INFINITY NOTES PROJECTILE MOTION

1. BASIC CONCEPT :

1.1 Projectile

Any object that is given an initial velocity obliquely, and that subsequently follows a path determined by the gravitational force (and no other force) acting on it, is called a **Projectile**.

Examples of projectile motion:

A cricket ball hit by the batsman for a six

A bullet fired from a gun.

A packet dropped from a plane; but the motion of the aeroplane itself is not projectile motion because there are forces other than gravity acting on it due to the thrust of its engine.

1.2 Assumptions of Projectile Motion

We shall consider only trajectories that are of sufficiently short range so that the gravitational force can be considered constant in both magnitude and direction.

All effects of air resistance will be ignored.

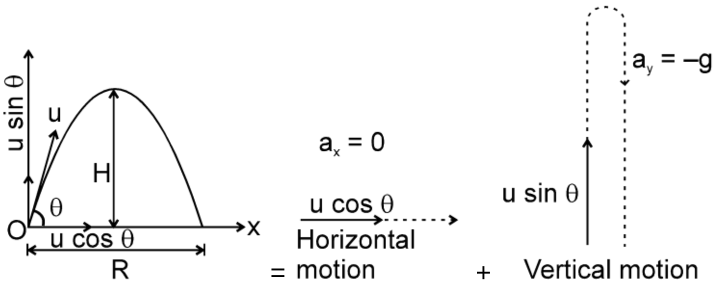
Earth is assumed to be flat.

1.3 Projectile Motion

The motion of projectile is known as projectile motion.

It is an example of two dimensional motion with constant acceleration.

Projectile motion is considered as combination of two simultaneous motions in mutually perpendicular directions which are completely independent from each other i.e. horizontal motion and vertical motion.

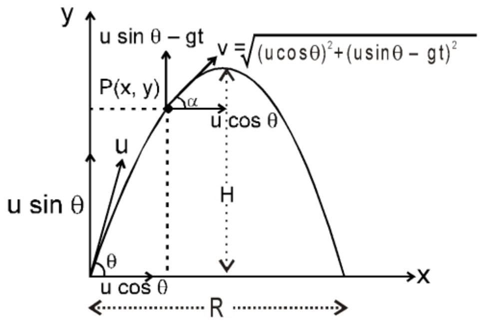


Parabolic path = vertical motion + horizontal motion.

Galileo's Statement :

Two perpendicular directions of motion are independent from each other. In other words any vector quantity directed along a direction remains unaffected by a vector perpendicular to it.

2. PROJECTILE THROWN AT AN ANGLE WITH HORIZONTAL



Consider a projectile thrown with a velocity u making an angle θ with the horizontal.

Initial velocity u is resolved in components in a coordinate system in which horizontal direction is taken as x-axis, vertical direction as y-axis and point of projection as origin.

$u_x = u \cos\theta$ $\qquad$ $u_y = u \sin\theta$

Again this projectile motion can be considered as the combination of horizontal and vertical motion. Therefore,

	Horizontal direction	Vertical direction
(a)	Initial velocity $u_x = u \cos\theta$	Initial velocity $u_y = u \sin\theta$
(b)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (down ward)
(c)	Velocity after time t, $v_x = u \cos\theta$	Velocity after time t,$v_y = u \sin\theta - gt$

2.1 Time of flight :

The displacement along vertical direction is zero for the complete flight.

Hence, along vertical direction net displacement = 0

$\Rightarrow \quad (u \sin\theta)\, T - \frac{1}{2} g T^2 = 0 \qquad \Rightarrow \qquad T = \frac{2u\sin\theta}{g}$

2.2 Horizontal range :

$R = u_x . T \qquad \Rightarrow \qquad R = u \cos\theta . \frac{2u\sin\theta}{g}$

$R = \frac{u^2 \sin 2\theta}{g}$

2.3 Maximum height :

At the highest point of its trajectory, particle moves horizontally, and hence vertical component of velocity is zero.

Using 3rd equation of motion i.e.

$v^2 = u^2 + 2as$

we have for vertical direction

$0 = u^2 \sin^2\theta - 2gH \qquad \Rightarrow \qquad H = \frac{u^2 \sin^2\theta}{2g}$

2.4 Resultant velocity :

$\vec{v} = v_x \hat{i} + v_y \hat{j} = u \cos\theta\, \hat{i} + (u \sin\theta - gt)\, \hat{j}$

Where, $|\,v\,| = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$ and $\tan\alpha = v_y / v_x$.

Also, $v\cos\alpha = u\cos\theta \quad \Rightarrow \quad v = \frac{u\cos\theta}{\cos\alpha}$

Note : 1. Results of article 2.2, and 2.3 are valid only if proejctile lands at same horizontal level from which it was projected.

2. Vertical component of velocity is positive when particle is moving up and vertical component of velocity is negative when particle is coming down if vertical upwards direction is taken as positive.

2.5 General result :

For maximum range $\theta = 45^o$

$R_{max} = \frac{u^2}{g} \qquad \Rightarrow \qquad H_{max} = \frac{R_{max}}{2}$

We get the same range for two angle of projections α and $(90 - \alpha)$ but in both cases, maximum heights attained by the particles are different.

This is because, $R = \frac{u^2 \sin 2\theta}{g}$, and $\sin 2(90 - \alpha) = \sin 180 - 2\alpha = \sin 2\alpha$

If $R = H$

i.e. $\dfrac{u^2 \sin 2\theta}{g} = \dfrac{u^2 \sin^2\theta}{2g}$ $\Rightarrow$ $\tan\theta = 4$

Range can also be expressed as

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u\sin\theta . u\cos\theta}{g} = \frac{2u_x u_y}{g}$$

Solved Examples

Example 1. A body is projected with a speed of 30 ms^{-1} at an angle of 30º with the vertical. Find the maximum height, time of flight and the horizontal range of the motion. [Take g = 10 m/s^2]

Solution. Here $u = 30$ ms^{-1}, Angle of projection, $\theta = 90 - 30 = 60º$

Maximum height,

$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{30^2 \sin^2 60º}{2\times 10} = \frac{900}{20}\times\frac{3}{4} = \frac{135}{4} \text{ m}$$

Time of flight,

$$T = \frac{2u\sin\theta}{g} = \frac{2\times 30\times \sin 60^0}{10} = 3\sqrt{3} \text{ sec.}$$

$$\text{Horizontal range} = R = \frac{u^2\sin 2\theta}{g} = \frac{30\times 30\times 2\sin 60º\cos 60º}{10} = 45\sqrt{3} \text{ m}$$

Example 2. A projectile is thrown with a speed of 100 m/s making an angle of 60^0 with the horizontal. Find the time after which its inclination with the horizontal is 45°?

