Fundamental of Mathematics - I

Fundamentals of Mathematics-I

He is unworthy of the name of man who is ignorant of the fact that the diagonal of square is incommensurable with its side **.......Plato**

SET

A set is a collection of well defined objects which are distinct from each other. Sets are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by small letters a, b, c .... etc.
If a is an element of a set A, then we write $a \in A$ and say a belongs to A.
If a does not belong to A then we write a A,
e.g. the collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

1. METHODS TO WRITE A SET :

(i) **Roster Method or Tabular Method :** In this method a set is described by listing elements, separated by commas and enclose then by curly brackets. Note that while writing the set in roster form, an element is not generally repeated e.g. the set of letters of word SCHOOL may be written as {S, C, H, O, L}.

(ii) **Set builder form (Property Method) :** In this we write down a property or rule which gives us all the element of the set.
$A = \{x : P(x)\}$ where P(x) is the property by which $x \in A$ and colon (:) stands for 'such that'

Example # 1 : Express set $A = \{x : x \in N \text{ and } x = 2^n \text{ for } n \in N\}$ in roster form
Solution : $A = \{2, 4, 8, 16, ...\}$

Example # 2 : Express set $B = \{x^3 : x < 5, x \in W\}$ in roster form
Solution : $B = \{0, 1, 8, 27, 64\}$

Example # 3 : Express set $A = \{0, 7, 26, 63, 124\}$ in set builder form
Solution : $A = \{x : x = n^3 - 1, n \in N, 1 \le n \le 5\}$

2. TYPES OF SETS :

(i) **Null set or empty set :** A set having no element in it is called an empty set or a null set or void set, it is denoted by φ or { }. A set consisting of at least one element is called a non-empty set or a non-void set.

(ii) **Singleton set :** A set consisting of a single element is called a singleton set.

(iii) **Finite set :** A set which has only finite number of elements is called a finite set.

(a) **Order of a finite set :** The number of distinct elements in a finite set A is called the order of this set and denoted by O(A) or n(A). It is also called cardinal number of the set.
e.g. $A = \{a, b, c, d\} \Rightarrow n(A) = 4$

(iv) **Infinite set :** A set which has an infinite number of elements is called an infinite set.

(v) **Equal sets :** Two sets A and B are said to be equal if every element of A is member of B, and every element of B is a member of A. If sets A and B are equal, we write A = B and if A and B are not equal then $A \neq B$

(vi) **Equivalent sets :** Two finite sets A and B are equivalent if their cardinal number is same
i.e. $n(A) = n(B)$
e.g. $A = \{1, 3, 5, 7\}, B = \{a, b, c, d\} \Rightarrow n(A) = 4$ and $n(B) = 4$
$\Rightarrow$ A and B are equivalent sets

Note - Equal sets are always equivalent but equivalent sets may not be equal

Example # 4 : Identify the type of set :

(i) $A = \{x \in W : 3 \leq x < 10\}$ (ii) $A = \{\alpha, \beta, \gamma, \delta\}$

(iii) $A = \{1, 0, -1, -2, -3, ...\}$ (iv) $A = \{1, 8, -2, 6, 5\}$ and $B = \{1, 8, -2, 1, 6, 5\}$

(v) $A = \{x : x$ is number of students in a class room$\}$

Solution : (i) finite set (ii) finite set (iii) infinite set (iv) equal sets (v) singleton set

Self Practice Problem :

(1) Write the set of all integers 'x' such that $-2 < x - 4 < 5$.

(2) Write the set $\{1, 2, 5, 10\}$ in set builder form.

(3) If $A = \{x : x^2 < 9, x \in Z\}$ and $B = \{-2, -1, 1, 2\}$ then find whether sets A and B are equal or not.

Ans. (1) $\{3, 4, 5, 6, 7, 8\}$

(2) $\{x : x$ is a natural number and a divisor of $10\}$

(3) Not equal sets

3. SUBSET AND SUPERSET :

Let A and B be two sets. If every element of A is an element of B then A is called a subset of B and B is called superset of A. We write it as $A \subseteq B$.

e.g. $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$ $\Rightarrow$ $A \subseteq B$

If A is not a subset of B then we write $A \not\subset B$

4. PROPER SUBSET :

If A is a subset of B but $A \neq B$ then A is a proper subset of B and we write $A \subset B$. Set A is not proper subset of A so this is improper subset of A

Note : (i) Every set is a subset of itself

(ii) Empty set φ is a subset of every set

(iii) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$

(iv) The total number of subsets of a finite set containing n elements is 2^n.

(v) Number of proper subsets of a set having n elements is $2^n - 1$.

(vi) Empty set φ is proper subset of every set except itself.

5. POWER SET :

Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

6. UNIVERSAL SET :

A set consisting of all possible elements which occur in the discussion is called a universal set and is denoted by U.

e.g. if $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 3, 5, 7\}$ then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

Example # 5 : Examine whether the following statements are true or false :

(i) $\{a\} \not\subset \{b, c, a\}$

(ii) $\{x, p\} \not\subset \{x : x$ is a consonant in the English alphabet$\}$

(iii) $\{\alpha, \beta, \gamma, \delta\} \subseteq \{\alpha, \beta, \varphi, \psi\}$

(iv) $\{a, b\} \in \{a, \{a\}, b, c\}$

Solution : (i) False as $\{a\}$ is subset of $\{b, c, a\}$

(ii) False as x, p are consonant

(iii) False as element γ, δ is not in the set $\{\alpha, \beta, \varphi, \psi\}$

(iv) False as $a, b \in \{a, \{a\}, b, c\}$ and $\{a, b\} \subseteq \{a, \{a\}, b, c\}$

Example # 6 : Find power set of set A = {1, 2, 3}

Solution : P(A) = {φ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

Example # 7 : If φ denotes null set then find

(a) P(φ) (b) P(P(φ)) (c) n(P(P(P(φ))))

(d) n(P(P(P(P(φ)))))

Solution : (a) P(φ) = {φ} (b) P(P(φ)) = {φ,{φ}}

(c) n(P(P(P(φ)))) = 2^2 = 4 (d) n(P(P(P(P(φ))))) = 2^4 = 16

Self Practice Problem :

(4) State true/false : A = {p, q, r, s}, B = {p, q, r, p, t} then A ⊆ B.

(5) State true/false : A = {p, q, r, s}, B = {s, r, q, p} then A ⊂ B.

