Sequence & Series

"1729 is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." ...... **S.Ramanujan**

1. Sequence :

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$\{f(1), f(2), f(3), ...\} = \{t_1, t_2, t_3, ...\} = \{t_n\}$

If all the elements of a sequence are real , it is said to be real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

(iii) 3, –9, 27, – 81, ...

Notes : (i) A sequence is said to be finite or infinite if it has finite or infinite number of terms respectively.

(ii) Series : By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ...a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

e.g. (i) 1 + 2 + 3 + 4 + ... + n

(ii) 2 + 4 + 8 + 16 + ...

(iii) – 1 + 3 – 9 + 27 – ....

(iii) Progression : Set of numbers, following a particular mathematical definition is called progression.

Example # 1 : Write down the sequence whose n_{th} term is

(i) $n \times 5_n$ (ii) $\dfrac{2+(-1)^n}{2^n}$

Solution : (i) Let $t_n = n \times 5_n$

put n = 1, 2, 3, 4, ... we get

$t_1 = 5, t_2 = 50, t_3 = 375$

so the sequence is 5, 50, 375 ......

(ii) Let $t_n = \dfrac{2+(-1)^n}{2^n}$

put n = 1, 2, 3, 4, ...

so the sequence is $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{1}{8}, \dfrac{3}{16},$ ....

2. Arithmetic Progression (A.P.) :

$a, a + d, a + 2d, ... ,a + (n - 1) d, ...$

Where a and d are first term and common difference respectively

$d = t_2 - t_1 = t_3 - t_2 = ... = t_n - t_{n-1}$

t_n (n^{th} term) $= a + (n - 1) d$, where $d = t_n - t_{n-1}$

S_n (sum of n terms) $= \dfrac{n}{2}[2a + (n - 1) d] = \dfrac{n}{2}[a + \ell]$ (Where ℓ is the last term)

Note : For any sequence $\{t_n\}$, whose sum of first r terms is S_r, r_{th} term $t_r = S_r - S_{r-1}$.

Example # 2 : If t_3 of an A.P. is 66 and $t_{21} = 30$, find t_2.

Solution : Let a be the first term and d be the common difference

so $t_3 = a + 2d = 66$ ....(i)

and $t_{21} = a + 20d = 30$ ....(ii)

equation (i) – (ii) we get

$18d = -36$

$\Rightarrow$ $d = -2$

$\therefore$ $a = 70$ so $t_2 = 70 + (-2) = 68$

Example # 3 : Find the number of terms in the sequence 6, 12, 18, ...,132.

Solution : $a = 6$, $d = 6$ so $132 = 6 + (n - 1)6$ $\Rightarrow$ $n = 22$

Example # 4 : Find the sum of all natural numbers divisible by 4, but less than 62.

Solution : All those numbers are 4, 8, 12, ..., 60

so $S = \frac{15}{2}(4 + 60) = 480$

Example # 5 : Find the sum of all the two digit natural numbers which on division by 6 leaves remainder 3.

Solution : All these numbers are 15, 21, 27, ...99.

$99 = 15 + (n - 1)\,6$

$\Rightarrow$ $n = 15$

so $S = \frac{15}{2}[15 + 99] = 855$

Example # 6 : The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11_{th} terms.

Solution : Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively,

then $\dfrac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \dfrac{7n+1}{4n+27}$ $\Rightarrow$ $\dfrac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \dfrac{7n+1}{4n+27}$

For ratio of 11_{th} terms

$\frac{n-1}{2} = 10$ $\Rightarrow$ $n = 21$

so ratio of 11_{th} terms is $= \dfrac{7(21)+1}{4(21)+27} = \dfrac{148}{111} = \dfrac{4}{3}$

Example # 7 : If sum of n terms of a sequence is given by $S_n = n^2 + n$, find its 50_{th} term.

Solution : Let t_n is n_{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= n^2 + n - (n-1)^2 - (n-1) = 2n$

so $t_{50} = 100$

Self practice problems :

(1) Which term of the sequence 2005, 2000, 1995, 1990, 1985, ... is the first negative term

(2) For an A.P. show that $t_m + t_{2n+m} = 2\,t_{m+n}$

(3) Find the maximum sum of the A.P. 40 + 38 + 36 + 34 + 32 + ...

Ans. (1) 403 (3) 420

Remarks :

(i) The first term and common difference of an A.P. can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow$ 2 b = a + c & if a, b, c, d are in A.P. $\Rightarrow$ a + d = b + c.

