Circle

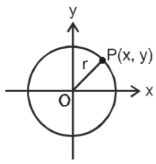
Four circles to the kissing come. The smaller are the benter. The bend is just the inverse of The distance from the centre. Through their intrigue left Euclid dumb There's now no need for rule of thumb. Since zero bend's a dead straight line And concave bends have minus sign. The sum of squares of all four bends Is half the square of their sum.

.... Soddy, Frederick

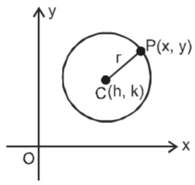
A circle is a locus of a point in a plane whose distance from a fixed point (called centre) is always constant (called radius).

1.. Equation of a circle in various forms :

(i) The circle with centre as origin & radius 'r' has the equation; $x^2 + y^2 = r^2$.



(ii) The circle with centre (h, k) & radius 'r' has the equation; $(x - h)^2 + (y - k)^2 = r^2$.



(iii) The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

with centre as (−g, −f) & radius = $\sqrt{g^2 + f^2 - c}$

Condition to define circle :-

$g^2 + f^2 - c > 0$ ⇒ real circle.

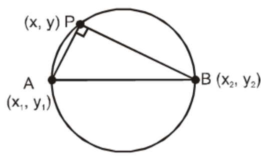
$g^2 + f^2 - c = 0$ ⇒ point circle.

$g^2 + f^2 - c < 0$ ⇒ imaginary circle, with real centre, that is (– g, – f)

Note : That every second degree equation in x & y, in which coefficient of x^2 is equal to coefficient of y^2 & the coefficient of xy is zero, always represents a circle.

(iv) The equation of circle with (x_1, y_1) & (x_2, y_2) as extremeties of its diameter is:

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.



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This is obtained by the fact that angle in a semicircle is a right angle.

∴ (Slope of PA) (Slope of PB) = – 1

⇒ $\frac{y - y_1}{x - x_1} . \frac{y - y_2}{x - x_2} = -1$ ⇒ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Note that this will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).

Example # 1 : Find the equation of the circle whose centre is (1, –2) and radius is 4.

Solution : The equation of the circle is $(x - 1)^2 + (y - (-2))^2 = 4^2$

$\Rightarrow$ $(x-1)^2 + (y+2)^2 = 16$ $\Rightarrow$ $x^2 + y^2 - 2x + 4y - 11 = 0$

Example # 2 : Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is (1, 1).

Solution : Let P be the point of intersection of the lines AB and LM whose equations are respectively

$3x - 2y - 1 = 0$ ....(i)

and $4x + y - 27 = 0$ ....(ii)

Solving (i) and (ii), we get x = 5, y = 7. So, coordinates of P are (5, 7). Let C(1, 1) be the centre of the circle. Since the circle passes through P, therefore

CP = radius = $\sqrt{(5-1)^2 + (7-1)^2} = \sqrt{16+36}$ $\Rightarrow$ radius = $\sqrt{52}$

Hence the equation of the required circle is $(x-1)^2 + (y-1)^2 = \left(\sqrt{52}\right)^2$

Example # 3 : Find the centre & radius of the circle whose equation is $x^2 + y^2 - 4x + 6y - 12 = 0$

Solution : Comparing it with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2g = -4$ $\Rightarrow$ $g = -2$

$2f = 6$ $\Rightarrow$ $f = 3$ & $c = -12$

$\therefore$centre is $(-g, -f)$ i.e. $(2, -3)$ and radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 + 12} = 5$

Example # 4: Find the equation of the circle, the coordinates of the end points of whose diameter are (–1, 2) and (4, –3)

Solution : We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1$, $x_2 = 4$, $y_1 = 2$ and $y_2 = -3$.

So, the equation of the required circle is

$(x+1)(x-4) + (y-2)(y+3) = 0$ $\Rightarrow$ $x^2 + y^2 - 3x + y - 10 = 0$.

Self practice problems :

(1) Find the equation of the circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ and whose centre is the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.

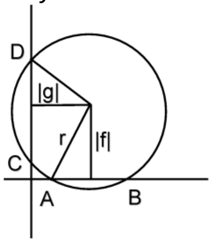
(2) Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6)

(3) Find the equation of a circle whose radius is 6 and the centre is at the origin.