(6) State true/false : [4, 15) ⊆ [–15, 15]

Ans. (4) False (5) False (6) True

7. SOME OPERATION ON SETS :

(i) Union of two sets : A ∪ B = {x : x ∈ A or x ∈ B}
e.g. A = {1, 2, 3}, B = {2, 3, 4} then A ∪ B = {1, 2, 3, 4}

(ii) Intersection of two sets : A ∩ B = {x : x ∈ A and x ∈ B}
e.g. A = {1, 2, 3}, B = {2, 3, 4} then A ∩ B = {2, 3}

(iii) Difference of two sets : A – B = {x : x ∈ A and x ∉ B}. It is also written as A∩B'.
Similarly B – A = B ∩ A' e.g. A = {1, 2, 3}, B = {2, 3, 4} ; A – B = {1}

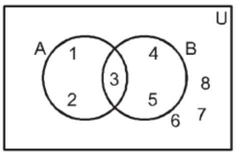
(iv) Symmetric difference of sets : It is denoted by A Δ B and A Δ B = (A – B) ∪ (B – A)

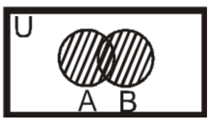
(v) Complement of a set : A' = {x : x ∉ A but x ∈ U} = U – A
e.g. U = {1, 2,........, 10}, A = {1, 2, 3, 4, 5} then A' = {6, 7, 8, 9, 10}

(vi) Disjoint sets : If A ∩ B = φ, then A, B are disjoint sets.
e.g. If A = {1, 2, 3}, B = {7, 8, 9} then A ∩ B = φ

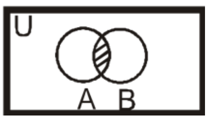
8. VENN DIAGRAM :

Most of the relationships between sets can be represented by means of diagrams which are known as venn diagrams. These diagrams consist of a rectangle for universal set and circles in the rectangle for subsets of universal set. The elements of the sets are written in respective circles.
For example If A = {1, 2, 3}, B = {3, 4, 5}, U = {1, 2, 3, 4, 5, 6, 7, 8} then their venn diagram is

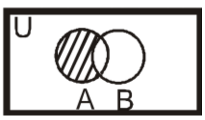




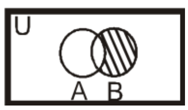
A ∪ B



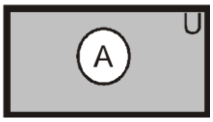
A ∩ B



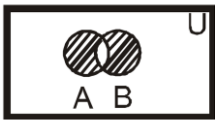
A – B



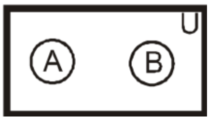
B – A



A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint

9. SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B, C are finite sets and U be the finite universal set then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) = n(A) - n(A \cap B)$

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(iv) Number of elements in exactly two of the sets A, B, C
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(v) Number of elements in exactly one of the sets A, B, C
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

10. LAWS OF ALGEBRA OF SETS (PROPERTIES OF SETS):

(i) **Commutative law :** $(A \cup B) = B \cup A$; $A \cap B = B \cap A$

(ii) **Associative law :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) **Distributive law :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) **De-morgan law :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(v) **Identity law :** $A \cap U = A$; $A \cup \varphi = A$

(vi) **Complement law :** $A \cup A' = U$, $A \cap A' = \varphi$, $(A')' = A$

(vii) **Idempotent law :** $A \cap A = A$, $A \cup A = A$

NOTE :

(i) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A \cap \varphi = \varphi$, $A \cup U = U$

Example # 8 : Let A = {1, 2, 3, 4, 5, 6} and B = {4, 5, 6, 7, 8, 9} then find $A \cup B$

Solution : $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Example # 9 : Let A = {1, 2, 3, 4, 5, 6}, B = {4, 5, 6, 7, 8, 9}. Find A – B and B – A.

Solution : $A - B = \{x : x \in A \text{ and } x \ \ B\} = \{1, 2, 3\}$ similarly $B - A = \{7, 8, 9\}$

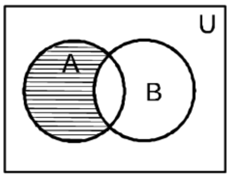
Example # 10 : State true or false :

(i) $A \cup A' = A$ (ii) $U \cap A = A$

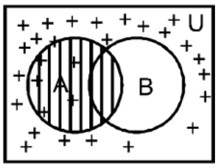
Solution : (i) false because $A \cup A' = U$ (ii) true as $U \cap A = A$

Example # 11 : Use Venn diagram to prove that $A - B = A \cap B'$.

Solution :



≡ A – B



|||| = A, ++ = B′

= A ∩ B′

From venn diagram we can conclude that A – B = A ∩ B′.

Example # 12: In a group of 60 students, 36 read English newspaper, 22 read Hindi newspaper and 12 read neither of the two. How many read both English & Hindi news papers ?

Solution : n(U) = 60, n(E) = 36, n(H) = 22

$n(E' \cap H') = 12 \Rightarrow n(E \cup H)' = 12$

$\Rightarrow \quad n(U) - n(E \cup H) = 12 \quad \Rightarrow \quad n(E \cup H) = 48$

$\Rightarrow \quad n(E) + n(H) - n(E \cap H) = 48 \quad \Rightarrow \quad n(E \cap H) = 58 - 48 = 10$

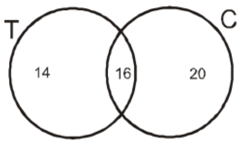
Example#13 : In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink tea and coffee both ? (ii) How many drink coffee but not tea ?

Solution : T : people drinking tea

C : people drinking coffee

(i) $n(T) = n(T - C) + n(T \cap C) \Rightarrow 30 = 14 + n(T \cap C) \Rightarrow n(T \cap C) = 16$



(ii) $n(C - T) = n(T \cup C) - n(T) = 50 - 30 = 20$

Self Practice Problem :

(7) Find A ∪ B if $A = \{x : x = 2n + 1, n \le 5, n \in N\}$ and $B = \{x : x = 3n - 2, n \le 4, n \in N\}$.

(8) Find A – (A – B) if A = {5, 9, 13, 17, 21} and B = {3, 6, 9, 12, 15, 18, 21, 24}

(9) Let A and B be two finite sets such that n(A – B) = 15, n(A ∪ B) = 90, n(A ∩ B) = 30. Find n(B)

(10) A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products ?

Ans. (7) {1, 3, 4, 5, 7, 9, 10, 11} (8) {9, 21}
(9) 75 (10) 170

11. Number system :

(i) Natural numbers : The counting numbers 1, 2, 3, 4, ... are called natural numbers. The set of natural numbers is denoted by N. Thus N = {1, 2, 3, 4, ...}.

