# Binomial Theorem

## 1. Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

## 2. Terminology used in binomial theorem :

**(i) Factorial notation :** $\underline{|n}$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3 \;.\; 2 \;.\; 1 \;; & \text{if } n \in N \\ 1 \;; & \text{if } n = 0 \end{cases}$$

**Note :** $n! = n\,.\,(n-1)!$ ; $n \in N$

**(ii) Mathematical meaning of $^nC_r$ :** The term $^nC_r$ denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\ r!}$, $n \in N$, $r \in W$, $0 \le r \le n$

**Note :** Other symbols of of $^nC_r$ are $\binom{n}{r}$ and C(n, r).

**(iii) Properties related to $^nC_r$ :**

(a) $^nC_r = {}^nC_{n-r}$

**Note :** If $^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(b) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(c) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(d) $^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = ............ = \frac{n(n-1)(n-2).........(n-(r-1))}{r\ (r-1)(r-2).......2\ .1}$

(e) Sum of two consecutive binomial coefficients

$^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \Rightarrow$ L.H.S. $= {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\ r!} + \frac{n!}{(n-r+1)!\ (r-1)!}$

$= \frac{n!}{(n-r)!\ (r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] = \frac{n!}{(n-r)!\ (r-1)!}\ \frac{(n+1)}{r(n-r+1)} = \frac{(n+1)!}{(n-r+1)!\ r!} = {}^{n+1}C_r =$ R.H.S.

(f) Ratio of two consecutive binomial coefficients

$\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(g) $^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = ............ = \frac{n(n-1)(n-2).........(n-(r-1))}{r\ (r-1)(r-2).......2\ .1}$

(h) If n and r are relatively prime, then $^nC_r$ is divisible by n. But converse is not necessarily true.

## 3. Statement of binomial theorem :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + ... + {}^nC_r\, x^{n-r} y^r + ...... + {}^nC_n\, x^0 y^n$

where $n \in N$

or $(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} y^r$

from the above expansion, we can find the following expansions

$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r \; x^r$

**Notes:**

(i) The number of terms in the binomial expansion $(a + b)^n$ is $n + 1$.

(ii) The sum of the indices of a and b in each term is n.

(iii) The binomial coefficients (${}^nC_0, {}^nC_1 ..........{}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because {}^nC_r = {}^nC_{n-r}\}$

(iv) General term in the expansion of $(x + y)^n$ can be considered as $T_{r+1} = {}^nC_r \, x^{n-r} y^r$

(v) The $r^{th}$ term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1} \, x^{r-1} y^{n-r+1}$

(vi) Middle term(s) in the expansion of $(x + y)^n$ can be calculated as below :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

**Example # 1 :** Expand the following binomials :

(i) $(x - 3)^5$ (ii) $\left(1-\frac{3x^2}{2}\right)^4$

**Solution :** (i) $(x - 3)^5 = {}^5C_0x^5 + {}^5C_1x^4 (-3)^1 + {}^5C_2 x^3 (-3)^2 + {}^5C_3 x^2 (-3)^3 + {}^5C_4 x (-3)^4 + {}^5C_5 (-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1-\frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^2}{2}\right) + {}^4C_2\left(-\frac{3x^2}{2}\right)^2 + {}^4C_3\left(-\frac{3x^2}{2}\right)^3 + {}^4C_4\left(-\frac{3x^2}{2}\right)^4$

$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$

**Example # 2 :** Expand the binomial $\left(\frac{2x}{3}+\frac{3y}{2}\right)^{20}$ up to four terms

**Solution :** $\left(\frac{2x}{3}+\frac{3y}{2}\right)^{20} = {}^{20}C_0\left(\frac{2x}{3}\right)^{20} + {}^{20}C_1\left(\frac{2x}{3}\right)^{19}\left(\frac{3y}{2}\right) + {}^{20}C_2\left(\frac{2x}{3}\right)^{18}\left(\frac{3y}{2}\right)^2 + {}^{20}C_3\left(\frac{2x}{3}\right)^{17}\left(\frac{3y}{2}\right)^3 + ....$

