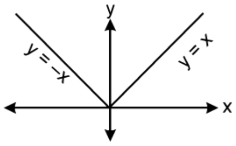
Fundamental of Mathematics - II

Fundamentals of Mathematics-II

He is unworthy of the name of man who is ignorant of the fact that the diagonal of square is incommensurable with its side
.......Plato

1. Absolute value function / modulus function :

The symbol of modulus function is f (x) = $|x|$ and is defined as: y = $|x|$ = $\begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$.



y = |x|

2. Properties of modulus :

For any a, b ∈ R

(i) $|a| \ge 0$ (ii) $|a| = |-a|$

(iii) $|a| \ge a$, $|a| \ge -a$ (iv) $|ab| = |a|\,|b|$

(v) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

(vi) $|a + b| \le |a| + |b|$; Equality holds when $ab \ge 0$

(vii) $|a - b| \ge ||a| - |b||$; Equality holds when $ab \ge 0$

Example # 1 : Solve the following linear equations

(i) $x\,|x| = 4$ (ii) $|x - 3| + 2|x + 1| = 4$

Solution : (i) $x|x| = 4$

If $x > 0$

∴ $x^2 = 4$ ⇒ $x = \pm 2$

∴ $x = 2$ (∵ $x \ge 0$)

If $x < 0$ ⇒ $-x^2 = 4$

⇒ $x^2 = -4$ which is not possible

(ii) $|x - 3| + 2|x + 1| = 4$

case I : If $x \le -1$

∴ $-(x - 3) - 2(x + 1) = 4$

⇒ $-x + 3 - 2x - 2 = 4$ ⇒ $-3x + 1 = 4$

⇒ $-3x = 3$ ⇒ $x = -1$

case II : If $-1 < x \le 3$

∴ $-(x - 3) + 2(x + 1) = 4$

⇒ $-x + 3 + 2x + 2 = 4$

⇒ $x = -1$ which is not possible

case III : If $x > 3$

$x - 3 + 2(x + 1) = 4$

$3x - 1 = 4$

⇒ $x = 5/3$ which is not possible

$\therefore \quad x = -1 \quad$ Ans.

3. Irrational Equations and Inequations :

(i) The equation $\sqrt{f(x)} = g(x)$, is equivalent to the following system
$f(x) = g^2(x) \quad \& \quad g(x) \geq 0$

(ii) The inequation $\sqrt{f(x)} < g(x)$, is equivalent to the following system
$f(x) < g^2(x) \quad \& \quad f(x) \geq 0 \quad \& \quad g(x) \geq 0$

(iii) The inequation $\sqrt{f(x)} > g(x)$, is equivalent to the following system
$g(x) \leq 0 \quad \& \quad f(x) \geq 0 \quad$ or $\quad g(x) \geq 0 \quad \& \quad f(x) > g^2(x)$

Example # 2 : Solve : $x + 2 > 2\sqrt{1-x^2}$

Solution : $4(1 - x^2) < (x + 2)^2$ and $x + 2 \geq 0$ & $1 - x^2 \geq 0$

$x \in \left(-\infty, \dfrac{-4}{5}\right) \cup (0, \infty)$ ...(1)

$x \in [-2, \infty)$ ...(2)

$x \in [-1, 1]$ ...(3)

$(1) \cap (2) \cap (3)$

$\left[-1, -\dfrac{4}{5}\right) \cup (0, 1]$

Self Practice Problem :

(1) $\sqrt{2x^2 + x - 6} < x$ (2) $\sqrt{5 - x} > x + 1$

(3) $x + 3 + \sqrt{x^2 + 4x - 5} > 0$ (4) $\sqrt{x} - \sqrt{4 - x} \geq 1$

Ans. (1) $\left[\dfrac{3}{2}, 2\right)$ (2) $(-\infty, 1)$ (3) $(-\infty, -1] \cup [5, \infty)$ (4) $\left[\dfrac{4+\sqrt{7}}{2}, 4\right]$

4. Greatest integer function or step up function :

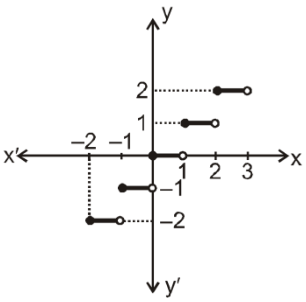
The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :

[3.2] = 3; [– 3.2] = – 4

for $-1 \leq x < 0$; [x] = – 1 ; for $0 \leq x < 1$; [x] = 0

for $1 \leq x < 2$; [x] = 1 ; for $2 \leq x < 3$; [x] = 2 and so on.

