1. Newton-Leibnitz formula :

Let $\frac{d}{dx}(F(x)) = f(x)\ \forall\ x \in (a, b)$. Then $\int_a^b f(x)\,dx = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$.

Note : (i) If a > b, then $\int_a^b f(x)\,dx = \lim_{x\to b^+} F(x) - \lim_{x\to a^-} F(x)$.

(ii) If F(x) is continuous at a and b, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Example # 1 : Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Solution : $\because \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = \left[\ell n(x+1) - \ell n(x+2)\right]_1^2$$

$$= \ell n3 - \ell n4 - \ell n2 + \ell n3 = \ell n\left(\frac{9}{8}\right)$$

Self practice problems :

Evaluate the following

(1) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$ (2) $\int_0^{\pi/4} \left(2\sec^2 x + x^3 + 2\right)dx$ (3) $\int_0^{\pi/3} \frac{x}{1+\sec x}\,dx$

Ans. (1) $5 - \frac{5}{2}\left(9\ell n\frac{5}{4} - \ell n\frac{3}{2}\right)$ (2) $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$

(3) $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\,\ell n\left(\frac{2}{\sqrt{3}}\right)$

2. Properties of Definite Integration :

Property (1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. definite integral is independent of variable of integration.

Property (2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property (3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b].

Example # 2 : If f(x) = $\begin{cases} x+5 & : \ x<2 \\ 2x^2+1 & : \ x \ge 2 \end{cases}$, then find $\int_0^4 f(x)\,dx$.

Solution : $\int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_0^2 (x+5)\,dx + \int_2^4 (2x^2+1)\,dx = \left[\frac{x^2}{2}+5x\right]_0^2 + \left[\frac{2x^3}{3}+x\right]_2^4$

$= (2 + 10) + \left(\frac{128}{3}+4\right) - \left(\frac{16}{3}+2\right) = 12 + \frac{112}{3} + 2 = 14 + \frac{112}{3} = \frac{154}{3}$

Example # 3 : Evaluate $\int_2^8 |x-5|\,dx$.

Solution : $\int_2^8 |x-5|\,dx = \int_2^5 (-x+5)\,dx + \int_5^8 (x-5)\,dx = 9$

Example # 4 : Show that $\int_0^2 (2x+1)\,dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Solution : L.H.S. = $x^2 + x]_0^2$ = 4 + 2 = 6

R.H.S. = 25 + 5 – 0 + (4 + 2) – (25 + 5) = 6

∴ L.H.S. = R.H.S

Self practice problems :

Evaluate the following

(4) $\int_0^2 |x^2+2x-3|\,dx$. (5) $\int_0^3 [x]\,dx$, where [x] is integral part of x.

(6) $\int_0^9 \left[\sqrt{t}\right] dt$.

Ans. (4) 4 (5) 3 (6) 13

Property (4) $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

Example # 5 : Evaluate $\int_0^1 \frac{e^{-x}dx}{1+e^x}\,dx$

Solution : I = $\int_0^1 \frac{e^{-x}dx}{1+e^x}\,dx = \int_0^1 \frac{dx}{e^x(1+e^x)}$ Put $e^x = t$ ∴ $e^x dx = dt$

$\int_0^1 \frac{dy}{t^2(t+1)} = \int_1^e \left(\frac{1}{1+t} - \frac{t-1}{t^2}\right) dt = \left|\log(1+t) - \log t - \frac{1}{t}\right|_1^e$

$= (\log(1 + e) - \log e - \frac{1}{e}) - (\log 2 - \log 1 - 1)$

$\log(1 + e) - \frac{1}{2} - \log 2$

Example # 6 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$.

Solution : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = 2\int_{0}^{\frac{\pi}{2}} \cos x \, dx = 2$ (∵ cos x is even function)

Example # 7 : Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx$.

Solution : Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right) \Rightarrow f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function ∴ $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

Self practice problems :

Evaluate the following

(7) $\int_{-1}^{1} |x| \, dx$. (8) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$. (9) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx$.

Ans. (7) 1 (8) 0 (9) 1

Property (5) $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

Further $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Example # 8 : Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)} dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)} dx = \frac{\pi}{4}$.