(iii) Three numbers in A.P. can be taken as $a - d, a, a + d$;
four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$;
five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$;
six numbers in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

(vi) The sum and difference of two AP's is an AP.

Example # 8 : The sum of three numbers in A.P. is 15 and the sum of their squares is 93, find them.

Solution : Let the numbers be $a - d,\ a,\ a + d$

so $3a = 15 \quad \Rightarrow \quad a = 5$

Also $(a - d)^2 + a^2 + (a + d)^2 = 93$.

$3a^2 + 2d^2 = 93$

$d^2 = 9 \quad \Rightarrow \quad d = \pm 3$

therefore numbers are 2, 5, 8

Example # 9: If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference≠0, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution : As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10$

Example # 10 : If a, b, c $\in R^+$ form a A.P. then prove that $a + \frac{1}{bc}$, $b + \frac{1}{ac}$, $c + \frac{1}{ab}$ are also in A.P.

Solution : a, b, c are in A.P.

$\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P. [Divided by abc]

$a + \frac{1}{bc}, b + \frac{1}{ac}, c + \frac{1}{ab}$ are in A.P. [$\because$ sum of two A.P.'s is also in A.P.]

Example # 11 : If $a^2(b + c), b^2(c + a), c^2(a + b)$ are in A.P. then prove that a, b, c in A.P.

Solution : $b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$

or $c(b^2 - a^2) + ab\,(b - a) = a\,(c^2 - b^2) + bc\,(c - b)$

or $(b - a)(ab + bc + ca) = (c - a)(ab + bc + ca)$

or $b - a = c - b$ or $2b = a + c$

hence a, b, c are in A.P.

3. Arithmetic mean (mean or average) (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n numbers $a_1, a_2, ..., a_n$ is; $A = \dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}$

4. n - Arithmetic means between two numbers :

If a, b are any two given numbers & a, $A_1, A_2, ..., A_n$, b are in A.P., then $A_1, A_2, ... A_n$ are the n A.M.'s between a & b.

$A_1 = a + \dfrac{b-a}{n+1}$, $A_2 = a + \dfrac{2(b-a)}{n+1}$,......, $A_n = a + \dfrac{n(b-a)}{n+1}$

Note : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b i.e. $\sum_{r=1}^{n} A_r = nA$, where A is the single A.M. between a & b i.e. $A = \dfrac{a+b}{2}$

Example # 12 : Between two numbers whose sum is 12, an even number of A.M.s is inserted, the sum of these means exceeds their number by 20. Find the number of means.

Solution : Let a and b be two numbers and 2n A.M.s are inserted between a and b, then

$\dfrac{2n}{2}(a + b) = 2n + 20$

$n\ 12 = 2n + 20$ $\left[\text{given } a + b = \dfrac{13}{6}\right]$ $\Rightarrow$ $n = 2$ $\therefore$ Number of means = 2

Example # 13 : Insert 30 A.M. between 4 and 128.

Solution : Here 4 is the first term and 128 is the 32_{nd} term of A.P. so $128 = 4 + (31)d$

$\Rightarrow$ $d = 4$

so the series is 4, 8, 12, 16,......., 128

$\therefore$ required means are 8, 12, 16,...,124

Self practice problems :

(4) If A.M. between p_{th} and q_{th} terms of an A.P. be equal to the A.M. between r_{th} and s_{th} terms of the A.P., then prove that $p + q = r + s$.

(5) If n A.M.s are inserted between 20 and 80 such that first mean : last mean = 1 : 3, find n.

(6) For what value of n, $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$, $a \neq b$ is the A.M. of a and b.

Ans. (5) n = 11 (6) n = 0

5. Geometric Progression (G.P.) :

$a, ar, ar^2, ar^3, ar^4, ...$ is a G.P. with 'a' as the first term & 'r' as common ratio.

$r = \dfrac{T_2}{T_1} = \dfrac{T_3}{T_2} = ... \dfrac{T_n}{T_{n-1}}$

(i) n_{th} term of GP $= a\,r^{n-1}$

(ii) Sum of the first n terms of GP

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} & , \quad r \neq 1 \\ na & , \quad r = 1 \end{cases}$$

(iii) Sum of an infinite terms of GP when $|r| < 1$.

$\because$ When $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore,

$$S_\infty = \frac{a}{1-r} \qquad (|r| < 1)$$

Example # 14 : The first term of a G.P. is 1. The sum of the third and fifth terms is 90, find the common ratio of G.P.