5. Graph of greatest integer function :



Fundamental of Mathematics - II

6. Properties of greatest integer function :

(i) $x - 1 < [x] \le x$

(ii) $[x \pm m] = [x] \pm m$ iff m is an integer.

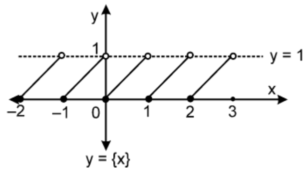
(iii) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(iv) $[x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

Note : $[mx] \ne m[x]$

7. Fractional part function :

It is defined as y = {x} = x − [x]. It is always non-negative and varies from [0, 1). The period of this function is 1 and graph of this function is as shown.



For example {2.1} = 2.1 – [2.1] = 2.1 – 2 = 0.1

{– 3.7} = – 3.7 – [– 3.7] = – 3.7 + 4 = 0.3

8. Properties of fractional part function :

(i) $\{x \pm m\} = \{x\}$ iff m is an integer

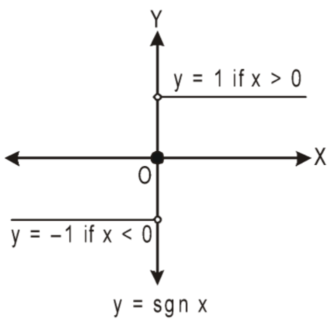
(ii) $\{x\} + \{-x\} = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$

Note: $\{mx\} \ne m\{x\}$

9. Signum function :

A function f (x) = sgn (x) is defined as follows :

$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



It is also written as $\text{sgn}(x) = \begin{cases} \dfrac{|x|}{x}; & x \ne 0 \\ 0; & x = 0 \end{cases}$

Note : ☞ $\text{sgn}(f(x)) = \begin{cases} \dfrac{|f(x)|}{f(x)}; & f(x) \ne 0 \\ 0; & f(x) = 0 \end{cases}$

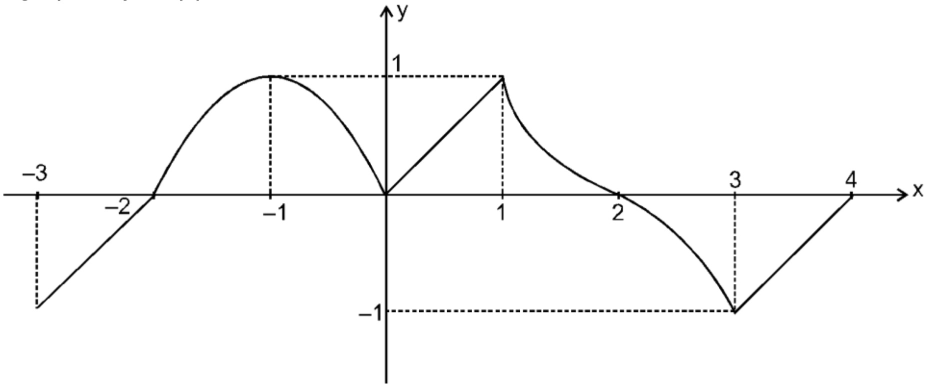
Example # 3 : Find x if $2 \le [x] \le 8$

Solution : x∈ [2, 9)

