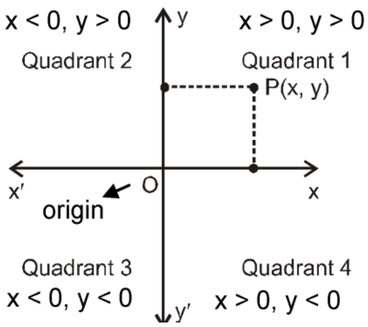
Straight Line

The knowledge of which geometry aims is the knowledge of the eternal...... Plato

1. Rectangular cartesian co-ordinate system :



2. Distance Formula :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is = $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.

Example # 1 : Find the value of x, if the distance between the points (x, –1) and (3, 2) is 5

Solution : Let P(x ,–1) and Q(3, 2) be the given points. Then PQ = 5 (given)

$\sqrt{(x-3)^2+(-1-2)^2} = 5 \Rightarrow (x-3)^2 + 9 = 25$

$\Rightarrow x = 7$ or $x = -1$

Self Practice Problem :

(1) Show that four points (0, –1), (6, 7) (–2, 3) and (8, 3) are the vertices of a rectangle.

3. Section Formula :

If P(x, y) divides the line joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then;

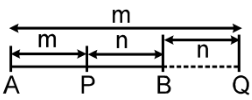
$x = \frac{mx_2 + nx_1}{m+n}$; $y = \frac{my_2 + ny_1}{m+n}$.

Notes :

(i) If $\frac{m}{n}$ is positive, the division is internal, but if $\frac{m}{n}$ is negative, the division is external.

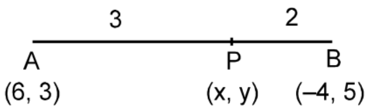
(ii) If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically, $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.



Example # 2 : Find the co-ordinates of the point which divides the line segment joining the points (6, 3) and (– 4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.

Solution : Let P (x, y) be the required point.

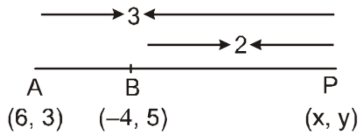


(i) For internal division :

$x = \dfrac{3 \times -4 + 2 \times 6}{3 + 2}$ and $y = \dfrac{3 \times 5 + 2 \times 3}{3 + 2}$ or x = 0 and $y = \dfrac{21}{5}$

So the co-ordinates of P are $\left(0, \dfrac{21}{5}\right)$

(ii) For external division

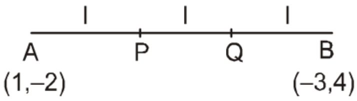


$x = \dfrac{3 \times -4 - 2 \times 6}{3 - 2}$ and $y = \dfrac{3 \times 5 - 2 \times 3}{3 - 2}$ or x = –24 and y = 9

So the co-ordinates of P are (–24, 9)

Example # 3 : Find the co-ordinates of points which trisect the line segment joining (1, – 2) and (5, 6).

Solution : Let A (1, –2) and B(–3, 4) be the given points. Let the points of trisection be P and Q. Then AP = PQ = QB = λ (say)



∴ PB = PQ + QB = 2λ and AQ = AP + PQ = 2λ

⇒ AP : PB = λ : 2λ = 1 : 2 and AQ : QB = 2λ : λ = 2 : 1

So P divides AB internally in the ratio 1 : 2 while Q divides internally in the ratio 2 : 1

∴ the co-ordinates of P are $\left(\dfrac{1 \times 5 + 2 \times 1}{1 + 2}, \dfrac{1 \times 6 + 2 \times -2}{1 + 2}\right)$ or $\left(\dfrac{7}{3}, \dfrac{2}{3}\right)$

and the co-ordinates of Q are $\left(\dfrac{2 \times 5 + 1 \times 1}{2 + 1}, \dfrac{2 \times 6 + 1 \times (-2)}{2 + 1}\right)$ or $\left(4, \dfrac{10}{2}\right)$

Hence, the points of trisection are $\left(\dfrac{7}{3}, \dfrac{2}{3}\right)$ and $\left(4, \dfrac{10}{3}\right)$

Self Practice Problems :

(2) In what ratio does the point (–1, –1) divide the line segment joining the points (4, 4) and (7, 7).

(3) The three vertices of a parallelogram taken in order are (–1, 0), (3, 1) and (2, 2) respectively. Find the co-ordinates of the fourth vertex.

