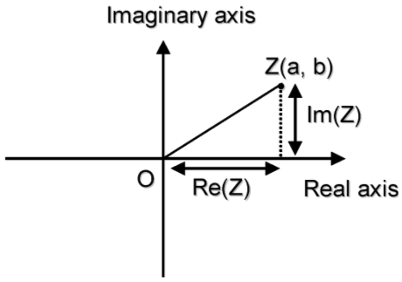
Complex Numbers

The shortest path between two truths in the real domain passes through the complex domain. ......Hadamard, Jacques

1. The complex number system :



Argand plane

Complex number is a point on Argand plane. It is represented by Z = a + i b, where i = $\sqrt{-1}$
'a' is called as real part of z which is denoted by (Re z) and 'b' is called as imaginary part of z, which is denoted by (Im z).
Any complex number is :

(i) Purely real, if b = 0

(ii) Imaginary, if b ≠ 0

(iii) Purely imaginary, if a = 0

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) i = $\sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non - negative.

(e) If z = a + ib, then a – ib is called complex conjugate of z and written as $\bar{z} = a - ib$

(f) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e. i > 0, 3 + i < 2 are meaningless.

Self Practice Problems :

(1) Write the following as complex number

(i) $\sqrt{-16}$ (ii) $\sqrt{x}$, (x > 0) (iii) $-b + \sqrt{-4ac}$, (a, c> 0)

(2) Write the following as complex number

(i) $\sqrt{x}$ (x < 0) (ii) roots of $x^2 - (2\cos\theta)x + 1 = 0$

Ans. (1) (i) 0 + 4i (ii) $\sqrt{x}$ + 0i (iii) $-b + i\sqrt{4ac}$

(2) (i) $0 + i\sqrt{-x}$ (ii) cos θ + i sin θ , cos θ – i sin θ

2. Algebraic operations :

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by – 1 when it occurs.

(i) Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)\, i$

(ii) Subtraction $(a + bi) – (c + di) = a + bi – c – di = (a – c) + (b – d)\, i$

(iii) Multiplication $(a + bi)\,(c + di) = ac + adi + bci + bdi^2 = (ac – bd) + (ad+ bc)i$

(iv) Division $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di} = \dfrac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$

$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless. In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Example # 1 : Find multiplicative inverse of 3 + 2i.

Solution : Let z be the multiplicative inverse of 3 + 2i. then

$\Rightarrow \quad z\,.\,(3 + 2i) = 1 \qquad \Rightarrow \quad z = \dfrac{1}{3+2i} = \dfrac{3-2i}{(3+2i)(3-2i)}$

$\Rightarrow \quad z = \dfrac{3}{13} - \dfrac{2}{13}\, i \qquad \Rightarrow \quad \left(\dfrac{3}{13} - \dfrac{2}{13}i\right)$

Self Practice Problem :

(3) Simplify $i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$, $n \in I$.

Ans. 0

3. Equality in complex number :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2 \quad \Leftrightarrow \quad \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

Example # 2 : Find the value of x and y for which $(2 + 3i)\, x^2 – (3 – 2i)\, y = 2x – 3y + 5i$ where $x, y \in R$.

Solution $(2 + 3i)x^2 – (3 – 2i)y = 2x – 3y + 5i$

$\Rightarrow \quad 2x^2 – 3y = 2x – 3y \qquad \Rightarrow \quad x^2 – x = 0$

$\Rightarrow \quad x = 0, 1 \qquad$ and $\qquad 3x^2 + 2y = 5$

$\Rightarrow \quad$ if $x = 0$, $y = \dfrac{5}{2}$ $\qquad$ and $\qquad$ if $x = 1$, $y = 1$

$\therefore \quad x = 0$, $y = \dfrac{5}{2}$ $\qquad$ and $\qquad x = 1$, $y = 1$

are two solutions of the given equation which can also be represented as $\left(0, \frac{5}{2}\right)$ & (1, 1)