10. Graphical transformation :

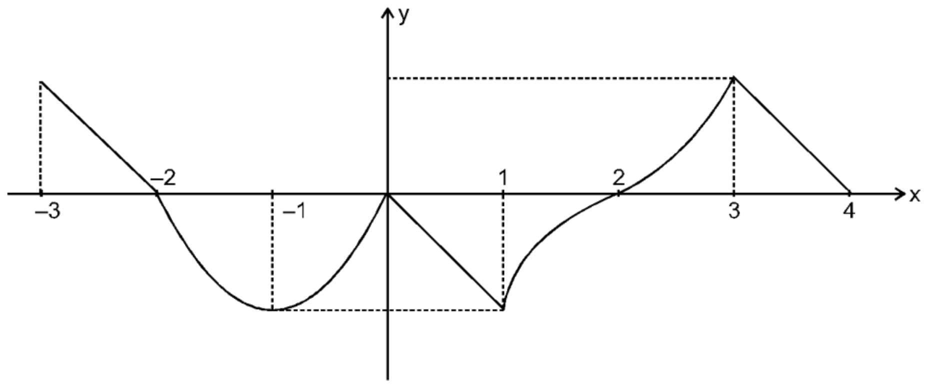
(i) Graphical transformations related to modulus :

If graph of y = f(x) is

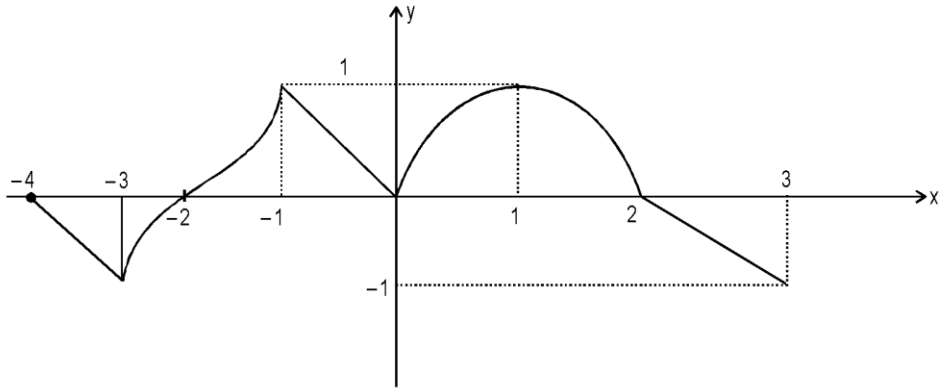


then graph of

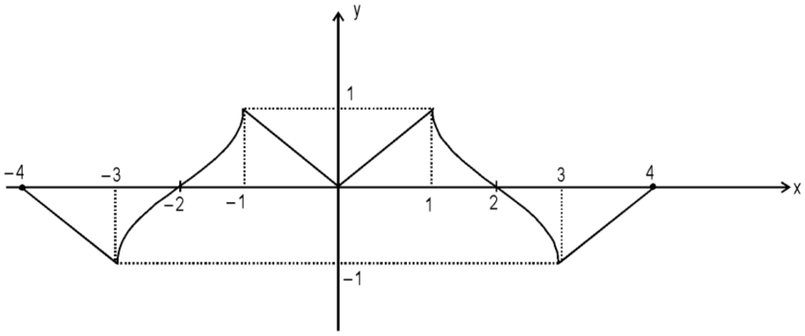
(a) y = – f(x) is



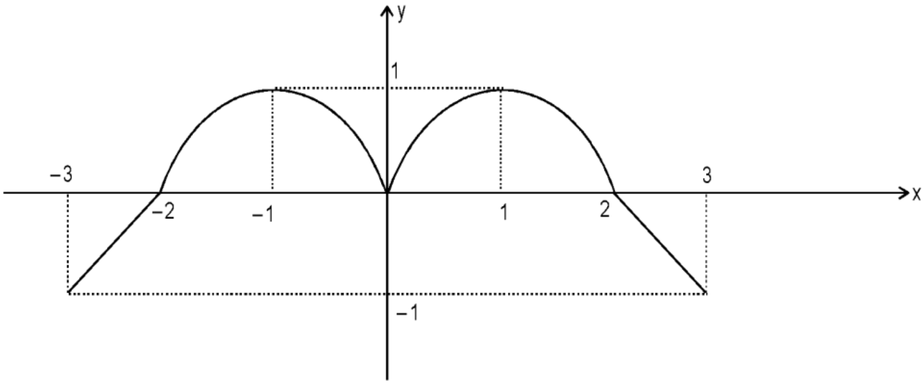
