Mathematical Reasoning

1. Statements (Proposition) :

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. "A sentence is called a mathematically acceptable statement or proposition if it is either true or false but not both". A statement is assumed to be either true or false. A true statement is known as a **valid statement** and a false statement is known as an **invalid statement**.

Example # 1 : Which of the following sentences are statements :
(i) Three plus two equals five.
(ii) The sum of two negative number is negative
(iii) Every square is a rectangle.

Solution : Each of these sentences is a true sentence therefore they all are statements.

Example # 2 : Which of the following sentences are statements :
(i) Three plus four equals six.
(ii) All prime numbers are odd.
(iii) Every relation is a function.

Solution : Each of these sentences is a false sentence therefore all of these are statements.

Example # 3 : Which of the following sentences are statements :
(i) The sum of x and y is greater than 0.
(ii) The square of a number is even.

Solution : Here, we are not in a position to determine whether it is true of false unless we know what the numbers are. Therefore these sentences are not a statement.

Example # 4 : Which of the following sentences are statements :
(i) Give me a glass of water.
(ii) Is every set finite ?
(iii) How beautiful ?
(iv) Tomorrow is Monday.
(v) May God bless you !

Solution : None of these sentences is a statement.

Note :
(a) Imperative (expresses a request or command), exclaimatory sentences (expresses some strong feeling), Interrogative sentences (asks some question), Optative sentences (blessing & wishes) are not considered as a statement in mathematical language.
(b) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.
(c) Scientifically established facts are considered true.
(d) Sentences involving word "here", "there" are not statements.

Self Practice Problems :

(1) Which of the following are statements :
(i) open the door (ii) square of an odd number is even

Ans. (1) (i) is not a statement (ii) is a statement

2. Truth table :

Truth table is that which gives truth values (The truth or falsity of a statement is called its truth value) of statements. It has a number of rows and columns.

Note that for **n** statements, there are 2^n rows.

(i) Truth table for single statement p:

Number of rows = $2^1 = 2$

p
T
F

(ii) Truth table for two statements p and q :

Number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(iii) Truth table for three statements p, q and r.

Number of rows = $2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

3. Negation of a statement :

The denial of a statement p is called its negation and is written as ~ p and read as 'not p'. Negation of any statement p is formed by writing "It is not the case that ...."

p	~p
T	F
F	T

Truth table

or "It is false that ...."

or inserting the word "not" in p.

Example # 5 : Write negation of following statements :

(i) "All cats scratch" (ii) $\sqrt{5}$ " is a rational number".

Solution : (i) Some cats do not scratch

OR

There exist a cat which does not scratch

OR

At least one cat does not scratch.

(ii) $\sqrt{5}$ is an irrational number

Self Practice Problems :

(2) Write negation of statement '2 + 2 = 7'

Ans. $2 + 2 \neq 7$

4. Compounds statements :

If a statement is combination of two or more statements, then it is said to be a compound statement. Each statement which form a compound statement is known as its sub-statement or component statement.

5. Basic connectives :

In the compound statement, two or more statements are connected by words like 'and', 'or', 'if . . . . . then', 'only if', 'if and only if', 'there exists', 'for all' etc. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words.

Basic logical connective	*Symbol*	*Compound statement*
AND	$\wedge$	Conjunction
OR	$\vee$	Disjunction
IF....... THEN	$\rightarrow$	Conditional statement
IF AND ONLY IF	$\leftrightarrow$	Biconditional statement

6. The word "AND" (Conjuction) :

Any two statements can be connected by the word "and" to form a compound statement. The compound statement with word "and" is true if all its component statements are true. The compound statement with word "and" is false if any or all of its component statements are false. The compound statement "p and q" is denoted by "p q".

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

7. The word "OR" (Disjunction) :

Any two statements can be connected by the word "OR" to form a compound statement. The compound statement with word "or" is true if any or all of its component statements are true. The compound statement with word "or" is false if all its component state ment are false. The compound statement "p or q" is denoted by "p v q."

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

8. Types of "OR" :

(i) **Exclusive OR :** If in statement $p \vee q$ i.e. p or q, happening of any one of p, q excludes the happening of the other then it is exclusive OR. Here both p and q cannot occur together. For example in statement "I will go to delhi either by bus or by train", the use of 'or' is exclusive.

(ii) **Inclusive OR :** If in statement p or q, both p and q can also occur together then it is inclusive OR. The statement 'In senior secondary exam, you can take optional subject as physical education or computers' is an example of use of inclusive OR.

Example # 6 : Find the truth value of the statement "2 divides 4 and 3 + 7 = 8"

Solution. 2 divides 4 is true and 3 + 7 = 8 is false. so given statement is false.