Solution : $a_3 + a_5 = 90$

$\Rightarrow$ $ar^2 + ar^4 = 90$ $\quad [\therefore \quad a = 1]$

$\Rightarrow$ $r^2 + r^4 = 90$

$\Rightarrow$ $r^4 + r^2 - 90 = 0$

$\Rightarrow$ $r^4 + 10r^2 - 9r^2 - 90 = 0$

$\Rightarrow$ $(r^2 + 10)(r^2 - 9) = 0$

$r^2 - 9 = 0 \quad \Rightarrow \quad r = \pm 3$

Example # 15 : The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution : Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow$ $r = \dfrac{r^2}{1-r}$ $\Rightarrow$ $r = \dfrac{1}{2}$,

Hence series is 1, $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8},$ ....∞

Example # 16 : Let $S = 1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots\ldots\ldots$, find the sum of

(i) first 20 terms of the series (ii) infinite terms of the series.

Solution : (i) $S_{20} = \dfrac{\left(1-\left(\frac{1}{3}\right)^{20}\right)}{1-\frac{1}{3}} = \dfrac{3^{20}-1}{2.3^{19}}$. (ii) $S_\infty = \dfrac{1}{1-\frac{1}{3}} = \dfrac{3}{2}$.

Self practice problems :

(7) Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.

(8) If the p^{th}, q^{th}, r^{th} terms of a G.P. be a, b, c respectively, prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

(9) A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 , then find the common ratio of the progression.

(10) The sum of infinite number of terms of a G.P. is 4, and the sum of their cubes is 192, find the series.

Ans. (7) 2, 6, 18, 54, 162, 486 (9) $\dfrac{S_2}{S_1}$. (10) 6, – 3, $\dfrac{3}{2}$,.........

Remarks :

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if $a_1, a_2, a_3, a_4, \ldots\ldots\ldots a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_2a_{n-1} = a_3\, a_{n-2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(ii) Any three consecutive terms of a G.P. can be taken as $\dfrac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as $\dfrac{a}{r^3}$, $\dfrac{a}{r}$, ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non−zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are two G.P's with common ratio r_1 and r_2 respectively, then the sequence $a_1b_1, a_2b_2, a_3b_3, ...$ is also a G.P. with common ratio $r_1 r_2$.

(vi) If $a_1, a_2, a_3, ...$ are in G.P. where each $a_i > 0$, then $\log a_1, \log a_2, \log a_3, ...$ are in A.P. and its converse is also true.

Example # 17 : Find three numbers in G.P. having sum 26 and product 216.

Solution : Let the three numbers be $\frac{a}{r}$, a, ar

so $a\left[\frac{1}{r}+1+r\right] = 26$ ....(i)

and $a^3 = 216 \Rightarrow a = 6$

so from (i) $6r^2 - 20r + 6 = 0 \Rightarrow r = 3, \frac{1}{3}$

Hence the three numbers are 2, 6, 18.

Example # 18 : Find the product of 11 terms in G.P. whose 6^{th} term is 5.

Solution : Using the property

$a_1a_{11} = a_2a_{10} = a_3a_9 = ... = a_6^2 = 25$

Hence product of terms $= 5^{11}$

Example # 19 : Using G.P. express $0.\overline{3}$ and $1.2\overline{3}$ as $\frac{p}{q}$ form.

Solution : Let $x = 0.\overline{3} = 0.3333 .... = 0.3 + 0.03 + 0.003 + 0.0003 + ...$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ... = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{9} = \frac{1}{3}.$$

Let $y = 1.2\overline{3}$

$= 1.233333 = 1.2 + 0.03 + 0.003 + 0.0003 + ...$

$$= 1.2 + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + ... = 1.2 + \frac{\frac{3}{10^2}}{1-\frac{1}{10}} = 1.2 + \frac{1}{30} = \frac{37}{30}$$

Example # 20 : Evaluate 5 + 55 + 555 + ... upto n terms.

Solution : Let S = 5 + 55 + 555 + ...upto n terms.

$= \frac{5}{9}$ [9 + 99 + 999 + ...]

$= \frac{5}{9}$ [(10 – 1) + (10^2 – 1) + (10^3 – 1) + ... + upto n terms]

$= \frac{5}{9}$ [10 + 10^2 + 10^3 + ...+ 10^n – n]

$$= \frac{5}{9}\left(\frac{10(10^n-1)}{9}-n\right) = \frac{5}{81}\,[10^{n+1} - 9n - 10]$$

6. Geometric means (mean proportional) (G.M.):

If a, b, c are in G.P., b is called as the G.M. of a & c.