(b) y = f(–x) is



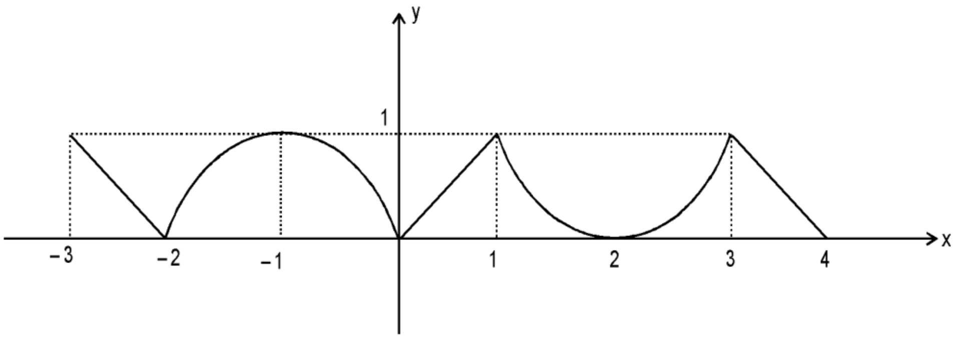
(c) y = f(|x|) is



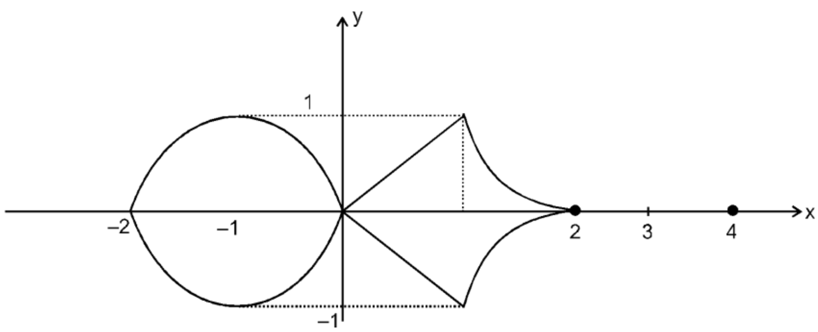
(d) y = f(–|x|) is



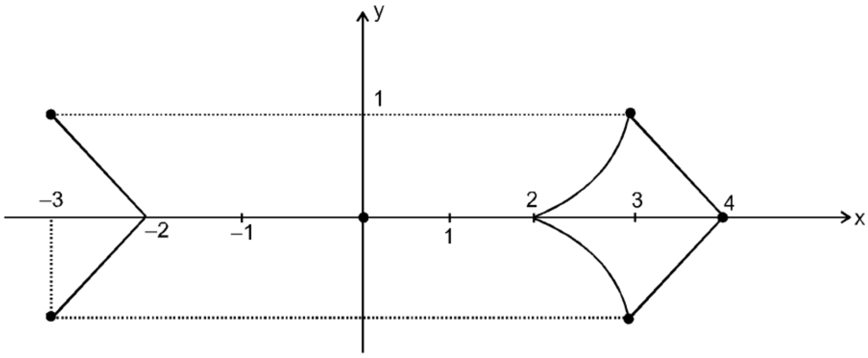
(e) y = |f(x)| is



(f) |y| = f(x) is

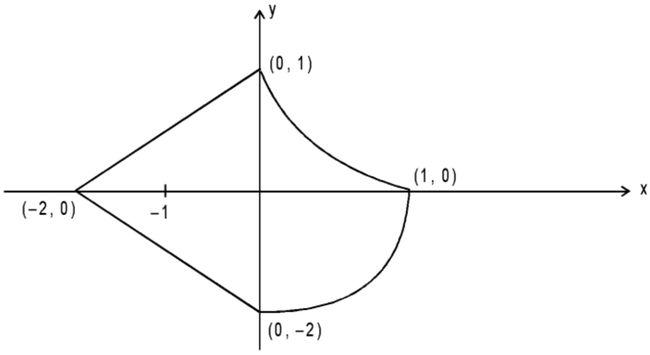


(g) |y| = – f(x)