Example # 3 : Find square root of 5 + 12i

Solution : Let $x + iy = \sqrt{5+12i}$ $\Rightarrow$ $5 + 12i = (x + iy)^2$

$5 + 12i = (x^2 - y^2) + 2ixy$ $\Rightarrow$ $x^2 - y^2 = 5$ .... (i)

and $2xy = 12$ . ....(ii)

Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$(x^2 + y^2)^2 = 5^2 + 12^2 = 169$

$x^2 + y^2 = 13$ ($\because$ $x^2 + y^2 > 0$) ....(iii)

Solving (i) & (iii), we get

$x^2 = 9$ and $y^2 = 4$ $\Rightarrow$ $x = \pm 3$ and $y = \pm 2$

from (ii), 2xy is positive, so x and y are of same sign

x = 3, y = 2 or x = –3 and y = – 2

hence, $\sqrt{5+12i} = \pm (3 + 2i)$

Example # 4 : Solve for z : $z^2 - (3 - 2i)z = 5i - 5$

Solution : $z^2 - (3 - 2i)z = 5i - 5$ $\Rightarrow$ $\dfrac{(3-2i) \pm \sqrt{(3-2i)^2 + 4(5i-5)}}{2}$

$z = \dfrac{(3-2i) \pm \sqrt{9-4-12i+20i-20}}{2} = \dfrac{(3-2i) \pm \sqrt{8i-15}}{2}$

Now, $\sqrt{-15+8i} = \pm \left\{\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}+(-15)\right\}}\right\} \pm i \left\{\sqrt{\frac{1}{2}\left\{\sqrt{(-15)^2+8^2}-(-15)\right\}}\right\} = \pm (1+4i)$

$z = \dfrac{3-2i \pm (1-4i)}{2}$ $\Rightarrow$ z = (2 + i) and (1 – 3i)

Self Practice Problem :

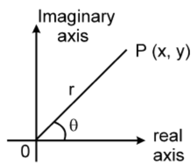
(4) Given that x, y ∈ R, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

Ans. (4) $x = K, y = \dfrac{3K}{2}$ $K \in R$

4. <u>Representation of a complex number</u> :

(i) Cartesian Form (Geometric Representation) :

Every complex number z = x + i y can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called the argument or amplitude.

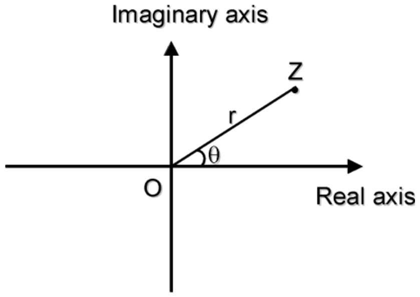
$|z| = \sqrt{x^2 + y^2}$ and $\tan \theta = \left(\dfrac{y}{x}\right)$ (angle made by OP with positive x–axis)

Note : (a) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta$; n ∈ I will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(b) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(c) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is only given by its modulus.

(ii) Trignometric/Polar Representation :



$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ

(iii) Euler's Formula :

$z = re^{i\theta}$, $|z| = r$, $\arg z = \theta$ $\Rightarrow$ $\bar{z} = re^{-i\theta}$

Proof of this formula is beyond scope of present discussion. A heuristic proof serving as motivation for this formula is by considering expansion.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

put $x = i\theta \Rightarrow e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots\right) = \cos\theta + i\sin\theta.$

Note: If θ is real then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\Rightarrow$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(iv) Vectorial Representation :

Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

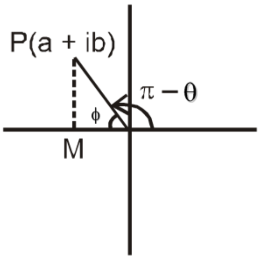
5. <u>Agrument of a complex number</u> :

Argument of a non-zero complex number P(z) is denoted and defined by arg(z) = angle which OP makes with the positive direction of real axis.

