Conic Section

Our notion of symmetry is derived from the human face. Hence, we demand symmetry horizontally and in breadth only, not vertically nor in depth ....***Pascal, Blaise***

This chapter focusses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone (not necessarily right circular) can be out in various ways by a plane, and thus different types of conic sections are obtained. Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

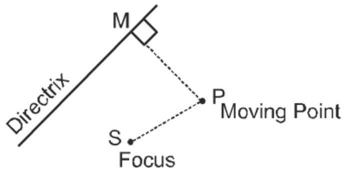
1. Conic sections :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(i) The fixed point is called the **Focus.**

(ii) The fixed straight line is called the **Directrix.**

(iii) The constant ratio is called the **Eccentricity** denoted by e.



$\frac{PS}{PM} = e$

(iv) The line passing through the focus & perpendicular to the directrix is called the **Axis**.

(v) A point of intersection of a conic with its axis is called a **Vertex**.

If S is (p, q) & directrix is $\ell x + my + n = 0$

then PS = $\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ & PM = $\frac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$\frac{PS}{PM} = e \quad \Rightarrow \quad (\ell^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(\ell x + my + n)^2$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

2. Section of right circular cone by different planes :

A right circular cone is as shown in the figure – 1

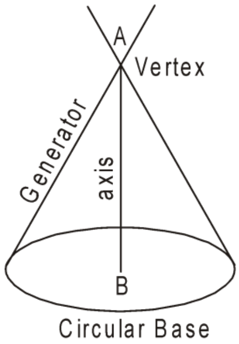


Figure 1

(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure - 2.

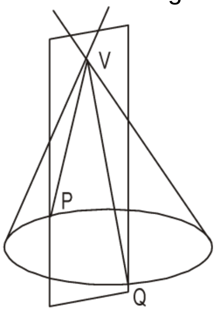


Figure –2

(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the figure − 3.

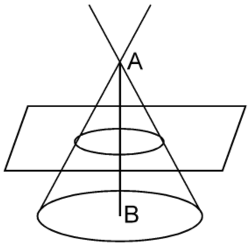


Figure- 3

(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the figure–4.

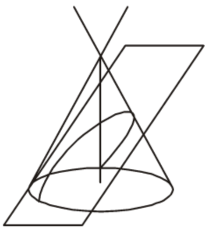


Figure-4

(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the **figure − 5 & 6**.

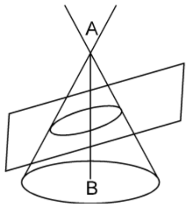


Figure -5

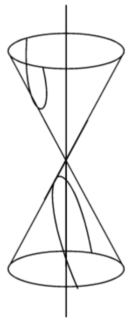
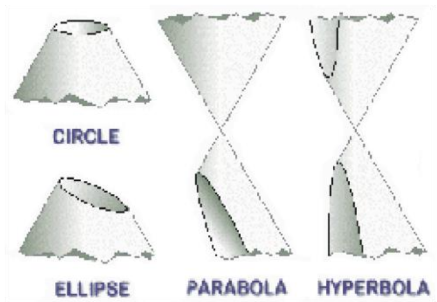


Figure -6

3D View :



3. Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

(i) Case (I) When The Focus Lies On The Directrix.

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if:

$e > 1 \equiv h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1 \equiv h^2 \geq ab$ the lines will coincident.

$e < 1 \equiv h^2 < ab$ the lines will be imaginary.

(ii) Case (II) When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1; \Delta \neq 0,$	$0 < e < 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab; a + b = 0$

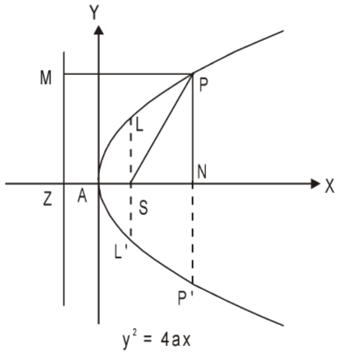
PARABOLA

1. Definition and terminology :

A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendiculardistance from a fixed straight line (directrix).Four standard forms of the parabola are

$y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$:



(i)	Vertex is (0, 0)	**(ii)**	focus is (a, 0)
(iii)	Axis is y = 0	**(iv)**	Directrix is x + a = 0

Focal Distance: The distance of a point on the parabola from the focus.

Focal Chord : A chord of the parabola, which passes through the focus.

Double Ordinate: A chord of the parabola perpendicular to the axis of the symmetry.

Latus Rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).

For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.

$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, − 2a).

NOTE :

(a) Perpendicular distance from focus on directrix = half the latus rectum.

(b) Vertex is middle point of the focus & the point of intersection of directrix & axis.

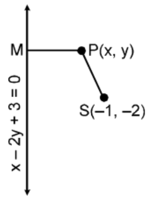
(c) Two parabolas are said to be equal if they have the same latus rectum.

Example#1: Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix is $x - 2y + 3 = 0$.

Solution : Let P(x, y) be any point on the parabola whose focus is S(– 1, – 2) and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,

SP = PM

$\Rightarrow \quad SP^2 = PM^2$



$\Rightarrow \quad (x + 1)^2 + (y + 2)^2 = \left(\dfrac{x - 2y + 3}{\sqrt{1+4}}\right)^2$

$\Rightarrow \quad 5\,[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$

$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

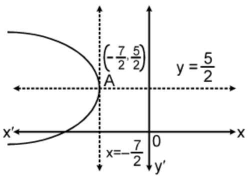
Example#2 : Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches. $4y^2 + 12x - 20y + 67 = 0$

Solution : The given equation is

$4y^2 + 12x - 20y + 67 = 0 \quad \Rightarrow \quad y^2 + 3x - 5y + \dfrac{67}{4} = 0$

$\Rightarrow \quad y^2 - 5y = -3x - \dfrac{67}{4} \quad \Rightarrow \quad y^2 - 5y + \left(\dfrac{5}{2}\right)^2 = -3x - \dfrac{67}{4} + \left(\dfrac{5}{2}\right)^2$

$\Rightarrow \left(y-\frac{5}{2}\right)^2 = -3x - \frac{42}{4} \quad \Rightarrow \left(y-\frac{5}{2}\right)^2 = -3\left(x+\frac{7}{2}\right)$ ....(i)



Let $x = X - \frac{7}{2}$, $y = Y + \frac{5}{2}$ ....(ii)

Using these relations, equation (i) reduces to

$Y^2 = -3X$ ....(iii)

This is of the form $Y^2 = -4aX$. On comparing, we get $4a = 3$

$\Rightarrow a = 3/4$.

Vertex - The coordinates of the vertex are (X = 0, Y = 0)

So, the coordinates of the vertex are

$\left(-\frac{7}{2},\frac{5}{2}\right)$ [Putting X = 0, Y = 0 in (ii)]

Axis

The equation of the axis of the parabola is Y = 0.

So, the equation of the axis is

$y = \frac{5}{2}$ [Putting Y = 0 in (ii)]

Focus-

The coordinates of the focus are (X = –a, Y = 0)

i.e. (X = – 3/4, Y = 0).

So, the coordinates of the focus are

(–17/4, 5/2) [Putting X = 3/4 in (ii)]

Directrix -

The equation of the directrix is X = a i.e. $X = \frac{3}{4}$.