Ans. (1) $x^2 + y^2 + 4x - 2y = 0$ (2) $x^2 + y^2 - 2x - 4y - 20 = 0$ (3) $x^2 + y^2 = 36$

2. Intercepts made by a circle on the axes :

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co−ordinate axes are $2\sqrt{g^2 - c}$ (on x-axis) & $2\sqrt{f^2 - c}$ (on y-axis) respectively.



If	$g^2 > c$	⇒	circle cuts the x-axis at two distinct points.
	$g^2 = c$	⇒	circle touches the x−axis.
	$g^2 < c$	⇒	circle lies completely above or below the x−axis.
	$f^2 > c$	⇒	circle cuts the y-axis at two distinct points.
	$f^2 = c$	⇒	circle touches the y−axis.
	$f^2 < c$	⇒	circle lies completely above or below the y-axis

Example # 5 : Find the equation of the circle touching the negative y-axis at a distance 5 from the origin and intercepting a length 8 on the x-axis.

Solution : Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it touches y-axis at (0, –5) and (0, – 3) lies on the circle.

∴ $c = f^2$ ...(i) $25 - 10f + f^2 = 0$ ....(ii)

From (i) and (ii), we get $25 - 10f + f^2 = 0 \Rightarrow (f - 5)^2 = 0 \Rightarrow f = 5$.

Putting f = 3 in (i) we obtain c = 9.

It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts length 8 on x-axis

∴ $2\sqrt{g^2 - c} = 8 \Rightarrow 2\sqrt{g^2 - 9} = 8 \Rightarrow g^2 - 9 = 16$

⇒ $g = \pm 5$

Hence, the required circle is $x^2 + y^2 \pm 10x + 10y + 9 = 0$.

Self practice problems :

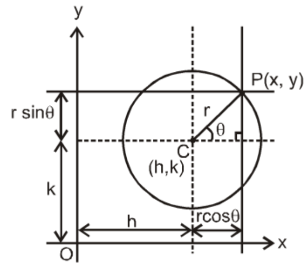
(4) Find the equation of a circle which touches the positive axis of y at a distance 3 from the origin and intercepts a distance 6 on the axis of x.

(5) Find the equation of a circle which touches positive y-axis at a distance of 2 units from the origin and cuts an intercept of 3 units with the positive direction of x-axis.

Ans. (4) $x^2 + y^2 \pm 6\sqrt{2}x - 6y + 9 = 0$ (5) $x^2 + y^2 - 5x - 4y + 4 = 0$

3. Parametric equations of a circle :

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are: $x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ where (h, k) is the centre, r is the radius & θ is a parameter.



Example # 6 : Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$

Solution : We have : $x^2 + y^2 - 4x - 2y + 1 = 0$

$\Rightarrow (x^2 - 4x) + (y^2 - 2y) = -1$

$\Rightarrow (x - 2)^2 + (y - 1)^2 = 2^2$

So, the parametric equations of this circle are

$x = 2 + 2\cos\theta,\ y = 1 + 2\sin\theta.$

Example # 7 : Find the equations of the following curves in cartesian form. Also, find the centre and radius of the circle $x = a + c\cos\theta,\ y = b + c\sin\theta$

Solution : We have : $x = a + c\cos\theta,\ y = b + c\sin\theta \Rightarrow \cos\theta = \frac{x-a}{c},\ \sin\theta = \frac{y-b}{c}$

$\Rightarrow \left(\frac{x-a}{c}\right)^2 + \left(\frac{y-b}{c}\right)^2 = \cos^2\theta + \sin^2\theta \Rightarrow (x-a)^2 + (y-b)^2 = c^2$

Clearly, it is a circle with centre at (a, b) and radius c.

