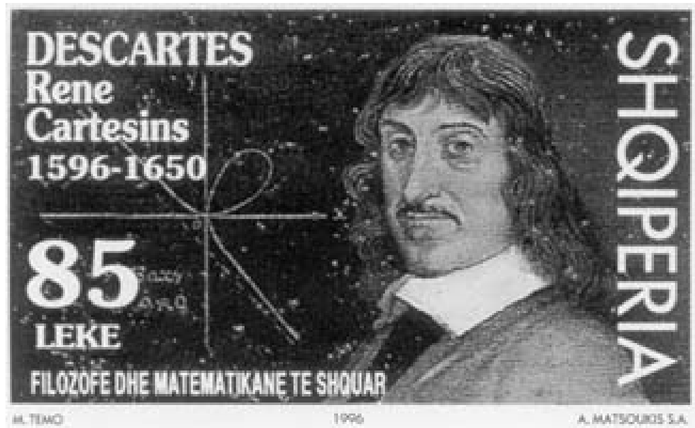
9. The graph is not a function. The person cannot be at three different places at the same time.

10. A **11.** E **12.** D **13.** F **14.** B

Line Segments Lesson #1:
Line Segments on a Cartesian Plane

Omit this page if you have studied Relations and Functions first.

The Cartesian Coordinate System

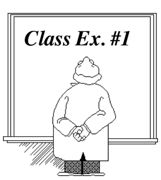


In mathematics "Cartesian" means relating to the French mathematician **René Descartes**. In 1637 he introduced the new idea of specifying the position of a point on a surface using two intersecting axes as measuring guides.

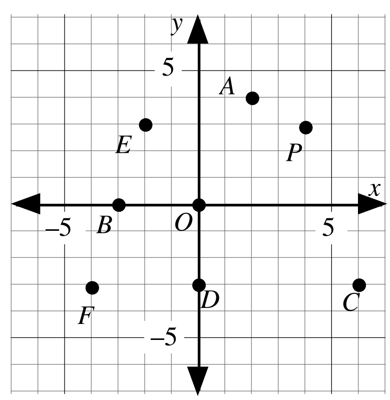
The modern Cartesian coordinate system in two dimensions is defined by two axes at right angles to each other forming a plane (called the *xy* plane). The horizontal axis is labelled *x* and the vertical axis is labelled *y*. All the points in a Cartesian coordinate system taken together form a **Cartesian plane**.

The point of intersection of the two axes is called the **origin**, usually labelled O.
On each axis a unit length is chosen and units are marked off to form a grid. To specify a particular point on the grid we use a unique ordered pair of numbers called coordinates. The first number in the ordered pair, called the ***x*-coordinate**, identifies the position with regard to the *x*-axis, while the second number, called the ***y*-coordinate**, identifies the position with regard to the *y*-axis. The point $P(4, 3)$ is shown on the grid below.

The intersection of the *x*-axis and *y*-axis creates four **quadrants**, numbered counterclockwise starting from the north-east quadrant.



a) Complete the following by writing the coordinates of the points represented by the letters on the grid.



A(*B*(*C*(
D(*O*(

b) Write the coordinates of the point in the second quadrant.

c) Write the coordinates of the point in quadrant III.

d) Complete the following table using "positive" or "negative".

Quadrant	*x*-coordinate	*y*-coordinate
I		
II		
III		
IV		

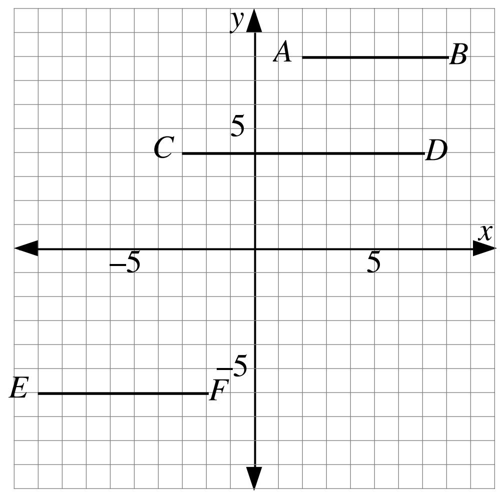
e) On the grid, plot the points $L(-4, -5)$ and $M(2, -5)$. Join *L* and *M* to form the horizontal line segment *LM*.

Line Segment

A line segment is the portion of a line between two points on the line.

If the endpoints of a line segment are A and B, we refer to it as line segment AB.
NOTE: Line segment AB may also be written as $\overline{AB}$.

Length of a Horizontal Line Segment



Consider the line segments shown on the grid.

a) Find the length of each line segment by counting.

- length of AB is _____ units.
- length of CD is _____ units.
- length of EF is _____ units.

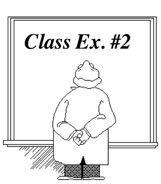
b) Determine the coordinates of the endpoints of each line segment.

- AB → A (,) B (,)
- CD → C (,) D (,)
- EF → E (,) F (,)

c) Complete the following.

- The difference in the x-coordinates, $x_B - x_A$, is _____ .
- The difference in the x-coordinates, $x_D - x_C$, is _____ .
- The difference in the x-coordinates, $x_F - x_E$, is _____ .

d) How can the coordinates of the end points of a horizontal line segment be used to find the length of the line segment?

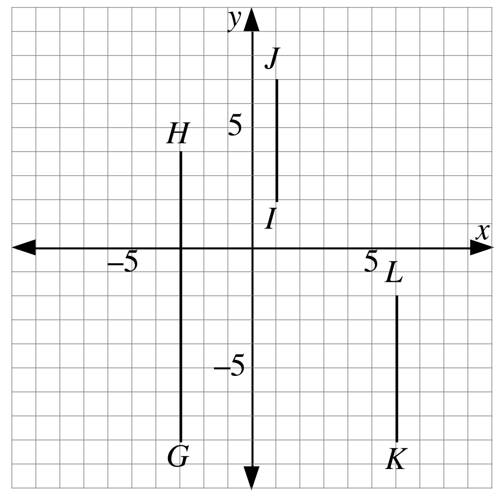


a) Line segment AB has endpoints $A(2, 8)$ to $B(-5, 8)$. Determine the length of $\overline{AB}$.

b) Calculate the line segment from $P(a - 2, b)$ to $Q(a + 4, b)$.

Length of a Vertical Line Segment

Consider the line segments shown on the grid.



a) Find the lengths of each line segment by counting.

- length of *GH* is _____ units.
- length of *IJ* is _____ units.
- length of *KL* is _____ units.

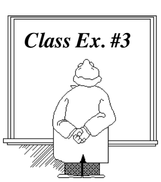
b) Determine the coordinates of the endpoints of each line segment.

- *GH* → *G* (,) *H*(,)
- *IJ* → *I* (,) *J* (,)
- *KL* → *K* (,) *L* (,)

c) Complete the following.

- The difference in the *y*-coordinates, $y_H - y_G$, is _____ .
- The difference in the *y*-coordinates, $y_J - y_I$, is _____ .
- The difference in the *y*-coordinates, $y_L - y_K$, is _____ .

d) How can the coordinates of the end points of a vertical line segment be used to find the length of the line segment?



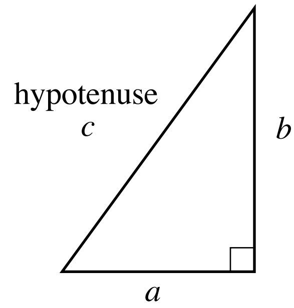
a) Line segment *RS* has endpoints $R(1, -4)$ to $S(1, -9)$. Determine the length of $\overline{RS}$.

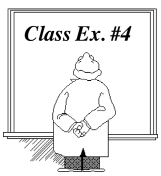
b) Calculate the line segment from $P(a\ , b)$ to $Q(a, b + 10)$.

Pythagorean Theorem Review

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

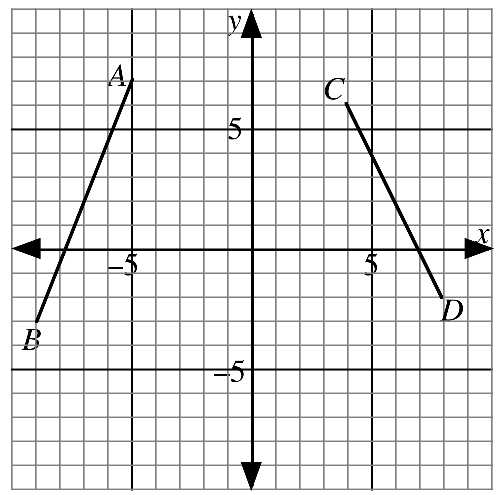
i.e. $a^2 + b^2 = c^2$





Use the Pythagorean theorem to determine the length of the line segments shown on the grid.

Answer as an exact value and as a decimal to the nearest tenth.

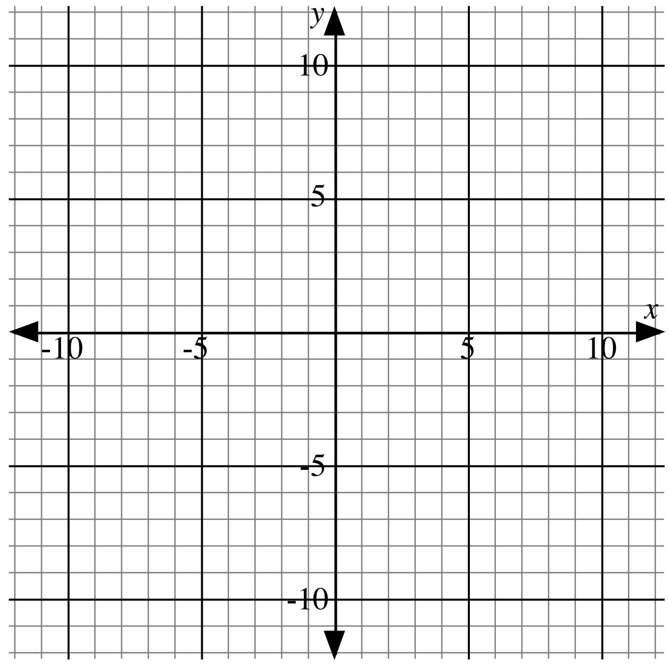


Complete Assignment Questions #1 - #13

Assignment

Omit question 1 if you have done Relations and Functions first.

1. The following is a scrambled message using ordered pairs. Plot the ordered pairs on the grid provided, unscramble the letters and find the message. The symbol • represents the beginning of a new letter.



• Join (5, –9) to (5, –5). Join (7, –9) to (5, –7) to (7, –5).

• Join (–11, –2) to (–10, 0) to (–9, 2) to (–8, 0) to (–7, –2). Join (–8, 0) to (–10, 0).

• Join (0, –10) to (0, –6) to (2, –6) to (2, –8) to (0, –8) to (2, –10).

• Join (2, 0) to (0, 0) to (0, –2) to (2, –2) to (2, –4) to (0, –4).

• Join (8, 5) to (8, 9). Join (10, 5) to (10, 9). Join (8, 7) to (10, 7).

• Join (–3, 2) to (–2, 6) to (–1, 4) to (0, 6) to (1, 2).

• Join (3, 8) to (5, 8). Join (4, 8) to (4, 4).

• Join (6, 1) to (4, 1) to (4, –3) to (6, –3).

• Join (–6, –6) to (–6, –10) to (–4, –10) to (–4, –6) to (–6, –6).

2. Determine the length of each line segment.

a) $A(2, 7)$ to $B(5, 7)$

b) $C(-5, 3)$ to $D(-5, 12)$

c) $E(2, -8)$ to $F(2, 3)$

d) $G(8, -12)$ to $H(-5, -12)$

3. Determine the length of each line segment.

a) $I(-3, -8)$ to $J(-3, -3)$

b) $K(7, -10)$ to $L(-35, -10)$

c) $M(-325, -892)$ to $N(255, -892)$

d) $P(7251, -1286)$ to $Q(7251, 1289)$

4. Determine the length of each line segment. Answer in simplest radical form

a) $R\left(0, \sqrt{8}\right)$ to $S\left(0, \sqrt{18}\right)$ **b)** $U\left(2\sqrt{3}, -1\right)$ to $V\left(-\sqrt{75}, -1\right)$

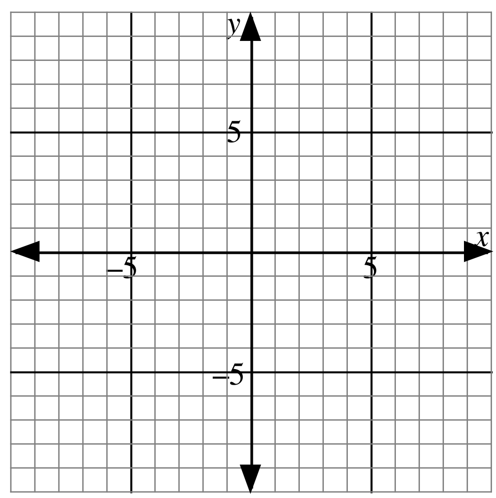
5. Determine whether each line segment is horizontal or vertical, and write an expression for its length.

a) $A(p, q)$ to $B(p - 4, q)$ **b)** $C(m - 3, n + 5)$ to $D(m - 3, n + 12)$

c) $J(a, b)$ to $K(c, b)$ where $a > c$ **d)** $M(s, t)$ to $N(s, z)$ where $t > z$

6. A triangle has vertices $P(-4, -3)$, $Q(9, -3)$ and $R(1, 5)$.

- Sketch the triangle on the grid
- Calculate the area of the triangle.

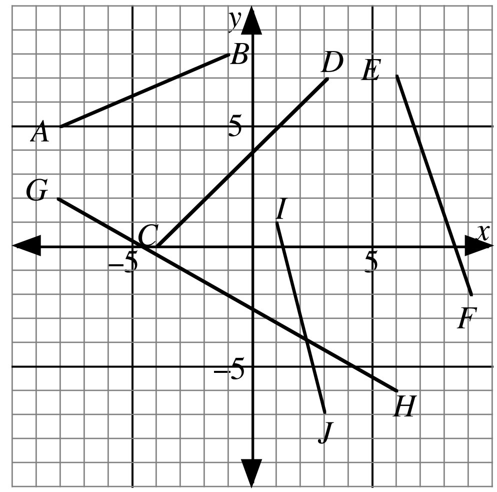


7. Use the Pythagorean theorem to determine the lengths of the line segments shown on the grid.

Give each answer as

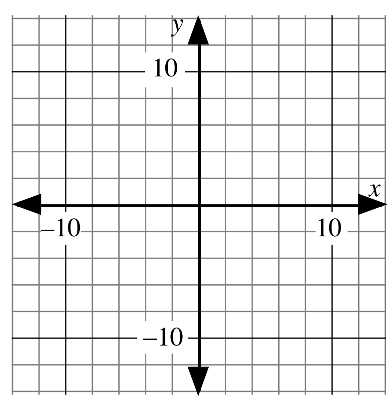
i) a mixed radical in simplest form

ii) a decimal to the nearest hundredth



8. List the coordinates of *A* and *B* from question #7.
How can they be used to find the length of *AB*?

9. On the grid, plot the points $P(-6, 6)$, $Q(-6, -10)$ and $R(8, -10)$



a) Determine, as an exact value, the distance from P to R.

b) Calculate the area and perimeter, to the nearest tenth, of triangle *PQR*

10. Rebecca uses quadrant I in a Cartesian plane to describe the location of the bases in a game of high school softball. The origin is at home plate, first base is at $(18, 0)$, and the distance between each base is 18 m. The pitcher's mound is located between home plate and second base.

a) State the coordinates of second base.

b) The pitcher stands on the mound 12 m from home plate. If she has to throw a ball to second base, what distance, to the nearest tenth of a metre, would she throw the ball?

c) Calculate the **exact** coordinates of the location of the pitcher.

Multiple Choice

11. Horizontal line segment *EF*, with endpoints $E(5,-1)$ and $F(x, y)$, has length 6 units. Which of the following must be true?

A. $x = 11$
B. $y = 5$ or -7
C. $x + y = 6$
D. $x = y$ or $x - y = 12$

12. *ABCD* is a square with vertices $\left(\sqrt{5}, 0\right), \left(0, \sqrt{5}\right), \left(-\sqrt{5}, 0\right)$, and $\left(0, -\sqrt{5}\right)$ respectively. The area of the square, in unit^2, is

A. 5
B. 10
C. 20
D. 100

Numerical Response

13. To the nearest tenth, the perimeter of triangle PQR with vertices, $P(3, 8)$, $Q(3, 0)$ and $R\ (-1, 8)$ is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. MATH ROCKS

2. a) 3 **b)** 9 **c)** 11 **d)** 13

3. a) 5 **b)** 42 **c)** 580 **d)** 2575

4. a) $\sqrt{2}$ **b)** $7\sqrt{3}$

5. a) horizontal, 4 **b)** vertical, 7 **c)** horizontal, $a - c$ **d)** vertical, $t - z$

6. 52 units^2

7. $AB = \sqrt{58} = 7.62$, $CD = 7\sqrt{2} = 9.90$, $EF = 3\sqrt{10} = 9.49$,
$GH = 2\sqrt{65} = 16.12$, $IJ = 2\sqrt{17} = 8.25$

8. $A(-8, 5)$, $B(-1, 8)$ $x_B - x_A = 7$, $y_B - y_A = 3$

$$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$$
$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

9. a) $2\sqrt{113}$ units **b)** area = 112.0 units^2, perimeter = 51.3 units

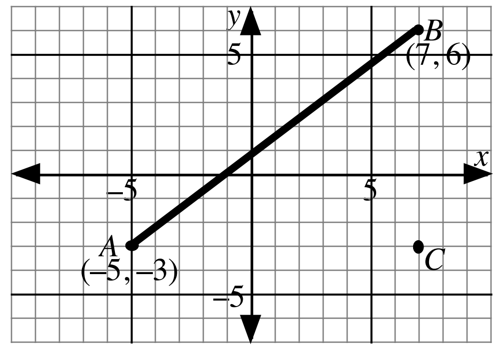
10. a) (18,18) **b)** 13.5 m **c)** $(6\sqrt{2}, 6\sqrt{2})$

11. D **12.** B **13.**

2	0	.	9

Line Segments Lesson #2:
The Distance Formula

Warm-Up



Consider line segment AB shown on the grid.

a) Use the Pythagorean theorem to show that the length of AB is 15 units.

b) Complete the following:

length of AC = $x_B - x_A$ = 7 − ___ = ____

length of CB = $y_B - y_A$ = ___ − ___ = ____

c) Complete the following to verify the length of AB.

(length of AB)^2 = (difference in x-coordinates of B and A)^2 + (difference in y-coordinates of B and A)^2

length of AB = $\sqrt{(\text{difference in } x\text{-coordinates of } B \text{ and } A)^2 + (\text{difference in } y\text{-coordinates of } B \text{ and } A)^2}$

length of AB = $\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

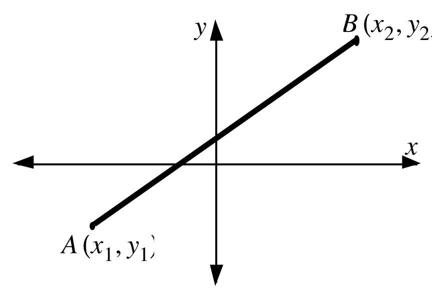
length of AB = $\sqrt{(7 - (-5))^2 \ + \ (\qquad\qquad)^2}$

length of AB = $\sqrt{12^2 + \qquad}$

length of AB = $\sqrt{\qquad}$

length of AB =

d) Use the same procedure to make a rule for finding the distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$.



(length of AB)^2 = (difference in x-coordinates of B and A)^2 + (difference in y-coordinates of B and A)^2

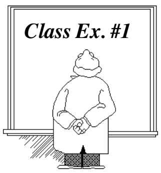
length of AB = $\sqrt{(\text{difference in } x\text{-coordinates of } B \text{ and } A)^2 + (\text{difference in } y\text{-coordinates of } B \text{ and } A)^2}$

length of AB = $\sqrt{(x_2 - \qquad)^2 \ + \ (\qquad\qquad)^2}$

The Distance Formula

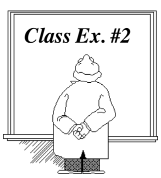
To find the distance, d, between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ use

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{or} \qquad d_{PQ} = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$



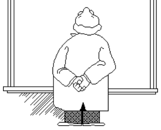
Find the exact length of the following line segments.

a) $P(2, 3)$ to $Q(10, –3)$ **b)** $G(–25, 3)$ to $H(–17, –5)$



Sometimes grids are superimposed on maps to find the distance between two locations. For example, to calculate the distance between two craters, Copernicus and Plato, on the Earth's moon, a coordinate system could be superimposed on a map of the moon. Ordered pairs would then be assigned to Copernicus and Plato to find the distance between the two craters. If the location of Copernicus on the coordinate grid is (–89, 226) and the location of Plato is (136, 179), calculate the distance between the two craters to the nearest unit.

Class Ex. #3



a) Explain how we can determine if a triangle is right angled if we know the length of each side.

b) L is the point $(0, 1)$, M is $(-3, -3)$ and N is $(-7, 0)$. Prove that ΔLMN is right angled.

Complete Assignment Questions #1 - #12

Assignment

1. Determine the exact distance between each pair of points.

a) $A(2, 0)$ and $B(7, 12)$

b) $C(3, 7)$ and $D(6, 11)$

c) $E(6, 1)$ and $F(-2, -3)$

d) $G(-3, -1)$ and $H(-12, 8)$

e) $I(-2, 5)$ and $J(22, -7)$

f) $K(-\frac{1}{2}, \frac{5}{2})$ and $L(1, 3)$

2. Determine the distance, to the nearest hundredth, between each pair of points.

a) $P(4, 0)$ and $Q(-2, -7)$

b) $R(-3, 7)$ and $S(-6, -11)$

c) $M(128, 0)$ and $N(-457, 752)$

d) $X(-2.3, 8.9)$ and $Y(-3.4, -6.8)$

3. Consider the points $P(-2, 2)$, $Q(1, 6)$ and $R(7, 14)$.

a) Calculate the lengths of PQ, QR and PR. What do you notice?

b) What does this mean with regard to the points P, Q and R?

4. Determine the exact distance between the points $A\left(\sqrt{12}, \sqrt{80}\right)$ and $B\left(\sqrt{75}, \sqrt{125}\right)$.

5. A is the point $(6, -2)$, B is $(4, 4)$ and C is $(-3, -5)$.

a) Calculate the exact lengths of AB, BC and AC in entire radical form.

b) Show how you can use the answers in a) to prove that angle BAC is a right angle.

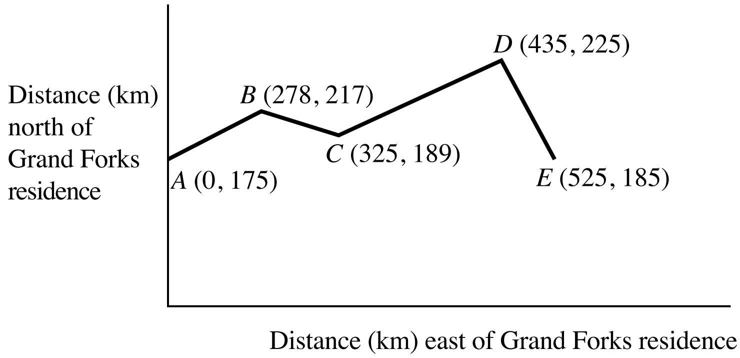
6. In a high school football game, the Chiefs' quarterback scrambles to the Chiefs' 7 yard line, 15 yards from the left sideline. From that position he throws the ball upfield. The pass is caught by a wide receiver who is on the Chiefs' 38 yard line, 4 yards from the left sideline.

a) The first quadrant of a coordinate grid is superimposed on the football field. State a location on the football field for the origin of the coordinate grid.

b) With reference to this origin give the location of the quarterback (when he throws the ball) and the wide receiver (when he catches the ball) as ordered pairs.

c) Determine the length of the pass (to the nearest yard).

7. A family moves from Grand Forks, North Dakota, to Toronto, Ontario. The family belongings were transported by van. The graph opposite shows <u>part of the</u> <u>route</u> taken by the van.



a) Describe the starting position, for the part of the route which is shown, relative to the family home in Grand Forks.

b) Calculate the distance, to the nearest km, travelled by the van from A to E.

c) The van used an average of one litre of gasoline for every five km travelled. If the average cost of gasoline for the trip was 65¢ per litre, calculate the amount spent on gasoline for the part of the route shown in a).

8. At the end of a high school soccer game, Jonas walks 6 blocks west and 5 blocks north from the corner of the soccer field to his house. Beverly walks 3 blocks east and 4 blocks south to reach her house from the same corner of the soccer field.

a) Taking the corner of the soccer field as the origin, list the coordinates of each home.

b) If a block represents 135 metres, determine the direct distance, to the nearest metre, between their homes.

Multiple Choice

9. The distance between the points $(2, -1)$ and $(6, 2)$ is

A. 5
B. $\sqrt{7}$
C. $\sqrt{17}$
D. 25

10. A circle, centre the origin, passes through the point $(-6, -8)$. The radius of the circle is

A. 6
B. 8
C. 10
D. 100

11. Which of the following points is equidistant from $A(-2, 7)$ and $B(-6, -5)$?

A. $(0, -4)$
B. $(7, -10)$
C. $(-1, 0)$
D. $(4, -5)$

Numerical Response

12. The diameter, to the nearest tenth, of the circle with centre $(-5, 9)$ which passes through $(4, -3)$ is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) 13 **b)** 5 **c)** $4\sqrt{5}$ **d)** $9\sqrt{2}$ **e)** $12\sqrt{5}$ **f)** $\frac{1}{2}\sqrt{10}$

2. a) 9.22 **b)** 18.25 **c)** 952.75 **d)** 15.74

3. a) $PQ = 5$, $QR = 10$, $PR = 15$. $PQ + QR = PR$ **b)** The points P, Q and R lie on a straight line.

4. $4\sqrt{2}$

5. a) $AB = \sqrt{40}$, $BC = \sqrt{130}$, $AC = \sqrt{90}$.
b) Since $BC^2 = AB^2 + AC^2$, triangle ABC must be right angled at A so angle BAC is a right angle.

6. a) The intersection of Chiefs' goal line and the left sideline **b)** $Q(15, 7)$ $W(4, 38)$ **c)** 33 yards
7. a) 175 kilometres north of Grand Forks. **b)** 550 km **c)** $71.50
8. a) $J(-6, 5)$ $B(3, -4)$ **b)** $9\sqrt{2}$ blocks = 1718 m.
9. A **10.** C **11.** C **12.**

3	0	.	0

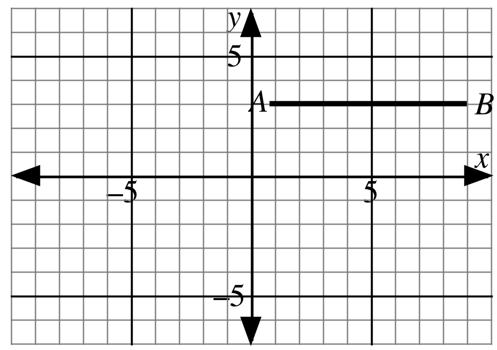
Line Segments Lesson #3:
The Midpoint of a Line Segment

Midpoint

The **midpoint, *M*,** of a line segment is the point at the centre of the line segment.

Warm-Up #1 *Midpoint of a Horizontal Line Segment*

Consider the line segment AB shown on the grid.

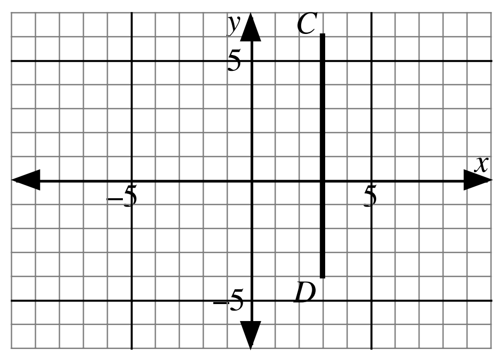


a) Determine the coordinates of the midpoint by counting. Label the midpoint M on the grid and list the coordinates beside it.

b) List the coordinates of point A and point B on the grid. How can the x-coordinates of points A and B be used to find the coordinates of the midpoint of a horizontal line?

Warm-Up #2 *Midpoint of a Vertical Line Segment*

Consider the line segment CD shown on the grid.

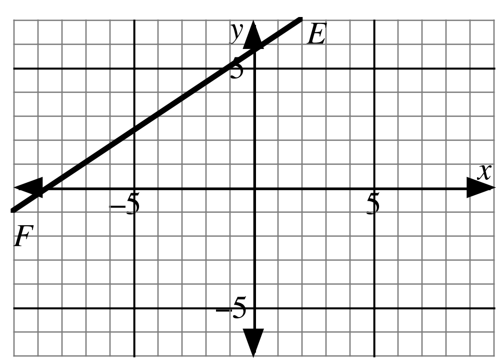


a) Determine the coordinates of the midpoint by counting. Label the midpoint M on the grid and list the coordinates beside it.

b) List the coordinates of point C and point D on the grid. How can the y-coordinates of points C and D be used to find the coordinates of the midpoint?

Warm-Up #3 *Midpoint of an Oblique (Diagonal) Line Segment*

Consider the line segment EF shown on the grid.



a) Use the results from Warm-Up #1 and Warm-Up #2 to find the midpoint of EF.

b) Express in words how to find the midpoint, M, of the line segment joining the points (x_1, y_1) and (x_2, y_2)

c) Complete the formula to express the relationship in b). $x_M =$ $\qquad\qquad$ $y_M =$

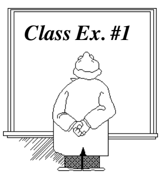
Midpoint of a Line Segment

Consider line segment PQ with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The midpoint, M, of the line segment has coordinates.

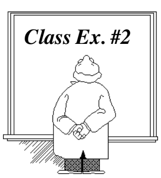
$$\boxed{M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$$

Line segment PQ can also be written as $\overline{PQ}$.



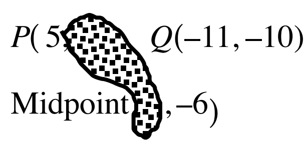
Determine the coordinates of the midpoint of the line segment with the given pair of endpoints.

a) $E(-5, 7)$, $F(-11, -2)$ **b)** $A(w + 3, 2w)$, $C(5w - 1, 7w + 1)$



Ruby was doing a question in her coordinate geometry homework and as a prank her little brother Max wrote over part of the question.

Calculate the missing coordinates.

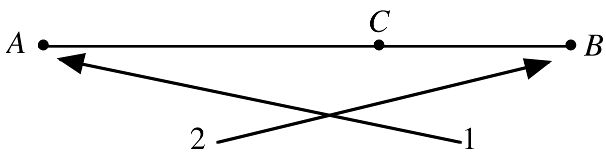


Complete Assignment Questions #1 - #15

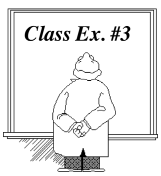
Extension - Dividing a Line Segment in the Ratio m to n.

Line segment AB is said to be divided in the ratio 2 to 1 by the point C if the ratio

$\dfrac{\text{length of } AC}{\text{length of } CB} = \dfrac{2}{1}$. In other words $AC = 2\,CB$.



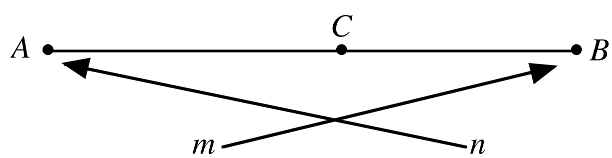
The coordinates of C can be found using the formula $x_C = \dfrac{x_A + 2x_B}{3}$ $\quad y_C = \dfrac{y_A + 2y_B}{3}$



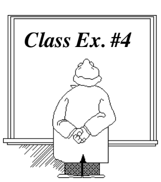
Calculate the coordinates of the point C which divides the line segment joining $A(4, 8)$ and $B(-5, 0)$ in the ratio 2 to 1.

General Formula

If line segment AB is divided in the ratio m to n by a point C, then the coordinates of C are given by $x_C = \dfrac{nx_A + mx_B}{m + n}$ $\quad y_C = \dfrac{ny_A + my_B}{m + n}$



The midpoint formula is a special case of the general formula where $m = n = 1$.



A and B have $(3, -2)$ and $(7, 10)$ respectively. Find the coordinates of the point, C, that divides line segment AB in the ratio 1 to 3.

Complete Assignment Questions #16 - #17

Assignment

1. Determine the coordinates of the midpoint of the line segment with the given pair of endpoints.

a) $A(2, 6),\ C(4, 16)$ **b)** $X(-3, -8),\ Y(-11, 0)$ **c)** $K(15, -17),\ L(-11, 3)$

d) $A(-25, 56),\ O(0, 0)$ **e)** $P(-2.5, 5.6), Q(1.5, -6.4)$ **f)** $E(-2, 7),\ F(6, 2)$

2. Determine the coordinates of the midpoint of the line segment with the given pair of endpoints.

a) $C(3x, 8y),\ D(7x, -4y)$ **b)** $S(a + b, a + 7b),\ T(a + b, a - 3b)$

c) $A(p + 3, q - 2),\ B(p - 1, q + 8),$ **d)** $U(m - 5n, m + n),\ V(3n - m, m - n),$

3. A circle has diameter AB, with $A\left(\sqrt{32}, \sqrt{28}\right)$, and $B\left(\sqrt{72}, \sqrt{112}\right)$. Find the coordinates of the centre of the circle, C, in simplest radical form.

4. In each case M is the midpoint of $\overline{AB}$. Determine the value of x.

a) $A(2, 6),\ B(6, x),\quad M(4, -1)$

b) $A(3, 6),\ B(x, 0),\quad M(0, 3)$

c) $A(x, 2),\ B(3x, 18),\quad M(-8, 10)$

d) $A(2, x + 1),\ B(-6, 2x - 7),\quad M(-2, x)$

5. A refinery is to be built halfway between the rural towns of Branton and Oilville.
A railway is to be built connecting the towns to the refinery.
On a Cartesian plane, Branton is located at (1232, 3421) and Oilville is located at (1548, 3753).

a) What are the coordinates of the refinery?

b) If the grid scale is 1 unit represents 100 metres, determine the length of the railway to the nearest kilometre.

6. Consider triangle ABC with vertices $A(-3, 6)$, $B(1, -2)$, $C(-11, 2)$

a) Calculate the length of each side of the triangle and explain why the triangle is right angled.

b) Calculate the coordinates of the midpoint, M, of the hypotenuse.

c) Calculate the distance from M to each of the vertices of the triangle. What do you notice?

7. In a relay race, a team of four athletes runs across the diagonal, AC, of a rectangular sports field. On a coordinate grid , the vertices of the rectangle are $A(4, 7)$, $B(240, 7)$, $C(240, 367)$ and $D(4, 367)$.
The four legs of the relay are of equal distance and none of the athletes can start until they have received the relay baton. Melanie runs the first leg from A to X, Alex runs from X to Y, Nick runs from Y to Z, and Rachel completes the race from Z to C.

a) Draw a diagram to represent the information.

b) Calculate the coordinates of X, Y and Z.

c) If the units for the coordinate system are metres, calculate the distance, to the nearest 0.1 m, that each athlete had to run.

Multiple Choice

8. $P(4, -8)$ and $Q(-2, 10)$ are the endpoints of a diameter of a circle. The coordinates of the centre of the circle are

A. $(-3, 9)$
B. $(2, 2)$
C. $(3, -9)$
D. $(1, 1)$

9. AB is a diameter of a circle, centre C. If $A(8, -6)$ and $C(5, -2)$ then B is the point

A. $(2, 2)$
B. $(6.5, -4)$
C. $(11, -10)$
D. $(13, -8)$

10. $\overline{PQ}$ has endpoints $P(-4, 9)$ and $Q(0, 5)$. Which of the following points is equidistant from $S(-6, -5)$ and M, the midpoint of $\overline{PQ}$?

A. $(0, -4)$
B. $(7, -10)$
C. $(-1, 0)$
D. $(4, -5)$

11. Which statement is always true?

A. Two line segments of equal length have the same midpoint.
B. Two line segments with the same midpoint are of equal length.
C. A point equidistant from the endpoints of a line segment is the midpoint.
D. None of the above statements is always true.

Numerical Response

12. The midpoint of line segment ST is $M\left(\frac{1}{2}, -4\right)$. If the coordinates of T are $(-3, 3)$ and the coordinates of S are (x, y), the value of x is _____.

(Record your answer in the numerical response box from left to right)

13. The point $M(a, 6)$ is the midpoint of $\overline{GH}$ with $G(22, b)$ and $H\,(6, -8)$. The value of $a + b$ is _____.

(Record your answer in the numerical response box from left to right)

14. The midpoint of line segment AB lies on the y-axis. A lies on the x-axis and B has coordinates $(-4, 5)$. The length of AB, to the nearest tenth, is _____.

(Record your answer in the numerical response box from left to right)

15. A **median** in a triangle is a line drawn from a vertex to the midpoint of the opposite side. ΔABC has vertices $A(0, 8)$, $B(-6, 2)$, and $C(10, 4)$. To the nearest tenth, the length of median AD is _____.

(Record your answer in the numerical response box from left to right)

Extension Questions.

16. Given that C divides AB in the given ratio, find the coordinates of C in each case.

a) $A(10, 6),\ B(-10, 16).$ $\quad \dfrac{AC}{CB} = \dfrac{1}{4}$

b) $A(-4, 8),\ B(-10, -4).$ $\quad \dfrac{AC}{CB} = \dfrac{1}{2}$

c) $A(8, -12),\ B(-2, 18)$ $\quad \dfrac{AC}{CB} = \dfrac{2}{3}$

d) $A(14, 0),\ B(-10, 12)$ $\quad \dfrac{AC}{CB} = \dfrac{2}{1}$

Use the following information to answer question 17.

- A **median** in a triangle is a line drawn from a vertex to the midpoint of the opposite side.
- The three medians in a triangle meet at the **centroid** (the point which represents the centre of gravity of the triangle).
- The centroid divides each median in the ratio 2 to 1 from the vertex.

17. Consider triangle PQR with vertices $P(-6, 4)$, $Q(10, -2)$ and $R(-10, -8)$.
A is the midpoint of QR, B is the midpoint of PR, and C is the midpoint of PQ.

a) Make a rough sketch of the triangle.
Draw and label the three medians PA, QB and RC
and mark the position of the centroid, G.

b) Find the coordinates of A.

c) Find the coordinates of the centroid, G.

d) Calculate the lengths of QG and GB and show that the ratio $\frac{QG}{GB} = \frac{2}{1}$.

Answer Key

1. a) $(3, 11)$ **b)** $(-7, -4)$ **c)** $(2, -7)$ **d)** $\left(-\frac{25}{2}, 28\right)$ **e)** $(-0.5, -0.4)$ **f)** $\left(2, \frac{9}{2}\right)$

2. a) $(5x, 2y)$ **b)** $(a + b, a + 2b)$ **c)** $(p + 1, q + 3)$ **d)** $(-n, m)$

3. $\left(5\sqrt{2}, 3\sqrt{7}\right)$ **4. a)** -8 **b)** -3 **c)** -4 **d)** 6 **5. a)** $(1390, 3587)$ **b)** 46 km

6. a) $AB = \sqrt{80} = 4\sqrt{5}$, $BC = \sqrt{160} = 4\sqrt{10}$, $AC = \sqrt{80} = 4\sqrt{5}$
Since $BC^2 = AB^2 + AC^2$, triangle ABC must be right angled at A
b) $M(-5,0)$ **c)** $MA = MB = MC = \sqrt{40} = 2\sqrt{10}$. M is equidistant from A, B and C.

7. b) $X(63, 97)$, $Y(122,187)$, $Z(181,277)$ **c)** 107.6 m

8. D **9.** A **10.** C **11.** D

12.

4			

13.

3	4		

14.

9	.	4	

15.

5	.	4	

16. a) $(6, 8)$ **b)** $(-6, 4)$ **c)** $(4,0)$ **d)** $(-2,8)$

17. b) $A(0, -5)$ **c)** $G(-2,-2)$ **d)** $B(-8,-2)$ $QG = 12$, $GB = 6$ so $\frac{QG}{GB} = \frac{2}{1}$

Line Segments Lesson #4:
Slope of a Line Segment

A trucker driving up a hill with a heavy load may be concerned with the steepness of the hill. When building a roof, a builder may be concerned with the steepness (or pitch) of the roof. A skier going down a hill may be concerned with the steepness of the ski hill.

In mathematics, the term **slope** is used to describe the steepness of a line segment.

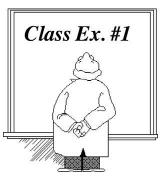
Slope of a Line Segment

The **slope** of a line segment is a measure of the steepness of the line segment.

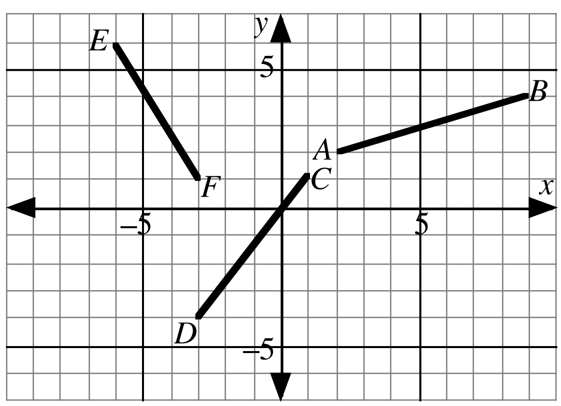
It is the ratio of **rise** (the change in vertical height between the endpoints) over **run** (the change in horizontal length between the endpoints.).

$$\textbf{Slope} = \frac{\textbf{rise}}{\textbf{run}}$$

- the **rise** is POSITIVE if we count UP, and NEGATIVE if we count DOWN.
- the **run** is POSITIVE if we count RIGHT, and NEGATIVE if we count LEFT.



Each line segment on the grid has endpoints with integer coordinates.
Complete the table below.

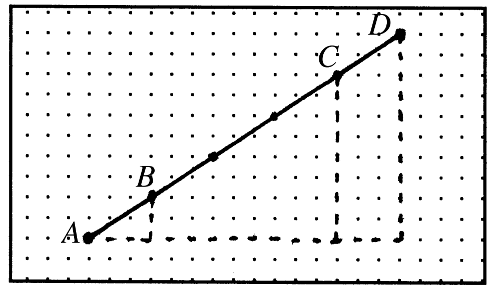


Line Segment	Rise	Run	Slope = $\frac{\text{Rise}}{\text{Run}}$
AB			
CD			
EF			

Investigation #1 — *Investigating the Slope of Line Segments*

a) Complete the chart. Write the slopes in simplest form.

Line Segment	Rise	Run	Slope = $\frac{\text{Rise}}{\text{Run}}$
AB			
AC			
AD			
BC			



b) How are the slopes of the line segments related?

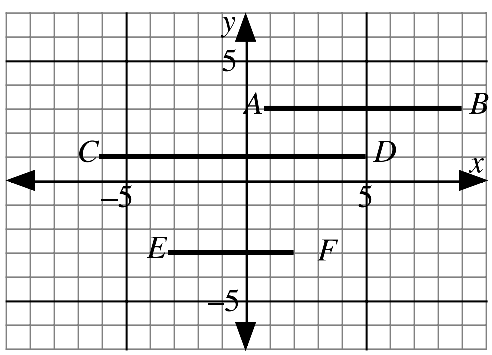
Slope of a Line

The slopes of all line segments on a line are equal. The slope of a line can be found using slope $= \frac{\text{rise}}{\text{run}}$ for any two points on the line.

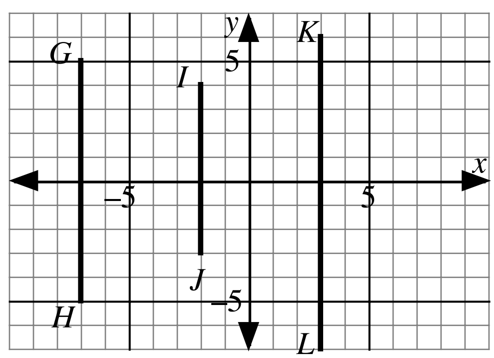
Investigation #2 — *Slopes of Horizontal and Vertical Line Segments*

Consider the line segments in Grid 1 and Grid 2 below.

Grid 1



Grid 2



a) Determine the slopes of all the line segments in Grid 1.

b) Determine the slopes of all the line segments in Grid 2.

c) Complete the following statements.

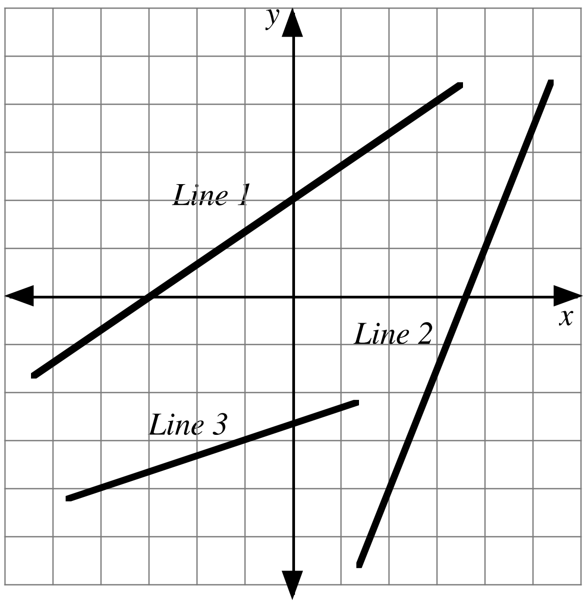
- Horizontal line segments have a slope of __________ .
- Vertical line segments have an _______________ slope.

Investigation #3 *Positive and Negative Slopes*

a) Each of the lines on the grids passes through at least two points with integer coordinates. Calculate the slope of each of the lines.

Remember: On a Cartesian Plane:
- the **rise** is POSITIVE if we count UP and NEGATIVE if we count DOWN.
- the **run** is POSITIVE if we count RIGHT and NEGATIVE if we count LEFT.

Grid 1



Grid 2

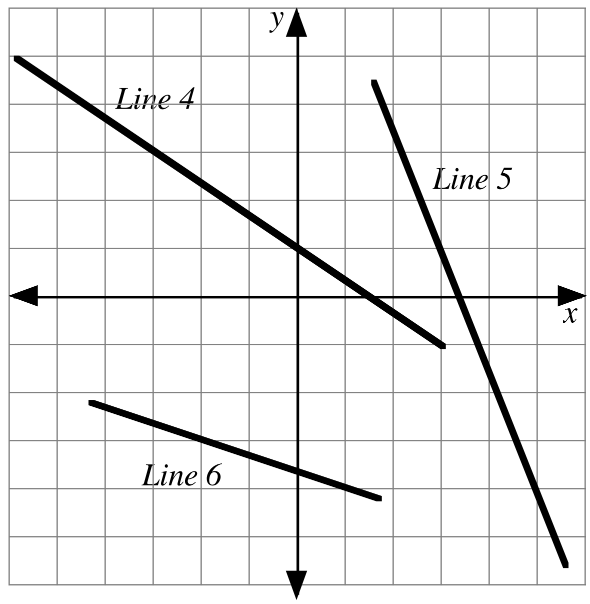


Table For Grid 1

Line	Slope
1	
2	
3	

Table For Grid 2

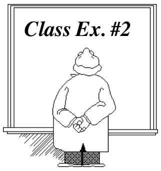
Line	Slope
4	
5	
6	

b) Compare the slopes of:

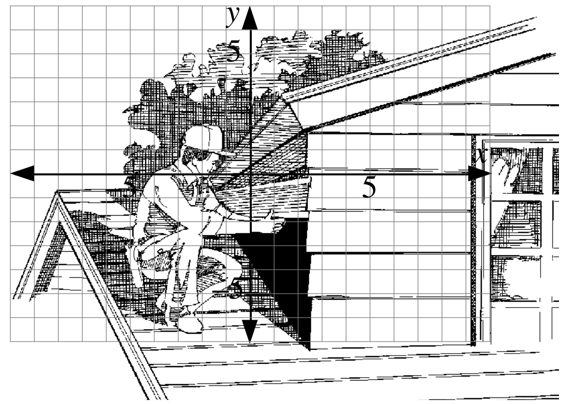
- Line 1 and Line 4
- Line 2 and Line 5
- Line 3 and Line 6

c) Complete the following statements.

- A line which rises from left to right has a ______________ slope.
- A line which falls from left to right has a ______________ slope.

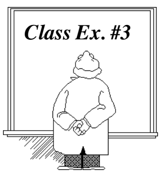


A grid has been superimposed on the sketch.



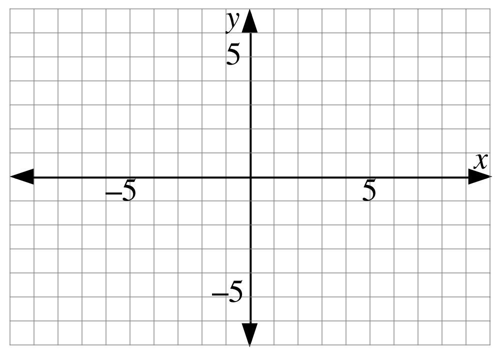
a) Estimate the pitch (slope) of the roof to the right of the worker's head.

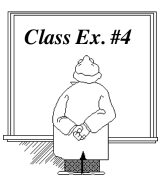
b) Could the grid be used to estimate the pitch of the roof the worker is standing on? Explain.



Draw a line segment on the grid which passes through the point (–4, 2) and has a slope of $-\frac{2}{3}$. The line segment must be long enough to cross both the *x*-axis and the *y*-axis.

Write the coordinates of three other points on the line segment which have integer coordinates.



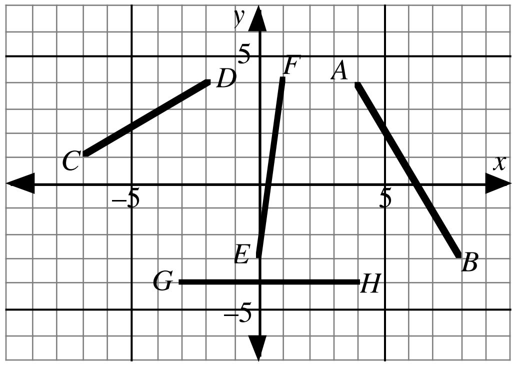


A line segment has a slope of $-\frac{5}{7}$ and a rise of 12. Calculate the run as an exact value.

Complete Assignment Questions #1 - #14

Assignment

1. Each line segment on the grid has endpoints with integer coordinates. Complete the table.



Line Segment	Rise	Run	Slope = $\frac{\text{Rise}}{\text{Run}}$
AB			
CD			
EF			
GH			

2. Each of the lines on the grid passes through at least two points with integer coordinates. Calculate the slope of each of the lines.

slope of Line 1 :

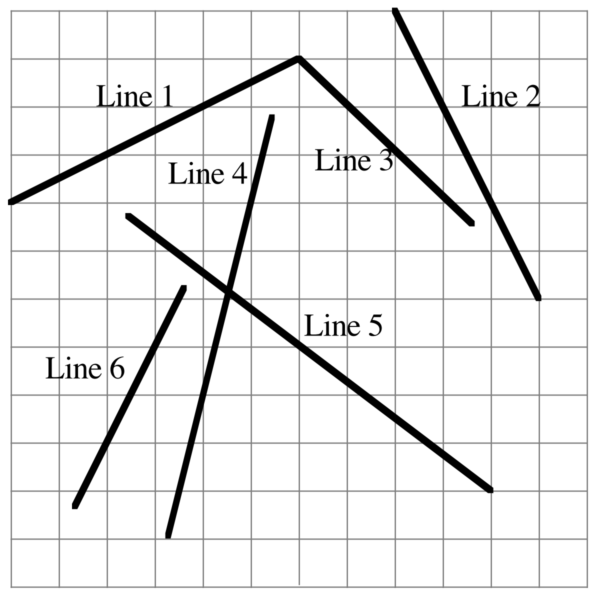
slope of Line 2:

slope of Line 3:

slope of Line 4:

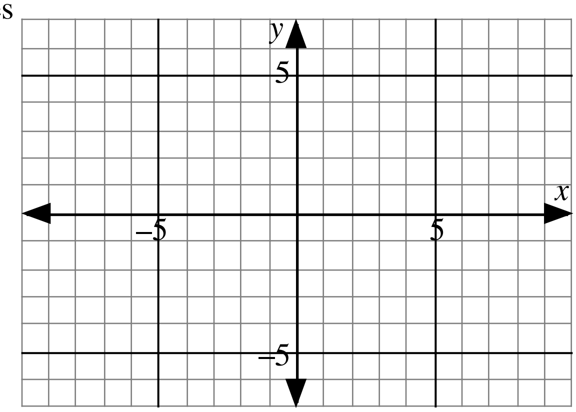
slope of Line 5:

slope of Line 6:



3. Draw a line segment on the grid which passes through the point (–5, –2) and has a slope of $\frac{2}{3}$. The line segment must be long enough to cross both the *x*-axis and the *y*-axis.

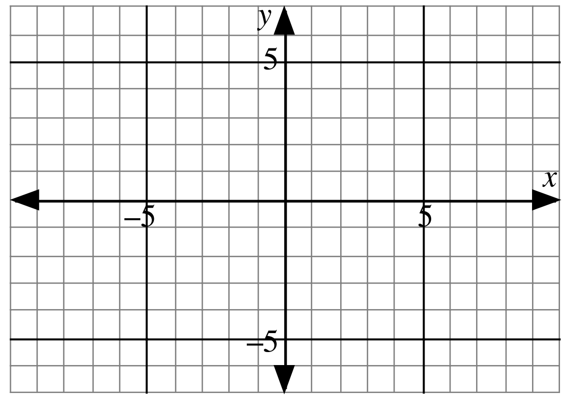
Write the coordinates of three other points on the line segment which have integer coordinates.



4. Repeat question #3 for line segments with the given slope passing through the given point.

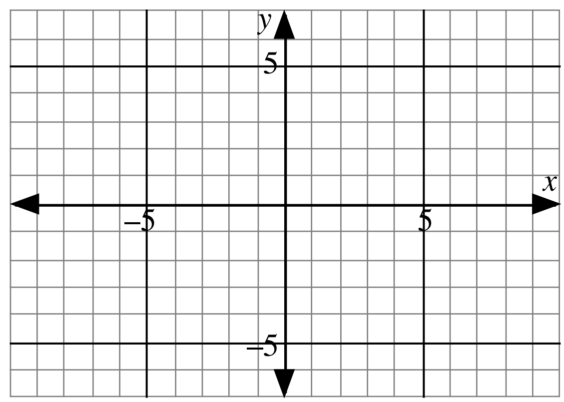
a) slope $= \frac{2}{5}$, $(2, 1)$

b) slope $= -\frac{1}{3}$, $(6, -3)$



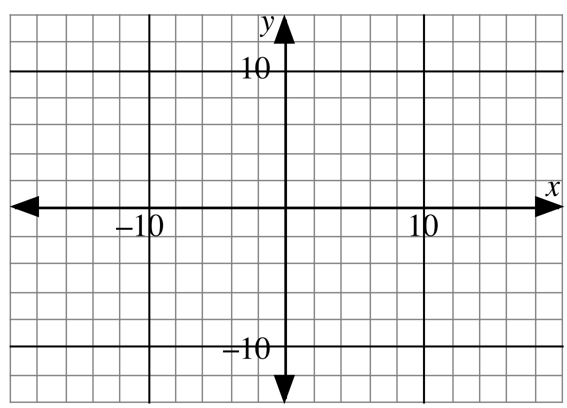
c) slope $= -\frac{4}{3}$, $(-9, 6)$

d) slope $= 4$, $(0, -7)$



e) slope $= -2$, $(4, -12)$

f) slope $= 0$, $(0, 6)$



5. P has coordinates $(-1, 2)$. Find two positions for point Q so that the slope of PQ is

a) 2 **b)** -3 **c)** $\frac{1}{3}$

d) $-\frac{2}{5}$ **e)** 0 **f)** undefined

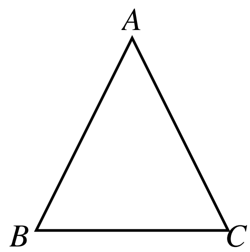
6. For each of the following, two of the measures, rise, run, and slope are given. Calculate the value of the third measure.

a) slope $= \dfrac{5}{7}$ and run $= 49$ **b)** slope $= -\dfrac{3}{8}$ and rise $= 15$

c) slope $= -\dfrac{6}{11}$ and run $= 33$ **d)** slope $= \dfrac{3}{4}$ and rise $= 15$

e) slope $= \dfrac{8}{7}$ and run $= 70$ **f)** slope $= -\dfrac{7}{2}$ and rise $= 21$

7. Triangle ABC is isoceles with $AB = AC$. $BC = 6.8$ cm. If the slope of $AC = -\frac{5}{4}$, calculate the area of the triangle.



8. A ramp which has been set up by skateboarders has a slope of $\frac{2}{3}$. If the ramp has a base length of 1.5 metres, calculate the height of the ramp.

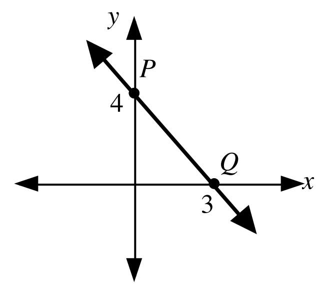
9. An exit ramp from a highway intersects an elevated country road that is 8 metres above the highway.

a) If a driver exits on to the ramp 44 metres before the elevated country road, determine the slope of the ramp.

b) Because of increased traffic, the exit ramp from the highway is lengthened. The slope of the new exit ramp is 0.12. At what distance before the elevated road will the exit ramp from the highway begin?

Multiple Choice

10. The slope of $\overline{PQ}$ is



A. $\frac{3}{4}$

B. $-\frac{3}{4}$

C. $\frac{4}{3}$

D. $-\frac{4}{3}$

11. Consider $\overline{AB}$ joining $A(6, -4)$ and $B(-4, -4)$ and $\overline{CD}$ joining $C(1, -9)$ and $D(1, 1)$. Which one of the following statements about these line segments is true?

A. $\overline{AB}$ and $\overline{CD}$ have the same slope and are equal in length

B. $\overline{AB}$ and $\overline{CD}$ have the same slope and are unequal in length

C. $\overline{CD}$ has a length of 10 units and a slope of zero.

D. $\overline{AB}$ and $\overline{CD}$ have the same midpoint and are equal in length.

12. P is a point in quadrant I, Q is a point in quadrant II, R is a point in quadrant III, and S is a point in quadrant IV.

Which one of the following statements must be true?

