Differential Equation

Differential Equation

1. Introduction :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

(i) **Ordinary Differential Equation :**
If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for Example $\frac{dy}{dx} + \frac{dz}{dx} = y + z$,

$\frac{dy}{dx} + xy = \sin x$, $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x$,

$k\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$, $y = x\frac{dy}{dx} + k\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

(ii) **Partial differential equation :** If the dependent variables depend on two or more independent variables, then it is known as partial differential equation

for Example $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y} = ax$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

2. Order and degree of a differential equation :

(i) **Order :** Order is the highest differential appearing in a differential equation.

(ii) **Degree :** It is determined by the highest degree of the highest order derivative present in it after the differential equation is cleared of radicals and fractions so far as the derivatives are concerned.

Note : In the differential equation, all the derivatives should be expressed in the polynomial form

$$f_1(x, y)\left[\frac{d^my}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \text{........ } f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

Example # 1: Find the order & degree of following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ (ii) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution : (i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}} \Rightarrow \left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right) = 1$

hence order is 1 and degree is 2

(ii) $\frac{dy}{dx} = \sin(x + y)$

hence order is 1 and degree is 1

Self Practice Problems :

(1) Find order and degree of the following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ (ii) $e^{\left(\frac{dy}{dx} - \frac{d^3y}{dx^3}\right)} = \ell n\left(\frac{d^5y}{dx^5} + 1\right)$ (iii) $\left[\left(\frac{dy}{dx}\right)^{1/2} + y\right]^2 = \frac{d^2y}{dx^2}$

Ans. (1) (i) order = 1, degree = 2
(ii) order = 5, degree = not applicable.
(iii) order = 2, degree = 2

3. Formation of differential equation :

Differential equation corresponding to a family of curve will have :

(i) Order exactly same as number of essential arbitrary constants in the equation of curve.

(ii) No arbitrary constant present in it.

The differential equation corresponding to a family of curve can be obtained by using the following
steps:

(a) Identify the number of essential arbitrary constants in equation of curve.

NOTE : If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant.

(b) Differentiate the equation of curve till the required order.

(c) Eliminate the arbitrary constant from the equation of curve and additional equations obtained in step (ii) above.

Example # 2 : Form a differential equation of family of straight lines passing through origin.

Solution : Family of straight lines passing through origin is y = mx where'm' is a parameter.

Differentiating w.r.t. x $\frac{dy}{dx} = m$

Eliminating 'm' from both equations, we obtain

$\frac{dy}{dx} = \frac{y}{x}$ which is the required differential equation.

Example # 3 : Form the differential equation of all circles touching the x-axis at the origin and centre on the y- axis

Solution : Such family of circle is given by $x^2 + (y - a)^2 = a^2 \therefore x^2 + y^2 - 2ay = 0$ ...(i)

differentiating, $2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$

or $x + y\frac{dy}{dx} = a\frac{dy}{dx}$

substituting the value of a in equation (i)

$\Rightarrow$ $(x^2 - y^2)\dfrac{dy}{dx} = 2xy$
(order is 1 again and degree 1)

Self Practice Problems :

(2) Obtain a differential equation of the family of curves y = a sin (bx + c) where a and c being arbitrary constant.

(3) Show that the differential equation of the system of parabolas $y^2 = 4a(x - b)$ is given by $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

(4) Form a differential equation of family of parabolas with focus as origin and axis of symmetry along the x-axis.

Ans. (2) $\dfrac{d^2y}{dx^2} + b^2y = 0$ (4) $y^2 = y^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx}$

4. Solving a differential equation :

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation

NOTE : The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be two types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive) :** A relation in x and y satisfying a given differential equation and involving exactly same number of arbitrary constants as order of differential equation.

(ii) **Particular Solution :** A solution obtained by assigning values to one or more than one arbitrary constant of general solution.