Self practice problems :

(6) Find the parametric equations of circle $x^2 + y^2 - 6x + 4y - 12 = 0$

(7) Find the cartesian equations of the curve $x = -2 + 3\cos\theta,\ y = 3 + 3\sin\theta$

Ans. (6) $x = 3 + 5\cos\theta,\ y = -2 + 5\sin\theta$ (7) $(x + 2)^2 + (y - 3)^2 = 9$

4. Power and position of a point with respect to a circle :

Power of the point (x_1, y_1) with respect to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ will be

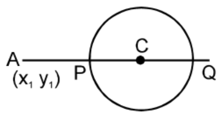
$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

(i) If $S_1 > 0 \Rightarrow$ Point will be outside the circle

(ii) If $S_1 = 0 \Rightarrow$ Point will be on the circle

(iii) If $S_1 < 0 \Rightarrow$ Point will be inside the circle

Note : The greatest & the least distance of a point A (lies outside the circle) from a circle with centre C & radius r is AC + r & AC − r respectively.



Example # 8 : Discuss the position of the points (1, 2) and (8, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

Solution : We have $x^2 + y^2 - 4x + 2y - 11 = 0$ or $S = 0$, where $S = x^2 + y^2 - 4x + 2y - 11$.

For the point (1, 2), we have $S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$

For the point (8, 0), we have $S_2 = 8^2 - 8 \times 4 - 11 > 0$

Hence, the point (1, 2) lies inside the circle and the point (8, 0) lies outside the circle.

Self practice problem :

(8) How are the points (0, 1) (3, 1) and (1, 3) situated with respect to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$?

Ans. (0, 1) lies on the circle ; (3, 1) lies outside the circle ; (1, 3) lies inside the circle.

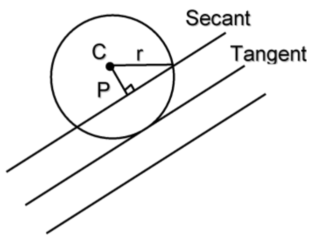
5. Line and a circle :

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.
(ii) $p = r \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)
(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.
(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then

(i) $c^2 < a^2 (1 + m^2) \Leftrightarrow$ the line is a secant of the circle.
(ii) $c^2 = a^2 (1 + m^2) \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)
(iii) $c^2 > a^2 (1 + m^2) \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.



These conditions can also be obtained by solving $y = mx + c$ with $x^2 + y^2 = a^2$ and making the discriminant of the quadratic greater than zero for secant, equal to zero for tangent and less the zero for the last case.

Example # 9 : For what value of c will the line $3y = 4x + c$ be a tangent to the circle $x^2 + y^2 = 5$?

Solution : We have : $3y = 4x + c$ or $4x - 3y + c = 0$ ......(i) and $x^2 + y^2 = 5$ ....(ii)

If the line (i) touches the circle (ii), then

length of the $\perp$ from the centre (0, 0) = radius of circle (ii)

$$\Rightarrow \left|\frac{3\times 0 - 4\times 0 + c}{\sqrt{4^2+3^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{c}{5}\right| = \sqrt{5} \Rightarrow \frac{c}{5} = \pm\sqrt{5} \Rightarrow c = \pm 5\sqrt{5}$$

Hence, the line (i) touches the circle (ii) for $c = \pm 5\sqrt{5}$

Self practice problem :

(9) For what value of λ, does the line $3x + 4y = \lambda$ touch the circle $x^2 + y^2 = 10x$.

Ans. 40, –10

6. Various forms of tangent :

(i) Slope form of tangent : $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Hence, equation of tangent is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$

(ii) Point form of tangent :

(a) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $x x_1 + y y_1 = a^2$.

(b) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is: $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$.

Note : In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$, xy by $\frac{x_1 y + x y_1}{2}$ and c remains as c.

(iii) **Parametric form of tangent :** The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$.

Note : The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Example # 10 : Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at (0, 0).

Solution : Equation of tangent is

$0 \times x + 0 \times y - \frac{30}{2}x + \frac{6}{2}y + 109 = 0 \Rightarrow -15x + 3y + 109 = 0$

Hence the required equation of tangent is $-15x + 3y + 109 = 0$

Example # 11 : Find the equation of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are perpendicular to the line $4x + 3y + 5 = 0$

Solution : Given circle is $x^2 + y^2 - 6x + 4y - 12 = 0$ ....(i)

and given line is $4x + 3y + 5 = 0$ ....(ii)

Centre of circle (i) is (3, –2) and its radius is 5. Equation of any line

$3x - 4y + k = 0$ perpendicular to the line (ii) ....(iii)

If line (iii) is tangent to circle (i) then

$\frac{|3.3 - 4.(-2) + k|}{5} = 5$ or $|17 + k| = 25$

or $17 + k = \pm 25$ $\therefore$ $k = 8, -42$

Hence equation of required tangents are $3x - 4y - 42 = 0$ and $3x - 4y + 8 = 0$

Self practice problem :

(10) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are

(i) parallel

(ii) perpendicular to the line $3x - 4y - 1 = 0$

Ans. (i) $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$

(ii) $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$

7. Normal :

If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is; $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$.