Example # 7 : Write component statements of the statement "All living things have two legs and two eyes".

Solution : Component statements are :

All living things have two legs

All living things have two eyes

Example # 8 : Find truth value of compound statement "All natural numbers are even or odd"

Solution : p : all natural numbers are even, q : all natural numbers are odd.

Here compound statement ($p \vee q$) is exclusive OR

Truth value of p is false and truth value of q is also false.

So truth value of compound statement ($p \vee q$) is false.

Self Practice Problems :

(3) Find the truth values of $\sim p \vee q$

(4) Find the truth values of the compound statement $(p \vee \sim r) \wedge (q \vee \sim r)$

Ans. (3)

p	q	~ p	~ p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(4)

p	q	r	~ r	p∨ ~ r	q∨ ~ r	(p∨ ~ r) ∧ (q∨ ~ r)
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	T	F	F	T	F
T	F	F	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	F	F
F	F	F	T	T	T	T

9. Implication :

There are three types of implications which are "if . . . . . then", "only if" and "if and only if".

10. Conditional connective 'IF . . . . THEN' :

If p and q are any two statements then the compound statement in the form "If p then q" is called a conditional statement. The statement "If p then q" is denoted by $p \rightarrow q$ or $p \Rightarrow q$ (to be read as p implies q). In the implication $p \rightarrow q$, p is called the antecedent (or the hypothesis) and q the consequent (or the conclusion)

If p then q reveals the following facts :

(i) p is a sufficient condition for q

(ii) q is a necessary condition for p

(iii) 'If p then q' has same meaning as that of 'p only if q'

(iv) $p \rightarrow q$ has same meaning as that of $\sim q \rightarrow \sim p$

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Truth table

Note : The conditional statement $p \to q$ is defined to be true except in case p is true and q is false.

Examples :

(i) If $x = 4$, then $x_2 = 16$

(ii) If ABCD is a parallelogram, then AB = CD

(iii) If Mumbai is in England, then 2 + 2 = 5

(iv) If Shikha works hard, then it will rain today.

We also need to be aware that in the English language, there are other ways for expressing the conditional statement $P \to Q$ other than "If P, then Q." Following are some common ways to express the conditional statement $P \to Q$ in the English language :

- If P, then Q.
- P implies Q.
- P only if Q.
- Q if P.
- Whenever P is true, Q is true.
- Q is true whenever P is true.
- Q is necessary for P. (This means that if P is true, then Q is necessarily true.)
- P is sufficient for Q. (This means that if you want Q to be true, it is sufficient to show that P is true.)

Converse of a conditional statement : If p and q are two statements, then converse of statement $p \Rightarrow q$ is $q \Rightarrow p$

Example # 9 : Write converse of statement "if two lines are parallel then they do not intersect in same plane"

Solution Converse of statement $p \Rightarrow q$ is $q \Rightarrow p$ so converse of given statement is "if two lines do not intersect in same plane then they are parallel".

11. Biconditional connective "IF and only IF" :

If p and q are any two statements then the compound statement in the form of "p if and only if q" is called a biconditional statement and is written in symbolic form as $p \leftrightarrow q$ or $p \Leftrightarrow q$.
Statement $p \leftrightarrow q$ reveals the following facts :

(i) p if and only if q

(ii) q if and only if p

(iii) p is necessary and sufficient condition for q

(iv) q is necessary and sufficient condition for p

p	q	$p \leftrightarrow q$	$q \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Truth table

Note : The biconditional statement $p \leftrightarrow q$ is defined to be true only when p and q have same truth value.

For Example : The following statements are biconditional statements

(i) A number is divisible by 3 if and only if the sum of the digits forming the number is divisible by 3.

(ii) One is less than seven if and only if two is less than eight.

(iii) A triangle is equilateral if and only if it is equiangular.

Example # 10 : Let p and q stand for the statement 'Bhopal is in M.P.' and '3 + 4 = 7' respectively. Describe the conditional statement $\sim p \rightarrow \sim q$.

Solution : $\sim p \rightarrow \sim q$: If Bhopal is not in M.P. then $3 + 4 \neq 7$

Example # 11 : Find the truth values of $(p \leftrightarrow \sim q) \leftrightarrow (q \rightarrow p)$.

Solution :

p	q	$\sim q$	$p \leftrightarrow \sim q$	$q \rightarrow p$	$(p \leftrightarrow \sim q) \leftrightarrow (q \rightarrow p)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Self Practice Problems :

(5) If statements p and q are respectively : 4 + 5 = 9 and 2 + 3 = 5, then write the conditional statement $p \rightarrow q$.