(ii) Whole numbers : Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W. Thus W = {0, 1, 2, ...}

(iii) Integer numbers : The numbers ... – 3, – 2, – 1, 0, 1, 2, 3 .... are called integer numbers and the set of integer numbers is denoted by I or Z. Thus I (or Z) = {.... – 3, – 2, – 1, 0, 1, 2, 3.....}

Note : (a) Positive integers $I_+ = \{1, 2, 3 ....\} = N$

(b) Negative integers $I_- = \{....., -3, -2, -1\}$.

(c) Non-negative integers (whole numbers) = {0, 1, 2, ...}.

(d) Non-positive integers = {......, –3, –2, –1, 0}.

(iv) Even integers : Integers which are divisible by 2 are called even integers.
e.g. 0, ± 2, ± 4,.......

(v) Odd integers : Integers which are not divisible by 2 are called odd integers.
e.g. ± 1, ± 3, ± 5, ± 7......

(vi) Prime numbers : Natural numbers which are divisible by 1 and itself only are called prime numbers.
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(vii) Composite numbers : Let 'a' be a natural number, 'a' is said to be composite, if it has atleast three distinct factors.
e.g. 4, 6, 8, 9, 10, 12, 14, 15 ....

Note : (a) 1 is neither a prime number nor a composite number.
(b) Numbers which are not prime are composite numbers (except 1).
(c) '4' is the smallest composite number.
(d) '2' is the only even prime number.

(viii) Co-prime numbers : Two natural numbers (not necessarily prime) are called coprime, if there H.C.F (Highest common factor) is one.
e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc.
These numbers are also called as **relatively prime** numbers.

Note : (a) Two prime number(s) are always co-prime but converse need not be true.
(b) Consecutive natural numbers are always co-prime numbers.

(ix) Twin prime numbers : If the difference between two prime numbers is two, then the numbers are called twin prime numbers.
e.g. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

Note : Number between twin prime numbers is divisible by 6 (except (3, 5)).

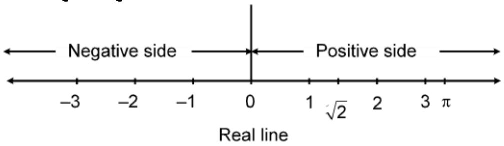
(x) Rational numbers : All the numbers that can be represented in the form $\frac{p}{q}$, where p and q are integers and q ≠ 0, are called rational numbers and their set is denoted by Q. Thus $Q = \{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\}$. It may be noted that every integer is a rational number since it can be written as . It may be noted that all recurring decimals are rational numbers.

Note : Maximum number of different decimal digits in $\frac{p}{q}$ is equal to q, i.e. $\frac{11}{9}$ will have maximum of 9 different decimal digits.

(xi) Irrational numbers : The numbers which can not be expressed in $\frac{p}{q}$ form where p, q ∈ I and q ≠ 0 i.e. the numbers which are not rational are called irrational numbers and their set is denoted by Q^c. (i.e. complementary set of Q) e..g. $\sqrt{2}$, $1 + \sqrt{3}$ etc. Irrational numbers can not be expressed as recurring decimals.

Note : e ≈ 2.71 is called Napier's constant and π ≈ 3.14 are irrational numbers.

(xii) Real numbers : Numbers which can be expressed on number line are called real numbers. The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^c$.



All real numbers follow the order property i.e. if there are two distinct real numbers a and b then either a < b or a > b.

Note : (a) Integers are rational numbers, but converse need not be true.
(b) Negative of an irrational number is an irrational number.
(c) Sum of a rational number and an irrational number is always an irrational number e.g. $2 + \sqrt{3}$
(d) The product of a non zero rational number & an irrational number will always be an irrational number.
(e) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.
(f) Sum, difference, product and quotient of two irrational numbers need not be a irrational number or we can say, result may be a rational number also.

(xiii) Complex number : A number of the form a + ib is called a complex number, where $a,b \in R$ and $i = \sqrt{-1}$. Complex number is usually denoted by Z and the set of complex number is represented by C. Thus $C = \{a + ib : a, b \in R \text{ and } i = \sqrt{-1}\}$

Note : It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

12. Divisibility test :

S.No.	Divisibility of	Test
1	2	The digit at the unit place of the number is divisible by 2.
2	3	The sum of digits of the number is divisible by 3.
3	4	The last two digits of the number together are divisible by 4.
4	5	The digit of the number at the unit place is either 0 or 5.
5	6	The digit at the unit place of the number is divisible by 2 & the sum of all digits of the number is divisible by 3.
6	8	The last 3 digits of the number all together are divisible by 8.
7	9	The sum of all it's digits is divisible by 9.
8	10	The digit at unit place is 0.
9	11	The difference between the sum of the digits at even places and the sum of digits at odd places is 0 or multiple of 11. e.g.1298, 1221, 123321, 12344321, 1234554321, 123456654321

13. Remainder theorem :

Let p(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If p(x) is divided by (x – a), then the remainder is equal to p(a).

14. Factor theorem :

Let p(x) be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that p(a) = 0, then (x – a) is a factor of p(x). Conversely, if (x – a) is a factor of p(x), then p(a) = 0.

Example # 14 : Show that (x – 4) is a factor of the polynomial $x^3 - 3x^2 + 4x - 32$.

Solution : Let $p(x) = x^3 - 3x^2 + 4x - 32$ be the given polynomial. By factor theorem, (x – a) is a factor of a polynomial p(x) iff p(a) = 0. Therefore, in order to prove that x – 4 is a factor of p(x), it is sufficient to show that p(4) = 0. Now, $p(x) = x^3 - 3x^2 + 4x - 32$

$\Rightarrow$ $p(4) = 4^3 - 3 \times 4^2 + 4 \times 4 - 32 = 0$

Hence, $(x - 4)$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 32$.

Example # 15 : Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

Solution : Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials. Then

$g(x) = x^2 - 3x + 2 = x^2 - 2x - x + 2 = x(x - 2) - 1(x - 2) = (x - 1)(x - 2)$

In order to prove that f(x) is exactly divisible by g(x), it is sufficient to prove that $x - 1$ and $x - 2$ are factors of f(x). For this it is sufficient to prove that $f(1) = 0$ and $f(2) = 0$.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow$ $f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2$

$f(1) = 0$

and, $f(2) = 2 \times 2^4 - 6 \times 2^3 + 3 \times 2^2 + 3 \times 2 - 2$

$f(2) = 0$

$\Rightarrow$ Hence $(x - 1)$ and $(x - 2)$ are factors of f(x).

$\Rightarrow$ $g(x) = (x - 1)(x - 2)$ is a factors of f(x).

Hence f(x) is exactly divisible by g(x).

Example # 16 : If the polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x - 4)$ leave the same remainder, then find value of k ?