Ans. (2) 5 : 8 externally (3) (–2, 1)

4. The ratio in which a given line divides the line segment joining two points :

Let the given line ax + by + c = 0 divide the line segment joining A(x_1, y_1) & B(x_2, y_2) in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then m/n is negative but if A & B are on opposite sides of the given line, then m/n is positive

Example # 4 : Find the ratio in which the line joining the points A (1, 2) and B(– 3, 4) is divided by the line x + y – 5 = 0.

Solution : Let the line x + y = 5 divides AB in the ratio k : 1 at P

∴ co-ordinate of P are $\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$

Since P lies on x + y – 5 = 0

∴ $\frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0$ ⇒ $k = -\frac{1}{2}$

∴ Required ratio is 1 : 2 extrenally.

Aliter Let the ratio is m : n

∴ $\frac{m}{n} = -\frac{(1\times1+1\times2-5)}{1\times(-3)+1\times4-5} = -\frac{1}{2}$ ∴ ratio is 1 : 2 externally. **Ans.** $\frac{18}{5}$

5. Area of a Triangle :

If A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) are the vertices of triangle ABC, then its area is equal to

Δ ABC = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense. The above formula will give a $(-)$ve area if the vertices (x_i, y_i), i = 1, 2, 3 are placed in the clockwise sense.

Note: Area of n-sided polygon formed by points (x_1, y_1) ; (x_2, y_2); ....;(x_n, y_n) is given by

$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots\ldots\ldots\ldots \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right)$.

Here vertices are taken in order.

Example # 5 : If the co-ordinates of two points A and B are (3, 4) and (5, –2) respectively. Find the co-ordinates of any point P if PA = PB and Area of ΔPAB = 10.

Solution : Let the co-ordinates of P be (x, y). Then

PA = PB ⇒ $PA^2 = PB^2$

⇒ $(x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$ ⇒ x – 3y – 1 = 0

Now, Area of ΔPAB = 10 ⇒ $\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix}$ = ± 10 ⇒ 6x + 2y – 26 = ± 20

⇒ 6x + 2y – 46 = 0 or 6x + 2y – 6 = 0 ⇒ 3x + y – 23 = 0 or 3x + y – 3 = 0

Solving x – 3 y – 1 = 0 and 3x + y – 23 = 0 we get x = 7, y = 2. Solving x – 3y – 1 = 0 and 3x + y – 3 = 0, we get x = 1, y = 0. Thus, the co-ordinates of P are (7, 2) or (1, 0)

Self Practice problems :

(4) The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on y = x + 3. Find the third vertex.

(5) The vertices of a quadrilateral are (6, 3), (–3, 5), (4, –2) and (x, 3x) and are denoted by A, B, C and D, respectively. Find the values of x so that the area of triangle ABC is double the area of triangle DBC.

Ans. (4) $\left(\frac{7}{2}, \frac{13}{2}\right)$ or $\left(-\frac{3}{2}, \frac{3}{2}\right)$ (5) $x = \frac{11}{8}$ or $-\frac{3}{8}$

6. Locus :

If a point moves under given condition or conditions, equation of the path traced by the point is called its locus.

Example # 6 : Find the equation to the locus of a point which moves so that

(i) Its distance from the point (a, 0) is always four times its distance from the axis of y.

(ii) Sum of the squares of its distances from the axes is equal to 3.

(iii) Its distance from x-axis is 3 times of its distance from y-axis.

Solution : Let the point be (h, k)

Distance of P from axis of y = |h|

Distance of P from (a, 0) = $\sqrt{(h-a)^2 + k^2}$

$\Rightarrow \sqrt{(h-a)^2 + k^2} = 4|h|$ $\Rightarrow$ $(h - a)^2 + k^2 = 16h^2$

$\Rightarrow h^2 - 2ah + a^2 + k^2 = 16h^2$ $\Rightarrow 15h^2 - k^2 + 2ah = a^2$

hence locus of P is $15x^2 - y^2 + 2ax = a^2$

(ii) Let the point be (h, k)

Distance of P from y-axis = |h|

Distance of P from x-aixs = |k|

$h^2+k^2 = 3$

hence Locus of P is $x^2+y^2 = 3$

(iii) Let the point be P(h, k)