If $OP = |z| = r$ and $\arg(z) = \theta$, then obviously $z = r(\cos\theta + i\sin\theta)$, called the polar form of z. 'Argument of z' would mean principal argument of z(i.e. argument lying in $(-\pi, \pi]$ unless the context requires otherwise. Thus argument of a complex number $z = a + ib = r(\cos\theta + i\sin\theta)$ is the value of θ satisfying $r\cos\theta = a$ and $r\sin\theta = b$.

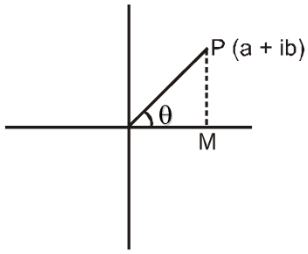
Let $\theta = \tan^{-1}\left|\frac{b}{a}\right|$

Second quadrant $a < 0, b > 0$



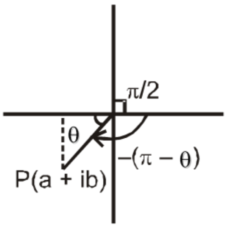
p.v. $\arg z = \pi - \theta = \pi - \tan^{-1}\left|\frac{b}{a}\right|$

First quadrant $a > 0, b > 0$



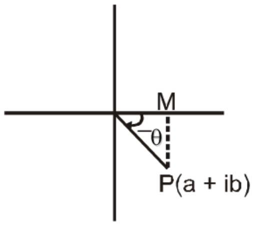
p.v. $\arg z = \theta = \tan^{-1}\left|\frac{b}{a}\right|$

Third quadrant $a < 0, b < 0$



p.v. $\arg z = -(\pi - \theta) = -\pi + \tan^{-1}\left|\frac{b}{a}\right|$

Fourth quadrant $a > 0, b < 0$



p.v. $\arg z = -\theta = -\tan^{-1}\left|\frac{b}{a}\right|$

Note : If $a > 0, b = 0$ then p.v. $\arg z = 0$

If $a < 0, b = 0$ then p.v. $\arg z = \pi$

If $a = 0, b > 0$ then p.v. $\arg z = \pi/2$

If $a = 0, b < 0$ then p.v. $\arg z = -\pi/2$

Example # 5 : Express the complex number $z = -1 + \sqrt{2}\,i$ in polar form.

Solution : $z = -1 + i\sqrt{2}$

$|z| = \sqrt{(-1)^2 + \left(\sqrt{2}\right)^2} = \sqrt{1+2} = \sqrt{3}$,

$\text{Arg } z = \pi - \tan^{-1}\left(\frac{\sqrt{2}}{1}\right) = \pi - \tan^{-1}(\sqrt{2}) = \theta$ (say)

$\therefore z = \sqrt{3}\,(\cos\theta + i\sin\theta)$ where $\theta = \pi - \tan^{-1}\sqrt{2}$

Self Practice Problems :

(5) Find the principal argument and $|z|$. If $z = \frac{-i(9+i)}{2-i}$

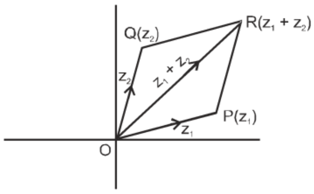
(6) Find the $|z|$ and principal argument of the complex number $z = 6(\cos 310^{\circ} - i\sin 310^{\circ})$

Ans. (5) $-\tan^{-1}\frac{17}{11}$, $\sqrt{\frac{82}{5}}$ (6) 6, 50°

6. Geometrical representation of fundamental operations :

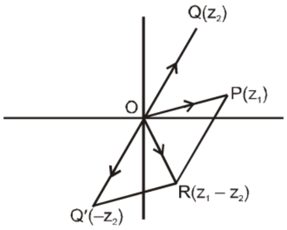
(i) Geometrical representation of addition :

Complex Numbers



If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

(ii) Geometric representation of substraction :



(iii) Modulus and argument of multiplication of two complex numbers :

Theorem : For any two complex numbers z_1, z_2 we have $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Proof : $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ $\Rightarrow$ $z_1z_2 = r_2r_2\, e^{i(\theta_1+\theta_2)}$

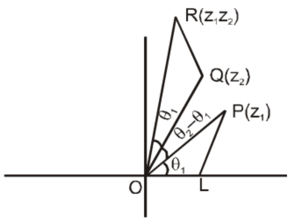
$\Rightarrow$ $|z_1z_2| = |z_1|\,|z_2|$ $\Rightarrow$ $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

i.e. to multiply two complex numbers, we multiply their absolute values and add their arguments.