$= \left(\frac{2x}{3}\right)^{20} + 20.\left(\frac{2}{3}\right)^{18} x^{19}y + 190 . \left(\frac{2}{3}\right)^{16} x^{18} y^2 + 1140 \left(\frac{2}{3}\right)^{14} x^{17} y^3 + .....$

**Example # 3 :** The number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is

(A) 21 (B) 31 (C) 41 (D) 61

**Solution :** $(1 - 3x + 3x^2 - x^3)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is 61.

**Example # 4 :** Find

(i) $26^{th}$ term of $(3x + 2y)^{30}$ (ii) $7^{th}$ term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

**Solution :** (i) $T_{25+1} = {}^{30}C_{25}\,(3x)^5\,(2y)^{25} = \frac{30!}{5!\ 25!}\,(3x)^5 \cdot (2y)^{25}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!\,6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$

**Example # 5 :** Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

**Solution :** The general term in the expansion of $\left(9^{1/4} + 8^{1/6}\right)^{1000}$ is

$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r = {}^{1000}C_r\ 3^{\frac{1000-r}{2}}\ 2^{\frac{r}{2}}$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ............, 1000}

Hence, number of rational terms is 501

**Example # 6 :** Find the middle term(s) in the expansion of

(i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

**Solution :** (i) $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means $T_8$ is middle term

$T_8 = {}^{14}C_7 \left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}\ x^{14}$.

(ii) $\left(3a - \frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2}+1\right)^{th}$.

It means $T_5$ & $T_6$ are middle terms

$T_5 = {}^9C_4\,(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}\ a^{17} \Rightarrow T_6 = {}^9C_5\,(3a)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}\ a^{19}$.

**Example # 7 :** Find the coefficient of $x^{32}$ and $x^{-17}$ in $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

**Solution :** Let (r + 1)th term contains $x^m$

$T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r = {}^{15}C_r\, x^{60-7r} (-1)^r$

(i) for $x^{32}$, $60 - 7r = 32 \Rightarrow 7r = 28 \Rightarrow r = 4$, so 5th term.

$T_5 = {}^{15}C_4\, x^{32} (-1)^4$

Hence, coefficient of $x^{32}$ is 1365

(ii) for $x^{-17}$, $60 - 7r = -17 \Rightarrow r = 11$, so 12th term.

$T_{12} = {}^{15}C_{11}\, x^{-17} (-1)^{11}$

Hence, coefficient of $x^{-17}$ is – 1365

**Note :** In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a + b)^n$

| If n | No. of greatest binomial coefficient | Greatest binomial coefficient |
|---|---|---|
| Even | 1 | ${}^nC_{n/2}$ |
| Odd | 2 | ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$ |

(Values of both these coefficients are equal )

**Self practice problems :**

(1) Write the first three terms in the expansion of $\left(2 - \frac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\frac{x^2}{3} + \frac{3}{x}\right)^5$.

(3) Find the term independent of x in $\left(x^2 - \frac{3}{x}\right)^9$

(4) The sum of all rational terms in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is

(A) 60 (B) 59 (C) 95 (D) 105

(5) Find the coefficient of $x^{-1}$ in $(1 + 3x^2 + x^4)\left(1 + \frac{1}{x}\right)^8$

(6) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

**Ans.** (1) $64 - 64y + \frac{80}{3}y^2$ (2) $\frac{x^{10}}{243} + \frac{5}{27}x^7 + \frac{10}{3}x^4 + 30x + \frac{135}{x^2} + \frac{243}{x^5}$.