So, the equation of the directrix is

$x = -\frac{11}{4}$ [Putting X = 3/4 in (ii)]

Latusrectum -

The length of the latusrectum of the given parabola is 4a = 3.

Self Practice Problems :

(1) Find the equation of the parabola whose focus is the point (0, 0)and whose directrix is the straight line $3x - 4y + 2 = 0$.

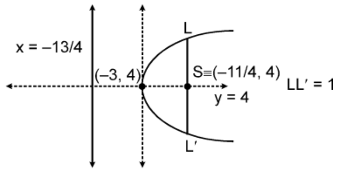
(2) Find the extremities of latus rectum of the parabola $y = x^2 - 2x + 3$.

(3) Find the latus rectum & equation of parabola whose vertex is origin & directrix is $x + y = 2$.

(4) Find the vertex, axis, focus, directrix, latusrectum of the parabola $y^2 - 8y - x + 19 = 0$. Also draw their roguht sketches.

Ans. (1) $16x^2 + 9y^2 + 24xy - 12x + 16y - 4 = 0$ (2) $\left(\frac{1}{2},\frac{9}{4}\right)\left(\frac{3}{2},\frac{9}{4}\right)$

(3) $4\sqrt{2}$, $x^2 + y^2 - 2xy + 8x + 8y = 0$

(4)


2. Parametric representation :

The simplest & the best form of representing the co−ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameterParametric form for :

$y^2 = -4ax$ $(-at^2, 2at)$
$x^2 = 4ay$ $(2at, at^2)$
$x^2 = -4ay$ $(2at, -at^2)$

Example # 3 : Find the parametric equation of the parabola $(x - 1)^2 = -16(y - 2)$

Solution : $\because$ $4a = -16$ $\Rightarrow$ $a = -4$, $y - 2 = at^2$

$x - 1 = 2at$ $\Rightarrow$ $x = 1 - 8t$, $y = 2 - 4t^2$

Self Practice Problems :

(5) Find the parametric equation of the parabola $x^2 = 4ay$

Ans. $x = 2at$, $y = at^2$.

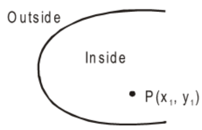
3. Position of a point relative to a parabola :

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

$S_1 : y_1^2 - 4ax_1$

$S_1 < 0 \rightarrow$ Inside

$S_1 > 0 \rightarrow$ Outside



Example # 4 : Check whether the point (3, 4) lies inside or outside the paabola $y^2 = 4x$.

Solution : $y^2 - 4x = 0$

$\because$ $S_1 \equiv y_1^2 - 4x_1 = 16 - 12 = 4 > 0$

$\therefore$ (3, 4) lies outside the parabola.

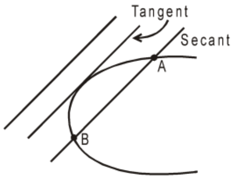
Self Practice Problems :

(6) Find the set of value's of α for which $(\alpha, -2 - \alpha)$ lies inside the parabola $y^2 + 4x = 0$.

Ans. $\alpha \in (-4 - 2\sqrt{3}, -4 + 2\sqrt{3})$

4. Line & a parabola :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm$



$\Rightarrow$ condition of tangency is, $c = a/m$.

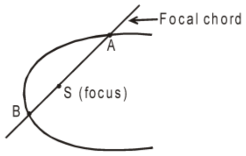
Length of the chord intercepted by the parabola

on the line $y = mx + c$ is :. $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$

NOTE :

(i) The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\,y + 2\,at_1 t_2 = 0$.

(ii) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co−ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



(iii) Length of the focal chord making an angle α with the x− axis is $4a\operatorname{cosec}^2\alpha$.

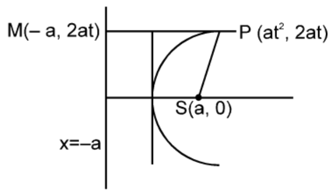
Example # 5 : Discuss the position of line $y = x + 1$ with respect to parabola $y_2 = 4x$.

Solution : Solving we get $(x + 1)_2 = 4x \Rightarrow (x - 1)_2 = 0$

so $y = x + 1$ is tangent to the parabola.

Example # 6 : Prove that focal distance of a point $P(at_2, 2at)$ on parabola $y_2=4ax\,(a>0)$ is $a(1+ t_2)$.

Solution :



$\because$ $PS = PM = a + at_2 \Rightarrow PS = a\,(1 + t_2)$.

Example # 7 : If the endpoint t_1, t_2 of a chord of the parabola $y_2 = x$ satisfy the relation $t_1\,t_2 = 8$ then prove that the chord always passes through a fixed point. Find the point?

Solution : Equation of chord joining $\left(\frac{1}{4}t_1^2, \frac{1}{2}t_1\right)$ and $\left(\frac{1}{4}t_2^2, \frac{1}{2}t_2\right)$ is

$y - \frac{1}{2}t_1 = \frac{2}{t_1+t_2}\left(x - \frac{1}{4}t_1^2\right) \Rightarrow (t_1 + t_2)\,y - \frac{1}{2}\,t_{12} - \frac{1}{2}\,t_1t_2 = 2x - \frac{1}{2}\,t_{12}$

$y = \frac{2}{t_1+t_2}\,(x + 2)$ $(\because t_1t_2 = 8)$

$\therefore$ This line passes through a fixed point $(-2, 0)$.

Example # 8 : Prove that the straight line y = mx + c touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \frac{a}{m}$

Solution : Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \frac{a}{m}$ $\Rightarrow$ $y = mx + a\left(m + \frac{1}{m}\right)$

but the given tangent is y = mx + c

$\therefore c = am + \frac{a}{m}$

Self Practice Problems :

(7) If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct point's then set of value's of 'λ' is

(8) Find the midpoint of the chord x + y = 2 of the parabola $y^2 = 4x$.

(9) If one end of focal chord of parabola $y^2 = 16x$ is (16, 16) then coordinate of other end is.

(10) If PSQ is focal chord of parabola $y^2 = 4ax$ (a > 0), where S is focus then prove that $\frac{1}{PS} + \frac{1}{SQ} = \frac{1}{a}$.

(11) Find the length of focal chord whose one end point is 't'.

Ans. (7) $(-\infty, 1/3)$ (8) (4, – 2) (9) (1, – 4) (11) $a\left(t + \frac{1}{t}\right)^2$

5. <u>Tangents to the parabola $y^2 = 4ax$</u> :

Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.

In replacement method, following changes are made to the second degree equation to obtain T.

$x^2 \to x x_1$, $y^2 \to y y_1$, $2xy \to xy_1 + x_1y$, $2x \to x + x_1$, $2y \to y + y_1$

So, it follows that the targents are :

(i) $y y_1 = 2a(x + x_1)$ at the point (x_1, y_1) ;

(ii) $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $ty = x + at^2$ at $(at^2, 2at)$.

Note : Point of intersection of the tangents at the point t_1 & t_2 is $\{ at_1t_2, a(t_1 + t_2) \}$.

Example # 9 : A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line y = 3x + 5. Find its equation and its point of contact.