Note :
(a) P.V. $\arg(z_1z_2) \neq$ P.V. $\arg(z_1)$ + P.V. $\arg(z_2)$
(b) $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$
(c) $\arg(z_1z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots + \arg z_n$

(iv) Geometrical representation of multiplication of complex numbers :

Let P, Q be represented by $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ repectively. To find point R representing complex number z_1z_2, we take a point L on real axis such that OL = 1 and draw triangle OQR similar to triangle OLP. Therefore



$\frac{OR}{OP} = \frac{OQ}{OL}$ $\Rightarrow$ OR = OP.OQ i.e. OR = r_1r_2 and $Q\hat{O}R = \theta_1$

$L\hat{O}R = L\hat{O}P + P\hat{O}Q + Q\hat{O}R = \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$

Hence, R is represented by $z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)}$

(v) Modulus and argument of division of two complex numbers :

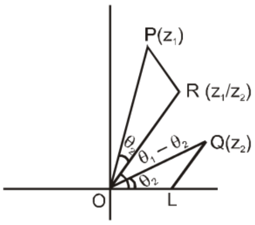
Theorem : If z_1 and z_2 ($\neq 0$) are two complex numbers, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note : P.V. arg $\left(\frac{z_1}{z_2}\right) \neq$ P.V. arg (z_1) – P.V. arg (z_2)

(vi) Geometrical representation of the division of complex numbers.

Let P, Q be represented by $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$ respectively. To find point R representing complex number $\frac{z_1}{z_2}$, we take a point L on real axis such that OL = 1 and draw a triangle OPR similar to OQL. Therefore $\frac{OP}{OQ} = \frac{OR}{OL}$ $\Rightarrow$ OR = $\frac{r_1}{r_2}$ and $L\hat{O}R = L\hat{O}P - R\hat{O}P = \theta_1 - \theta_2$



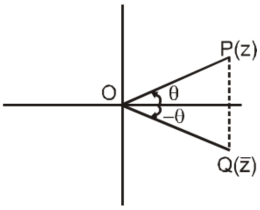
Hence, R is represented by $\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$

7. Conjugate of a complex number :

Conjugate of a complex number z = a + ib is denoted and defined by $\bar{z}$ = a – ib.

In a complex number if we replace i by – i, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.

Geometrical representation of conjugate of complex number.



$|z| = |\bar{z}|$

arg $\bar{z}$ = – arg (z)

General value of arg $\bar{z}$ = 2nπ – P.V. arg (z)

Properties :

(i) If z = x + iy, then x = $\frac{z+\bar{z}}{2}$, y = $\frac{z-\bar{z}}{2i}$

(ii) $z = \bar{z}$ $\Leftrightarrow$ z is purely real

(iii) $z + \bar{z} = 0$ $\Leftrightarrow$ z is purely imaginary

(iv) Relation between modulus and conjugate. $|z|^2 = z\bar{z}$

(v) $\bar{\bar{z}} = z$

(vi) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{(z_1\, z_2)} = \bar{z}_1\, \bar{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ ($z_2 \neq 0$)

Notes :

(a) Theorem : Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs

Proof : If z_0 is a root of $a_0z^n + a_1z^{n-1} + ... + a_{n-1}\, z + a_n = 0$,

$a_0, a_1, ... a_n \in R$, then $a_0z_0^n + a_1z_0^{n-1} + ... + a_{n-1}\, z_0 + a_n = 0$

