Indefinite Integration

But just as much as it is easy to find the differential of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not.

Bernoulli, Johann

If f & g are functions of x such that $g'(x) = f(x)$, then indefinite integration of f(x) with respect to x is defined and denoted as $\int f(x)\,dx = g(x) + C$, where C is called the **constant of integration**.

1. Standard formulae :

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\,|ax+b| + C$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a} + C;\ a > 0$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ell n\,|\sec(ax+b)| + C$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + C$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

(x) $\int \text{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$

(xi) $\int \sec(ax+b).\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

(xii) $\int \text{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{1}{a}\text{cosec}(ax+b) + C$

(xiii) $\int \sec x\,dx = \ell n\,|\sec x + \tan x| + C$ **OR** $\ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

(xiv) $\int \operatorname{cosec} x\,dx = \ell n\,|\operatorname{cosec}x - \cot x| + C$ **OR** $\ell n\left|\tan\frac{x}{2}\right| + C$ **OR** $-\ell n\,|\operatorname{cosec}x + \cot x| + C$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{x}{a}\tan^{-1}\frac{x}{a} + C$

(xvii) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right| + C$ **OR** $\sinh^{-1}\frac{x}{a} + C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right| + C$ **OR** $\cosh^{-1}\frac{x}{a} + C$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right| + C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right| + C$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-}\frac{x}{a} + C$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ell n\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ell n\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

2. **Theorems on integration :**

(i) $\int C\, f(x).dx = C\int f(x).dx$

(ii) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)\, dx$

(iii) $\int f(x)dx = g(x) + C_1 \Rightarrow \int f(ax+b)dx = \frac{g(ax+b)}{a} + C_2$

Example # 1 : Evaluate : $\int 4x^5\, dx$

Solution : $\int 4x^5\, dx = \frac{4}{6}x^6 + C = \frac{2}{3}x^6 + C.$

Example # 2 : Evaluate : $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

Solution : $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

$= \int x^3\, dx + \int 5x^2\, dx - \int 4dx + \int \frac{7}{x}dx + \int \frac{2}{\sqrt{x}}dx$

$= \int x^3\, dx + 5.\int x^2\, dx - 4.\int 1.\, dx + 7.\int \frac{1}{x}dx + 2.\int x^{-1/2}dx$

$= \frac{x^4}{4} + 5.\frac{x^3}{3} - 4x + 7\,\ell n\,|\,x\,| + 2\left(\frac{x^{1/2}}{1/2}\right) + C$

$= \frac{x^4}{4} + \frac{5}{3}x^3 - 4x + 7\,\ell n\,|\,x\,| + 4\sqrt{x} + C$

Example # 3 : Evaluate : $\int \left(e^{2\ell nx} + e^{a\ell nx} + e^{4\ell nx}\right)dx$, a > 0

Solution : $\int \left(e^{2\ell nx} + e^{a\ell nx} + e^{4\ell nx}\right)dx$

$= \int \left(e^{\ell nx^2} + e^{\ell nx^a} + e^{\ell nx^4}\right)dx$

$= \int (x^2 + x^a + x^4)\,dx = \frac{x^3}{3} + \frac{x^{a+1}}{a+1} + \frac{x^5}{5} + c$

Example # 4 : Evaluate : $\int \left(\frac{2^{x+1} - 5^{x-1}}{10^x}\right)dx$

Solution : $\int \frac{2^{(x+1)} - 5^{x-1}}{10^x}dx = \int \left[2\left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x\right]dx = \frac{2\left(\frac{1}{5}\right)^x}{\log_e\left(\frac{1}{5}\right)} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\log\left(\frac{1}{2}\right)} + C$

Example # 5 : Evaluate : $\int \sec^2 x \operatorname{cosec}^2 x\, dx$

Solution : $I = \int \sec^2 x \operatorname{cosec}^2 x = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} = \int \left(\sec^2 x + \operatorname{cosec}^2 x\right)dx = \tan x - \cot x + C$

Example # 6 : Evaluate : $\int \frac{(1+x)^3}{\sqrt{x}}\,dx$

Solution : $\int \frac{(1+x)^3}{\sqrt{x}}\,dx = \int \frac{1+3x+3x^2+x^3}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}} + 3\int x^{\frac{1}{2}}dx + 3\int x^{\frac{3}{2}}dx + \int x^{\frac{5}{2}}dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + C = 2\sqrt{x} + 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + C$