5. Solution methods of first order and first degree differential equations :

A differential equation of first order and first degree is of the type $\dfrac{dy}{dx} + f(x, y) = 0$, which can also be written as : Mdx + Ndy = 0, where M and N are functions of x and y.

(i) Variables separable :

If the differential equation can be put in the form, f(x) dx = φ(y) dy we say that variables are separable and solution can be obtained by integrating each side separately.

A general solution of this will be $\int f(x)\, dx = \int \phi(y)\, dy$ + c, where c is an arbitrary constant

Example # 4 : Solve the differential equation (1 + x) y dx = (y – 1) x dy

Solution : The equation can be written as -

$\left(\dfrac{1+x}{x}\right)dx = \left(\dfrac{y-1}{y}\right)dy \Rightarrow \int\left(\dfrac{1}{x}+1\right)dx = \int\left(1-\dfrac{1}{y}\right)dy$

$\ell n\, x + x = y - \ell n y + c \Rightarrow \ell n y + \ell n x = y - x + c \Rightarrow xy = ce^{y-x}$

Example # 5 : Solve $e^{\frac{dy}{dx}} = x + 1$, given that when x = 0, y = 3

Solution : $e^{\frac{dy}{dx}} = x + 1 \Rightarrow \frac{dy}{dx} = \ell n(x + 1)$

$\int dy = \int \ell n(x+1)\,dx \Rightarrow y = (x + 1)\ell n(x + 1) - x + c$

when x = 0, y = 3 gives c = 3

hence the solution is $y = (x + 1)\ell n(x + 1) - x + 3$

Example # 6: Solve the differential equation $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$

Solution : Differential equation can be rewritten as $xy\frac{dy}{dx} = (1 + y^2)\left(1+\frac{x}{1+x^2}\right)$

$\frac{y}{1+y^2}dy = \left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$

Integrating, we get

$\frac{1}{2}\ell n(1 + y^2) = \ell nx + \tan^{-1}x + \ell nc \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1}x}$

(ii) Equations reducible to the variables separable form :

If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is $\frac{dy}{dx} = f(ax + by + c)$ a, b ≠ 0. To solve this, put ax + by + c = t.

Example # 7 : Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution : Putting 4x + y + 1 = t

$4 + \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{dt}{dx} - 4$

Given equation becomes

$\frac{dt}{dx} - 4 = t^2 \Rightarrow \frac{dt}{t^2+4} = dx$ (Variables are separated)

Integrating both sides,

$\int\frac{dt}{4+t^2} = \int dx \Rightarrow \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

Example # 8 : Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution : $\frac{dy}{dx} = \sin(x + y)$

putting x + y = t

$\frac{dy}{dx} = \frac{dt}{dx} - 1 \quad \therefore \quad \frac{dt}{dx} - 1 = \sin t \Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \frac{dt}{1+\sin t} = dx$

Integrating both sides,

$\int \frac{dt}{1+\sin t} = \int dx \quad \Rightarrow \quad \int \frac{1-\sin t}{\cos^2 t}\, dt = x + c$

$\int (\sec^2 t - \sec t \tan t)\, dt = x + c$

$\tan t - \sec t = x + c$

$-\frac{1-\sin t}{\cos t} = x + c$

$\Rightarrow \quad \sin t - 1 = x \cos t + c \cos t$ substituting the value of t

$\sin (x + y) = x \cos (x + y) + c \cos (x + y) + 1$

Self Practice Problems :

Solve the following differential equation

(5) $x^2 y \frac{dy}{dx} = (x + 1)(y + 1)$

(6) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

(7) $xy \frac{dy}{dx} = 1 + x + y + xy$

(8) $\frac{dy}{dx} = 1 + e^{x-y}$

(9) $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

(10) $\frac{dy}{dx} = x \tan (y - x) + 1$

Ans. (5) $y - \ell n (y + 1) = \ell n x - \frac{1}{x} + c$

(6) $-\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$

(7) $y = x + \ell n |x(1 + y)| + c$

(8) $e^{y-x} = x + c$

(9) $\ell n \left| \tan \frac{x+y}{2} + 1 \right| = x + c$

(10) $\sin (y - x) = e^{\frac{x^2}{2} + c}$

(iii) Homogeneous differential equations :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equation and can be solved easily by putting y = vx.