By using property (vi) and (vii) we have $a_0\bar{z}_0^n + a_1\bar{z}_0^{n-1} + ... + a_{n-1}\bar{z}_0 + a_n = 0$

$\Rightarrow$ $\bar{z}_0$ is also a root. (If $w = f(z)$, then $\bar{w} = f(\bar{z})$)

(b) Theorem : $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + z_2\bar{z}_1)$

$= |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 \pm 2\,|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

Example # 6 : If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$

Solution : $\text{Re}\left(\dfrac{z-1}{z+1}\right) = 0$ $\Rightarrow$ $\dfrac{z-1}{z+1} + \overline{\left(\dfrac{z-1}{z+1}\right)} = 0$

$\Rightarrow$ $\dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1} = 0$ $\Rightarrow$ $z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0$

$\Rightarrow$ $z\bar{z} = 1$ $\Rightarrow$ $|z|^2 = 1$ $\Rightarrow$ $|z| = 1$

Self Practice Problems :

(7) Solve for z : $\bar{z} = i z^2$

(8) If $\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodulus and z_2 is not unimodulus then find $|z_1|$.

(9) If $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, then show that $f(z) = \bar{z}^2 + 2iz$

(10) If $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$ prove that $(x^2 + y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$

Ans. (7) $\pm\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i, 0, i$

(8) $|z_1| = 2$

8. <u>Distance, triangular inequality</u> :

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

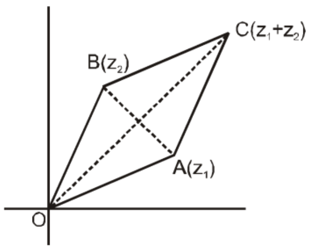
In triangle OAC

OC ≤ OA + AC

OA ≤ AC + OC

AC ≤ OA + OC

using these in equalities we have $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$



Similarly from triangle OAB

we have $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$

Note :

(i) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2 .

(ii) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

Example # 7 : If $|z_1 - 1| \le 1$, $|z_2 - 2| \le 2$, $|z_3 - 3| \le 3$, then find the greatest value of $|z_1 + z_2 + z_3|$

Solution : $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$\le |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

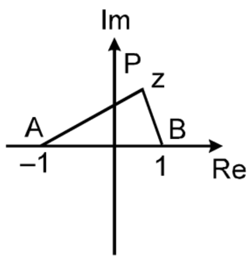
$\le 1 + 2 + 3 + 6 = 12$

hence greatest value is 12

Example # 8 : Find the minimum value of $|1 + z| + |1 - z|$.

Solution : $|1 + z| + |1 - z| \ge |1 + z + 1 - z|$ (triangle inequality)

$\Rightarrow$ $|1 + z| + |1 - z| \ge 2$



$\therefore$minimum value of $(|1 + z| + |1 - z|) = 2$

Geometrically $|z + 1| + |1 - z| = |z + 1| + |z - 1|$

which represents sum of distances of z from 1 and – 1

it can be seen easily that minimum (PA + PB) = AB = 2

Example # 9 : Find the greatest and least value of $|z_1 + z_2|$, if $z_1 = 24 + 7i$ and $|z_2| = 6$

Solution : $|z_1 + z_2| \le |z_1| + |z_2| = |24 + 7i| + 6 = 25 + 6 = 31$

also, $|z_1 + z_2| = |z_1 - (-z_2)| \le ||z_1| - |z_2||$

$|z_1 + z_2| \ge (25 - 6) = 19$

hence the least value of $|z_1 + z_2|$ is 19 and greatest value is 31.

Self Practice Problems :

(11) $|z - 3| < 1$ and $|z - 4i| > M$ then find the positive real value of M for which these exist at least one complex number z satisfy both the equation.

