Probability

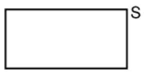
Lest men suspect your tale untrue. Keep probability in view **........Gay, John**

There are various phenomena in nature, leading to an outcome, which cannot be predicted apriori e.g. in tossing of a coin, a head or a tail may result. Probability theory aims at measuring the uncertainties of such outcomes.

1. Important terminology :

(i) Random experiment :
It is a process which results in an outcome which is one of the various possible outcomes that are known to us before hand e.g. throwing of a die is a random experiment as it leads to fall of one of the outcome from {1, 2, 3, 4, 5, 6}. Similarly taking a card from a pack of 52 cards is also a random experiment.

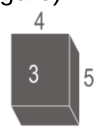
(ii) Sample space :
It is the set of all possible outcomes of a random experiment e.g. {H, T} is the sample space associated with tossing of a coin.
In set notation it can be interpreted as the universal set.



Example # 1 : Write the sample space of the experiment 'A coin is tossed and a die is thrown'.
Solution : The sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

Example # 2 : Write the sample space of the experiment 'A coin is tossed, if it shows head a coin tossed again else a die is thrown.
Solution : The sample space S = {HH, HT, T1, T2, T3, T4, T5, T6}

Example # 3 : Find the sample space associated with the experiment of rolling a pair of dice (plural of die) once. Also find the number of elements of the sample space.
Solution : Let one die be blue and the other be green. Suppose '1' appears on blue die and '2' appears on green die. We denote this outcome by an ordered pair (1, 2). Similarly, if '3' appears on blue die and '5' appears on green die, we denote this outcome by (3, 5) and so on. Thus, each outcome can be denoted by an ordered pair (x, y), where x is the number appeared on the first die (blue die) and y appeared on the second die (green die). Thus, the sample space is given by S = {(x, y) x is the number on blue die and y is the number on grey die}
We now list all the possible outcomes (figure)



	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

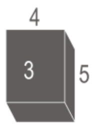


Figure Number of elements (outcomes) of the above sample space is 6 × 6 i.e., 36

Self practice problems :

(1) A coin is tossed twice, if the second throw results in head, a die is thrown then write sample space of the experiment.

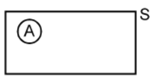
(2) An urn contains 3 red balls and 2 blue balls. Write sample space of the experiment 'Selection of a ball from the urn at random'.

Ans. (1) {HT, TT, HH1, HH2, HH3, HH4, HH5, HH6, TH1, TH2, TH3, TH4, TH5, TH6}.

(2) {R_1, R_2, R_3, B_1, B_2 }. (Here the balls are distinguished from one and other by naming red balls as R_1, R_2 and R_3 and the blue balls as B_1 and B_2.)

(iii) Event :

It is subset of sample space. e.g. getting a head in tossing a coin or getting a prime number in throwing a die. In general if a sample space consists 'n' elements, then a maximum of 2^n events can be associated with it.



(iv) Complement of event :

The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\bar{A}$ or A^c.

(v) Simple event :

If an event covers only one point of sample space, then it is called a simple event e.g. getting a head followed by a tail in throwing of a coin 2 times is a simple event.

(vi) Compound event :

When two or more than two events occur simultaneously, the event is said to be a compound event. Symbolically $A \cap B$ or AB represent the occurrence of both A & B simultaneously.

Note : "$A \cup B$" or A + B represent the occurrence of either A or B.

Example # 4 : Write down all the events of the experiment 'tossing of a coin'.

Solution : S = {H, T}

the events are φ , {H}, {T}, {H, T}

Example # 5 : A die is thrown. Let A be the event ' an odd number turns up' and B be the event 'a number divisible by 3 turns up'. Write the events (a) A or B (b) A and B

Solution : A = {1, 3, 5}, B = {3, 6}

∴ A or B = $A \cup B$ = {1, 3, 5, 6}

A and B = $A \cap B$ = {3}

Self practice problems :

(3) A coin is tossed and a die is thrown. Let A be the event 'H turns up on the coin and odd number turns up on the die' and B be the event ' T turns up on the coin and an even number turns up on the die'. Write the events (a) A or B (b) A and B.

(4) In tossing of two coins, let A = {HH, HT} and B = {HT, TT}. Then write the events (a) A or B (b) A and B.

Ans. (3) (a) {H1, H3, H5, T2, T4, T6} (b) φ

(4) (a) {HH, HT, TT} (b) {HT}

(vii) Equally likely events :

If events have same chance of occurrence, then they are said to be equally likely.

e.g

(i) In a single toss of a fair coin, the events {H} and {T} are equally likely.

(ii) In a single throw of an unbiased die the events {1}, {2}, {3} and {4}, are equally likely.

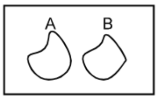
(iii) In tossing a biased coin the events {H} and {T} are not equally likely.