Solution. $u_x = 100 \times \cos 60° = 50$

$u_y = 100 \times \sin 60° = 50\sqrt{3}$

$v_y = u_y + a_y t = 50\sqrt{3} - gt$ and $v_x = u_x = 50$

When angle is 45° ,

$\tan 45° = \dfrac{v_y}{v_x}$ $\Rightarrow$ $v_y = v_x$

$\Rightarrow$ $50\sqrt{3} - gt = 50$

$\Rightarrow$ $50(\sqrt{3}-1) = gt$

$\Rightarrow$ $t = 5(\sqrt{3}-1)$ s

Example 3. A particle is thrown with initial speed u at an angle θ w.r.t. horizontal. Find the time after which velocity of the projectile becomes perpendicular to the initial velocity.

Solution. Initial velocity $\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

velocity after time t is given by

$$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\,\hat{j}$$

when the two velocities are perpendicular their dot product will be zero .

$$\vec{u}\,.\,\vec{v} = u^2\cos^2\theta + u^2\sin^2\theta - gt\,(u\sin\theta) = 0$$

$\Rightarrow$ $t = \dfrac{u}{g\sin\theta}$

Example 4. A large number of bullets are fired in all directions with the same speed v. What is the maximum area on the ground on which these bullets will spread ?

Solution. Maximum distance upto which a bullet can be fired is its maximum range, therefore

$R_{max} = \frac{v^2}{g}$ Maximum area = $\pi(R_{max})^2 = \frac{\pi v^4}{g^2}$

Example 5. The velocity of projection of a projectile is given by : $\vec{u} = 5\hat{i} + 10\hat{j}$. Find
(a) Time of flight, (b) Maximum height, (c) Range

Solution. We have $u_x = 5$ $u_y = 10$

(a) Time of flight = $\frac{2u\sin\theta}{g} = \frac{2u_y}{g} = \frac{2\times10}{10} = 2$ s

(b) Maximum height = $\frac{u^2\sin^2\theta}{2g} = \frac{{u_y}^2}{2g} = \frac{10\times10}{2\times10} = 5$ m

(c) Range = $\frac{2u\sin\theta . u\cos\theta}{g} = \frac{2u_x u_y}{g} = \frac{2\times10\times5}{10} = 10$ m

Self Practice Problem

1. A body is projected with a speed 'u' at an angle to the horizontal to have maximum range at the highest point the velocity is :
(1) zero (2) u (3) $\frac{u}{\sqrt{2}}$ (4) $\sqrt{2}$ u

2. If two stones projected from the same point with same initial speed but an angle $\pi/3$ and $\pi/6$ respectively have their ranges R_1 and R_2 then
(1) $R_1 = 2R_2$ (2) $R_1 = R_2$ (3) $R_1 = 5R_2$ (4) $R_1 = 25R_2$

3. The time of flight of projectile is 10 second and its range is 500m. The maximum height reached by it will be (g = 10 m/s^2)
(1) 25 m (2) 50 m (3) 82 m (4) 125 m

4. If four balls A, B, C, D are projected with same speed at angles of 15º, 30º, 45º and 60º with the horizontal respectively, the two balls which will fall at the same place will be
(1) A and B (2) A and D (3) B and D (4) A and C

5. A man can throw a stone 80 m. The maximum height to which it will rise in meters is :
(1) 10 m (2) 20 m (3) 40 m (4) 50 m

6. A body is projected at an angle of 30º to the horizontal with a speed of 40 m/s. The angle with the horizontal after 2 seconds will be
(1) 10º (2) 30º (3) 45º (4) 0º

Answer Key : **1.** (3) **2.** (2) **3.** (4) **4.** (3) **5.** (2) **6.** (4)

3. EQUATION OF TRAJECTORY

The path followed by a particle (here projectile) during its motion is called its **Trajectory**. Equation of trajectory is the relation between instantaneous coordinates (Here x & y coordinate) of the particle.

If we consider the horizontal direction,

$x = u_x.t$

$x = u \cos\theta. t$ ...(1)

For vertical direction : $y = u_y . t - 1/2\ gt^2$

$= u \sin\theta . t - 1/2\ gt^2$ ...(2)

Eliminating 't' from equation (1) & (2)

$$y = u\sin\theta \cdot \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\Rightarrow \quad y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

This is an equation of parabola called as trajectory equation of projectile motion.

Other forms of trajectory equation :

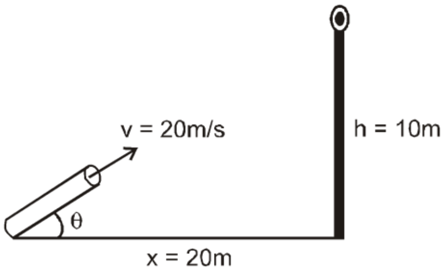
$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \quad \Rightarrow \quad y = x\tan\theta\left[1 - \frac{gx^2}{2u^2\cos^2\theta\tan\theta}\right]$$

$$\Rightarrow \quad y = x\tan\theta\left[1 - \frac{gx}{2u^2\sin\theta\cos\theta}\right] \quad \Rightarrow \quad y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

Solved Examples

Example 6 Find the value of θ in the diagram given below so that the projectile can hit the target.