Example # 9 : Solve $x^2 dy + y(x + y)\, dx = 0$

Solution : The given differential equation can be rewritten as

$\frac{dy}{dx} = \frac{-y(x+y)}{x^2}$ or $\frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$

putting y = vx $\quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Given equation transforms to $v + x\frac{dv}{dx} = -v - v^2$

$\Rightarrow \quad \int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x} \quad \Rightarrow \quad \frac{1}{2}\int \left[\frac{1}{v} - \frac{1}{v+2}\right] dv = -\int \frac{dx}{x}$

$\ell n|v| - \ell n|v + 2| = -2\ell n|x| + \ell n c \quad c > 0$

$\left|\frac{vx^2}{v+2}\right| = c \quad \Rightarrow \quad \left|\frac{x^2 y}{2x + y}\right| = c\ ;\ c > 0$

Example # 10 : Solve : $(x^2 - y^2)\, dx + 2xy\, dy = 0$ given that y = 1 when x = 1

Solution : $\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = -\frac{1-v^2}{2v}$

$\int \frac{2v}{1+v^2}\,dv = -\int \frac{dx}{x}$

$\ell n\,(1 + v^2) = -\ell nx + c$

at $\quad x = 1, y = 1 \quad \therefore \quad v = 1$

$\ell n\,2 = c$

$\therefore \quad \ell n\left\{\left(1+\frac{y^2}{x^2}\right) . \; x\right\} = \ell n2$

$x^2 + y^2 = 2x$

(iv) Equations reducible to the homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ ....(1)

can be made homogeneous (in new variables X and Y) by substituting x = X + h and y = Y + k,

where h and k are constants to obtain, $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$ . ...(2)

These constants are chosen such that ah + bk + c = 0, and Ah + Bk + C = 0. Thus we obtain the following differential equation $\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$.

The differential equation can now be solved by substituting Y = vX.

Example # 11 : Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

Solution : Let $\quad x = X + h, \quad y = Y + k$

$\frac{dy}{dX} = \frac{d}{dX}(Y + k)$

$\frac{dy}{dX} = \frac{dY}{dX}$ ....(i)

$= 1 + 0$ ....(ii)

on dividing (i) by (ii) $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore \quad \frac{dY}{dX} = \frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4} = \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$

h & k are such that $\quad h + 2k - 5 = 0$ & $2h + k - 4 = 0$

$h = 1, k = 2$

$\therefore \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ which is homogeneous differential equation.

Now, substituting Y = vX

$\frac{dY}{dX} = v + X\frac{dv}{dX}$ $\therefore$ $X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$

$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X}$

$\int \left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right) dv = \ell n\ X + c$

$\frac{1}{2}\ \ell n\ (v + 1) - \frac{3}{2}\ \ell n\ (1 - v) = \ell n\ X + c$

$\ell n \left|\frac{v+1}{(1-v)^3}\right| = \ell n X^2 + 2c$

$\frac{(Y+X)}{(X-Y)^3}\,\frac{X^2}{X^2} = e^{2c}$

$X + Y = c'(X - Y)^3$ where $e^{2c} = c_l$

$x - 1 + y - 2 = c'\ (x - 1 - y + 2)^3$

$x + y - 3 = c'\ (x - y + 1)^3$

Special case :

Case - 1

In equation (1) if $\frac{a}{A} = \frac{b}{B}$, then the substitution ax + by = v will reduce it to the form in which variables are separable.