Distance from x-axis = |k|

Distance from y-aixs = |h|

|k| = 3|h|

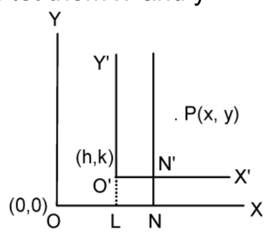
3h – k = 0 or 3h + k = 0

3x – y = 0 or 3x + y = 0

7. Shifting of origin :

Let O' be (h, k) at which we want to shift origin

Let P(x, y) be any point in the plane of the paper, and let its coordinates, referred to the original axes, be x and y and referred to the new axes let them x' and y'



The origin is therefore, transferred to the point (h,k) when we substitute for the coordinates x and y the quantities.

$x' = x - h$ and $y' = y - k$

The above article is true whether the axes be oblique or rectangular.

Example # 7 : Find the new coordinates of point (3,–4) if the origin is shifted to (1,2) by translation.

Solution : Since origin is shifted to x = 1 and y = 2

Hence $x - 1 = X$ and $y - 2 = Y$

Point (3,–4) is shifted to

$X = 3 - 1$, $Y = -4 - 2$

$X = 2$, $Y = -6$

Hence new coordinates are (2,–6)

Example # 8 : Find the newly transformed equation of the straight line $2x - 3y + 5 = 0$ is origin is shifted to (3, –1)

Solution : If origin is shifted (3,–1) $x - 3 = X, y - (-1) = Y$

$x = 3 + X$, $y = Y - 1$

New equation is $2(X + 3) - 3(Y - 1) + 5 = 0$

$2X - 3Y + 14 = 0$

Example # 9 : Find the point at which the origin be shifted so that the equation $x^2 + y^2 - 5x + 2y - 5 = 0$ has no first degree term.

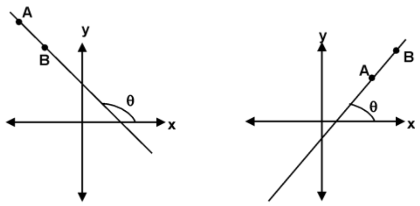
Solution : $x^2 + y^2 - 5x + 2y - 5 = 0 \Rightarrow \left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} + (y^2 + 2y + 1) - 1 - 5 = 0$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 + (y + 1)^2 - \frac{49}{4} = \theta$ $\Rightarrow X^2 + Y^2 = \frac{49}{4}$

If new equation to be formed has no first degree terms then shift the origin to

$x = \frac{5}{2}$, $y = -1 = \left(\frac{5}{2}, -1\right)$

8. <u>Slope Formula</u> :



Slope (m) = tanθ, where θ is the angle measured from positive x-axis.

If A (x_1, y_1) & B (x_2, y_2), $x_1 \neq x_2$, are points on a straight line, then the slope m of the line is given by

$m = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$.

Example # 10: What is the slope of a line whose inclination with the positive direction of x-axis is :

(i) 0º (ii) 90º (iii) 120º (iv) 150º

Solution : (i) Here $\theta = 0º$

Slope = $\tan\theta = \tan 0º = 0$.

(ii) Here $\theta = 90º$

∴ The slope of line is not defined.

(iii) Here $\theta = 120^\circ$

∴ Slope = $\tan\theta = \tan 120^\circ = \tan(180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$.

(iv) Here $\theta = 150^\circ$

∴ Slope = $\tan\theta = \tan 150^\circ = \tan(180^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$.

Example # 11: Find the slope of the line passing through the points :
(i) (1, 6) and (– 4, 2) (ii) (5, 9) and (2, 9)

Solution : (i) Let A = (1, 6) and B = (– 4, 2)

∴ Slope of AB = $\frac{2-6}{-4-1} = \frac{-4}{-5} = \frac{4}{5}$ $\left(\text{Using slope} = \frac{y_2 - y_1}{x_2 - x_1}\right)$

(ii) Let A = (5, 9), B = (2, 9)

∴ Slope of AB = $\frac{9-9}{2-5} = \frac{0}{-3} = 0$

Self practice problems :

(6) Find the value of x, if the slope of the line joining (1, 5) and (x, –7) is 4.