Solution. $y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

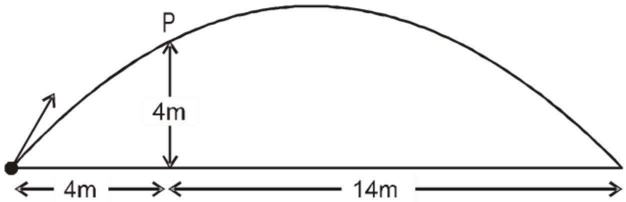
$$\Rightarrow \quad 10 = 20\tan\theta - \frac{5\times(20)^2}{(20)^2}(1+\tan^2\theta)$$

$$\Rightarrow \quad 2 = 4\tan\theta - (1+\tan^2\theta) \quad \Rightarrow \quad \tan^2\theta - 4\tan\theta + 3 = 0$$

$$\Rightarrow \quad (\tan\theta - 3)(\tan\theta - 1) = 0 \quad \Rightarrow \quad \tan\theta = 3, 1 \quad \Rightarrow \quad \theta = 45^\circ, \tan^{-1}(3)$$

Example 7 A ball is thrown from ground level so as to just clear a wall 4 m high at a distance of 4 m and falls at a distance of 14 m from the wall. Find the magnitude and direction of initial velocity of the ball.

Solution.



The ball passes through the point P(4, 4). Also range = 4 + 14 = 18 m.
The trajectory of the the ball is,

$$y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

Now $x = 4$m, $y = 4$m and $R = 18$ m

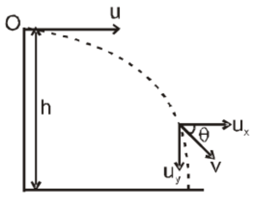
$$\therefore \quad 4 = 4\tan\theta\left[1 - \frac{4}{18}\right] = 4\tan\theta \cdot \frac{7}{9}$$

or $\tan\theta = \frac{9}{7}$ ⇒ $\theta = \tan^{-1}\frac{9}{7}$

And $R = \frac{2u^2 \sin\theta\cos\theta}{g}$

or $18 = \frac{2}{9.8} \times u^2 \times \frac{9}{\sqrt{130}} \times \frac{7}{\sqrt{130}}$ ⇒ $u = \sqrt{182}$

4. PROJECTILE THROWN PARALLEL TO THE HORIZONTAL FROM SOME HEIGHT



Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.

	Horizontal direction	**Vertical direction**
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

4.1 Time of flight :

This is equal to the time taken by the projectile to return to ground. From equation of motion

$S = ut + \frac{1}{2}at^2$, along vertical direction, we get

$-h = u_y t + \frac{1}{2}(-g)t^2$ ⇒ $h = \frac{1}{2}gt^2$ ⇒ $t = \sqrt{\frac{2h}{g}}$

4.2 Horizontal range :

Distance covered by the projectile along the horizontal direction between the point of projection to the point on the ground.

$R = u_x . t$

$R = u\sqrt{\frac{2h}{g}}$

4.3 Velocity at a general point P(x, y) :

$v = \sqrt{u_x^2 + u_y^2}$

Here horizontal velocity of the projectile after time t

$v_x = u$

velocity of projectile in vertical direction after time t

$v_y = 0 + (-g)t = -gt = gt$ (downward)

∴ $v = \sqrt{u^2 + g^2t^2}$ and $\tan\theta = v_y/v_x$

4.4 Velocity with which the projectile hits the ground :

$V_x = u$

$V_y^2 = 0^2 - 2g(-h)$

$V_y = \sqrt{2gh}$

$V = \sqrt{V_x^2 + V_y^2}$ $\Rightarrow$ $V = \sqrt{u^2 + 2gh}$

4.5 Trajectory equation :

The path traced by projectile is called the trajectory.

After time t,

$x = ut$ ....(1) $y = \frac{-1}{2} gt^2$ ....(2)

From equation (1)

$t = x/u$

Put value of t in equation (2)

$y = \frac{-1}{2} g . \frac{x^2}{u^2}$

This is trajectory equation of the particle projected horizontally from some height.

Examples based on horizontal projection from some height :

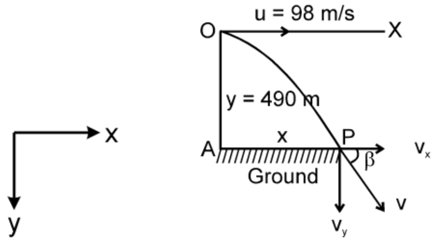
Solved Examples

Example 8. A projectile is fired horizontally with a speed of 98 ms^{-1} from the top of a hill 490 m high. Find (i) the time taken to reach the ground (ii) the distance of the target from the hill and (iii) the velocity with which the projectile hits the ground. (take g = 9.8 m/s^2)

Solution. (i) The projectile is fired from the top O of a hill with speed u = 98 ms^{-1} along the horizontal as shown as OX. It reaches the target P at vertical depth OA, in the coordinate system as shown, OA = y = 490 m

As, $y = \frac{1}{2} gt^2$ $\therefore$ $490 = \frac{1}{2} \times 9.8\, t^2$

or $t = \sqrt{100} = 10$ s.



(ii) Distance of the target from the hill is given by,

AP = x = Horizontal velocity × time = 98 × 10 = 980 m.

(iii) The horizontal and vertical components of velocity v of the projectile at point P are

$v_x = u = 98\ ms^{-1}$

$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$

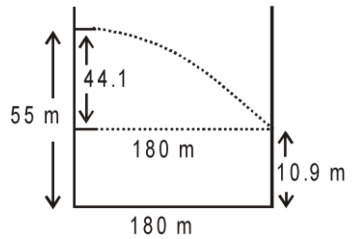
$V = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2}\ ms^{-1}$

Now if the resultant velocity v makes an angle β with the horizontal, then

$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$ $\therefore$ $\beta = 45^o$

Example 9 An object is thrown between two tall buildings. 180 m froms each other. The object is thrown horizontally from a window 55 m above ground from one building through a window 10.9 m above ground in the other building. Find out the speed of projection. (use g = 9.8 m/s^2)

Solution. $t = \sqrt{\dfrac{2h}{g}} = \sqrt{\dfrac{2 \times 44.1}{9.8}}$



$t = 3$ sec.