Example # 12 : Solve $\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$

Solution : $\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$ $\Rightarrow$ $yf'(x)dx - f(x)dy = y^2dx$

$\Rightarrow \frac{yf'(x)\ \ dx - f(x)dy}{y^2} = dx$ $\Rightarrow d\left[\frac{f(x)}{y}\right] = dx$

Integrating, we get

$\frac{f(x)}{y} = x + c$ or $f(x) = y(x + c)$

Case-2

In equation (1) if b + A = 0, then by simple cross multiplication equation (1) reduce in exact differential equation form

Example # 13 : Solve $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$

Solution : Cross multiplying,

$2xdy + y\ dy - dy = xdx - 2ydx + 5dx$

$2\ (xdy + y\ dx) + ydy - dy = xdx + 5\ dx$

$2\ d(xy) + y\ dy - dy = xdx + 5dx$

On integrating,

$$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$\Rightarrow \quad x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

Case-3

If the homogeneous equation is of the form :

$yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Self Practice Problems :

Solve the following differential equations

(11) $\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x$ given that $y = 0$ at $x = 1$

(12) $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$

(13) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y+3}$

(14) $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

(15) $\frac{dy}{dx} = \frac{3x+2y-5}{3y-2x+5}$

Ans. (11) $\sqrt{x^2+y^2} = e^{\frac{y}{x}\tan^{-1}\frac{y}{x}}$ (12) $x\sin\frac{y}{x} = C$

(13) $x + y = c\,(x - y + 6)^3$ (14) $3(2y - x) + \ell n\,(3x + 3y + 4) = C$

(15) $3x^2 + 4xy - 3y^2 - 10x - 10y = C$

(v) Linear differential equation of first order

The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y. (Where P and Q are functions of x only).

Integrating Factor (I.F.) : It is an expression which when multiplied to a differential equation converts it into an exact form.

I.F for linear differential equation = $e^{\int Pdx}$ (constant of integration will not be considered)

$\therefore$ after multiplying above equation by I.F it becomes;

$$\frac{dy}{dx}.\ e^{\int Pdx} + Py.\ e^{\int Pdx} = Q.\ e^{\int Pdx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(y.\ e^{\int Pdx}\right) = Q.\ e^{\int Pdx} \quad \Rightarrow \quad y.\ e^{\int Pdx} = \int Q.\ e^{\int Pdx} + C.$$

NOTE : Some times differential equation becomes linear, if x is taken as the dependent variable, and y as independent variable. The differential equation has then the following

form : $\frac{dx}{dy} + P_1x = Q_1$.

where P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1\,dy}$

Example # 14 : Solve $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3}$

Solution : $\frac{dy}{dx} + Py = Q \quad \Rightarrow \quad P = \frac{3x^2}{1+x^3}$

IF = $e^{\int P.dx} = e^{\int \frac{3x^2}{1+x^3}dx} = e^{\ell n(1+x^3)}$ = 1 + x^3

$\therefore$ General solution is

y(IF) = $\int Q(IF).dx + c$

y (1 + x^3) = $\int \frac{\sin^2 x}{1+x^3}$ (1 + x^3) dx + c

y(1 + x^3) = $\int \frac{1-\cos 2x}{2}$ dx + c

y(1 + x^3) = $\frac{1}{2}$ x – $\frac{\sin 2x}{4}$ + c

Example # 15 : Solve : x ℓnx $\frac{dy}{dx}$ + y = 2 ℓn x

Solution : $\frac{dy}{dx} + \frac{1}{x\ell nx}$ y = $\frac{2}{x}$

P = $\frac{1}{x\ell nx}$, Q = $\frac{2}{x}$

IF = $e^{\int P.dx} = e^{\int \frac{1}{x\ell nx}dx} = e^{\ell n(\ell nx)}$ = ℓn x

$\therefore$ General solution is

y. (ℓn x) = $\int \frac{2}{x}.\ell nx.dx + c$ $\Rightarrow$ y (ℓn x) = $(\ell n\ x)^2$ + c

