IDEAL FOR GCSE

Go From Zero to Hero!

Smash Exams!

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Get Your Self-Respect Back!

Amaze Your Teachers With Your Skills

Become A MathHacker

Paul Carson

THE ONLY TRULY PAINLESS WAY TO LEARN AND UNLOCK MATHS

Contents

Name/Copyright

Math-Hacker

Dedication

To Tymoteusz, I hope the world you grow up in teaches maths easily.

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Series Contents

Series of books and titles

Naked Numbers: The 3 Rules To Make Your Life Add Up**, Hodder Education, the Michel Thomas Method (2010)**

Maths in a Minute Series

Contents and numbering 2nd September 2013

Proverb

'Nil Satis Nisi Optimum'

(*Nothing but the best is good enough* - Everton FC proverb)

How To Best Use This Book

How To Best Use This Book

This book should be read in chapter order. Start at the beginning and work through to the end! At the end of some chapters there is a typical exam question for you to try. Make sure you do it. Then practice questions like these as much as you can.

Alongside the book there is a YouTube channel which demonstrates many of the techniques.

SUBSCRIBE

And you can see the techniques carried out in real time, as described by the book.

Finally, at the end is a guide to the tricky thing of taking an exam itself!

Read, understand, work through and master the skills. An A* will become incredibly easy to you and you’ll be amazed at how easy it can be!

Introduction to Multiplication Introduction

The idea of optimised mathematics.

This book is the culmination of years of research, experience, student feedback, a tireless search for better methods and a desire to make maths easy. This has taken fifteen years. In that time, my students have a 95% success rate, a greater understanding of maths, no misery or pain because maths is ‘boring’, their self-respect and pride has returned and of course, they see the world in a new, mathematical way.

They also realise that learning doesn’t have to be painful and it is possible to learn something that may appear to be difficult with ease and in relatively little time.

The method I have created is one that is holistic, containing as few methods as possible, subliminal, so that you learn advanced maths as you are doing the fundamentals, algebraic, meaning you think along abstract lines without initially realising, and joyful, because you can do things you hitherto thought impossible.

While reading this book, one question will constantly return to your mind.

‘WHY DON’T THEY TEACH THIS AT SCHOOL?’

I’ve been asked this many, many times, and there are a variety of answers.

For now, for you, you have in your hand the guide and passport to unlocking the secrets of maths and becoming one of its better users. You will be able to do things that will impress your friends, family, teachers and most of all yourself.

Take it on, be inspired, use the method for you and be a success!

The first chapter is all about multiplication.

Good luck. Have fun.

>>>
>>>

Chapter 1

There are loads of multiplication techniques out there.

Egyptian
Babylonian
Chinese
Russian
Napier's bones
Grid method
'Long' multiplication
The list goes on.

You could try each of these yourself and decide which is best.

What I want you to do is optimise. I want you to do multiplication in the most efficient, intuitive and mathematically advantageous way. A method that allows you to do five other things. And a half. But I'll get to that.

School teaches all sorts of ways with the hope that one will snag and you'll be able to do it. The most popular is the grid method.

Before we discuss these…

Why school techniques don't work

Some of their methods work. Of course they do. But why do people struggle so? Because the methods require memorisation of a number of steps, which, if any are wrong, makes the answer incorrect. Worse, it is not possible to know yourself if it is incorrect. You have to ask someone else. What kind of system is that?

Students find themselves asking:
'Is this right, Miss?'

Plus, because you're memorising steps, you don't really understand what is happening. And this leads to uncertainty. This means that you're being trained to act like a robot. And robots don't think for themselves. So how would you know if it was right or wrong? Only if your controller tells you so!

It is vital, really vital, that we know *why* we are doing things. This makes it easy to remember, more interesting to learn, and allows creativity of thought. And even improvements of existing systems.

I shall use an analogy to illustrate.

Do you have to remind yourself not to put your hand on a hot stove?

No?

Why not?

Because you understand the consequences if you were to do so.

You don't have to memorise a rule 'Never touch a hot stove' and not understand why. You already understand why. And so it is easy to remember and impossible to forget!

Recently, my curious cat decided to walk on my kitchen tops and over the cooker. Unfortunately the stove plate was still hot as I had just used it. There would have been no point 'teaching' him about this beforehand. But now he will never forget. I learnt maths in the same way...via painful failure! So you can avoid my mistakes from reading this course.

Understanding why something is makes it easy to remember and impossible to forget. That is how anything should be learned. And it works especially well with maths.

Let's start.

So in a book about multiplication, the first question has to be….

What is multiplication?

Before I just give you the answer, I want you to think about it a little bit. Like you've probably thought about the consequences of putting your hand on a hot stove, what you know, you won't forget. And if you come up with it yourself, you'll definitely remember it!

So, to make it a little more challenging and point you in the right direction as well, think for a moment, what is multiplication? But in your answer, you cannot, CANNOT, use the words
Multiply
Times
Product
By

Think about it. Think about it.

A bit more.

What have you got?

Multiplication is…..

Now in 13 years of tutoring, I have heard some varied answers. And I'm not going to embarrass my students by referring to them here. The overriding result, although not for everyone, is…actually, I don't know.

Isn't that something?

10 years in education (or more) and they don't know.

Is that their fault, or school's?

In my opinion it is school's. When I show you how easy this can be, you will be amazed that school can make it so hard.

What Multiplication Is

Chapter 1

What multiplication is

So, what did you come up with? Here's the answer:
Multiplication is just…..repeated addition.

That's it.

(Is that what you came up with? If not, don't worry.)
Let's look at this. Multiplication of two numbers tends to be thought of as 'timesing two numbers together'. The word 'times' here, which has morphed into a verb over the years, actually refers to the number of times we add. This is very important. It is how many times we add.

For example,

$3 \times 5 = 15$

Because we add 5… 3 times.

So above, you probably read that as 3 times 5. Now read it as
3 times (we add) 5.

3 x 5 = 3 times (we add) 5 = 5 + 5 + 5 = 15.

Another example

4 x 6 = 4 times (we add) 6 = 6 + 6 + 6 + 6 = 24

And so on!

5 x 7 = 5 times (we add) 7 = 7 + 7 + 7 + 7 + 7 = 35

How exciting. This is how easy it is.

Multiplication is just addition! So the ‘times’ is not another word for multiply, it is actually the number of times we add. You can probably see that this is Victorian sort of language, which has got dropped over the years. It sounds like some kind of proverb – 5 times we add 7. Obviously the ‘we add’ part has eroded away and now everyone thinks times = multiply, but it doesn’t.

Again, the ‘times’ is not another word for multiply, it is actually the number of times we add.

So what?

Now we know that we can never get a multiplication wrong. If you can add, you can multiply. You don’t actually need to memorise times tables anymore. You could work each one out every time if you wanted! The memorisation of times tables is okay when you understand why 5 x 7 = 35, but it’s basically useless if you don’t.

Now you do.

Now you can do any multiplication.

Any.

Because you know that you could just add over and over.

I thought you said this was going to be optimised?

It is! But we need to understand what we’re doing first.

Let’s look at some more examples.

7 x 9 = 7 times (we add) 9 = 9 + 9 + 9 + 9 + 9 + 9 + 9 = 63

8 x 12 = 8 times (we add) 12 = 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 = 96

14 x 7 = 14 times (we add) 7

But wait. Here’s another concept. It doesn’t matter which order we do it in.

For example, 3 x 5 = 15.

And, 5 x 3 = 5 times (we add) 3 = 3 + 3 + 3 + 3 + 3 = 15.

Which is quicker? Obviously we always want to add the lowest number of times. This is always quicker.

3 x 14 is much quicker than 14 x 3!

So always add the lowest number of times. Turn 14 x 7 into 7 x 14, giving
7 x 14 = 7 times (we add) 14 = 14 + 14 + 14 + 14 + 14 + 14 + 14 = 98

Is this the best way to multiply 2 numbers together? No. It doesn’t fulfil all of the requirements. We can’t even be sure it is correct, because we might have made a mistake while adding.

But we’re getting there.

New Method for Times Tables

Chapter 2

The Times Tables

The times tables are just a bunch of answers to questions that you can already find the answer to yourself, now you know that multiplication is just addition. At school, they don't tell you this, so they make you learn the tables by heart, which is boring. Because you don't know where these numbers came from, you tend to forget the answers. Remember, whenever we learn something new, always ask 'Why is it like that?' and that question can be more valuable than the actual fact you're learning. It can lead to other things and spark an interest. Learning a bunch of dry, boring facts is about as much fun as reading a dictionary in Greek.

So, as I said above, you don't HAVE to know the times tables anymore. So, don't worry about that. If you don't know one, just figure it out. Add as many times as you need. Is it quick? No, but it will get you there.

Another thing you can do is use reference points from ones that have snagged in your mind and you do know.

For example, let's say you know, from repetition, that 7 x 8 = 56. So when someone asks, what's 6 x 8? You think, well…

I know that multiplication is just addition. So 7 x 8 = 56 means
7 times (we add) 8.

If I add 8 six times instead, I therefore need to take away 8 from 56, because it is one too many!

So that leaves me with 7 x 8 = 56, 6 x 8 = 56 – 8 = 48.

So 6 times (we add) 8 must be equal to 48.

So that's ok if you have reference points.

If you don't have any at all, you can use the following system, which has three ways of doing it. Some prefer method 1, others method 2… It's up to you.

For the sake of optimisation, low number multiplication, like 4 x 5, could be done by addition. 4 times (we add) 5 = 5 + 5 + 5 + 5 = 20.

For higher number multiplication, like 7 x 9, say, addition here is a bit slow, so it'd be better to have a quick system.

The Nine Times Table system.

What we can do here is this:

Write two columns of numbers down, from 0-9 and then 9-0.

Left column
0
1
2
3
4
5
6
7
8
9

And right column

9
8
7
6
5
4

3
2
1
0

But write them next to each other!

09
18
27
36
45
54
63
72
81
90

And hey presto, we have the nine-times table.

To find 7 x 9, we just go to the 7th number from the top, (or the reverse, the fourth from the bottom), and that gives
09
18
27
36
45
54
63 7 (x 9)
72
81
90

Another example would be 6 x 9. Go to the sixth entry:
09
18
27
36
45

54 6 (x 9)
63
72
81
90

So that gives us answers for the nine times table!

What's the point of writing it all out? Isn't that slow?

It's not the quickest! So there's also a hand technique to do this too.

Hold both your hands up, palms facing you. Calling your left thumb 1, and your right thumb 10, you can find the answer to 7 x 9 by dropping (bending over) finger number 7.

Left of your dropped finger, you should count six fingers, including your thumb. Right of it, you should count 3.

So, that's 6 + 3……63.

So, 7 x 9 = 63.

Trying 6 x 9…drop your sixth finger. Left of that, you should count 5 fingers. Right…4.

So 5   +   4…… 54.

6 x 9 = 54.

System No 3.

A more advanced method is to use the pattern that we always take 1 away from the first digit in the answer, and notice that the answer always adds up to 9. What does that mean?

Looking above at the columns, we see that for each question, the answer always starts one less, *e.g.*

6 x 9 = 54

i.e. 5 is one less than 6.

The other thing to notice is that 5 + 4 = 9.
In fact, every answer in the column adds up to 9.
So instead, we could write

7 x 9 =
(first digit must be one less than 7, so)

7 x 9 = 6...

and the second digit must make the answer add to 9!

7 x 9 = 6(3), as 6 + 3 = 9, so the answer is 63.

We have noticed a pattern so we are applying it...and we can apply it for other times tables also.

Use any system, whichever you find best.

The Eight Times Table System.

Once you’ve become ultra-fluent at doing nines, it’s time to take on the challenge of the eights.

It may seem strange that I’m starting with high numbers like 9 and 8, but there is a method to my madness. Besides, now we know that it’s just addition, it doesn’t really matter where we start, does it?

So onto the 8s.

What we do here is bear two things in mind:

- • One, that the number 8 is one less than 9
- • Two, that the number 8 is two away from 10.

So, remember these two facts, and you can do the Eight Times Table System.

It is done in the following way.

Write two columns of numbers down, from 0-9 and then 9-0.

Left column
0
1
2
3
4
5
6
7
8
9

And right column

9
8
7
6
5
4
3
2
1
0

But write them next to each other!

09
18
27
36
45

54
63
72
81
90

Does this sound familiar?

That’s because it is. It is, of course, the same as for the nine times table.

To get an eight times table out of it, we remember our two facts.

- • One, that the number 8 is one less than 9
- • Two, that the number 8 is two away from 10.

Bearing these in mind then, let’s try to find 7 x 8.

First we write our column.

09
18
27
36
45
54
63
72
81
90

If we look at the seventh entry, of course it gives us the answer for 7 x 9. Not much use. But if we apply our two facts now, we can get the answer to 7 x 8.

So from the column.

09
18

27
36
45
54
63 …7
72
81
90

The seventh term is 63. What we do now is apply fact 1 on the number on the left.

That the number 8 is one less than 9

Applying this idea, we SUBTRACT 1 from the number on the left.

This gives 6 – 1 = 5. So the number on the left is now ….**5**.

Now we apply fact 2 to the number on the right.

Two, that the number 8 is two away from 10.

This distance from 10 is used to multiply the number on the right. So two away means,
3 x **2** = 6.

So the number on the right is **6**.

This gives that the number on the left and right together is 5 and 6 = **56**!

So **7 x 8 = 56**

Confused?

Read through that again. It may be an alien concept to use the nine times table to piggy back on to get the 8s, but it becomes a fluent way of doing it… and leads to the 7s and 6s in the same way. (Can you figure out how?)
Let's look at doing this on your hands.

Again, hold both your hands up, palms facing you. Calling your left thumb 1, and your right thumb 10, you can find the answer to 7 x 9 by dropping (bending over) finger number 7.

But of course, we're looking for 7 x 8.

So we apply the two facts.

Left of your dropped finger, finger number 7, we then drop another finger. This will be finger number 6. You should now have two fingers dropped, numbers 6 and 7. And left of these you should have 5 still up!

Right of these dropped fingers, you should have 3 still up. Remembering fact two we multiply by 2. So we have 5 + 3 x 2 = 56

Now try for 6 x 8.

(Now we know 7 x 8, we could just subtract 8, but for the sake of practice…)
Using the column method:

09
18
27
36
45
54 … 6
63
72
81
90

The sixth term is 54. So we take one from the left – which gives 5 – 1 = 4
And double the number on the right – which gives 4 x 2 = 8.

So our two numbers are 4 + 8 = 48

So, 6 x 8 = 48

To do this on your hands, you follow the same method, but this time drop your sixth finger, and then the fifth. You should have four left of your dropped fingers, and in this case, four right. The ones on the right are doubled, giving 4 + 8 = 48.

System No 3.

Again, we can use a more advanced method here, and noting the same facts, we can see that
7 x 8 = 7-2, for the first digit, which gives 5. So we have
7 x 8 = 5....

For the second digit, the answer must add up to...? What?

It must add up, not to 9, but to 8, as we are finding 7 times 8.

So to make 5 add up to 8, we need 3.

So,

7 x 8 = 5(3)

But this isn't correct...

Of course, we must multiply that second digit by 2.

7 x 8 = 5(3x2)

7 x 8 = 56

Again, use whichever system you like best.

The Seven Times Table System.

So, any guesses on how this works? Can you see a pattern developing?

Here our two important facts are: (can you think what they are before reading them?)

- • One, seven is two less than nine.
- • Two, seven is three away from ten.

What does this mean? Well, we're going to use the original column again, that of the nine times table, formed by writing 0-9 and 9-0. This time, we want to find 7 x 7.

09
18
27
36
45
54
63 … 7
72
81
90

The 7^{th} term is 63.

Remembering fact 1: 7 is two less than nine, we take 2 away from the number on the left, giving … 4.

Remembering fact 2: 7 is three away from 10, we multiply the number on the right by 3, giving….3 x 3 = 9.

So our answer is 4 + 9 = 49

To multiply by 3, you can add three times, or you can double and add once.

On your hands, it is the same idea…but bring 2 extra fingers down and then treble what is showing to the right.

So for 7 x 7, drop your seventh finger, and sixth, and fifth! On the left is four.

On the right, you'll have 3 fingers still up and multiply that by 3. Giving nine.

So, 49.

Again, the advanced system will have us subtracting 2 this time, but multiplying by 3:
7 x 7 = 7-3,
7 x 7 = 4...

We need 3 to get to 7 this time, so multiplying by 3

7 x 7 = 4(3 x 3) = 49.

So now we've got the following times tables practised:
1, 2, 3 & 7, 8, 9.

1 – 3 we just use addition, doubling or trebling.

7 – 9 we use the column method or the (same) hands method.

This takes care of a lot of the 'tricky' multiplications students come across. That just leaves 4, 5 and 6.

For the **six times table**, simply take away 3 and multiply by 4!

Ok, it's not as simple anymore.

But let's focus on the one multiplication in the six times table that everyone struggles with.

6 x 6 = 36

Anything above that, 6 x 7, 6 x 8 and so on, we can use our other systems above for.

Below that, 5 x 6, 4 x 6, we'll see shortly what to do about them!

So, 6 x 6...

Using System 3

6 x 6 = [6-4]
6 x 6 = 2....

Then we need to get to 6,

6 x 6 = 2....

2 + 4 = 6, so we need to put a 4 there, and then multiply it by 4 also...

6 x 6 = 2[4 x 4]

Which is 16...added on to 20, gives 36!

Ok, the system works. But it is a bit stretched.

We can use the other 2 systems in the same way.

For multiplication by 4, it's very simple.

Double, then double again!

So 12 x 4

Double 12 = 24

Double 24 = 48.

So 4 x 6 = 24

As double 6 is 12, and double 12 is 24.

So that means we can multiply every single digit from 1-9.

Master these thoroughly, and the rest will follow.

What Brackets Mean

Chapter 3

What Brackets Mean

What do they mean?

Let’s look at 8 x 12.

Do it in your head, or with paper and pencil.

What did you get?

Hopefully, you got 96.

But how did you do it?

Most students split the 12 up into 10 and 2, in this way
12 = 10 + 2

And we’re multiplying it by 8.

So that will look like

8 (10 + 2) = 96

So based on this, what do brackets mean?

Now the school definition is so wrong, I don’t like to repeat it, so let’s just examine what you did to get the answer 96.

What you likely did was say
8 x 10 = 80
8 x 2 = 16

and added them together (as multiplication is just addition!)
To give that 96.

So we multiplied each number in the brackets separately. So what do brackets mean then?

Logically, the symbol between the 8 and the 10 lead us to multiply them, so brackets must mean MULTIPLY.

NOTHING ELSE.

You may have heard other definitions, but I want you to forget them. They are wrong and useless.

If you thought of doing 8 x 12 in this way, well done! This is actually known as 'expanding a bracket'. In fact, it is an algebra technique. This means that if you thought of this you can intuitively do algebra. In fact, as we shall see, we can all do algebra, it has just never been pointed out to us that that's what we are doing.

So, from now on, brackets mean multiply.

So that means that when you see this
7 x 14

You could do

7(10 + 4) = 70 + 28 = 98

Since brackets mean multiply.

An alternative method would be to do this in a column
 14
x 7

And multiply 7 x 4 first
 14

⇩
x 7

28

But here we need to write the 8 and ‘carry the 2’, since it will add onto the tens unit.

So we want

14
x 7

$_2$ 8

Then, multiply the 1 x 7 = 7,
Adding the 2 from the 28 gives 9.

14
↴
x 7

98

So, we’ve learnt 2 things.

1. 1. Brackets mean...?
2. 2. How to multiply a 2 digit number by a single digit number.

2 x 2 Digit Numbers

Chapter 4

How to 2 x 2 digit numbers

The method for doing this is the most exciting and revolutionary part of this book. From this method, we will be able to do many more things, as well as multiply extremely easily. We will intuitively learn algebra and be able to do multiplications with joy instead of terror. In short, it satisfies all the requirements for optimised mathematics.

And it only takes a minute to learn. So this is Multiplication In A Minute...

So let's take a look.

First of all, the 2 x 1 digit method above is a subset of this method. So the beginning is the absolute same. However, the advantage with this method is that there is practically no working and the answer will just appear in front of you, working right to left.

The method takes 3 steps, so I'm going to explain each one. That means writing it out 3 times. In practice, we would only write out the question once.

Let's look at 14 x 21.

First, place it in a column.

```
  14
x 21
```

As above, start on the right hand column

  14
   ⇩
x 21

    4

which is 4 x 1, giving us 4.

Step 1 complete.

Step 2 is where you'll see the revolutionary difference that you won't have seen before.

  14
   ⤫
x 21

    4

Here, draw a small cross between the numbers and multiply along the lines of the cross.

So that gives:
1 x 1 = 1
4 x 2 = 8

Remember that multiplication is just addition, ADD these 2 answers in your mind. Really the goal is to have no working, so we need to store these in our memory for a moment.

So we have 8 + 1 = 9.

Giving

  14

x 21

  94

Step 3 we just multiply along the left column, just as we multiplied along the right.

  14
  ⇩
x 21

  294

Giving, of course, 1 x 2 = 2.

So 14 x 21 = 294.

No working was involved. We had some simple multiplications to make, one addition to store in our memory as we multiplied the 2 in the centre, and the answer appeared as if by magic.

Let’s try another.

```
  15      15      15
x 22     x22     x22
----    ----    ----
  ₁0     ₁30     330
```

Just to talk through this process.

5 x 2 = 10, written as $_{1}0$

5 x 2 + 1 x 2 = 12. Plus the carried 1, gives 13, written as $_{1}3$

Finally the left column gives 1 x 2 = 2, plus the 1 = 3.

So we clock up 330 as we move left.

How easy this is?

Are you thinking what all my students ask at this point?

“Why don’t they teach this at school?”

Let’s try another.

23 x 41

```
  23
x 41
```

$9_{1}43$
Here I’m leaving that ‘carry 1’ visible, but I ignore it in the final answer of 943.

Try these 5 and check your answers!

11 x 11
35 x 23
81 x 32
91 x 45
99 x 99

Answers are (including carries):
121
$8_{2}0_{1}5 = 805$
$25_{1}92 = 2592$
$40_{4}95 = 4095$
$98_{17}0_{8}1 = 9801$

The final one is effectively 81   162   81 but added up right to left, gives 9801.

There is a similar pattern in the first question, 11 x 11 = 1 2 1
Can you see the pattern?

What do you think 22 x 22 or 111 x 111 would be?**

Looking back now at 1 x 2 digit multiplication such as

8 x 12

we can see how this works with our 2 x 2 digits multiplication.

Again, putting the numbers in a column
 12
x 8

If we use the 2 x 2 method here, we can imagine that a 0 is in the place left of the 8, viz:
  12
x 08

---

and follow the procedure, giving
  12
x 08

 9$_1$6

As the right hand column gave 2 x 8 = 16
The middle cross part gave 1 x 8 + 2 x 0 = 8 + 0 = 8
And the left hand column gives 1 x 0 = 0.

We can see then that to do 1 x 2 digits multiplication, we don't need to bother with half of those four steps, as they always give zero. So we just do steps 1 and 2 to give the correct answer.

This explains how the method for 1 x 2 digits multiplication works.

Exam Question

380
x 72

------

3 x 3 Digit Numbers
Chapter 5

3 by 3 digit multiplication.

So far we have covered these types of multiplication then:
1 x 1 digit (also known as times tables) 1 x 2 digits
2 x 2 digits

So let's look at 3 x 3 digits, and then as a result, 3 x 2 digit multiplication.

An example of a 3 x 3 digit multiplication would be
123 x 421

Again, in school the method here is long and quite painful. It can take a long time, and there is a lot of working involved. It won't surprise you to know at this stage that I have eliminated all of that, with a similar method to the one in the previous chapter. It is also going to subliminally and intuitively teach you algebra.

So let's have a look.

An important part of mathematics is symmetry. And we see this in the 2 x 2 method. The steps are like a mirror image for the column multiplications, with step 2 being the line of symmetry. With 3 x 3, this simply gets extended further.

But there is a further, new step, which we need to introduce, which I call the 'Union Jack Situation'. This is where we will have to draw or imagine a Union Jack in the multiplication.

(Why a Union Jack? Because I'm British!)
So we going to see something like this:

*

(note there is no horizontal line required).

So for this method there are 5 steps. 4 of them will be exactly the same as you have already seen, the 5th but, middle, step is the ‘Union Jack Situation’.

Let’s take a look at the multiplication above:
123 x 421

Place in a column

  123
x 421

Step 1:

Exactly the same: right hand column.
  123
    ⇩
x 421

-----

3 x 1, gives

  123
    ⇩
x 421

    3

Next step, exactly the same: cross

  123
   ⤫
x 421

   3

So ignore the 1 and 4 on the left column, we pretend they're not there and then this gives, 2 x 1 + 3 x 2 = 8

So

  123
x 421

  83

So far, this has been exactly the same. Here, we see the new 'Union Jack Situation' come in.

We now place our union jack in between the numbers, viz:

  123
  ✱
x 421

  83

So we have 3 multiplications to do and remember 3 answers! After achieving fluency with the 2 x 2 method, this becomes easy.

So we have, multiplying along the lines,
1 x 1 + 3 x 4 + 2 x 2

= 1 + 12 + 4
= 17

So we now have

```
  123
x 421
```

$_{1}$783

So that’s 3 out of 5 steps!

Steps 4 and 5, as I mentioned above, now follow the symmetry of mathematics.

Step 4

```
  123
  X
x 421
```

$_{1}$783

Will not ignore the right hand column completely.

So we have 1 x 2 + 2 x 4 = 10

Adding the carry of 1, gives us 11.

```
  123
x 421
```

$_{1}$1$_{1}$783

Step 5 is simply multiplying the left column.

  123
  ⇩
x 421

$_{1}1_{1}783$

1 x 4 = 4
Plus the carried 1, = 5.

Giving us

  123
 x 421

$5_{1}1_{1}783$

Therefore

123 x 421 = 51 783.

I have gone through each step very slowly and carefully as it will be the first time you’ve seen it. With practice, you will be able to do this without writing any crosses or union jacks and just write down numbers, to lead up to the answer!

You can watch this live video of me doing a 3 x 3 digit multiplication of 124 x 132 at this YouTube link.

You’ll notice in the video that the student writes the additions for the ‘Union Jack Situation’ in the margin: you can do that to start, but you’ll find it easier to add on the fly, so you would go
“4 x 2 = 8, plus 4 x 2 = 8, makes 16, plus 1 x 1 makes 17.”

either out loud, or in your mind.

Try a few yourself!

124 x 132
311 x 243
124 x 241
111 x 111
542 x 231

Answers:

$16_{1}3_{1}68$ = 16 368
$7_{1}5_{1}573$ = 75 573
$29_{1}8_{1}84$ = 29 884
12 321
$12_{2}5_{2}2_{1}02$ = 125 202

Let’s talk about 111 x 111.

We notice that the answer is 12 321

How neat. But remember, multiplication is just addition, and in fact this answer reveals the number of additions required per step.

(The first and last step not included, as you can’t make an addition with one number! *I.e.* 5 + ... so this effectively means no additions.)
In fact this is something I was drawing your attention to earlier...

If the numbers are symmetrical, the multiplication is just the number of steps required, times by the numbers multiplied by themselves... *e.g.*

22 x 22

1 2 1 (the additions) x 4 (2 x 2)
= 4 8 4

Because

22

x 22

----

4 8 4

So notice again the middle number is double the outer columns, as we’re just adding that 4 twice.

33 x 33 then will be 9 18 9 = 9 $_1$8 9 = 1089.

And so on....

Or

  1111
x 1111

------

will be???

What?

Of course it is

1 234 321.

So this gives us an insight of how to proceed with 4-digit, 5-digit, 6-digit, as high as we want to go, multiplications. Essentially every time we increase the number of digits, we need to increase the number of additions.

But, before then, let’s look at a 3 x 2 digit multiplication.

So we have, for example

431 x 32

So we write as normal

 431
x 32

Again, we can imagine that there is a zero to the left of 32 and use the same technique. Again, the last 2 steps will not be fully necessary as we are multiplying by zero here.

So we have, after 3 steps upto and including the union jack situation,
 431
x 32

$_1$792

Here then, we only have to multiply 4 x 3

 431
 ↴
x 32

$_1$792

To finish off, as the rest will be 3 x 0 and 4 x 0.

So that gives

 431

x 32

13$_1$792

= 13 792

For 3 x 1 digit, it is even easier, we just multiply each top number by the single digit, *e.g.*

431
x 7

30$_2$17

i.e. 7 x 1 = 7
7 x 3 = 21 = $_2$1
7 x 4 = 28 + 2 = 30

= 3 017

Bigger Numbers

Chapter 6

Bigger numbers

To do more complicated multiplications, we follow the same method, and they are outlined here.

In practice however, no-one really does these types of multiplications manually, unless they are a good distance away from a calculator. I would, since I know a good method, but the world at large considers this kind of thing beyond them, so very much a calculator question. That is to say, you would never be expected to do one like this in any exam.

So I include this briefly for completeness, and also for a later reference to algebra in a (much) later book.

For 4 x 4 digit multiplications, there are now 7 steps. Again, we have the symmetry, so for six of the multiplications, they are techniques you have already used!

For the middle step, this time we need to extend our union jack out a bit.

For steps 1-3, it is column, cross, union jack.

For steps 5-7, it is the reverse: union jack, cross, column.

For step 4 we have two crosses with the same centre, but we multiply the inner two numbers with a cross, and the outer two numbers with a bigger cross.

For example, let’s say we have

1234 x 5678

1234

x 5678

For steps 1-3 we work left until we get the union jack situation.

Step 4 will involve 2 crosses, one like this
1234
X
x 5678

----

to multiply 2 x 7 and 3 x 6

But we also need a bigger cross on top to multiply 1 x 8 and 4 x 5.

We then add these 4 answers together!

From there we go on to step 5, union jack with the 3 numbers on the left
123
567

and continue left until we finish with the left column.

Try this yourself and see if you get the right answer!

What Multiplication Is For

Chapter 7

What Multiplication is For?

Okay, so far we have discovered an easy and exciting way to multiply, which will have a host of other benefits in the books to follow. But what exactly is the point of all this?

What is multiplication actually for? What is its application in everyday life? Or science? Or, hey, even maths?

This is a question I ask my students, and perhaps surprisingly, again, the answer is ‘I don’t know’. Yet, there is a huge emphasis on developing these skills in school, so much so that you spend YEARS learning maths.

So how come they don’t even know why they’re doing it or what for?

Let’s look at the answer to this question by looking at it from a different angle.

Answer this:

What shape do we get when we multiply 2 numbers together?

Shape?

Yes, what shape do we get?

(Have a think about it...then read forward).

This is a question that absolutely foxes my students. It’s a shame really, because the answer is very powerful, and has been lying under their noses for a long time.

Most students in the UK would be familiar with the grid method, *i.e.* they have done it at school, although this doesn’t mean that they can do it. However the grid method is based on this concept - it's just that students are never told this.

In fact, once I discussed this with one of my private students, she showed her teacher at school, and her teacher said she [the teacher] didn't know it.

So, you're probably dying to know the answer by now... did you get it?

The answer is a rectangle, or a square.

Look around you - what is the most common shape? In front of me I have a rectangular screen, I'm typing on a rectangular keyboard, my phone is rectangular, there are speakers that are rectangular, my desk... the list goes on.

So this is the most basic shape, and so comes from the most basic of operations.

If we think of a simple multiplication, like

2 x 3

= 6

What does this six represent?

Ok, we’ve said that if we have a bill of £3 added twice, the cost will be £6.

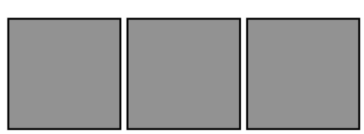
Or if I give my friend 3 sweets on two different occasions I’ll have given him a total of 6.

But what about shape-wise?

What does 6 represent in terms of the rectangle above?

What information do we get about the shape when we multiply these two numbers?

(Think before reading forward.)



As you can see, if we split the side that is 3 long into 1m parts, and split the side that is 2m long into 1m parts also, we get a rectangle that contains six squares.

In fact they contain six 1 metre squared squares (or 1 m^2).

What we find out about the rectangle then, is its area, or how much space it requires. Area is measured in squares. In other words, how many squares can fit in this shape?

So, when we multiply two (different) numbers together, we get a rectangle. And the information we find out about the rectangle is its area.

Also, to find the area of a rectangle, multiply its sides!

If the numbers are the same, what shape do we get?

Again have a think....

The answer is a special kind of rectangle - a square.

A rectangle where the sides are the same.

So for example, 7 x 7 = 49 will give you a square, the area of which is 49 m^2.

So in that square are 49 smaller squares of size 1 m^2!

What Is Division

What Is Division?

First of all, what is division?

In your answer, don't use 'divide', 'goes into', 'by' or 'reverse of multiplication'!

Again take a moment to have a think about this.

It forms the Second Rule of Maths, something which we will use repeatedly as we go through this course.

Like the first rule, it covers about 40% of maths so is absolutely vital to know.

So, what have you come up with?

Let's look at an example.

What is

$$\frac{6}{2} = ?$$

Hopefully you'll answer

3.

How did you get this?

If you thought

"How many 2s goes into 6?", answering, "3", then this is one way to divide certainly.

In fact you said to yourself

"What do I need to multiply 2 by to get 6?",
and of course the answer is 3.

Another way of writing this would be
2 x ? = 6.

What is the ?

This could also be written

$$2x = 6$$

(2 times something equals 6).

Already we have stumbled on algebra here, and if you thought process WAS 'what do I need to multiply 2 by...?' then you are intuitively doing algebra.

The 'x' is the question mark, and we want to know what multiplies by 2 to find 6.

I call this the 'algebra trick'. It is something we tend to intuitively do, and many, many of my students have the initial thought process on this question.

You may have thought

“6/2 means half of 6, and half of 6 is 3."

In that case, what is

$$\frac{12}{3} = ?$$

Or

$$\frac{72}{12} = ?$$

For these examples, you may find yourself reverting to this 'algebra trick' method of thinking
"Ok, what do I need to multiply/times 3 by to get 12?" or
"What do I need to times 12 by to get 72?"

Again these could be written

$$3x = 12$$

$$12x = 72$$

in which case

$$x = 4$$
$$x = 6$$

Or

$$\frac{12}{3} = 4$$

$$\frac{72}{12} = 6$$

We don't have to use algebraic notation to do division, I'm just making you aware that you are effectively doing algebra every time you answer a question similar to this.

(Many people say 'I can do arithmetic, but I could never do algebra'. In fact, they always do algebra since that is what dividing can be done by - and often is.)
So, you may have noticed that we haven't answered our original question at all, which was
'What is Division?'

What Is Division - Part 2

Perhaps you're convinced it is 'algebra' at this stage. In fact the use of algebra here is just turning the question into a multiplication (as they're much easier) and figuring out what it would be from there.

But what is division?

You may realise that division is the reverse of multiplication. However I said don't use that.

Still, we know what multiplication is, so if division is the reverse, what could that be?

Hopefully by now you're screaming at your screen - 'subtraction!'.

So Division is just Subtraction?

But how? Why is it subtraction?

Looking at 6/2 again, how could that be a subtraction?

Again, take a moment to think about this.

One way to solve this is to think about how a 5-year-old child would divide 6 by 2.

How would they do it?

They would probably say

"One for you, and one for me, one for you, one for me..." and so on.

Looking at 6/2

Let's say we have 6 coins to divide by 2.

If we say, one for you, one for me

That takes away 2 and leaves 4.

So 6 - 2 = 4

Repeat that process

One for you and one for me

4 - 2 = 2

Repeat again as there are coins left

2 - 2 = 0

And we get to zero.

How many subtractions did we have to make to get to zero?

Of course, three.

So division is just subtraction because it is the number of subtractions required to get to zero.

Again

6 - 2 = 4
4 - 2 = 2
2 - 2 = 0

Three subtractions.

Okay. But why do we stop at zero?

Again have a think.

We could continue to subtract 2 forever, theoretically.

But we don't. We know the answer is 3.

Why stop then?

Because division is also sharing. If we have 6 coins, or pizzas, or sweets, (or anything!) we are sharing them out.

As we take away from the pile, we get to the stage where there are none left, so we must stop!

In fact, 80% of maths is this idea of either adding to a pile repeatedly (multiplication), or taking away from a pile (the reverse, division) repeatedly.

It's hard to believe that all maths boils down to this, but it's true. The other 20%, the Third Rule of Maths, contains those two ideas within it as well.

But we'll come to that one later.

Three Types of Division

For now, let's return to looking at division, and why, if it is just subtracting from a pile repeatedly, it is not as accommodating as multiplication.

For a start, there are three types of division.

The type we've been looking at so far, of

$$\frac{6}{2}$$

$$\frac{12}{3}$$

and

$$\frac{72}{12}$$

I call a more than 1 type or > 1, because the answer you get is more than 1.

So

$$\frac{6}{2}$$

> 1

What do you think the other two types might be?

AGAIN, HAVE A THINK.

$\frac{6}{2}$

> 1

The other two types will be

< 1 (less than 1)

and

= 1 (equal to 1)

Can you think of an example of a division where the answer is equal to 1?

Please be careful here!

Some examples would be

$\frac{6}{6}$

$\frac{8}{8}$

$\frac{12}{12}$

Note they are all the same number.

6 x 1 = 6.

But also

6 - 6 = 0

It only takes one subtraction to get to zero, so the answer is 1.

Quite neat.

And finally, can you think of a division where the answer is less than 1 (< 1)?

Be really careful here.

Sometimes students will mention something like
"Hmm, a number less than 1? How about - 2?"

But we haven't mentioned negative numbers yet, so Let's Remain Positive!

"Ok, less than 1, that means a decimal? Like nought-point..."

And I want to stop you there.

Because I want you to

STAY AWAY FROM DECIMALS WHEN YOU'RE DOING DIVISION.

Decimals are a kind of default that students seem to switch to, when it comes to talking about numbers that are less than 1. However, as they are a human invention (more to come in the next books) they have nothing to do with mathematics. They actually make division harder, as we shall remark again later.

So we need a division with an answer that's less than 1 please.

Remember I said it wasn't very accommodating!

Since we seem to have eliminated some possibilities, like Sherlock Holmes, we can say, whatever remains, must be the answer!

So what's left?

How about

$$\frac{4}{6}$$

Here we see the number on top of the division is less than the one on the bottom. The complete reverse of a more than 1 type.

Some other examples would be

$$\frac{3}{9}$$
$$\frac{3}{4}$$
$$\frac{5}{7}$$

and so on.

Looking at

$$\frac{4}{6}$$

What does this equal?

Again, have a think about this one before answering.

In this situation, if we think of divisions as pizzas over people - I like to imagine a room which has become a food fight, with people throwing pizzas at each other (for which they'd have to go over peoples' heads), for the more than 1 types we have the simple situation of

6/2

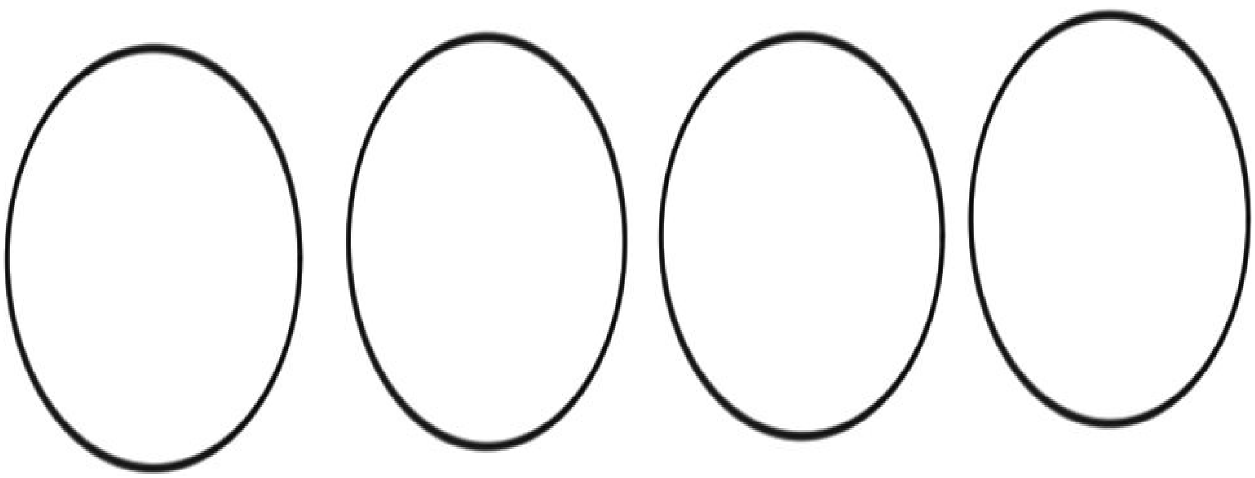
= 3

As 6 pizzas between 2 people would give them 3 each.

However, here we have a bad restaurant.

We only have 4 pizzas, but 6 people. This means they'd have 1 each, but 2 people don't get any (not allowed, or fair) or the pizzas must be cut up, meaning they'd have less than 1 each.

So how much would they get?

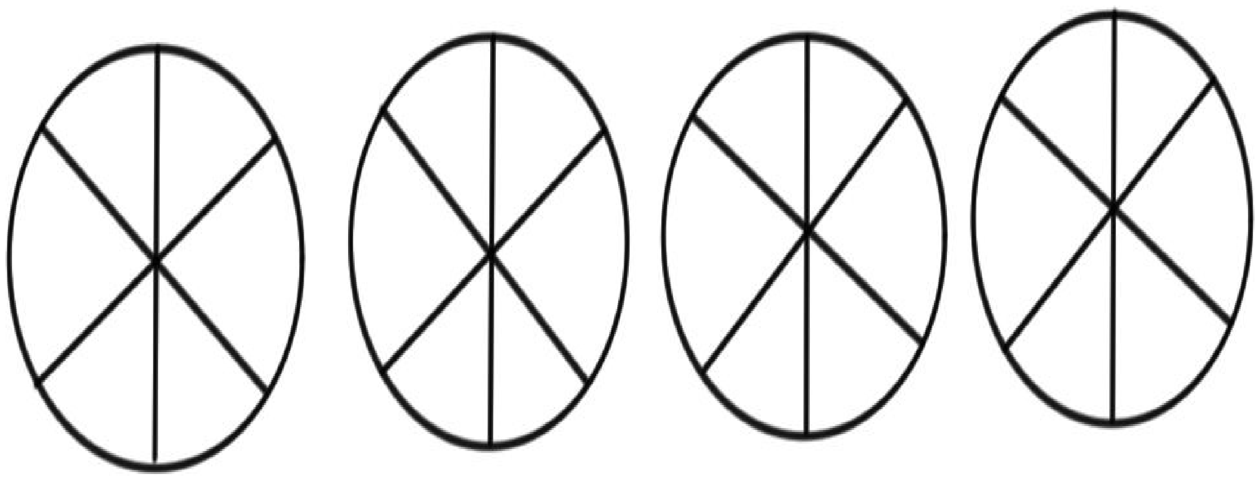


How could we fairly share these pizzas out?

We'd have to use a pizza-cutter, but what size would the pizzas be?

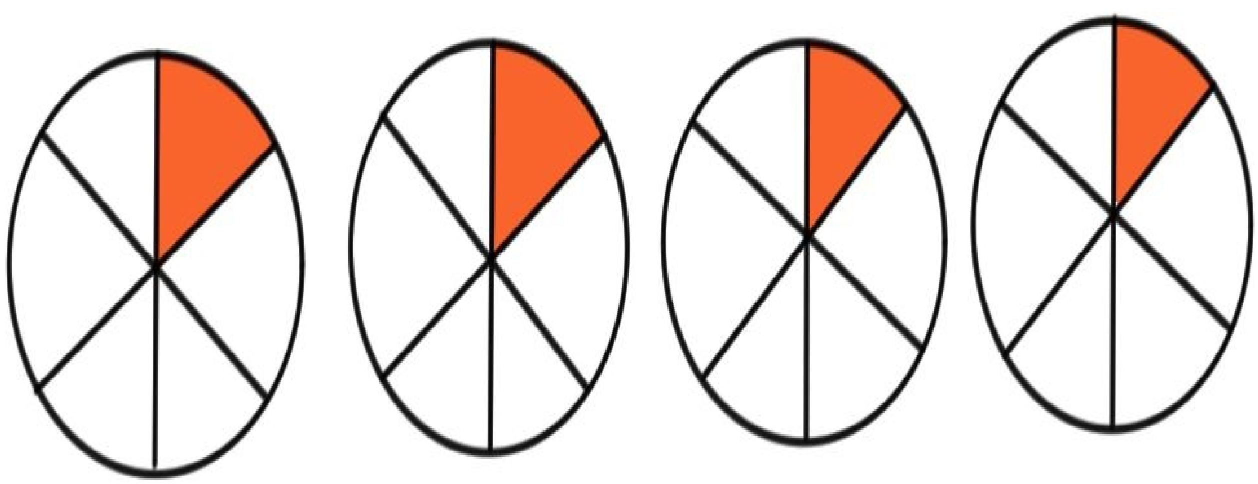
If we only had one pizza to share, and there were 6 people, we'd slice it up into 6 pieces.

So why don't we do that for all 6?

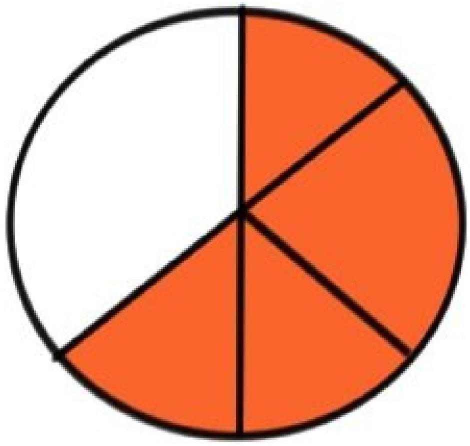


Imagine they were different flavours or something like that.

Then, we could say one person gets a slice from each, and puts them on their plate.



So the plate would look like this:



Which would have

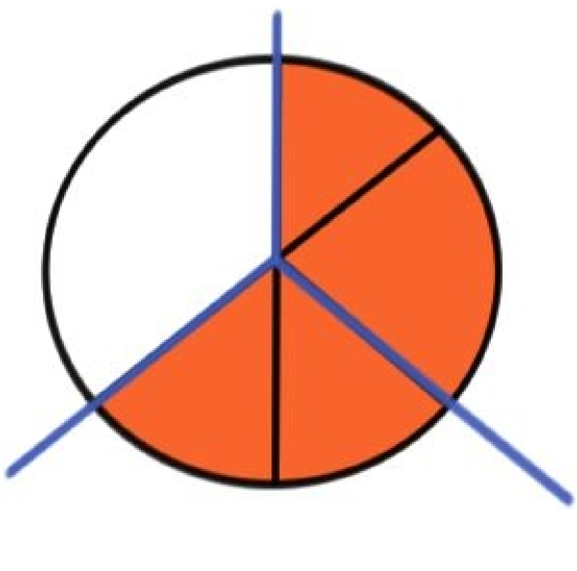
$\frac{4}{6}$

on it.

After all that, it would seem we have not made much progress.

$\frac{4}{6} = \frac{4}{6}$?

But, what if we were to divide the plate into 3 instead?



We'd see that

$\frac{2}{3}$

of the plate is covered.

So

$\frac{4}{6} = \frac{2}{3}$

(Four-sixths equals two-thirds).

Numerically, how did this happen?

How did 4 become 2 and 6 become 3?

We did NOT halve them, as they are equal to each other.

So what we actually did was

Find the Largest Number that Goes Into Both

4 and 6

Which was 2

And divide each one.

2 goes into 4 twice

2 goes into 6, 3 times.

So we end up with

$\frac{2}{3}$

as our answer.

This is called 'simplifying' a division that is less than 1.

THE REFLEX

In order to do divisions correctly every time, we need to develop a reflex to seeing one.

First of all, we need to identify what type it is
Is it more than 1, equal to 1 or less than 1?

Then we need to do what is necessary for that type. It's VERY important we develop this skill.

So looking at this table

< 1	= 1	< 1
Algebra trick or use DIJS	= 1	Simplify If Possible

If the division is a more than 1 type, we divide, using the algebra trick, or we could, although it is slow, use the fact that division is just subtraction and repeatedly take away until we reach zero and count the subtractions.

If it's equal to 1, that's easy! The answer is 1.

If it's less than 1, we simplify if possible.

Let's look at some examples of division, and you need to first identify the type, and then the answer.

$$\frac{32}{8}$$

$$\frac{8}{32}$$

$$\frac{5}{15}$$

$$\frac{10}{15}$$

$$\frac{3}{12}$$

$$\frac{9}{12}$$

$$\frac{24}{6}$$

and your answers

> 1, 4
< 1, 1/4
< 1, 1/3
< 1, 2/3
< 1, 1/4
< 1, 3/4
> 1, 4

Less-Straightforward Division

That covers almost all of division!

Remember, I said division is not as accommodating as multiplication for a reason, and here it is.

What is

$$\frac{25}{6} = ?$$

Here we note it's a more than 1 type, so we attempt the 'algebra trick'.

But immediately we note that it 'won't go in exactly'. So 6 x 4 = 24, 6 x 5 = 30, and 25 is in between.

What do we do here?

Let's try subtracting.

25 - 6 = 19
19 - 6 = 13
13 - 6 = 7
7 - 6 = 1

and we can't subtract more 6s as there is only one 'pizza' left.

So we've made four subtractions, with one pizza remaining.

We could write this

4, remainder 1
or

4 r 1 for short

What do we do with the remainder?

In practice, we'd share that out as well, using our pizza-cutter.

So 1/6 = one-sixth

In total then, everyone would get
4 & 1/6

which is written

$4\frac{1}{6}$

What's happened in effect is that the remainder has gone on top of the number that was dividing.

In practice, we don't want to subtract as it is too slow. It makes more sense to go to the nearest number to 25, in this case 24, note it is 4 x 6. Then count up to 25 (1) as our remainder.

Try

$$\frac{26}{6}$$

Here, again

4 x 6 = 24, so

4

and count 2 up to 26,

So

4 r 2

Divide the remainder by the number we're dividing by
$4\frac{2}{6}$

BUT whenever we see a division, we must always ask 'What type is it?' this is our REFLEX.

$$\frac{2}{6}$$

is a less than 1 type.

So how do we do those?

We simplify.

$$\frac{2}{6} = \frac{1}{3}$$

So the answer is

$4\frac{1}{3}$

Another way to think of it might have been
$$\frac{26}{6} = \frac{24+2}{6} = 4 + \frac{2}{6} = 4\frac{1}{3}$$

Either way is fine.

These types of divisions occur more often than nice ones that go in every time. The reason for this is due to the fact that it is more likely that it won't than it will. For example between 24 and 30 lies 25, 26, 27, 28 and 29.

So in those 7 numbers, if they were divided by 6, five of them don't go in exactly. 5 out of 7 don't go in, but 2 out of 7 do. So it's more likely we'll find divisions that won't go in. That is why it is so important to be able to do these. Plus, if the number we're dividing by is larger than 6, say 25, it becomes even more unlikely (24/25 possibilities in fact), that the 25 will go in.

Furthermore, if you try to use DECIMALS in a division, as you may find, it means you have to be very knowledgable about fractions to decimals, as you may have remainders such as

$$\frac{19}{21}$$

What's that as a decimal?

It doesn't matter what it is. It's easier and preferable to write

$$\frac{19}{21}$$

Let's make division more accommodating!

Finally, if you ask the average GCSE student what

$$\frac{23}{6}$$

is, I bet you they'll say...

'Er, 3-point-something...' and not know what it is.

Isn't that terrifying?

The Third Rule of Maths

The Third Rule of Maths

Before we move on to the final part of division, so-called 'Long Division', I want to take a look at the Third Rule of Maths, as it has come up a couple of times in our discussion so far.

You'll be glad to know this time I'm not going to ask you this time, but just tell you.

The Third Rule of Maths is

You Can Always Do The Reverse.

What does this mean?

Well, what it means is that everything in maths has a reverse to it. Everything you learn to do, can be done backwards. In maths, nothing is wasted, so you will end up doing it backwards as well in some way.

Everything has an opposite.

For example, what's the opposite of

addition?
multiplication?
squaring?
expanding a bracket (multiplying a bracket)?

They are

subtraction

division
square rooting (finding the number that was squared) factorising (essentially another word for division)
Everything has an opposite.

As I say, we have come across this already.

One example in this book has been the reverse of more than 1 types of division, which are less than 1 types.

Indeed, I've actually got an example of that in the questions to try section.

You may note that I asked you

$$\frac{32}{8}$$

and

$$\frac{8}{32}$$

and they are the reverse of each other.

The respective answers are

4 and

$$\frac{1}{4}$$

As a result, these numbers are considered the reverse of each other also, and the technical name is the

Reciprocal

of each other.

So the reciprocal of 4 is

$$\frac{1}{4}$$

The reciprocal of

$$\frac{2}{3}$$

Is

$$\frac{3}{2}$$

What is the reciprocal of

$$\frac{3}{4}$$

$$\frac{4}{3}$$

or

$1\frac{1}{3}$

In fact, the reciprocal of 2/3, is 3/2 =

$1\frac{1}{2}$

This concept of reciprocal is extremely important in itself, and will be used in a variety of contexts. However, for now let us note that it follows the Third Rule of Maths, and we have now completed all the Rules!

Everything from here on in will be using these rules in some way, and they will be mentioned a lot, as reference points, or places to fall back to if we get stuck.

So-Called Long Division

So-Called Long Division

When doing more than 1 types, we are going to come across harder questions than

$$\frac{32}{8}$$

or

$$\frac{24}{6}$$

We might have something like

$$\frac{288}{12}$$

here subtraction would be extremely lengthy (this really would be LONG division!) and using the algebra trick, though effective, would take a while to figure out, as our answer will be quite high.

As a result, 'long division' as it's called, breaks the question down using both techniques, part division is just subtraction, as we will see, and part 'algebra trick'. But an improvement on the school method would be to have a key that unlocks this division, so all we have to do is refer to the key and this will give the answer.

Let's look at the above example.

$$\frac{288}{12}$$

Since we're dividing by 12, we need to figure out, using the algebra trick, what number multiplies by 12 to get 288.

To do this, and to help us, we can use the 12 times table as the key to unlock this question.

12
24
36
48
60
72
84
96
108
120

which since Multiplication Is Just Addition, we can just add 12 each time. We could just keep adding up to 288, but this is just a low number to serve as an example. If it was 288456, adding would be a very lengthy manual process. For any question, the first 10 numbers will suffice.

We now use the concept of Division Is Just Subtraction also.

Writing the number in a sort of bus shelter

$\overline{)288}$

We ask does any number in the list go into 2?

$\overline{)\color{red}{2}88}$

No.

Any number in the list go into 28?

$\overline{)\color{red}{28}8}$

Yes, the second number, 24. It doesn't go in perfectly, but it is closest.

So we write 2 (as it's the second number) above the 8 in 28,

$$\begin{array}{l} 2 \\ \overline{)\color{red}{28}}\,8 \end{array}$$

and subtract this second number, leaving

$$\begin{array}{l} \;\;2 \\ \overline{)\,\color{red}{28}\,8} \\ -24 \\ \hline \;\;\;\;4 \end{array}$$

Then we repeat the process

Asking does any number in the list go into 4?

No.

So we need to bring that 8 down to create 48.

$$\begin{array}{l} \;\;2 \\ \overline{)\,\color{red}{28}\,8} \\ -24 \\ \hline \;\;\;\;48 \end{array}$$

Does any number in the list go into 48?

Yes! 48.

The fourth number in the list.

Again we write 4, this time

So

$$\begin{array}{l} \;\;24 \\ \overline{)\,288} \\ -24 \\ \hline \;\;\;\;48 \\ \;\;-48 \\ \hline \;\;\;\;\;\;0 \end{array}$$

We also subtract this 48, which leaves zero.

So the answer is 24.

$$\frac{288}{12} = 24$$

We have used a mixture of the algebra trick and DIJS to answer this. This is so-called ‘Long Division’.

However for me it's a mis-nomer. It's not really Long division compared to subtracting each time, and if we're clever, we can shorten it to 'Short Division'!

Instead of writing out the subtraction in full, we might have just done it on the fly, giving a much shorter calculation.

When we write the remainder, we could put it as a superscript instead of below the division, so it looks like this

$$\begin{array}{r} 24 \\ \overline{)28^{4}8} \end{array}$$

How does this work?

If you're interested in how this works, then using the example above, what has actually happened, when we subtracting the 24 for the first line, is that we're subtracting 240.

That's because we're multiplying it not by 2, but actually 20. 20 x 12 = 240. We put that 2 in the tens column, so it represents 20.

As a result, if we take this away from 288, we are left with 48. Then 4 x 12 = 48, giving

20 + 4 = 24.

It's just that we don't bother writing that zero down when we subtract 24 as it's a waste of time.

But it's important to know WHY we do things and not just HOW.

In primary school, this method is known as chunking. This is a more advanced form than that. Chunking is really, in my opinion, a poor method, as it is of course, division via subtraction, but taking 10 lots away each time. This can be sped up considerably by just making it 20, 30, 40, etc lots each time, which is not too big a leap to make. But in primary school, the HOW is more important than the WHY.

The **Current Maths Teaching System** is discussed, reviewed and critiqued in Book 45 of this series.

Checking your answer is correct.

How could we check the above answer was correct?

We have stated that

$$\frac{288}{12} = 24$$

How would we know for certain it was right?

Without using a calculator?

Since we know that division and multiplication are the reverse of each other, we could just multiply them. After all, that was the question. What multiplies by 12 to get 288? And we have said 24. So let's do that, using our method from Book 1.

```
  24
x 12
------
 288
```

We find it's correct!

Try the following and check the answers yourself.

$$\frac{576}{12}$$

$$\frac{432}{18}$$

$$\frac{3410}{25}$$

$$\frac{54625}{21}$$

Ans.

Check yourself!

Ha.

But let's do a worked example, as you will have noticed that they don't all 'go in exactly'.

3410/25

Writing the first 10 25s...

25
50
75
100
125
150
175
200
225
250

and using this key to unlock the question

$$\frac{3410}{25}$$

$$25\overline{)3410}$$

3 is not in that list.

34 is, but at 25, the first number.

$$\begin{array}{r} 1 \\ 25\overline{)3410} \\ -\,25 \\ \hline 9 \end{array}$$

Taking away that 25 leaves 9, but 9 is not in the list, so we need to bring down the 1

$$\begin{array}{r} 1 \\ 25\overline{)3410} \\ -\,25 \\ \hline 91 \end{array}$$

Here, the closest is 75, the third in the list, so we write

$$\begin{array}{r} 13 \\ 25\overline{)3410} \\ -\,25 \\ \hline 91 \\ -\,75 \\ \hline 16 \end{array}$$

Again, 16 is not in the list, so we bring down the zero....

$$\begin{array}{r} 13 \\ 25\overline{)3410} \end{array}$$

- 25

91
- 75

160

The closest is 150, the sixth

So we have

$$25\overline{)3410}^{\,136}$$

- 25

91
- 75

160
- 150

10

Since 10 is not in the list, *and* we have no new numbers to bring down, 10 is the remainder.

So the answer is 136 r 10

Or dividing the remainder by 25 also

$136^{10}/_{25}$

and then remembering to always REFLEX

we note

$$\frac{10}{25}=\frac{2}{5}$$

So the answer is

$136\frac{2}{5}$

To check this, we can ignore the 2/5 and multiply

136 x 25

Using the usual method

This gives

```
  136
 x 25
-------
```

$34_3 0_3 0$

Adding the remainder of 10 gives 3410. Therefore it is correct!

Introduction To Fractions

Introduction

Being able to add fractions is one of the skills required for a maths qualification. This is one example where there's no need for this in everyday life. However, there is a need to be able to do it in more advanced mathematics, such as algebraic fractions, when you come to study it in the future.

The skills you need at first are being able to

ADD

SUBTRACT

MULTIPLY

&

DIVIDE

Addition Of Fractions

Fractions.

POLITICIAN: *“The problem with education today is that students don’t know basic maths.”*
REPORTER: “Okay, so what’s two-thirds of three-quarters?”
POLITICIAN: *“That’s exactly the kind of question that can get us politicians into trouble...”*

...what happened when a politician was caught out.

In Division - In A Minute you saw three types of division, one of which was < 1.

This type is more commonly called ‘Fractions’ from the word fracture, to break into (equal) pieces.

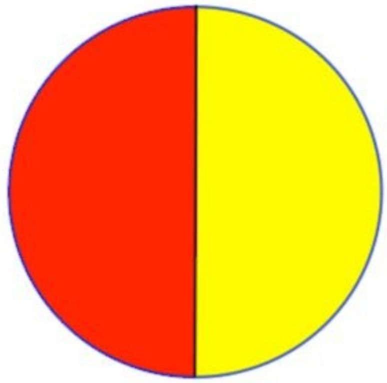
To be able to manipulate these, we usually try to simplify them first! Once that is done, we would like to also be able to add, subtract, multiply and divide them.

Adding fractions.

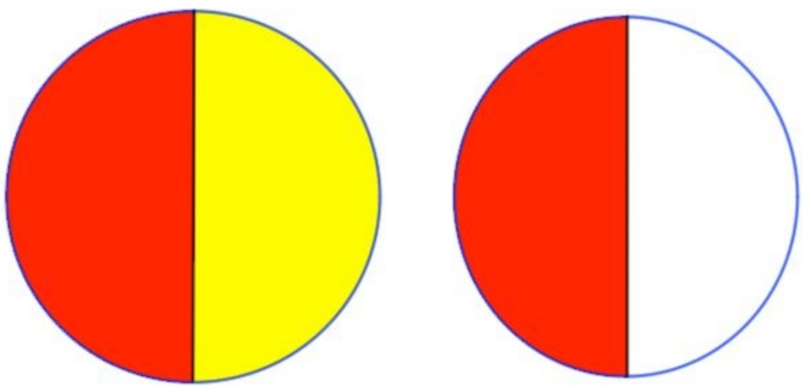
If the fractions are the same type, this is very easy. Just add as many as there are.

For example

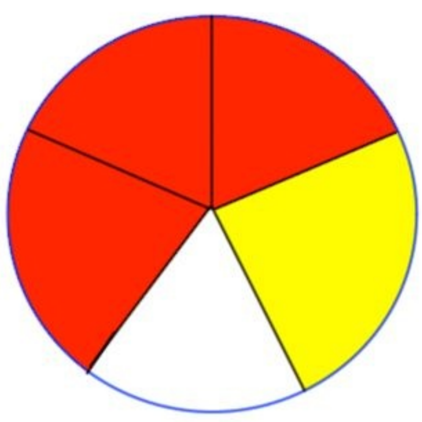
$$\frac{1}{2}+\frac{1}{2}=1$$



$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=1\tfrac{1}{2}$$



$$\frac{1}{5}+\frac{3}{5}=\frac{4}{5}$$



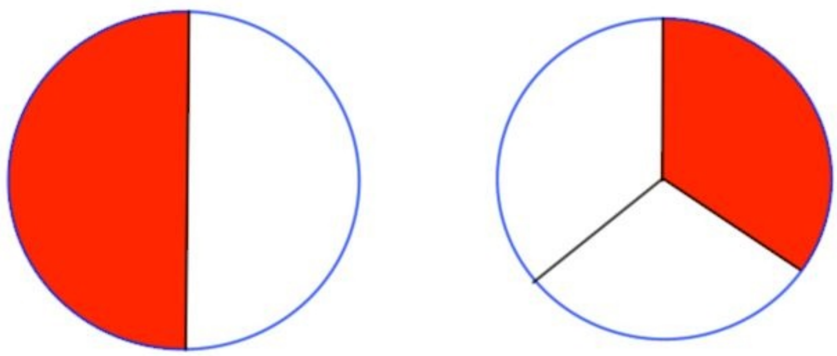
These are all fine, and nice and easy.

We can think of these as being similar to common currencies. For the last example, we could look at it as being
$1 + $3 = $4

The problem comes when we have fractions that aren't the same type.

For example

$$\frac{1}{2}+\frac{1}{3}$$



What is this equal to?

a. $\frac{2}{5}$
b. 0.something
c. $\frac{5}{6}$

It's certainly one of these!

But how do we find out?

Going back to the currencies analogy, the problem we have is that we are dealing with different type of fraction, or currency. One is divided by 2, the other by 3.

Let's think of this as an English tourist who goes to France.

In their wallet they have

£50 & €50.

How much do they have altogether?

Since it's impossible to add these up to get 100 europounds or something like that, what do we have to do? Intuitively, it may occur to you that we need to change one of the currencies over, so they are both the SAME.

To do this, let's say we change the pounds into Euros. So now we have
€75 + €50

Now they are both the SAME currency, we can ADD them.

€125

It's a similar issue with our fractions. They are both different currencies, so it is necessary to change them so they are the same. Otherwise, we can't add them up.

In fact, our example is slightly more complicated than that as we have to imagine it is an American tourist instead.

He visits England.
He visits France.

He returns to America to find he has
£50 & €50 in his wallet.

What thought will go through his mind?

How much that is in dollars!

So he has to change both over.

So £50 = $80
€50 = $55

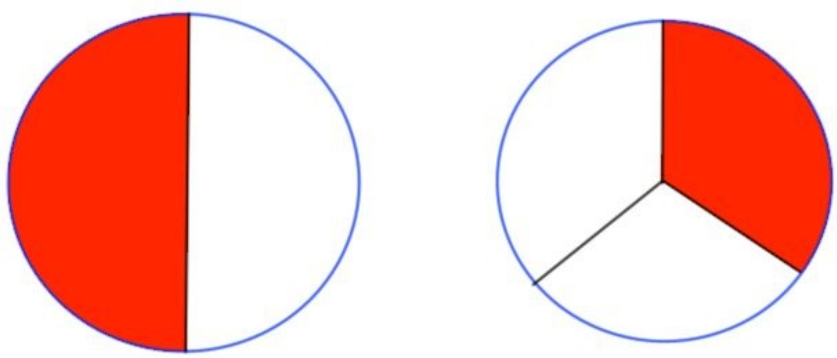
So now they're both the SAME currency we can ADD.