$R = uT$

$\dfrac{180}{3} = u$; $u = 60$ m/s

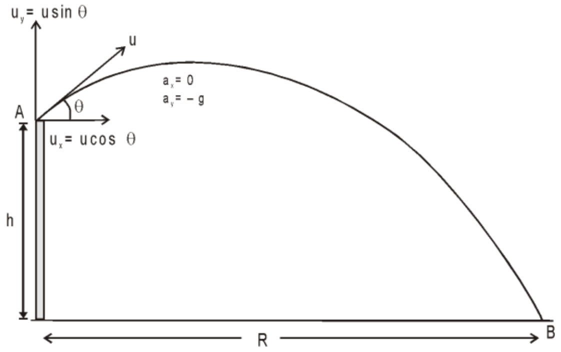
5. PROJECTION FROM A TOWER

Case (i) : Horizontal projection

$u_x = u$; $u_y = 0$; $a_y = -g$

This is same as previous section (section 4)

Case (ii) : Projection at an angle θ above horizontal



$u_x = u\cos\theta$;

$u_y = u\sin\theta$;

$a_y = -g$

Equation of motion between A & B (in Y direction)

$S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$

$S_y = u_y t + \dfrac{1}{2} a_y t^2 \quad \Rightarrow \quad -h = u\sin\theta\, t - \dfrac{1}{2} g t^2$

solving this equation we will get time of flight, T.

and range, $R = u_x T = u\cos\theta\, T$

Also, $v_y^2 = u_y^2 + 2a_y S_y$

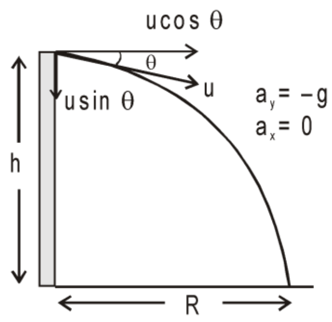
$= u^2 \sin^2\theta + 2gh$

$v_x = u\cos\theta$

$v_B = \sqrt{v_y^2 + v_x^2}$

$v_B = \sqrt{u^2 + 2gh}$

Case (iii) : Projection at an angle θ below horizontal



$u_x = u\cos\theta$;

$u_y = -\,u\sin\theta$;

$a_y = -\,g$

$S_y = u_y t + \frac{1}{2} a_y t^2$

$S_y = -h$, $u_y = -\,u\sin\theta$, $t = T$, $a_y = -\,g$

$\Rightarrow\ -h = -\,u\sin\theta\, T - \frac{1}{2} gT^2 \qquad \Rightarrow \qquad h = u\sin\theta + \frac{1}{2} gt^2$.

solving this equation we will get time of flight, T and range, $R = u_x T = u\cos\theta\, T$

$v_x = u\cos\theta$

$v_y^2 = u_y^2 + 2a_yS_y \qquad = u^2\sin^2\theta + 2(-g)(-h)$

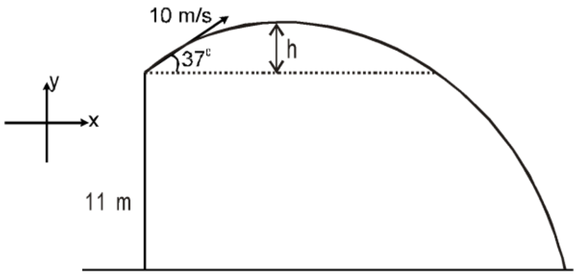
$v_y^2 = u^2\sin^2\theta + 2gh$

$v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$

Note :- objects thrown from same height in different directions with same initial speed will strike the ground with the same final speed. But the time of flight will be different.

Solved Examples

Example 10. From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37º as shown in figure. Find



(a) speed after 2s
(b) time of flight.
(c) horizontal range.
(d) the maximum height attained by the particle.
(e) speed just before striking the ground.

Sol. (a) Initial velocity in horizontal direction = 10 cos 37 = 8 m/s
Initial velocity in vertical direction = 10 sin 37º = 6 m/s
Speed after 2 seconds

$v = v_x\hat{i} + v_y\hat{j} = 8\hat{i} + (u_y + a_y t)\hat{j} = 8\hat{i} + (6 - 10\times 2)\hat{j} = 8\hat{i} - 14\hat{j}$

(b) $S_y = u_y t + \frac{1}{2} a_y t^2 \qquad \Rightarrow \qquad -11 = 6\times t + \frac{1}{2}\times(-10)\, t^2$

$5t^2 - 6t - 11 = 0 \quad \Rightarrow \quad (t+1)(5t - 11) = 0 \quad \Rightarrow \quad t = \frac{11}{5}$ sec.

(c) Range $= 8 \times \frac{11}{5} = \frac{88}{5}$ m

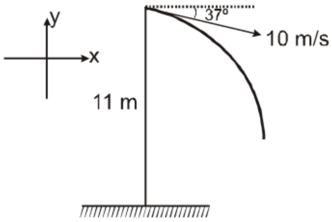
(d) Maximum height above the level of projection,

$h = \frac{u_y^2}{2g} = \frac{6^2}{2 \times 10} = 1.8$ m

$\therefore$ maximum height above ground $= 11 + 1.8 = 12.8$ m

(e) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11} \Rightarrow v = 8\sqrt{5}$ m/s

Example 11. From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37º as shown in figure. Find



(a) time of flight.
(b) horizontal range.
(c) speed just before striking the ground.

Solution. $u_x = 10 \cos 37^o = 8$ m/s , $u_y = -10 \sin 37^o = -6$ m/s

(a) $S_y = u_y t + \frac{1}{2} a_y t^2 \quad \Rightarrow \quad -11 = -6 \times t + \frac{1}{2} \times (-10) t^2$

$5t^2 + 6t - 11 = 0 \quad \Rightarrow \quad (t-1)(5t + 11) = 0 \quad \Rightarrow \quad t = 1$ sec

(b) Range $= 8 \times 1 = 8$ m

(c) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11} \qquad v = \sqrt{320}$ m/s $= 8\sqrt{5}$ m/s

Note : that in Example 10 and Example 11, objects thrown from same height in different directions with same initial speed strike the ground with the same final speed, but after different time intervals.