(6) Find the truth values of the compound statement $(p \wedge q) \rightarrow \sim p$

Ans. (5) If 4 + 5 = 9, then 2 + 3 = 5

(6)

p	q	$(p \wedge q) \rightarrow \sim p$
T	T	F
T	F	T
F	T	T
F	F	T

12. Tautology and fallacy :

(i) Tautology : This is a statement which is true for all truth values of its components.It is denoted by t. Consider truth table of $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

we observe that last column is always true. Hence $p \vee \sim p$ is a tautology.

(ii) Fallacy : This is statement which is false for all truth values of its components. It is denoted by f or c. Consider truth table of $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

We observe that last column is always false. Hence $p \wedge \sim p$ is a fallacy

13. Logically equivalent statements :

If truth values of statements p and q are same then they are logically equivalent and written as $p \equiv q$.

14. <u>Algebra of statements</u> :

If p, q, r are any three statements and t is a tautology, c is a contradiction then

(i) Commutative Law :

(a) $p \vee q = q \vee p$ (b) $p \wedge q = q \wedge p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

(ii) Associative Law :

(a) $p \vee (q \vee r) = (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

(iii) Distributive Law :

(a) $p \wedge (q\ \ r) = (p\ \ q)\ \ (p\ \ r)$ (b) $p \vee (q\ \ r) = (p\ \ q)\ \ (p\ \ r)$

(c) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$ (d) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

(iv) Identity Law :

(a) $p \vee t = t$ (b) $p \wedge t = p$ (c) $p \vee c = p$ (d) $p \wedge c = c$

p	t	c	$(p \wedge t)$	$(p \vee t)$	$(p \wedge c)$	$(p \vee c)$
T	T	F	T	T	F	T
F	T	F	F	T	F	F

(v) Complement Law :

(a) $p \vee (\sim p) = t$ (b) $p \wedge (\sim p) = c$ (c) $\sim t = c$

(d) $\sim c = t$ (e) $\sim(\sim p) = p$

p	$\sim p$	$(p \wedge \sim p)$	$(p \vee \sim p)$
T	F	F	T
F	T	F	T

(vi) Idempotent Law :

(a) $p \vee p = p$ (b) $p \wedge p = p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(vii) De Morgan's law :

(a) $\sim (p \vee q) = (\sim p) \wedge (\sim q)$ (b) $\sim (p \wedge q) = (\sim p) \vee (\sim q)$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(viii) Involution laws (or Double negation laws) :

$\sim(\sim p) \equiv p$

p	$\sim q$	$\sim (\sim p)$
T	F	T
F	T	F

(ix) Contrapositive Laws : $p \rightarrow q \equiv \sim q \rightarrow \sim p$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(x) Truth table of biconditional statement :

p	q	$p \leftrightarrow q$	$q \leftrightarrow p$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T	T
T	F	F	F	F	T	F
F	T	F	F	T	F	F
F	F	T	T	T	T	T

15. Negation of compound statements :

If p and q are two statements then

(i) Negation of conjunction : $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \vee q)$	$\sim (p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(iii) Negation of conditional : $\sim(p \rightarrow q) \equiv p \wedge \sim q$

p	q	$\sim q$	$(p \rightarrow q)$	$\sim (p \rightarrow q)$	$(p \wedge \sim q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) Negation of biconditional : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ or $p \leftrightarrow \sim q$

p	q	$\sim p$	$\sim q$	$(p \to q)$	$\sim(p \to q)$	$(p \wedge \sim q)$	$(p \leftrightarrow q)$	$\sim(p \leftrightarrow q)$	$p \leftrightarrow \sim q$	$q \wedge \sim p$	$(p \wedge \sim q) \vee (q \wedge \sim p)$
T	T	F	F	T	F	F	T	F	F	F	F
T	F	F	T	F	T	T	F	T	T	F	T
F	T	T	F	T	F	F	F	T	T	T	T
F	F	T	T	T	F	F	T	F	F	F	F

we know that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\therefore$ $\sim(p \leftrightarrow q) \equiv \sim[(p \to q) \wedge (q \to p)]$

$\equiv \sim(p \to q) \vee \sim(q \to p) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

SUMMARY

(i) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

(ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

(iii) $\sim(p \Rightarrow q) \equiv \sim(\sim p \vee q) = p \wedge (\sim q)$

(iv) $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ or $p \Leftrightarrow \sim q$ or $\sim p \Leftrightarrow q$

Note : $p \to q \equiv \sim p \vee q$

$p \to q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

Example # 12 : Write the negation of the following compound statements :

(i) All the students completed their homework and the teacher is present.

(ii) Square of an integer is positive or negative.