Example # 7 : Evaluate : $\int \frac{1}{4+9x^2}\,dx$

Solution : We have

$\int \frac{1}{4+9x^2}\,dx = \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2}\,dx = \frac{1}{9}\int \frac{1}{(2/3)^2+x^2}\,dx$

$= \frac{1}{9} . \frac{1}{(2/3)} \tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$

Example # 8 : Evaluate : $\int \cos x \cos 2x\,dx$

Solution : $\int \cos x \cos 2x\,dx = \frac{1}{2}\int 2\cos x \cos 2x\,dx = \frac{1}{2}\int (\cos 3x + \cos x)\,dx = \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C$

Self Practice Problems :

(1) Evaluate : $\int \tan^2 x\,dx$ (2) Evaluate : $\int \frac{1}{1+\sin x}\,dx$

Ans. (1) tanx – x + C (2) tanx – sec x + C

3. <u>Integration by substitution</u> :

If we substitution $\varphi(x) = t$ in an integral then

(i) everywhere x will be replaced in terms of new variable t.

(ii) dx also gets converted in terms of dt.

Example # 9 : Evaluate : $\int \frac{\sec^2 x}{3+\tan x}\,dx$

Solution : $I = \int \frac{\sec^2 x}{3+\tan x}\,dx$

Let 3 + tanx = t

$\Rightarrow \quad \sec^2 x\,dx = dt = \int \frac{dt}{t} = \ell n\, t + C = \ell n|(3 + \tan x)| + C$

Example # 10 : Evaluate : $\int \frac{1}{1+e^{-x}}\,dx$

Solution : $I = \int \frac{1}{1+e^{-x}} dx = \int \frac{e^x}{e^x+1} = \int \frac{\frac{d}{dx}(e^x+1)}{(e^x+1)} = \log_e|e^x + 1| + C$

Example # 11 : Evaluate : $\int \tan^4 x\,dx$

Solution : $\int \tan^4 x\,dx = \int \tan^2 x.\tan^2 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)dx \quad = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1)dx \quad = \frac{\tan^3 x}{3} - \tan x + x + C$

Example # 12 : Evaluate : $\int \frac{x}{x^4+x^2+1} dx$

Solution : We have,

$I = \int \frac{x}{x^4+x^2+1} dx = \int \frac{x}{(x^2)^2+x^2+1} dx$ {Put $x^2 = t \quad \Rightarrow \quad x.dx = \frac{dt}{2}$ }

$\Rightarrow \quad I = \frac{1}{2}\int \frac{1}{t^2+t+1} dt = \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} dt$

$= \frac{1}{2}.\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C.$

Note: (i) $\int [f(x)]^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{1-n}}{1-n} + C, n \neq 1$

(iii) $\int \frac{dx}{x(x^n+1)}$; $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

Self Practice Problems :

(3) Evaluate : $\int \frac{\sec^2 x}{1+\tan x} dx$ (4) Evaluate : $\int \frac{\sin(\ell n x)}{x} dx$

Ans. (3) $\ell n\,|1 + \tan x| + C$ (4) $-\cos(\ell n\,x) + C$

4. Integration by parts :

Product of two functions f(x) and g(x) can be integrate using formula :

$$\int (f(x)\, g(x))\, dx = f(x) \int (g(x))\, dx - \int \left(\frac{d}{dx}(f(x)) \int (g(x))\, dx \right) dx$$

(i) when you find integral $\int g(x)\, dx$ then it will **not** contain arbitarary constant.

(ii) $\int g(x)\, dx$ should be taken as same at both places.

(iii) The choice of f(x) and g(x) can be decided by ILATE guideline.
the function will come later is taken an integral function (g(x)).