(3) $28.3^7$ (4) B (5) 232

(6) ${}^{6n}C_{3n} \cdot x^{3n}$

## 4. Divisibility and remainder calculation :

Lets consider a number $(7)^{13}$, which can be written as $(8-1)^{13}$

Now $(8-1)^{13} = \underbrace{{}^{13}C_0(8)^{13} - {}^{13}C_1(8)^{12} + {}^{13}C_2(8)^{11} - {}^{13}C_3(8)^{10} + {}^{13}C_4(8)^9 - \ldots + {}^{13}C_{12}(8)}_{\text{Multiple of 8}} - {}^{13}C_{13}$

$\therefore (8-1)^{13} = 8I - 1$

$= 8(I-1) + 7$

Hence on dividing $(7)^{13}$ by 8, we get remainder 7.

**Example # 8 :** Prove that for each $n \in N$, $2^{3n} - 1$ is divisible by 7

**Solution :** $2^{3n} - 1 = (2^3)^n - 1 = (1 + 7)^n - 1$
$= [1 + {}^nC_1 (7) + {}^nC_2 (7)^2 \text{ ..........} + {}^nC_n (7)^n] - 1$
$= 7[{}^nC_1 + {}^nC_2\, 7 + \text{ ........ } {}^nC_n\, 7^{n-1}]$
$\Rightarrow$ $2^{3n} - 1$ is divisible by 7 for all $n \in N$

**Example # 9 :** Find the remainder when $5^{99}$ is divide by 8

**Solution :** $5^{99} = 5(5^2)^{49} = 5(24 + 1)^{49} = 5({}^{49}C_0\, 24^{49} + {}^{49}C_1\, 24^{49} \text{ ......... } {}^{49}C_{48}\, 24 + 1)$
hence remainder when $5^{99}$ is divided by 8 is 5

**Example # 10 :** Find the last two digits of the number $(17)^{10}$.

**Solution :** $(17)^{10} = (289)^5 = (290 - 1)^5$
$= {}^5C_0 (290)^5 - {}^5C_1 (290)^4 + \text{ ........ } + {}^5C_4 (290)^1 - {}^5C_5 (290)^0$
$= {}^5C_0 (290)^5 - {}^5C_1 . (290)^4 + \text{ .........} {}^5C_3 (290)^2 + 5 \times 290 - 1$
= A multiple of 1000 + 1449 Hence, last two digits are 49

**Note :** We can also conclude that last three digits are 449.

**Example # 11 :** Which number is larger $(1.01)^{1000000}$ or 10,000 ?

**Solution :** By Binomial Theorem
$(1.01)^{1000000} = (1 + 0.01)^{1000000}$
$= 1 + {}^{1000000}C_1 (0.01)$ + other positive terms
$= 1 + 1000000 \times 0.01$ + other positive terms
$= 1 + 10000$ + other positive terms
Hence $(1.01)^{1000000} > 10,000$

**Self practice problems :**

(7) If n is a positive integer, then show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

(8) What is the remainder when $7^{103}$ is divided by 25 .

(9) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(10) Which number is larger $(1.2)^{4000}$ or 800

**Ans.** (8) 18 (9) 1, 01, 001 (10) $(1.2)^{4000}$.

## 5. <u>Some standard expansions</u> :

**(i)** Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \text{ ..........} + {}^nC_r\, x^{n-r} y^r + \text{ ..........} + {}^nC_n\, x^0 y^n \quad \text{....(i)}$$

**(ii)** Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r\, (-1)^r x^{n-r} y^r$$
$$= {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y^1 + \text{...} + {}^nC_r (-1)^r x^{n-r} y^r + \text{...} + {}^nC_n (-1)^n x^0 y^n \quad \text{....(ii)}$$

**(iii)** Adding (i) & (ii), we get
$(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n y^0 + {}^nC_2\, x^{n-2} y^2 + \text{.........}]$

**(iv)** Subtracting (ii) from (i), we get
$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1} y^1 + {}^nC_3\, x^{n-3} y^3 + \text{.........}]$

## 6. Properties of binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ......... + C_rx^r + .......... + C_nx^n$ ......(1)

where $C_r$ denotes ${}^nC_r$

(i) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is $2^n$

Putting x = 1 in (1)