If a and c are both positive, then $b = \sqrt{ac}$ and if a and c are both negative, then $b = -\sqrt{ac}$.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

7. n-Geometric means between a, b :

If a, b are two given numbers & a, G_1, G_2,....., G_n, b are in G.P.. Then
G_1, G_2, G_3,...., G_n are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,......, $G_n = a(b/a)^{n/n+1}$

Note : The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$, where G is the single G.M. between a & b.

Example # 21 : Insert 5 G.M.s between 4 and 256.

Solution : Common ratio of the series is given by $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = (64)^{1/6} = 2$
Hence five G.M.s are 8, 16, 32, 64, 128.

Self practice problems :

(11) The sum of three numbers in G.P. is 70, if the two extremes be multiplied each by 4 and the mean by 5, the products are in A.P. Find the numbers.

(12) If $a = \underbrace{111\ldots\ldots1}_{55}$, $b = 1 + 10 + 10^2 + 10^3 + 10^4$ and $c = 1 + 10^5 + 10^{10} + \ldots.. + 10^{50}$, then prove that
(i) 'a' is a composite number (ii) a = bc.

Ans. (11) 10, 20, 40

8. Harmonic Progression (H.P.)

A sequence is said to be in H.P if the reciprocals of its terms are in A.P.. If the sequence a_1, a_2, a_3,...., a_n is in H.P. then $1/a_1$, $1/a_2$,...., $1/a_n$ is in A.P.

Note :

(i) Here we do not have the formula for the sum of the n terms of an H.P.. For H.P. whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

(ii) If a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

(iii) If a, b, c are in A.P. $\Rightarrow \dfrac{a-b}{b-c} = \dfrac{a}{a}$

(iv) If a, b, c are in G.P. $\Rightarrow \dfrac{a-b}{b-c} = \dfrac{a}{b}$

9. Harmonic Mean (H.M.) :

If a, b, c are in H.P., b is called as the H.M. between a & c, then b = $\dfrac{2ac}{a+c}$

If a_1, a_2 , ... a_n are 'n' non-zero numbers then H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

Example # 22 : If a > 1, b > 1, c > 1 are in G.P. then show that $\dfrac{1}{1+\log_e a}, \dfrac{1}{1+\log_e b}, \dfrac{1}{1+\log_e c}$ are in H.P.

Solution : $b^2 = ac$

or $2\log_e b = \log_e a + \log_e c$

Hence $\log_e a$, $\log_e b$, $\log_e c$ are in A.P.

$1 + \log_e a$, $1 + \log_e b$, $1 + \log_e c$ are in A.P.

$\dfrac{1}{1+\log_e a}, \dfrac{1}{1+\log_e b}, \dfrac{1}{1+\log_e c}$ are in H.P.

Example # 23 : Insert 4 H.M between $\dfrac{2}{3}$ and $\dfrac{2}{18}$.

Solution : Let 'd' be the common difference of corresponding A.P..

so $d = \dfrac{\frac{18}{2} - \frac{3}{2}}{5} = \dfrac{3}{2}$.

$\therefore$ $\dfrac{1}{H_1} = \dfrac{3}{2} + \dfrac{3}{2} = \dfrac{6}{2}$ or $H_1 = \dfrac{2}{6}$

$\dfrac{1}{H_2} = \dfrac{3}{2} + 3 = \dfrac{9}{2}$ or $H_2 = \dfrac{2}{9}$

$\dfrac{1}{H_3} = \dfrac{3}{2} + \dfrac{9}{2} = \dfrac{12}{2}$ or $H_3 = \dfrac{2}{12}$

$\dfrac{1}{H_4} = \dfrac{3}{2} + 6 = \dfrac{15}{2}$ or $H_4 = \dfrac{2}{15}$

Example # 24 : If p_{th}, q_{th}, r_{th} terms of an H.P. be a, b, c respectively, prove that

$(q – r)bc + (r – p) ac + (p – q) ab = 0$

Solution : Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $\dfrac{1}{a} = x + (p – 1)d$ ....(i)

$\dfrac{1}{b} = x + (q – 1) d$ ....(ii)

$\dfrac{1}{c} = x + (r – 1) d$ ....(iii)

(i) - (ii) $\Rightarrow$ $ab(p – q)d = b – a$ ....(iv)

(ii) - (iii) $\Rightarrow$ bc (q – r)d = c – b ....(v)
(iii) - (i) $\Rightarrow$ ac (r – p) d = a – c ....(vi)
(iv) + (v) + (vi) gives
bc (q – r) + ac(r – p) + ab (p – q) = 0.