Solution : Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)} = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)} dx$$

on adding, we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x) + g(\cos x)} + \frac{g(\cos x)}{g(\cos x) + g(\sin x)} \right) dx = \int_0^{\frac{\pi}{2}} dx \Rightarrow I = \frac{\pi}{4}$$

Self practice problems:

Evaluate the following

(10) $\int_0^{\pi} \frac{x}{1+\sin x} dx.$ (11) $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x + \cos x} dx.$ (12) $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx.$

(13) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$

Ans. (10) π (11) $\frac{\pi}{2\sqrt{2}} \log_e \left(1+\sqrt{2}\right)$ (12) $\frac{\pi^2}{16}$ (13) $\frac{\pi}{12}$

Property (6) $\int_0^{2a} f(x)\, dx = \int_0^{a} (f(x) + f(2a-x))\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Example # 9 : Evaluate $\int_0^{2\pi} \sin^{100} x \cos^{99} x\, dx$

Solution : $I = \int_0^{2\pi} \sin^{100} x \cos^{99} x\, dx$

here, $f(x) = \sin^{100} x \cos^{99} x$ for which $f(2\pi - x) = f(x)$

$I = 2\int_0^{\pi} \sin^{100} (\pi - x) \cos^{99} (\pi - x)\, dx$

$I = -2\int_0^{\pi} \sin^{100} x \cos^{99} x\, dx$

$-I = 2I$ ∴ $3I = 0$ $I = 0$

Example # 10 : Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} dx.$

Solution : Let $f(x) = \frac{1}{1+2\sin^2 x}$ ⇒ $f(\pi - x) = f(x)$ ⇒

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{1+\tan^2 x + 2\tan^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{1+3\tan^2 x}$$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\frac{\pi}{2}}$ $\because$ $\tan\frac{\pi}{2}$ is undefined, we take limit

$= \frac{2}{\sqrt{3}}\left[\operatorname*{Lt}_{x\to\frac{\pi}{2}^-}\tan^{-1}\left(\sqrt{3}\tan x\right)-\tan^{-1}\left(\sqrt{3}\tan 0\right)\right] = \frac{2}{\sqrt{3}}\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$

Alternatively : $\int_0^{\pi}\frac{dx}{1+2\sin^2 x} = \int_0^{\pi}\frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x+2}\,dx = \int_0^{\pi}\frac{\operatorname{cosec}^2 x\ dx}{\cot^2 x+3}$

Observe that we are not converting in terms of tan x as it is not continuous in (0, π)

$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi} = -\frac{1}{\sqrt{3}}\left[\operatorname*{Lt}_{x\to\pi^-}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)-\operatorname*{Lt}_{x\to 0^+}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]$

$= -\frac{1}{\sqrt{3}}\left[-\frac{\pi}{2}-\frac{\pi}{2}\right] = \frac{\pi}{\sqrt{3}}$

Example # 11 : Prove that $\int_0^{\frac{\pi}{2}}\ell n\ \sin x\ dx = \int_0^{\frac{\pi}{2}}\ell n\ \cos x\ dx = \int_0^{\frac{\pi}{2}}\ell n\ (\sin 2x)\ dx = -\frac{\pi}{2}\ell n\, 2$.

Solution : Let $I = \int_0^{\frac{\pi}{2}}\ell n\ \sin x\ dx$ ....(i)

$\Rightarrow$ $I = \int_0^{\frac{\pi}{2}}\ell n\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$ (by property P – 5)

$I = \int_0^{\frac{\pi}{2}}\ell n\ (\cos\ x)\ dx$ ....(ii)

Adding (i) and (ii)

$2I = \int_0^{\frac{\pi}{2}}\ell n\ (\sin\ x\ .\ \cos x)\ dx = \int_0^{\frac{\pi}{2}}\ell n\left(\frac{\sin 2x}{2}\right)dx$

$2I = \int_0^{\frac{\pi}{2}}\ell n\ (\sin\ 2x)\ dx - \int_0^{\frac{\pi}{2}}\ell n\ 2\ dx$ $\Rightarrow 2I = I_1 - \frac{\pi}{2}\ell n\, 2$ ...(iii)

where $I_1 = \int_0^{\frac{\pi}{2}}\ell n\ (\sin\ 2x)\ dx$

put 2x = t $\Rightarrow$ $dx = \frac{1}{2}dt$

L . L : x = 0 $\Rightarrow$ t = 0

U . L : $x = \frac{\pi}{2}$ $\Rightarrow$ t = π

$\Rightarrow$ $I_1 = \int_0^{\pi}\ell n\ (\sin\ t)\ .\ \frac{1}{2}dt = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}}\ell n\ (\sin\ t)\ dt$ (by using property P – 6)

$\Rightarrow$ $I_1 = I$ $\therefore$ (iii) gives $I = -\frac{\pi}{2}\ell n\, 2$

Self practice problems :

Evaluate the following

(14) $\int_0^1 \frac{\sin^{-1} x}{x}\, dx.$ (15) $\int_0^{\pi} x \ell n \sin x\, dx.$

Ans. (14) $\frac{\pi}{2} \ell n\ 2$ (15) $-\frac{\pi^2}{2} \ell n\ 2$

Property - (7)
Integration of Periodic functions :