Example # 12 : Find the equation of the normal to the circle $x^2 + y^2 = 25$ at the point (4, 3).

Solution : Since normal is line joining centre (0,0) and (4, 3)

Slope = $\frac{4}{3}$

Hence, the equation of the normal at (4, 3) is

$y - 0 = (4/3)(x - 0) \Rightarrow 3y = 4x$

Self practice problem :

(11) Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3).

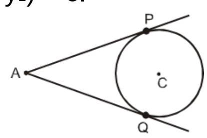
Ans $x - y + 1 = 0$

8. Pair of tangents from a point :

(i) **Joint equation :** The equation of a pair of tangents drawn from the point A (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

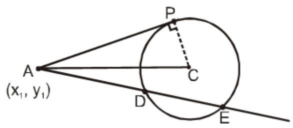


Also,

(a) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\frac{2RL}{L^2 - R^2}\right)$, where R and L are radius of the circle and length of tangent respectively.

(b) Equation of the circle circumscribing the triangle APQ is :
$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(ii) **Length of tangent (AP) :** The length of a tangent from an external point (x_1, y_1) to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$.



AP = length of tangent

$AP^2 = AD \cdot AE$

Example # 13 : Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1).

Solution : Given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ....(i)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5$

Clearly P lies outside the circle

and $T \equiv x \cdot 0 + y \cdot 1 - (x + 0) + 2(y + 1)$

i.e. $T \equiv -x + 3y + 2$.

Now equation of pair of tangents from P(0, 1) to circle (1) is $SS_1 = T^2$

or $5(x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ....(ii)

Separate equation of pair of tangents :

From (ii), $2x^2 + 3(y - 1)x - 2(2y^2 - 4y + 2) = 0$

$\therefore \quad x = \dfrac{-3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$

or $\quad 4x + 3y - 3 = \pm \sqrt{25y^2 - 50y + 25} = \pm 5(y - 1)$

$\therefore$ Separate equations of tangents are $2x - y + 1 = 0$ and $x + 2y - 2 = 0$

Example # 14 : Find the length of the tangent drawn from the point (1,2) to the circle $x^2 + y^2 + 6x - 4y - 2 = 0$

Solution : Given circle is $x^2 + y^2 + 6x - 4y - 2 = 0$ ....(i)

Given point is (1, 2). Let P = (1, 2)

Now length of the tangent from P(1, 2) to circle (i) = $\sqrt{1+4+6-8-2} = 1$

Self practice problems :

(12) Find the joint equation of the tangents through (7, 1) to the circle $x^2 + y^2 = 25$.

(13) Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

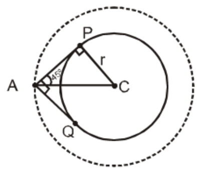
(14) If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, show that $15f^2 + 15g^2 - 64f + 4 = 0$

Ans. (12) $12x^2 - 12y^2 + 7xy - 175x - 25y + 625 = 0$

(13) 8 sq. units

9. Director circle :

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to times the original circle.



Proof :

AC = r cosec 45º = $\sqrt{2}$

Let C : $x^2 + y^2 = r^2$, equation of director circle will be $x^2 + y^2 = 2r^2$

Example # 15 : Find the equation of director circle of the circle $(x - 2)^2 + (y + 1)^2 = 8$.

Solution : Centre & radius of given circle are (2, –1) & $2\sqrt{2}$ respectively. Centre and radius of the director circle will be (2, –1) & $\sqrt{2} \times 2\sqrt{2} = 4$ respectively.

$\therefore$equation of director circle is $(x - 2)^2 + (y + 1)^2 = 16$

Self practice problem :

(15) Find the equation of director circle of the circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and area is 154 square units.