Self practice problems :

(13) If a, b, c be in H.P., show that a : a – b = a + c : a – c.

(14) If the ratio of H.M. between two positive numbers 'a' and 'b' (a > b) is to their G.M. as 12 to 13, prove that a : b is 9 : 4.

(15) If H be the harmonic mean of a and b, then find the value of $\frac{H}{2a}+\frac{H}{2b}-1$.

(16) If a, b, c, d are in H.P., then show that ab + bc + cd = 3ad

Ans. (15) 0

10. Arithmetico-geometric series :

A series, each term of which is formed by multiplying the corresponding terms of an A.P. & G.P. is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$
Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3..... are in G.P..

(i) Sum of n terms of an arithmetico-geometric series:

Let $S_n = a + (a + d)\, r + (a + 2d)\, r^2 + ... + [a + (n-1)d]\, r^{n-1}$, then

$$S_n = \frac{a}{1-r}+\frac{dr\left(1-r^{n-1}\right)}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{1-r}, \quad r \neq 1.$$

(ii) Sum to infinity: If $|r| < 1$ & $n \to \infty$, then $\lim_{n\to\infty} r^n = 0$ and $\lim_{n\to\infty} n.r^n = 0$ $\therefore$ $\mathbf{S_\infty} = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$

Example # 25 : Find the sum of the series $1+\frac{3}{5}+\frac{5}{5^2}+\frac{7}{5^3}+ ......$ to n terms.

Solution : Let $S = 1+\frac{3}{5}+\frac{5}{5^2}+\frac{7}{5^3}+ ...... +\frac{2n-1}{5^{n-1}}$ ....(i)

$\left(\frac{1}{5}\right)S = \frac{1}{5}+\frac{3}{5^2}+\frac{5}{5^3}+ .... +\frac{2n-3}{5^{n-1}}+\frac{2n-1}{5^n}$ ....(ii)

(i) – (ii) $\Rightarrow$

$\frac{4}{5}S = 1+\frac{2}{5}+\frac{2}{5^2}+\frac{2}{5^3}+ .... +\frac{2}{5^{n-1}}-\frac{2n-1}{5^n}$.

$\frac{4}{5}S = 1+\frac{\frac{2}{5}\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{1-\frac{1}{5}}-\frac{2n-1}{5^n} = 1+\frac{2}{5}\times\frac{1}{4}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]-\frac{2n-1}{5^n} = \frac{3}{2}-\frac{1}{2}\times\frac{1}{5^{n-1}}-\frac{2n-1}{5^n}$

Example # 26 : Evaluate $1 - 3x + 5x^2 + 7x^3 + ... + \infty$, where $|x| \leq 1$.

Solution : $S_\infty = 1 - 3x + 5x^2 - 7x^3 + ... \infty$ ....(i)

$-xS_\infty = - x + 3x^2 - 5x^3 + ... \infty$ ....(ii)

(i) – (ii)

$S_\infty (1 + x) = 1 - 2x + 2x^2 - 2x^3 + ... \infty \quad = 1 + \dfrac{2(-x)}{[1-(-x)]}$

$\Rightarrow \qquad S_\infty = \dfrac{1}{1+x} - \dfrac{2x}{(1+x)^2} = \dfrac{1-x}{(1+x)^2}$

Example # 27: Evaluate $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + ... \infty$.

Solution : Let $S = 1^2 + 2^2x + 3^2x^2 + ... \infty$ (i)

$xS = \quad 1^2x + 2^2x^2 + ... \infty$ (ii)

S(i) – (ii)

$S(1 - x) = 1^2 + (2^2 - 1^2)\, x + (3^2 - 2^2)x^2 + (4^2 - 3^2)x^3 + ... \infty$.

$S(1 - x) = 1 + 3x + 5x^2 + 7x^3 + ... \infty$.

This is an infiuite A.G.P. = $\dfrac{1}{1-x} + \dfrac{2x}{(1-x)^2} = \dfrac{1+x}{(1-x)^2} \Rightarrow \quad S = \dfrac{1+x}{(1-x)^3}$

Self practice problems :

(17) Evaluate : $1.2 + 2.2^2 + 3.2^3 + ... + 100.\ 2^{100}$

(18) Evaluate : $1 + 3x + 6x^2 + 10x^3 + ...$ upto infinite term, where $|\, x \,| < 1$.