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\, dx = n\int_0^{T} f(x)\, dx, n \in z$ (ii) $\int_a^{a+nT} f(x)\, dx = n\int_0^{T} f(x)\, dx, n \in z, a \in R$

(iii) $\int_{mT}^{nT} f(x)\, dx = (n-m)\int_0^{T} f(x)\, dx, m, n \in z$ (iv) $\int_{nT}^{a+nT} f(x)\, dx = \int_0^{a} f(x)\, dx, n \in z, a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\, dx = \int_a^{b} f(x)\, dx, n \in z, a, b \in R$

Example # 12 : Evaluate $\int_{-1}^{2} e^{\{x\}}\, dx.$

Solution : $\int_{-1}^{2} e^{\{x\}}\, dx = \int_{-1}^{-1+3} e^{\{x\}}\, dx = 3\int_0^{1} e^{\{x\}}\, dx = 3\int_0^{1} e^{x}\, dx = 3(e-1)$

Example # 13 : Evaluate $\int_0^{n\pi+v} |\cos x|\, dx$, $\frac{\pi}{2} < v < \pi$ and $n \in z$.

Solution : $\int_0^{n\pi+v} |\cos x|\, dx = \int_0^{v} |\cos x|\, dx + \int_v^{n\pi+v} |\cos x|\, dx$

$= \int_0^{\frac{\pi}{2}} \cos x\, dx - \int_{\pi/2}^{v} \cos x\, dx + n\int_0^{\pi} |\cos x|\, dx$

$= (1-0) - (\sin v - 1) + 2n\int_0^{\frac{\pi}{2}} \cos x\, dx = 2 - \sin v + 2n(1-0) = 2n + 2 - \sin v$

Self practice problems :

Evaluate the following

(16) $\int_{-1}^{2} e^{\{3x\}}\, dx.$ (17) $\int_0^{2000\pi} \frac{dx}{1+e^{\sin x}}\, dx.$ (18) $\int_{\pi}^{\frac{5\pi}{4}} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx.$

Ans. (16) 3 (e – 1) (17) 1000 π (18) $\frac{\pi}{4}$

3. Estimation of Integrals :

Method (1) If ψ(x) ≤ f(x) ≤ φ (x) for a ≤ x ≤ b, then

$\int_a^{b} \psi(x)\, dx \le \int_a^{b} f(x)\, dx \le \int_a^{b} \phi(x)\, dx$

Method (2) If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and

if f(x) is monotonically increasing in (a, b), then $f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

Example # 14 : Estimate the value of $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$.

Solution : Let $f(x) = \frac{\sin x}{x}$ $f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$

$\Rightarrow$ f(x) is monotonically decreasing function.

f(0) is not defined, so we evaluate

$\underset{x\to 0^+}{\text{Lt}} f(x) = \underset{x\to 0^+}{\text{Lt}} \frac{\sin x}{x} = 1$. Take $f(0) = \underset{x\to 0^+}{\text{Lt}} f(x) = 1$

$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow \frac{2}{\pi}.\left(\frac{\pi}{2} - 0\right) < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < 1.\left(\frac{\pi}{2} - 0\right) \Rightarrow 1 < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$

Example # 15 : Estimate the value of $\int_0^1 e^{x^2}dx$ by using $\int_0^1 e^x dx$.

Solution : For $x \in (0, 1)$, $e^{x^2} < e^x$

$\Rightarrow$ $1 \times 1 < \int_0^1 e^{x^2}dx < \int_0^1 e^x dx$

$1 < \int_0^1 e^{x^2}dx < e - 1$

Self practice problems :

(19) Prove the following : $\int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\,dx$

(20) Prove the following : $e^{-\frac{1}{4}} < \int_0^1 e^{x^2 - x}\,dx < 1$

(21) Prove the following : $1 < \int_0^{\pi/2} \sqrt{\sin x}\,dx < \sqrt{\frac{\pi}{2}}$

4. Leibnitz Theorem :

If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\frac{dF(x)}{dx} = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

Proof : Let $P(t) = \int f(t)\,dt$ $\Rightarrow$ $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt = P(h(x)) - P(g(x))$

$$\Rightarrow \quad \frac{dF(x)}{dx} = P'(h(x))\,h'(x) - P'(g(x))\,g'(x) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

Example # 16 : If $F(x) = \int_x^{x^2} \sqrt{\sin t}\; dt$, then find F'(x).

Solution : $F'(x) = 2x \cdot \sqrt{\sin x^2} - 1 \cdot \sqrt{\sin x}$

Example # 17 : Evaluate $\lim\limits_{x \to \infty} \dfrac{\left(\int_0^x e^{t^2}\, dt\right)^2}{\int_0^x e^{2t^2}\, dt}$.