A. Line segment PQ has a positive slope.

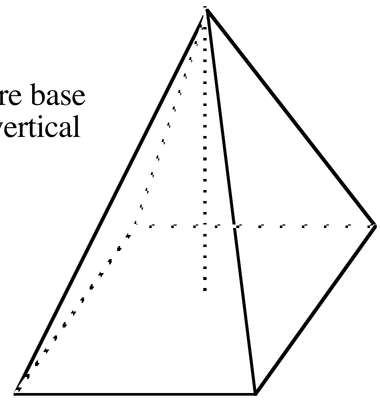
B. Line segment QR has a positive slope.

C. Line segment PR has a positive slope.

D. Line segment QS has a positive slope.

Use the following information to answer questions #13 and #14.

A pyramid has a square base of length 12 m and a vertical height of 15 m.



Numerical Response

13. A beetle starts to climb the pyramid starting from the midpoint of one of the sides. To the nearest tenth, the slope of the beetle's climb is _____.

(Record your answer in the numerical response box from left to right)

14. A fly starts to climb the pyramid along one of the edges. To the nearest tenth, the slope of the fly's climb is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

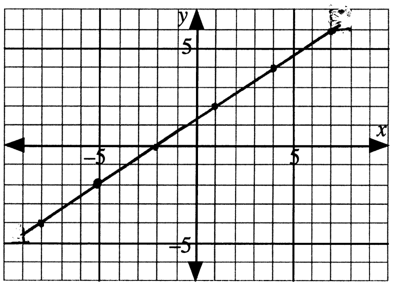
1.

Line Segment	Rise	Run	Slope = $\frac{\text{Rise}}{\text{Run}}$
AB	−7	4	$-7/4$
CD	3	5	$3/5$
EF	7	1	$7/1 = 7$
GH	0	7	$0/7 = 0$

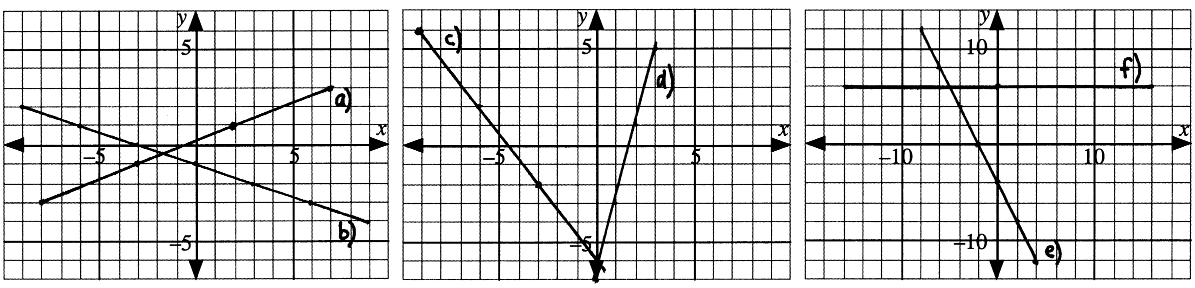
2. slope of line 1 = $\frac{1}{2}$, slope of line 2 = –2, slope of line 3 = –1

slope of line 4 = 4, slope of line 5 = $-\frac{3}{4}$, slope of line 6 = 2

3. Any three of (–8, –4), (–2, 0), (1, 2), (4, 4)



4.



a) (–8, –3), (–3, –1), (7, 3)

b) Any 3 of (–9, 2), (–6, 1), (–3, 0) (0, –1), (3, –2), (9, –4)

c) (–6, 2), (–3, –2), (0, –6)

d) (1, –3), (2, 1), (3, 5)

e) Many possible answers including (2, –8), (0, –4), (–2, 0)

f) Many possible answers including (1, 6), (2, 6), (3, 6)

5. Many possible answers including any two from:
a) (–3, –2), (–2, 0), (0, 4), (1, 6) **b)** (–3, 8), (–2, 5), (0, –1), (1, –4),
c) (2, 3), (5, 4), (–4, 1), (–7, 0) **d)** (–11, 6), (–6, 4), (4, 0), (9, –2)
e) (–3, 2), (–2, 2), (0, 2), (1, 2) **f)** (–1, 1), (–1, 0), (–1, –1), (–1, 3)

6. **a)** rise = 35 **b)** run = –40 **c)** rise = −18
d) run = 20 **e)** rise = 80 **f)** run = –6

7. **a)** 14.45 cm^2

8. 1 metre **9.** **a)** $\frac{2}{11}$ **b)** $66\frac{2}{3}$ metres.

10. D **11.** D **12.** C

13.

2	.	5	

14.

1	.	8	

Line Segments Lesson #5:
The Slope Formula

Warm-Up #1 *Review*

Complete the following statements.

a) Slope is the measure of the _______________ of a line.

b) Slope is the ratio of the vertical change (called the __________) over the horizontal change (called the __________) .

c) A line segment which rises from left to right has a ______________ slope.

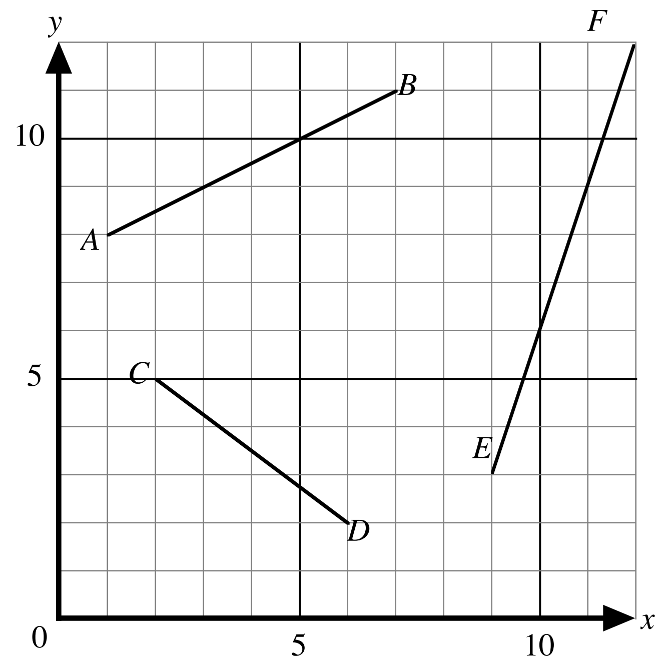
d) A line segment which falls from left to right has a ______________ slope.

e) A horizontal line segment has a slope of _____.

f) A vertical line segment has an ______________ slope.

g) The slopes of all line segments on a line are ________ .

Warm-Up #2 *Investigating the Slope Formula*



a) Calculate the slope of line segment AB using slope $= \dfrac{\text{rise}}{\text{run}}$.

b) List the coordinates of the endpoints of line segment AB.

$A($, $)$ $B($, $)$

c) How can the rise of line segment AB be determined using y_B and y_A?

d) How can the run of line segment AB be determined using x_A and x_B?

e) Use your results from c) and d) to write a formula which describes how the slope of line segment AB can be calculated using its endpoints.

f) Calculate the slope of line segment AB using the formula in e).

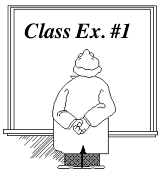
g) Calculate the slope of the line segments CD and EF using the method in a) and verify using the formula from e).

The Slope Formula

In mathematics the letter "*m*" is used to represent slope.

The slope of a line containing the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be calculated using

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{or} \qquad m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$$

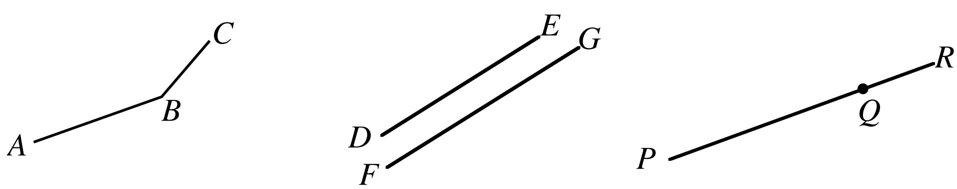


Find the slope of a line which passes through the points $G(-3, 8)$ and $H(7, -2)$.

$$m_{GH} = \frac{y_H - y_G}{x_H - x_G} =$$

Collinear Points

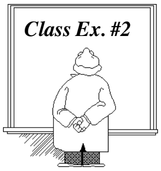
Two lines in a plane can either be



at an angle — parallel and distinct — parallel and form a straight line.

Points that lie on the same straight line are said to be **collinear**, i.e. P, Q, and R are collinear.

If three points P, Q, and R are collinear then $m_{PQ} = m_{QR} = m_{PR}$.
Proving that any two of these three slopes are equal is sufficient for the third to be equal and for the points to be collinear.



Consider points $A(5, -3)$, $B(2, 6)$ and $C(-7, 33)$.

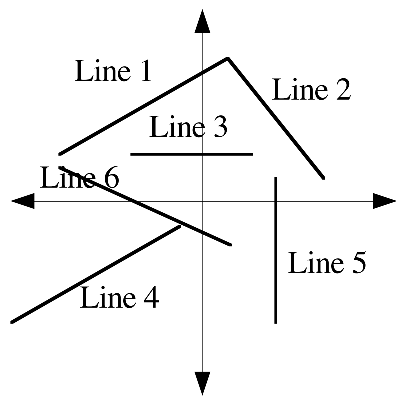
a) Prove that the points A, B, and C are collinear.

b) If the point $D(-4, y)$ lies on line segment AC find the value of y.

Complete Assignment Questions #1 - #12

Assignment

1. State whether the slope of each line is positive, negative, zero or undefined.



2. Use the slope formula to calculate the slope of the line segment with the given endpoints.

a) $A(12, -2)$ and $B(0, 3)$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} =$$

b) $C(-2, 3)$ and $D(2, -2)$

c) $P(-15, -2)$ and $O(0, 0)$

d) $S(36, -41)$ and $T(-20, -27)$

e) $U(-172, -56)$ and $V(-172, 32)$

f) $K(8, -41)$ and $L(397, -41)$

3. Use the slope formula to calculate the slope of the line passing through the given points.

a) $(3,-6)$ and $(8,4)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} =$$

b) $(-12,7)$ and $(0,-2)$

c) $(-3,-8)$ and $(1,5)$

d) $(21,1)$ and $(-4,-9)$

4. A coordinate grid is superimposed on a cross-section of a hill. The coordinates of the bottom and the top of a straight path up the hill are, respectively, $(3,2)$ and $(15,47)$, where the units are in metres.

a) Calculate the slope of the hill.

b) Calculate the coordinates of the midpoint of the path up the hill.

c) Calculate the length of the path to the nearest tenth of a metre.

5. Consider points $P(4,-9)$, $Q(-1,-7)$ and $R(-11,-3)$.

a) Use the slope formula to prove that the points P, Q and R are collinear.

b) Use the distance formula to prove that the points P, Q and R are collinear.

6. Consider points $A(8, -7)$, $B(-8, -3)$ and $C(-24, 1)$.

a) Prove that the points A, B and C are collinear.

b) Does the point $D(-2, -4)$ lie on line segment AC? Explain.

c) If the point $E(k, k)$ lies on line segment AC find the value of k.

7. A private jet has crashed in the desert at the point $P(-10, 17)$. A search party sets out in an all terrain vehicle from A_1 passing in a straight line through A_2. A helicopter sets out from B_1 and flies in a straight line through B_2.
If the search parties continue in these directions will either of them discover the crashed plane?

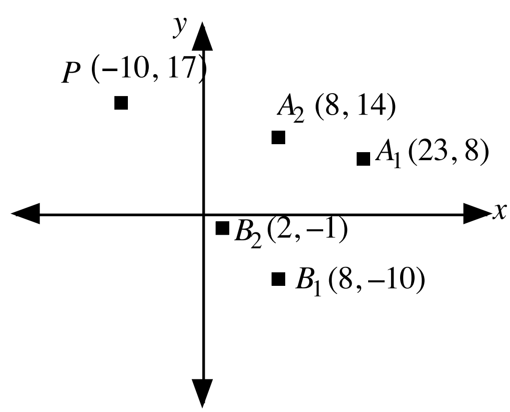
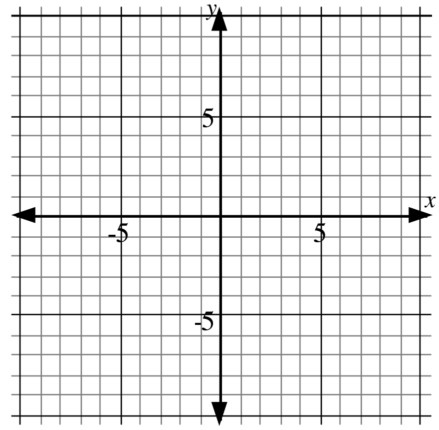


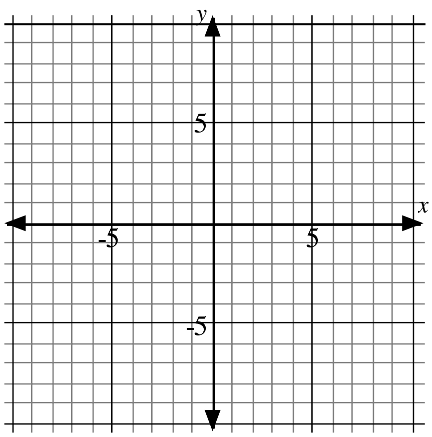
Diagram not to scale

8. The line segment joining each pair of points has the given slope. Determine each value of k and draw the line segment on the grid

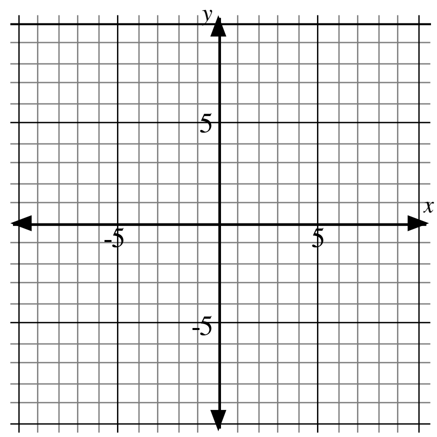
a) $S(4, 6)$ and $T(5, k)$ slope = 3



b) $L(k, -2)$ and $M(3, -7)$ slope = $-\frac{1}{2}$



c) $U(2, 5)$ and $V(k, 3)$ slope = $\frac{2}{7}$



Multiple Choice

9. The slope of the line segment joining $E(5,-1)$ and $F(3, 7)$, is

A. -3
B. -4
C. $-\frac{1}{3}$
D. $-\frac{1}{4}$

10. If the line segment joining $(2, 3)$ and $(8, k)$ has slope $-\frac{2}{3}$, then k =

A. -1
B. -3
C. -6
D. 7

11. One endpoint of a line segment is $(1, 6)$. The other endpoint is on the x-axis. If the slope of the line segment is -3, then the midpoint of the line segment is

A. $(4, 6)$
B. $(2, 3)$
C. $(-10, 3)$
D. $\left(\frac{1}{2}, \frac{15}{2}\right)$

12. $P(3, 6)$ $Q(8, -2)$ and $R(-6, 0)$ are the vertices of a triangle. The slope of the median PT, to the nearest tenth, is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. Line 1 - positive, Line 2 - negative, Line 3 - zero, Line 4 - positive, Line 5 - undefined, Line 6 - negative

2. **a)** $-\frac{5}{12}$ **b)** $-\frac{5}{4}$ **c)** $\frac{2}{15}$ **d)** $-\frac{1}{4}$ **e)** undefined **f)** 0

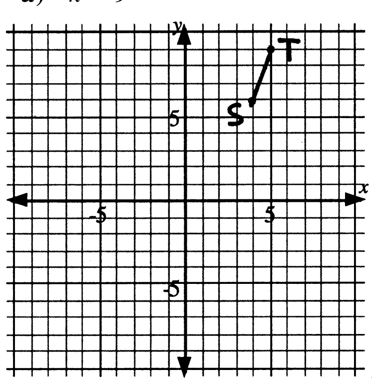
3. **a)** 2 **b)** $-\frac{3}{4}$ **c)** $\frac{13}{4}$ **d)** $\frac{2}{5}$ **4.** **a)** $\frac{15}{4}$ **b)** $\left(9, \frac{49}{2}\right)$ **c)** 46.6 m.

5. **a)** $m_{PQ} = -\frac{2}{5}$, $m_{QR} = -\frac{2}{5}$. Since $m_{PQ} = m_{QR}$, the points P, Q and R are collinear.
b) $PQ = \sqrt{29}$, $QR = 2\sqrt{29}$, $PR = 3\sqrt{29}$. Since $PQ + QR = PR$, the points P, Q and R are collinear.

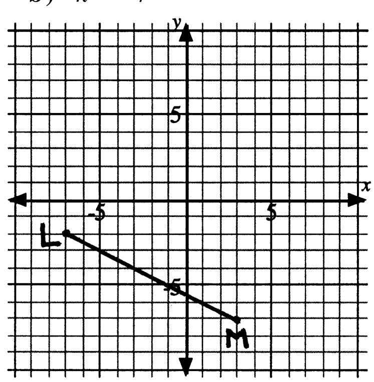
6. **a)** $m_{AB} = -\frac{1}{4}$, $m_{BC} = -\frac{1}{4}$. Since $m_{AB} = m_{BC}$, the points A, B and C are collinear.
b) $m_{AD} = -\frac{3}{10}$ Since $m_{AD} \neq m_{AB}$, the point D does not lie on line segment AC. **c)** $k = -4$

7. $m_{A_1A_2} = -\frac{2}{5}$, $m_{A_2P} = -\frac{1}{6}$. Since $m_{A_1A_2} \neq m_{A_2P}$, the search party in the all terrain vehicle will not discover the plane.
$m_{B_1B_2} = -\frac{3}{2}$, $m_{B_2P} = -\frac{3}{2}$. Since $m_{B_1B_2} = m_{B_2P}$, the search party in the helicopter will discover the plane.

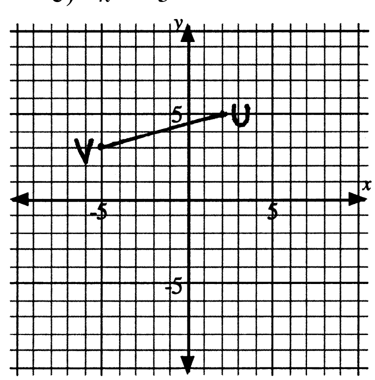
8. **a)** $k = 9$



b) $k = -7$



c) $k = -5$



9. B **10.** A **11.** B **12.**

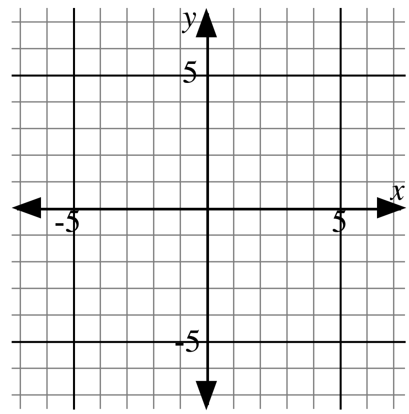
3	.	5	

Line Segments Lesson #6:
Parallel and Perpendicular Line Segments

Warm-Up #1 *Review: Transformations*

In earlier mathematics courses we studied transformations - translations, reflections, rotations and dilatations. In order to investigate parallel and perpendicular line segments we will review translations and rotations.

On the grid show the image of the point $A(2, 5)$ after the following transformations. In each case write the coordinates of the image.



a) translation 3 units right and 2 units up.

$A(2, 5) \rightarrow \quad B(\quad , \quad)$

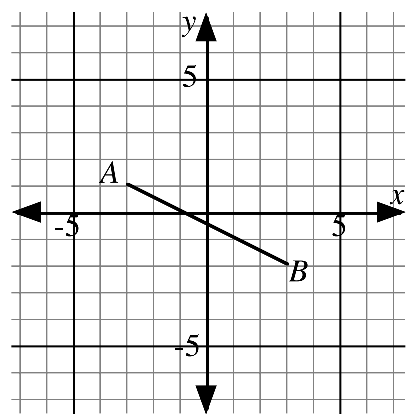
b) 90° clockwise rotation about the origin.

$A(2, 5) \rightarrow \quad C(\quad , \quad)$

c) 90° counterclockwise rotation about the origin.

$A(2, 5) \rightarrow \quad D(\quad , \quad)$

Warm-Up #2 *Investigating Parallel Line Segments*



a) On the grid show the image of line segment AB after the following transformations.

i) translation 4 units right and 1 unit down to form line segment CD

ii) translation 3 units left and 6 units up to form line segment EF

iii) translation 3 units down to form line segment GH

b) Calculate the slope of each of the line segments.

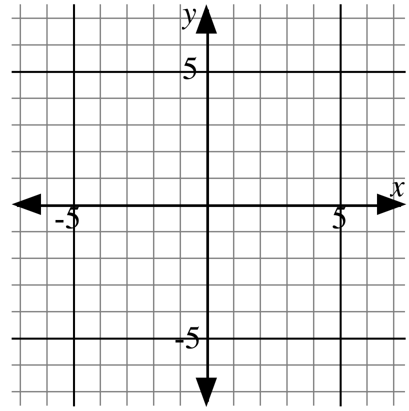
c) The four line segments are parallel. Make a conjecture about the slopes of parallel line segments.

Warm-Up #3 *Investigating Perpendicular Line Segments*

a) i) On the grid plot the point $A(5, 2)$ and draw the line joining the point to the origin, O.

ii) Rotate the line through an angle of 90° clockwise about O and show the image on the grid.

iii) Find the slopes of the two perpendicular lines and multiply them together.



b) Repeat part a) for the point $B(-6, 1)$.

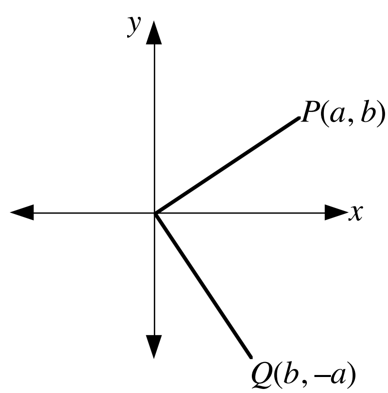
c) Make a conjecture about the slopes of perpendicular line segments.

d) Complete the following to prove the conjecture in c).
Under a rotation of 90° clockwise about O, $P(a, b) \to Q(b, -a)$.

$$m_{OP} = \frac{y_P - y_O}{x_P - x_O} = \frac{b - 0}{a - 0} = \frac{b}{a}$$

$$m_{OQ} = \frac{y_Q - y_O}{x_Q - x_O} = \frac{-a - 0}{b - 0} = -\frac{a}{b}$$

$$m_{OP} \times m_{OQ} =$$



Parallel Line Segments and Perpendicular Line Segments

Consider two line segments with slopes m_1 and m_2.

- The line segments are **parallel** if they have the same slope, i.e. $m_1 = m_2$.

- The line segments are **perpendicular** if the product of the slopes is -1,
  i.e. $m_1 \times m_2 = -1$ or $m_1 m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

  Note that for perpendicular line segments each slope is the negative reciprocal of the other, and neither of the slopes can equal zero.

Class Ex. #1

If P is the point $(4, 7)$ and Q is the point $(6, -2)$, find the slope of a line segment

a) parallel to line segment PQ **b)** perpendicular to line segment PQ

Class Ex. #2

Triangle LMN has coordinates $L(-4, 2)$, $M(-2, 7)$ and $N(1, 0)$. Use slopes to show that the triangle is right-angled at L.

Class Ex. #3

Two line segments have slopes of $-\dfrac{3}{4}$ and $\dfrac{k}{5}$ respectively. Find the value of k if the line segments are **a)** parallel **b)** perpendicular

Line Segment Summary

- distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- midpoint formula: $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Slope is the measure of the steepness of a line.
- $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$
- A line which **rises** from left to right has a **positive** slope.
- A line which **falls** from left to right has a **negative** slope.
- The slope of a **horizontal line** is **zero**.
- The slope of a **vertical line** is **undefined**.

Complete Assignment Questions #1 - #17

Assignment

1. The slopes of some line segments are given.

$m_{AB} = 6 \quad m_{CD} = \frac{1}{6} \quad m_{EF} = -6 \quad m_{GH} = 6 \quad m_{IJ} = -6 \quad m_{KL} = \frac{1}{6}$

Which pairs of lines are parallel to each other?

2. The slopes of some line segments are given.

$m_{RS} = -2 \quad m_{UV} = \frac{1}{4} \quad m_{EF} = 0.5 \quad m_{ZT} = 2$

$m_{PQ} = -4 \quad m_{KL} = -\frac{1}{2} \quad m_{MN} = 4 \quad m_{XY} = -\frac{1}{4}$

Which pairs of lines are perpendicular to each other?

3. In each case the slopes of parallel line segments are given. Determine the value of the variable.

a) $4, \frac{k}{3}$ **b)** $-2, \frac{2}{n}$

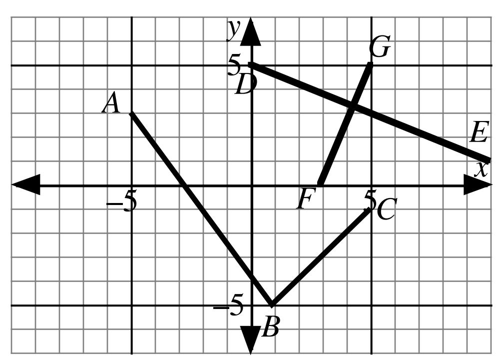
c) $\frac{5}{6}, 3m$ **d)** $\frac{3}{4}, -\frac{w}{6}$

4. In each case the slopes of perpendicular line segments are given. Determine the value of the variable.

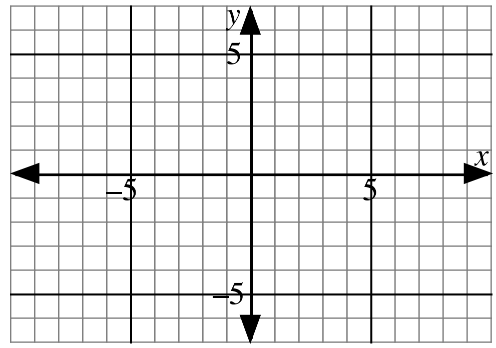
a) $\frac{1}{3}, 3h$ **b)** $4, \frac{8}{p}$

c) $-5, \frac{s}{2}$ **d)** $-\frac{3}{4}, -\frac{q}{6}$

5. The four line segments have endpoints with integer coordinates. In each case determine whether the two intersecting line segments are perpendicular.



6. $P(-4, 0)$ and $R(1, -3)$ are opposite vertices of a rhombus $PQRS$. Find the slope of diagonal QS.



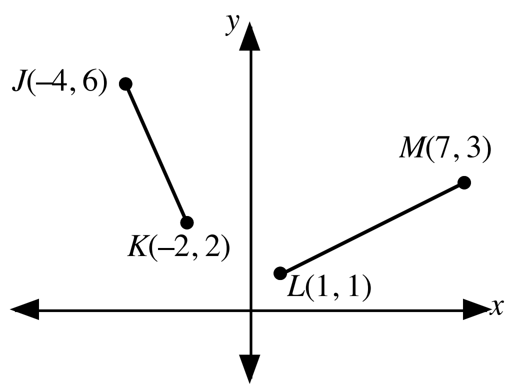
7. Given that A, B, and C are the points $(-3, 3)$, $(0, 6)$ and $(5, 1)$ respectively, prove that triangle ABC is right angled by using

a) the slope formula

b) the distance formula

8. **a)** Show that when line segments JK and ML are extended until they intersect, they will not meet at right angles.

b) If the y-coordinate of M is changed, the line segments, when extended, will meet at right angles. To what value should the y-coordinate of M be changed?



Multiple Choice

9. A and B are the points $(1, 2)$ and $(-2,3)$ respectively. A line perpendicular to AB will have slope

A. -3
B. $-\frac{1}{3}$
C. 3
D. $\frac{1}{3}$

Use the following information to answer questions #10-#12.

Line Segment AB	Line Segment PQ
$A(-2, 4)$ $B(2,-6)$	$P(7, 1)$ $Q(-3, -3)$

10. Which of the following statements is correct about the line segments?

A. The length of line segment AB is greater than the length of line segment PQ
B. The length of line segment AB is less than the length of line segment PQ
C. The length of line segment AB is equal to the length of line segment PQ
D. Not enough information is given to calculate the lengths of the line segments.

11. Which of the following statements is correct about the line segments?

A. The slope of line segment AB is positive and the slope of line segment PQ is negative.
B. The slope of line segment AB is positive and the slope of line segment PQ is positive.
C. Line segment AB is parallel to line segment PQ .
D. Line segment AB is perpendicular to line segment PQ .

12. Which of the following statements is correct about the line segments?

A. The midpoint of AB has an x-coordinate greater than the midpoint of PQ.
B. The midpoint of AB has an y-coordinate greater than the midpoint of PQ.
C. The midpoints of AB and PQ are the same point.
D. The line segment joining the midpoints is horizontal.

Use the following information to answer questions 13 – 16.

Quadrilateral $PQRS$ has vertices $P(-4,-6), Q(-6,-2), R(0,1)$, and $S(2,-3)$.

13. The slope and length of line segment PQ are respectively

A. -2 and $2\sqrt{5}$

B. $-\frac{1}{2}$ and $2\sqrt{5}$

C. -2 and $2\sqrt{29}$

D. $-\frac{1}{2}$ and $2\sqrt{29}$

14. The slope and length of line segment QR are respectively

A. 2 and $\sqrt{37}$

B. $\frac{1}{2}$ and $\sqrt{37}$

C. 2 and $3\sqrt{5}$

D. $\frac{1}{2}$ and $3\sqrt{5}$

15. Consider the following statements;

I. PQ is parallel to SR **II.** QR is perpendicular to SR

III. The lengths of PQ and SR are the same.

Which of the following is correct?

A. Statement **I** is false.

B. Statement **II** is false.

C. Statement **III** is false.

D. None of the above statements is false

16. Which of the following most completely describes Quadrilateral $PQRS$?

A. rectangle **B.** square **C.** parallelogram **D.** rhombus

Numerical Response

17. The line segment joining $U(-3, p)$ and $V(-6, 5)$ is perpendicular to the line segment joining $X(4, 2)$ and $Y(9, 0)$. The value of p, to the nearest tenth, is ______.

(Record your answer in the numerical response box from left to right)

Answer Key

1. AB and GH, CD and KL, EF and IJ.

2. RS and EF, UV and PQ, ZT and KL, MN and XY.

3. **a)** $k = 12$ **b)** $n = -1$ **c)** $m = \frac{5}{18}$ **d)** $w = -\frac{9}{2}$

4. **a)** $h = -1$ **b)** $p = -32$ **c)** $s = \frac{2}{5}$ **d)** $q = -8$

5. AB and BC are not perpendicular. DE and FG are perpendicular.

6. $m_{QS} = \frac{5}{3}$

7. **a)** $m_{AB} = 1, m_{BC} = -1$ Since the product of the slopes $= -1$, AB and BC are perpendicular. Triangle ABC is right angled at B.

b) $AB = \sqrt{18}$, $BC = \sqrt{50}$, $AC = \sqrt{68}$. $AC^2 = 68$. $AB^2 + BC^2 = 68$.
$AC^2 = AB^2 + BC^2$ so the Pythagorean theorem is satisfied and the triangle is right angled at B.

8. **a)** $M_{JK} = -2$, $M_L = \frac{1}{3}$. The product of the slopes does not equal -1. **b)** $y_M = 4$

9. C **10.** C **11.** D **12.** D **13.** A **14.** D **15.** D **16.** A **17.**

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Linear Functions and Equations Lesson #1: The Equation of a Line in Slope y-intercept Form → y = mx + b

Review

We have already learned that a **function** is a special type of relation.
Each element of the domain is related to only one element of the range.
In each ordered pair, for every value of the first coordinate there is only one value for the second coordinate. The graph of a function passes the vertical line test.

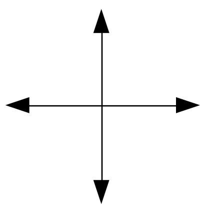
Consider the function $f(x) = 3x + 1$. This defines a formula for the function.
For example $f(2) = 3(2) + 1 = 7$ is the **value** of the function at $x = 2$.
$y = 3x + 1$ is the **equation of the graph** of the function.

Warm-Up #1

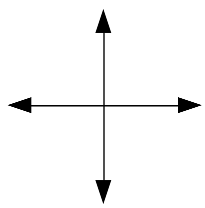
Investigating the Graphs of Linear and Non-Linear Functions

a) The equations of the graphs of some functions are given. In each case use a graphing calculator to sketch the graph of the function and make a rough sketch of the graph on the grid provided. Do not list any x- or y-intercepts.

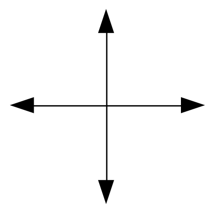
$y = 3x + 1$



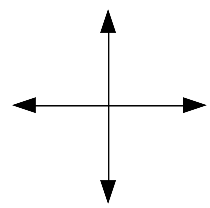
$y = -2x + 3$



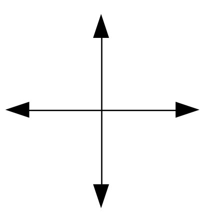
$y = -x^3 + 1$



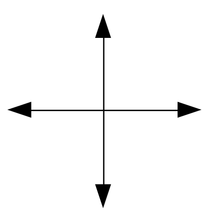
$y = x^2 - 3$



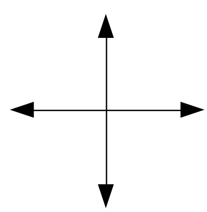
$y = \dfrac{1}{x}$



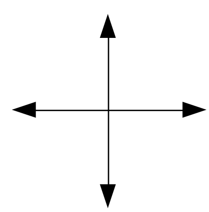
$y = \dfrac{1}{2}x - 2$



$y = 4x$



$y = 2^x$



b) List the equations of the graphs in the appropriate row.

LINEAR

NON-LINEAR

c) Compare the lists and write a rule which can be used to determine, from the equation, whether the graph is a straight line or not.

Linear Equation

A **linear equation** is an equation of the form $y = mx + b$, where $m, b \in \Re$.
The graph of a linear equation is a straight line.

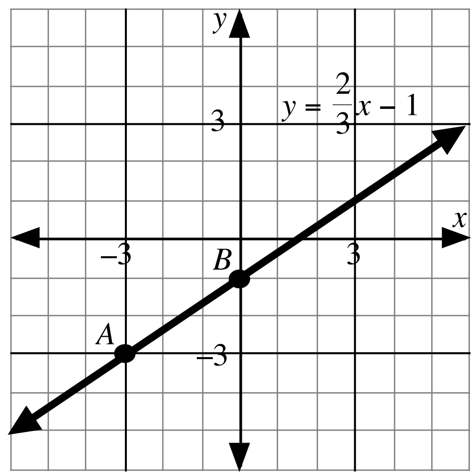
Linear Function

A **linear function** is a function of the form $f(x) = mx + b$, where $m, b \in \Re$.
The graph of a linear function is a straight line.

The emphasis of the next few lessons will be on linear equations.

Warm-Up #2

Investigating m and b in the equation $y = mx + b$.



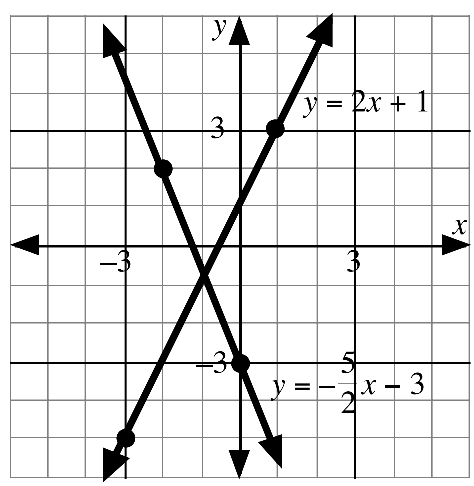
Jenine used a graphing calculator to sketch the graph of the linear equation $y = \frac{2}{3}x - 1$.
Her sketch is shown on the grid.

a) Use the sketch and points A and B to find the slope and y-intercept of the graph of $y = \frac{2}{3}x - 1$.

b) Compare the values found in a) with the coefficient of x and the constant term in the equation $y = \frac{2}{3}x - 1$.

c) Jenine sketched the graphs of two more linear equations. Use the grid to determine the slope and y-intercept of each graph

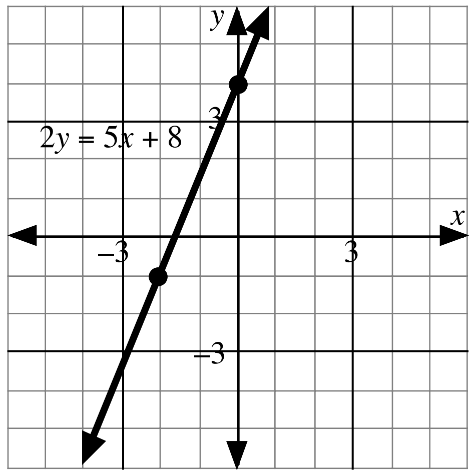
<u>equation</u>	<u>slope</u>	<u>y-intercept</u>
$y = 2x + 1$		
$y = -\frac{5}{2}x - 3$		



d) Make a conjecture about the slope and y-intercept of the graph of the linear equation $y = mx + b$.

Warm-Up #3

Hashib used a graphing calculator to graph the linear equation $2y = 5x + 8$.
The graph is shown on the grid.



a) Use the sketch to determine the slope and y-intercept of the graph of $2y = 5x + 8$.

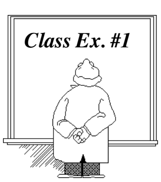
b) Explain why in this case the slope is not 5 (the coefficient of x) and the y-intercept is not 8 (the constant term).

Slope y-intercept Form of the Equation of a Line → y = mx + b

The graph of an equation in the form $\boldsymbol{y = mx + b}$ (or a function in the form $f(x) = mx + b$) is a straight line with **slope** $\boldsymbol{m}$ and ***y*-intercept** $\boldsymbol{b}$.

The equation $y = mx + b$ is known as the **slope *y*-intercept form** of the equation of a line.

The graph of an equation in this form can be drawn without making a table of values.

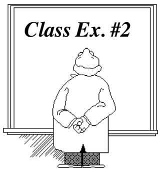


Determine the slope and y-intercept of the graph of each linear equation.

a) $y = 3x + 2$ **b)** $y = 7 - \frac{2}{3}x$ **c)** $6y = 8x + 1$

Graphing an Equation of the Form y = mx + b

In this section we will look at two ways of sketching the graph of a linear equation without using a graphing calculator or a table of values.



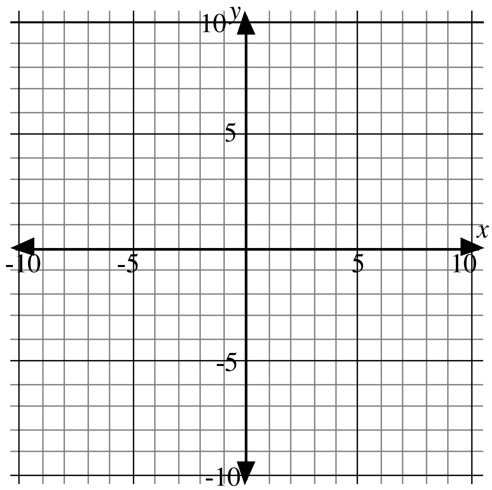
Consider the equation $y = 2x - 5$

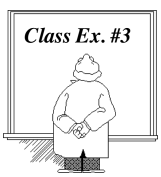
a) State the slope and y-intercept.

b) Mark the y-intercept on the grid.

c) Use the x-intercept and the formula slope $= \dfrac{\text{rise}}{\text{run}}$ to mark three other points on the grid. Join the points together and extend the line.

d) Verify the graph using a graphing calculator.





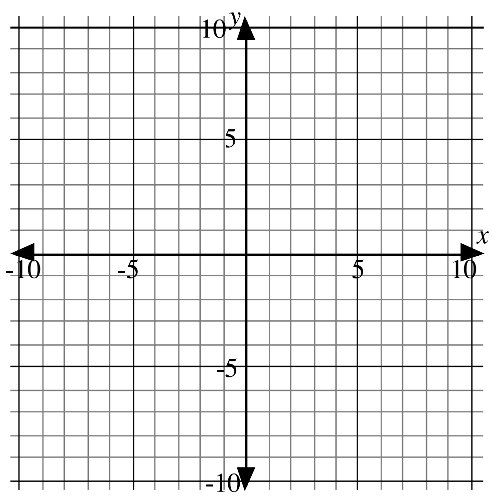
Consider the equation $y = \dfrac{2}{3}x - 6$

a) State the y-intercept.

b) Determine the x-intercept algebraically

c) Mark the x- and y-intercepts on the grid. Join the points together and extend the line.

d) Verify the graph and the intercepts using a graphing calculator.



Complete Assignment Questions #1 - #14

Assignment

1. Consider the following list of equations of the graphs of functions.

a) $y = 6x + 1$ **b)** $y = x^2$ **c)** $y = 3x^4 + 5$ **d)** $y = -\frac{1}{4}x - 8$

e) $y = 1 - x$ **f)** $y = \frac{2}{1 - x}$ **g)** $y = 4x$ **h)** $y = 4^x$

Without sketching the graphs, place the letters a) - h) in the appropriate row below.

LINEAR

NON-LINEAR

2. State the slope and y-intercept of the graph of each linear equation.

a) $y = 7x - 2$ **b)** $y = \frac{4}{3}x + 3$ **c)** $y = 6 - \frac{1}{6}x$ **d)** $4y = 6x + 8$ **e)** $y = ax + b$

3. Write the equation of each line with the given slope and y-intercept.

a) slope = 4, y-intercept = -9

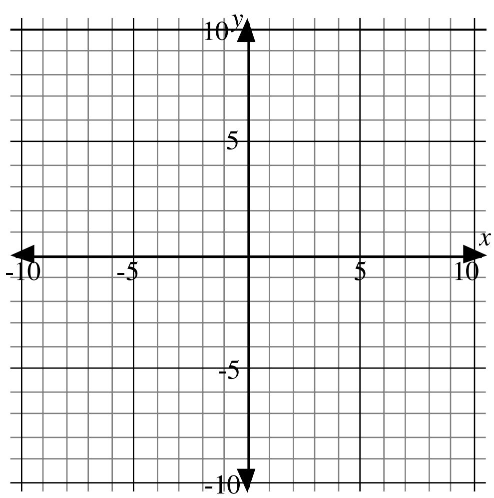
b) slope = $\frac{1}{5}$, y-intercept = $\frac{1}{2}$

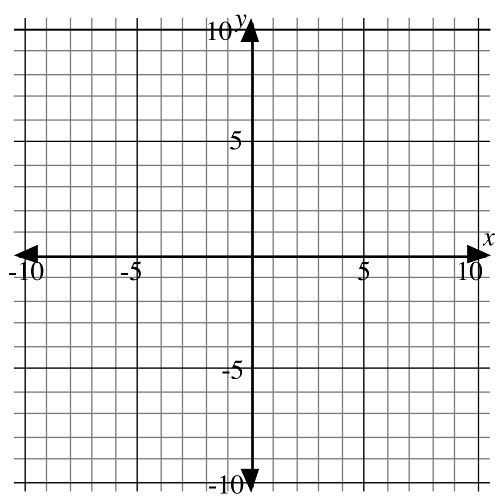
c) slope = -3, y-intercept = 0

d) slope = m, y-intercept = b

4. For each line state the slope and the y-intercept and graph the equation without using a graphing calculator.

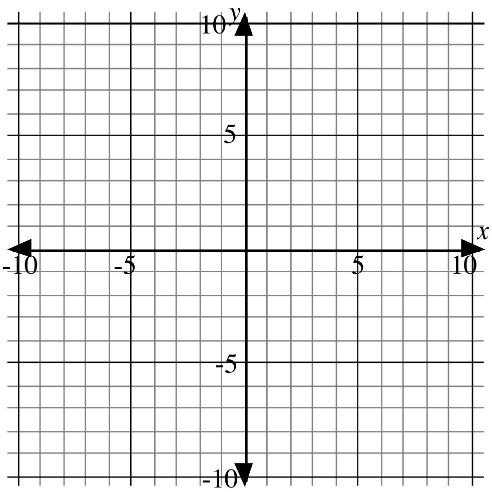
a) $y = \frac{1}{4}x + 2$ **b)** $y = -x - 1$ **c)** $y = -\frac{4}{3}x$ **d)** $y = 5$

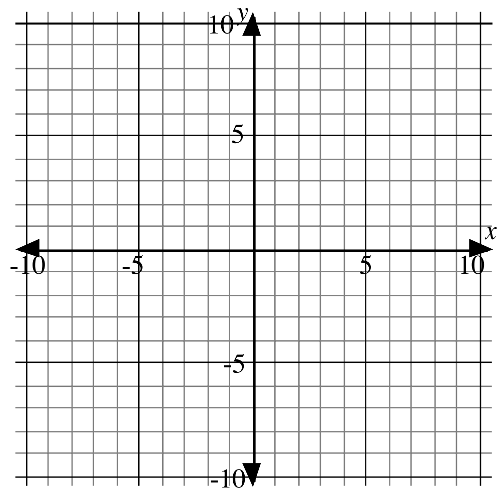




5. For each line state the y-intercept, determine the x-intercept algebraically, and graph the equation without using a graphing calculator.

a) $y = 2x + 6$ **b)** $y = -x - 4$ **c)** $y = \frac{6}{7}x - 6$ **d)** $y = -\frac{1}{2}x + 1$





6. Explain why the linear equation $y = 5x$ can be graphed using the method in question 4 but not by the method in question 5.

7. Consider the graph of the function with equation $y = x$.

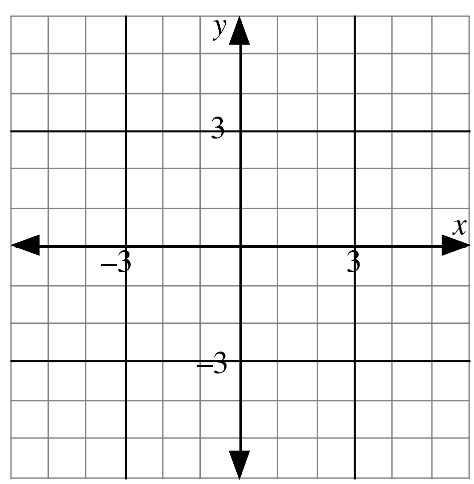
a) State the values of m and b.

b) Determine the x- and y-intercepts.

c) Sketch the graph on the grid provided without using a graphing calculator.

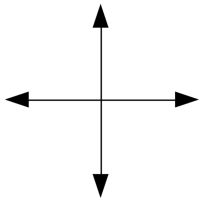
d) Determine the domain and range of the function.

e) Use a graphing calculator to graph the line $y = -x$ and sketch the graph on the grid.



8. Use a graphing calculator to sketch the graph of each of the following linear equations. Complete the table giving the x-intercept to the nearest hundredth.

i) $y = 7x - 8$

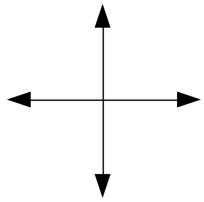


slope	
x-intercept	
y-intercept	

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin=
 Xmax=
 Xscl=
 Ymin=
 Ymax=
 Yscl=
 Xres=1
```

ii) $y = -\frac{31}{2}x - 25$

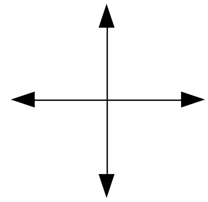


slope	
x-intercept	
y-intercept	

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin=
 Xmax=
 Xscl=
 Ymin=
 Ymax=
 Yscl=
 Xres=1
```

iii) $y = 75 - \frac{5}{3}x$



slope	
x-intercept	
y-intercept	

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin=
 Xmax=
 Xscl=
 Ymin=
 Ymax=
 Yscl=
 Xres=1
```

Multiple Choice

9. Which of the following does not represent the equation of a straight line?

A. $y = 3x$
B. $y = 11 - 3x$
C. $y = \frac{x}{3}$
D. all of the above represent the equation of a straight line

10. Which of the following statements is false for the line $y = -\frac{1}{2}x + 1$?

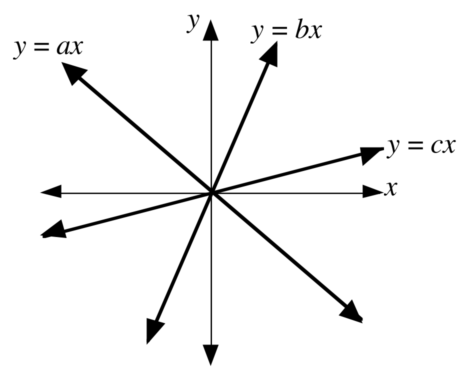
A. The graph of the line falls from left to right
B. The x-intercept is 2.
C. The graph passes through the point $(8, -3)$
D. The line is perpendicular to the line $y = -2x + 4$

11. Which of the following statements is true for the line $2y = \frac{1}{4}x + 6$?

A. The x-intercept is 24.
B. The y-intercept is 6.
C. The slope is $\frac{1}{8}$.
D. The graph passes through the point (–4, 5).

12. The lines $y = ax$, $y = bx$, and $y = cx$ are shown. Which of the following statements is true?

A. $a < b < c$
B. $a < c < b$
C. $c < a < b$
D. $c < b < a$



Use the following information to answer questions 13 and 14.

Consider the line with equation $y = 3x + 5$. The line intersects the x-axis at P and the y-axis at Q. Triangle POQ, where O is the origin, is formed.

Numerical Response

13. To the nearest tenth, the area, in square units, of triangle POQ is _____.

(Record your answer in the numerical response box from left to right)

14. To the nearest tenth, the perimeter of triangle POQ is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. LINEAR a), d), e), g). NON-LINEAR b), c), f), h).

2. **a.** slope = 7, y-int = –2 **b)** slope = $\frac{4}{3}$, y-int = 3

c) slope = $-\frac{1}{6}$, y-int = 6 **d)** slope = $\frac{3}{2}$, y-int = 2 **e)** slope = a, y-int = b

3. **a)** $y = 4x - 9$ **b)** $y = \frac{1}{5}x + \frac{1}{2}$ **c)** $y = -3x$ **d)** $y = mx + b$

4. **a)** slope = $\frac{1}{4}$, y-int = 2 **b)** slope = –1, y-int = –1

c) slope = $-\frac{4}{3}$, y-int = 0 **d)** slope = 0, y-int = 5

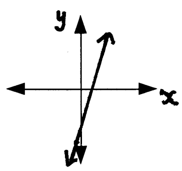
5. **a)** y-int = 6, x-int = –3 **b)** y-int = –4, x-int = –4 **c)** y-int = –6, x-int = 7 **d)** y-int = 1, x-int = 2

6. The method in #4 needs a point and a slope. We have point (0, 0) and slope = 5. The method in #5 needs two points to be joined. Since the x- and y-intercepts are the same point, the line cannot be drawn.

7. **a)** $m = 1$, $b = 0$ **b)** x-int = 0 and y-int = 0. **c)** **d)** $D = x \in \Re$ $R = y \in \Re$

8.

i) $y = 7x - 8$

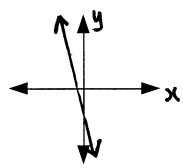


slope	7
x-intercept	1.14
y-intercept	–8

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin= -4
 Xmax= 4
 Xscl= 1
 Ymin= -10
 Ymax= 5
 Yscl= 2
 Xres=1
```

ii) $y = -\frac{31}{2}x - 25$

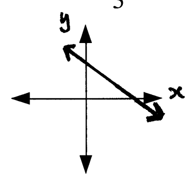


slope	$-\frac{31}{2}$
x-intercept	–1.61
y-intercept	–25

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin= -4
 Xmax= 4
 Xscl= 1
 Ymin= -40
 Ymax= 10
 Yscl= 10
 Xres=1
```

iii) $y = 75 - \frac{5}{3}x$



slope	$-5/3$
x-intercept	45
y-intercept	75

Graphing Window which includes both intercepts:

```
WINDOW
 Xmin= -10
 Xmax= 60
 Xscl= 10
 Ymin= -20
 Ymax= 100
 Yscl= 10
 Xres=1
```

9. D **10.** D **11.** C **12.** B

13.

4	.	2	

14.

1	1	.	9

Linear Functions and Equations Lesson #2:
Writing Equations using $y = mx + b$

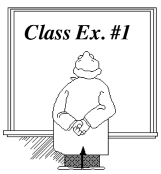
Review

We have learned that the graph of an equation in the form $y = mx + b$ is a straight line with slope m and y-intercept b.

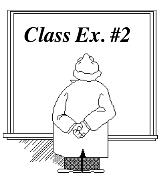
Using the Form $y = mx + b$ to Write the Equation of a Line

The form $y = mx + b$ can be used to determine the equation of a line when the following information is given:

- the slope of the line
- the y-intercept of the line.



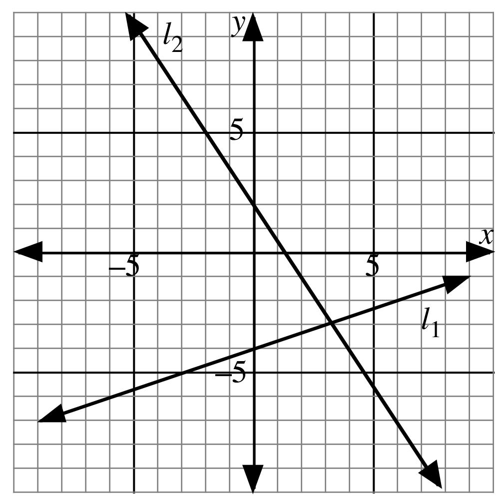
Write the equation of a line passing through the point $(0, 2)$ with slope $\frac{5}{2}$.

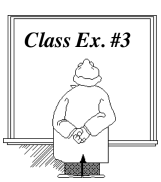


For each line shown on the grid

a) State the slope and y-intercept.

b) Determine the equation of the line.

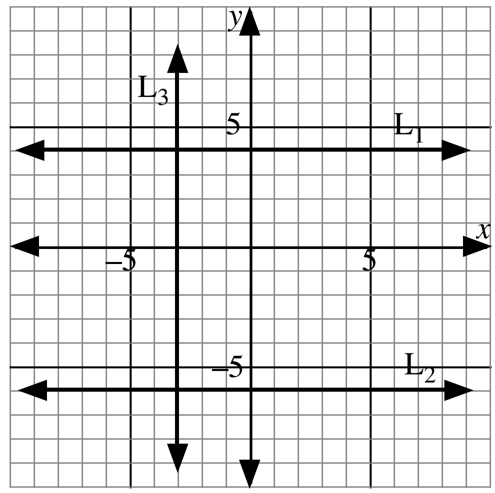




Determine the equation of a line which is perpendicular to the line $y = \frac{1}{3}x + 4$ and has the same y-intercept as $y = 6x - 7$.

Horizontal and Vertical Lines

a) State the slope and y-intercept of the horizontal line L_1 shown on the grid.



b) Use the form $y = mx + b$ to determine the equation of the horizontal line L_1 .

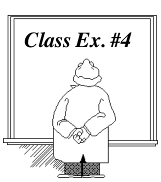
c) Predict the equation of the horizontal line L_2 . Use a graphing calculator to verify.

d) State the slope and y-intercept of the vertical line L_3 shown on the grid.

e) Why can we not use the form $y = mx + b$ to determine the equation of the vertical line L_3?

f) Predict the equation of the vertical line L_3 . Why can we not use a graphing calculator to verify?

The equation $y = k$ represents a horizontal line through $(0, k)$
The equation $x = k$ represents a vertical line through $(k, 0)$



Determine the equation of the line through the point $(-2, 8)$ and

a) parallel to the y-axis.

b) parallel to the x-axis.

Complete Assignment Questions #1 - #13

Assignment

1. Write the equation of each line:

a) with slope 4 and y-intercept –6

b) with a y-intercept of 3 and a slope of $-\frac{4}{3}$

c) passing through the origin with a slope of $-\frac{3}{5}$

d) with y-intercept –5 and parallel to $y = x$

e) with a y-intercept of –9 and perpendicular to $y = -\frac{2}{3}x + 7$

f) with the same y-intercept as $y = x + 2$ and parallel to $y = \frac{1}{4}x - 6$

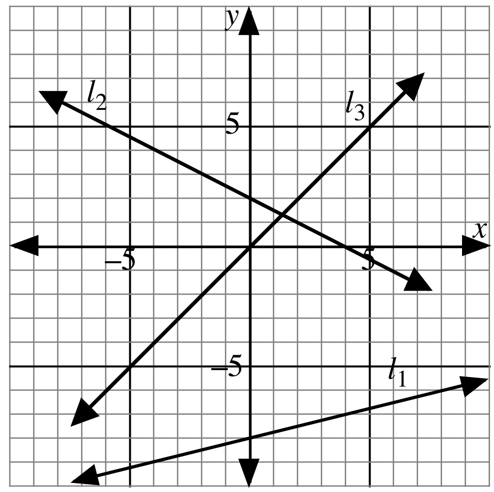
g) through the point $(0, 1)$ and perpendicular to $y = 4x - 2$

h) through the point $(0, 4)$ and parallel to $y = \frac{1}{10}x + 24$

i) with the same y-intercept as $y = 2x - 3$ and perpendicular to $y = \frac{7}{3}x - 2$

j) with the same y-intercept as $y = ax + b$ and perpendicular to $y = cx + d$

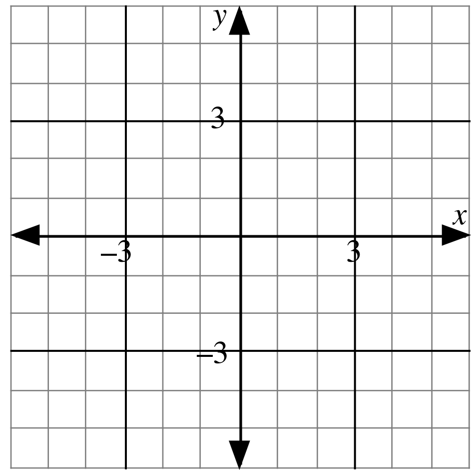
2. Determine the equation of each line shown on the grid.



3. Determine the equation of the line which passes through the point $(0, 6)$ and is parallel to the line which passes through $(1, 3)$ and $(4, -6)$

4. Determine the equation of the line which passes through the point $(0, -1)$ and is perpendicular to the line which passes through $(7, -2)$ and $(12, -3)$

5. Consider the graph of the function with equation $y = 2$.



a) State the values of m and b.

b) Sketch the graph on the grid provided.

c) State the x- and y-intercepts of the graph.

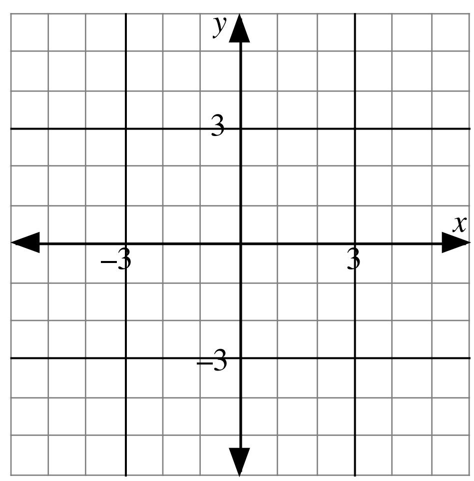
d) Determine the domain and range of the function.

e) On the same grid draw the line with equation $y = 2x - 4$ without using a graphing calculator.

f) State the coordinates of the point of intersection of the two lines.

g) On the grid draw the line with equation $y = -5$.

6. Consider the graph of the relation with equation $x = -4$.



a) Sketch the graph on the grid provided.

b) State the x- and y-intercepts of the graph.

c) Explain why the relation is not a function.

d) Explain why the $\boxed{\text{Y =}}$ editor key cannot be used to graph $x = -4$.

e) Determine the domain and range of $x = -4$.

f) On the grid draw the line with equation $x = 2$.

7. Write the equation of each line:

a) parallel to the x-axis through $(3, -9)$

b) parallel to the y-axis through $(3, -9)$

c) perpendicular to the x-axis through $(1, 4)$

d) perpendicular to the y-axis through $(1, 4)$

e) the x-axis

f) the y-axis

Multiple Choice

8. A line is parallel to the y-axis and passes through the point $(2, -7)$. The equation of the line is

A. $x = 2$
B. $x = -7$
C. $y = 2$
D. $y = -7$

9. A line is parallel to the x-axis and passes through the point $(-6, 10)$. The equation of the line is

A. $x = 10$
B. $x = -6$
C. $y = 10$
D. $y = -6$

10. The line through the origin, perpendicular to the line with equation $y = \frac{2}{3}x$, has equation

A. $y = \frac{2}{3}x$

B. $y = \frac{3}{2}x$

C. $y = -\frac{2}{3}x$

D. $y = -\frac{3}{2}x$

11. The point $(2, -1)$ lies on a line with slope 3. The y- intercept of the line is

A. -7
B. -5
C. 5
D. 7

Numerical Response

12. Consider the line which is perpendicular to the line $y = \frac{1}{3}x + 4$ and has the same y-intercept as $y = 6x - 7$. If the equation of this line is written in the form $y = mx + b$ then the exact value of $m - b$ is _____.

(Record your answer in the numerical response box from left to right)

13. Two perpendicular lines intersect on the y-axis. One line has equation $y = 4x + 6$. If the equation of the other line is $y = mx + b$ then the exact value of $m + b$ is _____.

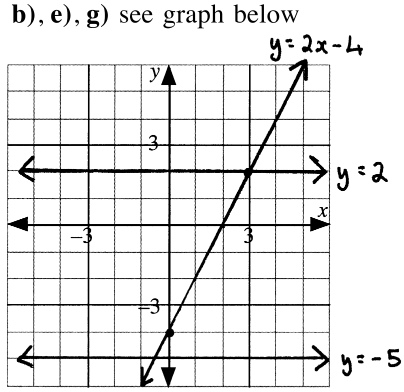
(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $y = 4x - 6$ **b)** $y = -\frac{4}{3}x + 3$ **c)** $y = -\frac{3}{5}x$ **d)** $y = x - 5$ **e)** $y = \frac{3}{2}x - 9$
f) $y = \frac{1}{4}x + 2$ **g)** $y = -\frac{1}{4}x + 1$ **h)** $y = \frac{1}{10}x + 4$ **i)** $y = -\frac{3}{7}x - 3$ **j)** $y = -\frac{1}{c}x + b$

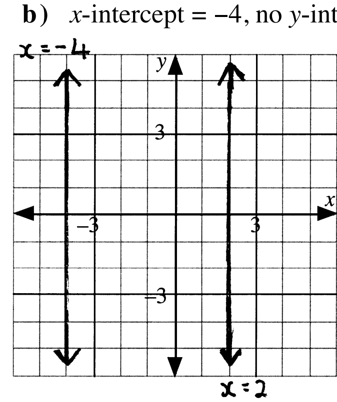
2. l_1: $y = \frac{1}{4}x - 8$ l_2: $y = -\frac{1}{2}x + 2$ l_3: $y = x$ **3.** $y = -3x + 6$ **4.** $y = 5x - 1$

5. a) $m = 0, b = 2$
b), e), g) see graph below



c) no x-intercept, y-intercept = 2
d) domain $x \in R$, range $y = 2$ **f)** (3, 2)

6. a), f) see graph below
b) x-intercept = −4, no y-intercept



c) When the input = −4, there are multiple values for the output. The graph of the relation does not pass the vertical line test.
d) The equation $x = -4$ cannot be written in the form y= . . . .
e) domain $x = -4$, range $y \in R$

7. a) $y = -9$ **b)** $x = 3$ **c)** $x = 1$
d) $y = 4$ **e)** $y = 0$ **f)** $x = 0$

8. A **9.** C **10.** D **11.** A

12.

4			

13.

5	.	7	5

Linear Functions and Equations Lesson #3:
The General Form Equation $Ax + By + C = 0$

In lesson #2 we used the form $y = mx + b$ to determine the equation of a line when given the slope of the line and the y-intercept.

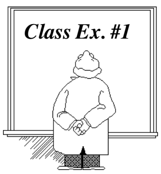
A linear equation written in the form $y = mx + b$ has slope m and y-intercept b.

General Form of the Equation of a Line → $Ax + By + C = 0$

The **general form** of the equation of a line is an equation where all the terms are collected to the left side of the equation.
The general form equation is $Ax + By + C = 0$, where A, B and C are expressed as integers if possible.

The general form of the equation of a line allows us to write equations for oblique lines, horizontal lines and vertical lines.

In some texts the form $Ax + By + C = 0$ is referred to as **standard form**.

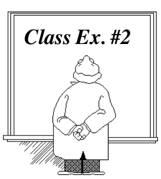


Convert the following equations from slope y-intercept form, $y = mx + b$, to general form $Ax + By + C = 0$, where A, B and C are integers.

a) $y = 5x - 8$ **b)** $y = \frac{2}{3}x + 7$ **c)** $y = -\frac{1}{4}x + \frac{3}{5}$

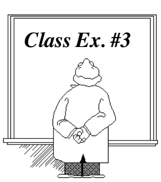
Determining the Slope and y-intercept from $Ax + By + C = 0$

Given the equation of a line in general form $Ax + By + C = 0$ the slope and y-intercept can be found by first converting the equation into slope y-intercept form $y = mx + b$.

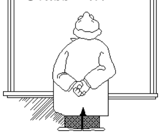


Determine the slope and y-intercept of the graph of the following lines.

a) $2x - 5y + 25 = 0$ **b)** $6x + 2y - 15 = 0$



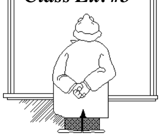
Given that the lines $3x - 4y + 8 = 0$ and $5x - ky - 6 = 0$ have the same y-intercept, determine the value of k.



Class Ex. #4

Which of the following lines is/are perpendicular to the line $4x - 2y + 9 = 0$?

i) $6x + 3y - 1 = 0$ **ii)** $x + 2y - 12 = 0$ **iii)** $5x + 10y = 0$



Class Ex. #5

Consider the graph of the linear equation $x + y + 6 = 0$.

a) Determine the x- and y- intercepts of the graph and calculate the slope of the line segment joining the intercepts.

b) Repeat part a) for the equation $x + y + 20 = 0$.

c) Repeat part a) for the equation $x + y + k = 0$ where $k \in I$. Comment on your findings.

d) Consider the statement "If the x- and y- intercepts on the graph of a linear equation are equal then the line has a slope of -1"

Is the statement always true, sometimes true, or never true? Explain.

Complete Assignment Questions #1 - #17

Assignment

1. Convert the following equations from slope y-intercept form, $y = mx + b$, to general form $Ax + By + C = 0$, where A, B and C are integers.

a) $y = 7x - 3$ **b)** $y = -2x + 9$ **c)** $y = mx + b$

d) $y = -\frac{3}{4}x + 5$ **e)** $y = \frac{2}{3}x + \frac{1}{6}$ **f)** $y = \frac{5}{3}x - \frac{1}{4}$

2. Determine the slope and y-intercept of the graph of the following lines.

a) $x + y - 11 = 0$ **b)** $3x - 2y + 30 = 0$ **c)** $8x - 3y - 3 = 0$

d) $3x + 6y - 7 = 0$ **e)** $8y = 4x + 32$ **f)** $4x + 3y = 12$

3. Determine the slope, y-intercept and x-intercept of the graph of the following lines.

a) $2x + y - 6 = 0$ **b)** $5x - 2y + 20 = 0$ **c)** $4x - 5y - 3 = 0$

4. Consider the lines $x - 2y + 1 = 0$ and $4x + ky - 8 = 0$.

a) If the lines have the same slope determine the value of k.

b) If the lines have the same y-intercept determine the value of k.

5. Consider the lines $3x - 5y - 15 = 0$ and $ax + 2y - 6 = 0$.

a) If the lines have the same slope determine the value of a.

b) If the lines have the same x-intercept, determine the value of a.

Multiple Choice

6. The slope of the line with equation $6x + 5y - 1 = 0$ is

A. $-\frac{6}{5}$

B. $-\frac{5}{6}$

C. $\frac{6}{5}$

D. $\frac{1}{5}$

7. Which line has a y-intercept of 1?

A. $x + 5y + 1 = 0$
B. $x + 3y + 3 = 0$
C. $x - 2y + 2 = 0$
D. $2y = 3x + 1$

8. The slope of a line perpendicular to the line $x + 3y + 8 = 0$ is

A. -8

B. $-\frac{1}{3}$

C. $\frac{1}{3}$

D. 3

9. The line $2y + 3x + 6 = 0$ intersects the y-axis at P.
The slope of the line joining P to $Q(6, -2)$ is

A. $-\frac{5}{6}$

B. $\frac{1}{6}$

C. $-\frac{1}{6}$

D. $-\frac{2}{3}$

10. The slope of the line $Ax + By + C = 0$ is

A. A

B. $\frac{A}{B}$

C. $-\frac{A}{B}$

D. $-\frac{C}{B}$

11. The lines with equations $ay = 4x + 9$ and $y = 5x - 7$ are perpendicular. The value of a is

A. $\frac{4}{5}$
B. $-\frac{4}{5}$
C. $-\frac{5}{4}$
D. -20

12. Which of the following is/are true for all lines of the form $kx + 4y - 8 = 0$ where $k \in R$?
The lines have: 1) the same slope 2) the same y-intercept 3) the same x-intercept

A. 1), 2) and 3)
B. 1) only
C. 2) only
D. 3) only

13. Which of the following is/are true for the lines $2x - y + 10 = 0$ and $x - 2y + 5 = 0$?

They: 1) are perpendicular 2) have the same y-intercept 3) have the same x-intercept

A. 1) and 3) only
B. 1) only
C. 3) only
D. some other combination of 1), 2) and 3).

14. Which of the following is/are true for the line $2y = 5x + 12$?

1) a slope of 5 2) parallel to $10x - 4y + 13 = 0$ 3) passes through $(-2, 1)$

A. 1) and 2) only
B. 1) and 3) only
C. 2) and 3) only
D. some other combination of 1), 2) and 3).

15. The equations of four straight lines are

1) $7x - y = 0$ 2) $7x + y - 6 = 0$ 3) $x - 7y + 4 = 0$ 4) $x + 7y - 2 = 0$

Which pairs of lines are perpendicular?

A. 1) and 2) only
B. 1) and 4) only
C. both 1) and 4) and 2) and 3)
D. both 1) and 2) and 2) and 3)

16. Which of the following lines is/are perpendicular to the line $9x + y + 2 = 0$?

i) $9y + x = 2$ **ii**) $9y - x = 2$ **iii**) $y = 9x + 2$ **iv**) $9y = x - 2$

A. i) and iii) only
B. ii) only
C. iv) only
D. some other combination of i), ii), iii) and iv)

Numerical Response

17. Given that the line joining the points $(2, 3)$ and $(8, -q)$, where $q \in W$, is perpendicular to the line $3x - 2y - 5 = 0$, then the value of q is ________ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $7x - y - 3 = 0$ **b)** $2x + y - 9 = 0$ **c)** $mx - y + b = 0$
d) $3x + 4y - 20 = 0$ **e)** $4x - 6y + 1 = 0$ **f)** $20x - 12y - 3 = 0$

2. **a)** slope $= -1$, y–int $= 11$ **b)** slope $= \frac{3}{2}$, y–int $= 15$ **c)** slope $= \frac{8}{3}$, y–int $= -1$
d) slope $= -\frac{1}{2}$, y–int $= \frac{7}{6}$ **e)** slope $= \frac{1}{2}$, y–int $= 4$ **f)** slope $= -\frac{4}{3}$, y–int $= 4$

3. **a)** slope $= -2$, y–int $= 6$, x–int $= 3$
b) slope $= \frac{5}{2}$, y–int $= 10$, x–int $= -4$
c) slope $= \frac{4}{5}$, y–int $= -\frac{3}{5}$, x–int $= \frac{3}{4}$

4. **a)** -8 **b)** 16 **5.** **a)** $-\frac{6}{5}$ **b)** $\frac{6}{5}$

6. A **7.** C **8.** D **9.** B

10. C **11.** D **12.** C **13.** C

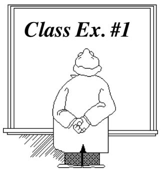
14. C **15.** C **16.** D **17.**

1			

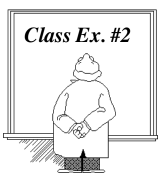
Linear Functions and Equations Lesson #4:
Writing Equations using $Ax + By + C = 0$

In lesson #1 we used the form $y = mx + b$ to determine the equation of a line when given the slope of the line and the y-intercept.

In this section we may need to calculate the slope and/or y-intercept before using $y = mx + b$.



Write the equation of a line which is perpendicular to $2x + 5y - 7 = 0$ and with the same y-intercept as $2x + y - 6 = 0$.
Answer in slope y-intercept form and in general form.



Given $P(8, -3)$ and $Q(-2, -8)$, determine the equation, in general form, of a line passing through the two points.

An alternative method for finding the equation of a line given two points on the line will be given in the next lesson.

Complete Assignment Questions #1 - #12

Assignment

1. Write the equation, in slope y-intercept form, of a line parallel to $2x - 3y + 9 = 0$ and with the same y-intercept as $22x - 3y - 18 = 0$.

2. Write the equation, in general form, of a line perpendicular to $3x - 2y + 5 = 0$ and with the same y-intercept as $3x - y + 18 = 0$.

3. Write the equation, in general form, of a line perpendicular to $x - 4y + 8 = 0$ and with the same x-intercept as $4x - 3y + 4 = 0$.

4. Find the equation of the line, in slope y-intercept form, which passes through the following points.

a) $(7, 5)$ and $(3, 1)$ **b)** $(4, -10)$ and $(3, -12)$

5. Write the equation, in general form, of a line passing through the given points.

a) $(-3, 4)$ and $(11, 11)$ **b)** $(10, -15)$ and $(-2, -12)$

6. A line, l, is perpendicular to a line which passes through the points $E(-4, 5)$ and $F(8, -1)$. l passes through the point $D(7, 9)$. Determine the equation of l in general form.

Multiple Choice

7. Which of the following is the equation of a line perpendicular to $5y + x + 6 = 0$?

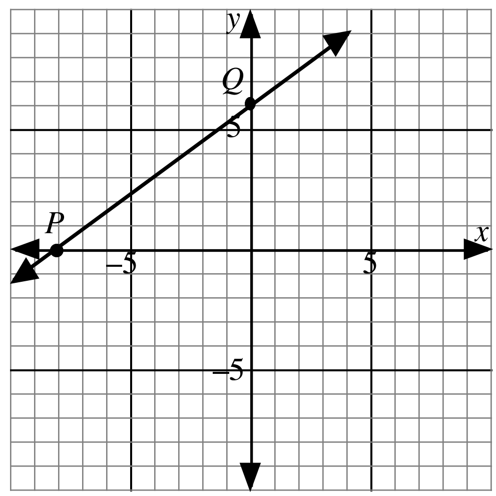
A. $y = 5x$
B. $y = x$
C. $y = \frac{1}{5}x$
D. $y = -\frac{1}{5}x$

8. The point of intersection of the line $9x - 3y + 9 = 0$ and the y-axis is

A. $(0, 9)$
B. $(0, 3)$
C. $(0, -1)$
D. $(0, -3)$

9. The equation of the line PQ in the diagram is

A. $3x + 4y + 24 = 0$
B. $3x + 4y + 32 = 0$
C. $3x - 4y + 24 = 0$
D. $3x - 4y + 32 = 0$



10. Line L has equation $5x - 3y + 21 = 0$. A is the point $(-6, -3)$, B is $(3, -2)$ and C is $(-3, 2)$. Which of these points lie on line L?

A. A only
B. A and B only
C. A and C only
D. B and C only

11. If the lines $ax + by + c = 0$ and $dx + ey + f = 0$ are parallel then

A. $ae - bd = 0$
B. $ae + bd = 0$
C. $ad - be = 0$
D. $ad + be = 0$

12. Match each equation on the left with the correct characteristic of the graph of the equation on the right. Each characteristic may be used once, more than once, or not at all.

i) $6x - 2y + 5 = 0$

ii) $2x - 5y = 0$

iii) $x + 3y + 6 = 0$

iv) $x - 4y + 10 = 0$

v) $2x - y - 5 = 0$

A. Slope $= -\frac{1}{3}$

B. y-intercept $= -\frac{5}{2}$

C. Passes through $(-10, -4)$

D. Slope $= 0$

E. y-intercept $= \frac{5}{2}$

F. Perpendicular to $y = \frac{5}{2}x - 3$

G. x-intercept $= \frac{5}{2}$

Answer Key

1. $y = \frac{2}{3}x - 6$

2. $2x + 3y - 54 = 0$

3. $4x + y + 4 = 0$

4. **a)** $y = x - 2$ **b)** $y = 2x - 18$

5. **a)** $x - 2y + 11 = 0$ **b)** $x + 4y + 50 = 0$

6. $2x - y - 5 = 0$

7. A

8. B

9. C

10. C

11. A

12. **i)** E **ii)** C **iii)** A **iv)** E **v)** G

Linear Functions and Equations Lesson #5:
Point-Slope Form → $y - y_1 = m(x - x_1)$

Review

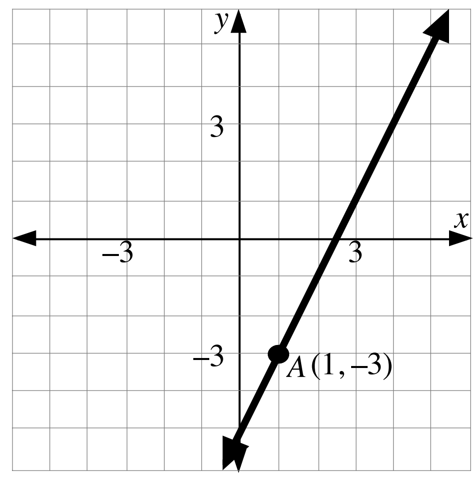
Complete:

a) The general form of an equation of a line is ___________________ .

b) The slope y-intercept form of the equation of a line is __________________ .

Equation of a Line Given the Slope of the Line and a Point on the Line

Consider the line with slope 2 passing through the point $A(1, -3)$. The line is shown on the grid.



Our objective is to determine the equation of the line. In other words, to find a relation between x and y which is satisfied by every point (x, y) on the line.

Let $P(x, y)$ be any point on the line except A.

Using the slope formula we have

$$\frac{y_P - y_A}{x_P - x_A} = m_{AP}$$

$$\frac{y - (-3)}{x - 1} = 2$$

Cross multiply and solve for y to determine the equation of the line in the form $y = mx + b$.

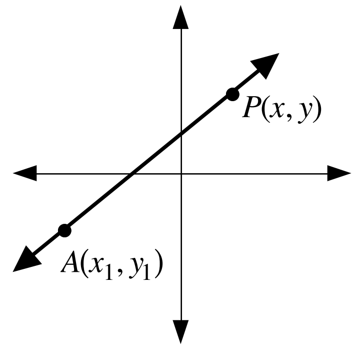
At this point in the exploration, the equation above is valid for all points on the line except A.

Note that the coordinates of A also satisfy the equation, so that it is the equation of all points on the line.

In the next section we will use the same procedure to develop a formula for the equation of any line given the slope of the line and the point on the line.

The Equation of the Line with slope m through the point (x_1, y_1)

Consider the line with slope m passing through the point with coordinates (x_1, y_1).
We will use the same procedure as above to show that the equation of the line can be expressed in the form $y - y_1 = m(x - x_1)$.



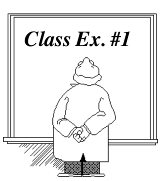
Let $P(x, y)$ be any point on the line distinct from A.

Using the slope formula we have

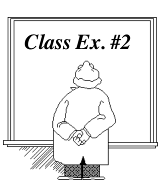
$$m_{AP} = \frac{y_P - y_A}{x_P - x_A} \qquad \text{so} \qquad m = \frac{y - \quad}{x - \quad}$$

Point-Slope Equation of a Line → $y - y_1 = m(x - x_1)$

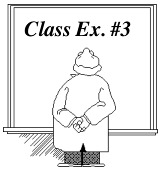
- The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) represents a point on the line.
- To determine the equation of a line in higher grade math courses the point-slope equation, $y - y_1 = m(x - x_1)$, is used more frequently than the slope-y-intercept equation, $y = mx + b$.
- Note that <u>the point-slope equation is used when we have the slope of a line and the coordinates of any point on the line</u>. When using this method to determine the equation of a line it is usual to give the final equation in general form, $Ax + By + C = 0$, or in slope-y-intercept form, $y = mx + b$.



Find the equation of a line passing through the point $(-3, 5)$ which has a slope of 4. Answer in slope y-intercept form.

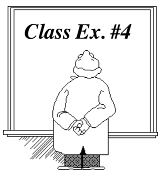


Determine the equation, in general form, of a line with slope $-\frac{2}{3}$ passing through the point $(2, -5)$.

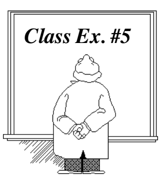


The point-slope equation of a line is $y + 2 = -\frac{3}{5}(x - 8)$. State the slope and the coordinates of the point which was used to write this equation.

We know that the point-slope equation is used when we have the slope of a line and the coordinates of any point on the line. In many cases, either the point or the slope of the line has to be determined from the information given before the equation can be used.



Determine the equation, in general form, of a line through the point $(5, 0)$ and perpendicular to the line with equation $3x - 5y + 17 = 0$.



Given $P(3, -1)$ and $Q(-2, -6)$, determine the equation, in general form, of a line passing through the two points.

Complete Assignment Questions #1 - #16

Assignment

In this assignment write the equation in general form unless otherwise indicated.

1. Find the equation, in slope y-intercept form, of the line through the given point and with the given slope.

a) $(2,4),\ \ 6$ **b)** $(2,-1),\ \ 2$ **c)** $(0,4),\ \ -2$

d) $(-6,2),\ \ \frac{1}{2}$ **e)** $(-7,-7),\ \ 1$ **f)** $(0,b),\ \ m$

2. Find the equation, in general form, of the line through the given point and with the given slope.

a) $(6,1),\ \ 3$ **b)** $(2,-5),\ \ \frac{1}{4}$ **c)** $(-4,2),\ \ -\frac{1}{3}$

d) $(-9,-2),\ \ \frac{2}{5}$ **e)** $(0,-8),\ \ -\frac{3}{4}$ **f)** $(0,0),\ \ \frac{4}{3}$

3. The point-slope equation of a line is given. State the slope and the coordinates of the point which was used to write the equation.

a) $y - 9 = -\frac{11}{3}(x + 3)$ **b)** $y + 3 = \frac{1}{2}x$ **c)** $y - 8 = -2(x - 6)$

d) $y = 3(x + 12)$ **e)** $y - 9 = -\frac{5}{3}x$ **f)** $y = \frac{1}{2}x$

4. Find the equation of the line through each pair of points.

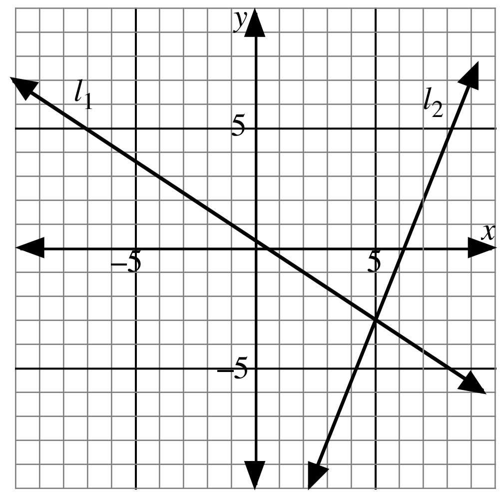
a) $(7, 5)$ and $(6, 1)$ **b)** $(3, -7)$ and $(-5, 9)$ **c)** $(-3, 4)$ and $(11, 25)$

d) $(10, -15)$ and $(-2, -12)$ **e)** $(4, -7)$ and $(3, -7)$ **f)** $(-5, -8)$ and $(-4, -10)$

5. Which of the lines in #4 are
i) parallel **ii)** perpendicular?

6. Two lines have been drawn on the grid. Each line passes through at least two points with integer coordinates.

Determine the equation of each line.



7. Write the equation of each line in general form.

a) with slope $\frac{2}{7}$ and an x-intercept of –6

b) with a y-intercept of $-\frac{8}{3}$ and a slope of 7

c) through the point (2, 0) and perpendicular to $3x - 5y + 19 = 0$

d) through the point (3, –6) and parallel to $5x + 3y + 9 = 0$

8. A child with a fixed amount of money can buy 2 bags of chips and 5 cans of pop or 3 bags of chips and 2 cans of pop. A linear relationship exists between the number of bags of chips, x, and the number of cans of pop, y, which can be bought.

a) Wrtie the coordinates of two points which lie on the graph of this linear relationship.

b) Determine the equation of the linear relationship.

Multiple Choice

9. The equation of the line through the point $(7, -4)$ perpendicular to the line with equation $5x - 4y + 13 = 0$ can be written in the form

A. $y + 4 = \frac{5}{4}(x - 7)$

B. $y - 4 = -\frac{4}{5}(x + 7)$

C. $y + 4 = -\frac{4}{5}(x - 7)$

D. $y + 4 = \frac{4}{5}(x - 7)$

10. The equation of the line passing through the point $(4, 2)$ with slope -3 is

A. $3x + y - 14 = 0$
B. $3x + y + 10 = 0$
C. $3x + y - 10 = 0$
D. $3x + y + 14 = 0$

11. A line passing through the point $(0, 3)$ is perpendicular to the line $x - 2y - 5 = 0$. The equation of the line is

A. $2x + y - 3 = 0$
B. $2x + y + 3 = 0$
C. $x - 2y + 6 = 0$
D. $2x - y + 3 = 0$

12. The line passing through the points $(-5, -2)$ and $(-2, -1)$ has equation

A. $x + y + 3 = 0$
B. $x + 3y + 5 = 0$
C. $x - 3y + 1 = 0$
D. $x - 3y - 1 = 0$

13. A line passing through the point $(0, 3)$ is parallel to the line $x - 2y - 5 = 0$. The equation of the line is

A. $2x + y - 3 = 0$
B. $2x + y + 3 = 0$
C. $x - 2y + 6 = 0$
D. $2x - y + 3 = 0$

14. The image of $y = 2x + 7$ after a counterclockwise rotation of 90° about the origin is

A. $y = -\frac{1}{2}x + \frac{7}{2}$
B. $y = \frac{1}{2}x - \frac{7}{2}$
C. $y = -\frac{1}{2}x - \frac{7}{2}$
D. $y = -2x - 7$

Numerical Response

15. The line through the points $(-3, 4)$ and $(-1, -2)$ has equation $y + ax + b = 0$ where a and b are integers. The value of $a + b$, is _____.

(Record your answer in the numerical response box from left to right)

16. The equation of the line $2x + 5y - 5 = 0$ after a reflection in the y-axis is of the form $Ax + By + 5 = 0$. The value of $A - B$ is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $y = 6x - 8$ **b)** $y = 2x - 5$ **c)** $y = -2x + 4$ **d)** $y = \frac{1}{2}x + 5$ **e)** $y = x$ **f)** $y = mx + b$

2. a) $3x - y - 17 = 0$ **b)** $x - 4y - 22 = 0$ **c)** $x + 3y - 2 = 0$
d) $2x - 5y + 8 = 0$ **e)** $3x + 4y + 32 = 0$ **f)** $4x - 3y = 0$

3. a) $m = -\frac{11}{3}$, $P(-3, 9)$ **b)** $m = \frac{1}{2}$, $P(0, -3)$ **c)** $m = -2$, $P(6, 8)$ **d)** $m = 3$, $P(-12, 0)$
e) $m = -\frac{5}{3}$, $P(0, 9)$ **f)** $m = \frac{1}{2}$, $P(0, 0)$

4. a) $4x - y - 23 = 0$ **b)** $2x + y + 1 = 0$ **c)** $3x - 2y + 17 = 0$
d) $x + 4y + 50 = 0$ **e)** $y + 7 = 0$ **f)** $2x + y + 18 = 0$

5. i) b and f **ii)** a and d

6. $l_1 \Rightarrow 2x + 3y - 1 = 0$ or $y = -\frac{2}{3}x + \frac{1}{3}$ $l_2 \Rightarrow 5x - 2y - 31 = 0$ or $y = \frac{5}{2}x - \frac{31}{2}$

7. a) $2x - 7y + 12 = 0$ **b)** $21x - 3y - 8 = 0$ **c)** $5x + 3y - 10 = 0$ **d)** $5x + 3y + 3 = 0$

8. a) (2, 5) and (3, 2) **b)** $3x + y - 11 = 0$

9. C **10.** A **11.** A **12.** D **13.** C **14.** C

15.

8			

16.

7			

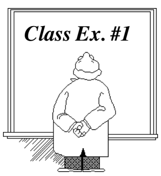
Linear Functions Lesson #6:
Further Practice with Linear Equations

Writing Linear Equations

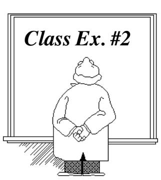
Linear equations can be written in different forms:

$Ax + By + C = 0$	$\rightarrow$	General form of a linear equation.
$y = mx + b$	$\rightarrow$	Slope y-intercept form of a linear equation.
$y - y_1 = m(x - x_1)$	$\rightarrow$	Point-slope form of a linear equation.

In this lesson we will discuss more complex problems involving linear equations including as enrichment some extended response problems requiring the intersection point of two lines. An algebraic technique for finding the intersection point (system of equations) is covered in a higher level math course. The method used here is the INTERSECT feature on the graphing calculator.



Find the equation of the line perpendicular to the line $5x - 7y - 10 = 0$ and with the same x-intercept as $x - 2y - 12 = 0$.



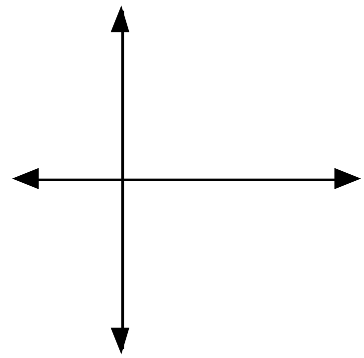
Consider the points $P(-7, -2)$, $Q(2, 1)$, $R(-2, -7)$ and $S(8, 3)$

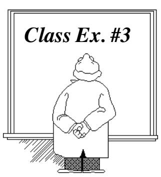
a) Show that the equation of the line, L_1, through S perpendicular to PQ is $y = -3x + 27$.

b) Determine the equation of the line, L_2, through R parallel to PQ.

Enrichment

c) Use a graphing calculator to determine the intersection point of lines L_1 and L_2.
State a suitable window and show both lines on the grid.





$ABCD$ is a square. A is $(5, 1)$, $B(9, 4)$ and $C(6, 8)$.

a) Find the coordinates of D.

b) Determine the equations of the diagonals of the square and graphically determine their point of intersection. (***Enrichment***)

c) Explain how we can use the equations of the diagonals to prove that the diagonals intersect at right angles.

Complete Assignment Questions #1 - #10

Assignment

1. Write the equation of each line in general form :

a) perpendicular to $y = x$ and with the same x-intercept as $y = 2x + 10$

b) parallel to $2x - 3y + 7 = 0$ and with the same y-intercept as $5x - 3y - 12 = 0$.

c) perpendicular to $6x - 2y + 5 = 0$ and with the same y-intercept as $x - y + 8 = 0$.

d) with the same x-intercept as $9x - 2y + 18 = 0$ and through the point $(4, -5)$.

2. Line l contains the point $A(7, 9)$ and is parallel to a line which contains the points $B(-4, 5)$ and $C(8, -1)$. Determine the equation of line l in the form $y = mx + b$.

3. A, B, C and D are the points $(8, -2)$, $(6, 14)$, $(14, 22)$ and $(11, 28)$ respectively.

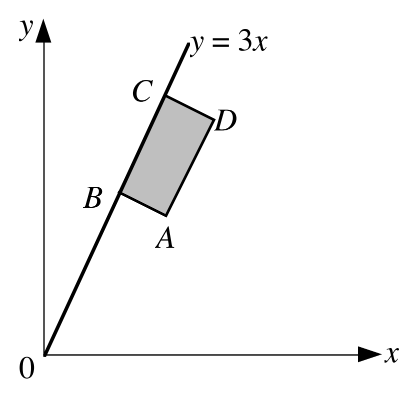
a) Determine the equation of AC.

b) Determine the equation of the line through D parallel to AB.

c) The line through D parallel to AB meets AC at K. Use a graphing calculator to determine the coordinates of K. (***Enrichment***)
State a suitable window and show both lines on a grid.

4. A Cartesian plane is placed on a plan of a farm. The farmhouse is at the origin and $ABCD$ represents a rectangular field of wheat. A farm road, with equation $y = 3x$, runs from the farmhouse along one side of the field.

a) If the point A has coordinates $(2, 4)$ determine the equation of AD.



b) Determine the equation of AB.

5. P, Q and R are the points $(-3, 5)$, $(2, -5)$ and $(4, 1)$ respectively.

a) Determine the equation of the line PQ.

b) Find S, the point of intersection of PQ and the y-axis.

c) Show that RS is perpendicular to PQ.

d) Show that M, the midpoint of QR, is equidistant from Q, R and S.

6. Consider the line with equation $4x + y - 34 = 0$.

a) Determine the equation of the line through the origin perpendicular to the given line.

b) Find the point of intersection of these two lines and hence calculate the shortest distance from the origin to the given line.

7. *RSTV* is a parallelogram. R is $(1, -1)$, $S(5, 1)$ and $T(3, 3)$.

a) Find the coordinates of V.

b) Determine the equations of the diagonals RT and SV of the rectangle.

c) Use b) to determine if the diagonals of the parallelogram intersect at right angles.

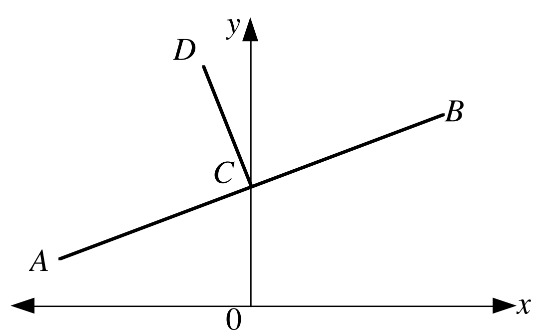
d) Graphically determine the point of intersection of the diagonals. (***Enrichment***)

Multiple Choice

8. The equation of AB is $x - 2y + 4 = 0$. AB cuts the y-axis at C. CD is perpendicular to AB.

The equation of CD is

A. $x + 2y - 2 = 0$
B. $2x + y - 2 = 0$
C. $2x - y + 2 = 0$
D. $2x + y - 4 = 0$



9. Consider triangle PQR in which side PQ has slope $\frac{1}{3}$ and R has coordinates $(-4, 7)$. The equation of the altitude from R to PQ (the line drawn from R to PQ, perpendicular to PQ), is

A. $x + 3y = 25$
B. $3x + y = 19$
C. $3x + y = -5$
D. $3x + y = -19$

Extension Question

10. The perpendicular bisector of a line segment AB is a line at right angles to AB, passing through the midpoint of AB.

a) If A is the point $(-6, 8)$ and B is $(-10, -4)$ determine the equation, in general form, of the perpendicular bisector of AB.

b) If C is $(6, -4)$ determine the point of intersection, P, of the perpendicular bisectors of AC and BC.

c) Verify that the coordinates of P satisfy the equation in a).
What does this mean about the three perpendicular bisectors of triangle ABC?

Answer Key

1. **a)** $x + y + 5 = 0$ **b)** $2x - 3y - 12 = 0$ **c)** $x + 3y - 24 = 0$ **d)** $5x + 6y + 10 = 0$

2. $y = -\frac{1}{2}x + \frac{25}{2}$

3. **a)** $y = 4x - 34$ or $4x - y - 34 = 0$
b) $y = -8x + 116$ or $8x + y - 116 = 0$
c) x:[–2, 25, 5] y:[–2, 25, 5] $K\left(\frac{25}{2}, 16\right)$

4. **a)** $y = 3x - 2$ or $3x - y - 2 = 0$ **b)** $y = -\frac{1}{3}x + \frac{14}{3}$ or $x + 3y - 14 = 0$

5. **a)** $y = -2x - 1$ or $2x + y + 1 = 0$ **b)** $S(0, -1)$
c) $m_{RS} = \frac{1}{2}$, $m_{PQ} = -2$, product of slopes = –1
d) $M(3, -2)$, $MQ = MR = MS = \sqrt{10}$

6. **a)** $y = \frac{1}{4}x$ or $x - 4y = 0$ **b)** $2\sqrt{17}$ units

7. **a)** $V(-1, 1)$ **b)** $RT \Rightarrow y = 2x - 3$ or $2x - y - 3 = 0$
$SV \Rightarrow y = 1$ or $y - 1 = 0$
c) No, $m_{RT} = 2$ and $m_{SV} = 0$ and the products of the slope do not equal –1.
d) (2, 1)

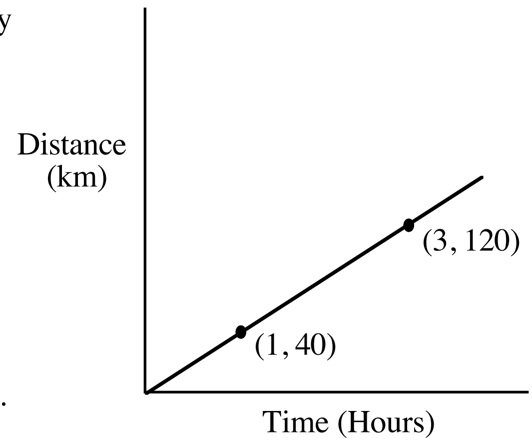
8. B **9.** C

10. **a)** $x + 3y + 2 = 0$ **b)** (–2, 0) **c)** They intersect at the same point.

Linear Functions and Equations Lesson #7:
Slope as a Rate of Change

Warm-Up #1

The graph shown represents the distance travelled by a car as a function of time.



a) Complete the following to calculate the slope of the line.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} =$$

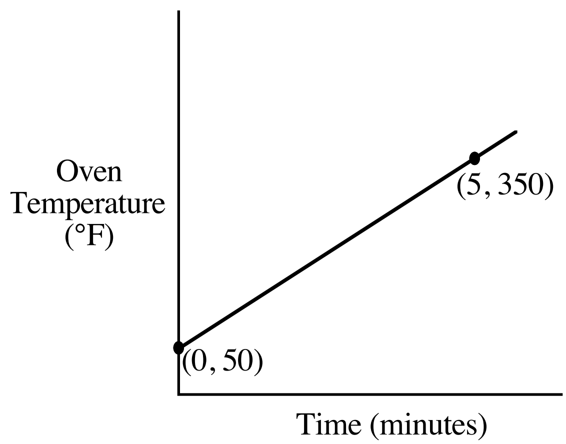
b) The slope represents a **rate of change** - a change in distance divided by a change in time.

Complete the following statement by choosing the correct alternative and filling in the blank.

The distance is (increasing/decreasing) at the rate of ______ km per hour

Warm-Up #2

The graph shown represents the temperature of an oven as a function of time.



a) Calculate the slope of the line.

b) The slope represents a **rate of change** - a change in temperature divided by a change in time.

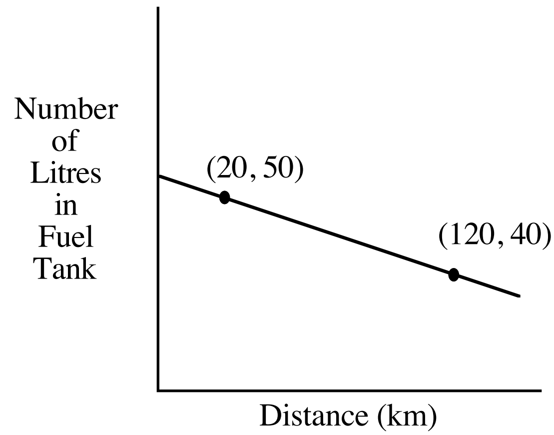
What units are used to represent this rate of change?

c) Complete the following statement.

The temperature is (increasing/decreasing) at the rate of ______ ____ per ______ .

Warm-Up #3

The graph shown represents the amount of fuel in the gas tank of a car as a function of the distance travelled.



a) Calculate the slope of the line.

b) The slope represents a **rate of change** - a change in the amount of fuel in the fuel tank divided by a change in distance.

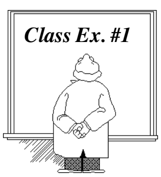
What units are used to represent this rate of change?

c) Complete the following statements.

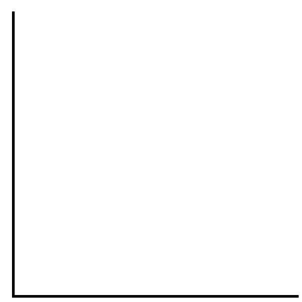
The amount of fuel in the tank is (increasing/decreasing).

The rate of change of fuel in the fuel tank is ______ ____ per ______ .

The amount of fuel is decreasing at the rate of ______ ____ per ______ .



Tyrone is paid a base salary per week plus commission for selling electrical appliances. Last week, his sales totalled \$3500 and he earned \$620. This week, he earned \$680 for sales of \$4250.



a) On a grid plot ordered pairs to represent this information.

b) Calculate the slope of the line segment joining the ordered pairs.

c) Explain what the slope of the graph represents.

d) Determine, as a percent, the rate of commission which Tyrone is paid.

e) Calculate his weekly base salary.

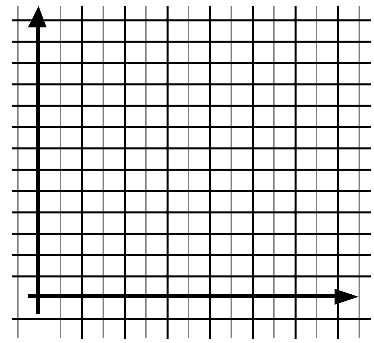
f) How does the answer to e) relate to the graph?

g) Two weeks ago Tyrone earned \$486. Calculate his sales for that week.

Average Speed

John is taking part in a long distance car race. After 3 hours he had travelled 270 km and after 6 hours he had travelled 630 km.

a) On a grid plot ordered pairs to represent this information.



b) Calculate the slope of the line segment joining the ordered pairs.

c) The slope of the line segment represents a rate of change **- a change in distance divided by a change in time**.
This rate is the **average speed** between the two points.
State the average speed of the car from 3 h to 6 h using appropriate units.

d) On the grid plot the point (0, 0) and determine the average speed of the car during the first 3 h of the race.

e) By looking at the grid and without doing any calculations how can we tell that the average speed during the first 3 h was less than the average speed during the next 3 h?

On a graph of distance as a function of time, the slope of a line segment joining two points represents the average speed between the two points.

Complete Assignment Questions #1 - #10

Assignment

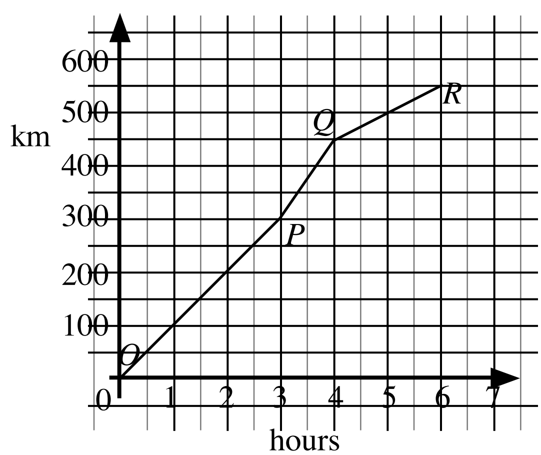
1. The distances and times are recorded at certain points on a journey.
Calculate the average speed;

a) between O and P

b) between P and Q

c) between Q and R

d) for the whole journey.



2. Absolute Value Computer Company was formed in 2002. By Jan 2004, the company had sold 520 000 computers and by July 2004, the company had sold 610 000 computers. Calculate the average rate of change stating appropriate units.

3. In 2003, the transit authority in a large city reported 17 678 465 passenger journeys. In 1998, the number of passenger journeys was 21 520 075. Calculate the average rate of change stating appropriate units.

4. Shanna is paid a base salary per month plus commission for working in a clothing store. In January her sales totalled \$10500 and she earned \$2460. In February her sales totalled \$9350 and she earned \$2322.

a) On a grid plot ordered pairs to represent this information.

b) Calculate the slope of the line segment joining the ordered pairs.

c) Explain what the slope of the graph represents.

d) Determine, as a percent, the rate of commission which Shanna is paid.

e) Calculate her monthly base salary.

f) How does the answer to e) relate to the graph?

g) In March, her sales were \$11 200. Calculate her earnings for March.

h) In April her rate of commission was increased by 1%. If her earnings were \$70 less than in March, calculate the value of her sales in April.

5. Water is leaking out of the bottom of a barrel at a constant rate. After 3 min the water level is 58 cm and after 8 min the water level is 23 cm.

a) On a grid plot ordered pairs (time, water level) to represent this information.

b) Calculate the slope of the line segment joining the ordered pairs.

c) Explain what the slope of the graph represents.

d) Complete the following.

The water level is changing at the rate of ______ ____ per ______

e) Determine an equation to represent this information in the form $W = mt + b$, where W represents the water level in cm and t represents the time in minutes.

f) After how many minutes will the barrel be empty?

g) What was the water level in the barrel when it started leaking?

h) The barrel is cylindrical in shape with a radius of 20 cm.

i) Calculate the volume (in terms of π) of water in the barrel after 3 min and after 8 min.

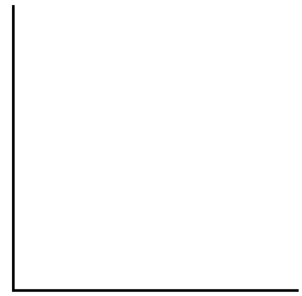
ii) Calculate the rate (in terms of π) at which water is leaking out of the barrel.

6. Jack rented a car from Absolute Value Rent-a-Car Company. After driving for two hours the odometer reading was 21 328 and after five hours the odometer reading was 21 604. His journey was completed after six hours.

a) On a grid plot ordered pairs to represent this information.

b) Calculate the slope of the line segment joining the ordered pairs.

c) Explain what the slope of the graph represents.



d) Assuming a constant rate of driving for the whole journey, determine the odometer reading at the start of the journey.

e) Determine an equation to represent this information in the form $f(t) = mt + b$, where $f(t)$ represents the odometer reading after t hours.

f) State an appropriate domain and range for f.

7. To test the gas consumption of a new SUV, Jana filled up the gas tank of 58 L and drove the SUV until it was empty. She drove the SUV for 464 km.

a) Sketch the graph on the grid provided with distance travelled on the horizontal axis.

b) Write an equation in the form $y = mx + b$ which represents the volume of fuel in the tank as a function of distance.

c) State the slope of the graph and explain what it represents.

d) Determine the distance travelled on 12 litres of gas.

e) How many litres of gas are used by the SUV when it has travelled 200 km?

8. After 7 days of heavy rainstorms the water level in a river peaked at 2.85 m above the regular level. Four days later the water dropped to 2.25 m above the regular level.

a) Assuming the water level falls at a constant rate, determine a function $h(t)$ which describes the height of the river above regular level as a function of time. Take $t = 0$ at peak water level.

b) State the slope of the graph of the function and explain what it represents.

c) Determine an appropriate domain and range for h.

Multiple Choice

9. A repair company charges a fixed call-out fee for any service call plus a fixed rate per hour for the length of the repair. A three hour repair costs \$155 and a four and a half hour repair costs \$215. The fixed call-out fee and the cost of a seven hour repair are respectively

A. \$35 and \$285
B. \$40 and \$285
C. \$35 and \$315
D. \$40 and \$315

Numerical Response

10. The temperature at the top of a mine shaft is 18° C. 250 metres below the surface, the temperature is 18.8° C. To the nearest tenth, the rate of temperature increase in ° C per km is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** 100 km/h **b)** 150 km/h **c)** 50 km/h **d)** $91\frac{2}{3}$ km/h

2. 15 000 computers per month

3. –768322 passenger journeys per year

4. **b)** 0.12 **c)** the rate of commission (earnings per sales)
d) 12% **e)** \$1200
f) it is the intercept on the vertical axis **g)** \$2544 **h)** \$9800

5. **b)** –7
c) the rate at which the water level is changing in cm/min
d) –7 cm/min **e)** $W = -7t + 79$ **f)** $11\frac{2}{7}$ min
g) 79 cm **h)** i) 23200π cm^3, 9200π cm^3 ii) 2800π cm^3/min

6. **b)** 92
c) the average speed between 2h and 5h is 92 km/h
d) 21144 **e)** $f(t) = 92t + 21144$
f) domain $0 \leq t \leq 6, t \in R$, range $21144 \leq f(t) \leq 21696, f(t) \in R$

7. **b)** $y = -\frac{1}{8}x + 58$
c) slope = $-\frac{1}{8}$, the volume of fuel in the gas tank is decreasing at the rate of $\frac{1}{8}$ L/km
d) 96 km **e)** 25 litres

8. **a)** $h(t) = -0.15t + 2.85$
b) slope = –0.15, it represents the rate at which the water level is changing in metres per day
c) domain $0 \leq t \leq 19, t \in R$, range $0 \leq h(t) \leq 2.85, h(t) \in R$

9. C **10.**

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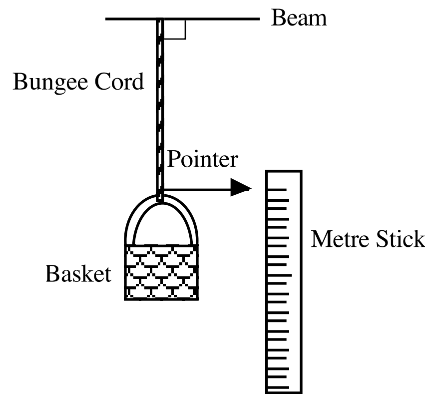
Linear Functions and Equations Lesson #8:
Direct Variation

Direct Variation as an Application of Linear Functions

In this lesson we will introduce the concept of direct variation as an application of linear functions.

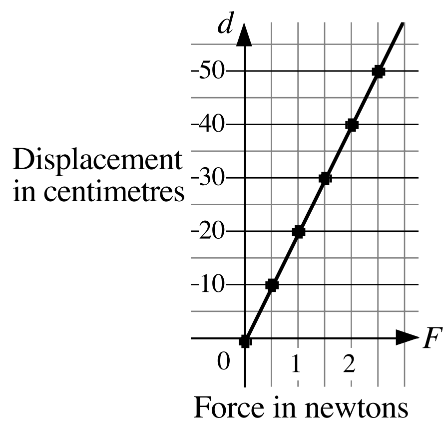
Investigation