(19) Sum to n terms of the series : $1 + 2\left(1+\dfrac{1}{n}\right) + 3\left(1+\dfrac{1}{n}\right)^2 + ...$

Ans. (17) $99.2^{101} + 2$. (18) $\dfrac{1}{(1-x)^3}$ (19) n^2

11. Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.)

(ii) For n positive numbers $a_1, a_2, a_3, ...a_n$ A.M. $\geq$ G.M. $\geq$ H.M.

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. = $(a_1\, a_2\, a_3\, ...a_n)^{1/n}$ and their

H.M. = $\dfrac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}}$

It can be shown that

A.M. $\geq$ G.M. $\geq$ H.M. and equality holds at either places iff

$a_1 = a_2 = a_3 = ...= a_n$

Example # 28 : The A.M. of two numbers exceeds the G.M. by $\dfrac{3}{2}$ and the G.M. exceeds the H.M. by $\dfrac{6}{5}$; find the numbers.

Solution : Let the numbers be a and b, now using the relation

$G^2 = AH = \left(G+\frac{3}{2}\right)\left(G-\frac{6}{5}\right) = G^2 + \frac{3}{10}G - \frac{9}{5} \Rightarrow G = 6$

i.e. $ab = 36$ also $a + b = 15$

Hence the two numbers are 3 and 12.

Example # 29 : Prove that $e^x + e^{-x} \geq 2$

Solution : Using the relation A.M. ≥ G.M. we have

$\frac{e^x + e^{-x}}{2} \geq (e^x \times e^{-x})^{1/2} \Rightarrow e^x + e^{-x} \geq 2.$

Example # 30 : If $n > 0$, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution : Using the relation A.M. ≥ G.M. on the numbers $1, 2, 2^2, 2^3, \ldots\ldots\ldots, 2^{n-1}$, we have

$\frac{1+2+2^2+\ldots\ldots+2^{n-1}}{n} > (1.2.\,2^2.\,2^3.\,\ldots\ldots\ldots\,.2^{n-1})^{1/n}$

Equality does not hold as all the numbers are not equal.

$\Rightarrow \frac{2^n-1}{2-1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \Rightarrow 2^n - 1 > n.\,2^{\frac{(n-1)}{2}} \Rightarrow 2^n > 1 + n.\,2^{\frac{(n-1)}{2}}$

Example # 31 : Find the greatest value of xyz for positive value of x, y, z subject to the condition $xy + yz + zx = 12$.

Solution : Using the relation A.M. ≥ G.M.

$\frac{xy+yz+zx}{3} \geq (x^2\, y^2\, z^2)^{1/3} \Rightarrow 4 \geq (x\,y\,z)^{2/3} \Rightarrow xyz \leq 8$

Example # 32 : If a, b, c are in H.P. and they are distinct and positive, then prove that $a^n + c^n > 2b^n$

Solution : Let a^n and c^n be two numbers

then $\frac{a^n+c^n}{2} > (a^n\, c^n)^{1/2}$

$a^n + c^n > 2\,(ac)^{n/2}$ ....(i)

Also G.M. > H.M.

i.e. $\sqrt{ac} > b$, $(ac)^{n/2} > b^n$ ....(ii)

hence from (i) and (ii), we get $a^n + c^n > 2b^n$

Self practice problems :

(20) If a, b, c are real and distinct, then show that $a^2(1 + b^2) + b^2(1 + c^2) + c^2(1 + a^2) > 6abc$

(21) Prove that $n^n \geq 1\,.\,3\,.\,5\,\ldots\ldots\ldots(2n-1)$, $n \in N$

(22) If a, b, c, d be four distinct positive quantities in G.P., then show that

(i) $a + d > b + c$

(ii) $\frac{1}{ab} + \frac{1}{cd} > 2\left(\frac{1}{bd} + \frac{1}{ac} - \frac{1}{ad}\right)$

(23) Prove that ΔABC is an equilateral triangle iff $\tan A + \tan B + \tan C = 3\sqrt{3}$

(24) If $a, b, c > 0$, prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7\, a^4\, b^4\, c^4$

Results :

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$.

(ii) $\sum_{r=1}^{n} k\, a_r = k \sum_{r=1}^{n} a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k + \ldots\ldots\ldots$ n times $= nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Example # 33 : Find the sum of the series to n terms whose general term is 2n + 1.