${}^nC_0 + {}^nC_1 + {}^nC_2 + ........ + {}^nC_n = 2^n$ ......(2)

or $\sum_{r=0}^{n} {}^nC_r = 2^n$

(ii) Again putting x = –1 in (1), we get

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ............. + (-1)^n\, {}^nC_n = 0$ ......(3)

or $\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$

(iii) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to $2^{n-1}$.

from (2) and (3)

${}^nC_0 + {}^nC_2 + {}^nC_4 + ................ = {}^nC_1 + {}^nC_3 + {}^nC_5 + ................ = 2^{n-1}$

**Example # 12 :** If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ............. + C_nx^n$, then show that

(i) $C_0 + 3C_1 + 3^2C_2 + .......... + 3^n C_n = 4^n$.

(ii) $C_0 + 2C_1 + 3.\, C_2 + ........ + (n + 1)\, C_n = 2^{n-1}\, (n + 2)$.

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ......... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$.

**Solution :** (i) $(1 + x)^n = C_0 + C_1 x + C_2x^2 + ........... + C_nx^n$

put x = 3

$C_0 + 3 .\, C_1 + 3^2 .\, C_2 + .......... + 3^n .\, C_n$ **= 4^n**

(ii) **I Method : By Summation**

L. H.S. $= {}^nC_0 + 2.\, {}^nC_1 + 3 .\, {}^nC_2 + ........ + (n + 1).\, {}^nC_n$.

$= \sum_{r=0}^{n} (r+1)\, .\, {}^nC_r = \sum_{r=0}^{n} r.\, {}^nC_r + \sum_{r=0}^{n} {}^nC_r$

$= n \sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r = n .\, 2^{n-1} + 2^n = 2^{n-1}\, (n + 2)$. RHS

**II Method : By Differentiation**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ........... + C_nx^n$

Multiplying both sides by x,

$x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ........ + C_n x^{n+1}$.

Differentiating both sides

$(1 + x)^n + x\, n\, (1 + x)^{n-1} = C_0 + 2.\, C_1x + 3 .\, C_2x^2 + ....... + (n + 1)C_nx^n$.

putting x = 1, we get

$C_0 + 2.C_1 + 3 .\, C_2 + ...... + (n + 1)\, C_n = 2^n + n .\, 2^{n-1}$

$C_0 + 2.C_1 + 3 .\, C_2 + ...... + (n + 1)\, C_n = 2^{n-1}\, (n + 2)$ **Proved**

(iii) **I Method : By Summation**

L.H.S. = $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots\dots + (-1)^n.\frac{C_n}{n+1}$

$= \sum_{r=0}^{n}(-1)^r \cdot \frac{{}^nC_r}{r+1}$

$= \frac{1}{n+1} \cdot \sum_{r=0}^{n}(-1)^r \; {}^{n+1}C_{r+1}$ $\left\{\frac{n+1}{r+1} \cdot {}^nC_r = {}^{n+1}C_{r+1}\right\}$

$= \frac{1}{n+1}\left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \dots\dots\dots\dots + (-1)^n \cdot {}^{n+1}C_{n+1}\right]$

$= \frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \dots\dots\dots + (-1)^n \cdot {}^{n+1}C_{n+1} + {}^{n+1}C_0\right]$

$= \frac{1}{n+1}$ = R.H.S. , since $\left\{-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \dots + (-1)^n \; {}^{n+1}C_{n+1} = 0\right\}$

**II Method : By Integration**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_n x^n$.

Integrating both sides, within the limits – 1 to 0.

$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \dots\dots + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$

$\frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \dots\dots + (-1)^{n+1}\frac{C_n}{n+1}\right]$

$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots\dots\dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$ Proved

**Example # 13 :** If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + \dots\dots + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + \dots\dots\dots + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1. C_0^2 + 3 . C_1^2 + 5. C_2^2 + \dots\dots\dots + (2n + 1) . C_n^2 . = 2n. {}^{2n-1}C_n + {}^{2n}C_n$.