An experiment was set up to investigate the relationship between the stretch (or displacement) in a length of bungee cord and the force applied to the cord. The force is provided by the weight of a known mass placed in a basket attached to the free end of the bungee cord. The displacement is measured using a metre stick with the pointer at zero when the basket is empty.

The following measurements are taken and the ordered pairs are plotted on the grid.

Force (*F*) in newtons	0	0.5	1	1.5	2	2.5
Displacement (*d*) in centimetres	0	10	20	30	40	50



Notice that if the force is doubled, the displacement is doubled. If the force is halved, then the displacement is halved, etc. In earlier math courses we learned that two quantities related in this way are in proportion, and that problems involving the two quantities could be solved using a ratio or rate method.

a) For each ordered pair (excluding the origin) divide the displacement by the force and complete the following statement.

For each ordered pair (F, d) the ratio $\frac{d}{F} =$

b) The points when plotted on a grid lie on a straight line through the origin. The equation of the line is of the form $d = kF$, (similar to $y = mx$ on the x,y Cartesian plane). The value of k may be found by using the slope of the graph, or by using the ordered pairs in a).

Complete the statement: The equation relating d and F is $d =$

In the previous example, the quantities d and F were in **direct proportion**. When one quantity is multiplied by a factor, the other quantity is multiplied by the same factor. We say that the displacement **varies directly** as the force.

The Language of Direct Variation

The example above can be described in the language of variation as follows:

1. d varies directly as F (or simply d varies as F)

2. $d \propto F$ (read as in 1).

3. $d = kF$, where k is a constant called the **constant of variation** which is determined from the data.

4. The displacement varies directly as the force.

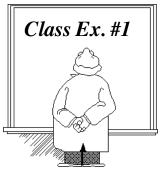
Direct Variation

Consider two variable quantities x and y.

- If y varies directly as x then $\frac{y}{x} = k$ or $y = kx$ where k is a constant.
- The relationship between two quantities has a **direct variation** when the ratio between the two variable values remains constant.
- The notation $y \propto x$ means "y varies directly as x" and is expressed as $y = kx$.
- The phrases
    - "y is directly proportional to x"
    - "y varies directly as x", and,
    - "y varies as x"

  are equivalant.

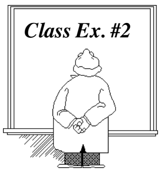
- The graph of a direct variation is a straight line passing through the origin.
- The ordered pairs (x, y) of the graph of a direct variation are such that the ratio $\frac{y}{x}$ is constant.
- If two points (x_1, y_1) and (x_2, y_2) lie on the graph of a direct variation, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.



Write each of the following in the forms $y \propto x$ and $y = kx$.

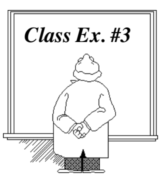
a) The cost, $\$C$, of a car journey varies directly as the number, n, of km travelled.

b) The electrical resistance, R, of a wire of constant diameter varies directly as its length L.



Complete the table if $y \propto x$. State the constant of variation.

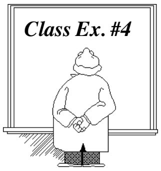
x	10		5		35
y	2	8		9	



In each case, determine the constant of variation, write a formula relating the two variables and solve the problem.

a) y varies directly as x. When $y = 42$, $x = 7$. Find y when $x = 14$.

b) $a \propto b$. When $a = 4$, $b = 32$. Find b when $a = 22$.



In downtown Calgary, parking fees in a privately owned lot vary directly with the length of time parked. Adriana paid \$1.25 for 50 minutes.

a) Introduce variables to represent the variable quantities and determine the equation of the variation.

b) Sketch the graph of the function without the aid of a graphing calculator.

c) How much would Adriana have been charged if she parked for 1 hour and 10 minutes?

d) Rachel's parking fee was \$8.50. Determine the length of time her car was parked in the lot.

Complete Assignment Questions #1 - #12

Assignment

1. Write each of the following in the forms $y \propto x$ and $y = kx$.

a) The perimeter P units of a square varies directly as the side length L units.

b) At a constant speed, the distance s km travelled by a car varies directly as the time t hours.

c) The amount, A dollars, an athlete collects in a walk-a-thon is directly proportional to the number, n km walked.

d) Under certain conditions, the volume V cm^3 of a gas varies directly as its absolute temperature T in °K.

e) The cost C dollars of transporting goods varies as the distance d km.

f) The area, A, of a rectangle is directly proportional to its length, l, when the width is constant.

2. The values representing the variables in each of the following tables are in direct variation. Calculate the constant of variation.

a) $y \propto x$

x	y
5	15
20	60
80	240
320	960

b) $b \propto a$

a	b
0.15	3
0.65	13
1.5	30
2.7	54

c) $i \propto g$

g	i
25	10
50	20
60	24
85	34

3. In each case complete the table if $y \propto x$ and state the constant of variation.

a)

x	4		10		
y		21	30	9	16.5

b)

x	2	6	9	15		
y		21			0.7	35

4. In each case, determine the constant of variation, write a formula relating the two variables and solve the problem.

a) y varies directly as x. When $y = 24, x = 6$. Find y when $x = 3$.

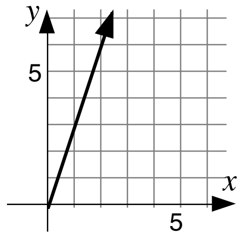
b) $a \propto b$. When $a = 20, b = 12$. Find b when $a = 15$.

c) $w \propto y$. When $w = 32, y = 12$. Find w when $y = 4$.

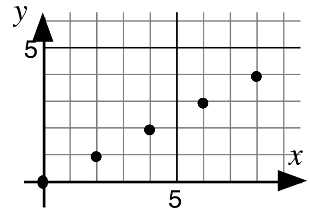
d) y is directly proportional with x. When $y = 72, x = 12$. Find x when $y = 6$.

5. The values representing the variables in each of the following graphs are directly proportional. Calculate the constant in each.

a)



b)



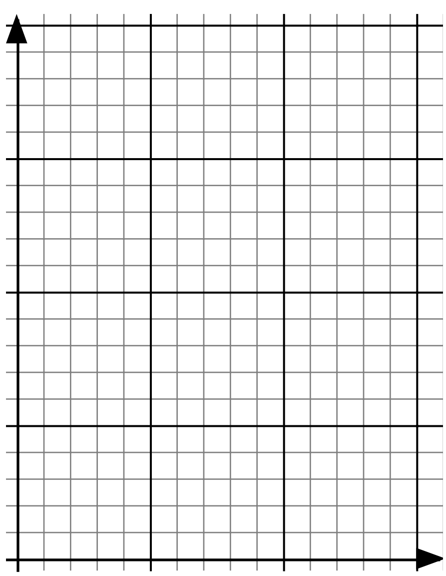
6. The following data (rounded to the nearest tenth) has been collected regarding the radius, circumference, and area of several circles.

radius (cm)	3	5	7	9	12
circumference (cm)	18.8	31.4	44.0	56.5	75.4
area (cm^2)	28.2	78.5	153.9	254.5	452.4

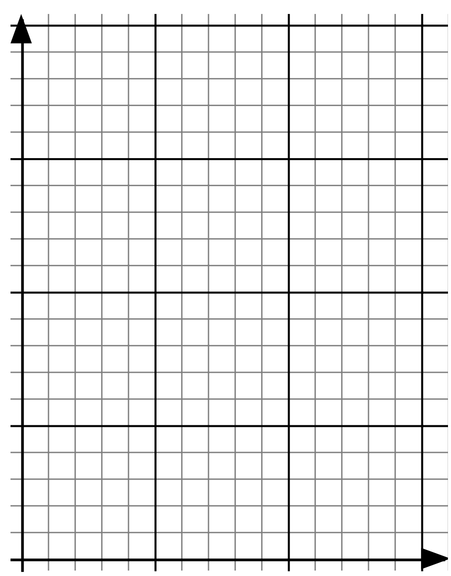
a) Plot the radius (x-axis) and the circumference (y-axis) on Grid 1.

b) Plot the radius (x-axis) and the area (y-axis) on Grid 2.

Grid 1



Grid 2



c) Which of the two graphs shows direct variation? Explain.

7. During a period of heavy construction, the cost of building a sidewalk in Rocky Vistas, a subdivision of a major city, varies directly as its length. A sidewalk 2000 m long costs \$72 000 to build.

a) Determine the cost of a sidewalk 3750 m long.

b) Determine the length of a sidewalk which cost \$99 216.

8. In an electrical circuit the voltage varies directly as the current. If the voltage is 6 V, then the current is 8 A.

a) Determine the voltage if current is 26 A.

b) Determine the current if the voltage is 15 V.

c) What happens to the voltage if the current is halved?

Multiple Choice

9. y varies directly as x, and $y = 0.5$ when $x = 7.5$. The constant of variation is

A. $\frac{1}{15}$

B. 3.75

C. 7.5

D. 15

10. p varies directly as q, and $q = 0.2$ when $p = 0.06$. When $p = 0.4$, the value of q is

A. $\frac{4}{3}$

B. $\frac{3}{4}$

C. 0.12

D. 0.2

Numerical Response

11. Given that y varies directly as x, the blank in the table is _____.

(Record your answer in the numerical response box from left to right)

x	7	5
y		37.5

12. The number of articles produced by a machine varies directly as the time the machine is operating. Yesterday the machine operator switched the machine on at 8am and by the time the machine was switched off at lunchtime(11.30 am) 266 articles had been produced. If the machine operated for a further 4 hours in the afternoon, the total number of articles produced in the day was _____

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $P \propto L,\ P = kL$ **b)** $s \propto t,\ s = kt$ **c)** $A \propto n,\ A = kn$
d) $V \propto T,\ V = kT$ **e)** $C \propto d,\ C = kd$ **f)** $A \propto l,\ A = kl$

2. **a)** 3 **b)** 20 **c)** $\frac{2}{5}$

3. **a)** see table below **b)** see table below

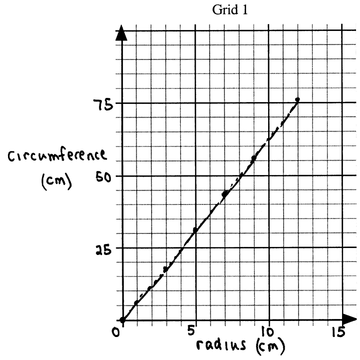
x	4	7	10	3	5.5
y	12	21	30	9	16.5

x	2	6	9	15	0.2	10
y	7	21	31.5	52.5	0.7	35

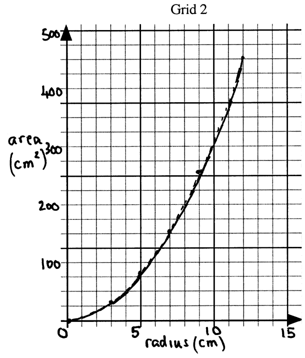
4. **a)** 12 **b)** 9 **c)** $\frac{32}{3}$ **d)** 1 **5.** **a)** 3 **b)** $\frac{1}{2}$

6. **a)** see graph below **b)** see graph below **c)** <u>Grid 1</u> shows direct variation - a straight line through the origin.
<u>Grid 2</u> does not show direct variation.

Grid 1



Grid 2



7. **a)** \$135 000 **b)** 2756 m.

8. **a)** 19.5 V **b)** 20 A **c)** the voltage is halved

9. A **10.** A **11.**

5	2	.	5

12.

5	7	0	

Linear Functions and Equations Lesson #9:
Partial Variation

Partial Variation as an Application of Linear Functions

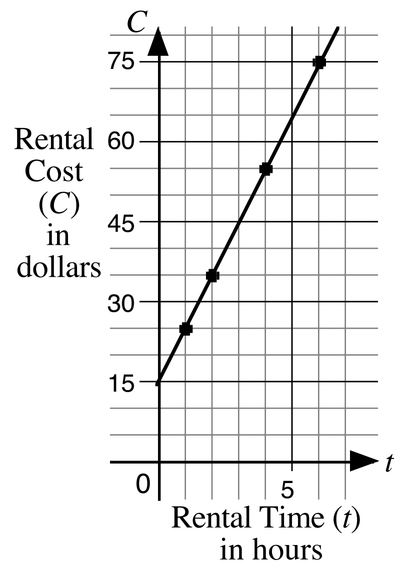
In this lesson we will introduce the concept of **partial variation** as an application of linear functions.

Investigation

When renting a removal van there is a fixed daily charge, plus an amount for every hour the van is rented.

The following table relates the rental cost, C dollars, and the rental time, t hours. The ordered pairs from the table are plotted on the grid.

Rental Time (*t*) in hours	1	2	4	6
Rental cost (*C*) in dollars	25	35	55	75



Notice that if the time is doubled, the cost is not doubled so the ratio $\frac{C}{t}$ does not equal a constant. **This relationship is not a direct variation**.

a) In what way is the graph in this investigation different from the graph of a direct variation?

b) A straight line graph of this form has an equation of the form $y = mx + b$ or in this case $C = mt + b$.

Use the graph to determine the values of m and b and the equation of the relation.

c) How much is the fixed daily charge?

d) How much is the hourly rental charge?

In the previous example, the cost C was **partly constant** (\$15) and **partly varied with the time** (\$10 per hour).

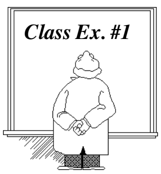
We say that C **varies partially** as t and that C and t are in **partial variation**.

Partial Variation

Consider two variable quantities x and y.

- If y varies partially as x then $y = kx + b$ where b is a constant and k is the constant of variation.
- The graph of a partial variation is a straight line not passing through the origin. The equation of the graph is of the form $y = kx + b$.

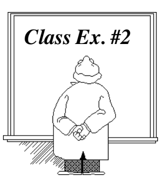
  Note that we use k rather than m in the equation to remind us that this is the constant of variation. The value of k can be determined from the slope of the graph or from two ordered pairs using the slope formula. The value of b is the intercept on the vertical axis. It can also be found by making the independent variable zero in the equation.



y varies partially as x. When $x = 0$, $y = 7$ and when $x = 5$, $y = 27$.

a) The equation relating x and y is of the form $y = kx + b$.
Determine the values of k and b.

b) State the constant of variation.



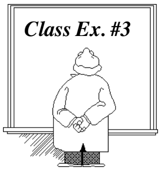
Complete the tables if y varies partially with x. State the constant of variation.

a)

x	0	3	5	
y	6	18		9

b)

x	0	3	5		42
y		18	22	37	



Maria works in a department store. She is paid a base salary of \$412 per week plus 6% commission on all her sales.

a) Determine the partial variation equation.

b) Sketch a graph of the partial variation on a grid.

c) How much did she earn last week when her sales were \$5250?

d) What would her weekly sales have to be for her to earn \$1000 in a week?

Complete Assignment Questions #1 - #10

Extension - Other Forms of Variation

In this section we extend the curriculum to include direct variation involving powers and roots, inverse variation and joint variation.

Direct variation involving powers and roots.

eg. i) the volume of a sphere varies directly as the cube of its radius. $V \propto r^3$

ii) the time of swing of a pendulum varies as the square root of its length. $t \propto \sqrt{L}$

Inverse variation

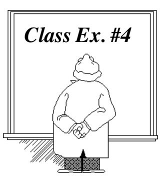
eg. i) the time taken to travel a fixed distance varies inversely as the average speed. $t \propto \dfrac{1}{s}$

ii) the intensity of light varies inversely with the square of the distance from the light source. $I \propto \dfrac{1}{d^2}$

Joint variation

eg. i) the area of a triangle varies jointly with its base and its height $A \propto bh$

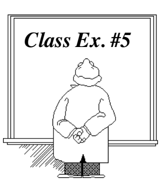
ii) the electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. $R \propto \dfrac{L}{D^2}$



Two electical charges attract one another with a force, F units, which varies inversely as the square of the distance, d units, between them.

a) If $F = 48.6$ when $d = 2$, find a formula connecting F and d.

b) Evaluate F when $d = 3$.



The pressure, P pascals, on a disc immersed in a liquid varies as the depth, d metres, and the square of the radius, r metres, of the disc. If the pressure is 6250 pascals when the depth is 4 m and the radius is 2.5 m, calculate the pressure when the depth and radius are both 3 m.

Complete Assignment Questions #11 - #18

Assignment

1. A partial variation is given by $y = 5x + 2$. Calculate

a) y when $x = 35$ **b)** y when $x = 5.5$ **c)** x when $y = 26$ **d)** x when $y = 5.5$

2. y varies partially as x. When $x = 0, y = 24$ and when $x = 4, y = 34$.

a) The equation relating x and y is of the form $y = kx + b$.
Determine the values of k and b.

b) State the constant of variation and calculate the value of y when $x = 8$.

3. Complete the tables if y varies partially with x. State the constant of variation.

a)

x	0	2	7	
y	9	21		75

b)

x	0	6	10		34
y		32	34	40	

4. The graph of a partial variation passes through the points (4, 7) and (11, 28).
Determine the equation of the partial variation.

5. We have met the following scenarios in previous lessons.

a) "Triple A Car Rental charges \$100 per rental plus10¢ per km."

Write a partial variation equation for the total cost, T dollars, of renting a car where n km are travelled.

b) "Student Council is planning a school dance and sells tickets at \$5 per student. The expenses for the dance are \$800."

Write a partial variation equation for the profit, P dollars, realized if n tickets are sold.

c) Consider the vertical intercepts of the graphs of the partial variation equations in a) and b). With reference to the scenarios, explain why one graph has a positive intercept and the other has a negative intercept.

6. The cost of renting a piece of farm machinery is partly constant and varies partly with the number of hours for which it is rented. A 12 hour rental costs \$660 and a 30 hour rental costs \$1290.

a) On a grid plot ordered pairs for the above scenario.

b) Determine a partial variation equation for the total cost, C dollars, of renting the machinery for n hours.

c) How much is the i) fixed rental charge ii) hourly rental charge?

d) If the rental cost was \$800, for how long was the equipment rented?

7. Jaime works in a shoe store. He earns 8% commission on all his sales and he is paid \$6.60 per hour for every hour he works. He is employed to work 35 hours in a week.

a) Determine the partial variation equation for the amount of money he earns in a week.

b) How much would he earn in a week in which his sales were \$4560?

c) What would his weekly sales have to be for him to earn \$1000 in a week?

d) During the course of the last year Jaime's average sales have been \$4200 per week. The manager of the store offers him an alternative wage structure of \$4 per hour plus 12% commission. Should Jaime accept the new wage structure? Explain.

8. The total cost, C dollars, of producing a school magazine depends on a fixed production cost of \$950 and a variable cost of 65 cents per copy. The school council are doing some calculations before deciding on the selling price per magazine. Any profit made is to be donated to charity.

a) Write a partial variation equation for the total cost of producing n magazines.

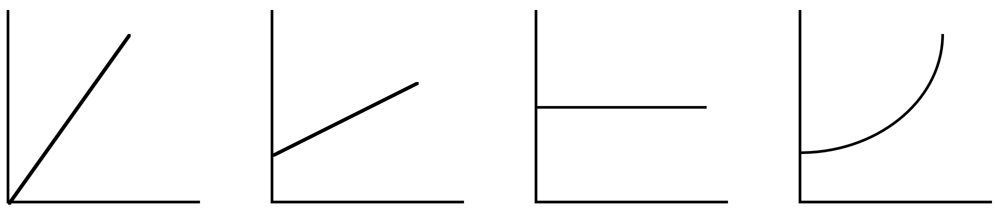
b) If 800 magazines are produced and sold, suggest a selling price per magazine which would result in at least \$2500 dollars being donated to charity.

Multiple Choice

9. y varies partially as x. When $x = 4$, $y = 40$ and when $x = 0$, $y = 20$. The constant of variation is

A. 20
B. 10
C. 5
D. 0.2

10. The graph which represents a partial variation is



A. **B.** **C.** **D.**

Extension Questions.

11. Express each of the following as an equation using k as the constant of variation.

a) L varies as the cube root of S

b) P varies inversely as V.

c) T varies as D and inversely as S.

d) W varies jointly as L and the square of d.

12. The speed, S, at which a car tyre is liable to aquaplane varies directly as the square root of the tyre pressure P.

a) Determine an equation connecting S and P if $S = 84$ when $P = 36$.

b) Calculate the value of S when $P = 25$.

13. For square prisms of fixed volume, the height, h cm, varies inversely as the square of the length of the side, s cm.

a) If $h = 36$ when $s = 2$, find a formula connecting h and s.

b) Calculate the height of the prism if the length of the side is 9 cm.

c) Calculate the length of the side if the prism has a height of 9 cm.

14. The conductance, G, of a solution varies directly as the cross-sectional area, A and inversely as the length, L.

a) Calculate the constant of variation if $G = 1.6$ when $A = 2$ and $L = 3$.

b) Calculate G when $A = 3$ and $L = 4$.

15. q varies inversely as the square root of p. If $q = 8$ when $p = 4$ then when $p = 1$, the value of q is

A. 128
B. 16
C. 4
D. 0.5

16. y varies as the square of x. When $x = 2^{-2}$, $y = 2$. When $x = 2$, y equals

A. 2^{-4}
B. 2^{-2}
C. 2^{4}
D. 2^{7}

17. z varies as the square of x and inversely as y. When $x = 3$ and $y = 4$, z is 18. The value of z when $x = 1$ and $y = 1$ is

A. $\frac{1}{2}$
B. 2
C. 8
D. 24

18. p varies inversely as the square of q. When $q = 8, p = 2$. The value of p when $q = 2$ is

A. $4\sqrt{2}$
B. $8\sqrt{2}$
C. 32
D. 64

Answer Key

1. a) 177 **b)** 29.5 **c)** 4.8 **d)** 0.7

2. a) $k = \frac{5}{2}$, $b = 24$ **b)** constant of variation $= \frac{5}{2}$, $y = 44$

3. a) see table below, constant of variation = 6 **b)** see table below, constant of variation $= \frac{1}{2}$

x	0	2	7	11
y	9	21	51	75

x	0	6	10	22	34
y	29	32	34	40	46

4. $y = 3x - 5$

5. a) $T = 0.1n + 100$ **b)** $P = 5n - 800$
c) In a) vertical intercept = 100.
This is positive since there is a fixed charge of \$100, which does not depend on the number of km travelled.
In b) vertical intercept = –800.
This is negative since there is a loss of \$800 if no tickets are sold.

6. b) $C = 35n + 240$ **c)** i) \$240 ii) \$35 **d)** 16 hours

7. a) $A = 0.08s + 231$ **b)** \$595.80 **c)** \$9612.50
d) present wage structure \$567, new wage structure \$644. He should accept the new wage structure.

8. a) $C = 0.65n + 950$ **b)** about \$5 **9.** C **10.** B

11.a) $L = k\sqrt[3]{S}$ **b)** $P = \frac{k}{V}$ **c)** $T = \frac{kD}{S}$ **d)** $W = kLd^2$

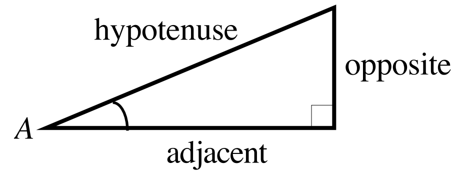
12.a) $S = 14\sqrt{P}$ **b)** 70 **13.a)** $h = \frac{144}{s^2}$ **b)** $\frac{16}{9}$ cm **c)** 4 cm

14.a) 2.4 **b)** 1.8 **15.** B **16.** D **17.** C **18.** C

Trigonometry Lesson #1:
Right Triangle Trigonometry

Warm-Up *Review*

The diagram shows a right triangle with $\angle A$ indicated. The hypotenuse and the sides opposite and adjacent to $\angle A$ are also indicated.



In a previous math course we studied the three primary trigonometric ratios - the **sine ratio**, the **cosine ratio** and the **tangent ratio**.

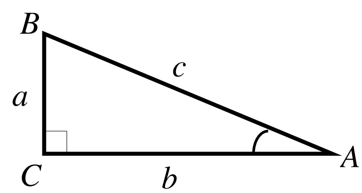
Complete for $\angle A$.

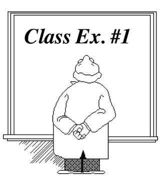
$\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\qquad$ $\cos A =$ $\qquad$ $\tan A =$

These ratios can be memorized by the acronym ***SOHCAHTOA***.

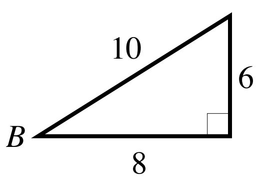
Pythagorean Theorem

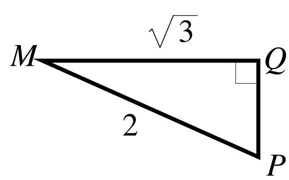
In right triangle ABC, $a^2 + b^2 = c^2$

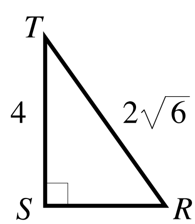




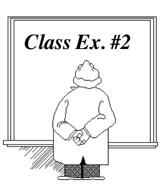
Express the following trigonometric ratios in simplest form.





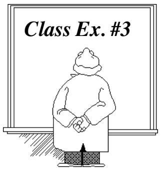


a) $\sin B =$ $\qquad$ **b)** $\cos P =$ $\qquad$ **c)** $\tan R =$



Use a calculator to determine the value of each trigonometric ratio to four decimal places.

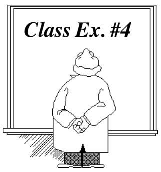
a) $\sin 32° =$ $\qquad$ **b)** $\tan 71° =$ $\qquad$ **c)** $\cos 11° =$



In each case determine the acute angle A to the nearest degree.

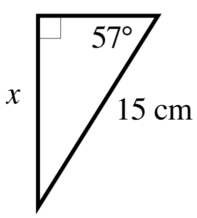
a) $\sin A = 0.2376$ **b)** $\cos A = \frac{3}{8}$ **c)** $\tan A = 1$

$\angle A =$

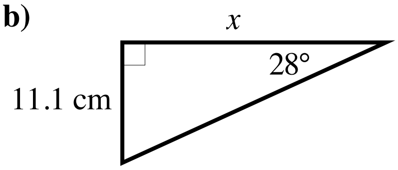


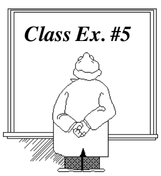
Determine the length of the indicated side to the nearest 0.1 cm.

a)



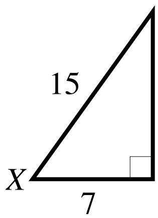
b)



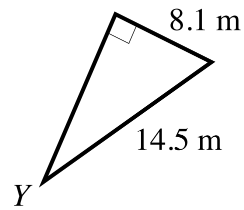


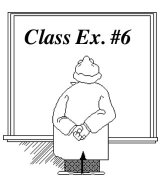
Determine the measure of the indicated angle to the nearest degree.

a)



b)



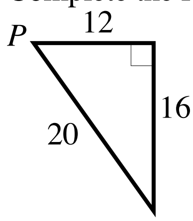


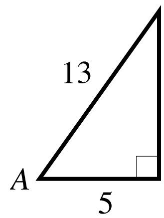
From the top of a building a surveyor determines the angle of depression of a parked car on the street below to be 34°. If the building is 28 m high calculate the distance from the base of the building to the parked car. Answer to the nearest metre.

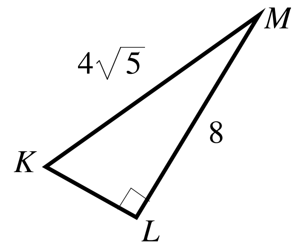
Complete Assignment Questions #1 - #14

Assignment

1. Complete the following for the indicated trigonometric ratios.







a) $\cos P =$ **b)** $\tan A =$ **c)** $\sin M =$

2. Use a calculator to determine the value of each trigonometric ratio to four decimal places.

a) $\sin 68° =$ **b)** $\tan 30° =$ **c)** $\cos 19° =$

d) $\cos 22° =$ **e)** $\tan 85° =$ **f)** $\sin 7° =$

3. In each case determine the indicated acute angle to the nearest degree.

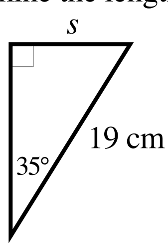
a) $\sin A = 0.6789$ **b)** $\cos X = 0.1234$ **c)** $\tan P = 0.55$

$\angle A =$

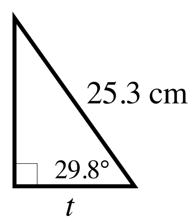
d) $\sin K = \frac{\sqrt{2}}{2}$ **e)** $\cos M = \frac{7}{24}$ **f)** $\tan R = \sqrt{3}$

4. Determine the length of the indicated side to the nearest 0.1 cm.

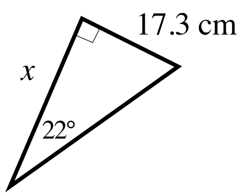
a)



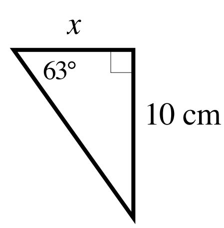
b)



c)

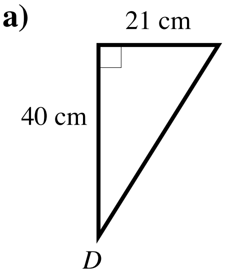


d)

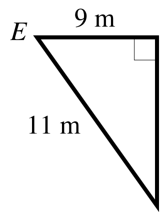


5. Determine the measure of the indicated angle to the nearest degree.

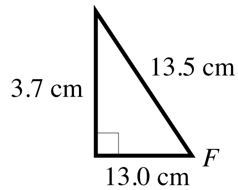
a)



b)

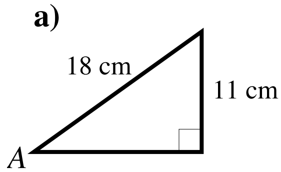


c)

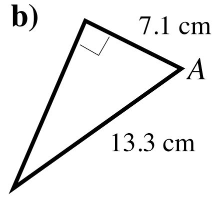


6. Solve for angle A to the nearest 0.1°.

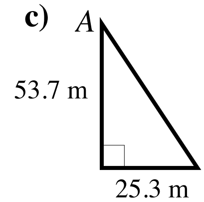
a)



b)



c)

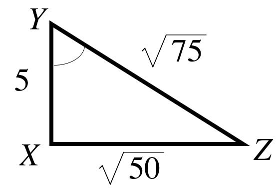


7. Jacob has been given the task of determining the height of a building. He walks 30 m away from the base of the building and uses a clinometer to measure the angle of elevation of the top of the building to be 58°.
Calculate the height of the building to the nearest metre.

8. From the top of a vertical cliff 120 metres above sea level, Susan measures the angle of depression of a boat in the water to be 37°. To the nearest metre, determine the distance between the boat and the base of the cliff.

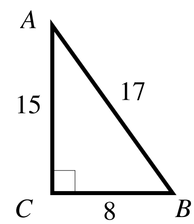
Multiple Choice

9. In ΔXYZ, $XY = 5$ units, $XZ = \sqrt{50}$ units and $YZ = \sqrt{75}$ units. Cos Y is



A. $\sqrt{2}$
B. $\sqrt{3}$
C. $\frac{\sqrt{3}}{3}$
D. $\frac{\sqrt{6}}{3}$

10. For the right angled triangle ABC, only one of the following ratios is correct. The correct ratio is

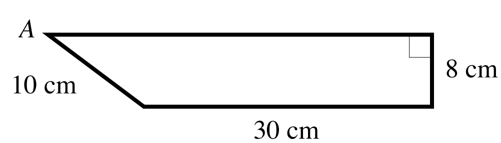


A. $\sin A = \frac{8}{15}$
B. $\cos A = \frac{8}{17}$
C. $\tan B = \frac{8}{15}$
D. $\sin B = \frac{15}{17}$

11. $ABCD$ is a rhombus with diagonals meeting at O. $AC = 4$ cm and $BD = 6$ cm. Sin $\angle ABO$ is

A. $\frac{3\sqrt{13}}{13}$
B. $\frac{2\sqrt{13}}{13}$
C. $\frac{\sqrt{13}}{2}$
D. $\frac{\sqrt{13}}{3}$

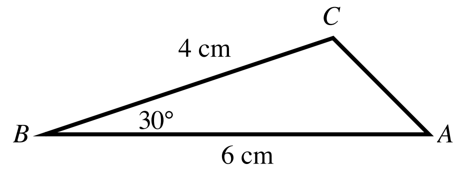
12. In the figure cos A is equal to



A. $\frac{3}{5}$
B. $\frac{4}{5}$
C. $\frac{4}{15}$
D. cannot be calculated from the given information

Numerical Response

13. To the nearest whole number, the area of triangle ABC in cm^2, is _______ .



(Record your answer in the numerical response box from left to right)

14. A corner flag in a World Cup soccer match is 5 feet high. At game time, the flag casts a shadow which is 3.2 feet long. To the nearest 0.1 degree, the angle of elevation of the sun is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $\frac{3}{5}$ **b)** $\frac{12}{5}$ **c)** $\frac{\sqrt{5}}{5}$

2. **a)** 0.9272 **b)** 0.5774 **c)** 0.9455 **d)** 0.9272 **e)** 11.4301 **f)** 0.1219

3. **a)** 43° **b)** 83° **c)** 29° **d)** 45° **e)** 73° **f)** 60°

4. **a)** 10.9 cm **b)** 22.0 cm **c)** 42.8 cm **d)** 5.1 cm

5. **a)** 28° **b)** 35° **c)** 16°

6. **a)** 37.7° **b)** 57.7° **c)** 25.2° **7.** 48 m **8.** 159 m

9. C **10.** D **11.** B **12.** A

13.

6			

14.

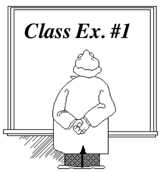
5	7	.	4

Trigonometry Lesson #2: Problems Involving Two Right Triangles

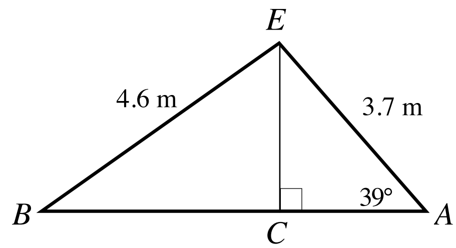
Warm-Up

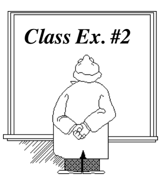
In this lesson we deal with problems involving two right triangles. In many cases we can determine the value of a quantity from one right triangle and then use that value in a second right triangle to determine the solution to a problem. When this situation arises do not round intermediate answers. Only the final solution should be rounded.

Problems in Two Dimensions

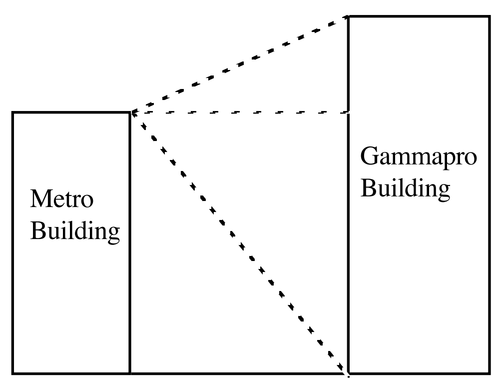


Calculate the measure of $\angle BEC$ to the nearest degree.





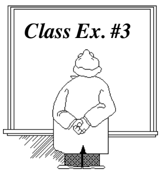
In January 2003 the tallest building in Rockyville was the Metro Building. Recently a developer was commissioned by the Gammapro Oil Company to build a taller building next to the Metro Building. From the top of the Metro Building the angle of elevation of the top of the Gammapro Building is 24° and the angle of depression to the foot of the Gammapro Building is 56°. If the buildings are 45 m apart determine the height of each building to the nearest metre.



Complete Assignment Questions #1 - #5

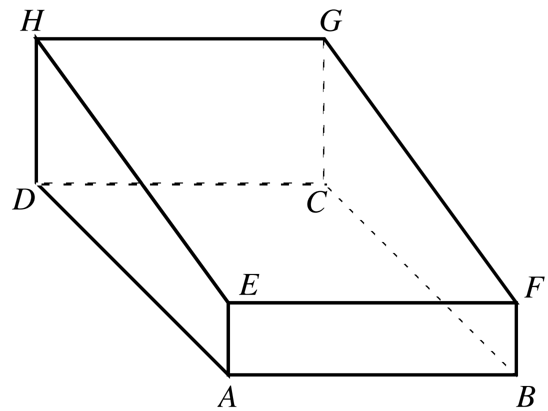
Problems in Three Dimensions

As an aid to understanding a problem in three dimensions, we sketch a two dimensional representation of the triangle we are working with.



The solid in the diagram was formed from a rectangular prism by removing a wedge.

$HD = GC =$ 5 cm. $EA = FB = 2$ cm.
$AD =$ 6 cm and $AB = 8$ cm.



a) Name four rectangles in the diagram.

b) Calculate the measure of $\angle HEA$ (nearest degree).

c) Calculate the measure of angle HFE (nearest degree).

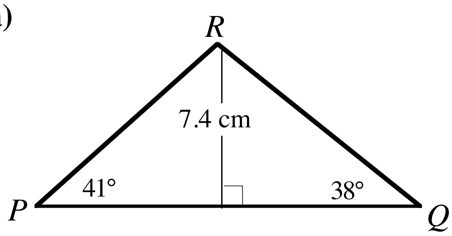
d) Calculate the measure of angle HBD (nearest degree).

Complete Assignment Quesitons #6 - #16

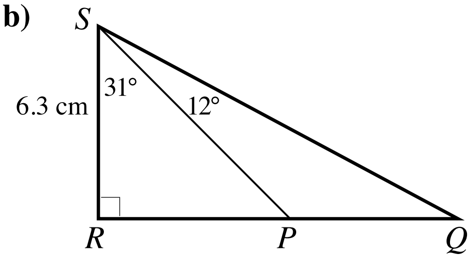
Assignment

1. Determine the length of PQ, to the nearest 0.1 cm.

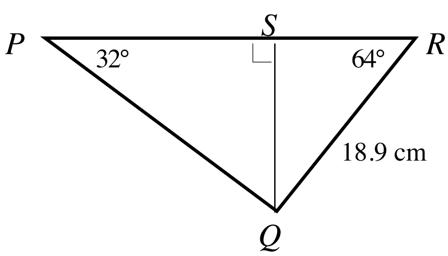
a)



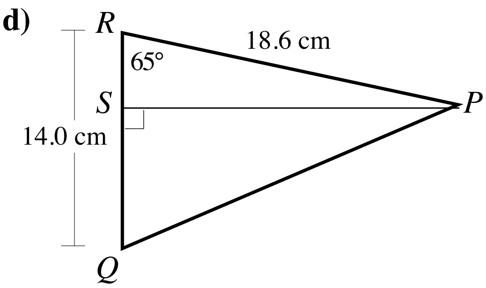
b)



c)

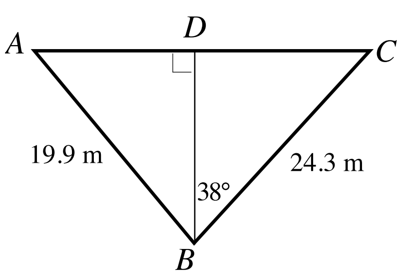


d)

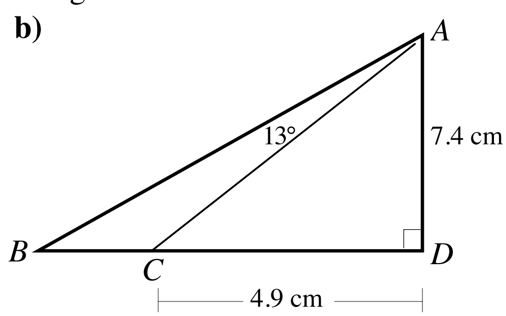


2. Determine the measure of angle ABC, to the nearest degree.

a)

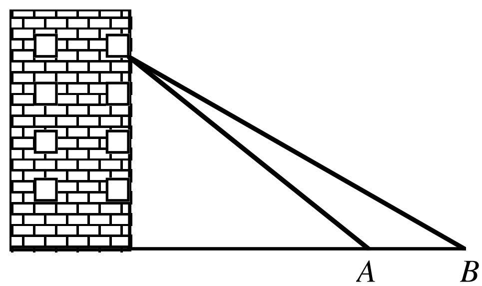


b)



3. From the top of a cliff 110 m high an observer sees two boats, one directly behind the other, heading for shore. The angle of depression from the observer to the boat farther from the observer is 48° and the angle of depression to the nearer boat is 57°.
Calculate the distance between the boats, to the nearest metre.

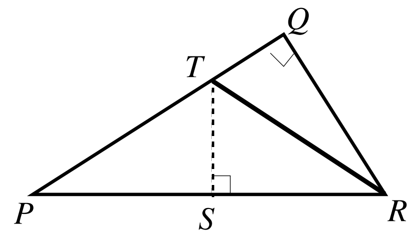
4. The diagram shows two marathon runners, *A* and *B*, heading towards the finish line of a race. From an apartment window 80 metres above the ground and 20 metres behind the finish line, Tony measures the angle of depression of the runners to be 28° and 24° respectively.



a) Calculate the distance between the runners to the nearest metre.

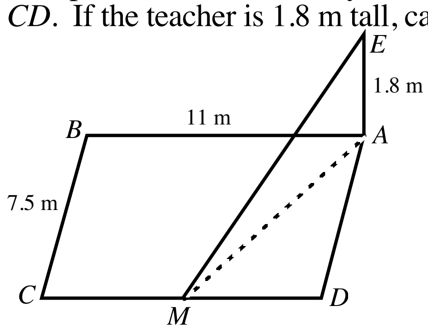
b) *A* is travelling at a constant speed of 4.5 m/s while *B* is travelling at a constant speed of 5.1 m/s. Which runner will finish the race first?

5. In the diagram $PQ = 16$ m, $QR = 12$ m and $PT = TR$.
a) Calculate the measure of $\angle QRT$ to the nearest 0.1 °.

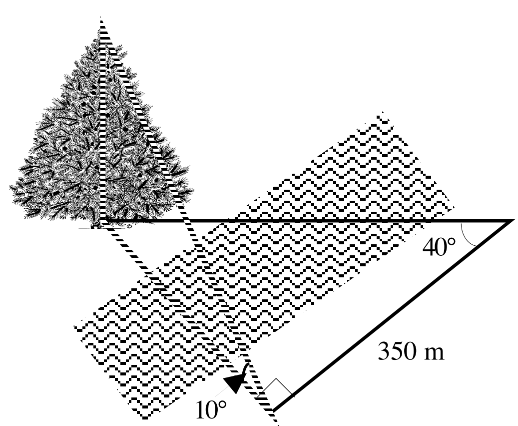


b) Calculate the length of PT to the nearest 0.1 m.

6. In the diagram *ABCD* represents a rectangular sandbox for kindergarten children to play in. A teacher stands at the corner of the area to supervise the children. At a certain time of day the tip of the shadow cast by the teacher on the play area is exactly at *M* , the midpoint of *CD*. If the teacher is 1.8 m tall, calculate the measure of angle *EMA*, to the nearest degree.

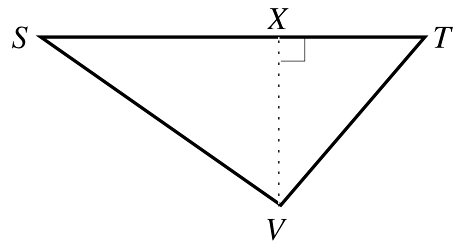


7. In order to find the height of a tree on the opposite bank of a river Jenny makes the measurements shown on the diagram. Calculate the height of the tree to the nearest metre.



8. In the diagram ST is part of a straight road and V is the position of a cellphone tower, 12 m high. ΔSVT is a right triangle with $ST = 61$ m and $SV = 51$ m.

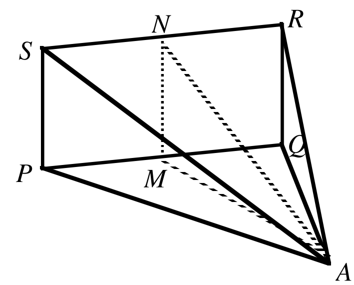
A surveyor is positioned at the edge of the roadway closest to the cellphone tower.



a) Calculate the distance, to the nearest metre, between the surveyor and the cell phone tower.

b) Calculate the angle of elevation, to the nearest degree, of the top of the tower from the surveyor.

9. $PQRS$ is a rectangular plate held vertically between plates APQ and ASR.
APQ lies in a horizontal plane through PQ.
$AP = AQ = 40$ cm, $PS = 15$ cm and $RS = \frac{1}{2}$ m.
M is the midpoint of PQ and N is the midpoint of RS.

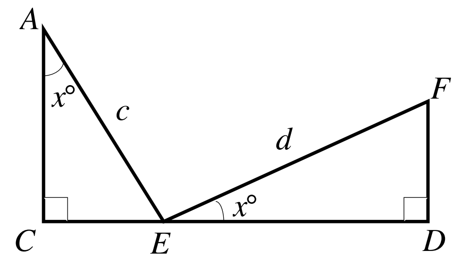


a) Calculate the length of AM to the nearest 0.1 cm.

b) The angle between the plates ASR and $PQRS$ is defined to be $\angle ANM$. Calculate the measure of this angle to the nearest tenth of a degree.

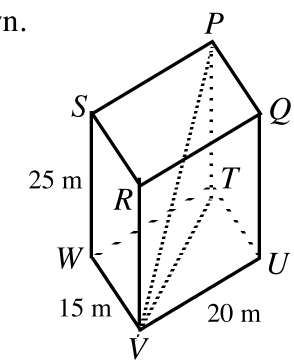
Multiple Choice

10. From the diagram the length of CD is



A. $c \sin x° + d \cos x°$
B. $c \cos x° + d \sin x°$
C. $(c + d) \cos x°$
D. $(c + d) \sin x°$

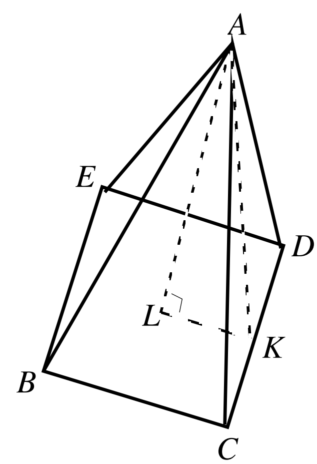
11. A rectangular prism has dimensions 15 m, 20 m and 25 m as shown. The angle between diagonal VP and the plane $TUVW$ is defined to be angle PVT. The tangent of this angle is



A. $\frac{3}{4}$
B. 1
C. $\frac{5}{4}$
D. $\frac{5}{3}$

Numerical Response

12. $ABCDE$ is a square based pyramid with vertical height 14 cm. K is the midpoint of CD and angle $AKL = 70°$.
To the nearest degree, the measure of angle ACK is _____.



(Record your answer in the numerical response box from left to right)

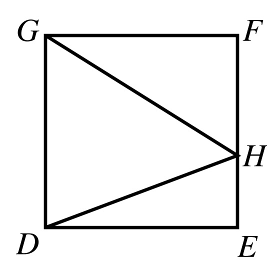
13. From a point 240 m above sea level, a coastguard measures the angle of depression of a ship due west of him as 27°. Another ship, due west of the coastguard, and 750 m behind the first ship, comes into view. To the nearest degree, the angle of depression of the second ship from the coastguard is _____.

(Record your answer in the numerical response box from left to right)

14. Δ *ABC* is isosceles with $AB = CB$. The area of the triangle is 30 cm^2 and $AC = 12$ cm. The measure of angle *ABC*, to the nearest degree, is _____.

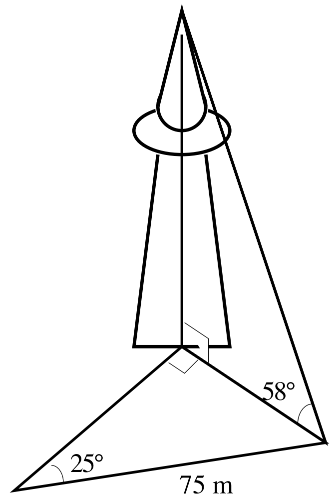
(Record your answer in the numerical response box from left to right)

15. In the diagram, *DEFG* is a square of side 12 cm and $GH = 15$ cm. To the nearest degree, the difference between the measures of $\angle HGF$ and $\angle HDE$ is _____ .



(Record your answer in the numerical response box from left to right)

16. In the diagram the height of the tower, to the nearest metre, is _____.



(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** 18.0 cm **b)** 2.1 cm **c)** 32.1 cm **d)** 17.9 cm

2. **a)** 54° **b)** 43°

3. 28 m

4. **a)** 29 m **b)** A 29 sec B 31 sec so A finishes first.

5. **a)** 16.2° **b)** 12.5 m

6. 11°

7. 52

8. **a)** 28 m **b)** 23°

9. **a)** 31.2 cm **b)** 64.3°

10. A **11.** B

12.

7	1		

13.

1	1		

14.

1	0	0	

15.

2	3		

16.

5	1		

Trigonometry Lesson #3:
.Sine and Cosine Ratios for Angles from 0° to 180°

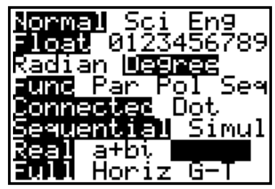
The enrichment lesson in this unit extends the sine and cosine ratios for angles from 0° to 360°. **Students intending to study this enrichment lesson should omit lesson #3 entirely and move straight to Lesson #3 Extension.**

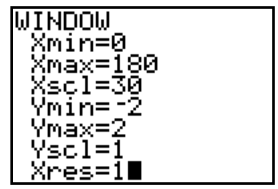
Warm-Up #1

a) Use a calculator to find the value of sin 30° and sin 150° and compare the answers.

b) Compare the value of cos 40° and cos 140° (to four decimal places).

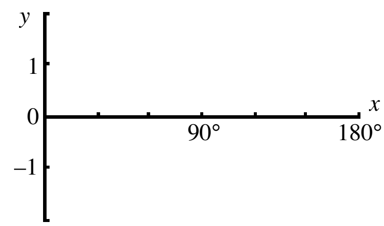
- To explain the above answers we will construct and analyze the graphs of $y = \sin x$ and $y = \cos x$ from 0° to 180°.
- Use the following mode and window settings.





Warm-Up #2 *Exploring the Graph of y = sin x*

a) Sketch the graph of $y = \sin x$, $0 \leq x \leq 180°$, on the grid. Show the results of Warm-Up #1 a) on the grid.



b) Use the trace feature to complete the table below to four decimal places where necessary.

Press [Trace], enter the value of x,

then press [Enter] to find the value of y.

x (angle in degrees)	**y (sine ratio)**
0°	
30°	
45°	
60°	
90°	

x (angle in degrees)	**y (sine ratio)**
120°	
135°	
150°	
180°	

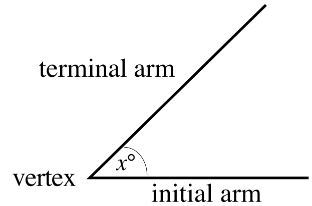
c) Without using a calculator state two angles (not in the table) which have the same value for the sine ratio.

Complete Assignment Question #1

Warm-Up #1 and Assignment question #1 showed a relationship between pairs of angles. This relationship can be investigated further using the concepts of **rotation** angles and **reference** angles.

Rotation Angle

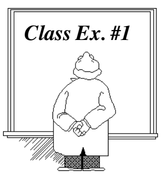
A **rotation angle** is formed by rotating an <u>initial arm</u> through an angle x° about a fixed point (the vertex) to the <u>terminal arm</u>.



A **positive angle** results from a counter clockwise rotation.

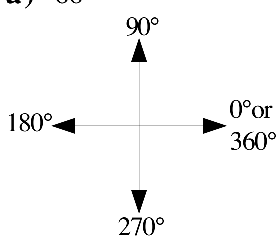
A **negative angle** results from a clockwise rotation. Negative rotation angles will be studied in more advanced math courses.

If the vertex is at the origin and the initial arm is along the x–axis the rotation angle is said be in **standard position**.

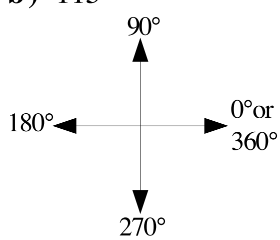


Draw the following rotation angles in standard position.

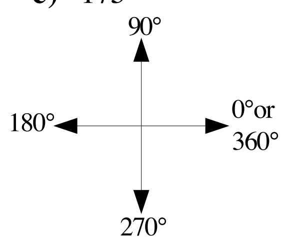
a) 60°



b) 115°

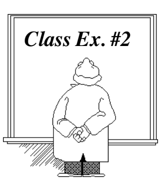


c) 175°



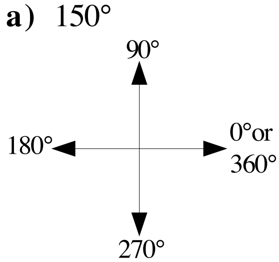
Reference Angles

In order to investigate pairs of angles with identical trigonometric ratios, we introduce the concept of a **reference angle**. A **reference angle** is the acute angle formed between the terminal arm of the rotation angle and the x-axis.

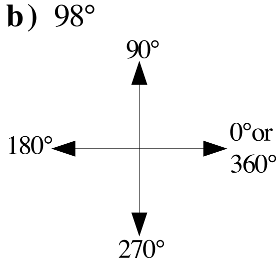


In each case, sketch the rotation angle and state the reference angle.

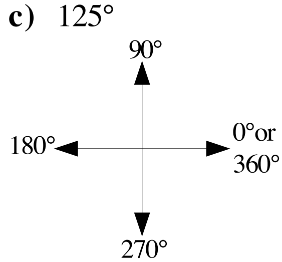
a) 150°



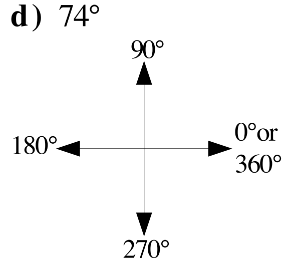
b) 98°



c) 125°



d) 74°



Complete Assignment Questions #2 - #4

Warm-Up #2 *Positive and Negative Values for the Sine and Cosine Ratios*

The diagram below shows the values of the sine and cosine ratios from the earlier part of this lesson placed in their respective quadrants.

$y = \sin x$

Quadrant 2

x (angle)	y (ratio)	sign +/–
120°	0.8660	
135°	0.7071	
150°	0.5	
180°	0	

Quadrant 1

x (angle)	y (ratio)	sign +/–
0°	0	
30°	0.5	
45°	0.7071	
60°	0.8660	
90°	1	

Quadrant 3 **Quadrant 4**

$y = \cos x$

Quadrant 2

x (angle)	y (ratio)	sign +/–
120°	–0.5	
135°	–0.7071	
150°	–0.8660	
180°	–1	

Quadrant 1

x (angle)	y (ratio)	sign +/–
0°	1	
30°	0.8660	
45°	0.7071	
60°	0.5	
90°	0	

Quadrant 3 **Quadrant 4**

Sine and cosine ratios for angles in Quadrants 3 and 4 are discussed in enrichment lesson #7.

a) Complete the diagram above by entering the sign (+/–) of the sine or cosine ratio in each quadrant.

b) Complete the following statements using the results from a).

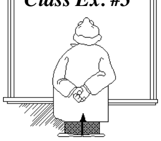
i) Sine ratios have **positive** values in quadrants _____ and _____ .

ii) Cosine ratios have **positive** values in quadrant _____ .

iii) Cosine ratios have **negative** values in quadrant_____ .

These results should be memorized.

Class Ex. #3



Without using a calculator or a graph, state whether the following ratios are positive or negative.

a) sin 155° **b)** cos 170° **c)** sin 17° **d)** cos 35°

Solving Equations Involving Sine or Cosine

We can use the concepts of reference angles and signs of the trigonometric ratio to solve equations involving sine or cosine. The following procedure may be used to solve an equation such as $\sin x = 0.5$, where $0° \leq x \leq 180°$.

Step 1: Determine the quadrant(s) the angle will be in by looking at the sign of the ratio.

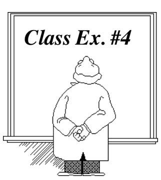
Step 2: Determine the reference angle (always between 0° and 90°) and draw a rough sketch in the appropriate quadrant(s). The reference angle is found as follows:

Use $\boxed{2^{\text{nd}}}$ $\boxed{\text{sin}}$ or $\boxed{2^{\text{nd}}}$ $\boxed{\text{cos}}$ of the **absolute value** of the given quantity.

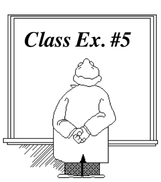
Step 3: Determine the rotation angle(s) using the reference angle and the quadrant(s).

- Always check the given domain to determine which quadrants are valid in the calculation. Sometimes the domain is restricted to $90° \leq x \leq 180°$.

- Pressing $\boxed{\text{sin}}$ or $\boxed{\text{cos}}$ then the angle $\boxed{\text{Enter}}$ gives the value of the trigonometric ratio for that angle, whereas pressing $\boxed{2^{\text{nd}}}$ $\boxed{\text{sin}}$ or $\boxed{2^{\text{nd}}}$ $\boxed{\text{cos}}$ then the given value $\boxed{\text{Enter}}$ determines the reference angle.

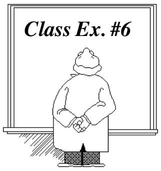


Use the above procedure to solve $\sin x = 0.5$, where $0° \leq x \leq 180°$.



Find the measure of x to the nearest whole number, where $0° \leq x \leq 180°$.

a) $\cos x = 0.7880$ **b)** $\sin x = 0.8090$ **c)** $\cos x = -0.8090$



Solve the following equations, to the nearest degree for $0° \le \theta \le 180°$.

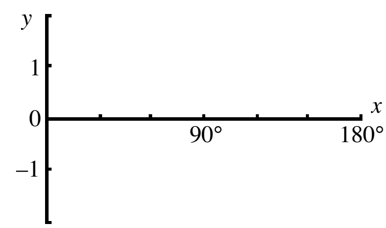
a) $2 \cos \theta - 1 = 0$ **b)** $3 \sin \theta - \sqrt{3} = 0$

Complete Assignment Questions #5 - #12

Assignment

1. Use a graphing calculator with the settings for Warm-Up #1 to answer the following.

a) Sketch the graph of $y = \cos x$ on the grid.



b) Complete the table below to four decimal places where necessary.

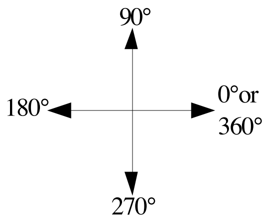
x **(angle in degrees)**	***y*** **(cosine ratio)**
0°	
30°	
45°	
60°	
90°	

x **(angle in degrees)**	***y*** **(cosine ratio)**
120°	
135°	
150°	
180°	

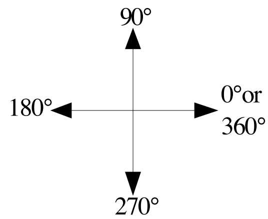
c) Without using a calculator state two angles (not in the table) which have cosine ratios which are equal in value, but opposite in sign.

2. In each case, sketch the rotation angle and state the reference angle.

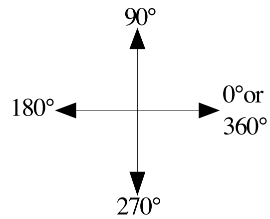
a) 45°



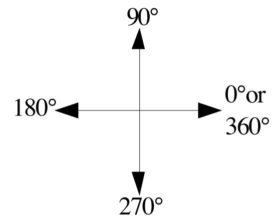
b) 92°



c) 162°



d) 90°



3. Find the reference angle for the following rotation angles.

a) 135° **b)** 25° **c)** 112° **d)** 159°

e) 95° **f)** 179° **g)** 0° **h)** 90°

4. Complete the following tables given the reference angle and the quadrant.

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		
30°	2		
4°	2		
89°	1		
89°	2		
90°	between 1 and 2		

5. Find the measure of x where $0° \le x \le 180°$ to the nearest degree.

a) $\cos x = 0.8088$ **b)** $\sin x = 0.2431$ **c)** $\sin x = 0.7071$

d) $\cos x = -0.4668$ **e)** $\sin x = 0.6485$ **f)** $\cos x = 0.6485$

g) $\cos x = -0.0602$ **h)** $\sin x = -0.0856$ **i)** $\sin x = 0.9900$

6. Solve for x in each of the following where $90° \le x \le 180°$.

a) $\cos x = 0.8088$ **b)** $\sin x = 0.2431$ **c)** $\cos x = -0.7071$

7. Find the measure of θ where $0° \le \theta \le 180°$ to the nearest degree.

a) $\sin \theta = 0$ **b)** $\cos \theta = 0$ **c)** $\sin \theta = 1$

d) $\cos \theta = 1$ **e)** $\cos \theta = -1$ **f)** $\sin \theta = -1$

8. Evaluate to 4 decimal places.

a) $\sin 75°$ **b)** $\sin 105°$ **c)** $\cos 48.2°$

d) $\cos 2°$ **e)** $\sin 130°$ **f)** $\cos 122.9°$

9. Solve the following equations, to the nearest degree, where the variables are defined on a domain of 0° to 180°

a) $2 \sin a - 1 = 0$ **b)** $2 \cos \theta - \sqrt{2} = 0$ **c)** $2 \sin b = \sqrt{3}$

10. Given that $(\cos x)^2$ can be written as $\cos^2 x$, solve the following equations, to the nearest degree, on the given domain.

a) $\cos^2 x = \frac{1}{16}$, $0° \le x \le 180°$ **b)** $2 \sin^2 \theta = 1$, $0° \le \theta \le 180°$

Multiple Choice

11. Without using a calculator, which of the following is equal to $-\cos 170°$?

A. $\cos 100°$
B. $\cos 170°$
C. $\cos 10°$
D. $-\cos 10°$

Use the following information to answer question #12

To solve the equation $0.1663 = \sin x$, a student displayed the graph of $y = \sin x$ on a calculator using degree mode and the window shown.

