Calculus required continuity, and continuity was supposed to require the infinitely little; But nobody could discover what the infinitely little might be .... **Russell, Bertrand**

1. Definition : (Limit of a function f(x) is said to exist, as x → a when)

$\lim\limits_{h \to 0^-} f(a - h) = \lim\limits_{h \to 0^-} f(a + h)$ = Finite

(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at x = a. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '$-\infty$', then it is said to be infinite limit.

Remember, '$x \to a$' means that x is approaching to 'a' but not equal to 'a'.

2. Fundamental theorems on limits :

Let $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = m$. If ℓ & m are finite, then:

(i) $\lim\limits_{x\to a} \{ f(x) \pm g(x) \} = \ell \pm m$

(ii) $\lim\limits_{x\to a} \{ f(x).\ g(x) \} = \ell . m$

(iii) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$, provided $m \neq 0$

(iv) $\lim\limits_{x\to a} k f(x) = k \quad f(x) = k\ell$; where k is a constant.

(v) $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f is continuous at g (x) = m.

Example # 1 : Evaluate the following limits : -

(i) $\lim\limits_{x\to 2} (x + 2)$ (ii) $\lim\limits_{x\to 0} \cos(\sin x)$

Solution : (i) x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim\limits_{x\to 2} (x + 2) = 2 + 2 = 4$

(ii) $\lim\limits_{x\to 0} \cos(\sin x) = \cos\left(\lim\limits_{x\to 0} \sin x\right) = \cos 0 = 1$

Self practice problems :

Evaluate the following limits : -

(1) $\lim\limits_{x\to 2} x(x - 1)$ (2) $\lim\limits_{x\to 2} \dfrac{x^2+4}{x+2}$

Ans. (1) 2 (2) 2

3. Indeterminate forms :

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\mathbf{0 \times \infty, \infty - \infty, \infty^0, 0^0, 1^\infty}$

· Consider $f(x) = \dfrac{x^2-4}{x-2}$. Here $\lim\limits_{x\to 2} x^2 - 4 = 0$ and $\lim\limits_{x\to 2} x - 2 = 0$

$\therefore \lim\limits_{x \to 2} f(x)$ has an indeterminate form of the type $\frac{0}{0}$.

- $\lim\limits_{x \to \infty} \frac{\ell nx}{x}$ has an indeterminate form of type $\frac{\infty}{\infty}$.
- $\lim\limits_{x \to 0} (1 + x)^{1/x}$ is an indeterminate form of the type 1^{∞}
- $\lim\limits_{x \to 0} \frac{x}{x}$ is an indeterminate form whereas $\lim\limits_{x \to 0} \frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function)

  i.e. $\frac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form whereas $\frac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

  similarly $(\text{tending to one})^{\text{tending to } \infty}$ is indeterminate form whereas $(\text{exactly one})^{\text{tending to } \infty}$ is not an indeterminate form, its value is one.
- $\lim\limits_{x \to 0} \frac{[\sin^2 x]}{x}$ (where [.] represents greatest integer function) is not of indeterminate form as numerator is exact zero & denominator is approaching zero.

NOTE :

(i) $\infty + \infty = \infty$ (ii) $\infty \times \infty = \infty$

(iii) $\frac{a}{\infty} = 0$, if a is finite. (iv) $\frac{a}{0}$ is not defined for any $a \in R$.

4. Methods of removing indeterminancy :

Basic methods of removing indeterminancy are

(A) Factorisation (B) Rationalisation

(C) Using standard limits (D) Substitution

(E) Expansion of functions.

(i) Factorisation method :

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Example # 2 : (i) $\lim\limits_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3}$ (ii) $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{2\ (2x-3)}{x^3 - 3x^2 + 2x}\right]$

Solution : (i) $\lim\limits_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim\limits_{x \to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

(ii) $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{2\ (2x-3)}{x^3 - 3x^2 + 2x}\right] = \lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{2\ (2x-3)}{x(x-1)(x-2)}\right]$

$= \lim\limits_{x \to 2} \left[\frac{x(x-1) - 2(2x-3)}{x(x-1)(x-2)}\right] = \lim\limits_{x \to 2} \left[\frac{x^2 - 5x + 6}{x(x-1)(x-2)}\right]$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

(ii) Rationalisation method :-

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Example # 3 : Evaluate :

(i) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$ (ii) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution : (i) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\;\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}}\;\frac{\sqrt{1+x^3}+\sqrt{1+x}}{\sqrt{1+x^3}+\sqrt{1+x}}$

$$\lim_{x\to 0}\frac{1+x^2-(1+x)}{1+x^3-(1+x)}\;\frac{\sqrt{1+x^3}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}} \Rightarrow \frac{(x)(x-1)}{(x)(x-1)(x+1)} = 1$$

(ii) The form of the given limit is $\frac{0}{0}$ when $x \to 0$. Rationalising the numerator, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1+x)-(1-x)}{x\ (\sqrt{1+x}+\sqrt{1-x})}\right] = \lim_{x\to 0}\left[\frac{2x}{x\ (\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

Self practice problems :

Evaluate the following limits : -

(3) $\lim_{x\to 1}\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$ (4) $\lim_{x\to\frac{\pi}{2}}\frac{1-(\sin x)^{1/3}}{1-(\sin x)^{2/3}}$ (5) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

(6) $\lim_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ (7) $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}}$

Ans. (3) $\frac{-1}{10}$ (4) $\frac{1}{2}$ (5) $\frac{1}{2\sqrt{x}}$ (6) $\frac{1}{4a\sqrt{a-b}}$ (7) 0

5. <u>Standard limits</u> :

(i) (a) $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$ [Where x is measured in radians]

(b) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$

(c) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$; $\lim\limits_{x\to 0} (1+ax)^{\frac{1}{x}} = e^a$

(d) $\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$; $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$

(e) $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$; $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \log_e a = \ell n a$, $a > 0$

(f) $\lim\limits_{x\to 0} \frac{\ell n(1+x)}{x} = 1$

(g) $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

(ii) If $f(x) \to 0$, when $x \to a$, then

(a) $\lim\limits_{x\to a} \frac{\sin f(x)}{f(x)} = 1$ (b) $\lim\limits_{x\to a} \cos f(x) = 1$

(c) $\lim\limits_{x\to a} \frac{\tan f(x)}{f(x)} = 1$ (d) $\lim\limits_{x\to a} \frac{e^{f(x)}-1}{f(x)} = 1$

(e) $\lim\limits_{x\to a} \frac{b^{f(x)}-1}{f(x)} = \ell n\, b$, $(b > 0)$ (f) $\lim\limits_{x\to a} \frac{\ell n(1+f(x))}{f(x)} = 1$

(g) $\lim\limits_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

(iii) $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} \varphi(x) = B$ (a finite quantity), then $\lim\limits_{x\to a} [f(x)]^{\varphi(x)} = A^B$.