Solution : Slope of required tangent's are $m = \frac{3 \pm 1}{1 \mp 3}$

$m_1 = -2$, $m_2 = \frac{1}{2}$

$\because$ Equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

$\therefore$ tangent's $y = -2x - 1$ at $\left(\frac{1}{2}, -2\right)$

$y = \frac{1}{2}x + 4$ at $(8, 8)$

Example # 10 : Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is $y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\therefore\ 10 = 4m + \frac{9}{4m} \Rightarrow 16m^2 - 40m + 9 = 0 \Rightarrow m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$.

Example # 11 : Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution : Equation of tangent to $y^2 = 4ax$ is

$y = mx + \frac{a}{m}$ ....(i)

Equation of tangent to $x^2 = 4by$ is

$x = m_1y + \frac{b}{m_1} \Rightarrow y = \frac{1}{m_1}x - \frac{b}{(m_1)^2}$ ....(ii)

for common tangent, (i) & (ii) must represent same line.

$\therefore\ \frac{1}{m_1} = m$ & $\frac{a}{m} = -\frac{b}{m_1^2}$

$\Rightarrow\ \frac{a}{m} = -bm^2 \Rightarrow m = \left(-\frac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is $y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}$

Self Practice Problems :

(12) Find equation tangent to parabola $y^2 = 4x$ whose intercept on y–axis is 2.

(13) Prove that perpendicular drawn from focus upon any tangent of a parabola lies on the tangent at the vertex.

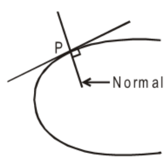
(14) Prove that image of focus in any tangent to parabola lies on its directrix.

(15) Prove that the area of triangle formed by three tangents to the parabola $y^2 = 4ax$ is half the area of triangle formed by their points of contacts.

Ans. (12) $y = \frac{x}{2} + 2$

6. Normals to the parabola $y^2 = 4ax$:

Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1} \Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ;

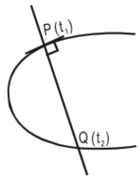
(ii) $y = mx - 2am - am_3$ at $(am_2, -2am)$

(iii) $y + tx = 2at + at_3$ at $(at_2, 2at)$.

NOTE :

(a) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.



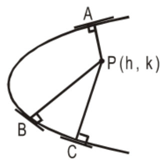
(c) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

(d) If normal are drawn from a point P(h, k) to the parabola $y_2 = 4ax$ then

$k = mh - 2am - am_3$ i.e. $am_3 + m(2a - h) + k = 0$.

$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = -\frac{k}{a}$.

Where m_1, m_2, & m_3 are the slopes of the three concurrent normals. Note that



A, B, C → Conormal points

⇒ algebraic sum of the slopes of the three concurrent normals is zero.

⇒ algebraic sum of the ordinates of the three conormal points on the parabola is zero

⇒ Centroid of the Δ formed by three co−normal points lies on the x−axis.

Example # 12 : Find the locus of the point N from which 3 normals are drawn to the parabola $y_2 = 4ax$ are such that

(i) Two of them are equally inclined to x-axis

(ii) Two of them are perpendicular to each other

Solution : Equation of normal to $y_2 = 4ax$ is

$y = mx - 2am - am_3$

Let the normal passes through N(h, k)

$\therefore k = mh - 2am - am_3$ ⇒ $am_3 + (2a - h)m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's of above equation

$\therefore$ $m_1 + m_2 + m_3 = 0$ ......(i)

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$ ......(ii)

$m_1m_2m_3 = -\frac{k}{a}$ ......(iii)

(i) If two normal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore$ $m_3 = 0$ ⇒ $y = 0$

(ii) If two normal's are perpendicular

$\therefore$ $m_1 m_2 = -1$

from (3) $m_3 = \frac{k}{a}$ .....(iv)

from (2) $-1 + \frac{k}{a}(m_1 + m_2) = \frac{2a-h}{a}$ .....(v)

from (1) $m_1 + m_2 = -\frac{k}{a}$ .....(vi)

from (5) & (6), we get

$$-1 - \frac{k^2}{a} = 2 - \frac{h}{a}$$

$$y_2 = a(x - 3a)$$

Self Practice Problems :

(16) Find the points of the parabola $y_2 = 4ax$ at which the normal is inclined at 30° to the axis.

(17) If the normal at point P(1, 2) on the parabola $y_2 = 4x$ cuts it again at point Q then Q = ?

(18) Find the length of normal chord at point ‘t’ to the parabola $y_2 = 4ax$.

(19) If normal chord at a point 't' on the parabola $y_2 = 4ax$ subtends a right angle at the vertex then prove that $t_2 = 2$

(20) Prove that the chord of the parabola $y_2=4ax$, whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$, is a normal to the curve and that its length is $6\sqrt{3}a$.

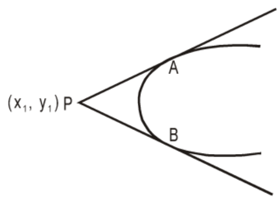
(21) If the normals at 3 points P, Q & R are concurrent, then show that
(i) The sum of slopes of normals is zero, (ii) Sum of ordinates of points P, Q, R is zero
(iii) The centroid of ΔPQR lies on the axis of parabola.

Ans. (16) $\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right), \left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$ (17) (9, – 6) (18) $\ell = \frac{4a(t^2+1)^{\frac{3}{2}}}{t^2}$

7. Pair of tangents :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by: $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y\, y_1 - 2a(x + x_1)$.



Example # 13 : Write the equation of pair of tangents to the parabola $y_2 = 4x$ drawn from a point P(–1, 2)

Solution : We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore$ $(y_2 - 4x)(4 + 4) = (2y - 2(x - 1))_2$

$\Rightarrow$ $8y_2 - 32x = 4y_2 + 4x_2 + 4 - 8xy + 8y - 8x$

$\Rightarrow$ $y_2 - x_2 + 2xy - 6x - 2y = 1$

Example # 14 : Find the directrix of parabola such that from point P tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that $m_1 m_2 = -2$

Solution : Equation of tangent to $y^2 = 4ax$, is

$y = mx + \frac{a}{m}$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

$m_1 m_2 = -2 \quad \Rightarrow \quad a = -2$

so directrix is x = 2

Self Practice Problems :

(22) If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.

(i) $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$ (a constant) (ii) $\cos\theta_1 \cos\theta_2 = \lambda$ (a constant)

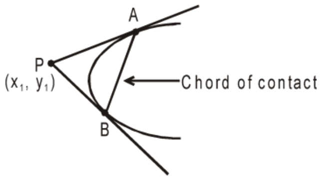
Ans. (i) $y^2 - 2ax = \lambda x^2$, (ii) $x^2 = \lambda^2 \{(x - a)^2 + y^2\}$

8. Director circle :

Locus of the point of intersection of the perpendicular tangents to a curve is called the Director Circle. For parabola $y^2 = 4ax$ it's equation is x + a = 0 which is parabola's own directrix.