I → Inverse function

L → Logarithmic function

A → Algebraic function

T → Trigonometric function

E → Exponential function

Example # 13 : Evaluate : $\int x \log_e x\, dx$

Solution : Let I = $\int x \log_e x\, dx$

$\log_e x \int x\, dx - \int \left\{ \frac{d}{dx}(\log x) \int x dx \right\} dx$

$= \log_e x \left(\frac{x^2}{2} \right) - \int \frac{1}{x} \times \frac{x^2}{2}\, dx \quad = \frac{x^2}{2} \log_e x - \frac{x^2}{4} + C$

Example # 14 : Evaluate : $\int x\, \ell n(1+x)\, dx$

Solution : Let I = $\int x\, \ell n(1+x)\, dx$

$= \ell n\,(x+1) \,.\, \frac{x^2}{2} - \int \frac{1}{x+1} \,.\, \frac{x^2}{2}\, dx$

$= \frac{x^2}{2}\, \ell n\,(x+1) - \frac{1}{2} \int \frac{x^2}{x+1}\, dx = \frac{x^2}{2}\, \ell n\,(x+1) - \frac{1}{2} \int \frac{x^2 - 1 + 1}{x+1}\, dx$

$= \frac{x^2}{2}\, \ell n\,(x+1) - \frac{1}{2} \int \left(\frac{x^2 - 1}{x+1} + \frac{1}{x+1} \right) dx$

$= \frac{x^2}{2}\, \ell n\,(x+1) - \frac{1}{2} \int \left((x-1) + \frac{1}{x+1} \right) dx$

$= \frac{x^2}{2}\, \ell n\,(x+1) - \frac{1}{2} \left[\frac{x^2}{2} - x + \ell n\, |x+1| \right] + C$

Example # 15 : Evaluate : $\int e^{2x} \sin 2x\, dx$ dx

Solution : We know that $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}$ (asinbx – bcosbx) + C

a = 2 and b = 2 = $\frac{e^{2x}}{8}$ (2sin2x – 2cos2x) + C

Note :

(i) $\int e^x [f(x) + f'(x)]\, dx = e^x. f(x) + C$ (ii) $\int [f(x) + xf'(x)]\, dx = x f(x) + C$

Example # 16 : Evaluate : $\int \left[\ell n\ (\ell nx) + \frac{1}{(\ell nx)^2}\right]$ dx

Solution : Let I = $\int \left[\ell n\ (\ell nx) + \frac{1}{(\ell nx)^2}\right]$ dx {put $x = e^t \Rightarrow dx = e^t\, dt$}

$\therefore$ I = $\int e^t \left(\ell nt + \frac{1}{t^2}\right)$ dt $\int e^t \left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)$ = dt

= $e^t \left(\ell nt - \frac{1}{t}\right)$ + C = x $\left[\ell n\ (\ell nx) - \frac{1}{\ell nx}\right]$ + C

Self Practice Problems :

(5) Evaluate : $\int x \sin x\, dx$ (6) Evaluate : $\int x^2 e^x\, dx$

Ans. (5) – x cosx + sin x + C (6) $x^2 e^x - 2xe^x + 2e^x + C$

5. Integration of rational algebraic functions by using partial fractions :

(i) Partial Fractions :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x.
Let degree of f(x) < degree of g(x) [if it is not so, divide f(x) by g(x) until the degree of numerator becomes less than that of denominator]
Apply the concept of partial fractions as below:

CASE I :

When denominator is expressible as the product of non-repeating linear factors.
Let $g(x) = (x\ \ a_1)(x - a_2) .....(x - a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} \equiv \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$$

where $A_1, A_2, ... A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, ..., a_n$.

CASE II :

When the denominator g(x) is expressible as the product of the linear factors such that some of them are repeating.

Example $\frac{1}{g(x)} = \frac{1}{(x-a)^k (x-a_1)(x-a_2).......(x-a_r)}$ this can be expressed as

$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + ...+ \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + ... + \frac{B_r}{(x-a_r)}$

Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by equating the coefficient of same power of x.

The following example illustrate the procedure.

CASE III :

When some of the factors of denominator g(x) are quadratic but non-repeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$ The following example illustrates the procedure.