Example # 4 : Evaluate : $\lim\limits_{x\to 0} \frac{(1+x)^n-1}{x}$

Solution : $\lim\limits_{x\to 0} \frac{(1+x)^n-1}{x} = \lim\limits_{x\to 0} \frac{(1+x)^n-1}{(1+x)-1} = n$

Example # 5 : Evaluate : $\lim\limits_{x\to 0} \frac{e^{6x}-1}{2x}$

Solution : $\lim\limits_{x\to 0} \frac{e^{6x}-1}{2x} = \lim\limits_{x\to 0} 3\, \frac{e^{6x}-1}{6x} = 3.$

Example # 6 : Evaluate : $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3}$

Solution : $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{\tan x(1-\cos x)}{x^3} = \lim\limits_{x\to 0} \frac{\tan x \,.\, 2\sin^2 \frac{x}{2}}{x^3}$

$$= \lim\limits_{x\to 0} \frac{1}{2} \,.\, \frac{\tan x}{x} \,.\, \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.$$

Example # 7 : Evaluate : $\lim\limits_{x\to 0} \frac{\tan x}{\tan 5x}$

Solution : $\lim\limits_{x\to 0}\frac{\tan x}{\tan 5x} = \lim\limits_{x\to 0}\left[\frac{\tan x}{x}\cdot\frac{x}{5x}\cdot\frac{5x}{\tan 5x}\right] = \left[\lim\limits_{x\to 0}\frac{\tan x}{x}\right].\frac{1}{5}.\left[\lim\limits_{5x\to 0}\frac{5x}{\tan 5x}\right] = \frac{1}{5}$

Example # 8 : Evaluate : $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Solution : $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^{\lim\limits_{x\to\infty}\frac{2}{x}.x} = e^2.$

Example # 9 : Evaluate : (i) $\lim\limits_{x\to 3}\frac{e^x-e^3}{x-3}$ (ii) $\lim\limits_{x\to 0}\frac{x(e^x-1)}{1-\cos x}$

Solution : (i) Put y = x – 3. So, as x → 3 ⇒ y → 0. Thus

$$\lim_{x\to 3}\frac{e^x-e^3}{x-3} = \lim_{y\to 0}\frac{e^{3+y}-e^3}{y} = \lim_{y\to 0}\frac{e^3\ .\ e^y-e^3}{y} = e^3\lim_{y\to 0}\frac{e^y-1}{y} = e^3\ .\ 1 = e^3$$

(ii) $$\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0}\frac{x(e^x-1)}{2\sin^2\frac{x}{2}} = \frac{1}{2}.\lim_{x\to 0}\left[\frac{e^x-1}{x}\ .\ \frac{x^2}{\sin^2\frac{x}{2}}\right] = 2.$$

Self practice problems :

Evaluate the following limits : -

(8) $\lim\limits_{x\to 0}\frac{\sin 2x}{x}$ (9) $\lim\limits_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$

(10) $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$ (11) $\lim\limits_{x\to 0}\frac{5^x-9^x}{x}$

(12) $\lim\limits_{x\to\infty}(1+a^2)^x\sin\frac{b}{(1+a^2)^x}$, where a ≠ 0

Ans. (8) 2 (9) $\frac{1}{32}$ (10) does not exist (11) $\ell n\frac{5}{9}$

(12) b

6. Use of substitution in solving limit problems :

Sometimes in solving limit problem we convert $\lim\limits_{x\to a}f(x)$ into $\lim\limits_{h\to 0}f(a+h)$ or $\lim\limits_{h\to 0}f(a-h)$ according as need of the problem. (here h is approaching to zero.)

Example # 10 : Evaluate $\lim\limits_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}$

Solution : Put $x = \frac{\pi}{4} + h$

∵ $x \to \frac{\pi}{4} \Rightarrow h \to 0$

$$\lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)} = \lim_{h\to 0}\frac{1-\frac{1+\tan h}{1-\tan h}}{1-\sin h-\cos h} = \lim_{h\to 0}\frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$= \lim_{h\to 0} \frac{-2\tan h}{2\sin\frac{h}{2}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)} = \lim_{h\to 0} \frac{-2\frac{\tanh}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)} = \frac{-2}{-1} = 2.$$

7. Limits using expansion :

(i) $a^x = 1+\frac{x\ \ell n\ a}{1!}+\frac{x^2\,\ell n^2 a}{2!}+\frac{x^3\,\ell n^3 a}{3!}+.........,\quad a > 0$

(ii) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......$

(iii) $\ell n\,(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.........,\quad \text{for } -1 < x \le 1$

(iv) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.....$

(v) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.....$

(vi) $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+......$

(vii) for $|x| < 1$, $n \in R$; $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3 + ...\infty$

Example # 11 : Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Solution : $\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}.....\infty\right)}{x^3} = \frac{\left(\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}.....\infty\right)}{x^3} = \frac{1}{6}$

Example # 12 : Evaluate $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Solution : $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} = \lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+.......\right)-\left(x-\frac{x^3}{3\,!}+......\right)}{x^3} = \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$.

Example # 13 : Evaluate $\lim_{x\to 1}\frac{(7+x)^{\frac{1}{3}}-2}{x-1}$

Solution : Put $x = 1 + h$

$$\lim_{h\to 0}\frac{(8+h)^{\frac{1}{3}}-2}{h}$$

$$= \lim_{h\to 0} \frac{2.\left(1+\frac{h}{8}\right)^{\frac{1}{3}}-2}{h} = \lim_{h\to 0} \frac{2\left\{1+\frac{1}{3}\ .\ \frac{h}{8}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{h}{8}\right)^2}{1\ .\ 2}+.......-1\right\}}{h} = \lim_{h\to 0} 2 \times \frac{1}{24} = \frac{1}{12}$$

Example # 14 : Evaluate $\lim_{x\to 0} \frac{\ell n(1+x)-\sin x+\frac{x^2}{2}}{x\tan x\ \sin x}$

Solution : $\lim_{x\to 0} \frac{\ell n(1+x)-\sin x+\frac{x^2}{2}}{x\tan x\ \sin x} = \lim_{x\to 0} \frac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-.....\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right)+\frac{x^2}{2}}{x^3\ .\ \frac{\tan x}{x}\ .\ \frac{\sin x}{x}} = \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

Example # 15 : Evaluate $\lim_{x\to 0} \frac{e-(1+x)^{\frac{1}{x}}}{\tan x}$

Solution : $\lim_{x\to 0} \frac{e-(1+x)^{\frac{1}{x}}}{\tan x} = \lim_{x\to 0} \frac{e-e\left(1-\frac{x}{2}+......\right)}{\tan x} = \lim_{x\to 0} \frac{e}{2} \times \frac{x}{\tan x} = \frac{e}{2}$

Example # 16 : Find the values of a,b and c so that $\lim_{x\to 0} \frac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$

Solution : $\lim_{x\to 0} \frac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$ .....(1)

at $x \to 0$ numerator must be equal to zero

$\therefore$ $a-b+c=0$ $\Rightarrow$ $b=a+c$ .....(2)

From (1) & (2), $\lim_{x\to 0} \frac{ae^x-(a+c)\cos x+ce^{-x}}{x\sin x} = 2$

$$\Rightarrow \lim_{x\to 0} \frac{a\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......\right)-(a+c)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)+c\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+......\right)}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right)} = 2$$

$$\Rightarrow \lim_{x\to 0} \frac{\frac{(a-c)}{x}+(a+c)+\frac{x}{3!}(a-c)+.....}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)} = 2$$

Since R.H.S is finite,

$\therefore$ $a-c=0$

$\therefore$ $a=c$, then $\frac{0+2a+0+....}{1} = 2$

$\therefore$ $a=1$ then $c=1$

From (2), $b=a+c=1+1=2$

8. Limit when $x \to \infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x \to \infty$, we use to substitute $y = \frac{1}{x}$ and in this case $y \to 0^+$.