(viii) Mutually exclusive / disjoint / incompatible events :

Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other i.e. both cannot occur simultaneously. In the vein diagram the events A and B are mutually exclusive. Mathematically, we write $A \cap B = \varphi$

Events $A_1, A_2, A_3, ... A_n$ are said to be mutually exclusive events iff

$A_i \cap A_j = \varphi \ \forall \ i, j \in \{1, 2, ..., n\}$ where $i \neq j$



Note : If $A_i \cap A_j = \varphi \ \forall \ i, j \in \{1, 2, ..., n\}$ where $i \neq j$, then $A_1 \cap A_2 \cap A_3 .... \cap A_n = \varphi$ but converse need not to be true.

Example # 6 : In a single toss of a coin find whether the events {H}, {T} are mutually exclusive or not.

Solution : Since $\{H\} \cap \{T\} = \varphi$, $\therefore$ the events are mutually exclusive.

Example # 7 : In a single throw of a die, find whether the events {1, 2}, {2, 3} are mutually exclusive or not.

Solution : Since $\{1, 2\} \cap \{2, 3\} = \{2\} \neq \varphi$

$\therefore$ the events are not mutually exclusive.

Self practice problems :

(5) In throwing of a die write whether the events 'Coming up of an odd number' and 'Coming up of an even number' are mutually exclusive or not.

(6) An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events :

A : the sum is greater than 8.

B : 2 occurs on either die.

C : the sum is at least 7 and a multiple of 3.

Also, find $A \cap B$, $B \cap C$ and $A \cap C$.

Are (i) A and B mutually exclusive ?

(ii) B and C mutually exclusive ?

(iii) A and C mutually exclusive ?

Ans. (5) Yes

(6) A = {(3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)}

B = {(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (2, 1), (2, 3), (2, 4). (2, 5), (2, 6)}

C = {(3, 6), (6, 3), (5, 4), (4, 5), (6, 6)}

$A \cap B = \varphi$, $B \cap C = \varphi$, $A \cap C$ = {(3, 6), (6, 3), (5, 4), (4, 5), (6, 6)}

(i) Yes (ii) Yes (iii) No.

(ix) Exhaustive system of events :

If each outcome of an experiment is associated with at least one of the events E_1, E_2, E_3, ...E_n, then collectively the events are said to be exhaustive. Mathematically we write $E_1 \cup E_2 \cup E_3 ....E_n = S$. (Sample space)

Example # 8 : In throwing of a die, let A be the event 'even number turns up', B be the event 'an odd prime turns up' and C be the event 'a numbers less than 4 turns up'. Find whether the events A, B and C form an exhaustive system or not.

Solution : $A \equiv \{2, 4, 6\}$, $B \equiv \{3, 5\}$ and $C \equiv \{1, 2, 3\}$.

Clearly $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\} = S$. Hence the system of events is exhaustive.

Example # 9 : Three coins are tossed. Describe

(i) two events A and B which are mutually exclusive

(ii) three events A, B and C which are mutually exclusive and exhaustive.

(iii) two events A and B which are not mutually exclusive.

(iv) two events A and B which are mutually exclusive but not exhaustive.

(v) three events A, B and C which are mutually exclusive but not exhaustive.

Solution.

(i) A : "getting at least two heads" B : "getting at least two tails"

(ii) A : "getting at most one heads" B : "getting exactly two heads"
C : "getting exactly three heads"

(iii) A : "getting at most two tails" B : "getting exactly two heads"

(iv) A : "getting exactly one head" B : "getting exactly two heads"

(v) A : " getting exactly one tail" B : "getting exactly two tails"
C : "getting exactly three tails"

[Note : There may be other cases also]

Self practice problems :

(7) In throwing of a die which of the following pair of events are mutually exclusive ?
(a) the events 'coming up of an odd number' and 'coming up of an even number'
(b) the events 'coming up of an odd number' and 'coming up of a number ≥ 4'

(8) In throwing of a die which of the following system of events are exhaustive ?
(a) the events 'an odd number turns up', 'a number ≤ 4 turns up' and 'the number 5 turns up'.
(b) the events 'a number ≤ 4 turns up', 'a number > 4 turns up'.
(c) the events 'an even number turns up', 'a number divisible by 3 turns up', 'number 1 or 2 turns up' and 'the number 6 turns up'.

Ans. (7) (a) (8) (b)

2. Classical (a priori) definition of probability :

If an experiment results in a total of (m + n) outcomes which are equally likely and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A', denoted by P(A), is defined by $\frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$ i.e. $P(A) = \frac{m}{m+n}$. We say that odds in favour of 'A' are m : n, while odds against 'A' are n : m.