CASE IV :

When some of the factors of the denominator g(x) are quadratic and repeating

fractions of the form $\left\{\frac{A_0(2ax+b)}{ax^2+bx+c} + \frac{A_1}{ax^2+bx+c}\right\} + \left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2} + \frac{A_2}{(ax^2+bx+c)^2}\right\}$

$+ ....+ \left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k} + \frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$

Example # 17 : Evaluate $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)} dx$

Solution : Let $\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$\Rightarrow \frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$

Putting x = 1, $-6A = 1 \Rightarrow A = -\frac{1}{6}$

Putting x = 3, $10C = 5 \Rightarrow C = \frac{1}{2}$

Putting x = –2, $15B = 5 \Rightarrow B = -\frac{1}{3}$

So $= -\frac{1}{6}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$

$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log_e|x+2| + \frac{1}{2}\log_e|x-3| + C$

Example # 18 : Resolve $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Solution : Here the given function is an improper rational function (i.e. degree of numerator > degree of denominator) .On dividing we get

$\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1+\frac{(-x+4)}{(x^2-5x+6)}$ ....(i)

we have, $\frac{-x+4}{x^2-5x+6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$, then

$-x+4 = A(x-3) + B(x-2)$ ....(ii)

Putting $x-3=0$ or $x=3$ in (ii), we get

$1 = B(1) \Rightarrow B = 1.$

Putting $x-2=0$ or $x=2$ in (ii), we get

$2 = A(2-3) \Rightarrow A = -2$

$\therefore \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2} + \frac{1}{x-3}$

Hence $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1-\frac{2}{x-2}+\frac{1}{x-3}$

Example # 19 : Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate $\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

Solution : Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} + \frac{A_4}{x+2}$

$\Rightarrow \quad 3x-2 = A_1(x-1)(x+1)(x+2) + A_2(x+1)(x+2)$
$+ A_3(x-1)^2(x+2) + A_4(x-1)^2(x+1)$ ....(i)

Putting $x-1=0$ or, $x=1$ in (i) we get

$1 = A_2(1+1)(1+2) \Rightarrow A_2 = \frac{1}{6}$

Putting $x+1=0$ or, $x=-1$ in (i) we get

$-5 = A_3(-2)^2(-1+2) \Rightarrow A_3 = -\frac{5}{4}$

Putting $x+2=0$ or, $x=-2$ in (i) we get

$-8 = A_4 (-3)^2 (-1) \Rightarrow A_4 = \frac{8}{9}$

Now equating coefficient of x^3 on both sides, we get $0 = A_1 + A_3 + A_4$

$\Rightarrow \quad A_1 = -A_3 - A_4 = \frac{5}{4} - \frac{8}{9} = \frac{13}{36}$

$\therefore \frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{13}{36(x-1)} + \frac{1}{6(x-1)^2} - \frac{5}{4(x+1)} + \frac{8}{9(x+2)}$

and hence $\int \frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

$= \frac{13}{36} \ell n\,|x-1| - \frac{1}{6(x-1)} - \frac{5}{4} \ell n\,|x+1| + \frac{8}{9} \ell n\,|x+2| + C$

Example # 20 : Evaluate $\int \frac{x^2}{(x^2+4)(x^2+1)} dx$

Solution : $\int \frac{x^2}{(x^2+4)(x^2+1)} dx = \frac{1}{3} \int \left[\frac{4}{x^2+4} - \frac{1}{x^2+1}\right] dx$

$= \frac{4}{3} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3} \tan^{-1}x + C = \frac{2}{3} \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3} \tan^{-1}x + C$

Example # 21 : Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution : Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting x = 1 in (i), we get $-1 = A(1 + 1)^2 \Rightarrow A = -$

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0,\ C - B = 0,\ 2A + B - C + D = 0,\ C + E - B - D = 2$ and $A - C - E = -3$.

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$B = \frac{1}{4} = C,\ D = \frac{1}{4}$ and $E = \frac{5}{2}$ $\therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$

Example # 22 : Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Solution : We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

Then, $2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ...(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting x = 0 in (i), we get A – C = 0 ⇒ C = A = $\frac{2}{3}$
Putting x = – 1 in (i), we get –2 = A + 2B – 2 C.