```
WINDOW
 Xmin=0
 Xmax=180
 Xscl=30
 Ymin=-2
 Ymax=2
 Yscl=1
 Xres=1
```

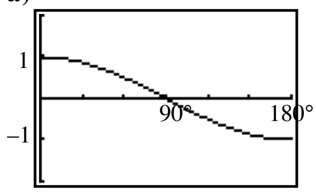
Numerical Response

12. Using the graph on the calculator display the student solved the equation $0.1663 = \sin x$ and found two correct solutions to the nearest degree. One solution, a, is the greater value, and the other solution b, is the smaller value.
To the nearest whole number, the value of $a - b$ is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. a)



b)

x (angle in degrees)	*y* (cosine ratio)
0°	1
30°	0.8660
45°	0.7071
60°	0.5
90°	0

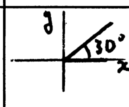
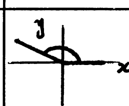
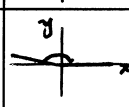
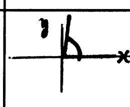
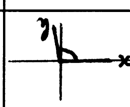
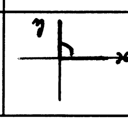
x (angle in degrees)	*y* (cosine ratio)
120°	–0.5
135°	–0.7071
150°	–0.8660
180°	–1

c) answers may vary: 10°, 170°

2. a) 45° **b)** 88° **c)** 18° **d)** 90°

3. a) 145° **b)** 25° **c)** 68° **d)** 21°
e) 85° **f)** 1° **g)** 0° **h)** 90°

4. See table below

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		30°
30°	2		150°
4°	2		176°
89°	1		89°
89°	2		91°
90°	between 1 and 2		90°

5. a) 36° **b)** 14°, 166° **c)** 45°, 135° **d)** 118°
e) 40°, 140° **f)** 50° **g)** 93° **h)** no solution **i)** 82°, 98°

6. a) no solution **b)** 166° **c)** 135°

7. a) 0°, 180° **b)** 90° **c)** 90° **d)** 0° **e)** 180° **f)** no soluton

8. a) 0.9659 **b)** 0.9659 **c)** 0.6665 **d)** 0.9993 **e)** 0.7660 **f)** –0.5432

9. a) 30°, 150° **b)** 45° **c)** 60°, 120°

10. a) 76°, 104° **b)** 45°, 135°

11. C **12.**

1	6	1	

Trigonometry Lesson #3 Extension:
Sine and Cosine Ratios for Angles from 0° to 360°

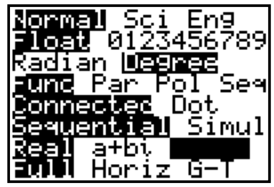
If you have studied Lesson #3 omit this lesson and move to Lesson #4.

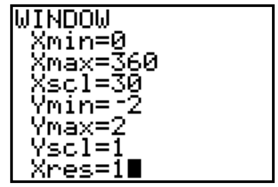
Warm-Up #1

a) Use a calculator to find the value of sin 30° and sin 150° and compare the answers.

b) Compare the value of cos 40° and cos 140° (to four decimal places).

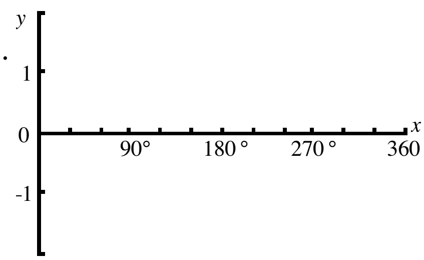
- To show why the above answers are the same, we will construct and analyze the graphs of $y = \sin x$ and $y = \cos x$ from 0° to 360°.
- Use the following mode and window settings.





Warm-Up #2 *Exploring the Graph of* $y = \sin x$

a) Sketch the graph of $y = \sin x$, $0 \le x \le 360°$, on the grid. Show the results of Warm-Up #1 a) on the grid.



b) Use the trace feature to complete the table below to four decimal places where necessary.
Press **Trace**, enter the value of x,
then press **Enter** to find the value of y.

x **(angle in degrees)**	***y*** **(sine ratio)**
0°	
30°	
45°	
60°	
90°	

x **(angle in degrees)**	***y*** **(sine ratio)**
120°	
135°	
150°	
180°	

x **(angle in degrees)**	***y*** **(sine ratio)**
210°	
225°	
240°	
270°	

x **(angle in degrees)**	***y*** **(sine ratio)**
300°	
315°	
330°	
360°	

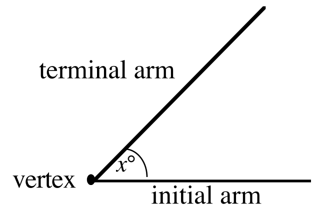
c) Without using a calculator, state two angles (not in the table) which have:
i) the same positive value for the sine ratio.
ii) the same negative value for the sine ratio.

Complete Assignment Question #1

Warm-Up #1 and Assignment question #1 showed a relationship between pairs of angles. This relationship can be investigated further using the concepts of **rotation** angles and **reference** angles.

Rotation Angle

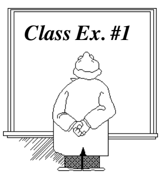
A **rotation angle** is formed by rotating an initial arm through an angle $x°$ about a fixed point (the vertex) to the terminal arm.



A **positive angle** results from a counter clockwise rotation.

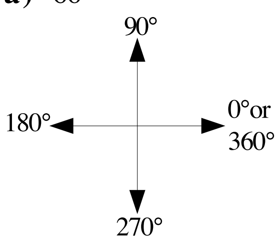
A **negative angle** results from a clockwise rotation. Negative rotation angles will be studied in more advanced math courses.

If the vertex is at the origin and the initial arm is along the x–axis the rotation angle is said be in **standard position**.

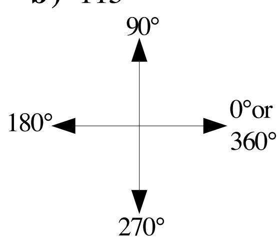


Draw the following rotation angles in standard position.

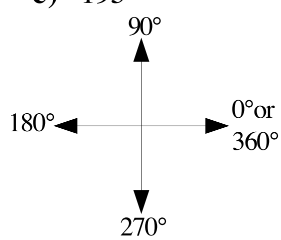
a) 60°



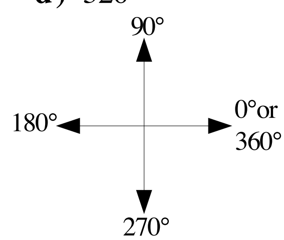
b) 115°



c) 195°

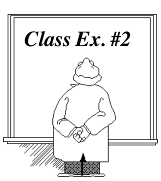


d) 320°



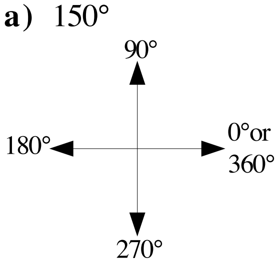
Reference Angles

In order to investigate pairs of angles with identical trigonometric ratios, we introduce the concept of a **reference angle**. A **reference angle** is the acute angle formed between the terminal arm of the rotation angle and the x-axis.

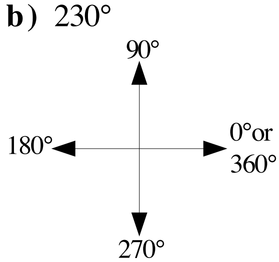


In each case, sketch the rotation angle and state the reference angle.

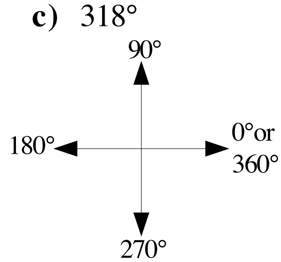
a) 150°



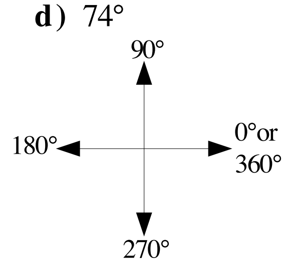
b) 230°



c) 318°



d) 74°



Complete Assignment Questions #2 - #4

Warm-Up #2 — *Positive and Negative Values for the Sine and Cosine Ratios*

The diagram below shows the values of the sine and cosine ratios from the earlier part of this lesson placed in their respective quadrants.

$y = \sin x$

Quadrant 2

x (angle)	y (ratio)	sign +/–
120°	0.8660	
135°	0.7071	
150°	0.5	
180°	0	

Quadrant 1

x (angle)	y (ratio)	sign +/–
0°	0	
30°	0.5	
45°	0.7071	
60°	0.8660	
90°	1	

Quadrant 3

x (angle)	y (ratio)	sign +/–
210°	–0.5	
225°	–0.7071	
240°	–0.8660	
270°	–1	

Quadrant 4

x (angle)	y (ratio)	sign +/–
300°	–0.8660	
315°	–0.7071	
330°	–0.5	
360°	0	

$y = \cos x$

Quadrant 2

x (angle)	y (ratio)	sign +/–
120°	–0.5	
135°	–0.7071	
150°	–0.8660	
180°	–1	

Quadrant 1

x (angle)	y (ratio)	sign +/–
0°	1	
30°	0.8660	
45°	0.7071	
60°	0.5	
90°	0	

Quadrant 3

x (angle)	y (ratio)	sign +/–
210°	–0.8660	
225°	–0.7071	
240°	–0.5	
270°	0	

Quadrant 4

x (angle)	y (ratio)	sign +/–
300°	0.5	
315°	0.7071	
330°	0.8660	
360°	1	

a) Complete the diagram above by entering the sign (+/–) of the sine or cosine ratio in each quadrant.

b) Complete the following statements using the results from a).

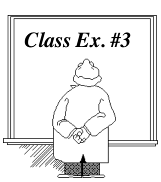
i) Sine ratios have **positive** values in quadrants _____ and _____ .

ii) Cosine ratios have **positive** values in quadrants _____ and _____ .

iii) Sine ratios have **negative** values in quadrants _____ and _____ .

iv) Cosine ratios have **negative** values in quadrants _____ and _____ .

These results should be memorized.



Without using a calculator or a graph, state whether the following ratios are positive or negative.

a) sin 155° **b)** cos 170° **c)** sin 275° **d)** cos 206°

Solving Equations Involving Sine or Cosine

We can use the concepts of reference angles and signs of the trigonometric ratio to solve equations involving sine or cosine. The following procedure may be used to solve an equation such as $\sin x = 0.5$, where $0° \le x \le 360°$.

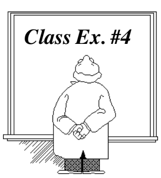
Step 1: Determine the quadrant(s) the angle will be in by looking at the sign of the ratio.

Step 2: Determine the reference angle (always between 0° and 90°) and draw a rough sketch in the appropriate quadrant(s). The reference angle is found as follows:
Use $\boxed{\text{2nd}}$ $\boxed{\text{sin}}$ or $\boxed{\text{2nd}}$ $\boxed{\text{cos}}$ of the **absolute value** of the given quantity.

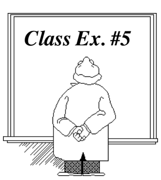
Step 3: Determine the rotation angle(s) using the reference angle and the quadrant(s).

- Always check the given domain to determine which quadrants are valid in the calculation. Sometimes the domain is restricted to $0° \le x \le 180°$, or $90° \le x \le 180°$.

- Pressing $\boxed{\text{sin}}$ or $\boxed{\text{cos}}$ then the angle $\boxed{\text{Enter}}$ gives the value of the trigonometric ratio for that angle,
  whereas
  pressing $\boxed{\text{2nd}}$ $\boxed{\text{sin}}$ or $\boxed{\text{2nd}}$ $\boxed{\text{cos}}$ then the given value $\boxed{\text{Enter}}$ determines the reference angle.

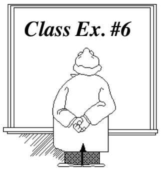


Use the above procedure to solve $\sin x = 0.5$, where $0° \le x \le 360°$.



Find the measure of x to the nearest whole number, where $0° \le x \le 360°$.

a) $\cos x = 0.7880$ **b)** $\sin x = -0.8090$ **c)** $\cos x = -0.8090$



Solve the following equations, to the nearest degree for $0° \le \theta \le 180°$.

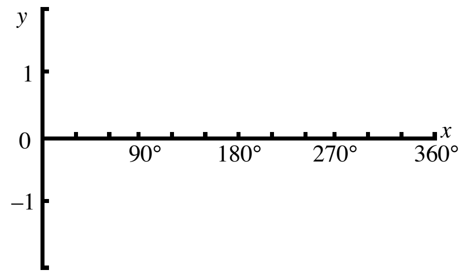
a) $2\cos\theta - 1 = 0$ **b)** $3\sin\theta - \sqrt{3} = 0$

Complete Assignment Questions #5 - #13

Assignment

1. Use a graphing calculator with the settings for Warm-Up #1 to answer the following.

a) Sketch the graph of $y = \cos x, 0 \le x \le 360°$, on the grid.



b) Complete the table below to four decimal places where necessary.

x (angle in degrees)	*y* (cosine ratio)
0°	
30°	
45°	
60°	
90°	

x (angle in degrees)	*y* (cosine ratio)
120°	
135°	
150°	
180°	

x (angle in degrees)	*y* (cosine ratio)
210°	
225°	
240°	
270°	

x (angle in degrees)	*y* (cosine ratio)
300°	
315°	
330°	
360°	

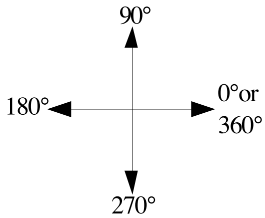
c) Without using a calculator, state two angles (not in the table) which have:

i) the same positive value for the cosine ratio.

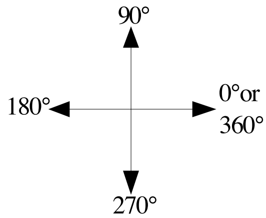
ii) the same negative value for the cosine ratio.

2. In each case, sketch the rotation angle and state the reference angle.

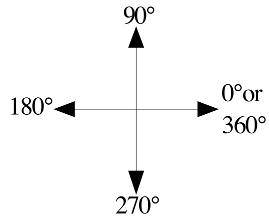
a) 45°



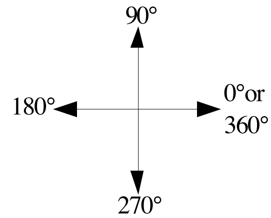
b) 270°



c) 200°



d) 92°



3. Find the reference angle for the following rotation angles.

a) 305° **b)** 25° **c)** 112° **d)** 209°

e) 280° **f)** 179° **g)** 358° **h)** 271°

i) 0° **j)** 90° **k)** 270° **l)** 360°

4. Complete the following tables given the reference angle and the quadrant.

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	2		
45°	4		
60°	1		
25°	3		
15°	4		
36°	3		

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		
30°	4		
4°	3		
89°	2		
0°	between 2 and 3		
90°	between 1 and 2		

5. Find the measure of x where $0° \le x \le 360°$ to the nearest degree.

a) $\cos x = 0.8088$ **b)** $\sin x = -0.2431$ **c)** $\sin x = 0.7071$

d) $\cos x = -0.4668$ **e)** $\sin x = 0.6485$ **f)** $\cos x = 0.6485$

g) $\cos x = -0.0602$ **h)** $\sin x = -0.0856$ **i)** $\sin x = 0.9900$

6. Solve for x in each of the following where $0° \le x \le 180°$.

a) $\cos x = 0.8088$ **b)** $\sin x = 0.2431$ **c)** $\cos x = -0.7071$

d) $\cos x = -0.0515$ **e)** $\sin x = 0.0755$ **f)** $\cos x = -0.0889$

7. Find the measure of θ where $0° \leq \theta \leq 360°$ to the nearest degree.

a) $\sin \theta = 0$ **b)** $\cos \theta = 0$ **c)** $\sin \theta = 1$

d) $\cos \theta = 1$ **e)** $\cos \theta = -1$ **f)** $\sin \theta = -1$

8. Evaluate to 4 decimal places.

a) sin 75° **b)** sin 105° **c)** cos 48.2°

d) cos 192° **e)** sin 330° **f)** cos 122.9°

9. Solve the following equations, to the nearest degree, where the variables are defined on a domain of 0° to 180°

a) $2 \sin a + 1 = 0$ **b)** $2 \cos \theta - \sqrt{2} = 0$ **c)** $2 \sin b = \sqrt{3}$

10. Given that $(\cos x)^2$ can be written as $\cos^2 x$, solve the following equations, to the nearest degree, on the given domain.

a) $\cos^2 x = \frac{1}{16},\ 0° \leq x \leq 360°$ **b)** $2 \sin^2 \theta = 1,\ 0° \leq \theta \leq 360°$

c) $2 \cos^2 x = \sqrt{3},\ 0° \leq x \leq 180°$

11. Find x in the following ($0° \le x \le 360°$). Calculate angles to the nearest degree and sides to 4 decimal places where necessary.

a) $\sin 150° = x$ **b)** $\cos x = -1.7737$ **c)** $0.3456 = \sin x$

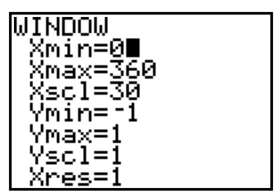
d) $\cos 93° = x$ **e)** $-0.0089 = \cos x$ **f)** $\sin x = -0.4226$

Multiple Choice **12.** Without using a calculator, which of the following is equal to $-\cos 170°$?

A. $\cos 100°$
B. $\cos 170°$
C. $\cos 10°$
D. $-\cos 10°$

Use the following information to answer question #13

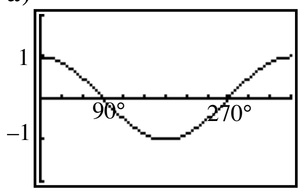
To solve the equation $0.1663 = \cos x$, a student displayed the graph of $y = \cos x$ on a calculator using degree mode and the window shown.



Numerical Response **13.** Using the graph on the calculator display the student solved the equation $0.1663 = \cos x$ and found two correct solutions to the nearest degree. One solution, a, is the greater value, and the other solution b, is the smaller value.
To the nearest whole number, the value of $a - b$ is _____ .
(Record your answer in the numerical response box from left to right)

Answer Key

1. a)



b)

x (angle in degrees)	y (cosine ratio)
0°	1
30°	0.8660
45°	0.7071
60°	0.5
90°	0

x (angle in degrees)	y (cosine ratio)
120°	–0.5
135°	–0.7071
150°	–0.8660
180°	–1

x (angle in degrees)	y (cosine ratio)
210°	–0.8660
225°	-0.7071
240°	–0.5
270°	0

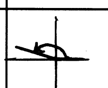
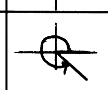
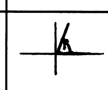
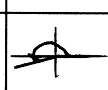
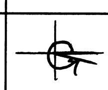
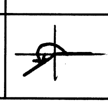
x (angle in degrees)	y (cosine ratio)
300°	0.5
315°	0.7071
330°	0.8660
360°	1

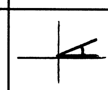
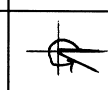
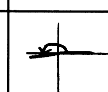
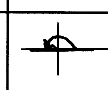
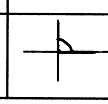
c) answers may vary
i) 70°, 290°
ii) 130°, 230°

2. a) 45° **b)** 90° **c)** 20° **d)** 88°

3. a) 55° **b)** 25° **c)** 68° **d)** 29° **e)** 80° **f)** 1°
g) 2° **h)** 89° **i)** 0° **j)** 90° **k)** 90° **l)** 0°

4.

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	2		150°
45°	4		315°
60°	1		60°
25°	3		205°
15°	4		345°
36°	3		216°

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		30°
30°	4		330°
4°	3		184°
89°	2		91°
0°	between 2 and 3		180
90°	between 1 and 2		90°

5. a) 36°, 324° **b)** 194°, 346° **c)** 45°, 135° **d)** 118°, 242°
e) 40°, 140° **f)** 50°, 310° **g)** 93°, 267° **h)** 185°, 355° **i)** 82°, 98°

6. a) 36° **b)** 14°, 166° **c)** 135° **d)** 93° **e)** 4°, 176° **f)** 95°

7. a) 0°, 180°, 360° **b)** 90°, 270° **c)** 90° **d)** 0°, 360° **e)** 180° **f)** 270°

8. a) 0.9659 **b)** 0.9659 **c)** 0.6665 **d)** –0.9781 **e)** –0.5000 **f)** –0.5432

9. a) no solution **b)** 45° **c)** 60°, 120°

10. a) 76°, 104°, 256°, 284° **b)** 45°, 135°, 225°, 315° **c)** 21°, 159°
11. a) 0.5 **b)** no solution **c)** 20°, 160° **d)** –0.0523 **e)** 91°, 269° **f)** 205°, 335°
12. C **13.**

2	0	0	

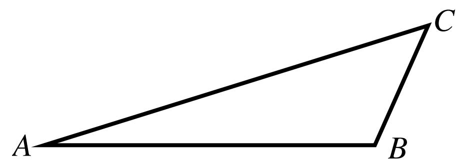
Trigonometry Lesson #4: The Sine Law

Warm-Up

Often in trigonometry it is convenient to use the following notation.

In triangle ABC, represent

the length of the side opposite angle A by a,
the length of the side opposite angle B by b,
and the length of the side opposite angle C by c.



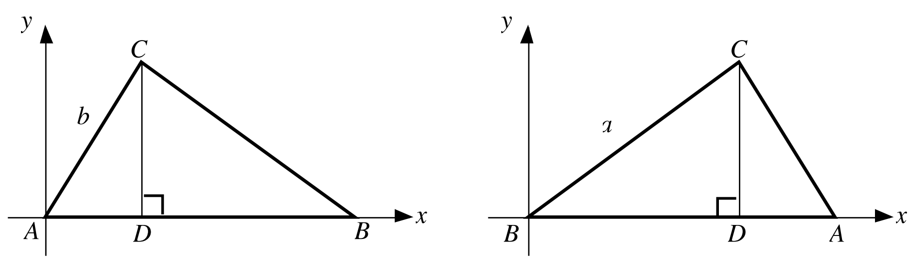
In the next three lessons, we focus on solving triangles which are not right angled and in which *SOHCAHTOA* is not valid.

The Sine Law

In every triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Proof of the Sine Law

The diagrams show the same triangle ABC placed with base AB on the x–axis.
In diagram i) the origin is at A, and in diagram ii) the origin is at B.
The line CD is drawn perpendicular to AB.



Complete the following work to show that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

In i) $\sin A = \dfrac{CD}{AC} = \dfrac{CD}{b}$ In ii) $\sin B = \qquad =$

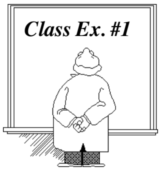
$CD =$ $CD =$

It follows that $b \sin A =$

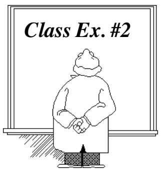
Dividing both sides by $\sin A \sin B$ gives the result

Repeating the work above with AC placed on the x–axis would give the result $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

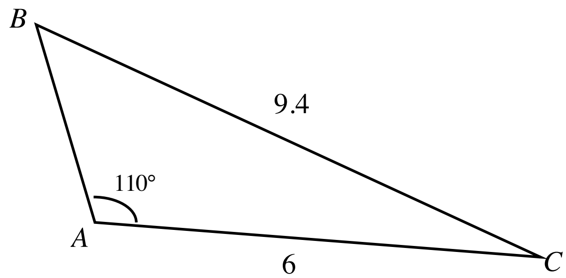
Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

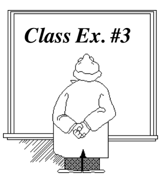


In ΔABC, angle $C = 40°$, angle $B = 48°$ and $c = 5.8$ cm.
Calculate b to the nearest tenth of a cm.



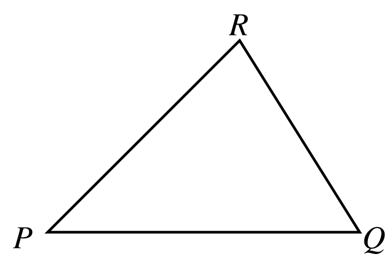
Use the sine law in the triangle shown to determine the measure of $\angle ACB$ to the nearest degree.





A surveyor measures a base line PQ 440 m long. He takes measurements of a landmark R from P and Q, and finds that $\angle QPR = 46°$ and $\angle PQR = 75°$.

a) Calculate the perimeter of ΔPQR to the nearest metre.



b) Calculate the area of ΔPQR to the nearest square metre.

Complete Assignment Questions #1 - #11

Extension

The Sine Law - the Ambiguous Case

Triangles can be constructed given three measurements, one of which must be the length of a side. The following explorations can be completed using a compass and protractor set or software such as *Geometer's Sketchpad*.

Exploration #1 *Constructing Congruent Triangles*

SSS - Sketch ΔPQR with $PQ = 3$ cm, $PR = 4$ cm, and $QR = 6$ cm. Is it possible to construct a triangle with the same measurements which is not congruent to the first one?

SAS - Sketch ΔPQR with $PQ = 5$ cm, $QR = 4$ cm, and $\angle PQR = 40°$. Is it possible to construct a triangle with the same measurements and angle which is not congruent to the first one?

ASA - Sketch ΔPQR with $PQ = 3$ cm, $\angle PQR = 40°$and $\angle QPR = 30°$. Is it possible to construct a triangle the same measurements and two angles which is not congruent to the first one?

SSA - Sketch ΔPQR with $PQ = 6$ cm, $PR = 4$ cm, and $\angle PQR = 40°$. Is it possible to construct a triangle with the same measurements and angle which is not congruent to the first one?

Exploration #2 *Constructing Triangles using SSA*

In Exploration #1 we found that we could not draw triangles which were not congruent in the case of SSS, SAS, and ASA.

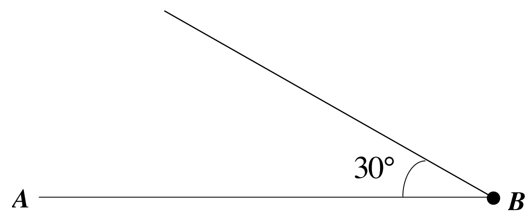
However, in the case of SSA we found that we could draw two different triangles when given two sides and an angle.

Given SSA in a triangle there are actually three possible situations.
Use the diagrams below to explore the three different cases.

ΔABC has AB = 6 cm and $\angle ABC = 30°$.

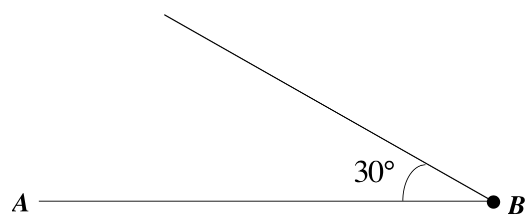
Case 1 *Two Possible Triangles*

Use compasses to show that for a given length of AC there are two possible positions for C.



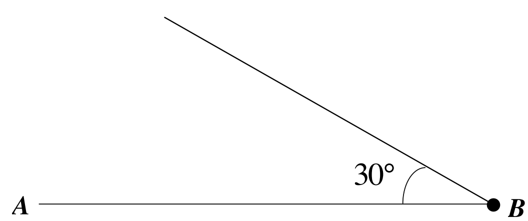
Case 2 *One Unique Triangle*

Use compasses to show that for a given length of AC there is only one position for C.

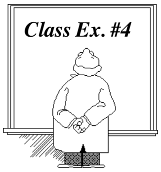


Case 3 *No Possible Triangle*

Use compasses to show that for a given length of AC there are no possible positions for C.

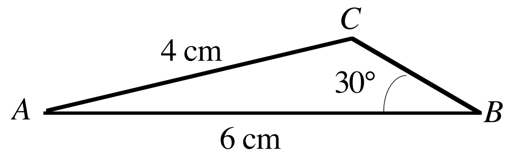


Ambiguous Case of the Sine Law



ΔABC has $AB = 6$ cm, $AC = 4$ cm and $\angle ABC = 30°$. Complete the following procedure which uses the sine law to determine the measure of $\angle ACB$.

Step 1: Construct a diagram.
(At this stage we do not know if the diagram is unique or not)



Step 2: Use the Sine Law to determine sin C.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$\sin C = 0.75$

Note that the original triangle we constructed could have been obtuse angled at A.

Step 3: Solve $\sin C = 0.75$

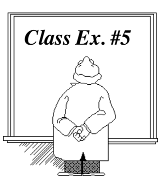
Reference angle is 49°

$\angle C = 49°$ or $\angle C =$ ______

$\angle ACB = 49°$ or ______

Note that since $\angle B = 30°$, $\angle C$ must be less then $180° - 30° = 150°$ in order to form a triangle.

- If the reference angle θ in Step 3 is greater than the given angle in ΔABC then there will be two possible values for the required angle, θ and $180° - \theta$. This is known as the Ambiguous Case.
- If the reference angle θ is less then the given angle in ΔABC there will only be one value for the required angle.



Find the measure of $\angle C$ in the following triangles.

a) ΔABC where $\angle A = 50°$, $a = 7.5$ cm and $c = 9.5$ cm

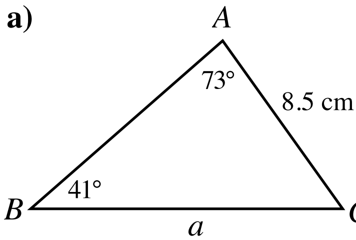
b) ΔABC where $\angle A = 50°$, $a = 9.5$ cm and $c = 7.5$ cm

Complete Assignment Questions #12 - #16

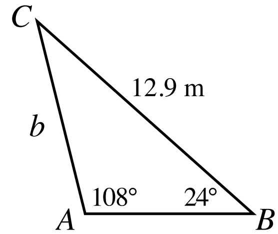
Assignment

1. In each case find the length of the indicated side, to the nearest tenth.

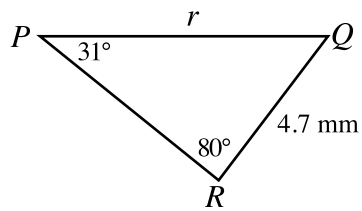
a)



b)

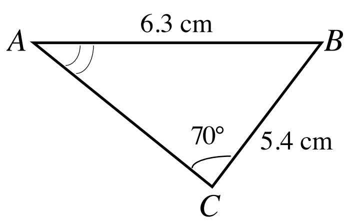


c)

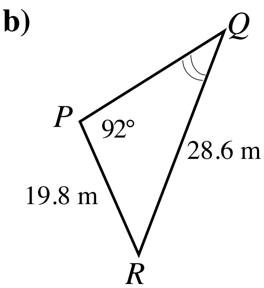


2. In each case find the measure of the indicated angle, to the nearest degree.

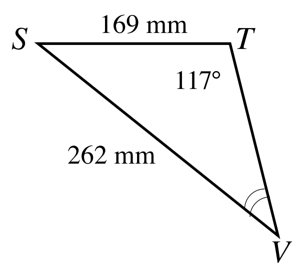
a)



b)



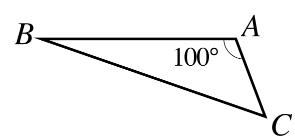
c)



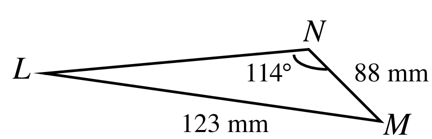
3. In ΔABC, angle $A = 49°$, angle $B = 57°$ and $a = 8$. Calculate b, to the nearest tenth.

4. In ΔABC, angle $A = 53°$, angle $B = 61°$ and $b = 2.8$. Calculate a, to the nearest tenth.

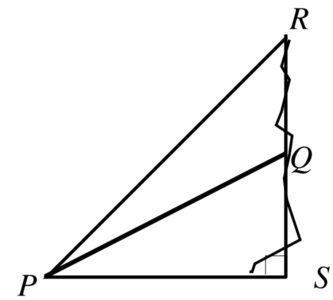
5. In ΔABC, angle $A = 100°$, $a = 7.9$ and $b = 4.5$.
Calculate angle B, to the nearest degree.



6. In ΔLMN, angle $LNM = 114°$, $LM = 123$ mm and $MN = 88$ mm. Calculate $\angle LMN$, to the nearest degree.



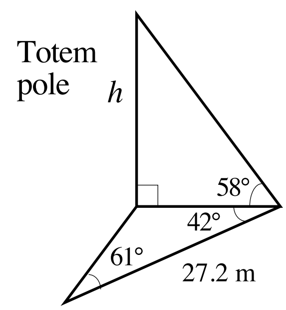
7. P and Q are two bases for a mountain climb. PQ is 600 m and QR is a vertical stretch of a rock face. The angle of elevation of Q from P is 31° and the angle of elevation of R from P is 41°.



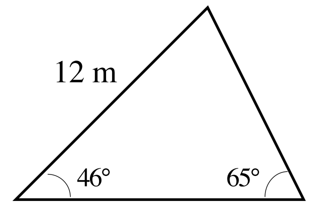
a) Mark these measurements on the diagram and state the measure of angle PRQ.

b) Use the sine law in ΔPQR to calculate the height of the vertical climb, QR, to the nearest metre.

8. On June 30, 1956, the world's largest free standing totem pole was erected in Beacon Hill Park in Victoria. Recently a surveyor took measurements to verify the height of the totem pole.
Use the measurements to find the height of the totem pole, to the nearest metre.



9. Three students are trying to determine the area of the triangle in the diagram. Each student is given a different formula with which to determine the area. The area of the triangle is 53.3 m^2.



Show how each student arrived at this answer.

Student #1: Use the formula $A = \frac{1}{2}bh$, where b is the length of the base and h is the vertical height.

Student #2: Use Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the three sides and s is the semi-perimeter of the triangle.

Student #3: Use the formula $A = \frac{1}{2}ab\sin C$, where a and b are the lengths of two sides and angle C is the angle containing the sides a and b.

Multiple Choice

10. In triangle PQR, angle $P = 20°$, angle $R = 150°$ and $QR = 6$ m. The length in m of PQ is

A. 4.1
B. 8.8
C. 15.2
D. 17.3

11. Triangle LMN is obtuse angled at M and $\angle MLN = 40°$. Sin LNM is equal to

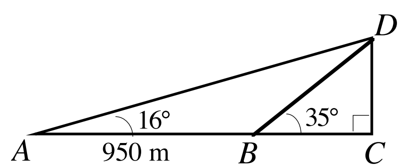
A. $\dfrac{MN}{LM \sin 40°}$
B. $\dfrac{LM}{MN \sin 40°}$
C. $\dfrac{LM \sin 40°}{MN}$
D. $\dfrac{MN \sin 40°}{LM}$

12. In ΔABC, $\angle A = 30°$, $BC = 10$ units and $AC = 15$ units. If $\angle B$ is acute-angled, then $\angle$C is

A. 19.4°
B. 48.6°
C. 101.4°
D. 130.6°

Numerical Response

13. From a point A, level with the foot of a hill, the angle of elevation of the top of the hill is 16°. From a point B, 950 metres nearer the foot of the hill, the angle of elevation of the top is 35°.
The height of the hill, DC, to the nearest metre, is ____ .



(Record your answer in the numerical response box from left to right)

Extension Questions.

14. Find the measure of $\angle C$ in the following triangles.

a) ΔABC where $\angle A = 31°$, $a = 4.5$ cm and $c = 4.9$ cm

b) ΔABC where $\angle A = 61°$, $a = 7.5$ cm and $c = 5.8$ cm

15. In ΔLMN, angle $LMN = 42°$, $LN = 32$ mm and $LM = 42$ mm. Calculate angle LNM, to the nearest degree.

Multiple Choice

16. In triangle PQR, angle $P = 30°$, $QR = 5$ cm, and $PR = 7.5$ cm. The measure of angle PQR is

A. 19.4°
B. 19.4° or 161.6°
C. 48.6°
D. 48.6° or 131.4°

Answer Key

1. a) 12.4 cm **b)** 5.5 m **c)** 9.0 mm **2. a)** 54° **b)** 44° **c)** 35°

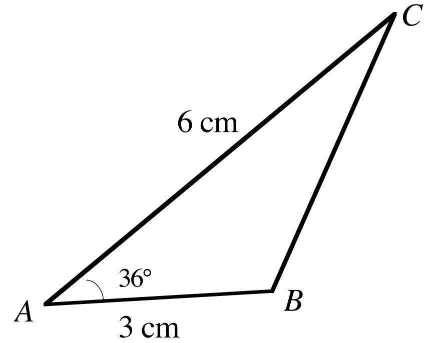
3. 8.9 **4.** 2.6 **5.** 34° **6.** 25° **7. a)** 49° **b)** 138 m

8. 39 m **10.** B **11.** C **12.** C **13.**

4	6	1	

14. a) 34° or 146° **b)** 43° **15.** 61° or 119° **16.** D

Trigonometry Lesson #5:
The Cosine Law



Warm-Up

Consider triangle ABC in which $\angle A = 36°$, $AB = 3$ cm and $AC = 6$ cm. What happens when you apply the sine law to determine the length of BC?

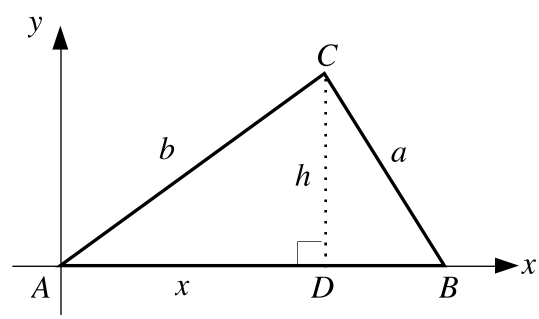
In the example above the sine law is **not** applicable if we are given the length of two sides and the contained angle.

In such a situation the **cosine law** can be applied to solve the problem.

The Cosine Law

In every triangle ABC, $\quad a^2 = b^2 + c^2 - 2bc \cos A$.

Proof of the Cosine Law



- The diagram shows triangle ABC placed with base AB on the x–axis and A at the origin.
- The line CD is drawn perpendicular to AB and is h units in length.
- $AD = x$ units so $DB = c - x$ units.

Complete the following work to show that $\quad a^2 = b^2 + c^2 - 2bc \cos A$.

In $\Delta ADC \quad \cos A = \dfrac{AD}{AC} = \dfrac{x}{b}$

so $x =$

In $\Delta BDC \quad BC^2 = CD^2 + DB^2$

$$a^2 = h^2 + (c - x)^2$$

$$a^2 = h^2 + c^2 - 2cx + x^2$$

$$a^2 = (h^2 + x^2) + c^2 - 2cx$$

$$a^2 = \qquad + c^2 - 2c(\qquad)$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

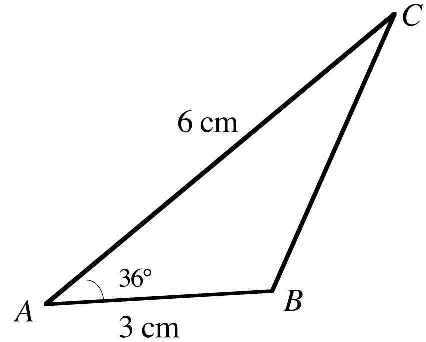
By placing AC and then BC on the x–axis similar equations can be derived.

$$b^2 = c^2 + a^2 - 2ca \cos B \qquad\qquad c^2 = a^2 + b^2 - 2ab \cos C$$

This version of the cosine law can be used in any triangle if we are given the length of two sides and the contained angle.

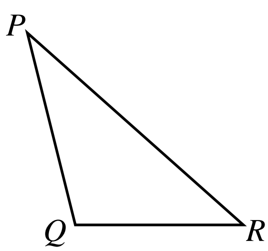
Class Ex. #1

Consider the Warm-up where ΔABC has $\angle A = 36°$, $AB = 3$ cm and $AC = 6$ cm. Determine the length of BC, to the nearest tenth of a cm, using the cosine law.



Class Ex. #2

Find the length, to the nearest tenth of a cm, of the third side of ΔPQR if $QP = 1.7$ cm, $QR = 3.1$ cm and $\angle PQR = 110°$.



Class Ex. #3

Bellevue is 30 km north of Ayr and Churchville is 18 km northwest of Ayr. Calculate the distance between Bellevue and Churchville to the nearest km.

Complete Assignment Questions #1 - #5

Alternative Form of the Cosine Law

The equation $a^2 = b^2 + c^2 - 2bc \cos A$

can be rearranged to the form $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

This form of the cosine law can be used to determine any angle in a triangle when we are given the length of all three sides.

Class Ex. #4

Complete the following for triangle ABC.

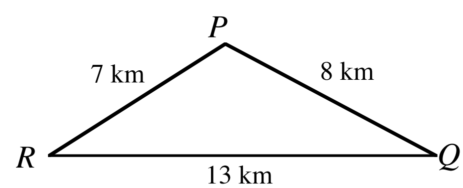
a) $\cos B =$ **b)** $\cos C =$

Class Ex. #5

Determine the largest angle in ΔABC if $a = 14.7$, $b = 8.9$, and $c = 12.6$.

Class Ex. #6

Two ships set sail from port heading in different directions. The first ship sails 7 km to R and the second ship sails 8 km to Q. If the distance between R and Q is 13 km, determine the angle between the directions of the two ships.



Complete Assignment Questions #6 - #13

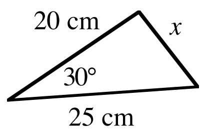
Assignment

1. Complete the following for triangle *STV*.

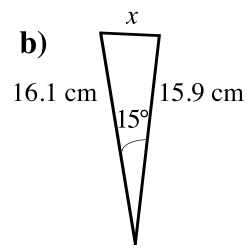
a) $s^2 =$ **b)** $v^2 =$

2. In each case find the length of the indicated side, to the nearest 0.1 cm.

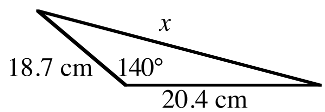
a)



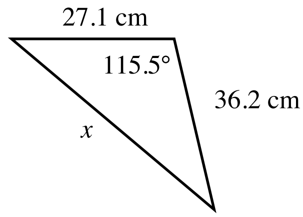
b)



c)

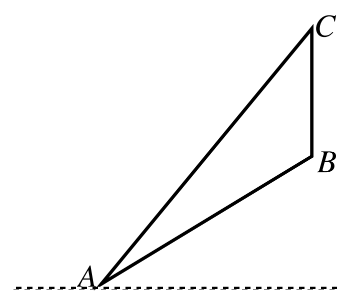


d)



3. In ΔABC, angle $A = 49°$, $b = 24$ and $c = 37$. Calculate a to the nearest whole number.

4. In the diagram, *AB* represents part of a road constructed on the incline of a hill. *BC* represents a telephone pole 7.5 m tall at the side of the road. A guide wire attached to the top of the pole is joined to the ground at *A*. If *AB* = 11.4 m and $\angle ABC = 135°$, determine the length of the guide wire to the nearest 0.1 m



5. Solve triangle ABC in which $AB = 4.5$ cm, $BC = 7.8$ cm and angle $ABC = 79°$.
Round sides to the nearest tenth of a cm and angles to the nearest tenth of a degree.

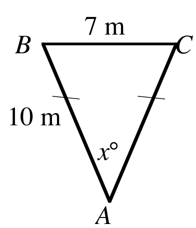
6. Complete the following for triangle DEF.

a) $\cos E =$

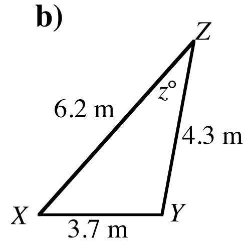
b) $\cos F =$

7. In each case find the measure of the indicated angle, to the nearest degree.

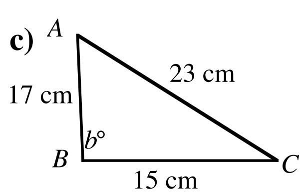
a)



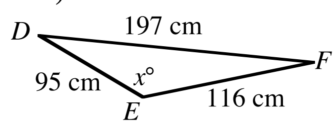
b)



c)

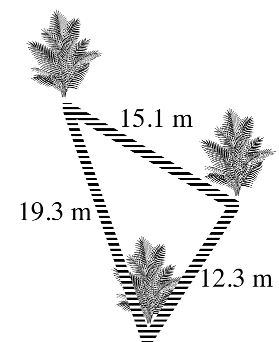


d)

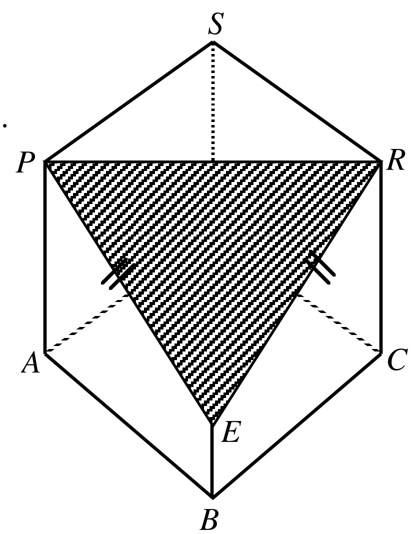


8. On a circular sheet of plastic on radius 21 cm, two holes have been drilled, one at the centre O, and one at C on the circumference of the circle. A third hole is to be drilled 17 cm from C and 12 cm from O. A triangle is drawn between the three holes. Calculate the measure of the angle formed at the centre of the circle.

9. Anwar and Ingrid have three trees in their garden. The trees form a triangle as shown in the diagram. Determine the smallest angle between the trees.



10. The solid in the diagram was formed by removing a corner from a cube of 24 cm. The length of EB is 6 cm.

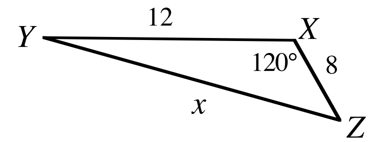


a) Calculate, to the nearest tenth, the lengths of PE and PR.

b) Calculate the measure of angle PER, to the nearest degree.

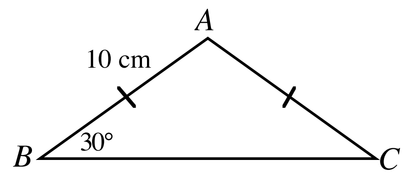
Multiple Choice

11. The value of x^2 is



A. 112
B. 304
C. $208 - 96\sqrt{3}$
D. $208 + 96\sqrt{3}$

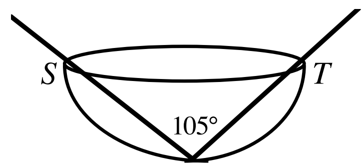
12. The length of BC in cm is



A. $5\sqrt{3}$
B. 10
C. $10\sqrt{3}$
D. 20

Numerical Response

13. The diagram shows a glass bowl with two chop-sticks resting on the rim at points S and T. The lengths of the parts of the chop-sticks inside the bowl are 9 cm and 11.5 cm respectively.



The length of ST, to the nearest tenth of a cm, is _____ .
(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $s^2 = t^2 + v^2 - 2tv\cos S$ **b)** $v^2 = s^2 + t^2 - 2st\cos V$

2. a) 12.6 cm **b)** 4.2 cm **c)** 36.7 cm **d)** 53.8 cm **3.** 28 **4.** 17.5

5. $\angle ABC = 79°$, $\angle BAC = 68.5°$, $\angle ACB = 32.5°$, $AC = 8.2$ cm, $BC = 7.8$ cm, $AB = 4.5$ cm.
Answers may vary slightly depending on method.

6. a) $\cos E = \dfrac{d^2 + f^2 - e^2}{2df}$ **b)** $\cos F = \dfrac{d^2 + e^2 - f^2}{2de}$

7. a) 41° **b)** 36° **c)** 92° **d)** 138° **8.** 54° **9.** 40°

10. a) $PE = 30.0$ cm, $PR = 33.9$ cm **b)** 69°

11. B **12.** C **13.**

1	6	.	3

Trigonometry Lesson #6:
Problems Involving the Sine Law and the Cosine Law

Problems in Trigonometry can be solved using *SOHCAHTOA*, the Sine Law, the Cosine Law or a combination of these. Use the following to determine which method is appropriate.

1. In a right triangle use *SOHCAHTOA*.

2. In a non-right triangle use:

i) the Cosine Law if you are given; • all three sides (SSS)

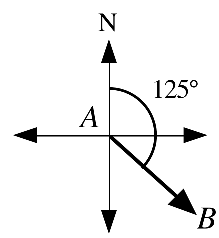
or

• two sides and the contained angle (SAS)

ii) the Sine Law in all other cases.

There are many practical examples in which students have to choose the appropriate method for solution. We introduce the concept of bearings since it provides further applications of the sine and cosine law. The assignment questions include questions on bearings and questions on other topics.

Bearings

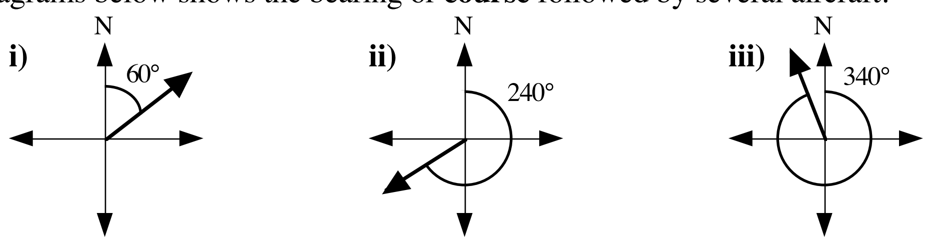


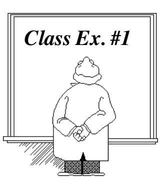
• The **bearing** of one point from a second point is a way of giving the direction. The diagram shows two points A and B and the North-South line through A.

• The bearing of B from A is the measure of angle NAB, i.e. 125°.

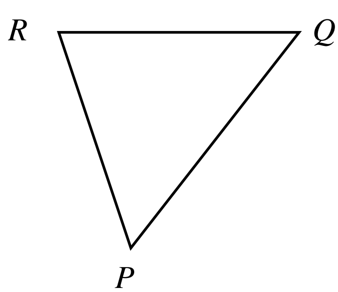
• Bearings are measured from North in a clockwise direction.

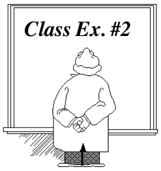
• The diagrams below shows the bearing or **course** followed by several aircraft.





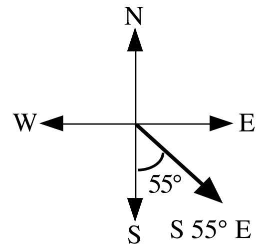
A liner leaves a port P and sails 15 km on a course of 37° to a position Q where it changes course to 270° and sails 12 km to a position R. Complete the sketch to illustrate this information and calculate the distance and course the liner must sail to return from R to P.





Two aircraft *A* and *B* leave an airport at the same time. *A* flies on a course of 90° at 700 km/h, and *B* flies on a course of 290° at 600 km/h. Calculate their distance apart after 12 minutes to the nearest kilometre.

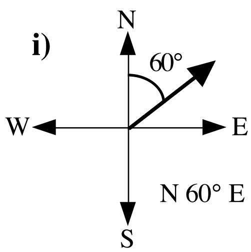
An Alternative Way of Describing a Direction

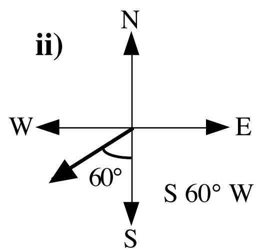


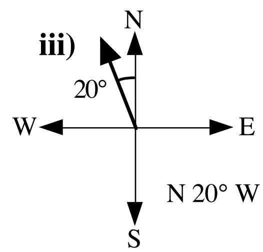
There are other ways of describing a course. A bearing of 125° is the same as 55° E of S (read as "55° East of South")

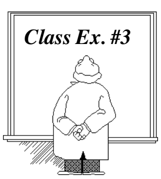
The direction can also be given as S 55° E.

The diagrams below show the course followed by the aircraft on the previous page.









A ship observes a lighthouse in a direction 50° W of N. After sailing 36 km in a direction 35° W of S the lighthouse is observed in a direction 15° E of N.

a) Draw a sketch showing the information given.

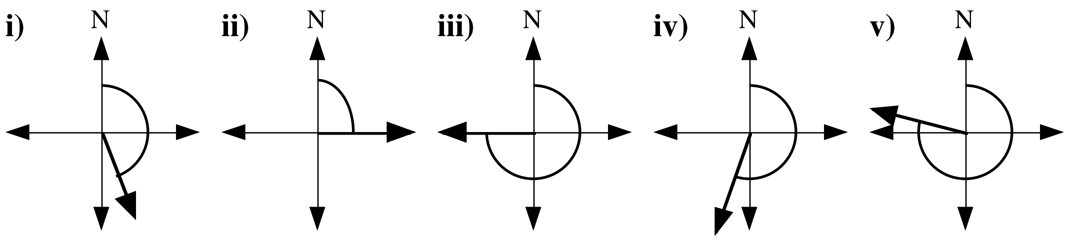
b) Calculate the distance of the ship from the lighthouse when the second observation is made.

Complete Assignment Questions #1 - #18

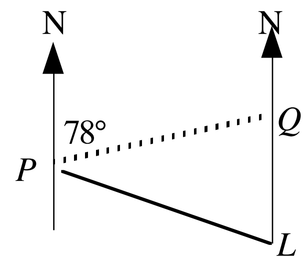
Assignment

1. Pair the following bearings with the correct diagram.

a) 270° **b)** 160° **c)** 300° **d)** 90° **e)** 220°



2. A ship is steaming at 16 km/h on a course of 78°, illustrated by the dotted line in the diagram. *L* represents the position of a lighthouse. At 0800 hours the ship is at *P*, which is on a bearing of 314° from *L* and one hour later it is at *Q* which is due north of *L*.



a) Determine the measures of angles *PLQ* and *LPQ*.

b) Calculate the distance *PL*, to the nearest kilometre.

c) At what time, to the nearest minute, is the ship nearest to the lighthouse?

3. PQ is a line on a stage. A spotlight S is mounted 8.5 m directly above P and is inclined at 45° to the vertical so that it shines on R, the midpoint of PQ.

a) Draw a diagram to represent this information and calculate the length of PQ.

b) To the nearest degree, calculate the measure of the angle which the spotlight must be turned to shine on Q.

4. At 12 noon a ship observes a lighthouse at a distance of 15 km in a direction of N50°E. It sails at 15 km/h in a direction S35°W. Find the distance and direction of the lighthouse from the ship at 3 pm. Answer to the nearest whole number.

5. Two spruce trees are 100m apart. From the point on the ground halfway between the trees the angles of elevation to the tops of the trees are 21° and 39°. Determine the distance, to the nearest metre, between the tops of the two trees.

6. Point A is 31.4 km due North of B. Point C is on a bearing of 57° from A and 32° from B.

a) To the nearest 0.1 km, determine the distance from B to C.

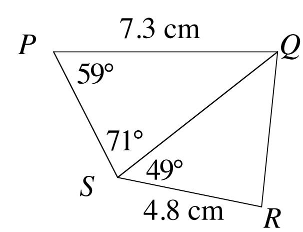
b) The line joining C to A is extended to the point D, where D is 24.7 km from A. Calculate the distance from D to B, to the nearest 0.1 km

7. Triangle ABC is drawn on a coordinate grid with $A(5, 7)$, $B(-1, 4)$ and $C(0, -3)$.

a) Use the distance formula to calculate the length of each side of the triangle in simplest radical form.

b) Calculate the measure of angle ACB to the nearest degree.

8. Use the information provided in the diagram to answer the following questions.



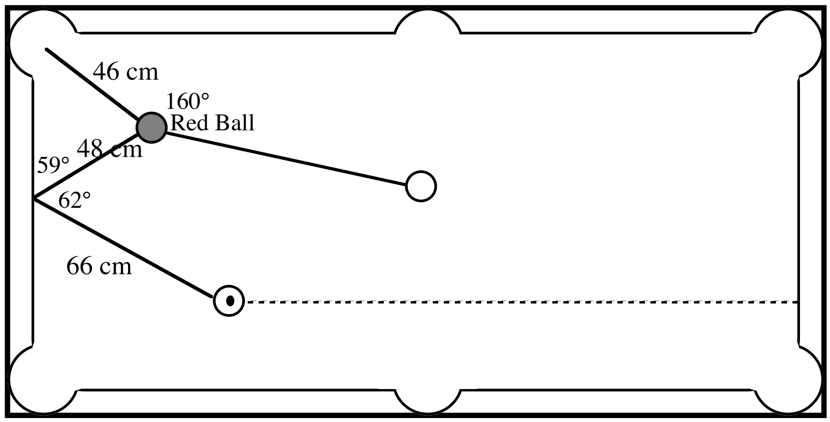
a) Calculate the length of PS, to the nearest 0.1 cm.

b) Calculate the length of QR, to the nearest 0.1 cm.

c) Calculate the area of quadrilateral $PQRS$, to the nearest square centimetre.

9. Billiards is a game like pool or snooker which is played by two players on a rectangular table 3.66 metres long by 1.86 metres wide. Three balls are used - white, spot white and red. The object of the game is to score points by pocketing balls (called hazards) or by hitting both other balls (called cannons).

In the diagram Bob propels his white ball on to the red ball which goes in to the corner pocket (scoring 3 points). The white ball deflects off the red ball on to the left cushion, rebounds, and strikes his opponent's spot white (scoring a further 2 points).



a) If the measurements are as in the diagram calculate the distance, to the nearest cm, between the red ball and the spot white ball before Bob attempts his shot.

b) After the spot white is hit by the white ball it travels parallel to the bottom cushion until it stops just touching the right cushion. To the nearest 10 cm, calculate the distance travelled by the spot white (ignore the width of the ball in the calculation).

Multiple Choice

In questions 10 -12 you are to decide which is the most appropriate method for solving the problem.

10. Yachts in a race have to sail a triangular course. First they sail in a direction of S45°E for 8 km. They change direction and sail on a course of N45°E. The last part of the course is to return to the start by sailing due West. How far was the second part of the course? The most appropriate method for solving this problem is

A. *SOHCAHTOA*
B. the Sine Law
C. the Cosine Law
D. the problem cannot be solved without further information.

11. In ΔABC, $BA = 9$ cm, $AC = 13$ cm and $\angle ABC = 113°$. Calculate the measure of $\angle BCA$. The most appropriate method for solving this problem is

A. *SOHCAHTOA*
B. the Sine Law
C. the Cosine Law
D. the problem cannot be solved without further information.

12. A pilot leaves base flying on a bearing of 340°. After 30 minutes he changes course to 108° and flies in this direction until he is due north of base. How far does he have to fly South to return to base? The most appropriate method for solving this problem is

A. *SOHCAHTOA*
B. the Sine Law
C. the Cosine Law
D. the problem cannot be solved without further information.

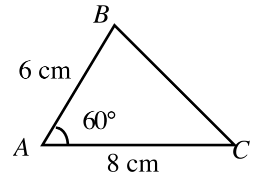
Numerical Response

13. Two aircraft X and Y leave an airport at the same time. X flies on a course of 70° at 720 km/h, and Y flies on a course of 350° at 600 km/h. To the nearest kilometre, the distance between the aircraft after 5 minutes is _____

(Record your answer in the numerical response box from left to right)

Use the following information to answer questions #14 and #15

In triangle ABC, angle $BAC = 60°$, $AB = 6$ cm, and $AC = 8$ cm.



14. The length of BC, to the nearest tenth of a cm, is _____

(Record your answer in the numerical response box from left to right)

15. The ratio $\dfrac{\sin C}{\sin B}$, to the nearest hundredth, is _____

(Record your answer in the numerical response box from left to right)

Use the following information to answer questions #16 - #18

The minute hand of a clock is 12 cm long and the hour hand is 10 cm long.

16. To the nearest degree, the angle between these hands at 5 o'clock is _____ .

(Record your answer in the numerical response box from left to right)

17. To the nearest degree, the angle between these hands at 7:30 pm is _____ .

(Record your answer in the numerical response box from left to right)

18. To the nearest 0.1 cm, the distance between the tips of the hands at 9:30 pm is _____ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** iii) **b)** i) **c)** v) **d)** ii) **e)** iv)

2. **a)** 46°, 56° **b)** 22 km **c)** 0846

3. **a)** 17 m **b)** 18°

4. 60 km in a direction N 39° E

5. 102 m

6. **a)** 62.3 km **b)** 27.4 km

7. **a)** $AB = 3\sqrt{5}$, $AC = 5\sqrt{5}$, $BC = 5\sqrt{2}$ **b)** 35°

8. **a)** 5.9 cm **b)** 5.0 cm **c)** 30 cm^2

9. **a)** 61 cm **b)** 310 cm

10. A **11.** B **12.** D

13.

7	1		

14.

7	.	2	

15.

0	.	7	5

16.

1	5	0	

17.

4	5		

18.

1	7	.	5

Sequences and Data Tables Lesson #1:
Tables

GST, PST, and Harmonized Sales Taxes

GST is a government tax placed on goods and services. Stores are required to collect GST on behalf of, and then submit to, **the federal government**. The rate is 7% for all provinces and territories.

PST is a provincial sales tax placed on goods and services and charged everywhere except for Alberta and the territories. Stores are required to collect PST on behalf of, and then submit to, **the provincial government**. In all provinces (except Quebec and Prince Edward Island) that PST is collected, the tax is charged on the selling price of the item before GST is applied and GST is charged on the selling price before PST is applied. This avoids GST being charged on PST. In Quebec and Prince Edward Island, the PST is charged on the selling price PLUS the GST, thus charging GST on PST.

Harmonized Sales Tax (HST) is charged in New Brunswick, Nova Scotia, and Newfoundland and Labrador. This tax replaces the GST and PST by charging 15% on goods and services and is <u>included</u> in the selling price of the item. The HST is composed of the federal GST (7%) and a provincial component of 8%. It is not added separately at the cash register like GST and PST.

The table below shows GST, PST, and HST for provinces and territories in Canada, 2004.

Province or Territory	**GST**	**PST**	**HST**
Alberta	7%	no PST	not applicable
British Columbia	7%	7.5%	not applicable
Manitoba	7%	7%	not applicable
Northwest Territories	7%	no PST	not applicable
Nunavut	7%	no PST	
Ontario	7%	8%	not applicable
Saskatchewan	7%	7%	not applicable
Yukon	7%	no PST	not applicable
*Prince Edward Island	7%	10%	not applicable
*Quebec	7%	7.5%	not applicable
**New Brunswick	-	-	15%
**Newfoundland & Labrador	-	-	15%
**Nova Scotia	-	-	15%

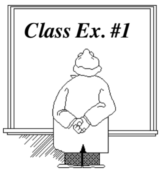
* In Prince Edward Island and Quebec, PST is calculated <u>after</u> the GST has been added to an item.

** In Newfoundland and Labrador, New Brunswick, and Nova Scotia, the harmonized sales tax replaces GST and PST. <u>The tax is included in the selling price.</u>

For the most recent GST, PST, and HST rates go to the website:
http://www.taxtips.ca/provincial_sales_tax.htm

In this unit we will use the term **selling price** to mean the price that the item is advertised for sale in the shop. In New Brunswick, Newfoundland & Labrador, and Nova Scotia this price includes the harmonized sales tax. In all other provinces and territories the selling price is the price before taxes are added.

The **selling price** can also be referred to as the **advertised price** or the **ticket price**.



Giancarlo and Gina travelled to Nunavut and bought a gift for their parents. The advertised selling price was $50.00.

a) Giancarlo calculated the final price by multiplying $50 by 7% and then adding the tax to the $50. Complete his work.

$$\text{Total price} = \begin{matrix}\text{Price}\\ \text{before taxes}\end{matrix} + \left(0.07 \times \begin{matrix}\text{Price}\\ \text{before taxes}\end{matrix}\right)$$

$$= \qquad + \ (0.07 \times \qquad)$$

$$\text{Total Price} = \qquad + \qquad =$$

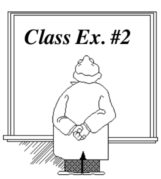
b) Gina calculated the final price by multiplying $50 by 1.07. Use factoring to complete the following to show why her method worked.

$$\text{Total price} = \begin{matrix}\text{Price}\\ \text{before taxes}\end{matrix} + \left(0.07 \times \begin{matrix}\text{Price}\\ \text{before taxes}\end{matrix}\right)$$

$$= 50 + (0.07 \times 50)$$

$$= 50(1 + \qquad)$$

$$\text{Total Price} = 50(\qquad) =$$



Jane visited Alberta, Manitoba, Prince Edward Island, and Nova Scotia during her holiday. In each province she bought a T-shirt as a souvenir. The selling price of the shirt in each province was $25.00. Complete the table.

Province	Selling Price	GST	PST	HST	Total Price
Alberta	$25.00				
Manitoba	$25.00				
Prince Edward Island	$25.00				
Nova Scotia	$25.00				

Complete Assignment Questions #1 - #13

Assignment

1. Brennon was hired to take digital photographs depicting each province and territory in Canada. He decided that the photographs he took in each province/territory would be printed in the same province/territory. He researched the cost of printing thirty 8x10 enlargements in each province. The information is shown in the table.

a) Complete the table.
(Remember: Some selling prices below include taxes, eg. Nova Scotia.)

Province/Territory	Selling Price	GST	PST	HST	Total Price
British Columbia	$69.05				
Alberta	$68.00				
Saskatchewan	$70.99				
Manitoba	$69.50				
Ontario	$68.75				
Quebec	$67.75				
Northwest Territories	$72.50				
Yukon	$70.00				
New Brunswick	$80.21				
Nova Scotia	$79.35				
Prince Edward Island	$68.51				
Nunavut	$72.00				
Newfoundland & Labrador	$80.21				

b) List the four provinces/territories where the selling price before taxes was the least expensive.

c) List the three provinces/territories where the selling price before taxes was the most expensive.

d) List the four provinces/territories where the total price for developing was the least expensive. Compare your answer with b).

e) List the three provinces/territories where the total price for developing was the most expensive. Compare your answer with c).

2. Albert bought a home stereo system from an audio store in Quebec City. The stereo system was advertised at $1789 plus taxes.

a) Calculate the taxes he paid. **b)** Determine the total cost of the stereo system.

3. Nicole bought a boat for $17 899 including taxes in Nova Scotia. What was the cost of her boat before taxes?

4. Dunigan and Pat bought a HDTV plasma television in Vancouver for $6 523 including taxes. If they paid $427.27 in PST, what was the selling price of the television before any taxes were added?

5. The first quarter sales of magazines for teenagers are given in the table.

a) How many magazines were sold in the first quarter?

b) What is the total sales revenue for all the magazines?

Magazine Title	Unit Price	Number Sold	Sales Revenue
Teen Tech	$3.25	265	
Lion Beat	$3.75	275	
Yo Music	$3.95	258	
New Wave	$4.95	329	
High School Clues	$5.75	299	

c) Calculate the final cost, including taxes, of buying *Teen Tech* magazine if the magazine is bought in:

i) Saskatchewan **ii)** Prince Edward Island

6. Danker bought a Vancouver Canucks jersey in British Columbia which was discounted by $20.50. Complete the table

Regular Price	Discount	GST	PST	Total
-	$20.50	$5.74	$6.15	$93.89

Use the following information to answer the next question.

Brady bought a Calgary Flames jersey in Alberta. It was discounted by $12.75. The following table shows some information on the transaction.

Regular Price	Discount	GST	PST	Total
-	$12.75	-	not applicable	$99.99

7. a) Calculate the regular selling price of the jersey, to the nearest cent.

b) Determine the amount of GST, to the nearest cent, paid on the jersey.

8. Carol, who lives in Edmonton, was vacationing in Quebec and bought a camera for 10% off the regular sale price of $575.00. Upon return to Edmonton, Carol was allowed to claim back the provincial sales tax she paid in Quebec. What is the amount, to the nearest cent, that Carol should claim?

9. Rose bought a native wolf sculpture for 15% off the regular price of $365.00 in Kelowna, B.C. Calculate the final price, to the nearest cent, she paid for the wolf sculpture.

Use the following information to answer questions #10 - #13

Standings for a high school football league near the end of a season are shown in the following table.

Team	Wins	Losses	Ties	Points
Chiefs	8	0	3	19
Cobras	9	2	0	18
Eagles	7	1	3	17
Jacks	7	2	2	16
Knights	6	2	3	15
Trojans	5	1	5	15

10. How many wins did the Knights have?

11. Determine how many points are awarded for i) a win ii) a loss iii) a tie