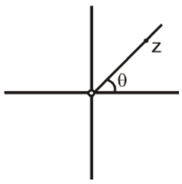
(12) If z lies on circle $|z| = 2$, then show that $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \le \dfrac{1}{3}$

Ans. (11) $M \in (0, 6)$

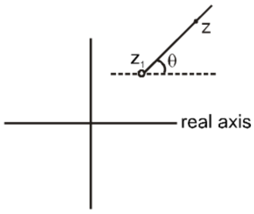
9. Rotation :

(i) Important results :

(a) arg $z = \theta$ represents points (non-zero) on ray eminating from origin making an angle θ with positive direction of real axis

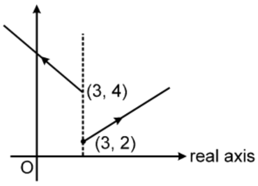


(b) arg $(z - z_1) = \theta$ represents points ($\neq z_1$) on ray eminating from z_1 making an angle θ with positive direction of real axis



Example # 10 : Solve for z, which satisfy Arg (z – 3 – 2i) = $\frac{\pi}{6}$ and Arg (z – 3 – 4i) = $\frac{2\pi}{3}$.

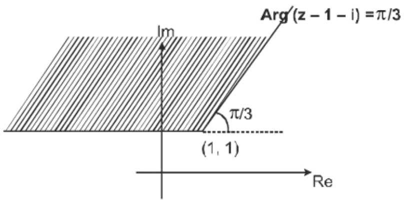
Solution : From the figure, it is clear that there is no z, which satisfy both ray



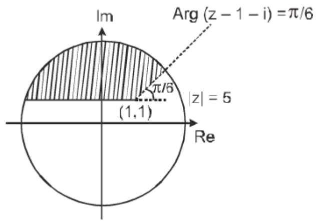
Example # 11 : Sketch the region given by

(i) Arg (z – 1 – i) ≥ π/3 (ii) |z| ≤ 5 & Arg (z – i – 1) > π/6

Solution : (i)



(ii)



Self Practice Problems :

(13) Sketch the region given by

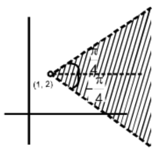
(i) |Arg (z – i – 2)| < π/4 (ii) Arg (z + 1 – i) ≤ π/6

(14) Consider the region |z – 15i| ≤ 10. Find the point in the region which has

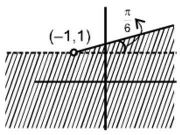
(i) max |z| (ii) min |z|

(iii) max arg (z) (iv) min arg (z)

Ans. (13) (i)



(ii)

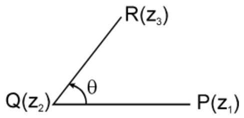


(14) (i) 25i (ii) 5 i (iii) $\pi - \tan^{-1}\frac{\sqrt{5}}{2}$ (iv) $\tan^{-1}\frac{\sqrt{5}}{2}$

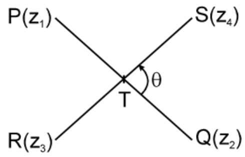
10. Rotation theorem :

(i) If P(z_1) and Q(z_2) are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1 e^{i\theta}$ where θ = ∠POQ

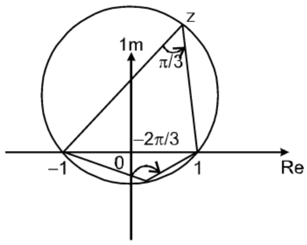
(ii) If P(z_1), Q(z_2) and R(z_3) are three complex numbers and ∠PQR = θ, then $\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$



(iii) If P(z_1), Q(z_2), R(z_3) and S(z_4) are four complex numbers and ∠STQ = θ, then $\frac{z_3 - z_4}{z_1 - z_2}$ $= \left|\frac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$



Example # 12 : If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ then interpret the locus.



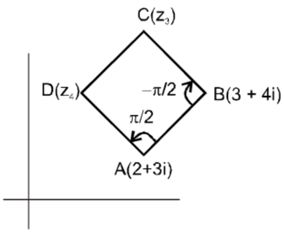
Solution : $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$

$\Rightarrow \arg\left(\frac{1-z}{-1-z}\right) = \frac{\pi}{3}$

Here $\arg\left(\frac{1-z}{-1-z}\right)$ represents the angle between lines joining –1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

Example # 13 : If A(2 + 3i) and B(3 + 4i) are two vertices of a square ABCD (taken in anticlock wise order) then find C and D.