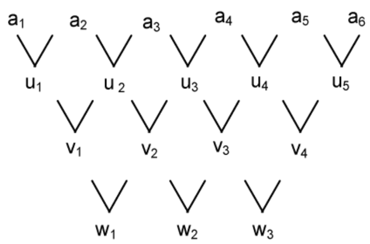
Solution : $S_n = \Sigma T_n = \sum(2n + 1) = 2\Sigma n + \Sigma 1 = \frac{2(n+1)n}{2} + n = n^2 + 2n$

Example # 34 : $T_k = k^2 + 2^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k = \frac{n(n+1)(2n+1)}{6} + \frac{2(2^n - 1)}{2-1} = \frac{n(n+1)(2n+1)}{6} + 2^{n+1} - 2$.

12. Method of difference for finding n^{th} term :

Let. $a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots\ldots$ be a sequence such that



Where $u_m = a_{m+1} - a_m$ is 1st difference $\forall m = 1, 2, 3\ldots\ldots$
$v_m = u_{m+1} - u_m$ is 2nd difference $\forall m = 1, 2, 3\ldots\ldots$
$w_m = v_{m+1} - v_m$ is 3nd difference $\forall m = 1, 2, 3\ldots\ldots$
and so on.

Case-I : If k^{th} difference are all equal then a_n = polynomial in n of degree k
i.e $a_n = b_0 n^k + b_1 n^{k-1} + \ldots\ldots + b_k$

Case-II : If k^{th} difference are in GP with common ratio $r (r \neq 1)$ then $a_n = \lambda r^{n-1}$ + polynomial in n of degree (k–1)

Example # 35 : Find the sum to n-terms 2 + 5 + 10 + 17 + 26 + ...

Solution : Let $S = 2 + 5 + 10 + 17 + 26 + \ldots\ldots + T_n$ ....(i)
$S = 2 + 5 + 10 + 17 + 26 + \ldots\ldots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒ $T_n = 2 + (3 + 5 + 7 + ... + (T_n - T_{n-1}))$

$= 2 + \frac{n-1}{2}[6 + (n-2)2]$

$= 2 + (n-1)(n+1)$

$= n^2 + 1$

Hence $S = \sum (n^2 + 1)$

$\frac{n(n+1)(2n+1)}{6} + n \Rightarrow n\left[\frac{2n^2+3n+7}{6}\right]$

Example # 36 : Find the nth term and the sum of n term of the series 2 + 12 + 36 + 80 + 150 + 252 + ...

Solution : Let $S = 2 + 12 + 36 + 80 + 150 + 252 + ...+T_n$ ....(i)

$S = \quad 2 + 12 + 36 + 80 + 150 + 252 + ...+T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒ $T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ ....(iii)

$T_n = \quad 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) ⇒ $T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ...$

$= [4 + (n-1)6] = n[3n-1] \quad T_n - T_{n-1} = 3n^2 - n$

∴ general term of given series is $\sum (T_n - T_{n-1}) = \sum (3n^2 - n) = n^3 + n^2$.

Hence sum of this series is

$S = \sum n^3 + \sum n^2$

$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{12}(3n^2 + 7n + 2) = \frac{1}{12} n(n+1)(n+2)(3n+1)$

Example # 37 : Find the sum to n-terms 1 + 2 + 5 + 14 + 41 + ...

Solution : Let $S = 1 + 2 + 5 + 14 + 41 + ...+ T_n$ ....(i)

$S = 1 + 2 + 5 + 14 + 41 + ...+ T_{n-1} + T_n$ ....(ii)

(i) – (ii) ⇒ $T_n = 1 + (1 + 3 + 9 + ... + T_n - T_{n-1})$

$T_n = 1 + \left(\frac{3^{n-1}-1}{3-1}\right)$

$T_n = \frac{3^{n-1}+1}{2}$

So $S = \sum T_n = (\Sigma 3^{n-1} + \Sigma 1)$

$= \frac{1}{2}\left[\frac{3^n-1}{2} + n\right] = \frac{3^n + 2n - 1}{4}$

13. Method of difference for finding s_n:

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$t_1 = f(1) - f(0)$,

$t_2 = f(2) - f(1)$,

$t_n = f(n) - f(n-1) \quad \Rightarrow \quad S_n = f(n) - f(0)$

Example # 38 : Find the sum of n-terms of the series 1.2.3 + 2.3.4 + 3.4.5 + ...