Note that $P(\bar{A})$ or P(A') or $P(A^c)$, i.e. probability of non-occurrence of A = $\frac{n}{m+n} = 1 - P(A)$ In the above we shall denote the number of out comes favourable to the event A by n(A) and the total number of out comes in the sample space S by n(S).

$\therefore$ P(A) = $\frac{n(A)}{n(S)}$.

Example # 10 : In throwing of a fair die find the probability of the event of getting a prime numbers.

Solution : Sample space S = {1, 2, 3, 4, 5, 6} ; event A = {2, 3, 5}

$\therefore$ n(A) = 3 and n(S) = 6

$\therefore$ P(A) = $\frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

Example # 11 : In throwing of a fair die, find the probability of turning up of an odd number ≥ 4.

Solution : S = {1, 2, 3, 4, 5, 6}

Let E be the event 'turning up of an odd number ≥ 4' then E = {5}

$\therefore$ P(E) = $\frac{n\,(E)}{n\,(S)} = \frac{1}{6}$.

Example # 12 : In throwing a pair of fair dice, find the probability of getting a total of 9.

Solution : When a pair of dice is thrown the sample space consists

{(1, 1) (1, 2) .... (1, 6)

(2, 1,) (2, 2,)......... (2, 6)

....

....

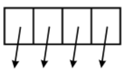
(6, 1), (6, 2) .... (6, 6)}

Note that (1, 2) and (2, 1) are considered as separate points to make each outcome as equally likely.To get a total of '8', favourable outcomes are, (3, 6) (4, 5) (5, 4) and (6, 3)

Hence required probability = $\frac{4}{36} = \frac{1}{9}$

Example # 13 : A four digit number is formed using the digits 0, 1, 2, 3, 4 without repetition. Find the probability that it is divisible by 3.

Solution : Total 4 digit numbers formed



4 × 4 × 3 × 2 = 96

Each of these 96 numbers are equally likely & mutually exclusive of each other.

now, A number is divisible by 3, If sum of digits is divisible by 3

we can either use 0, 2, 3, 4 or 0, 1, 2, 3

so total favorable cases = (3 × 3 × 2) × 2 = 36

probability = $\frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{36}{96} = \frac{3}{8}$

Self practice problems :

(9) A bag contains 4 white, 3 red and 2 blue balls. A ball is drawn at random. Find the probability of the event

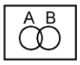
(a) the ball drawn is white or red (b) the ball drawn is white as well as red.

(10) In throwing a pair of fair dice find the probability of the events ' a total of of less than or equal to 9"'.

Ans. (9) (a) 7/9 (b) 0 (10) 5/36.

3. Addition theorem of probability :

If ‘A’ and ‘B’ are any two events associated with an experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



De Morgan’s laws : If A & B are two subsets of a universal set U, then

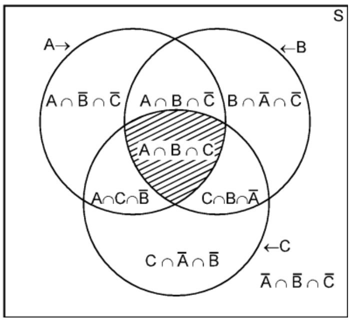
(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$

Distributive laws : (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For any three events A, B and C we have the figure



(i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P(exactly two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P(exactly one of A, B, C occur) =

$P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Example # 14 : A bag contains 4 white, 3red and 4 green balls. A ball is drawn at random. Find the probability of the event ‘the ball drawn is white or green’.

Solution : Let A be the event ‘the ball drawn is white’ and B be the event ‘the ball drawn is green’.P(The ball drawn is white or green) $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{8}{11}$

Example # 15: In throwing of a die, let A be the event ‘an odd number turns up’, B be the event ‘a number divisible by 3 turns up’ and C be the event ‘a number ≤ 5 turns up’. Then find the probability that exactly two of A, B and C occur.

Solution : Event A = {1, 3, 5}, event B = {3, 6} and event C = {1, 2, 3, 4, 5}

∴ $A \cap B = \{3\}$, $B \cap C = \{3\}$, $A \cap C = \{1, 3, 5\}$ and $A \cap B \cap C = \{3\}$.

Thus P(exactly two of A, B and C occur)

$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$

$= \frac{1}{6} + \frac{1}{6} + \frac{3}{6} - 3 \times \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Self practice problems :

(11) In throwing of a die, let A be the event ‘an odd number turns up’, B be the event ‘a number divisible by 3 turns up’ and C be the event ‘a number ≤ 4 turns up’. Then find the probability that atleast two of A, B and C occur.

(12) In the problem number 11, find the probability that exactly one of A, B and C occurs.

Ans. (11) $\frac{1}{3}$ (12) $\frac{2}{3}$

4. Conditional probability :

If A and B are two events, then $P(A/B) = \frac{P(A \cap B)}{P(B)}$

Note that for mutually exclusive events P(A/B) = 0.