Solution : $P(4) = 64k + 48 - 3 = 64k + 45$

$Q(4) = 128 - 20 + k = k + 108$

given $P(4) = Q(4)$ $\therefore$ $64k + 45 = k + 108$ $\Rightarrow$ $63k = 63 \Rightarrow k = 1$

Example # 17 : A polynomial in x of second degree which will vanish at x = 4 and x = 2 and will have value 3 when x = 5 respectively, is in the form of $ax^2 + bx + c$. Find value of $a + b + c$

Solution : Hence p(x) vanish at x = 2 and x = 4 , so

$p(x) = a(x - 2)(x - 4)$

Given that $p(5) = 3$, $p(5) = a(5 - 2)(5 - 4) = 3$, $3a - 3 = 0$

$a = 1$ so,

$p(x) = (x - 2)(x - 4) = x^2 - 6x + 8$. $\Rightarrow$ $a + b + c = 3$

Self practice problems :

(11) Determine the remainder when the polynomial $P(x) = x^4 - 3x^2 + 2x + 1$ is divided by $x - 1$

(12) Find the value of a, if $x - a$ is a factor of $x^3 - a^2x + x + 2$.

(13) Using factor theorem, show that $a - b$, $b - c$ and $c - a$ are the factors of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

(14) Find a polynomial in x of the third degree which will vanish when x = 1 & x = - 2 and will have the values 4 & 28 when x = - 1 and x = 2 respectively ?

Ans. (11) 1 (12) $a = -2$ (14) $f(x) = 3x^3 + 4x^2 - 5x - 2$

15. Some important identities :

(i) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(ii) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(iii) $a^2 - b^2 = (a + b)(a - b)$

(iv) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(v) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(vi) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(vii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(viii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

(ix) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(x) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

(xi) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(xii) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

Example # 18 : If $\left(a+\frac{1}{a}\right)^2 = 3$, then find value of $a^3 + \frac{1}{a^3}$

Solution : $a + \frac{1}{a} = \pm\sqrt{3} \Rightarrow a^3 + \frac{1}{a^3} = \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right) = \pm 3\sqrt{3} \pm 3\sqrt{3} = 0.$

Example # 19 : Show that the expression, $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz).(y^2 - zx).(z^2 - xy)$ is a perfect square and find its square root.

Solution : $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)(z^2 - xy)$

$= a^3 + b^3 + c^3 - 3abc$ where $a = x^2 - yz$, $b = y^2 - zx$, $c = z^2 - xy$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[(x^2 - yz - y^2 + zx)^2 + (y^2 - zx - z^2 + xy)^2 + (z^2 - xy - x^2 + yz)^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)[\{x^2 - y^2 + z(x - y)\}^2 + \{y^2 - z^2 + x(y - z)\}^2 + \{z^2 - x^2 + y(z - x)\}^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)^2[(x - y)^2 + (y - z)^2 + (z - x)^2]$

$= (x + y + z)^2(x^2 + y^2 + z^2 - xy - yz - zx)^2 = (x^3 + y^3 + z^3 - 3xyz)^2$ (which is a perfect square)

its square roots are) $\pm(x^3 + y^3 + z^3 - 3xyz$

Self practice problems :

(15) If x, y, z are all different real numbers, then prove that

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2} = \left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2.$$

(16) Factorise the expression, $(x + y + z)^3 - x^3 - y^3 - z^3$ into linear factors.

(17) Factorize

(i) $1 + x^4 + x^8$ (ii) $x^4 + 4$

Ans. (16) $3(x + y)(y + z)(z + x)$

(17) (i) $(x^4 - x^2 + 1)(x^2 + x + 1)(x^2 - x + 1)$ (ii) $(x^2 - 2x + 2)(x^2 + 2x + 2)$

16. Definition of indices and its Lows :

If 'a' is any non zero real or imaginary number and 'm' is the positive integer, then $a^m = a. a. a. ...a$ (m times). Here a is called the base and m is called the index, power or exponent.

(i) $a^0 = 1$, $(a \neq 0)$

(ii) $a^{-m} = \dfrac{1}{a^m}$, $(a \neq 0)$

(iii) $a^{m+n} = a^m . a^n$, where m and n are rational numbers

(iv) $a^{m-n} = \dfrac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(v) $(a^m)^n = a^{mn}$

(vi) $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

Example # 20 : Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4 \left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

Solution : $a^{9\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)4} . a^{9\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)4} = a^2 . a^2 = a^4$.

Example # 21 : Simplify $a\left(\dfrac{\sqrt{a} + \sqrt{b}}{2b\sqrt{a}}\right)^{-1} + b\left(\dfrac{\sqrt{a} + \sqrt{b}}{2a\sqrt{b}}\right)^{-1}$

Solution : The given expression is equal to

$$a\left(\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) + b\left(\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = 2ab\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = 2ab$$

Example # 22 : Evaluate $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

Solution : $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}} = \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4+3+2\sqrt{12}}}}$

$= \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4}+\sqrt{3}}} = \sqrt{3+\sqrt{3}+\sqrt{4+2\sqrt{3}}} = \sqrt{3+\sqrt{3}+\sqrt{3}+1} = \sqrt{4+2\sqrt{3}} = \sqrt{3} + 1$

Example # 23 : Find rational numbers a and b, such that $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}} = a + b\sqrt{5}$

Solution : $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \dfrac{4+3\sqrt{5}}{4+3\sqrt{5}} = a + b\sqrt{5}$

$\dfrac{61+24\sqrt{5}}{-29} = a + b\sqrt{5} \Rightarrow a = -\dfrac{61}{29}$, $b = -\dfrac{24}{29}$

Self practice problem :

(18) Find the value of

(i) $\left(\dfrac{1}{3}\right)^{-10}.27^{-3} + \left(\dfrac{1}{5}\right)^{-4}.(25)^{-2} + \left(64^{-\frac{1}{9}}\right)^{-3}$ (ii) $\dfrac{\left(5\sqrt{3}+\sqrt{50}\right)\left(5-\sqrt{24}\right)}{\sqrt{75}-5\sqrt{2}}$

Ans. (18) (i) 8 (ii) 1

17. Ratio :

(i) If A and B be two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\dfrac{A}{B}$ (This may be an integer or fraction)

(ii) A ratio may represented in a number of ways e.g. $\dfrac{a}{b} = \dfrac{ma}{mb} = \dfrac{na}{nb} =$ ..... where m, n,..... are non-zero numbers.

(iii) To compare two or more ratio, reduced them to common denominator.

(iv) Ratio between two ratios may be represented as the ratio of two integers e.g. $\dfrac{a}{b} : \dfrac{c}{d} : \dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \dfrac{ad}{bc}$ or ad : bc.

(v) Ratios are compounded by multiplying them together i.e. $\dfrac{a}{b} . \dfrac{c}{d} . \dfrac{e}{f}$ .... $= \dfrac{ace}{bdf}$ ....