**Solution :** (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots\dots\dots + C_n x^n$. ........(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots\dots\dots + C_n x^0$ ........(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + \dots\dots\dots + C_nx^n)(C_0x^n + C_1x^{n-1} + \dots\dots\dots + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of $x^n$,

$C_0^2 + C_1^2 + C_2^2 + \dots\dots\dots + C_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of $x^{n-2}$ or $x^{n+2}$ both sides,

$C_0C_2 + C_1C_3 + C_2C_4 + \dots\dots\dots + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. = $1. C_0^2 + 3. C_1^2 + 5. C_2^2 + \dots\dots\dots + (2n + 1) C_n^2$.

$= \sum_{r=0}^{n}(2r+1)\; {}^nC_r^2 = \sum_{r=0}^{n} 2.r \cdot ({}^nC_r)^2 + \sum_{r=0}^{n}({}^nC_r)^2$

$= 2\sum_{r=1}^{n} . \; n \cdot {}^{n-1}C_{r-1} \; {}^nC_r + {}^{2n}C_n$

$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots\ldots\ldots. {}^nC_n\, x^n$ ..........(i)

$(x + 1)^{n-1} = {}^{n-1}C_0\, x^{n-1} + {}^{n-1}C_1\, x^{n-2} + \ldots\ldots\ldots + {}^{n-1}C_{n-1} x^0$ .........(ii)

Multiplying (i) and (ii) and comparing coeffcients of $x^n$.

${}^{n-1}C_0 \,.\, {}^nC_1 + {}^{n-1}C_1 \,.\, {}^nC_2 + \ldots\ldots\ldots\ldots + {}^{n-1}C_{n-1} \,.\, {}^nC_n = {}^{2n-1}C_n$

$\sum_{r=0}^{n} {}^{n-1}C_{r-1} \,.\, {}^nC_r = {}^{2n-1}C_n$

Hence, required summation is $2n.\, {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

Hence, required summation is $2n.\, {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

**II <u>Method : By Differentiation</u>**

$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + \ldots\ldots\ldots\ldots. + C_n\, x^{2n}$

Multiplying both sides by x

$x(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots\ldots\ldots + C_nx^{2n+1}$.

Differentiating both sides

$x \,.\, n\, (1 + x^2)^{n-1} \,.\, 2x + (1 + x^2)^n = C_0 + 3.\, C_1x^2 + 5.\, C_2\, x^4 + \ldots.. + (2n + 1)\, C_n\, x^{2n}$......(i)

$(x^2 + 1)^n = C_0\, x^{2n} + C_1\, x^{2n-2} + C_2\, x^{2n-4} + \ldots\ldots\ldots + C_n$ ........(ii)

Multiplying (i) & (ii)

$(C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots\ldots + (2n + 1)\, C_n\, x^{2n})\, (C_0\, x^{2n} + C_1x^{2n-2} + \ldots\ldots\ldots\ldots + C_n)$

$= 2n\, x^2\, (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$

comparing coefficient of $x^{2n}$,

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots\ldots\ldots + (2n + 1)\, C_n^2 = 2n \,.\, {}^{2n-1}C_{n-1} + {}^{2n}C_n$.

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots\ldots\ldots + (2n + 1)\, C_n^2 = 2n \,.\, {}^{2n-1}C_n + {}^{2n}C_n$. Proved

**Example # 14 :** Find the summation of the following series –

(i) ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots\ldots. + {}^nC_m$

(ii) ${}^nC_3 + 2 \,.\, {}^{n+1}C_3 + 3.\, {}^{n+2}C_3 + \ldots\ldots\ldots + n \,.\, {}^{2n-1}C_3$

**Solution :** (i) I **Method :** Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots\ldots. + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + \ldots\ldots\ldots\ldots. + {}^nC_m$ $\{\because\ {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + \ldots\ldots\ldots\ldots\ldots\ldots. + {}^nC_m$

$= {}^{m+3}C_{m+1} + \ldots\ldots\ldots\ldots + {}^nC_m = {}^nC_{m+1} + {}^nC_m =$ $\mathbf{{}^{n+1}C_{m+1}}$