9. Chord of contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.



Note : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$

Example # 15: If the line x – y – 5 = 0 intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution : Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0 \quad .....(i)$

But given is x – y – 5 = 0

$\therefore \frac{\sqrt{3}}{2} = \frac{-k}{-1} = \frac{4h}{-5} \quad \Rightarrow \quad h = -5, k = 4 \quad \therefore \quad$ point $\equiv (-5, 4)$

Self Practice Problems :

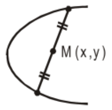
(23) Prove that locus of a point whose chord of contact w.r.t. parabola passes through focus is directrix

(24) If from a variable point 'P' on the line x – 2y + 1 = 0 pair of tangent's are drawn to the parabola $y^2 = 8x$ then prove that chord of contact passes through a fixed point, also find that point.

Ans. (25) (1, 8)

10. Chord with a given middle point :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is



(x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1) \equiv T = S_1$

Example # 16 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given point (p, q).

Solution : Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,
so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.
Since it passes through (p, q)
$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$
$\therefore$ Required locus is
$y^2 - 2ax - qy + 2ap = 0$.

Example # 17 : Find the locus of middle point of the chord of the parabola $y_2 = 4ax$ which are parallel to the line $x + 2y = 0$

Solution : Let P(h, k) be the mid point of chord of parabola $y_2 = 4ax$,
so equation of chord is $yk - 2a(x + h) = k_2 - 4ah$.
but slope $= \frac{2a}{k} = m$ $\frac{k}{2} = \frac{-2a}{1}$
$\therefore$locus is $k = -4a$ $y = 4a$

Self Practice Problems :

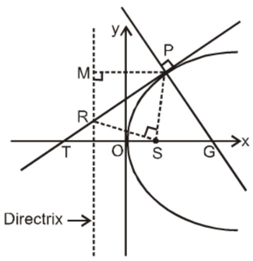
(25) Find the equation of chord of parabola $y_2 = 4x$ whose mid point is (4, 2).

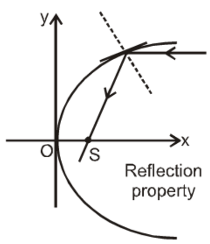
(26) Find the locus of mid - point of chord of parabola $y_2 = 4ax$ which touches the parabola $x_2 = 4by$.

Ans. (25) $x - y - 2 = 0$ (26) $y(2ax - y_2) = 4a_2b$

11. Important Highlights :

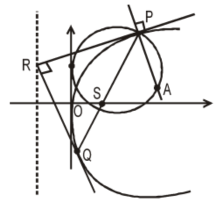
(i) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.



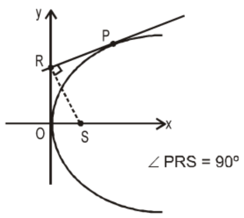


(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus. See figure above.

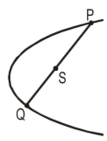
(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex.



(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.



(v) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



$$\frac{2(PS)(SQ)}{PS + SQ} = 2a$$

(vi) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

ELLIPSE

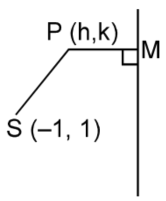
In this chapter we are going to discuss in detail the nature of path in which on planets move around the sun. They follow on elliptical path with the sun at one of its foci. Let us look at the definition of ellipse.

1. Definitions :

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e), which is less than one.
The fixed point is called - **focus**
The fixed line is called -**directrix.**
The constant ratio is called - **eccentricity**, it is denoted by **'e'**.

Example # 18 : Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Solution : Let $P \equiv (h, k)$ be moving point, $e = \frac{PS}{PM} = \frac{1}{2} \Rightarrow (h + 1)^2 + (k - 1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$



$\Rightarrow$ locus of P(h, k) is $8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$

$7x^2 + 7y^2 + 2xy + 10x – 10 y + 7 = 0$.

Note : The general equation of a conic with focus (p, q) & directrix $\ell x + my + n = 0$ is:

$(\ell^2 + m^2) [(x − p)^2 + (y − q)^2] = e^2 (\ell x + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represent ellipse if $0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$

Self Practice Problems :

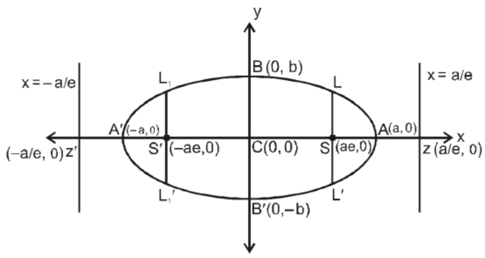
(27) Find the equation to the ellipse whose focus is (0, 0) directrix is $x + y – 1 = 0$ and $e = \frac{1}{\sqrt{2}}$.

Ans. $3x^2 + 3y^2 – 2xy + 2x + 2y – 1 = 0$.

2. Standard equation :

Standard equation of an ellipse referred to its principal axes along the co−ordinate axes is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2 (1 − e^2)$.



(i) Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

(ii) Focii : $S \equiv (ae, 0)$ & $S' \equiv (−ae, 0)$.

(iii) Equations of Directrices : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(iv) Major Axis : The line segment A'A in which the focii S' & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

(v) Minor Axis : The y−axis intersects the ellipse in the points $B' \equiv (0, −b)$ & $B \equiv (0, b)$. The line segment B'B is of length 2b (b < a) is called the minor axis of the ellipse.

(vi) Principal Axis : The major & minor axes together are called principal axis of the ellipse.

(vii) Vertices : Point of intersection of ellipse with major axis. $A' \equiv (− a, 0)$ & $A \equiv (a, 0)$.

(viii) Focal Chord : A chord which passes through a focus is called a focal chord.

(ix) Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

(x) Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

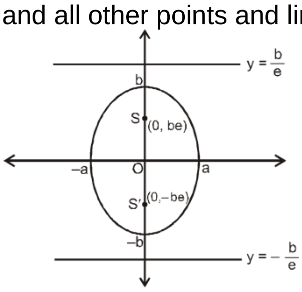
Length of latus rectum **(LL')** = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a\left(1-e^2\right)$

= 2e (distance from focus to the corresponding directrix)

Centre : The point which bisects every chord of the conic drawn through it, is called the centre of the conic. C ≡ (0, 0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Note : (i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.



Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Foci (0, ± be) Directrices : $y = \pm \frac{b}{e}$

$a^2 = b^2 (1 - e^2)$, a < b. ⇒ $e = \sqrt{1-\frac{a^2}{b^2}}$

Vertices (0, ± b) ; .R. y = ± be

ℓ (L·R.) = $\frac{2a^2}{b}$, centre : (0, 0)

Example # 19 : Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Solution : Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

∴ $\frac{4}{a^2} + \frac{4}{b^2} = 1$ ....(i)

and $\frac{9}{a^2} + \frac{1}{b^2} = 1$ ....(ii)

from (i) – 4 (ii), we get

$\frac{4-36}{a^2} = 1 - 4 \Rightarrow a^2 = \frac{32}{3}$

from (i), we get

$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32} \Rightarrow b^2 = \frac{32}{5}$ ∴ Ellipse is $3x^2 + 5y^2 = 32$

Example # 20 : Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Solution : Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ ae = 4 and e = $\frac{1}{3}$ (Given)

$\therefore$ a = 12 and $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 144\left(1 - \frac{1}{9}\right)$ $\Rightarrow$ $b^2 = 16 \times 8$ $\Rightarrow$ $b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Example # 21 : If lenght of latus rectum is equal to half of semi major axis then find the eccentricity of ellipse

Solution : Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

$\frac{2b^2}{a} = \frac{a}{2} \Rightarrow \frac{b^2}{a^2} = \frac{1}{4} \Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{3}}{2}$

Example # 22 : Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus-rectum and eccentricity of an ellipse $\frac{(x-2)^2}{25} + \frac{(y-4)^2}{16} = 1$.