Self Practice Problem

7. An aeroplane is moving with a horizontal velocity u at a height h above the ground, if a packet is dropped from it the speed of the packet when it reaches the ground will be :

(1) $\sqrt{u^2 + 2gh}$ (2) $\sqrt{2gh}$ (3) $\sqrt{u^2 - 2gh}$ (4) 2gh

8. A ball is thrown horizontally and the same time another ball is dropped down from the top of a tower

(A) Both the balls will reach the ground at the same time
(B) Both will strike the ground with the same velocity

(1) A is true and B is false (2) A is true and B is true
(3) A is false and B is true (4) A is false and B is false

9. A body is thrown downward at an angle of 30º with the horizontal from the top of a tower 160m high. If its initial speed is 40 m/s the time taken to reach the ground will be :

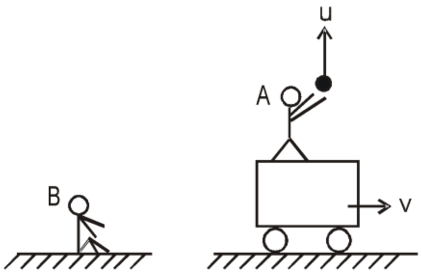
(1) 2s (2) 3s (3) 4s (4) 5s

10. From the top of a tower of height h a body of mass m is projected in the horizontal direction with a velocity v, it falls on the ground at a distance x from the tower if a body of mass 2m is projected from the top of another tower of height 2h in the horizontal direction so that it falls on the ground at a distance 2x from the tower, the horizontal velocity of the second body is :

(1) 2v (2) $\sqrt{2}$ v (3) v/2 (4) $v/\sqrt{2}$

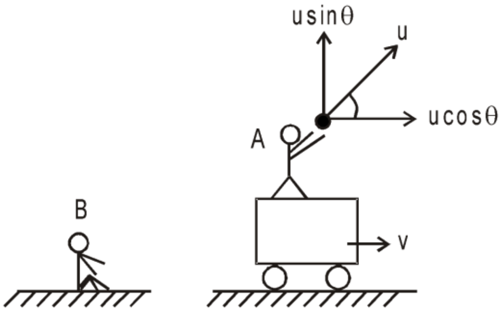
Answer Key : **7.** (1) **8.** (1) **9.** (3) **10.** (2)

6. PROJECTION FROM A MOVING PLATFORM

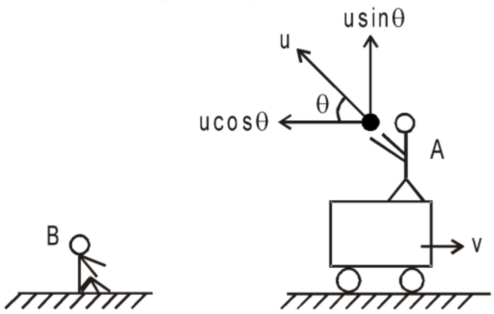


CASE (1) : When a ball is thrown upward from a truck moving with uniform speed, then observer A standing in the truck, will see the ball moving in straight vertical line (upward & downward).
The observer B sitting on road, will see the ball move in a parabolic path. The horizontal speed of the ball is equal to the speed of the truck.

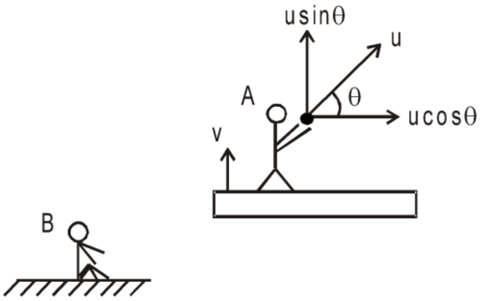
CASE (2) : When a ball is thrown at some angle 'θ' in the direction of motion of the truck, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the truck, is ucosθ, and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is
$u_x = u\cos\theta + v$ and $u_y = u\sin\theta$ respectively.



CASE (3) : When a ball is thrown at some angle 'θ' in the opposite direction of motion of the truck, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the truck, is ucosθ, and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is
$u_x = u\cos\theta - v$ and $u_y = u\sin\theta$ respectively.

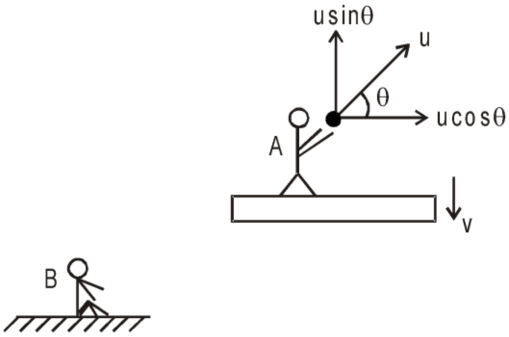


CASE (4) : When a ball is thrown at some angle 'θ' from a platform moving with speed v upwards, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the moving platform, is ucosθ and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is
$u_x = u\cos\theta$ and $u_y = u\sin\theta + v$ respectively.



CASE (5) : When a ball is thrown at some angle 'θ' from a platform moving with speed v downwards, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the moving platform, is ucosθ and usinθ respectively.

Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is u_x = ucosθ and u_y=usinθ – v respectively.



Solved Examples

Example 12. A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of 1 m/s² and the projection speed in the vertical direction is 9.8 m/s. How far behind the boy will the ball fall on the car ?