Solution : $\lim\limits_{x \to \infty} \dfrac{\left(\int_0^x e^{t^2}\, dt\right)^2}{\int_0^x e^{2t^2}\, dt}$ $\quad\left(\dfrac{\infty}{\infty} \text{ form}\right)$

Applying L' Hospital rule

$$= \lim_{x \to \infty} \frac{2 \cdot \int_0^x e^{t^2} dt \cdot e^{x^2}}{1 \cdot e^{2x^2}} = \lim_{x \to \infty} \frac{2 \cdot \int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x \to \infty} \frac{2 \cdot e^{x^2}}{2x \cdot e^{x^2}} = 0$$

Self practice Problems :

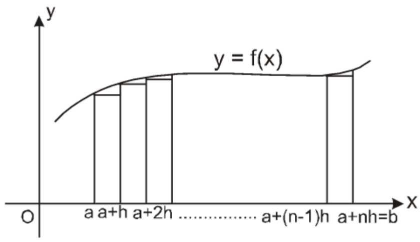
(22) If $f(x) = \int_0^{x^3} \sqrt{\cos t}\; dt$, find f'(x). (23) If $f(x) = \int_x^{x^2} x^2 \sin t\; dt$, then find f'(x).

(24) Evaluate $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} \cos t^2\; dt}{x \sin x}$.

Ans. (22) $3x^2 \sqrt{\cos x^3}$ (23) $x^2 (2x \sin x^2 - \sin x) + (\cos x - \cos x^2)\, 2x$ (24) 1

5. Definite Integral as a limit of sum :

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\frac{b-a}{n} = h$.
Let S_n denotes the area of these n rectangles.
Then, $S_n = hf(a) + hf(a + h) + hf(a + 2h) + \ldots\ldots\ldots + hf(a + (n - 1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\,dx = \mathop{Lt}_{n\to\infty} S_n$

$$\int_a^b f(x)\,dx = \mathop{Lt}_{n\to\infty}\sum_{r=0}^{n-1} h\ f(a+rh) = \mathop{Lt}_{n\to\infty}\sum_{r=0}^{n-1}\left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)\ r}{n}\right)$$

Note : 1. We can also write

S_n = hf(a + h) + hf (a + 2h) + ...+ hf(a + nh) and

$$\int_a^b f(x)\,dx = \mathop{Lt}_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right) r\right)$$

2. If a = 0, b = 1, $\int_0^1 f(x)\,dx = \mathop{Lt}_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as defînte integral :

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\mathop{Lt}_{n\to\infty}\Sigma$ by $\int$

Step 2. Evaluate $\mathop{Lt}_{n\to\infty}\left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For Example $\mathop{Lt}_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)\,dx$ $\left(\because \mathop{Lt}_{n\to\infty}\left(\frac{r}{n}\right)\Big|_{r=1} = 0,\ \mathop{Lt}_{n\to\infty}\left(\frac{r}{n}\right)\Big|_{r=np} = p\right)$

Example # 18 : Evaluate $\mathop{Lt}_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+4)}..........\frac{1}{6n^2}\right]$

Solution : The given limit is $\mathop{Lt}_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+4)}..........\frac{1}{6n^2}\right]$

$$\mathop{Lt}_{n\to\infty}\sum_{r=1}^{n} n\frac{1}{(n+r)(n+2r)}$$

$$\mathop{Lt}_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)(1+2\frac{r}{n})} = \int_0^1\frac{dx}{(1+x)(1+2x)} = \int_0^1 \frac{-1}{1+x}+\frac{2}{1+2x}\,dx$$

$[-\log(1+x) + \log(1+2x)]_0^1$

= [(–log2 + log3) – (–log1 + log1)] = log (3/2)

Example # 19 : Evaluate $\mathop{Lt}_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+.........+\frac{3}{5n}\right]$.

Solution : $$\mathop{Lt}_{n\to\infty}\sum_{r=1}^{2n}\frac{n+r}{n^2+r^2} = \mathop{Lt}_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$\because \quad \lim_{n\to\infty}\left(\frac{r}{n}\right) = 0$, when r = 1, lower limit = 0

and $\lim_{n\to\infty}\left(\frac{r}{n}\right) = \lim_{n\to\infty}\left(\frac{2n}{n}\right) = 2$, when r = 2n, upper limit = 2

$$\int_0^2 \frac{1+x}{1+x^2}\,dx = \int_0^2 \frac{1}{1+x^2}\,dx + \frac{1}{2}\int_0^2 \frac{2x}{1+x^2}\,dx$$

$$= [\tan^{-1}x]_0^2 + \left[\frac{1}{2}\log_e(1+x^2)\right]_0^2 = \tan^{-1}2 + \frac{1}{2}\ell n\,5$$

Example # 20 : Evaluate $\lim_{n\to\infty}\frac{(1^2+2^2+3^2....n^2)(1^3+2^3+3^3.....n^3)}{1^6+2^6+3^6.....n^6}$.