12. A proposal has been made to change the way points are awarded. Under the proposal a win earns 3 points, a tie earns 1 point , and a losses earns no points. If the proposed system of awarding points was applied to this league, how would the standings change?

Numerical Response

13. Of the total games played, the percent, to the nearest tenth, that were ties is ________ .

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) See table

b) Alberta, Quebec, Prince Edward Island, Ontario

c) Saskatchewan, Northwest Territories, Nunavut

d) Alberta, Northwest Territories, Yukon, Nunavut

e) Saskatchewan, Prince Edward Island, New Brunswick and Newfoundland & Labrador tied for third most expensive

Province/Territory	Selling Price	GST	PST	HST	Total
British Columbia	$69.05	$4.83	$5.18	-	$79.06
Alberta	$68.00	$4.76	-	-	$72.76
Saskatchewan	$70.99	$4.97	$4.97	-	$80.93
Manitoba	$69.50	$4.87	$4.87	-	$79.23
Ontario	$68.75	$4.81	$5.50		$79.06
Quebec	$67.75	$4.74	$5.44	-	$77.93
Northwest Territories	$72.50	$5.08	-	-	$77.58
Yukon	$70.00	$4.90	-	-	$74.90
New Brunswick	$80.21	-	-	$10.46	$80.21
Nova Scotia	$79.35	-	-	$10.35	$79.35
Prince Edward Island	$68.51	$4.80	$7.33		$80.64
Nunavut	$72.00	$5.04	-	-	$77.04
Newfoundland & Labrador	$80.21	-	-	$10.46	$80.21

2. a) GST = $125.23 PST = $143.57 **b)** $2,057.80 **3.** $15, 564.35 **4.** $5, 696.94

5. a) 1426

b) $6259.40

c) i) $3.71 **ii)** $3.83

Magazine Title	Unit Price	Number Sold	Sales Revenue
Teen Tech	$3.25	265	$861.25
Lion Beat	$3.75	275	$1,031.25
Yo Music	$3.95	258	$1,019.10
New Wave	$4.95	329	$1,628.55
High School Clues	$5.75	299	$1,719.25

6. $102.50

7. a) $106.20 **b)** $6.54

8. $41.53 **9.** $355.24

10. 6 **11. i)** 2 **ii)** 0 **iii)** 1

12. The Chiefs and Cobras would be tied for first place with 27 points each,
the Eagles (24 points) and Jacks (23 points) would be in third and fourth place,
and the Trojans (20 points) would be in last place by themselves instead of tied for last place
with the Knights (21 points)

13.

2	4	.	2

Sequences and Data Tables Lesson #2:
Recursive and Nonrecursive Tables

Warm-Up #1

The table provides some information about the number of students in Grades 9 - 12 in a rural high school.

Grade	# of Males	# of Females	Total
9	68		121
10	57	71	
11		64	128
12	69	125	

a) Complete the table.

b) Can you complete the table by starting with any of the four rows?

c) Do the calculations in any row depend on calculations from other rows?

Warm-Up #2

The table provides some information about the odometer reading of a rental truck.

Day	Odometer Reading at Start of Day	Distance Travelled	Odometer Reading at End of Day
Monday	14637	201	
Tuesday			15037
Wednesday		326	
Thursday		235	
Friday			16024

a) Complete the table.

b) Can you complete the table by starting with any of the four rows?

c) Do the calculations in any row depend on calculations from other rows?

Recursive and Nonrecursive Tables

A **recursive table** is a table in which some values in a given row depend on values in the previous row as in Warm-up # _____ .

A **non recursive table** is a table in which the calculations are independent from row to row as in Warm-up # _____ .

Warm-Up #3

Toran invested $8000 in a **simple interest** investment account for 5 years. A table which shows the progress of the investment is provided. The **opening value** in each year is the amount on which interest is calculated for that year. The closing value in each year is the value of the investment at the end of the year.

a) Complete the table.

b) Does the opening value of Toran's account in any year depend on the previous closing value of the account?

Toran's Simple Interest Investment Account

Year	Opening Value	Interest Rate(%)	Interest Earned	Closing Value
1999	$8000.00	4.00	$320.00	$8320.00
2000	$8000.00	3.00	$240.00	$8560.00
2001	$8000.00	5.00	$400.00	
2002		3.75		
2003		3.50		

c) Describe 2 methods of calculating the interest earned from 1999 to 2003.

Abra invested $8000 in a **compound interest** investment account for 5 years with the same annual interest rates as Toran.

d) Complete the table.

e) Does the opening value of Abra's account in any year depend on the previous closing value of the account?

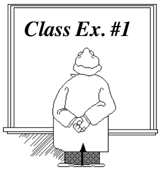
f) Calculate the total interest earned from 1999 to 2003.

Abra's Compound Interest Investment Account

Year	Opening Value	Interest Rate(%)	Interest Earned	Closing Value
1999	$8000.00	4.00	$320.00	$8320.00
2000	$8320.00	3.00	$249.60	$8569.60
2001	$8569.60	5.00	$428.48	
2002		3.75		
2003		3.50		

g) Describe how the two tables differ relative to the opening values and closing values.

h) Use your answer in g) to describe the difference between **simple** interest and **compound** interest.



Terri and Lynol want to save $15 000 to renovate their home. They open an account to deposit $3300 at the beginning of every year towards their renovation. Interest is compounded annually every year. The interest on the account pays 5% of the balance at the end of the first year, 4.75% of the balance at the end of the second year, 5.25% of the balance at the end of the third year, and 4.25% of the balance at the end of the fourth year.

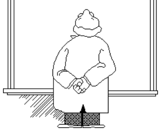
a) Create a table to determine if Terri and Lynol will have enough money to renovate their home after 4 years. Use the following headings for your columns - Year, Opening Balance ($), Yearly Deposit ($), Opening Balance + Yearly Deposit ($), Interest Rate (%), Interest Earned($), Closing Balance ($).

b) Is the table in a) a recursive table? Explain.

c) Describe how to calculate the closing balance in year 3.

d) Explain how to calculate the interest earned and the closing balance at the end of year 1.

Class Ex. #2



Rachel takes out a loan to buy a car. The principal amount of the loan is $12 000 at an annual interest rate of 5% per year compounded annually. The loan is for five years and the bank has informed her that her annual payments are $2771.70 per year, payable at the end of the year. The loans officer has provided her with the following information.

Year	Opening Balance	Interest Rate	Interest Charged	Annual Payment	Closing Balance
1	$12,000.00	5.00%	$600.00	$2,771.70	$9,828.30
2	$9,828.30	5.00%	$491.42	$2,771.70	$7,548.02
3	$7,548.02	5.00%	$377.40	$2,771.70	$5,153.72
4	$5,153.72	5.00%	$257.69	$2,771.70	$2,639.70
5	$2,639.70	5.00%	$131.99	$2,771.70	-$0.01

a) Describe how to determine the Opening Balance, Interest Charged, and Closing Balance for year two.

Opeining Balance =

Interest Charged =

Closing Balance =

b) The closing balance of all loan transactions **must** equal $0.00 at the end of the term of the loan. Describe what change the bank will make to the table to have a closing balance of $0.00 at the end of year 5.

c) Rachel decides she cannot afford to make the payments suggested by the loans officer. The loans officer suggests an reduced annual payment of $2074 over a longer period of time. Modify the table above by completing the table below to reflect this change.

Year	Opening Balance	Interest Rate	Interest Charged	Annual Payment	Closing Balance
1	$12,000.00	5.00%		$2,074.00	
2		5.00%			
3		5.00%			
		5.00%			
		5.00%			
		5.00%			
		5.00%			$0.00

d) Rachel asked her friends, Renee and Jordon, to calculate the real cost of her car using the table in c).
Renee calculated it using $12 000 + interest charged over 7 years. Jordon calculated the cost by taking the annual payment multiplied by 7.

i) Calculate the real cost of Rachel's car using both Renee's and Jordon's methods.

<u>Renee's Method</u> | <u>Jordon's Method</u>

ii) Explain who made the error in their calculation. Correct their mistake.

e) Describe mathematically a disadvantage and an advantage of both loan tables for Rachel.

Complete Assignment Questions #1 - #8

Assignment

1. Carrie wants to invests a sum of money for 8 years. She has a choice of investing it in a stock which has a return with simple interest and one which earns compound interest. If the interest rates are the same between the stocks, explain to her which one she should invest in and why.

2. Veronica invested $9000 in a simple interest savings bond which yielded a return of 8.25% interest in the first year, 6.5% interest in the second year, and 7.3% interest in each of the third and fourth years.
Create a table which shows the interest earned and closing balance of the four year investment. Use the following headings for your columns - Year, Opening Value of Bond ($), Interest Rate (%), Interest Earned ($), Closing Value of Bond ($).

3. Glen invested \$8700 in a compound interest savings bond which yielded a return of 5.25% interest in the first year, 4.5% interest in the second year, and 6.2% interest in each of the third and fourth years.

Create a table which shows the interest earned and closing balance of the four year investment. Use the following headings for your columns - Year, Opening Value of Bond (\$), Interest Rate (%), Interest Earned (\$), Closing Value of Bond (\$).

4. Explain which table is recursive between assignment questions #2 and #3.

5. Connor takes out a loan for \$15 000 to buy a car. The bank will finance the loan at 8% per year compounded annually for 5 years. His yearly payments will be \$3756.85. Complete the loan table below. Remember that the closing balance at the end of year 5 must equal \$0.

Year	Opening Balance (\$)	Interest Rate(%)	Interest Charged (\$)	Annual Payment (\$)	Closing Balance (\$)

6. Refer to the table in Class Ex. #2 c).
Rachel's stock investments performed better than expected and she was able to apply some of her extra income against her loan. She added \$500 to her annual payment at the end of year 1 and \$1225 to her annual payment at the end of year 3. She also received a bonus of \$1806.58 from her employer in year 4 which she also applied against her loan at the end of year four. Complete the chart and use it to answer the following.

Year	Opening Balance	Interest Rate(%)	Interest Charged	Annual Payment	Extra Payment	Closing Balance

a) Use the headings to describe how obtain the closing balance at the end of year 3.

b) Determine the remaining balance on her loan at the end of year 3.

c) How much sooner she will pay off her loan?

7. Jonathon wants to save \$20 000 to put as a deposit towards a house. He opens an interest bearing account compounded annually with a starting balance of \$1500. He deposits \$6300 at the beginning of every year. In the first year the interest on the account is 3.75% of the balance at the end of the year. For the second year it is 5.25%, 4.75% for the third year, and 4.25% for the fourth year.

a) Create a table to determine Jonathon will have enough money as a deposit. Use the following headings for the columns - Year, Opening Balance (\$), Deposit (\$), Opening Balance + Deposit (\$), Interest Rate (%), Interest Earned (\$), Closing Balance (\$).

b) Describe how to calculate the closing balance in year 2 given the closing balance in year 1.

c) Suppose the interest rate was changed to 3.50% in year 4.
Write a new row for year 4 below.

8. A moving company buys a large trailer in 2003 and a semi-truck in 2004. The company took out two separate loans. The tables below show the progress of the loans based on projected interest rates.

Trailer Loan

Year	Opening Balance	Interest Rate(%)	Interest Paid ($)	Yearly Payment	Closing Balance
2003	$85,000.00	5.75%	$4,887.50	$17,000.00	$72,887.50
2004	$72,887.50	5.75%	$4,191.03	$17,000.00	$60,078.53
2005	$60,078.53	5.75%	$3,454.52	$17,000.00	$46,533.05
2006	$46,533.05	5.75%	$2,675.65	$17,000.00	$32,208.70

Semi-Truck Loan

Year	Opening Balance	Interest Rate(%)	Interest Paid ($)	Yearly Payment	Closing Balance
2004	$125,000.00	6.00%	$7,500.00	$21,000.00	$111,500.00
2005	$111,500.00	6.00%	$6,690.00	$21,000.00	$97,190.00
2006	$97,190.00	6.00%	$5,831.40	$21,000.00	$82,021.40
2007	$82,021.40	6.00%	$4,921.28	$21,000.00	$65,942.68

a) Consider the following scenario.

At the end of year 2005, interest rates dropped to 5.25%. As a result the owner of the company went to his loans officer at the bank to consolidate (to combine) the two loans. He also increased his combined yearly payments by $2767.25.

Combine the two tables into one using the table below.

Year	Opening Balance	Interest Rate	Interest Paid	Yearly Payment	Closing Balance
2006					

b) What was the balance left to be paid on the consolidated loan at the end of year 2008?

c) How long will it take to pay off the consolidated loan?

9. Abacus High School owns a school van that staff members may use for field trips and extra-curricular activities. Each staff member who uses the van is supposed to record the odometer readings and other information shown on the table.

Date	*Starting Odometer Reading*	*Ending Odometer Reading*	*Distance Travelled*
Oct. 8	35 237.2		121.5
		35 768.3	
Oct. 15			28
	35 796.3	35 813.5	

a) The school manager, Verne, maintains the account for usage of the van. Some staff members did not fill in all the information required. Enter values in the table which have enough information to be calculated.

b) Is this a recursive or nonrecursive table? Explain.

10. Mikala works full time at a video store. She is paid \$6.50 per hour for regular hours and time and a half for overtime. Her employer pays her overtime for any time worked over 8 hours per day. The table shows the number of hours that she worked in one week.

	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
Total Hours	5	9.5	7.25	8.75	6	9	0	45.5
Regular Hours								
Overtime Hours								

Calculate Mikala's;

a) regular earnings

b) overtime earnings

c) overall earnings

Answer Key

1. Carrie should invest in the stock which earns compound interest because she will earn more money on this investment.
Compound interest pays interest on the interest made the previous pay period. Simple interest does not.

2. See table at the side.
Table style may vary.

Year	Opening Value of Bond ($)	Interest Rate(%)	Interest Earned($)	Closing Value of Bond($)
1	9,000.00	8.25	742.50	9,742.50
2	9,000.00	6.50	585.00	10,327.50
3	9,000.00	7.30	657.00	10,984.50
4	9,000.00	7.30	657.00	11,641.50

3. See table at the side.
Table style may vary.

Year	Opening Value of Bond	Interest Rate	Interest Earned	Closing Value of Bond
1	\$8,700.00	5.25%	\$456.75	\$9,156.75
2	\$9,156.75	4.50%	\$412.05	\$9,568.80
3	\$9,568.80	6.20%	\$593.27	\$10,162.07
4	\$10,162.07	6.20%	\$630.05	\$10,792.12

4. Assignment question #3 is recursive because the closing value of the previous year is the opening value of the next year.

5. See table at the side. Table style may vary.

Year	Opening Balance ($)	Interest Rate(%)	Interest Charged ($)	Annual Payment ($)	Closing Balance ($)
1	15,000.00	8	1,200.00	3,756.85	12,443.15
2	12,443.15	8	995.45	3,756.85	9,681.75
3	9,681.75	8	774.54	3,756.85	6,699.44
4	6,699.44	8	535.96	3,756.85	3,478.55
5	3,478.55	8	278.28	3,756.83	0.00

6. **a)** Opening Balance
+ Interest Charged
– Annual Payment
– Extra Payment

b) $5,576.97

c) 2 years sooner.

Year	Opening Balance	Interest Rate	Interest Charged	Annual Payment	Extra Payment	Closing Balance
1	$12,000.00	5.00%	$600.00	$2,074.00	$500.00	$10,026.00
2	$10,026.00	5.00%	$501.30	$2,074.00	$0.00	$8,453.30
3	$8,453.30	5.00%	$422.67	$2,074.00	$1,225.00	$5,576.97
4	$5,576.97	5.00%	$278.85	$2,074.00	$1,806.58	$1,975.23
5	$1,975.23	5.00%	$98.76	$2,073.99	$0.00	$0.00

7. **a)** See table at side. Table style may vary.

b) Opening Balance
+ Deposit
+ Interest Earned

Year	Opening Balance	Deposit	Opening Balance + Deposit	Interest Rate (%)	Interest Earned	Closing Balance
1	$1,500.00	$6,300.00	$7,800.00	$3.75	$292.50	$8,092.50
2	$8,092.50	$6,300.00	$14,392.50	$5.25	$755.61	$15,148.11
3	$15,148.11	$6,300.00	$21,448.11	$4.75	$1,018.79	$22,466.89
4	$22,466.89	$6,300.00	$28,766.89	$4.25	$1,222.59	$29,989.48

c) See table at side. Table style may vary.

Year	Opening Balance	Deposit	Opening Balance + Deposit	Interest Rate (%)	Interest Earned	Closing Balance
4	$22,466.89	$6,300.00	$28,766.89	3.5	$1,006.84	$29,773.73

8. **a)** See table at side

b) $38, 733.68

c) 4 years, end of year 2009

Year	Opening Balance	Interest Rate	Interest Paid	Yearly Payment	Closing Balance
2006	$143,723.05	5.25%	$7,545.46	$40,767.25	$110,501.26
2007	$110,501.26	5.25%	$5,801.32	$40,767.25	$75,535.33
2008	$75,535.33	5.25%	$3,965.60	$40,767.25	$38,733.68
2009	$38,733.68	5.25%	$2,033.52	$40,767.20	$0.00

9. **a)** See table at the side

b) Recursive table because the ending odometer reading is the starting odometer reading for the next date.

Date	*Starting Odometer Reading*	*Ending Odometer Reading*	*Distance Travelled*
Oct. 8	35 237.2	35 358.7	121.5
	35 358.7	35 768.3	409.6
Oct. 15	35 768.3	35 796.3	28
	35 796.3	35 813.5	17.2

10. a) $274.63 **b)** $31.69 **c)** $306.32

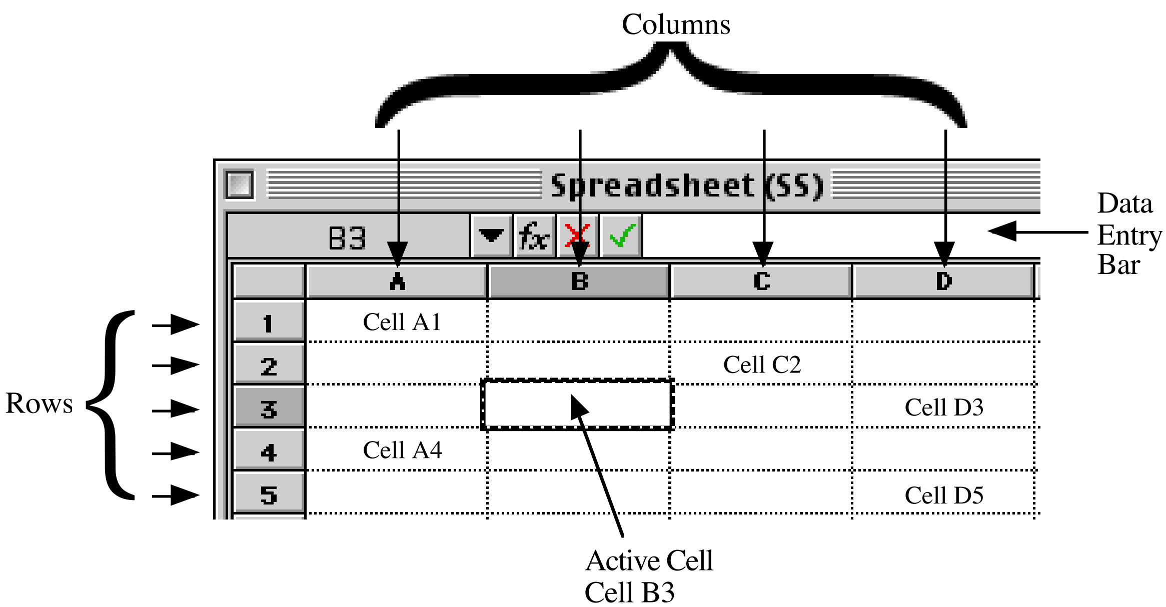
	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
Total Hours	5	9.5	7.25	8.75	6	9	0	45.5
Regular Hours	5	8	7.25	8	6	8	0	42.25
Overtime Hours	0	1.5	0	0.75	0	1	0	3.25

Sequences and Data Tables Lesson #3:
Spreadsheets

Spreadsheets

Spreadsheets, like tables, are a useful and effective way to represent large amounts of data. They are exceptional for exploring patterns and creating tables and graphs, because they perform many calculations at the same time with the use of a computer or calculator.

The diagram shown illustrates the main parts of a spreadsheet.



- This lesson requires the use of a computer or graphing calculator with spreadsheet programs such as *Microsoft Excel* (computer), *Appleworks Spreadsheet* (computer), or *Cell-Sheet* for the TI-83 graphing calculator (not like the table/list feature).

Warm-Up #1

Create 2 spreadsheets:
- A GST spreadsheet from \$5 to \$75 counting by 5's showing the values.
- A GST spreadsheet from \$5 to \$75 counting by 5's showing the formulas used.

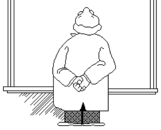
Use the headings: Amount, GST, and Total in the column.
Use formulas wherever possible.

Warm-Up #2

Modify the spreadsheets in Investigation #1 doing the following:

- Change to values from 12 to 132 counting by 12's under Amount.
- Include PST from B.C. of 7.5%.

Class Ex. #1



Consider the table shown from Warm-Up #3 in Lesson 2.

Toran's Simple Interest Investment Account

Year	Opening Value	Interest Rate(%)	Interest Earned	Closing Value
1999	$8000.00	4.00	$320.00	$8320.00
2000	$8000.00	3.00	$240.00	$8560.00
2001	$8000.00	5.00	$400.00	
2002		3.75		
2003		3.50		

Abra's Compound Interest Investment Account

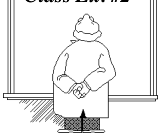
Year	Opening Value	Interest Rate(%)	Interest Earned	Closing Value
1999	$8000.00	4.00	$320.00	$8320.00
2000	$8320.00	3.00	$249.60	$8569.60
2001	$8569.60	5.00	$428.48	
2002		3.75		
2003		3.50		

a) Use a spreadsheet to complete both tables. Start with cell A1 as the Year, B1 as the Opening Value, etc.

b) For each table state a spreadsheet formula which can be used in the following cells. Include the headings in your spreadsheets in Row 1.

i) B3 **ii)** D2 **iii)** E4

Class Ex. #2



Michael took out a student loan of $15 000 to attend university. Upon graduation and finding employment, Michael made arrangements with a loans officer to pay off the loan. The spreadsheet below shows part of the loan payments Michael agreed to make.

	A	B	C	D	E	F
1	Year	Opening Balance ($)	Interest Rate(%)	Interest Charged ($)	Annual Payment ($)	Closing Balance ($)
2	1	$15,000.00	6.00%	$900.00	$2,687.03	$13,212.97
3	2	$13,212.97	6.00%	$792.78	$2,687.03	$11,318.72
4	3	$11,318.72	6.00%			
5	4					
6	5					
7	6					
8	7					

a) Write a spreadsheet formula which can be used in the following cells.

i) B3 **ii)** D3 **iii)** E3 **iv)** F3

b) Use a spreadsheet to determine the total number of years to pay off the student loan.

c) What is the final annual payment so the closing balance is $0.00?

Assignment

1. Complete the following statements.

a) A spreadsheet is made up of rectangular _________ .

b) A cell on a spreadsheet is named by the ____________ of the column followed by the ____________ of the row.

c) The cell which is currently selected on a spreadsheet is called the ____________ cell.

d) The location on a spreadsheet which shows the contents of a cell, or the formulas of the spreadsheet is called the ________ _________ _________ .

This remainder of this assignment requires the use of a computer or graphing calculator with spreadsheet programs such as *Microsoft Excel* (computer), *Appleworks Spreadsheet* (computer), or *Cell-Sheet* for the TI-83 graphing calculator.

2. Varon invested \$12,750 in a simple interest savings bond which yielded a return of 3.25% interest in the first year, 3.75% interest in each of the second and third years, and 3.15% the fourth year.

a) Create a spreadsheet which shows the interest earned and closing balance of the four year investment. Use the following headings for your columns - Year, Opening Value of Bond (\$), Interest Rate (%), Interest Earned (\$), Closing Value of Bond (\$).

b) Modify the spreadsheet to display the formulas used in a).

3. Helen invested \$5250 in a compound interest savings bond which yielded a return of 5.00% interest compounded annually.

a) Create a spreadsheet which shows the interest earned and closing balance of the 10 year investment. Hint: Copy your spreadsheet from question #2 and modify it for this question. Remember that this type of spreadsheet is recursive.

b) Write possible spreadsheet formula for each cell in the third row.

4. Anton wants to invest money in a compound interest savings account. He opens an account with a starting balance of \$3250. He then deposits \$3000 at the beginning of every year in the account. The interest is compounded annually. The bank pays an interest rate of 3.75% of the balance at the end of the first year, 3.85% for the second year, 4.05% for the third year, and 3.95% for the fourth year.

a) Create a spreadsheet table for four years. Use the following headings for your columns - Year, Opening Balance (\$), Deposit (\$), Opening Balance + Deposit (\$), Interest Rate (%), Interest Earned(\$), Closing Balance (\$).

b) Copy the spreadsheet in a) and change the cells so that all the formulas are displayed.

5. <u>NOTE:</u> The solution to this question is not included in the answer key.
Consider the scenario from assignment question #8 from Lesson #2 shown below.
A moving company buys a large trailer in 2003 and a semi-truck in 2004. The company took out two separate loans. The loans are partially illustrated in the tables below.

Trailer Loan

Year	Opening Balance	Interest Rate(%)	Interest Paid ($)	Yearly Payment	Closing Balance
2003	$85,000.00	5.75%	$4,887.50	$17,000.00	$72,887.50
2004	$72,887.50	5.75%	$4,191.03	$17,000.00	$60,078.53
2005	$60,078.53	5.75%	$3,454.52	$17,000.00	$46,533.05
2006	$46,533.05	5.75%	$2,675.65	$17,000.00	$32,208.70

Semi-Truck Loan

Year	Opening Balance	Interest Rate(%)	Interest Paid ($)	Yearly Payment	Closing Balance
2004	$125,000.00	6.00%	$7,500.00	$21,000.00	$111,500.00
2005	$111,500.00	6.00%	$6,690.00	$21,000.00	$97,190.00
2006	$97,190.00	6.00%	$5,831.40	$21,000.00	$82,021.40
2007	$82,021.40	6.00%	$4,921.28	$21,000.00	$65,942.68

At the end of year 2006, interest rates dropped to 5.15%. At this time the company bought a moving van for $75 000. As a result the owner of the company went to his loans officer at the bank to consolidate the two loans and include the money for the moving van.
He also increased his combined yearly payments by $15 551.20 .

a) Create two spreadsheets for this scenario; one which displays the values and one which displays the formulas used in the cells.

b) • On a new spreadsheet, modify the spreadsheet in a) to determine the yearly payment of the combined loan so that the balance of the loan will be paid off by the year 2010.

• How much more interest will this cost the company?

c) The company makes an extra payment at the end of 2008 of $12 000.

• On a new spreadsheet modify the spreadsheet from a) to show this change.
Create two spreadsheets for this scenario; one which displays the values and one which displays the formulas used in the cells.

• How much money did the company save by making this extra payment?

Extension

NOTE: The solution to question #6 is not included in the answer key.

6. Kalin deposits $250 at the beginning of every month in an account that earns 12% per year **compounded monthly** for 2 years.

Note: The interest earned each month is calculated by taking the interest rate and dividing by the number of times the account is compounded in one year.

In this case that is $\frac{12}{12} = 1\%$ per month. This needs to be considered in your formula to calculate the interest earned each pay period.

a) Construct a spreadsheet using the following headings; Month, Opening Balance, Deposit, Interest rate per month, Interest Earned, Closing Balance. Use formulas whenever possible.

b) Modify your spreadsheet to show a interest rate of 9% per year compounded monthly.

NOTE: The solution to question #7 is not included in the answer key.

7. Mulva acquired a credit card for emergency purposes. The card charges interest on any balance owing after a payment has been made. The interest rate for the card is 21% per year compounded monthly.

Mulva is on a budget and makes regular payments of $500 on her credit card. She had some car repairs done and they cost her $2000. Assume she does not use the card for any other purchases and that she makes her first payment before the payment due date.

a) Use the headings shown below to construct a spreadsheet to show Mulva's credit card statement until she has paid off the card. Adjust her last payment so the balance owing at the end is $0.

	A	B	C	D	E	F
1	Month	Opening Balance	Purchases	Payment	Interest Charged	Closing Balance
2	1	$0.00	$2000.00	$500.00		

b) How much money did Mulva pay in interest for the car repairs?

Assignment Key

1 . a) cells **b)** letter, number **c)** active **d)** data entry bar

2 . a) See table below. Chart style may vary.

	A	B	C	D	E
1	Year	Opening Value ($)	Interest Rate(%)	Interest Earned ($)	Closing Value
2	1	12,750.00	3.25	414.38	$13,164.38
3	2	12,750.00	3.75	478.12	$13,642.50
4	3	12,750.00	3.75	478.12	$14,120.62
5	4	12,750.00	3.15	401.62	$14,522.25

b) See table below. Formulas may vary.

	A	B	C	D	E
1	Year	Opening Value ($)	Interest Rate(%)	Interest Earned ($)	Closing Value
2	1	12,750.00	3.25	=B2*C2/100	=B2+D2
3	=A2+1	=B2	3.75	=B3*C3/100	=E2+D3
4	=A3+1	=B3	3.75	=B4*C4/100	=E3+D4
5	=A4+1	=B4	3.15	=B5*C5/100	=E4+D5

3. a) See table below. Formulas may vary. For instance, unlike the spreadsheet solution to question #2, this spreadsheet uses the percent feature and converts the percent to decimal for automatic calculations.

	A	B	C	D	E
1	Year	Opening Value of Bond	Interest Rate	Amount of Interest	Closing Value of Bond
2	1	$5,250.00	5.00%	$262.50	$5,512.50
3	2	$5,512.50	5.00%	$275.62	$5,788.12
4	3	$5,788.12	5.00%	$289.41	$6,077.53
5	4	$6,077.53	5.00%	$303.88	$6,381.41
6	5	$6,381.41	5.00%	$319.07	$6,700.48
7	6	$6,700.48	5.00%	$335.02	$7,035.50
8	7	$7,035.50	5.00%	$351.78	$7,387.28
9	8	$7,387.28	5.00%	$369.36	$7,756.64
10	9	$7,756.64	5.00%	$387.83	$8,144.47
11	10	$8,144.47	5.00%	$407.22	$8,551.70

b) A3 → =A2+1
B3 → =E2
C3 → =C2
(Copy C2 and use fill down command from C2 to C11)
D3 → =B3*C3 (Since the percent feature is enabled the percent is automatically converted to a decimal and therefore we do not divide by 100 like the spreadsheet solution example \in quest #2.
E3 →=B3+D3

4. a) See table below. Chart style may vary.

	A	B	C	D	E	F
1	Year	Opening Balance	Deposit	Interest Rate(%)	Interest Earned	Closing Balance
2	1	$3,250.00	$3,000.00	3.75	$234.38	$6,484.38
3	2	$6,484.38	$3,000.00	3.85	$365.15	$9,849.52
4	3	$9,849.52	$3,000.00	4.05	$520.41	$13,369.93
5	4	$13,369.93	$3,000.00	3.95	$646.61	$17,016.54

b) See table below. Formulas may vary.
D2 answer may vary slightly depending on the program used→ =(B2+C2)*D2/100

	A	B	C	D	E	F
1	Year	Opening Balance	Deposit	Interest Rate(%)	Interest Earned	Closing Balance
2	1	$3,250.00	$3,000.00	3.75	=(B2+C2)*D2/100	=B2+C2+E2
3	=A2+1	=F2	=C2	3.85	=(B3+C3)*D3/100	=B3+C3+E3
4	=A3+1	=F3	=C3	4.05	=(B4+C4)*D4/100	=B4+C4+E4
5	=A4+1	=F4	=C4	3.95	=(B5+C5)*D5/100	=B5+C5+E5

Sequences and Data Tables Lesson #4:
Patterns and Sequences

Warm-Up #1

Imagine as a student sitting in the stands at a football game, you hear the quarterback yell out a sequence of a colour and numbers: "Blue, 2, 5, 8". Then you notice player #2 carries the ball through hole #5 using play #8. The sequence of numbers is 2, 5, 8. But can you imagine what would happen if the numbers were given in a different order? It would be a completely different play. The ***order*** in a sequence is important.

A Sequence of Numbers

A **sequence** of numbers is a set of numbers arranged in a definite order.

We are already familiar with sequences of numbers.
For example:

i) the sequence of odd numbers less than 10, written as $1, 3, 5, 7, 9$

ii) the sequence of odd numbers less than 100, written as $1, 3, 5, 7 \, .\,.\,.\,.\,. \, 97, 99$

iii) the sequence of odd numbers, written as $1, 3, 5, 7 \, .\,.\,.\,.\,.\,.\,.\,.\,.$

The sequences in i) and ii) above are examples of **finite sequences** - they contain a specific number of terms.
The sequence in iii) is an **infinite sequence**.

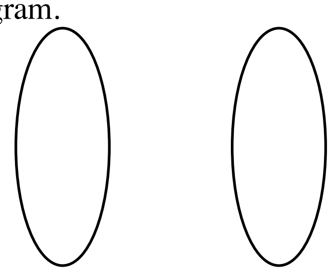
Sequences consist of **terms.** In the sequences above the first term, written t_1 , is ____
the second term, t_2, is ____ , the third term, ____ , is ____ etc.

A Sequence as a Mapping

Consider the sequence $1, 3, 5, 7, 9$ where $t_1 = 1$, $t_2 = 3$, $t_3 = 5$, $t_4 = 7$, $t_5 = 9$.

We can regard this as a mapping from a domain $\{1, 2, 3, 4, 5\}$ to a range $\{1, 3, 5, 7, 9\}$ in which $1 \to 1$, $2 \to 3$, $3 \to 5$, $4 \to 7, 5 \to 9$.

Show the mapping in an arrow diagram.



Every sequence of numbers can be regarded as a mapping with
domain - the set of natural numbers, $N = \{1, 2, 3, 4, \, .\,.\,.\,. \}$ and
range - the terms of the sequence.

A Sequence as a Function

Since each element of the domain is related to exactly one element of the range, the relationship between the set of natural numbers and the terms of a sequence is a functional relationship.

A sequence is a function, where the domain is the set of natural numbers and the values of the range are the terms of the sequence.

A function can often be written using function notation or by a formula or equation and the same is true of many sequences.

Consider the sequence of odd numbers $1, 3, 5, 7 \dots\dots$

We shall try to determine a rule connecting the elements of the domain and the elements of the range. We know that $1 \to 1$, $2 \to 3$, $3 \to 5$, etc.
The elements of the domain increase by 1 whereas the elements of the range increase by 2. This suggests that the rule must include multiplication by a factor of 2.

Notice that $t_1 = 2(\mathbf{1}) - 1 = 1$ $t_2 = 2(\mathbf{2}) - 1 = 3$ $t_3 = 2(\mathbf{3}) - 1 = 5$ etc.

Complete the following;

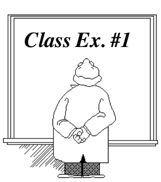
$t_7 = 2(\quad) - 1 =$ $t_{50} = \qquad =$ $t_n =$

We have discovered a formula that can be used to determine the value of any term provide we know the position of the term in the sequence.

The term t_n, the nth term, is known as the **general term** of the sequence, and the formula for the nth term is called the **general formula**.

If we let f represent the function for the sequence of odd numbers we have, in function notation, $f(n) = 2n - 1, n \in N$ with domain the set of natural numbers and range the terms of the sequence.

In this case the function is a linear function of the form $f(x) = mx + b$.



A sequence is defined by the general term $t_n = 2n^3 - n^2$.

a) List the first two terms of the sequence **b)** Evaluate t_7

Class Ex. #2

In each case state the next two terms of the sequence and give a formula for the general term.

a) $3, 9, 15, 21 \ldots$ **b)** $3, 9, 27, 81 \ldots$ **c)** $\sqrt{2}, 2, 2\sqrt{2} \ldots$

Class Ex. #3

Consider the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$

a) Determine the next two terms of the sequence.

b) Explain how you determined the answers in a)

Note

The sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ is an example of a **Fibonacci Sequence**. Such sequences occur as natural phenomena in nature in such things as seed growth, leaves on stems, petals on flowers, etc. Internet research on Fibonacci sequences is an interesting exercise.

If we try to find a general term formula for the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ in terms of n we will not be successful and yet the sequence can be simply defined in part b) above.

This leads us to an alternative form of general formula - a recursive definition of the general term.

A Recursive Formula for the General Term

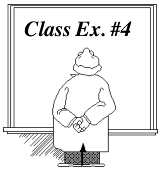
A recursive formula describes a sequence where the first term is given and each term after the first is defined in terms of the previous one. Occasionally, as is the case in the Fibonacci sequence, more than one term is required to define the sequence.

Generally, a recursive formula consists of the following three parts;

i) the value of the first term

ii) a formula connecting t_n and t_{n-1} or t_{n+1} and t_n

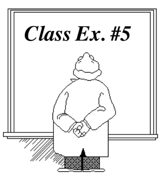
iii) the values of n for which the formula is valid.



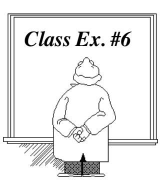
Find the first three terms of the given sequences.

a) $\begin{cases} t_1 = 3 \\ t_n = 5t_{n-1}\ ,\ n \geq 2, n \in N \end{cases}$

b) $\begin{cases} t_1 = 0.5 \\ t_{n+1} = 4t_n + 2,\quad n \in N \end{cases}$



A sequence is defined by the formula $t_1 = \frac{1}{2},\ t_n = \left(n + \frac{3}{2}\right) + t_{n-1}, n > 1,\ n \in N$.
Determine the fourth term.



State a recursive formula for the general term of the following sequences.

a) $2, 7, 12, 17,\ ...$

b) $2, 6, 18, 54,\ ...$

Complete Assignment Questions #1 - #9

Assignment

1. Find the first three terms of each sequence from the formula for the general term.

a) $t_n = 1 - 4n$ **b)** $t_n = (n-3)^2$ **c)** $t_k = (-2)^k$ **d)** $t_n = 3(2^n)$

e) $t_n = 2 + \dfrac{1}{n}$ **f)** $t_n = \dfrac{3n+1}{2n-1}$ **g)** $t_n = \dfrac{n^2}{4} - \dfrac{4}{n^3}$

2. In each case state the next two terms of the sequence and give a formula for the general term.

a) $1, 2, 3, \ldots$ **b)** $2, 4, 6, \ldots$ **c)** $2, 4, 8, \ldots$

d) $1, 4, 9, 16, \ldots$ **e)** $1, 8, 27, 64, \ldots$ **f)** $5, 25, 125, \ldots$

g) $\sqrt{5}, 5, 5\sqrt{5}, 25, \ldots$ **h)** $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, \ldots$

3. Find the first three terms of the sequence defined by each recursive formula.

a) $\begin{cases} t_1 = 5 \\ t_{n+1} = 2t_n - 3, \ n \in N \end{cases}$

b) $\begin{cases} t_1 = -3 \\ t_n = 4t_{n-1} + 7, n \geq 2, n \in N \end{cases}$

c) $\begin{cases} t_1 = -4 \\ t_n = t_{n-1} - (3n)^n, n > 1, n \in N \end{cases}$

d) $\begin{cases} t_1 = \dfrac{3}{4} \\ t_{n+1} = \left(\dfrac{n}{n+1}\right)t_n, \ n \in N \end{cases}$

4. State a recursive formula for the general term of the sequences in Class Example #2. In each case state the formula for t_{n+1} in terms of t_n.

a) $3, 9, 15, 21 \ldots$ **b)** $3, 9, 27, 81 \ldots$ **c)** $\sqrt{2}, 2, 2\sqrt{2} \ \ldots$

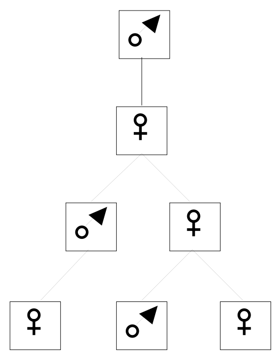
5. State a recursive formula for the general term of the following sequences In each case state the formula for t_n in terms of t_{n-1}.

a) $3, 6, 9, 12, \ldots$ **b)** $c, c+d, c+2d, c+3d, \ldots$

6. If $t_n = 5n + 1$, prove that $t_{n+1} - t_n = 5$.

7. The reproduction of bees in nature follows an interesting sequence. Whereas the female bee has two parents (a mother and a father), the male bee has only one parent (a mother).

a) Complete three more generations of the family tree for the male bee below.



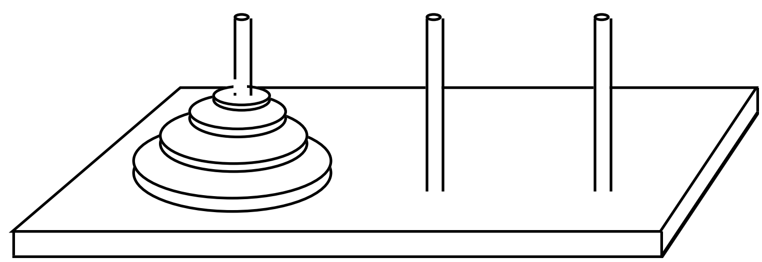
The bee in the first row is the child of the bee in the second row.

b) What sequence does the pattern in a) follow?

c) State a recursive formula to describe this situation.

Use the following information to answer questions #8 and #9.

In the puzzle called the "Towers of Hanoi" a person is asked to move rings from one peg and stack them in order on another peg. You can make as many moves as you want, but only one ring per move. Also, no ring may be placed on top of a smaller ring. For example, the minimum number of moves required to move 1 ring is 1 move, 2 rings is 3 moves, 3 rings is 7 moves, 4 rings is 15 moves, and 5 rings is 31 moves.



Multiple Choice

8. The minimum number of moves required to move 6 rings is

A. 45

B. 55

C. 63

D. 65

Numerical Response

9. The formula for this sequence is $t_n = a^n + b$.
The value of $a + b$, to the nearest hundredth, is ______ .
(Record your answer in the numerical response box from left to right)

Answer Key

1a) $-3, -7, -11$ **b)** $4, 1, 0$ **c)** $-2, 4, -8$ **d)** $6, 12, 24$ **e)** $3, \frac{5}{2}, \frac{7}{3}$ **f)** $4, \frac{7}{3}, 2$ **g)** $-\frac{15}{4}, \frac{1}{2}, \frac{227}{108}$

2a) $4, 5;\ t_n = n$ **b)** $8, 10;\ t_n = 2n$ **c)** $16, 32;\ t_n = 2^n$ **d)** $25, 36;\ t_n = n^2$
e) $125, 216;\ t_n = n^3$ **f)** $625, 3125;\ t_n = 5^n$ **g)** $25\sqrt{5}, 125;\ t_n = \left(\sqrt{5}\right)^n$ **h)** $\frac{6}{243}, \frac{7}{729};\ t_n = \frac{n+1}{3^n}$

3a) $5, 7, 11$ **b)** $-3, -5, -13$ **c)** $-4, -40, -769$ **d)** $\frac{3}{4}, \frac{3}{8}, \frac{1}{4}$

4a) $t_1 = 3, t_{n+1} = t_n + 6, n \geq 2, n \in N$ **b)** $t_1 = 3, t_{n+1} = 3t_n, n \geq 2, n \in N$
c) $t_1 = \sqrt{2}, t_{n+1} = \sqrt{2}\, t_n, n \geq 2, n \in N$

5a) $t_1 = 3, t_n = t_{n-1} + 3, n \geq 2, n \in N$ **b)** $t_1 = c, t_n = t_{n-1} + d, n \geq 2, n \in N$

7b) 1, 1, 2, 3, 5, 8 Fibonacci sequence **c)** $t_{n+2} = t_n + t_{n+1}, t_1 = 1, t_2 = 1, n \in N$

8. C **9.**

1	.	0	0

Sequences and Data Tables Lesson #5:
Arithmetic Sequences

There are different types of sequences. In this course we focus on two types of sequence, the arithmetic sequence and the geometric sequence.

Sequences

Arithmetic Sequences ***Geometric Sequences*** ***Other Sequences***

In this lesson we will study arithmetic sequences.

Arithmetic Sequence

An **arithmetic sequence** is a sequence where each term is formed from the preceding term by ***adding*** a constant (positive or negative).

Complete the following for the sequence 7, 10, 13, 16, ...

- Each term is determined by adding ____ to the previous term.
- Calculate the differences; $t_2 - t_1 =$ ____ $t_3 - t_2 =$ ____ $t_4 - t_3 =$ ____

Notice that there is a **common difference** between successive terms.

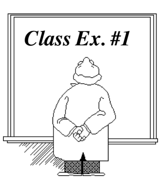
The common difference in this example is ____.

Finding a Common Difference

To find a common difference in an arithmetic sequence we can subtract any term from the term before it.

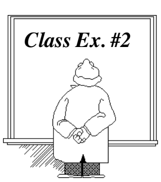
For example $t_2 - t_1$ = common difference, or
$t_5 - t_4$ = common difference, etc.

$$\boxed{\text{common difference} = t_n - t_{n-1}}$$



Consider the sequence 16, 13, 10, 7, ...

The **common difference** in the sequence is ______ .

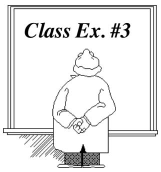


For each of the following sequences:
- Determine which are arithmetic sequences, and,
- Find the common difference for those sequences which are arithmetic.

a) $2, 4, 6, 8, ...$ **b)** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} ...$ **c)** $-10, -4, 2, 8, ...$ **d)** $4, 8, 16, 32, ...$

In an arithmetic sequence we often use the following terminology.

The first term in an arithmetic sequence is represented by a and the common difference is represented by d.



State the values of a and d in the following sequences.

i) $-8, 2, 12, 22, ...$ **ii)** $15, 10, 5, 0 \ldots$

Warm-Up ***Investigating the General Formula for an Arithmetic Sequence***

Consider the sequence $2, 12, 22, 32, 42 \ldots$

a) State the following
$t_1 =$ $t_2 =$ $t_3 =$ $t_4 =$ $t_5 =$ $a =$ $d =$

b) Complete the following pattern which describes each term in the sequence in terms of the first term, a, and the common difference, d.

$t_1 = 2$	$t_1 = a$
$t_2 = 2 + 1(10) = 12$	$t_2 = a + (1)d$
$t_3 = 2 + 2(10) = 22$	$t_3 = a + 2d$
$t_4 =$	$t_4 =$
$t_5 =$	$t_5 =$
$t_{30} =$	$t_{30} =$
$t_n =$	$t_n =$

c) The general term in b) has been described in terms of n, the position of the term. This is the non-recursive formula for the general term.

Determine the recursive formula for the general term.

General Term Formulas for an Arithmetic Sequence

The **general term formula in non-recursive form** for an arithmetic sequence is:

$$\boxed{t_n = a + (n - 1)d}$$

where, t_n is the general term of the arithmetic sequence in non-recursive form
a is the first term
d is the common difference
n is the position of the term in the sequence

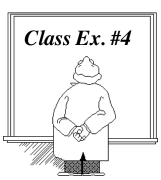
The **general term formula in recursive form** for an arithmetic sequence is:

$$\boxed{t_1 = a, \ t_n = t_{n-1} + d}\text{, where } n \geq 2, n \in N.$$

where, t_n is the general term of the arithmetic sequence in recursive form
a is the first term
d is the common difference
n is the position of the term in the sequence

Unless otherwise instructed, assume that when you are asked to determine a general term it is the non-recursive form which is required.

The general arithmetic sequence is $a, a + d, a + 2d, a + 3d, \ldots\ldots a + (n - 1)d \ldots\ldots$



Consider the arithmetic sequence $-6, -1, 4, 9, \ldots$.

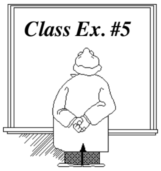
a) Determine the formula in non-recursive form for the general term of the sequence.

b) Determine the formula in recursive form for the general term of the sequence.

c) Determine the twelfth term using a).

d) Determine the twelfth term using b).

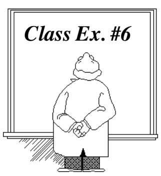
e) Which version of the formula for the general term is easier to use if n is large?



Find the number of terms in the arithmetic sequence $3, -1, -5, \ldots, -117$.

Arithmetic Means

The terms placed between two non-consecutive terms of an arithmetic sequence are called **arithmetic means**. For example, in the sequence $5, 10, 15, 20$, the numbers 10 and 15 are arithmetic means between 5 and 20. In order to determine arithmetic means between two given terms it is helpful to think of the two given terms as the *first* and *last* terms of a sequence.

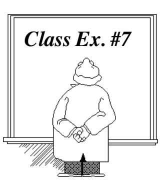


Place three arithmetic means between -4 and 8.

Solving Sequence Problems Where Both "a" and "d" Are Unknown

The next two examples involve solving sequence problems where both the first term and the common difference are unknown.

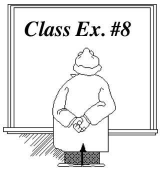
The second example can be solved by a modification of the method in class example #6 or by a very simple system of equations which can be regarded as an extension to the curriculum.



Consider the sequence $x + 2, 3x - 1$, and $2x + 1$.

a) Determine the value of x such that $x + 2, 3x - 1$, and $2x + 1$ form an arithmetic sequence.

b) Determine the numerical value of the three terms.



The third and eighth terms of an arithmetic sequence are 12 and –18, respectively.

a) Use arithmetic means to determine the fifth term of the sequence.

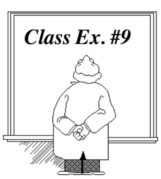
b) State the first term, a, and the common difference, d, of the sequence.

c) Complete the following $\dfrac{t_8 - t_3}{8 - 3} = \underline{\qquad\qquad} =$

d) Write t_3 and t_8 in terms of a and d and prove that $\dfrac{t_8 - t_3}{8 - 3} = d$.

e) Suggest a formula for finding the common difference of a sequence if you are given the value of the p th term and the q th term.

Extension



The third and eighth terms of an arithmetic sequence are 12 and –18, respectively. Use linear systems to determine the fifth term of the sequence.

Complete Assignment Questions #1 - #18

Assignment

1. For the following arithmetic sequences:
i) Determine the common difference. **ii)** Find the next three terms of the sequence.

a) $8, 14, 20, \ldots$ **b)** $-5, 7, 19, \ldots$ **c)** $70, 53, 36, 19, \ldots$

d) $7.1, 4.2, 1.3, \ldots$ **e)** $\frac{2}{3}, \frac{1}{15}, -\frac{8}{15}, \ldots$ **f)** $a, a + d, a + 2d, \ldots$

g) $-2x + 3y, -5x + y, -8x - y, \ldots$ **h)** $4 + 2\sqrt{2},\ 6 + 3\sqrt{2},\ 8 + 4\sqrt{2}, \ldots.$

2. In each of the following sequences, the value of one term is given. Write the missing terms of the sequence if the common difference is as indicated.

a) ____, ____, 0, ____, ____: $d = 3$ **b)** ____, ____, ____, –3, ____: $d = -7$

c) ____, ____, 1, ____, ____: $d = -2$ **d)** ____, ____, ____, ____, 15: $d = 2.5$

3. Find the first four terms of the arithmetic sequences with given term and common difference, d.

a) $t_1 = 5, d = 6$ **b)** $t_3 = 15, d = -2$ **c)** $t_5 = 20, d = -1$

4. Find the recursive formula for t_n.

a) $-5, 3, 11, 19, \ldots$ **b)** $-21, -36, -51, -66, \ldots$

5. Consider the sequence $12, 5, -2, -9, \ldots$

a) Determine the formula in non-recursive form for the general term of the sequence.

b) Determine the formula in recursive form for the general term of the sequence.

c) Determine the nineteenth term.

6. Find the indicated terms in each arithmetic sequence.

a) $-1, -4, -7, -10, \ldots \; t_5,\ t_{24},\ t_n$

b) $-21, -6, 9, 24, \ldots \; t_{10},\ t_{90},\ t_n$

c) $-b, 2a - b, 4a - b, 6a - b, \ldots \; t_{12},\ t_n$

d) $6 + 2\sqrt{3}, 3 + \sqrt{3}, 0, \ldots \; t_5,\ t_n$

7. Find the number of terms in each sequence.

a) $4, 7, 10, \ldots, 49$

b) $-52, -56, -60, \ldots, -148$

8. a) How many multiples of 5 are there from 25 to 315, inclusive?

b) How many multiples of 7 are there between 51 and 275?

c) How many multiples of 12 are there between 179 and 892?

9. a) Place five arithmetic means between 20 and –76.

b) Place four arithmetic means between –24 and –104.

10. The terms $2x + 3$, $3x + 1$, and $8x - 1$ are consecutive terms in an arithmetic sequence. Calculate the value of x and state the three terms.

11. The terms $x + 3$, $3x - 1$, and $7x - 2$ are consecutive terms in an arithmetic sequence. Calculate the value of x and determine the general term of the sequence.

12. In an arithmetic sequence, the seventh term is 3 and the sixteenth term is 9.

a) Use arithmetic means to find the common difference and the first term of the sequence.

b) Find t_{19} and state the general term of the sequence.

Multiple Choice

13. Which of the following represents an arithmetic sequence with a common difference of –4?

A. 8, 4, 2, 1 . . . **B.** 20, 24, 28, 32 . . . **C.** 32, –8, 2, –0.5 . . . **D.** 20, 16, 12, 8 . . .

14. $p - 1,\ p + 3,\ 3p - 1$, in that order, form an arithmetic sequence.
Which of the following is/are true about p?

1. p is even 2. p is odd 3. p is a perfect square

A. 1 only
B. 1 and 3 only
C. 2 only