Solution : Let x – 2 = X, y – 4 = Y, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is Y = 0 $\Rightarrow$ y = 4.

equation of minor axis is X = 0 $\Rightarrow$ x = 2.

centre (X = 0, Y = 0) $\Rightarrow$ x = 2, y = 4

C ≡ (2, 4)

Length of semi-major axis a = 5

Length of major axis 2a = 10

Length of semi-minor axis b = 4

Length of minor axis = 2b = 8.

Let 'e' be eccentricity

$\therefore$ $b^2 = a^2(1 - e^2)$

$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25 - 16}{25}} = \frac{3}{5}$.

Length of latus rectum = LL' = $\frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

Co-ordinates focii are X = ± ae, Y = 0

$\Rightarrow$ S ≡ (X = 3, Y = 0) & S' ≡ (X = –3, Y = 0)

$\Rightarrow$ S ≡ (5, 4) & S' ≡ (–1, 4)

Co-ordinate of vertices

Extremities of major axis A ≡ (X = a, Y = 0) & A' ≡ (X = – a, Y = 0)

⇒ A ≡ (x = 7, y = 4) & A' = (x = – 3, y = 4)

A ≡ (7, 4) & A' ≡ (– 3, 4)

Extremities of minor axis B ≡ (X = 0, Y = b) & B' ≡ (X = 0, Y = – b)

B ≡ (x = 2, y = 8) & B' ≡ (x = 2, y = – 0)

B ≡ (2, 8) & B' ≡ (2, 0)

Equation of directrix $X = \pm \frac{a}{e}$

$x - 2 = \pm \frac{25}{3}$ ⇒ $x = \frac{31}{3}$ & $x = -\frac{-19}{3}$

Self Practice Problems :

(28) Find the equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$ respectively.

(29) Find the eccentricity of ellipse whose minor axis is double the latus rectum.

(30) Find the co-ordinates of the focii of the ellipse $4x_2 + 9y_2 = 1$.

(31) Find the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passing through (2, 1) and having eccentricity $\frac{1}{2}$

Ans. (28) $3(x - 3y + 3)_2 + 2(3x + y - 1)_2 = 180$,

$21x_2 - 6xy + 29y_2 + 6x - 58y - 151 = 0$.

(29) $\frac{\sqrt{3}}{2}$ (30) $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$ (31) $3x_2 + 4y_2 = 16$

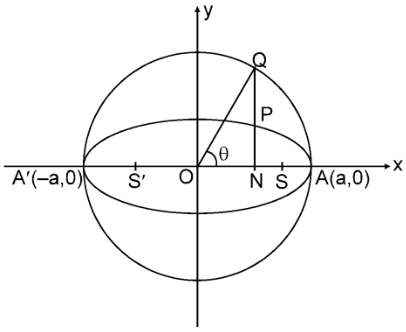
3. Auxiliary circle / eccentric angle :

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x − axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. ‘θ’ is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \le \pi)$.

Q ≡ (a cosθ , a sinθ)

P ≡ (a cosθ , b sinθ)



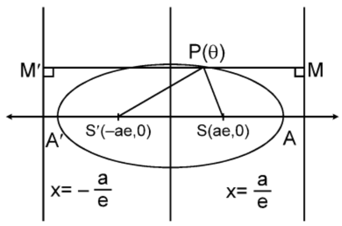
Focal property : summation of distance of any point from two focus is equal to lenght of major axis

Proof : Let 'e' be the eccentricity of ellipse.

∴ PS = e . PM

$= e\left(\frac{a}{e} - a\cos\theta\right)$

PS = (a – a e cosθ)



and PS' = e. PM' = $e\left(a\cos\theta + \frac{a}{e}\right)$

PS' = a + ae cosθ

∴ focal distance are (a ± ae cosθ)

PS + PS' = 2a, PS + PS' = AA' = lenght of major axis

Self Practice Problems :

(32) Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is α

(33) Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre is 2.

Ans. (32) $r = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ (33) $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

4. Parametric representation :

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then;

Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by .

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

Example # 23 : Write the equation of chord of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Solution : Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} + \frac{y}{4}.\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$$\frac{x}{5}.\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}.\sin\left(\frac{3\pi}{4}\right) = 0 \Rightarrow -\frac{x}{5} + \frac{y}{4} = 0 \Rightarrow 4x = 5y$$

Self Practice Problems :

(34) Show that the sum of squares of reciprocals of two perpendicular diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is constant. Find the constant also.

Ans. $\frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$

5. Position of a point w.r.t. an ellipse :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ > < or = 0.

Example # 24: Check whether the point P(5, 4) lies inside or outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution : $S_1 \equiv \frac{25}{25} + \frac{16}{16} - 1 = 1 + 1 - 1 > 0$

$\therefore$ Point P ≡ (5, 4) lies outside the ellipse.

6. Line and an ellipse :

The line y = mx + c meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

Example # 25 : Find the set of value(s) of 'λ' for which the line 3x – 4y + λ = 0 lies outside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution : Solving given line with ellipse, we get $\frac{(4y-\lambda)^2}{9\times16} + \frac{y^2}{9} = 1$

$\frac{2y^2}{9} - \frac{y\lambda}{18} + \frac{\lambda^2}{144} - 1 = 0$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ D < 0

$\Rightarrow \frac{\lambda^2}{(18)^2} - \frac{8}{9}.\left(\frac{\lambda^2}{144} - 1\right) < 0 \quad \Rightarrow \quad \lambda \in (-\infty, -12\sqrt{2}) \cup (12\sqrt{2}, \infty)$

Self Practice Problems :

(35) Find the value of 'λ' for which 2x – y + λ = 0 touches the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Ans. $\lambda = \pm\sqrt{109}$

7. Tangents :

(i) Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.

(ii) Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(iii) Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$.

Note :

(a) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.These tangents touches the ellipse at extremities of a diameter.

(b) Point of intersection of the tangents at the point α & β is, $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(c) The eccentric angles of the points of contact of two parallel tangents differ by π.

Example # 26 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line x + y = 3.

Solution : Slope of tangent = m = 1

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2 + 3}$

$\because$ m = 1

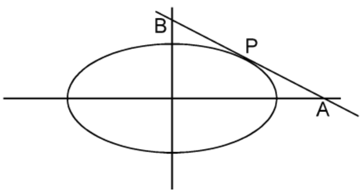
$\therefore$ $y = x \pm \sqrt{7}$

Example # 27 : A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the co-ordinate axes at A and B respectively. If P divides AB in the ratio 3 : 1, then find the locus of point P.

Solution : Let P ≡ (a cosθ, b sinθ)

$\therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$



A ≡ (a secθ, 0)

B ≡ (0, b cosecθ)

$\because$ P divide AB internally in the ratio 3 : 1

$h = \frac{a}{4\cos\theta}$; $k = \frac{3a}{4\sin\theta}$

$\Rightarrow \quad \cos\theta = \frac{a}{4h}, \quad \sin\theta = \frac{3a}{4k}$

then the locus is $\frac{a^2}{16x^2} + \frac{9a^2}{16y^2} = 1$

Self Practice Problems :

(36) Show that the locus of the point of intersection of the tangents at the extremities of any focal chord of an ellipse is the directrix corresponding to the focus.