Example # 16: If P(A/B) = 0.2 and P(B) = 0.5 and P(A) = 0.2. Find $P(\bar{A} \cap B)$.

Solution : $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Also $P(A/B) = \frac{P(A \cap B)}{P(B)}$ $\Rightarrow$ $P(A \cap B) = 0.1$

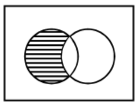
From given data, $P(\bar{A} \cap B) = 0.5 - 0.1 = 0.4$

Example # 17 : If P(A) = 0.25, P(B) = 0.5 and P(A ∩ B) = 0.14, find probability that neither 'A' nor 'B' occurs. Also find $P\left(A \cap \bar{B}\right)$

Solution : We have to find $P\left(\bar{A} \cap \bar{B}\right) = 1 = 1 - P(A \cup B)$ (by De-Morgan's law)

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

putting data we get, $P\left(\bar{A} \cap \bar{B}\right) = 0.39$



The shaded region denotes the simultaneous occurrence of A and $\bar{B}$

Hence $P\left(A \cap \bar{B}\right) = P(A) - P(A \cap B) = 0.11$

Self practice problem :

(13) If P $P(\bar{A}/\bar{B}) = 0.2$, $P(A \cup B) = 0.9$ then find s $P(A \cap \bar{B})$

Ans. 0.4

5. Independent and dependent events :

If two events are such that occurence or non-occurence of one does not affect the chances of occurence or non-occurence of the other event, then the events are said to be independent. Mathematically : if $P(A \cap B) = P(A)\,P(B)$, then A and B are independent.

Notes: (i) If A and B are independent, then

(a) A' and B' are independent,

(b) A and B' are independent and

(c) A' and B are independent.

(ii) If A and B are independent, then P(A / B) = P(A).

If events are not independent then they are said to be dependent.

6. Independency of three or more events :

Three events A, B & C are independent if & only if all the following conditions hold :

$P(A \cap B) = P(A) . P(B)$; $P(B \cap C) = P(B) . P(C)$

$P(C \cap A) = P(C) . P(A)$; $P(A \cap B \cap C) = P(A) . P(B) . P(C)$

Example#18: A pair of fair coins is tossed yielding the equiprobable space S = {HH, HT, TH, TT}. Consider the events:
A = {head on first coin} = {HH, HT}, B = {head on second coin} = {HH, TH}
C = {head on exactly one coin} = {HT, TH}
Then check whether A, B, C are independent or not.

Solution : $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$.

Also $P(A \cap B) = \frac{1}{4} = P(A)\,P(B)$, $P(A \cap C) = \frac{1}{4}$

$= P(A)\,P(C)$, $P(B \cap C) = \frac{1}{4} = P(B)\,P(C)$

but $P(A \cap B \cap C) = 0 \neq P(A)\,P(B)\,P(C)$

∴ A, B & C are not independent

Example # 19 : In drawing two balls from a box containing 6 red and 4 white balls without replacement, which of the following pairs is independent ?
(a) Red on first draw and red on second draw
(b) Red on first draw and white on second draw

Solution : Let E be the event 'Red on first draw', F be the event 'Red on second draw' and G be the event 'white on second draw'.

$P(E) = \frac{6}{10}$, $P(F) = \frac{6}{10}$, $P(G) = \frac{4}{10}$

(a) $P(E \cap F) = \frac{{}^6P_2}{{}^{10}P_2} = \frac{1}{3}$ ⇒ $P(E) . P(F) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25} \neq \frac{1}{3}$

∴ E and F are not independent

(b) $P(E) . P(G) = \frac{6}{10} \times \frac{4}{10} = \frac{6}{25}$ ⇒ $P(E \cap G) = \frac{{}^6P_1 \times {}^4P_1}{{}^{10}P_2} = \frac{4}{15}$

∴ $P(E) . P(G) \neq P(E \cap G)$ ∴ E and G are not independent

Example # 20: A speaks truth in 60% of the cases and b in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

Solution : Let E be the event that A speaks truth and F be the event that B speaks truth. Then E and F are independent events such that

$P(E) = \frac{60}{100} = \frac{3}{5}$ F and $P(F) = \frac{90}{100} = \frac{9}{10}$

A and B will contradict each other in narrating the same fact in the following mutually exclusive ways :

(i) A speaks truth and B tells a lie i.e. $E \cap \bar{F}$

(ii) A tells a lie and B speaks truth lie i.e. $\bar{E} \cap F$

∴ P(A and B contradict each other)

= $P(I \text{ or } II) = (I \cup II)$ = $P[(E \cap \bar{F}) \cup (\bar{E} \cap F)]$

= $P(E \cap \bar{F}) + P(\bar{E} \cap F)$ [$\because E \cap \bar{F}$ and $\bar{E} \cap F$ are mutually exclusive]

= $P(E)\,P(\bar{F}) + P(\bar{E})\,P(F)$ [$\because$ E and F are in dep.]