Example # 17 : Evaluate $\lim\limits_{x\to\infty} x \sin \frac{1}{x}$

Solution : $\lim\limits_{x\to\infty} x \sin \frac{1}{x} = \lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

Example # 18 : Evaluate $\lim\limits_{x\to\infty} \frac{x^2-2}{3x^2+4}$

Solution : $\lim\limits_{x\to\infty} \frac{x^2-2}{3x^2+4} = \lim\limits_{x\to\infty} \frac{1-\frac{2}{x^2}}{3+\frac{4}{x^2}} = \frac{1}{3}$.

Example # 19 : Evaluate $\lim\limits_{x\to\infty} \frac{x^3+7x+5}{4x^3+x^4+1}$

Solution : $\lim\limits_{x\to\infty} \frac{x^3+7x+5}{4x^3+x^4+1} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}+\frac{7}{x^3}+\frac{5}{x^4}}{\frac{4}{x}+1+\frac{1}{x^4}} = 0$

Example # 20 : Evaluate $\lim\limits_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x-2}$

Solution : $\lim\limits_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x-2}$ (Put $x = -\frac{1}{t}$, $x \to -\infty \Rightarrow t \to 0^+$)

$= \lim\limits_{t\to0^+} \frac{\sqrt{3+2t^2}\cdot\frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim\limits_{t\to0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)}\frac{t}{|t|} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$.

9. Some important facts :

(i) $\lim\limits_{x\to\infty} \frac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \frac{x}{e^x} = 0$ (iii) $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$

(iv) $\lim\limits_{x\to\infty} \frac{(\ell nx)^n}{x} = 0$ (v) $\lim\limits_{x\to0^+} x(\ell nx)^n = 0$

As $x \to \infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ and $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h \to 0^+$.

Example # 21 : Evaluate $\lim\limits_{x\to\infty} \frac{x^{1000}}{e^x}$

Solution : $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x} = 0$

10. Limits of form 1^∞, 0^0, ∞^0

(i) All these forms can be converted into $\dfrac{0}{0}$ form in the following ways

(a) If $x \to 1$, $y \to \infty$, then $z = (x)^y$ is of 1^∞ form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{\ell n x}{\frac{1}{y}} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

As $y \to \infty \Rightarrow \dfrac{1}{y} \to 0$ and $x \to 1 \Rightarrow \ell n x \to 0$

(b) If $x \to 0$, $y \to 0$, then $z = x^y$ is of (0^0) form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell n x}} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

(c) If $x \to \infty$, $y \to 0$, then $z = x^y$ is of $(\infty)^0$ form

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{\frac{1}{\ell n x}} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

(ii) $(1)^\infty$ type of problems can be solved by the following method

(a) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\lim\limits_{x\to a} [f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \lim_{x\to a} [1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.\,g(x)} = \lim_{x\to a}\left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)\,g(x)} = e^{\lim\limits_{x\to a}[f(x)-1]\,g(x)}$$

Example # 22 : Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-2}\right)^{2x-1}$

Solution : Since it is in the form of 1^∞

$$\lim_{x\to\infty}\left(\frac{x+1}{x-2}\right)^{2x-1} = \exp\left(\lim_{x\to\infty}\left(\frac{x+1-x+2}{x-2}\right)(2x-1)\right) = e^6$$

Example # 23 : Evaluate $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution : Since it is in the form of 1^∞ so $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$

$$= e^{2\times\frac{\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

Example # 24 : Evaluate $\lim\limits_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Solution : $\lim\limits_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put $x = a + h$

$= \lim\limits_{h \to 0} \left(1 + \frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2} + \frac{\pi h}{2a}\right)} = \lim\limits_{h \to 0} \left(1 + \frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)} = e^{\lim\limits_{h\to0} -\cot\frac{\pi h}{2a}\cdot\left(1+\frac{h}{a+h}-1\right)}$

$= e^{\lim\limits_{h\to0} -\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right)\cdot\frac{\frac{2a}{\pi}}{a+h}} = e^{-\frac{2}{\pi}}$

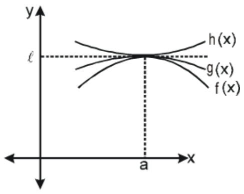
Example # 25 : Evaluate $\lim\limits_{x\to0^+} x^x$

Solution : Let $y = \lim\limits_{x\to0^+} x^x$

$\Rightarrow \quad \ell n\, y = \lim\limits_{x\to0^+} x\, \ell n\, x = \lim\limits_{x\to0^+} -\frac{\ell n\frac{1}{x}}{\frac{1}{x}} = 0$, as $\frac{1}{x} \to \infty \quad \Rightarrow \quad y = 1$

11. Sandwitch theorem or squeeze play theorem :

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that



$\lim\limits_{x\to a} f(x)\ \ell = \lim\limits_{x\to a} = h(x)$, Then $\lim\limits_{x\to a} g(x) = \ell$.

Example # 26 : Evaluate $\lim\limits_{n\to\infty} \frac{[x]+[2x]+[3x]+....+[nx]}{n^2}$, where [.] denotes greatest integer function.

Solution : We know that, $x - 1 < [x] \le x$

$2x - 1 < [2x] \le 2x$

$3x - 1 < [3x] \le 3x$

. . .

. . .

$nx - 1 < [nx] \le nx$

$\therefore \quad (x + 2x + 3x + ... + nx) - n < [x] + [2x] + ... +[nx] \le (x + 2x + ... + nx)$

$\Rightarrow \quad \frac{xn(n+1)}{2} - n < \sum\limits_{r=1}^{n}[rx] \le \frac{x.n(n+1)}{2}$

$\Rightarrow \quad \lim\limits_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \lim\limits_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} \le \lim\limits_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right)$

$\Rightarrow \quad \frac{x}{2} < \lim\limits_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} \le \frac{x}{2} \quad \therefore \lim\limits_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} = \frac{x}{2}$

Aliter : We know that $[x] = x - \{x\}$

$\sum_{r=1}^{n}[rx]$ = [x] + [2x] + ... + [nx]

= (x + 2x + 3x + ... + nx) – ({x} + {2x} + ... + {nx}) $= \frac{xn(n+1)}{2}$ – ({x} + {2x} + .. + {nx})

$\therefore \quad \frac{1}{n^2}\sum_{r=1}^{n}[rx] = \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{\{x\}+\{2x\}+....+\{nx\}}{n^2}$