(vi) If a : b is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$ ..... etc.

(vii) If a : b is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

18. Proportion :

When two ratios are equal, then the four quantities compositing them are said to be proportional. If $\dfrac{a}{b} = \dfrac{c}{d}$, then it is written as a : b = c : d ⇒ a : b :: c : d

(i) 'a' and 'd' are known as extremes and 'b and c' are known as means.

(ii) An important property of proportion : Product of extremes = product of means.

(iii) If a : b = c : d, then
b : a = d : c (Invertando) i.e. ⇒

(iv) If a : b = c : d, then

a : c = b : d (Alternando)i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{b}{a}=\frac{d}{c}$

(v) If a : b = c : d, then

= (Componendo) i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{c}=\frac{b}{d}$

(vi) If a : b = c : d, then

= (Dividendo) i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{b}+1=\frac{c}{d}+1$

(vii) If a : b = c : d, then

$\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{b}+1=\frac{c}{d}+1$ $\Rightarrow$ $\frac{a+b}{b}=\frac{c+d}{d}$ ....(1)

$\frac{a}{b}-1=\frac{c}{d}-1$ $\Rightarrow$ $\frac{a-b}{b}=\frac{c-d}{d}$ ....(2)

Dividing equation (1) & (2) we obtain $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

Example # 24: What term must be added to each term of the ratio 5 : 37 to make it equal to 1 : 3?

Solution : Let x be added to each term of the ratio 5 : 37.

Then $\frac{x+5}{x+37}=\frac{1}{3}$

$\Rightarrow$ 3x + 15 = x + 37$\Rightarrow$ x = 11

Example # 25 : If x : y = 3 : 4, then find the ratio of 7x – 4y : 3x + y

Solution : $\frac{x}{y}=\frac{3}{4}$ $\therefore$ 4x = 3y $\Rightarrow$ x = $\frac{3}{4}$ y

Now $\frac{7x-4y}{3x+y}=\frac{7\ .\ \frac{3}{4}y-4y}{3\ .\ \frac{3}{4}y\ +\ y}$ (putting the value of x)

$=\frac{\frac{21}{4}y-4y}{\frac{9}{4}y\ +\ y}=\frac{5y}{13y}=\frac{5}{13}$

Example # 26 : If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then show that $\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$.

Solution : $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=$ k (constant) $\therefore$ x = ak; y = bk; z = ck

Substituting these values of x, y, z in the given expression

$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$

we obtain

L.H.S. $= \frac{a^3k^3 + a^3}{a^2k^2 + a^2} + \frac{b^3k^3 + b^3}{b^2k^2 + b^2} + \frac{c^3k^3 + c^3}{c^2k^2 + c^2} = \frac{a^3(k^3+1)}{a^2(k^2+1)} + \frac{b^3(k^3+1)}{b^2(k^3+1)} + \frac{c^3(k^3+1)}{c^2(k^2+1)}$

$= \frac{a(k^3+1)}{k^2+1} + \frac{b(k^3+1)}{k^2+1} + \frac{c(k^3+1)}{k^2+1} = \frac{(k^3+1)}{(k^2+1)} . (a + b + c)$

Now R.H.S $= \frac{(ak+bk+ck)^3 + (a+b+c)^3}{(ak+bk+ck)^2 + (a+b+c)^2} = \frac{k^3(a+b+c)^3 + (a+b+c)^3}{k^2(a+b+c)^2 + (a+b+c)^2}$

$= \frac{(k^3+1)(a+b+c)^3}{(k^2+1)(a+b+c)^2} = \frac{(k^3+1)}{(k^2+1)} . (a + b + c)$

We see that L.H.S. = R.H.S.

Example # 27 : If a, b, c, d, e are in continued proportion, then prove that

$(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2)\ (b^2 + c^2 + d^2 + e^2)$

Solution : If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}$, then we have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{\sqrt{(a^2+b^2+c^2+d^2)}}{\sqrt{(b^2+c^2+d^2+e^2)}} = k$ (say)

i.e. $a = bk$ $\therefore$ $ab = b^2k$

$b = ck$ $\therefore$ $bc = c^2k$

$c = dk$ $\therefore$ $cd = d^2k$

$d = ek$ $\therefore$ $de = e^2k$

Again $(a^2 + b^2 + c^2 + d^2) = k^2 (b^2 + c^2 + d^2 + e^2)$ ....(i)

Now L.H.S. $= (ab + bc + cd + de)^2$

$= (kb^2 + kc^2 + kd^2 + ke^2)^2$

$= k^2 (b^2 + c^2 + d^2 + e^2)^2$

$= k^2 (b^2 + c^2 + d^2 + e^2) (b^2 + c^2 + d^2 + e^2)$

$= (a^2 + b^2 + c^2 + d^2) (b^2 + c^2 + d^2 + e^2)$ (Note) (use (i))

Hence $(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2) (b^2 + c^2 + d^2 + e^2)$

Example # 28 : Solve the equation $\frac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \frac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$

Solution : $\frac{3x^4 + x^2 - 2x - 3}{3x^4 - x^2 + 2x + 3} = \frac{5x^4 + 2x^2 - 7x + 3}{5x^4 - 2x^2 + 7x - 3}$

By the process of componendo and dividendo, we have

$\frac{3x^4}{x^2 - 2x - 3} = \frac{5x^4}{2x^2 - 7x + 3}$

or $3x^4 (2x^2 - 7x + 3) - 5x^4(x^2 - 2x - 3) = 0$

or $x^4 [6x^2 - 21x + 9 - 5x^2 + 10x + 15] = 0$

or $x^4 (x^2 - 11x + 24) = 0$

$\therefore$ $x = 0$ or $x^2 - 11x + 24 = 0$

$x = 0$ or $(x - 8) (x - 3) = 0$

$\therefore$ $x = 0, 8, 3$

Self practice problems :

(19) If $\frac{a}{b} = \frac{2}{3}$ and $\frac{b}{c} = \frac{4}{5}$, then find value of $\frac{a+b}{b+c}$.

(20) If sum of two numbers is c and their quotient is $\frac{p}{q}$, then find numbers.

(21) If $(a_2 + b_2 + c_2)(x_2 + y_2 + z_2) = (ax + by + cz)_2$, show that x : a = y : b = z : c.

(22) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then find the value of $\frac{2\ a^4\ b^2\ +\ 3\ a^2\ c^2\ -\ 5\ e^4\ f}{2\ b^6\ +\ 3b^2\ d^2\ -\ 5f^5}$ in terms of a and b.