Solution : The given limit is $\lim_{n\to\infty}\frac{\sum_{r=1}^{n} r^2\sum_{r=1}^{n} r^3}{\sum_{r=1}^{n} r^6}$

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\times\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^6} = \frac{\int_0^1 x^2dx\int_0^1 x^3dx}{\int_0^1 x^6dx} = \frac{\left[\frac{x^3}{3}\right]_0^1\left[\frac{x^4}{4}\right]_0^1}{\left[\frac{x^7}{7}\right]_0^1} = \frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{7}} = \frac{7}{12}$$

Self practice Problems :

Evaluate the following limits

(25) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+....+\frac{1}{\sqrt{n^2+n^2}}\right]$

(26) $\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.......+\frac{1}{5n}\right]$

(27) $\lim_{n\to\infty}\frac{1}{n^2}\left[\sin^3\frac{\pi}{4n}+2\sin^3\frac{2\pi}{4n}+3\sin^3\frac{3\pi}{4n}+........+n\sin^3\frac{n\pi}{4n}\right]$

(28) $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

(29) $\lim_{n\to\infty}\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+......+\sqrt{\frac{n}{n+3(n-1)}}\right]$

Ans. (25) $2\left(\sqrt{2}-1\right)$ (26) $\ell n\,5$ (27) $\frac{\sqrt{2}}{9\pi^2}(52-15\pi)$

(28) $\frac{\pi}{2}$ (29) 2

6. Reduction formulae in definite Integrals :

(i) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Proof : $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$I_n = \left[-\sin^{n-1} x \ \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \ \sin^{n-2} x \ . \ \cos^2 x \ dx$

$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \ . \ (1-\sin^2 x) \ dx \quad = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \ dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \ dx$

$I_n + (n-1) I_n = (n-1) I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

(ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx$, then $I_n + I_{n-2} = \frac{1}{n-1}$

(ii) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \ . \ \cos^n x \ dx$, then $I_{m,n} = \frac{m-1}{m+n} I_{m-2,\,n}$

7. Walli's Formula

Let, $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \ . \ \cos^n x \ dx$,

$$I_{m,n} = \begin{cases} \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........} \dfrac{\pi}{2} & \text{when both m, n are even} \\ \dfrac{(m-1)\ (m-3)\ (m-5)\ ....(n-1)\ (n-3)\ (n-5).......}{(m+n)\ (m+n-2)\ (m+n-4)........} & \text{otherwise} \end{cases}$$

Example # 21 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^2 x(\sin x + \cos x) \ dx$.

Solution : Given integral = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \ \cos^2 x \ dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$ ($\because$ $\sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even) $= 2. \frac{1.2}{5.3.1} = \frac{4}{15}$

Example # 22 : Evaluate $\int_0^{\pi} x \sin^5 x \cos^6 x \ dx$.

Solution : Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x \ dx$

$$I = \int_0^{\pi} (\pi - x)\ \sin^5(\pi - x)\ \cos^6(\pi - x)\ dx$$

$$= \pi \int_0^{\pi} \sin^5 x . \cos^6 x\, dx - \int_0^{\pi} x \sin^5 x\ .\ \cos^6 x\ dx \qquad \Rightarrow 2I = \pi\ .\ 2 \int_0^{\frac{\pi}{2}} \sin^5 x\ .\ \cos^6 x\ dx$$

$$I = \pi\ \frac{4\ .\ 2\ .\ 5\ .\ 3\ .\ 1}{11\ .\ 9\ .\ 7\ .\ 5\ .\ 3\ .\ 1} \qquad I = \frac{8\pi}{693}$$

Example # 23 : Evaluate $\int_0^1 x^3(1-x)^5 dx$.