= $\frac{3}{5} \times \left(1-\frac{9}{10}\right) + \left(1-\frac{3}{5}\right) \times \frac{9}{10} = \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{21}{50}$

Example # 21 : A box contains 5 bulbs of which two are defective. Test is carried on bulbs one by one untill the two defective bulbs are found out. Find the probability that the process stops after

(i) Second test (ii) Third test

Solution : (i) Process will stop after second test. Only if the first and second bulb are both found to be defective

probability = $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$ (Obviously the bulbs drawn are not kept back.)

(ii) Process will stop after third test when either

(a) DND $\rightarrow \frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{10}$ Here 'D' stands for defective and 'N' is for not defective.

(b) NDD $\rightarrow \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$;k (c) NNN $\rightarrow \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

hence required probability = $\frac{3}{10}$

Example # 22 : If E_1 and E_2 are two events such that $P(E_1) = \frac{1}{4}$; $P(E_2) = \frac{1}{2}$; $P\left(\frac{E_1}{E_2}\right) = \frac{1}{4}$, then choose the correct options.

(i) E_1 and E_2 are independent (ii) E_1 and E_2 are exhaustive

(iii) E_1 and E_2 are mutually exclusive (iv) E_1 & E_2 are dependent

Also find $P\left(\frac{\bar{E}_1}{E_2}\right)$ and $\left(\frac{E_2}{\bar{E}_1}\right)$

Solution : Since $P\left(\frac{E_1}{E_2}\right) = P(E_1)$ $\Rightarrow$ E_1 and E_2 are independent of each other

Also since $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1)\,.\,P(E_2) \neq 1$

Hence events are not exhaustive. Independent events can't be mutually exclusive.

Hence only (i) is correct

Further since E_1 & E_2 are independent; E1 and $\bar{E}_2$ or $\bar{E}_1$, E_2 are $\bar{E}_1$, $\bar{E}_2$ are also independent.

Hence $P\left(\frac{\bar{E}_1}{E_2}\right) = P(\bar{E}_1) = \frac{3}{4}$ and $P\left(\frac{E_2}{\bar{E}_1}\right) = P(E_2) = \frac{1}{2}$

Example # 23 : If cards are drawn one by one from a well shuffled pack of 52 cards without replacement, until an ace appears, find the probability that the fourth card is the first ace to appear.

Solution : Probability of selecting 3 non-Ace and 1 Ace out of 52 cards is equal to $\frac{{}^{48}C_3 \times {}^{4}C_1}{{}^{52}C_4}$

Since we want 4th card to be first ace, we will also have to consider the arrangement, Now 4 cards in sample space can be arranged in 4! ways and, favorable they can be arranged in 3 ! ways as we want 4th position to be occupied by ace

Hence required probability = $\dfrac{^{48}C_3 \times ^4C_1}{^{52}C_4} \times \dfrac{3!}{4!}$

Aliter : 'NNNA' is the arrangement then we desire in taking out cards, one by one

Hence required chance is $\dfrac{48}{52} \times \dfrac{47}{51} \times \dfrac{46}{50} \times \dfrac{4}{49}$

Self practice problems :

(14) An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting
(i) 2 red balls (ii) 2 blue balls (iii) one red and one blue ball

(15) Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
(i) the problem is solved (ii) exactly one of them solves the problem.

(16) In throwing a pair of dies find the probability of getting an odd number on the first die and a total of 7 on both the dies.

(17) In throwing of a pair of dies, find the probability of getting a boublet or a total of 4.

(18) A bag contains 8 marbles of which 3 are blue and 5 are red. One marble is drawn at random, its colour is noted and the marble is replaced in the bag. A marble is again drawn from the bag and its colour is noted. Find the probability that the marbles will be
(i) blue followed by red (ii) blue and red in any order (iii) of the same colour.

(19) A coin is tossed thrice. In which of the following cases are the events E and F independent ?
(i) E : "the first throw results in head". F : "the last throw result in tail".
(ii) E : "the number of heads is two". F : "the last throw result in head".
(iii) E : "the number of heads is odd ". F : "the number of tails is odd".

Ans. (14) (i) $\frac{49}{121}$ (ii) $\frac{16}{121}$ (iii) $\frac{56}{121}$

(15) (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$ (16) $\frac{1}{12}$ (17) $\frac{2}{9}$

(18) (i) $\frac{15}{64}$ (ii) $\frac{15}{32}$ (iii) $\frac{17}{32}$ (19) (i)

7. Binomial probability theorem :

The probability of getting exactly r success in n trials of such an experiment is $_nC_r\, p^r\, q^{n-r}$, where 'p' is the probability of a success and q is the probability of a failure in one particular experiment. Note that p + q = 1.