**II Method**

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots. + {}^nC_m$

The above series can be obtained by writing the coefficient of $x^m$ in

$(1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots\ldots. + (1 + x)^n$

= =

= coefficient of $x^m$ in $\dfrac{(1+x)^{n+1}}{x} - \dfrac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$

**(ii)** ${}^{n}C_3 + 2 \,.\, {}^{n+1}C_3 + 3 \,.\, {}^{n+2}C_3 + .......... + n \,.\, {}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of $x^3$ in

$(1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3 \,.\, (1 + x)^{n+2} + ........... + n \,.\, (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3.\, (1 + x)^{n+2} + ........... + n\, (1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2\,(1 + x)^{n+2} + ............ + (n - 1)\,(1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$-xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + ............. + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n\,(1 + x)^{2n}$$

$$S = \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of $x^3$ in S)

$$x^3 : \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^{n}C_5 + n \,.\, {}^{2n}C_4$

**Self practice problem :**

(11) Prove the following

(i) $C_0 + 3C_1 + 5C_2 + ............. + (2n + 1)\, C_n = 2^n\, (n + 1)$

(ii) $4C_0 + \frac{4^2}{2} \,.\, C_1 + \frac{4^3}{3}\, C_2 + ............. + \frac{4^{n+1}}{n+1}\, C_n = \frac{5^{n+1} - 1}{n+1}$

(iii) ${}^{n}C_0 \,.\, {}^{n+1}C_n + {}^{n}C_1 \,.\, {}^{n}C_{n-1} + {}^{n}C_2 \,.\, {}^{n-1}C_{n-2} + ........... + {}^{n}C_n \,.\, {}^{1}C_0 = 2^{n-1}\, (n + 2)$

(iv) ${}^{2}C_2 + {}^{3}C_2 + ......... + {}^{n}C_2 = {}^{n+1}C_3$

## 7. Binomial theorem for negative and fractional indices :

If $n \in R$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}\, x^2 + \frac{n(n-1)(n-2)}{3\,!}\, x^3 + ...............$

$................. + \frac{n(n-1)(n-2).......(n-r+1)}{r\,!}\, x^r + ................... \infty.$

**Remarks**

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^{n}C_r$ cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x + y)^n = \begin{cases} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n\;.\;\frac{y}{x}+\frac{n\;(n-1)}{2\;!}\left(\frac{y}{x}\right)^2+.....\right\} & \text{if } \left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n\;.\;\frac{x}{y}+\frac{n\;(n-1)}{2\;!}\left(\frac{x}{y}\right)^2+.....\right\} & \text{if } \left|\frac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2).........(n-r+1)}{r\,!}\, x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ ($x^2$ and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + .......... + (-1)^r x^r + .........\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + .......... + x^r + .........\infty$

**Example # 15:** Prove that the coefficient of $x^r$ in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

**Solution:** $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r\,!}(-x)^r$$

$$= (-1)^r \frac{n(n+1)(n+2)......(n+r-1)}{r\,!}(-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r\,!}x^r$$

$$= \frac{(n-1)!\; n(n+1)......(n+r-1)}{(n-1)\,!\; r\,!}x^r$$

Hence, coefficient of $x^r$ is $\frac{(n+r-1)!}{(n-1)!\; r!} = {}^{n+r-1}C_r$ Proved

**Example # 16 :** If x is so small such that its square and higher powers may be neglected, then find the value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

**Solution :** $$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right) = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$

**Self practice problems :**

(12) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(13) If $y = \frac{2}{5} + \frac{1.3}{2\,!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3\,!}\left(\frac{2}{5}\right)^3 + ............$, then find the value of $y^2 + 2y$

(14) The coefficient of $x^{100}$ in $\frac{3-5x}{(1-x)^2}$ is

(1) 100 (2) –57 (3) –197 (4) 53

**Ans.** (12) $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ (13) 4 (14) (3)