TOTAL = $135

So we have to do this in exactly the same way with adding fractions. Once we can, it's great as it sets up to be able to subtract, multiply and divide with ease.

Looking again at

$$\frac{1}{2}+\frac{1}{3}$$



we need to think of these as being the Pound and Euro as above. To add them we need to convert them to a 3rd currency, like our American tourist does. Once we've achieved that we're finished!

It's a 3-step process.

1. Write out the times tables of the two bottom numbers

2 4 6 8 10 12...
3 6 9 12 15....

Look for a common number in both lists.

Here we see there are two. Both 6 and 12. However we always choose the lowest. 6. This will be our 3rd currency!

1. Convert both $\frac{1}{2}$ and $\frac{1}{3}$ into 6

At first you put

$$\frac{1}{2} = \frac{?}{6}$$

We need to convert 2 to 6. What do we need to multiply by 2 to get 6?

Three.

So we need to have

$$\frac{1}{2} \times \frac{}{3} = \frac{?}{6}$$

and since we have 3 on the bottom, we also put it on top (why?)

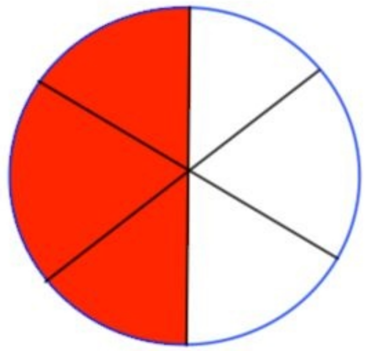
$$\frac{1}{2} \times \frac{3}{3} = \frac{?}{6}$$

and since 1 x 2 = 2

$$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$$

So

$$\frac{1}{2} = \frac{3}{6}$$



We then apply the same method to $\frac{1}{3}$

$$\frac{1}{3} = \frac{?}{6}$$

What do we need to multiply by 3 to get 6?

Two.

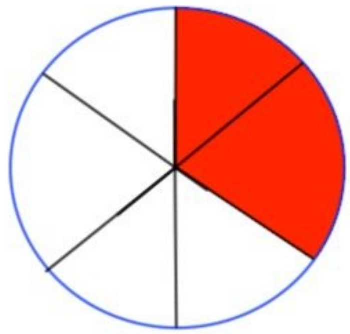
So we have

$$\frac{1}{3} \times \frac{}{2} = \frac{?}{6}$$

Again, we put 2 on the top (why?)

$$\frac{1}{3} \times \frac{2}{2} = \frac{?}{6}$$

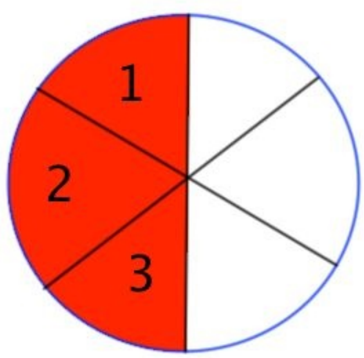
$$\frac{1}{3} \times \frac{2}{2} = \frac{2}{6}$$

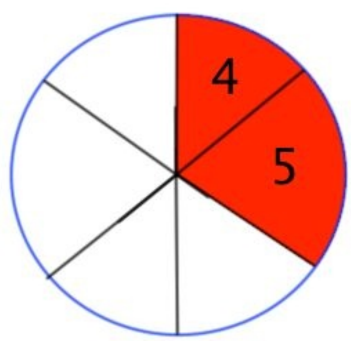


1. ADD.

So now we have our original fractions in their new currencies, which are both the same. As a result we can add them

$$\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$





A few questions.

Did you notice that we were effectively doing algebra when we asked the question ‘What do we need to multiply 2 by to get 6?’ This was very similar to the Algebra Trick we saw in Division - In A Minute.

Why doesn’t 6 + 6 make 12? Because remember it’s like they are the same currency. So adding $2 and $3 makes $5, not 5 double dollars or anything like that. Second, we spent ages trying to find it. Plus, look at the picture of this, we can see it will be 5/6 from that.

Is there a quicker way to do this? Funnily enough, yes there is! We will see that later. However I want you to understand why it will work first.

What type of division is 5/6? Less than 1. Did you remember the REFLEX? So what do we need to do for these types? Right, simplify. Can we do that? No. So the answer remains the same.

Try

$$\frac{1}{3}+\frac{1}{4}$$

Ans:

$$\frac{7}{12}$$

How?

3 6 9 12 15 18...
4 8 12

Lowest common number is 12. Note that I stopped once I reached a common number as it saves time.

We need to convert both fractions to twelfths /12

To make $\frac{1}{3}$ become a twelfth we need

$$\frac{1}{3}\times\frac{}{4}=\frac{?}{12}$$

$$\frac{1}{3}\times\frac{4}{4}=\frac{4}{12}$$

and $\frac{1}{4}$ becomes

$$\frac{1}{4}\times\frac{}{3}=\frac{?}{12}$$

$$\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$$

Adding together

$$\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

Interesting Note.

One thing you may have noticed is that I'm asking you to contradict an explicit instruction from Division - In A Minute.

Normally if we see a less than 1 (< 1) type, we would do what?

Simplify.

Here though, when we our fractions are converted into

$\frac{4}{12}$ and

$\frac{3}{12}$

as above, I'm suddenly asking you to completely ignore that. This is the only time!

There's no point simplifying these as we will just end up back where we started, with

$\frac{1}{3}$

and

$$\frac{1}{4}$$

In fact, when we are doing addition of fractions, we are in fact doing the complete reverse of simplifying. We are following the Third Rule. Everything has a reverse. This includes simplifying. This process of changing the fractions to have a common number on the bottom is the exact opposite of simplifying them. Instead of dividing top and bottom by the largest number that goes into both, we are in fact multiplying them by the same amount.

Everything has a reverse, and also nothing is wasted. So we must use it somewhere. This is where.

Subtraction of Fractions

Subtraction of Fractions

BONUS NO. 1

YOU ALREADY KNOW HOW TO DO IT.

Why?

Because it's exactly the same.

In order to subtract

$$\frac{1}{2} - \frac{1}{3}$$

We follow the same process exactly. If we think back to the analogy of the different currencies, by taking away one from another, we are trying to spend Euros when we have Pounds. So we need to make them the same currency before we can spend.

So it will follow the same as above

$$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$$

$$\frac{1}{3} \times \frac{2}{2} = \frac{2}{6}$$

Then at the end we simply subtract instead

$$\frac{3}{6} - \frac{2}{6} = \frac{1}{6}.$$

Try

$$\frac{1}{3} - \frac{1}{4}$$

Ans:

$$\frac{1}{12}$$

Simple!

Multiplication of Fractions

Multiplication of Fractions

BONUS NO. 2

YOU ALREADY KNOW HOW TO DO IT!

Why?

Because it isn’t the same as adding and subtracting!

Eh?

In fact, when we did adding and subtracting, we picked up the skill we need in order to multiply.

I didn’t mention it at the time, but if we look at the currency conversion process, we can see it in action.

$$\frac{1}{2}\times\frac{3}{3}=\frac{3}{6}$$

$$\frac{1}{3}\times\frac{2}{2}=\frac{2}{6}$$

I just took it for granted that you would see that, for the first conversion
1 x 3 = 3

and

2 x 3 = 6

and each would change with no problem.

Although we are multiplying by

$$\frac{3}{3}$$

which is effectively 1, this is still written in the format of a fraction, and this is how we would also multiply any two fractions.

So you already know how to do it.

$$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$

Since

1 x 1 = 1
2 x 3 = 6

That's it!

It is ironic that in school they spend all year learning how to add and subtract fractions, then when they come to multiplication they announce

'Sorry everyone, this is completely DIFFERENT!'

When it isn't. You intuitively and subliminally pick up the skill when you learn to add and subtract.

Finally, another way of saying 'times' when we are doing fractions is

$$\frac{1}{2}\,of\,\frac{1}{3}$$

=

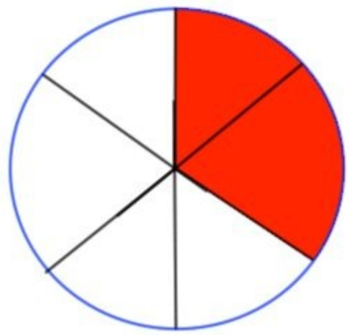
$$\frac{1}{6}$$

also.

So we might say

A half times a third or
A half of a third.

If we look at what a half of a third actually is



We can see that it is indeed

$$\frac{1}{6}$$

How about

$$\frac{2}{3} of \frac{3}{4}?$$

Multiplying

$$\frac{2}{3}\times\frac{3}{4}$$

$$=\frac{6}{12}$$

Remembering the REFLEX

$$=\frac{1}{2}$$

So why are simplifying now when I told you not to bother earlier?!?

Whenever we get a final answer, we should attempt to simplify, for adding and subtracting also. It's just that we haven't had an answer like that yet.

For adding and subtracting, reverse simplifying is vital to how the method works. In multiplication, and we will see also, division, making the fraction simpler makes the multiplication easier.

So if you had

$$\frac{4}{12}\times\frac{3}{12}$$

it's easier to simplify

$$\frac{1}{3}\times\frac{1}{4}$$

=

$$\frac{1}{12}$$

Than have

$$\frac{12}{144}$$

and then simplify afterwards, as you would have to for any type of calculation when you get the final answer.

In the question

$$\frac{2}{3}\times\frac{3}{4}$$

however, they are already as simplified as they can be.

Dividing Fractions

Dividing Fractions

BONUS NO. 3

YOU CAN ALREADY DO IT!

(Ok, almost).

In fact, we noticed in Division - In A Minute, that division is not very accommodating. So a division of a division is even less so. As a result, we switch it to something easier.

If we look at our methods so far, we see that for

Adding, we multiplied twice
Subtraction, we multiplied twice (being the same) Multiplication, we multiplied (not surprisingly!)
So for division, we turn that into a multiplication also.

One, we already know how to do it, and two, it's easy.

So to make it ready for this process, we note that division is the reverse of multiplication.

As a result, we need to make a reversal of the second fraction.

Can you remember the reciprocal from Division - In A Minute?

The reciprocal was the reverse number, or the division flipped around.

So the reciprocal of $\frac{1}{3}$ was $\frac{3}{1}$ or 3.

If we divide

$$\frac{1}{2} \div \frac{1}{3}$$

we switch this to

$$\frac{1}{2} \times \frac{3}{1}$$

by changing the second fraction to its reciprocal.

Multiplying,

$$= \frac{3}{2}$$

Then doing that division by recognising its type (> 1)
The answer is

$$1\tfrac{1}{2}$$

This is a lot easier than trying to imagine how many thirds are in a half!

Or subtracting a third from a half repeatedly.

For more complicated fractions this comes into its own as a method.

$$\frac{2}{3} \div \frac{3}{4}$$

It's hard to imagine or calculate how many $\frac{3}{4}$ are in $\frac{2}{3}$!

So if we do this 'times and flip' method

$$\frac{2}{3} \times \frac{4}{3}$$

$$= \frac{8}{9}$$

"Mixed Numbers"

Mixed Numbers

A 'mixed number' is a term I never use. The way I like to think of them is 'the answer to a division that is more than 1 (> 1)'.

For example,

$$\frac{11}{4}$$

$$= 2\,{}^{3}\!/_{4}$$

as we saw from Division - In A Minute
This answer is known as a mixed number in school, as it is a mix of a whole number and a fraction. However it is more useful to think of it as 'the answer to the division..'

$$\frac{11}{4}$$

The reason for that is that if we try to multiply this mixed number with another, the easiest way to do this would be to revert it to its original division.

So let's say we want to multiply

$$2\,{}^{3}\!/_{4} \times 3\,{}^{1}\!/_{8}$$

Instead of doing a complicated calculation using the usual multiplication method we saw in Book 1, which would look like

$$\begin{array}{r} 2\,{}^{3}\!/_{4} \\ \text{x}\ \ 3\,{}^{1}\!/_{8} \end{array}$$

----

and do the 3 steps - this would actually be quite complicated, and unnecessarily so.

What we can do instead is to revert each 'mixed number' to its original division. In other words, reverse the process - following the Third Rule - of division on them. The reverse of division is multiplication.

$2\frac{3}{4}$

came from the fact that 4 went into 11 twice, and had 3 remainder.

To do this it will become

4 x 2, + 3 = 11

divided by 4

and $3\frac{1}{8}$ will become
8 x 3, + 1 = 25

divided by 8

To give

$$\frac{11}{4} \times \frac{25}{8}$$

Once we get the numbers into this format, it is very simple as we can just multiply top and bottom as we would do for any multiplication of fractions.

11 x 25 = 275
4 x 8 = 32

The answer is

$$\frac{275}{32}$$

And following the REFLEX, what type of division is this?

> 1

So we have to divide 275 by 32

$$8\frac{19}{32}$$

Ans.

When we have 'mixed numbers' then, just reverse the process to turn them into their original divisions, via multiplication. Once that is complete, use the usual method to multiply fractions.

One other way to turn them into their original divisions is to think about how many parts each one is made of.

For example, $3\frac{1}{8}$ we could think instead of having 8 eighths per whole. So 3 'wholes' plus 1 eighth would be 25 eighths.

$2\frac{3}{4}$ would be 4 quarters per whole. That's 8 quarters plus 3 quarters is 11 quarters.

Either way is fine. It's often best to be able to do both ways.

Division of Mixed Numbers

Division of Mixed Numbers

So now we're going to be a division of a division! Ring any bells? So we follow the same method.

Let's say we have

$$2\frac{3}{4} \div 3\frac{1}{8}$$

Turning them into their original divisions

$$\frac{11}{4} \div \frac{25}{8}$$

And following the 'times and flip' method gives

$$\frac{11}{4} \times \frac{8}{25}$$

$$= \frac{88}{100}$$

Again, what type of division is this?

< 1

Simplifying,

$$= \frac{22}{25}$$

Ans.

How can we check these answers?

For the first we could divide the answer we got ($8\frac{19}{32}$) by $2\,{}^{3}\!/_{4}$ and it should give $3\frac{1}{8}$. To do this we follow the method of division of mixed numbers just outlined.

To check the answer $\frac{22}{25}$ is correct, we just multiply it by $3\frac{1}{8}$ or $\frac{25}{8}$. It should give $2\frac{3}{4}$.

Addition of Mixed Numbers

Addition of Mixed Numbers

To add these, they follow exactly the same rule as the very first example we did in this book. However we also add the whole numbers.

So

$$4\frac{1}{2}+2\frac{1}{3}$$

Would be

6 (from 4 + 2)
and

$$\frac{5}{6}$$

So

$$6\frac{5}{6}$$

Ans.

Subtraction of Mixed Numbers

If we are subtracting these same two numbers, we can do exactly the same, but subtract.

Therefore we would have
2 (from 4 - 2)

and

$$\frac{1}{6}$$

So

$$2\frac{1}{6}$$

What about

$$4\frac{1}{2} - 2\frac{2}{3}$$

Here we have a little problem, because $\frac{2}{3}$ is more than $\frac{1}{2}$.

This means that when we subtract it, we're going to have to take away from the whole number also. This is something that it is very important to be aware of, but you will be fine as long as you keep the whole numbers and fractions separate.

For

4 - 2 = 2

and

$$\frac{1}{2} - \frac{2}{3}$$

would become

$$\frac{3}{6} - \frac{4}{6}$$

this would equal what?

Ans

$$-\frac{1}{6}$$

As a result, we then need to do

$$2-\frac{1}{6}$$

=

$$\frac{12}{6}-\frac{1}{6}$$

(by switching the 2 into sixths also)

$$=\frac{11}{6}$$

Now we have our answer, we then apply the REFLEX

It's a > 1 type, so we have

$$=1\frac{5}{6}$$

Ans.

Ultra-Fast Ways of Adding and Subtracting Fractions

Ultra-Fast Ways of Adding and Subtracting Fractions - ADVANCED

This method should only be attempted once you've fully understood everything previous to this chapter.

The methods above, though correct, can be laborious, and unfortunately, do not apply to algebra.

As a result, we need a method that will work for both numbers and algebraic fractions in the future. For now I'll just explain how to do it with numbers, with algebraic fractions explained in a later book.

Going back to the example of

$$\frac{1}{2}+\frac{1}{3}$$

Instead of following the method, what we can do instead is notice that actually, the lowest common number just happens to be the two bottom numbers multiplied together. This is not always the simplest case, but it can always be used. Plus, we may have noticed that the new values, when converted to

$$\frac{3}{6}$$

and

$$\frac{2}{6}$$

include the numbers 2 and 3, just like those original bottom numbers!

Is there a pattern here? Of course.

What we can do instead is to multiply the bottom two numbers to get 6.

For the top values, we 'cross-multiply' and get

$$\frac{3\times1}{6}+\frac{2\times1}{6}$$

These add to

$$\frac{5}{6}$$

Or all in one

$$\frac{3+2}{6}$$

For subtraction we can use the exact same technique.

$$\frac{1}{2}-\frac{1}{3}$$

gives

$$\frac{3-2}{6}$$

If the numbers are not one on top, it doesn't matter, we must just multiply.

$$\frac{1}{2}+\frac{2}{3}$$

$$\frac{3\times 1}{6}+\frac{2\times 2}{6}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$$

Using the REFLEX

> 1 type

$1\frac{1}{6}$

Ans.

We don't need a faster method for multiplying and dividing as they're already the fastest.

That's fractions!

Are we ever going to use this in real-life? No!

But we will use it extensively in algebra, where similarly, division is not very accommodating, and a fluency in it, and especially with algebraic fractions, is vitally important.

Decimals

Decimals – In A Minute

Every word in maths has an origin in Greek or Latin, because it is such an old subject. Also, mathematical papers were written in Latin up until relatively recently. So the word ‘decimal’ has a meaning, which, once we understand, can be used to make using decimals easier.

First, since English is a mixture of languages, we can look at similar words to figure out what it means!

So what does decade mean?
Decathlon?
Decagon?
December*?

Before reading on, have a think what they mean and what is the connection between them.

So?

The answer is ten.

Ten years
Ten events
Ten sides
And tenth month*

*December means tenth month then… but hang on, it’s the twelfth month! In fact, four of our months are mis-named. September, October and November too. (7^{th}, 8^{th} and 9^{th}). Why? It’s because these names are Roman. They come from a time when the main god of the Romans was Mars – hence March. That then, was the first month, which means those names would fit perfectly. When March was made the third month, for some reason we kept the same names even though they were wrong. And they've been wrong for a very long time….I honestly don't know why they were chosen and why they haven't been changed!

So the word decimal means ten. We operate in what's known as a decimal-base system, *ie.* Everything revolves around the number ten. You're probably intuitively aware of this already, as you may have noticed that when counting, we start again at ten every time…and the cycle goes around again.

So bearing this in mind, what is the connection between these numbers?

4.50
45.00
0.45

(Apart from them all containing 4s and 5s!)

They are all either ten times bigger or ten times smaller than one another.

These numbers all contain a 'decimal point'. The reason it's called that is because the only thing that can move a decimal point is the number ten. If you multiply or divide by ten, it will move the decimal point right, or left.

So what happens to the point if we multiply by ten?

It moves one to the right.

So, using the third rule, which is?

You can always do the reverse, if we divide by ten, the decimal point must move one to the…?

Left.

So 4.50 x 10 = 45.00

Note that we don't need to write zeroes after a decimal point if it is the last number. However, I'm writing these so they are like money to you, and make them easier to make sense of. For example, £4.50 x 10 = £45.00.

What happens then if we multiply by ten again?

The decimal place moves to the right one more.

So if we go back to my first question, multiplying by ten and multiplying by ten again, what is that the same as?

Multiplying by 100.

So multiplying by 100 moves the decimal place - 2 places. This has great significance for future concepts, such as indices and logarithms, but for now, let's just look at the effect of dividing by 100.

If you can always do the reverse, then dividing by 100 will do what to the decimal point?

Move it 2 places to the left.

This is a very important concept, which we will come across in **Percentages – In A Minute**, so please take careful note.

Dividing by 100 moves the decimal place 2 to the left.

Great.

Let's quickly look at:

Adding and subtracting decimals.

Again, we can look at these like money. ALWAYS make sure the decimal point is ‘lined up’, so that if we have a problem such as this:
4.56
+ 2.34

We don’t have problems because the numbers are in the wrong column.

This should read

  4.56
+2.34

Which can then be added column by column, carrying as necessary.

Subtraction is exactly the same. Really it is no different than adding or subtracting if the decimal point wasn’t even there, the only difference being we MUST MAKE SURE IT IS LINED UP.

Great.

Let’s look at:

Multiplying decimals

What is multiplication?

Addition, of course.

So how are we going to use this concept in the multiplication of something like 1.4 x 2.1?

Whenever we are multiplying, we must be adding somewhere. That is why it's the first rule of maths! But how can you add 1.4 times? The answer is, you can't! But you will see that there will be an addition involved!

So let's do it.

To do this multiplication, we do the method I taught you in **Multiplication – In A Minute.** Remember I said it had many other uses? Well, here's one!

So we use exactly the same method, but importantly to start off with, completely ignore the decimal points.

So in other words look at it like this

```
  14
X 21
```

That's not so scary is it?

No problem for you.

You should have got

294.

Now, all we have to do is bring those decimal points back in.

So

```
  1.4
X 2.1
```

294

We now ask, how many decimal *places* are in each number? In 1.4, there is one. In 2.1 there is also one.

(Remember I'm asking how many places, not points. *E.g.* 6.47 has two decimal places, 3.141 has 3 decimal places, etc.)
So we have

1.4……. 1 d.p.
X 2.1……. 1 d.p

294

So, remembering that multiplication is just addition (!), we just add the number of places! So 1 + 1 = 2. So this means the answer must have 2 decimal places in it.

So we give it 2 decimal places viz:
2.94

That's the answer!

So try these yourself – no calculator required!!!!

4.3 x 2.3 4.1 x 3.7 3.2 x 0.31 0.43 x 0.23

You should have got

9.89

15.17

9.92

0.0989

I hope you did these using my method from **Multiplication – In A Minute.** If not, get hold of it and look how easy it could have been for you!

What's interesting is that for two of these questions, I didn't use that method. In fact, for questions 1 and 2 I used an even quicker method, because I noticed something interesting about them. To find out how to do these even more quickly, you can get yourself a hold of **Squaring – In A Minute!** This explains an amazing squaring technique I realised when I was 17 and I've been using ever since. It can also be applied to calculations such as 1 and 2 above.

Notice that last result. Did you realise it is exactly the same calculation as the first one?

Key Change Effect

The Key Change Effect

Have you ever listened to a pop song? Have you ever noticed they go on a bit at the end, so for a bit of colour, they will apply an effect to regurgitate the song and make it more lively. Do they sing new lyrics? New melody? No…they use a key change.

This is a device where they play/sing the same song and words, but using different notes. So it still sounds the same, but it is also different!

A good example is 'Man In The Mirror' by Michael Jackson. Listen at [2:30 - 3:00]. Suddenly there's a big jump, but the chorus is still the same. It gives life and energy to the song, but nothing else is different.

Incidentally, other songs that include key changes include: Penny Lane by The Beatles
Build Me Up Buttercup by the Foundations Can't Fight The Moonlight by LeAnn Rimes I Will Always Love You by Whitney Houston Livin' On A Prayer by Bon Jovi Bohemian Rhapsody by Queen
My Heart Will Go On by Celine Dion Hallejulah by Alexandra Burke

So what's this got to do with maths? And decimals!!??

When we multiply or divide numbers, the answers are always the same, no matter how many decimal places or zeroes are in there.

Look at the two examples above. For each we get the result 989. We get 2 versions of it, *i.e.* 9.89 and 0.0989, but those 989s are still there. They will always be there! All that happens is a kind of accounting where we have to add the decimal places and make sure the right number is in the answer.

So when I told you that you know how to multiply any two numbers, it is now most certainly true, because now you know that the answer to every question is always the SAME it's just a matter of adding decimal places to make sure it's right. That's the only difference!

This is just like in pop songs. They always have the same melody. Even when there's a 'key change', the underlying notes are different (position of a decimal place), but the tune (the answer) is still the same!

This concept can be utilised in many part of maths, such as this, standard form, percentages, division...it's extremely powerful.

Reverse Situation - taking the Key Change Effect further

Reverse Situation - taking the Key Change Effect further.

What about multiplication of numbers that don’t have decimals, but in fact, the complete reverse, a bunch of zeroes?

Let’s start with:

  430
x 230

Again, this must have the answer 989 in it somewhere.

The question is, what about the zeroes?

So, since multiplication is addition, we just add how many there are!

There are 2 altogether, so the answer will be
98 900

Another example

  14 000 000 (6 zeroes) x 210 000   (4 zeroes) --------

Again, this must have 294 (from earlier), but we just need to add those zeroes on.

So it’s going to be

294 with 10 zeroes,

2 940 000 000 000

Easy!

What if there's a mix of zeroes and decimal points in the question??

If this is the case, remember the **Key Change Effect**.

So

   0.43
x 230000

Again will give 989.

So what about the zeroes and decimals? Simple, we add on the zeroes first...

9 890 000

and THEN put in the decimal places (2)
So

9 890 0.00

Or

98 900.

This means that questions like
   0.0000014
x 21000000

----

may have previously seemed impossible. But now, they should be simple.

294 (with 6 zeroes and 7 decimal places) gives

294000000

29.4

In fact, there's a standard form for doing this kind of question, which, in fact, is called Standard Form. Since the **Key Change Effect** says that the numbers are more important than the zeroes or decimals, these can be separated away and we see that an addition or subtraction is required, *e.g.* in the last one we had 7 (d.p.) - 6 (zeroes) = 1 d.p.

However, this is covered in **Standard Form - In A Minute.**

Division of Decimals

Division of Decimals

Again, this is nothing to be scared of. We have come across division in '**Division - In A Minute**' so we know that there are how many types?

Three!

So there will be 3 types of division when we use decimals.

Again, the Key Change effect is obeyed, so we will see the same answer in each question. However, we must account for the decimals, or zeroes!

So let's start with

$$\frac{4.2}{7} \qquad \frac{4.2}{0.7} \qquad \frac{42}{0.7}$$

These are all effectively

$$\frac{42}{7}$$

which is 6.

So we know the answer is 6 to all, but we need to account for the decimals. First write like this,

$$\frac{4.2}{7} = 6 \qquad \frac{4.2}{0.7} = 6 \qquad \frac{42}{0.7} = 6$$

Accounting for decimals, we noticed when we were multiplying, that we ADDED the number of decimal places.

What do you think we do when we're dividing?

(Hint: What is division?)

Of course, we subtract.

So we have,

$\frac{4.2}{7} = 6$ Has 1 d.p over 0 d.p.'s

$\frac{4.2}{0.7} = 6$ Has 1 d.p over 1 d.p.

$\frac{42}{0.7} = 6$ Has 0 d.p's over 1 d.p.

Looking at

$$\frac{4.2}{7}$$

We have 1 - 0 = 1.

So the answer must have 1 decimal place!

$$\frac{4.2}{7} = 0.6$$

Looking at

$$\frac{4.2}{0.7}$$

Here we have 1 - 1 = 0

So the answer will have no decimal places

$$\frac{4.2}{0.7} = 6$$

Looking at the final example, we see a strange thing...

$$\frac{42}{0.7}$$

We see 0 - 1 = - 1 decimal places!

What can this mean?

Have a think first!

What this will mean is that we need to reduce the number of decimal places by 1.

So 6 actually means 6.0

So we need to move in the reverse direction (of course), and move the decimal place one to the right.
So the answer is 60.

$$\frac{42}{0.7} = 60$$

Also if we think about this in two other ways, we can confirm this.

1. If we were to subtract 0.7 from 42, since 0.7 is less than 1, it is going to take more than 42 attempts, *i.e.* 60.
2. If we multiply 0.7 x 60, using the Key Change Effect, we get 6 x 7 = 42 and this would become

420

42.0

= 42

That is, the number we were dividing.

So it works!

So we can think of negative decimal places as the number of places you have to move the point to the right. Positive decimal places, as in example 1, moves the point to the left.

Percentages - In A Minute

Percentages - In A Minute

As we noticed in Decimals - In A Minute, every word in maths has some Latin or Greek origin, as the subject is so old.

The word percentage, for example, comes from Latin/French. Do you know what ‘cent’ means?

Think of centimetres, dollars/euros and cents, or centipede!

It means ‘100’.

The ‘per’ part means ‘for’ or ‘out of’. So we have that a percentage means ‘out of 100’.

This is the first introduction to an important mathematical idea. We compare what we are measuring with the number 1.

This is technically known as ‘normalisation’ and although I avoid jargon in this series of books like the plague, this one (pun, ha ha) is so important that I will refer to it again and again now and in future books.

This idea of measurement compared to one crops up in a variety of contexts in maths and once we understand its importance and significance, it makes some advanced concepts easier to understand also.

So percentage means a measure out of 100. For example, let’s say you were doing a survey, asking 100 people if they liked chocolate.
Let’s say that 80 say yes, and 20 say no.

So what percentage of people like chocolate?

$$\frac{80}{100}$$

people liked it, so 80%.

In other words the percentage sign is replacing the $\overline{100}$ to say, ‘out of 100’.

What about if you got up the next day and thought ‘I’ll only ask 50 people today’?

So you ask 50 people, this time 35 say yes.

So what percentage of people like chocolate?

This time, it’s

$$\frac{35}{50}$$

But this does not mean 35%!

Here we have to scale it up to imagine we had asked 100 people, so since we only asked 50 (half of 100), we need to double that to make it 100.

In this case then,

$$\frac{35}{50} \times \frac{2}{2} = \frac{70}{100}$$

= 70%

Since we have to double 50, we must do that for 35 also (why?).

This is the first abstract concept in mathematics. Even though you haven’t asked 100 people, you’re taking a measurement as if you had. So this is the *normalisation* process which allows us to compare to yesterday’s survey. We can now compare like with like.

It is something that is used widely, for example, in recording the profits a company makes compared to another - 10% for one, is not the same as 10% for another, in terms of the actual profit they made, but they must be growing at the same rate if they make the same percentage profit.

Since 100% is considered to be a ‘whole’ as they say in school, that is to say, 1, we are measuring 2 things compared to 1, which means their original size doesn’t matter, it’s just how much that has changed compared to someone else. Like the example above, where 2 different companies will likely have a different amount of profit to each other last year, but this year they both grew 10% of that.

Let’s try a few examples of this:

Write the following survey results (or divisions) as percentages.

1. 1) $\frac{20}{25}$

2) $\frac{18}{25}$

3) $\frac{14}{50}$

4) $\frac{68}{200}$

5) $\frac{12.4}{10}$

You’ll note that some percentages don’t have a maximum at 100! You may have thought they did, for example, many people mock competitors on reality shows (and such like) who say they’re going ‘to give 110%!!!’, as if that’s possible. But it is possible, since what they mean is that they will give 10% more than they normally do. We can all do that. We’ll also see this concept in compound interest later, to make a calculation a little shorter.

Finding a Percentage of Any Number

Finding a Percentage of Any Number

Let’s talk about how this is done at school and then, in only a minute, you’ll learn how to do it in seconds instead. You’ll also learn how to do percentages in your head, and learn why the answer to every multiplication question you’ve ever done has the potential to be the answer to a percentage question.

First of all, what are we talking about?

What do we mean by find a percentage of a number?

What we mean by this is to find a proportion of it, some fraction of it, or even some multiple of that number.

For example, 50% of 80 means half of 80 = 40.

200% of 120 means double 120 = 240.

So the fact that we are trying to find a fraction or a multiple of a number by finding its percentage can be very useful to us.

However, let’s quickly look at the school method.

For example, let’s say you want to find 14% of 21.

The school method is to either say ‘divide 14 by 100, and multiply by 21’ or to break the calculation up into finding
10%, which equals

$$\frac{21}{10} = 2.1$$

And

1% which equals

$$\frac{21}{100} = 0.21$$

(from above, dividing by ten again) = 0.21

Therefore, 14% can be found by adding 10% and 4 x 1%
2.1 + (4 x 0.21) = 2.1 + 0.84 = 2.94

This is a bit slow. However, the method works.

My version is more suitable to someone who has read and learnt from 'Multiplication In A Minute'.

As above, I noted that we are trying to find a fraction or a multiple of the number. This is a useful idea because we realise that we can simply, to find a percentage of any number, utilise the 'Key Change' effect once again, and multiply the numbers together!

So (from Multiplication In A Minute)
14% of 21

=

  14
x 21

 294

Then DIVIDE by 100 (from Decimals In A Minute)

$$\frac{294}{100} = 2.94$$

So we instantly (almost) get the answer!

So to find any percentage of any number, we need to multiply them together and simply divide by 100.

Simple!

Calculate Percentages Mentally

You can see that this technique is so much easier than the usual school method!

Again we notice a pattern. All multiplications give the same answer, it's just a matter of magnitude.

If we multiply two numbers together we get a rectangle. If we divide by 100 we get what they are as a percentage of each other.

This means that multiplications that you've been carrying around for years in your head are also all the answers to every percentage question. Think about it - you've known these answers to the following for years, but no-one ever pointed it out.

What do I mean? I'm now going to challenge you to do some percentages in your head. I'll ask you a question, and in a few seconds you should be able to come up with the answer. These are questions that would normally confound the average school pupil.

Let's see. Remember, multiply and then divide by 100.

Ready?

So...

3% of 5?

Did you get it?

When I'm tutoring privately, sometimes students fall back on using the old school technique here. Did you? Or did you multiply them together (15) and then divide by 100? (0.15).

Did you do that? Do you see how easy it is? Why didn't they just say this at school eh?

Now you may have learned from your mistake if you stumbled on that one.

How about

4% of 6?

5% of 7?

6% of 8?

7% of 9?

Did you get them in a few seconds?

Ans:

0.24
0.35
0.48

0.63

All these 'times tables' are the answers to all percentage questions also!
Excitingly easy isn't it?
And this is true of every multiplication question you've ever done and ever will do. To get the percentage, just divide by 100.

Useful percentages

Useful percentages.

What about 5% of 3?

What is this?

Or 12% of 8?

You'll note that they are exactly the same answer, because the multiplication doesn't 'know' which one has the percentage sign after it. So it's the same result via the key change effect.

So this means we can do some more complicated looking percentages with ease, by switching them round to something easier.

For example, 50% of anything is half of it, as we saw above.

So 50% of 22 = 11.

What about 22% of 50? Instead of multiplying these together, we can notice that if we switch these around, to 50% of 22, we'll get the same answer! So it may seem a bit tricky, but of course 22% of 50 = 11.

What about 9.8% of 50? Easy, 4.9.

What about 12.26% of 50? Easy, 6.13.

We can also do this with 25%, 75%, 20%, 200%, any percentage that is also an easy fraction.

So, for 25% of something is a quarter of it.

So 8% of 25 will equal 2 (as we've switched them around!)
Or 4.8% of 25? 1.2 of course.

35% of 20? 20% would be one-fifth of it (i.e. divide by 5), so we're looking at 35/5 = 7.

And so on...

The goal is to achieve fluency and self-assuredness in something that only minutes ago seemed impossible. Impress your friends!

Before we looked at 14% of 180.

What's 180% of 14?

If you use the same method, you get....

What?

Of course, the same answer. In the multiplication, it doesn't know which is the percentage and which isn't! So you get 2 for the price of 1. This is something we can really use to our advantage.

Let's say we need to find 22.4% of 50.

Ouch, multiplying them together may seem a little tedious. Can we get the answer immediately, without multiplying them?

Remember 22.4% of 50 = 50% of 22.4 as the multiplication doesn't know which is the percentage! Of course, 50% of something is just half of it, so we can just half 22.4...

Ans:

11.2

Seemingly difficult percentages are made easy.

How about 21% of 21? Easy, we have a squaring technique for that one, available in Squaring & Area - In A Minute!

Or you could just do

```
   21
X 21
------
441
```

In the usual way.

Dividing by 100 to make it a percentage,
Ans:

4.41

Calculating Discounts

Calculating Discounts

When we go shopping and there is a sale on, the percentages and the new prices are usually already calculated for us. However, what if we're the ones deciding! So we need to be able to calculate the discounts on things we buy (or sell).

Here's a simple example, let's say that you want to buy some shoes, which are normally £30. Today there's a 10% discount. So the new price will be 10% off, which is £3

30 x 10 = 300
300/100 = 3

and we can subtract that from the Original Price
30 - 3 = £27

So we know the Sale Price.

But is there a faster and more intuitive way to achieve this result?
What is the percentage is a bit more complicated (say, 19%)?
What is the price is more complicated (say, £37)?

Is there a faster way?

Funnily enough there is.

What we can do instead is realise that we can do the reverse. When we're subtracting 10% off something, as in the first example, we are effectively saying, 'What is 90% of the Original Price?'. In other words, since we're subtracting 10% from 100% (the Original Price), we're left with 90%.

So in fact, we could have found 90% of 30

  90
x 30

2700

Divide by 100

$$\frac{2700}{100} = 27$$

= £27

And found the final price much more quickly.

This technique really comes into its own with more complicated situations.

For example,
Let’s say you want to buy a £76 pair of shoes, but today there’s a 22% discount.
If we do this the traditional way, it’s a bit complicated, as we’ll end up with a nasty subtraction.

Viz:

22% of 78

22 x 78 = 1716
Divide by 100

$$\frac{1716}{100} = 17.16$$

THEN we have to subtract that from £76...

So we have

76.00

- 17.16

....which is unpleasant!

How about we reverse finding the percentage and then subtracting to:
Subtract, then find the percentage.

So subtraction wise, subtract 22 from 100.
100 - 22 = 78%
(since we want 78% of the Original Price).

So we find 78% of 76 using the usual method

$$\begin{array}{r} 78 \\ \times\ 76 \\ \hline \end{array}$$

$59_{10}2_{4}8$

Divide by 100 and we get

£59.28

Much more intuitive, much easier.

So instead of finding the percentage and then subtracting our answer, it is much EASIER to subtract the percentage from 100%, the whole cost, and then find our new percentage!

This way we avoid a nasty subtraction, every time.

Instead of multiplying by 22%, using 78% instead is known as using a MULTIPLIER.

Here are a few to try

1. A pair of shoes are £62 with a 34% discount, what is the Sale Price?
2. An iPod is £180 with a 22% discount, what is the Sale Price?
3. A car is £8050, with a 11% discount, what is the Sale Price?

Ans:

1. £40.92
2. £140.40
3. £7,164.50

Real Life example - using the Multiplier in reality

Real Life example - using the Multiplier in reality - ADVANCED

I was once looking to rent out a house of mine. An agent offered to do this for me, and wanted to take a fee.

To calculate how much it would cost, and how much I'd have left (the important part for me!) he did in the following way.

The rental was £500.

The fee was 12.5% of the rental.

So that's

12.5

x 500

62.50

On top of that I had to pay VAT, which is 20% of the agent's fee for his services.

So he then calculated

62.50

x 20

£12.50

So in total he'd have to charge £62.50 + £12.50 = £75

He then subtracted that from £500 to give
£425!

After all that work, I asked him if he wanted a way to calculate the final rental without these calculatory gymnastics.

Can you figure out how to do it in go?

First of all, we can spot something special about the VAT calculation.

The agent calculated 20% of the amount he was going to charge me (£62.50), but what he might have done is calculate 20% of his percentage charge (12.5%).

12.5

x 20

2.50%

As we're finding a percentage of a percentage, the answer is a percentage too!

So what we're saying here is that VAT is an extra 2.5% to add onto his 12.5% agent fee.

So his total charge is 15%, including VAT.

So he could just find 15% of 500 (£75) and subtract that.

Even better though would be to use the MULTIPLIER method above, and subtract that 15% from 100% = 85%.

So that

 85
x 500

£42500

Dividing by 100

= £425

And in fact, from then on in, he could just use 85 for ALL CASES, no matter what the rent was.

He couldn't believe it!

Reverse Percentage

Reverse Percentage

As the name suggests, we're going to do the complete reverse. Most people don't understand this one straight away, so don't worry to much if you don't! However, there is a reason why I kept using 'Original Price' and 'Sale Price' above.

To calculate a discount, we go from the Original Price to the Sale Price.

Reverse percentage situations go from the Sale Price to the Original Price!

In other words, this is somewhat algebraic, we have the answer, but what was the question?

So we know the final price of the item, but what was it before?

So, to start with an example,

Let's say that you bought a jacket for £54 (the Sale Price). This INCLUDED a 10% discount. What was the price before the discount was applied (the Original Price).

That is, we're looking for a figure more than £54.

The common thing people do here is to find 10% of £54 and then add it on,
10% = £5.40,
Original Price = £59.40

However, this can't be right, as if we find 10% of £59.40 and subtract it from here, (or, as above, find 90% of £59.40) we get
90% x 59.40 = £53.46

which is not £54, the original amount we started from.

For this to work, if we remove the 10% discount from the Original Price, it must equal £54.

So what has gone wrong?

First of all, as we noted above, removing 10% from the Original Price means that the Sale Price now equals 90% of the Original Price.

So in fact, £54 does not equal 100%, from which we can find 10% and add it on, but only 90%

I.e.

90% = 54
100% = ?

So we need to go from 90% to 100%.

To achieve this, instead of dividing by 10 to get 10%, or multiplying by 10 and dividing by 100, we need to actually find 10% of 54 by dividing by NINE!

Since $\frac{90\%}{9} = 10\%$

Therefore $\frac{54}{9} = 6 = 10\%$

And as 10% = 6, we can add that on to £54, to give £60 or we can multiply 10 x 10% = 100% = 10 x £6 = £60

Either is fine.

More Advanced Explanation

Effectively we have,

$$\frac{90}{100} = \frac{54}{x}$$

and we need to figure out x.

If we make x the subject, first by using the reciprocal

$$\frac{100}{90} = \frac{x}{54}$$

Then multiplying

$$\frac{10}{9} \times 54$$

we get x = £60

Let’s try another example:

Imagine the Sale Price is £64, but this INCLUDES a 20% discount. What was the Original Price?

Have a go yourself...

So this time

64 = 80%

So dividing by 8 to get 10% = 8.

Therefore Original Price was £80.

Or using the algebra advanced method

$$\frac{80}{100} = \frac{64}{x}$$

Flipping

$$\frac{100}{80} = \frac{x}{64}$$

Multiplying

$$\frac{100}{80} = \frac{5}{4}$$

by 64

gives £80!

Compound Interest

Compound Interest
- Or How to Get Rich or Poor and Die Tryin'

Imagine you have some savings in a bank.

Let's say £1000.

What happens if you leave it there for a year?

At the end of the year, you are awarded interest from the bank. This is calculated via a percentage.

For example, let's say the bank offered 6% interest.

At the end of one year, the interest you would be given is
  1000
x 6

60.00

So your new balance would be

£1060.

What happens after another year?

There are two possibilities. If the bank uses 'simple interest' calculations (which, in reality, none do), you would be awarded another £60, as it is 6% of the original amount.

However, what happens in reality is that you are given 6% of the NEW amount of £1060.

So that's

1060
x 6

63.60

A little bit more than £60.

What happens is that not only do you get your 6% of £1000 (£60), but you also get 6% of that interest of £60 you earned last year (£3.60).

So you're earning interest on your interest, as well as your starting amount, which is known as the Principal.

Once you start earning interest on interest, that is when you see significant gains in your savings.

The amount you have in total grows larger as it grows larger! Because you are always calculating a percentage of a growing amount, the interest grows each year. As a result, your percentage calculation accelerates more quickly and surprisingly large growth can be found.

For example, as a thought experiment, how long do you think it will be before your savings double from £1000 to £2000?

With simple interest, it would be

$$\frac{£1{,}000}{60}$$

years which is 16 or so years.

However, with compound interest, it will take only 12 years to double. This follows the **RULE OF 72**.

The Rule of 72 is a financial formula which allows you to calculate how long an investment will take to double, given the interest rate.

So for example, if your interest rate is 6%, then

$$\frac{72}{6} = 12$$

So you divide 72 by your interest rate and the answer is how long it will take.

It's a rough and ready reckoner to give you an idea of how long it would take, assuming a constant interest rate and no further deposits or withdrawals on your account..

As you can see, the higher the interest rate, the sooner it doubles. For example, if you could get 12% return on your investment, this would take 72/12 = 6 years to double.

This is why the world's stock markets, banks and private investors fight tooth and nail to get the best deals and investments they can. The better the return, the sooner it will double.

What Happens After Doubling The First Time Leads To Great Wealth

Let's look at our original investment of £1000 and imagine it is left untouched for 12 years at 6%.

As a result, it will be equal to £2000.

How long will take to double to £4000?

Amazingly, only 12 years again.

How long will it take to double to £8000?

Again, 12 years!

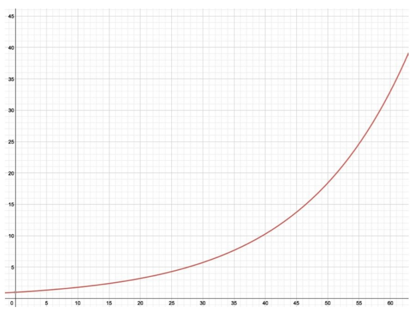
12 years later, it will be £16000.

12 years later, it will be £32000!

So, if you have £1000, and can wait 60 years, you'll end up with 32 times as much as you invested!

That's incredible growth.

You can see this in the graph below.



So the earlier the start, the better. So if you're 1 and you're reading this, start saving!

However, in reality you will probably add to your £1000 through life as you go and although this is a more complicated calculation, you can see that it will significantly turbo boost your savings and it will grow extremely well.

Although it is nice to have 32 times your investment, remember this would be eroded by inflation (for the same reasons, as this is compounded every year as well) and it may be that life is more expensive in 60 years time anyway! So you have to stay ahead of the curve.

That's why pension funds tend to invest in the stock market, rather than savings as they have the potential for higher returns.

So the higher the interest rate, the faster the rate of growth of your investment.

Compound interest has an incredible effect on growth, and indeed nature follows the same growth law in many different species.

One example of incredible growth would be this idea.

Imagine if you had the 'infernal' DeLorean time machine that featured in Back To The Future (1985) and you went back to 1 A.D. to invest £1 of gold in the Bank of Rome. Let's say you agree a 6% interest rate.

You jump back in your DeLorean and come back to today (this calculation will be based on 2013).

How much do you think your investment will be worth?

A. £100,000
B. £10,000,000
C. £1000
D. None of the above.

The answer?

Well, let's figure it out. How many years will it take to double from £1 to £2 at 6%? Of course, we have seen from earlier that it will be 12 years.

So in the year 13 AD, your investment will be worth £2. In 12 years after that, it will double again to £4. And so on every 12 years.

So really, the question is how many times does your investment have the chance to double in 2012 years?

To find that, we just divide 2012 by 12 =

$$167\frac{2}{3}$$

So let's say 167.

So doubling that many times is the same as multiplying by 2 that many times. In other words, 2^{167} x £1.

What's 2^{167}?

1.87×10^{50} (This is ‘Standard Form’, to be examined in a later book, Standard Form - In A Minute).

In other words, nearly 2 followed by 50 zeroes, or

£200,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000

Wow!!

So the answer was D.

Did you imagine it was going to be that much? That is the power of compound interest.

Chess and how it was invented

There's a story that the inventor of chess was allowed to showcase his invention to his Emperor.

The Emperor quickly became addicted to the game, and wanted to reward the inventor. He offered him anything he wanted.

The canny inventor asked for 'One grain of rice for the first square, two for the second, four for the third and so on...'. The Emperor was amazed 'Is that all you want?'.

'Yes.' replied the inventor.
How much rice did the inventor receive?

Let's look at the 64th square on the board. On that square, there will be 2x2x2x2x2x2....63 times. This will be the number of grains on this square alone. On the 63rd square, will be 2x2x2x2...62 times and so on.

What is 2^{63}?

9×10^{18} grains of rice. Again, this is 9 followed by 18 zeroes.

And the 63rd square would have 4.5×10^{18} grains. And so on. This would have been more rice than was in the entire country! And just for two squares!

History records that the Emperor decided to cut off his head instead! Much cheaper.

The Trouble With Tribbles

In ***Star Trek***, there were some cute and furry animals called Tribbles. In one episode they invade the ship because they reproduce so quickly. They have a litter of ten every 12 hours. If one gets on the ship, how many Tribbles will there be after 3 days?

Spock claims he knows exactly that there are 1 771 561 already. Is he correct?

It sounds too large!

After 12 hours there will be 11.
After 12 hours, each one of these will produce ten, meaning there will now be 110 plus the original 11 altogether = 121.
After another 12 hours, each of these will produce ten, making 1210 plus the original 121 altogether = 1331.
After another 12 hours, each of these will produce ten, making 13310 plus the original 1331 altogether =14 641.
After another 12 hours (!), each of these will produce ten, making 146410 plus the original 14641 altogether = 161 051.
For the final 12 hours, each of these will produce ten, making 1610510 plus the original 161051 altogether = 1 771 561!

How Spock did it was to just realise that every 12 hours the amount of Tribbles grew by 1100%. That is to say always 11 times more than previously, as each one made ten, plus the original 1. A growth rate of 1100% comes from r being 10 in a compound interest formula, which we will see in a later chapter. For now,

Essentially we have

$$NumberTribbles = 1(1+10)^6$$

In other words, just

$$11^6$$

= 1 771 561

Incredible growth. Of course Spock was correct.

How many will there by after another 12 hours?

How many were there after 18 hours?

Ans:

Multiply by 11 again.

1 771 561 x 11

19 487 171

Hence the rush to do something!

After 18 hours, it will still be the same as after 12 hours, so it is

11.

How could we figure out when the number reaches any number? Let's say we knew the capacity of the ship for Tribbles was 200 million. How would we know when that was without having to calculate the value for every 12 hours?

We will see how to do this in Logarithms - In A MInute & Functions - In A Minute.

Finally, the numbers that have appeared in this calculation are very interesting for two reasons.

11
121
1331
14641
……….

First of all, they are formed by multiplying 11 repeatedly. Doing this generates a triangle of values where it seems that the middle value is generated by adding the two above it.

1   1
1 2 1

That's interesting! This is from using the method of multiplication where you can see how many additions are required as you work along, as we saw in Multiplication - In A Minute, and will examine again in How the InAMin Method Works.

If we continue…

1 2 1
13 3 1

And so on.

Here we are generating a triangle of values which is famously known as Pascal's triangle. This has many, many applications, and we will look at this again in Binomial Theorem - In A Minute.

The Flip Side

The Flip Side

This is great if you have savings and you want to become a millionaire, and you're prepared to wait.

But what if you borrow the money?

If you borrow the money from a bank, you're going to be subject to the same interest calculations that they so generously gave you if/when you had money saved with them. The only problem is that if the bank was lending money out at the same rate of interest that it paid savers, it wouldn't make any money itself. In real life, they may charge a few times more.

For example, you may get a savings rate of 5%, but a loan rate of 10%!

Common mistake - How much more percent is 10% than 5%?

Most people will think, 'Well, it's only 5% more, can't be that bad'. In fact, it is 100% more!

Why?

Because imagine you saved and borrowed £1,000 at the same time.

If you did that you would earn £50 on your savings, but have to pay £100 on your loan - twice as much. Or 100% of 50 extra than you received on your savings. So small changes in percentages can make a big difference!

So let's say you've been offered a loan rate of 10%. Should you take the loan?

Well if you calculate the doubling time, you'll see it will be $\frac{72}{10}$ or just over 7 years.

If you take the loan over 7 years, you're going to be looking at a repayment of around double what you borrowed!

And what did you borrow the money for?

If it was for clothes, a car, a holiday.... these are all things that DEPRECIATE hugely (especially a holiday!), so you end up paying double for something you can't really sell on for a decent price, 7 years later.

In reality, 10% is a fairly low interest rate for a loan. Some credit cards can be 20%. Store cards such as in big chain stores with luxury items, can be 29.9%. If you have poor credit, *i.e.* you don't pay your bills regularly, and you want a credit card, you may have to get one that offers around 40-50%. Ironically, you're going to find it harder to repay these types of loans and therefore, even harder to prove that you are 'creditworthy' and able to be offered lower rates of interest in the future.

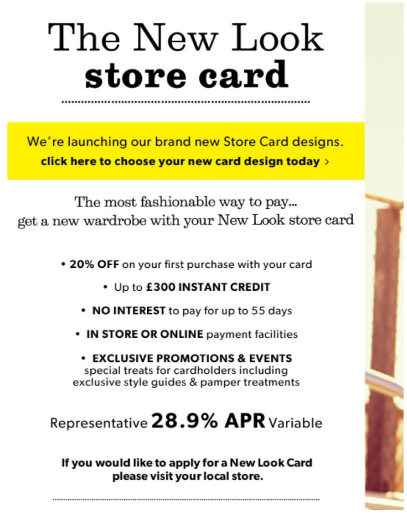
Let's take an example of a 20% credit card

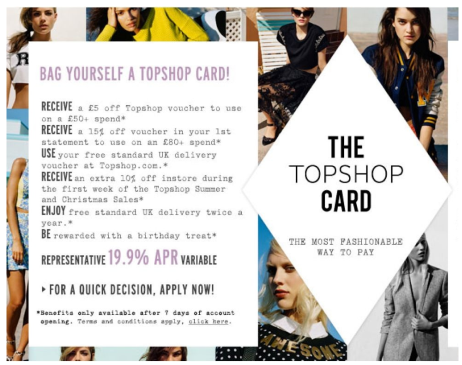
$$\frac{72}{20} = 3\frac{3}{5}$$

If you don't repay this card at all, within 3 and a bit years, you'll owe double what you borrowed!

This is why banks love credit cards! They have lots of lovely tempting offers, such as a personal card, a gold card, 'cashback' deals, reward points, air miles...

Teen clothes shops are great at this. They have customers who are keen to be fashionable and usually don't understand how compound interest works. That's why they offer 'pink' store cards and even have a 10% discount on your first purchase! But in reality they offer interest rates of 20-30% and are really selling you money as well as clothes.





So the power of compound interest can make you very rich, or very poor. It's up to you!

Fast calculation of compound interest

Fast calculation of compound interest

If we want to calculate compound interest, we find it's a bit repetitive. It's essentially the same step over and over for as many years (or times) as we compound. This kind of calculation is more computational, and a software program or a calculator is better suited to this kind of repetition.

As long as we know what we are doing, it is ok for the computer to do it for us!

So if we want to find an answer quickly, we can use the compound interest formula.

This is

$$A = P(1+r)^t$$

Where A = final amount, P = principal, or starting amount, r = rate of interest, expressed as a decimal, and t = time in years
So for example, a loan or savings of £5000 at 12% over 3 years would be written

$$A = 5000(1+0.12)^3$$

Here we have a calculation we've not seen before. Technically known as a 'binomial expansion', which in English, means 2 numbers in a bracket multiplied. Binomial expansion is an A-level topic in the UK, and is not in the scope of this book. For this calculation only, however, you are able to add the 1 and r (0.12) together.

$$A = 5000(1.12)^3$$

Following BIDMAS, we first calculate $(1.12)^3$ and then multiply by 5000.

In reality, the indices calculation is telling you the growth over that period, which is similar to the idea of the rule of 72. The actual starting amount doesn't really matter, all that matters is the growth rate.

In the earlier examples we saw that you could get up to 32 times your starting investment. In other words, the large part of that

($\frac{31}{32}$ of it)

is interest. So it almost doesn't matter what you start with. The 'DeLorean' example started off with a pound!

Depreciation - the reverse of Compound Interest

Depreciation - the reverse of Compound Interest

Early I referred to depreciation of the things you buy. What this means is that they are worth less (or sometimes, even worthless) some time after you've bought them. Common examples are cars. They are usually a depreciating item unless you buy a limited edition Aston Martin or something.

For example, let's say you buy a car for £9,000. And that this car will depreciate in value by 10% each year.

That means after year 1

9000
x 10

900

It will lose £900.

The year after.....

8100
x 10

810

It will lose another £810, bring the value down to £7,290.

After another year
7290
x 10

729

Bringing the value down from £7290 to £6561.

And so on.

Can you see a similarity with discounts in these calculations?

You might realise that depreciation is a bit like a compounded discount. Each year, a possible purchaser gets a 10% discount on the purchase of your car.

So you may recall that a better way to calculate that would be...?

Yes, instead of using 10% each time, use 100 - 10 = 90% each time. You will see in the above calculation that it suddenly makes sense and all the answers look very familiar as they are multiples of 9!

(In fact, they are $9^2, 9^3, 9^4$ ...)
So again

  9000
x 90

8100

  8100
x 90

7290

  7290
x 90

6561

and so on...

The fast method

We can also use the compound interest formula here, but in reverse, since depreciation is the reverse of compound interest.

The formula is

$$A = P(1 - r)^t$$

Notice we have a minus.

So it would be for the above calculation

$$A = 9000(1 - 0.1)^3$$

$$A = 9000(0.9)^3$$

= £6,561

Which you can see is the same as multiplying by 90%, 3 times over, as above.

Writing A Division As A Percentage

Writing A Division As A Percentage

As in the original introduction to percentages, on surveys on chocolate(!), I used some simple examples with round numbers to turn our survey into a percentage, such as

$\frac{40}{50}$ and so on.

For this method to work, we simply use the idea of making the 50 out of 100 instead, so as a result, we need to multiply 50 by 2. We then must do this for 40 (why?).

[If you're not sure why, see 'Fractions - In A Minute']

So

$$\frac{40}{50} \times \frac{2}{2} = \frac{80}{100}$$

= 80%

So this method is absolutely fine for simple examples with nice round numbers.

However, what about trickier examples, such as

$$\frac{24}{80}, \frac{31}{72}, or \frac{67}{84}$$

or even

$$\frac{128}{23}$$

How would we write these as percentages?

Bearing in mind the third rule of maths here (which is?)

We need to do the reverse.

Up to now then we have entirely been finding a percentage of a number, and that meant we had to

MULTIPLY

then

DIVIDE BY 100

So to completely reverse that would mean

MULTIPLY BY 100

then

DIVIDE

So, looking at $\frac{24}{80}$ as an example,

$$\frac{24}{80} \times 100 = \frac{2400}{80}$$

So we have multiplied by 100.

Then just perform the division (as in ‘Division - In A Minute’)
We need to do

$$\frac{2400}{80} = \frac{240}{8} = 30$$

Therefore $\frac{24}{80}$ = 30%

Example 2

$$\frac{31}{72}$$

Multiply by 100

$$= \frac{3100}{72}$$

Then do the division

From Division - In A Minute you'll recognise this doesn't 'go in' exactly, so there'll be a remainder. Using the technique that is taught in that book for so-called 'long' division, we get

$$43\frac{4}{72}$$

$$43\frac{1}{18}\%$$

There's no need to write this as a decimal.

Example 3

$$\frac{67}{84}$$

Same technique!

$$\frac{6700}{84}$$

$$67\frac{64}{67}\%$$

[Very near to its square number, funnily enough! We'll see another way to tackle a question like this in 'Squaring - In A Minute'.]

In this case, we can see the advantage of not using decimals in division, as it's preferable not to have to figure out what that fraction would be as a decimal.

And finally,

Example 4

$$\frac{128}{23}$$

$$\frac{12800}{23}$$

$$= 556\frac{12}{23}$$

$$= 556\frac{12}{23}\%$$

Introduction to Negative Numbers

Introduction

Up to now in the books preceding this one, all the numbers we have considered are positive. According to the Third Rule of Maths however, there must be a reverse of this situation.

These are negative numbers.

We are going to examine in this book where they occur in everyday life, how to add and subtract, multiply and divide them, and look at an example of what to do with what I might call 'double negatives', where we see more than one.

Everyday life

Everyday life

First of all, where do we see negative numbers in everyday life? In fact, not in many places!

Here are some examples most people think of:
Temperature
Debt/Overdraft
Financial losses/shares going down in value In subtraction

I write in subtraction there, because a sum like
5 - 4 = 1

Could be considered to be

5 + (- 1) = 4

However, most intuitively for many people is to think of negative numbers in temperature, since they usually spoken of every day (especially in England!) and we also have our fair share of them in the Winter.

So let's say it is

- 7 degrees

and it gets warmer by 4 degrees. What is the new temperature?

- 3 degrees.

Why? Because - 7 + 4 = - 3

In other words, an increase in temperature (getting warmer) means we're adding. You might want to think of the thermometer (or number line) scale to help with this.

What if it is

- 5 degrees

and it gets colder by 3 degrees?

It will be

- 8 degrees.

Since we're doing the reverse, we'll head in the opposite direction.

Again we can think of that thermometer.

Let's try one more.

Let's say it is

4 degrees

and it gets colder by 9 degrees, what is the new temperature?

It will be

- 5 degrees.

A clever way to do these questions would actually be to turn them around, and they will give the same answer. Sometimes this can help.

As in the above example, if we looked at

4 - 9 as

9 - 4

numerically, we'd get the same answer (5). But we just need to remember to make it negative (- 5).

If we look at the first example, - 7 + 4, turning that around
7 - 4 = 3

gives us, numerically, the right answer, again we must recall to make it negative.

Multiplying Negative Numbers

Multiplying Negative Numbers

So let's look at multiplying now.

As we've seen so far, and should be extremely familiar with (!), Multiplication is Just Addition, so in multiplying negative numbers, we are going to see this borne out.

What's

4 x - 5

You may recall from school (or not) what the rules are here, but of course I want to know and for you to know, why those rules are there at all.

You may have thought, it's either

20
or
- 20

But which?

Let's use the first rule of maths, MIJA, to figure it out.

Since we're being told to add - 5 four times, let's do that.

- 5 - 5 - 5 - 5 = - 20

If we add - 5 four times, as above, we get - 20.

Why? Imagine it like this. Let's say the temperature was 0 degrees. Let's say after every hour, it decreases by 5 degrees. So after 4 of these hourly drops, the temperature will be - 20 degrees.

That's it.

In general, the rule is

"A plus times a minus is a minus."

So now we now that rule, which is a plus. Or a minus. Ha ha.

What about this situation?

- 4 x - 5

What will this be?

We can look at this in a couple of ways, analysing what it means, or using the rules of maths.

If we do the latter first, we can say that this is the reverse of the previous example. The 4 has become a - 4, so we are adding - 4 times.

That's quite a weird thing to think of. We have always, up to now, thought of adding a positive number of times, and it is strange to do the reverse of this. In fact, it is physically impossible to imagine.

However, it must have a meaning (or answer) because the rules say so.

If we look at it from the other point of view - what is its meaning? - we can get a better idea of this.

Let's imagine that the example

4 x - 5 can be thought of in the following way.

You have a mobile phone contract, and each month you are charged £5 for its use.

This is all fine, expect that you have nothing in your account at the start of this process! So each month, you go £5 overdrawn, or after four months, your account would be
- £20.

You can think of these bills as being money taken away from you each time.

Let's say that your boyfriend/girlfriend/husband/wife/mother/father/son/daughter hears about this, and feels somewhat concerned/worried/terrified/extremely angry.

So they offer to pay the next four months for you.

So not only are they paying your bills (money being taken away from you), they will do it for four months (taking away the number of months you have to pay).

As a result of this arrangement, how much money will you save?

If you're not paying your £5 bill a month (- 5), for four months (- 4), you'll save

£20.

So here we see that

- 4 x - 5 = 20

Or in general,

"A minus times a minus is a plus."

So now we know that rule. Which is a plus. (Okay, already done that joke).

At school they have a rhyme which goes

"A Minus times a Minus is a Plus,
The REASON for which, we do not discuss."

Which you'll probably find is true. Students do occasionally ask why this is, but teachers don't know do they? I once worked in a school where one teacher came to the staffroom and starting to complain that the students were asking questions and one was, why was a minus times a minus a plus? I empathised, and then asked, as if curious myself, 'Yeah, why is it?' and she tried to evade the question. I asked again and she admitted she didn't know.

Why are we teaching it if we don't know why it is? And then complaining if the students ask?

Division of Negative Numbers

Division of Negative Numbers

What about

- $\frac{-20}{-4}$

Here we see a minus divided by a minus.

What do you think the answer will be?

Neatly, the rules for division are the same as for multiplication, so for this question
A minus divided by a minus gives a plus.

So

$$\frac{-20}{-4} = 5$$

This is because if we multiply, from the algebra trick (see ‘Division - In A Minute’)

- -4×5

we get

- -20

For the situation when the signs are different,

$$\frac{20}{-4}$$

This equals

- -5

as we have a plus divided by a minus (or it can be a minus divided by a plus) and this will give a minus.

Plus

- $-4 \times -5 = 20$
-
- as above.

Final Problem

Final Problem

What if we have

$-7-(-2)$?

Here we have two minuses beside each other. What shall we do here?

If we think back to 'Multiplication - In A Minute' we saw that Brackets Mean...?

Multiply.

So we see above that we have a bracket sign, and outside it a minus sign. Inside the bracket we have another minus sign.

Since a minus times a minus is a plus, this now becomes
$-7+2$

Which then becomes

- -5

This is a situation that comes up in Standard Form and Algebra, and will be covered later in more depth.

But in algebra, let's say

$x=-2$

and we have an expression that says

- $-7-x$

If

$x = -2$

Then

- $-7 - x = -5$

Introduction to Squaring & Area

Introduction

In ‘Multiplication - In A Minute’ we saw that when you multiply 2 numbers together it will give you a shape.

Can you recall what it was?

What information did we find out about the shape?

These ideas are about to be extended slightly further as we look at why...

Squaring is called ‘Squaring'

Squaring is called ‘Squaring’

Why is squaring called ‘squaring’?

We saw in ‘Multiplication - In A Minute’ that when we multiply two numbers we get a shape.

It comes as a surprise to many to find out that the shape you get is a rectangle. However, if the numbers are the same, we get a special kind of rectangle, called a square.

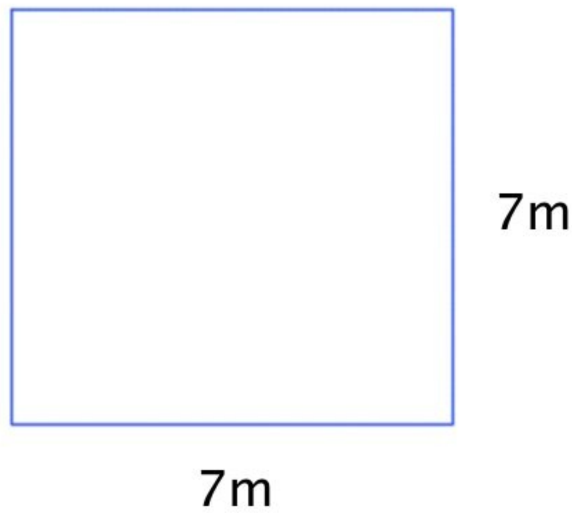
And that’s why squaring is called squaring.