⇒ $-2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

∴ $\frac{2x}{x^3-1} = \frac{2}{3} . \frac{1}{x-1} + \frac{(-\ 2/3)\ \ x+2/3}{x^2+x+1}$ or $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$

Self Practice Problems :

(7) (i) Evaluate : $\int \frac{1}{(x+2)(x+3)} dx$ (ii) Evaluate : $\int \frac{dx}{(x+1)(x^2+1)}$

Ans. (7) (i) $\ell n \left|\frac{x+2}{x+3}\right| + C$ (ii) $\frac{1}{2}\ \ell n\ |x+1| - \frac{1}{4}\ \ell n\ (x^2+1) + \frac{1}{2} \tan^{-1}(x) + C$

6. Integration of type $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

Example # 23 : Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Solution : We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\ dx = \int \sqrt{(x+1)^2+2^2}$

$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2} . (2)^2\ \ell n\ |(x+1) + \sqrt{(x+1)^2+2^2}\ | + C$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\ \ell n\ |(x+1) + \sqrt{x^2+2x+5}\ | + C$

Example # 24 : Evaluate : $\int \frac{1}{x^2-2x+3}\ dx$

Solution : $I = \int \frac{1}{x^2-2x+3}\ dx = \int \frac{1}{(x-1)^2+2}\ dx = \int \frac{1}{(x-1)^2+(\sqrt{2})^2}\ dx = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$

Example # 25 : Evaluate : $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx$

Solution : $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx = \int \frac{1}{\sqrt{-\{x^2-8x-33\}}}\ dx = \int \frac{1}{\sqrt{-\{x^2-8x+16-49\}}}\ dx$

$= \int \frac{1}{\sqrt{-\{(x-4)^2-7^2\}}}\ dx = \int \frac{1}{\sqrt{7^2-(x-4)^2}}\ dx = \sin^{-1}\left(\frac{x-4}{7}\right) + C$

Self Practice Problems :

(8) Evaluate : $\int \frac{1}{2x^2+x-1}\ dx$ (9) Evaluate : $\int \frac{1}{\sqrt{2x^2+3x-2}}\ dx$

Ans. (8) $\frac{1}{3}\ \ell n \left|\frac{2x-1}{2x+2}\right| + C$ (9) $\frac{1}{\sqrt{2}}\ \ell n \left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x-1}\right| + C$

7. Integration of type $\int \frac{px+q}{ax^2+bx+c}\,dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$, $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Express px + q = A (differential co–efficient of denominator) + B.

Example # 26 : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Solution : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\,dx = \int \frac{2x+4}{\sqrt{x^2+4x+1}}\,dx - \int \frac{1}{\sqrt{x^2+4x+1}}\,dx$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}}\,dx$, where $t = (x^2 + 4x + 1)$ for I_{st} integral

$= 2\sqrt{t} - \ell n\,|\,(x+2) + |+\sqrt{x^2+4x+1}$ C $= 2\sqrt{x^2+4x+1} - \ell n\,|x+2+\sqrt{x^2+4x+1}|+C$

Example # 27 : Evaluate : $\int (x-5)\sqrt{x^2+x}\,dx$

Solution : Let $(x-5) = \lambda.\frac{d}{dx}(x^2+x) + \mu$. Then, $x - 5 = \lambda(2x+1) + \mu$.

Comparing coefficients of like powers of x, we get

$1 = 2\lambda$ and $\lambda + \mu = -5 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\frac{11}{2}$

Hence, $\int (x-5)\sqrt{x^2+x}\,dx = \int \left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}\,dx$

$= \int \frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int \sqrt{x^2+x}\,dx$

$= \frac{1}{2}\int \sqrt{t}\,dt - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$ (where $t = x^2 + x$ for first integral)

$= \frac{1}{2}.\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.\right.$

$\left.\left. - \frac{1}{2}.\left(\frac{1}{2}\right)^2 \ell n \left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}\right] + C$

$= \frac{1}{3}t^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$

$= \frac{1}{3}(x^2+x)^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$

Self Practice Problems :

(10) Evaluate : $\int \frac{x+1}{x^2+x+3}\,dx$ (11) Evaluate : $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$

(12) Evaluate : $\int (x-1)\sqrt{1+x+x^2}\,dx$

Ans. (10) $\frac{1}{2}\ell n\,|x^2+x+3| + \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$

(11) $2\sqrt{3x^2-5x+1} + C$

(12) $\frac{1}{3}(x^2+x+1)^{3/2} - \frac{3}{8}(2x+1)\sqrt{1+x+x^2} - \frac{9}{16}\ell n(2x+1+2\sqrt{x^2+x+1})+C$

8. Integration of trigonometric functions :

(i) $\int \frac{dx}{a+b\sin^2 x}$ OR $\int \frac{dx}{a+b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put $\tan x = t$.