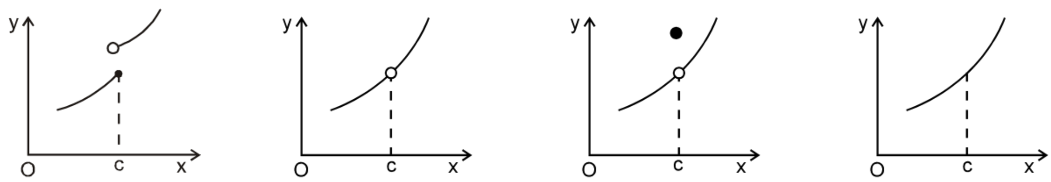
Since, $0 \le \{rx\} < 1$, $\quad\therefore\quad 0 \le \sum_{r=1}^{n}\{rx\} < n \quad \Rightarrow \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2} = 0$

$\therefore \quad \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2} = \lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) - \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2} \Rightarrow \lim_{n\to\infty}\frac{\sum_{r=1}^{n}[rx]}{n^2} = \frac{x}{2}$

12. Continuity & Derivability :

A function f(x) is said to be continuous at x = c, if $\text{Limit}_{x\to c}$ f(x) = f(c) i.e. f is continuous at x = c if $\text{Limit}_{h\to 0^+}$ f(c – h) = $\text{Limit}_{h\to 0^+}$ f(c+h) = f(c).

If a function f (x) is continuous at x = c, the graph of f (x) at the corresponding point (c, f (c)) will not be broken. But if f (x) is discontinuous at x = c, the graph will be broken when x = c



(i) (ii) (iii) (iv)

((i), (ii) and (iii) are discontinuous at x = c)
((iv) is continuous at x = c)

A function f can be discontinuous due to any of the following three reasons:

(i) $\text{Limit}_{x\to c}$ f(x) does not exist i.e. $\text{Limit}_{x\to c^-}$ f(x) $\ne$ $\text{Limit}_{x\to c^+}$ f (x) **[figure (i)]**

(ii) f (x) is not defined at x = c **[figure (ii)]**

(iii) $\text{Limit}_{x\to c}$ f(x) $\ne$ f (c) **[figure (iii)]**

Geometrically, the graph of the function will exhibit a break at x= c.

Example # 27 : If f(x) = $\begin{cases}\sin\frac{\pi x}{2}, & x<1\\ [x], & x\ge 1\end{cases}$, then find whether f(x) is continuous or not at x = 1, where [.] is greatest integer function.

Solution : f(x) = $\begin{cases}\sin\frac{\pi x}{2}, & x<1\\ [x], & x\ge 1\end{cases}$

For continuity at x = 1, we determine f(1), $\lim_{x\to 1^-}$ f(x) and $\lim_{x\to 1^+}$ f(x).

Now, f(1) = [1] = 1

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}[x] = 1$

so $f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1

Self practice problems :

(13) If possible find value of λ for which f(x) is continuous at $x = \frac{\pi}{2}$

$$f(x) = \begin{cases} \dfrac{1-\sin x}{1+\cos 2x}, & x < \dfrac{\pi}{2} \\ \lambda \quad , & x = \dfrac{\pi}{2} \\ \dfrac{\sqrt{2x-\pi}}{\sqrt{4+\sqrt{2x-\pi}}-2} \quad , & x > \dfrac{\pi}{2} \end{cases}$$

(14) Find the values of a and b such that the function

$$f(x) = \begin{cases} x + a\sqrt{2} \quad \sin x & ; \quad 0 \le x < \dfrac{\pi}{4} \\ 2x\cot x + b & ; \quad \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \\ a\cos 2x - b\sin x & ; \quad \dfrac{\pi}{2} < x \le \pi \end{cases}$$

is continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

(15) If $f(x) = \begin{cases} (1+ax)^{1/x} & ; \quad x < 0 \\ b & ; \quad x = 0 \\ \dfrac{(x+c)^{1/3}-1}{x} & ; \quad x > 0 \end{cases}$, then find the values of a, b, c, for which f(x) is continuous at x = 0

Ans. (13) discontinuous (14) $a = \frac{\pi}{6}$, $b = \frac{-\pi}{12}$ (15) $a = -\ell n\ 3$, $b = \frac{1}{3}$, c = 1

13. Theorems on continuity :

(i) If f & g are two functions which are continuous at x = c, then the functions defined by:

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\, f(x)$, K is any real number ; $F_3(x) = f(x).g(x)$ are also continuousat x = c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(ii) If f(x) is continuous & g(x) is discontinuous at x = a, then the product function

$\varphi(x) = f(x).\, g(x)$ may or may not be continuous but sum or difference function

$\varphi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at x = a.

e.g. f (x) = x & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

(iii) If f (x) and g(x) both are discontinuous at x = a, then the product function φ (x) = f(x). g(x) is not necessarily be discontinuous at x = a.

e.g. f (x) = g(x) = $\begin{cases} 1 & , \quad x \geq 0 \\ -1 & , \quad x < 0 \end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

Example # 28 : If f(x) = [sin(x–1)] – {sin(x–1)}. Comment on continuity of f(x) at x = $\frac{\pi}{2} + 1$ (where [**.**] denotes G.I.F. and { **.** } denotes fractional part function).

Solution : f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [sin (x – 1)] + {sin (x – 1)} = sin (x – 1)

which is continuous at x = $\frac{\pi}{2}$ + 1

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at x = $\frac{\pi}{2}$ + 1

∴ At most one of f(x) or g(x) can be continuous at x = $\frac{\pi}{2}$ + 1

As g(x) is continuous at x = $\frac{\pi}{2}$ + 1, therefore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluating $\lim\limits_{x \to \frac{\pi}{2}+1}$ f(x) and f$\left(\frac{\pi}{2}+1\right)$.

14. Continuity of composite functions :

If f is continuous at x = c and g is continuous at x = f(c), then the composite g[f(x)] is continuous at x = c. eg. f(x) = $\frac{x \sin x}{x^2 + 2}$ & g(x) = $|x|$ are continuous at x = 0, hence the composite function (gof) (x) = $\left|\frac{x \sin x}{x^2 + 2}\right|$ will also be continuous at x = 0.

Self practice problem :

(16) $f(x) = \begin{cases} 1 + x^3 & , \quad x < 0 \\ x^2 - 1 & , \quad x \geq 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{\frac{1}{3}} & , \quad x < 0 \\ (x+1)^{\frac{1}{2}} & , \quad x \geq 0 \end{cases}$

Then define fog (x) and comment on the continuity of gof(x) at x = 1

Ans. [fog(x) = x, x ∈ R and gof(x) is discontinous at x = 1]

15. Continuity in an Interval :

(i) A function f is said to be continuous in (a, b) if f is continuous at each & every point ∈(a, b).

(ii) A function f is said to be continuous in a closed interval [a, b] if:

(a) f is continuous in the open interval (a, b),

(b) f is right continuous at 'a' i.e. $\text{Limit}_{x \to a^+}$ f(x) = f(a) = a finite quantity and

(c) f is left continuous at 'b' i.e. $\text{Limit}_{x \to b^-}$ f(x) = f(b) = a finite quantity.