Because you get a square.

So if we have

$7^2 = 7 \times 7 = 49$

We have a square shape.



What information do we get about this square?

You may guess from the title of this book!

However, as with rectangles, when we square a number we also found out the area of the square.

So if the square was 7m long and wide, the area would be $49m^2$.

If we had a rectangle that was 3m x 4m (you see now why rectangles are described in this way?), its area would be $12m^2$.

So the unit of area is m^2.

Why?

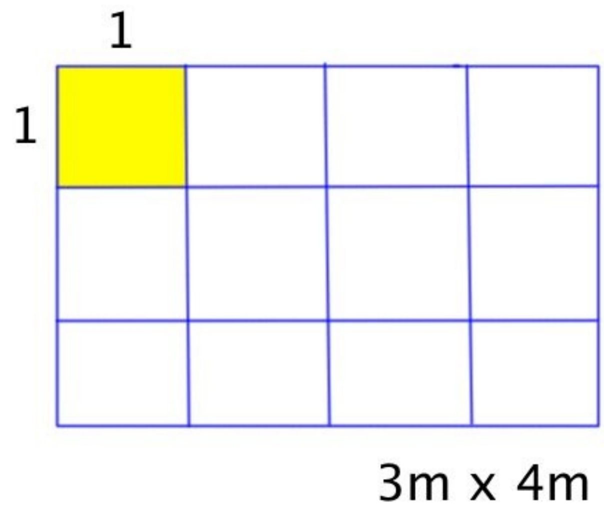
Why is the unit for area m^2?

Have a think about this one...

If we look at the rectangle again,

It might be a bit confusing. Why do we have a measure using a square (m^2) when we're finding the area of a rectangle?

Well, if we divide the rectangle up into 1m lengths, we see this



And each small square is the size 1m x 1m. They're literally a metre square. And how many fit into the rectangle?

12.

So area is the measure of how many 1 metre squares you can fit into it. So it doesn't matter what the shape is, all that matters is how many squares you can fit in.

In fact, it doesn't have to be a metre - it can be any measurement; kilometre, mile, inch... but metres are most used in science and engineering.

As far as the universe is concerned, there is no right measurement. As we shall see later.

The Squaring System

The Squaring System or
How to Square Any Number In Your Head in Seconds

Now we know what squaring is, we find it very useful to be able to square numbers quickly. Numbers that end in 5 especially.

I discovered this method when I was 17, while I was messing around with squares in my head, as you do. I've used it non-stop since.

To start with, let's try 21^2

We could use the normal method, as in 'Multiplication - In A Minute'. This would give

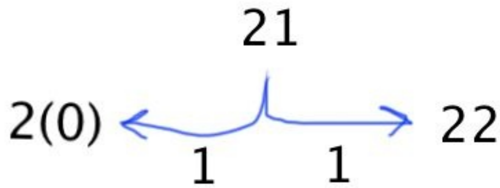
```
  21
x 21
-------
 441
```

That's great. However, sometimes it is more convenient to use your head, and sometimes easier to use my system, which is outlined below.

21^2

First of all, find the nearest ten.

This is 20.



This is one away from 21. Using symmetry, go the reverse direction the same distance.

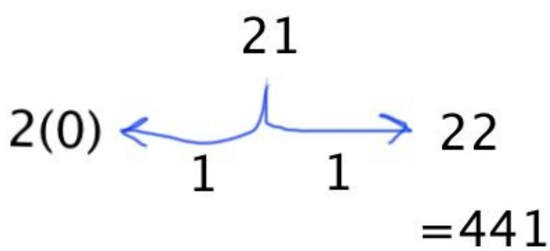
This takes us to 22.

Here we are now practically finished.

The point of this system is that we can ignore the 0 on the 20.

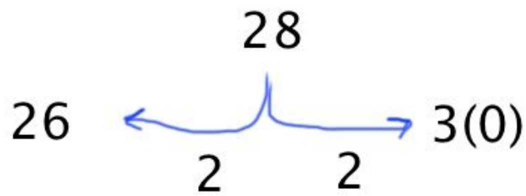
All we have to do is multiply 1 x 1, or square, 1.

And then we multiply 2 x 22.



So we have 441!

Let’s try 28^2



This time, the nearest ten is 30. This time it's 2 away. Going in the reverse direction gives us 26.

So we now square 2, and multiply 3 x 26 which gives
784.

Easy!

Next.

To impress your friends, try

97^2

Nearest ten is 100. Ignore that zero again to give 10.

So we have

$94 \text{ x } 10 + 3^2$

9409.

Easy again!

This works for any number.

However, these are the difficult ones. The easy ones are the squares, as I mentioned above, that end in 5.

So let's look at 35^2

Which is the nearest ten here? It's a delightful coincidence that we have two nearest tens here. So we can ignore both zeroes.

So we get

3 x 4 + 5 x 5

1225.

We see this for all numbers that end in 5. Just multiply the number you have by its next, add 25 and you have the answer.

For example, 75^2 will be
7 (from 70) x 8 (its next) + 25

5625.

95^2

90025

45^2

2025

15^2

225.

All delightfully simple. Amazing they didn't show you this at school, eh?

Finally we can then apply the 'Key Change Effect' from 'Decimals - In A Minute'. Remember, the answers will always be the same, no matter what.

So

4.5^2

Will yield

2025

But we just need to shuffle the decimal places accordingly.

What will the answer be? Where will the decimal point go?

As we squaring 4.5, we are doing

4.5 x 4.5

And adding the number of places (since Multiplication is Just Addition) we have 2 places to insert.

So the answer is

20.25

Plus we're multiply numbers that are between 4 and 5, so we're expecting something between 16 & 25.

This will work for any, so

0.21^2

will be

0.0441

And 0.000097^2

which probably seemed impossible not too long ago..?

Just

9409

But with the right number of places. How many?

12!

So it will be

0.000000009409

Extending the Square System Further

Sometimes when I tell my students that I can square numbers in my head, they instantly give me a challenge to prove me wrong!

Usually, it goes fine.

A couple of times, a student has asked a question that I initially thought I couldn't do, but actually, quickly realised I could.

The first was

'Ok then, what's 175^2?'

While the student was getting up to fetch her calculator, I gave her the answer.

30 625.

How did I do this?

My normal method would work fine, but I would have to do 17 x 18, which is ok. I could do $17^2 = 289$, and add on a 17 = 306. Then add on a 25... 30 625.

But this wouldn't be very quick.

How do you think I could improve this?

What I quickly realised was that, because I knew the algebra behind this method (also figured out, eventually) I didn't have to go to the nearest ten.

That’s the beauty of algebra!

I just needed to go to some easy point, and the same distance back again. I realised for this question, 200 was excellent.

So I went 25 to the right to get to 200.

And 25 to the left took me to 150.

So I had

150 x 200 + 25^2

Knowing the Key Change Effect, I realised the first multiplication was just 15 x 2 = 30, plus three zeroes.

I could quickly calculate $25^2 = 625$

This give 30 625.

My student was pretty impressed!

Another example was when I mentioned it, a student said

‘Ok then, what’s 1024^2 ? ’

Again I thought, ‘Er, that’s way too big!’ but I had learned from the above experience that I didn’t have to use the nearest ten.

So what do you think I did here?

You might have realised that once again I got lucky. He had chosen a number close to 1000. So I used that, giving
1000 x 1048 + 24^2

So all I had to do was find 24^2 and add it on to

1048___

Which is?

So I got

1 048 576

Again this took 8 or 9 seconds, and he thought I'd made it up. So he pulled the calculator out to find it was correct.

Incidentally that number might be familiar to you if you know about computers.

1024 is the number of bytes in a kilobyte (2^{10}, *i.e.* 2 x 2 x 2 x 2.... ten times)
And

1 048 576

the number in a megabyte. (2^{20} *i.e.* 2 x 2 x 2 x 2... twenty times)
Try some squares yourself

31^2

42^2

89^2

71^2

Squares ending in 4.

The actual square that made me see this pattern existed was
14^2

I noticed that if you subtract 4^2 from its answer, 196, you get 180. Which was what you would get if you took and added 4 to 14, and multiplied them. So I kind of reverse engineered it.

Interestingly, numbers that end in 4, or are 4 from the ten above, are a little more tricky.

When you do them, do the 4^2 first, and immediately write or remember 6 carry 1, as you have to add 1 to your answer.

For example, in the above you have to add 1 to 1 x 18 to get
19(6)

So, as above try

24^2

This will be

2 x 28 + 16

So it will be

$57_16 = 576$

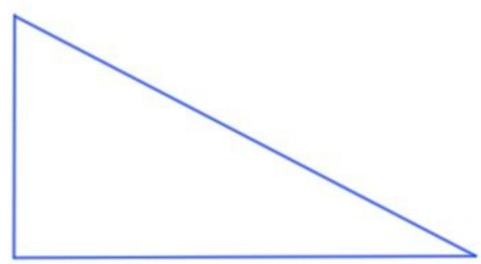
Calculation of Areas

Calculation of Areas

We’re now going to look at some easy ways to calculate some areas of different shapes.

We already know how to calculate the area of a square or a rectangle! Every multiplication you’ve ever done has effectively been the answer to that.

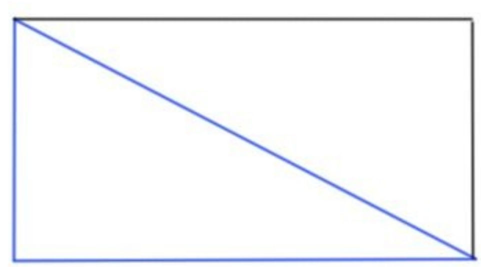
Let’s look at a triangle



This is nice and simple.

A triangle is always half a rectangle.

So we just find the area of the rectangle as normal, and halve it.

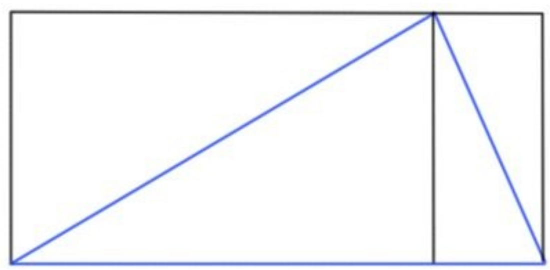


The length and width must be perpendicular (at right angles, or 90 degrees to each other), just like with a rectangle, for this to work.

As a result, the ‘school’ formula for this is base x height, a way to keep the measurements perpendicular.

In any case, we are just finding the area of a rectangle, then halving.

It doesn’t matter if the triangle looks a bit strange either, this will always work. From the image below, you can see that it will be half of 2 rectangles, which is still half.



So

Area of a triangle

$$A_{triangle} = \frac{b \times h}{2}$$

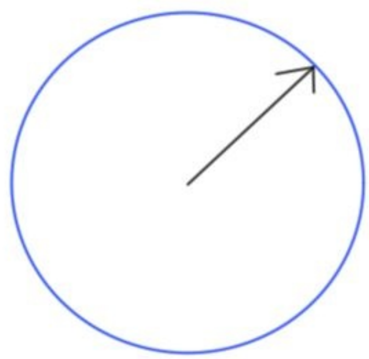
Circles.

“You can’t fit a square peg into a round hole.”

With circles we have a small problem, because we measure area in squares. But as you can see, or might have heard of, you can’t fit a square peg into a round hole.

Let’s look first at the circumference of a circle.

This is the distance you’d have to walk if you walked around in a circle!



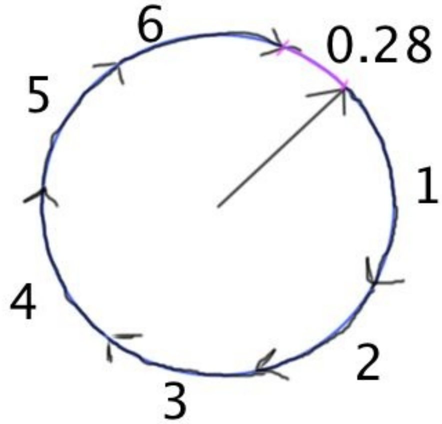
What’s the name of the line going from the centre to the edge?

The radius.

It’s called this because it radiates out from the centre. ‘Radiators’ have that name for the same reason - and you may know ‘radiation’!

How do we calculate the circumference?

If we were to use the radius as a measuring stick, and pick it up and droop it around the circle, we would find that 6.28 radii would fit around



So the circumference is equal to 6.28 radii.

The use of the radius as a measurement is more preferable than the 'diameter' - twice the radius - and we will see why later.

The formula will be

$$C = 6.28 \times r$$

Now we have stumbled over something quite interesting here. The number 6.28 is very important in maths. But also we have come across a new concept.

When we know that the size of the circumference is linked directly to the size of the radius, we call that '*proportional to*'.

So we would say

The circumference is proportional to the radius.

However, what we note is that it is not a one-to-one relationship.

If the radius is 1m, the circumference isn't 1m. But 6.28 times bigger.

This 6.28 then is technically called

'The Constant of Proportionality'

which is a terrifyingly large name. I never use jargon but this is one concept - as well as *'normalisation'* - that is so important I must include.

What it says is, whatever the radius is, the circumference will be 6.28 times bigger.

In fact, 6.28 is not quite right. The exact figure turns out to be
$2 \times \pi$

where $\pi = 3.141...$, pronounced 'pi', is this true constant of proportionality associated with circles and oscillations. It is a number interwoven into the fabric of the universe, and we will see it many times again.

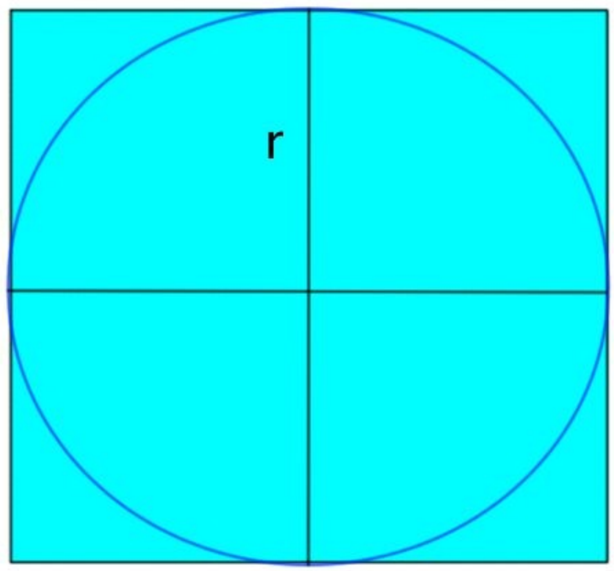
So the correct formula is

$C = 2\pi r$

(Where we have letters next to each other in algebra or formulae, this means they are multiplied)
So that's the circumference.

For the area, we need to fit squares in.

The unit for measure for a circle is the radius, and we see we have a problem.

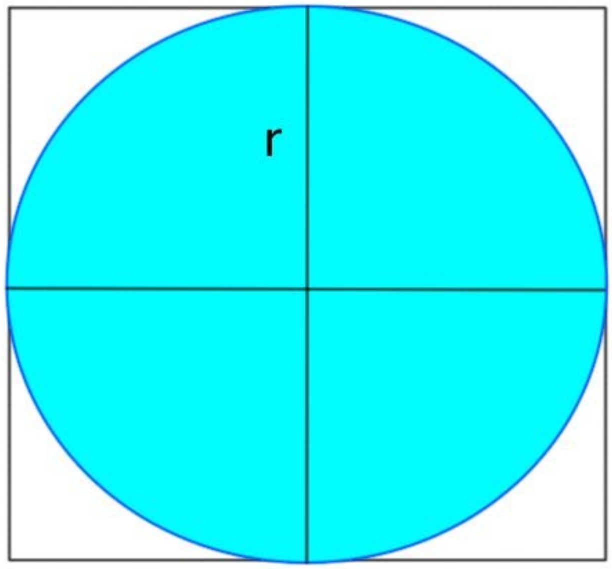


Each square is r^2

So the Area is proportional to r^2.

But how many?

We can see that four is too many, as it covers more than the area of the circle.



It turns out the numbers of squares we need to cover the circle is exactly 3.14, or π.

In fact it is possible to prove a relationship between these two formulae, but it is beyond the scope of this book. It will appear in a later text.

So the formula for the area is

$A_{Circle} = \pi r^2$

For example, if the radius is 10m, the Circumference will be
$= 20\pi$

And the area

$= 100\pi$

it is usually better to keep it in terms of pi.

However, it you can't, an answer to 2 decimal places will suffice.

62.83

314.16

For area, we will always see that the answer will never be a nice round number. Why? Because you can't fit a square peg into a round hole! We can't put a round number of squares into a circle, so it'll always be a decimal, like $314.16m^2$

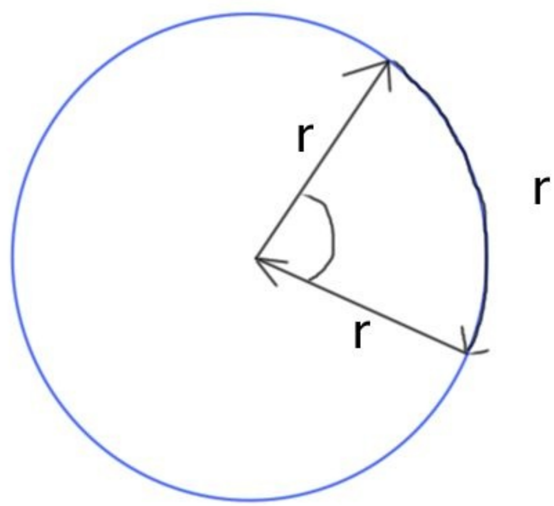
.

A New Angle

A New Angle

If we look back to calculating the circumference, we can see something interesting.

When we drop our first radius around the circumference, as if we are on our way to finding its length, how about we draw a line back to the centre?



Then what we have is an angle created by the radius itself.

In fact, this is the angle that the universe prefers. The use of degrees is a human invention (once again), which is associated with things being a multiple of 6. (That’s why we have 12 months a year, 12 inches in a foot, 60 seconds in a minute, 60 minutes an hour, 24 hours a day...etc). However the universe is unconcerned with human inventions, so in fact degrees don’t fit very well into the scheme of things.

It turns out that this ‘radius angle’ is better.

In fact, it is called a ‘radius angle’ or RADIus ANgle...RADIAN. for short.

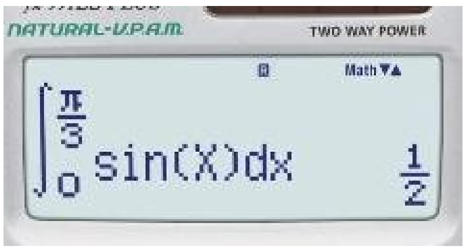
And because it is created by the radius on the circumference, can you guess/calculate how many radians are in a circle (or 360 degrees)? Of course, it is 2π also. Very neat!

So there are 2π radians in a circle.

If we want to know the value of 1 radian, we just divide 360 by 2π. This is around 57.3 degrees.

So that gives you an idea.

You'll notice on your calculator that it will be set to degrees as a default. However you can always tell a true mathematician. When you switch his calculator on (if they use one), it will be set to radians [R].



Introduction to Cubing

In Squaring - In A Minute, we saw that if we multiply the same number by itself this is called squaring.

If we multiply the same number by itself three times, this is called cubing a number.

For example

$7^3 = 7 \times 7 \times 7$

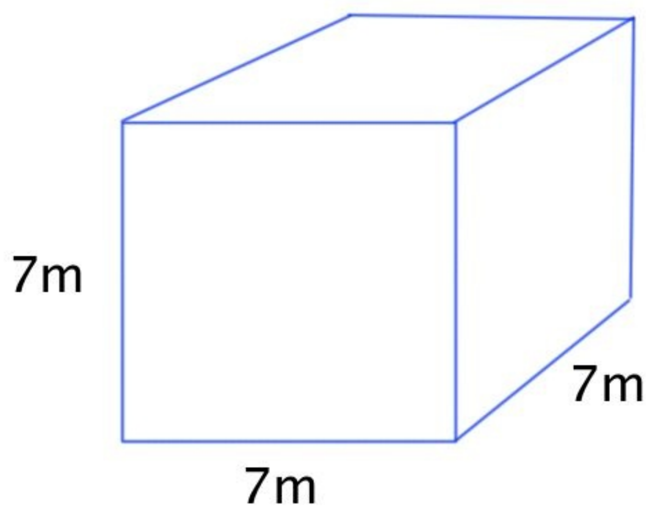
is seven cubed.

Why is cubing called cubing?

Like in Squaring - In A Minute, the answer lies in the name.

Because you get a cube.

For the 7 x 7 part, we will get a square. But when that is multiplied by 7, we have no choice but to go ‘into’ the page, into the 3rd dimension, and that will create a cube.



Hence multiplying a number by itself 3 times is called cubing.

What information do we get about this cube?

For when we squared a number, we found the area. What will we get here?

If we think of the square of one of the sides moving back to the far side of the cube, we will see, as a result of this, that the square will take up space. The question is how much?

So what we find out is the capacity, or volume of the cube. We find how much space it takes up, or how much it can hold inside it.

The unit for volume is m^3.

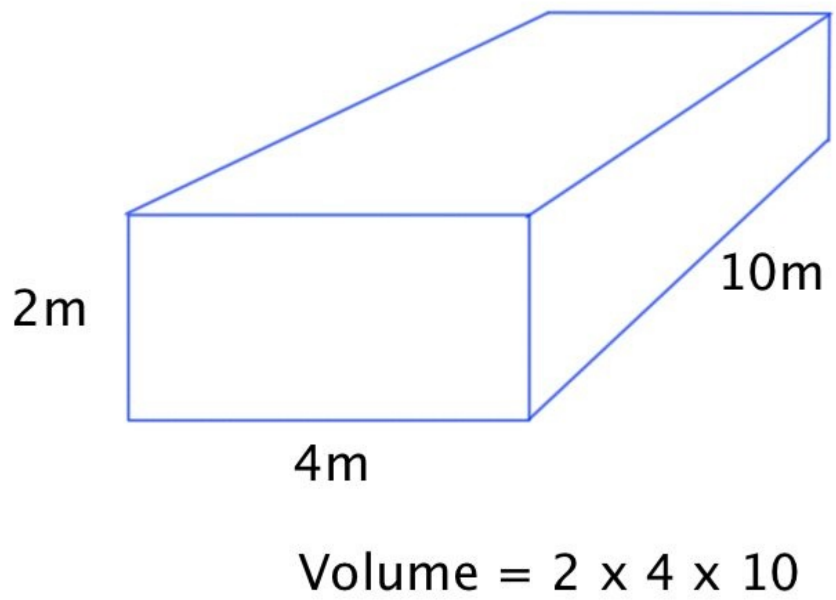
So the volume of this cube will be

$$7^3 = 7 \times 7 \times 7 = 343m^3$$

Why is the unit for volume m^3?

Like with squaring (again), this is a measure of how many cubes, one metre high, wide and deep, would fit inside our larger cube. In this case, 343 would fit. To get an idea of what a one metre cube looks like, think of a washing machine or dishwasher. 343 of these would fit into our cube.

Just like rectangles are more common than squares, cuboids are more common than cubes. A cuboid is a shape with 3 dimensions, at least two of which are different. Look around you and you'll them everywhere. Maybe even this device you're holding is a cuboid! The room you're in? Your laptop, books, remote control, television, mobile phone...



The volume of this will also be measured in m^3.

Any shape with any capacity will be measured in m^3. Just like m^2 is used to measure the area of any shape - even circles - we will later find the volume of shapes like cones and spheres.

How Engines Work

How Engines Work

What does volume have to do with car engines?

You may have heard people say ‘My car’s a 3-litre’ or ‘This bike is a 125’.
What are they talking about?

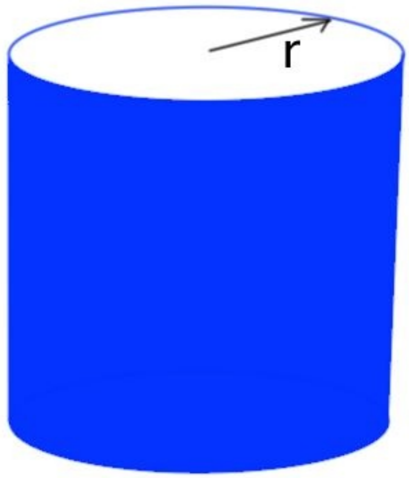
The *size* of the engine. This is measured by the volume of the cylinders.

An engine works like a bicycle, but without someone pedalling up and down! But if you imagine one leg pushing down, and then the other, this is the motion that an engine goes through. Instead there are cylinders that contain the legs (known as pistons) that move up and down.

To get them to do this, fuel is injected above them and in the small gap between the top of the cylinder and the piston, a spark makes the fuel/air mix explode and expand rapidly. This forces the piston down. As it goes down, the other goes up, and at the top of its piston, another explosion forces that down and so on...

The bigger the person cycling, the stronger the muscle and the faster they go. The bigger the piston, the more powerful the engine. So these pistons are measured by the size of the cylinders in the engine.

To calculate the volume of a cylinder, we use the 2D shape of a circle, and multiply by its height.



The formula will be the area,

$A = \pi r^2$

Multiplying this by its height, will give the volume, which will be
$V = \pi r^2 h$

If the radius of the circle is measured in cm, the unit will be cm^3. Also known as cubic centimetres, or cc for short.

You may have heard of 125cc, 600cc or 1100cc motorbikes. Small cars will have an engine of around 1400cc. Because 1000cc is one litre, people will instead say ‘1.4 litre engine’ or ‘3-litre engine’, meaning 3000cc.

Again, because you can’t fit square pegs into round holes, the numbers are not nice and round like this in reality, but they are rounded for convenience.

Surface Area of a Cylinder

The surface area is how much wrapping paper you'd need to use to wrap a cylinder as a present!

For a cylinder we have the area of the 2 circles, top and bottom, so that's $2\pi r^2$.

For the centre, if were to roll it out, in what's known as a development, or net, we'd see it was just a rectangle. To get an idea of this, if you take the wrapper off a tin of cat food, or bottle of coke, you'll see it's just rectangular in shape. The height of the wrapper will be the height of the tin, and what would the width be?

Because it's a circle that's been rolled out, it is in fact = $2\pi r$.

So the area of the rectangle will be $2\pi rh$.

Therefore the surface area of a cylinder will be

$$A_{SurfCyl} = 2\pi r^2 + 2\pi rh$$

This can be 'factorised', which means common letters are taken to one side. It's another name for division.

It will become

$$A_{SurfCyl} = 2\pi r(r + h)$$

Use of Letters

Use of Letters

You may have noticed that I sneaked in a use of letters there. With the mention of the circumference and area of a circle came the use of r and π.

'I thought we were just using numbers at this stage!?'

You are probably crying out. In fact remember these letters represent numbers. It's just that we haven't decided to call it a particular number, just yet, so while we decide, we put a letter there.

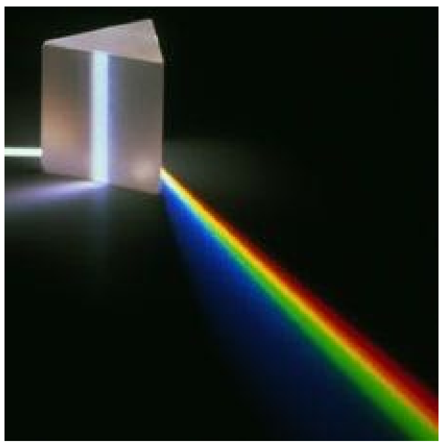
Don't panic!

We will see them used a little more as we look at...

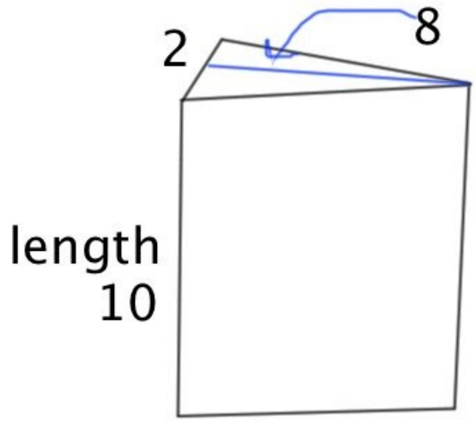
Volumes of Other Shapes

Volumes of Other Shapes

A prism is like a triangular cuboid. Isaac Newton famously used one to discover that white light was made of seven colours (rainbow). You might also be familiar with a certain Swiss chocolate company that makes similar shape bars! They have rectangular lengths, but triangle sides.



To find the volume of this shape, again we find the 2D area of the side - a triangle - and then multiply by its depth, or height. To find the area of a triangle, we just need its base and height, as from Squaring - In A Minute.



The volume of this prism will be $80cm^3$

If the triangle is slanted back, that is not at right angle to the shape, then this is a different fish-kettle. This is the shape you see on those famous Swiss bars. Can you figure out how to calculate the volume yourself?

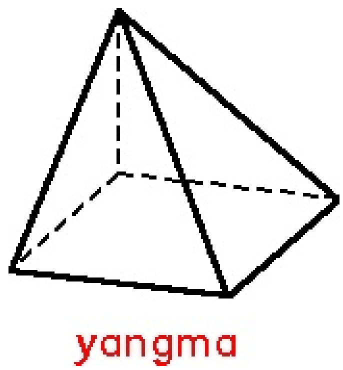
To find the volume here we just take the average length or depth of the prism, and use that.

This is because the sides are like a trapezium, or trapezia, which to find their area we also take the average height or width.

Volume of a Pyramid

The Pyramid is a bit different to a cylinder in that it doesn't have a constant size 2-dimensional area which we can just multiply by the depth or length. It has a square as a base, but that square tapers to a point directly above the centre of the square. As a result, this isn't as simple to calculate.

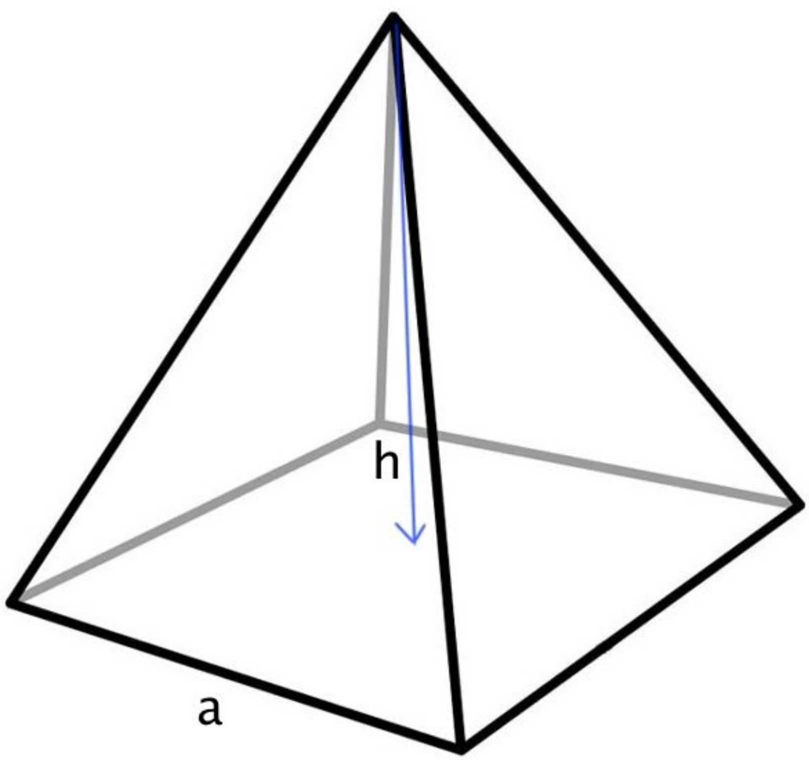
However, if you take a yangma, a pyramid where the vertex is over one of the corners of the pyramid, and take 3 of these, they will fit together to form a whole cube. So that implies that a pyramid with its vertex directly above the centre will be similar.



Not quite.

If the height of the pyramid is different to the length of the sides of the base, the formula will actually be

$$V_{Pyramid} = \frac{1}{3}a^2h$$



Volume of a Cone

This can be calculated by using the cylinder formula, and dividing by 3. This may seem very similar to the relationship between a cube/cuboid and a pyramid.

It is outside the scope of this book to explain why, as it relies on more advanced maths. However, it will be explained in a later book. To 'prove' it is true, watch this amateurish, but accurate, video.

$$V_{Cone} = \frac{1}{3}\pi r^2 h$$

Volume of A Sphere

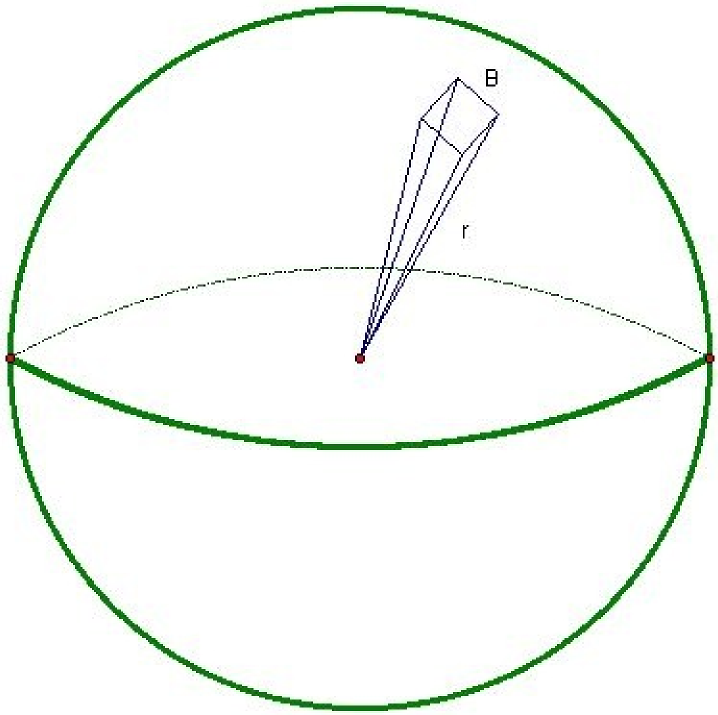
Like a circle for area, this volume is directly proportional to the radius. However, as it is a volume, the radius is cubed. So here we're looking at fitting into cubes into a sphere. This is the 3-dimensional equivalent of *'You Can't Fit a Square Peg Into A Round Hole'* as we saw in calculating an area for a circle.

As a result, you can't get a round number of cubes in a sphere. Imagine trying to make a sphere out of dice. As a result, once again we'll have a decimal.

The formula

$$V_{Sphere} = \frac{4}{3}\pi r^3$$

To calculate the formula here, there are a number of strategies, but one we can look at is to use the idea that the sphere is made of lots of pyramids with their vertices at the centre of the sphere.



If we were to calculate the volume of each pyramid we could calculate the volume of the spheres.

The formula for a pyramid was

$$V = \frac{1}{3}a^2h$$

The height of all these pyramids is the radius of the sphere, so we can change that to

$$V = \frac{1}{3}a^2r$$

So if we could find the area of the bases, we'd be finished.

The area of all of the bases would be just the surface area of the sphere. So if we found that we'd know that when multiplied by $\frac{1}{3}r$ we'd have the volume.

However, calculating the surface area of a sphere is outside the scope of this book also! Ha. Sorry. But again it will be in a future book. Still, the formula is recognisable. It turns out to be four circles.

$$A_{SurfSphere} = 4\pi r^2$$

So if we multiply this, as it's equal to the total area of all the bases of the pyramids, by $\frac{1}{3}r$, we find

$$V_{Sphere} = (4\pi r^2)(\frac{1}{3}r)$$

$$V_{Sphere} = \frac{4}{3}\pi r^3$$

Introduction to Indices

Introduction

Indices, or powers, are a subject that is unnecessarily complicated at school, where often they use general definitions that don't explain in simple terms what the rules really are. I've seen students stare at them in the hope they will seep in. This book will give you the understanding and skills to find them really easy, as well as the fact it follows the series method of using the Three Rules of Maths.

In that sense, Indices are one of the best examples of the Three Rules in action as they follow them exactly, making it easy to remember, anticipate and understand them. Developing your Indices skills will lead to you being able to do more advanced maths where indices (and its reverse, logarithms) are absolutely crucial.

Indices - In A Minute

Indices - In A Minute

7 Rules of Indices

In Squaring - In A Minute, we saw 7^2
In Cubing - In A Minute, we saw 7^3

Imagine now that we multiply these together.

What would we get?

$7^2 \times 7^3$

Instead of multiplying these out to 49 x 343, it's more useful to keep the answer in terms of 7.

So, what do you think is the answer?

You should have got

$7^2 \times 7^3 = 7^5$

Why did you get that? How did you do it?

Of course, from the rules of maths, we're multiplying here, and of course, Multiplication is Just Addition!

So therefore, if we're multiplying there must be an addition, somewhere, and in this case we see that the powers add.

So 2 + 3 = 5.

So $7^2 \times 7^3 = 7^5$

Another way we can look at this is to think of what each term equals in terms of 7

So

$7^2 \times 7^3$

= (7 x 7) x (7 x 7 x 7)

Which is really just 7 x 7 x 7 x 7 x 7 = 7^5

So instead of writing out this number of 7s every time, we can take a short-cut and add the powers.

This is a skill we have actually come across already in 'Decimals -In A Minute', and we'll see it developed in 'Standard Form - In A Minute' too with a very important use.

Another example would be what if it were

$7^{18} \times 7^{21}$

We wouldn't want to write out 7 x 7 x 7 x 7.... 39 times! So it makes sense to simply add.

Giving 7^{39}

So this is the first rule of indices, and of course, it follows the rules of maths. There are seven rules of indices, but don't worry, they all follow the rules we've already encountered, and actually, rule 7 is just an extension of rule 6, so really they're just the same.

Can you guess what the second rule would be?

Second Rule

Naturally, it will be that Division is just....?

So if we have

$$\frac{7^3}{7^2}$$

this equals?

$$\frac{7^3}{7^2} = 7^1 = 7$$

as anything to the power 1 is itself. In fact, every number we've come across so far could be said to have been to the power 1.

So why is division subtraction in this case?

If we figure it out from the example above, we see that

$$\frac{7 \times 7 \times 7}{7 \times 7}$$

will leave us with 7/7 = 1, twice, so that we end up
1 x 1 x 7 = 7

So since the numbers cancel each other out, we'll see this always happen, and division will be subtraction, just like the second rule of maths.

Now that we're doing a division, we have to recall everything we know about it. How many types of division are there?

Of course, you'll recall that there are 3.

And what are they?

Hopefully you'll remember that they are

< 1 = 1 and > 1!

In the example above we see that the answer is 7, which means that must have been the> 1 type.

Therefore there must be a rule for each of = 1 and < 1 types.

So, rule 3 will be about this.

Third Rule

What powers must we have on these 7s to make sure the answer is = 1?

$$\frac{7^?}{7^?}$$

What could it be?

There are an infinite number of answers to this one, but some people would say, 1 and 1, or 2 and 2, or 3 and 3.... it doesn't matter what, as long as they're the same.

Let's say they are 3 and 3.

This gives

$$\frac{7^3}{7^3}$$

Of course this equals 1.

However, if we follow the rule that division is just subtraction, then what power would 7 have after this subtraction?

$$\frac{7^3}{7^3} = 7^{3-3} = 7^0$$

We end up then with a number to the power 0, which is pretty hard to imagine. This is the second abstract concept of mathematics (the first being that we can use a measure out of 100 to measure things that are not out of 100 - see Percentages - In A Minute).

However, logically we have proven that it must equal one, as the division that led to 7^0 came from an expression of one.

Another example would be

$7^2 \times 7^0 = 7^2$

as 2 + 0 = 2

Therefore 7^0 has had no effect on the original number (7^2), being multiplied. For that to be true, 7^0 must equal 1.

This is the third rule of indices.

In general, everything to the power zero is equal to one.

$105^0 = 1$

$90.111^0 = 1$

$a^0 = 1$

$House^0 = 1$

ANYTHING!

Fourth Rule

Fourth rule

So far we’ve covered the cases of >1 and = 1, so now we need to look at a division where the answer is <1.

So, what powers could we have so that the answer is less than 1?

$$\frac{7^?}{7^?}$$

Just like a <1 type of division, we’d be looking for powers where the number on top is less than the bottom.

So let’s try 2 and 3, so

$$\frac{7^2}{7^3}$$

What would this equal?

By using Division is just Subtraction, we’d have that

$$\frac{7^2}{7^3} = 7^{-1}$$

But what does this mean?

Think for a moment, before reading on.

We can figure it out. Similar to the case for the >1 type, we can write this as

$$\frac{7 \times 7}{7 \times 7 \times 7}$$

and cancel 7s.

This gives

$$\frac{1}{7}$$

which of course is less than 1.

In other words we’ve found the exact reverse of the more than 1 type, which, from Division - In A Minute, you may recall as the ‘reciprocal’.

So something to the power of -1 gives you the reciprocal of it.

What about

$$7^{-2}$$

What would this be equal to?

Many, many students think this will be

$$\frac{2}{7}$$

But it’s not.

If we think again as the power of -2 being there because there must have been an extra 2 7s on the bottom of the division, something like

$$\frac{7^2}{7^4}$$

which would indeed give

$$\frac{7^2}{7^4} = 7^{-2}$$

So if we cancel these as above we'd be left with

$$\frac{7^2}{7^4} = \frac{1}{7^2}$$

The power to the - 2 has created the same effect (the reciprocal) but it is also squared.

If we look at what happened between

$$7^{-2}$$

and

$$\frac{1}{7^2}$$

We can see that the minus sign led to a division with a one at the top, the seven going to the bottom and the power becoming the reverse sign and going down too.

So in this case, subtraction (the minus sign in the power) has led to division! In other words, as Rule 3, the reverse.

What about

7^{-3}

4^{-4}

$(7^2)^{-1}$

Ans:

$\frac{1}{7^3}$

$\frac{1}{4^4}$

$\frac{1}{7^2}$

The first four rules then have been about Multiplication being addition and Division being subtraction (but as usual, 3 types of division has meant a rule each).

On to the fifth rule!

Fifth Rule of Indices

Fifth Rule of Indices

What is this equal to?

$(7^2)^3$

(Seven squared in a bracket, cubed)
On first seeing this, you may recall what brackets mean, from Multiplication - In A Minute.

They mean..?

Multiply.

So since multiplication is just addition, we just add, right? However this is not the case.

The 7^2 is cubed, so we have
$7^2 \times 7^2 \times 7^2 = 7^6$

In other words, to get the answer more quickly, we could have just multiplied the powers
2 x 3 = 6.

Indeed, brackets mean multiply! But the multiplication symbol, in indices, means add.

What would

$(7^4)^5$

be?

Ans.

7^{20}

That’s the fifth rule of Indices.

If we look back to the last question in Rule 4, we see brackets there also. In fact we could also have just multiplied the numbers and got

$(7^2)^{-1}$

$7^{-2} = \frac{1}{7^2}$

So we can think of any negative power as a positive number multiplied by - 1, meaning take its reciprocal, and the power will be positive.

E.g.

$10^{-4} = (10^4)^{-1} = \frac{1}{10^4}$

The Sixth Rule

The Sixth Rule is also part of the Seventh Rule. The Seventh rule is just a logical extension of Rule 6, so we have two rules for the price of 1.

Here, we are going to use the Third Rule of Maths, that 'everything has a reverse' and ask, so far we have used whole numbers only for powers. What about the reverse - non-whole numbers?

What would it mean if we had

$$25^{\frac{1}{2}}$$

What effect would having a number to the power of one half have?

Following the reverse rule,

What is the reverse of one half?

Up to now, we've seen that the power 2 squares the number (or multiplies it by itself). So what effect would the reciprocal of 2 have?

Remembering the reciprocal of 2 is one-half, we might expect it to have the reverse effect of squaring, that is to say, to square-root the number.

This turns out to be the case!

$$25^{\frac{1}{2}} = \sqrt{25} = 5$$

So we find that a fractional power means to root!

For 1/2 it means square root.

What would 1/3 be?

The reciprocal is 3, which would have the effect to cubing, so 1/3 must be to cube root!

So $27^{\frac{1}{3}} = 3$

That is, 3, when cubed, gives 27. Cube rooting 27 takes us back to 3 (i.e. doing the reverse).

Putting a number to a power and then rooting it returns us to the original number. It is not too surprising then that the powers required to do this are reciprocals of one another.

We can also test this.

If we square $25^{\frac{1}{2}}$, *i.e.* multiply it by itself, it should cancel the square root out.

$$25^{\frac{1}{2}} \times 25^{\frac{1}{2}} = 25^1 = 25$$

So yes, it works beautifully.

Try these.

$8^{\frac{1}{3}}$

$81^{\frac{1}{4}}$

$1024^{\frac{1}{10}}$

Before we move onto rule 7, how could this idea be expanded?

How could we ‘complicate’ these roots? Any thoughts?

Ans:

2
3

2

Rule 7 of Indices

Rule 7 of Indices

If we were to complicate these roots, all we would have to do is change that number 1 on every one we've seen so far, for another number.

In Rule 6 we saw that a fraction as a power means root.

But there are more fractions than ones with just a 1 on top.

There are, for example, 2/3, 4/7, 3/8...an infinite number of possibilities.

So what would this be?

$$27^{\frac{2}{3}}$$

To figure it out, we could utilise the first rule of indices, that Multiplication is just Addition.

This would split this up into two parts that we've already come across.

$$27^{\frac{2}{3}} = 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$$

And we know that

$$27^{\frac{1}{3}} = 3$$

Therefore $27^{\frac{2}{3}}$

= 3 x 3 = 9

In other words, the cube root of 27, squared, or
27 squared, then cube rooted.

Either is possible (why?).

So $27^{\frac{2}{3}} = \sqrt[3]{27^2}$

The Seventh Rule then, is the extension of the idea in the 6th rule, that what if the top of the fraction isn't 1? Then we can see in the example above that it had the effect of squaring (multiplying the root by itself).

So, in the fractional power, the top number is the power, and the bottom number is the root.

So $81^{\frac{3}{4}}$

would mean

$\sqrt[4]{81^3}$

which can be read as the 4root of 81 cubed or

81 cubed, then 4th rooted

Both of which equal 27.

We can also have the reverse of this, of course, such as

$16^{\frac{3}{2}}$

Which would mean?

$\sqrt{16^3}$

Which would equal 64.

Some to try:

$32^{\frac{3}{5}}$

$729^{\frac{2}{3}}$

$3125^{\frac{2}{5}}$

Ans.

8

81

25

They are the Seven Rules of Indices. They are extremely important for more advanced maths, such as logarithms, algebra and calculus.

Algebraic Indices

In fact, as we will see in later books on algebra, algebraic indices will follow exactly the same rules, and even though I have mostly used 7 throughout, this could be replaced with a letter (such as t), which would all give exactly the same answers.

For instance, try these.

$t^2 \times t^3$

$\frac{t^5}{t^2}$

t^0

t^{-1}

$(t^2)^3$

$(2t^2)^3$

$t^{\frac{1}{2}}$

If you can do these, you can do algebra! And calculus too.

Ans.

t^5

t^3

1

$\frac{1}{t}$

t^6

Not $2t^6$ a common mistake, but $8t^6$. The '2' in the brackets has a power as well. What is it? This must be multiplied by 3 also. The 2 is actually
2^1

So that it becomes

2^3t^6

$8t^6$

This is important for something later called a Binomial Expansion.

$\sqrt{t}$

And now you can do algebraic indices as well!

NOTE

Something interesting to note. Even though you may feel you've not come across indices before, in fact you have. In the book 'Squaring & Area - In A Minute', we looked at
3m x 4m

To find the area of a rectangle. At the moment we just view that as 3 metres by 4 metres. Its answer, $12m^2$ we look upon as its area.

Let's look at this slightly differently. Now we know the rules of indices, let's apply them.

$3m^1 \times 4m^1 = 12m^2$

It's no surprise that we get m^2 as 1 + 1 = 2! But we don't tend to think of it that way when we're practically using maths to actually find something, in this case, area. The point is that maths has a practical application which can be used to understand the theory, but we can also use the theory to put into practice (3rd rule!). This is something we'll do at a more advanced level.

Zen maths

Another way of looking at it is a bit like a perspective question. For example, does the sun rise or does the Earth turn to see the sun? In reality the Earth turns, but we never look at a sun-rise in that way. Sometimes equations or methods can be looked at it many ways to have many meanings, and it can be useful to keep your mind open to that.

Surds

When we were using powers as whole numbers, every time we did that we found the numbers would square, cube or anything else easily.

This isn’t true when reversing that process, rooting. This is true of square roots particularly, and is a concept you’ll come across a lot.

Since most numbers aren’t square numbers, if we square root them, we going to end up with a number containing a decimal. In fact this isn’t quite accurate either, as these square roots have an infinite number of decimal places.

What we find is that actually, as in division, maths doesn’t like decimal places very much. It becomes necessary (and far easier) to be able to write simpler forms of these square roots which are easier to manipulate and 100% accurate.

For example

$$\sqrt{72}$$

will be a number between 8 and 9, but how to simplify it to make it easier to use without decimals?

What we can do is a trick based on reverses.

If we write 72 as a product (two or more numbers multiplied together), and choose one of them to be a square number, we could partially square root it. Then the root 72 would be a multiple of these square roots - and we can work with that.

So $\sqrt{72}$ could be written
$\sqrt{36 \times 2}$

(since 36 is square, 6 x 6)

We could then write

$$\sqrt{36}\times\sqrt{2}$$

Then square rooting

$6 \times \sqrt{2}$

So $\sqrt{72}$ is actually 6 lots of root 2.

These non-squareroot-able numbers are technically called ‘surds’, almost like they are ‘absurd’.

They are also known as irrational numbers. This doesn’t mean they are crazy and don’t think about the major decisions in their lives, but that they can’t be written as a ratio (or division) of two numbers. Numbers that can be are called ‘rational’.

So 3/4 is rational

$\sqrt{45}$ is irrational.

Rationalising The Denominator

To change a number from irrational to rational is, not surprisingly, called ‘rationalising’ it.

This is an important method for divisions that contain an irrational number, as it other wise makes it impossible to add them to other divisions and obtain a reasonable answer.

For example

$$\frac{3}{\sqrt{3}}$$

is an example of a division that contains an irrational number.

To rationalise it, we just multiply the top and bottom by $\sqrt{3}$

This will give

$$\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{3\sqrt{3}}{3} = \sqrt{3}$$

which we can then use this in something like

$$\frac{3}{\sqrt{3}} + \frac{1}{2}$$

as we'll only end up with a fraction of an irrational number (permissible) instead of an answer that contains an irrational number in the division (harder to deal with).

So the above would equal

$$\frac{2\sqrt{3}+1}{2}$$

This could then be used elsewhere too, as it has a non-surd denominator.

Introduction to Standard Form

Introduction

When scientists or engineers want to deal with large or small numbers, such as 3 000 000 000 or 0.00000037, these can be unwieldy. As a result, a 'standard form' of expressing these in an easier way has been developed. In this book we will see how it works, why it is better and how they are used by scientists and engineers to make their lives easier.

Standard Form - The Short Way of Writing Large Numbers

Standard Form - The Short Way of Writing Large Numbers

Looking at a number like

3 000 000

we can write this in a shorter format.

Using the fact that Multiplication is just Addition, we can think of it as

3 x 1 000 000

instead.

Imagine if we had 3 suitcases of 1 million pounds in. They add to £3million.

This is even more unwieldy. However if we use our newly learned methods from Indices - In A Minute, we know that we can use a trick here.

What we do is to use powers of ten as a shortcut. Let's look at these

$10^0 = 1$

$10^1 = 10$

$10^2 = 100$

$10^3 = 1000$

$10^4 = 10000$

$10^5 = 100000$

$10^6 = 1000000$

is actually a thousand thousand, but we call it a million.

$10^9 = 1000000000$

is a billion, or a thousand thousand thousand.

Notice that thousand appears twice now before the final thousand. If we think of this as being ‘2 thousands before the thousand’ you may see why it is called a ‘billion’ as ‘bi’ means 2.

‘Million’ was 1 thousand before the thousand. ‘Mille’ is French/Latin for a thousand.

Notice also that the number of zeroes corresponds to the power. Note for

$10^0 = 1$

There are no zeroes. This is a further proof that any number to the power zero is 1.

If we have

$10^{12} = 1000000000000$

we’ll have 3 thousands before the final thousand. This is called a ‘trillion’ or ‘trillion’. We have ‘tri’ means 3. As in tri-angle.

In any case, we can write these large numbers in this shorter format.

If we replace 1 000 000 with 10^6

We would then convert

3 000 000

to

3×10^6

This is standard form. It is also known as scientific notation.

The first number must be between 1 and 10 and the second term must be 10^n where n is any whole number.

Some more examples would be

7 000

$= 7 \times 10^3$

800 000

$= 8 \times 10^5$

and

200

$= 2 \times 10^2$

What about

78 000

What would that be?

If we take it as

78 x 1000

we then have

78×10^3

But this breaks the rule that the first number must be between 1 and 10. So to achieve this we must have
$7.8 \times 10^1 \times 10^3$

We can then add that 10 on to the 10^3, since Multiplication is just Addition.

So

7.8×10^4 (since $10^1 \times 10^3 = 10^4$)
It is now in standard form.

Another way of doing that is to just count the number of places from between 7 and 8 to the end of the number.

4

78 000

What about

152 000

23 000

298 000 000

1.52×10^5

2.3×10^4

2.98×10^8

Standard Form for Small Numbers

Standard Form for Small Numbers

As the Third Rule goes, we must have a reverse to this, so we have standard form for small numbers also.

Numbers like

0.00007

are as unwieldy as their large counterparts.

Using the same model, but in reverse, we think of

0.00007

as

7 x 0.00001

and then we just need to replace
0.00001

with a power of ten.

What will we use for this?

Looking again at powers of ten, but in reverse this time,
$10^0 = 1$

$10^{-1} = \frac{1}{10} = 0.1$

$10^{-2} = \frac{1}{100} = 0.01$

$$10^{-3} = \frac{1}{1000} = 0.001$$

$$10^{-4} = \frac{1}{10000} = 0.0001$$

we see that if we read these numbers backwards, the number is described.

For example, one-thousandth,

$$\frac{1}{1000} = 0.001$$

If we ignore that point, and read it backwards, we see 1000!

So a true reverse.

It is very easy then to write these decimals when we know their fractions. Just reverse the number and put a decimal point after the first zero.

One-millionth would be

$$\frac{1}{1000000}$$

000 000 1

= 0.000001

Easy.

We can replace our

0.00001 from

7 x 0.00001

with

10^{-5}

So we have

7×10^{-5}

Try these

0.004

0.00045

0.0000037

These become

4×10^{-3}

4.5×10^{-4}

3.7×10^{-6}

Multiplying Numbers In Standard Form

Multiplying Numbers in Standard Form

When we looked at Decimals - In A Min, we met the Key Change Effect. This was the idea that the decimal places when multiplying or dividing didn't matter, the answer we get is always the same.

At the time we just added the number of decimal places when adding, since Multiplication is just Addition, and subtracted the number of places when dividing, since Division is just Subtraction.

What we were really doing was a more rough-and-ready version of Standard Form. We can now employ those tactics far more easily by multiplying (or dividing) the numbers as usual, while adding (or subtracting) the powers of ten.

For example,

$2 \times 10^3 \times 3 \times 10^4$

Multiply the numbers, ignoring the 10s
2 x 3 = 6

and then add the powers
$10^3 \times 10^4 = 10^7$

Ans.

6×10^7

If we had done this the rough-and-ready way it would have been
2000 x 30000

2 x 3 = 6

and add the 7 zeroes

60 000 000.

Standard Form does that for us without the zeroes.

Another example

$1.4\times10^{4}\times2.1\times10^{-7}$

Again, multiply 1.4 x 2.1

We employ the Key Change Effect here, ignoring the decimal points, to give

```
  14
x 21
-----
 294
```

Adding the two places (1 + 1), gives 2.94

Adding the powers gives
$10^{4}\times10^{-7}=10^{-3}$

Ans.

2.94×10^{-3}

or

0.00294

Dividing Numbers In Standard Form

Dividing

This is the exact same process.

$$\frac{6\times10^{7}}{3\times10^{4}}$$

Divide the numbers

$$\frac{6}{3}=2$$

and subtract the powers

$$\frac{10^{7}}{10^{4}}=10^{3}$$

Giving

$$2\times10^{3}$$

Because numbers are written between the ranges of 1-10, this (usually) makes division easier.

Try

$$\frac{7.2\times10^{5}}{6\times10^{9}}$$

Doing

$$\frac{7.2}{6}$$

From Decimals - In A Minute we saw this would be

$$\frac{72}{6}=12$$

Then adding the one decimal place (as 1 - 0 = 1)
Gives

1.2

And subtracting the powers gives
10^{-4}

So our answer will be

1.2×10^{-4}

How about

$$\frac{7.2 \times 10^{5}}{6 \times 10^{-3}}$$

Here we again get 1.2 from dividing the numbers.

What will we get from the powers?

We will have - 5 - (- 3)

Which, from Negative Numbers - In A Minute, we saw will be..?

= - 2

So we have

1.2×10^{-2}

Typical Example Question

Typical Example Question

The Sun is 150 000 000 kilometres from Earth. The speed of light, c, is 300 000 000 metres per second.

Using standard form, calculate how long, in minutes, light takes to reach the Earth from the Sun.

Have a go at this yourself.

Answer:

Putting both our numbers into standard form

150 000 000

1.5×10^{8}

and

300 000 000

3×10^{8}

However, we need to make them the same unit. The speed is in metres per second, where the distance is in kilometres. As there are a thousand metres in a kilometre, we multiply by 1000 or, 10^{3}

$1.5 \times 10^{8} \times 10^{3}$

$= 1.5 \times 10^{11}$

If we think of a speed, it is distance divided by time. Miles per hour, or metres per second, tell us how far we go per second. If you travelled at 60 miles per hour for one hour, you'd travel 60 miles.

So to calculate time, we need to use the formula

$$s = \frac{d}{t}$$

and re-arrange for t.

This is examined in a later book - Changing the Subject - but for now we'll say it gives

$$t = \frac{d}{s}$$

So we have

$$\frac{1.5 \times 10^{11}}{3 \times 10^{8}}$$

Doing

$$\frac{1.5}{3}$$

$$\frac{15}{3} = 5$$

$$\frac{1.5}{3} = 0.5$$

and subtracting the powers

Gives

$$10^{3}$$

So we have

0.5×10^3

but this isn't standard form (why?).

The number must be between 1 and 10. Before we had this issue with 78, and had to make it 7.8. To make that happen we used 7.8 x 10 and it gave an extra ten to the power.

Here it is the reverse. We have to make the 0.5 bigger, to 5, and to do that requires multiplying by ten.

We take that ten from the ones we already have in 10^3, so we end up with
$0.5 \times 10 \times 10^2$

5×10^2

= 500 seconds

Since there are 60 seconds in a minute
$\frac{500}{60}$

$8\frac{1}{3}\text{min}$

Ans.

This is a small example of where standard form might be used in science. With numbers that are, well, astronomical!

Further Examples of use of Standard Form in Science

Further Examples of use of Standard Form in Science & Engineering

Many constants and data are represented by standard form.

For example, the charge on the electron is

0.00000000000000000016 Coulombs.

This is very difficult to use.

So instead

$1.602 \times 10^{-19}\,Coulombs$

is used.

The mass of an electron is even worse.

$9.109 \times 10^{-31}\,kg$

Here is a data sheet of some physical constants. Note they are largely in standard form.

Name	Symbol	Value	Units
Gravitational constant	G	6.673×10^{-11}	$m^3\,kg^{-1}\,s^{-2}$
Permittivity of the vacuum	ϵ_o	8.854×10^{-12}	$C^2\,N^{-1}\,m^{-2}$
Permeability of the vacuum	μ_o	$4\pi \times 10^{-7}$	W m
Elementary charge	e	1.602×10^{-19}	C
Speed of light in vacuum	c	2.998×10^{8}	$m\,s^{-1}$
Planck constant	h	6.626×10^{-34}	J s
Reduced Planck constant	$\hbar \equiv h/2\pi$	1.055×10^{-34}	J s
Boltzmann constant	k	1.381×10^{-23}	$m^2\,kg\,s^{-2}\,K^{-1}$
Stefan–Boltzmann constant	σ_{SB}	5.670×10^{-8}	$W\,m^{-2}K^{-4}$
Thomson cross-section	σ_e	6.652×10^{-29}	m^2
Proton mass	m_p	1.673×10^{-27}	kg
Electron mass	m_e	9.109×10^{-31}	kg

Standard Form is used in engineering in powers of 3. This is something you are already familiar with.

Have you heard of a kilometre? Kilobyte? Kilogram?

This is for $10^3 = 1000$ of each.

For $10^6 = 1000000$ of each, we have

Megabyte and

Megagram

For $10^9 = 1000000000$

we have

Gigabyte (or gig)

In the film '***Back To The Future***', the car uses a nuclear reactor to generate the 1.21 *GigaWatts* of power needed for time travel. The producers were not scientists and the consultant they hired pronounced it with a J, as in Jigawatt!

This is for large numbers.

For small numbers we have

millimetre - 10^{-3}
micrometre, microwave, 10^{-6}
nanometre, 10^{-9}

Often then we talk in terms of these units such as

‘42 kilometres’, rather than ‘42 thousand metres’.

‘550 nanometres’, not $5.5\times10^{-7}\,metres$.

It just makes it easier to say and communicate.

It is very useful to be familiar with standard form to be able to study subjects like engineering and science.

Introduction to Logarithms

Introduction

Once upon a time, there was a scientific problem. How could calculations be done more quickly? Intelligent scientists were having to spend their time doing long calculations in arithmetic, and this was diverting them from their task of doing any useful science.

One solution would have been to have someone else do it for you, of course. But this still took time. What was needed was a quick way to multiply and, especially, divide, large or small numbers in seconds, instead of an hour or more.

Division especially, as we have seen it is not as accommodating as multiplication.

To achieve this, one man began to notice a pattern, and from this discovery and his extremely hard work, this led to the invention of the world's first computer, which could be held in the palm of your hand. Amazingly, this was in 1630.

For centuries, this method has been used to calculate quickly and easily, only ceasing with the invention of the modern computer and electronic calculator in the 20th century.

So why should we bother learning about it?

Because this discovery - logarithms - give us yet another tool in algebra and for solutions for much more complicated things than multiplying 2 numbers. It is easiest to understand them first as a tool for calculation, then it's a short hop to their use in algebra.

In any case, it's a fascinating story of how this invention in 1614 (exactly 400 years ago, as this book is published!) led to NASA engineers and astronauts using the same techniques to solve problems on the fly in missions to the moon over 350 years later.

What Are Logarithms?

What are logarithms?

Firstly, let's see how they were discovered and all will become clear.

As with most discoveries, a single question was asked.

The question was

If

$10^1 = 10$

And

$10^2 = 100$

Can any number, such as 37, say, be written as 10 to the power something?

It would have to be a number between 1 and 2 of course, that makes logical sense.

But what would it be?

It turns out, with much calculation, that the answer is
$37 = 10^{1.568}$

And so any number can be written in this way.

$5 = 10^{0.699}$

$81 = 10^{1.908}$

$129 = 10^{2.111}$

Here's a table. Note the bottom figure for 37.

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
10	.0000	0043	0086	0128	0170						4	8	13	17	21	25	29	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	.0414	0453	0492	0531	0569						4	8	12	16	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	.0792	0828	0864	0899	0934						4	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	.1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	6	10	13	16	19	22	26	29
14	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	10	12	14	17	19	22
19	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	6	9	11	13	15	18	20
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	11	13	15	17
23	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	5	7	9	11	13	14	16
24	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	13	14	16
25	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	6	8	10	11	13	14
27	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	10	11	13	14
28	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	8	10	11	13
31	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	13
32	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	10	12
33	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	12
35	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	8	10	11
36	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	10	11

Ok, great.

So we can describe numbers in terms of powers of ten.

What's the significance of that?

What it meant was we could do a neat trick. We could express the first rule of maths - that Multiplication is just Addition - in an extraordinary way.

What if we wanted to multiply 37 x 81?

Of course, we could do that with the method outlined in 'Multiplication - In A Minute', in seconds, but, how could we do it with the new expressions above?

We could re-write 37 x 81 as

$$10^{1.568} \times 10^{1.908}$$

And using the fact that multiplication was just addition as the first rule of indices, add the powers.

This would give

$$37 \times 81 = 10^{3.476}$$

However, we need to figure out what that equals!

Looking again at our answer

$$37 \times 81 = 10^{3.476}$$

We can start to deconstruct this using the 'Key Change Effect' and the first rule of maths.

$$10^{3.476} = 10^{0.476} \times 10^{3}$$

(since the powers add)

And this is the key change effect in action, since the 10^3 part is only acting as "multiplies by a thousand".

So we're just interested in the

$$10^{0.476}$$

If we look at the log table, how it works is that it gives every number between 1 and 10 as its power of 10.

For example,

$$3.7 = 10^{0.568}$$

In fact, we know

$$37 = 10^{1.568}$$

as it's

$$3.7 \times 10 = 10^{0.568} \times 10^{1}$$

i.e. the powers added.

So this table, if we ran our finger down it, would have somewhere on it, 0.476.

However, we don't actually need to do this, as using the third rule, of course we have the reverse!

And someone realised this and produced a table that gives all these values in reverse.

That is, if you have the power, what would its number be?

In the first table, we had the number and we wanted to know its power.

In this table, we have the power, and want to know its number.