Example # 16 : Solve the differential equation

t (1 + t^2) dx = (x + xt^2 – t^2) dt and it given that x = – π/4 at t = 1

Solution : t (1 + t^2) dx = [x (1 + t^2) – t^2] dt

$\frac{dx}{dt} = \frac{x}{t} - \frac{t}{(1+t^2)}$ $\Rightarrow$ $\frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$

which is linear in x

Here, P = – $\frac{1}{t}$, Q = – $\frac{t}{1+t^2}$

IF = $e^{-\int \frac{1}{t}dt}$ = $e^{-\ell nt}$ = $\frac{1}{t}$

$\therefore$ General solution is -

x . $\frac{1}{t}$ = $\int \frac{1}{t} \cdot \left(-\frac{t}{1+t^2}\right)$ dt + c

$\frac{x}{t}$ = – $\tan^{-1}$ t + c

putting x = – π/4, t = 1

– π/4 = – π/4 + c $\Rightarrow$ c = 0

$\therefore$ x = – t $\tan^{-1}$ t

(vi) Equations reducible to linear form

(a) By change of variable.

Often differential equation can be reduced to linear form by appropriate substitution of the non-linear term

Example # 17 : Solve : $y \sin x \frac{dy}{dx} = \cos x (\sin x - y^2)$

Solution : The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$2y \frac{dy}{dx} = \frac{dz}{dx}$

differential equation becomes

$\frac{\sin x}{2} \frac{dz}{dx} + \cos x . z = \sin x \cos x$

$\frac{dz}{dx} + 2 \cot x . z = 2 \cos x$ which is linear in $\frac{dz}{dx}$

IF $= e^{\int 2\cot x \, dx} = e^{2 \ell n \sin x} = \sin^2 x$ $\therefore$ General solution is -

$z . \sin^2 x = \int 2 \cos x . \sin^2 x . dx + c$ $y^2 \sin^2 x = \frac{2}{3} \sin^3 x + c$

(b) Bernoulli's equation :

Equations of the form $\frac{dy}{dx} + Py = Q.y^n$, $n \neq 0$ and $n \neq 1$

where P and Q are functions of x, is called Bernoulli's equation and can be made linear in v by dividing by y^n and putting $y^{-n+1} = v$. Now its solution can be obtained as linear differential equation in v.

e.g. : $2 \sin x \frac{dy}{dx} - y \cos x = xy^3 e^x$.

Example # 18 : Solve : $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

Solution : Dividing both sides by y^2

$\frac{1}{y^2} \frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2}$ ..... (1)

Putting $\frac{1}{y} = t$

$-\frac{1}{y^2} \frac{dy}{dx} = \frac{dt}{dx}$ $\therefore$ differential equation (1) becomes,

$-\frac{dt}{dx} - \frac{t}{x} = \frac{1}{x^2}$

$\frac{dt}{dx} + \frac{t}{x} = -\frac{1}{x^2}$ which is linear differential equation in $\frac{dt}{dx}$

IF $= e^{\int \frac{1}{x} dx} = e^{\ell n x} = x$ $\therefore$ General solution is -

$t . x = \int -\frac{1}{x^2} . x \, dx + c$ $\Rightarrow$ $tx = -\ell n x + c$ $\Rightarrow$ $\frac{x}{y} = -\ell n x + c$

Self Practice Problems :

Solve following differential equations

(16) $x(x^2 + 1)\frac{dy}{dx} = y(1 - x^2) + x^2 \ell nx$ (17) $(x + 2y^3)\frac{dy}{dx} = y$

(18) $x\frac{dy}{dx} + y = y^2 \ell n\, x$ (19) $xy^2\left(\frac{dy}{dx}\right) - 2y^3 = 2x^3$ given y = 1 at x = 1

Ans. (16) $\left(\frac{x^2+1}{x}\right)$ $y = x\,\ell n\, x - x + c$ (17) $x = y(c + y^2)$

(18) $y(1 + cx + \ell nx) = 1$ (19) $y^3 + 2x^3 = 3x^6$

6. <u>Exact differential equation</u> :

The differential equation $M + N\frac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form f(x, y) = c

e.g. $y^2\, dy + x\, dx + \frac{dx}{x} = 0$ is an exact differential equation.