Example 24 : A pair of dice is thrown 5 times. Find the probability of getting a doublet twice.

Solution : In a single throw of a pair of dice probability of getting a doublet is $\frac{1}{6}$

onsidering it to be a success, p = $\frac{1}{6}$ $\therefore$q = 1 – $\frac{1}{6}$ = number of success r = 2

$\therefore$ $P(r = 2) = {}_5C_2\, p^2\, q^3 = 10 . \left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$

Example # 25 : A pair of dice is thrown 4 times. If getting ‘a total of 9’ in a single throw is considered as a success then find the probability of getting ‘a total of 9’ thrice.

Solution : p = probability of getting ‘a total of 9’ = $\frac{4}{36} = \frac{1}{9}$

$\therefore$ $q = 1 – \frac{1}{9} = \frac{8}{9}$ $\therefore$ r = 3, n= 4 $\therefore$ $P(r = 3) = {}_4C_3\, p^3\, q = 4 \times \left(\frac{1}{9}\right)^3 . \frac{8}{9} = \frac{32}{6561}$

Example # 26 : In an examination of 10 multiple choice questions (1 or more can be correct out of 4 options). A student decides to mark the answers at random. Find the probability that he gets exactly two questions correct.

Solution : A student can mark 15 different answers to a MCQ with 4 option i.e. ${}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4 = 15$

Hence if he marks the answer at random, chance that his answer is correct = $\frac{1}{15}$ and being incorrecting $\frac{14}{15}$. Thus p = $\frac{1}{15}$, q = $\frac{14}{15}$.

$P\text{ (2 success)} = {}_{10}C_2 \times \left(\frac{1}{15}\right)^2 \times \left(\frac{14}{15}\right)^8$

Example # 27 : A family has three children. Event ‘A’ is that family has at most one boy, Event ‘B’ is that family has at least one boy and one girl, Event ‘C’ is that the family has at most one girl. Find whether events ‘A’ and ‘B’ are independent. Also find whether A, B, C are independent or not.

Solution : A family of three children can have

(i) All 3 boys (ii) 2 boys + 1 girl (iii) 1 boy + 2 girls (iv) 3 girls

(i) $P\text{ (3 boys)} = {}_3C_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ (Since each child is equally likely to be a boy or a girl)

(ii) $P\text{(2 boys+1girl)} = {}_3C_1 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{3}{8}$ (Note that there are three cases BBG, BGB, GBB)

(iii) $P\text{ (1 boy + 2 girls)} = {}_3C_2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = \frac{3}{8}$

(iv) $P\text{ (3 girls)} = \frac{1}{8}$

Event ‘A’ is associated with (iii) & (iv). Hence P(A) = $\frac{1}{2}$

Event ‘B’ is associated with (ii) & (iii). Hence P(B) = $\frac{3}{4}$

Event ‘C’ is associated with (i) & (ii). Hence P(C) = $\frac{1}{2}$

$P(A \cap B) = P(iii) = \frac{3}{8} = P(A) . P(B)$. Hence A and B are independent of each other
$P(A \cap C) = 0 \neq P(A) . P(C)$. Hence A, B, C are not independent

Self practice problems :

(20) A box contains 2 red and 3 blue balls. Two balls are drawn successively without replacement. If getting 'a red ball on first draw and a blue ball on second draw' is considered a success, then find the probability of 2 successes in 3 performances.

(21) Probability that a bulb produced by a factory will fuse after an year of use is 0.2. Find the probability that out of 5 such bulbs not more than 1 bulb will fuse after an year of use.

Ans. (20) 189 (21) $\frac{2304}{3125}$

8. <u>Total probability theorem</u> :

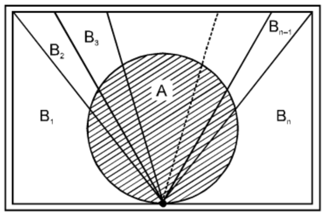
If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, ..., B_n$ and the probabilities $P(A/B_1), P(A/B_2) .... P(A/B_n)$ are known, then

$$P(A) = \sum_{i=1}^{n} P(B_i) . P(A/B_i)$$

Proof :

The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, ..., B_n$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup .... \cup (A \cap B_n)$



$P(A) = P(A \cap B_1) + P(A \cap B_2) + ... + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$

$P(A \cap B_i) = P(A) . P(B_i/A) = P(B_i) . P(A/B_i)$

$\therefore \quad P(A) = \sum_{i=1}^{n} P(B_i) . P(A/B_i)$

Example # 28 : Box - I contains 5 red and 4 white balls while box - II contains 4 red and 2 white balls. A biased coin is tossed if head appears with probability of $\frac{1}{3}$, on getting a head a ball is drawn from box - I else a ball is drawn from box - II. Find the probability that the ball drawn is white.