Solution : Let affix of C and D are z_3 and z_4 respectively.
Considering ∠DAB = 90º and AD = AB



we get $\frac{z_4 - (2+3i)}{(3+4i)-(2+3i)} = \frac{AD}{AB} e^{\frac{i\pi}{2}}$

$\Rightarrow$ $z_4 - (2 + 3i) = (1 + i)\, i$

$\Rightarrow$ $z_4 = 2 + 3i + i - 1 = 1 + 4i$

and $\frac{z_3 - (3+4i)}{(2+3i)-(3+4i)} = \frac{CB}{AB} e^{-\frac{i\pi}{2}}$ $\Rightarrow$ $z_3 = 3 + 4i - (1 + i)(-i) \Rightarrow z_3 = 3 + 4i + i - 1 = 2 + 5i$

Self Practice Problems :

(15) z_1, z_2, z_3, z_4 are the vertices of a square taken in anticlockwise order then prove that
$2z_2 = (1 + i)\, z_1 + (1 - i)\, z_3$

(16) Check that z_1z_2 and z_3z_4 are parallel or not,
where $z_1 = 1 + i$ $z_3 = 4 + 2i$
$z_2 = 2 - i$ $z_4 = 1 - i$

(17) P is a point on the argand diagram on the circle with OP as diameter, two point Q and R are taken such that ∠POQ = ∠QOR = θ. If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$

(18) If a, b, c ; u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r) a + rb$, $w = (1 - r) u + rv$ where r is a complex number show that the two triangles are similiar.

Ans. (16) z_1z_2 and z_3z_4 are not parallel.

11. Demoivre's theorem :

Case I

Statement : If n is any integer then

(i) $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

(ii) $(\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2)(\cos\theta_3 + i \sin\theta_3)(\cos\theta_4 + i \sin\theta_4) \ldots (\cos\theta_n + i \sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$

Case II

Statement : If $p, q \in Z$ and $q \neq 0$ then $(\cos\theta + i \sin\theta)^{p/q}$

$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i \sin\left(\frac{2k\pi + p\theta}{q}\right)$ where $k = 0, 1, 2, 3, \ldots, q - 1$

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

Self Practice Problems :

(19) Prove the identities :

(a) $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos\theta$;

(b) $(\sin 5\theta) / (\sin\theta) = 16 \cos^4\theta - 12 \cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi \ldots$

(20) Prove the identities

(a) $\sin^3\theta = \frac{3}{4} \sin\theta - \frac{1}{4} \sin 3\theta$ (b) $\cos^4\theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$

12. Cube roots of unity :

(i) The cube roots of unity are 1, $\frac{-1 + i\sqrt{3}}{2}$, $\frac{-1 - i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :

$\cos 0 + i \sin 0$; $\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(v) The following factorisation should be remembered :

(a, b, c ∈ R & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$;

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$ $a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(v) The following factorisation should be remembered :

(a, b, c ∈ R & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$;

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$ $\quad a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Example # 14 : Find the value of $\omega^{192} + \omega^{194}$

Solution : $\omega^{192} + \omega^{194} = 1 + \omega^2 = -\omega$

Example # 15 : If 1, ω, ω^2 are cube roots of unity, then prove that

(i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

(ii) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$

(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... to 2n factors $= 2^{2n}$

Solution : (i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = (-2\omega)(-2\omega^2) = 4$

(ii) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5 = -32(\omega^2 + \omega) = 32$

(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= ((1 - \omega)(1 - \omega^2))^2$

$= (1 + 1 + 1)^2 = 9$

(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... to 2n factors

$=(-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$............ to 2n factors

$= (4)(4)$.......... to n factors

$= 4^n = 2^{2n}$

Self Practice Problems :

(21) Find $\sum_{r=0}^{10}(1 + \omega^r + \omega^{2r})$

(22) It is given that n is an odd integer greater than three, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x + 1)^n - x^n - 1$