(iii) All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

On the basis of above facts continuity of a function should be checked at the following points

(a) Continuity of a function should be checked at the points where definition of a function changes.

(b) Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

(c) Continuity of sgn (f(x)) should be checked at the points where f(x) = 0 (if f(x) = 0 in any open interval containing a, then x = a is not a point of discontinuity)

(d) In case of composite function f(g(x)) continuity should be checked at all possible points of discontinuity of g(x) and at the points where g(x) = c, where x = c is a possible point of discontinuity of f(x).

Example # 29 : If $f(x) = \begin{cases} [\sin \pi x] & ; \quad 0 \le x < 1 \\ \left\{x - \frac{2}{3}\right\} \cdot \text{sgn}\left(x - \frac{5}{4}\right) & ; \quad 1 \le x \le 2 \end{cases}$, where { . } represents fractional part function and [.] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

Solution : (i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2

(ii) For [sin πx], continuity should be checked at all values of x at which sin πx ∈ I

i.e. $x = 0, \frac{1}{2}$

(iii) For , $\left\{x - \frac{2}{3}\right\} \cdot \text{sgn}\left(x - \frac{5}{4}\right)$ continuity should be checked when $x - \frac{5}{4} = 0$

(as sgn (x) is discontinuous at x = 0) i.e. $x = \frac{5}{4}$ and when $x - \frac{2}{3} \in I$

i.e. $x = \frac{5}{3}$ (as {x} is discontinuous when x ∈ I)

∴ overall discontinuity should be checked at $x = 0, \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$ and 2
check the discontinuity your self.

discontinuous at $x = \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$

Example # 30 : If $f(x) = \frac{x+3}{x-3}$ and $g(x) = \frac{1}{x-7}$, then discuss the continuity of f(x), g(x) and fog (x).

Solution : $f(x) = \frac{x+3}{x-3}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 3
∴f is discontinuous at x = 3

$g(x) = \frac{1}{x-7}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 7

∴ g is discontinuous at x = 7

Now fog (x) will be discontinuous at

(i) x = 7 (point of discontinuity of g(x))

(ii) $g(x) = 3$ (when $g(x)$ = point of discontinuity of $f(x)$)

if $g(x) = 3 \Rightarrow \frac{1}{x-7} = 3 \Rightarrow x = 22/3$

$\therefore$ discontinuity of $fog(x)$ should be checked at $x = 7$ and $x = 22/3$

$$fog(x) = \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3}$$

fog (7) is not defined

$$\lim_{x\to 7} fog(x) = \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = \lim_{x\to 7} \frac{1+3x-21}{1-3x+21} = 1$$

$\therefore$fog (x) is discontinuous at x = 7 and it is removable discontinuity at x = 7

fog (22/3) = not defined

$$\lim_{x\to \frac{22}{3}^+} fog(x) = \lim_{x\to \frac{22}{3}^+} \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = \infty$$

$$\lim_{x\to \frac{22}{3}^-} fog(x) = \lim_{x\to \frac{22}{3}^-} \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = -\infty$$

$\therefore$ fog (x) is discontinuous at x = 22/3 and it is non removable discontinuity of IInd kind.

Self practice problem :

(17) If $f(x) = \begin{cases} [\ell n\ x]\ .\ \text{sgn}\left(\left\{x-\frac{1}{2}\right\}\right) ; & 1 < x \le 3 \\ \{x^2\} ; & 3 < x \le 3.5 \end{cases}$. Find the points where the continuity of f(x), should be checked, where [.] is greatest integer function and {.} fractional part function.

Ans. $\{ 1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5 \}$

16. <u>Intermediate value theorem</u> :

A function f which is continuous in $[a,b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

Example # 31 : Show that the function $f(x) = (x - a)^2 (x - b)^2 + x$ take the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

Solution : f(a) = a ; f(b) = b. Also f is continuous in [a, b] and $\frac{a+b}{2} \in [a, b]$. Hence using intermediate value theorem, there exist atleast one $c \in [a, b]$ such that $f(c) = \frac{a+b}{2}$.

Self practice problem :

(18) If $f(x) = xe^x - 2$, then show that f(x) = 0 has exactly one root in the interval (0, 1).

Example # 32 : Let $f(x) = \lim\limits_{n\to\infty} \frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Solution : Let $f(x) = \lim\limits_{n\to\infty} \frac{1}{1+n\sin^2 x} \Rightarrow f\left(\frac{\pi}{4}\right) = \lim\limits_{n\to\infty} \frac{1}{1+n.\sin^2\frac{\pi}{4}} = \lim\limits_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$

Now

$f(0) = \lim\limits_{n\to\infty} \frac{1}{n.\sin^2(0)+1} = \frac{1}{1+0} = 1$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left[\lim\limits_{n\to\infty} \frac{1}{1+n\ \sin^2 x}\right] = 0$

{here $\sin^2 x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes 0}

∴f(x) is not continuous at x = 0

Self practice problem :

(19) If $f(x) = \lim\limits_{n\to\infty} (1 + x)^n$.

Comment on the continuity of f(x) at x = 0 and explain $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

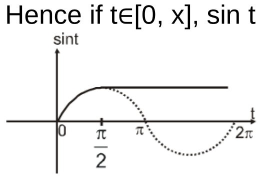
Ans. Discontinous (non-removable)

Example # 33 : f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π discuss the continuity of this function at $x = \frac{\pi}{2}$

Solution : f(x) = maximum (sin t, 0 ≤ t ≤ x), 0 ≤ x ≤ 2π

if $x \in \left[0, \frac{\pi}{2}\right]$, sin t is increasing function

Hence if t∈[0, x], sin t will attain its maximum value at t = x.



∴f(x) = sin x if $x \in \left[0, \frac{\pi}{2}\right]$ if $x \in \left(\frac{\pi}{2}, 2\pi\right]$ and t ∈ [0, x]

then sin t will attain its maximum value when $t = \frac{\pi}{2}$

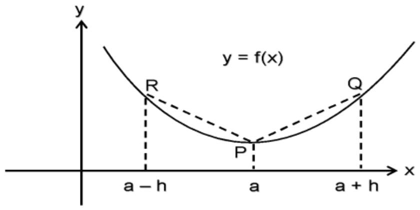
∴ $f(x) = \sin\frac{\pi}{2} = 1$ if $x \in \left(\frac{\pi}{2}, 2\pi\right]$ ∴ $f(x) = \begin{cases} \sin x , & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 1 , & \text{if } x \in \left(\frac{\pi}{2}, 2\pi\right] \end{cases}$

Now $f\left(\frac{\pi}{2}\right) = 1 \Rightarrow \lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-} \sin x = 1$

$\lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} 1 = 1$ as f(π/2) = L.H.L. = R.H.L.

$\therefore$ f(x) is continuous at x = $\frac{\pi}{2}$.

17. Differentiability of a function at a point :



(i) The right hand derivative (R.H.D) of f (x) at x = a denoted by f '(a_+) is defined by slop of PQ

$\therefore$ R.H.D. = f '(a_+) = $\underset{h\to 0^+}{\text{Limit}} \frac{f(a+h)-f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative (L.H.D) of f(x) at x = a denoted by f'(a_-) is defined by slop of PR

$\therefore$ L.H.D. = f ' (a_-) = $\underset{h\to 0^+}{\text{Limit}} \frac{f(a-h)-f(a)}{-h}$, provided the limit exists.