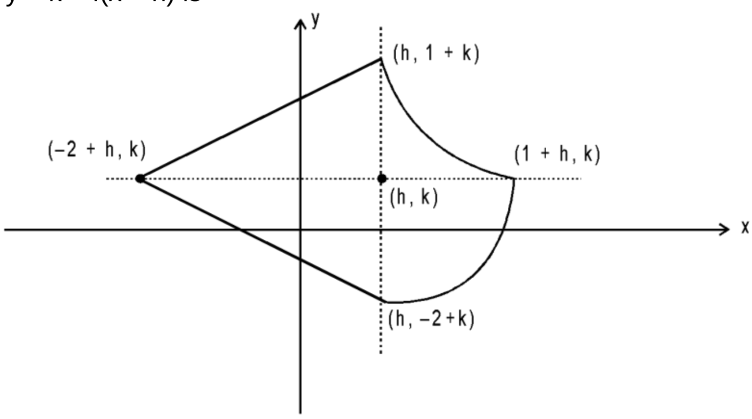
(ii) Other Graphical Trasformations :

If graph of y = f(x) is

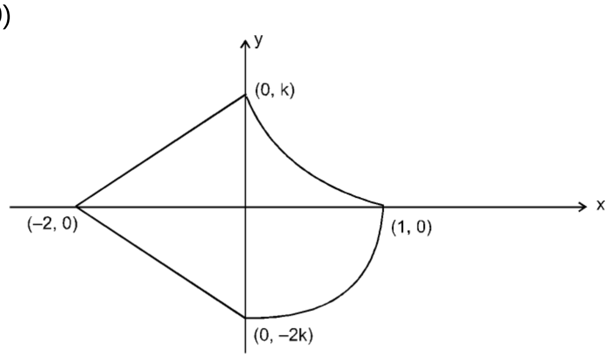


then graph of

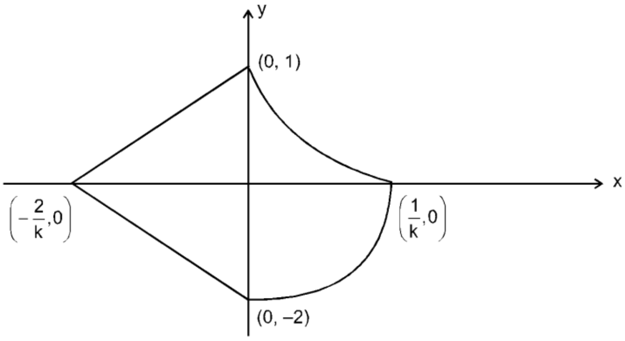
(a) y – k = f(x – h) is

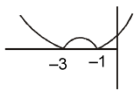


(b) y = kf(x) is
(k > 0)

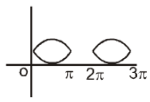


(c) y = f(kx) is
(k > 0)

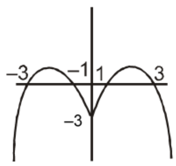




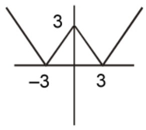
Example # 4 : $y = |x^2 + 4x + 3|$ **Solution :**



Example # 5 : $|y-1| = \sin x$ **Solution :**

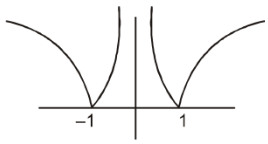


Example # 6 : $y = -x^2 + 4|x| - 3$ **Solution :**

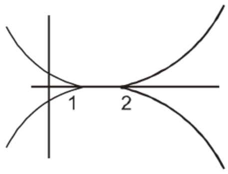


Example # 7 : $y = ||x| - 3|$ **Solution :**

Example # 8 : $y = \sin\left(\frac{x}{3}\right)$ **Solution :** period is 6π



Example # 9 : $y = |-\ell n\,|-x||$ **Solution :**



Example # 10 : $|y| = x^2 - 3x + 2$

Solution :