Solution : Put $x = \sin^2\theta \Rightarrow dx = 2 \sin \theta \cos \theta\, d\theta$

L . L : $x = 0 \Rightarrow \theta = 0$

U.L. : $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \int_0^1 x^3(1-x)^5 dx = \int_0^{\frac{\pi}{2}} \sin^6 \theta (\cos^2 \theta)^5\ 2\ .\ \sin \theta\ .\ \cos \theta\, d\theta$$

$$= 2\ .\int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^{11} \theta\ d\theta = 2\ .\ \frac{6\ .\ 4\ .\ 2\ .\ 10\ .\ 8\ .\ 6\ .\ 4\ .\ 2}{18\ .\ 16\ .\ 14\ .\ 12\ .\ 10\ .\ 8\ .\ 6\ .\ 4\ .\ 2} = \frac{1}{504}$$

Self practice Problems:

Evaluate the following

(30) $\int_0^{\frac{\pi}{2}} \sin^5 x\ dx$. (31) $\int_0^{\frac{\pi}{2}} \sin^5 x\ \cos^4 x\ dx$. (32) $\int_0^1 x^6\ \sin^{-1} x\ dx$.

(33) $\int_0^a x\ \left(a^2 - x^2\right)^{\frac{7}{2}}\ dx$. (34) $\int_0^2 x^{3/2}\ \sqrt{2-x}\ dx$.

Ans. (30) $\frac{8}{15}$ (31) $\frac{8}{315}$ (32) $\frac{\pi}{14} - \frac{16}{245}$ (33) $\frac{a^9}{9}$ (34) $\frac{\pi}{2}$

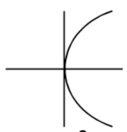
8. Area under the curve :

(i) Curve-tracing :

To find approximate shape of a curve, the following phrases are suggested :

(a) Symmetry:

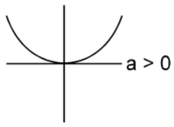
- **Symmetry about x-axis :**
  If all the powers of 'y' in the equation are even then the curve (graph) is symmetrical about the x-axis.



a > 0

E.g. : $y^2 = 4ax$.

- **Symmetry about y-axis :**
  If all the powers of 'x' in the equation are even then the curve (graph) is symmetrical about the y-axis.

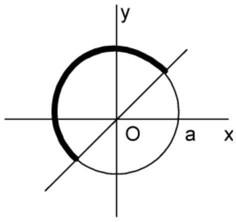


E.g. : $x^2 = 4ay$.

- **Symmetry about both axis :**
  If all the powers of 'x' and 'y' in the equation are even, then the curve (graph) is symmetrical about the axis of 'x' as well as 'y' .

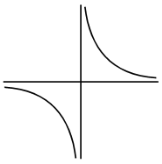
E.g. : $x^2 + y^2 = a^2$.

- **Symmetry about the line y = x :**
  If the equation of the curve remain unchanged on interchanging 'x' and 'y', then the curve (graph) is symmetrical about the line y = x.



E.g. : $x^2 + y^2 = a^2$

- **Symmetry in opposite quadrants :**
  If the equation of the curve (graph) remain unaltered when 'x' and 'y' are replaced by '–x' and '–y' respectively, then there is symmetry in opposite quadrants.



E.g. : $xy = c^2$

(b) Find the points where the curve crosses the x-axis and the y-axis.

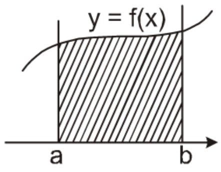
(c) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Examine intervals when f(x) is increasing or decreasing

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$

(ii) Area included between the curve y = f(x), x-axis and the ordinates x = a, x = b

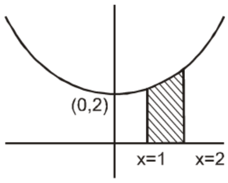
(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $\int_a^b f(x)\, dx$



Graph of y = f(x)

Example # 24 : Find the area enclosed between the curve $y = x^2 + 2$, x-axis, $x = 1$ and $x = 2$.

Solution: Graph of $y = x^2 + 2$



Area = $\int_1^2 \left(x^2 + 2\right) dx = \left[\frac{x^3}{3} + 2x\right]_1^2 = \frac{13}{3}$

Example # 25 : Find area bounded by the curve $y = \ell n\, x + \tan^{-1} x$ and x-axis between ordinates $x = 1$ and $x = 2$.

Solution : $y = \ell n\, x + \tan^{-1} x$

Domain $x > 0$, $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

y is increasing and $x = 1$, $y = \frac{\pi}{4}$ $\Rightarrow$ y is positive in [1, 2]

$\therefore$ Required area = $\int_1^2 (\ell n\, x + \tan^{-1} x)\, dx$

$= \left[x\ \ell n\ x - x + x \tan^{-1} x - \frac{1}{2}\ \ell n\ (1 + x^2)\right]_1^2$

$= 2\, \ell n\, 2 - 2 + 2 \tan^{-1} 2 - \frac{1}{2}\, \ell n\, 5 - 0 + 1 - \tan^{-1} 1 + \frac{1}{2}\, \ell n\, 2$

$= \frac{5}{2}\, \ell n\, 2 - \frac{1}{2}\, \ell n\, 5 + 2 \tan^{-1} 2 - \frac{\pi}{4} - 1$

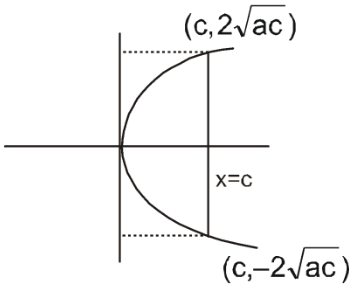
Note : If a function is known to be positive valued then graph is not necessary.