NOTE: (i) The necessary condition for (1) to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) For finding the solution of exact differential equation, following results on exact differentials should be remembered :

(a) $xdy + y\,dx = d(xy)$ (b) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$ (c) $2(x\,dx + y\,dy) = d(x^2 + y^2)$

(d) $\frac{xdy - ydx}{xy} = d\left(\ell n\frac{y}{x}\right)$ (e) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ (f) $\frac{xdy + ydx}{xy} = d(\ell n\, xy)$

(g) $\frac{xdy + ydx}{x^2 y^2} = d\left(-\frac{1}{xy}\right)$

Example # 19: Solve $xdx + ydy = \frac{xdy - ydx}{x^2 + y^2}$

Solution : The differential equation can be written as $\frac{1}{2}d(x^2 + y^2) = d\{\tan^{-1}(y / x)\}$

Integrating, we get

$$\frac{1}{2}(x^2 + y^2) = \tan^{-1}(y / x) + c$$

Example # 20 : Solve : $(2x\,\ell ny)\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

Solution : The given equation can be written as

$$\ell ny\,(2x)\,dx + x^2\left(\frac{dy}{y}\right) + 3y^2\,dy = 0$$

$\Rightarrow \quad \ell ny \; d(x^2) + x^2 \, d(\ell ny) + d(y^3) = 0$

$\Rightarrow \quad d(x^2 \, \ell ny) + d(y^3) = 0$ Now integrating each term, we get

$\Rightarrow \quad x^2 \, \ell ny + y^3 = c$

Self Practice Problems :

(20) Solve : $xdy + ydx + xy \, e^y \, dy = 0$

(21) Solve : $ye^{-x/y} \, dx - (xe^{-x/y} + y^3) \, dy = 0$

Ans. (20) $\ell n(xy) + e^y = c$ (21) $2e^{-x/y} + y^2 = c$

7. Geometrical application of differential equation :

Form a differential equation from a given geometrical problem. Often following formulae are useful to remember

(i) Length of tangent (L_T) = $\left|\dfrac{y\sqrt{1+m^2}}{m}\right|$

(ii) Length of normal (L_N) = $\left| y\sqrt{1+m^2} \right|$

(iii) Length of sub-tangent (L_{ST}) = $\left|\dfrac{y}{m}\right|$

(iv) Length of subnormal (L_{SN}) = $|my|$

where y is the ordinate of the point, m is the slope of the tangent = $\left(\dfrac{dy}{dx}\right)$

Example # 21 : Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Solution : Let the equation of the curve be y = f(x). P(x, y) be any point on the curve.

Slope of the tangent at P(x, y) is $\dfrac{dy}{dx} = m$

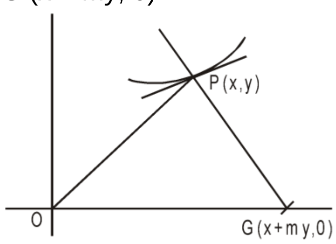
$\therefore$ Slope of the normal at P is

$m' = -\dfrac{1}{m}$

Equation of the normal at 'P'

$Y - y = -\dfrac{1}{m}(X - x)$

Co-ordinates of G (x + my, 0)



Now, $OP^2 = PG^2$

$x^2 + y^2 = m^2y^2 + y^2$

$m = \pm \dfrac{x}{y}$ $\quad\quad \dfrac{dy}{dx} = \pm \dfrac{x}{y}$

Taking as positive sign

$\dfrac{dy}{dx} = \dfrac{x}{y} \quad \Rightarrow \quad y\,.\,dy = x\,.\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + \lambda \quad \Rightarrow \quad x^2 - y^2 = -2\lambda$