Solution : Let A be the event 'a multiple of 3 turns up on the die' and R be the event 'the ball drawn is white' then P (ball drawn is white)

$= P(A) . P(R / A) + P(\bar{A}) \; P(R / \bar{A}) = \frac{1}{3} \times \frac{4}{9} + \left(1 - \frac{1}{3}\right)\frac{1}{3} = \frac{10}{27}$

Example # 29 : Cards of an ordinary deck of playing cards are placed into two heaps. Heap - I consists of all the red cards and heap - II consists of all the black cards. A heap is chosen at random and a card is drawn, find the probability that the card drawn is a king or a queen.

Solution : Let I and II be the events that heap - I and heap - II are choosen respectively. Then

$P(I) = P(II) = \frac{1}{2}$

Let K be the event 'the card drawn is a king or queen'

$\therefore \quad P(K / I) = \frac{4}{26}$ and $P(K / II) = \frac{4}{26}$

∴ P(K) = P (I) P(K / I) + P(II) P(K / II) = $\frac{1}{2} \times \frac{4}{26} + \frac{1}{2} \times \frac{4}{26} = \frac{2}{13}$.

Self practice problems :

(22) Box - I contains 3 red and 2 blue balls while box - II contains 2 red and 3 blue balls. A fair coin is tossed. If it turns up head, a ball is drawn from box - I, else a ball is drawn from box - II . Find the probability that the ball drawn is red.

(23) There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. Find the probability of selecting a brilliant student.

Ans. (22) $\frac{1}{2}$ (23) $\frac{17}{125}$

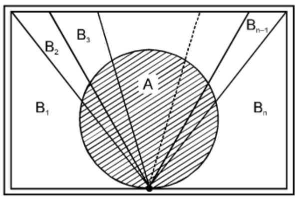
9. Baye's Theorem :

If an event A can occur with one of the n mutually exclusive and exhaustive events B_1, B_2, ..., B_n and the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known, then

$$P(B_i / A) = \frac{P(B_i) \,.\, P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \,.\, P(A/B_i)}$$

Proof :

The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, \ldots, B_n$



$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)$

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$

Now,

$P(A \cap B_i) = P(A) \,.\, P(B_i/A) = P(B_i) \,.\, P(A/B_i)$

$$P(B_i/A) = \frac{P(B_i) \,.\, P(A/B_i)}{P(A)} = \frac{P(B_i) \,.\, P(A/B_i)}{\sum_{i=1}^{n} P(A \cap B_i)}$$

$$P(B_i/A) = \frac{P(B_i) \,.\, P(A/B_i)}{\sum P(B_i) \,.\, P(A/B_i)}$$

Example # 30 : There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. A student is a chosen and is found to be brilliant, find the probability that the chosen student is from class XI.

Solution : Let E and F be the events 'Class XI is chosen' and 'Class XII is chosen' respectively.

Then $P(E) = \frac{2}{5}$, $P(F) = \frac{3}{5}$

et A be the event 'Student chosen is brilliant'.

then $P(A / E) = \frac{5}{50}$ and $P(A / F) = \frac{8}{50}$.

∴ $P(A) = P(E) \,.\, P(A / E) + P(F) \,.\, P(A / F) = \frac{2}{5} \,.\, \frac{5}{50} + \frac{3}{5} \,.\, \frac{8}{50} = \frac{34}{250}$.

$\therefore P(E / A) = \dfrac{P(E).P(A/E)}{P(E).P(A/E)+P(F).P(A/F)} = \dfrac{5}{17}$.

Example # 31 : A bag contains 6 white and an unknown number of black balls (≤ 3). Balls are drawn one by one with replacement from this bag twice and is found to be white on both occassion. Find the probability that the bag had exactly '3' Black balls.

Solution : Apriori, we can think of the following possibilies

(i)	E_1	6W	,	0 B
(ii)	E_2	6W	,	1 B
(iii)	E_3	6W	,	2 B
(iv)	E_4	6W	,	3 B

Clearly $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \dfrac{1}{4}$

Let 'A' be the event that two balls drawn one by one with replacement are both white therefore we have to find $P\left(\dfrac{E_4}{A}\right)$

By Baye's theorem $P\left(\dfrac{E_4}{A}\right) = \dfrac{P\left(\dfrac{A}{E_4}\right)\times P(E_4)}{P\left(\dfrac{A}{E_1}\right)\times P(E_1)+P\left(\dfrac{A}{E_2}\right).P(E_2)+P\left(\dfrac{A}{E_3}\right).P(E_3)+P\left(\dfrac{A}{E_4}\right).P(E_4)}$