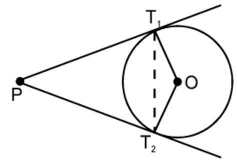
Ans. $(x + 3)^2 + (y - 2)^2 = 98$

10. Chord of contact :

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is:

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Notes : Here R = radius; L = length of tangent.



(i) Chord of contact exists only if the point 'P' is not inside.

(ii) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(iii) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$

Example # 16 : Find the equation of the chord of contact of the tangents drawn from (3, 0) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$

Solution : Given circle is $x^2 + y^2 - 2x + 4y + 7 = 0$ ....(i)

Let P = (3, 0)

For point P (3, 0), $x^2 + y^2 - 2x + 4y + 7 = 9 - 6 + 7 = 10 > 0$

Hence point P lies outside the circle

For point P (3, 0), $T = x \cdot 3 + y \cdot 0 - (x + 3) + 2y + 7$

i.e. $T = 2x + 2y + 4$

Now equation of the chord of contact of point P(3, 0) w.r.t. circle (i) will be

$x + y + 2 = 0$

Self practice problems :

(16) Find the co-ordinates of the point of intersection of tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$

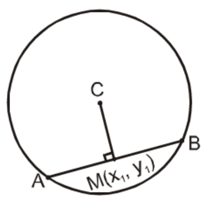
(17) Find the area of the triangle formed by the tangents drawn from the point (4, 6) to the circle $x^2 + y^2 = 25$ and their chord of contact.

(18) Find the equation of chord of contact of the circle $x^2 + y^2 - 4x + 3y - 1 = 0$ with respect to the point (1, –2)

Ans. (16) (1, – 2) (17) $\dfrac{405\sqrt{3}}{52}$ sq. unit; $4x + 6y - 25 = 0$ (18) $2x + y + 12 = 0$

11. Equation of the chord with a given middle point :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.



Notes :

(i) The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

(ii) The chord passing through a point ' M ' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

Example # 17: Find the equation of the shortest chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, which passes through point (1, –1)

Solution: Equation of given circle is $S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$

Let $L \equiv (1, -1)$

For point L(1, –1), $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$ and

$T \equiv x.1 + y(-1) + 3(x + 1) + 4(y - 1) - 11$ i.e. $T \equiv 4x + 3y - 12$

Now equation of the chord of circle (i) whose middle point is L(1, –1) is

$T = S_1$ or $4x + 3y - 12 = -11$ or $4x + 3y - 1 = 0$

Second Method: Let C be the centre of the given circle, then $C \equiv (-3, -4)$. $L \equiv (1, -1)$ slope of $CL = \frac{-4+1}{-3-1} = \frac{3}{4}$

$\therefore$Equation of chord of circle whose middle point is L, is $y + 1 = -\frac{4}{3}(x-1)$

($\because$ chord is perpendicular to CL) or $4x + 3y - 1 = 0$

Self practice problems :

(19) Find the equation of that chord of the circle $x^2 + y^2 = 15$, which is bisected at (3, 2)

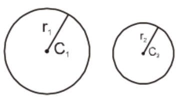
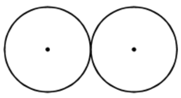
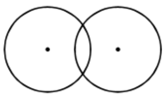
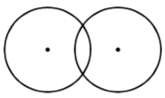
(20) Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $y = x + 2$.

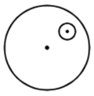
Ans. (19) $3x + 2y - 13 = 0$ (20) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

12. Equation of the chord joining two points of circle :

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points P(α) and Q(β) on it is given by the equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is $x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}$.

13. Common tangents to two circles :

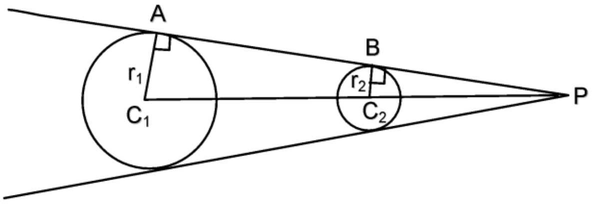
	Case	Number of Tangents	Condition
(i)		4 common tangents (2 direct and 2 transverse)	$r_1 + r_2 < C_1C_2$.
(ii)		3 common tangents.	$r_1 + r_2 = C_1C_2$.
(iii)		2 common tangents.	$\lvert r_1 - r_2\rvert < C_1C_2 < r_1 + r_2$
(iv)		1 common tangent.	$\lvert r_1 - r_2\rvert = C_1C_2$.