ANTILOGARITHMS 10^x

x	0	1	2	3	4	5	6	7	8	9	Δ_m	1	2	3	4	5	6	7	8	9
											+	ADD								
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	2	0	0	1	1	1	1	1	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	2	0	0	1	1	1	1	1	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	2	0	0	1	1	1	1	1	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	2	0	0	1	1	1	1	1	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	3	0	1	1	1	1	2	2	2	3
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	3	0	1	1	1	1	2	2	2	3
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	3	0	1	1	1	1	2	2	2	3
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	3	0	1	1	1	1	2	2	2	3
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	3	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	3	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	3	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	3	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	3	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	3	0	1	1	1	2	2	2	2	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	3	0	1	1	1	2	2	2	2	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	3	0	1	1	1	2	2	2	2	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	3	0	1	1	1	2	2	2	2	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	4	0	1	1	2	2	2	3	3	4
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	4	0	1	1	2	2	2	3	3	4
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	4	0	1	1	2	2	2	3	3	4
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	4	0	1	1	2	2	2	3	3	4
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	4	0	1	1	2	2	2	3	3	4
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	4	0	1	1	2	2	2	3	3	4
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	4	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	4	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	4	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	4	0	1	1	2	2	2	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	4	0	1	1	2	2	2	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	4	0	1	1	2	2	2	3	3	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	4	0	1	1	2	2	2	3	3	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	5	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	5	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	5	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	5	1	1	2	2	3	3	4	4	5
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	5	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	5	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	5	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	6	1	1	2	2	3	4	4	5	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	6	1	1	2	2	3	4	4	5	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	6	1	1	2	2	3	4	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	6	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	6	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	6	1	1	2	2	3	4	4	5	5
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	6	1	1	2	2	3	4	4	5	5
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	6	1	1	2	2	3	4	4	5	5
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	7	1	1	2	3	3	4	5	6	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	7	1	1	2	3	3	4	5	6	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	7	1	1	2	3	3	4	5	6	6

Scanning down the left hand side, we find 0.47, and then 6 along to the right
This gives

2 992

as the answer.

So 37 x 81 = 2 992

If we do the actual calculation, as a check

We get

  37
x 81

2997

So is it perfect? No, there's a slight error, 5/2997 = 1/600 = 0.16%.

This is acceptable for the time saved.

The method really comes into its own for larger numbers which would take extremely long to multiply.

But *where it really comes into its own* is in the field of division.

For example, as above I had to divide 5 by 2997. Not very nice! What about

$$\frac{2439}{183}$$

It's a slog, using old-fashioned methods.

But using logarithms, we can turn that division into a...

SUBTRACTION.

So we find out what 2439 is

$2439 = 10^{3.387}$

and

$183 = 10^{2.262}$

As we're dividing, we subtract the powers, giving
$10^{1.125}$

$10^{1.125} = 10^{0.125} \times 10^{1}$

(From ‘Indices - In A Minute’)

and looking that up on the reverse table

ANTILOGARITHMS 10^x

x	0	1	2	3	4	5	6	7	8	9	Δ_m	1	2	3	4	5	6	7	8	9
											+	ADD								
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	2	0	0	1	1	1	1	1	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	2	0	0	1	1	1	1	1	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	2	0	0	1	1	1	1	1	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	2	0	0	1	1	1	1	1	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	3	0	1	1	1	1	2	2	2	3
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	3	0	1	1	1	1	2	2	2	3
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	3	0	1	1	1	1	2	2	2	3
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	3	0	1	1	1	1	2	2	2	3
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	3	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	3	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	3	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	3	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	3	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	3	0	1	1	1	2	2	2	2	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	3	0	1	1	1	2	2	2	2	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	3	0	1	1	1	2	2	2	2	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	3	0	1	1	1	2	2	2	2	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	4	0	1	1	2	2	2	3	3	4
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	4	0	1	1	2	2	2	3	3	4
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	4	0	1	1	2	2	2	3	3	4
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	4	0	1	1	2	2	2	3	3	4
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	4	0	1	1	2	2	2	3	3	4
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	4	0	1	1	2	2	2	3	3	4
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	4	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	4	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	4	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	4	0	1	1	2	2	2	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	4	0	1	1	2	2	2	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	4	0	1	1	2	2	2	3	3	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	4	0	1	1	2	2	2	3	3	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	5	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	5	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	5	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	5	1	1	2	2	3	3	4	4	5
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	5	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	5	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	5	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	6	1	1	2	2	3	4	4	5	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	6	1	1	2	2	3	4	4	5	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	6	1	1	2	2	3	4	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	6	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	6	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	6	1	1	2	2	3	4	4	5	5
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	6	1	1	2	2	3	4	4	5	5
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	6	1	1	2	2	3	4	4	5	5
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	7	1	1	2	3	3	4	5	6	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	7	1	1	2	3	3	4	5	6	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	7	1	1	2	3	3	4	5	6	6

(Where we're looking for 0.125 as that's the only part we're concerned with)
We find

1.334

and multiplying by 10 for the 10^1 part, gives
13.34

If we do the division with more modern methods, we get the answer
13.33.

Amazing!

That hard work has been significantly reduced and we get an answer in no time.

The Name

We begin to realise as we do this that the fact it's to the power of ten doesn't really matter. It only acts as 'Key Changer', making it magnitudes of ten smaller or bigger.

As a result, the notation that is used is not the same as I've been using above, as actually, it is the POWERS that are all important, and we could do with those being a bit bigger, and the 10, as a consequence of its lack of importance, being a lot smaller, in order to read them more easily.

So the way these powers are actually written is different.

They are also given a different name, which you would have guessed by now, is ***logarithms***.

You may see in the word 'arithm' coming from arithmos, Greek for number. Arithmetic comes from the same root.

'Log' comes from logos, Greek for proportion. The concept was a ratio or proportion of a number. In other words, the relationship between its value and its power.

So for our

$37 = 10^{1.568}$

is actually written

$\log_{10} 37 = 1.568$

As we can see the 10 is reduced in size and the power is much bigger.

The way to read this is

'What power do we need to make 10 become 37? We need the power 1.568'.

or

$\log_{10} 183 = 2.262$

'What power do we need to make 10 become 183? We need the power 2.262'.

Nowadays we don't need to carry tables around with us (unfortunately*), and all these values are programmed into calculators.

As a result, if you look at your 'log' button on the calculator, it will give you the power you need for any number you want.

The log button doesn't say log 10, but it is taken to be the 'common logarithm'. If you look at the inverse/2nd function/reverse button on the top left, you'll notice that it's

10^x

which means the opposite of a logarithm, that is a power.

E.g. type in 'log' 37 and it'll give you 1.568...

Do 2nd function, Ans, and it'll return you to 37, as

$10^{1.568} = 37$

Of course, we don't need to do calculations by logarithms any more, as we have the calculator to do it for us in any case! But that's not the only reason we use logarithms, as will be covered later. I want you to understand them first, and this approach leads to an intuitive understanding which we can apply later.

*Why 'unfortunately'?

Because the great thing about the table above is that it gives a sense of scale of what logarithms (or anything else that uses tables) looks like. You can see the lowest value, the highest value, and get some sense or feel of what a logarithm is and how it behaves. When you enter a value into a calculator one at a time, the brain has no opportunity to see any pattern, so although it is extremely efficient, something is lost. And when information of any kind is lost, it's much harder to get it back again. We will see this again in the future trigonometry books.

Touch Base

Do we always have to use 10 as our number?

The technical name for using 10 as our number is actually called the base. So we're using base 10 as our 'basis'. Probably why it's called the base.

So up to now our logarithms have been entirely to base 10. But in fact we can have any base, just like we can have any power.

For example, when we looked at indices in 'Indices - In A Minute', we used 7 a lot, and we could have base 7 if we wanted.

$\log_7 49 = 2$

and what would be

$\log_7 343 = ?$

Of course, since

$7^3 = 343$

$\log_7 343 = 3$

and so on. Try these

$\log_2 8$

$\log_5 25$

$\log_3 81$

All these questions are just saying is

What power do we need to make, say, 5 become 25? Of course it is 2.

Ans:

3
2

4

Is there anything you notice about our choice of numbers above? If not, don't worry, I'll come to it later.

CALCULATOR TIP: Modern calculators have a button when you can insert any base and find the power you need for any number, it looks like this.

This is handy for solving equations later.

One base we can have is very special indeed.

When we talked about compound interest in 'Percentages - In A Minute', one thing I didn't examine is what would happen when the time between compounds was shortened? What if we reduced it to almost nothing?

For example, let’s say you start off with £1, at 10% for ten years.

The formula for this would be

$$A = 1(1 + \frac{1}{10})^{10}$$

This would give £2.59. In other words, we have 2.59 times more than we started with. Or it grew 2.59 times the starting amount.

However, this is taking that we compounded once a year. This doesn’t seem fair, for 11 months and 30 days of the year, it earned no interest. So if it was made that it was compounded after 6 months instead, it would have earned 5%. And it would, for the ten years, be compounded 20 times. So this would give

$$A = 1(1 + \frac{1}{20})^{20}$$

$$= 2.65$$

A higher amount. Or it would be 2.65 times what we started off with.

If we continue this to shorter time periods and thereby more compounds for the 10 years, the amount will hit a limit of natural growth.

For 100 compounds, or 1% each time,

$$A = 2.71$$

For 1000, or 0.1% each time,

$$A = 2.717$$

And with a high enough number, A grows to

$$2.71828\ldots$$

This number is found in nature too.

If we look at a mollusc, and its ratio between each spiral of growth, as it grows larger, it grows larger - a compounded effect of course. And if we were to measure the distances between each spiral, we'd see a sequence where each distance is 2.71828 times bigger than the previous.

This number, which was called e, is prevalent in many natural situations, such as radioactive decay, population growth, cooling and many others.

This number then, has been called a natural number.

And, in the case of logarithms, if we use it as a base, we have
$\log_e$

which is called a natural logarithm.

In latin, this becomes logarithm naturalis, or ln for short.

So

$\ln 2.71828... = 1$

That is,

$e^1 = 2.71828...$

This particular base has many functions and uses within more advanced maths.
If you like, you can use it entirely from now on.

Logarithms in everyday life

Logarithms in everyday life

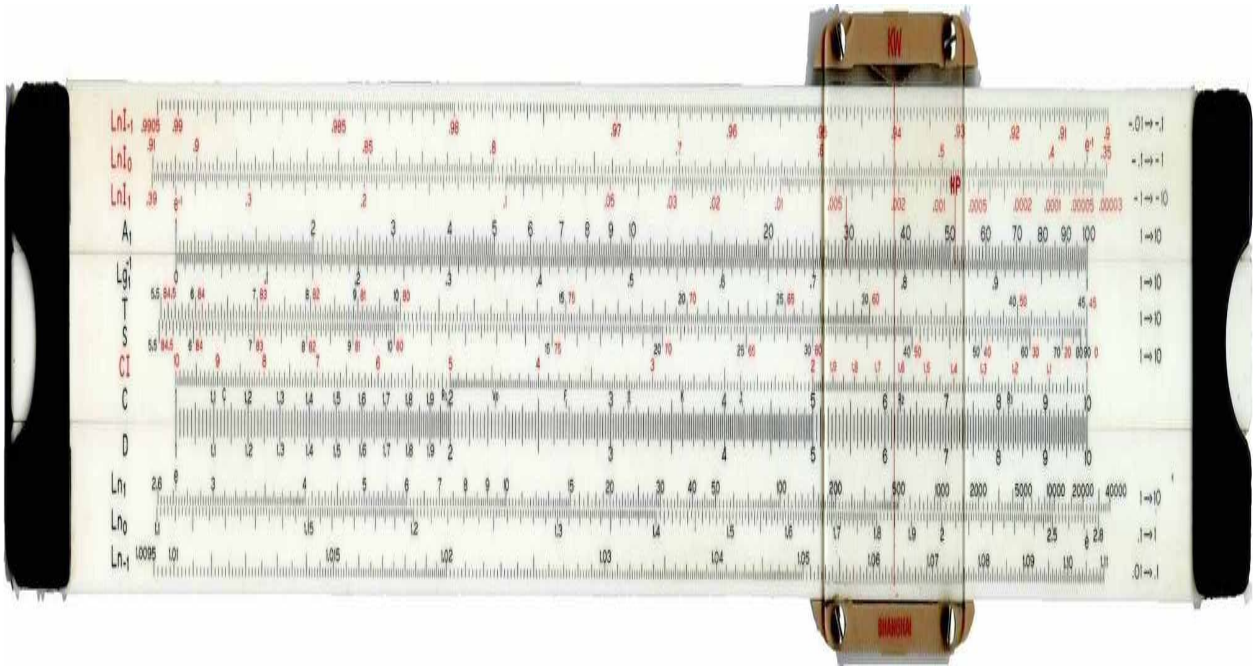
Slide Rule

In the film Apollo 13, starring Tom Hanks, the story is about a moon mission that suffered a mechanical failure and the resulting race to get the astronauts back alive, despite them being over 200,000 miles away and having a crippled ship.

At one stage they have to use the lunar module as a lifeboat, and as a result, all the calculations have to change. To do these new calculations, NASA controllers use something called a 'Slide Rule'. These are the days before electronic calculators, remember.

What is a slide rule?

A slide rule is a calculator which uses the concept above of the fact that Multiplication is just addition and division is just subtraction.



This probably looks quite complicated! But the lines C and D allow you to multiply and divide.

For example, let's say you want to multiply 14 x 21.

What you end up doing to find this is simply sliding a rule a certain distance. That is, making an addition.

You slide the C to 14, look at 21 on C, and below that will be the answer 294.

To do a division, you just do the reverse, sliding the rule back, and thereby making a subtraction in its length.

The slide rule is essentially an early and quite ingenious version of an electronic calculator, as it's mobile, practical and has many functions.

A certain generation of readers might remember what a slide rule is. These days, with the prevalence and superiority of the electronic calculator, they have been completely forgotten about. Think VHS cassettes these days. Or analogue mobile phones.

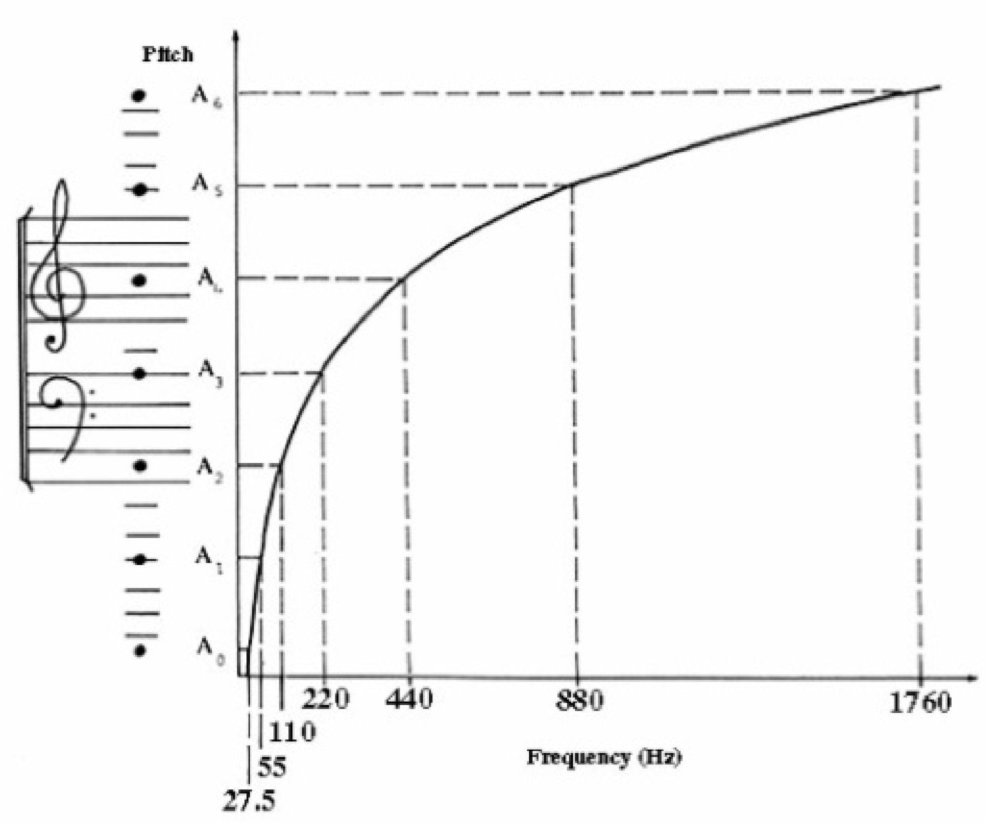
I can't quote the song for copyright reasons, but a song in the 60s, by Sam Cooke, called 'What a wonderful world it would be' where he declares his ignorance of what a slide rule is for. He knew that his listeners would know what a slide rule was, but nowadays no-one does. You can find them in museums these days - see the Science Museum in London for example. If you'd like to own one, visit antique auctions or eBay!

These days people love their retro flavour so much that they have used modern technology to create one that you can use online!

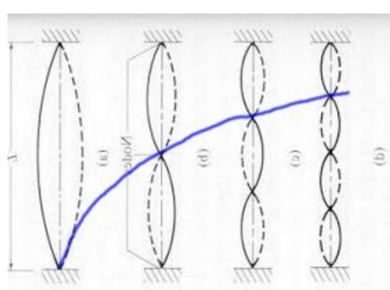
Of course if you’re going to the trouble to program a computer to create a calculator, it’s like using a Ferrari as a wheelbarrow. However you can get a taste at that site.

Piano

In music, the notes follow a logarithmic scale. In other words, they double in frequency every octave. So the space between the lowest A and the next A is twice the frequency. The next A is twice the frequency again. And so on. This graph illustrates the relationship.

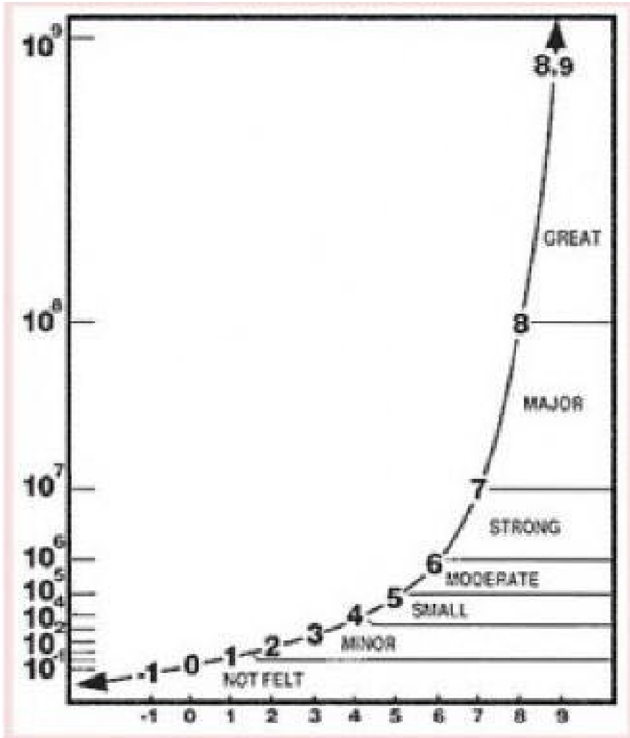


Another way to look at it is below. The lowest frequency has the simplest vibration. As they increase in 'nodes', the frequency doubles. This is what happens with the strings on a piano (harp). Drawing a line through the first node from the top also shows a logarithmic graph.

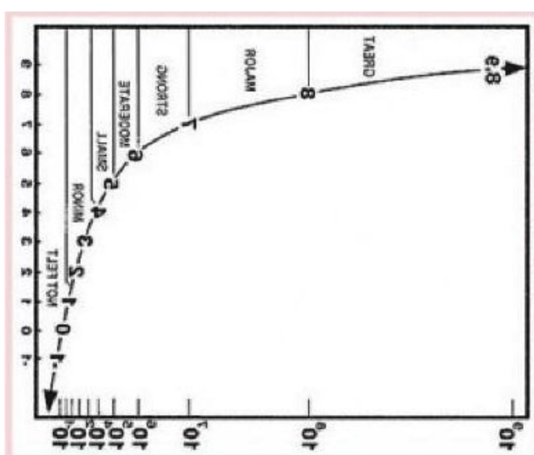


Richter

The Richter scale is a measure of the strength of earthquakes. The difference between each level is not 1, but actually 10 times bigger than the previous. This leads to a graph that looks like this



This is in fact, an exponential graph, which of course, because of the third rule of maths, is the reverse of a logarithmic graph. On the left you can see 10 to the power increasing on the left. If this graph is rotated and flipped, we get its inverse, which will look more familiar.



The Rules of Logarithms

The Rules of Logarithms

If you've read 'Indices - In A Minute', you'll have come across the rules of indices, and seen how they follow the rules of maths.

Logarithms, being the reverse of indices, have rules that are the exact reverse of the rules of indices! It all follows naturally, as everything must obey the rules of maths.

The third rule of maths, that everything has a reverse is therefore true here. The rules of indices have a reverse, and they are the rules of logarithms, since logs are the reverse of indices.

So all we have to do is reverse the rules of indices.

Rule 1: For indices, multiplication is just addition.

Therefore for logarithms, addition is just multiplication!

Let's look at an example.

$\log_{10} 10 = 1$

If we add these, by this new rule, the answer should be
$\log_{10} 10 + \log_{10} 10 = \log_{10} 100$

Is this true?

$\log_{10} 10 = 1$

And

$\log_{10} 100 = 2$

So what the above says is

$\log_{10} 10 + \log_{10} 10 = \log_{10} 100$

1 + 1 = 2

and this is true, of course.

So yes, addition of logs leads to their being multiplied.

Rule 2

So, can you guess rule 2?

For indices, it is of course

Division is just subtraction.

So for logarithms?

Naturally, it is subtraction is just division.

For example

$$\log_{10} 100 - \log_{10} 10 = \log_{10} 10$$

Here we see that

2 - 1 = 1

So subtraction has led to a division, *i.e.* 100/10 = 10

Another example would be

$$\log_{10} 250 - \log_{10} 10 = \log_{10} 25$$

It doesn't seem to make sense at first, but remember that we are dealing with powers essentially, so they operate via addition and subtraction.

Rule 3

In indices, we saw that any number to the power 0 is equal to 1.

$7^0 = 1$

$37^0 = 1$

$t^0 = 1$

It doesn't matter what.

In logs, we have the reverse, that

$\log_{10} 1 = 0$

That log of 1 is zero, no matter what the base.

$\log_7 1 = 0$

$\log_{37} 1 = 0$

$\log_t 1 = 0$

$\ln 1 = 0$

since we are saying, 'what power do we need to make 10 equal to 1?'. Of course, it is zero.

Rule 4

As I asked you about earlier, did you see a pattern in the questions
$\log_2 8$

$\log_5 25$

$\log_3 81$

and the answers were

3
2
and
4

Another way to write

$\log_2 8$

would be

$\log_2 2^3$

Since

$2^3 = 8$

And funnily enough, the answer is 3.

Of course, logs ask the question 'what power do we need..?' and if we're trying to find a power for a number where we already have the base, as in this example, not surprisingly, the answer is the power!

Remember, the log is saying, the power you need to make 2 become 8 is 3. If we had noticed that 8 is 2^3 in any case, we'd know the answer. But more generally, this leads us to a new log rule.

The rule is, that if you have a power in the log, you can bring it forward to the front.

Let's look at that with this example.

$\log_2 2^3$ can then be written
$3 \times \log_2 2$

Since
$\log_2 2 = 1$
we have

3 x 1 = 3.

And we get 3 as our answer. So that means it is permissible to bring that power forward.

Looking at the other two examples,

$\log_5 25 =$

can be written

$\log_5 5^2$

which can be

$2\log_5 5$

And,

$\log_3 81$

$\log_3 3^4$

$4\log_3 3$

Which is true.

We can also do the reverse - if we have a number at the front, we can put it up as a power.

So

$4\log_2 8$

Can become

$\log_2 8^4$

They are the four rules of logarithms!

However, there are a few other situations to consider which are the reverse of indices.

What if we have a negative logarithm?

E.g.

$-\log_2 8$

What would this be equal to?

Since we can use rule 4 to bring the number in front of a logarithm to become a power, we get
$\log_2 8^{-1}$

Which is

$\log_2 \frac{1}{8}$

From the Rules of Indices

which is

$\log_2 \frac{1}{2^3}$

= -3

And vice versa. If we have a negative power, or a division in the logarithm, we can make it negative.

E.g.

$\log_{10} 5^{-1}$

$= -\log_{10} 5$

And

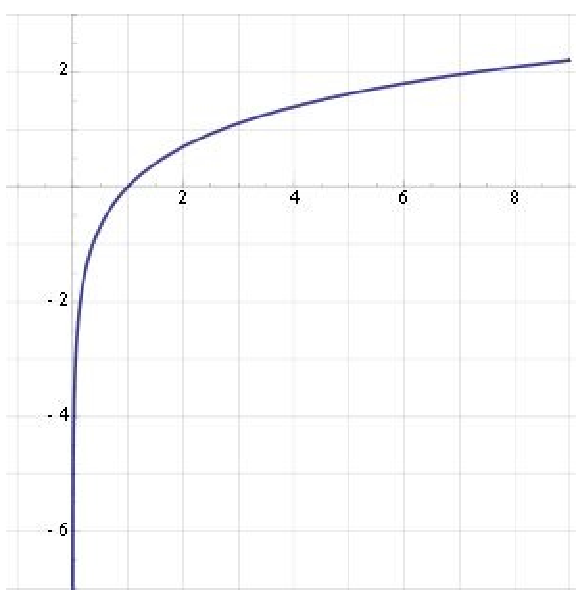
$\log_{10} \frac{1}{4} = -\log_{10} 4$

If we think about why this is, we know that
$\log_{10} 1 = 0$

so that any number that is less than 1, such as 1/4, will be negative.

Any number that is greater than 1, will be positive.

A graph of this is below.



We see that it is positive after 1 (that is, the y value is positive) and negative before 1.

Introduction & Puzzles

Introduction

One of the main things I've found as a private tutor for 14 years is that students are not taught to think. To figure out what to do, when they don't know what to do.

This leads to a significant problem.

Students can know what to do, but they don't know what they are doing.

What I mean by this is that they are happy not to understand why something is, in order to be able to do it.

This is a particular issue in maths.

Another problem is how they behave when they get stuck. They stop.

In fact, they need to adapt the mantra 'I don't know, but can I figure it out?'.

To encourage this, and to get my students to think, I regularly ask them puzzles. It will be worth your while to try these to practice your problem solving skills, out of the box thinking and mental perseverance.

You can find Hints in the next chapter. Solutions in the chapter after.

Puzzle No. 1

You have a 3 litre jug and a 5 litre jug. They are sat beside an infinite pool of water. All you have to do is fill the 5 litre jug with exactly 4 litres. There are no markings on the jugs, and you only know for certain if they are full or empty.

Puzzle No.2

A man steals 2 valuable gold icons from a tribe in the jungle. They each weigh 8 kg. He comes to a rope bridge. He is being chased by the tribe members and has no time but to cross the bridge. The bridge only allows a weight limit of 90kg. He weighs 80kg. How does he get across successfully?

Puzzle No. 3

A man has 2 hourglasses. One measure 4 minutes, the other 7 minutes. How can he use them to measure exactly 9 minutes?

Puzzle No. 4

What stays in the corner, but can go around the world?

Puzzle No. 5

Every day a woman, who lives on the 20th floor, gets in the lift, goes to ground floor and walks to work. On her return she always gets out at the 12th floor, and walks the steps the rest of the way. She hates exercise. Why does she do it?


14
13
12
11
10
9
8
6
4

Puzzle No. 6

A man is walking through a desert. As he reaches the summit of a sand dune, he notices a man lying below. There are no tracks around him. He runs down to the man to discover he is dead, naked, and has half a matchstick clutched in his fingers.

How did he get there?
Why was he naked?
What was the significance of the stick?

Puzzle No. 7

A father and his son are driving in their car when they are involved in an accident. Both are injured and are taken away by separate ambulances. The father to one hospital, the son to another. The son is in a bad way and is immediately taken to the operating theatre. There, the surgeon walks in and says 'That's my son'. How is this possible?

Puzzle No. 8

A man is digging in his garden and he comes across a coin. On it is some writing and a face. The writing says 37 BC. How does he know it's a fake?

Puzzle No. 9

A man walks into a bar and asks for a glass of water. The barman immediately pulls out a gun and points it at the man's head. The man says 'Thank you' and walks out. Why?


UNDER LICENSE OF
Carl Walther Waffenfabrik Ulm/Do.
Modell PPK/S Cal.9mm kurz/.380ACP
WALTHER
WALTHER

Hints

Hints

Hint Puzzle 1

Start by filling the 3L jug, there are only so many options from there.

Hint Puzzle 2

He has to reduce the weight, temporarily.

Hint Puzzle 3

This is starting from now to in 9 minutes time. Start both off at the same time.
How can you measure 1 minute?

Hint Puzzle 4

Sometimes you lick it, sometimes you stick it.

Hint Puzzle 5

On Saturday nights, when she goes out, she comes home and goes to the 16th floor...

Hint Puzzle 6

He must have got there from the sky, since no vehicles or tracks are around.

Hint Puzzle 7

Not an alien, or a stepfather, or a uncle, or any man.

Hint Puzzle 8

Who made the coin?

Hint Puzzle 9

Why would he want water? How do you feel when someone points a gun to your head?

Solutions

Solutions

Solution to Puzzle 1

There are 2 solutions. There are a limited number of things you can do with so few items, so firstly you can fill the 3L jug, empty it into the 5L. Repeat. This will leave you with 1L in the 3L jug. Empty the 5L jug. Put the 1L into the 5L jug. Then put 3L more into the 5L jug!

Other method.

Fill the 5L jug. Pour 3L into the 3L jug. This will leave 2L in the 5L jug. Empty the 3L jug and put in the 2L. Now fill the 5L jug. Empty 1L into the 3L jug to fill it. This will leave exactly 4L. This was in the film 'Die Hard - With A Vengeance'.

Solution to Puzzle 2

He juggles them.

Solution to Puzzle 3

Turn both over. At 4 minutes turn the 4minuter over. At 7 minutes turn the 7minuter over. When the 4minuter runs out again, turn over the 7 minuter as well. We will know it has ran for one minute (since 7 minutes) so it will run for one more, taking it to 9 minutes.

Solution to Puzzle 4

A stamp.

Solution to Puzzle 5

She's short. In the morning she can reach the G button, but at night, she can't select 20, only being able to reach 12.

Solution to Puzzle 6

A hot air balloon
It was having problems, so to reduce weight the passengers stripped off.
This didn't work, so they drew straws to decide who should jump.

Solution to Puzzle 7

The surgeon was his mother. (Tests sexism)

Solution to Puzzle 8

How did the coin-maker know Jesus would be along in 37 years?

Solution to Puzzle 9

He had hiccups. He wanted water, but the barman realised a scare would prove as effective.

Introduction to Sequences

Introduction

Sequences of numbers have fascinated for millennia.

The most basic sequence of all, counting, forms the foundation for algebra, which we will discover in this book. This will then springboard through the rest of algebra with ease and understanding.

Sequences - In A Minute

Sequences

A sequence is a list of numbers that follow a certain rule. You will be already familiar with these if you have ever done times tables.

For example

2, 4, 6, 8, 10....

you'll likely recognise as the 2 times table.

This follows the rule that you just add 2 each time.

Here's another

3, 6, 9, 12, 15....

This, the 3 times table, follows the rule that you add 3 each time. This just follows from the First Rule of Maths, that **Multiplication Is Just Addition**.

As you can see, the list for each would be infinitely long if it were not for the dots at the end. In fact, mathematicians have a short cut to describe these sequences, and these will be used from now on.

For the 2 times table, this is known as 2n.

'n' means the counting numbers. In other words, the numbers that a child would count and start from 1.

1, 2, 3, 4, 5,... and so on.

The two times table, or 2n, is formed by multiplying 2 by each of these numbers
2 x 1 = 2
2 x 2 = 4
2 x 3 = 6

and so on.

Using this more compact form, 2n, describes the whole thing in one simple expression. It takes care of every value up to infinity.

It is known as the ‘nth term’.

So the nth term for the first sequence is 2n. This will give us any term in the sequence, we only have to decide which one we wanted, and multiply it by 2.

Let’s say we want the 37th term. So that will be 2(37) = 74.

The nth term for the second sequence - the three times table - would be?

3n.

These are two very simple examples of sequences, they will be very familiar to you from a young age. However they are now 2n and 3n.

Most sequences are not this straightforward.

An example would be

3, 5, 7, 9, 11...

Here we immediately see that it’s not a ‘times table’. The sequence starts at 3, but doesn’t follow the usual rule of just adding 3.

So what would the ‘nth term’ of this sequence be?

What’s most important is that we examine this from the point of view of the Rules of Maths (like most things!). What we notice is that it increases by the same amount each time. It increases by 2 each time, or 2 is being added every time to each term. Therefore, this has the same definition as the two times table, or 2n. BUT this isn’t 2n! 2n starts at 2. This starts at 3.

What this means is that it must be 2n, but since it starts one away from 2, it must be
2n + 1

Each term is the two times table, plus one.

What about these 3 sequences?

What are the nth terms?

4, 7, 10, 13, 16...

4, 11, 18, 25, 32....

2, 0, -2, -4, -6....

Ans:

3n + 1

7n - 3

and

- 2n - 4 or 4 - 2n

We write that last as 4 - 2n as it's less cluttered, with one less sign, plus it's more intuitive, as it's easy to see that the first term will be 4 - 2 = 2. This sequence follows the Third Rule of Maths. Normally we think of positive times tables since that's what we're very used to. But of course we can have the reverse of that, and have negative ones. The third sequence could be thought of as the -2 times table, plus 4.

All these sequences are called linear sequences. This is because they go up (or down) by the same amount each time. If a graph were to be made of them, they would form a straight line, hence linear.

Again we look at the Third Rule, and conclude that there must be a reverse to this kind of sequences. So the reverse of LINEAR sequences would be..?

Non-linear Sequences

Yes, **NON**-linear sequences!

Here's one. Can you figure out the nth term?

1, 4, 9, 16, 25...

You may notice that the difference between them is different this time. This is how we know it is non-linear. What else? The difference each time seems to increase by the same amount though!

Difference = 3, 5, 7, 9... so that increases by 2 each time. Hmm, interesting!

Have you figured out the nth term? Confident?

Do you recognise these numbers at all? Seen them anywhere before?

You might recognise them as being square numbers. That is the areas of squares with sides of 1, 2, 3, 4, 5... and so on.

That means that they are the areas of squares from the sequence n.

Therefore this nth term will be n^2

This non-linear sequence has a special name - a quadratic sequence - since we will get a shape with four (quad) sides from it. This is just a complicated way of saying that if we multiply 2 numbers together we get a rectangle or a square (just as we noted in Book 1, Multiplication - In A Minute).

This will not give a straight line when graphed so that's why, as well as not having the same amount between terms, this is called non-linear. This is a circular argument, since this will not give a straight line when graphed because it doesn't have the same amount between terms... and it doesn't have the same amount between each term because it won't give a straight line when graphed...

Any power on n, such as $n^3, n^4 or, n^8$, are non-linear. So this is a big family compared to our linear sequences, of just n^1, or n.

Some examples would be...

n^3

1, 8, 27, 64, 625....

n^8

1, 256, 6561, 65536...

A Man Called Al

A Man Called Al

A couple of thousand years ago, all the above was known about. People were merrily using these sequences and it was all well understood.

A question was asked by a man called Al. What if we could use different numbers in the sequence than just the list in n = 1, 2, 3, 4...?

For example, looking at 2n + 1, this generates
3, 5, 7, 9...

as we use n = 1, 2, 3, 4...

But what would happen if we used 1.5? That value is halfway between 1 and 2. Would the answer we get be halfway between 3 and 5?

Let's see

2(1.5) + 1 = 4

Yes! It is halfway!

So it turns out we can use any number, not just from the sequence n, and get answers that will fit on the line, between our current answers. This was expanded further.

What about negative numbers?

What about negative decimals?

Trying

-2

2(-2) + 1 = - 3

adding 2 to this repeatedly takes us to -1, 1, 3, 5... and yes back on to that sequence! So we know that fits.

What about

- 1.5?

2(-1.5) + 1 = - 2

An answer that lies between our answer above, - 3, and the next term, - 1. This tells us that the decimals work too.

So what happened was that n became expanded to an infinite amount of possible values apart from just positive whole numbers as it was before. This made the discrete (steps) become continuous (no gaps). It's like the difference between stairs and a flat slope. You are stuck where on the stairs you can step, only at certain points. The possible locations are discrete. On a slope you can place your foot anywhere. The possible locations are continuous.

This discovery led to a new name being given to all possible numbers. They decided they couldn't use n anymore, so they said that this new 'any number' would be called...

x

Sequences like

2n + 1

became

$2x+1$

instead.

Meaning any value can be used for x

This led to ‘true’ straight lines when graphed as the values between the steps were actual values which were known to be true. Before this was ignored when the dots were joined. We’ll see more of this in ‘Gradient/Equation of A Straight Line - In A Minute’.

Oh, this man called Al.

His surname was Gebra.

Hence Algebra!

Ok, not really. Algebra means ‘restoration’ which doesn’t precisely translate to what we are doing now. However we will see later that we could interpret this as using the Third Rule of Maths, doing the reverse.

Child, 12, Amazes Teacher

Child, 12, Amazes Teacher

In a German schoolroom, a tired and bored teacher takes his class for another day. Wanting half an hour of peace, he decides to give the class yet another boring problem to solve.

"Add up all the numbers from 1 to 100. Quietly."

With a sigh, the class got started.

After 2 minutes one pupil raised his hand.

"What?" asked the teacher.

"I've finished", said the boy.

"You can't have. Show me."

Indeed the boy had finished. He showed his work to the teacher. In it was a ultra fast way to add up sequences.

What had the boy figured out?

Can you figure out a fast way to add up 1 to 100, apart from adding 1 + 2 + 3 + 4 and so on?

Have a think!

The boy, Carl, had realised that there's no rule that says you have to add them up in order. He also knew well that Multiplication was just addition. So if he could turn this addition into a multiplication, it would be an easy task. But how?

He noticed that he had

1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100

to do.

Adding the two outer numbers, gives 1 + 100 = 101
Then the next two 2 + 99 = 101
Then the next two 3 + 98 = 101
And so on.

Each addition gives 101.

As there are one hundred numbers, that makes 50 additions.

50 additions of 101 will be
50 x 101

101
x 50

5050

So the answer is 5 050!

Try this yourself with 1 - 10

1 + 10 = 11
2 + 9 = 11

there will be 5 of these
5 x 11 = 55.

Easy.

So it turns out all we do in general is
Add 1 to the final value (100 + 1, or 10 + 1) and multiply it by half the number there are (50 or 5).

So 1 - 16

will be 17 x 8

 17
x 8

136

For odd numbers

1 - 11

It will still work.

12 x 5.5

  55
x 12

660

One decimal place (as in ‘Decimals - In A Minute’).

66.0

or just

66

We know 1-10 is 55, so 1-11 will be 55 + 11 = 66. Yes it works!

The boy, Carl, went on to become a famous mathematician and scientist.

His full name was

Carl Friedrich Gauss.

The unit of magnetic flux density, the gauss, is named after him.

Example Questions

In Maths questions you can be asked to find the nth term, such as we have seen, or to write a sequence given the nth term. Here's an example of both.

Find the nth term
2, 8, 18, 32, 50....

Write out the first four terms of the sequence 5 - 3n

Ans:

$2n^2$

2, -1, -4, -7...

Introduction to Gradient

Introduction

Having looked at sequences, we know look at the equation of a straight line. This is the first stepping stone to understanding and being able to do algebra. Follow on from here, understand it well, and you will be well on your way to becoming a mathematician!

What Is Gradient?

Imagine you are a climbing a hill. Is it easy or difficult?

If easy, why?

If difficult, why?

It may be because you're unfit, but assuming not (!), then it may well be due to the steepness of the hill.

The steeper it is, the harder it is to walk up.

Imagine you are climbing this hill.

I call you up and ask you what you are up to.

You say *'I'm climbing a hill! It's very steep.'*

I ask

'How steep?'

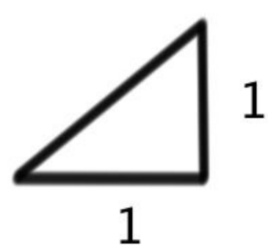
What would you say then? Does 'very' really qualify? What I think is very steep may not be the same for you as for me. So, we need to *attach a number* to this so that we both know how steep it is and what that means.

How to Calculate Steepness

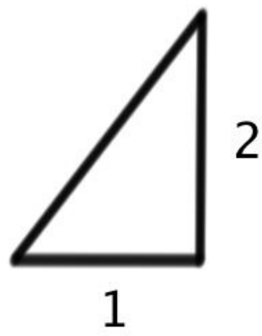
Steepness of a hill, or its gradient, is calculated in a very simple manner.

For every step you go along horizontally, how many steps do you have to go up?

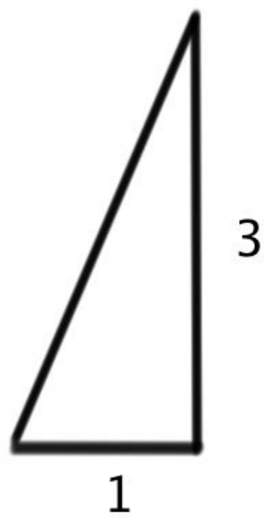
For one step up



For two steps



For three steps



So as you walk one along, the amount you have to go up is its gradient.

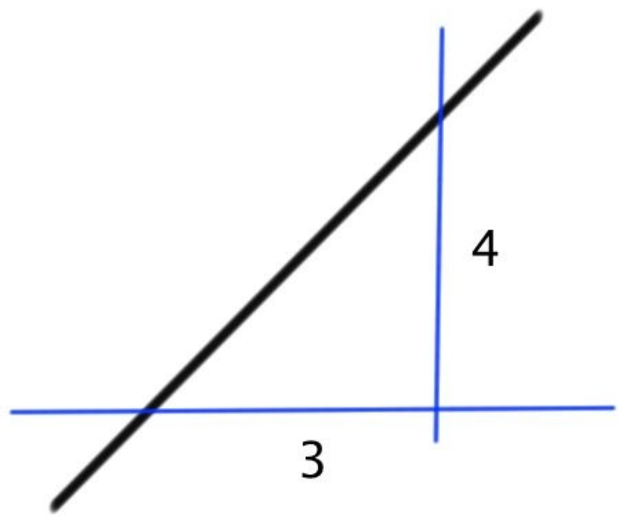
If you measure it over more than one horizontal step, you have to divide the vertical measurement by the horizontal
So

$$gradient = \frac{up}{across}$$

The reason we do this is to normalise the gradient. Since it is a measure per horizontal step, we need to reduce it in size so that the base of the triangle is only

1. We have seen normalisation before, in Percentages - In A Minute.

To calculate ANY gradient, we simply drop a vertical line anywhere. We then put a horizontal line across anywhere. Measuring the sides of this new triangle, and dividing (as we want the base to be 1), we find its gradient.



Here we need to do

$$gradient = \frac{4}{3}$$

To see what it would be for just one step, not 3. So we get

$$1\frac{1}{3}$$

Although sometimes we would leave it as

$$\frac{4}{3}$$

Since gradient is calculated using a division (unless the base is 1), how many types of gradient do you think there will be?

Three.

Since we have three types of division, from Division - In A Minute, we will have three types of gradient, which will be
> 1

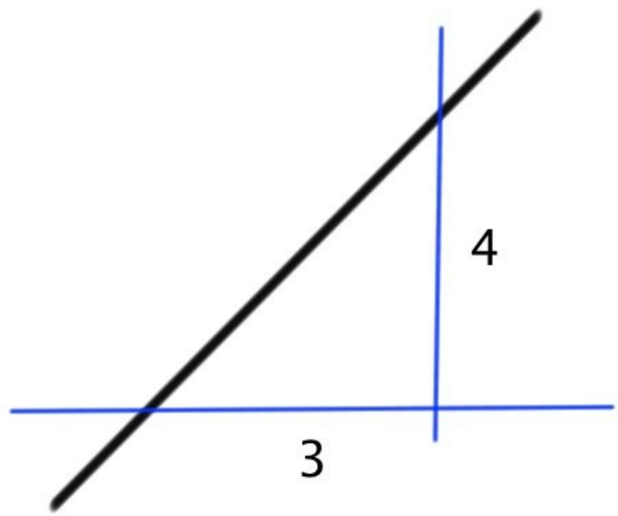
= 1

&

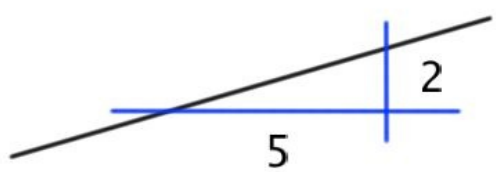
< 1

An example of > 1

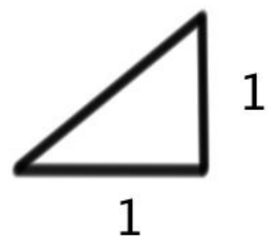
Is the one we’ve just looked at:



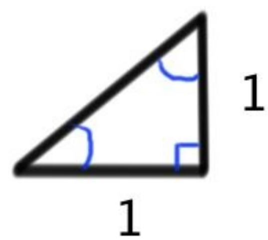
An example of < 1



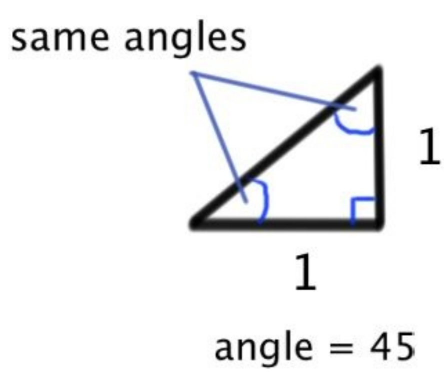
An example of = 1



For = 1, we might notice that the sides of the triangle will be the same. As a result this is an isosceles triangle. The triangles we create for calculating gradient are all ‘right-angled triangles’, in other words, they have an angle of 90 degrees. What would the other two angles be in this example?



Since the angles must add up to 180 degrees, as in any triangle (why?), the other two angles must add up to 90. Since it is isosceles, they will be the same, so a gradient of 1 means it has an angle of 45 degrees. **This is very important and is extremely useful to remember.**



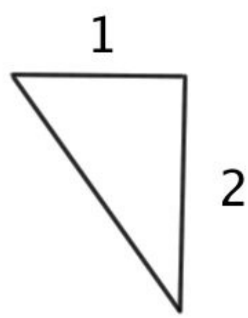
The letter we use to denote gradient is *m*, which is apparently short for *monter*, the French for ‘to climb’. You might like to think of it as being short for mountain.

So we’ve looked at all three types of gradient. However, here comes the third rule. We can always do the reverse! So we also have another type of gradient. Hills go up, but they also go down. So we also have negative gradients.

These are measured in exactly the same way, except we say for every step we go along, it is how many DOWN we have to go.

For example, if m = - 2

For every one we go along, we go down 2.



Just as for positive gradients, there are three types.

These will be... what?

Of course,

< - 1

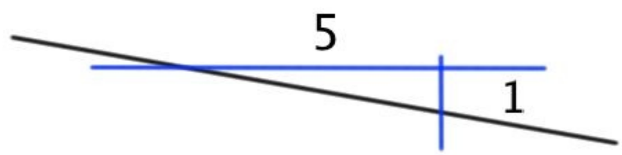
= - 1

&

> - 1

Bear in mind that the ‘more than’ type means a fraction here, and the ‘less-than’ means a whole number greater than 1 (or a mixed number), but negative. So this is also the reverse!

An example



For every 5 we go along, we go down one. So this will be negative. Dividing by 5 to make it only one horizontal step…

$$m = \frac{1}{5}$$

$$m = -\frac{1}{5}$$

As it’s negative. This is > - 1.

So they are all our types of gradients.

The Equation of A Straight Line

Where We Use Gradients

In ‘Sequences - In A Minute’, we saw how times tables were generalised into sequences. Instead of saying ‘the two times table’, mathematicians say ‘2n’. This had a variation, where we could have something like

$2n+1$

where this was the two times table, with 1 added to each term.

Can you remember what these sequences were called?

These were

Linear sequences

as they increased (or decreased) by a set amount. In this case, two.

We then saw that the development of algebra led to n being replaced by

x

where x is any number.

This led to a new equation for the above

$y=2x+1$

where

$2x+1$

is our original sequence, but including any value for x.

This meant that we had an infinity of possible answers for an infinite amount of possible inputs. The result of this was to have a constant, continuous set of inputs and outputs that led to a straight line which was connected by dots rather than huge leaps.

y

in

$y = 2x + 1$

is all the possible values.

For example, if $x = -2$

$y = 2(-2) + 1$

$y = -3$

So y is just the answer we get.

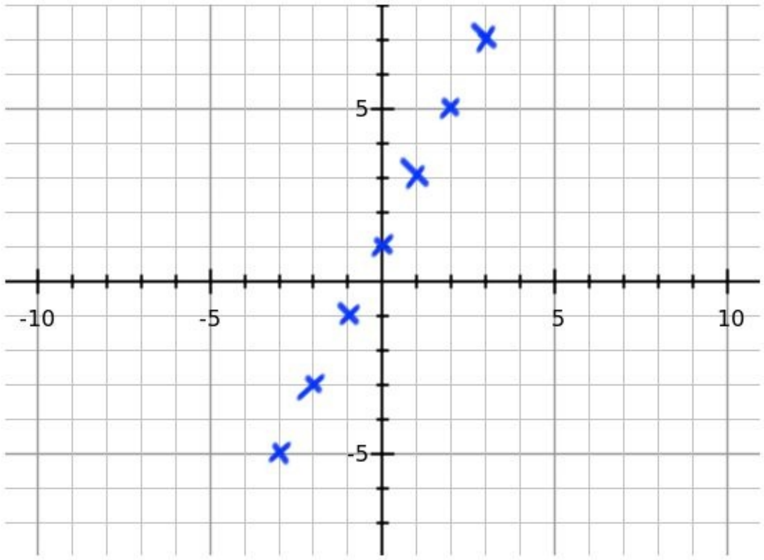
If we draw a table of this

x	- 3	- 2	- 1	**0**	**1**	**2**	**3**
$y = 2x + 1$	- 5	- 3	- 1	**1**	**3**	**5**	**7**

We can choose any values for x we like and calculate what the y values would be.

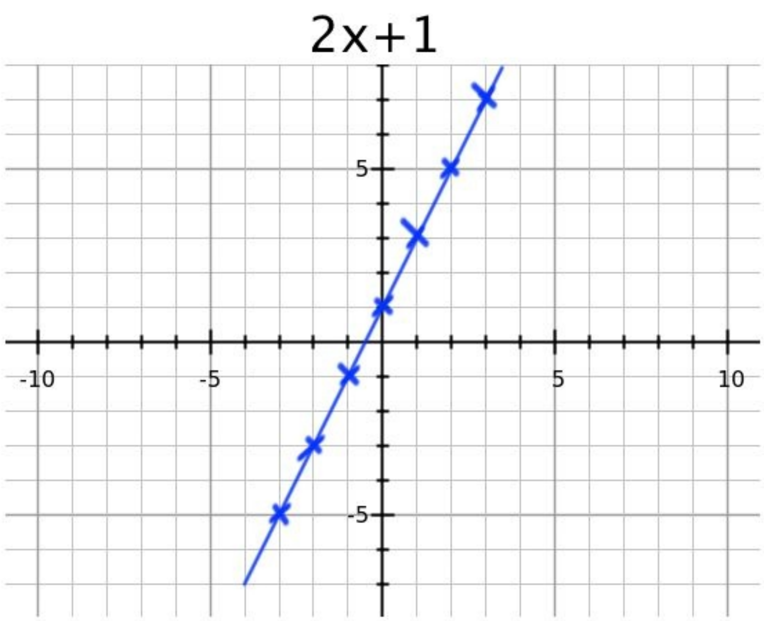
If we use round numbers we'll see that they follow the two times table, plus 1, from 2n+ 1, of course!

Once we have all these values, what we can do is plot them on a graph. This graph is technically known as the 'Cartesian Axis'. In other words it is two number lines. One for x and one for y. They cross each other at 90 degrees and at 0 for each.



Once we plot these points we can do a ‘dot-to-dot’, you may remember from your primary school days, and join them up!

This gives a straight line.

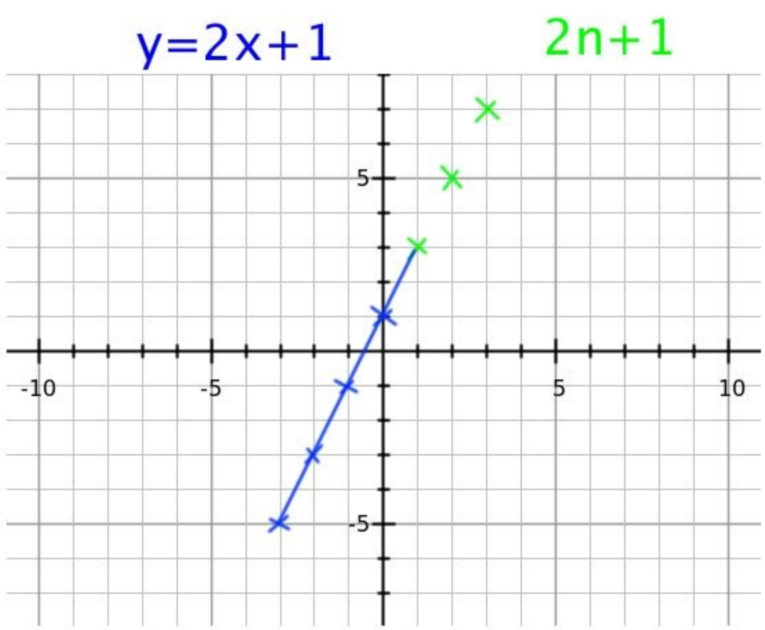


Not surprisingly then, this

$y = 2x + 1$

is called an *'Equation of A Straight Line'*.

This line actually represents all the possible values that x and y can take. Remember we said that now we are using x there are an infinite number of possibilities for x and y. So this is a true line. The previous graph only demonstrated this for 2n + 1 really. Here is a comparison.



What I would like you to do now is try a couple yourself.

Using graph paper, draw an x and y axis.

Then do three tables like these

tables

$y = 2x + 1$

$y = 3x + 2$

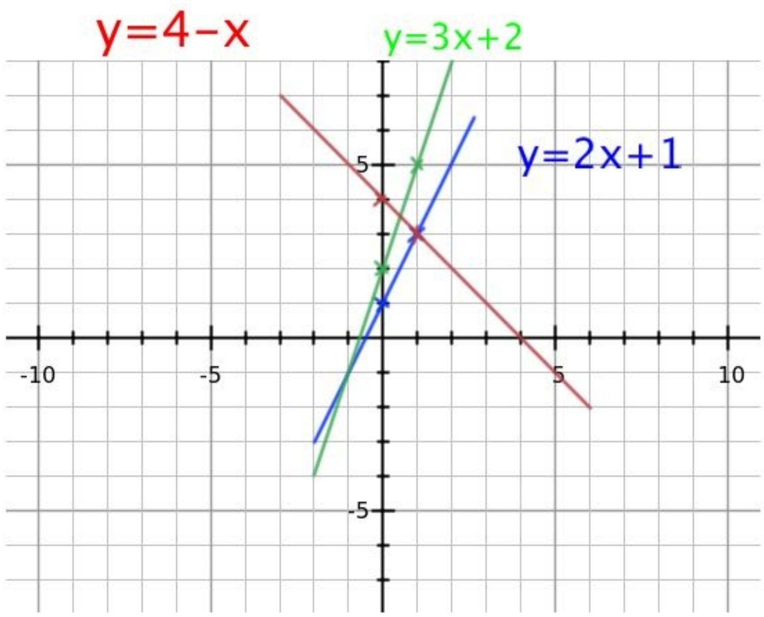
$y = 4 - x$

And calculate the y values

Having done that, plot the points. Then read on further to see if you are correct.

Make sure you do this exercise, as you need this sheet to make further calculations later in this book!

Did you get these?



Often people get the last one wrong.

Calculating the Gradient and ‘Cut

Calculating the gradient and ‘cut’.

Straight lines such as these have 2 characteristics, the gradient, which we’ve already met, and something I call the ‘cut’. This is the place where the line cuts the y-axis. Please calculate the gradients and cuts for all three of your straight lines.

Remember to calculate the gradient, just draw a horizontal and vertical line, and

$$gradient = \frac{up}{across}$$

Fill in this table, and see if you can spot any pattern?

	m	c
$y = 2x + 1$	**2**	**1**
$y = 3x + 2$	**3**	**2**
$y = 4 - x$	- 1	**4**

You might be able to see that the numbers in the equations and the columns m and c match up!

The numbers multiplied by x are the gradients, the numbers NOT multiplied by x are the cuts.

For the negative one, it’s the reverse. Of course we can always do the reverse...

This means that the information about the straight line’s characteristics is in the equation.

As a result, we don’t really need to make tables for straight lines. Since we know its gradient and cut, we know a starting point, its cut, and from there we can go along one, and up or down whatever the gradient is.

This will give a second point. Since two points is all we need for a straight line, we can just join them and extend either side.

If you look again at the graphs above, you’ll notice that I only made 2 points for each line. That was because I knew the gradient and cut of each line.

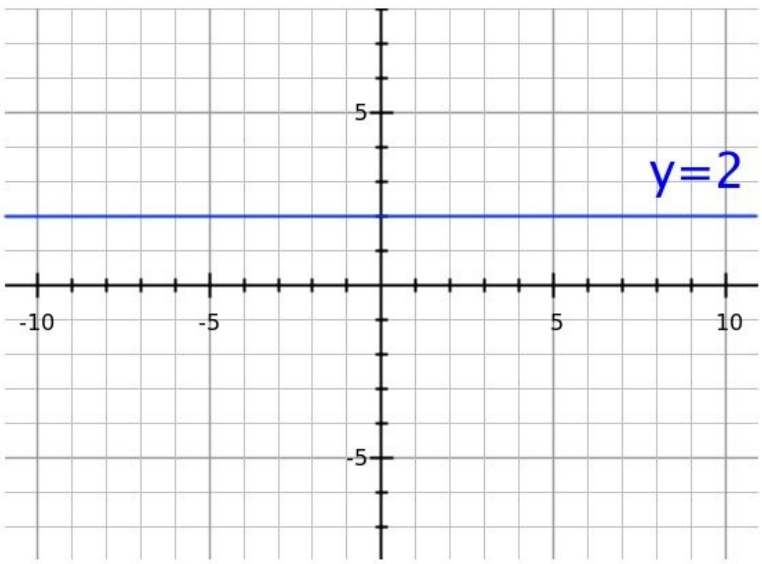
In practice, this is what mathematicians do. In an exam, you’d be expected to fill out a table.

Other Straight Line Graphs

Other Straight Line Graphs

$y = 2$

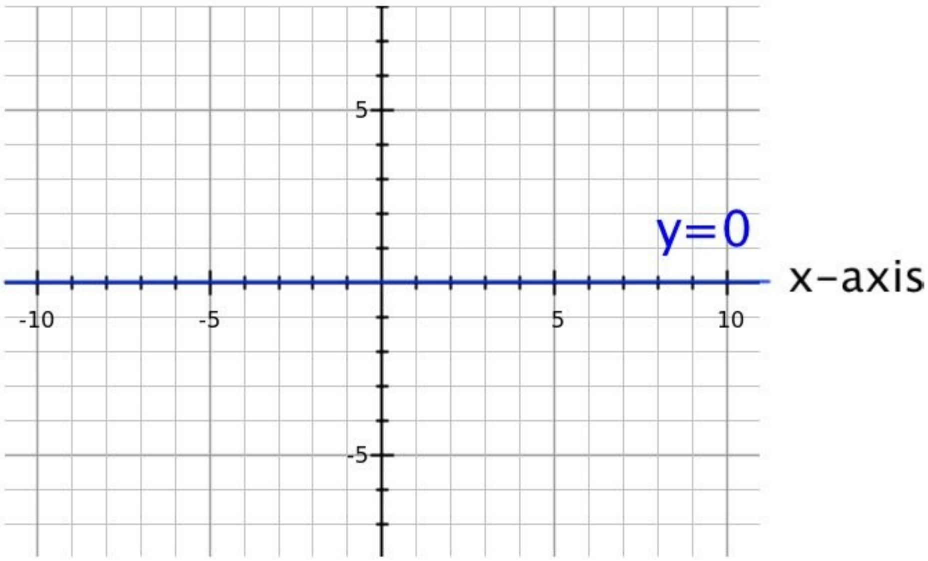
If m = 0, and we have a cut of 2, we'll have the equation $y = 2$ and this looks like this.



Note there is no gradient, so it is horizontal.

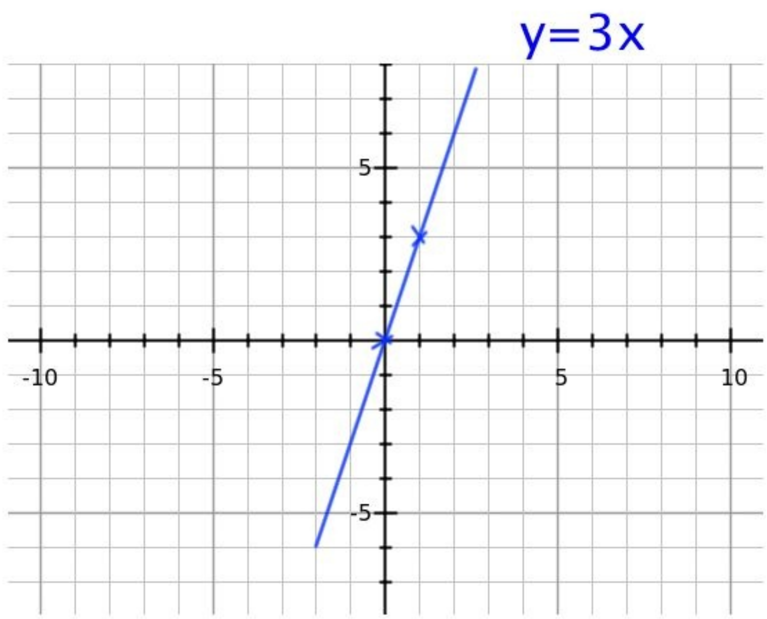
$y = 0$

This has no gradient or cut, so in fact it is the equation of the x-axis!



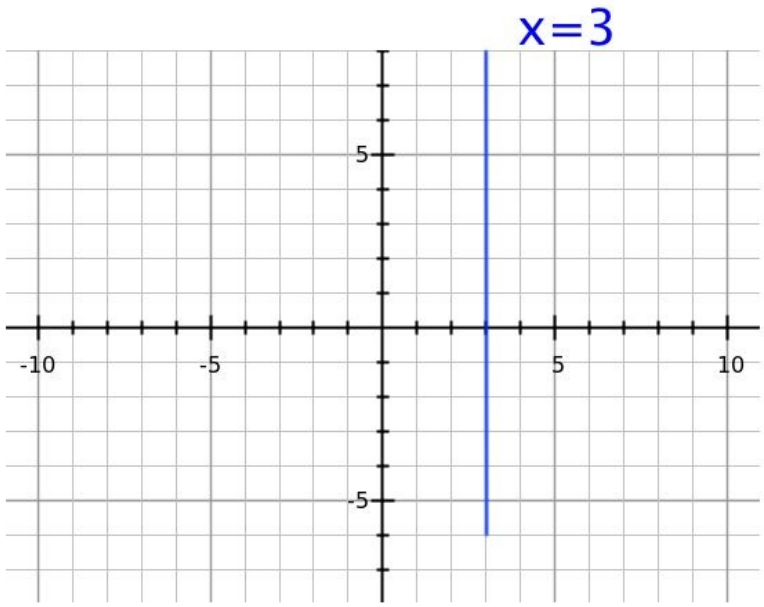
$y = 3x$

This has no cut, but a gradient of 3.



$x = 3$

Here, the equation does not begin y = ... so it is all the places that x = 3, which is a vertical line of points.



Read on to

Simultaneous Equations - In A Minute

to see what we will do with all of these lines.

Introduction to Simultaneous Equations

Introduction

Having looked at sequences and the equation of a straight line, we're now ready to do something with these which is of a practical or abstract use. These form the building blocks for solving all sorts of maths problems, and once we master these, we can learn how to do the same with quadratics, cubics or higher, as well as solve problems in calculus, which is especially useful.

In this book you will learn that it is quite straight-forward, not entirely abstract, and will leave you wondering why they made it so difficult at school!

Simultaneous Equations

Simultaneous Equations

Before we look at what these are, let's look at

The Three Types of Algebra

As you can see from the title, the good news is that there are only 3 types of algebra. The even better news is that we've come across one already, in Book 2, Division - In A Minute.

We saw a question like

$$\frac{6}{2}$$

and asked how to do it.

The first, intuitive instinct was to ask
'How many times does 2 go into 6?' or in other words,
'What do I have to multiply 2 by to get 6?'

and we called that the Algebra Trick, as it was asking
2 x ? = 6

or

$$2x = 6$$

what's x?

This is also the first type of algebra.

So Type 1

$$4x = 24$$

What’s x?

x must be 6 as

4 x 6 = 24

Remember, when we see letters next to each other, like $4x$ or ab, this means $4\times x$ or $a\times b$.

Another one

$3x=12$

$x=4$

Because 3 x 4 = 12

and finally

$2x=4$

$x=2$

Because 2 x 2 = 4

All very simple.

So these are actually doing the reverse of divisions. We’re just saying what we would multiply.

Another way to do these would be, of course, to do the reverse, and actually DO THE DIVISION, although this just leads us to asking what we would multiply it by, which is absolutely crazy! You turn it into a division, just to ask what would multiply it! Not surprisingly then, this is the method favoured in schools.

For example,

$4x=12$

You could turn that into

$$x = \frac{12}{4}$$

By dividing both sides by 4.

So now you ask...

4 x ? = 12?

So to answer this you need to think of what multiplies to get 12! And of course it is 3.

But if we had just done that from the start, where we had

$$4x = 12$$

and asked, 4 x ? = 12, we could have got to 3 sooner.

So there's ABSOLUTELY no need to do this division exercise. It's a complete waste of time.