(37) Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

(38) Prove that the portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

(39) Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. (39) $\frac{2a^3}{\sqrt{a^2 - b^2}}$

8. Normals :

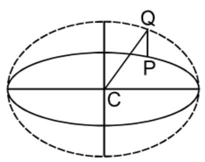
(i) Equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is;
$ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)\,m}{\sqrt{a^2 + b^2m^2}}$.

Example # 28 : P and Q are corresponding points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Solution :



Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ $Q \equiv (a\cos\theta, a\sin\theta)$

Equation of normal at P is

$(a\sec\theta)\,x - (b\operatorname{cosec}\theta)\,y = a^2 - b^2$ ....(i)

equation of CQ is $y = \tan\theta \, . \, x$ ....(ii)

Solving equation (i) & (ii), we get $(a - b)\,x = (a^2 - b^2)\cos\theta$

$x = (a + b)\cos\theta$, & $y = (a + b)\sin\theta$

$\therefore R \equiv ((a + b)\cos\theta, (a + b)\sin\theta)$ $\therefore$ $CR = a + b$

Self Practice Problems :

(40) Find the value(s) of 'k' for which the line x + y = k is a normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Ans. $k = \pm\sqrt{\dfrac{\left(a^2-b^2\right)^2}{a^2+b^2}}$

9. Pair of tangents :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$S \equiv \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} - 1$; $S_1 = \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2} - 1$; $T \equiv \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$

Example # 29 : How many real tangents can be drawn from the point (4, 3) to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Solution : Given point $P \equiv (4, 3)$

ellipse $S \equiv \dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = 0$

$\because S_1 \equiv \dfrac{16}{16} + \dfrac{9}{9} - 1 = 1 > 0$ $\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

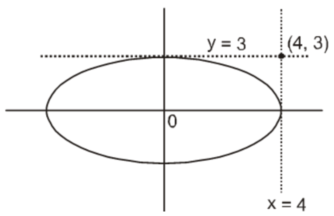
Equation of pair of tangents is $SS_1 = T_2$

$\Rightarrow \left(\dfrac{x^2}{16}+\dfrac{y^2}{9}-1\right) . 1 = \left(\dfrac{4x}{16}+\dfrac{3y}{9}-1\right)^2 \Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = \dfrac{x^2}{16} + \dfrac{y^2}{9} + 1 + \dfrac{xy}{6} - \dfrac{x}{2} - \dfrac{2y}{3}$

$\Rightarrow$ $-xy + 3x + 4y - 12 = 0$ $\Rightarrow$ $(4 - x)(y - 3) = 0$ $\Rightarrow$ $x = 4$ & $y = 3$

and angle between them = $\dfrac{\pi}{2}$

Alternative



By direct observation

x = 4, y = 3 are tangents.

Example # 30 : Find the locus of point of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents equation of pair of tangents is $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right) = \left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$

$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right) + ... = 0$ ....(i)

Since equation (i) represents two perpendicular lines

$\therefore \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right) = 0$

$\Rightarrow k^2 - b^2 + h^2 - a^2 = 0$ $\Rightarrow$ locus is $x^2 + y^2 = a^2 + b^2$

Self Practice Problems :

(41) Find the locus of point of intersection of the tangents drawn at the extremities of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. $x = \pm \frac{a}{e}$

10. Director circle :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axes.

Example # 31: An ellipse slides between two perpendicular lines. Show that the locus of its centre is a circle.

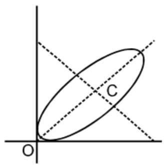
Solution : Let length of semi-major and semi-minor axis are 'a' and 'b' and centre is C≡(h, k)

Since ellipse slides between two perpendicular lines, therefore point of intersection of two perpendicular tangents lies on director circle.

Let us consider two perpendicular lines as x & y axes

$\therefore$ point of intersection is origin O ≡ (0, 0)

$\therefore$ OC = radius of director circle



$\therefore \sqrt{h^2+k^2} = \sqrt{a^2+b^2}$

$\Rightarrow$ locus of C ≡ (h, k) is $\Rightarrow x^2 + y^2 = a^2 + b^2$ which is a circle

Self Practice Problems :

(42) A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

11. Chord of contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $T = 0$, where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

Self Practice Problems :

(43) Find the locus of point of intersection of tangents at the extremities of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(44) Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Ans. (43) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$ (44) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$

12. Chord with a given middle point :

Equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$; $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

Self Practice Problems :

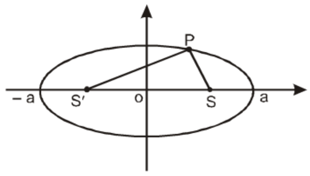
(45) Find the equation of the chord $\frac{x^2}{36} + \frac{y^2}{9} = 1$ which is bisected at (2, 1).

Ans. $x + 2y = 4$

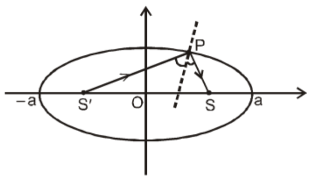
13. Important highlights :

Refering to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

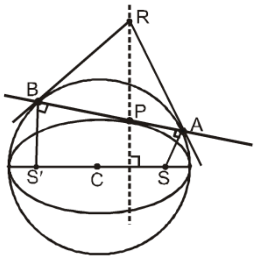
(i) If P be any point on the ellipse with S & S' as its foci then $\ell(SP) + \ell(S'P) = 2a$.



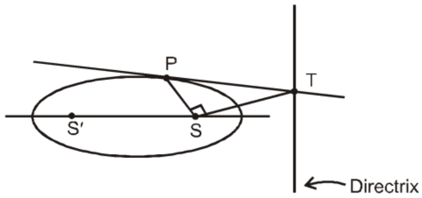
(ii) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice−versa. Hence we can deduce that the straight
lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



(ii) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle.



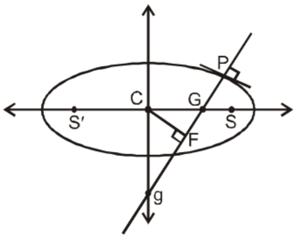
(iv) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.



(v) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(a) PF. PG = b^2 (b) PF. Pg = a^2

(c) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.



[where S and S' are the focii of the ellipse]

(vi) The circle on any focal distance as diameter touches the auxiliary circle.

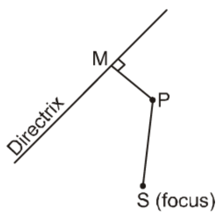
HYPERBOLA

Hyperbolic curves are of special importance in the field of science and technology especially astronomy and space studies. In this chapter we are going to study the characteristics of such curves.

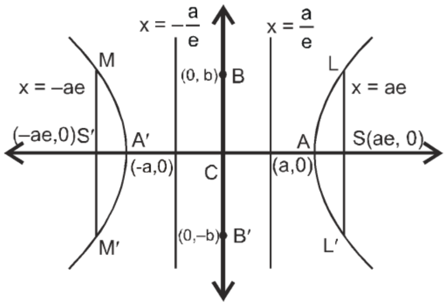
1. Definition :

A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is
a fixed constant greater than 1.