(v)  No common tangent. $C_1 C_2 < |r_1 - r_2|$.

(Here C_1C_2 is distance between centres of two circles.)

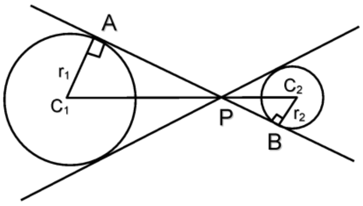
Notes:

(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.



$$\frac{C_1P}{PC_2} = \frac{r_1}{r_2}$$

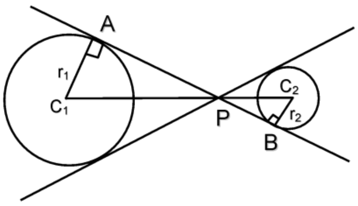
(ii) Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.



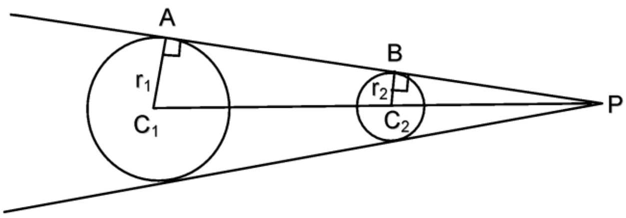
$$\frac{C_1P}{PC_2} = \frac{r_1}{r_2}$$

(iii) Length of an internal (or transverse) common tangent to the two circles are given by

$L_{ext}(AB) = \sqrt{d^2 - (r_1 - r_2)^2}$



(iv) Length of an external (or direct) common tangent $L_{int}(AB) = \sqrt{d^2 - (r_1 + r_2)^2}$,where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.



Example # 18 : Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Solution : Given circles are $x^2 + y^2 - 2x - 4y = 0$ ....(i)

and $x^2 + y^2 - 8y - 4 = 0$ ....(ii)

Let A and B be the centres and r_1 and r_2 the radii of circles (i) and (ii) respectively, then

$A \equiv (1, 2)$, $B \equiv (0, 4)$, $r_1 = \sqrt{5}$, $r_2 = 2\sqrt{5}$

Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and $r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = \sqrt{5}$

Thus $AB = |r_1 - r_2|$, hence the two circles touch each other internally.

Self practice problem :

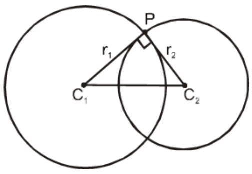
(21) Find the position of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ with respect to each other.

Ans. (21) One circle lies completely outside the other circle.

14. Orthogonality of two circles :

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Proof :



$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2$

$\Rightarrow$ $(g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$ $\Rightarrow$ $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Example # 19 : Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Solution : Given circles are $x^2 + y^2 + 3x - 5y + 6 = 0$ ....(i)

and $4x^2 + 4y^2 - 28x + 29 = 0$

or $x^2 + y^2 - 7x + \frac{29}{4} = 0.$ ....(ii)

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(iii)

Since circle (iii) cuts circles (i) and (ii) orthogonally

$\therefore$ $2g\left(\frac{3}{2}\right) + 2f\left(-\frac{5}{2}\right) = c + 6$ or $3g - 5f = c + 6$ ....(iv)

and $2g\left(-\frac{7}{2}\right) + 2f.0 = c + \frac{29}{4}$ or $-7g = c + \frac{29}{4}$ ....(v)

From (iv) & (v), we get $10g - 5f = -\frac{5}{4}$

or $\quad 40g - 20f = -5.$ ....(vi)

Given line is $3x + 4y = -1$ ....(vii)

Since centre $(-g, -f)$ of circle (iii) lies on line (vii),

$\therefore \quad -3g - 4f = -1$ ....(viii)

Solving (vi) & (viii), we get $g = 0$, $f = \frac{1}{4}$

$\therefore$ from (5), $c = -\frac{29}{4}$ $\quad \therefore$ from (iii), required circle is

$$x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0 \quad \text{or} \quad 4(x^2 + y^2) + 2y - 29 = 0$$

Self practice problems :

(22) For what value of k the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$ cut orthogonally.