A function f(x) is said to be differentiable at x = a if f '(a_+) = f ' (a_-) = finite

By definition f '(a) = $\underset{h\to 0^+}{\text{Limit}} \frac{f(a+h)-f(a)}{h}$

Example # 34 : Comment on the differentiability of f(x) = $\begin{cases} x, & x<1 \\ x^2, & x\ge 1 \end{cases}$ at x = 1.

Solution : R.H.D. = f' (1_+) = $\underset{h\to 0^+}{\text{Limit}} \frac{f(1+h)-f(1)}{h}$

$= \underset{h\to 0^+}{\text{Limit}} \frac{(1+h)^2-1}{h} = \underset{h\to 0^+}{\text{Limit}} \frac{1+h^2+2h-1}{h} = \underset{h\to 0^+}{\text{Limit}} (h+2) = 2$

L.H.D. = f'(1_-) $= \underset{h\to 0^+}{\text{Limit}} \frac{f(1-h)-f(1)}{-h} = \underset{h\to 0^+}{\text{Limit}} \frac{1-h-1}{-h} = 1$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

Example # 35 : Discuss differentiability of f(x) = $\begin{cases} \frac{\sin x^2}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ at x = 0.

Solution : For continuity $\lim\limits_{x\to 0}$ f(x) = $\lim\limits_{h\to 0} \frac{\sin h^2}{h} = \frac{h.\sin h^2}{h^2} = 0$

Hence f(x) is continuous at x = 0

Also. f'(0_+) = $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{\sin h^2}{h^2} = 1$

and f '(0_-) = $\lim\limits_{h\to 0} \frac{f(-h)-f(0)}{-h} = \lim\limits_{h\to 0} \frac{\sin h^2}{h^2} = 1$

Thus, f(x) is differentiable at x = 0

Example # 36 : Discuss the differentiability of f(x) = $\begin{cases} x\sin(\ell n x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at x = 0

Solution : for continuity

$f(0^+) = \lim_{h \to 0} h \sin(\ell n h^2) = 0 \times$ (any value between –1 and 1) = 0

$f(0^-) = \lim_{h \to 0} (-h) \sin(\ell n h^2) = 0 \times$ (any value between –1 and 1) = 0

hence f(x) is continuous at x = 0

for differentiability

$f'(0^+) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h\sin(\ell n h^2) - 0}{h} = \lim_{h \to 0} \sin(\ell n h^2)$

⇒ any value between –1 and 1

Hence, f ' (0) do not take any fixed value hence f(x) is not differentiable at x = 0

Self practice problems :

(20) If f(x) = $\begin{cases} [2x] + x, & x < 1 \\ \{x\} + 1, & x \geq 1 \end{cases}$, then comment on the continuity and differentiable at x = 1, where [.] is greatest integer function and {.} is fractional part function.

(21) If f(x) = $\begin{cases} x\tan^{-1} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then comment on the derivability of f(x) at x = 0.

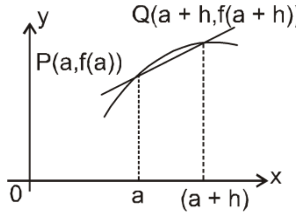
Ans. (20) Discontinuous and non-differentiable at x = 1

(21) non-differentiable at x = 0

18. Concept of tangent and its association with derivability :

Tangent :- The tangent is defined as the limiting case of a chord or a secant.

slope of the line joining (a,f(a)) and (a + h, f(a + h)) = $\frac{f(a+h) - f(a)}{h}$]



Slope of tangent at P = f'(a) = $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

The tangent to the graph of a continuous function f at the point P(a, f(a)) is

(i) the line through P with slope f'(a) if f'(a) exists ;

(ii) the line x = a if L.H.D. and R.H.D. both are either ∞ or – ∞.

If neither (i) nor (ii) holds then the graph of f does not have a tangent at the point P.

In case (i) the equation of tangent is y – f(a) = f'(a) (x – a).

In case (ii) it is x = a

Note : (i) tangent is also defined as the line joining two infinitesimally close points on a curve.

(ii) A function is said to be derivable at x = a if there exist a tangent of finite slope at that point.$f'(a^+) = f'(a^-)$ = finite value

(iii) $y = x^3$ has x-axis as tangent at origin.

(iv) $y = |x|$ does not have tangent at x = 0 as L.H.D. ≠ R.H.D.

Example # 37 : Find the equation of tangent to $y = (x)^{1/3}$ at x = 1 and x = 0.

Solution : At x = 1 Here $f(x) = (x)^{1/3}$

L.H.D = $f'(1^-) = \lim_{h \to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h \to 0^+} \frac{(1-h)^{1/3}-1}{-h} = \frac{1}{3}$

R.H.D. = $f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^{1/3}-1}{h} = \frac{1}{3}$

As R.H.D. = L.H.D. = $\frac{1}{3}$

∴ slope of tangent = $\frac{1}{3}$ ∴ $y - f(1) = \frac{1}{3}(x-1)$

$y - 1 = \frac{1}{3}(x-1)$ ⇒ $3y - x = 2$ is tangent to $y = x^{1/3}$ at (1, 1)

At x = 0

L.H.D. = $f'(0^-) = \lim_{h \to 0^+} \frac{(0-h)^{1/3}-0}{-h} = +\infty$

R.H.D. = $f'(0^+) = \lim_{h \to 0^-} \frac{(0+h)^{1/3}-0}{h} = +\infty$

As L.H.D. and R.H.D are infinite. y = f(x) will have a vertical tangent at origin.

∴ x = 0 is the tangent to $y = x^{1/3}$ at origin.

Self practice problem :

(22) If possible find the equation of tangent to the following curves at the given points.

(i) $y = x^3 + 3x^2 + 28x + 1$ at x = 0.

(ii) $y = (x-8)^{2/3}$ at x = 8.

Ans. (i) y = 28x + 1 (ii) x = 8

19. Relation between differentiability & continuity :

(i) If f '(a) exists, then f(x) is continuous at x = a.

(ii) If f(x) is differentiable at every point of its domain of definition, then it is continuous in that domain.

Note : The converse of the above result is not true i.e. "If 'f' is continuous at x = a, then 'f' is differentiable at x = a is not true.

e.g. the functions $f(x) = |x-2|$ is continuous at x = 2 but not differentiable at x = 2.

If f(x) is a function such that R.H.D = $f'(a^+) = \ell$ and L.H.D. = $f'(a^-) = m$.

Case - I

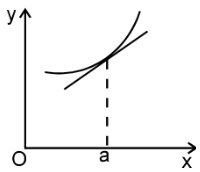
If $\ell = m$ = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II

if $\ell \neq m$ = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

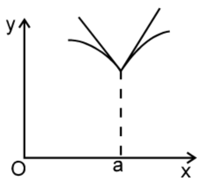
Case - III

If at least one of the ℓ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



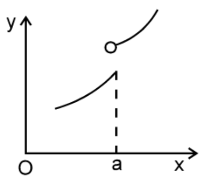
(i)

continuous and differentiable



(ii)

continuous but not differentiable



(iii)

neither continuous nor differentiable

Example # 38 : If f(x) is differentiable at x = a, prove that it will be continuous at x = a.