Solution. Let the initial velocity of car be 'u'.

time of flight, $t = \frac{2u_y}{g} = 2$

where u_y = component of velocity in vertical direction

distance travelled by car $x_c = u \times 2 + \frac{1}{2} \times 1 \times 2^2 = 2u + 2$

distance travelled by ball $x_b = u \times 2$

$x_c - x_b = 2u + 2 - 2u = 2m$ **Ans.**

Example 13. A fighter plane moving with a speed of $50\sqrt{2}$ m/s upward at an angle of 45º with the vertical, releases a bomb from height 1000 m above the ground. Find

(a) time of flight

(b) maximum height of the bomb above ground

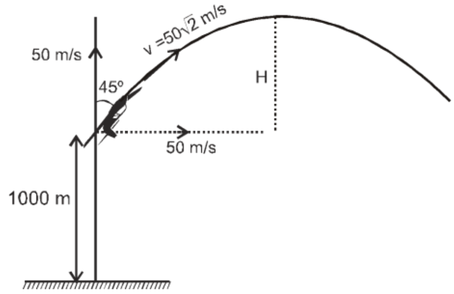
Solution. (a) $y = u_y t + \frac{1}{2}a_y t^2$

$-1000 = 50t - \frac{1}{2} \times 10 \times t^2$

$t^2 - 10t - 200 = 0$

$(t - 20)(t + 10) = 0$

$t = 20$ sec



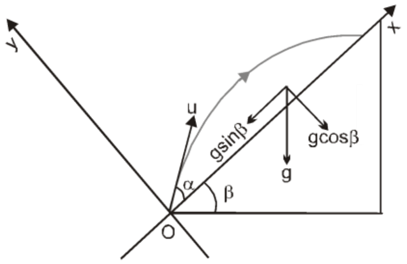
(b) $H = \frac{u_y^2}{2g} = \frac{50^2}{2g} = \frac{50 \times 50}{20} = 125$ m

Hence maximum height above ground. H = 1000 + 125 = 1125 m

7. PROJECTION ON AN INCLINED PLANE

Case (i) : Particle is projected up the incline

Here α is angle of projection w.r.t. the inclined plane. x and y axis are taken along and perpendicular to the incline as shown in the diagram. In this case: $a_x = -g\sin\beta$



$u_x = u\cos\alpha$

$a_y = -g\cos\beta$

$u_y = u\sin\alpha$

Time of flight (T) : When the particle strikes the inclined plane y becomes zero

$$y = u_y t + \frac{1}{2} a_y t^2$$

$\Rightarrow \quad 0 = u\sin\alpha T - \frac{1}{2} g\cos\beta T^2 \qquad \Rightarrow \quad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$

Where $u_\perp$ and $g_\perp$ are component of u and g perpendicular to the incline.

Maximum height (H) :

When half of the time is elasped y coordinate is equal to maximum height of the projectile

$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\cos\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2 \qquad \Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2_\perp}{2g_\perp}$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle

$x = u_x t + \frac{1}{2} a_x t^2 \qquad \Rightarrow \quad R = u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$

$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$

Case (ii) : Particle is projected down the incline

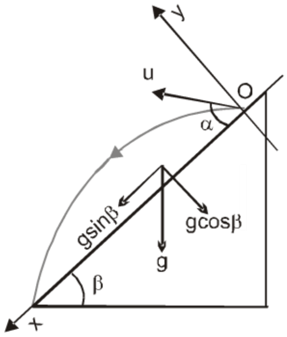
In this case :

$a_x = g\sin\beta$

$u_x = u\cos\alpha$

$a_y = -g\cos\beta$

$u_y = u\sin\alpha$



Time of flight (T) : When the particle strikes the inclined plane y coordinate becomes zero

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow \quad 0 = u\sin\alpha T - \frac{1}{2} g\cos\beta T^2$$

$$\Rightarrow \quad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$$

Maximum height (H) :

When half of the time is elasped y coordinate is equal to maximum height of the projectile

$$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2} g\sin\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad R = u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) + \frac{1}{2} g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$$

Standard results for projectile motion on an inclined plane

	Up the Incline	Down the Incline
Range	$\frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$	$\frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$
Time of flight	$\frac{2u\sin\alpha}{g\cos\beta}$	$\frac{2u\sin\alpha}{g\cos\beta}$

Angle of projection for maximum range	$\frac{\pi}{4}-\frac{\beta}{2}$	$\frac{\pi}{4}+\frac{\beta}{2}$
Maximum Range	$\frac{u^2}{g(1+\sin\beta)}$	$\frac{u^2}{g(1-\sin\beta)}$

Here α is the angle of projection with the incline and β is the angle of incline.

Note: For a given speed, the direction which gives the maximum range of the projectile on an incline, bisects the angle between the incline and the vertical, for upward or downward projection.

Solved Examples

Example 14. A bullet is fired from the bottom of the inclined plane at angle $\theta = 37°$ with the inclined plane. The angle of incline is 30° with the horizontal. Find (i) the position of the maximum height of the bullet from the inclined plane. (ii) Time of light (iii) Horizontal range along the incline. (iv) For what value of θ will range be maximum. (v) Maximum range.

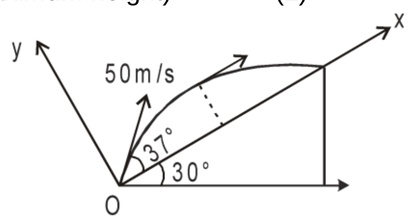
Solution. (i) Taking axis system as shown in figure

At highest point $V_y = 0$

$V_y^2 = U_y^2 + 2a_y y$

$0 = (30)^2 - 2g\cos30^o y$

$y = 30\sqrt{3}$ (maximum height) ....(1)



(ii) Again for x coordinate

$V_y = U_y + a_y t$

$0 = 30 - g\cos30^o \times t \Rightarrow t = 2\sqrt{3}$

$T = 2 \times 2\sqrt{3}$ sec Time of flight

(iii) $x = U_x t + \frac{1}{2}a_x t^2$

$x = 40 \times 4\sqrt{3} - \frac{1}{2}g\sin30^o \times (4\sqrt{3})^2$

$x = 40\,(4\sqrt{3} - 3)$ m Range

(iv) $\frac{\pi}{4} - \frac{30^\circ}{2} = 45^o - 15^o = 30^o$

(v) $\frac{u^2}{g(1+\sin\beta)} = \frac{50\times50}{10\left(1+\frac{1}{2}\right)} = \frac{2500}{15} = \frac{500}{3}$ m