Unless.

Unless unless unless.

What if it was

$$4x = 13$$

then what?

We can still do it this way.

4 x ? = 13

leads to think, at least 3, with 1 remainder.

In other words

3 r 1

And so that becomes

$$3\frac{1}{4}$$

That’s a small jump to doing these questions ‘on the fly’ instead of going through the rigmarole of writing

$$x = \frac{13}{4}$$

and then going through exactly the same thought process.

Okay. What about

$$8x = 2$$

Then what!?!

Here we see we may struggle to think of

8 x ? = 2

as it is clearly going to be less than 1.

So here, and only here, do we do that division. Why? Because we can clearly see that it’s a less than 1 type, and these are not solved by doing a multiplication or the Algebra Trick.

As a result, we have to make it in a division so that we can simplify it.

This gives

$$x = \frac{2}{8}$$

simplifying,

$$x = \frac{1}{4}$$

And that makes sense. Since

$$8 \times \frac{1}{4} = 2$$

The great advantage of these **Three Types of Algebra** is that it is IMPOSSIBLE to get things wrong, as we can just place our answer back into the original question, and see if it works or not.

$$8x = 2$$

When

$$x = \frac{1}{4}$$

If it works it's correct.

If it doesn't, we've gone wrong somewhere.

That's all there is to Type 1 Algebra!

Type 2 Algebra

Type 2 Algebra

This is just Type 1 algebra with an extra bit. That's all.

So for example

$$2x + 7 = 15$$

At school we're taught a complicated method for this now, but let's just look at what this says.

It says

‘Something plus 7 equals 15.’

Ok.

So we need to figure out what the ‘something’ is.

What must that something be equal to?

Since we know that to make 15, you need to add 8 to 7, that ‘something’ must be 8, right?

So

8 + 7 = 15

Therefore that something equals 8

So

$2x = 8$

What's happened now is that from Type 2 algebra, we've got a Type 1.

And that's our goal. If we have Type 2, make it Type 1, as these are simple. This will always be the goal.

So

$2x = 8$

We say

‘2 x ? = 8’

and that must be

$x = 4$

Another example

$3x - 9 = 30$

First thought must be

What minus 9 equals 30?

Of course it is 39.

So we write

$3x = 39$

as this must logically be true.

So we have our Type 1

$x = 13$

We can test both our answers too.

$2x + 7 = 15, x = 4$

2(4) + 7 = 8 + 7 = 15 - yes that's correct!

$3x - 9 = 30, x = 13$

3(13) - 9 = 39 - 9 = 30 - yes that's correct!

Simple again!

So that's Type 2 algebra.

Type 3 Algebra

Type 3 Algebra

This is just Type 2 with an extra bit!

So we need to cascade down the Types. Change

Type 3 into Type 2
Type 2 into Type 1.

Let's look at an example.

$3x+1=x+6$

- Type 3 has inserted an extra x term on the right hand side here. Remember though that

1, we need to change it to Type 2

2, both sides are equal

To make it Type 2, we need to get rid of that extra bit on the right hand side. It's really ruined things. To do that we can think of picking it up or rubbing it off that side. However, since both sides are equal, we must do the same thing to the other side.

For example if we had

$4+5+2=3+6+2$

This is true.

If we remove the 2 from the right hand side, we need to do that to the other, or it won't be true anymore..

$4+5+2 \neq 3+6$

but

4 + 5 = 3 + 6

So back to our example

$3x+1=x+6$

If we remove an x from the right, we must from the left.

Since there are $3x's$ on the left, when we take one away, we are left with 2.

$2x+1=6$

This is now Type 2.

So we ask

‘What plus 1 equals 6?’

Obviously, it is 5

So therefore

$2x=5$

and now it is Type 1

$x=2\frac{1}{2}$

To check:

$3\left(2\frac{1}{2}\right)+1=2\frac{1}{2}+6$

$8\frac{1}{2}=8\frac{1}{2}$

which is true!

Another example

$4x+3=-3x+5$

Again, this is a Type 3. We want to get rid of that extra x on the right hand side, to make it into a Type 2.

However, this is slightly different. We notice that there's a minus sign. So what do we do here?

Just the reverse!

Before, with a positive x, we subtracted. We took the x we didn't want away. Here, we just add! Essentially, we are trying to get a zero. Before we took away so there was none there. Here we have to add to make $-3x$ become zero. So we need to add... $3x$ of course!

If we add $3x$ to the right hand side to achieve this, we must also do the same to the left to equalise things.

So

$4x+3=-3x+5$

becomes

$7x+3=5$

We have Type 2

$7x=2$

We have Type 1

$x=\frac{2}{7}$

Finished.

To check

$$4\left(\frac{2}{7}\right)+3=-3\left(\frac{2}{7}\right)+5$$

$$4\frac{1}{7}=4\frac{1}{7}$$

And that’s Type 3 algebra!

What We’re Doing This For

What We’re Doing This For

So why bother trying to find x? All we seem to be doing is unpicking an unnecessarily complicated puzzle for no reason.

Well, there is a reason!

Here’s what it’s for.

If we look at the first example, our Type 1,
$2x=6$

what does this mean apart from
‘2 times something equals 6’?

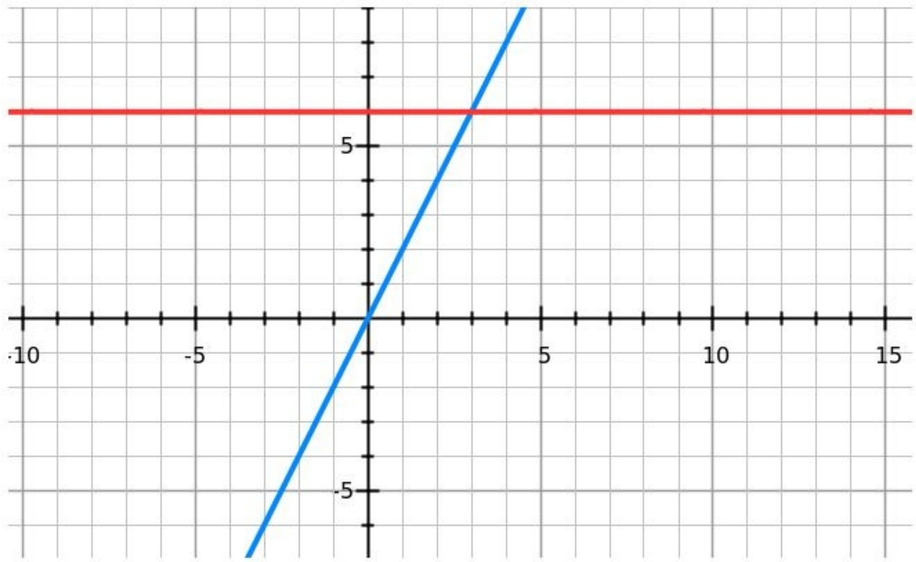
If we think back to the Equation of A Straight Line, in the previous book, we could sketch (or plot) two straight lines
$y=2x$

Blue

and

$y = 6$

Red



If we do this we can see that they cross each other. The question that interests mathematicians is ‘where do they cross?’.

To find this, we could look at the graph, and read off the co-ordinates. However this isn’t always accurate, especially if you draw as badly as me!

So a better, exact, way to do it would be to notice something about

$y = 2x$ and $y = 6$

What do you notice about them?

Do they have anything in common?

Of course they both equal y.

What can we do with this? Let’s look at these two examples
6 + 4 = 10

2 x 5 = 10

Notice we’ve written two ways to describe ten. So we could write

6 + 4 = 2 x 5

and not even mention ten!

So we can do the same with our equations
$y = 2x$ and $y = 6$

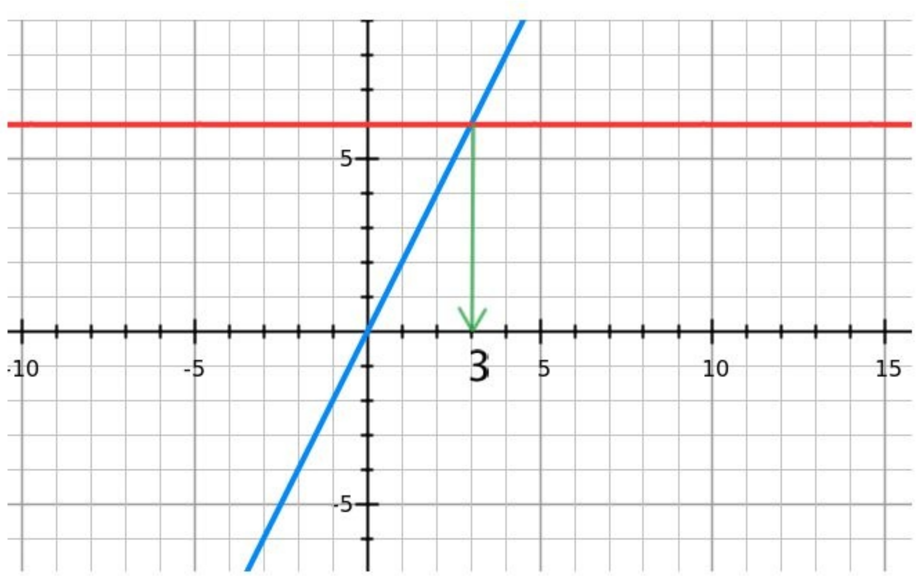
and write that as they both equal y, they must equal each other.

So

$2x = 6$

This is Type 1 algebra, and ‘solving’ it, will tell us where they two lines cross
$x = 3$

If we look on the graph, we can see this is true.



Because we know y = 6 the co-ordinates where they cross will be
(3, 6).

And we’ve found where they cross.

That’s just one thing we can do with the Three Types of Algebra.

Another Example

$$2x + 1 = 6$$

If we solve this we find

$$2x = 5$$

and

$$x = 2\frac{1}{2}$$

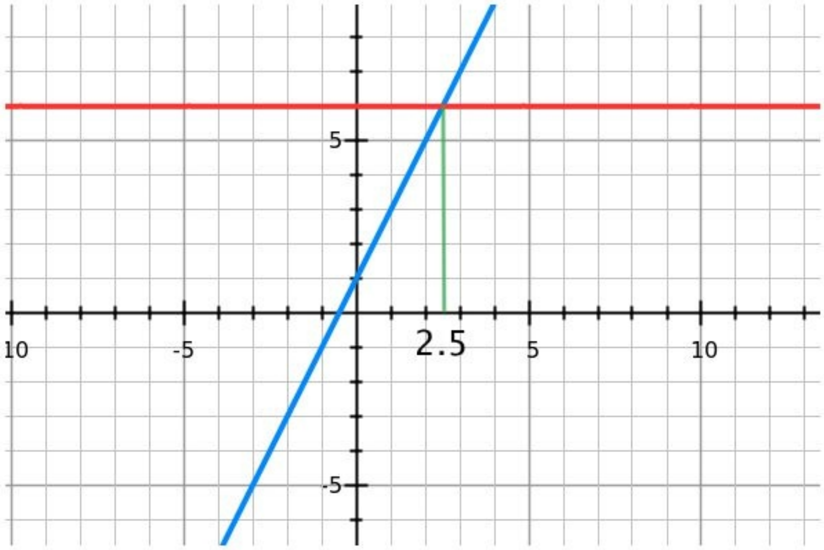
If we sketch the lines

$$y = 2x + 1$$

and

$$y = 6$$

We will have



And we can see that they cross at

$$x = 2\frac{1}{2}$$

Finally

$4x+3=-3x+5$

Solving, as above, gives

$7x=2$

$x=\frac{2}{7}$

Again, sketching

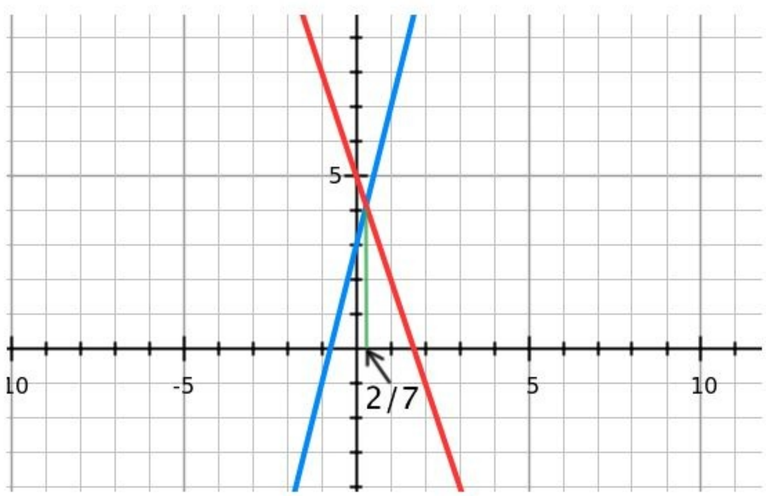
$y=4x+3$

&

$y=-3x+5$

we see that they cross at

$x=\frac{2}{7}$



Real-Life Example of Simultaneous Equations

Real-Life Example of Simultaneous Equations

So what's the point of drawing graphs to find where they intersect? Why do we care?

Often in real life, we don't get to use algebra very much. However there have been a couple of occasions in my life where I have.

Here is one.

Years ago, when the internet began, there was something called the 'Internet Cafe'. Here you could surf the internet while downing a coffee.

I went to one while visiting a city. They had two options,

1. Pay as you go, for £3 an hour
2. Become a member for £5 and get £1 an hour for ever.

Which is the best option?

The question really is, how much time do you want to use the computers for? If you want to use it for an hour, we can see it'd be cheaper to pay as you go. But at what point would it be cheaper to become a member?

If we describe each situation as a straight line equation, we find
Option 1

Cost = 3t

The cost will be 3 pounds per hour, and if t=1, or 1 hour, the cost will be £3. In equation of a line terms, the gradient is 3, since this is essentially the three times table.

Option 2

Cost = 1t + 5

Here we have an initial cost (or cut) of £5. Then for every hour after that we pay £1.

If we make these slightly more mathematical

$$C = 3t$$
$$C = t + 5$$

Since they both equal C, we can make them equal to each other
$$3t = t + 5$$

And solve for t to find at what point it would be cheaper to become a member (or the break-even point).

This is Type 3 algebra, since t is on both sides
$$2t = 5$$

$$t = 2\frac{1}{2}$$

So after $2\,{}^1\!/_2$ hours, it would be cheaper to be a member. If you want to use the internet for less than that, go Pay as You Go, but otherwise, being a member is better.

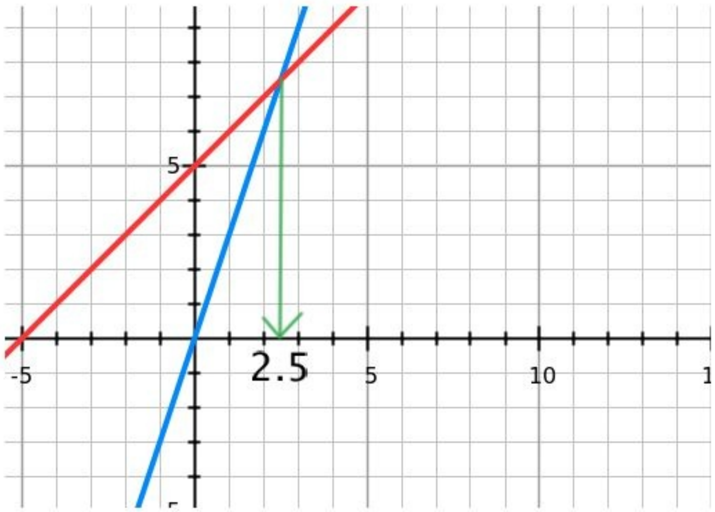
We can see this on a graph also.

$$C = 3t$$

Blue line

$C = t + 5$

Red line



Simultaneous Equations - School Style

Simultaneous Equations - School Style

If you're reading this for exam purposes, you will need to know a different technique for finding where two lines cross each other.

For more complicated situations, far in advance of A-level maths, this method forms the basis to solve them. However, for some reason, this method is still required at GCSE.

What you will be given is the equations of two straight lines, as above, but in a different format, such as
$2x + y = 6$

all this really is

$y = -2x + 6$

that is to say, it is a straight line with gradient of - 2 and cut of 6.

If we have another straight line, such as
$3x - y = 14$

which is really

$y = 3x - 14$

we could find out where they cross by making them equal to each other as we did above.

However, since you get them in the format
$2x + y = 6$
$3x - y = 14$

We can do them in a different way.

Instead of the above method, we could simply add the equations.

This would give

$$2x + y = 6$$

$$+$$

$$3x - y = 14$$

$$\overline{5x + 0 = 20}$$

Or

$5x = 20$

Then we have Type 1 algebra, to give

$x = 4$

Since we now know that $x = 4$,

$2(4) + y = 6$

$8 + y = 6$

$y = -2$

Co-ordinates

$(4, -2)$

That's it!

If the original signs had been the opposite, we could have just done the reverse, and subtracted.

So if it was

$2x + y = 6$

$3x + y = 14$

We could have subtracted instead, and this would lead to

$2x + y = 6$

-

$3x + y = 14$

$-x + 0 = -8$

$x = 8$

There is one other variation.

It was just lucky that the last example added nicely. But what if they don't?

In this example, we can't use that technique.

$2x + 3y = 8$

$5x + 2y = 9$

In this case we need to make either the x's or the y's the same number and then we can subtract the two equations.

Here the lowest numbers are the y's, (3 and 2). Because of this we can use a technique we learnt in Fractions - In A Minute, where we find the lowest common number of the two times tables.

2 4 6 8 10 12

3 6

which turns out to be **6.**

So we can make the y's equal to 6 by multiplication (just like with Fractions).

$$2x+3y=8 \ldots \times 2$$
$$4x+6y=16$$

and

$$5x+2y=9 \ldots \times 3$$

$$15x+6y=27$$

Now they both have equal y terms.

$$4x+6y=16$$
$$15x+6y=27$$

If we subtract these, but first change the order for convenience

$$15x+6y=27$$

-

$$4x+6y=16$$

$$11x+0=11$$

$$x=1$$

to find y

$$2(1)+3y=8$$

$$3y=6$$

$$y=2$$

Introduction to Quadratic Equations

To start this book, I want to ask you a *couple of unusual questions.*

One, is there gravity in space?

Yes or No?

When the space shuttle or rocket takes off, what path does it take to get into space?

SPACE

Bear in mind your answers to these questions, as we begin to learn about...

Multiplying Brackets

Multiplying Brackets

What’s

5(7) =

35

Because ‘brackets mean multiply’!

What about if we split the 7 up into
5(5 + 2) =

what then?

The best thing to do here is to multiply straight out, to get
5(5 + 2) = 25 + 10 = 35

Why not just add the 5 + 2? Because I want you to see that you should never use that method. Brackets mean multiply, so multiply!

For example, it doesn’t take much to show that adding doesn’t always work.

$5(x+1) = 5x+5$

It is impossible to add the $x+1$. Since this will be the case for the majority of the time, never add.

Multiplying a bracket like this is also known as ‘expanding’ a bracket.
Sometimes you’ll see a question like
Expand

$4(2x-3)$

and the answer will be

$8x - 12$

Because you simply multiply.

Try these

$3(x - 2)$

$4(4x - 9)$

$3(a + b)$

Ans.

$3x - 6$

$16x - 36$

$3a + 3b$

Factorising

Doing the Reverse

Okay, so we can multiply (or expand) a bracket. Of course, according to the Third Rule, there must be a reverse to this.

Since we are multiplying, the reverse of it is clearly dividing. However this is known as ‘factorising’. But factorising and dividing are the same word! For instance,
If we divide

$$\frac{12}{4}$$

we are factorising it, because we want to know
$4 \times ? = 12$

In other words, what is the other factor (apart from 4) that goes into 12. This is of course 3. So we have factorised 12, to find 3, using another factor. This is an important idea. Please hold on to it.

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

As above, we factorised (divided) 12, using a factor (4). From that we found another factor (3). Both 4 and 3 divide into 12.

To do the same with

$$3x - 6$$

we need to do the same. A factor of both $3x$ and -6 is needed to be able to divide or factorise this expression.

We already know it is 3 because we have done the original question(!), but we can see that both $3x$ and -6 can be divided by 3.

So we take 3 to one side, and write a bracket...

$3($

We then fill in the bracket using the idea of

What times by 3 to get $3x$?

x

So we now have

$3(x$

And what times by 3 to get - 6?

-2

So we have

$3(x-2)$

and we close the bracket as there are no more terms.

And we end up with what we started with.

For

$3a+3b$

Again we can see that it will be 3 as a ‘common factor’ as it is called. The number or term that will divide into both.

Placing 3 in front of a bracket

$3($

and asking

What times 3 gives $3a$

a

$3(a$

And what times 3 gives $3b$

b

So we have

$3(a+b)$

Interesting note.

If we only have letters to factorise, this is actually EASIER than with numbers, as we don’t have to find highest common numbers or anything like that.

For instance

$ab + ac + ad$

The common factor is a

$a($

And running through the same questions will give
$a(b + c + d)$

Much easier.

Multiplying - In A Minute

Multiplying 2 Brackets

Going back to

$5(5+2)$

where all of this started from, we've learnt to multiply that out (or expand) and to do the reverse, divide (or factorise).

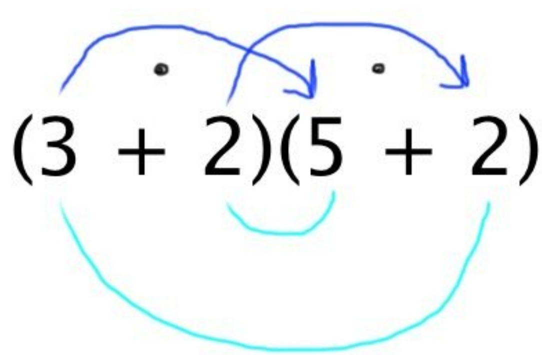
If we take this a little further by writing the 5 as
$(3+2)(5+2)$

What should we do here?

I've already asked you not to add what's inside the brackets, so don't do that! We know the answer will be 35, but how do we do this?

The school method is to use the smiley face or FOIL technique. By now you'll have realised that I never use the school method, as I've invented something better! So how?

For now, let's look at the smiley face method.



This will give

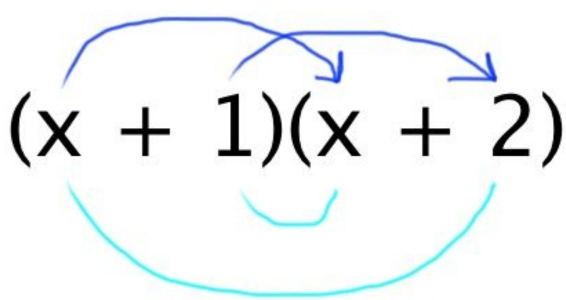
$15+4+10+6=35$

That’s fine.

What about for

$(x+1)(x+2)$

Same process



$x^2+2+1x+2x$

$=x^2+3x+2$

Notice I’ve added these up afterwards in order. The order we use in mathematics is to do with powers. So we have the highest power first (x^2), then the next ($3x$, which is x^1) and then no x at all. Actually this is $2x^0$, but what is x^0 equal to?

$x^0=1$

So it is 2 x 1 = 2.

So a disadvantage with the smiley face method is having to put these in order. This is known as ‘degree order’.

My method

This should be very familiar to you by now, especially if you have read 'Multiplication - In A Minute'.

Instead of writing the brackets in a row,
$(x+1)(x+2)$

Write them in a column

$$\begin{array}{r}(x+1)\\ \times(x+2)\end{array}$$

And using exactly the same multiplication technique you've used countless times since Book 1, multiply the brackets!

$$\begin{array}{r}(x+1)\\ \Downarrow\\ \times(x+2)\\ \text{------------}\\ +2\end{array}$$

then...

$$\begin{array}{r}(x+1)\\ \times\\ \times(x+2)\end{array}$$

$3x+2$

The cross here will be

$2x+1x=3x$

And finally the left column

$$\begin{array}{l} \quad (x+1) \\ \quad\ \Downarrow \\ \times(x+2) \end{array}$$

$x^2 + 3x + 2$

Much easier, and note that the terms drop out in degree order (every time).

Try a few yourself now.

$(x+3)(x+4)$

$(x+3)(x-4)$

$(x-2)(x-3)$

Ans.

$x^2 + 7x + 12$

$x^2 - x - 12$

$x^2 - 5x + 6$

Each time you should find they easily drop out.

Here is the last one as a worked example to check your working (initially placed in height order to show you which to do first. Don't do it like this… just the last line is all that is necessary).

$$\begin{array}{l} \quad (x-2) \\ \times(x-3) \end{array}$$

$$\begin{array}{l} \qquad\quad +6 \\ \quad -5x \end{array}$$

x^2

$x^2 - 5x + 6$

What We're Doing This For

What We're Doing This For

Okay, so we can multiply brackets now, great. What is it for?

Now the new expressions you have got are called quadratics. What I want you to do is to sketch our first one,

$x^2 + 3x + 2$

to do this, we again - as in Equation of A Straight Line - make y equal to it and use a table of values, such as this one

$y = x^2 + 3x + 2$

x	- 3	- 2	- 1	**0**	**1**	**2**	**3**	**4**
$y = x^2 + 3x + 2$							**20**	

We calculate the y values by inserting the x values into the equation, just like we did for the Equation of A Straight Line

For example

When $x = 3$

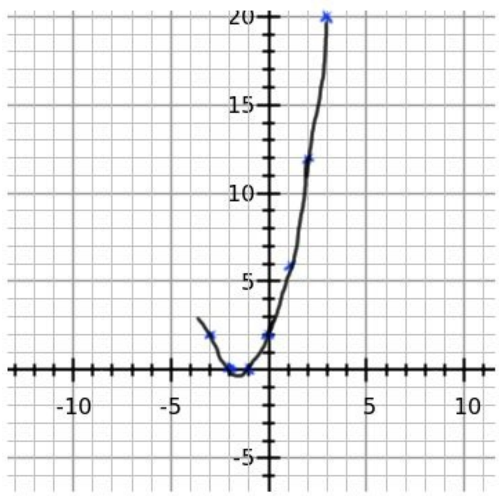
$y = (3)^2 + 3(3) + 2 = 20$

Fill in the rest of the table and plot the graph.

When you draw the lines between the dots, make them curvy!

x	- 3	- 2						

			- 1	**0**	**1**	**2**	**3**	**4**
$y = x^2 + 3x + 2$	**2**	**0**	**0**	**2**	**6**	**12**	**20**	**30**



(Note I didn't put the final (4, 30) co-ordinate in also.

We go back now to the questions I asked you at the start of the book.

Is there gravity in space?

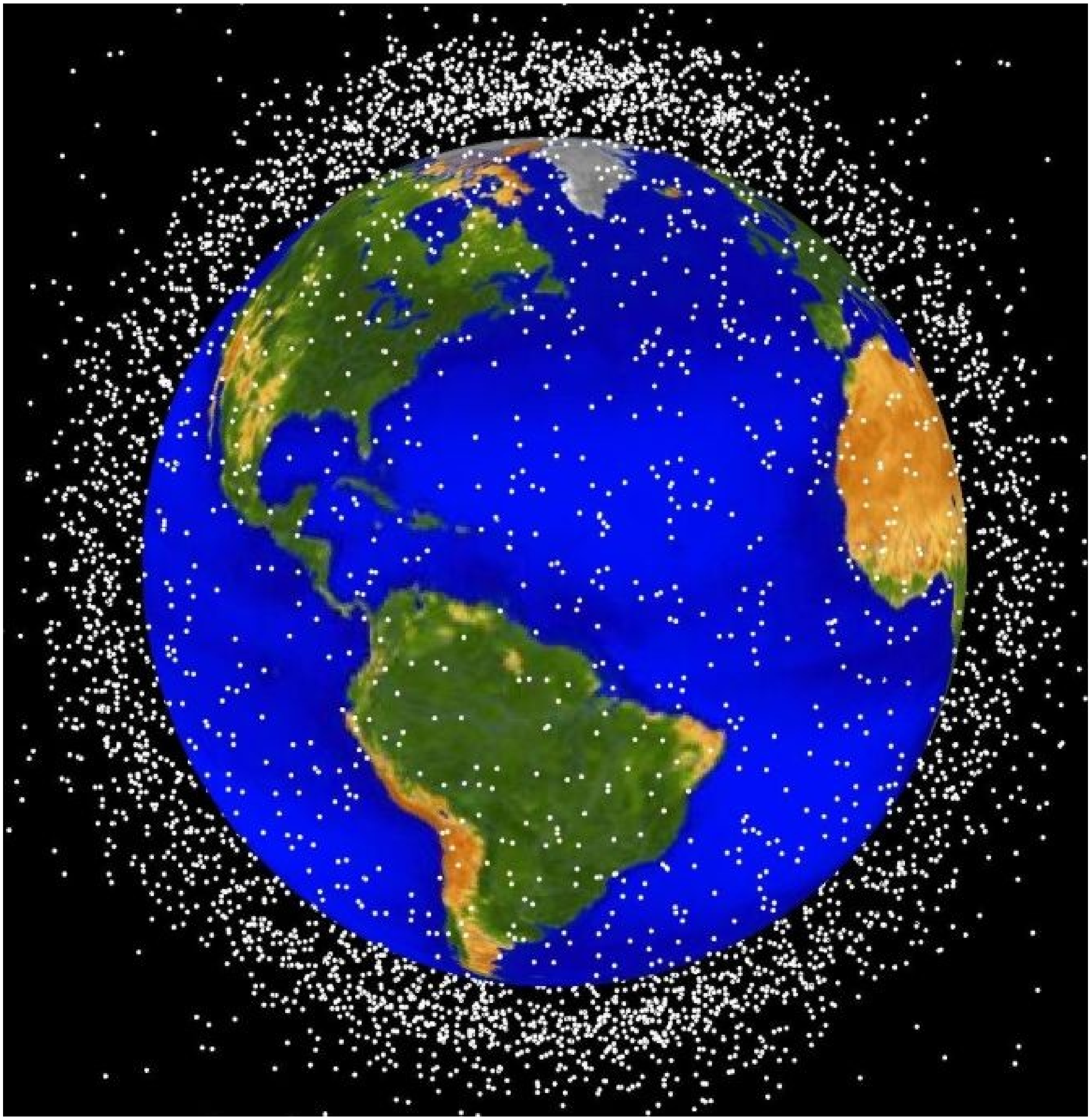
Most people say no. Why?

Because astronauts float and their teachers have told them there is no gravity in space. So is this correct?

Er, no.

There is gravity everywhere!

It is gravity keeping the moon in its orbit, gravity keeping the Earth in its orbit around the Sun, and gravity keeping the Sun in its orbit around the centre of the galaxy! Plus there are hundreds of satellites in orbit, how are they staying up there..?

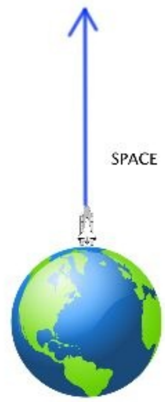


So how come astronauts float?

Let's look at the second question first.

What path does the space shuttle take to get into space?

Again, people usually say, ‘it goes straight up’.



Is this what you said?

In fact, this would be the worst way for that to happen. If it went straight up, it would eventually run out of fuel, slow down (due to gravity pulling it back down again, as we’ve just established THERE IS gravity in space), come to a stop, and a bit like Wily Coyote, pause, and then plummet downwards.

What happens in fact is that it gets thrown.

If you throw a ball, maybe it will go 20-30 metres. That is because gravity pulls it down before it has a chance to get anywhere. If you threw it faster, it would go further before gravity pulled it down.

Imagine you threw it so fast, that by the time it fell down, it had reached the horizon. In fact you have to throw it at 5 miles per second/8 kilometres per second for that to happen.

What would happen is that as the Earth is round, the ground curves away. So the ball would fall, but as the Earth curves away, it doesn’t hit it.

You can find this explained in my blog, here

You can see this demonstrated in this time lapse picture. This is probably my favourite picture. You can see the curve that is so beautiful. And notice it goes down, completely contrary to what you would think. It falls off the Earth.

The shape of this curve is the same as a quadratic. If you turn your graph upside down, you should see the same shape.

It is called a Parabola.

And it is described by a quadratic equation. And this is just ONE of the many reasons why we study quadratics - although this is my favourite.

Why are astronauts 'weightless'?

They're not. They, like the space shuttle or space station they are in, are also falling towards the Earth, because of gravity. They are also traveling at 8 kilometres per second.

Because they are falling at the same rate as the shuttle, effectively they are skydiving but inside a lift that is also skydiving. So to them it appears they can just float around. The real issue is that to feel weight you need to have the floor pushing up against you. If that's falling away from you, there's nothing to push on you.

So, as the lift is falling, there is nothing to push back on their feet where they are standing. Relatively to them though, everything is still, and they are able to 'float'. In fact, if you were stood still by the shuttle, it would rush past you at around 17,500 mph as it falls towards Earth. To them, they are falling at the same rate as the shuttle so they are able to move around. It's a like a very long skydive without any air resistance in a lift that is doing the same thing at the same time.

You feel the same thing in a lift yourself. When it initially drops, to go down, you feel weightless for a millisecond. When it decelerates to a stop, it makes you feel slightly heavier.

Notice on the graph you have drawn there is a cut. Where is it?

At $y = 2$

So $c = 2$

If we look at the original equation again, we see that the number not times by x is 2. This will be the cut. This is always the case, and is the same as we saw for the Equation of A Straight Line.

$y = x^2 + 3x + 2$

2 is the cut.

We can now employ the Third Rule of maths and do the reverse. To find this quadratic, we multiplied out those brackets.

We're now going to reverse that process to change our quadratic
$x^2 + 3x + 2$

back to the brackets.

How will we do this?

The problem we have is that there is no common factor. x only appears in 2 of the terms. As a result, this can seem like a tricky problem. However, since we know the origin of this expression (from multiplying), we can figure out how to get back to those brackets.

Remember they came from the column multiplication technique. As a result, we know that the 2 was due to a multiplication, and the $3x$ came from an addition.

In other words, we need to find 2 numbers that multiply to make 2 and add to make 3.

The x^2 just comes from the left column, $x \times x$

2 numbers that multiply to make 2 can be

1 x 2
Or - 1 x - 2

Remember to get in the habit of always writing the negative numbers as well, since it could well be them.

Adding these numbers

1 + 2 = 3

- Or -1 - 2 = - 3

So it must be both positive 1 & 2.

This means we have

$x^2 + 3x + 2$
$(x+1)(x+2)$

and the expression is factorised.

How could we check this was correct?

Of course, multiply them again!

$$\begin{array}{l} \;\;(x+1) \\ \times(x+2) \end{array}$$

----

x^2+3x+2

which is correct.

So now we can go in both directions.

If you're given brackets you can multiply them to get a quadratic.

If you're given a quadratic expression, you can factorise (divide) it to get the original brackets.

Try these

$x^2+7x+12$

$x^2-7x+12$

x^2+5x+6

x^2-x-2

Ans.

$(x+3)(x+4)$

$(x-3)(x-4)$

$(x+3)(x+2)$

$(x+1)(x-2)$

Brackets don’t have to be in any particular order. (Why?)

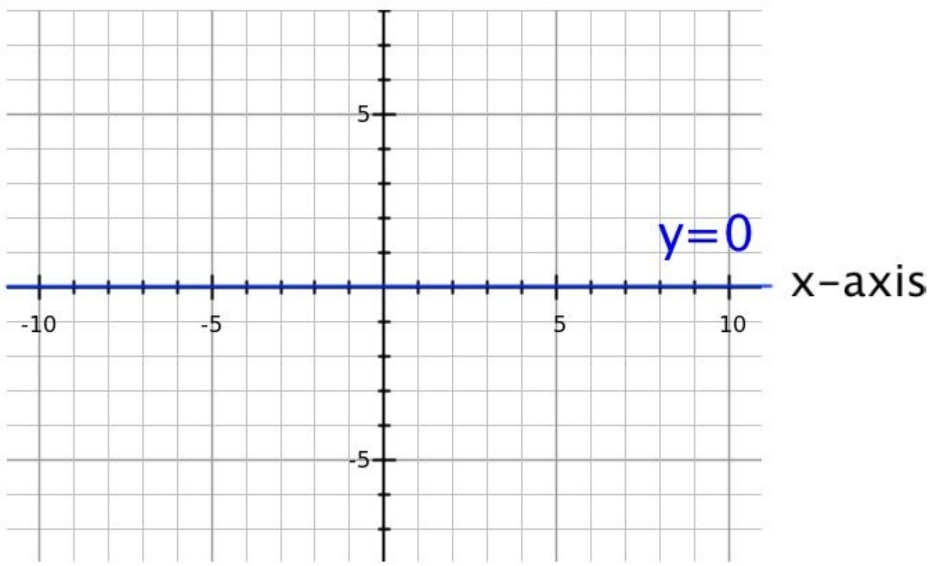
Solving A Quadratic

Now we can factorise a quadratic, there is something else we can do.

From Equation of A Straight Line, can you remember what the equation of the x-axis is?

You might recall it is

$y = 0$



In ‘Simultaneous Equations’ we saw that if we have two equations like
$y = 2x$

and

$y = 6$

we can make them equal to each other to find out where they cross. Or, if we’re asked to solve algebra like this, this is one of the things we’re doing it for.

$2x = 6$

If we want to find out where the quadratic crosses the x-axis, we can use the two equations of each for this.

$$y = x^2 + 3x + 2$$

and

$y = 0$

and make them equal to each other.

$x^2 + 3x + 2 = 0$

since they both equal y

Fine so far.

But if we want to know what the value of x will be to make the expression on the left equal to zero, we're going to struggle a bit. We could guess what it might be... say, using $x = 1$

$1^2 + 3(1) + 2 = 6 \neq 0$

and keep guessing. But guesswork is unmathematical. **NEVER GUESS.**

(In this case we could refer to your original table, and you'll find it there. But we want a method that doesn't rely on this.)
Now we know how to factorise the quadratic, we could do that.

So

$x^2 + 3x + 2 = 0$

becomes

$$(x+1)(x+2)=0$$

which is still true.

Remembering what brackets mean, this says that these two things multiplied together equals zero.

So what does each one equal?

Each bracket must equal zero too.

It's a neat trick to tell us what x will be.

So

$$x+1=0$$

and

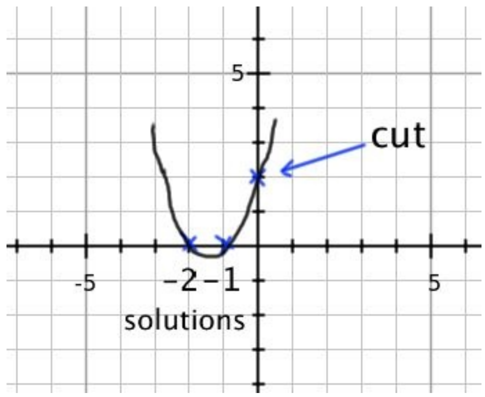
$$x + 2 = 0$$

this means that

$x = -1$

and
$x = -2$

So now we know where our quadratic crosses the x-axis, and the y-axis. We know the shape (parabolic) too, so it's easy for us to sketch a quadratic without having to calculate a table of values - because all we have to do is solve for x to find out where it crosses the x-axis!



This was our goal. To be able to create, factorise, solve and sketch a quadratic.

Now we can do that.

Try this one yourself.

$y = x^2 + 4x + 3$

Factorise

Solve
Sketch.

Also, try to find the '*minimum value*'. This is the location of the lowest point on the curve.

Answer below.

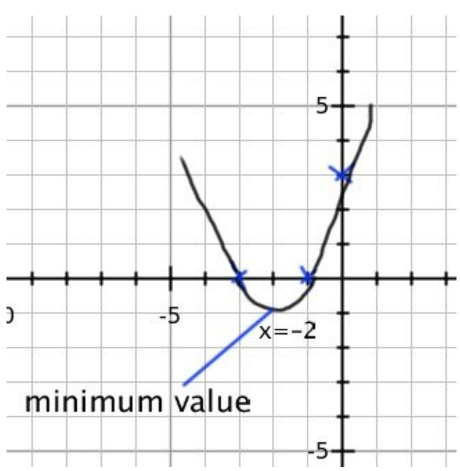
Did you find the minimum value?

It is

- $x = -2$

There are actually four different ways to find this value. One of the ways is to use our solutions. Because the quadratic is symmetrical, the lowest point is always halfway in between the two solutions.

Once we've solved the quadratic, it is then easy to find the minimum value as it will be halfway between the solutions.



How could we find the y-value of this point?

One way to do it would be to use the equation of the quadratic.

$y = x^2 + 4x + 3$

since it begins ‘y equals...’

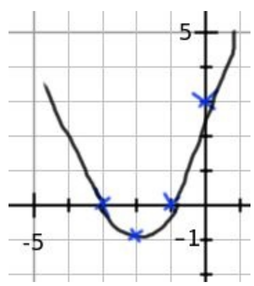
all we have to do is substitute in the x-value.

$y = (-2)^2 + 4(-2) + 3$

$= -1$

So the co-ordinate for the minimum value is
(- 2, - 1)

Now we know four points where the quadratic goes through. As a result, just like for the equation of a straight line, we don’t need to actually do a table of values in order to sketch one. If we have enough points, and we know the shape (always a parabola) we can just sketch through the four points, as below.



Do the same with these questions

$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 - x - 2$

Non-factorisable Quadratics

Non-factorisable Quadratics

Sometimes you'll be asked to solve a quadratic that isn't factorisable. However it will cross through the x-axis, so you'll be asked to find out where.

To be able to do this we use a quadratic formula, which comes from the general expression of a quadratic.

This is

$ax^2 + bx + c$

for example,

$x^2 + 3x + 2$

would have

$a = 1$
$b = 3$
$c = 2$

The c here is of course the cut!

Another example

$9x^2 + 29x - 28$

$a = 9$
$b = 29$
$c = -28$

If we equate the general expression to zero, to find where it crosses the x-axis, we will have
$ax^2 + bx + c = 0$

We want to know what x will be in this to make the equation equal to zero.

What we can do is to 're-arrange' the equation so we have x in terms of everything else. The derivation of this is left to the book ‘Changing The Subject’ as this is what we’re effectively doing...

You may have come across this formula already. In any case, the format I use is different to the school one (this may not come as a surprise by now!) If we use my format, we actually get a bonus piece of information, plus it tells you something very interesting about quadratics and squares in general.

So here it is

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

I appreciate it probably looks a bit scary! But we’ll look at it bit by bit.

Firstly, why is there a plus and a minus in front of the square root?

To answer this, what is the square root of 9, or $\sqrt{9}$

Everyone usually says 3. But no, this isn’t the answer.

It is 3... or - 3!

Because $-3 \times -3 = 9$, then - 3 is the square root also.

As a result, any square root has two possible answers. So that’s why that’s in the formula. Use this from now on for your square roots in general!

Where this formula differs from the one you’ll usually find in textbooks is to have the

$$\frac{-b}{2a}$$

The reason I write it like this is because it will give us the minimum value!

For example, for

$y = x^2 + 3x + 2$

We found that the minimum was

$-\frac{3}{2} = -1\frac{1}{2}$

If we had used the formula the

$\frac{-b}{2a}$

would give us

- $-\frac{3}{2} = -1\frac{1}{2}$

That is, it gives you the minimum too!

In fact, from now on you can easily find the minimum of a quadratic in this way.

Another example would be

$y = x^2 + 4x + 3$

What is the minimum here?

$\frac{-b}{2a}$

$-\frac{4}{2} = -2$

as we saw for yourself when you did it.

If we do an example we can see how the formula works.

$x^2 + 3x + 2$

would have

$$a = 1$$
$$b = 3$$
$$c = 2$$

substituting into the formula gives

$$x = \frac{-3}{2(1)} \pm \frac{\sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-3}{2} \pm \frac{\sqrt{9-8}}{2}$$

$$x = \frac{-3}{2} \pm \frac{\sqrt{1}}{2} = x = \frac{-3}{2} \pm \frac{1}{2}$$

$$x = -\frac{3}{2} - \frac{1}{2} = -2$$

OR

$$x = -\frac{3}{2} + \frac{1}{2} = -1$$

we have the minimum value

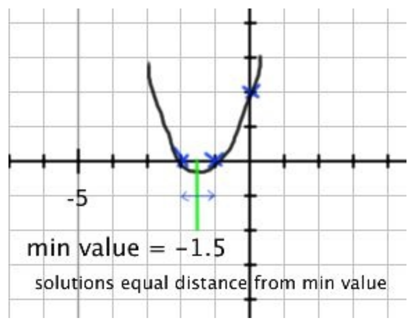
- min = $-\frac{3}{2} = -1\frac{1}{2}$

- and then either side of that the same distance lie the solutions. Because we use the minimum value as our starting point, we see the symmetry of the parobola, quadratic and square numbers all in one.

Here we see that each solution is the same distance

$$\left(\frac{1}{2}\right)$$

away from the minimum value. This will always be the case for any solution for any quadratic. Once we find the minimum, which is easy to do, we can then go right and left the same distance to find the solutions. If we use the formula in this way, we get 3 pieces of information instead of just 2 (as in the school method). That's why I prefer you to use my version of the formula.



Try the formula on these questions

$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 - x - 2$

Ans.

Now if we try

$$x^2 + 5x + 4$$

Working this out, we see that in the square root, we have a square number. In fact, this is because so far I've only asked you to use the formula on factorisable quadratics. If the number in the square root is a square number, they end up cancelling each other out, and we always get a round number. So we have a minimum value, and a 'nice' number right and left of it.

$$x = -\frac{5}{2} \pm \frac{\sqrt{25-16}}{2}$$
$$= -\frac{5}{2} \pm \frac{\sqrt{9}}{2}$$
$$= -\frac{5}{2} \pm \frac{3}{2}$$

In reality, we use the formula to solve non-factorisable quadratics. We should always try to factorise it first, and only if that fails do we try the formula.

For instance,

$$x^2 + 7x + 1 = 0$$

This quadratic isn't factorisable, because there isn't two numbers that multiply to get 1 but add to get 7. As a result, the solutions will be surds. Let's see what that means.

$$a = 1$$
$$b = 7$$
$$c = 1$$

$$x = -\frac{7}{2} \pm \frac{\sqrt{7^2 - 4(1)(1)}}{2}$$

We find that the minimum is

$$-\frac{7}{2}$$

and that the solutions will be

$$\pm\frac{\sqrt{45}}{2}$$

right and left of it.

$$\sqrt{45}$$

is a surd (from Indices - In A Minute) and we can simplify this to get

$$\frac{3\sqrt{5}}{2}$$

This is the exact answer. This also tells us that it wasn't factorisable.

If we use a calculator to find what this will be we'll see that we have

$$x=-\frac{7}{2}\pm 3.35$$

-

$$x=-3.5\pm 3.35$$

Gives that

-

$$x=-3.5+3.35=-0.15$$

-
- Or that,
-

$$x=-3.5-3.35=-6.85$$

-

There would be no way we'd think of these numbers to factorise it! This is where the formula comes in.

Try these yourself

Solve and find the minimum value of these:

$x^2 + 4x + 2$

$x^2 - 9x + 4$

$x^2 - 9x - 14$

Ans.

$x = -2 \pm \sqrt{2}$

$$x = \frac{9}{2} \pm \frac{\sqrt{73}}{2}$$

$$x = \frac{9}{2} \pm \frac{\sqrt{137}}{2}$$

Note we can keep these last two they way they are as the numbers in the square roots are primes, and will not simplify.

Any Type of Quadratic

When a is not equal to 1

Now we know how to tackle all quadratic of every kind, except one.

So far we've looked at quadratics where a is always 1, *i.e.* ones like
$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 - x - 2$

We can have any type here though, such as
$2x^2 + 7x + 3$

Again, we try to factorise these first. To do this we follow the same method of finding two numbers to multiply to make 3 but add to make 7.

Of course that won't work!

However, this is where the $2x^2$ part comes in.

Because we have this instead of just the usual x^2, we also have to find 2 numbers that multiply to get 2.

So this adds an extra complication, but all we have to do is match up the right numbers.

To get 3, we would need

3 x 1
Or - 3 x - 1

and to get 2, we would need

2 x 1
Or - 2 x - 1

Since it is entirely positive, we can take it (to begin with) that to get 2, we would use 2 x 1.

So we have

$$2x^2 + 7x + 3$$
$$(2x + ?)(x + ?)$$

To fill in the rest, let's try 3 in the first bracket and 1 in the second.

$$(2x + 3)(x + 1)$$

If we multiply this we get

$$\begin{array}{r} (2x+3) \\ \times(x+1) \end{array}$$

$$2x^2 + 5x + 3$$

Which is almost correct!

However, we want $7x$ in the middle.

If we try 1 in the first bracket and 3 in the second, we get

$$\begin{array}{l} (2x+1) \\ \times(x+3) \end{array}$$

$$2x^2 + 7x + 3$$

Which is what we're looking for, so we know they're the correct brackets.

$$2x^2 + 7x + 3$$
$$(2x + 1)(x + 3)$$

With practice and familiarity, you will get to a stage where you'll begin to do this process in your head, but for now trying different possibilities is fine.

To go ahead and solve this - that is, find out where the quadratic crosses the x-axis, we just equate it to ... what?

$y = 0$

So we have

$$2x^2 + 7x + 3 = 0$$
$$(2x+1)(x+3) = 0$$

So

$$2x + 1 = 0$$

And

$$x+3=0$$

$$x=-\frac{1}{2}$$

$$x=-3$$

Another Example

Another example

$3x^2 + 13x + 4$

To get four, there are four possibilities.

4 x 1

- 4 x - 1

2 x 2

- 2 x - 2

To get 3, there are two

3 x 1

- 3 x - 1

Again, since they're all positive terms in the quadratic, we can start with positive values.

Try

$(3x + 2)(x + 2)$

This would give

$3x^2 + 8x + 4$

which isn't right.

We need to get $13x$.

Since 3 x 4 = 12, which is near to 13, let's try that combination.

$(3x+1)(x+4)$

gives

$3x^2+13x+4$

which is correct.

It is useful to see what we are aiming for and try to 'make our way there' by using two of our numbers to do it. It may give us a pointer in the right direction.

Here's one to try that came up in an actual GCSE exam. (I know, because I was taking it!)
It's quite hard, so take your time over it. In fact it's one of the hardest I have seen in an exam.

$9x^2+29x-28$

Ans.

$(9x-7)(x+4)$

(In the exam itself I didn't actually spot this and went via the 'back door' by using the quadratic formula. If you try this you'll see why it's not an easy option, but do-able. Remember I didn't have a calculator. Notice that the number in the square root is a square number. What does that mean?)

Finally we can look at this type using the *formula* but one that isn't factorisable, such as

$2x^2 + 4x - 4$

$a = 2$
$b = 4$
$c = -4$

This would give

$$x = \frac{-4}{2(2)} \pm \frac{\sqrt{16 - 4(2)(-4)}}{2(2)}$$

$$x = -1 \pm \frac{\sqrt{16 + 32}}{4}$$

$$x = -1 \pm \frac{4\sqrt{3}}{4}$$

$$x = -1 \pm \sqrt{3}$$

The minimum value would be at

$x = -1$

and the two solutions are

$\sqrt{3}$

right and left of that.

Completing the Square

Completing the Square

When we factorise and solve a quadratic, we get 2 pieces of information about it. The two solutions and the minimum value, as it is half-way between the solutions. When we use the quadratic formula, we get the same information (if we use it my way).

There's a third way to get this information, but it also gives us a couple of bits extra. As a result, this is the method I tend to use if I want to know everything about a quadratic I'm using.

So what do we do? Let's say we want to find out all about this quadratic
$y = x^2 + 3x + 2$

Instead of factorising or using a formula, we almost take half of it and write
$(x+1.5)^2$

It's like half the x^2, and half the 3.

What we do now is multiply out the brackets and see what we would get.

$(x+1.5)$
$\times(x+1.5)$

$x^2 + 3x + 2.25$

(this is where knowing how to square numbers that end in 5 comes in so useful. Do you remember this from Squaring - In A Minute?)
Note that

$(x+1.5)^2$

isn't

$x^2 + 1.5^2$

which is a common mistake.

Our answer of

$$x^2 + 3x + 2.25$$

is very similar to the original quadratic!

The first two terms are the same, but the number is slightly out.

So if we subtract 0.25 from these brackets, it would be the same.

Giving

$$(x + 1.5)^2 - 0.25$$

So this is the square completed.

I'll talk a bit more about why it is called this later on in the book.

If we were to multiply this back out again, it would look like

$$y = x^2 + 3x + 2$$

So this is just another way of saying the same thing.

What information does this give us?

Firstly, it gives us the minimum value. If we take the opposite sign of the number in the brackets, we instantly know what it is.

$$x = -1.5$$

which is true, from what we saw earlier.

Here is the bonus.

The number outside the brackets, - 0.25, is the y-value of the minimum value. So we get the whole thing in one. The minimum value coordinate is then
(- 1.5, - 0.25)

note we don't use the opposite here, just exactly what it is. There's a reason the y value is negative too - do you know why?

So immediately, we have the minimum value - and its y-coordinate.

To find the solutions, we just equate this all to zero.

We do that because we want to know where the line

$$y = x^2 + 3x + 2$$

and

$$y = 0$$

cross each other.

$$y = x^2 + 3x + 2 = 0$$

which is

$$(x + 1.5)^2 - 0.25 = 0$$

We now shuffle this equation around to find x

$$(x + 1.5)^2 = 0.25$$

Square root both sides

$$x + 1.5 = \pm\sqrt{0.25}$$

$$x = -1.5 \pm \sqrt{0.25}$$

which is

$$x = -1.5 \pm \frac{1}{2}$$

which is

$$x = -2$$
$$x = -1$$

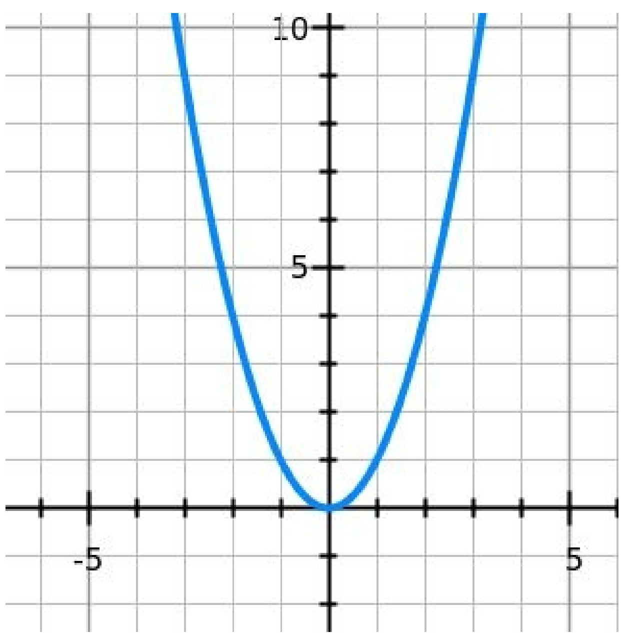
as we saw earlier.

We've got the solutions, both coordinates of the minimum value and now we can find something else!

We've looked at lots of quadratics, but actually they all form part of the same template. They use the same model and every quadratic we've looked at is just a variation of that.

The fundamental quadratic they all come from is

$$y = x^2$$



If you were to place a mouse on this and move it around the graph, it would still be $y = x^2$ in shape, but it would have different solutions and minimum values.

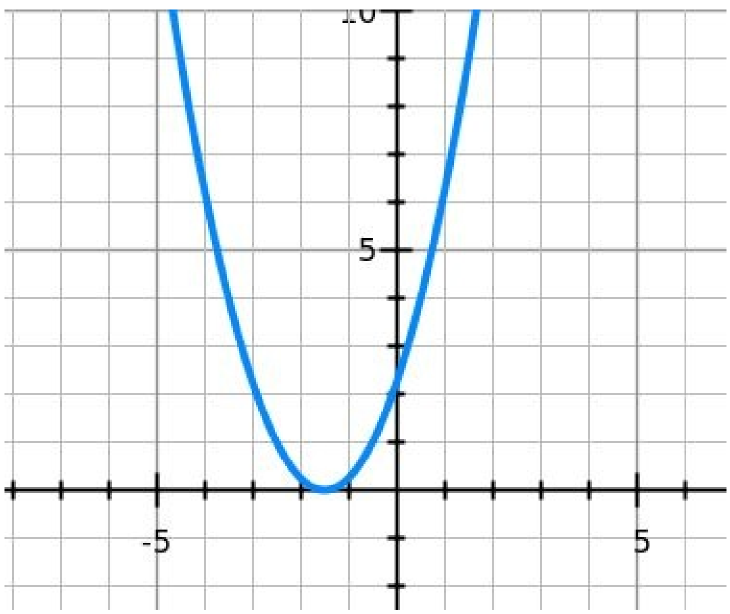
All quadratics are in this exact situation.

Looking at our completed square

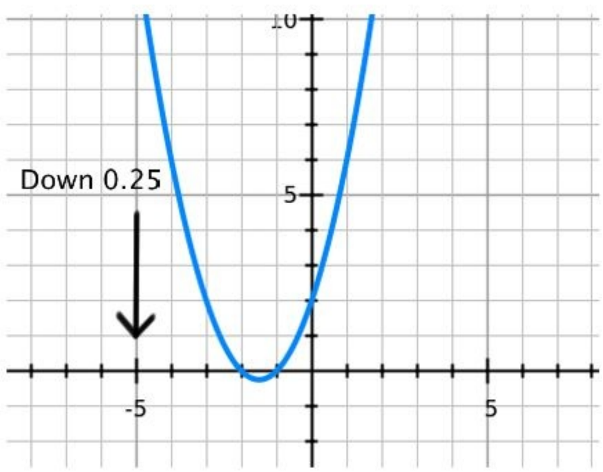
$(x+1.5)^2 - 0.25$

What this is says is that our $y = x^2$ has had two moves.

The first is a move to the left of 1.5



and the second is a move down of 0.25



So we end up with our original quadratic
$y = x^2 + 3x + 2$

which has a cut of 2 and solutions at - 1 and - 2, as well as that minimum at
$x = -1.5, y = -0.25$

What our completed square has told us is the TRANSFORMATION that x^2 has made to look like
$y = x^2 + 3x + 2$

which can be of a lot of use.

Not surprisingly this is the same as the minimum value because we are actually charting the movement of the quadratic from the origin (0, 0) to where it finishes. This will always be the minimum.

We use this concept of ‘transformation’ in other areas of maths - this is the first time we come across it. I will refer to it again when we look at transformations.

This is written as
[- 1.5, - 0.25]

Try this one yourself.

$y = x^2 + 5x + 6$

Answer.

Working through

$(x + 2.5)^2 - 0.25$

It is just a coincidence this is the same y-value!

From this we know the minimum value

(- 2.5, - 0.25)

and the transformation

[- 2.5, - 0.25]

and to find the solutions

Equate to zero

$$(x+2.5)^2 - 0.25 = 0$$

$$(x+2.5)^2 = 0.25$$

$$x + 2.5 = \pm\sqrt{0.25}$$

$$x = -2.5 \pm \sqrt{0.25}$$

Which gives

$$x = -3$$

Or

$$x = -2$$

We start to see a pattern developing here.

We can see that the solutions will always be the x - minimum value plus or minus the square root of the (positive) y - minimum value. So in practice we can just jump straight to that. In an exam you have to show each line of working.

For example, we could have

$$y = x^2 + 7x + 6$$

The Completed Square

$$(x+3.5)^2 - 6.25$$

Therefore the solutions will be the x - minimum value, plus or minus the square root of the (positive) y - minimum value.

$$x = -3.5 \pm \sqrt{6.25}$$

$$x = -1, x = -6$$

Easy!

An Even Easier Way To Find the Y-Minimum Value

So far to find the y-value we've just squared the brackets to see what that would give us. Then we just need to make an adjustment to match with the original equation.

In the previous example, the completed square multiplied out would give
$x^2 + 7x + 12.25$

So we need to subtract 6.25 from that to get back down to 6.

That's fine.

A quicker way is just to square that number only. Because the first two terms are always the same (that's the idea), the only difference will ever be the number.
So if we just squared 3.5

$$3.5^2 = 12.25$$

We then have the number we need.

There's a second reason for this. The actual location of the minimum value is x = −3.5

If we put this into the original equation, like we did when we used the solutions to find the minimum value, we find that the middle term is always double the first term. So it is working backwards on us for no reason - *I.e.* It's a waste of time to do it that way.

$$(-3.5)^2 + 7(-3.5) + 6$$

$12.25 - 24.5 + 6$

$= -6.25$

The second term is exactly double the first, meaning we just make our value negative, just to add positive 6. We can avoid all this working out by just squaring the number in the original completed square bracket, then decided what we need to get down to our number, as above.

Solve Quadratics In Your Head!

Now we know the two concepts above, we can apply both to solve a quadratic in our head.

Using completing the square, let's say we want to solve
$x^2 + 7x + 10 = 0$

Immediately we complete the square, giving
$(x + 3.5)^2$

$x = -3.5$

For the minimum value.

And to calculate the y-value, as we just saw, we just square this

12.25

And we have to subtract −2.25 to get to 10, so that's our y-value.

So now we can just jump straight to our solutions
$x = -3.5 \pm \sqrt{2.25}$

The number in the square root is a square number itself (15^2), so it is easily square rooted in your mind to 1.5

Our solutions will be

$x = -3.5 \pm 1.5$

$x = -5, -2$

Practicing a few of these will make it easy for you to do in your head.

When *a* is Not Equal to 1

Again, as with factorising or using the formula, we may want to complete the square on a quadratic where *a* isn't 1.

For instance,

$2x^2 + 8x + 5$

Here, just factorise the first two terms first, to give
$2(x^2 + 4x) + 5$

Then we just complete the square on what is in the brackets
$2(x+2)^2$

This will give

$2(x^2 + 4x + 4)$

$2x^2 + 8x + 8$

So almost the same.

We just need to minus 3 to get to 5, so we have
$2(x+2)^2 - 3$

And the square is completed.

Our minimum value will be

(- 2, - 3)

And the solutions will be

$$2(x+2)^2 = 3$$
$$(x+2)^2 = \frac{3}{2}$$
$$x = -2 \pm \sqrt{\frac{3}{2}}$$

The only difference we have to make here is to our y-value. Because this is $2x^2$ instead of our usual x^2, the quadratic stretches upwards. As a result, our solutions will be closer to the minimum value. Just remember to divide by *a* before square rooting, if *a* is not equal to 1!

A Closer Look At The Quadratic Formula

A Closer Look At the Formula

Looking again at the quadratic formula, there is another piece of information we can get from it.

(Amazing, eh, how much information is in quadratics?)

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

In the square root part, there is a

$$b^2 - 4ac$$

As we saw in our examples, if this number is positive, we had two solutions.

(If the number was positive and a square number itself, the quadratic was factorisable too).

What if the number isn't positive?

What if it is zero or negative? Then what?

If the number is zero, as in this example

$$x^2 + 4x + 4$$

$$a = 1$$
$$b = 4$$
$$c = 4$$

$$b^2 - 4ac$$

$$(4)^2 - 4(1)(4) = 0$$

We end up with only 1 solution.

We can see this from the formula

$$x = -\frac{4}{2} \pm \frac{\sqrt{0}}{2}$$

The right hand side

$$\frac{\sqrt{0}}{2}$$

is zero

So we only have

$$x = -\frac{4}{2}$$

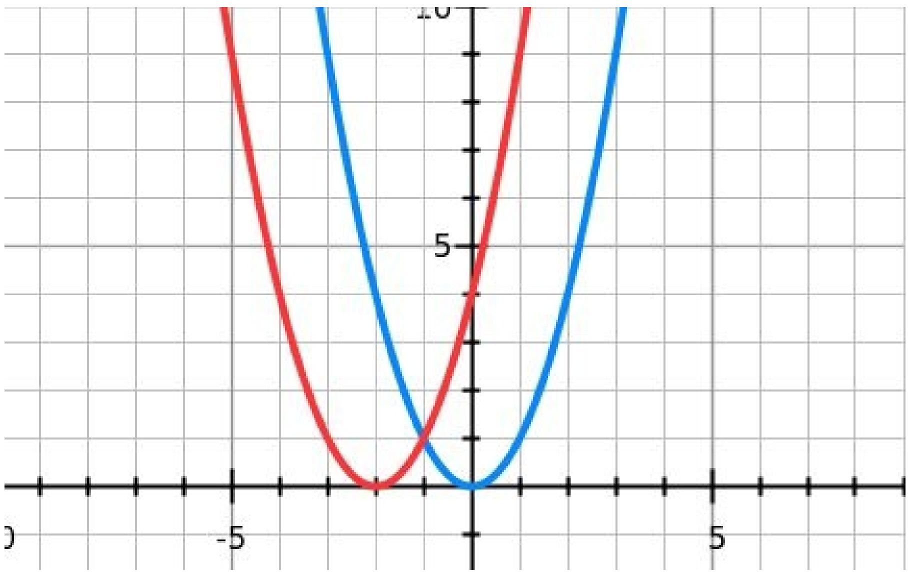
$$x = -2$$

What does this mean?

It means that the quadratic sits on the x-axis at this point, only touching at one co-ordinate.

To show this, if we complete the square on it we find,
$(x+2)^2$

There is no y-value because we don't need to make any adjustment up or down. So the transformation of this quadratic is from $y = x^2$ and two to the left.



We can see that it has only one solution.

*It is interesting to note that y is also zero here.

Finally, what if the square root is negative, as in this example
$y = x^2 + 2x + 10$

$b^2 - 4ac$

$(2)^2 - 4(1)(10)$

$= -36$

Since it is impossible to square root a negative number*, there is no answer to this. So there are no solutions, so the quadratic doesn't cross the x-axis at all!