Solution : $f'(a^+) = \lim_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h} = \ell$

$\lim_{h\to 0^-}$ [f(a+h) – f(a)] = $h\ell$ as $h \to 0$ and ℓ is finite, then $\lim_{h\to 0^-}$ f(a + h) – f(a) = 0

$\Rightarrow \lim_{h\to 0^+}$ f (a + h) = f(a).

Similarly $\lim_{h\to 0^+}$ [f(a – h)– f(a)] = – $h\ell$ $\Rightarrow$ $\lim_{h\to 0^+}$ f(a – h) = f(a)

$\therefore$ $\lim_{h\to 0^-}$ f(a + h) = $\lim_{h\to 0^-}$ f(a) = $\lim_{h\to 0^+}$ f(a – h)

Hence, f(x) is continuous.

Example # 39 : If $f(x) = \begin{cases} x^2\,\text{sgn}[x] + \{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$, comment on the continuity and differentiability of f(x), where [.] is greatest integer function and {.} is fractional part function, at x = 1, 2.

Solution : Continuity at x = 1

$\lim_{x\to 1^+}$ f(x) = $\lim_{x\to 1^+}$ (x^2 sgn[x] + {x}) = 1 + 0 = 1

$\lim_{x\to 1^-}$ f(x) = $\lim_{x\to 1^-}$ (x^2 sgn [x] + {x}) =1 sgn (0) + 1 = 1

$\because$ f(1) = 1

$\therefore$ L.H.L = R.H.L = f(1). Hence f(x) is continuous at x = 1.

Now for differentiability,

R.H.D. = $f'(1_+) = \lim_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \dfrac{(1+h)^2 \operatorname{sgn}[1+h] + \{1+h\} - 1}{h}$

$= \lim_{h\to 0^-} \dfrac{(1+h)^2 + h - 1}{h} = \lim_{h\to 0^-} \dfrac{1+h^2+2h+h-1}{h} = \lim_{h\to 0^-} \dfrac{h^2+3h}{h} = 3$

and L.H.D. = $f'(1_-) = \lim_{h\to 0^-} \dfrac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^-} \dfrac{(1-h)^2 \operatorname{sgn}[1-h] + 1 - h - 1}{-h} = 1$

$\Rightarrow \quad f'(1_+) \neq f'(1_-)$.

Hence f(x) is non differentiable at x = 1.

Now at x = 2

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 \operatorname{sgn}[x] + \{x\}) = 4 \,.\, 1 + 1 = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (\sin x + |x - 3|) = 1 + \sin 2$

Hence L.H.L ≠ R.H.L

Hence f(x) is discontinuous at x = 2 and then f(x) also be non differentaible at x = 2.

Self practice problem :

(23) If $f(x) = \begin{cases} \left(\dfrac{e^{[x]} + |x| - 1}{[x] + \{2x\}}\right) & x \neq 0 \\ 1/2 & x = 0 \end{cases}$, comment on the continuity at x = 0 and differentiability at x = 0, where [.] is greatest integer function and {.} is fractional part function.

Ans. discontinuous hence non-differentiable at x = 0

20. Differentiability of sum, product & composition of functions :

(i) If f(x) & g(x) are differentiable at x = a, then the functions f(x) ± g(x), f(x). g(x) will also be differentiable at x = a & if g (a) ≠ 0, then the function f(x)/g(x) will also be differentiable at x = a.

(ii) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = |x| and g(x) = x^2.

(iii) If f(x) & g(x) both are not differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = |x| & g(x) = |x|.

(iv) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = |x| & g(x) = −|x|.

(v) If f is differentiable at x = a, then $\lim_{h\to 0} \dfrac{f(a+g(h)) - f(a+p(h))}{g(h)-p(h)} = f'(a)$, where $\lim_{h\to 0} p(h) = \lim_{h\to 0} g(h) = 0$

Example # 40 : Discuss the differentiability of f(x) = x + |x| at x = 0

Solution : As we know that f(x) + g(x) will be non-differentiable at x = a if f(x) is differentiable but g(x) is non-differentiable at the same point.

Here, x is differentiable but |x| is non-differentiable at x = 0.

Hence, given function is Non-differentiable at x = 0.

Example # 41 : Discuss the differentiability of f(x) = x|x|

Solution : $\because$ $f(x) = \begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$

R.H.D = $\dfrac{h|h|-0}{h} = 0$

L.H.D = $\dfrac{-h|-h|-0}{-h} = 0$

Differentiable at x = 0

Example # 42 : If f(x) is differentiable and g(x) is differentiable, then prove that f(x) . g(x) will be differentiable.

Solution : Given, f(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{f(a+h)-f(a)}{h} = f'(a)$

g(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{g(a+h)-g(a)}{h} = g'(a)$

let p(x) = f(x) . g(x)

Now, $\lim\limits_{h \to 0^+} \dfrac{p(a+h)-p(a)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{f(a+h).g(a+h)-f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{f(a+h)g(a+h)+f(a+h).g(a)-f(a+h).g(a)-f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \left[\dfrac{f(a+h)\ \ (g\ \ (a+h)-g(a))}{h} + \dfrac{g\ \ (a)(f(a+h)-f(a))}{h}\right]$

$= \lim\limits_{h \to 0^+} \left[f(a+h).\dfrac{g(a+h)-g(a)}{h} + g(a).\ \dfrac{f(a+h)-f(a)}{h}\right]$

= f(a) . g'(a) + g(a) f'(a) = p' (a)

Hence p(x) is differentiable.

21. Differentiability over an Interval :

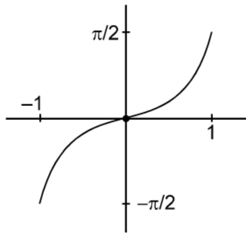
f (x) is said to be differentiable over an open interval if it is differentiable at each point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if:

(i) for the points a and b, $f'(a_+)$ and $f'(b_-)$ exist finitely

(ii) for any point c such that a < c < b, $f'(c_+)$ & $f'(c_-)$ exist finitely and are equal.

All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin_{-1} x$.



Non differentiable at x = 1 & x = – 1

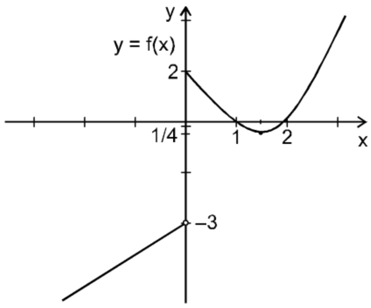
Note : Derivability should be checked at following points

(i) At all points where continuity is required to be checked.