Self Practice Problem

11. A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is $\alpha = 30°$, and the angle of this barrel to the horizontal is $\beta = 60^o$. The Initial velocity v of the shell is 21 m/sec. Then distance of point from the gun at which shell will fall-

(1) 10 m (2) 20 m (3) 30 m (4) 40 m

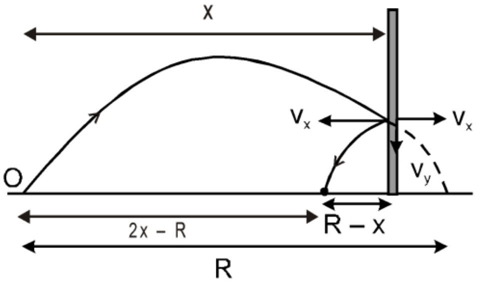
Answer Key : 11. (3)

Elastic collision of a projectile with a wall :

Suppose a projectile is projected with speed u at an angle θ from point O on the ground. Range of the projectile is R. A vertical, smooth wall is present in the path of the projectile at a distance x from the point O. The collision of the projectile with the wall is elastic. Due to collision, direction of x component of velocity is reversed but its magnitude remains the same and y component of velocity remains unchanged. Therefore the remaining distance (R – x) is covered in the backward direction and the projectile lands at a distance of R – x from the wall. Also time of flight and maximum height depends only on y component of velocity, hence they do not change despite collision with the vertical, smooth and elastic wall.

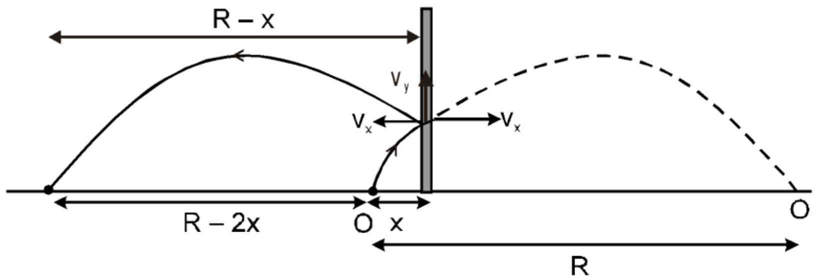
Case I : If $x \geq \frac{R}{2}$

Here distance of landing place of projectile from its point of projection is 2x – R.



Case II : If $x < \frac{R}{2}$

Here distance of landing place of projectile from its point of projection is R – 2x.



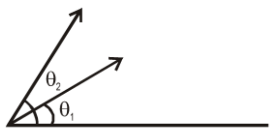
Solved Examples

Example 15. Two projectiles are thrown with different speeds and at different angles so as to cover the same maximum height. Find out the sum of the times taken by each to the reach to highest point, if time of flight is T.

Answer. Total time taken by either of the projectile.

Solution. $H_1 = H_2$ (given)

$$\frac{u_1^2 \sin^2\theta_1}{2g} = \frac{u_2^2 \sin^2\theta_2}{2g}$$



$u_1^2 \sin^2\theta_1 = u_2^2 \sin^2\theta_2$ .... (1)

at maximum height final velocity = 0

$v^2 = u_1^2 - 2gH_1$

$U_1^2 = 2gH_1$ similarly $U_2^2 = 2gH_2$

$U_1 = U_2$

on putting in equation (1)

$\therefore \quad u_1^2 \sin^2\theta_1 = u_2^2 \sin^2\theta_2 \Rightarrow \quad \theta_1 = \theta_2$

$T_1 = \frac{2u_1 \sin\theta_1}{g}$ $\Rightarrow$ $T_2 = \frac{2u_2 \sin\theta_2}{g}$ $\therefore$ $T_1 = T_2$

Time taken to reach the maximum height by 1st projectile = $\frac{T_1}{2}$

Time taken to reach the maximum height by 2nd projectile = $\frac{T_2}{2}$

$\therefore$ Sum of time taken by each to reach highest point = $\frac{T_1}{2} + \frac{T_2}{2} = 2\,\frac{T_1}{2}$ (or $2\,\frac{T_2}{2}$)

= T_1 (or T_2)

Total time taken by either of the projectile

Example 16. A particle is projected with speed 10 m/s at an angle 60^0 with horizontal. Find :

(a) time of flight (b) range

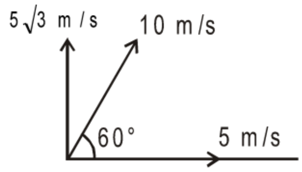
(c) maximum height (d) velocity of particle after one second.

(e) velocity when height of the particle is 1 m

Answer. (a) $\sqrt{3}$ sec. (b) $5\sqrt{3}$ m

(c) $\frac{15}{4}$ m (d) $10\sqrt{2-\sqrt{3}}$ m/s (e) $\vec{v} = 5\hat{i} \pm \sqrt{55}\,\hat{j}$

Solution. (a) $T = \frac{2u\sin\theta}{g} = \frac{2\times5\sqrt{3}}{10} = \sqrt{3}$ sec.



(b) Range = $\frac{u^2 \sin 2\theta}{g} = \frac{10\times10\times2\times\sin60^\circ\cos60^\circ}{10}$

$= 0.20 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = 5\sqrt{3}$ m.