(7) What is the inclination of a line whose slope is

(i) 0 (ii) 1 (iii) –1 (iv) $-1/\sqrt{3}$

Ans. (6) – 2 (7) (i) 0º, (ii) 45º, (iii) 135º, (iv) 150º

9. Condition of collinearity of three points :

Points A (x_1, y_1), B (x_2, y_2), C(x_3, y_3) are collinear if

(i) $m_{AB} = m_{BC} = m_{CA}$ i.e. $\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \left(\frac{y_2 - y_3}{x_2 - x_3}\right)$

(ii) Δ ABC = 0 i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) AC = AB + BC or AB ~ BC

(iv) A divides the line segment BC in some ratio with respect to x-coordinates and y-coordinates.

Example # 12 : Show that the points (1, 1), (2, 3) and (3, 5) are collinear.

Solution : Let (1, 1) (2, 3) and (3, 5) be the co-ordinates of the points A, B and C respectively.

Slope of AB = $\frac{3-1}{2-1} = 2$ and Slope of BC = $\frac{5-3}{3-2} = 2$

∴ Slope of AB = slope of BC ∴ AB & BC are parallel

∴ A, B, C are collinear because B is on both lines AB and BC.

Self practice problem :

(8) Prove that the points (a, 0), (0, b) and (1, 1) are collinear if $\frac{1}{a} + \frac{1}{b} = 1$

10. Equation of a Straight Line in various forms :

(i) General Form : ax + by + c = 0 is the equation of a straight line in the general form

In this case, slope of line = $-\frac{a}{b}$

x - intercept = $-\frac{c}{a}$, y - intercept = $-\frac{c}{b}$

(ii) **Point - Slope form :** $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

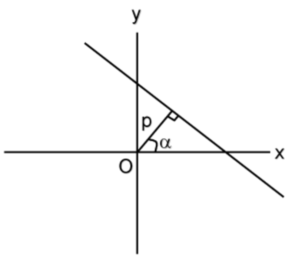
(iii) **Slope-intercept form :** $y = mx + c$ is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis.

(iv) **Two point form :** $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

(v) **Determinant form :** Equation of line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(vi) **Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

(vii) **Perpendicular/Normal form :** $x\cos\alpha + y\sin\alpha = p$ (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x–axis.



(viii) **Parametric form of straight line :** P (r) = (x, y) = $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ or $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ is the equation of the line in parametric form, where 'r' is the parameter whose absolute value is the distance of any point (x, y) on the line from the fixed point (x_1, y_1) on the line.

Remark : The above form is derived from point-slope form of line.

$y - y_1 = m(x - x_1)$ where $m = \tan\theta$ $\Rightarrow$ $y - y_1 = \frac{\sin\theta}{\cos\theta}(x - x_1)$.

Example # 13: Find the equation of a line passing through (2, –3) and inclined at an angle of 135º with the positive direction of x-axis.

Solution : Here, m = slope of the line = tan 135º = tan (90º + 45º) = – cot 45º = –1, $x_1 = 2$, $y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or $y + 3 = -x + 2$ or $x + y + 1 = 0$

Example # 14 : Find the equation of a line with slope –1 and cutting off an intercept of 4 units on positive direction of y-axis.

Solution : Here m = –1 and c = 4. So, the equation of the line is y = mx + c

i.e. $y = -x + 4$ or $x + y - 4 = 0$

Example # 15 : Find the equation of the line joining the points (– 1, 3) and (4, – 2)

Solution : Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{3-(-2)}{-1-4}(x + 1) \quad \Rightarrow \quad y - 3 = -x - 1 \quad \Rightarrow \quad x + y - 2 = 0$$

Example #16 : Find the equation of line passing through (2, 4) & (– 1, 3).

Solution : $\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad x - 3y + 10 = 0$

Example #17 : Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

Solution : Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ....(i)

This passes through (3, 4), therefore $\frac{3}{a} + \frac{4}{b} = 1$ ....(ii)

It is given that a + b = 14 ⇒ b = 14 – a.

Putting b = 14 – a in (ii), we get $\frac{3}{a} + \frac{4}{14-a} = 1$

⇒ $a^2 - 13a + 42 = 0$ ⇒ $(a - 7)(a - 6) = 0$ ⇒ a = 7, 6

For a = 7, b = 14 – 7 = 7 and for a = 6, b = 14 – 6 = 8.

Putting the values of a and b in (i), we get the equations of the lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or $x + y = 7$ and $4x + 3y = 24$

Example # 18 : Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30º with the positive direction of the x-axis.