D. 2 and 3 only

15. Two students are asked to write the first four terms of an arithmetic sequence.
Rob writes the sequence –14, –6, 2, 10 . . . .
Jason writes the sequence 166, 162, 158, 154 . . . . .
Which statement is true about the fifteenth term of these sequences?

A. t_{15} is the same in each sequence
B. t_{15} is smaller in Rob's sequence
C. t_{15} is smaller in Jason's sequence
D. there is not enough information to answer the question

Numerical Response

16. Twenty-seven arithmetic means are inserted between the first and last terms of a sequence. The number of terms in the sequence is _______ .
(Record your answer in the numerical response box from left to right)

17. A sequence is given by the recursion formula $t_1 = 3, t_2 = 6;\ t_{n+2} = t_{n+1} + t_n$.
The fifth term of the sequence is _______ .
(Record your answer in the numerical response box from left to right)

Extension

18. Use linear systems to find a, d and t_n for each sequence given each pair of terms.

a) $t_5 = 21, t_{10} = 41$ **b)** $t_4 = -9, t_{15} = -31$

Answer Key

1. **a)** i) 6 ii) $t_4 = 26, t_5 = 32, t_6 = 38$ **b)** i) 12 ii) $t_4 = 31, t_5 = 43, t_6 = 55$
c) i) -17 ii) $t_5 = 2, t_6 = -15, t_7 = -32$ **d)** i) -2.9 ii) $t_4 = -1.6, t_5 = -4.5, t_6 = -7.4$
e) i) $-3/5$ ii) $t_4 = -17/15, t_5 = -26/15, t_6 = -7/3$ **f)** i) d ii) $t_4 = a + 3d, t_5 = a + 4d, t_6 = a + 5d$
g) i) $-3x - 2y$, ii) $t_4 = -11x - 3y, t_5 = -14x - 5y, t_6 = -17x - 7y$
h) i) $2 + \sqrt{2}$ ii) $t_4 = 10 + 5\sqrt{2}, t_5 = 12 + 6\sqrt{2}, t_6 = 14 + 7\sqrt{2}$

2. **a)** $-6, -3, 0, 3, 6$ **b)** $18, 11, 4, -3, -10$ **c)** $5, 3, 1, -1, -3$ **d)** $5, 7.5, 10, 12.5, 15$

3. **a)** 5, 11, 17, 23 **b)** 19, 17, 15, 13 **c)** 24, 23, 22, 21

4. **a)** $t_1 = -5, t_n = t_{n-1} + 8, n \geq 2, n \in N$ **b)** $t_1 = -21, t_n = t_{n-1} - 15, n \geq 2, n \in N$

5. **a)** $t_n = 19 - 7n$ **b)** $t_1 = 12, t_n = t_{n-1} - 7, n \geq 2, n \in N$ **c)** -114

6. **a)** $-13, -70, t_n = 2 - 3n$ **b)** $114, 1314,\ t_n = 15n - 36$ **c)** $22a - b, t_n = 2an - 2a - b$
d) $-6 - 2\sqrt{3}, t_n = \left(-3 - \sqrt{3}\right)n + 3\sqrt{3} + 9$

7. **a)** 16 **b)** 25 **8.** **a)** 59 **b)** 32 **c)** 60

9. **a)** $4, -12, -28, -44, -60$ **b)** $-40, -56, -72, -88$ **10.** $x = 0;\ 3, 1, -1$

11. $x = -\frac{3}{2};\ t_n = \frac{17}{2} - 7n$ **12.** **a)** $d = \frac{2}{3}, a = -1;$ **b)** $t_{19} = 11,\ t_n = \frac{2}{3}n - \frac{5}{3}$

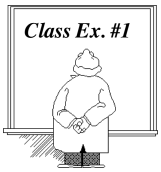
13. D **14.** B **15.** B **16.** | 2 | 9 | | | **17.** | 2 | 4 | | |

18.a) $a = 5, d = 4, \quad t_n = 4n + 1$ **b)** $a = -3, d = -2, \quad t_n = -2n - 1$

Sequences and Data Tables Lesson #6:
Arithmetic Growth

Generating Number Patterns exhibiting Arithmetic Growth

Many real-life scenarios can be represented by a pattern of numbers which exhibits arithmetic growth.



In the first year of employment, Jane is paid $28 000 per year. She receives annual increments of $1500 per year until the end of her twelfth year.

a) State the first four terms of an arithmetic sequence which represents her annual salary during her years of employment.

b) Use the general term formula to calculate her salary in the tenth year.

Relating Arithmetic Sequences to Linear Functions

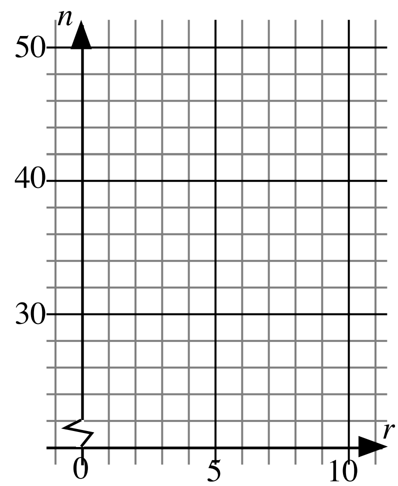
We can relate arithmetic sequences to linear functions over the natural numbers. Consider the following example:

A pile of bricks is arranged in rows. There are 28 bricks in the first row and the number of bricks in each row is two more than in the previous row.

a) Complete the table of values showing the number of bricks in each of the first 10 rows.

Row Number, r	Number of Bricks, n

b) Plot the ordered pairs on the grid. and classify the relationship as linear or non-linear.



c) Determine if the range is an arithmetic sequence.

d) State the domain of the relationship.

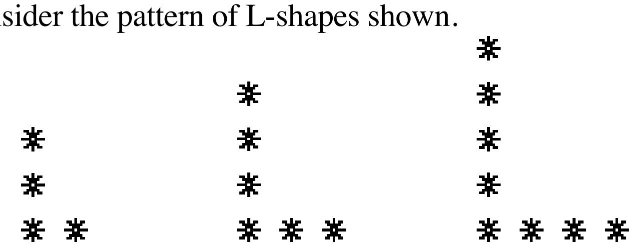
e) Why does the graph not have an intercept on the vertical axis?

f) Write the equation for the number of bricks in a row, n, as a function of the row number, r.

Complete Assignment Questions #1 - #11

Assignment

1. Consider the pattern of L-shapes shown.



a) State the number of stars in each of the first four patterns.

b) How many stars are in the 34th pattern?

2. The manager of a condo development receives a base salary of \$15 000 per year plus \$800 for every condo unit sold.

a) Write the first four terms of the arithmetic sequence for the manager's earnings if 1, 2, 3, ... units are sold.

b) Determine the formula for the general term of the arithmetic sequence in the form $t_n = a + (n - 1)d$.

c) Use the general term formula to calculate his earnings if:

i) 23 **ii)** 54 units are sold.

d) Write the equation for the manager's earnings, E, as a linear function of the number, n, of units sold. Write the equation in the form $E = mn + b$.

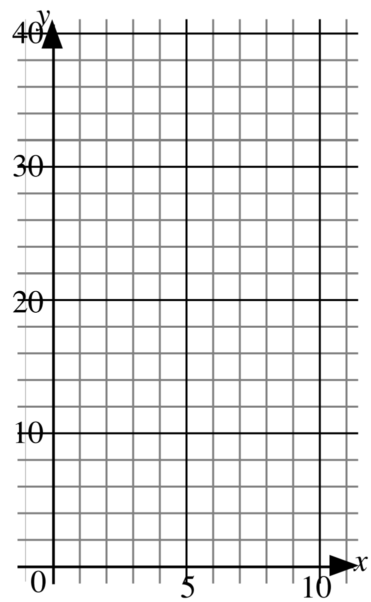
e) Use the linear function in d) to calculate his earnings if:

i) 23 **ii)** 54 units are sold.

3. For a forthcoming horticultural exhibition, bulbs were planted in rows. The number of bulbs in each row forms an arithmetic sequence. There are 58 bulbs in the eighth row and 107 bulbs in the fifteenth row.

How many bulbs in total are in the first three rows?

4. Consider the linear function with equation $y = 3x + 5$.



a) Sketch the graph of the linear function on the grid.

b) Restrict the domain to the set of natural numbers. Mark with dots points on the graph which represent the function on the restricted domain.

c) Write the first five elements of the range in numerical order.

d) Show that the elements of the range form an arithmetic sequence and state the common difference.

5. Charity starts a new job as a geologist in the oil industry. Her rate of pay for the first year is \$36 000 with an increase of \$2750 per year thereafter

a) Calculate her rate of pay in the seventh year.

b) In which year will she first earn more than \$60 000?

6. A sports utility vehicle sells for \$35 000. The vehicle depreciates \$5000 the first year and \$2400 each year thereafter. Calculate the value at the vehicle at the end of the eleventh year.

7. Chairs in an auditorium are arranged in rows in such a way that the first two rows each have the same number of chairs. The third and fourth rows each have three more chairs than the first and second row, the fifth and sixth rows each have three more chairs than the third and fourth row, etc. The sequence of number of chairs for every second row forms an arithmetic sequence. The first two rows each have 27 chairs, and the last two rows each have 114 chairs.

a) How many rows of chairs are there?

b) How many chairs are in the **i)** thirteenth **ii)** thirtieth row?

8. A banquet is to be arranged to celebrate the success of a high school volleyball team. The cost of the banquet is \$250 plus \$25 per person attending. Some statements are made about the relationship in which total cost, C, is a function of the number, n, attending.

Which of the following statements are false?

A. The function represents a partial variation

B. The function can be represented by an arithmetic sequence.

C. The equation for the function is $C = 250 + 25(n - 1)$

Answer **D** if none of the statements A, B or C is false.

Multiple Choice

9. As part of a fitness program, Melissa walks for 35 minutes on day 1 and increases the walking time by 8 minutes each day. She completes the fitness program on the day she first spends more than 3 hours walking. The program is completed on day _____ .

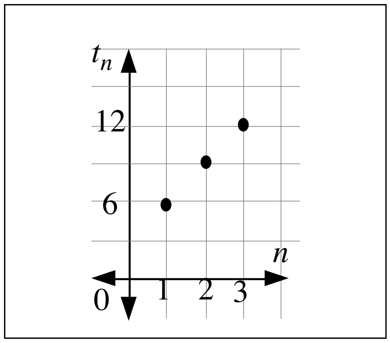
A. 17

B. 18

C. 19

D. 20

Use the following graph to answer questions #10 and #11.



10. The graph above represents arithmetic growth. If the graph continued indefinitely, then the fifteenth term of the sequence is

A. 48
B. 51
C. 90
D. 153

Numerical Response **11.** The formula for this sequence is $t_n = mn + b$.
The value of b is ______ .
(Record your answer in the numerical response box from left to right)

Answer Key

1. a) 4, 6, 8, 10 **b)** 70

2. a) 15800, 16600,17400,18200 **b)** $t_n = 15800 + (n - 1)(800)$
c) i) \$33400 ii) \$58200 **d)** $E = 800n + 15000$ **e)** i) \$33400 ii) \$58200

3. 48 **4. c)** 8, 11, 14, 17, 20 **d)** common difference = 3

5. a) \$52500 **b)** year 10 **6.** \$6000 **7. a)** 60 **b) i)** 45 **ii)** 69

8. C **9.** D **10.** A **11.** 3

Sequences and Data Tables Lesson #7:
Arithmetic Series

Arithmetic Series

When the terms of an arithmetic sequence are added, the result is known as an **arithmetic series**.

For example; 3, 5, 7, 9, 11 → arithmetic sequence

3 + 5 + 7 + 9 + 11 → arithmetic series.

The symbol, S_n is used to represent the sum of n terms of an arithmetic series.

In the example above $S_5 =$ _____ .

Warm-Up #1

Non-Recursive General Formula for the sum of an Arithmetic Series

To illustrate the method for determining a formula for the sum of n terms of an arithmetic series, the story of the great mathematician Karl Gauss (1777-1855) is frequently told.

> When Karl was about 10 years old he was placed in Master Buttner's arithmetic class. As Master Buttner often did, he gave his class long arithmetic problems to keep them quiet for a time. On one particular day, Master Buttner asked his class to add the whole numbers from 1 to 100. While all the students began to work madly on this assignment, Karl laid his slate on the desk and informed Master Buttner he was finished. Master Buttner asked Karl what his answer was. To Master Buttner's astonishment Karl responded a correct answer of 5050.

Here we apply a method similar to his to determine the answer.

$$S_{100} = 1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots\ldots + 96 + 97 + 98 + 99 + 100.$$

$$S_{100} = 100 + 99 + 98 + 97 + 96 + \ldots\ldots\ldots\ldots + 5 + 4 + 3 + 2 + 1.$$

Add the rows and complete the work to show that $S_{100} = 5050$

$$2\,S_{100} = 101 + 101 +$$

Non-Recursive Formulas for the sum of an Arithmetic Series

In an arithmetic series of n terms, the last term, $t_n = a + (n - 1)d$.
Since the common difference is d then the second last term can be written as $t_n - d$.

$$S_n = a + (a + d) + (a + 2d) + \dots + (t_n - 2d) + (t_n - d) + t_n$$

or

$$S_n = t_n + (t_n - d) + (t_n - 2d) + \dots + (a + 2d) + (a + d) + a$$

Adding these two lines together gives

$$2S_n = (a + t_n) + (a + t_n) + (a + t_n) + \dots + (a + t_n) + (a + t_n) + (a + t_n)$$

$$2S_n = n(a + t_n)$$

Dividing by 2 gives the **non-recursive formula for the sum of *n* terms of an arithmetic series.**

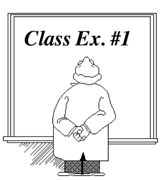
$$\boxed{S_n = \frac{n(a + t_n)}{2}}$$

where

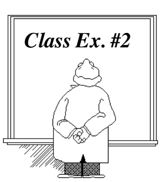
S_n= sum of the first n terms n = position of the term
t_n = the last term a = first term

However, when the last term of the series is not known, another **non-recursive formula for the sum of *n* terms of an arithmetic series** can be formed by replacing t_n by $a + (n - 1)d$ to give

$$\boxed{S_n = \frac{n[2a + (n - 1)d]}{2}}$$



Determine the sum of the first fourteen terms of the arithmetic series 9 + 15 + 21 + ...



Find the sum of the terms in the sequence 17, 12, 7, ... −38.

Complete Assignment Questions #1 - #3

Warm-Up #2 *Investigating a Recursive Formula for an Arithmetic Series*

Corrie was given two questions on a sequence assignment. The first question is listed below.

"Find the first four terms of the series defined by $S_n = 2n^2 - n$."

a) Complete her work below to find the first four terms.

$S_n = 2n^2 - n.$

$S_1 = 2(1)^2 - 1 = 1 \qquad a = t_1 \qquad \Rightarrow t_1 = S_1 \qquad \therefore t_1 = 1$

$S_2 = 2(2)^2 - 2 = 6 \quad \Rightarrow S_2 = S_1 + t_2 \Rightarrow t_2 = S_2 - S_1 \Rightarrow t_2 = 6 - 1 \quad \therefore t_2 =$

$S_3 = 2(3)^2 - 3 = 15 \quad \Rightarrow S_3 = \_\_ + \quad \Rightarrow t_3 = \_\_ - \_\_ \Rightarrow t_3 = \quad - \quad \therefore t_3 =$

$S_4 = \qquad\qquad = \quad \Rightarrow S_4 = \_\_ + \_\_ \Rightarrow t_4 = \qquad \Rightarrow t_4 = \qquad \therefore t_4 =$

b) Express t_{10} in terms of S.

c) Express t_n in terms of S.

d) The second question Corrie received was "Find t_n if $S_n = 2n^2 - n$."

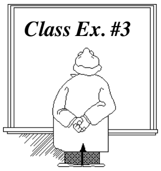
i) Find t_n using the formula $t_n = a + (n - 1)d$

ii) Find t_n using the formula in c)

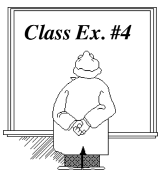
The Recursive Formula for an Arithmetic Series

The recursive formula for an arithmetic series is

$$\boxed{S_1 = a,\ t_n = S_n - S_{n-1},\ n \geq 2, n \in I}$$



For a certain arithmetic series $S_n = \frac{1}{2}n(11 - n)$. Find the first four terms of the corresponding arithmetic sequence.



Peter starts work at a salary of \$16 000 per annum. He receives annual increases of \$850. He works for the firm for twelve years.

a) Calculate his salary in the twelfth year.

b) How much has he earned in total over the twelve years?

Complete Assignment Questions #1 - #14

Assignment

1. Find the sum of each series.

a) $2 + 3 + 4 + \ldots$ (first 30 terms)

b) $(-8) + (-4) + 0 + \ldots$ (first 27 terms)

c) $2.5 + 2.7 + 2.9 + \ldots$ to 16 terms

d) $\frac{5}{2} + \frac{11}{6} + \frac{7}{6} + \frac{1}{2} + \ldots$ to 12 terms

2. Find the sum of each arithmetic series given the first and last terms.

a) $a = 8, t_{15} = 120$

b) $a = -11, t_{23} = -253$

3. Find the sum of each series.

a) $11 + 23 + 35 + 47 + \ldots + 179$

b) $29 + 21 + 13 + 5 + \ldots -27$

4. Consider the series defined by $S_n = 3n - 1$.

a) Find the first four terms of the series

b) Is the sequence arithmetic? Explain.

5. Consider the series defined by $S_n = 3n^2 - n$.

a) Find the first four terms of the series.

b) Determine the eighth term of the corresponding sequence.

6. As payment for his daughter's yard work, a father agrees to give his daughter an allowance of \$3.50 in the first week of the year with an increase of 50 cents each week until the last week of the year.

a) How much money did she receive for an allowance in the last week of the year?

b) What was the total amount of money her father gave her in allowances for the year?

7. Joe from Perfection Millworks has fourteen counter tops to deliver to fourteen floors in an office building. Because of the size of the counter tops, he can only get one counter top into the elevator at a time. He starts at the main floor and takes the first counter top to the first floor, returns to the main floor, picks up the second counter top and takes it to the second floor, and so on. He continues in this way until all fourteen parcels have been delivered. If the distance between floors is exactly 7 m, how far has the elevator travelled when Joe has delivered all the counter tops and returns to the main floor?

8. Bob "Bubbles" Burble was asked on a math assignment to find the sum of eight arithmetic means placed between –15 and 12. Bubbles proceeded to find the eight arithmetic means and determine the sum. Along comes Sally "Sequential" Sequence and asks Bubbles why he is first finding all the means. Sally told Bubbles he does not have to find the means at all! Is Sally correct in stating that Bubbles does not have to find all the means? If so, show clearly Sally's method and determine the sum.

Use the following information to answer questions #9 and #10

Mary Ann was a statistician. She was paid according to a salary grid with annual increases from year one to year ten. The sequence of salaries from years one to ten formed an arithmetic sequence. After year ten she reached her maximum salary. She earned \$65 328 in the fifth year and \$81 276 in the ninth year. She worked as a statistician for twelve years.

Multiple Choice

9. Mary Ann's pay in her first year of work was

A. \$49 380
B. \$53 367
C. \$57 354
D. none of the above

10. The total amount that Mary Ann earned as a statistician was

A. \$85 263
B. \$673 215
C. \$843 741
D. \$1 346 430

11. Which of the following is an arithmetic series?

A. $1 + 4 + 9 + 16 + 25$
B. $1, 3, 5, 7, \ldots.$
C. $6 + 2 + (-2) + (-6)$
D. $8, -8, 8, -8$

12. The sum of the first 100 terms of the arithmetic series $3 + 1 + (-1) + (-3) + \ldots$ is

A. –1020
B. –1005
C. –9705
D. –9600

Use the following information to answer questions #13 - #14

> A child arranges animal blocks in rows on a floor. There are 64 animal blocks in the fifth row and 92 animal blocks in the ninth and last row. Assume that the number of animal blocks from row to row form an arithmetic sequence.

Numerical Response

13. The number of animal blocks in the first row is _____ .
(Record your answer in the numerical response box from left to right)

14. The total number of blocks used in the arrangement is _____ .
(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** 495 **b)** 1188 **c)** 64 **d)** –14
2. **a)** 960 **b)** –3036 **3.** **a)** 1425 **b)** 8
4. **a)** 2, 3, 3, 3 **b)** no, because there is no common difference between successive terms.
5. **a)** 2, 8, 14, 20 **b)** 44
6. \$29, \$845 **7.** 1470 m
8. Sally is correct. We are given: i) the first and last terms, ii) the number of terms - 10 (eight arithmetic means plus the first and last term) and iii) the series is arithmetic. ∴ given this information, all Bubbles has to do is substitute the appropriate numbers in the general arithmetic series formula $S_n = \dfrac{n(a + t_n)}{2}$ to get the answer S_{10}. Subtract the first and last terms to get the answer of –12.
9. A **10.** C **11.** C **12.** D

13.

3	6		

14.

5	7	6	

Sequence and Data Tables Lesson #8:
Geometric Growth

Warm-Up

Consider the following two sequences: **i)** $2, 4, 6, 8, ...$ **ii)** $2, 4, 8, 16, ...$

a) Explain how to calculate t_5 and t_6 for each sequence. State the value of each term.

b) Determine the formula in recursive form for the general term of each sequence.

Geometric Sequence

A sequence where each term is obtained by **multiplying** the preceding term by a constant is called a **geometric sequence**.

For example, $3, 6, 12, 24, ...$ is a geometric sequence.

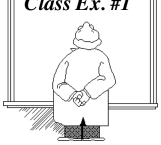
The value of the constant can be found by dividing term 2 by term 1 or term 3 by term 2 etc.

In the example the constant is ____.

The constant in a geometric sequence is called the **common ratio** and can be found by the following formula:

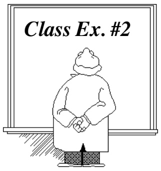
$$\boxed{\textbf{common ratio} = \mathbf{r} = \frac{t_n}{t_{n-1}}} \quad \text{where } n \geq 2, n \in N$$

Class Ex. #1



For the following sequences:
- Identify the type of sequence (arithmetic or geometric)
- State the common difference or common ratio depending on the sequence
- State the fifth and sixth term for each.

a) $8, 24, 72, 216, ...$ **b)** $108, 72, 36, 0, ...$ **c)** $-3, 2, -\frac{4}{3}, \frac{8}{9}, ...$



$x + 3$, x, and $x - 5$ are three terms in a geometric sequence. Use the concept of common ratio to determine the value of the three terms.

Complete Assignment Questions #1 - #5

Geometric Growth

The following exploration illustrates the difference between arithmetic growth and geometric growth.

Simple Interest and Compound Interest

In simple interest the principal at the beginning of the second year is the same as the principal at the beginning of the first year.

In compound interest the interest earned during the first year is added to the original principal to form a different principal for each year.

Exploration

A bank offers two types of savings bond:

- Regular Savings Bond which pays simple interest at 9% per year
- Compound Savings Bond which pays interest at 9% per year compounded annually.

Simple Interest - Arithmetic Growth

The simple interest each year is 9% of \$5000 = \$450.

The arithmetic growth factor is 450 because the amounts form an arithmetic sequence with a common difference of \$450. The amount at the end of each successive year is found by adding \$450.

Find the value of the bond at the end of each of the first 8 years, and complete the table.

End of Year	Amount(\$)
1	5450
2	5900
3	
4	
5	
6	
7	
8	

Compound Interest - Geometric Growth

Complete the following to determine the compound interest and the value of the bond.

End of Year 1: Value of Bond = Principal + Interest

$= 5000 + (0.09)(5000)$
$= 5000(1 + 0.09)$ factor out 5000
$= 5000(1.09)$

End of Year 2: Value of Bond = Principal + Interest

$= 5000(1.09) + (0.09)\,[5000(1.09)]$
$= 5000(1.09)(1 + 0.09)$ factor out 5000(1.09)
$= 5000(1.09)(1.09)$
$= 5000(1.09)^2$

End of Year 3: Value of Bond = Principal + Interest

=

=

=

=

The geometric growth factor is 1.09 because the amounts form a geometric sequence with a common ratio of 1.09. The amount at the end of each successive year is found by multiplying by 1.09. The value at the end of each year is 1.09 times the value at the end of the previous year. Find the value of the bond at the end of each of the first 8 years, and complete the table.

Term	End of Year	Value of Bond	Amount ($)
t_1	1	5000(1.09)	5450
t_2	2	$5000(1.09)^2$	
t_3			
t_4			
t_5			
t_6			
t_7			
t_8			
t_n			

Geometric Decay

Consider the following scenario. Hamish invested $1000 in the stock market. Unfortunately the value of his investment decreased by 10% each year

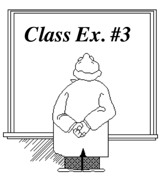
Complete the table to determine the value of his investment at the end of each year.

End of Year	1	2	3	4
Value of Investment ($)	900			

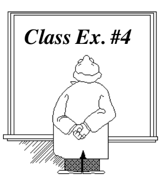
The values form a geometric sequence. Calculate the value of the common ratio.

Complete : The geometric growth factor is _____.

Geometric growth where the factor is between 0 and 1 is often called **geometric decay**.



Babar deposits $7 000 into a bank account which pays interest compounded annually at 5%. How much does Babar have in the bank account at the end of each of the first three years.



State the growth factor in each of the following situations.

a) The rate of inflation is increasing by 3.5% each year.

b) The number of fish in a lake is decreasing by 2% each year.

c) The number of rabbits in a population is doubling each year

Complete Assignment Questions #6 - #11

Extension

In the extension we introduce two formulas which are often used when dealing with geometric growth

i) The Non-Recursive Formula for the General Term of a Geometric Sequence

ii) The Compound Interest Formula

The Non-Recursive Formula for the General Term of a Geometric Sequence

Consider the following scenario:
A university marine biology student studies the growth of a strand of moss by observing the change in its weight every day. The table shows the mass of the moss in grams at 12 noon on each of the first four days.

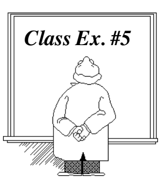
Days	1	2	3	4
Mass (g)	8	12	18	27

a) The masses form a geometric sequence. State the common ratio.

b) Let a = the first term, and r = the common ratio. Complete the following table:

t_1	First term = 8	$t_1 = a$
t_2	$8 \times 1.5 = 12$	$t_2 = ar$
t_3	$8 \times 1.5 \times 1.5 = 18$	$t_3 = ar^2$
t_4	$8 \times 1.5 \times 1.5 \times$ ____ $=$	$t_4 =$
t_5		$t_5 =$
t_n		$t_n =$

c) The non-recursive geometric formula for a geometric sequence is $t_n =$



Lucasito drops a rubber ball from the top of a building 20 m high. Each time the ball bounces it bounces back up to 80% of its previous height. Calculate, to the nearest cm, the height of the ball after the fifteenth bounce.

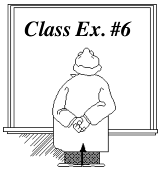
The Compound Interest Formula

In the example on compound interest on the third page of this lesson, when \$5000 was invested at 9% for n years , we had the general term $t_n = 5000(1.09)^n$.

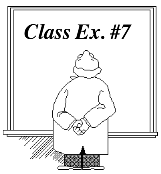
This is an example of the compound interest formula $\boldsymbol{A = P(1 + i)^n}$

where A = final value of the investment (or Amount)
P = initial investment (or Principal)
i = annual interest rate (as a decimal)
n = number of years

Compounding semi-annually, monthly etc is covered in the next math course.



Kirsten deposits \$4 000 into a bank account which earns 3.5% p.a.(per annum) compounded yearly. How much does Kirsten have in the bank account after 6 years?



A certain brand of sports utility vehicle, initial value \$48 000, depreciates 15% per year. Find the value, to the nearest dollar, of the sports utility vehicle after 7 years by

a) using $t_n = ar^{n-1}$ **b)** using $A = P(1 + i)^n$

Complete Assignment Questions #12 - #15

Assignment

1. Consider the sequence 3, 6, ...

Is there enough information to determine whether the sequence is arithmetic or geometric? Explain.

2. For the following sequences:
- Identify the type of sequence (arithmetic or geometric or neither)
- State the common difference or common ratio (where relevant)
- Determine the next two terms

a) $4, 12, 36, 108, ...$ **b)** $512, 64, 8, 1, ...$ **c)** $2, 4, 7, 11, 16, ...$

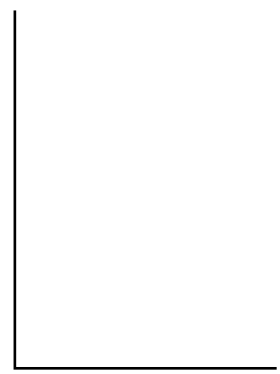
d) $6, 6\sqrt{2}, 12, ...$ **e)** $3, 6, 9, ...$ **f)** $b, bc, bc^2, ...$

g) $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, ...$ **h)** $\frac{3}{7}, \frac{6}{21}, \frac{12}{63}, ...$ **i)** $0.4, 0.3, 0.225, ...$

3. The number of bacteria is tripling every hour.

a) Starting with 1 bacterium at 1 pm, write a sequence for the number of bacteria at 1 pm, 2 pm, 3 pm, up to 7 pm.

b) Plot the ordered pairs (time, # bacteria) on the grid. Does the graph represent a linear relationship?

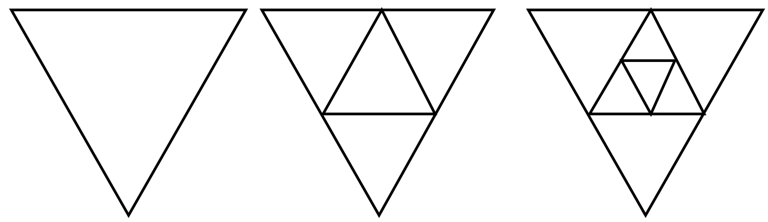


4. The first three terms of a geometric sequence can be written in the form $x + 75$, $x - 25$, and $x - 45$.

a) Use the concept of common ratio to determine the value of x

b) State the common ratio and the value of the first five terms.

5. An equilateral triangle (triangle 1) has sides with a length of 30 cm each.
A triangle (triangle 2) is placed in the middle of the larger triangle by joining the midpoints of the sides of the previous triangle.
The pattern is continued as illustrated in the diagram.



a) Write the first five terms of the sequence for the lengths of the sides of the triangles.

b) Find the perimeter of the tenth triangle in the sequence.

c) How does the area of triangle 7 compare to the area of triangle 9?
(Note: You do not require to calculate the area of any triangles)

6. State the growth factor in each of the following situations.

a) Canada's population is increasing by 1% per year

b) My car is depreciating in value by 12% per year.

c) Henri receives a salary increase of 2.4% per year

d) Newspaper readership is declining by $\frac{1}{2}$% per year.

7. A pendulum swings through an arc of 120 cm (Swing #1). With each further swing the arc length is reduced by 15%.

a) State the growth factor.

b) Calculate the length of the arc in Swing #4.

8. Didra deposits $3 500 into a bank account which earns 6% p.a. compounded annually. How much does Didra have in the bank account after 4 years?

Multiple Choice

9. Consider two geometric sequences X and Y.

Sequence X $48, -24, 12, -6, \ldots$ Sequence Y $-\frac{3}{64}, -\frac{3}{32}, -\frac{3}{16}, -\frac{3}{8}, \ldots$

Which statement is true?

A. The common ratios of both sequences are the same

B. Both sequences contain a term with value –3

C. The sixth term of sequence X is equal to the sixth term of sequence Y

D. The eighth term of sequence X is smaller than the eighth term of sequence Y

10. Which of the following when graphed does not result in a straight line?

A. direct variation **B.** partial variation **C.** geometric growth **D.** arithmetic growth

Numerical Response

11. The first three terms in a geometric sequence are x, $x - 5$ and $x - 9$. The exact value of the fourth term of the sequence is ______ .

(Record your answer in the numerical response box from left to right)

Extension Questions

12. Use the general term formula to determine the value of the given term

a) $32, 16, 8, 4 \ldots$ term 9

b) $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \ldots$ term 8

c) $5, 5\sqrt{3}, 15, \ldots.$ term 8

d) $\frac{1}{48}, \frac{1}{24}, \frac{1}{12}, \ldots$ term 11

13. Determine the number of terms in the sequence $32, 64, 128 \ldots, 16\,384$.

14. For each compound interest investment determine the amount to the nearest dollar.

a) \$8000 at 5% p.a. for 6 years

b) \$750 at 8% p.a. for 5 years

15. A truck depreciates in value by 15% per year. If the truck is valued at \$40000 on Aug 1, 2003 what will its value be, to the nearest dollar, on Aug 1, 2009?

Answer Key

1. No. More than two terms are required to determine a common difference or a common ratio.

2. **a)** geometric, $r = 3$, terms are 324, 972 **b)** geometric, $r = \frac{1}{8}$, terms are $\frac{1}{8}, \frac{1}{64}$

c) neither **d)** geometric, $r = \sqrt{2}$, terms are $12\sqrt{2}$, 24 **e)** arithmetic, $d = 3$, terms are 12, 15

f) geometric, $r = c$, terms are bc^3, bc^4 **g)** arithmetic, $d = \frac{1}{4}$, terms are 1, $\frac{5}{4}$

h) geometric, $r = \frac{2}{3}$, terms are $\frac{24}{189}, \frac{48}{567}$ **i)** geometric, $r = 0.75$, terms are 0.16875, 0.1265625

3. **a)** 1, 3, 9, 27, 81, 243, 729 **b)** no

4. **a)** 50 **b)** $r = \frac{1}{5}$, terms are 125, 25, 5, 1, $\frac{1}{5}$

5. **a)** 30, 15, $\frac{15}{2}, \frac{15}{4}, \frac{15}{8}$ **b)** $\frac{45}{256}$ cm **c)** It is 16 times as large

6. **a)** 1.01 **b)** 0.88 **c)** 1.024 **d)** 0.995 **7.** **a)** 0.85 **b)** 73.695 cm

8. \$4418.67 **9.** C **10.** C **11.**

1	2	.	8

12. **a)** $\frac{1}{8}$ **b)** $-\frac{1}{384}$ **c)** $135\sqrt{3}$ **d)** $\frac{64}{3}$

13. 10 **14.** **a)** \$10 721 **b)** \$1 102 **15.** \$15 086

Sequence and Data Tables Lesson #9:
Sampling Techniques

Warm-Up

In this unit we have analyzed numerical data and looked for patterns, trends and relationships. In this lesson we focus on how data is collected.

Data can be collected from a **population** or from a **sample** of a population.

- **Population** - the complete collection of all items in a study.

  Suppose that the study consisted of a series of questions which could be answered using a **survey**. A **census** is where all the members of the population are surveyed.
  If a census is not possible (too costly, too time-consuming etc.) then a **sample** of the population would be surveyed.

- **Sample** - a portion picked out of the population by one of various techniques.

  Data collected from a sample is used to represent and make **inferences** and generalisations about the population. The sample must be chosen carefully so that it is representative of the population otherwise inferences made from the sample are unreliable.
  Any sample chosen should be free from **selection bias** - no element of the population should be excluded from a chance of being in the sample.
  The larger the **sample size** the more reliable are the inferences made from the sample.

Probability Sampling Techniques

The purpose of probability sampling is to get a sample that represents the population from which it was drawn. The four probability sampling techniques that we discuss all possess an element of randomness i.e. every element of the population has a chance of being selected before the sampling begins.

We will use the following scenario to investigate different sampling techniques.

Scenario

Lunchtime at AVP High School (Grade 9 -12) is getting so crowded that administration are proposing a split lunch hour. Under the proposal, senior students will have lunch from 11am to 12 noon and classes from 12 noon to 1pm and junior students will have classes from 11am to 12 noon and lunch from 12 noon to 1pm. Before implementing the proposal, administration decides to get feedback from the stakeholders in the school, one of whom is the student body.

In this case the population is the entire student body of 1000 students. We assume that the whole population is not surveyed and that a sample is to be taken from the student body.

Number of Students	Grade 9	Grade 10	Grade 11	Grade 12
Male	150	150	100	90
Female	150	130	120	110

Random Sampling

Random sampling (or **simple random sampling**) is a sampling technique in which each member of the population has an equal probability of being included in the sample.

Suggest how this might be done in the AVP scenario or using your own school.

Stratified Sampling

Stratified sampling (or **stratified random sampling**) involves first dividing the population into sub-groups or strata based on some known demographic (age, gender, etc.) and then take a simple random sample from each sub-group. This technique is useful when we want to ensure that all sub-groups are represented in the sample. The proportion of each sub-group in the sample should be similar to the proportion of each sub-group in the population.

Suggest how this might be done in the AVP scenario or using your own school.

Systematic Sampling

Systematic sampling is a sampling technique in which the elements of the population are placed in a list and every *n* th element is included in the sample. To make the sample random in nature a starting point from 1 to *n* would be randomly chosen.

Suggest how this might be done in the AVP scenario or using your own school.

Cluster Sampling

Cluster sampling is a sampling technique that involves dividing the population into groups or clusters, randomly selecting a certain number of clusters and then sampling every element in each cluster.

Suggest how this might be done in the AVP scenario or using your own school.

Non-Probability Sampling Techniques

These techniques are not based on probability. Inferences made from such samples may be unreliable.

Convenience Sampling

Convenience sampling is a sampling technique that does not involve randomness. The sample is taken from the most convenient elements of the population available.

Suggest how this might be done in the AVP scenario or using your own school.

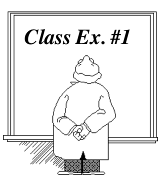
Volunteer Sampling

Volunteers are asked to participate in the sample.

Bias

Selection Bias

No element of the population should be excluded from a chance of being in the sample.



A local newspaper conducted a survey to decide what kind of items to put in their entertainment and arts section. They handed out a questionnaire to 100 people as they leave an operatic performance.

a) Describe the population of interest in this survey

b) Why are the results of this survey subject to selection bias?

c) Suggest an alternative sampling method that would more accurately reflect the views of the population.

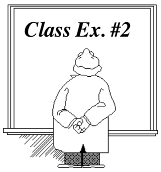
Non-Response Bias

A questionnaire may be sent out to 1000 homes. If a large number of people selected for the sample do not complete the questionnaire the results obtained may be distorted. Maybe only those with strong views on an issue will respond. Their views may be different from the general population.

Question Bias or Interviewer Bias

A question is phrased in a particular way to try to influence the answer.

eg “You don’t believe that “Fire In The City” is a good movie do you? “
Answer NO yes

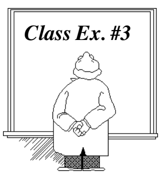


Reword the following survey questions to remove question bias.

a) “You don’t believe that “Fire In The City” is a good movie do you?”

b) “Aren’t the Canucks just the best hockey team in Canada?”

c) “Choose your favourite between awesome rock music and boring country music?”



A representative sample of 1000 voters in a Calgary riding were asked which issues the government needed to focus on in the next year.

The results are given below.

Education	498	Health	503	Transportation	721
Environment	84	Taxation	296		

Comment on the following inferences.

i) The figures add up to more than 1000 so the data has been incorrectly tabulated.

ii) Transportation is the issue the voters feel the government needs to make its number one priority.

iii) Most voters in the riding don’t really care about the environment.

iv) Voters in the riding think the government should focus more on health than on education.

Complete Assignment Questions #1 - #14

Assignment

1. Students were asked by their teacher to determine which of the five local radio stations in the city is the most popular.

Gina interviewed every fourth person in a line up for tickets to a rap concert. She interviewed 300 people. Her data is summarized below.

WRAP 177	WMTL 69	WCTY 24	WRCK 24	WNWS 6

Chloe interviewed every fourth person at an entrance to a large shopping mall. She interviewed 300 people. Her data is summarized below.

WRAP 24	WMTL 45	WCTY 81	WRCK 90	WNWS 60

a) Calculate the percentage of students in each survey who preferred WRAP.

b) Explain the huge difference in the percentages in a).

c) Which survey is more likely to represent the opinions of the population in the city?

d) Assuming that each of the 235 000 radio listeners in the city has a most popular station, estimate the number who prefer the country music station WCTY.

2. Reword the following survey questions to remove question bias.

a) "Wasn't "Seinfeld" the best comedy show ever?"

b) "Basketball is so much better than football, isn't it?"

c) "Do you prefer exciting contact sports or dull nature walks?"

3. The local government is considering providing free lunches to Grade 1 students. A questionnaire appeared in a local newspaper and readers were invited to reply whether they thought that providing free lunches was a good idea. Of the 1326 replies, 1306 were in favour of the proposal.

a) What inferences about the population can be drawn from the survey?

b) Why might the results of the survey be biased?

4. Students in a math class were asked to collect data for a statistics project. The topic was the most popular type of books read by students in the school. As usual Dave was late in getting down to his homework and at the last minute he came up with the following survey question. "Do you prefer marvelous mystery novels or stupid science fiction books?" He asked six of his closest friends and handed the data in to his teacher. He did not get a very good mark!

a) There are at least four areas in which Dave's work can be improved. List four things that statistically Dave did wrong.

b) Addressing each of Dave's errors., explain how you might have collected the data for the project.

Multiple Choice

5. You are trying to determine what percentage of students in your school drive to school. Which of the following sampling techniques would give the best estimate?

A. You survey every fifth student that passes you in the hallway.

B. You randomly select one classroom to visit and ask all the students in that class.

C. You survey all the students in your English class.

D. You survey 12 of your best friends.

6. In which of the following scenarios would cluster sampling be the most suitable sampling technique?

A. You want to determine the proportion of grade 10 students in your high school who are left handed.

B. You want to determine the proportion of grade 10 students in your math class who are left handed.

C. You want to determine the proportion of grade 10 students in the high schools of Alberta and British Columbia who are left handed.

D. You want to determine the proportion of students in your high school who are left handed.

7. Which type of bias is present in a census?

A. selection bias **B.** non-response bias **C.** question bias **D.** no bias

Use the following answer key to answer questions #8-#12.

A. Systematic Sampling **B.** Convenience Sampling **C.** Cluster Sampling
D. Simple Random Sampling **E.** Stratified Random Sampling

8. With which sampling technique does each member of the population have an equal chance of being chosen?

9. Which sampling technique are you using if you use for your survey the first name on each page of the phone book?

10. The population consists of 40 men and 60 women. Which sampling technique involves surveying 8 men and 12 women?

11. Which sampling technique does not involve a degree of randomness ?

12. You collect data about every tenth car that passes your local coffee shop. Which sampling technique is involved?

13. Which statement is true?

A. A school has 800 students, 300 in Grade 10, 300 in Grade 11 and 200 in Grade 12. In a stratified sample of size 100, one third of the sample should be Grade 10 students.

B. Cluster sampling is biased because not every member of the population has a chance of being chosen in the sample.

C. In sampling by volunteers every member of the population has an equal probability of being chosen in the sample.

D. The greater the percentage of the population that are surveyed in a representative sample, the more reliable are the inferences made from the data.

14. A school has 1000 students, 400 in Grade 10, 400 in Grade 11 and 200 in Grade 12. Which of the following is a systematic sampling procedure to collect data from 10% of the students?

A. Interview 40 students chosen at random from Grade 10, 40 students from Grade 11, and 20 students from Grade 12.

B. Interview 50 males and 50 females chosen at random.

C. Obtain a printout of the names of all 1000 students arranged in alphabetical order. Interview every tenth person on the list.

D. Obtain a printout of the names of all 1000 students arranged in alphabetical order. Select 100 students randomly from the list.

Answer Key

1. **a)** Gina 59% Chloe 8% **b)** WRAP is probably a rap station. Gina' data is collected at a rap concert and so is biased in favour of that station. **c)** Chloe's survey **d)** 63 450

2. **a)** "Was Seinfeld the best comedy show ever?" **b)** "Is basketball better than football?"
c) "Do you prefer contact sports or nature walks?"

3. **a)** Most people think that providing free lunches for Grade 1 students is a good idea
b) The survey may be influenced by non-response bias. Only those who feel strongly about the issue are likely to reply to a newspaper questionnaire. Most of the general population would not reply but may hold differing views. It is likely that the percentage of the population who support the issue is less than the percentage in the sample who support the issue.

4. **a)** i) He only asks about two types of books. ii) The question is biased in favour of mystery novels.
iii) The sample is likely not representative of the population - it is a convenience sample.
iv) The sample size is too small

b) Answers will vary but should include a question like 'Which is your favourite type of book to read?" possibly with answers to choose from. It should also include a sampling technique without bias and representative of the population. The sample size needs to be much larger.

5. A	**6.** C	**7.** D	**8.** D	**9.** A
10. E	**11.** B	**12.** A	**13.** D	**14.** C

Rational Expressions Lesson #1:
Simplifying Rational Expressions - Part One

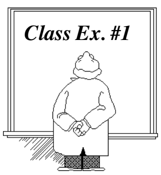
Warm-Up #1 *Factoring Review*

Recall the following methods for factoring polynomial expressions:

i) greatest common factor **ii)** difference of squares **iii)** factoring trinomials by inspection

iv) factoring trinomials by decomposition **v)** grouping

The key to success in operating with rational expressions in this unit lies in our ability to factor polynomials.



Factor

a) $10x^3 - 40x$ **b)** $x^2 + 12x - 45$ **c)** $8x^2 - 14x - 15$

Complete Assignment Question #1-#2

Warm-Up #2

A single variable **rational expression** is an algebraic fraction in which the numerator and denominator are both polynomials.

eg $\dfrac{x-3}{x^2+1}$, $\dfrac{7}{2y+5}$ etc.

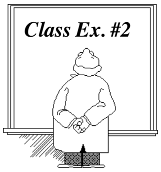
If the denominator is a real number and does not contain a variable then the fraction could be written as a polynomial.

eg $\dfrac{x+4}{2}$ can be written as $\dfrac{1}{2}x + 2$.

If both the numerator and denominator are real numbers and do not contain a variable then the fraction is a rational number.

eg $\dfrac{8}{5}$

Equivalent Forms of a Rational Expression



Consider the rational expressions

$\frac{2x+2}{x^2+3x+2}$ and $\frac{2}{x+2}$.

a) Complete the table.

Value of x	Value of $\frac{2x+2}{x^2+3x+2}$	Value of $\frac{2}{x+2}$
0		
1		
2		
3		
4		

b) What can we say about the values of the rational expressions $\frac{2x+2}{x^2+3x+2}$ and $\frac{2}{x+2}$ when x is replaced by 0, 1, 2 , 3, or 4?

c) The expressions $\frac{2x+2}{x^2+3x+2}$ and $\frac{2}{x+2}$ are known as **equivalent forms** of a rational expression. To explain why they are equivalent read the following procedure and complete after step ii).

i) write the numerator, $2x+2$, and the denominator, x^2+3x+2 in factored form

ii) reduce the rational expression by dividing out a common factor, called **cancelling factors** and show that $\frac{2x+2}{x^2+3x+2}$ can be reduced to $\frac{2}{x+2}$.

Complete:

$$\frac{2x+2}{x^2+3x+2} = \frac{2(\quad+\quad)}{(\qquad)(\qquad)} =$$

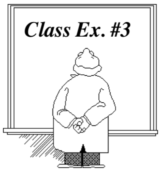
- When $\frac{2x+2}{x^2+3x+2}$ is written in the form $\frac{2}{x+2}$ it is said to be in **lowest terms** or **simplest form**.

- $\frac{2}{x+2}$ cannot be further reduced by cancelling terms. **The two 2's cannot** be reduced.

  ie $\frac{2}{x+2}$ is not equivalent to $\frac{1}{x+1}$ (Replace x by any permissible value to verify this)

- **To reduce fractions we cancel factors, not terms.**

Nonpermissible Values



a) Complete the table.

Write the value as **not defined** if the value cannot be calculated.

Value of x	Value of $\dfrac{2x+2}{x^2+3x+2}$	Value of $\dfrac{2}{x+2}$
0		
–1		
–2		
–3		

b) For which value(s) of x is the expression $\dfrac{2x+2}{x^2+3x+2}$ not defined?

c) For which value(s) of x is the expression $\dfrac{2}{x+2}$ not defined?

d) Why do the values in b) and c) result in the expressions not being defined?

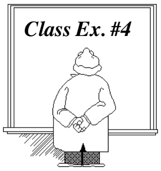
Values of the variable which result in the value of a rational expression not being defined are called **nonpermissible values.** These values are known as the **restrictions** on the variable.

Nonpermissible values are values of the variable which make the denominator equal to zero.

Note that although $\dfrac{2x+2}{x^2+3x+2}$ and $\dfrac{2}{x+2}$ are equivalent forms of a rational expression they have different restrictions on the value of x.

The restrictions must be determined before dividing out common factors.

In an earlier unit involving division of polynomials we stated that nonpermissible values of the variable were present even though they were not stated in the answers. Be aware that nonpermissible values are present each time we divide by an expression containing a variable.



Express in simplest form stating the nonpermissible values of the variable.

a) $\dfrac{12x^2}{2x}$ **b)** $\dfrac{(a+1)(a-6)}{(a+7)(a+1)}$ **c)** $\dfrac{y+4}{y^2-y-20}$ **d)** $\dfrac{x^2+11x+28}{x^2-49}$

Complete Assignment Questions #3 - #11

Assignment

1. Factor the following polynomial expressions.