$P\left(\dfrac{A}{E_4}\right) = \dfrac{6}{9}\times\dfrac{6}{9}$;

$P\left(\dfrac{A}{E_3}\right) = \dfrac{6}{8}\times\dfrac{6}{8}$; $P\left(\dfrac{A}{E_2}\right) = \dfrac{6}{7}\times\dfrac{6}{7}$;

$P\left(\dfrac{A}{E_1}\right) = \dfrac{6}{6}\times\dfrac{6}{6}$;

Putting values $P\left(\dfrac{E_4}{A}\right) = \dfrac{\dfrac{1}{81}}{\dfrac{1}{81}+\dfrac{1}{64}+\dfrac{1}{49}+\dfrac{1}{36}}$

Self practice problems :

(24) Box-I contains 3 red and 2 blue balls whilest box-II contains 2 red and 3 blue balls. A fair coin is tossed. If it turns up head, a ball is drawn from box-I, else a ball is drawn from box-II. If the ball drawn is red, then find the probability that the ball is drawn from box-II.

(25) Cards of an ordinary deck of playing cards are placed into two heaps. Heap - I consists of all the red cards and heap - II consists of all the black cards. A heap is chosen at random and a card is drawn, if the card drawn is found to be a king, find the probability that the card drawn is from the heap - II.

Ans. (24) $\dfrac{3}{5}$ (25) $\dfrac{1}{2}$

10. Probability distribution :

(i) A probability distribution spells out how a total probability of 1 is distributed over several values of a random variable (i.e. how .... possibilities)

(ii) Mean of any probability distribution of a random variable is given by :

$\mu = \frac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$ (Since $\Sigma p_i = 1$)

(iii) Variance of a random variable is given by, $\sigma^2 = \Sigma (x_i - \mu)^2 . p_i$

$\therefore$ $\sigma^2 = \Sigma p_i x_i^2 - \mu^2$ (Note that SD = $+\sqrt{\sigma^2}$)

(iv) The probability distribution for a binomial variate 'X' is given by :

$P(X = r) = {}^nC_r\, p^r q^{n-r}$ where P(X = r) is the probability of r successes. (recurrence The recurrence formula $\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1} . \frac{p}{q}$ is very helpful for quickly computing P(1) . P(2) . P(3) etc. if P(0) is known.

Mean of Binomial Probability Distribution = np ; variance of Binomial Probability Distribution = npq.

(v) If p represents a person's chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

Example # 32 : A pair of dice is thrown 5 times. If getting a doublet is considered as a success, then find the mean and variance of successes.

Solution : In a single throw of a pair of dice, probability of getting a doublet = $\frac{1}{6}$

considering it to be a success, p = $\frac{1}{6}$ $\therefore$ $q = 1 - \frac{1}{6} = \frac{5}{6}$

mean = $5 \times \frac{1}{6} = \frac{5}{6}$ variance= $5 \times \frac{1}{6} . \frac{5}{6} = \frac{25}{36}$

Example # 33 : A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of successes.

Solution : p = probability of getting a total of 9 = $\frac{4}{36} = \frac{1}{9}$

$\therefore$ $q = 1 - \frac{1}{9} = \frac{8}{9}$ $\therefore$ mean= np = $4 \times \frac{1}{9} = \frac{4}{9}$

variance = npq = $4 \times \frac{1}{9} \times \frac{8}{9} = \frac{32}{81}$.

Example # 34 : Difference between mean and variance of a Binomial variate is '1' and difference between their squares is '11'. Find the probability of getting exactly three success

Solution : Mean = np & variance = npq

therefore, $np - npq = 1$ ....(i)

$n^2p^2 - n^2p^2q^2 = 11$ ....(ii)

Also, we know that p + q = 1 ....(iii)

Divide equation (ii) by square of (i) and solve, we get, $q = \frac{5}{6}$, $p = \frac{1}{6}$ and n = 36

Hence probability of '3' success = ${}_{36}C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{33}$

Self practice problems :

(26) A box contains 2 red and 3 blue balls. Two balls are drawn successively without replacement. If getting 'a red ball on first draw and a blue ball on second draw' is considered a success, then find the mean and variance of successes.

(27) Probability that a bulb produced by a factory will fuse after an year of use is 0.2. If fusing of a bulb is considered an failure, find the mean and variance of successes for a sample of 10 bulbs.

(28) A random variable X is specified by the following distribution law :

X	2	3	4
P(X = x)	0.3	0.4	0.3

Then the variance of this distribution is :

(1) 0.6 (2) 0.7 (3) 0.77 (4) 1.55

Ans. (26) mean = 2.1, σ_2 = .63

(27) mean = 8 and variance = 1.6