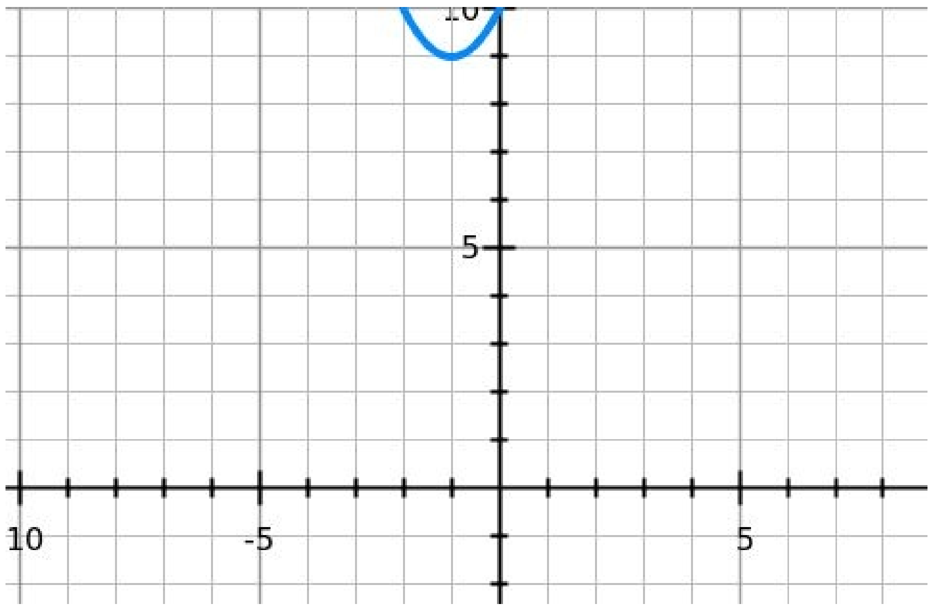
Using

$b^2 - 4ac$

in this way is called using the ***discriminant.***

This can check for us - quite quickly - whether the quadratic has any solutions, and if so, how many.

We can see this is true from its graph.



However, we can check whether there are solutions much more quickly in a much simpler way.

Instead of using the *discriminant*, we could just use the y-value in the completed square.

If it is negative, there are 2 solutions.
If it is zero, there is 1 solution (it sits on the x-axis) If it is positive, there are no solutions.

This is because the y-value is part of the minimum value. If it sits below the x-axis, there must be 2 solutions. If it sits on the x-axis, at $y=0$, that must be 1 solution, and if it is above the x-axis, there can't be any solutions.

It is easier to find y, as all we have to do is
Square half the x-coefficient, and see how far that is from c, as we saw in an earlier chapter.

For example, for

$y=x^2+2x+10$

$(1)^2 = 1$

And we need positive 9 to get to 10. Therefore the minimum is 9 whole units above the x-axis. There is no way this quadratic has solutions!

Try this technique for the following

$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 + 6x + 9$

$x^2 + 8x + 9$

Answers

Yes, two

Yes, two

Yes, two

Yes, one

No

From now on you can very easily and very quickly calculate whether a quadratic has solutions. *Complete the square* on it and it will tell you

- the x and y co-ordinates of the minimum value
- the transformation it has undergone
- whether it has any solutions, and if so how many

- what the solutions are

This is why completing the square is SO useful.

Final example

Doing a Quadratic - In A Minute!

$x^2 + 7x + 12$

Completing the Square

$(x + 3.5)^2 - 0.25$

Minimum - (- 3.5, - 0.25)
Transformation - [- 3.5, - 0.25]
Solutions, YES, two
They are
$x = -3.5 \pm \sqrt{0.25}$

(From the earlier chapter - minimum value plus or minus the square root of the positive y-value)
$x = -3, -4$

What Multiplying Two Expressions Together Will Give

What Multiplying Two Expressions Together Will Give

When we looked at multiplying brackets such as
$(x+1)(x+2)$

or

$(x+6)(x-1)$

we of course, got a quadratic.

But what shape does that give?

Your first thought might be a parabola.

However, it also gives another shape.

We are multiplying two expressions together. They actually represent numbers - it's just that we don't know what the numbers are.

Thinking back to Multiplication - In A Minute, when we multiply two numbers together, what shape do we get?

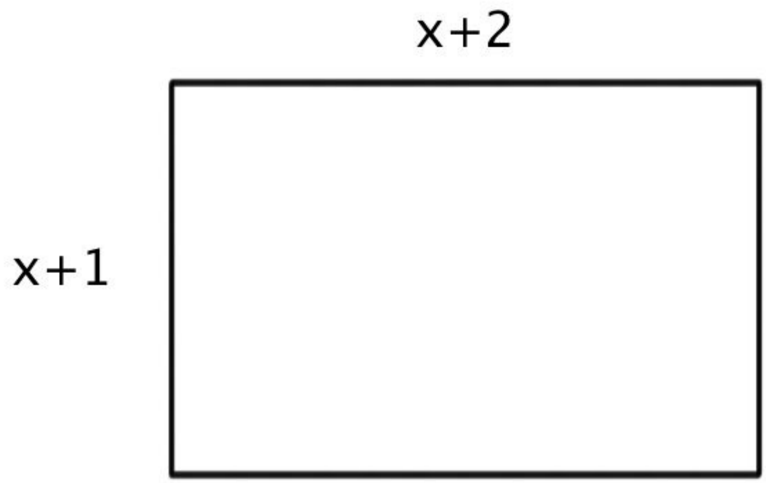
A rectangle.

So our expressions above, when multiplied will give a rectangle.

For example

$(x+1)(x+2)$

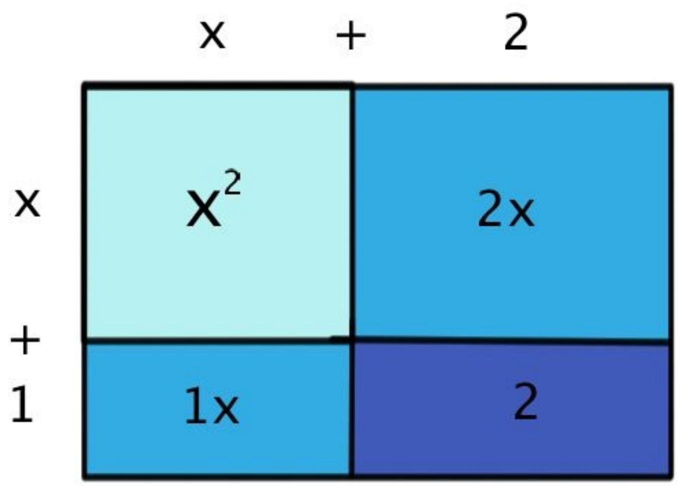
Looks like this



And when multiplied out gives

$x^2 + 3x + 2$

and this will look like this in the rectangle



Since the light blue shape has sides of x and x
The blue shapes have sides of 1 and x and 2 and x respectively, and The purple shape has sides of 1 and 2.

These ‘shapes’ are all rectangles and one square (light blue).

The information we get about the rectangle is the area of it.

So remember, when we are solving for x, not only are we trying to find where the curve crosses the x-axis, but we're trying to find what the lengths would be, given the area. When we do this, we set the quadratic equal to zero. This means we're saying the area is zero, so what would the lengths be? Of course, each length must be zero for that to happen, so that's why when we have

$$(x+1)(x+2)=0$$

$$(x+2)=0$$

x must be - 2 to make zero.

If the area of the rectangle isn't zero, we can also find out what the area of the sides will be.

Let's say we have

$$x^2+5x+4=10$$

So the rectangle has an area of 10.

$$x^2+5x+4=10$$
$$(x+1)(x+4)=10$$

These are the lengths of the sides that give 10.

What would x be to make this work? (We can probably see this quite easily in this example, but just to see how we would do it if it was more complicated.) To find this, we solve the quadratic as normal.

However, to do this, we need to have the situation of there being no area or having the equation equal to zero.

As a result, we minus 10 from both sides.

$$x^2+5x-6=0$$

and factorise and solve as usual

$$(x-1)(x+6)=0$$

$$x=1, x=-6$$

Looking again at our original quadratic,

$$x^2+5x+4=10$$
$$(x+1)(x+4)=10$$

We can’t use -6 here, as this wouldn’t make sense, as both lengths would be negative. So we take the value $x=1$

This tells us that the two sides of the rectangle will be
(1 + 1)(1 + 4)

2 x 5

This of course, multiplies to make 10!

Why Is it Called ‘Completing The Square'?

Why Is It Called Completing the Square?

As we have just seen, when we multiply brackets together, we get the shape of a rectangle.

Completing the Square is turning that rectangle into a square.

However, even though it is possible to turn the area of a rectangle into a square (via square rooting), this doesn’t happen when we complete the square because of the way we do it.

As we just sort of half the expression, this isn’t the same as square rooting it.
For example, if we complete the square on
$x^2 + 3x + 2$

this is

$$(x+1.5)^2 - 0.25$$

This isn’t

$$\sqrt{x^2 + 3x + 2}$$

Actually when we complete the square, we do get a square of course, since that is what

$$(x+1.5)^2$$

gives.

But since it has the extra term, it either takes away or adds to a square, so our shape is ‘a square and a bit’ or ‘a square with a bit taken away’.

However, if we complete the square on this

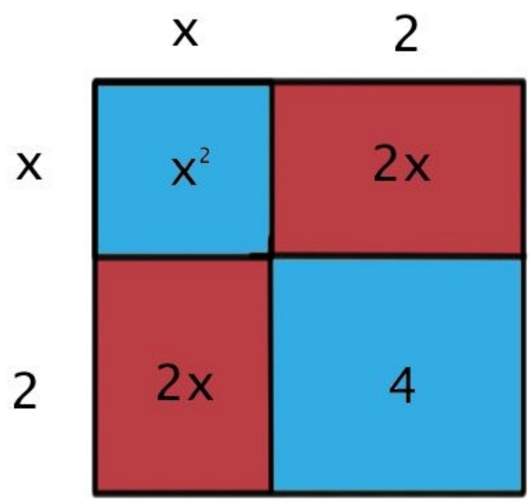
$$x^2 + 4x + 4$$

we get

$(x+2)^2$

which is a square.

This is because the original quadratic was a square itself



This is technically known as a ‘binomial square’.

This is related to the transformation of the curve $y=x^2$ too. If the original quadratic was a square, the transformation will just be left or right.

If it was a rectangle, it can be both left or right, and up or down.

Simultaneous Equations - Extended

Simultaneous Equations - Extended

To solve a quadratic, we are trying to find where it crosses the x-axis. To do this we equate it to the equation of the x-axis,

$y = 0$

As we have done many times already.

However, there is nothing to say we can't find where the quadratic crosses any line. Here are three examples.

Example 1

Another horizontal line on the graph.

Let's say we want to know where

$y = x^2 + 5x + 6$

crosses the line

$y = 2$

Equating

$x^2 + 5x + 6 = 2$

At this stage we need to make it equal to zero, like the area situation in the previous chapter.

This gives

$x^2 + 5x + 4 = 0$

by taking 2 from both sides.

We then factorise and solve as usual

$x = -4$

$x = -1$

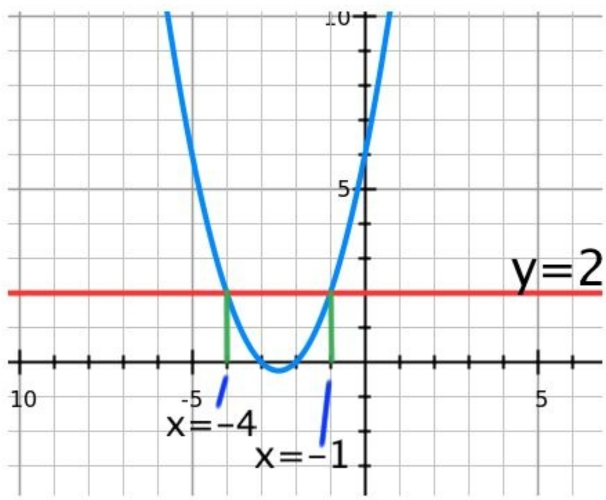
So we find that the curve crosses the line

$y = 2$

At the points,

$x = -4$

$x = -1$



And we can see this is true.

Example 2

Let's say we we want to find where a quadratic crosses a straight line which isn't horizontal.

For example let's say we want to find where
$y = x^2 + 6x + 4$

crosses

$y = 2x + 1$

Again, because we want to know where they cross, we equate them
x$x^2 + 6x + 4 = 2x + 1$

And again, we want to have zero on the right hand side, since this is our trick.

This gives

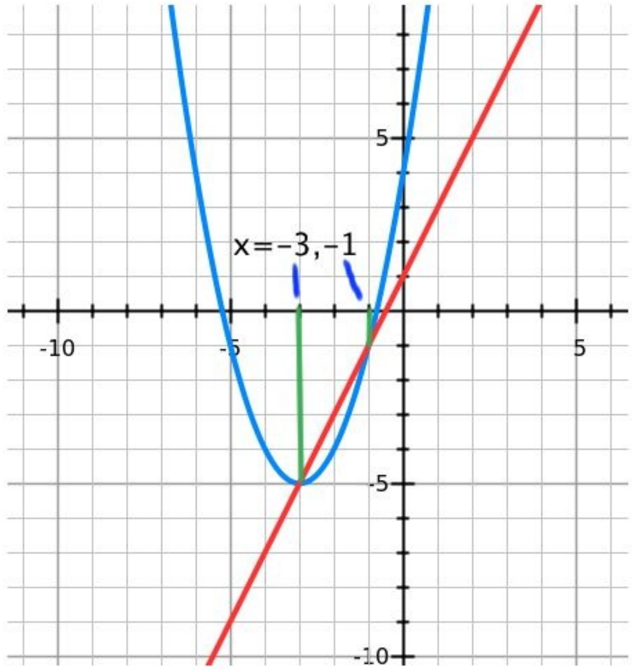
$x^2 + 4x + 3 = 0$

We factorise and solve as usual

$(x+1)(x+3) = 0$

$x = -3$
$x = -1$

Looking at the graph



We see this is true.

Example 3

Finally, we can even see where a quadratic crosses another quadratic!

If we have

$$y = x^2 + 3x + 2$$

and

$$y = x^2 + 5x + 6$$

Equating these

$$x^2 + 5x + 6 = x^2 + 3x + 2$$

Because they both have x^2 terms, we can cancel these to give
$5x+6=3x+2$

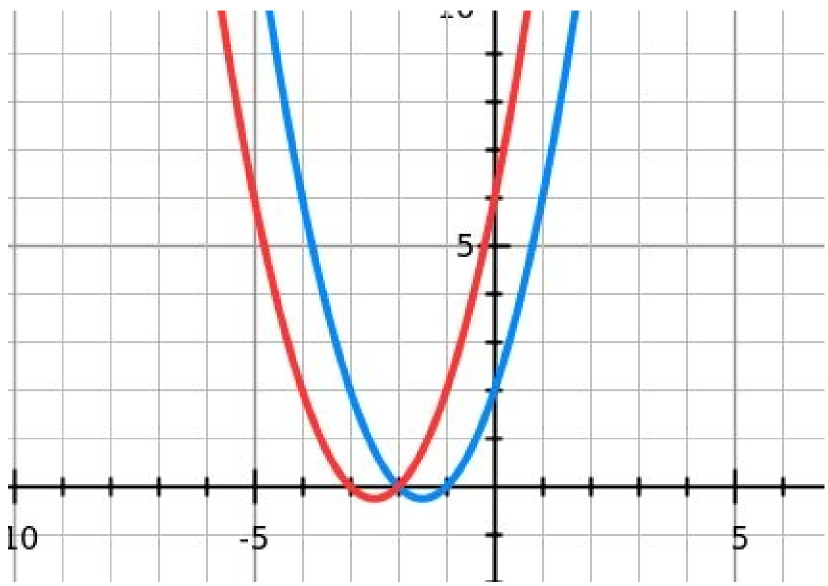
This is Type 3 algebra, and to solve we then have
$2x+6=2$

$2x=-4$

$x=-2$

So they cross at

$x=-2$



Epilogue

Epilogue

Quadratics are of a lot of importance in mathematics. Master the skills in this book. Now you can do a Quadratic - In A Minute. It will give you the tools you need for now and for more advanced maths. You have to deal with quadratics over and over again, not just to solve for x to find where it crosses the x-axis, but in the following areas.

Trigonometry

Matrices
Differential Equations Differentiation

and many others.

Try to achieve a fluency with them so you can do them with ease. Now you understand how they work, and the different ways of solving and finding out information about them, you're well on your way to being able to tackle all of maths.

Introduction to Inequalities

Introduction

Inequalities are the reverse situation of equations. Up to now, we've tried to solve equations to find out the value of x, and it always gives one (or more) specific values. However, we can also look at the reverse scenario, where, using the Third Rule, we see what happens when one side *doesn't* equal the other.

In this book we will look at the Three Types of Algebra to see how this works. We'll also see how it works with quadratics.

Three Types of Algebra for Inequalities

Three Types of Algebra for Inequalities

In Simultaneous Equations - In A Minute, we first met the Three Types of Algebra. For example, something like

$2x = 6$

With inequalities we look at the same situation, but when they DON'T equal each other. We could have

$2x < 6$

$2x \leq 6$

$2x > 6$

or

$2x \geq 6$

What do these all mean?

Let’s tackle one at a time.

$2x < 6$

Means ‘when is 2x less than 6?’

We solve this as usual, giving

$x < 3$

The answer then is when x is less than 3.

For

$2x \leq 6$

Same again, but this time when x is less than or equal to 6

So for this it would be

$x \leq 3$

Or in other words when x is less than or equal to 3.

For the reverse situations,

$2x > 6$

Here we are asking when 2x is more than 6.

Solving as usual

$x > 3$

The answer is when x is more than 3.

So when x is more than 3, 2x is not surprisingly, more than 6.

Simple stuff!

Finally

When

$2x \geq 6$

$x \geq 3$

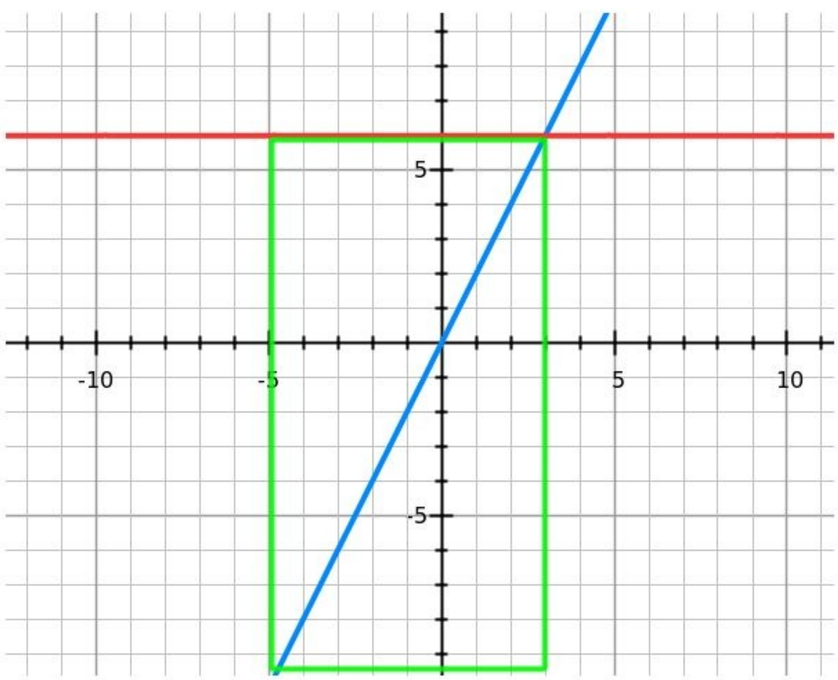
So when x is equal to or more than 3, 2x is equal or more than 6.

They are all the types of inequalities there are.

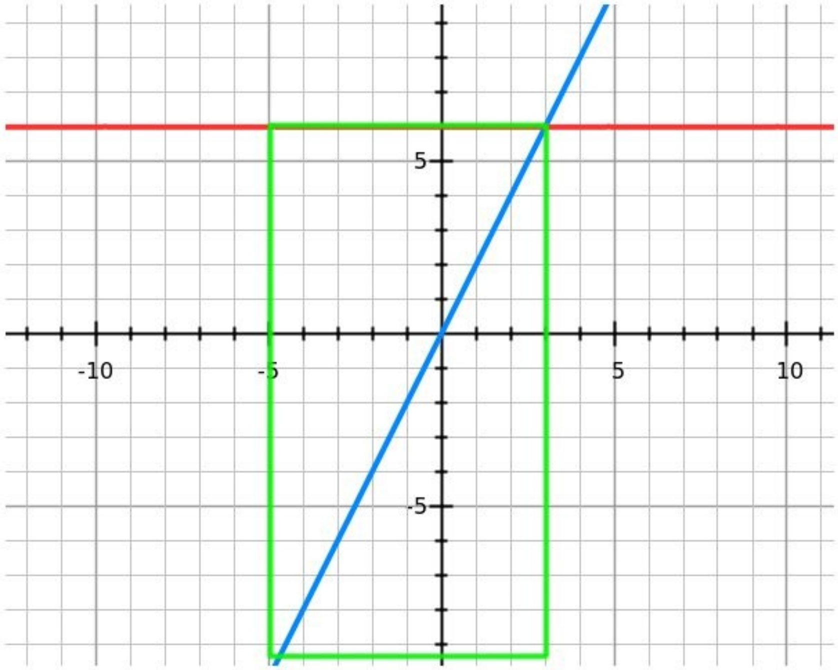
To see what this looks like on a graph, here's a graph of each situation.

As with simultaneous equations, these all represent straight lines. Instead of trying to find out where they cross, we can see where, in fact, they do not cross! We are going to find regions instead of specific points.

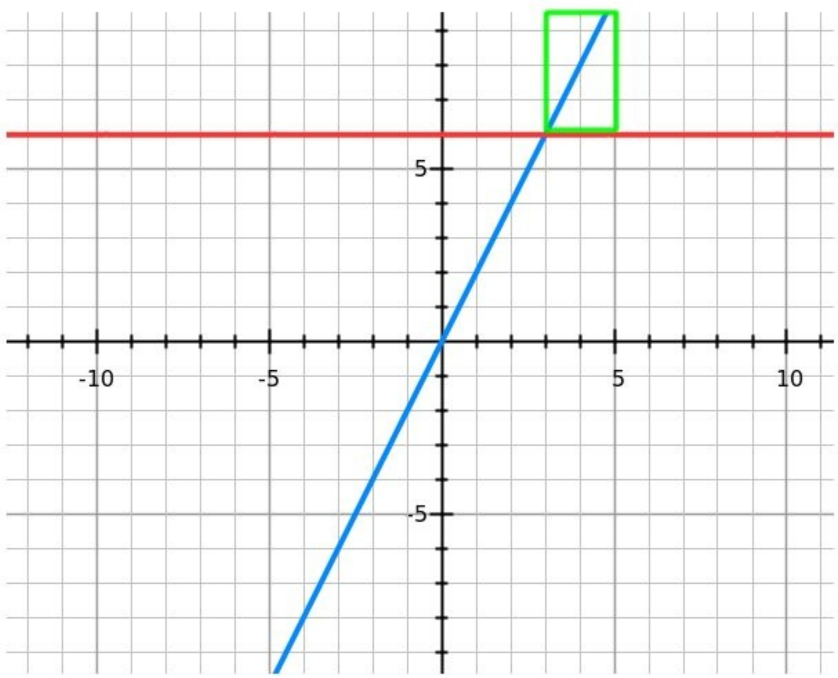
$2x < 6$



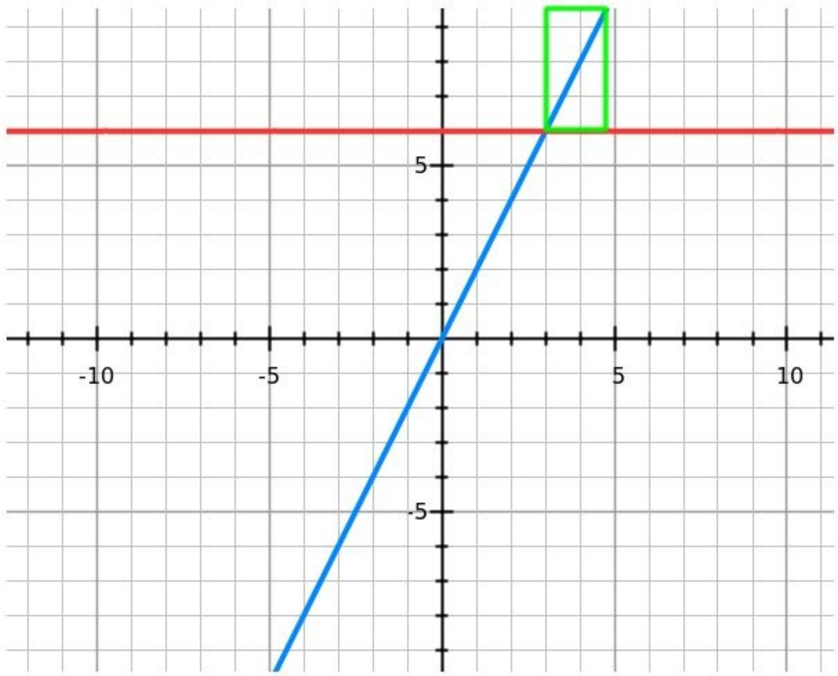
$2x \leq 6$



$2x > 6$



$2x \geq 6$



5
-5
-10
-5
5
10

Type 2 Algebra

Type 2 Algebra

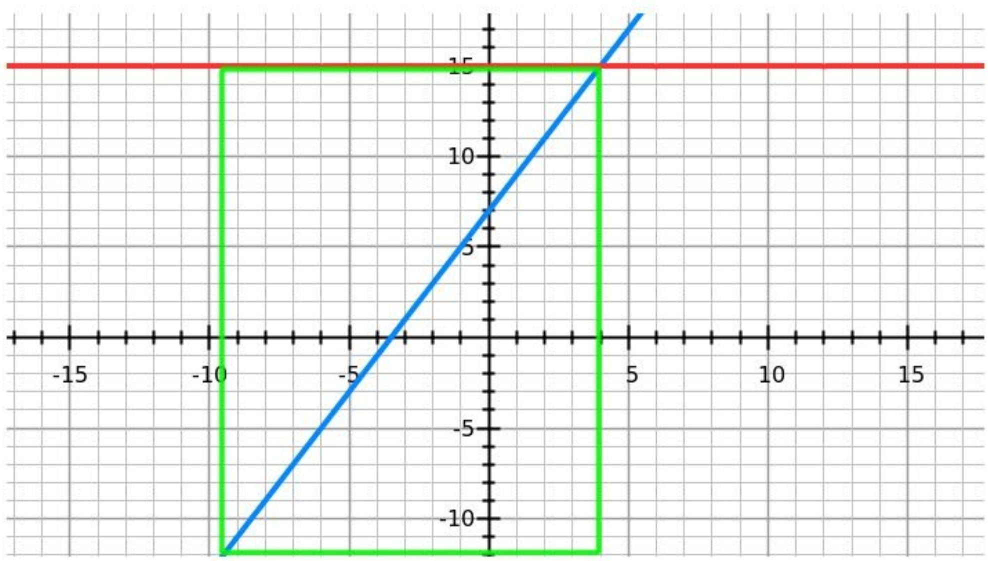
$2x + 7 < 15$

This is exactly the same treatment as we had in Simultaneous Equations, giving

$2x < 8$

$x < 4$

We can see this on a graph



Type 3 Algebra

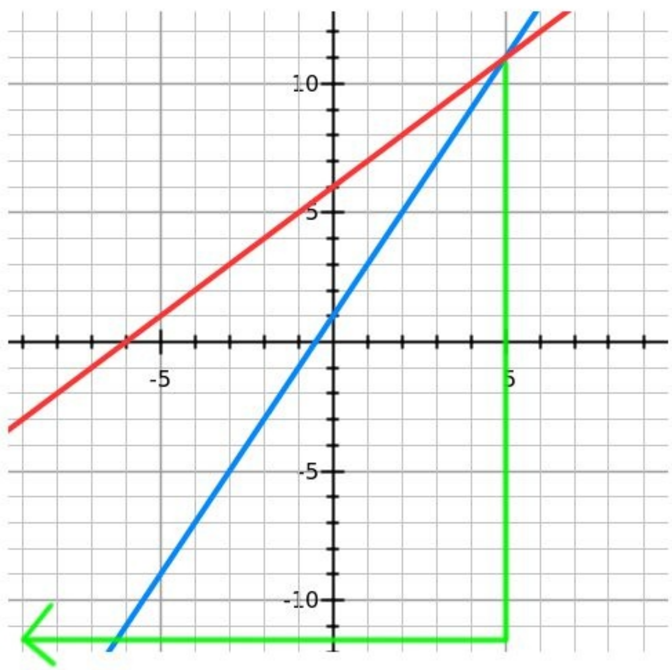
Type 3 Algebra

$2x+1<x+6$

Again, exactly the same as with Simultaneous Equations!

$x+1<6$

$x<5$



When the Inequality Sign Flips

When the Inequality Sign Flips

If we think back to Gradient, we discovered that there were three types of negative gradient, and these were

$m > -1$

$m = -1$

and

$m < -1$

Remember I pointed out that 'fractions' were in the > -1 region, which is the opposite to the positive gradient situation. For normal division and positive gradients, we saw fractions when the division was < 1.

This direction flip is something we need to bear in mind when we are solving inequalities. For example

$\frac{2}{3} < 1$

But if I multiply the left hand side by a negative (minus 1), this gives

$-\frac{2}{3} > -1$

The sign has to flip for this to make sense.

When solving, if we had

$7 - x < 5$

We can see the answer will have to be when

$x > 2$

How do we get to this?

If we subtract 7 from both sides

- $-x < -2$

Multiplying both sides by minus 1

$x < 2$

wouldn't make sense!

If x was less than 2, we would have
$7-1<5$

which isn't true.

If the sign flips in our answer to
$x>2$

Now we have

$$7 - 3 < 5$$

Which makes sense.

The treatment of inequalities is exactly the same as with simultaneous equations, except for this one detail. Watch out for it!

Inequalities of Quadratics

Inequalities of Quadratics

The two simplest situations we can find with quadratics is where the quadratic is either negative or positive.

For each, we just have to bear in mind a different way of writing for each.

For example,
Let's say we have our usual quadratics of

$x^2 + 3x + 2$

Where is it positive?

If we set it to

$$x^2 + 3x + 2 > 0$$

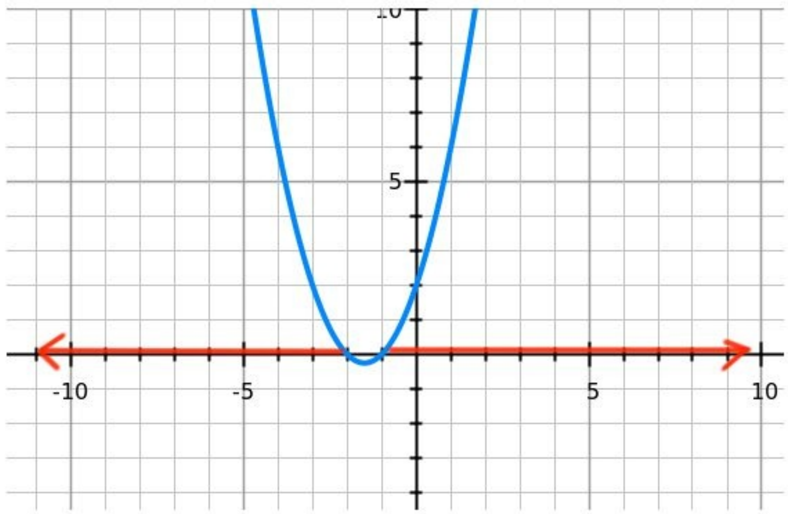
That's what this means.

From here, we solve as usual, by any means we like (factorising, formula, or completing the square).

This gives solutions of

$x = -1$
&
$x = -2$

So we know that the quadratic is positive to the left and right of these values.



We'd write this as

$x < -2$

&

$x > -1$

(be careful to write the lowest number first!)
If we wanted to find the same thing for when the quadratic was negative, we would have

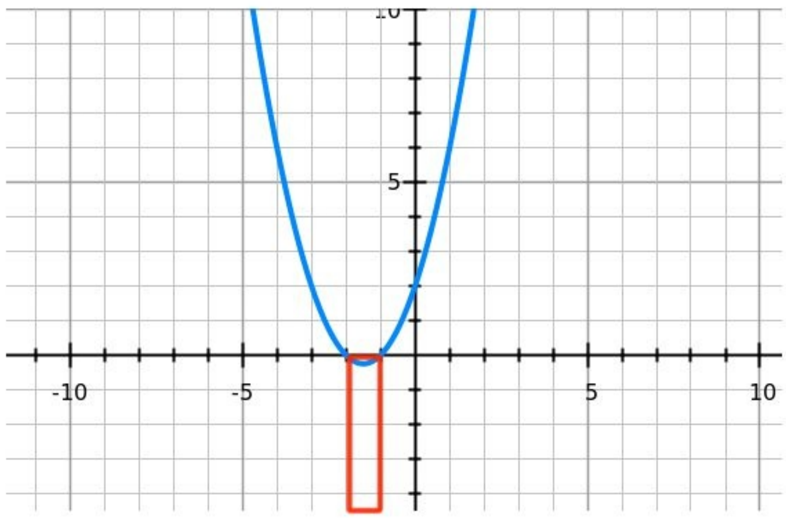
$x^2 + 3x + 2 < 0$

And we'd follow the same method.

Then we'd have a different way of writing as the answers wouldn't be two different regions. In fact, we are just describing one.

This becomes

- $-2 < x < -1$



We can see the region here where the quadratic is negative. This has two borders, so we just write x to be inside them.

Quadratics and Straight Lines

Quadratics and Straight Lines

Of course the x-axis is a straight line. The previous example is really our template for all of these questions.

If we want to know where a straight line is less than a quadratic, again, we can see this on a graph.

However, for accuracy, and speed, we can do this algebraically.

Let's say we want to know where

$y = 2x + 1$

is less than

$y = x^2 + 9x + 11$

Again this would be written

$$2x+1<x^2+9x+11$$

or we can write this in reverse

$$x^2+9x+11>2x+1$$

To find these regions, we just treat it the same as in Quadratics - In A Minute, where we wanted to know where they crossed each other.

Firstly we move

$$2x+1$$

over to the left-hand side to give

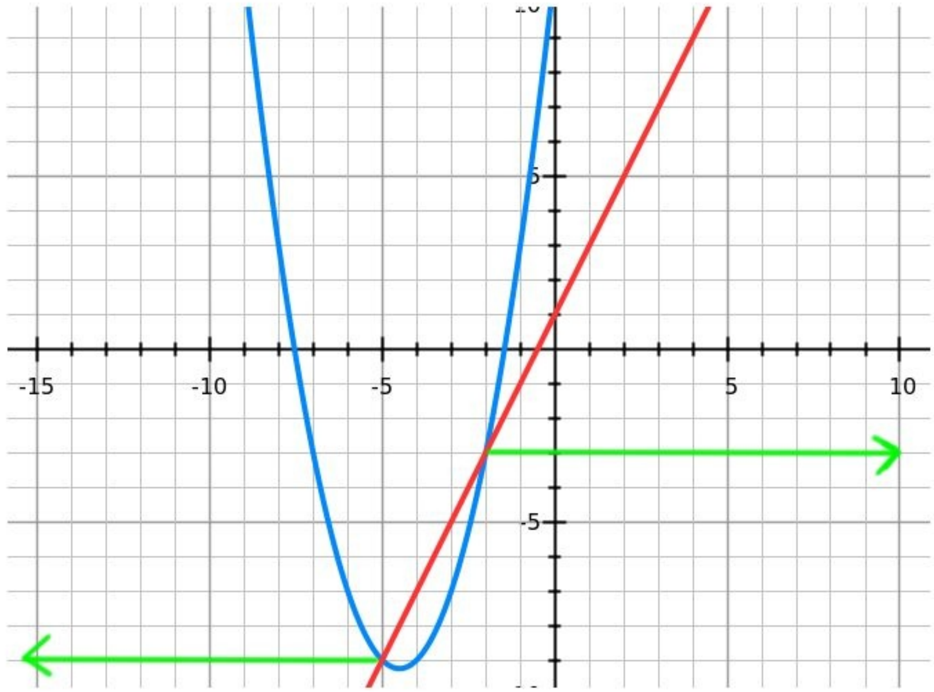
$$x^2+7x+10>0$$

And solve as normal

$$x = -2$$
$$x = -1$$

This then gives

$$x < -5$$
$$x > -2$$



In between - 5 and - 2, we can see that

$$y = 2x + 1$$

is below the quadratic.

Introduction to Changing The Subject

Introduction

Changing the subject…

This doesn't mean let's stop doing maths and talk about something else!

This is sometimes called 'Rearranging Formulae' in schools. I don't call it that because it's a title that is not all-encompassing. In other words, if you learn to change the subject, you can do it with anything, it doesn't particularly have to be a formula. In fact we'll see that we've come across this concept already in the Three Types of Algebra.

In this book we're going to examine changing the subject from simple equations to more complicated expressions. To do this we'll see how we can use BIDMAS - you'll see that this can be used in an unexpected way!

After that we'll look at how solving our 3 types of algebra is essentially changing the subject. We will then look at complicated situations where it's not clear what our first step should be.

Finally, we'll look at some famous formulae in science and re-arrange them.

Using BIDMAS to Change the Subject

Using BIDMAS to Change the Subject

First of all, what is the subject?

If we look at these few equations, the subject is the letter that everything else is equal to.

So for

$E = mc^2$

E is the subject

$v = u + at$

v

is the subject

$T = 2\pi\sqrt{\frac{l}{g}}$

T

is the subject.

To change the subject, this means putting a different letter from the equation in the place of the original. To do this, we must follow some mathematical rules.

You may have come across

BIDMAS

before.

I have not mentioned it as yet in my course. This is because I don’t believe it’s entirely true. The letters stand for
BRACKETS INDICES DIVISION MULTIPLICATION ADDITION SUBTRACTION

and denote the order you should do things if you’re not sure.

For example

3 + 5 x 8

Should be

3 + 40 = 43

as you should do the multiplication before the addition.

So far you’ll have noticed we’ve survived not using this acronym. That’s because I’ve explained certain things in a different way. Another reason I’ve not referred to it is that it has B at the start. This is usually taken to mean that you should ‘do brackets first’ which as we’ve demonstrated in an earlier book, isn’t true.

So to use BIDMAS

I want you to drop the B!

So now only IDMAS.

This is now correct, as we’ll see again.

So IDMAS tells us the normal order of things to do in a question in arithmetic or algebra. That’s fine.

When we change the subject, we follow the Third Rule, and do IDMAS in reverse, in other words
SAMDI

And follow it along.

Examples

Let's look at a basic example
$a = b + c$

What's the subject?

a

Let's say we want to make b the subject.

What we do is run through SAMDI and ask

Is there anything Subtracting from b, the subject?

No.

Anything Adding?

Yes. What? c.

So to get rid of c, we do the complete opposite of adding, and subtract. It then goes on the OTHER side.

This gives

$a - c = b$

We subtract c, and put it with a on the other side.

In this case, this leaves b on its own, so we now have b as the subject, which was our goal.

By convention, we place the subject first, so we write
$b = a - c$

Finished!

Let’s look at another example

$a = b - c$

Again, what is the subject?

a

Let’s make b the subject.

Running through SAMDI

Is there anything being subtracted from b?

Yes. c

So let’s do the opposite and add c to the other side.

$a + c = b$

Again, we have b on its own so it is now the subject.

By convention

$b = a + c$

As you can see, we just run through IDMAS backwards, in reverse, and this will change the subject for us.

Let's try

$a = bc$

Again, a is the current subject, let's make it b

Is anything subtracted from b?
No.

Is there anything added to b?

No.

Is there anything multiplied to b?

Yes. c

So we do the reverse, and DIVIDE by c on the other side.

This gives

$$\frac{a}{c} = b$$

Again, by convention

$$b = \frac{a}{c}$$

Finally

Let’s say we have

$$a = \frac{b}{c}$$

and we try to make b the subject.

Running through IDMAS

Anything subtracted from b? No.
Anything added? No.
Anything multiplied? No.
Anything divided? Yes! c

So we do the reverse, and MULTIPLY by c.

This gives

$$ac = b$$

and by convention

$b = ac$

The last letter, indices, means that there is a power on the letter we want to make as the subject, so we must reverse this.

For a famous equation like

$E = mc^2$

and we want to make c the subject, we have a problem as it has a square on it!

But we follow the process as normal.

Anything subtracted from c?
Anything added?
Anything multiplied? Yes. m.

So we divide on the other side giving

$\frac{E}{m} = c^2$

Anything divided? No.

Any indices? Yes. The square.

So the opposite is square rooting.

This gives

$$\sqrt{\frac{E}{m}} = c$$

and by convention

$$c = \sqrt{\frac{E}{m}}$$

How We've Used This Before

We have actually seen everything we’ve done so far in earlier books, I just didn’t explicitly say ‘this is changing the subject’.

For example

$$2x = 6$$

If we say we want to make x the subject, that is, solve this equation, and we follow IDMAS in reverse we could do it that way. Because they are numbers though, and I've asked you to tackle this kind of question intuitively, this hasn't come up.

We could say
Is anything subtracting from x? No.
Anything added? No.
Anything multiplied? Yes. 2.

So divide by 2 on the other side and we have
$x = 3$

So our Three Types of Algebra have all been an exercise in changing the subject.

Exceptions To The Rule

Exceptions To The Rule

There are 3 main exceptions to following IDMAS backwards, and they are little hurdles that need to be recognised and jumped over before using IDMAS at all.

Exception No.1

There is a division in the equation.

For example

$$a = \frac{b+c}{t}$$

and we want to make c the subject.

Because of that division of t, we can't do anything yet. You may remember from Division - In A Minute that division isn't very accommodating. This is why!

So the first step is to GET RID OF THE DIVISION

And do the reverse to give

$$at = b + c$$

We now can follow IDMAS/SAMDI.

$$c = at - b$$

Exception No. 2

Brackets are in the equation.

(this is something we saw in 'Completing the Square', as I will describe later).

$a(b+c)=d$

Let's say we want to make c the subject.

The problem we have is that c is inside brackets, so we have to get it out of there.

So, before using IDMAS/SAMDI, we must multiply the brackets!

This gives

$ab+ac=d$

We now follow IDMAS/SAMDI

To give

$$ac=d-ab$$
$$c=\frac{d-ab}{a}$$

In the previous example, if we wanted to make a the subject, instead of multiplying the brackets, we could be a little cleverer and just divide. This solves our question in one go.

$a(b+c)=d$

$$a=\frac{d}{b+c}$$

Exception No. 3

The *'repeated factor'*.

What this means is that the subject we want to change our equation to appears twice.

For example

$ab + bc = d$

And we want to make b the subject.

What would we do here?

Really, we only want one b. To get this, all we have to do is factorise, to give $b(a + c) = d$

and then, as in the previous example

$$b = \frac{d}{a + c}$$

We could then have combinations of these exceptions. So something like

$$\frac{a}{a + c} = b + c$$

and we want to make a the subject.

GETTING RID OF DIVISION first,

$a = (a + c)(b + c) = ab + ac + bc + cc$

We now have three a's!

Bringing them all together

$a - ab - ac = bc + cc$

Factorising

$a(1 - b - c) = bc + cc$

and dividing by the bracket

$$a = \frac{bc + cc}{1 - b - c}$$

We could also factorise the top line as it contains c twice.

$$a = \frac{c(b + c)}{1 - b - c}$$

Here I've factorised the top line - but why? Why bother? This is answered in the book 'Algebraic Fractions - In A Minute'.

Another example would be

$$(a + c)^2 = b$$

Let's say we want to make a the subject. Again, this is trapped in a bracket, but worse, this bracket is also squared. Before we can apply IDMAS/SAMDI, we need to square root just to make it a bracket at all!

$$a + c = \sqrt{b}$$

$$a = -c + \sqrt{b}$$

This is what we saw in 'Completing the Square' which led to a fast way to solve a quadratic, by square rooting the left-hand side to make x the subject.

$$y = x^2 + 3x + 2 = 0$$

which is

$$(x + 1.5)^2 - 0.25 = 0$$

We now shuffle this equation around to find x

$$(x + 1.5)^2 = 0.25$$

Square root both sides

$$x + 1.5 = \pm\sqrt{0.25}$$

$$x = -1.5 \pm \sqrt{0.25}$$

which is

$$x = -1.5 \pm \frac{1}{2}$$

Here we are making x the subject, although it is locked up in a bracket and squared. You can see we follow IDMAS in reverse to make it the subject, as soon as we get rid of that squared bracket.

Famous Science Formulae Rearranged

Famous Science Formulae Rearranged

$F = ma$

This is Newton's 2nd law of motion, which tells us how much force you get from an accelerating mass.

Making a the subject

$$\frac{F}{m} = a$$

$$a = \frac{F}{m}$$

$$F = \frac{Gm_1m_2}{r^2}$$

This is Newton's law of Gravitation, which calculates the gravitation force between two bodies (things) that have mass (almost everything!).

Making G the subject…

$$G = \frac{Fr^2}{m_1m_2}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

This is the formula which describes simple harmonic motion. In other words, if a pendulum oscillates regularly, we can calculate how long it will take based on its length.

Making g the subject

$$T^2 = 4\pi^2 \frac{l}{g}$$

$$g = 4\pi^2 \frac{l}{T^2}$$

Introduction to Cubics

Introduction

In Quadratics - In A Minute we saw what happened when we multiplied two brackets. This gave a rectangle, and a parabola, and a new expression, which began
$x^2 + ...$

In Cubics, we will look at what happens when we take this one step further, and multiply by another bracket. We will see a new and simple way to do this, which was taken from Multiplying - In A Minute.

From then on we'll see what shape we get this time, what information we get about a cubic, how to factorise and solve them, and whether we can find their minimum (and maximum!) values.

This book is suitable for A-level in UK and will demonstrate advanced techniques to achieve things that take much longer using traditional school methods.

Firstly, let's look at multiplying 3 brackets!

Multiplying Three Brackets

Multiplying Three Brackets

Here, let's say we choose to multiply
$(x+1)(x-2)(x+3)$

For the first two brackets, we could multiply them using the column method from Quadratics, viz:
$(x+1)$

$(x-2)$

Giving

$x^2 - x - 2$

This would then need to be multiplied by

$(x+3)$

Instead of multiplying each term and collecting like terms, we can do this by the 3 x 3 method of the 'Union Jack Situation' we saw in Multiplying, viz:
$x^2 - x - 2$

$\times (x+3)$

giving

$x^3 + 2x^2 - 5x - 6$

If you're not sure how I did this, here's a reminder of that method for numbers.

…So for this method there are 5 steps. 4 of them will be exactly the same as you have already seen, the 5th but, middle, step is the 'Union Jack Situation'.

Let’s take a look at the multiplication above:

123 x 421

Place in a column

  123
x 421

Step 1:

Exactly the same: right hand column.
  123
    ⇩
x 421

-----

3 x 1, gives

  123
    ⇩
x 421

    3

Next step, exactly the same: cross
  123

✕
x 421

3

So ignore the 1 and 4 on the left column, we pretend they're not there and then this gives, 2 x 1 + 3 x 2 = 8

So

123
x 421

83

So far, this has been exactly the same. Here, we see the new 'Union Jack Situation' come in.

We now place our union jack in between the numbers, viz:

123
✱
x 421

83

So we have 3 multiplications to do and remember 3 answers! After achieving fluency with the 2 x 2 method, this becomes easy.

So we have, multiplying along the lines,
1 x 1 + 3 x 4 + 2 x 2
= 1 + 12 + 4

= 17

So we now have

123
x 421

${}_{1}783$

So that’s 3 out of 5 steps!

Steps 4 and 5, as I mentioned above, now follow the symmetry of mathematics.

Step 4

123
X
x 421

${}_{1}783$

Will not ignore the right hand column completely.

So we have 1 x 2 + 2 x 4 = 10

Adding the carry of 1, gives us 11.

123
x 421

${}_{1}1{}_{1}783$

Step 5 is simply multiplying the left column.

  123
  ⇩
x 421

${}_{1}1{}_{1}783$

1 x 4 = 4
Plus the carried 1, = 5.

Giving us

   123
 x 421

$5{}_{1}1{}_{1}783$

Therefore

123 x 421 = 51 783.

I have gone through each step very slowly and carefully as it will be the first time you've seen it. With practice, you will be able to do this without writing any crosses or union jacks and just write down numbers, to lead up to the answer!

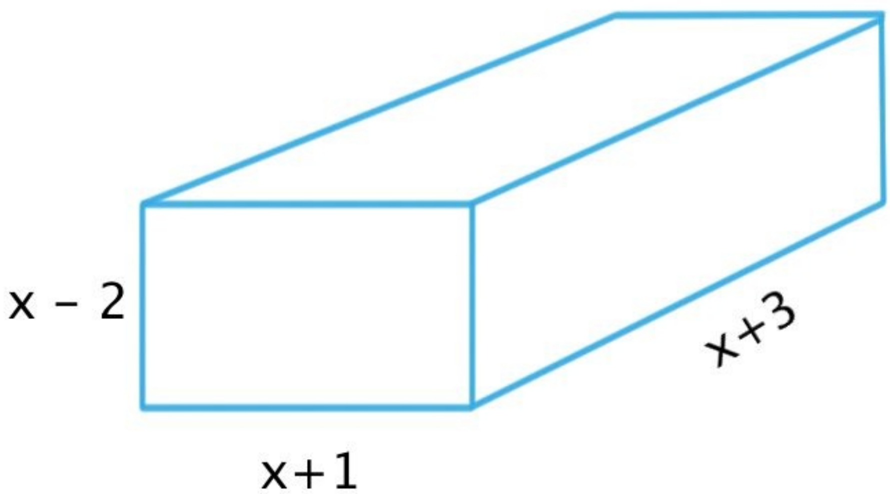
You can watch this live video of me doing a 3 x 3 digit multiplication of 124 x 132 at this YouTube link.

This is a cubic. So called as its highest power is x^3.

We saw in **Quadratics** that two brackets gave both a rectangle and a parabola. What shape do you think this will give?

This will give a

cuboid



And what information will that give us about the cuboid?

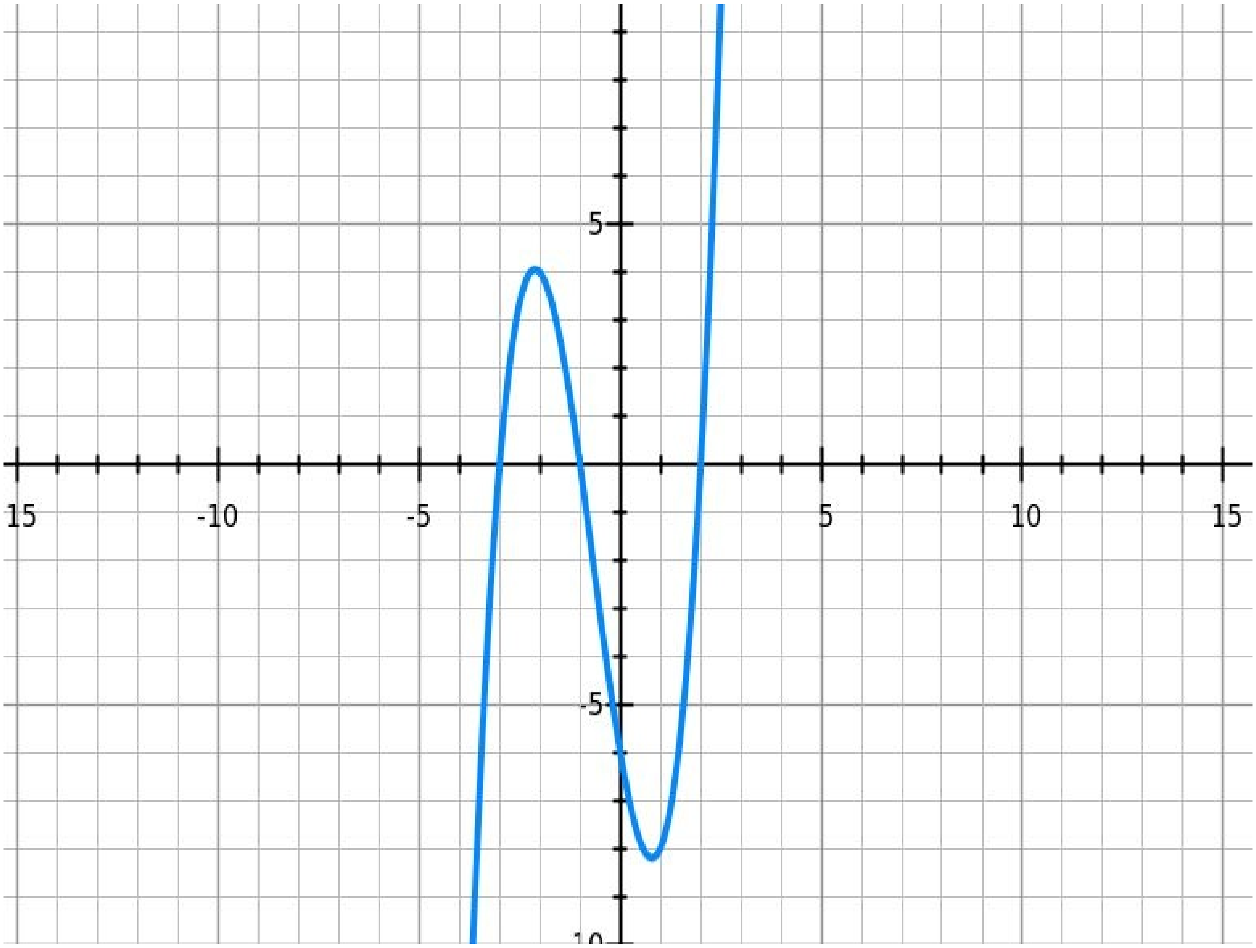
The volume.

The image on the graph we can find by plotting points using a table.

$y = x^3 + 2x^2 - 5x - 6$

x	- 3	- 2	- 1	0	1	2
$y = x^3 + 2x^2 - 5x - 6$	0	4	0	- 6	-8	0

If we plot these points, we'll see it gives an unusual looking graph, which isn't symmetrical.



This is because it is an ‘odd’ function. A quadratic, being symmetrical, is an even function.   This will be examined in more detail in ‘Functions - In A Minute’.

We can see from this table where the solutions of the cubic are here. y = 0 when x equals - 3, - 1 and 2.

Once we know how to multiply these brackets out, using the Third Rule, we will want to know how to return to the brackets, or factorise, this cubic.

Factorising a Cubic

Factorising a Cubic

As we saw in Quadratics,

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

So to factorise this cubic, we need to find one factor, and using that find the others.

If we demonstrate this with the same cubic we've just found, we can see how it works.

We know in this cubic that the solutions are

$x = -3$

$x = -1$

$x = 2$

This means that

$x + 3 = 0$

$x + 1 = 0$

$x - 2 = 0$

As x has these values.

In other words, these are the factors of the cubic. You'll note that these are exactly the same as the brackets we started off from in the previous chapter. *These brackets multiply to form the cubic*, so they are its factors.

But what if you don't have the solutions to start off with? What if you want to find the solutions without drawing the graph?

Again,

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

So our first step will be to find at least one of the factors of the cubic. To do this, since we know that they are linked to the solutions, we could just 'try' solutions and see if they give zero.

Here we should be a little clever. Don't just try ANY solution. Use numbers that are factors of the 'cut'. The number on the end of the cubic, not multiplied by x . Our number for c is

- c = - 6

So try
1, -1
2, -2
3, -3
and
6, -6

So let's start...

Let

$x = 1$

$y = (1)^3 + 2(1)^2 - 5(1) - 6$

we see that this doesn't give zero. In fact, it gives - 8 as we saw for our table.

Let's try

$x = -1$

This gives

$y = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

This does give zero, so we know that

$x = -1$

is a solution and

$x + 1 = 0$

$(x+1)$

is a factor.

Now we have one factor, we can use it to find the others.

Just like the algebra trick for division, we can ask, what times by $(x+1)$ to give the cubic?

So if we write

$(x+1)(.........)$

We can fill the second bracket with a quadratic which will multiply to give that cubic.

The first term will be x^2, in order to give x^3 when they multiply.

$(x+1)(x^2.......)$

$$(x + 1)(x^2 \qquad\qquad)$$

The final term, the number, will have to multiply by 1 to get - 6.

Therefore it must be - 6

$$(x + 1)(x^2 \quad -6)$$

$(x+1)(x^2 ......-6)$

The middle term of the quadratic we can find in two different ways.

Way 1.

We need to see how many x^2's we need to make the same number in the cubic.

In the cubic we have

$2x^2$

In the multiplication we already have

$1 \times x^2 = x^2$

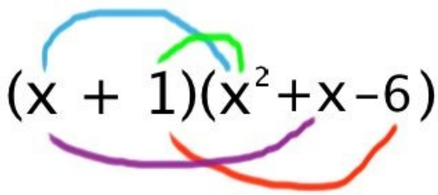
$$(x + 1)(x^2 \quad -6)$$

So we'll need another x^2 to make it to

$2x^2$

In the middle then we'll have

$(x+1)(x^2+x-6)$



That’s it!

We’ve found the other factor that multiplies by
$(x+1)$

to get

x^3+2x^2-5x-6

or

Way 2.

We can see how many $x's$ we need to get the same in the cubic.

In the cubic we have

- $-5x$

To get this, we can look at our - 6 and note that when it multiplies by x, we get

- $-6x$

So we need just a single positive x to get it to

- $-5x$

Hence we just need x in the centre.

Factorising a Cubic 2

Now to find the other two solutions/factors, we factorise/solve the quadratic we now have.

$(x^2 + x - 6)$

This gives

$(x+3)(x-2)$

From **Quadratics - In A Minute**

From here, we have factorised the cubic entirely. This gives
$(x+1)(x+3)(x-2)$

Equating the cubic to zero will tell us where it intersects the x-axis, as again, we saw with Quadratics.

Therefore

$x^3+2x^2-5x-6=0$

and we can write

$$(x+1)(x+3)(x-2)=0$$

since this is like a 'pre-multiplied' version of that same cubic.

If these terms all multiply to equal zero, as in Quadratics, we can say that each bracket must equal zero.

Or, if these brackets multiply to form a cubic, which as volume, then if the volume of the shape is zero, then its dimensions must be zero.

So

$$x+3=0$$

$$x+1=0$$

$$x-2=0$$

As we saw already.

Then for these to make sense, x must be equal to

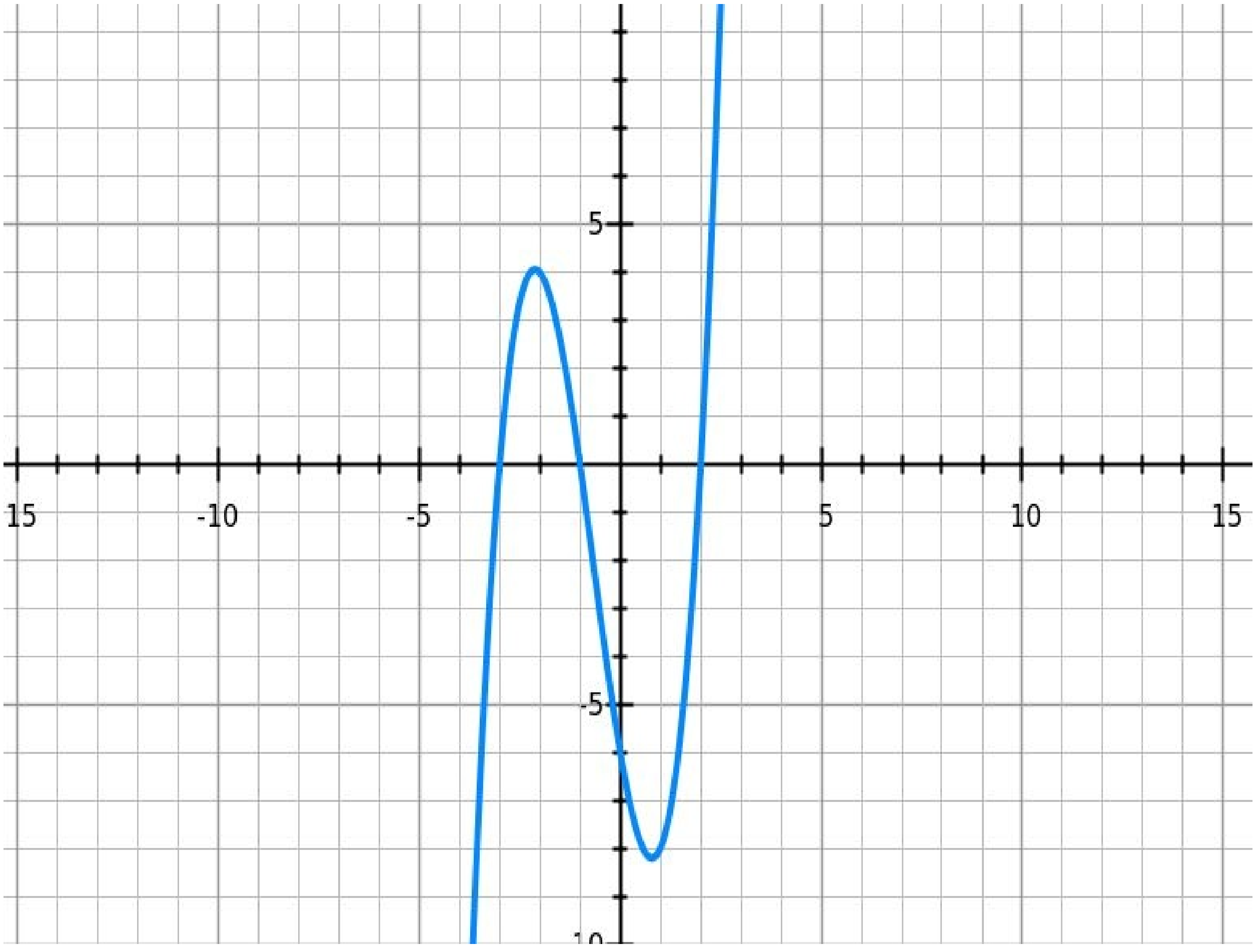
$$x=-3$$

$$x=-1$$

$$x=2$$

which is where the cubic crosses the x-axis.

We can see this on the graph too.



This idea that when you multiply factors together to get an answer is technically known as the ‘Factor Theorem’ which is just a complicated way of saying that *To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.*

That is, it is possible to factorise a cubic... with its factors. That’s all.

Remains of The Day

Remains of The Day

So of course, we have the reverse of this situation.

Either an expression like

$(x+1)$

will be a factor of a cubic, or it won't.

If it's not, all that happens is that it won't divide in exactly.

This is just like

$$\frac{15}{4}$$

Four is not a factor of 15, as it doesn't go in exactly. We see this because when we divide them there is a remainder.

Similarly for algebra, if we were to divide our cubic
$$x^3 + 2x^2 - 5x - 6$$

by

$$(x+4)$$

because it's not a factor (as we know now) it won't divide in exactly. As a result there will be a remainder.

The imaginatively titled **'Remainder Theorem'** is the name for this situation. To find the remainder, we just use the same method for when we wanted to check whether an expression was a factor, by writing it as
$$x + 4 = 0$$

So we try

$$x = -4$$

in the cubic.

If we try this we get

$$y = (-4)^3 + 2(-4)^2 - 5(-4) - 6$$

this gives

$$y = -18$$

And since this isn't zero, we know that

$(x+4)$

isn't a factor.

If we had done a division where the cubic was divided by
$(x+4)$

we would have seen that this was our remainder.

Hence the name 'Remainder Theorem'.

In other words, *if our remainder is zero,*

it is a factor

If our remainder is non-zero,

it is NOT a factor.

You've been doing this with numbers for years, so this is the same concept.

As yet, I haven't described how to actually do this.

How to divide a cubic by an expression such as

$(x+4)$

The reason I haven't explained it is because I don't want you to!

There is a way of finding the answer without having to divide at all.

Again, we just need to imagine what we need to multiply
$(x+4)$

by.

First of all, we need to find the remainder.

We do this, as we've said, by substituting

$x = -4$

This gave

$$y = -18$$

as we saw.

If we think back to our division example above with numbers

$$\frac{15}{4}$$

We know the remainder is 3.

If we subtract that from 15 we get 12.

THEN we can figure out what multiplies 4 to get 12 (also 3).

Since we subtracted the remainder, we were able to perform the division.

We do the same for the cubic.

We subtract the remainder from it, giving

$$x^3 + 2x^2 - 5x - 6 - (-18)$$

giving

$$x^3 + 2x^2 - 5x + 12$$

(Since that minus times a minus is a plus, from Negative Numbers - In A Minute)
We can then figure out what multiplies

$$(x+4)$$

to get this.

We use the same bracket technique

$$(x+4)(..........)$$

and fill the second bracket with terms.

Again, to start with

$$(x+4)(x^2 ....)$$

As they multiply to get x^3

To get 12, we'll need + 3

So we have

$(x+4)(x^2....+3)$

And finally to get the x term in the centre, we will need

- $-2x$

To reduce our x^2 terms to $2x^2$.