(ii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put tan $\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x+b.\sin x+c}{\ell.\cos x+m.\sin x+n}dx$

Express Nr ≡ A(Dr) + B $\frac{d}{dx}$ (Dr) + C & proceed.

Example # 28 : Evaluate : $\int \frac{3\sin x+2\cos x}{4\cos x+5\sin x}\,dx$

Solution : $I = \int \frac{3\sin x+2\cos x}{4\cos x+5\sin x}\,dx$

Let $3\sin x + 2\cos x = \lambda(4\cos x + 5\sin x) + \mu \frac{d}{dx}(4\cos x + 5\sin x)$

⇒ $3\sin x + 2\cos x = \lambda(4\cos x\ \ 5\sin x) + \mu(5\cos x - 4\sin x)$

comparing coefficients of sinx and cosx

$4\lambda + 5\mu = 2$

$5\lambda - 4\mu = 3$

$\lambda = \frac{23}{41}$ and m = – $\frac{2}{41}$

$I = \frac{23}{41} \int 1.dx - \frac{2}{41} \int \frac{5\cos x - 4\sin x}{4\cos x + 5\sin x} dx$

$= \frac{23}{41} x - \frac{2}{41}$ λn|4cosx + 5sibx| + C

Example # 29 : Evaluate : $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx$

Solution : We have,

$I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx$

Let 3 cos x + 2 = λ (sin x + 2 cos x + 3) + μ (cos x – 2 sin x) + ν

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

λ – 2μ = 0, 2λ + μ = 3, 3λ + ν = 2 ⇒ $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

∴ $I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3} dx$

⇒ $I = \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3} dx + \nu \int \frac{1}{\sin x + 2\cos x + 3} dx$

⇒ $I = \lambda x + \mu \log | \sin x + 2\cos x + 3 | + \nu I_1$

where $I_1 = \int \frac{1}{\sin x + 2\cos x + 3} dx$

Putting, $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$, $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3} dx = \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)} dx$

$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5} dx$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2 \frac{x}{2} = dt$ or $\sec^2 \frac{x}{2} dx = 2\,dt$, we get

$I_1 = \int \frac{2dt}{t^2 + 2t + 5} = 2 \int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2} \tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$

Hence, $I = \lambda x + \mu \log | \sin x + 2\cos x + 3 | + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right) + C$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Example # 30 : Evaluate : $\int \frac{dx}{1+3\cos^2 x}$

Solution : Multiply Nr. & Dr. of given integral by $\sec^2 x$, we get

$$I = \int \frac{\sec^2 x \; dx}{\tan^2 x + 4} = \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$$

Self Practice Problems :

(13) Evaluate : $\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\, dx$

Ans. (13) $\frac{40}{41}x + \frac{9}{41}\ell n\,|5\sin x + 4\cos x| + C$

9. Integration of type $\int \sin^m x.\cos^n x dx$

Case - I

If m and n are even natural number then converts higher power into higher angles.

Case - II

If at least one of m or n is odd natural number then if m is odd put cosx = t and vice-versa.

Case - III

When m + n is a negative even integer then put tan x = t.

Example # 31 : Evaluate : $\int \sin^5 x \cos^4 x\, dx$

Solution : Let $I = \int \sin^5 x \cos^4 x\, dx$ put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow \quad I = -\int (1-t^2)^2 . t^4 . dt = -\int (t^4 - 2t^2 + 1)\, t^4\, dt = -\int (t^8 - 2t^6 + t^4)\, dt$$

$$= -\frac{t^9}{9} + \frac{2t^7}{7} - \frac{t^5}{5} + C = -\frac{\cos^9 x}{9} + 2\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

Example # 32 : Evaluate : $\int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}}$

Solution : $I = \int \frac{dx}{\sin^{\frac{11}{3}} x \cos^{\frac{1}{3}} x}$

Divide and multiply by $\cos^{11/3}x$ = $\int \frac{dx}{\tan^{\frac{11}{3}}x\cos^4 x}$ = $\int \frac{(1+\tan^2 x)\sec^2 x}{\tan^{\frac{11}{3}}x\,dx}$

= $\int \frac{(1+t^2)}{t^{\frac{11}{3}}}$ dt [put tanx = t] = $-\frac{3}{8}t^{-8/3} - \frac{3}{2}t^{-2/3} + C$ (where t = tanx)

10. Integration of type :

$\frac{x^2 \pm 1}{x^4 + Kx^4 + 1}$ dx where K is any constant. Divide Nr & Dr by x² & put $x \pm \frac{1}{x}$ = t.