(c) maximum height $H = \frac{u^2 \sin^2\theta}{2g} = \frac{10\times10\times.3}{2\times10\times4} = \frac{15}{4}$ m

(d) velocity at any time 't'

$\vec{v} = v_x\hat{i} + v_y\hat{j}$ $\Rightarrow$ $\vec{v}_x = \vec{u}_x$ $\Rightarrow$ $v_x = 5$

$\vec{v}_y = \vec{u}_y + \vec{a}_y t$ $\Rightarrow$ $v_y = 5\sqrt{3} - 10 \times 1$

$\vec{v} = 5\hat{i} + (5\sqrt{3} - 10)\hat{j}$ $\Rightarrow$ $v = 10\left(\sqrt{2-\sqrt{3}}\right)$ m/s

(e) $v^2 = u^2 + 2gh$

velocity at any height 'h' is

$\vec{v} = v_x\hat{i} + v_y\hat{j}$

$v_x = u_x = 5$

$v_y = u^2_y - 2gh = \left(5\sqrt{3}\right)^2 - 2 \times 10 \times 1$

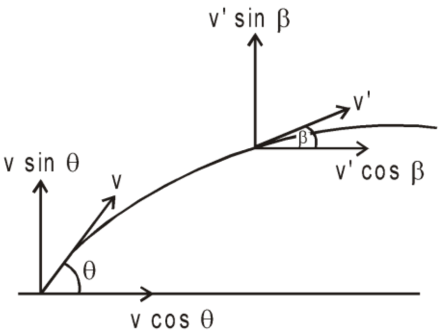
$v_y = \sqrt{55}$ $\Rightarrow$ $\vec{v} = 5\hat{i} \pm \sqrt{55}\,\hat{j}$

Example 17. A stone is thrown with a velocity v at angle θ with horizontal. Find its speed when it makes an angle β with the horizontal.

Answer. $\frac{v \cos \beta}{\cos \theta}$

Solution. $v \cos \theta = v' \cos \beta$

$v' = \frac{v \cos \theta}{\cos \beta}$



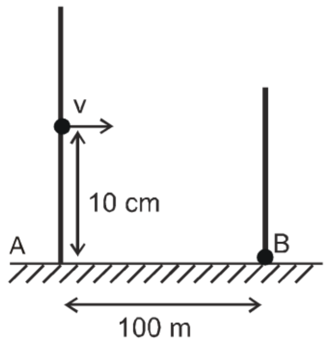
Example 18. Two paper screens A and B are separated by a distance of 100 m. A bullet pierces A and then B.The hole in B is 10 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting the screen A, calculate the velocity of the bullet when it hits the screen A. Neglect the resistance of paper and air.

Answer. 700 m/s

Solution. Equation of motion in x direction

$100 = v \times t$

$t = \frac{100}{v}$ .... (1)



in y direction

$0.1 = 1/2 \times 9.8 \times t^2$ .... (2)

$0.1 = 1/2 \times 9.8 \times (100/v)^2$

From equation (1) & (2)

on solving we get $U = 700$ m/s

Example 19. Two stones A and B are projected simultaneously from the top of a 100 m high tower. Stone B is projected horizontally with speed 10 m/s, and stone A is dropped from the tower. Find out the following ($g = 10$ m/s^2)

(a) time of flight of the two stone (b) distance between two stones after 3 sec.
(c) angle of strike with ground (d) horizontal range of particle B.

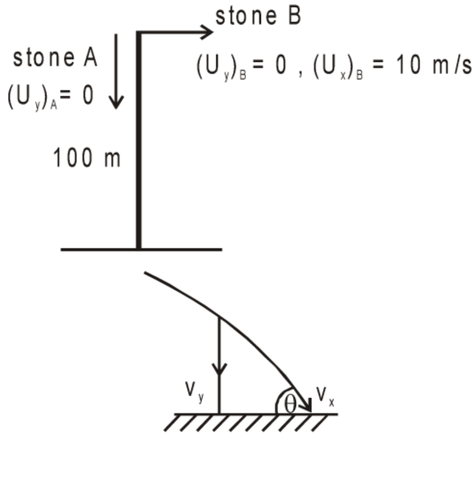
Answer. (a) $2\sqrt{5}$ sec. (b) $x_B = 30$ m, $y_B = 45$ (c) $\tan^{-1} 2\sqrt{5}$ (d) $20\sqrt{5}$ m

Solution. (a) To calculate time of flight (for both stone)

apply equation of motion in y direction

$100 = 1/2\ gt^2$

$t = 2\sqrt{5}$ sec.



(b) $X_B = 10 \times 3 = 30$ mt

$Y_B = 1/2 \times g \times t^2$

$= 1/2 \times 10 \times 3 \times 3$

$Y_B = 45$ mt

distance between two stones after 3 sec. $X_B = 30$, $Y_B = 45$

(c) angle of stricking with ground

$v_y^2 = u_y^2 + 2gh = 0 + 2 \times 10 \times 100$

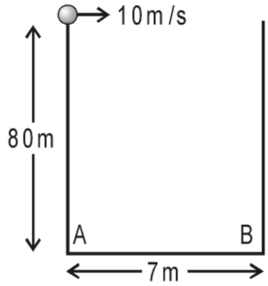
$v_y = 20\sqrt{5}$ m/s $\Rightarrow$ $v_x = 10$ m/s

$\because \vec{v} = v_x\hat{i} + v_y\hat{j}$ $\Rightarrow$ $\tan\theta = \dfrac{v_y}{v_x}$

$\theta = \tan^{-1}\left(\dfrac{20\sqrt{5}}{10}\right) = \tan^{-1}(2\sqrt{5})$

(d) Horizontal range of particle 'B' $X_B = 10 \times (2\sqrt{5}) = 20\sqrt{5}$ mt

Example 20. A ball is projected horizontally from top of a 80 m deep well with velocity 10 m/s. Then particle will fall on the bottom at a distance of (all the collisions with the wall are elastic).



(1) 5 m from A (2) 5 m from B (3) 2 m from A (4) 2 m from B

Answer. (2) 5 m from B (3) 2 m from A

Solution. Total time taken by the ball to reach at bottom = $\sqrt{\dfrac{2H}{g}} = \sqrt{\dfrac{2x80}{10}}$ = 4 sec.

Let time taken in one collision is t

Then t x 10 = 7

t = .7 sec.

No. of collisions = $\dfrac{4}{.7} = 5\dfrac{5}{7}$ (5th collisions from wall B)

Horizontal distance travelled in between 2 successive collisions = 7 m

$\therefore$ Horizontal distance travelled in 5/7 part of collisions = $\dfrac{5}{7}$ x7 = 5 m

Distance from A is 2 m. **Ans.**