Ans. (19) $\frac{20}{27}$ (20) $\frac{pc}{p+q}, \frac{qc}{p+q}$ (22) $\frac{a^4}{b^4}$

19. <u>Cross multiplication</u> :

If two equations containing three unknowns are

$a_1x + b_1y + c_1z = 0$ ....(i)

$a_2x + b_2y + c_2z = 0$ ....(ii)

Then by the rule of cross multiplication

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \qquad ....(iii)$$

In order to write down the denominators of x, y and z in (iii) apply the following rule,
"write down the coefficients of x, y and z in order beginning with the coefficients of y and repeat them as in the diagram"

$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

Multiply the coefficients across in the way indicated by the arrows; remembering that informing the products any one obtained by descending is positive and any one obtained by ascending is negative.

Example # 29 : Find the ratios of x : y : z from the equations 7x = 4y + 8z, 3z = 12x + 11y.

Solution : By transposition we have 7x – 4y – 8z = 0, 12x + 11y – 3z = 0,
Write down the coefficients, thus

$$\begin{array}{cccc} -4 & -8 & 7 & -4 \\ 11 & -3 & 12 & 11, \end{array}$$

hence we obtain the products

(– 4) × (– 3) – 11 × (– 8), (– 8) × 12 – (– 3) × 7, 7 × 11 – 12 × (– 4), or100,– 75, 125

$$\therefore \frac{x}{100} = \frac{y}{-75} = \frac{z}{125}, \text{ that is, } \frac{x}{4} = \frac{y}{-3} = \frac{z}{5}.$$

Example # 30 : Eliminate x, y, z from the equations

$a_1x + b_1y + c_1z = 0$ ....(1)

$a_2x + b_2y + c_2z = 0$ ....(2)

$a_3x + b_3y + c_3z = 0$ ....(3)

From (2) and (3), by cross multiplication, $\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2}$;

denoting each of these ratios by k, by multiplying up, substituting in(1), and dividing through out by k, we obtain

$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

This relation is called the **eliminant** of the given equations.

20. Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains.

If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

Name	Representation	Discription
Open Interval	(a, b)	$\{x : a < x < b\}$ i.e. end points are not included.
Close Interval	[a, b]	$\{x : a \le x \le b\}$ i.e. end points are also included. This is possible only when both a and b are finite.
Open - Closed Interval	(a, b]	$\{x : a < x \le b\}$ i.e. a is excluded and b is included.
Close - Open Interval	[a, b)	$\{x : a \le x < b\}$ i.e. a is included and b is excluded.

Note : (i) The infinite intervals are defined as follows :

(a) $(a, \infty) = \{x : x > a\}$ (b) $[a, \infty) = \{x : x \ge a\}$

(c) $(-\infty, b) = \{x : x < b\}$ (d) $(\infty, b] = \{x : x \le b\}$ (e) $(-\infty, \infty) = \{x : x \in R\}$

(ii) $x \in \{1, 2\}$ denotes some particular values of x, i.e. x = 1, 2

(iii) If there is no value of x, then we say $x \in \varphi$ (null set)

21. General method to solve inequalities :

(Method of intervals (Wavy curve method)

Let $g(x) = \left(\frac{(x-b_1)^{k_1}(x-b_2)^{k_2} --- (x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2} --- (x-a_n)^{r_n}}\right)$ ... (i)

Where k_1, k_2 .....k_n and r_1, r_2 ....r_n $\in$ N and b_1, b_2.....b_n and a_1, a_2 ......a_n are real numbers.

Then to solve the inequality following steps are taken.

Steps : -

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function.

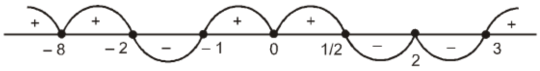
(ii) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.

(iii) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

(iv) Thus we consider all the intervals. The solution of the g(x) > 0 is the union of the intervals in which we have put the plus sign and the solution of g(x) < 0 is the union of all intervals in which we have put the minus sign.

Example # 31: Solve the inequality if $f(x) = \frac{(x-2)^{10}(x+1)^3\left(x-\frac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ is > 0 or < 0.

Solution : Let $f(x) = \dfrac{(x-2)^{10}(x+1)^3\left(x-\frac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ the poles and zeros are 0, 3, – 2,– 1, $\frac{1}{2}$, – 8, 2



If f(x) > 0, then $x \in (-\infty, -8) \cup (-8, -2) \cup (-1, 0) \cup \left(0, \frac{1}{2}\right) \cup (3, \infty)$

and if f(x) < 0, then $x \in (-2, -1) \cup \left(\frac{1}{2}, 2\right) \cup (2, 3)$ **Ans.**

22. Introduction to various types of functions :

(i) Polynomial Function :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ where n is a **non negative integer** and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note : ☞There are two polynomial functions, satisfying the relation; f(x).f(1/x) = f(x) + f(1/x), which are $f(x) = 1 \pm x^n$ (will be discussed in detail under the chapter " Functions")

(ii) Constant function :

A function f : A → B is said to be a constant function, if every element of A has the same f image in B.Thus f : A → B; f(x) = c, ∀ x ∈ A, c ∈ B is a constant function.

(iii) Identity function :

The function f : A → A defined by f(x) = x ∀ x ∈ A is called the identity function on A and is denoted by I_A. It can be observed that identity function is a bijection.

(iv) Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \ldots\ldots. + P_{n-1}(x)\, y + P_n(x) = 0$ where n is a positive integer and $P_0(x)$, $P_1(x)$....... are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

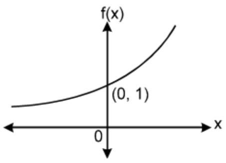
Note : ☞ All polynomial functions are algebraic but not the converse.

☞ A function that is not algebraic is called Transcendental Function.

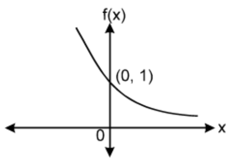
(v) Exponential function

A function $f(x) = a^x = e^{x \ell n a}$ ($a > 0, a \neq 1, x \in R$) is called an exponential function. Graph of exponential function can be as follows :

Case - I
For a > 1



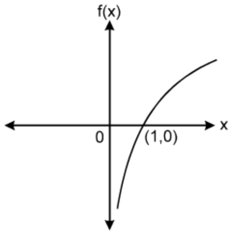
Case - II
For 0 < a < 1



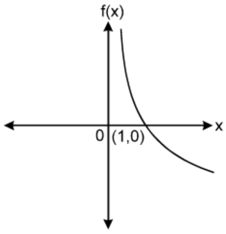
(vi) Lograithmic Function :

$f(x) = \log_a x$ ($x > 0$, $a > 0$ & $a \neq 1$) is called lograthmic function Graphof lograithmic function can be as follows :

Case- I
For a > 1



Case- II
For 0 < a < 1



23. Logarithm of a number :

The logarithm of the number N to the base ' a ' is the exponent indicating the power to which the base ' a ' must be raised to obtain the number N. This number is designated as $\log_a N$. Hence:

$\mathbf{\log_a N = x \Leftrightarrow a^x = N}$ **, a > 0, a ≠ 1 & N > 0**

If a = 10, then we write log b rather than $\log_{10} b$.