Example # 33 : Evaluate : $\frac{1-x^2}{1+x^2+x^4}$ dx

Solution : Let $I = \frac{1-x^2}{1+x^2+x^4}$ dx $= -\frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$ {put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$}

$\therefore I = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\ell n\left|\frac{t-1}{t+1}\right| + C = -\frac{1}{2}\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

Self Practice Problems :

(14) Evaluate : $\int \frac{x^2-1}{x^4-7x^2+1}$ dx (15) Evaluate : $\int \sqrt{\tan x}$ dx

Ans. (14) $\frac{1}{6}\ell n\left|\frac{x+\frac{1}{x}-3}{x+\frac{1}{x}+3}\right| + C$

(15) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ell n\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + C$ where $y = \sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}$

11. Integration of type :

(i) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ OR $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ Put px + q = t^2.

(ii) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put ax + b = $\frac{1}{t}$;

(iii) $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put x = $\frac{1}{t}$

Example # 34 : Evaluate : $\int \frac{x+1}{(x-1)\sqrt{x+2}}$ dx

Solution : $I = \int \frac{x+1}{(x-1)\sqrt{x+2}}\,dx$

Let $x + 2 = t^2$ or $dx = 2t\,dt$

$I = \int \frac{t^2-1}{(t^2-3)t} \times 2t\,dt$

$2\int \frac{t^2-3+2}{(t^2-3)} = 2\int \left(1+\frac{2}{t^2-3}\right) dt = 2t + \frac{2}{\sqrt{3}} \log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C = 2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right| + C$

Example # 35 : Evaluate : $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Solution : Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Putting $x + 1 = t^2$, and $dx = 2t\,dt$, we get $I = \int \frac{(t^2+1)\,2t\,dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$

$\Rightarrow \quad 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt \qquad \{\text{put } t-\frac{1}{t} = u\}$

$= 2\int \frac{du}{u^2+\left(\sqrt{3}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\} + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\} + C$

Example # 36 : Evaluate $\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$

Solution : Let $x + 1 = \frac{1}{t}$ $\quad dx = -\frac{1}{t^2}\,dt$

$I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}} \left(-\frac{1}{t^2}\right) dt = \int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-\frac{1}{2}}\,dt = -\frac{(1-2t)^{\frac{1}{2}}}{(-2)\times\frac{1}{2}} + C = \sqrt{1-2t} + C$

$= \sqrt{1-\frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C$

Example # 37 : Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Put $x = \frac{1}{t}$ $\Rightarrow$ $dx = -\frac{1}{t^2}\,dt$ $\Rightarrow$ $I = \int t\frac{dt}{(t^2+1)\sqrt{t^2-1}}$ $\quad \{\text{put } t^2 - 1 = y^2 \Rightarrow t\,dt = y\,dy\}$

$\Rightarrow \quad I = -\int \frac{y\ dy}{(y^2+2)\,y} = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

Self Practice Problems :

(16) Evaluate : $\int \frac{dx}{(x+2)\sqrt{x+1}}$ (17) Evaluate : $\int \frac{dx}{(x^2+5x+6)\sqrt{x+1}}$

(18) Evaluate : $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$ (19) Evaluate : $\int \frac{dx}{(2x^2+1)\sqrt{1-x^2}}$

(20) Evaluate : $\int \frac{dx}{(x^2+2x+2)\sqrt{x^2+2x-4}}$

Ans. (16) $2\tan^{-1}\left(\sqrt{x+1}\right) + C$ (17) $2\tan^{-1}\left(\sqrt{x+1}\right) - \sqrt{2}\tan^{-1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right) + C$