Giving

$$(x+4)(x^2-2x+3)$$

If we now add on our remainder, the answer to the division of the original cubic by

$$(x+4)$$

will be

$(x^2 - 2x + 3) - 18$

Which we can check by multiplying out again
$(x+4)(x^2 - 2x + 3)$

$(x^2 - 2x + 3)$
X $\quad (x+4)$

which gives

$x^3 + 2x^2 - 5x + 12$

if we then add the remainder of - 18

we get

the original cubic

$$x^3 + 2x^2 - 5x + 12$$

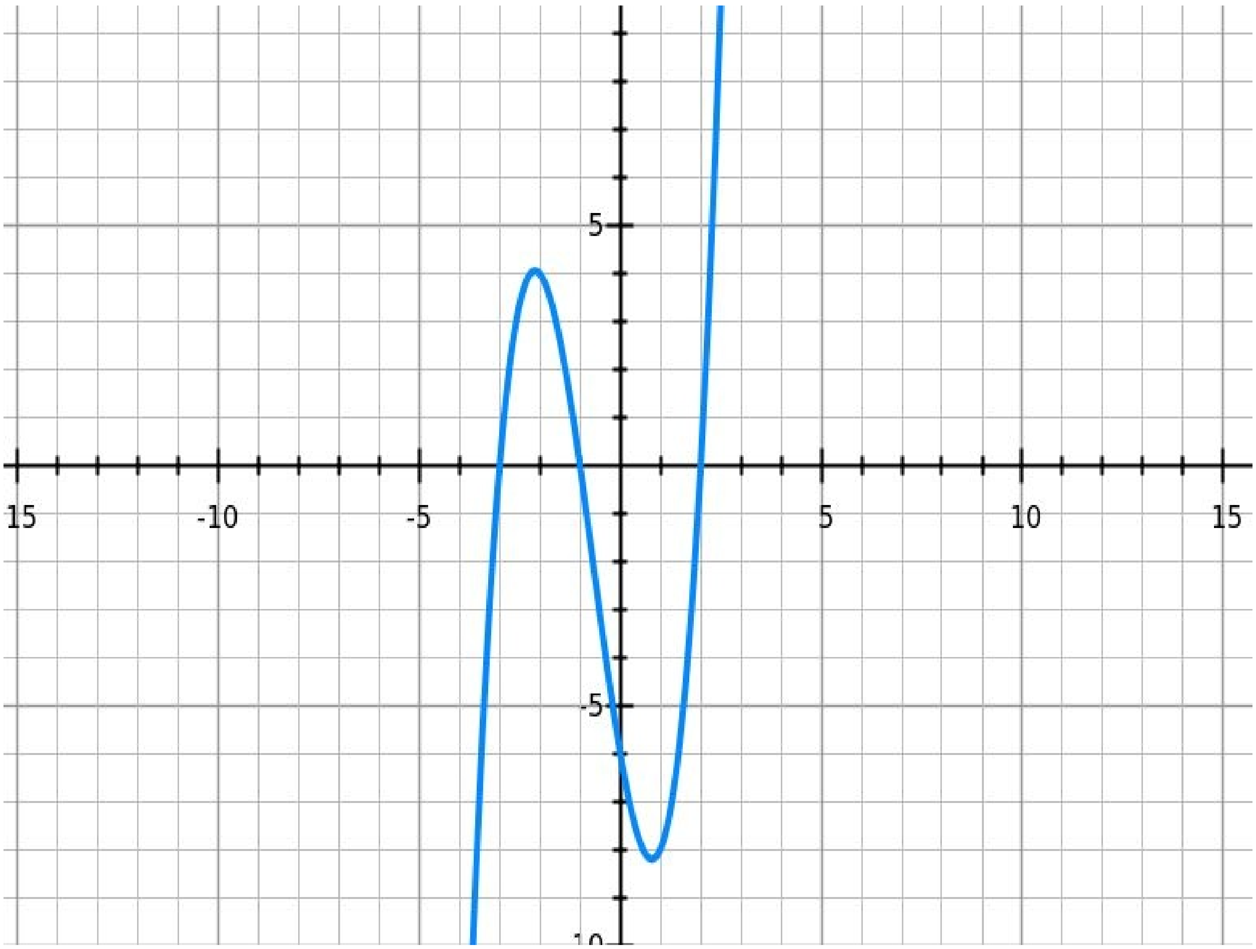
We can divide a cubic like this for any expression.

We DO NOT need to use long division. EVER!

Finding the maximum or minimum values

Finding the maximum or minimum values

You can see from the graph of the cubic that we have a maximum and minimum value.



In Quadratics, we spent a lot of time finding different and easy ways to find the minimum value (or maximum, depending on whether it was a negative quadratic). For cubics, there’s no magic algebraic way of doing it. You can’t do it in your head like you can for Quadratics!

This is because a cubic is an ‘odd’ function. By that we mean the reverse of ‘even’. An even function is symmetrical. An odd function is not. As a result, we have no algebraic way of finding these points.

But it IS possible to find them.

To find out how, read Book 25 - Gradient/Differentiation 1.

Introduction to Advanced Mental Multiplication

Introduction to Advanced Multiplication Techniques

In this book we're going to examine some ultra-fast ways to multiply numbers. In the original book, Multiplication - In A Minute, we saw a technique to apply for every situation for the times tables, and then for numbers that are 2 digits in size, as well as 3, and 4 digits.

We then saw in further books this method being applied for decimals, percentages, standard form and quadratics. Later in the book, I will explain the algebra behind how this works. This is what I call 'Algebraic Arithmetic'. However I don't call it this at the start as it's somewhat of a scary name. However, that is why the book has this subtitle.

We shall now 'move on' from this technique a little, by seeing if sometimes we can calculate mentally (in our heads) instead of relying on pencil and paper.

For this we're going to rely on one method for this for both single digit and double digit numbers.

It is a method taught in 'Squaring - In A Minute', but I will outline it again here. We'll then see it applied to everything.

Once we've looked at examples such as

7 x 9

and

19 x 23

we’ll look at squaring larger numbers manually.

We’ll also then look at ‘doing the reverse’ of this, square rooting these back again. Once we understand how this works, we’re away.

After that, we’ll look at a way of becoming more familiar with square numbers, by using them to find other square numbers.

We’ll examine how we can check our answers are correct, in seconds, and without having to divide.

We’ll then use this technique to perform a couple of magic tricks - useful for parties! This will have your friends scratching their heads and thinking you are some kind of wizard.

To finish, we’ll look at the algebra behind these systems, and see how they work. We will find them very simple, and you’ll likely be amazed they are not more widely known.

The Times Tables

The Times Tables

In Book 1, we saw how to find the answers to times tables - by which I mean single digit numbers, such as 7 x 9 - using 3 possible methods, whatever was your preference.

In this more advanced book we're going to use square numbers to find them all.

In other words, the numbers from this table in red. When the same numbers are multiplied, instead of getting a rectangle, we get a square.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Firstly, you may not know the square numbers up to ten. That's fine. We can use my squaring system for this also. Here's a brief reminder. For example:
For

7^2

We go to the nearest ten, which for numbers we’re dealing with at the moment, is always 10.

This is 3 away.

We then go in the opposite direction for the same distance.

This takes us to 4.

We multiply together, giving

4 x 10 = 40

The distance we had to move to the nearest ten, in this case 3, we square and add on.

$3^2 = 9$

40 + 9 = 49.

So that’s

7^2

We can use this for all the squares from 6 onwards.

The first 5 you just need to learn. We had to know 3-squared for this one! You can do this by adding repeatedly every time, since Multiplication Is Just Addition. There is also a way of doing it using our above method. However this doesn’t use the nearest ten (why not?).
What we could is always move a distance of 2, so we know we always add on 4.

For 4 x 4

We could do 2 x 6 + 4 = 16

For 3 x 3

1 x 5 + 4 = 9

For 2 x 2 of course we know it is 4, since that is our usual square to use!

And 1 x 1... hopefully you know that one by now, since we've seen it in AREA, as a definition for a metre square.

Using Squares For The Times Tables

Let's say we want to find

5 x 7

All we do is look for the number exactly in between 5 and 7.

This is 6.

We *square* this.

$6^2 = 36$

We then *square and subtract* the difference we had to go to get from 5 or 7 to 6. This is 1.

$1^2 = 1$

36 - 1 = 35

Another example

5 x 9

Here our middle number is 7.

We square this.

$7^2 = 49$

We then square the difference and subtract.

$2^2 = 4$

49 - 4 = 45

How about

8 x 12

We square 10.

$10^2 = 100$

Square and subtract the difference, 4,
100 - 4 = 96

As you can see we can use this method quite easily, especially if we know the square numbers. As a result, out of all of those numbers in the table, you only have to know the squares!

Another one

3 x 5

$4^2 - 1 = 15$

3 x 7

$5^2 - 4 = 21$

7 x 11

$9^2 - 4 = 77$

We can do any multiplication in this way. And in our heads!

Odd Number Differences

So far, you may have noticed that the differences are always a symmetrical distance apart, so that when we find the number in the middle, it is always a round number, rather than something like 7.5.

We can tackle this problem in one of two ways. The first way is recommended, but if you want to have some fun, use the second method.

Let's say we want to do

7 x 8

Probably the best way would be to square either number and then add or subtract as necessary.

So

$7^2 = 49$

So 7 x 8 = 56

Since we add on an extra 7.

Or we could do that with 8^2

Giving

64 - 8 = 56

Another example

5 x 8

Here our middle number is 6.5. If instead we make it easier by looking at

6 x 8

We could then do

$7^2 - 1 = 48$

6 x 8 = 48

So

5 x 8 = 40

This is all done in your head, and comes with practice and fluency.

Another way to do it would be to use the fact that we know how to square numbers ending in 5. Again, looking at Squaring - In A Minute
We can do

75^2

by again looking at the nearest ten.

This is 70 AND 80.

Then we just do

7 x 8 + 25

= 5625

6.5^2

would be

42.25

For our example above

5 x 8

We could do our middle number squared and then subtract the square of the difference.

Again this difference will end in 5, so this is just another simple one to do.

For 5 x 8

We have

$6.5^2 - 1.5^2$

42.25 - 2.25 = 40

This will always work out neatly like this.

4 x 7

$5.5^2 - 1.5^2$

30.25 - 2.25 = 28

4 x 9

$6.5^2 - 2.5^2$

42.25 - 6.25 = 36

Once we master this squaring technique, we can easily see how to calculate all our times tables, just based on that diagonal in that grid, plus knowing squares that end in 5.

In practice, you might just want to use the first method outlined for odd differences as it is the simplest.

But this second method can also be used for two-digit numbers also.

How to Multiply Two Digit Numbers In Your Head

How to Multiply Two Digit Numbers In Your Head

The previous chapter took care of all single digit multiplications. We saw we could tackle these in a couple of ways, and it is quite surprising, I think, that you can take these different routes, but end up at the same destination. That is the beauty of arithmetic, algebra and mathematics.

To do two digit numbers, such as

17 x 23

We can use exactly the same method. Again, we need to rely on our squaring method from Squaring - In A Minute.

Make sure you achieve fluency with this.

For example for

17 x 23

This is quite easy.

Our middle number is 20.

Squaring this

$20^2 = 400$

We then subtract the square of the difference

3^2

400 - 9 = 391

Finished!

How about

29 x 33

Middle number is 31.

Square this.

$31^2 = 961$

Subtract the square of the difference (4)
961 - 4 = 957

The real advantage of this method is being able to do this mentally. You can just quickly think of the square and you're nearly at the answer.

Another example

41 x 49

Easy...

$45^2 - 16$

2025 - 16

= 2009

76 x 82

We can think of as $79^2 - 9$

6241-9

= 6 232

And so on.

For numbers that are far apart, such as
78 x 28

We can again use the same method, but with a larger difference between them.

In this case we have a difference of 50

So that’s

$53^2 - 25^2$

2809 - 625

= 2 184

Again, familiarity with squares and my squaring system is vital here.

Another...

94 x 38

Difference = 56

So half-way point is 38 + 28 = 66

Squaring...

$66^2 - 28^2$

4356 - 784

= 3 572

37 x 27

$32^2 - 5^2$

1024 - 25

= 999

107 x 85

$96^2 - 11^2$

9216 - 121

= 9 095

Odd Number Differences

Let's say like the last example we have
107 x 86

In that case we can do

107 x 85 + 107

Calculating

= 9 095 + 107

= 9 202

Or

23 x 28

We can do 24 x 28 - 28

$26^2 - 4 - 28$

676 - 32

= 644

Finally

93 x 38

We can do

94 x 38 - 38

$66^2 - 28^2 - 38$

4356 - 784 - 38

= 3 534

Once you become familiar with the squares, these multiplications become easier and easier.

We start to read multiplications in terms of the squares needed instead of the standard way.

For example

35 x 43 can now read

$38^2 - 9$

and

57 x 61

$59^2 - 4$

Squaring Large Numbers

Squaring Large Numbers

In Squaring - In A Minute, we looked at squaring up to around 105, or thereabouts. I also hinted it was possible to square larger numbers.

We can do something like
218^2

By doing

$2 \times 236(00) + 18^2$

47200 + 324

47 524

Simple!

Or

397^2

4 x 394(00) + 9

160000 - 2400 + 9 = 157 609

Here I did 4 x 400(00) and subtracted 6 400s, or you could just do
394
$\times 4$

$15_3 7_1 600$

- +9

----

157 609

So we can do any square by just using a different ‘nearest ten’, that is, using anything as our focal point to square around.

One final example

825^2

8 x 850(00) + 625

= 680 625

It is amazing how quickly this method works, with some basic times tables and a few squares as our tools!

Discovering More Square Numbers

One good way to find more square numbers is to multiply two square numbers together.

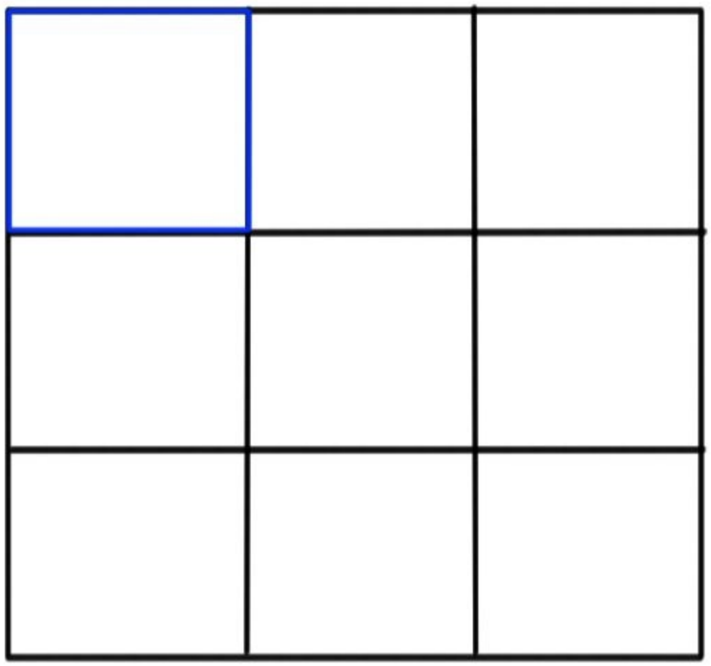
For example,

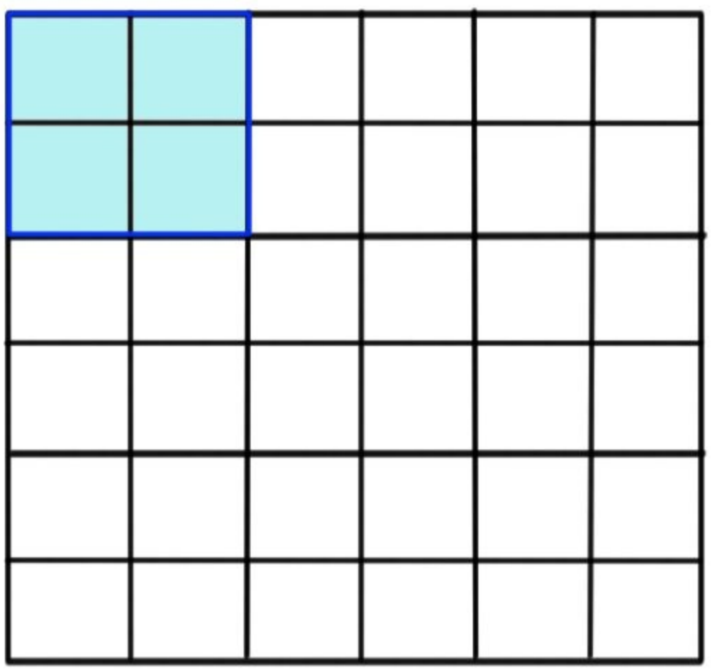
4 and 9

are both square numbers. When we multiply them, we get another square number.

4 x 9 = 36

This is because we can fit more squares into the original square.





If we want to have a general ‘look around’ we can keep doing this to see what square numbers there are out there.

For example,

9 x 49 = 441

25 x 81 = 2025

And so on.

All these answers will be squares.

And to find out what squares they are is very simple.

All we have to do is square root each number and multiply them.

For example

25 x 81

5 x 9 = 45

So

$45^2 = 25 \times 81 = 2025$

Or

9 x 49

3 x 7 = 21

$21^2 = 9 \times 49 = 441$

We can discover whatever we like, and it’s a very simple way to have a look for ourselves, out of interest.

Finding Square Roots from Square Numbers

Finding Square Roots from Square Numbers

Once we get some large square numbers like
2025

How can we reverse this process, like the Third Rule says we can, to find out the original square root?

As we've just seen, we can do it if we know the origin of the number from multiplying two other squares.

But what if you know it is a square number, but you don't know its origin?

For example

4225

is square. What is its square root?

To find this, we look at the ending.

We see it ends in 25. This immediately tells us that the square will end in 5.

Looking at 42, we note that

$6^2 = 36$

and

$7^2 = 49$

49 is too large, so it must be 6.

So the square root is

65!

Easy.

What about this square number?

1156?

The number ends in 6, which tells us that it must have been squared from a number ending in 4, since

$4^2 = 16$

Again, we know that

$3^2 = 9$

and

$4^2 = 16$

So

$30^2 = 900$

$40^2 = 1600$

Therefore it must be

34.

However, couldn't it be

36?

This would also generate a number ending in 6.

To test this, we just square it back to see if it gives us 1156.

34^2

$34^2 = 3 \times 380 + 16$

1140 + 16.

1156

So it was 34^2

For all squares that don’t end in 5, there will be two possibilities, so we must try and confirm it is one or the other.

Now try

1849

This ends in 9, implying our square will end in 3, or 7.

We know that

$40^2 = 1600$

$50^2 = 2500$

Since it lies in between, it must be 43 or 47.

Trying

43^2

4 x 46 + 9

1849

So it is 43.

Trying

8464

This ends in 4 so it must be a square ending in 2 or 8.

$90^2 = 8100$

So it must be

92 or 98

Trying

92^2

9 x 94 + 4

= 8464

It is 92

Finally

9801

This ends in 1, so we know our square ends in 1 or 9.

$90^2 = 8100$

and

$100^2 = 10000$

It must be 91 or 99.

Trying

91^2

9 x 92 + 1

8281

(this seemed unlikely also, since the number is so close to 100^2)
Trying

99^2

98 x 10 + 1

9801

Therefore it is 99.

Numbers that aren't square numbers are harder to square root, although it is possible to do it arithmetically. It is usually done algebraically and this is for a later book.

Arithmetically if we wanted to square root
200

We would think of the nearest square number
This is

$14^2 = 196$

So we know it is around 14.

Trying

14.1^2

This gives

  141
x 141

19881

Which, when adjusted for decimal places (see Decimals - In A Minute)
Becomes

198.81

Trying

  142
x 142

20164

Is over 200.

So we know it’s around 14.1 to 14.2

From here we could try

14.15

and so on, to try to get closer to 100.

This is fairly laborious, but do-able.

I leave this for you to try.

Four-digit multiplication is in ‘Multiplication - In A Minute’!

We can also use this technique to find

$\sqrt{2}$

To find this we can multiply by 100

and find

$\sqrt{200}$

this we have just done, and found it was

14.1

So

$\sqrt{2} = 1.41$

To find

$\sqrt{3}$

we can find the square root of 300 in a similar fashion.

Again, I leave this as an exercise...

Start at

$17^2 = 289$

The Christmas Party Where I Was Called A Wizard

The Christmas Party Where I Was Called A Wizard

One Christmas, all of my friends and family had a party trick prepared for entertainment. My one was a maths trick of course!

I asked them to choose any four-digit number. They said
3214

I wrote it down and I asked for another one.

2312

I wrote this down.

Giving them a calculator, I said
‘Now I want you to multiply these on this calculator - but DON’T tell me the answer.’

So they did it.

‘Now, read out the answer, in any order, and leave one number out. I will tell you what number is missing!’

They read out

8 6 7 0 4 7

I announced the missing number was...

3!

They were amazed!

Then I said

‘That was nothing! Here’s a better trick. This time, let’s choose a number together.’

Asking one person at a time for a single digit, they gave me

3 2 7

and I said,

‘And I’ll add 6’

giving

3276

Then I said

‘Now multiply this by ANY 4-digit number, but DON’T tell me this either! Again, read out the answer in any order, leave a number out, and I will tell you the missing number’.

My friends were flabbergasted. Their faces looking doubtful, but amused, they wondered how I could possibly do this.

They read out the numbers...

6 5 3 1 1 9 1...

I looked thoughtful and said..

‘1!’

They were thoroughly astounded!

I will show you how to perform this trick. But first we need to know...

How To Check Multiplications Are Correct - In Seconds

How To Check Multiplications Are Correct - In Seconds

When we carry out a multiplication, how can we check it is correct? Apart from using a calculator, which seems to defeat the object of the exercise.

We could do a division, and that would tell us.

So, as a simple example,
4 x 3 = 12

If we divided

$$\frac{12}{4}$$

we should get 3.

However the problem is to get this answer we do a multiplication, what I call the Algebra Trick, so we're just doing the same question again effectively.

What we need is an easy, independent method to confirm our answer.

The one we can use is called the Digit Sum Method.

For example,

Let's say we're doing

14 x 21

and we get

294

as our answer

To check this is correct, all we have to do is sum the digits (and a couple of steps extra).

$$\begin{array}{r} 14 = 5 \\ \times\ 21 = 3 \end{array}$$

Adding 1 and 4 gives 5 and 2 and 1 gives 3.

Since we are multiplying, we multiply these numbers.

We get

5 x 3 = 15.

Because this is called the DIGIT SUM method, we keep adding until we get a single digit.

This gives

1 + 5 = **6**

And this is our Magic Number for the Question.

All we have to do is see if this matches our answer.

If we add up our answer’s digits, we get
2 + 9 + 4 = 15

1 + 5 = **6**

That’s correct!

The two numbers match up!

Let’s try

32 x 38

Using the squaring technique, this gives
$35^2 - 9$

$1225 - 9$

= 1216

Checking

3 + 2 = 5
3 + 8 = 11 = 1 + 1 = 2

Multiplying

5 x 2 = 10 = 1 + 0 = **1**

Does the answer add to 1?

1 + 2 + 1 + 6 = 10 = 1 + 0 = **1**

Yes!

It is correct.

Throwing OUT 9s

One neat thing in this system is that if we come across a 9, we can ignore it. This is because 9 = 0. It has no value.

Looking at the above example, we got
1216.

The

2, 1, and 6, add up to 9. If we had ignored them, we would have just been left with 1.

So, ignoring 9s works.

This is called ‘*Throwing out 9s*’.

Now run back through all the multiplications in this book and check whether they are correct using this method.

Use it yourself from now on for a super-fast check to see if your working is correct!

The Christmas Party Magic Trick

The Christmas Party Magic Trick

Once you know this method, you can demonstrate an excellent trick.

At a dinner-party, or such like, ask for someone to provide a calculator, pencil and paper. The calculator is for them.

Ask them to choose a random 3 or 4 digit number - it doesn't matter!

Let's say they randomly choose

2401

then ask for another - from someone else!

Let's say they go for

9832

Ok. At this stage, quickly add these numbers up and multiply their digits. So that would be
2401 = 7
9832 = 22 = 4

You can even write 7 and 4 here on the page, as you add. Away from or opposite to the number you've added. Smart friends will figure out it's an addition. Otherwise, it won't mean anything to them. Or if you want to be more subtle, I hold my 7th finger and my 4th finger with my thumbs.

Do 7 x 4 = 28 = 10 = 1.

Your magic number is 1.

What we do as they say these numbers is write them down and pretend to be deep in thought.

You now tell them you want them to calculate the answer on their calculator. They will tell you every number in the answer in any order, except one of them. Whatever is missing - you will tell them!

They then begin to read out the numbers.

Let’s say they give you

6 6 6 2 3 2 0

What number is missing?

As they are reading these numbers out, add them up.

6 + 6 + 6 + 2 + 3 + 2 + 0 = 25

This equals 7

However, we want 1.

So the number that must be missing will get us from 7 to 1.

To get from 7 to 1 is only 3. Why?

7 + 3 = 10

1 + 0 = 1

So we just need 3.

Say, “I think the number that is missing is... [pause]...3!”

And they will be amazed.

It doesn’t matter what numbers they give you. As long as one is left out.

Let’s say they give you

6 6 6 2 3 2 3

And you add these up.

These will give

28

2 + 8 = 10

1 + 0 = 1

Aha! So here we have a problem! We've reached 1, even though that's what we want.

So this means that the number that is missing must be 0. Or... 9. Remember in this system, 9=0.

So here you say... "Well, this is a hard one. You've chosen well there. The number missing could be one of two!! I'm going to go for..."

and then choose one.

If you're lucky, it will be correct.

If it is wrong... play along as if it is very funny how wrong you are.

Then turn around and say... "Ok...I'm going to go for [the other one]".

They'll still be amazed.

If they're not so impressed by this, and you've just been genuinely unlucky, then promise them that you will show them a greater trick.

THIS TIME, you only want to know one four-digit number. They can multiply it by ANYTHING and you will perform the same trick EVEN THOUGH YOU DON'T KNOW WHAT THEY CHOSE!

So here's how to do it.

For this to work, it must be a multiple of 9.

As we saw in the Digit Sum Method, if a number contains a 9, we can ignore it and 'throw it out'.

This is true even if it's in the multiplication.

For example

9 x 5

The answer is

45

this sums to 9.

18 x 5

The answer

90

This sums to 9.

18 sums to 9.

(1 + 8)

So if there is a nine involved in the question, the answer must sum to 9 too.

So let each person choose a single number this time, and say 3 people choose

4 2 1

Here, you have to make it 9, so you say
“Ok, and I’ll add a 2”

This will give

4212

Which sums to 9.

NO MATTER WHAT THEY MULTIPLY IT BY NOW, the answer will sum to 9 also.

So instruct them to multiply it by any other 4 digit number they choose, but NOT to tell you.

Let’s say they choose 1234

(It doesn’t matter what they choose, but to show you how it works.)
The answer they will get will be

5 197 608

They read out...

‘8 0 6 9 1 5...what’s missing?’

Adding these as they go, you get
8 + 0 + 6 + 9 + 1 + 5 = 29

2 + 9 = 11 = 2

So you are seven short!

So you have a deep think.

At this stage, they don’t believe you know the answer to this. How can you? You don’t even know what was in the question! How could you know the answer?

Then say
‘Hmm. Tough. I reckon... it’s 7!’

And they will be amazed again!

Again, it may happen that they choose all numbers except 9 or 0. Again, you’ll have to do the ‘one of two’ remark. You’d be unlucky for this to happen twice. But don’t worry. They will be amazed at how you did it!

Remember, never explain a magic trick. Leave it at that.

Algebra Behind the Multiplication Method

Algebra Behind the Multiplication Method

In Book 1 and throughout the series, we used my refined method for multiplication which goes as follows.

The method takes 3 steps, so I'm going to explain each one. That means writing it out 3 times. In practice, we would only write out the question once.

Let's look at 14 x 21.

First, place it in a column.

```
  14
x 21
----
```

As above, start on the right hand column

```
  14
   ⇩
x 21
----
   4
```

which is 4 x 1, giving us 4.

Step 1 complete.

Step 2 is where you'll see the revolutionary difference that you won't have seen before.

14
✕
x 21

4

Here, draw a small cross between the numbers and multiply along the lines of the cross.

So that gives:
1 x 1 = 1
4 x 2 = 8

Remember that multiplication is just addition, ADD these 2 answers in your mind. Really the goal is to have no working, so we need to store these in our memory for a moment.

So we have 8 + 1 = 9.

Giving

   14
x 21

   94

Step 3 we just multiply along the left column, just as we multiplied along the right.

   14
  ⇩
x 21

  294

Giving, of course, 1 x 2 = 2.

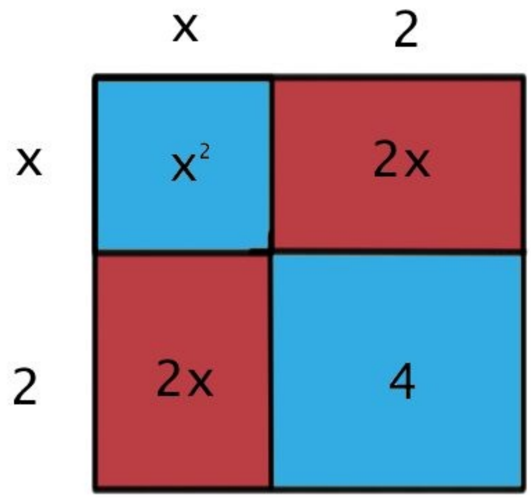
So 14 x 21 = 294.

No working was involved. We had some simple multiplications to make, one addition to store in our memory as we multiplied the 2 in the centre, and the answer appeared as if by magic.

How does this work?

We're now going to look at the algebra behind how this works, to see why I call it 'Algebraic Arithmetic'.

First of all, you may recognise this from Quadratics - In A Min



This was the Binomial Square which came from multiplying brackets
$(x+2)^2$

$= (x+2)(x+2)$

Once I saw this, I realised it could be applied to multiplication of numbers also.

If we think of 14 x 21 as tens and units, we can change our $(x+2)$ to

$(t+u)^2$

In other words

$$(t+u)$$
$$\times(t+u)$$

----

where the tens unit is being replaced by t and the unit is replaced by u. In practice, these numbers will be different, most likely, but the beauty of it is that it doesn't matter whether they are different are not, the algebra holds true.

We saw above that these multiplied by doing the right column, then the cross, then the left column.

If we do that here, we get

$$u^2$$

for the right column

$$tu + tu = 2tu$$

for the cross

and

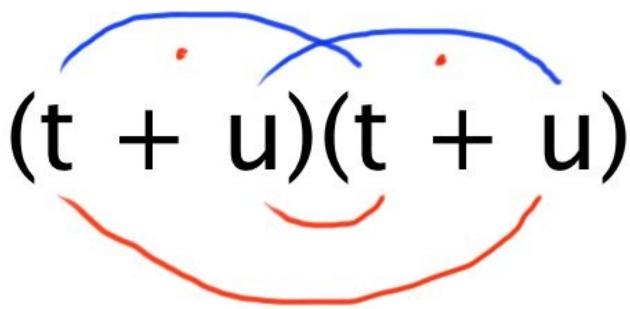
t^2

for the left column

$t^2 + 2tu + u^2$

Which is a binomial square, just like the image above.

This is in fact the expansion of two brackets given by FOIL or the smiley face method also.



What is clever is that we can call t and u anything and it will still work. We still get a binomial square algebraically. However because those values are different we get a rectangle in practice.

We can see then that we are using the concept of a binomial square for ALL multiplications, and it doesn't matter that the numbers are different.

We have unit times unit.

The cross of u times t added to u times t is the same as multiplying it by 2 if they are the same. If not, we add them.

And finally we have ten times ten.

I first came across the binomial square when I was researching Maria Montessori. Montessori was an educator from Italy who had a different way of going about education (sound familiar?). I was curious to see how she taught maths. I spoke to a teacher who showed me a curious cube.

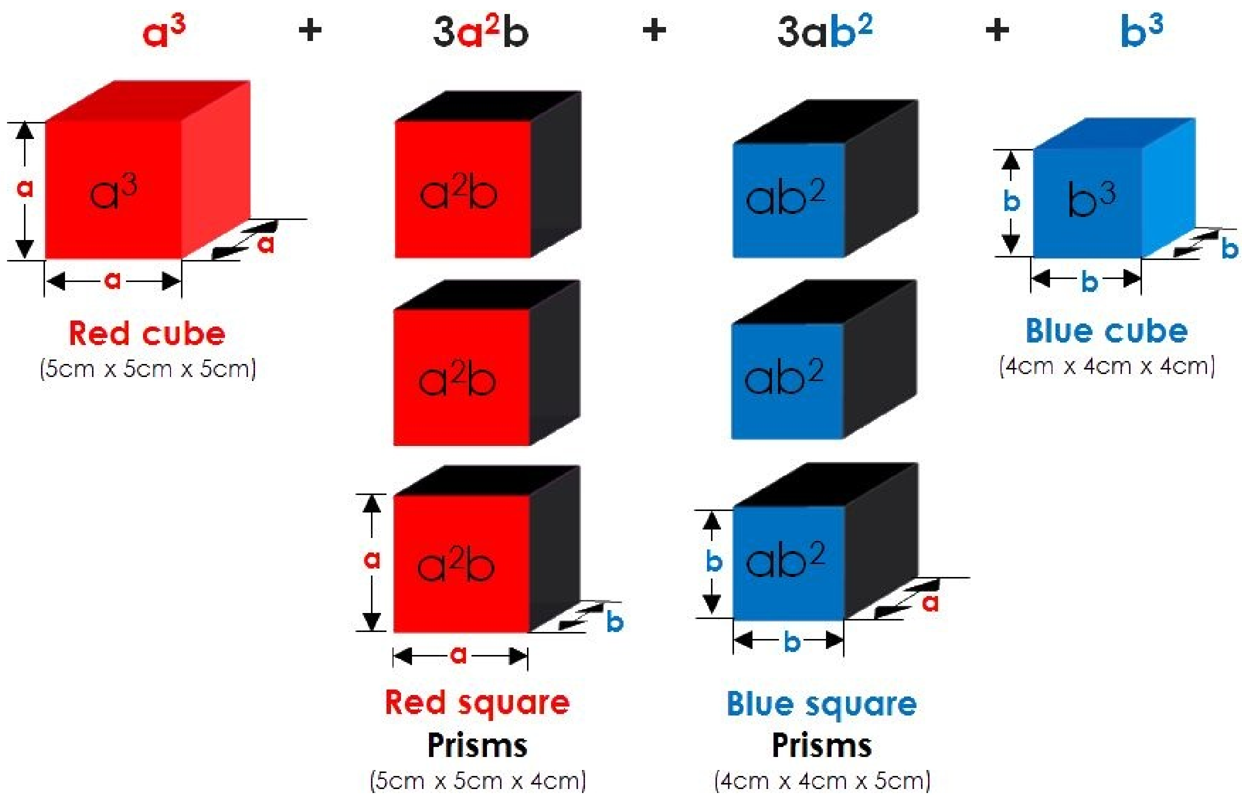
This was something that 5-year-olds put together. After I played around with it I realised it was a binomial cube.

This is a 3-D realisation of this expression.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Which comes from just multiplying brackets!

And how it works…all these terms add up to make a larger cube. Just like our normal multiplication of two numbers form a square.



Montessori was a genius to think of this. This is A-level material, being done by 5 year-olds! I was amazed. The teacher herself didn't understand it, but children can put it together with ease.

This led me to think of the square (which features on the top of the box) and then how to multiply numbers using this.

Algebra Behind The Squaring System
Algebra Behind The Squaring System

When I was 17, I was idly thinking of

$14^2 = 196$

(I don't know why!)

I thought that $4^2 = 16$, and that if you subtract that from 196 you have 180.

I thought that was funny. That is

10 x 18.

Which are both equidistant from 14, the number I was squaring.

I stumbled on my squaring system.

I then applied the concept to other squares I knew, such as

$12^2 = 144$

And realised that worked too.

10 x 14 + 2 x 2

I had discovered something! I showed my maths teacher at school. She wasn't impressed. 'Now prove it', she said.

Here's the algebra behind it.

When we use the method, the first step is to go to the nearest ten.

If we choose 21 to be our number to square, our nearest ten is 20. For that we need to go 1.

So let's call $x = 21$ and $y = 1$

What we do is subtract 1 to get 20 and add 1 to get 22. Then we add the number we've moved squared onto our answer.

Algebraically this is

$$(x-y)(x+y)+y^2$$
$$20\times 22+1^2$$

Multiplying these brackets using the column method, as well as adding y^2

$$\begin{array}{r} (x-y) \\ \times(x+y) \end{array}$$

$$x^2+(xy-xy)-y^2+y^2$$

The $xy's$ cancel.

The $y^2's$ cancel!

This just leaves x^2.

The number we were trying to square!

And that's the algebra.

Has this been discovered before? Yes, it has. I understand it was discovered by Vedic maths, but information is sketchy.

One thing I was very proud of was that, by coming up with this method, I had outdone Richard Feynman. Feynman is a big hero and idol of mine, and his books taught me a lot about physics and how to learn (especially). In one of his parables, he mentions that a colleague of his could 'square numbers around 50'. His method for this was the same as mine. Feynman never seemed to realise this could be extended further for all squares. I only discovered this story a year or so after I'd discovered the method myself.

If you want to read the story yourself, I strongly recommend 'Surely You're Joking, Mr Feynman', a book that influenced me greatly.

Algebra Behind Advanced Multiplication Using Squares

Algebra Behind Advanced Multiplication Using Squares

Once we know the squaring system, we can combine that with a concept in maths known as the **Difference of Two Squares**.

This is what we've used for our multiplications earlier in this book.

It comes from

$(x-y)(x+y)$

As we saw from my squaring system, this is the first part of it. What is special about it is that it always simplifies to

$x^2 - y^2$

which is why it is called the Difference of Two Squares!

This concept was what we used for a multiplication like

27 x 37

Choosing $x = 32$, gives that $y = 5$, so we have
(32 - 5)(32 + 5)

This will multiply out as

$32^2 - 5^2$

from the Difference of Two Squares.

(this is why you should always multiply brackets!)
This gives

1024 - 25

= 999

Introduction to Gradient/Tangent

Introduction

In the book Gradient/Equation of A Straight Line, we saw that it was a short step from using the times tables to become straight lines. These straight lines had gradients, which, it turned out, happened to be the very times tables that formed the sequences for the values of the table for the equation of a straight line!

We then saw that we didn't need these tables at all, as the information contained within the equation gave us everything we needed to know, the gradient and the cut.

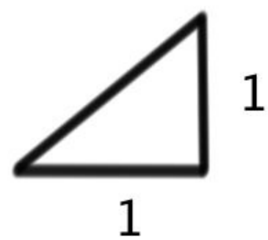
The gradient is something we're going to expand on here and see what else can be done with it other than measuring the steepness of straight lines. In fact, we are going to do something similar, but use a different measure of gradient. It is just like switching from using kilometres to miles (or from degrees to radians - Book 21!) or vice versa. The thing we're measuring will be the same (steepness), but we'll just use a different unit.

Concept Connected To Gradient

Concept Connected To Gradient

So let’s look at this different way to express steepness.

As we saw in earlier books, the definition of gradient was that for every one you went along, the amount you go up (or down) is the gradient. So if you took one step along, and one step up, the gradient would be one.



Do you remember what this concept of using one as a measure was called? It was called...

Normalisation.

We saw in percentages this use of one (100%) as a benchmark measure, even if we’re not measuring something out of 100. In units of distance, area and volume, we saw they were based on one (1m, 1m x 1m, and 1m x 1m x 1m). We saw it again in gradient with the idea being for every one you go along...the amount you go up (or down) is how steep your line or hill is. This comparison to 1 is extremely important for the topic of trigonometry which we are beginning now.

Here is a real life example of normalisation courtesy of a film called Spinal Tap, called, It Goes Up to 11.

The interviewer naturally and intuitively tries to normalise the amplifier by making the maximum 10, but the musician doesn’t get what he’s on about.

However, going back to gradient, what concept, do you think, is similar or connected to gradient?

What other way could we measure steepness?

If you’re not sure, think of this

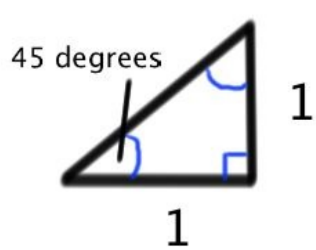
When we see a plane taking off, it doesn’t take off at a gradient, but at an...

ANGLE.

So another way of measuring could be using angles and degrees.

So angle is another measure.

So then, gradient and angle are connected. Whatever a gradient is, there must be some angle connected to it. In fact, I alluded to this in the book Gradient/Eqn of A Straight Line when I pointed out that a gradient of 1 has an angle of 45 degrees, as it is an isosceles triangle.



So the natural question is does every gradient have an associated angle? (Did you wonder this?)

To get a visual idea of what this is like, we could draw a graph from a table of values that we can calculate easily ourselves.

Let’s look at the following angles, and then figure out their gradients.

0°

45°

90°

135°

180°

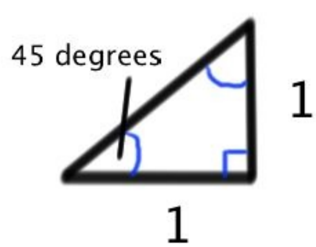
225°

and then make a graph of gradient versus angle and see what that would look like. We could also use the graph to find other values of either angle or gradient, depending on which we’re trying to convert to.

Okay, so zero degrees would be flat.

What will be its gradient? Zero! For every one we go along, we go up zero.

45 degrees we have already examined, the gradient is 1.



90 degrees? What will this be?

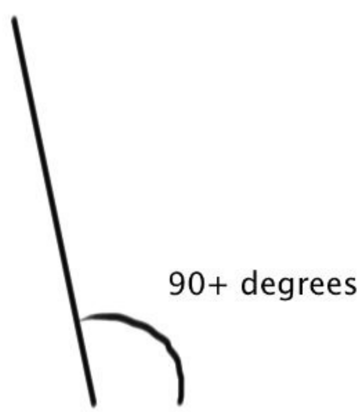
As we can see it just goes straight up! What value will that take? If we think of gradient as being the difficulty of climbing a hill, we saw with 45 degrees that it wasn't too bad. For every one we went along, we went one up. For 90 degrees, how difficult is this to climb? Impossible? And what number would that be?

Infinity - which has the symbol

∞

Which is a strange graph! It goes off towards the edge of the universe (to infinity and beyond, ha ha)! However, we know this is true via calculation - 89.99999 degrees is a very large number too.

After 90 degrees, what happens?



As we saw with straight lines, we know have a negative gradient. It is like going down hill. For every one you go along right, it goes down (negative). So just after 90 degrees the graph goes from plus infinity to minus infinity! Very unusual graph.

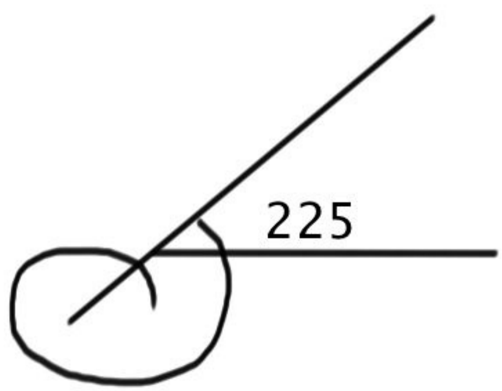
Then we have our next value. 135 degrees.

Just like the third rule, we see this is the reverse of 45 degrees

and for every one we go along, we go down one, so the gradient is - 1.

At 180 degrees, again we see that the gradient is zero.

At 225, we see that this is just 45 degrees again.

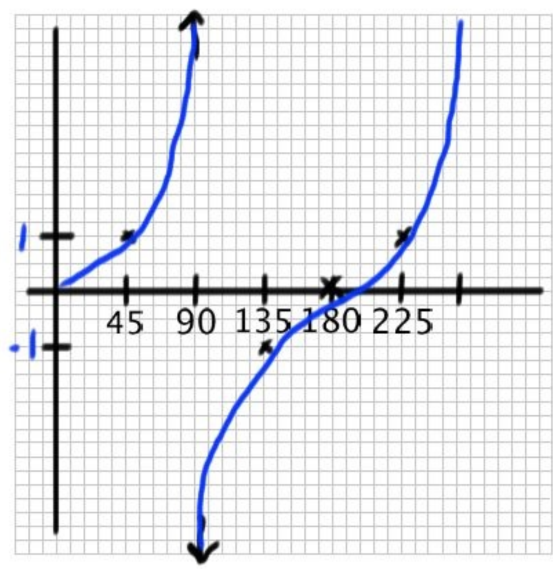


and the cycle repeats.

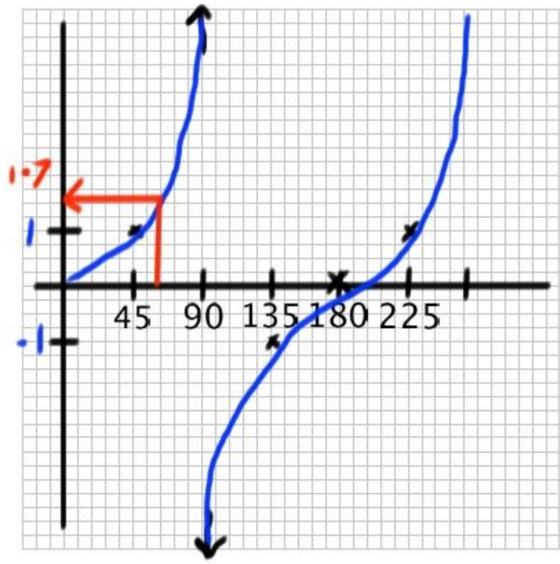
We can now create a table of values

Angle	0	45	90	135	180	225
Gradient	0	1	∞	-1	0	1

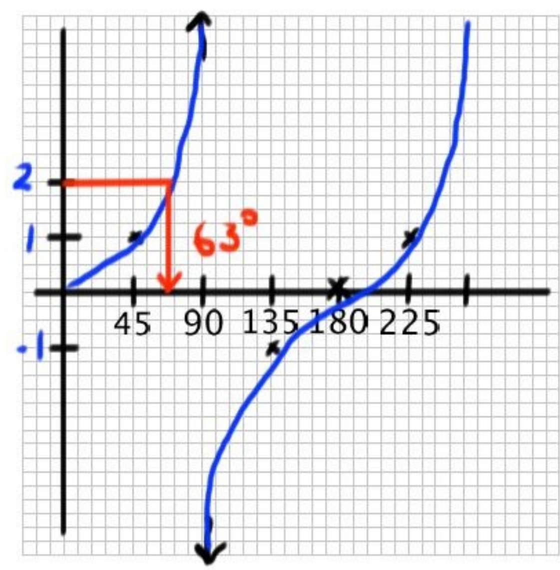
and then draw a 'Gradient Graph' which will show equivalent values of gradients and angles up to 360 degrees, or just 180 since it repeats.



From this graph we could find any gradient for any angle. So let's say you had an angle of 60 degrees and wanted to know what its gradient was. All you do is read up and across and you get about 1.7.



And of course, we can do the reverse. If we have a gradient, we can use it to find an angle. For example, a gradient of 2 would give an angle of around 63 degrees.



So we have a handy-ish device for conversion between the two. Obviously this is not very practical (or accurate) and a table of complete values is available in this list.

Angle	Gradient	Angle	Gradient	Angle	Gradient
0	0.0000				
1	0.0174	31	0.6008	61	1.8040
2	0.0349	32	0.6248	62	1.8807
3	0.0524	33	0.6494	63	1.9626
4	0.0699	34	0.6745	64	2.0603
5	0.0874	35	0.7002	65	2.1445
6	0.1051	36	0.7265	66	2.2460
7	0.1227	37	0.7535	67	2.3558
8	0.1405	38	0.7812	68	2.4750
9	0.1583	39	0.8097	69	2.6050
10	0.1763	40	0.8390	70	2.7474
11	0.1943	41	0.8692	71	2.9042
12	0.2125	42	0.9004	72	3.0776
13	0.2308	43	0.9325	73	3.2708
14	0.2493	44	0.9656	74	3.4874
15	0.2679	45	1.0000	75	3.7320
16	0.2867	46	1.0355	76	4.0107
17	0.3057	47	1.0723	77	4.3314
18	0.3249	48	1.1106	78	4.7046
19	0.3443	49	1.1503	79	5.1445
20	0.3639	50	1.1917	80	5.6712
21	0.3838	51	1.2348	81	6.3137
22	0.4040	52	1.2799	82	7.1153
23	0.4244	53	1.3270	83	8.1443
24	0.4452	54	1.3763	84	9.5143
25	0.4663	55	1.4281	85	11.4300
26	0.4877	56	1.4825	86	14.3006
27	0.5095	57	1.5398	87	19.0811
28	0.5317	58	1.6003	88	28.6362
29	0.5543	59	1.6642	89	57.2899
30	0.5773	60	1.7320	90	------------

We can see some examples in this list of our calculations above, which I've ringed in red.

Note that as the angle increases towards 90 the values get higher in bigger leaps and that there is no value for 90 degrees.

In fact, in truth, we don't need this table either. Of course, this information is saved on your calculator. You just have to know where it is.

An important idea here is that the word 'Gradient' is very closely linked to the word 'Tangent' in maths (we'll see why later) and in fact, I've been lying. This is not called a Gradient Graph, but a **Tangent** Graph. They are effectively the same.

So... if you press the TAN button on your calculator, imagine it says GRADIENT because it will give you the gradient of any angle you put in.

Pressing

TAN 45

Will give you an answer of 1. Why? Because 45 degrees is a gradient of 1! So you're effectively saying Gradient 45?

TAN 60

will give you

$\sqrt{3}$

which is 1.73

Try

TAN 90

You'll see it doesn't like that.

To do the reverse (third rule!) we need to tell the calculator what we're doing, so we just need to use the 2nd function or shift button to reverse the process. You can see this in the picture above.

The symbol for that is

$\tan^{-1}$

Which means, of course the reverse (or more correctly the inverse) of Tangent.

In this case, we have the gradient to start with, and we want to know the angle.

So if we do

$\tan^{-1} 1$

we get

45 degrees of course.

$\tan^{-1} 2$

Means what angle is a gradient of 2?

And we get

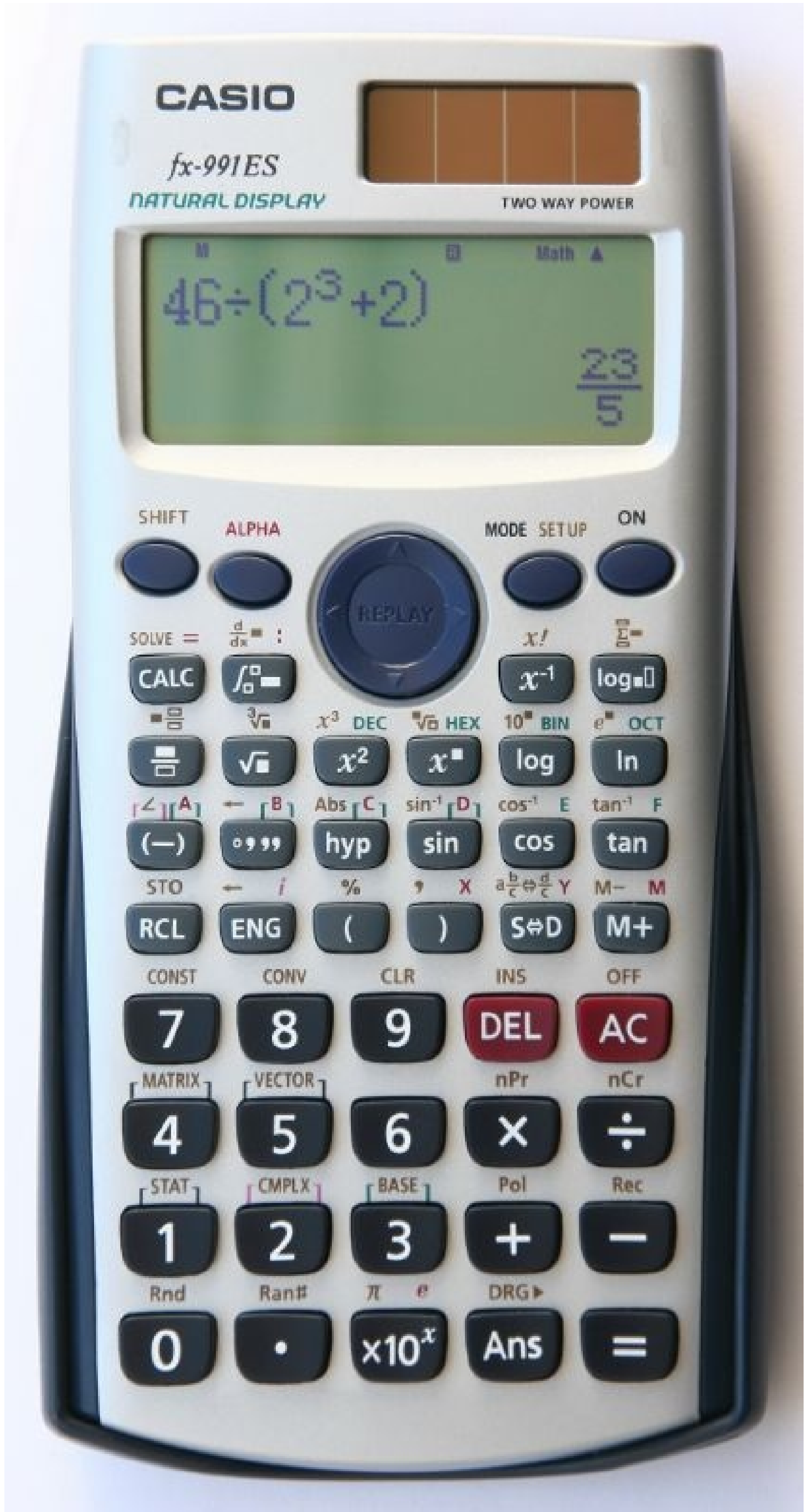
63.4 degrees.

And so on.

So your calculator has all these values stored in its memory. (It doesn't calculate them.)
A quick note about a calculator to buy.

I recommend the

Casio Fx - 991 ES or ES Plus



Because it’s full of functions that are very convenient for use in mathematics. If you’re not using this calculator, you need to get it, as it is permitted for use in school exams like GCSE and A-Level, and can do many functions the average school calculator can't. This makes it great for CHECKING your answer in an exam (not doing it for you!). In the exam you can check if you are correct, but you'll not receive full marks if you just write an answer. But if you made a silly mistake in your working, and got a different answer, the calculator will tell you!

I write more about this in a chapter of my book - Guarantee An A*… in the exam technique section.

Putting Our Values To Use

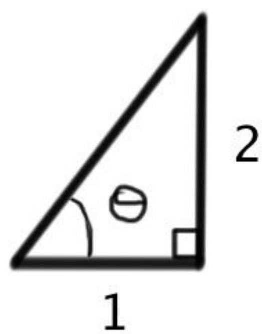
Putting Our Values To Use

So what’s the point of all this? Why is it useful to be able to switch between an angle and a gradient and vice versa?

If we look at our gradient ‘triangle’ which is formed when we try to calculate a gradient, we see we get a... right angled triangle!

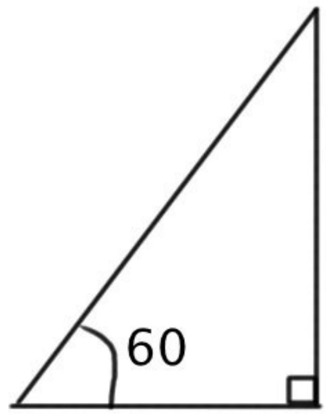
That means that if we have just a triangle, and not a line, slope or hill, we can use the same tool to find out information about this triangle as with the line.

For example, if we had a triangle with a gradient of 2, we would also be able to tell that its angle would be 63.4 degrees.



Similarly, if we had its angle, we could find its gradient. If the angle was 60 degrees, we would know that it would have sides that divide to give an answer of

$\sqrt{3}$

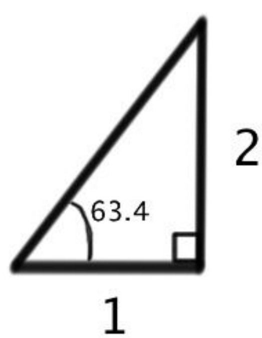


From this we can have 3 possible scenarios.

1 We can find an angle given the sides of the triangle (the gradient) 2 We can find one of the sides of the triangle given the angle and one side 3 We can find one of the other sides of the triangle given the angle and one side
An example follows of each.

Angle

Given the gradient, or sides of a triangle, we can find the angle using our graph/table/calculator.



Here we have a gradient of 2.

Therefore, if we call the angle theta θ (rhymes with eater), we can find it just by saying that

$$\tan\theta = 2$$

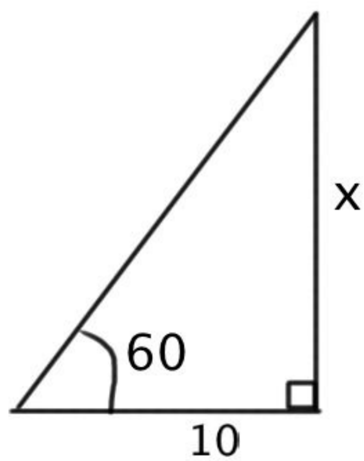
$$\tan^{-1} 2 = \theta$$

$$\theta = 63.4°$$

Note that the values of θ and 2 interchange when we use the inverse version of tangent.

Sides

In this scenario, we have the angle, but we don’t know one of the sides.



So if the angle is 60 degrees, we know we can find the gradient, since it is

TAN 60 =

$$\sqrt{3}$$

which is 1.73

However, since we also know that

$$\frac{x}{10} = gradient$$

we have two expressions which we can put equal to each other.

Tan 60 = gradient = 1.73

$$\frac{x}{10} = gradient$$

$$\frac{x}{10} = \sqrt{3} = 1.73$$

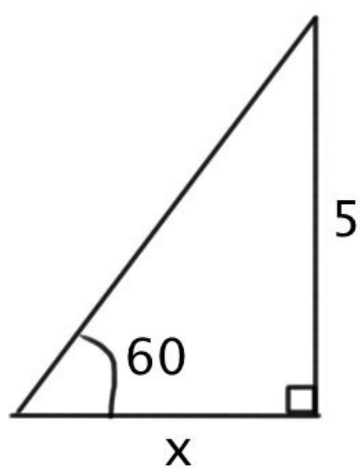
Therefore,

x = 17.3

Finished!

In the opposite scenario, we don't know the bottom of the triangle (the width) but we know the height.

For the same problem with the same angle,



We will have

Tan 60 = 1.73

$$\frac{5}{x} = \sqrt{3}$$

Therefore we need to use the reciprocal to solve this giving

$$\frac{x}{5} = \frac{1}{\sqrt{3}}$$

and multiplying by 5

Gives

$$x = \frac{5}{\sqrt{3}}$$

Or

$$x = 2.89$$

These are the three types of problems we could have.

This is entering a new world. **Trigonometry.** This is Greek for three-tri, line-gon, measure-metry.

In other words, trying to find the lengths of the lines in triangles. We can also find angles of course!

For gradient/tangent, we divide the height by the width. We need to do that to 'normalise' the gradient, so that the base is 1. Naturally, there are other ways to divide two sides of a triangle. The first of which would be to divide the width by the height, which of course, is the complete reverse of gradient.

This is known as the

Co-tangent

Which kind of makes sense!

All that means is that it is the RECIPROCAL of tangent, or gradient. So

$$\cot\theta = \frac{1}{\tan\theta}$$

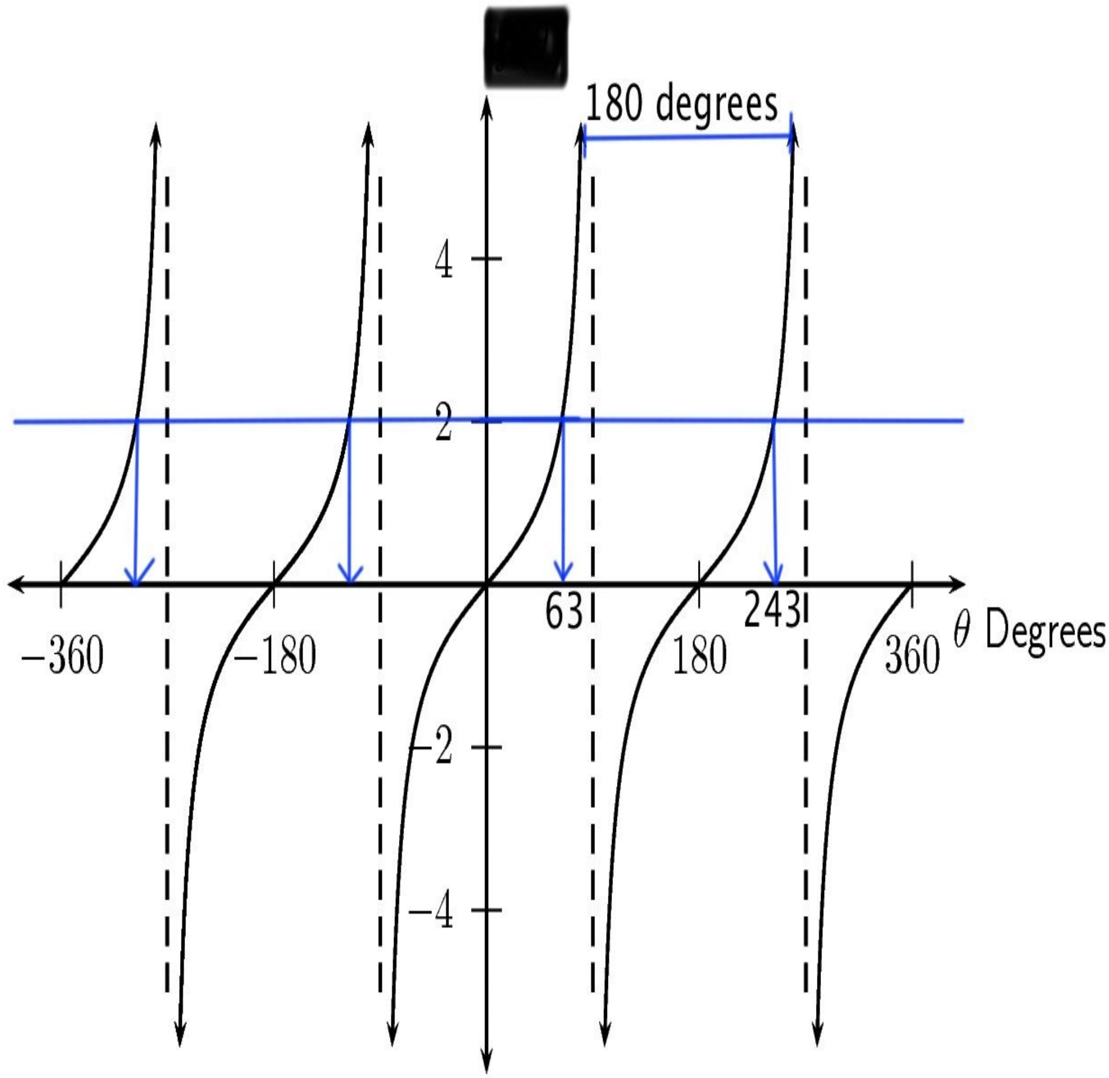
There are four other possible ways to divide the sides of a triangle, which we'll look at in the next book.

Secondary Solutions

Secondary Solutions

If we look again at the graph of gradient, SORRY, tangent, we see that we can read off our values for gradient or angle, depending on which we want. However, being maths, we can generalise and realise that for any value of a gradient, there'll actually be more than one answer. Because our gradient goes around in a circle, this means that every 180 degrees it's going to repeat. We saw this with our first manual graph, where we saw that 0 and 180 degrees had a gradient of zero, and 45 and 225 degrees had a gradient of 1. As a result, we are just going round and round in a circle as we move to the right (or left) on the graph. Consequently, the values will repeat over and over again ad infinitum.

Therefore, our answers so far have been incorrect.



If we want to solve

$$\tan\theta = 2$$

(i.e. find the angle when the gradient is 2)

there are an INFINITE number of possible answers, since only the first is the one we'd use in reality, doesn't mean that is all of them, and we have seen that if we rotate 180 and 360 degrees we'll get the same answer again. Since there's nothing stopping us rotating repeatedly, the answer here would actually be

$63.4 \pm 180n°$

where n, as we saw in Sequences - In A Minute, was the counting numbers, 1, 2, 3,...etc.

So our possible (positive) answers are, to begin with
63.4, 243.4, 423.4, 603.4,....

and the same for negative, just subtracting 180 degrees each time.

We can see this on the graph too.

So we’ve now generalised beyond a right-angled triangle. We often do this in mathematics!

Use of Radians

Just like switching from miles to kilometres, we can do everything so far in Radians instead of degrees. In fact, mathematically, this is preferable.

All this means doing is switching your calculator to Radians and using a tan graph with radians as a measure instead.

Introduction to Sine & Cosine

Introduction

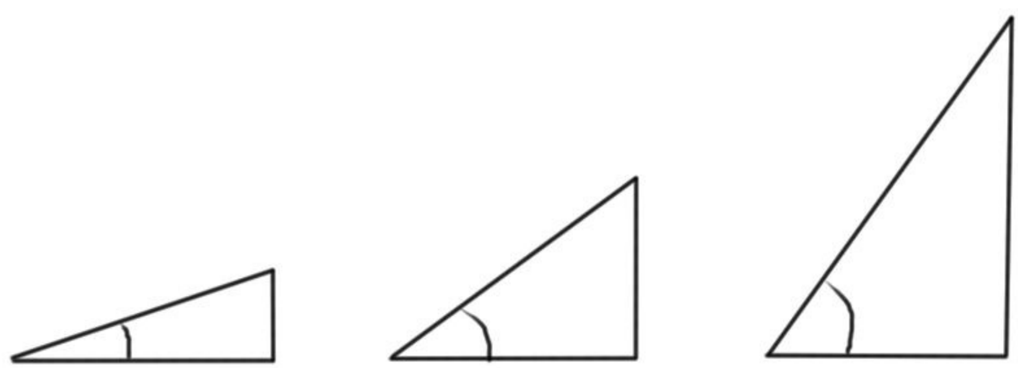
In Gradient/Tangent In A Minute, we saw that we used the concept of gradient to find angles of triangles, and of course, the reverse, finding sides if we had the angle.

In Sine and Cosine, we will see this concept extended to the other sides of a triangle. We noted that there are 6 possible ways to divide the three sides of a triangle, with the previous book looking at the first two. We will look at the other four in this book.

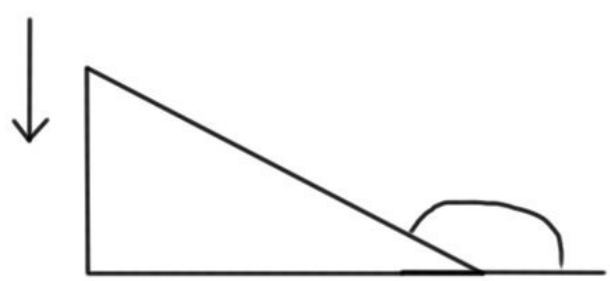
Normalising the Hypotenuse

Normalising the Hypotenuse

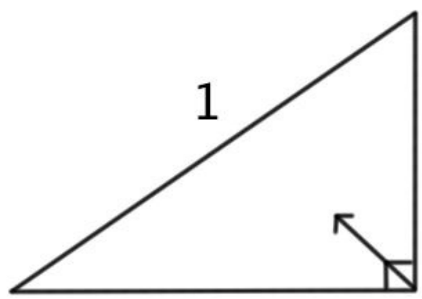
To start with, we are going to look at the height of the triangle. If we look at a variety of possibilities of angle, we will see that as the angle of a triangle gets larger, the height gets larger too.