Solution : Here p = 3, α = 30º

∴ Equation of the line in the normal form is

$x \cos 30^\circ + y \sin 30^\circ = 3$ or $x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3$ or $\sqrt{3}x + y = 6$

Example # 19 : Find slope, x-intercept & y-intercept of the line 2x – 3y + 5 = 0.

Solution : Here, a = 2, b = – 3, c = 5 ∴ slope $= -\frac{a}{b} = \frac{2}{3}$

x-intercept $= -\frac{c}{a} = -\frac{5}{2}$ ⇒ y-intercept $= \frac{5}{3}$

Example # 20 : Find the equation of the line through the point A(2, 3) and making an angle of 45º with the x-axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0

Solution : The equation of a line through A and making an angle of 45º with the x-axis is

$\frac{x-2}{\cos 45^o} = \frac{y-3}{\sin 45^o}$ or $\frac{\frac{x-2}{1}}{\sqrt{2}} = \frac{\frac{y-3}{1}}{\sqrt{2}}$ or $x - y + 1 = 0$

Suppose this line meets the line x + y + 1 = 0 at P such that AP = r. Then the co-ordinates of P are given by

$\frac{x-2}{\cos 45^o} = \frac{y-3}{\sin 45^o} = r \Rightarrow x = 2 + r\cos 45^o,\ y = 3 + r\sin 45^o$

$\Rightarrow x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$

Thus, the co-ordinates of P are $\left(2+\frac{r}{\sqrt{2}},\ 3+\frac{r}{\sqrt{2}}\right)$

Since P lies on x + y + 1 = 0, so $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2} \Rightarrow$ length AP = | r | = $3\sqrt{2}$

Thus, the length of the intercept = $3\sqrt{2}$.

Self practice problems :

(9) Find the equations of the sides of the triangle whose vertices are (–1, 8), (4, –2) and (–5, –3). Also find the equation of the median through (–1, 8)

(10) Find the equation of the passing through (– 2, 3) & (– 1, – 1).

(11) Find the equation of the line through (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.

(12) The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150º with the positive direction of y-axis. Find the equation of the line.

(13) Find the slope, x-intercept & y-intercept of the line 3x – 5y – 8 = 0.

(14) A straight line is drawn through the point A$\left(\sqrt{3},\ 2\right)$ making an angle of π/6 with positive direction of the x-axis. If it meets the straight line $\sqrt{3x}$ – 4y + 8 = 0 in B, find the distance between A and B.

Ans.

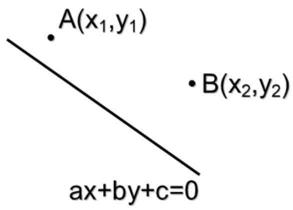
(9) 2x + y – 6 = 0, x – 9y – 22 = 0, 11x – 4y + 43 = 0, 21x + y + 13 = 0

(10) 4x + y + 5 = 0 (11) 3x + 2y = 12.

(12) $\sqrt{3}$ x + y – 14 = 0 (13) $\frac{3}{5}, \frac{8}{3}, -\frac{8}{5}$ (14) 6 units

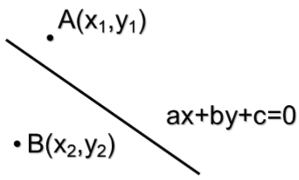
11. Position of the point (x_1, y_1) relative to the line ax + by + c = 0 :

(i)



$$\frac{ax_1+by_1+c}{ax_2+by_2+c}>0$$

(ii)



$$\frac{ax_1+by_1+c}{ax_2+by_2+c}<0$$

Example # 21 : Show that (1, 4) and (–8, –4) lie on the opposite sides of the line x + 3y + 7 = 0.

Solution : At (1, 4), the value of x + 3y + 7 = 1 + 3(4) + 7 = 20 > 0.
At (–8, – 4), the value of x + 3y + 7 = – 8 – 12 + 7 < 0
∴ The points (1, 4) and (–8, – 4) are on the opposite sides of the given line.

Self practice problems :

(15) Are the points (3, – 4) and (2, 6) on the same or opposite side of the line 3x – 4y = 8 ?

(16) Which one of the points (1, 1), (–1, 2) and (2, 3) lies on the side of the line 4x + 3y – 5 = 0 on which the origin lies?