$\frac{PS}{PM}$ = e >1, e – eccentricity



2. Standard equation & definition(s)



Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

(i) **Eccentricity (e) :** $e^2 = 1 + \frac{b^2}{a^2}$

(ii) **Foci :** S ≡ (ae, 0) & S' ≡ (− ae, 0).

(iii) **Equations of directrices :** $x = \frac{a}{e}$ & $x = -\frac{a}{e}$

(iv) **Transverse axis :** The line segment A'A of length 2a in which the foci S' & S both lie is called the transverse axis of the hyperbola.

(v) **Conjugate axis :** The line segment B'B of length 2b between the two points B' ≡ (0, − b) & B ≡ (0, b) is called as the conjugate axis of the hyperbola.

(vi) **Principal axes :** The transverse & conjugate axis together are called principal axes of the hyperbola.

(vii) **Vertices :** A ≡ (a, 0) & A' ≡ (− a, 0)

(viii) **Focal chord :** A chord which passes through a focus is called a focal chord**.**

(ix) **Double ordinate :** A chord perpendicular to the transverse axis is called a double ordinate**.**

(x) **Latus rectum :** Focal chord perpendicular to the transverse axis is called latus rectum.
Its length (ℓ) is given by

$$\ell = \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$$

Note : (a) Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

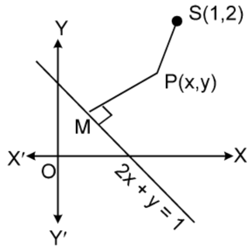
(b) End points of latus rectum are $L \equiv \left(ae, \frac{b^2}{a}\right)$, $L' \equiv \left(ae, -\frac{b^2}{a}\right)$, $M \equiv \left(-ae, \frac{b^2}{a}\right)$, $M' \equiv \left(-ae, -\frac{b^2}{a}\right)$

(xi) **Centre:** The point which bisects every chord of the conic, drawn through it, is called the centre of the conic. C ≡ (0,0) the origin is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(xii) **General note :** Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

Example # 32 : Find the equation of the hyperbola whose directrix is $3x + 4y = 2$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Solution :



Let P(x,y) be any point on the hyperbola.
Draw PM perpendicular from P on the directrix.
Then by definition SP = e PM
$\Rightarrow$ $(SP)^2 = e^2 (PM)^2$

$$\Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{3x+4y-2}{5}\right\}^2$$

which is the required hyperbola.

Example # 33: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution: Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus–rectum = $\frac{2b^2}{a}$. According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow$ $2b^2 = a^2$ $(\because$ $b^2 = a^2(e^2 - 1))$

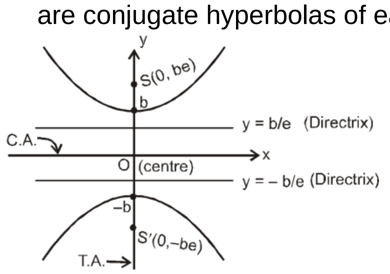
$\Rightarrow$ $2a^2(e^2 - 1) = a^2$ $\Rightarrow$ $2e^2 - 2 = 1$ $\Rightarrow$ $e^2 = \frac{3}{2}$

$\therefore$ $e = \sqrt{\frac{3}{2}}$ Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

3. <u>Conjugate hyperbola</u> :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **conjugate hyperbolas** of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.



Equation : $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$a^2 = b^2 (e^2 - 1) \Rightarrow e = \sqrt{1+\frac{a^2}{b^2}}$

Vertices$(0, \pm b)$; ℓ (L.R.) = $\frac{2a^2}{b}$

Note :

(a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similar if they have the same eccentricity.

(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Example # 34 : Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Solution : The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore$ $a^2 = 9, b^2 = 16 \Rightarrow a = 3, b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1+\frac{a^2}{b^2}\right)} = \sqrt{\left(1+\frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum = $\frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices : The equation of directrices are

$y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{4}{(5/4)} \Rightarrow y = \pm \frac{16}{5}$

Self Practice Problems :

(46) Find the equation of the hyperbola whose foci are (6, 4) and (– 4, 4) and eccentricity is 2.

(47) Obtain the equation of a hyperbola with coordinates axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.

(48) The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. Find the equation of the hyperbola if its eccentricity is 2.

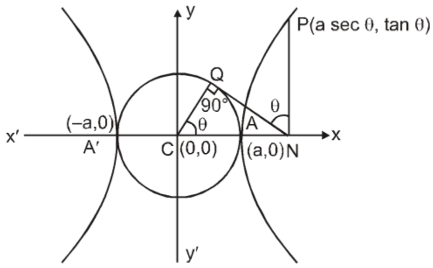
Ans. (46) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ (47) $\frac{x^2}{16}-\frac{y^2}{9}=1$, $\frac{y^2}{16}-\frac{x^2}{9}=1$

(48) $3x^2 - y^2 - 12 = 0$.

4. Auxiliary circle :

A circle drawn with centre C and transverse axis as a diameter is called the **auxiliary circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the following figure that P & Q are called the **"corresponding points"** of the hyperbola & the auxiliary circle.



5. Parametric representation :

The equations x = a sec θ & y = b tan θ together represent the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where θ is a parameter.

Note that if P(θ) ≡ (a sec θ, b tan θ) is on the hyperbola then,

Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the hyperbola joining the two points P(α) & Q(β) is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$$

6. Position of a point 'P' w.r.t. a hyperbola :

The quantity $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is positive, zero or negative according as the point (x_1, y_1) lies inside, on or outside the curve.

Example # 35 : Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution : Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,

So the point (5,–4) lies inside the hyperbola $9x^2 - y^2 = 1$..

7. Line and a hyperbola :

The straight line y = mx + c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as : c^2 > or = or < $a^2m^2 - b^2$, respectively.

8. Tangents :

(i) **Slope form :** $y = mx \pm \sqrt{a^2m - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm'.

(ii) **Point form:** Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(iii) **Parametric form:** Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point. $(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Note : **(a)** Point of intersection of the tangents at $P(\theta_1)$ & $Q(\theta_2)$ is $\left(a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}},\ b\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)$

(b) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 & θ_2) are parallel.

(c) There are two parallel tangents having the same slope m. These tangent touches the hyperbola at the extremities of a diameter.

Example # 36 : For what value of λ for which line $2x + y + \lambda = 0$ touches the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Solution : This line will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$

$\lambda^2 = 16 \times 4 - 9$

$\lambda^2 = 55 \quad \Rightarrow \quad \lambda = \sqrt{55}$

Self Practice problems :

(49) For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

(50) Find the equation of the tangent to the hyperbola $x^2 - y^2 = 1$ which is parallel to the line $4y = 5x + 7$.