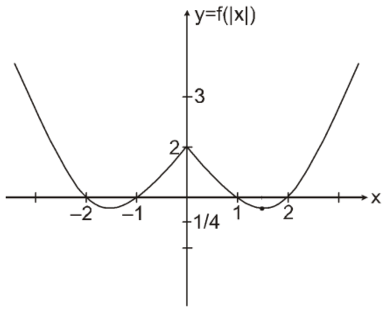
(ii) At the critical points of modulus and inverse trigonometric function.

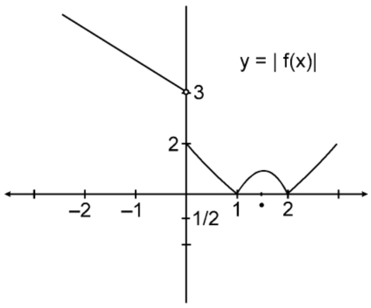
Example # 43 : If $f(x) = \begin{cases} x-3 & , \quad x<0 \\ x^2-3x+2 & , \quad x \ge 0 \end{cases}$ and g(x) = f(|x|) + |f(x)|, then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)|.

Solution :



Graph of f(|x|) and |f(x)|





If f(|x|) and |f(x)| are continous, then g(x) is continuous. At x = 0 f(|x|) is continuous, and |f(x)| is discontinuous therefore g(x) is discontineous at x = 0.

∴ g(x) is non differentiable at x = 0, 1, 2, (find the reason yourself).

Example # 44 : If $f(x) = \begin{cases} |x|-3 \; ; & |x| \ge 1 \\ -x^2-1 \; ; & |x| < 1 \end{cases}$, write the doubtful points of differentiability for f(x).

Solution : x = 1 and x = – 1 are doubtful points as the definition of the function changes at these points. Also. x = 0 is a doubtful points because of |x|.

Self practice problems:

(24) If f(x) = [x] + [1 – x] , – 1 ≤ x ≤ 3, then draw its graph and comment on the continuity and differentiability of f(x), where [.] is greatest integer function.

(25) If $f(x) = \begin{cases} |1-4x^2| \;, & 0 \le x < 1 \\ [x^2-2x] \;, & 1 \le x \le 2 \end{cases}$, then draw the graph of f(x) and comment on the differentiability and continuity of f(x), where [.] is greatest integer function.

Ans. (24) f(x) is discontinuous at x = –1, 0, 1, 2, 3 hence non-differentiable.

(25) f(x) is discontinuous at x = 1, 2 & non differentiable at x = $\frac{1}{2}$, 1, 2.

22. Problems of finding functions satisfying given conditions :

If f(x) is non-zero & non-constant function satisfying the given conditions, function can be found as below.

Condition to be satisfied	Required function
f(x + y) = f(x) + f(y)	f(x) = kx
f(x + y) = f (x). f (y)	$f(x) = a^{kx}$.
f(xy) = f (x) + f (y)	f(x) = k ln x
f(xy) = f (x). f(y)	$f(x) = x^n$, n ∈ R
f(x) is a polynomial function and $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$	$f(x) = x^n + 1$ or $f(x) = -x^n + 1$, n ∈ N

Example # 45 : If f(x) is a function satisfies the relation for all x, y ∈ R, f(x + y) = f(x) + f(y) and if f'(0) = 2 and function is differentiable every where, then find f(x).

Solution : $f'(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x)+f(h)-f(x)-f(0)}{h}$ (∵ f(0) = 0)

$= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = f'(0)$

f'(x) = 2 ⇒ $\int f'(x)\ dx = \int 2\ dx$

f(x) = 2x + c

∵ f(0) = 2.0 + c as f(0) = 0

∴ c = 0 ∴ f(x) = 2x

Second Method :

Since f(x + y) = f(x) + f(y) is true for all values of x and y is independent of differentiating both sides w.r.t x (here y is constant with respect to x).

f'(x + y) = f'(x)

put x = 0

f'(y) = f'(0) ⇒ $\int f'(y)\ dy = \int 2\ dy$

f(y) = 2y + c ⇒ f(0) = 0 + c = 0

$\therefore$ c = 0

$\therefore$ f(y) = 2y $\Rightarrow$ f(x) = 2x.

Example # 46 : f(x + y) = f(x) . f(y) $\forall$ x, y $\in$ R and f(x) is a differentiable function and f'(0) = 1, f(x) $\neq$ 0 for any x. Find f(x)

Solution : f(x) is a differentiable function

$\therefore$ $f'(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x).f(h)-f(x).f(0)}{h}$ $(\because f(0) = 1)$

$= \lim_{h\to 0^-} \frac{f(x).(f(h)-f(0))}{h}$ = f(x) . f'(0) = f(x) $\therefore$ f'(x) = f(x)

$\therefore$ $\int \frac{f'(x)}{f(x)}\, dx = \int 1\, dx$

$\Rightarrow$ ℓn f(x) = x + c $\therefore$ ℓn 1 = 0 + c

$\Rightarrow$ c = 0 $\therefore$ ℓn f(x) = x $\Rightarrow$ f(x) = e^x

Example # 47 : $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ $\forall$ x, y $\in$ R and f(0) = 1 and f'(0) = – 1 and function is differentiable for all x, then find f(x).

Solution : $f'(x) = \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f\left(\frac{2x+0}{2}\right)}{h} = \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2} - \frac{f(2x)+f(0)}{2}}{h}$

$= \lim_{h\to 0} \frac{f(2h)-f(0)}{2h}$ = f'(0) = – 1

f'(x) = – 1

integrating both sides, we get

f(x) = – x + c

$\because$ c = + 1 (as f(0) = 1) $\therefore$ f(x) = – x + 1 = 1 – x

Example # 48 : Evaluate $\lim_{h\to 0} \frac{f(a+2h)-f(a-3h)}{h}$, if f'(a) = 3

Solution : $\because$ $\lim_{h\to 0} \frac{f(a+2h)-f(a-3h)}{h} = \lim_{h\to 0} \frac{f(a+2h)-f(a-3h)}{5h} . 5$

= f'(a) × 5 = 3 × 5 = 15

Example # 49 : If f(x) is a polynomial function satisfying f(x) . $f\left(\frac{1}{x}\right)$ = f(x) + $f\left(\frac{1}{x}\right)$ $\forall$ x $\in$ R – {0} and f(2) = 9, then find f (3)

Solution : f(x) = 1 ± x^n

As f(2) = 9 $\therefore$ f(x) = 1 + x^3

Hence f(3) = 1 + 3^3 = 28

Self practice problems :

(26) $f\left(\frac{x}{y}\right)$ = f(x) – f(y) $\forall$ x , y $\in$ R^+ and f'(1) = 1 , then show that f(x) = ℓnx.

(27) If f(x) and g(x) are differentiable, then prove that f(x) ± g(x) will be differentiable.

(28) If f'(2) = 4, then find the value of $\lim_{h\to 0} \frac{f(2+h)-f(2+\sinh)}{h.\ \sinh.\tanh}$.

(29) If f(x) is a polynomial function satisfying $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ $\forall\ x \in R - \{0\}$ and f(3) = – 8, then find f(4)

(30) If f(x + y) = f(x) . f(y) for all real x, y and f(0) ≠ 0, then prove that the function, $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

Ans. (28) 2/3 (29) – 15