Ans. (15) Opposite sides (16) (–1, 2)

12. Linear Inequalities :

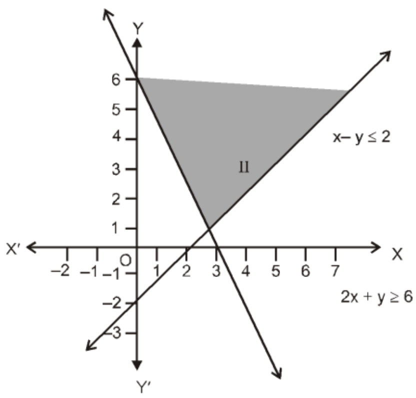
Linear inequalities ax+by+c>0 or ax+by+c<0 represents region above or below the line ax+by+c=0, which can be divided by taking one particular point in the region.

Example # 22 : Solve the following system of linear inequalities graphically.

2x + y ≥ 6 ....(1)

x – y ≤ 2 ....(2)

Solution : The graph of linear equation 2x + y = 6
is drawn in fig. We note that solution of inequality (1) is represented by the shaded region above the line 2x + y = 6, including the point on the line
On the same set of axes, we draw graph of the equation x – y = 2 as shown in fig. Then we note that inequality (2) represents the shaded region above



the line $x - y = 2$ including the points on the line.

Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities.

Example # 23 : Solve the following system of inequalities graphically

$x + 2y \leq 8$ .....(1)

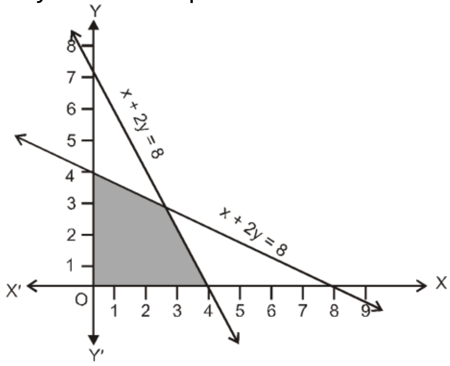
$2x + y \leq 8$ .....(2)

$x \geq 0$ .....(3)

$y \geq 0$ .....(4)

Solution : We draw the graphs of the lines $x + 2y = 8$ and $2x + y = 8$. The inequality (1) and (2) represent the region below the two lines, including the point on the respective lines

Since $x \geq 0$, $y \geq 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequalities.



13. Angle between two straight lines in terms of their slopes :

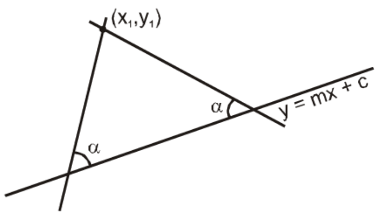
If m_1 & m_2 are the slopes of two intersecting straight lines ($m_1 m_2 \neq -1$) & θ is the acute angle between them, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Notes : (i) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0; L_2 = 0; L_3 = 0$ where $m_1 > m_2 > m_3$, then the tangent of interior angles of the Δ ABC formed by these lines are given by,

$\tan A = \dfrac{m_1 - m_2}{1 + m_1 m_2}$; $\tan B = \dfrac{m_2 - m_3}{1 + m_2 m_3}$ & $\tan C = \dfrac{m_3 - m_1}{1 + m_3 m_1}$

(ii) The equation of lines passing through point (x_1, y_1) and making angle α with the line $y = mx + c$ are given by :

$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.



Example # 24 : The acute angle between two lines is $\pi/4$ and slope of one of them is 1/2. Find the slope of the other line.

Solution : If θ be the acute angle between the lines with slopes m_1 and m_2, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $\theta = \frac{\pi}{4}$ and $m_1 = \frac{1}{2}$ $\therefore$ $\tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2}m_2}\right|$

$\Rightarrow$ $1 = \left|\frac{1-2m_2}{2+m_2}\right|$ $\Rightarrow$ $\frac{1-2m_2}{2+m_2} = +1$ or -1

Now $\frac{1-2m_2}{2+m_2} = 1$ $\Rightarrow$ $m_2 = -\frac{1}{3}$ and $\frac{1-2m_2}{2+m_2} = -1$ $\Rightarrow$ $m_2 = 3$.