If a = e, we write $\ln b$ rather than $\log_e b$. Here ' e ' is called as Napier's base & has numerical value equal to 2.7182.

Remember

$\log_{10}2$ 0.3010 ; $\log_{10}3$ 0.4771

$\ell n\ 2$ 0.693 ; $\ell n\ 10$ 2.303

24. Fundamental logarithmic identity :

$a^{\log_a N} = N$, a > 0, a ≠ 1 & N > 0

The principal properties of logarithm

Let M & N are arbitrary positive numbers, a > 0, a ≠ 1, b > 0, b ≠ 1 and α, β are any real numbers, then :

(i) $\log_a(M.N) = \log_a M + \log_a N$; in general $\log_a(x_1 x_2 \ldots\ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots\ldots + \log_a x_n$

(ii) $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$

(iii) $\log_a M^\alpha = \alpha . \log_a M$

(iv) $\log_{a^\beta} M = \frac{1}{\beta} \log_a M$

(v) $\log_b M = \frac{\log_a M}{\log_a b}$ (base changing theorem)

Note : ◆ $\log_a 1 = 0$ ◆ $\log_a a = 1$ ◆ $\log_{\frac{1}{a}} a = -1$

◆ $\log_b a = \frac{1}{\log_a b}$ ◆ $a^x = e^{x\,\ell n\, a}$ ◆ $a^{\log_c b} = b^{\log_c a}$

Note : (i) If the number and the base are on the same side of the unity, then the logarithm is positive.

(ii) If the number and the base are on the opposite sides of unity, then the logarithm is negative.

Example # 32 : Find the value of the followings :

(i) $\log_a 2 + \log_a\left(1+\frac{1}{2}\right) + \log_a\left(1+\frac{1}{3}\right) + \ldots\ldots + \log_a\left(1+\frac{1}{n}\right)$

(ii) $\log_2 72 + \log_2\left(\frac{32}{81}\right) + \log_2\left(\frac{9}{64}\right)$

(iii) $7^{\frac{1}{\log_{25} 49}}$

Solution : (i) $\log_a 2 + \log_a\left(1+\frac{1}{2}\right) + \log_a\left(1+\frac{1}{3}\right) + \ldots\ldots + \log_a\left(1+\frac{1}{n}\right)$

$= \log_a\left(\frac{2}{1}\right) + \log_a\left(\frac{3}{2}\right) + \ldots\ldots + \log_a\left(\frac{n+1}{n}\right) = \log_a\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n+1}{n}\right) = \log_a (n+1)$

(ii) $\log_2 72 + \log_2\left(\frac{32}{81}\right) + \log_2\left(\frac{9}{64}\right) = \log_2\left\{2^3.3^2.\frac{2^5}{3^4}.\frac{3^2}{2^6}\right\} = \log_2 4 = 2$

(iii) $7^{\frac{1}{\log_{25} 49}} = 7^{\log_{49} 25} = 7^{\frac{2}{2}\log_7 5} = 5^{\log_7 7} = 5$

Self practice problems :

(23) Find the value of the followings :

(i) $\log_{49} 343$ (ii) $4\log_{27} 243$ (iii) $\log_{\frac{1}{100}} 1000$

(iv) $\log_{(7-4\sqrt{3})}(7+4\sqrt{3})$ (v) $\log_{125} 625$

(24) $\log_8 9.\log_9 10 \ldots\ldots\ldots \log_{63} 64$

(25) Find the value of log cot1° + log cot2° + log cot3° + ... + log cot89°

Ans. (23) (i) $\frac{3}{2}$ (ii) $\frac{20}{3}$ (iii) $-\frac{3}{2}$ (iv) –1 (v) $\frac{4}{3}$

(24) 2 (25) 0

25. Logarithmic Equation :

The equality $\log_a x = \log_a y$ is possible if and only if x = y i.e.

$\log_a x = \log_a y \Leftrightarrow x = y$

Always check validity of given equation, $(x > 0, y > 0, a > 0, a \neq 1)$

Example # 33 : Solve $\log_x(4x - 3) = 2$

Solution : Domain : $x > 0, 4x - 3 > 0, x \neq 1$

Hence $4x - 3 = x_2 \Rightarrow x_2 - 4x + 3 = 0$

x = 3 or x = 1 (rejected as not in domain) ⇒ so x = 3 only

Exmaple # 34 : Solve $\log_2 (\log_3\{\log_5 (x_2 + 4)\} = 0$

Solution : $\log_3(\log_5(x_2 + 4)\} = 2° = 1 \Rightarrow \log_5(x_2 + 4) = 3_1 = 3$

$\Rightarrow (x_2 + 4) = 5_3 = 125 \Rightarrow x_2 = 121 \Rightarrow x = \pm 11$

Example # 35 : Solve $\log_2(x_2) + \log_2 (x + 2) = 4$

Solution : $\log_2 (x_2(x + 2)) = 4 \Rightarrow x_3 + 2x_2 - 16 = 0 \Rightarrow (x - 2)\underbrace{(x^2+4x+8)}_{D<0} = 0$ x = 2

Self practice problems :

Solve following equations

(26) $3^{3\log_3 x} = 27$ (27) $(\log_{10}x)_2 - (\log_{10}x) - 6 = 0$

(28) $3(\log_7 x + \log_x 7) = 10$ (29) $(x+2)^{\log_2(x+2)} = 8(x+2)^2$

Ans. (26) x = 3 (27) $x = 10_3, \frac{1}{10^2}$

(28) $x = 343, \sqrt[3]{7}$ (29) $x = 6$ or $-3/2$

26. Logarithmic inequality :

Let 'a' is a real number such that

(i) If $a > 1$, then $\log_a x > \log_a y$ $\Rightarrow$ $x > y$

(ii) If $a > 1$, then $\log_a x < \alpha$ $\Rightarrow$ $0 < x < a^\alpha$

(iii) If $a > 1$, then $\log_a x > \alpha$ $\Rightarrow$ $x > a^\alpha$

(iv) If $0 < a < 1$, then $\log_a x > \log_a y$ $\Rightarrow$ $0 < x < y$

(v) If $0 < a < 1$, then $\log_a x < \alpha$ $\Rightarrow$ $x > a^\alpha$

Form - I : $f(x) > 0$, $g(x) > 0$, $g(x) \neq 1$

Form		Collection of system
(a) $\log_{g(x)} f(x) \geq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad g(x) > 1 \\ 0 < f(x) \leq 1 \quad , \quad 0 < g(x) < 1 \end{cases}$
(b) $\log_{g(x)} f(x) \leq 0$	$\Leftrightarrow$	$\begin{cases} f(x) \geq 1 \quad , \quad 0 < g(x) < 1 \\ 0 < f(x) \leq 1 \quad , \quad g(x) > 1 \end{cases}$
(c) $\log_{g(x)} f(x) \geq a$	$\Leftrightarrow$	$\begin{cases} f(x) \geq (g(x))^a \quad , \quad g(x) > 1 \\ 0 < f(x) \leq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$
(d) $\log_{g(x)} f(x) \leq a$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq (g(x))^a \quad , \quad g(x) > 1 \\ f(x) \geq (g(x))^a \quad , \quad 0 < g(x) < 1 \end{cases}$