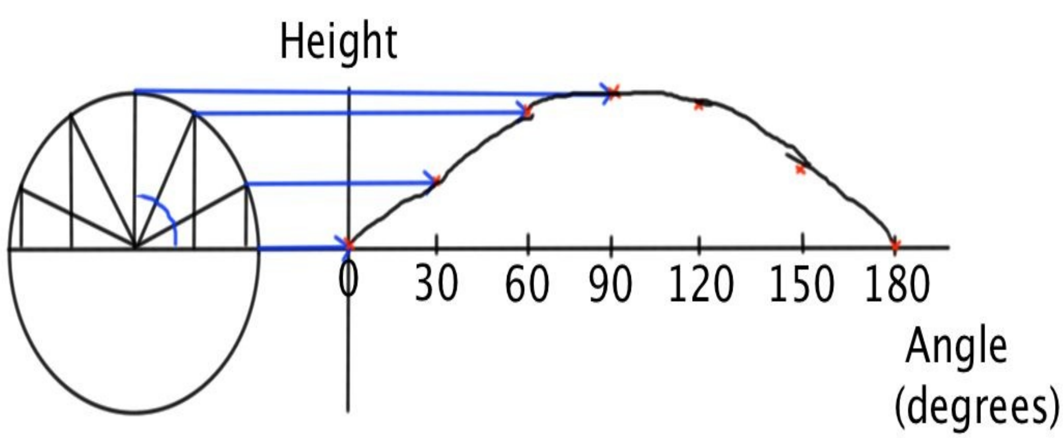
If we extend this for more than 90 degrees, we will see this then comes back down again. (The reverse, of course.)



In Gradient/Tangent, our definition was that for every one we went along horizontally, how many we went up (or down) was the gradient. In other words, the base of the triangle was 'normalised'. This meant it had to be one - even if we had to divide to make it one. Looking at height, we see that it varies as angle increases. This time we will normalise (make one) the *hypotenuse* of the triangle. This is the name of the longest side of the triangle directly opposite the right angle.

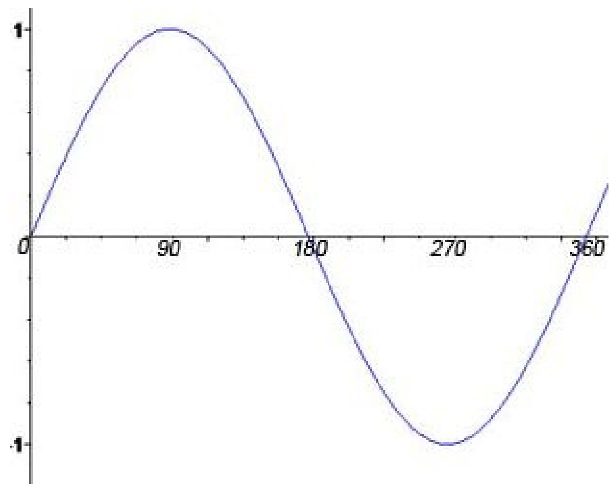


What we can now do is develop a graph where we look at the height of the triangle as it goes from 0 degrees to 180 degrees, in 30 degree intervals, and with a hypotenuse of 1 throughout. You will see that this is basically a circle with a radius of 1, but we are concerned with the triangles within!



You can see how I did this from a video, here.

If we were to keep going around the full 360, the bottom half of the circle would just be the reverse (yet again) and negative, so the full graph would give this.



From this graph we can find a relationship between the angle of a triangle and its height. If we know the angle of the triangle, we can find its height. If we know the height of the triangle we can find its angle.

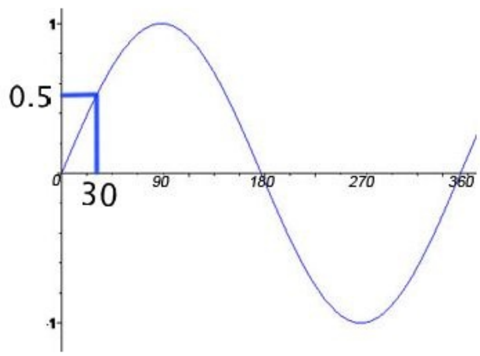
This is known as a Sine Graph. You can think of sine meaning height.

Again, like Tangent, we could use this graph or a table of values, but our calculator has these values stored.

We can see the graph oscillates up and down. This is due to the fact that the angle is going around in a circle.

There's an excellent video of this here.

For example, as we can see from our graph, an angle of 30 degrees gives a height of 0.5.



So Height at 30 degrees is 0.5. Or we would write

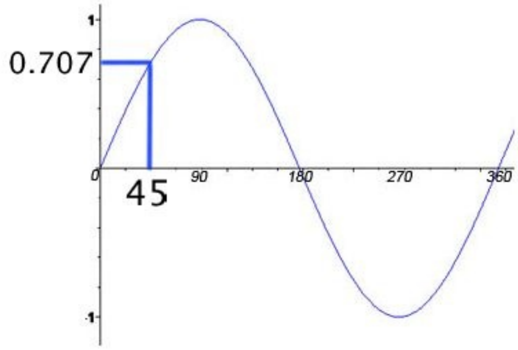
SIN 30 = 0.5

This is only if the length of the hypotenuse of the triangle is one. If the hypotenuse was double that (2), the height would be doubled too. In fact, we can also think of this 'height' value as being the fraction of the size of the hypotenuse. So if the triangle had an angle of 30 degrees, and a hypotenuse of 10, its height would be 5.

In reverse, a height of

$$\frac{\sqrt{2}}{2}$$

gives an angle of 45 degrees.



We tell the calculator we are doing the reverse by using 2nd function -

$\sin^{-1}$

$$\sin^{-1}\frac{\sqrt{2}}{2} = 45$$

Which we write

45°

Putting Our Values To Use 1

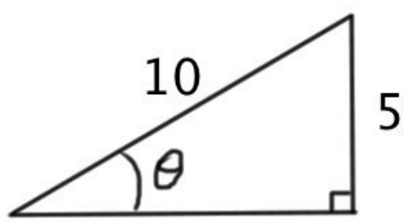
Putting Our Values To Use 1

From Gradient/Tangent, we saw that there were three possibilities. We could either find the angle given the sides, or sides given the angle! In Sine we have exactly the same concept, but different sides.

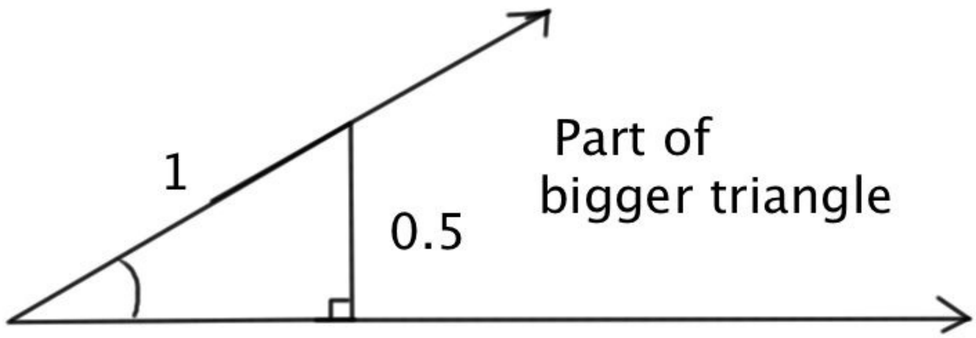
The two sides are of course the height of the triangle and its hypotenuse.

ANGLE

Let’s say we have a triangle with a height of 5 and a hypotenuse of length 10.



First of all, we have to divide that hypotenuse by 10 to make it 1. We therefore have to do the same to the height, to make it a similar triangle. This gives a height of 0.5.



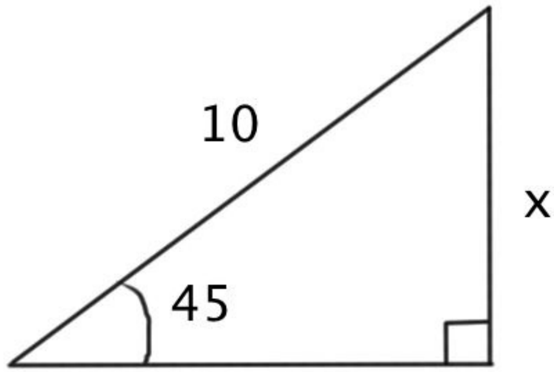
Now we know the actual ‘height’ of the triangle since we have normalised the hypotenuse, we can now use this to find the angle.

Since we are going backwards from the height to the angle, we use the 2nd function;
$\sin^{-1} 0.5 = 30°$

And so we have found our angle.

SIDE 1

Let’s say we know our angle, and we know the length of the hypotenuse this time.



Here then we can find the height of the triangle straight away.

Height at 45 degrees:

Type

$\sin 45$

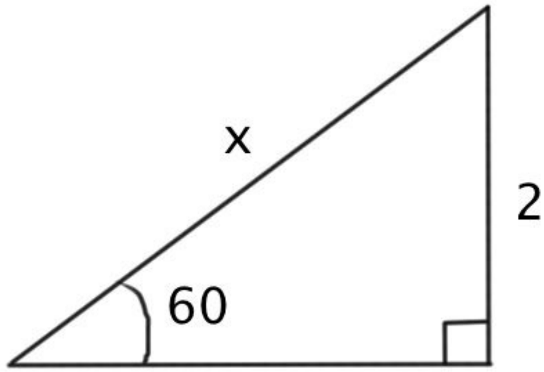
$= \frac{\sqrt{2}}{2} = 0.707$

If our hypotenuse was 1, this would be our height. But the hypotenuse is 10. So we need to make the height ten times bigger, giving
7.07

SIDE 2

For the other side, the situation is reversed, where again we know the angle and the height, but we don't know the length of the hypotenuse.

If we have that the angle is 60 degrees, and the height is 2, obviously we know the height in two different ways here!



However, our height measurement is if the hypotenuse is 1. So there's no way that the hypotenuse is 1 because our height is more than 1 too. Therefore we need to see what the height would be if the hypotenuse is 1.

To find this, again we just type

$\sin 60$

You can think of this as 'height at 60'

And this gives

$= \frac{\sqrt{3}}{2} = 0.866$

However, our height is 2 so that means we need to normalise this hypotenuse.

To do that, we need to divide the height by our hypotenuse, which will be

$\frac{2}{x}$

As we know that this will equal the height too, we set them equal to each other.

$\frac{2}{x} = \frac{\sqrt{3}}{2}$

As with Tangent, we then just use the reciprocal

$$\frac{x}{2} = \frac{2}{\sqrt{3}}$$

And multiply by 2

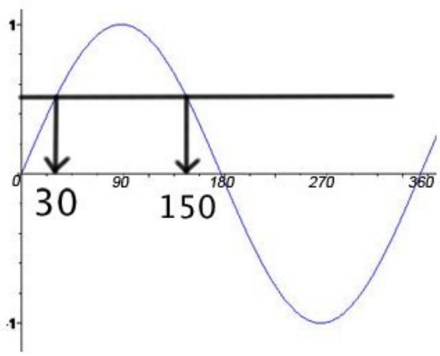
$$x = \frac{4}{\sqrt{3}} = 2.31$$

Secondary Solutions

Secondary Solutions

As we saw with Tangent, there are other solutions other than solving for the angle in a triangle.

If we look at the graph, we can see that if we extend the line out further both sides of the y-axis, the line will cross the curve again (and again).



Looking at the next solution to the right, how can we figure out what it would be?

Because it is a symmetrical curve, we can see that the second solution will exactly mirror our first.

So if we have that

$\sin\theta = 0.5$

This will give 30 degrees as a primary solution. This is what your calculator will give, or what you would expect for a normal triangle problem. Therefore, the second solution will be 180 degrees minus 30 degrees.

This will give 150 degrees.

In fact, like tangent, there are an infinite number of solutions. As we can see they will cycle around every 360 degrees, so our solutions will actually be $30 \pm 360n°$

$150 \pm 360n°$

Normally we limit the range in order to just get between one or two cycles of the sine wave, viz:
$0 \leq \theta \leq 360$

would give just

$30°, 150°$

as answers.

CALCULATOR NOTE!

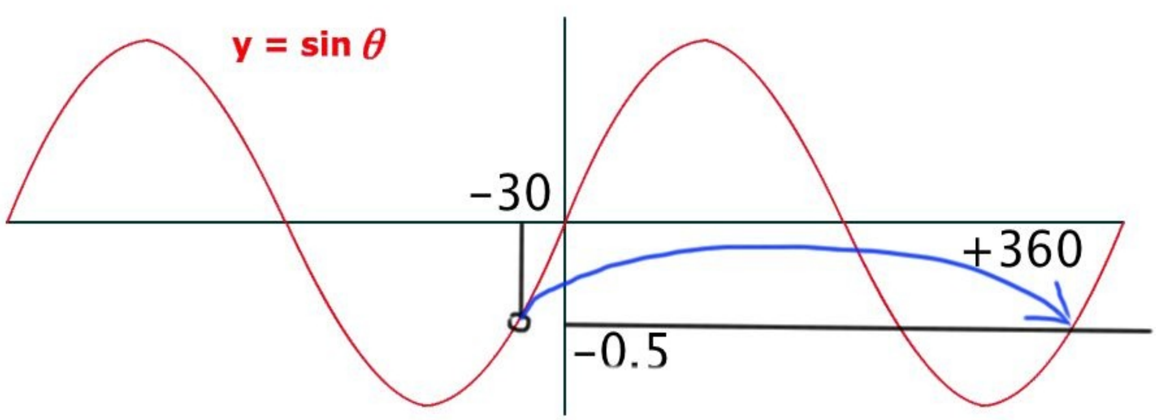
If you type in a value for

$\sin\theta = -0.5$

The calculator gives you a negative answer of - 30 degrees! If you're looking between a range of

$0 \leq \theta \leq 360$

then obviously this lies outside the range. To find your values, you need to add 360 degrees and work out the second one from there using symmetry.



This will give

330.

The reason the calculator gives you the ‘wrong’ value is because it’s programmed to give values between

$-180 \le \theta \le 180$

it doesn’t know that you only want positive values!

So watch out for that.

Using Sine To Find the Area of A Triangle

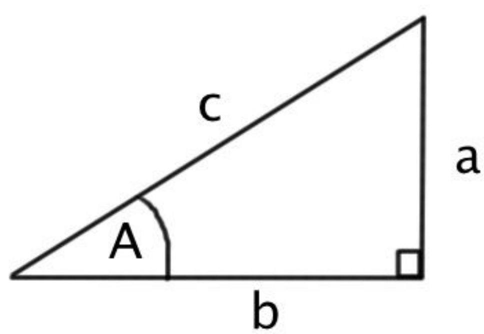
Using Sine To Find the Area of A Triangle

Because Sine means ‘Height’ of a triangle, we can also use it to find the area of a triangle. As we have seen in Squaring & Area - In A Minute, a triangle’s area is calculated just by halving a rectangle. One of the sides of a rectangle is its height.

We have seen that multiplying 2 numbers gives a rectangle. Halving that will give a triangle. The ‘formula’ will be

$$A_{Triangle} = \frac{1}{2}bh$$

Since we can now replace height with Sine, we generate a new formula, which means that we don’t even need the height any more! We can use the angle instead.



In this triangle, Angle A is opposite side a (the height) and b is the base (makes sense!) and c is the hypotenuse.

As we said, is c is 1, then we can use Sine.

If c isn’t 1, we have to normalise by dividing by c.

So we find that

$$\sin A = \frac{a}{c}$$

The height of this triangle is then

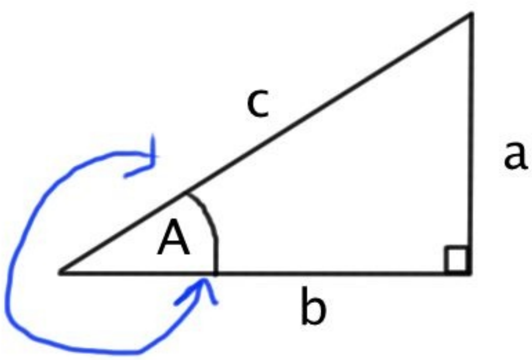
$$a = c \sin A$$

If we replace the h in the area formula with this we get

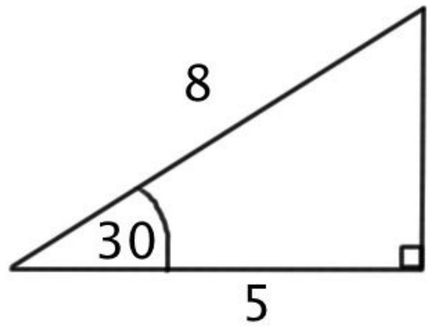
$$A_{Triangle} = \frac{1}{2} bc \sin A$$

Which given the lengths b and c, and the angle A, we can use to find the Area of the triangle.

In fact, the information we've got is all around one corner. If we have this info, we can find the area.



Example



To find the area of this triangle, we just substitute values into our formula.

$$A_{Triangle} = \frac{1}{2} bc \sin A$$

since

b = 5
c = 8
A = 30
Sin A = 0.5

We get

A = 10 square units

The Other Two Sides of The Triangle

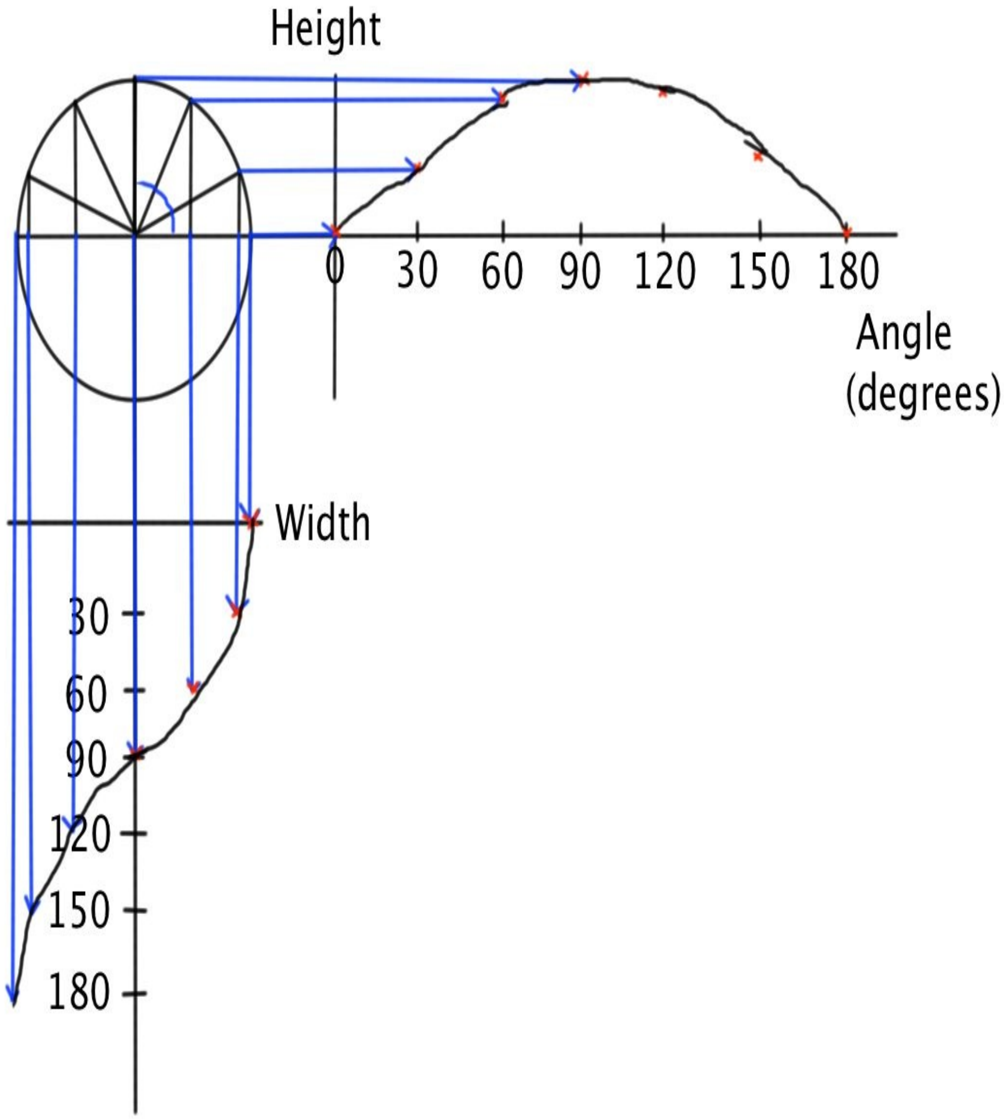
The Other Two Sides of The Triangle

Up to now we've looked at the gradient, and how this is led us to being able to find the angle.

We've also looked at the height of the triangle, and again, this has led us to find the angle.

We're now going to look at what is at ninety degrees to the height, the width, and again, use this to find the angle.

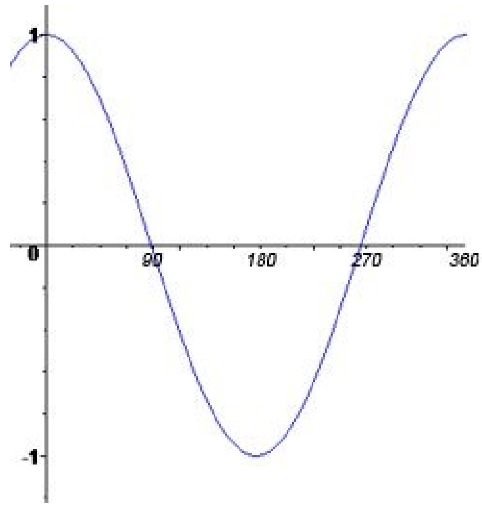
Since the width is at ninety degrees, we can look at the same circle as for the height, but work downwards. In other words, as if it was rotated ninety degrees! We move along in 30 degree increments again and see what graph this gives us.



We can see that at an angle of zero, the width is at a maximum and this works its way down to zero as the angle rotates around to 90.

It then continues as we get to 180, and at that point, does the exact reverse.

If we now rotate this graph 90 degrees we get this graph in a more familiar orientation.



As we can see, this is actually the sine graph, but advanced 90 degrees onwards. As a result, this is called the

Co-sine graph

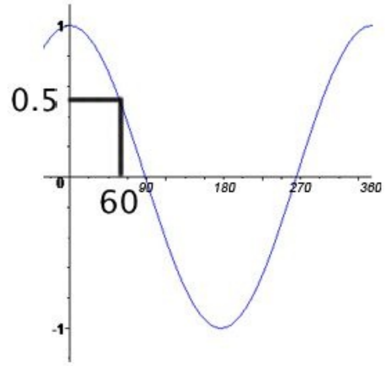
As it is just sine but moved along 90. Perhaps not surprising considering we were looking for the width, which is 90 degrees from the height!

Just with the sine graph, we can find values now of the angle or the width, depending on what information we have to start with.

For example, if we have the angle of a triangle, we can use that to find the width of the triangle.

Again, we are normalising the hypotenuse, so it must be one. And again the value for the width we get is a *fraction* of the hypotenuse, so if it isn't one, it will be that fraction of whatever it is!

So let's say we get that for angle of 60 degrees, we find the width is 0.5.



If the hypotenuse is 1, the width will indeed by one-half. But if the hypotenuse is 3, it will be half of *that*, so it will be 1.5.

So we now have a graph for the width of the triangle, and we can see that this varies with angle, with a maximum at 0 degrees and zero at 90 degrees, the complete REVERSE of Sine.

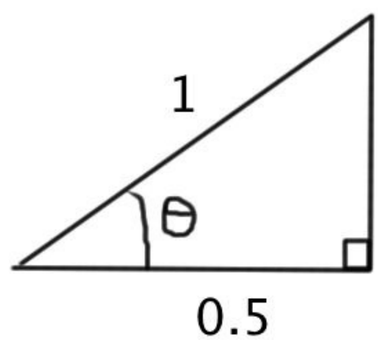
Putting Our Values To Use 2

Putting Our Values To Use 2

Just as with sine, we can now use this graph to find the angle or width in the same 3 possible scenarios.

1 We have the angle and we can find a side (width) 2 We have the angle and we can find the other side (hypotenuse) 3 We have both sides so we can find the angle.

We will now see this in action.



In this triangle, we have the width of the triangle, and the hypotenuse is normalised, *i.e.* it is one.

To find the angle, we just type

$\cos\theta = 0.5$

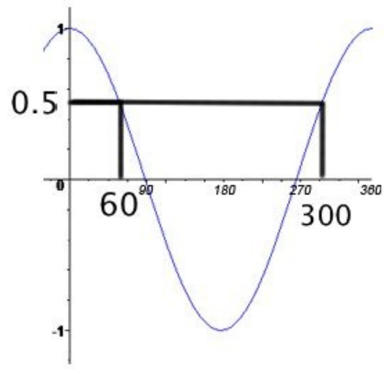
$\cos^{-1} 0.5 = \theta$

and this will give

$\theta = 60°$

So now we know the angle for that triangle.

We could also expand this idea more generally and actually find secondary solutions. This would give, looking at the graph



$300°$

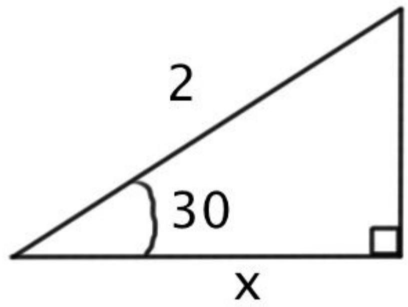
As cosine is symmetrical at 180 degrees.

In fact, as there are an infinite number of solutions, the values would be
$60° \pm 360n$
$300° \pm 360n$

But again these are often limited to a range.

SIDE 1

If we had the hypotenuse, which isn’t 1, and an angle, such as this example



From the angle we could find the width, x.

So

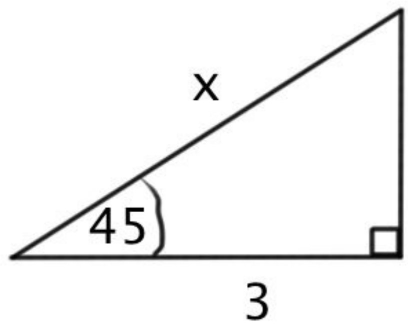
$$\cos 30 = \frac{\sqrt{3}}{2}$$

However, this is the width per 1 length of the hypotenuse. We know the hypotenuse is 2, so that means we have to double the width to get the right answer for this triangle.

$$\cos 30 = \frac{\sqrt{3}}{2} = \frac{x}{2}$$

$$x = \sqrt{3} = 1.73$$

SIDE 2



This is the reverse problem, where we have an angle and the width, but not the hypotenuse.

In this situation we know the width because it's given, but our value of

Cos 45 gives a different width.

That's because it's width per 1 length of the hypotenuse. So we know it isn't 1!

Therefore to find it we

$$\cos 45 = \frac{\sqrt{2}}{2} = \frac{3}{x}$$

Using the reciprocal,

$$\cos 45 = \frac{x}{3} = \frac{2}{\sqrt{2}}$$

And multiplying by 3

$$x = \frac{6}{\sqrt{2}}$$

$$x = 4.24$$

The Relationship Between Sine, Cosine, Tangent, Gradient

The Relationship Between Sine, Cosine, Tangent, Gradient, Height & Width

We have known for a long time that

$$\frac{height}{width} = gradient$$

But now, we know new definitions for these three terms
tangent = gradient
sine = height
cosine = width

Therefore, replacing these, we now know that

$$\frac{\sin}{\cos} = \tan$$

or more correctly

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

This is known as a true mathematical fact. This is absolutely and fundamentally true and can never be disproved. It is true on Earth and every part of the universe. It is part of the fabric of the universe and science. It is amazing that human beings have discovered it!

It is specifically known as a ‘trigonometric identity’.

So if we look back at the graphs of sine and cosine, if we were to somehow divide them, they would give a tangent graph.

Weird, huh?

Reverse of Sine

Reverse of Sine & Cosine

As with tangent, both sine and cosine have a reverse.

These are found by their reciprocals.

The reverse of

$\sin\theta$

is

Cosecant, or cosec for short,

as it is

$$\frac{1}{\sin\theta}$$

and the reverse of

$$\cos\theta$$

is

Secant, or sec

as it is

$$\frac{1}{\cos\theta}$$

These are just the final two ways to divide two sides of a triangle.

Cosecant

is just the

$$\frac{hypotenuse}{height}$$

and

secant

is just

$$\frac{hypotenuse}{width}$$

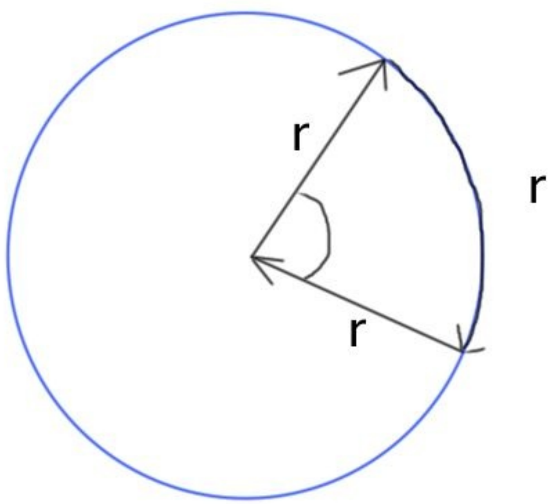
This is all the possible ways that two sides of a triangle can be divided!

A New Angle

A New Angle

If we look back to calculating the circumference, we can see something interesting.

When we drop our first radius around the circumference, as if we are on our way to finding its length, how about we draw a line back to the centre?



Then what we have is an angle created by the radius itself.

In fact, this is the angle that the universe prefers. The use of degrees is a human invention (once again), which is associated with things being a multiple of 6. (That’s why we have 12 months a year, 12 inches in a foot, 60 seconds in a minute, 60 minutes an hour, 24 hours a day...etc). However the universe is unconcerned with human inventions, so in fact degrees don’t fit very well into the scheme of things.

It turns out that this ‘radius angle’ is better.

In fact, it is called a ‘radius angle’ or RADIus ANgle...RADIAN. for short.

And because it is created by the radius on the circumference, can you guess/calculate how many radians are in a circle (or 360 degrees)? Of course, it is 2π also. Very neat!

So there are 2π radians in a circle.

If we want to know the value of 1 radian, we just divide 360 by 2π. This is around 57.3 degrees.

So that gives you an idea.

You'll notice on your calculator that it will be set to degrees as a default. However you can always tell a true mathematician. When you switch his calculator on (if they use one), it will be set to radians [R].

As a result, everything we have done in the proceeding two books can be done using Radians, and in fact, it is preferable to do so. For example, in the graphic above, although you may not recognise the squiggly long S on the left, the other terms,

$Sinx$

And

$\frac{\pi}{3}$

You will have come across by now.

$\frac{\pi}{3}$

Represents an angle of

$60°$

Since it is

$$\frac{2\pi}{6}$$

Which simplifies to

$$\frac{\pi}{3}$$

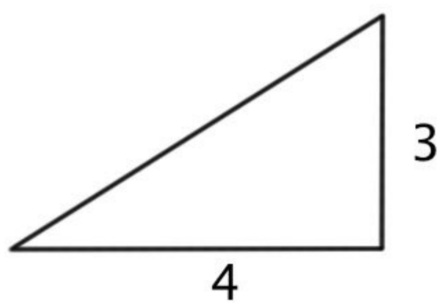
Pythagoras’ theorem

Pythagoras’ theorem

So far in right angled triangles we’ve been able to find other information about a triangle as long as we have 2 pieces of information about it.

Another variation of this is to have the gradient, but instead of using that to find the angle, we can use it to find the other side.

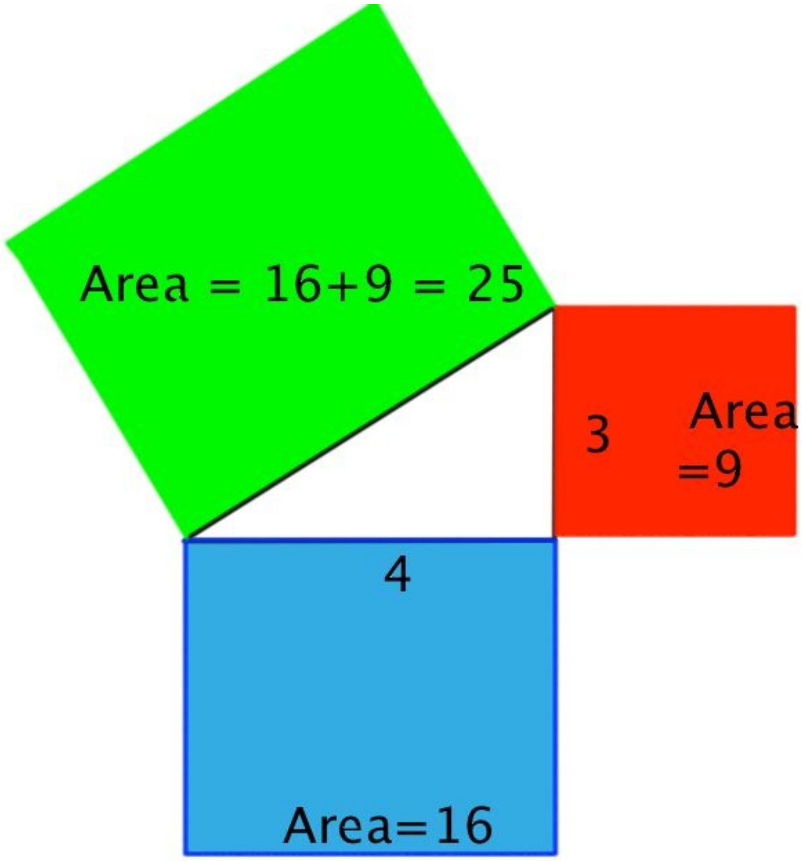
At the moment, if I asked you to find the length of the hypotenuse in this triangle, you could do it, but only by using the concepts of tangent and sine (or cosine). You would first find the angle then use that to find the hypotenuse, since you know the height (or width).



However, if we have BOTH measurements (height and width) of the gradient, we could just skip this and find the hypotenuse in one go.

How to do this was discovered by Pythagoras.

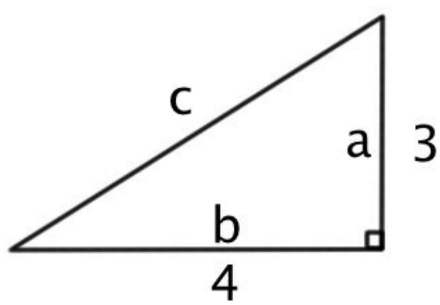
He realised that a right angled triangle had the property that the area of a square on one side, plus the area of another square on the other, would add together to form the area of a square on the hypotenuse.



Or in other words,

The square of the hypotenuse is equal to the sum of the squares of the other two sides.

Writing this a formula, and calling c the hypotenuse
$c^2 = a^2 + b^2$



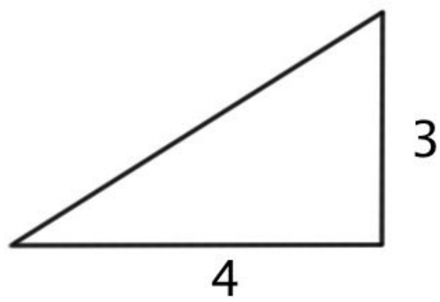
Or, making it as easy as ABC

$$a^2 + b^2 = c^2$$

From this we can find the hypotenuse from our two sides!

Example

a = 3 b = 4



To find c, we know that

$c^2 = a^2 + b^2$

Substituting

$c^2 = 3^2 + 4^2$

= 9 + 16
= 25

If

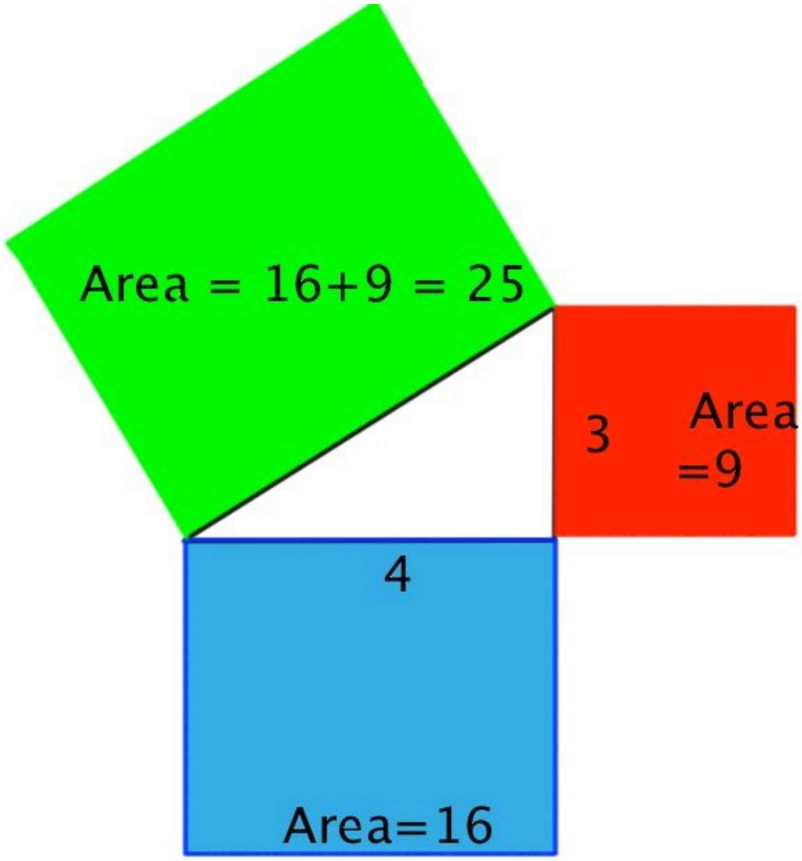
$c^2 = 25$

From Changing The Subject - In A Minute, we need to square root
$c = \sqrt{25} = 5$

So

c = 5

And that’s doing Pythagoras’ Theorem!



In fact, this is a famous triangle in mathematics, as it is common result. It's also a nice pattern... 3, 4, 5!

It just so happens that the squares of 3 and 4 add to form the square of 5.

Another example of this is 5, 12, 13.

This is because

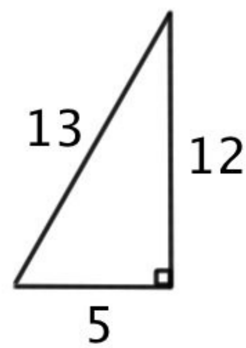
$$5^2 = 25$$
$$12^2 = 144$$

adding together

= 169

which happens to be

13^2



Most of the time, they will not be such round numbers.

If we had

a = 4 and b = 6

We get

16 + 36

= 52

$c^2 = 52$

$c = \sqrt{52}$

$= 2\sqrt{13}$

Which is not a round number! This will often be the case.

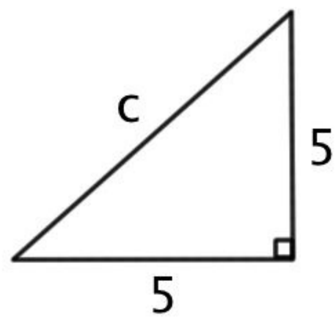
Special Situation

Special Situation

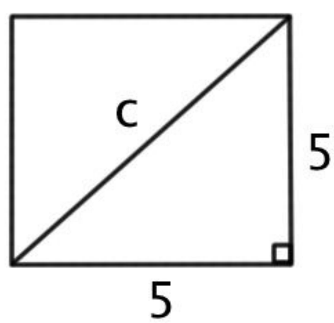
Let’s examine a situation where we have a triangle that is isosceles.

Let’s say we had a triangle where

a = 5 and b = 5



This could also be the diagonal length of a square.



To find c we use the same method

$c^2 = 25 + 25$

= 50

$c = \sqrt{50}$

$= 5\sqrt{2}$

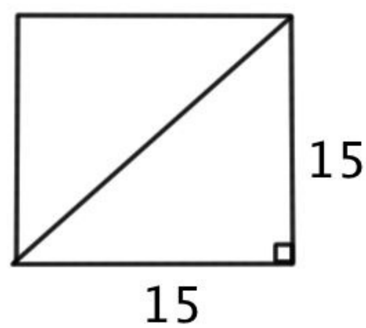
So we see the original length (5) is multiplied by root 2.

This will be true no matter what the length. As long as a and b are the same. In other words, the length of a diagonal of a square is just

$\sqrt{2}$

times by the side of the square.

What is the length of this diagonal?



Reverse Situation

Reverse Situation

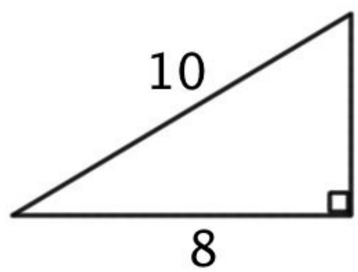
The reverse of this would be where we have the hypotenuse, but not one of the sides. From this can we find the side's length?

Of course, we can. We just reverse the formula, by changing the subject.

Let's say we're trying to find a.

And we have that

c = 10 and b = 8



Pythagoras' theorem is
$c^2 = a^2 + b^2$

Since we want a, we change the subject by following the rules set out in Changing The Subject - In A Minute

giving

$c^2 - b^2 = a^2$

$a^2 = c^2 - b^2$

$= 10^2 - 8^2$

= 100 - 64

= 36

$a^2 = 36$

a = 6

So we just do the exact reverse.

The Pythagorean Theorem Is Not Just For Triangles Once

The Pythagorean Theorem Is Not Just For Triangles

Once people learn Pythagoras' theorem they think it's only ever used in this way. In fact, the Pythagorean Theorem forms the final part of what I call the '**Trinity**' of A-Level, or advanced maths, because it is a concept that is used over and over again in a variety of surprising ways.

The Trinity of A-Level Maths is

Quadratics
Gradient/Tangent
Pythagoras' Theorem

Once they are understood and can be used fluently, these 3 concepts form a huge part of A-Level and are used repeatedly.

Now you're reading Pythagoras' Theorem - In A Minute you have now completed the third item in the Trinity and you are fully prepared for A-Level!

In this book, we will examine 3 other ways in which Pythagoras' Theorem is used.

No. 1

Combining Trigonometry with Pythagoras' Theorem Gives a New Equation

In a right-angled triangle, we now know that

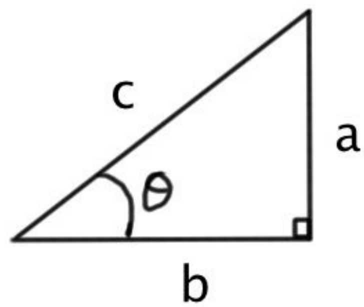
$c^2 = a^2 + b^2$

However, we also know that

$\sin\theta = \frac{a}{c}$

and

$\cos\theta = \frac{b}{c}$



So

$a = c\sin\theta$

and

$b = c\cos\theta$

If we substitute our values for a and b into Pythagoras, we get

$c^2 = (c\sin\theta)^2 + (c\cos\theta)^2$

$c^2 = c^2\sin^2\theta + c^2\cos^2\theta$

This is where

$\sin\theta \times \sin\theta = \sin^2\theta$

Which is how it is written.

If we factorise, since c is common on the RHS.

$$c^2 = c^2(\sin^2\theta + \cos^2\theta)$$

We see that

$$\sin^2\theta + \cos^2\theta = 1$$

As otherwise this expression wouldn't be true, since c-squared would be altered by multiplying it by any other value.

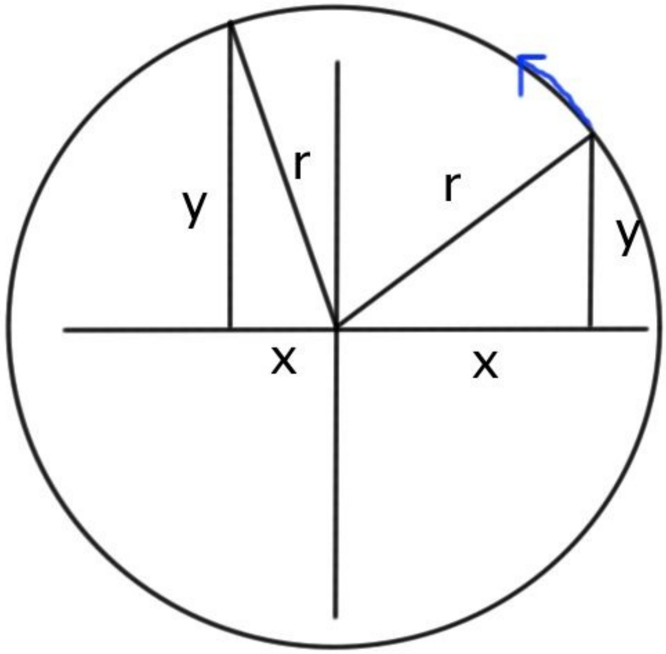
So this is yet another complicated way of writing 1!

This gives a new equation, but which is actually called an identity, which is a true mathematical fact. This identity helps us to solve problems in trigonometry.

No. 2

We Derive The Equation of A Circle

If we rename the sides of our triangle, x, y and r, we see that at any point as r rotates around the centre of the origin, Pythagoras' theorem holds true.



$x^2 + y^2 = r^2$

If we sweep around the whole 360, we get a path which gives a circle. So Pythagoras' theorem also gives the Equation of A Circle!

Where r is its radius.

No. 3

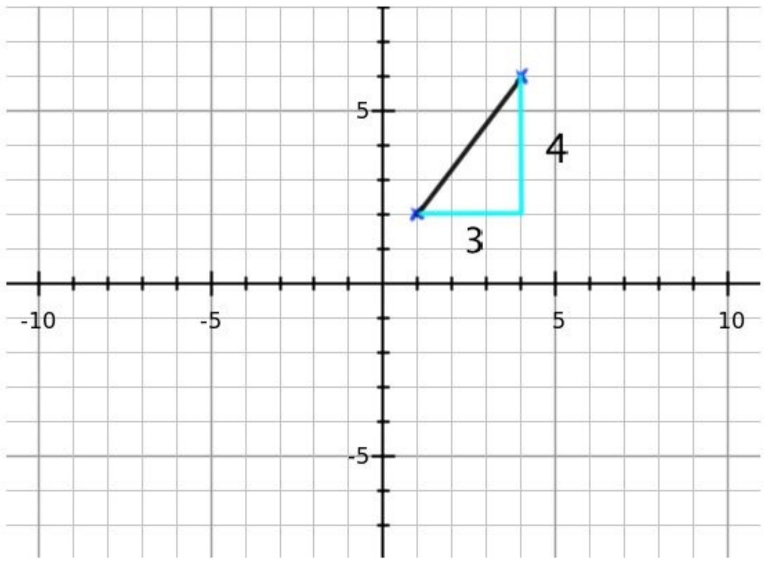
Finding the Length of A Line

If we have a line with co-ordinates on a graph, we can find its length, even though we have no way of measuring it.

Let's say we have the points

(1, 2) and (4, 6)

Between which is a straight line.



Because we have rulers that are at right angles, we can't measure anything that is diagonal. However, this is where Pythagoras comes in!

If we draw a triangle underneath this line, we can find the length of ITS sides very easily.

We can then use Pythagoras.

In this example, we can see it will be a 3, 4, 5 triangle.

So our length will be 5.

We will be using all of these ideas a lot in the future.

Introduction to Sine & Cosine Rules

Introduction

In the previous 3 books we looked at right-angled triangles exclusively, and we were able to find sides and angles, as long as we had at least 2 of each!

In this book we'll look at what to do if we have to deal with non right-angled triangles. We'll see that our first strategy, the Sine Rule, can deal with almost any situation. However, there is one exception, and this will be dealt with using the Cosine Rule.

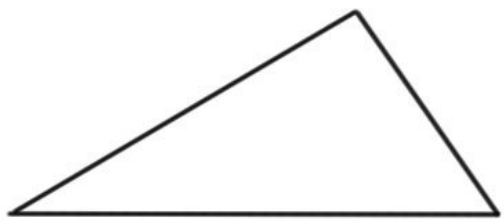
We'll look at how each rule comes about and a few examples of how to apply them.

The Sine Rule

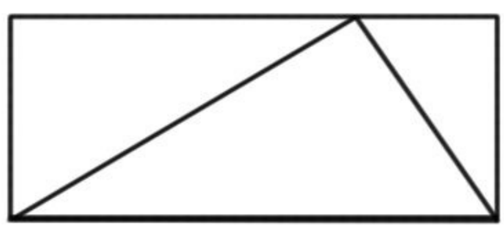
The Sine Rule

If we have a right-angled triangle, Tangent, Sine and Cosine can be used to handle all situations. This all stemmed from the concept of gradient. If we don't have an angle, we can use Pythagoras' theorem to find another length. We really have a few options.

However, if we have a non right-angled triangle, we can't use any of these strategies. So it is necessary to derive a method for this.



If we look again at the area of a triangle, we know, from the beginning, that multiplying two numbers together gives a rectangle. So halving this will give a triangle.



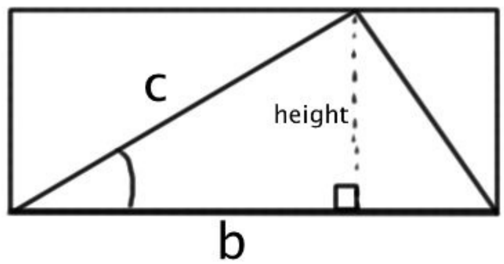
So our formula for Area is

$$A_{triangle} = \frac{1}{2}bh$$

and when we replaced the height with our new definition of 'height', which is a normalised length when the hypotenuse is 1, we get

$$A_{triangle} = \frac{1}{2}bc\sin A$$

So starting at angle A, and looking at the two lengths that form the angle, will give us the area.

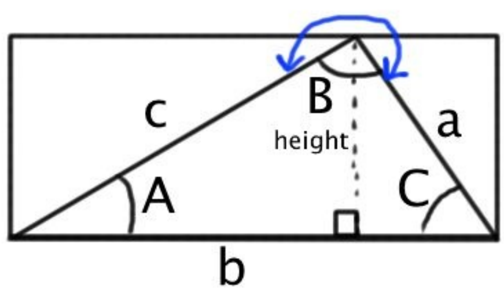


What's great about this formula is that it doesn't have to be a right-angled triangle.

Because the triangle has 3 vertices (sharp points!) we can work our way around each vertex and attach a formula for each.

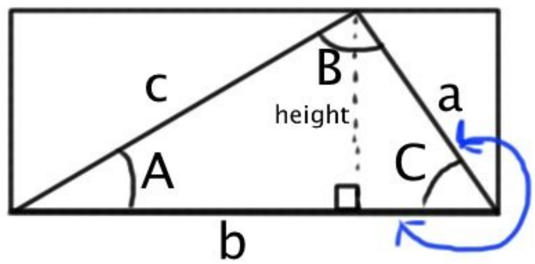
So we'd also have

$$A_{triangle} = \frac{1}{2}ac\sin B$$



and

$$A_{triangle} = \frac{1}{2}ab\sin C$$



And since they will all have the same area, we could put them equal to each other.

$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

So we have three alternate ways to write the Area. Note that each contains the 3 different ways to write abc in order.

If we manipulate this a little, we can develop a formula.

Multiplying through by

$$\frac{2}{abc}$$

We get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We now have relationships between an angle and its opposite side. This essentially tells us that the bigger the opposite side, the bigger the angle there must be.

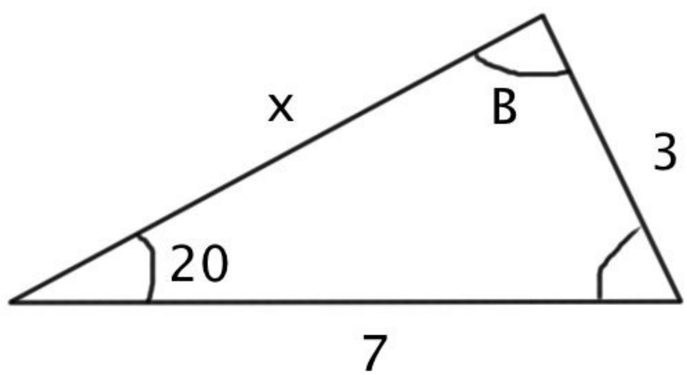
The three sides of a triangle are all equal to each other in this way. Very neat.

We can also write this in its reverse format (of course!) and using the reciprocal, gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Using this new formula, we can now use it in a similar way as using Tangent, Sine and Cosine, to find sides or angles, depending on what information we have to start with.

Example.



In this example we can find B.

We know that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Calling Angle A = 20, and its side will be a = 3

So Angle B faces side of length 7, so we'll say that b = 7.

This is all we need to use the Sine Rule. We don't need to know C or x.

So we have

$$\frac{\sin 20}{3} = \frac{\sin B}{7}$$

Re-arranging to make B the subject, gives

$$\sin B = \frac{7 \times \sin 20}{3}$$

This gives that

$$\sin B = 0.798$$

With right-angled triangles, this would tell us that the height of a triangle would be 0.798 times the hypotenuse, but this is the clever thing - we're not doing a right-angled triangle here, but it will still give us a value… as if it were.

$$\sin^{-1} 0.798 = 52.9°$$

$$B = 52.9°$$

Right-Angled Triangles
Right-Angled Triangles

We can even use this for right-angled triangles if we wanted. One of the angles will likely be

$$\sin 90 = 1$$

So there isn't much point.

We could just use Tangent, Sine or Cosine. The Sine Rule is a more general approach to all problems, whereas using the trig functions of Tangent, Sine & Cosine is for more particular cases - in particular when we have a right-angled triangle. A reductive case.

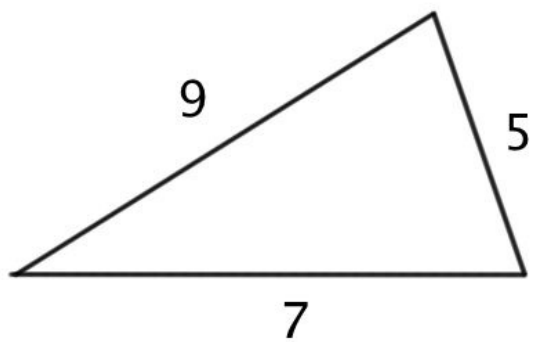
However, I did have a student once who only used the Sine Rule for everything. She never saw that she was repeating work unnecessarily by always finding Sin 90 - which is what just using the trig functions does for you - but she was happy. So I leave it to you the choice!

Having said that, it's important as we progress to realise that the trig functions are not just used for solving triangles - that is just one application of them. Knowing and understanding them will be extremely useful later. And that's why I gave you a glimpse of this by showing you 'Secondary Solutions' in the earlier books.

One Situation Remains - The Cosine Rule

One Situation Remains - The Cosine Rule

We are able to use the Sine Rule for virtually every possibility. As long as we don't have all angles or all sides! If we have a triangle and we only know what sides or angles it has, the Sine Rule won't help us at all.



As a result, we can use another rule for just this situation. As you may have guessed from the title of this book, it is the Cosine Rule.

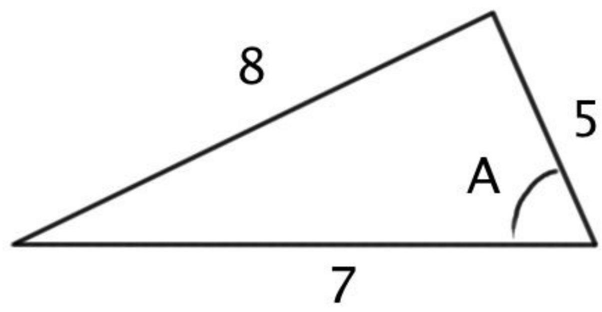
The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

I'll explain where it comes from at the end of the book.

You can probably see that it contains hints of Pythagoras' theorem and of course, a Cosine, and indeed, it is a mixture of trig functions and Pythagoras.

Let's see it being used.



In this triangle we know all the sides, so, normally, to find angle A would be impossible. But we can use the Cosine Rule here, by substituting values.

First, let's re-arrange the equation to put the angle up front.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{7^2 + 5^2 - 8^2}{2(7)(5)}$$

$$= \frac{10}{70} = \frac{1}{7}$$

$$\cos^{-1}\frac{1}{7} = A$$

$$A = 81.8°$$

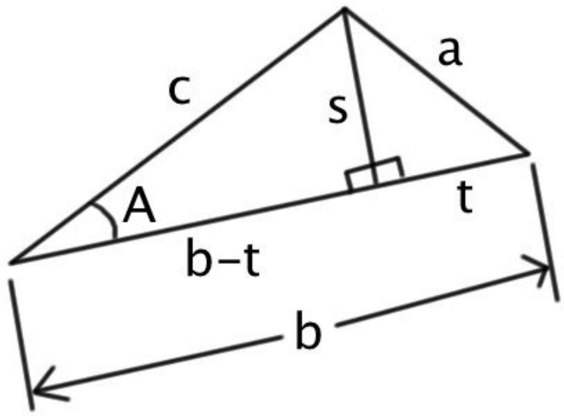
Another issue is when we have sides and angles, but they don't face each other. Again, this is what the Sine Rule depends on, and we could use the Cosine Rule instead.

So we now have a way of doing every type of triangle!

Derivation of Cosine Rule

Derivation of Cosine Rule (Not Essential)

Let’s say we have a triangle like this one.



We can see that there are a few relationships we can make.

We can see that

$$t^2 + s^2 = a^2$$

And

$$\sin A = \frac{s}{c}$$

$$\cos A = \frac{b-t}{c}$$

So that

$$c\cos A = b - t$$

And

$t = b - c\cos A$

And

$s = c\sin A$

If we now substitute s and t into the top formula, Pythagoras' theorem, we get
$a^2 = t^2 + s^2$

$a^2 = (b - c\cos A)^2 + (c\sin A)^2$

To do this we use the column method, viz:

$$\begin{array}{l} b - c\cos A \\ \times b - c\cos A \\ \hline b^2 - 2bc\cos A + c^2\cos^2 A \end{array}$$

Which gives

$$a^2 = b^2 - 2bc\cos A + c^2\cos^2 A + c^2\sin^2 A$$

Factorising all of this

$$a^2 = b^2 - 2bc\cos A + c^2(\cos^2 A + \sin^2 A)$$

We note that we know the value of

$$\cos^2 A + \sin^2 A$$

We saw in Pythagoras' Theorem - In A Minute, that this must be equal to one.

Therefore

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Where we have just switched round
$-2bc\cos A + c^2$

And that’s the Cosine Rule!

As you can see it is just a mixture of Pythagoras’ theorem and trigonometric functions, sine and cosine, blended together to find out something new!

Exam Technique

Exam technique

In schools, although the end result of all of your study of mathematics is an exam, they never seem to give you any strategies for taking one! As a result, most students end up just doing question 1, then question 2, and so on. This seems like the normal thing to do.

But there is a far superior strategy, which, although seems to waste time initially, actually saves you time and guarantees you will pass.

The problem with the strategy above is that if you come across a ‘hard’ question then you can end up spending a lot of time on something that you may not even get correct in the end!

Plus, as you are going along, it’s like you are walking through a jungle, hacking away at the foliage, desperate to find a clearing for a break or civilisation.

What would be more useful would be to have a plan view of the jungle so you’d know when it comes to an end and where the less dense foliage is!

To achieve that, what you do is before writing ANYTHING, read through the paper. This will immediately give you a view of how many questions there are, which are easy and which are hard, and most importantly, your brain will subconsciously begin working on them. Plus knowing what task is ahead of you is a great way of calming any nerves at the beginning. If you never know your task in full for the length of the whole exam, you’re never sure that it isn’t suddenly going to get worse. It’s like waiting. If it’s a countdown, you know how long it will be, that’s fine. When you don’t know how long you’re waiting for, it seems to stretch out forever!

Then, tick off the easy ones! Whichever you think “I’m glad that’s there, I can do it”, tick it off.

As you’re doing this, you’ll see and hear in your periphery everyone else scribbling furiously. Don’t worry about that. Let them go ahead with the hacking. You’ve got a better plan.

When you’ve finished that, which takes 3-4 minutes, then DO THE EASY ONES FIRST. There’s no law that says you have to do them in order. Go through the jungle the way you know better - through the less dense foliage! Do all those easy ones and get those marks in the bank! Isn’t that how you’re going to pass this thing?

When all those easy ones are done, you can now turn your attention to the harder questions.

And this is the brilliant thing. In the time you’ve been warming up, doing those easier questions, your brain has been hard at work in its subconscious on those harder questions! So I bet you that when you look at them again they’ll be just that little bit easier! And that means you’ll do better in this exam than ever before.

GCSE exam links

THE END

THE END.

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About the author

Paul Carson is an author and private maths tutor. Despite hating maths at school, he loved science enough that he studied it at university. Then, he found that he needed maths! Figuring it out himself, he invented a new method which led to him studying maths at degree level. Now, using the very same approach 95% of his students achieve a pass rate at GCSE - compared with 54% using the traditional curriculum method.

Paul lives a quiet, understated life in Lincoln with his young daughter. Beneath this regular exterior however, hides a shockingly smart man whose brain power is not down to the good fortune of expensive schooling or an exceptional memory, but 100% work.

His first book - Naked Numbers: The 3 Rules To Make Your Life Add Up - has been designed to transform the lives of numbers-phobics by stripping down maths into 3 simple rules. By exposing the secret workings of the subject, it allows users to confidently solve almost any maths problem.

His new series, In A Minute, has been designed to allow people to find bite sized chunks of his method at a low cost. It also contains improvements from his original course.

He is now studying mechanical engineering. For fun.

Q&A with Paul Carson

What inspired you to redesign mathematics?

I've always been inspired by great teachers (Stand and Deliver, Dead Poets Society etc) and I wanted to emulate them. I was amazed when I saw a documentary about Michel Thomas, in 1997, who claimed you could learn French in a few days. I knew it was true from the small snippet he gave of his method. When I studied maths, I thought it could be made simpler in the same way. When I tutored maths I discovered that the majority of students don't understand maths, they just memorise it in the hope of fooling the examiner into thinking they do! But that strategy isn't very effective as it more often than not fails, and if it succeeds, you end up not knowing anything anyway! I decided to create a course that would be intuitive, simple and exciting. That is how my students respond--they are amazed it can be so easy after the difficult methods taught at school.

What are the magic 3 rules?

He Ha ha, you'll have to buy the course to find out! It is best if you interact to find the three rules yourself. That way you'll always know them. And what you know, you don't forget.

How can your method and mathematics change your life?

There are two kinds of learning, informational learning, where you just learn data or facts, and transformational learning, where what you learn actually changes you. Maths In A Minute achieves a transformation in the student because they start to see the world in a different way. They come to realise that it is necessary to understand what you're doing, no matter what subject, and it gives them a love for learning (which is the opposite that traditional schools, by and large, tend to achieve...everyone can't wait to finish).

Maths wise, it makes the student appreciate they CAN do it, and if they can do something they previously thought impossible, it gives them huge confidence and self-esteem, which can carry over into other activities or challenges they take on in life. For me, it has given me the confidence to tackle a degree in engineering, knowing my maths is very strong!

Give us your top maths trick?

There are so many it is hard to know which one to choose.

Have you always been brilliant at maths?

No, definitely not! I used to hate maths (mainly because, I didn't understand it). However, for some weird reason I can't explain, I never gave up trying to understand it and eventually, through determination and finding a good book or two (which are hard to find and never on any reading list), I figured it out in the end. But I was on the brink of being expelled by my university. In fact I was averaging around 27% most of the time. But once I figured it out my grades trebled to 80%. So, because I struggled with maths I can empathise as to why my students do too. And this has been very effective, with a 95% success rate, and many happy students (and parents). Often students tell me that their teacher doesn't understand why they (the student) don't understand it...but I do, because I was sat there once too.

What makes you smile?

My daughter playing, singing or messing around with our kitten.

What keeps you awake at night?

Thinking about how to make algebra simpler. (Yes I know, sad).

Where is your favourite place in the world?

I haven't travelled as much as I would like, but my favourite city is Barcelona. My favourite getaway is Connemara, Galway, which has amazing scenery.

What is your TV guilty pleasure?

I've always been pretty inspired by Star Trek: The Next Generation. In fact that's one of the reasons I'm becoming an engineer! (You asked!) **What song would be the soundtrack to your life?**

The Impossible Dream from The Man from La Mancha.

Describe your dream dinner party line-up.

Richard P. Feynman, Jaime Escalante, Michel Thomas, Allen Carr, John Lennon, Andrew Matthews and Sir Patrick Stewart.

What's your signature dish?

I had a bit of training as a chef in France years ago so sometimes I throw a gourmet meal together, but I tend to do simpler meals like pasta or a stir fry.

What is your earliest memory?

Playing football with my Dad.

What were your aspirations when you were a teenager?

I wanted to be a fighter pilot in the RAF and fly Tornados. I still do. I would love to fly in a Tornado--that is one of my dreams still.

What's your favourite time of year?

Summer, I love nothing more than a beautiful sunny day.

What has been your proudest moment?

Apart from the birth of my daughter (now 4), I would have to say becoming an author with Hodder! But I was also proud to join the Royal Auxiliary Air Force and achieve my first rank.

What would be your specialist subject on Mastermind?

Either Richard Feynman, Michel Thomas or the Tornado jet.

If you were a superhero, what would be your super power, and how would you use it?

Create food and give it to the starving.

What one thing do you do better than anyone else you know?

That's a tough one, I know a lot of people! I suppose it would have to be...tutor! I'm considered to be quite good at that.

What's your motto for life?

Sometimes people say “Weather's bad today!" but I think "Actually, it's another beautiful day." Plus, “Don't Give Up.”