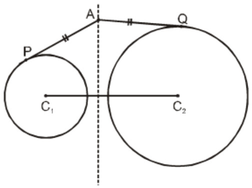
(23) Find the equation to the circle which passes through the origin and has its centre on the line $x + y + 4 = 0$ and cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

Ans. (22) -18 (23) $3x^2 + 3y^2 + 4x + 20y = 0$

15. <u>Radical axis and radical centre</u> :

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given by

$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.$



The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

Notes : (i) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(ii) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

(iii) If two circles intersect, then the radical axis is the common chord of the two circles.

(iv) If two circles touch each other, then the radical axis is the common tangent of the two circles at the common point of contact.

(v) Radical axis is always perpendicular to the line joining the centres of the two circles.

(vi) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(vii) Radical axis bisects a common tangent between the two circles.

(viii) Pairs of circles which do not have radical axis are concentric.

Example # 20: Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$x^2 + y^2 = 1$

$x^2 + y^2 - 8x + 15 = 0$

$x^2 + y^2 + 10y + 24 = 0$

Solution : Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of x^2 and y^2 be each unity. Subtracting in pairs the three radical axes are

$x - 2 = 0$

$8x + 10y + 9 = 0$

we get the point $\left(2, -\frac{5}{2}\right)$ which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

Example # 21 : Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents.

Solution : Given circles are $S_1 \equiv x^2 + y^2 - 12 = 0$ .... (i)

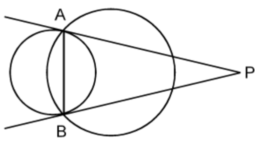
and $S_2 = x^2 + y^2 - 5x + 3y - 2 = 0$ .... (ii)

Now equation of common chord of circle (i) and (ii) is

$S_1 - S_2 = 0$ i.e. $5x - 3y - 10 = 0$ .... (iii)

Let this line meet circle (i) [or (ii)] at A and B

Let the tangents to circle (i) at A and B meet at P(α, β), then AB will be the chord of contact of the tangents to the circle (i) from P, therefore equation of AB will be



$x\alpha + y\beta - 12 = 0$ .... (iv)

Now lines (iii) and (iv) are same, therefore, equations (iii) and (iv) are identical

$\therefore \quad \frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10} \qquad \therefore \quad \alpha = 6, \beta = -\frac{18}{5}$

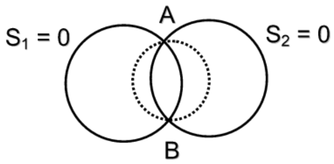
Hence P = $\left(6, -\frac{18}{5}\right)$

Self practice problem :

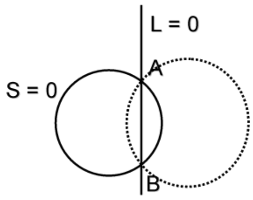
(24) Find the point from which the tangents to the three circles $x^2 + y^2 - 4x + 7 = 0$, $2x^2 + 2y^2 - 3x + 5y + 9 = 0$ and $x^2 + y^2 + y = 0$ are equal in length. Find also this length.

Ans. (24) (2, – 1) ; 2.

16. Family of Circles :

(i) 

Any circle through A and B can be considered as $S_1 + K S_2 = 0$
($K \neq -1$, provided the co−efficient of x^2 & y^2 in S_1 & S_2 are same)

(ii) 

Any circle through A and B can be considered as $S + K L = 0$

(iii) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K = 0, \text{ where } K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \text{ is a parameter.}$$

(iv) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K(y - y_1 - m(x - x_1)) = 0$, where K is a parameter.

(v) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided co−efficient of $xy = 0$ and co−efficient of x^2 = co−efficient of y^2.

(vi) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $u L_1L_3 + \lambda L_2L_4 = 0$ where values of u & λ can be found out by using condition that co−efficient of x^2 = co−efficient of y^2 and co−efficient of $xy = 0$.