Solution : Let T_r be the general term of the series

So $T_r = r(r + 1)(r + 2)$.

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r + 3) - (r - 1)]$

so $T_r = \frac{r}{4}(r + 1)(r + 2)[(r + 3) - (r - 1)]$

$= \frac{1}{4}[r(r + 1)(r + 2)(r + 3) - (r - 1)r(r + 1)(r + 2)]$.

Let $f(r) = \frac{1}{4} r(r + 1)(r + 2)(r + 3)$

so $T_r = [f(r) - f(r - 1)]$.

Now $S = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$T_1 = \frac{1}{4}[1 . 2 . 3 . 4 - 0]$

$T_2 = \frac{1}{4}[2 . 3 . 4 . 5 - 1 . 2 . 3 . 4]$

$T_n = \frac{1}{4}[n(n+1)(n + 2)(n + 3) - (n - 1)n(n + 1)(n + 2)]$

$\therefore S = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$

Hence sum of series is $f(n) - f(0)$.

Example # 39 : Sum to n terms of the series $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \ldots\ldots$

Solution : Let T_r be the general term of the series

$T_r = \frac{1}{(1+rx)(1+(r+1)x)}$ So $T_r = \frac{1}{x}\left[\frac{(1+(r+1)x)-(1+rx)}{(1+rx)(1+(r+1)x)}\right] = \frac{1}{x}\left[\frac{1}{1+rx} - \frac{1}{1+(r+1)x}\right]$

$T_r = f(r) - f(r + 1)$ $\therefore$ $S = \sum T_r = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$= \frac{1}{x}\left[\frac{1}{1+x} - \frac{1}{1+(n+1)x}\right] = \frac{n}{(1+x)[1+(n+1)x]}$

Example # 40 : Sum to n terms of the series $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \ldots\ldots$

Solution : Let $T_r = \frac{r+3}{r(r+1)(r+2)} = \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)}$

$= \left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$

$\therefore$ $S = \left[\frac{1}{2} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{5}{4} - \frac{1}{n+2}\left[1 + \frac{3}{2(n+1)}\right]$

$= \frac{5}{4} - \frac{1}{2(n+1)(n+2)}[2n + 5]$

Example # 41 : Find the general term and sum of n terms of the series 9 + 16 + 29 + 54 + 103 + ...

Solution : Let $S = 9 + 16 + 29 + 54 + 103 + \ldots\ldots\ldots + T_n$ ....(i)

$S = \quad 9 + 16 + 29 + 54 + 103 + \ldots\ldots + T_{n-1} + T_n$ ....(ii)

(i) – (ii) $\Rightarrow T_n = 9 + 7 + 13 + 25 + 49 + \ldots\ldots\ldots\ldots\ldots + (T_n - T_{n-1})$ ....(iii)

$T_n = \quad 9 + 7 + 13 + 25 + 49 + \ldots\ldots\ldots + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ....(iv)

(iii) – (iv) $\Rightarrow T_n - T_{n-1} = 9 + (-2) + \underbrace{6 + 12 + 24 + \ldots\ldots}_{(n-2)\text{ terms}} = 7 + 6\,[2^{n-2} - 1] = 6(2)^{n-2} + 1.$

$\therefore$ General term is $T_n = 6(2)^{n-1} + n + 2$

Also sum $S = \sum T_n$

$$= 6\sum 2^{n-1} + \sum n + \sum 2 = 6 . \frac{(2^n - 1)}{2 - 1} + \frac{n\,(n+1)}{2} + 2n = 6(2^n - 1) + \frac{n(n+5)}{2}$$

Self practice problem :

(25) Sum to n terms the following series

(i) $4 + 14 + 30 + 52 + 80 + 114 + \ldots\ldots\ldots$

(ii) $2 + 5 + 12 + 31 + 86 + \ldots\ldots\ldots\ldots\ldots$

(iii) $\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + \ldots\ldots\ldots$

(iv) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots\ldots\ldots$

(v) $1 . 5 . 9 + 2 . 6 . 10 + 3 . 7 . 11 + \ldots\ldots\ldots$

Ans. (25) (i) $n(n + 1)^2$ (ii) $\frac{3^n + n^2 + n - 1}{2}$

(iii) $\frac{2n}{n+1}$ (iv) $\frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$

(v) $\frac{n}{4}\,(n + 1)\,(n + 8)\,(n + 9)$