Example # 26 : The area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by the double ordinate and its distance from the vertex. Find the value of k ?

Solution : Consider $y^2 = 4ax$, $a > 0$ and $x = c$

Area by double ordinate = $2\int_0^c 2\sqrt{a}\sqrt{x}\ dx = \frac{8}{3}\sqrt{a}\ c^{3/2}$

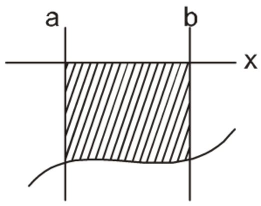
Area by double ordinate = k (Area of rectangle)



Figure

$$\frac{8}{3}\sqrt{a}\ \ c^{3/2} = k\ \ 4\sqrt{a}\ \ c^{3/2} \Rightarrow k = \frac{2}{3}$$

(b) If f(x) < 0 for x ∈ [a, b], then area bounded by curve y = f(x), x-axis, x = a and x = b is $-\int_a^b f(x)\ dx$



Graph of y = f(x)

Example # 27 : Find area bounded by y = $\log_{\frac{1}{2}} x$ and x-axis between x = 1 and x = 2

Solution : A rough graph of y = $\log_{\frac{1}{2}} x$ is as follows

Area = $-\int_1^2 \log_{\frac{1}{2}} x\ dx = -\int_1^2 \log_e x\ .\ \log_{\frac{1}{2}} e\ dx$

$= -\log_{\frac{1}{2}} e\ .\ [x\ \ \log_e x - x]_1^2$

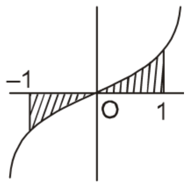
$= -\log_{\frac{1}{2}} e\ .\ (2\log_e 2 - 2 - 0 + 1)$ $\qquad = -\log_{\frac{1}{2}} e\ .\ (2\log_e 2 - 1)$

Note :– If y = f(x) does not change sign in [a, b], then area bounded by y = f(x), x-axis between ordinates x = a, x = b is $\left|\int_a^b f(x)\ dx\right|$

(c) If f(x) $\geq$ 0 for x ∈ [a,c] and f(x) $\leq$ 0 for x ∈ [c,b] (a < c < b) then area bounded by curve y = f(x) and x-axis between x = a and x = b is $\int_a^c f(x)\ dx - \int_c^b f(x)\ dx$

Example # 28 : Find the area bounded by y = x^3 and x- axis between ordinates x = –1 and x = 1

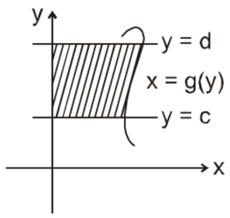
Solution : Required area = $\int_{-1}^{0} -x^3 dx + \int_0^1 x^3 dx = \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_0^1$



Graph of y = x^3

$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$

Note : Most general formula for area bounded by curve y = f(x) and x- axis between ordinates x = a and x = b is $\int_a^b |f(x)|\,dx$



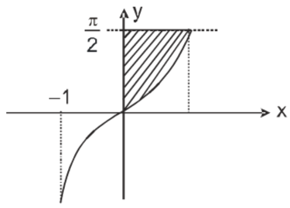
Graph of x = g(y)

Example # 29 : Find area bounded between y = $\sin^{-1}x$ and y–axis between y = 0 and y = $\frac{\pi}{2}$.

Solution : y = $\sin^{-1} x$

$\Rightarrow$ x = sin y

Required area = $\int_0^{\frac{\pi}{2}} \sin y \; dy$



Graph of y = $\sin^{-1}x$

= $\left.-\cos y\right]_0^{\frac{\pi}{2}}$ = – (0 – 1) = 1

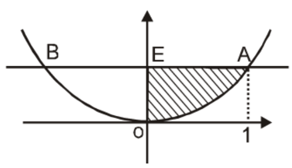
Note : The area in above Example can also evaluated by integration with respect to x.

Area = (area of rectangle formed by x = 0, y = 0 , x = 1, y = $\frac{\pi}{2}$) – (area bounded by y = $\sin^{-1}x$, x–axis between x = 0 and x = 1)

$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \left(x\sin^{-1}x + \sqrt{1-x^2}\right)_0^1 = \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1\right) = 1$

Example # 30 : Find the area bounded by the parabola x^2 = y, y-axis and the line y = 1.