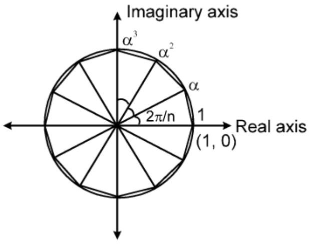
(23) If $x = a + b$, $y = a\alpha + b\beta$, $z = a\beta + b\alpha$ where α, β are imaginary cube roots of unity show that $xyz = a^3 + b^3$

(24) If $x^2 - x + 1 = 0$, then find the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$

Ans. (21) 12 (24) 8

13. n^{th} roots of unity :

If 1, α_1, α_2, α_3..... α_{n-1} are the n, n^{th} root of unity then :



(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2)......(1 - \alpha_{n-1}) = n$ &
$(1 + \alpha_1)(1 + \alpha_2).......(1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1.\ \alpha_1.\ \alpha_2.\ \alpha_3.........\ \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Example # 16 : Find the roots of the equation $z^6 + 64 = 0$ where real part is positive.

Solution : $z^6 = -64$

$z^6 = 2^6 . e^{i(2n+1)\pi}$ n = 0, 1, 2, 3, 4, 5 $\Rightarrow$ $z = 2\ e^{i(2n+1)\frac{\pi}{6}}$

$\therefore$ $z = 2e^{i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{7\pi}{6}}, 2e^{i\frac{3\pi}{2}}, 2e^{i\frac{11\pi}{6}}$

$\therefore$ roots with +ve real part are $= 2e^{\frac{i\pi}{6}}, 2e^{i\frac{11\pi}{6}}$

14. Geometrical properties :

(i) Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$z = \dfrac{mz_2 + nz_1}{m+n}$ where m, n > 0

If C divides PQ in the ratio m : n externally then $z = \dfrac{mz_2 - nz_1}{m-n}$

Notes :

(a) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively anda, b, c are the length of sides then,

- Centroid of the Δ ABC $= \dfrac{z_1 + z_2 + z_3}{3}$:
- Incentre of Δ ABC $= (az_1 + bz_2 + cz_3) / (a + b + c)$

(b) amp(z) = θ is a ray eminating from the origin inclined at an angle θ to the positive x− axis.

(c) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(d) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(e) The equation of circle having centre z_0 & radius ρ is : $|z - z_0| = \rho$

(f) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is arg $\dfrac{z - z_2}{z - z_1} = \pm \dfrac{\pi}{2}$

(g) Arg $\left(\frac{z - z_1}{z - z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$ (ii) Pair of ray if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.

(h) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(i) If $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is a circle.

(j) If $\big||z - z_1| - |z - z_2|\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

Self Practice Problem :

25. Match the following columns :

	Column - I		**Column - II**
(i)	If $\| z - 3+2i \| - \| z + i \| = 0$, then locus of z represents	(i)	Circle
(ii)	If arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then locus of z represents...	(ii)	Straight line
(iii)	if $\| z - 8 - 2i \| + \| z - 5 - 6i \| = 5$ then locus of z represents	(iii)	Ellipse
(iv)	If arg $\left(\frac{z-3+4i}{z+2-5i}\right) = \frac{5\pi}{6}$, then locus of z represents	(iv)	Hyperbola
(v)	If $\| z - 1 \| + \| z + i \| = 10$ then locus of z represents	(v)	Major Arc
(vi)	$\| z - 3 + i \| - \| z + 2 - i \| = 1$ then locus of z represents	(vi)	Minor arc
(vii)	$\| z - 3i \| = 25$ then locus of z represents......	(vii)	Perpendicular bisector of a line segment
(viii)	arg $\left(\frac{z-3+5i}{z+i}\right) = \pi$ then locus of z represents......	(viii)	Line segment

Ans. Column I	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
ColumnII	(vii)	(v)	(viii)	(vi)	(iii)	(iv)	(i)	(viii)