Ans. (49) $\lambda = \pm 2\sqrt{5}$ (50) $4y = 5x \pm 3$

9. Normals :

(i) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P (x_1, y_1) on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(ii) The equation of the normal at the point P $(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

(iii) Equation of normals in terms of its slope 'm' are $y = mx \pm \frac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$

Self Practice problems :

(51) Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

(52) Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Ans. (52) $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = x^2y^2 (a^2 + b^2)^2$

10. Pair of tangents :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$; $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$; $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Example # 37 : How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Solution : Given point $P \equiv (4, 3)$

Hyperbola $S \equiv \frac{x^2}{16} - \frac{y^2}{9} - 1 = 0$

$\because$ $S_1 \equiv \frac{16}{16} - \frac{9}{9} - 1 = -1 < 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$\Rightarrow$ $\left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right) . (-1) = \left(\frac{4x}{16} - \frac{3y}{9} - 1\right)^2$

$\Rightarrow$ $-\frac{x^2}{16} + \frac{y^2}{9} + 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3}$

$\Rightarrow$ $3x^2 - 4xy - 12x + 16y = 0$ $\Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

11. Director circle :

The locus of the point of intersection of the tangents which are at right angle is known as the director circle of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, then the director circle is real.

If $b^2 = a^2$ (i.e. rectangular hyperbola), then the radius of the director circle is zero and it reduces to a point circle at the origin. In this case centre is the only point from which two perpendicular tangents can be drawn on the curve.

If $b^2 > a^2$, then the radius of the director circle is imaginary, so that there is no such circle and so no pair of tangents at right angle can be drawn to the curve.

12. Chord of contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is T = 0, where T = $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Example # 38: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution: Let P ≡ (h, k) be the point of intersection of tangents at A & B

$\therefore$Equation of chord of contact AB is $\frac{xh}{a^2} - \frac{yk}{b^2} = 1$ ......(i)

Which touches the parabola

Equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m} \Rightarrow mx - y = -\frac{a}{m}$ ......(ii)

equation (i) & (ii) as must be same

$\therefore \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(-\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \Rightarrow m = \frac{h}{k}\frac{b^2}{a^2}$ & $m = -\frac{ak}{b^2}$

$\therefore \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \Rightarrow$ locus of P is $y^2 = -\frac{b^4}{a^3} . x$

13. Chord with a given middle point :

Equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is **$T = S_1$**, where

$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$; $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Example # 39 : Find the locus of the mid - point of chords of the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

Solution : Let P ≡ (h, k) be the mid-point

∴ equation of chord whose mid-point is given is $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$

$$\frac{6x}{25} - \frac{4y}{9} = \frac{x^2}{25} - \frac{y^2}{9} - 1$$

Self Practice Problems :

(53) Find the equation of the chord $\frac{x^2}{36} - \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(54) Find the point 'P' from which pair of tangents PA & PB are drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ in such a way that (5, 2) bisect AB

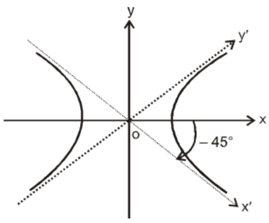
Ans. (53) $x = 2y$ (54) $\left(\frac{20}{3}, \frac{8}{3}\right)$

14. Rectangular hyperbola (equilateral hyperbola) :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Since $a = b$

equation becomes $x^2 - y^2 = a^2$ ⇒ $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

Rectangular hyperbola ($xy = c^2$) :

Vertices : (c, c) & $(-c, -c)$;

Foci : $(\sqrt{2}\,c, \sqrt{2}\,c)$ & $(-\sqrt{2}\,c, -\sqrt{2}\,c)$,

Directrices : $x + y = \pm \sqrt{2}\,c$

Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1 t_2 y = c(t_1 + t_2)$.

Equation of the tangent at $P(x_1, y_1)$ is $\left(\frac{x}{x_1} + \frac{y}{y_1}\right) = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

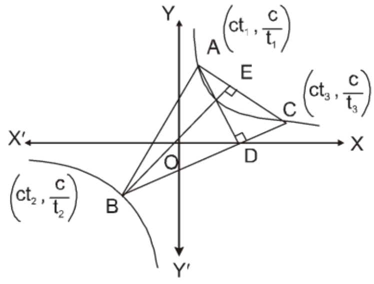
Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Example # 40 : A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution : Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now slope of BC is $\frac{c(t_3 - t_2)}{c(t_2 - t_3)t_2 t_3} = -\frac{1}{t_2 t_3}$

$\therefore$ Slope of AD is t_2t_3

$\therefore$ Slope of AD is t_2t_3

Equation of Altitude AD is $y - \frac{c}{t_1} = t_2t_3(x - ct_1)$

or $t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3$ .....(1)

Similarly equation of altitude BE is

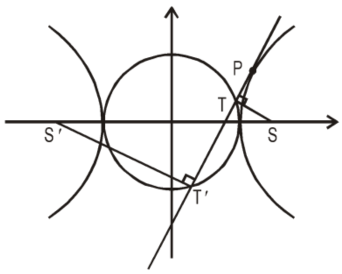
$t_2y - c = x\, t_1t_2t_3 - ct_1t_2^2t_3$ ......(2)

Solving (1) and (2), we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ which lies on $xy = c^2$.

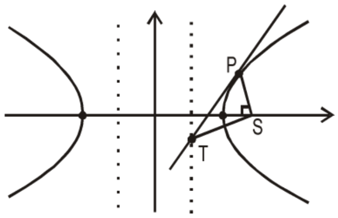
15. Important results :

(i) Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$

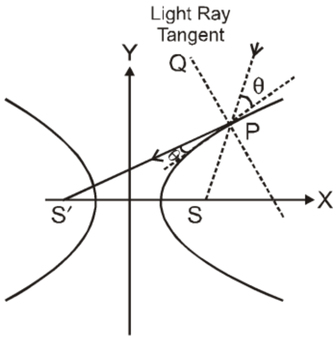
(ii) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



(iii) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.



(iv) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



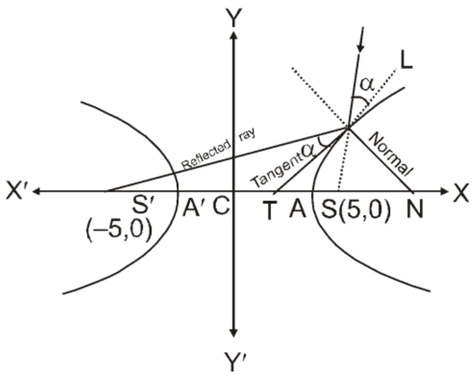
(v) Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}$ = 1 (a > k > b > 0) are confocal and therefore orthogonal.

A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\ t_2\ t_3}, -ct_1\ t_2\ t_3\right)$.

Example # 41 : A ray originating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lying in first quadrant.

Solution : Given hyperbola is $9x^2 - 16y^2 = 144$. This equation can be rewritten as $\frac{x^2}{16}-\frac{y^2}{9}=1$ ....(1)
Since x co–ordinate of P is 8. Let y
co–ordinate of P is α.

$\because$ (8,α) lies on (1)



$\therefore$ $\frac{64}{16}-\frac{\alpha^2}{9}=1$ ⇒ $\alpha^2 = 27$ ⇒ $\alpha = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence co-ordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passes through P $(8, 3\sqrt{3})$ and S'(–5,0)

$\therefore$ Its equation is $y - 3\sqrt{3} = \frac{0 - 3\sqrt{3}}{-5 - 8}(x - 8)$ or $13y - 39\sqrt{3} = 3\sqrt{3}x - 24\sqrt{3}$

or $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$.