(18) $\sin^{-1}\left(\frac{\frac{3}{2}-\frac{1}{x+1}}{\frac{\sqrt{5}}{2}}\right) + C$ (19) $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{3}\,x}\right) + C$

(20) $-\frac{1}{2\sqrt{6}}\ \ell n\left(\frac{\sqrt{x^2+2x-4}-\sqrt{6}\,(x+1)}{\sqrt{x^2+2x-4}+\sqrt{6}\,(x+1)}\right) + C$

12. Integration of type :

(i) $\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}dx$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(ii) $\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}dx$ dx; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

(iii) $\int \frac{dx}{(x-\alpha)(x-\beta)}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

13. Reduction formula of :

$\int \tan^n x\ dx$ $\int \cot^n x\ dx$, $\int \sec^n x\ dx$, $\int \text{cosec}^n x\ dx$

(i) $I_n = \int \tan^n x\ dx = \int \tan^2 x\ \tan^{n-2} x\ dx = \int (\sec^2 x - 1)\tan^{n-2} x\ dx$

$\Rightarrow I_n = \int \sec^2 x \tan^{n-2} x\ dx - I_{n-2} \Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

(ii) $I_n = \int \cot^n x\ dx = \int \cot^2 x\ .\ \cot^{n-2} x\ dx = \int (\text{cosec}^2 x - 1)\cot^{n-2} x\ dx$

$\Rightarrow I_n = \int \text{cosec}^2 x \cot^{n-2} x\ dx - I_{n-2} \Rightarrow I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

(iii) $I_n = \int \sec^n x \, dx = \int \sec^2 x \ \sec^{n-2} x \, dx$

$\Rightarrow$ $I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2) \sec^{n-3} x . \sec x \tan x \, dx.$

$\Rightarrow$ $I_n = \tan x \sec^{n-2} x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \, dx$

$\Rightarrow$ $(n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}$ $\Rightarrow$ $I_n = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

(iv) $I_n = \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^2 x \ \operatorname{cosec}^{n-2} x \, dx$

$\Rightarrow$ $I_n = -\cot x \operatorname{cosec}^{n-2} x + \int (\cot x)(n-2) (-\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot x) \, dx$

$\Rightarrow$ $-\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \cot^2 x \operatorname{cosec}^{n-2} x \, dx$

$\Rightarrow$ $I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec}^{n-2} x \, dx$

$\Rightarrow$ $(n-1) I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}$ $\Rightarrow$ $I_n = \dfrac{\cot x \operatorname{cosec}^{n-2} x}{-(n-1)} + \dfrac{n-2}{n-1} I_{n-2}$

Example # 38 : Obtain reduction formula for $I_n = \int \sin^n x \, dx$. Hence evaluate $\int \sin^4 x \, dx$

Solution : $I_n = \int \underset{\text{II}}{(\sin x)} \underset{\text{I}}{(\sin x)^{n-1}} \, dx$

$= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \cos^2 x \, dx$

$= -\cos x (\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} (1 - \sin^2 x) \, dx$

$I_n = -\cos x (\sin x)^{n-1} + (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow$ $I_n = -\dfrac{\cos x (\sin x)^{n-1}}{n} + \dfrac{(n-1)}{n} I_{n-2}$ $(n \geq 2)$

Hence $I_4 = -\dfrac{\cos x (\sin x)^3}{4} + \dfrac{3}{4}\left(-\dfrac{\cos x (\sin x)}{2} + \dfrac{1}{2} x\right) + C$

Self Practice Problems :

(21) Evaluate : $\int \sqrt{\dfrac{x-3}{x-4}} \, dx$

(22) Evaluate : $\int \dfrac{dx}{[(x-1)(2-x)]^{3/2}}$

(23) Evaluate : $\int \dfrac{dx}{[(x+2)^8 (x-1)^6]^{1/7}}$

Ans. (21) $\sqrt{(x-3)(x-4)} + \ell n\left(\sqrt{x-3} + \sqrt{x-4}\right) + C$ (22) $2\left(\sqrt{\dfrac{x-1}{2-x}} - \sqrt{\dfrac{2-x}{x-1}}\right) + C$

(23) $\dfrac{7}{3}\left(\dfrac{x-1}{x+2}\right)^{1/7} + C$