(iii) If my car is not in workshop, then I can go college.

(iv) ABC is an equilateral triangle if and only if it is equiangular

Solution : (i) The component statements of the given statement are :

p : all the students completed their homework.

q : The teacher is present.

The given statement is p and q. so its negation is $\sim p$ or $\sim q$ = Some of the students did not complete their home work or the teacher is not present.

(ii) The component statement of the given statements are :

p : Square of an integer is positive.

q : Square of an integer is negative.

The given statement is p or q. so its negation is $\sim p$ and $\sim q$ = Their exists an integer whose square is neither positive nor negative.

(iii) Consider the following statements :

p : My car is not in workshop

q : I can go to college

The given statement in symbolic form is $p \to q$

Now, $\sim(p \to q)$ $p \wedge (\sim q)$

$\Rightarrow$ $\sim(p \to q)$: My car is not in workshop and I cannot go to college.

Hence the negation of the given statements is “My car is not in workshop and i can not go to college”.

(iv) Consider the following statements :

p : ABC is an equilateral triangle.

q : It is equiangular

Clearly, the given statement is symbolic form is $p \leftrightarrow q$.

Now, $\sim(p \leftrightarrow q) \cong (p \wedge \sim q) \vee (\sim p \wedge q)$

$\Rightarrow$ $\sim(p \rightarrow q)$: Either ABC is an equilateral triangle and it is not equiangular or ABC is not an equilateral triangle and it is equiangular.

Example # 13 : Show that $p \rightarrow (p \vee q)$ is a tautology

Solution :

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Example # 14 : By using laws of algebra of statements show that $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$.

Solution. $(p \vee q) \wedge \sim p \equiv (\sim p) \wedge (p \wedge q)$

$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$

$\equiv f \vee (\sim p \wedge q)$

$\equiv \sim p \wedge q$

Example # 15 : Find the negation of statement $p \wedge \sim q$

Solution : Negation of $(p \wedge \sim q) \equiv \sim(p \wedge \sim q)$ $\equiv \sim p \vee \sim \sim q \equiv \sim p \vee q$

Self Practice Problems :

(7) Show that $[(\sim p \wedge q) \wedge (q \wedge r)] \wedge \sim q$ is a fallacy.

(8) By using laws of algebra of statements show that $p \wedge (p \vee q) \equiv p \vee (p \wedge q)$.

(9) By using laws of algebra of statements show that $p \vee (p \wedge q) \equiv p$.

(10) Find the negation of statement of $(p \rightarrow q) \rightarrow (q \rightarrow p)$.

Ans. (10) $(\sim p \vee q) \wedge (q \wedge \sim p)$

16. Contrapositive of a conditional statement :

If p and q are two statements then

Let $p \Rightarrow q$ Then
(Contrapositive of $p \Rightarrow q$) is $(\sim q \Rightarrow \sim p)$

Note : A statement and its contrapositive convey the same meaning.

Example # 16 : Write the contrapositive of the following statement : “If Mohan is poet, then he is poor”

Solution : Consider the following statements :

p : Mohan is a poet

q : Mohan is poor

Clearly, the given statement in symbolic form is $p \rightarrow q$. Therefore, its contrapositive is given by $\sim q \rightarrow \sim p$.

Now, $\sim p$: Mohan is not a poet.

$\sim q$: Mohan is not poor.

$\therefore \sim q \rightarrow \sim p$: If Mohan is not poor, then he is not a poet.

Hence the contrapositive of the given statement is “If Mohan is not poor, then he is not a poet”.

17. Quantifiers :

There exists (atleast one), For all (for every)

Quantifiers are phrases like "There exists" ($\exists$) and " For all" ($\forall$). In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S satisfies that property. While "There exists" can be interpreted as saying that "atleast one" member of the given set S satisfies that property.

Example :

P : For every real number x, $x^2 \geq 0$.

Q : There exists a natural no, which is even and prime.

Negation of statement that says "for every" can be interpreted as saying that atleast one member of given set S does not satisfies that property.

(using " There exists")

Example :

~P : There exists a real number x, such that $x^2 < 0$.

Negation of statement that says "There exists" can be interpreted as saying that every member of given set S does not satisfies that property. (Using "For every")

Example :

~Q : Every natural number is not even or not prime.

Self Practice Problems :

(11) Negate the following sentences using quantifiers.

(i) There exists $x \in Z$, such that $x^2 + x = 0$ (ii) For all $x \in R$, $\sqrt{x}$ is a real number.

Ans. (i) For all $x \in Z$, $x^2 + x \neq 0$.

(ii) There exists $x \in R$ such that $\sqrt{x}$ is not a real number.