Example # 22 : Find the equations of the circles passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose centre's absica is 3

Solution : Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$

or $(x^2 + y^2 - 2x - 4y - 4) + \lambda(x^2 + y^2 - 10x - 12y + 40) = 0$

or $(x^2 + y^2) - 2\dfrac{(1+5\lambda)}{1+\lambda}x - 2\dfrac{(2+6\lambda)}{1+\lambda}y + \dfrac{40\lambda - 4}{1+\lambda} = 0$ ....(i)

$\dfrac{1+5\lambda}{1+\lambda} = 3 \Rightarrow 1 + 5\lambda = 3 + 3\lambda \Rightarrow 2\lambda = 2$; $\lambda = 1$

so the circle is $x^2 + y^2 - 6x - 8y + 18 = 0$

Example # 23: Find the equations of circles which touches $2x - y + 3 = 0$ and pass through the points of intersection of the line $x + 2y - 1 = 0$ and the circle $x^2 + y^2 - 2x + 1 = 0$.

Solution : The required circle by $S + \lambda P = 0$ is

$x^2 + y^2 - 2x + 1 + \lambda (x + 2y - 1) = 0$ or $x^2 + y^2 - x (2 - \lambda) + 2\lambda y + (1 - \lambda) = 0$

centre $(-g, -f)$ is $\left(\frac{2-\lambda}{2}, -\lambda\right)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{(2-\lambda)^2}{4} + \lambda^2 - (1-\lambda)} = \frac{1}{2}\sqrt{5\lambda^2} = \frac{\sqrt{5}}{2}|\lambda|$.

Since the circle touches the line $2x - y + 3 = 0$ therefore perpendicular from centre is equal to radius $\left|\frac{2.((2-\lambda)/2) - (-\lambda) + 3}{\sqrt{5}}\right| = \frac{|\lambda|}{2}\sqrt{5}$ $\therefore \lambda = \pm 2$

Putting the values of λ in (i) the required circles are

$x^2 + y^2 + 4y - 1 = 0$

$x^2 + y^2 - 4x - 4y + 3 = 0$.

Example # 24 : Find the equation of circle passing through the points A(1, 1) & B(2, 2) and whose centre's absieas is 5.

Solution : Equation of AB is $x - y = 0$

$\therefore$ equation of circle is

$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$

or $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

$\frac{-(\lambda - 3)}{2} = 5 \Rightarrow -\lambda + 3 = 10 \Rightarrow \lambda = -7$

so the equation of circle is $x^2 + y^2 - 10x + 4y + 4 = 0$

Example # 25 : Find the equation of the circle passing through the point (1, –2) and touching the line $x + y = 0$ at the point (0, 0).

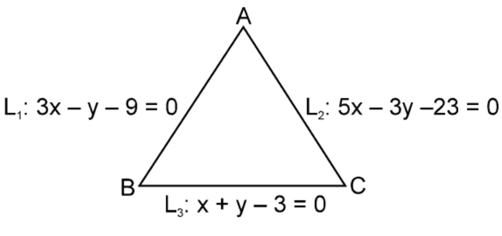
Solution : Equation of circle is $x^2 + y^2 + \lambda(x + y) = 0$

Since it passes through the point (1, –2), $1 + 4 + \lambda(1 - 2) = 0 \Rightarrow \lambda = 5$

$\therefore$circle is $x^2 + y^2 + 5x + 5y = 0$

Example # 26 : Find the equation of circle circumcscribing the triangle whose sides are

$3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$.



Solution :

$L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$(3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$(15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69)$
$+ \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$(5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu) + y(50 - 14\lambda - 6\mu)$
$+ (207 + 69\lambda + 27\mu) = 0$ ....(i)

coefficient of x^2 = coefficient of y^2

$\Rightarrow \quad 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$

$2\lambda + \mu + 3 = 0$ ....(ii)

coefficient of xy = 0

$\Rightarrow \quad \lambda + \mu - 7 = 0$ ....(iii)

Solving (ii) and (iii), we have

$\lambda = -10, \mu = 17$

Putting these values of λ & μ in equation (i), we get $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

Self practice problems :

(25) Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line y = x.

(26) Find the equation of circle circumscribing the quadrilateral whose sides are $5x + 3y = 9x = 3y$, $2x = y$ and $x + 4y + 2 = 0$.

Ans. (25) $7x^2 + 7y^2 - 10x - 10y - 12 = 0$ (26) $9x^2 + 9y^2 - 20x + 15y = 0$