Solution : Graph of y = x^2

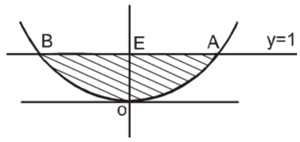


Area OEBO = Area OAEO = $\int_0^1 |x|\,dy = \int_0^1 \sqrt{y}\,dy = \frac{2}{3}$

Example # 31 : Find the area bounded by the parabola x^2 = y and line y = 1.

Solution : Graph of y = x^2

Required area is area OABO

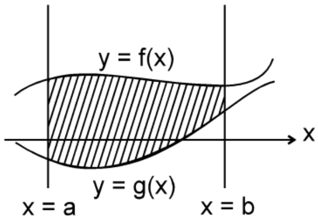


= 2 area (OAEO) = $2\int_0^1 |x|\, dy = 2\int_0^1 \sqrt{y}\ dy = \frac{4}{3}$.

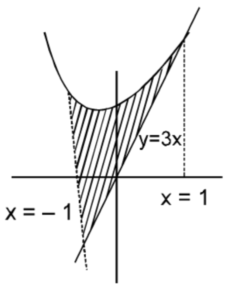
Note : General formula for area bounded by curve x = g(y) and y–axis between abscissa y = c and y = d is $\int_{y=c}^{d} |g(y)|\, dy$

(iii) **Area between two curves**

If f(x) $\geq$ g(x) for x∈[a,b] then area bounded by curves (graph) y = f(x) and y = g(x) between ordinates x = a and x = b is $\int_a^b (f(x)-g(x))\ dx$.



Example # 32 : Find the area enclosed by curve (graph) y = x^2 + x + 1 and its tangent at (1,3) between ordinates x = – 1 and x = 1.



Solution : $\frac{dy}{dx} = 2x + 1$

$\frac{dy}{dx} = 3$ at x = 1

Equation of tangent is

y – 3 = 3 (x – 1)
y = 3x

Required area $= \int_{-1}^{1} (x^2+x+1-3x)\ dx = \int_{-1}^{1} (x^2-2x+1)\ dx = \left.\frac{x^3}{3}-x^2+x\right]_{-1}^{1}$

$= \left(\frac{1}{3}-1+1\right) - \left(-\frac{1}{3}-1-1\right) = \frac{2}{3} + 2 = \frac{8}{3}$

Note : Area bounded by curves y = f(x) and y = g (x) between ordinates x = a and x = b is $\int_a^b |f(x)-g(x)|\, dx$.

Example # 33 : Find the area of the region bounded by y = sin x, y = cos x and ordinates x = 0, x = π/2

Solution : $\int_0^{\pi/2} |\sin x - \cos x|\ dx$

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = 2(\sqrt{2} - 1)$$

Self practice problems :

(35) Find the area between curve $y = x^2 - 3x + 2$ and x–axis
(i) bounded between $x = 1$ and $x = 2$.
(ii) bound between $x = 0$ and $x = 2$.

(36) Find the area included between curves $y = 2x - x^2$ and $y + 3 = 0$.

(37) Find area between curves $y = x^2$ and $y = 3x - 2$ from $x = 0$ to $x = 2$.

(38) Find the area of the region bounded by the x-axis and the curves defined by $y = \tan x$, $\left(\text{where } \frac{-\pi}{3} \le x \le \frac{\pi}{3}\right)$ and $y = \cot x$ $\left(\text{where } \frac{\pi}{6} \le x \le \frac{2\pi}{3}\right)$.

(39) Curves $y = \sin x$ and $y = \cos x$ intersect at infinite number of points forming regions of equal area between them calculate area of one such region.

(40) Find the area included between $y = \tan^{-1}x$, $y = \cot^{-1}x$ and y–axis.

(41) Find area common to circle $x^2 + y^2 = 2$ and the parabola $y^2 = x$.

(42) Find the area bounded by the curve $|y| + \frac{1}{2} = e^{-|x|}$.

(43) Find are bounded by $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$, $x \ge 0$.

Ans. (35) (i) $\frac{1}{6}$ (ii) 1 (36) $\frac{32}{3}$ (37) 1

(38) $\ell n \frac{3}{2}$ (39) $2\sqrt{2}$ (40) $\ell n2$ (41) $\frac{\pi}{3} - \frac{\sqrt{3}}{2} - \frac{2}{3}$

(42) $2(1 - \ell n2)$ (43) $\left(\frac{3\pi - 8}{6}\right)a^2$