$x^2 - y^2 = c$ (Rectangular hyperbola)

Again taking as –ve sign

$\dfrac{dy}{dx} = -\dfrac{x}{y} \quad \Rightarrow \quad y\,dy = -x\,dx$

$\dfrac{y^2}{2} = -\dfrac{x^2}{2} + \lambda' \quad \Rightarrow \quad x^2 + y^2 = 2\lambda'$

$x^2 + y^2 = c'$ (circle)

Example # 22 : Find the equation of the curve such that the square of the intercept cut off by any tangent from the y-axis is equal to the product of the coordinates of the point of tangency

Solution : Equation of tangent at any point (x, y) is $Y - y = \dfrac{dy}{dx}(X - x)$

intercept on y-axis is given by putting x = 0 $\quad \therefore \quad$ y-intercept $= y - x\dfrac{dy}{dx}$

according to the question

$\left(y - x\dfrac{dy}{dx}\right)^2 = xy \quad \Rightarrow \quad y - x\dfrac{dy}{dx} = \pm\sqrt{xy}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y \pm \sqrt{xy}}{x}$

Let $y = vx \quad \Rightarrow \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

hence $v + x\dfrac{dv}{dx} = v \pm \sqrt{v} \quad \Rightarrow \quad \pm\displaystyle\int \dfrac{dv}{\sqrt{v}} = \int \dfrac{dx}{x}$

$\Rightarrow \quad \pm 2\sqrt{v} = \ell nx + \ell nc \quad \Rightarrow \quad cx = e^{\pm 2\sqrt{v}}$

$\Rightarrow \quad cx = e^{\pm 2\sqrt{(y/x)}}$

Example # 23: Show that (4x + 3y + 1) dx + (3x + 2y + 1) dy = 0 represents a hyperbola having the lines x + y = 0 and 2x + y + 1 = 0 as asymptotes

Solution : $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$

$4xdx + 3\,(y\,dx + x\,dy) + dx + 2y\,dy + dy = 0$

Integrating each term,

$2x^2 + 3\,xy + x + y^2 + y + c = 0$

$2x^2 + 3xy + y^2 + x + y + c = 0$

which is the equation of hyperbola when $h^2 > ab$ & $\Delta \neq 0$.

Now, combined equation of its asymptotes is -

$$2x^2 + 3xy + y^2 + x + y + \lambda = 0$$

which is pair of straight lines

$\therefore \quad \Delta = 0$

$$\Rightarrow \quad 2.1\,\lambda + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} - 2 \cdot \frac{1}{4} - 1 \cdot \frac{1}{4} - \lambda\,\frac{9}{4} = 0 \quad \Rightarrow \quad \lambda = 0$$

$\therefore \quad 2x^2 + 3xy + y^2 + x + y = 0$

$(x + y)(2x + y) + (x + y) = 0$

$(x + y)(2x + y + 1) = 0$

$x + y = 0 \quad$ or $\quad 2x + y + 1 = 0$

Example # 24 : The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1)

Solution : Let P (x, y) be any point on the curve

Equation of tangent at 'P' is -

$Y - y = m(X - x)$

$mX - Y + y - mx = 0$

Now, $\left(\dfrac{y - mx}{\sqrt{1 + m^2}}\right) = x$

$y^2 + m^2x^2 - 2mxy = x^2(1 + m^2)$

$\dfrac{y^2 - x^2}{2xy} = \dfrac{dy}{dx}$ which is homogeneous equation

Putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \therefore \qquad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} \qquad \int \frac{2v}{v^2 + 1}\,dv = -\int \frac{dx}{x}$$

$$\ell n(v^2 + 1) = -\ell n\,x + \ell n\,c$$

$$x\left(\frac{y^2}{x^2} + 1\right) = c$$

Curve is passing through (1, 1)

$\therefore \quad c = 2$

$x^2 + y^2 - 2x = 0$