a) $7y - 49$ **b)** $4a^2 + 16$ **c)** $4a^2 - 16$ **d)** $-3c - 27c^2$

e) $50x^2 - 18$ **f)** $100 - t^2$ **g)** $a^4 - 16$ **h)** $3x^5 - 768x$

2. Factor.

a) $x^2 + 10x + 21$ **b)** $y^2 - 4y + 4$ **c)** $t^2 - t - 72$ **d)** $b^3 + 3b^2 - 40b$

e) $3a^2 - 5a - 2$ **f)** $8p^2 + 22p + 9$ **g)** $10p^2 + 25p - 15$ **h)** $27x^2 - 36x + 12$

3. Which of the following expressions are single variable rational expressions?

a) $-\dfrac{5}{3}$ **b)** $-\dfrac{5}{3x}$ **c)** $\dfrac{7b-1}{(b-1)^3}$ **d)** $\dfrac{v+4}{w+4}$ **e)** $\dfrac{\pi}{2}$

4. Determine the nonpermissible values of the variable.

a) $\dfrac{6}{8x-7}$ **b)** $\dfrac{y}{10y+20}$ **c)** $\dfrac{5a}{5-a}$ **d)** $\dfrac{a^2+7a+12}{(a+4)(a+5)}$ **e)** $\dfrac{12y^2-2}{y}$

f) $\dfrac{1+16x^2}{1-16x^2}$ **g)** $\dfrac{40p^3-4}{8q^3}$ **h)** $\dfrac{3}{x^2+13x+12}$ **i)** $\dfrac{d}{d^2-8d+16}$

5. Express in simplest form stating the nonpermissible values of the variable.

a) $\dfrac{4ab}{16a}$ **b)** $\dfrac{25x^3y^4}{5y^9}$ **c)** $\dfrac{(a+3)(a-8)}{(a+1)(a-8)}$ **d)** $\dfrac{(x+7)(x-2)}{x(x-2)(x+14)}$

e) $\dfrac{y+9}{y^2-81}$ **f)** $\dfrac{25y^2-36}{5y+6}$ **g)** $\dfrac{64-9p^2}{(8-3p)(3+8p)}$ **h)** $\dfrac{x^2-100}{(x+10)^2}$

6. The area of a soccer field is represented by $a^2 - 12a + 32$ square metres.

a) Find a simplified expression for the length of the field if the width can be represented by $a - 8$ metres.

b) Calculate the area of the field if $a = 90$.

7. Reduce to lowest terms stating the restrictions on the variable.

a) $\dfrac{(t+3)^2}{(t+1)(t+3)}$ **b)** $\dfrac{x^2-1}{x^2+2x+1}$ **c)** $\dfrac{e^2+2e-35}{e^2+14e+49}$ **d)** $\dfrac{m^2-2m-15}{m^2+12m+27}$

e) $\dfrac{y^2+4y}{y^2-16}$ **f)** $\dfrac{x^2+9x-22}{x^2+12x+11}$ **g)** $\dfrac{a^2+11a+10}{a^2+8a-20}$ **h)** $\dfrac{p^2+5p+6}{p^2-4}$

Multiple Choice

8. When simplified the rational expression $\dfrac{a^2+a-2}{a^2-1}$ can be reduced to

A. $\dfrac{a-2}{-1}$

B. $\dfrac{a-2}{a-1}$

C. $\dfrac{a+2}{a+1}$

D. $\dfrac{a-2}{a+1}$

9. $\dfrac{(x-y)^2}{x^2-y^2}$ is equivalent to

A. 0

B. 1

C. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2$

D. $\dfrac{x-y}{x+y}$

10. In the rational expression $\dfrac{a-3}{a(a+7)}$ the nonpermissible value(s) of a are

A. 3, −7
B. 0, 3, −7
C. 0, −7
D. −7

Numerical Response

11. The rational expressions $\dfrac{x^2+13x+40}{x^2-13x+40}$ and $\dfrac{64+x^2}{64-x^2}$ have one nonpermissible value in common.
To the nearest tenth, this nonpermissible value is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $7(y-7)$ **b)** $4(a^2+4)$ **c)** $4(a-2)(a+2)$ **d)** $-3c(1+9c)$ **e)** $2(5x-3)(5x+3)$
f) $(10-t)(10+t)$ **g)** $(a-2)(a+2)(a^2+4)$ **h)** $3x(x-4)(x+4)(x^2+16)$

2. **a)** $(x+7)(x+3)$ **b)** $(y-2)^2$ **c)** $(t-9)(t+8)$ **d)** $b(b+8)(b-5)$
e) $(3a+1)(a-2)$ **f)** $(4p+9)(2p+1)$ **g)** $5(2p-1)(p+3)$ **h)** $3(3x-2)^2$

3. b and c.

4. **a)** $x \neq \frac{7}{8}$ **b)** $y \neq -2$ **c)** $a \neq 5$ **d)** $a \neq -5, -4$ **e)** $y \neq 0$
f) $x \neq \pm\frac{1}{4}$ **g)** $q \neq 0$ **h)** $x \neq -12, -1$ **i)** $d \neq 4$

5. **a)** $\frac{b}{4}$. $a \neq 0$ **b)** $\frac{5x^3}{y^5}, y \neq 0$ **c)** $\frac{a+3}{a+1}, a \neq -1, 8$ **d)** $\frac{x+7}{x(x+14)}, x \neq -14, 0, 2$
e) $\frac{1}{y-9}, y \neq \pm 9$ **f)** $5y-6, y \neq -\frac{6}{5}$ **g)** $\frac{8+3p}{3+8p}, p \neq -\frac{3}{8}, \frac{8}{3}$ **h)** $\frac{x-10}{x+10}, x \neq -10$

6. **a)** $a-4$ metres **b)** 7052 square metres

7. **a)** $\frac{t+3}{t+1}, t \neq -1, -3$ **b)** $\frac{x-1}{x+1}, x \neq -1$ **c)** $\frac{e-5}{e+7}, e \neq -7$ **d)** $\frac{m-5}{m+9}, m \neq -9, -3$
e) $\frac{y}{y-4}, y \neq \pm 4$ **f)** $\frac{x-2}{x+1}, x \neq -11, -1$ **g)** $\frac{a+1}{a-2}, a \neq -10, 2$ **h)** $\frac{p+3}{p-2}, p \neq \pm 2$

8. C **9.** D **10.** C **11.**

8	.	0	

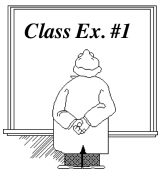
Rational Expressions Lesson #2:
Simplifying Rational Expressions Part Two

Warm-Up

In this lesson we extend the method of simplifying rational expression to more complex examples.

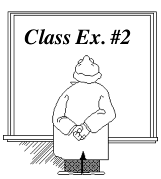
Recall that when we divide by an expression containing a variable there are restrictions on the value which can be replaced for the variable.
These are called nonpermissible values of the variable.

Recall also that reducing rational expressions involves cancelling factors and not terms.



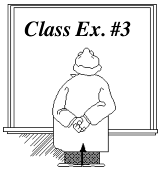
Reduce to lowest terms stating the restrictions on the variable.

a) $\dfrac{4t^3 - 9t}{2t^2 - 3t}$ **b)** $\dfrac{2x^2 + 5x - 3}{2x^2 + x - 1}$ **c)** $\dfrac{a^2 + 2a - 8}{a^4 - 20a^2 + 64}$



Express in simplest form stating the values of the variable for which the expression is not defined.

a) $\dfrac{c - 4}{4 - c}$ **b)** $\dfrac{2p^3 - 4p^2}{16 - 8p}$ **c)** $\dfrac{1 - 4x^2}{6x^2 - 5x - 4}$



The area (in m^2) of a rectangular field can be represented by the expression $12a^2 + 25a - 7$ and the length (in m) of the field can be represented by $3a + 7$.

a) Write and simplify a rational expression which represents the width of the field.

b) If the perimeter of the field is 54 metres, determine the value of a.

c) The field has to be treated with fertilizer at a cost of \$2.40 per square metre.
Calculate the cost of the treatment.

Complete Assignment Questions #1 - #11

Assignment

1. Reduce to lowest terms stating the restrictions on the variable.

a) $\dfrac{5a^3 - 15a^2}{30a}$ **b)** $\dfrac{7x}{7x - 21}$ **c)** $\dfrac{6a - 3}{8a - 4}$

d) $-\dfrac{4a - 12}{a - 3}$ **e)** $-\dfrac{a^2}{a^2 + a}$ **f)** $\dfrac{3t^2 - 75}{(t + 3)(t - 5)}$

g) $\dfrac{2 - r}{r - 2}$ **h)** $-\dfrac{9a^2 - 1}{1 - 3a}$ **i)** $\dfrac{2b^2 - 18b}{b(b - 9)^2}$

2. Express in simplest form stating the values of the variable for which the expression is not defined.

a) $\dfrac{t^2 + 4t + 4}{2t^2 + 10t + 12}$ **b)** $\dfrac{2x^2 + 5x - 3}{4x - 2}$ **c)** $\dfrac{2y^2 - 3y - 2}{2y^2 - y - 6}$

3. Express in simplest form stating the values of the variable for which the expression is not defined.

a) $\dfrac{3t^2 - 5t - 12}{2t^2 - 6t}$ **b)** $\dfrac{9v^2 - 6v + 1}{12v^2 - 13v + 3}$ **c)** $\dfrac{6y^2 - 13y + 6}{3y^2 + 10y - 8}$

4. Express in simplest form stating the restrictions on the variable

a) $\dfrac{32 - 2a^2}{2a^2 + 4a - 16}$ **b)** $\dfrac{4 - 4x^2}{8x^3 + 8x^2 - 16x}$

5. Express in simplest form stating the restrictions on the variable

a) $\dfrac{2 - x - x^2}{x^4 - 5x^2 + 4}$

b) $\dfrac{16x^4 - y^4}{8x^3 + 4x^2y + 2xy^2 + y^3}$

6. A rectangular prism has a length of p cm and a width and height that are each 2 cm less than the length.

a) Write an expression in terms of p for the surface area of the prism and express the surface area in simplest factored form.

b) The rectangular prism has two square faces. Write an expression in simplest factored form which represents the total length of the edges which make up these two squares.

c) If the ratio of the surface area in a) to the edge length in b) is 10:1, find the volume of the prism.

7. Consider the rectangle shown.

a) Write and simplify an expression for the length of the rectangle.

area
$24x^3 - 54x^2 - 15x$

$6x^2 - 15x$

b) Determine the perimeter of the rectangle if $x = 2\sqrt{2}$ cm. Give the answer in simplest radical form.

8. Write a rational expression in x with a numerator of 1 and a denominator written as a integral polynomial so that the nonpermissible values are

a) $x \neq 2, 3$ **b)** $x \neq -2, 0$ **c)** $x \neq -\frac{3}{4}, \frac{1}{3}$ **d)** $x \neq \pm 2, 0$

Multiple Choice

9. With appropriate restrictions, the simplified form of $\dfrac{x^2 - 121}{3x^2 + 29x - 44}$ is

A. $\dfrac{x - 11}{3x - 4}$

B. $\dfrac{x - 11}{3x + 4}$

C. $\dfrac{x + 11}{3x - 4}$

D. $\dfrac{x + 11}{3x + 4}$

10. The reduced form of $\dfrac{15 + 2y - y^2}{3y^2 - 16y + 5}$ is

A. $\dfrac{3 - y}{3y - 1}$

B. $\dfrac{-3 - y}{3y + 1}$

C. $\dfrac{3 + y}{3y - 1}$

D. $\dfrac{3 + y}{1 - 3y}$

Numerical Response

11. If the rational expression $\dfrac{8a^2 + 22a + k}{2a^2 - 11a - 21}$, where k is a constant, reduces to $\dfrac{4a + 5}{a - 7}$, then the value of k, to the nearest whole number, is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $\frac{a(a-3)}{6}, a \neq 0$ **b)** $\frac{x}{x-3}, x \neq 3$ **c)** $\frac{3}{4}, a \neq \frac{1}{2}$ **d)** $-4, a \neq 3$ **e)** $-\frac{a}{a+1}, a \neq -1, 0$

f) $\frac{3(t+5)}{t+3}, t \neq -3, 5$ **g)** $-1, r \neq 2$ **h)** $3a+1, a \neq \frac{1}{3}$ **i)** $\frac{2}{b-9}, b \neq 0, 9$

2. a) $\frac{t+2}{2(t+3)}, t \neq -3, -2$ **b)** $\frac{x+3}{2}, x \neq \frac{1}{2}$ **c)** $\frac{2y+1}{2y+3}, y \neq -\frac{3}{2}, 2$

3. a) $\frac{3t+4}{2t}, t \neq 0, 3$ **b)** $\frac{3v-1}{4v-3}, v \neq \frac{1}{3}, \frac{3}{4}$ **c)** $\frac{2y-3}{y+4}, y \neq -4, \frac{2}{3}$

4. a) $\frac{4-a}{a-2}, a \neq -4, 2$ **b)** $\frac{-1-x}{2x(x+2)}, x \neq -2, 0, 1$

5. a) $-\frac{1}{(x+1)(x-2)}, x \neq \pm 2, \pm 1$ **b)** $2x-y, x \neq -\frac{1}{2}y$

6. a) $2(3p-2)(p-2)$ cm^2 **b)** $8(p-2)$ cm **c)** 2016 cm^3 $(p = 14)$

7. a) $4x+1$ **b)** $98 - 44\sqrt{2}$ cm

8. a) $\frac{1}{x^2-5x+6}$ **b)** $\frac{1}{x^2+2x}$ **c)** $\frac{1}{12x^2+5x-3}$ **d)** $\frac{1}{x^3-4x}$

9. A **10.** D **11.**

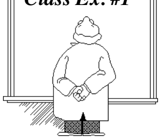
1	5		

Rational Expressions Lesson #3:
Addition and Subtraction of Rational Expressions
Part 1

Warm-Up #1 *Addition and Subtraction of Rational Numbers*

Recall these steps for adding or subtraction rational numbers.

1. Determine the lowest common denominator (LCD) for the rational numbers.
2. Express each rational number as an equivalent rational number with the LCD as the denominator.
3. Combine the rational numbers by adding/subtracting numerators.
4. Reduce to lowest terms, if possible.


Class Ex. #1

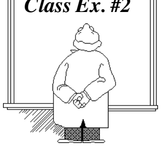
Add or subtract as indicated.

a) $\frac{2}{3} + \frac{2}{5}$ **b)** $\frac{5}{6} - \frac{1}{4}$ **c)** $\frac{7}{8} - \frac{2}{3} + \frac{5}{12}$

Addition/Subtraction of Single Variable Rational Expressions

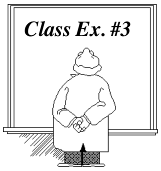
The method for addition and subtraction of rational expressions is identical to the method described for addition and subtraction of rational numbers. Recall that when we deal with rational expressions with a variable in the denominator there are restrictions on the variable.

Addition/Subtraction with Non-Variable Denominators


Class Ex. #2

Simplify.

a) $\frac{3x}{4} + \frac{x}{5} - \frac{7x}{10}$ **b)** $\frac{3a-1}{3} + \frac{4a+5}{6}$ **c)** $\frac{8y-3}{8} - \frac{2y+1}{3}$

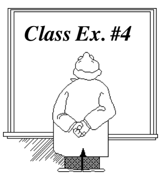


Simplify.

a) $\dfrac{7x+3}{2} + 3x - \dfrac{x-3}{8}$ **b)** $4 - \dfrac{y-4}{9} - \dfrac{5-3y}{3}$

Complete Assignment Questions #1 - #3

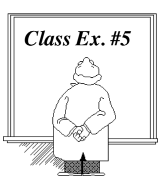
Addition/Subtraction with Common Denominators



Simplify. Express answers in lowest terms and indicate nonpermissible values.

a) $\dfrac{2}{x} + \dfrac{7}{x} - \dfrac{1}{x}$ **b)** $\dfrac{5a+3}{2a} + \dfrac{a-6}{2a}$ **c)** $\dfrac{3y}{3y-1} - \dfrac{1}{3y-1}$ **d)** $\dfrac{4x+10}{x+3} - \dfrac{2x+4}{x+3}$

Addition/Subtraction with Different Monomial Denominators



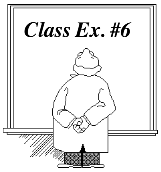
Simplify. Express answers in lowest terms and state any restrictions on the variables.

a) $\dfrac{5}{3p} + \dfrac{2}{7p}$ **b)** $\dfrac{1}{3y} + \dfrac{6}{5}$ **c)** $\dfrac{2x-5}{6x} - \dfrac{3x-2}{3x}$ **d)** $\dfrac{a+7}{2a^2} - \dfrac{3}{5a}$

Complete Assignment Questions #4 - #6

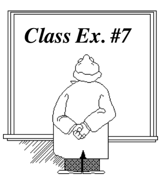
Addition/Subtraction with Different Monomial/Binomial Denominators

In this section we will add or subtract rational expressions with monomial or binomial denominators with no factor in common.



Simplify. Express answers in lowest terms and indicate nonpermissible values.

a) $\dfrac{x-9}{2x} + \dfrac{3x}{x-4}$ **b)** $\dfrac{4}{2y+5} - \dfrac{1}{y-3}$ **c)** $\dfrac{2x+1}{x-5} - \dfrac{x-4}{x+1}$



Perform the indicated operation. State the nonpermissible values.

$\dfrac{3}{x+2} - \dfrac{2}{x+1} + \dfrac{1}{x}$

Complete Assignment Questions #7 - #12

Assignment

1. Add or subtract as indicated.

a) $\frac{5}{8} + \frac{3}{4}$ **b)** $\frac{4}{7} - \frac{2}{5}$ **c)** $\frac{7}{9} - \frac{1}{3} + 2$ **d)** $\frac{3}{2} - \frac{4}{3} + \frac{5}{4}$

2. Simplify.

a) $\frac{4x}{5} + \frac{3x}{10} - \frac{2x}{3}$ **b)** $\frac{c}{2} - \frac{c+2}{6}$ **c)** $\frac{a+2}{3} + \frac{a-3}{5}$

d) $\frac{t-2}{4} - \frac{t-3}{5}$ **e)** $\frac{2y-3}{4} - \frac{y+4}{7}$ **f)** $\frac{2x-3}{3} - \frac{5-2x}{9}$

3. Simplify.

a) $\frac{x}{4} + \frac{x+3}{6} + \frac{3x}{2}$ **b)** $\frac{4-2p}{3} + \frac{7-3p}{4} - \frac{p}{5}$ **c)** $\frac{6x+3}{5} - \frac{2x+1}{2} - \frac{x-3}{10}$

d) $2 - \dfrac{y-5}{5} + \dfrac{6y}{7}$ **e)** $\dfrac{3a+4}{12} + \dfrac{5-4a}{18} - 1$ **f)** $\dfrac{t}{7} - t - \dfrac{t-3}{3}$

4. Simplify. Express answers in lowest terms and indicate nonpermissible values.

a) $\dfrac{2}{y} + \dfrac{3}{y} + \dfrac{4}{y}$ **b)** $\dfrac{1}{2x} + \dfrac{3}{2x} - \dfrac{5}{2x}$ **c)** $\dfrac{7y}{3y+8} - \dfrac{4y}{3y+8}$

d) $\dfrac{a+2}{a^2} - \dfrac{2-a}{a^2}$ **e)** $\dfrac{4b+1}{b+3} - \dfrac{2b-5}{3+b}$ **f)** $\dfrac{15x}{4(3x+5)} + \dfrac{25}{4(3x+5)}$

5. Simplify. Express answers in lowest terms and state any restrictions on the variables.

a) $\dfrac{1}{4x} + \dfrac{1}{2}$ **b)** $\dfrac{1}{3a} - \dfrac{1}{4a}$ **c)** $\dfrac{1}{3t} + \dfrac{1}{4t} + \dfrac{1}{5t}$ **d)** $\dfrac{1}{2t} - \dfrac{2}{3t} - \dfrac{3}{4t}$

e) $\dfrac{6}{5x} + \dfrac{2}{3x}$ **f)** $\dfrac{7}{5p} - \dfrac{5}{7p}$ **g)** $\dfrac{2}{x} - \dfrac{3}{2x} + \dfrac{4}{3x} - \dfrac{5}{4x}$ **h)** $\dfrac{3}{x} + 1$

6. Simplify. Express answers in lowest terms and state any restrictions on the variables.

a) $\dfrac{9}{2x} + \dfrac{1}{x^2}$ **b)** $\dfrac{3}{4a^2} - \dfrac{5}{3a}$ **c)** $\dfrac{8}{3b^2} + \dfrac{7}{b^3}$ **d)** $\dfrac{4}{3c^2} - \dfrac{5}{2c^3} + \dfrac{6}{c^4}$

7. Simplify. Express answers in lowest terms and indicate nonpermissible values.

a) $\dfrac{1}{a+1} + \dfrac{1}{a-1}$ **b)** $\dfrac{2}{b+3} + \dfrac{3}{b+2}$ **c)** $\dfrac{5}{x+2} - \dfrac{2}{x+5}$

d) $\dfrac{4}{x-3} + \dfrac{6}{x-1}$ **e)** $\dfrac{3}{y+2} - \dfrac{1}{y-7}$ **f)** $\dfrac{5t}{2t+1} - \dfrac{3t}{4t+1}$

8. Simplify.

a) $\dfrac{x-5}{3} + \dfrac{4x}{x-2}$ **b)** $\dfrac{p-1}{p+2} + \dfrac{p+2}{p+3}$ **c)** $\dfrac{2x-1}{x+2} - \dfrac{x+2}{2x-1}$

d) $\dfrac{2}{2x-3} + \dfrac{3}{3x-2} + \dfrac{4}{4x-1}$

e) $\dfrac{2}{t} - \dfrac{t+3}{t+2} - \dfrac{t+4}{t+3}$

Multiple Choice

9. $\dfrac{a}{a+2} + \dfrac{2}{a+2}$, $a \neq -2$, is equal to

A. $\dfrac{2a}{a+2}$

B. $\dfrac{a^2}{a+2}$

C. 1

D. $a+2$

10. For all $t \neq \pm 1$, the reduced form of $\dfrac{t+1}{t-1} - \dfrac{t-1}{t+1}$ is

A. 1

B. $\dfrac{2}{t^2-1}$

C. $\dfrac{2t}{t^2-1}$

D. $\dfrac{4t}{t^2-1}$

11. A rectangle has length $\dfrac{1}{x}$ cm and width $\dfrac{1}{x+1}$ cm. The perimeter of the rectangle (in cm) is

A. $\dfrac{4}{4x+2}$

B. $\dfrac{4x+2}{x(x+1)}$

C. $\dfrac{2x+1}{x(x+1)}$

D. $\dfrac{1}{x(x+1)}$

Numerical Response

12. For some whole number k, the reduced form of $\dfrac{x-k}{2x}+\dfrac{x+1}{4x}$ is $\dfrac{3x-5}{4x}$.
The value of k is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. **a)** $\frac{11}{8}$ **b)** $\frac{6}{35}$ **c)** $\frac{22}{9}$ **d)** $\frac{17}{12}$

2. **a)** $\frac{13x}{30}$ **b)** $\frac{c-1}{3}$ **c)** $\frac{8a+1}{15}$ **d)** $\frac{t+2}{20}$ **e)** $\frac{10y-37}{28}$ **f)** $\frac{8x-14}{9}$

3. **a)** $\frac{23x+6}{12}$ **b)** $\frac{185-97p}{60}$ **c)** $\frac{x+4}{10}$ **d)** $\frac{23y+105}{35}$ **e)** $\frac{a-14}{36}$ **f)** $\frac{21-25t}{21}$

4. **a)** $\frac{9}{y}, y \neq 0$ **b)** $-\frac{1}{2x}, x \neq 0$ **c)** $\frac{3y}{3y+8}, y \neq -\frac{8}{3}$ **d)** $\frac{2}{a}, a \neq 0$ **e)** $2, b \neq -3$ **f)** $\frac{5}{4}, x \neq -\frac{5}{3}$

5. **a)** $\frac{1+2x}{4x}, x \neq 0$ **b)** $\frac{1}{12a}, a \neq 0$ **c)** $\frac{47}{60t}, t \neq 0$ **d)** $-\frac{11}{12t}, t \neq 0$
e) $\frac{28}{15x}, x \neq 0$ **f)** $\frac{24}{35p}, p \neq 0$ **g)** $\frac{7}{12x}, x \neq 0$ **h)** $\frac{3+x}{x}, x \neq 0$

6. **a)** $\frac{9x+2}{2x^2}, x \neq 0$ **b)** $\frac{9-20a}{12a^2}, a \neq 0$ **c)** $\frac{8b+21}{3b^3}, b \neq 0$ **d)** $\frac{8c^2-15c+36}{6c^4}, c \neq 0.$

7. **a)** $\frac{2a}{(a-1)(a+1)}, a \neq \pm 1$ **b)** $\frac{5b+13}{(b+3)(b+2)}, b \neq -3, -2$ **c)** $\frac{3x+21}{(x+2)(x+5)}, x \neq -5, -2$
d) $\frac{10x-22}{(x-3)(x-1)}, x \neq 1, 3$ **e)** $\frac{2y-23}{(y+2)(y-7)}, y \neq -2, 7$ **f)** $\frac{14t^2+2t}{(2t+1)(4t+1)}, t \neq -\frac{1}{2}, -\frac{1}{4}$

8. **a)** $\frac{x^2+5x+10}{3(x-2)}, x \neq 2$ **b)** $\frac{2p^2+6p+1}{(p+2)(p+3)}, p \neq -3, -2$ **c)** $\frac{3x^2-8x-3}{(x+2)(2x-1)}, x \neq -2, \frac{1}{2}$
d) $\frac{72x^2-116x+37}{(2x-3)(3x-2)(4x-1)}, x \neq \frac{1}{4}, \frac{2}{3}, \frac{3}{2}$ **e)** $\frac{-2t^3-10t^2-7t+12}{t(t+2)(t+3)}, t \neq -3, -2, 0$

9. C **10.** D **11.** B **12.**

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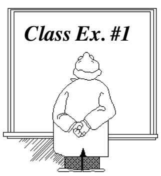
Rational Expressions Lesson #4:
Addition and Subtraction of Rational Expressions
Part 2

Denominators with Factors in Common

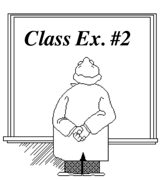
In this lesson we will add/subtract rational expressions where the denominators are different but have a common monomial or binomial factor.

It is important to factor the denominators in the rational expressions (if possible) **before** beginning to add or subtract.

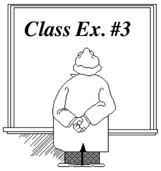


Perform the indicated operations. Express final answers in lowest terms, and indicate the nonpermissible values.

a) $\dfrac{3}{5x} - \dfrac{3}{10x}$ **b)** $\dfrac{4}{5x+5} + \dfrac{3}{2x+2}$ **c)** $\dfrac{1}{x^2} - \dfrac{1}{x^2+2x}$



Simplify, stating restrictions on the value of x. $\dfrac{5}{(x+1)(x-2)} + \dfrac{2}{(x+4)(x-2)}$

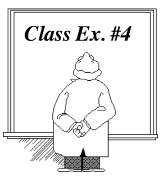


Simplify $\dfrac{4}{p^2 - 1} + \dfrac{2}{p + 1}$. State the nonpermissible values for p.

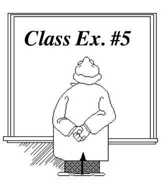
Notice that in Class Example #3, the numerator of the answer had a factor in common with the denominator. This resulted in a further reduction which simplified the answer.
We must always check to see that our answers are in fully-reduced form.

Complete Assignment Questions #1 - #2

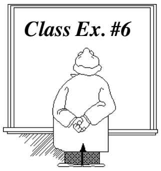
Trinomial Denominators



Simplify: **a)** $\dfrac{2}{x + 1} - \dfrac{x - 1}{x^2 - 2x - 3}$ **b)** $\dfrac{1}{y^2 - 3y + 2} + \dfrac{3}{y^2 + y - 2}$



Simplify $\dfrac{x^2 - 3x + 2}{x^2 - 5x + 4} - \dfrac{x^2 + 10x + 24}{x^2 + 8x + 12}$.



Show that $\dfrac{2a+7}{a^2+7a+12} + \dfrac{2a}{9-a^2}$ can be reduced to $\dfrac{-7}{(a+4)(a-3)}$

Complete Assignment Questions #3 - #12

Assignment

1. Perform the indicated operations. Express final answers in lowest terms, and indicate the nonpermissible values.

a) $\dfrac{1}{a} - \dfrac{1}{6a}$ **b)** $\dfrac{2}{5x-15} + \dfrac{3}{2x-6}$ **c)** $\dfrac{3}{4x+2} - \dfrac{1}{6x+3}$

d) $\dfrac{1}{x^2-3x} - \dfrac{1}{x}$ **e)** $\dfrac{y}{8-6y} + \dfrac{2y}{20-15y}$ **f)** $\dfrac{4}{b} - \dfrac{1}{b^3-b}$

2. Perform the indicated operations. Express final answers in lowest terms, and indicate the nonpermissible values.

a) $\dfrac{1}{(x-1)(x-2)} - \dfrac{1}{(x-2)(x-3)}$

b) $\dfrac{4}{a(a+4)} + \dfrac{3}{a(a-3)}$

c) $\dfrac{7}{(x-2)(x+5)} - \dfrac{8}{(x+5)(x-3)}$

d) $\dfrac{2}{x(x-1)(x+1)} - \dfrac{1}{x(x-1)(x+2)}$

3. Simplify.

a) $\dfrac{1}{x^2+2x+1} - \dfrac{1}{x+1}$

b) $\dfrac{1}{y+2} - \dfrac{1}{y^2-4}$

c) $\dfrac{2}{t^2-1} + \dfrac{1}{t+1}$

4. Perform the indicated operations. Express final answers in lowest terms, and indicate the nonpermissible values.

a) $\dfrac{1}{x^2 - x - 2} - \dfrac{1}{x^2 + 4x + 3}$

b) $\dfrac{3}{t^2 - 7t + 10} - \dfrac{2}{t^2 - 6t + 8}$

c) $\dfrac{2x}{x^2 - 3x - 88} - \dfrac{2x - 1}{x^2 - 10x - 11}$

d) $\dfrac{12y}{y^2 - 8y - 20} - \dfrac{7y}{y^2 - 13y + 30}$

5. Simplify, stating the nonpermissible values.

a) $\dfrac{2x + 3}{5x - 25} + \dfrac{x - 4}{20 - 9x + x^2}$

b) $\dfrac{4x}{2x^2 - 5x - 3} - \dfrac{1 - 2x}{9 - x^2}$

6. Simplify, stating the restrictions on x.

a) $\dfrac{x^2 - x - 12}{x^2 - 8x + 16} - \dfrac{x^2 + 5x - 14}{x^2 + 10x + 21}$

b) $\dfrac{2x^2 - x - 3}{2x^2 + 7x - 15} + \dfrac{x^2 + 16x + 63}{x^2 + 12x + 35}$

c) $\dfrac{x^2 - 9}{x^2 - x - 12} - \dfrac{x^2 - 5x - 14}{x^2 - 4x - 21}$

d) $\dfrac{4x^2 - 4x - 3}{4x^2 - 1} - \dfrac{x^2 - 4x - 96}{x^2 + 4x - 32}$

7. Simplify.

a) $\dfrac{2}{2a+3}+\dfrac{8}{4a^2+4a-3}$

b) $\dfrac{2}{6b^2-5b-4}-\dfrac{3}{9b^2-16}$

8. A helicopter left Calgary and travelled 135 km west into the Rocky Mountains at an average speed of $2x^2+3x$ km/h. The return journey was at an average speed of $4x^2-9$ km/h.

a) Write and simplify an expression for the total flying time, in hours.

b) If the value of x is 6, determine the total flying time.

Multiple Choice

9. For all $x \neq \pm 6$, the sum $\dfrac{3}{x^2 - 36} + \dfrac{2}{x - 6}$ is equal to

A. $\dfrac{5x + 12}{x^2 - 36}$

B. $\dfrac{5x - 12}{(x - 6)^2}$

C. $\dfrac{2x - 9}{x^2 - 36}$

D. $\dfrac{2x + 15}{x^2 - 36}$

10. A simplified form of $\dfrac{3}{x - 7} - \dfrac{5}{7 - x}$, $x \neq 7$, is

A. $\dfrac{8}{x - 7}$

B. $\dfrac{-2}{x - 7}$

C. $\dfrac{8}{7 - x}$

D. $\dfrac{-2}{7 - x}$

11. Consider the nonpermissible values for the addition $\dfrac{5y}{6y^2 - 7y - 3} + \dfrac{4y - 3}{2y^2 - 15y + 18}$.

The product of the nonpermissible values is

A. -3

B. $-\dfrac{9}{2}$

C. $-\dfrac{4}{3}$

D. 3

Numerical Response

12. When simplified, the difference $\dfrac{5}{x^2 - 7x + 12} - \dfrac{3}{x^2 - x - 12}$ can be written in the form $\dfrac{Ax + B}{(x-4)(x-3)(x+3)}$, where A and B are integers. The value of $B - A$ is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) $\dfrac{5}{6a}, a \neq 0$ b) $\dfrac{19}{10(x-3)}, x \neq 3$ c) $\dfrac{7}{6(2x+1)}, x \neq -\dfrac{1}{2}$

d) $\dfrac{4-x}{x(x-3)}, x \neq 0, 3$ e) $\dfrac{9y}{10(4-3y)}, y \neq \dfrac{4}{3}$ f) $\dfrac{4b^2-5}{b(b-1)(b+1)}, b \neq 0, \pm 1$

2. a) $-\dfrac{2}{(x-1)(x-2)(x-3)}, x \neq 1, 2, 3$ b) $\dfrac{7}{(a+4)(a-3)}, a \neq -4, 0, 3$

c) $-\dfrac{1}{(x-2)(x-3)}, x \neq -5, 2, 3$ d) $\dfrac{x+3}{x(x-1)(x+1)(x+2)}, x \neq -2, \pm 1, 0$

3. a) $-\dfrac{x}{(x+1)^2}, x \neq -1$ b) $\dfrac{y-3}{(y+2)(y-2)}, y \neq \pm 2$ c) $\dfrac{1}{t-1}, t \neq \pm 1$

4. a) $\dfrac{5}{(x-2)(x+1)(x+3)}, x \neq -3, -1, 2$ b) $\dfrac{1}{(t-5)(t-4)}, t \neq 2, 4, 5$

c) $\dfrac{8-13x}{(x-11)(x+8)(x+1)}, x \neq -8, -1, 11$ d) $\dfrac{5y}{(y+2)(y-3)}, y \neq -2, 3, 10$

5. a) $\dfrac{2x+8}{5(x-5)}, x \neq 4, 5$ b) $\dfrac{12x+1}{(2x+1)(x-3)(x+3)}, x \neq -\dfrac{1}{2}, \pm 3$

6. a) $\dfrac{12x+1}{(x-4)(x+3)}, x \neq -7, -3, 4$ b) $2, x \neq -7, -5, \dfrac{3}{2}$

c) $\dfrac{2x-1}{(x-4)(x+3)}, x \neq -3, 4, 7$ d) $\dfrac{14x}{(2x-1)(x-4)}, x \neq -8, \pm\dfrac{1}{2}, 4$

7. a) $\dfrac{2}{2a-1}, a \neq -\dfrac{3}{2}, \dfrac{1}{2}$ b) $\dfrac{5}{(3b-4)(2b+1)(3b+4)}, b \neq -\dfrac{1}{2}, \pm\dfrac{4}{3}$

8. a) $\dfrac{405x-405}{x(2x+3)(2x-3)}, x \neq 0, \pm\dfrac{3}{2}$ b) 2.5 h

9. D **10.** A **11.** A **12.**

2	2		

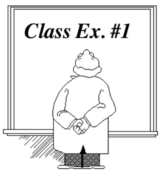
Rational Expressions Lesson #5:
Multiplication of Rational Expressions

Warm-Up #1 *Review - Multiplication of Rational Numbers*

Recall these steps for multiplying rational numbers.

1. Consider the factors of the numerator and of the denominator.
2. If there are factors common to the numerator and the denominator reduce by dividing out the common factors.
3. Multiply all the numerators together and multiply all the denominators together.

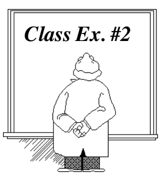
   Note that steps 2 and 3 may be interchanged.



Multiply: **a)** $\frac{7}{10} \times \frac{1}{5}$ **b)** $\frac{9}{10} \times \frac{5}{6}$

Warm-Up #2 *Review - Multiplication of Monomials*

The above method can be extended to multiplication of monomials containing variables.

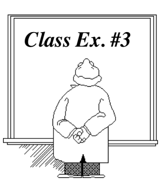


Simplify. State the restrictions on the variables.

a) $\frac{12xy}{4z} \times \frac{3xz^2}{y}$ **b)** $\frac{5a^2b^2c^4}{14b^2cd} \times \frac{35b}{40a^3c}$

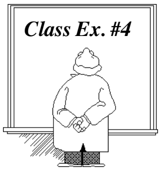
Multiplication of Single Variable Rational Expressions

The method for multiplication of rational expressions is similar to the method described for multiplication of rational numbers. The first step is usually to factor the numerator and denominator of each rational expression.

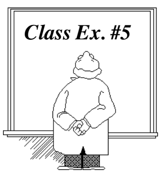


Simplify. State the restrictions on the variable.

a) $\frac{(x+1)}{(x-2)(x+3)} \times \frac{2(x+3)}{x(x+1)}$ **b)** $\frac{4x+16}{14x-7} \times \frac{2x-1}{(x+4)^2}$



Simplify $\dfrac{2m^3 - 4m^2}{3m^2 - 9m} \times \dfrac{m^2 - m - 6}{m^2 - 4}$. State the restrictions on the variable.



Simplify $\left(\dfrac{a^2 + 8a + 15}{6a^2 + 21a + 9}\right)\left(\dfrac{a - 4a^3}{2a^2 + 9a - 5}\right)$. State the nonpermissible values.

Complete Assignment Questions #1 - #8

Assignment

1. Simplify. State the restrictions on the variables.

a) $\dfrac{8a^2b^2c}{12abc^2} \times \dfrac{12a^2c}{6bc}$

b) $\dfrac{9x^4y^3}{12x^5} \times \dfrac{48x^2y^3}{14y} \times \dfrac{6x}{27y^4}$

2. Simplify. State the restrictions on the variable.

a) $\dfrac{15a^2(a-1)}{8(2a+3)} \times \dfrac{10(2a+3)}{3a}$

b) $\dfrac{7x(x+2)(x-3)}{21(x-7)(x+7)} \times \dfrac{(x+7)^2(x-7)}{2x(x-3)}$

c) $\dfrac{6y-30}{(y-1)} \times \dfrac{5y-5}{3y^2-15y}$

d) $\dfrac{10x+2}{5x-1} \times \dfrac{x-1}{35x+7}$

3. Simplify. State the nonpermissible values.

a) $\dfrac{x^2-9}{6x+24} \times \dfrac{10x+40}{x(x+3)}$

b) $\dfrac{4a^2-1}{4a^2-16} \times \dfrac{2-a}{2a-1}$

c) $\dfrac{x^2+5x+6}{3x} \times \dfrac{6x}{x^2+9x+14}$

d) $\dfrac{2y^3-4y^2}{3y^2-9y} \times \dfrac{y^2-y-6}{y^2-4}$

4. Simplify. State the nonpermissible values.

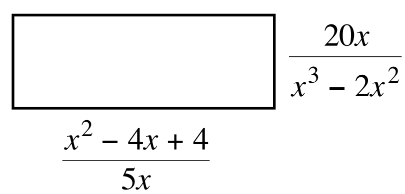
a) $\left(\dfrac{x^2 - 3x + 2}{x^2 + 3x - 4}\right)\left(\dfrac{x^2 + 9x + 20}{x^2 + x - 6}\right)$

b) $\left(\dfrac{3t^2 + 3t - 6}{2t^2 - 2t - 4}\right)\left(\dfrac{4t^2 + 4t - 24}{3t^2 + 6t - 9}\right)$

c) $\dfrac{x^2 - 6x}{x^2 + 5x} \times \dfrac{x^2 + 7x + 10}{18 - 3x}$

d) $\dfrac{a^2 - 6a + 8}{2a^2 - 8a} \times \dfrac{a^2 - a}{8a^2 + 28} \times \dfrac{12a^2 + 42}{2a}$

5. Consider the rectangle shown



a) Write and simplify an expression for the area of the rectangle.

b) Calculate the *exact* area if $x = 4\sqrt{5}$ cm.

Multiple Choice **6.** For all $x \neq 1, \pm\frac{7}{3}$, $\dfrac{(3x-7)^3}{3x^2-10x+7} \times \dfrac{4-4x}{9x^2-49}$ reduces to

A. -4

B. $\dfrac{4(3x+7)}{3x-7}$

C. $\dfrac{4(3x-7)}{(3x+7)}$

D. $-\dfrac{4(3x-7)}{3x+7}$

Numerical Response **7.** For the appropriate restrictions, the product $\left(\dfrac{12x-24}{3x^2-12}\right)\left(\dfrac{6x^2+30x+36}{2x+6}\right)$ reduces to a whole number, k. The value of k is _____.

(Record your answer in the numerical response box from left to right)

Extension Question.

8. Simplify. State the nonpermissible values.

a) $\dfrac{2x^2-8y^2}{12x+6y} \times \dfrac{18x^2+9xy}{6x+12y}$

b) $\dfrac{a^2+6ab}{a^2-3ab-4b^2} \times \dfrac{a^2-7ab+12b^2}{a^2+3ab-18b^2}$

c) $\dfrac{p^2 + 2pq - 15q^2}{3p^2 - 33pq + 84q^2} \times \dfrac{12q^2 + qp - p^2}{2p^2 + 16pq + 30q^2}$ **d)** $\dfrac{12y^2 + yx - 6x^2}{12y^2 - 5yx - 2x^2} \times \dfrac{8y^2 + 2yx - 21x^2}{8y^2 - 10yx - 3x^2}$

Answer Key

1. a) $\dfrac{4a^3}{3c}$, a ≠ 0, b ≠ 0, c ≠ 0 **b)** $\dfrac{4x^2y}{7}$, $x \neq 0, y \neq 0$

2. a) $\dfrac{25a(a-1)}{4}, a \neq -\dfrac{3}{2}, 0$ **b)** $\dfrac{(x+2)(x+7)}{6}, x \neq 0, 3, \pm 7$

c) $\dfrac{10}{y}, y \neq 0, 1, 5$ **d)** $\dfrac{2(x-1)}{7(5x-1)}, x \neq \pm\dfrac{1}{5}$

3. a) $\dfrac{5(x-3)}{3x}, x \neq -4, -3, 0$ **b)** $\dfrac{-2a-1}{4(a+2)}, a \neq \pm 2, \dfrac{1}{2}$

c) $\dfrac{2(x+3)}{x+7}, x \neq -7, -2, 0$ **d)** $\dfrac{2y}{3}, y \neq \pm 2, 0, 3$

4. a) $\dfrac{x+5}{x+3}, x \neq -4, -3, 1, 2$ **b)** $\dfrac{2(t+2)}{t+1}, t \neq -3, \pm 1, 2$

c) $\dfrac{-x-2}{3}, x \neq -5, 0, 6$ **d)** $\dfrac{3(a-1)(a-2)}{8a}, a \neq 0, 4$

5. a) $\dfrac{4(x-2)}{x^2}$ **b)** $\dfrac{2\sqrt{5}-1}{10}$ cm^2 **6.** D **7.**

1	2		

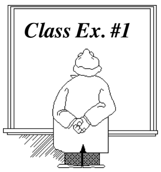
8. a) $\dfrac{x(x-2y)}{2}, x \neq -\dfrac{1}{2}y, -2y$ **b)** $\dfrac{a}{a+b}, a \neq -b, -6b, 3b, 4b$

c) $\dfrac{3q-p}{6(p-7q)}, p \neq -5q, -3q, 4q, 7q$ **d)** $\dfrac{(4y+3x)(4y+7x)}{(4y+x)^2}, y \neq -\dfrac{1}{4}x, \dfrac{2}{3}x, \dfrac{3}{2}x$

Rational Expressions Lesson #6:
Division of Rational Expressions

Warm-Up #1 *Review - Division of Rational Numbers*

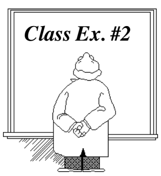
Recall that the procedure for dividing by a rational number is to multiply by the reciprocal of the rational number.



Divide: **a)** $\frac{7}{10} \div \frac{3}{14}$ **b)** $\frac{6}{5} \div \frac{9}{10} \times \frac{1}{20}$ **c)** $\frac{6}{5} \div \left(\frac{9}{10} \times \frac{1}{20}\right)$

Warm-Up #2 *Review - Division of Monomials*

The above method can be extended to division of monomials containing variables. Remember to invert the divisor and multiply.



Simplify. At this stage do not state the restrictions on the variables.

a) $\frac{16a}{9b^2} \div \frac{32a^2}{15b}$ **b)** $\frac{-5xy^3}{7xz^3} \times \frac{2z}{15x} \div \frac{10x^2y^2}{-21z^4}$

Nonpermissible Values in Division of Rational Expressions

Consider the division $\frac{a}{b} \div \frac{c}{d}$ where a, b, c and d are variables.

For the rational expression $\frac{a}{b}$ the nonpermissible value is ______.

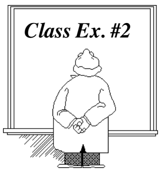
For the rational expression $\frac{c}{d}$ the nonpermissible value is ______.

The first step in simplifying $\frac{a}{b} \div \frac{c}{d}$ is to invert the divisor and multiply to obtain $\frac{a}{b} \times \frac{d}{c}$.

This introduces another nonpermissible value ________.

For a division of the type $\frac{a}{b} \div \frac{c}{d}$ we need to consider nonpermissible values at b, c and d.

Any variable which appears in the denominator at **any** stage in the simplification should be considered for nonpermissible values.



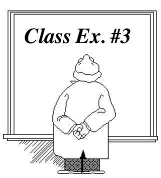
State the restrictions on the variables in Class Example 2.

a) $\dfrac{16a}{9b^2} \div \dfrac{32a^2}{15b}$

b) $\dfrac{-5xy^3}{7xz^3} \times \dfrac{2z}{15x} \div \dfrac{10x^2y^2}{-21z^4}$

Division of Single Variable Rational Expressions

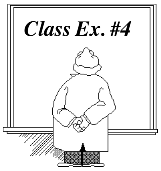
The method for division of rational expressions is similar to the method described for division of rational numbers. The first step is usually to invert the divisor and multiply. Then follow the procedure for multiplication of rational expressions. Nonpermissible values occur when a variable is present in the denominator at any stage in the simplification.



Simplify. State the restrictions on the variable.

a) $\dfrac{(x+1)}{(x-2)(x+3)} \div \dfrac{2(x+1)}{x(x+3)}$

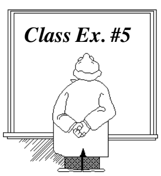
b) $\dfrac{\dfrac{4x+12}{3x+12}}{\dfrac{3x^2+9x}{(x+4)^2}}$



Perform the indicated operations for each of the following expressions.
Express final answers in lowest terms and identify the nonpermissible values.

a) $\dfrac{4x^2 - 12x}{x^2 - 9} \div \dfrac{7x^3 + 7x^2}{x^2 + 4x + 3}$

b) $\dfrac{20m^2 + 30m}{9 - 4m^2} \div \left(\dfrac{11m^3 - 11m}{2m^2 - m - 3} \times \dfrac{2m + 3}{m - 1} \right)$



Simplify $\dfrac{\dfrac{10}{a} - \dfrac{12}{2a + 1}}{\dfrac{5}{a} + 4}$. State the restrictions on the value of a.

Complete Assignment Questions #1 - #10

Assignment

1. Simplify. State the restrictions on the variables.

a) $\dfrac{3a^2bc}{10bc^2} \div \dfrac{12a^2b^2c}{6bc}$

b) $\dfrac{8x^2y^3}{-9x^3y} \div \dfrac{-15x^2y}{14y^3} \div \dfrac{7x}{-6xy^4}$

c) $\dfrac{\dfrac{2xy}{5x^2y^2}}{\dfrac{10x^2y}{15y}}$

d) $\dfrac{-5m^3n}{2p} \div \left(\dfrac{8p^3}{10m} \div \dfrac{4p}{15n}\right)$

2. Simplify. State the nonpermissible values.

a) $\dfrac{(3x+5)^2}{x^2-49} \div \dfrac{(3x+5)(x+1)}{x-7}$

b) $\dfrac{4y+20}{5y-20} \div \dfrac{2y^2-50}{y^2-16}$

c) $\dfrac{(p-6)(p+2)}{p(p+1)} \div \dfrac{36-p^2}{p^2+p}$

d) $\dfrac{\dfrac{a^2-81}{9a}}{(a-9)^2}$

3. Simplify

a) $\dfrac{a^2 - 3a - 10}{a^2 - 5a + 6} \div \dfrac{a^2 + a - 30}{a^2 + 4a - 12}$

b) $\dfrac{x^2 + 13x + 36}{x^2 - 4} \div \dfrac{x^2 - 6x - 40}{x^2 - 8x - 20}$

c) $\dfrac{\dfrac{y^3 + 4y^2 - 32y}{y^2 - 64}}{y - 4}$

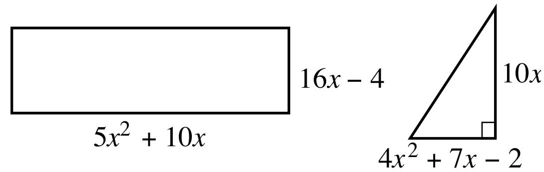
d) $\dfrac{x^2 + 14x + 49}{\dfrac{x^2 + 5x - 14}{x^2 - 2x}}$

4. Simplify

a) $\dfrac{2a^2 - 3a - 9}{8a^2 + 14a + 3} \div \dfrac{3a^2 - 7a - 6}{8a^2 + 14a + 3}$

b) $\dfrac{16x^2 + 8x + 1}{x^2 + 6x - 27} \div \dfrac{8x^2 + 22x + 5}{2x^2 - x - 15}$

5. The rectangle shown has length $5x^2 + 10x$ cm and width $16x - 4$ cm. The triangle has base $4x^2 + 7x - 2$ cm and height $10x$ cm.



Write and simplify an expression that represents the ratio of the area of the rectangle to the area of the triangle.

6. Simplify.

a) $\dfrac{5 - \dfrac{1}{a}}{5 + \dfrac{1}{a}}$ **b)** $\dfrac{8 + \dfrac{4}{x}}{4 - \dfrac{1}{x^2}}$ **c)** $\dfrac{\dfrac{3}{p^2} - \dfrac{1}{p^2 - 4}}{1 - \dfrac{6}{p^2}}$

7. Simplify. State the nonpermissible values.

a) $\dfrac{a-1}{a+4} \div \dfrac{a^2 + 6a + 5}{a^2 - 1} \times \dfrac{a^2 + 3a - 4}{a^2 - 2a + 1}$ **b)** $\dfrac{a-1}{a+4} \div \left(\dfrac{a^2 + 6a + 5}{a^2 - 1} \times \dfrac{a^2 + 3a - 4}{a^2 - 2a + 1}\right)$

8. Simplify $\left(\dfrac{x}{x+1} \times \dfrac{3}{3-x}\right) - \left(\dfrac{1}{x+1} \div \dfrac{2}{x-3}\right)$

Multiple Choice

9. Consider the division $\dfrac{a^2 - a - 12}{a^2 - 4a + 4} \div \dfrac{a^2 - 10a + 24}{a^2 - a - 2}$. The number of nonpermissible values is

A. 2
B. 3
C. 4
D. more than 4

Numerical Response

10. When simplified, the complex fraction $\dfrac{\dfrac{10x^2 - x - 3}{2x^2 - 5x - 3}}{\dfrac{5x - 3}{2x^2 - 18}}$ reduces to a linear expression of the form $Ax + B$. The value of $A + B$ is _____.

(Record your answer in the numerical response box from left to right)

Extension Question.

11. Simplify

a) $\dfrac{a^2 - 9y^2}{a^2 - 2ay - 3y^2} \div \dfrac{a^2 + 3ay}{4a^2 + 7ay + 3y^2}$ **b)** $\dfrac{x^4 - 5x^2y^2 + 4y^4}{x^2 + 3xy + 2y^2} \div \dfrac{x^2 - 4xy + 4y^2}{5x - 10y}$

Answer Key

1. a) $\dfrac{3}{20bc}$, $a \neq 0, b \neq 0, c \neq 0$ **b)** $-\dfrac{32y^8}{45x^3}$, $x \neq 0, y \neq 0$

c) $\dfrac{3}{5x^3y}$, $x \neq 0, y \neq 0$ **d)** $-\dfrac{5m^4}{6p^3}$, $m \neq 0, n \neq 0, p \neq 0$

2. a) $\dfrac{3x + 5}{(x + 7)(x + 1)}$, $x \neq \pm 7, -\dfrac{5}{3}, -1$ **b)** $\dfrac{2y + 8}{5(y - 5)}$, $y \neq \pm 5, \pm 4$

c) $\dfrac{-p - 2}{p + 6}$, $p \neq \pm 6, -1, 0$ **d)** $\dfrac{a + 9}{9a(a - 9)}$, $a \neq 9, 0$

3. a) $\dfrac{a + 2}{a - 3}$, $a \neq -6, 2, 3, 5$ **b)** $\dfrac{x + 9}{x - 2}$, $x \neq \pm 2, -4, 10$

c) $\dfrac{y}{y - 8}$, $y \neq \pm 8, 4$ **d)** $x(x + 7)$, $x \neq -7, 0, 2$

4. a) $\dfrac{2a + 3}{3a + 2}$, $a \neq -\dfrac{3}{2}, -\dfrac{2}{3}, -\dfrac{1}{4}, 3$ **b)** $\dfrac{4x + 1}{x + 9}$, $x \neq -9, -\dfrac{5}{2}, -\dfrac{1}{4}, 3$

5. 4 to 1 **6. a)** $\dfrac{5a - 1}{5a + 1}$, $a \neq 0, -\dfrac{1}{5}$ **b)** $\dfrac{4x}{2x - 1}$, $x \neq \pm\dfrac{1}{2}, 0$ **c)** $\dfrac{2}{p^2 - 4}$, $p \neq \pm 2, \pm\sqrt{6}, 0$

7. a) $\dfrac{a - 1}{a + 5}$, $a \neq -5, -4, \pm 1$ **b)** $\dfrac{(a - 1)^3}{(a + 4)^2(a + 5)}$, $a \neq -5, -4, \pm 1$

8. a) $\dfrac{x^2 + 9}{2(x + 1)(3 - x)}$, $x \neq -1, 3$ **9.** C **10.**

8			

11. a) $\dfrac{(4a + 3y)}{a}$, $a \neq \pm 3y, -y, -\dfrac{3}{4}y, 0$ **b)** $5(x - y)$, $x \neq \pm 2y, -y$

Rational Expressions Lesson #7:
Rational Equations

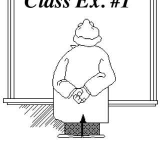
Warm-Up #1 *Review - Solving Equations*

To **solve** an equation means to find the value(s) of the variable which **satisfy** the equation (ie which make the equation true). In the process of solving an equation the same operation must be applied to both sides of the equation.

Recall these steps from previous math courses for solving single-variable linear equations.

1. If the equation contains any terms in the denominator, multiply both sides of the equation by the lowest common multiple of the denominators.
2. If the equation contains any brackets, remove the brackets by using the distributive law.
3. Isolate all terms containing the variable to one side of the equation and all terms not containing the variable to the other side of the equation.
4. Combine like terms. Note that steps 3 and 4 may be interchanged.
5. Divide both sides by the coefficient of the variable to obtain the solution.
6. Verify the solution by substituting the value obtained in the original equation.

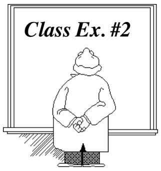
Class Ex. #1



Solve and verify.

a) $5x - 2 = 2(6 + x)$ **b)** $\frac{x}{8} - \frac{x}{4} = 2$ **c)** $a - \frac{2a - 1}{5} = 3 - \frac{a + 14}{3}$

In some cases although the original equation may not be linear, the solution process eventually requires us to solve a linear equation. See class example #2.

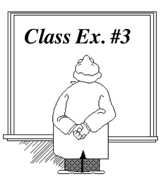


Solve $(2x - 1)(x - 1) - 2 = (3x + 2)(x - 4) - x^2$.

Complete Assignment Questions #1 - #5

Solving Rational Equations

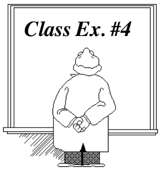
In this section we will solve rational equations which reduce to linear equations. When verifying solutions note that the solution **cannot be a nonpermissible value** since this would result in division by zero. This means that the original rational equation has no solution.



Solve and verify.

a) $4 + \dfrac{2}{x} = 7 + \dfrac{3}{x}$

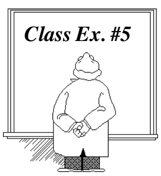
b) $\dfrac{5}{x+1} = \dfrac{2}{x+2}$



Solve.

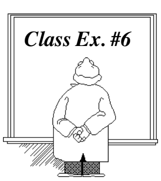
a) $\dfrac{4x-2}{2x+3} = \dfrac{6x-1}{3x+5}$

b) $\dfrac{5t-2}{t+1} - \dfrac{2t-1}{t-3} = 3$



Solve and verify $\dfrac{1}{x-4} + \dfrac{2}{x+4} = \dfrac{5}{x^2-16}$.

An Equation with No Solution



Show that the equation $\dfrac{8x+10}{x-3} - 4 = \dfrac{10x+4}{x-3}$ has no solution.

Complete Assignment Questions #6 - #10

Assignment

In this assignment a written verification is only required where indicated. All solutions must be checked for nonpermissible values.

1. Solve and verify.
a) $3x - 12 = x$ **b)** $4y + 3 = 9$ **c)** $5t - 7 + 3t = 7t + 11 - t$

2. Solve.
a) $5x + 4 = -2x + 6$ **b)** $8 - 3y = 2y - 3$ **c)** $4(x - 5) = 2 + 3(x + 9)$

3. Solve.
a) $5x - 2(x + 1) = x + 3(2x - 3)$ **b)** $-2(3 - 2x) + 4 = x - (2 + x)$

4. Solve.
a) $(3a - 5)(2a + 1) = 5a(a - 2) + a^2$ **b)** $2 - (x - 3)(x - 8) = (2 - x)(2 + x)$

5. Solve and verify.

a) $\frac{y}{2} - \frac{y}{5} = 18$ **b)** $2a - \frac{a+2}{3} = \frac{a+3}{4}$ **c)** $\frac{1}{5}(3x+1) - 7 = \frac{1}{2}(x-1) - 2x$

6. Solve and verify

a) $\frac{6a+3}{2a-3} = \frac{3}{2}$ **b)** $\frac{2}{m+1} = \frac{8m}{m+1} - 3$ **c)** $\frac{5a-3}{a+7} = \frac{5a-14}{a+1}$

7. Solve

a) $\dfrac{2x+1}{x-3} - \dfrac{4x-1}{2x-3} = 0$

b) $\dfrac{6y-2}{3y-2} - \dfrac{2y+6}{y+6} = 0$

c) $\dfrac{4a+9}{2a} - \dfrac{3}{4} = 2$

d) $\dfrac{5}{3x-1} + \dfrac{3x}{3x+1} = 1$

e) $\dfrac{8x}{2x+3} - \dfrac{x+3}{x+7} = 3$

f) $\dfrac{4x+3}{2x-1} - 2 = \dfrac{6x+2}{2x-1}$

Multiple Choice

8. If $\dfrac{2}{4-y} = 3$, then y equals

A. $\dfrac{5}{2}$

B. $\dfrac{10}{3}$

C. -2

D. $\dfrac{14}{3}$

9. The solution to the equation $\dfrac{7}{a+6} - \dfrac{3}{a} = \dfrac{4}{a+6}$ is

A. $a = 18$

B. $a = -6$

C. $a = 0$

D. there is no solution.

Numerical Response

10. The solution to the rational equation $\dfrac{4}{x} + \dfrac{2x}{x-4} - 2 = 0$, to the nearest tenth, is _____.

(Record your answer in the numerical response box from left to right)

Answer Key

1. a) 6 **b)** $\dfrac{3}{2}$ **c)** 9 **2. a)** $\dfrac{2}{7}$ **b)** $\dfrac{11}{5}$ **c)** 49 **3. a)** $\dfrac{7}{4}$ **b)** 0

4. a) $\dfrac{5}{3}$ **b)** $\dfrac{26}{11}$ **5. a)** 60 **b)** 1 **c)** 3 **6. a)** $-\dfrac{5}{2}$ **b)** 1 **c)** 5

7. a) $\dfrac{2}{3}$ **b)** 0 **c)** 6 **d)** $-\dfrac{1}{2}$ **e)** -18 **f)** no solution $\left(\dfrac{1}{2} \text{ is a nonpermissible value}\right)$

8. B **9.** D **10.**

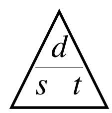
1	.	3	

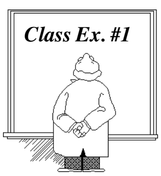
Rational Expressions Lesson #8:
Solving Problems Involving Rational Equations

Guideline for Solving Problems

1. Read the problem carefully and understand what is being asked.
2. Introduce a variable to represent an unknown quantity (usually the quantity that is being asked for).
3. Write an algebraic equation (in this case a rational equation) to represent the given information.
4. Solve the equation.
5. State the solution to the problem. Check that the solution "makes sense".

Problems Involving Distance, Speed, and Time





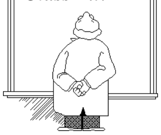
Competing in an endurance race, Shannon cycled for 120 km, then swam for 12 km. Her average cycling speed was eight times faster than her average swimming speed. Shannon took nine hours to complete the race.

a) If her average swimming speed is s km/hr , use the information above to complete the table.

	Distance	Speed	Time
Cycle			
Swim		s	

b) Calculate her average swimming speed.

Class Ex. #2



A plane flew from Red Deer to Winnipeg, a flying distance of 1260 km. On the return journey, due to a strong head wind, the plane travelled 1200 km in the same time it took to complete the outward journey. On the outward journey, the plane was able to maintain an average speed 20 km/hr greater than on the return journey.

a) Calculate the average speed of the plane from Winnipeg to Red Deer.

b) Calculate the total flying time for the round trip

Complete Assignment Questions #1 - #10

Assignment

1. Evan drove 308 km in the same time that Meghan drove 329 km. If Meghan drove on average 6 km/h faster than Evan, calculate her average speed and the time taken for the journey.

2. Erin Airlines has a fleet of airplanes whose average speed is 4 times the average speed of the Derailer passenger train. A Derailer train requires 12 hours more than an Erin airplane to travel a distance of 2000km. Calculate the average speed of each mode of transport.

3. On average, Exante Express trains are 50 km/hr faster than Paral passenger trains. A Paral train requires 60% more time than an Exante train to travel 1800 km from Matsay to Rawindi.

a) Calculate the average speed of each train.

b) Calculate the time it takes each train for the journey.

4. Govinda has a cardiovascular routine where he walks for 3km, runs for $7\frac{1}{2}$ km and then walks for an additional 4 km. He runs $2\frac{1}{2}$ times as fast as he walks and the total time taken for his routine is 2 hours. How fast does he walk?

5. Al and Bob, who live in North Vancouver are Seattle Mariners fans. They regularly drive the 264 km from their home to the ballpark in Seattle. On one particular day Bob drove to the game. On the return journey Al was able to increase their average speed by 10% and save 18 minutes on the travelling time.

a) Calculate the average speed at which Bob drove to the game.

b) Calculate the time it took Al to drive back from the game.

6. To prevent grounding, a cruise ship anchors 18 km away from a river port. To transport the passengers to the port, the crew uses smaller boats. The smaller boats travel 12 km downstream the same time it takes them to travel 8 km upstream.

a) If the speed of the current is 6 km/hr, write expressions for the speed of the boat travelling upstream and travelling downstream.

b) Calculate the time it takes for the small boats to travel upstream from the cruise ship to the port.

7. A rectangular flower bed at a garden centre has an area of 144 m^2. During a redesign of the garden centre the dimensions of the rectangular flower bed are altered but the area is unchanged. The width is doubled and the length is decreased by 12m. Calculate the dimensions of the redesigned flower bed.

8. Part of a student's midterm Math10 report card is shown. Before her mother could analyze the report she spilled some coffee over it and could not read one of the figures.

The student's mother asked her if she could calculate the mark possible for the quiz on radicals. Show how she could calculate the possible mark for the radical quiz if each of the quizzes are equally weighted.

Quiz	**Actual Mark**	**Total Possible Mark**
Polynomials	21	30
Factoring	38	50
Radicals	15	
Exponents	29	40

Average mark for quizzes is 64%

Use the following information to answer questions 9 and 10.

Kelcie drove from Edmonton Airport to downtown Calgary, a distance of 340 km, in the same time that Nick drove from Calgary Airport to downtown Edmonton, a distance of 360 km. Nick's average speed was 6 km/h faster than Kelcie's average speed.

Multiple Choice

9. If Nick's average speed is denoted by *s* km/h then the equation which can be used to determine the value of *s* is

A. $\dfrac{340}{s} = \dfrac{360}{s-6}$

B. $\dfrac{340}{s} = \dfrac{360}{s+6}$

C. $\dfrac{340}{s-6} = \dfrac{360}{s}$

D. $\dfrac{340}{s+6} = \dfrac{360}{s}$

Numerical Response

10. The number of minutes taken for each journey, to the nearest minute, is _____.

(Record your answer in the numerical response box from left to right)

Extension Questions. These questions can only be solved by students who have learned to solve simple quadratic equations in Enrichment Lesson on page 89.

11. Two consecutive even whole numbers are selected. The difference between the reciprocals of the two numbers is $\dfrac{1}{60}$. Determine the numbers.

12. St. Albert students' council is travelling from Edmonton to Winnipeg for the Students' Council National conference. From the travel budget allowed for the trip, the St. Albert Students' Council has two options. They can leave tonight by bus, or they can save three hours by leaving tomorrow morning and using the express train which travels 25 km/hr faster than the bus.

If the distance between Edmonton and Winnipeg is 1500 km, determine how long it would take to travel by express train.

13. The formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ concerning resistance in an electrical circuit is used in Physics. If $R = 3$ ohms and R_1 is 8 ohms more than R_2, calculate the resistances R_1 and R_2.

14. A plane flew from Red Deer to Winnipeg, flying distance of 1260 km. On the return journey, due to a strong head wind, the average flying speed was 90 km/hr slower than on the outward journey. The time taken for the return journey was 20 minutes more than for the outward journey.

a) Calculate the time taken for the journey from Red Deer to Winnipeg.

b) Calculate the average speed of the journey from Winnipeg to Red Deer.

Answer Key

1. 94 km/hr, and 3.5 hours **2.** Erin airplane 500 km/h, Derailer train 125 km/h

3. **a)** Paral $83\frac{1}{3}$ km/hr Exante $133\frac{1}{3}$ km/hr. **b)** Paral 21 hours 36 minutes Exante 13 hours 30 minutes

4. 5 km/hr **5.** **a)** 80 km/hr **b)** 3 hours **6.** 45 minutes

7. 12m x 12m **8.** 40 **9.** C **10.**

2	0	0	

11. 10, 12 **12.** 12 hours **13.** 12 ohms, 4 ohms **14.** **a)** 2 hours **b)** 540 km/hr

Notes Page 2

Notes Page 4

Notes Page 9