From - II : When the inequality of the form

Form		Collection of system
(a) $\log_{\varphi(x)} f(x) \geq \log_{\varphi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x) \geq g(x), \quad \phi(x) > 1, \\ 0 < f(x) \leq g(x) \;;\; 0 < \phi(x) < 1 \end{cases}$
(b) $\log_{\varphi(x)} f(x) \leq \log_{\varphi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} 0 < f(x) \leq g(x), \quad \phi(x) > 1, \\ f(x) \geq g(x) > 0, \quad 0 < \phi(x) < 1 \end{cases}$

Example # 36 : Solve the logarithmic inequality $\log_{\frac{1}{5}} (2x^2 + 7x + 7)$.

Solution : Since $\log_{\frac{1}{5}} 1 = 0$, the given inequality can be written as $\log_{\frac{1}{5}} (2x^2 + 7x + 7) \geq \log_{\frac{1}{5}} 1$ when the domain of the function is taken into account the inequality is equivalent to the system of ineqaulities.

$$\begin{cases} 2x^2 + 7x + 7 > 0 \\ 2x^2 + 7x + 7 \leq 1 \end{cases}$$

Solving the ineqaualities by using method of intervals $x \in \left[-2, \frac{-3}{2}\right]$

Example # 37 : Solve the inequality $\log_{\frac{1}{3}} (5x - 1) > 0$.

Solution : by using the basic property of logarithm.

$$\begin{cases}5x-1<1\\5x-1>0\end{cases} \Rightarrow \begin{cases}5x<2 & x<\dfrac{2}{5}\\ & \Rightarrow\\5x>1 & x>\dfrac{1}{5}\end{cases}$$

$\Rightarrow$ The solution of the inequality is given by $\left(\dfrac{1}{5}, \dfrac{2}{5}\right)$

Self practice problem :

(30) Solve the following inequalities

(i) $\log_2 (x^2 - 4x + 5) > 1$ (ii) $\log_{0.2} (x^2 - x - 2) > \log_{0.2} (-x^2 + 2x + 3)$

Ans. (i) $(-\infty, 1) \cup (3, \infty)$ (ii) $\left(2, \dfrac{5}{2}\right)$

27. Characteristic & Mantissa

$[\log_a N]$ is called characteristic of log of N with base 'a'. It is always an integer.

$\{\log_a N\}$ is called mantissa of log of N with base 'a'. Mantissa $\in [0, 1)$

characteristic of log of 1 with base 10 = 0
characteristic of log of 10 with base 10 = 1
characteristic of log of 100 with base 10 = 2
characteristic of log of 1000 with base 10 = 3
characteristic of log of 83.5609 with base 10 = 1
characteristic of log of 613.0965 with base 10 = 2

Interval	Characteristic (Base 10)	Number of digits in number	No. of integers in the interval
[1, 10)	0	1	$9 = 9 \times 10^0$
[10, 100)	1	2	$90 = 9 \times 10^1$
[100, 1000)	2	3	$900 = 9 \times 10^2$
[1000, 10000)	3	4	$9000 = 9 \times 10^3$
	|	|	|
	n	(n+1)	9×10^n

Note : If characteristic of a number (base of log is 10) is found to be n, then there would be (n + 1) digits in that number.

Characteristic of log of $\dfrac{1}{10} = 0.1$ with base 10 = – 1

Characteristic of log of $\dfrac{1}{100} = 0.01$ with base 10 = – 2

Characteristic of log of $\dfrac{1}{1000} = 0.001$ with base 10 = – 3

Interval	Characteristic (Base 10)	No. of zeros immediately after decimal	No. of integers reciprocal of which lies in interval.
$\left[\frac{1}{10}, 1\right)$	–1	0	$9 = 9 \times 10^{1-1}$
$\left[\frac{1}{100}, \frac{1}{10}\right) \equiv [0.01, 0.1)$	–2	1	$90 = 9 \times 10^{2-1}$
$\left[\frac{1}{10^3}, \frac{1}{10^2}\right) \equiv [0.001, 0.01)$	–3	2	$900 = 9 \times 10^{3-1}$
[0.0001, 0.001)	–4	3	$9000 = 9 \times 10^{4-1}$
	|	|	|
	–n	(n–1)	$9 \times 10^{n-1}$

Note :

If characteristic of a number (base of log is 10) is found to be –n, then there would be (n –1) zeros immediately after decimal before first significant digit.

Example # 38 : Find the total number of digits in the number 18^{50}.

(Given $\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$)

Solution : $N = 18^{50}$

$\log_{10}N = 50 \log_{10}18 = 50\ (0.6020 + 0.4771) = 50(1.0791) = 53.9550$

Characterstic = $[\log_{10}N] = 53$

No. of digits = 53 + 1 = 54

Self practice problem :

(31) Find the total number of zeros immediately after the decimal in 6^{-200} .

Ans. 155

28. Expansion of a determinat :

(i) Expansion of two order determinant : $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

(ii) Expansion of 3^{rd} order determinant :

(a) With respect to first row :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1\ (a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_3a_3)$$

(b) With respect to second column :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\,b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -\,b_1(a_2c_3 - a_3c_2) + b_2\ (a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

Similarly a determinant can be expanded with respect to any row or column.

Example # 39: Find the value of the determinant. $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & -3 & 2 \end{vmatrix}$

Solution : Expanding the determinant w.r.t first row

$$\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & -3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 5 & 7 \\ -3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 6 & 7 \\ 1 & 2 \end{vmatrix} + 4 \begin{vmatrix} 6 & 5 \\ 1 & -3 \end{vmatrix} = 2(10 + 21) - 3\ (12 - 7) + 4\ (-18 - 5)$$

$$= 62 - 15 - 92 = -\,45$$