$\therefore$ The slope of the other line is either – 1/3 or 3

Example # 25 : Find the equation of the straight line which passes through the origin and making angle 60º with the line $x - \sqrt{3}\,y = 0$

Solution : Given line is $x - \sqrt{3}\,y = 0$. $\Rightarrow$ $\tan\theta = \frac{1}{\sqrt{3}}$ $\Rightarrow \theta = 30^\circ$

so, α can be 90º or 30º – 60º = –30º

so line can be $x = 0$ or $y = -\frac{1}{\sqrt{3}}x$

Self practice problem :

(17) A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is x + y = 2. Find the equation of the other sides of the triangle.

Ans. (17) $(2 - \sqrt{3})x - y + 2\sqrt{3} - 1 = 0$ and $(2 + \sqrt{3})\,x - y - 2\sqrt{3} - 1 = 0$.

14. Parallel Lines :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where 'd' is a parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

15. Perpendicular Lines :

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form $y = -\frac{1}{m}x + d$, where 'd' is any parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where 'k' is any parameter.

Example # 26 : Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line 3x + 2y + 5 = 0

Solution : The equation of a line perpendicular to 3x + 2y + 5 = 0 is

$$2x - 3y + \lambda = 0 \qquad \text{...........(i)}$$

This passes through the point (3, 4)

$\therefore \quad 3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting λ = 6 in (i), we get 2x – 3y + 6 = 0, which is the required equation.

Aliter The slope of the given line is –3/2. Since the required line is perpendicular to the given line. So, the slope of the required line is 2/3. As it passes through (3, 4). So, its equation is $y - 4 = \frac{2}{3}(x - 3)$ or 2x – 3y + 6 = 0

Self practice problems :

(18) Find the equation of the perpendicular bisector of the line segment joining the points A(2, 3) and B (6, –5).

(19) Find the equation of a straight line which cuts off an intercept of length 3 on y-axis and is parallel to the line joining the points (3, –2) and (1, 4).

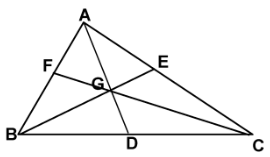
(20) The vertices of a triangle are A(10, 4), B (–4, 9) and C(–2, –1). Find the equation of its altitudes. Also find their intersecting point.

Ans. (18) x – 2 y – 6 = 0 (19) 3x + y – 3 = 0

(20) x – 5y + 10 = 0, 12x + 5y + 3 = 0, 14x – 5y + 23 = 0, $\left(-1, \frac{9}{5}\right)$.

16. Centroid, Circumcentre, Orthocentre, Incentre & Excentre :

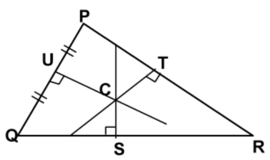
(i) Centroid : Intersecting point of medians



where D, E, and F are mid points of sides

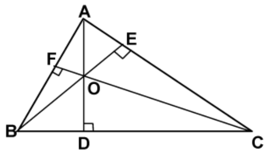
Centroid G ≡ $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

(ii) Circumcentre : Intersecting point of perpendicular bisectors of sides of triangle

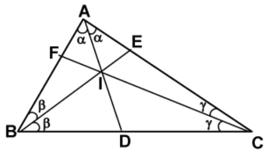


where S, T, and U are mid points of sides

(iii) Orthocentre : Intersecting point of altitudes



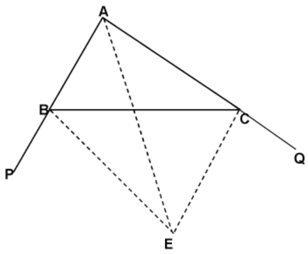
(iv) **Incentre :** Intersecting point of angle bisectors



Incentre I $\equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$, where a, b and c are sides BC, AC and AB respectively.

Also, $\frac{AI}{ID} = \frac{b+c}{a}$, $\frac{BI}{IE} = \frac{a+c}{b}$ and $\frac{CI}{IF} = \frac{a+b}{c}$

(v) **Ex-centre :** Intersecting point of an internal angle bisector and externel angle bisectors of other two vertices.



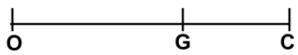
$\angle BAE = \angle CAE$; $\angle CBE = \angle PBE$; $\angle BCE = \angle QCE$

Excentre (to A) $I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$ and so on, where a, b and c are sides BC, AC and AB respectively

Properties :

(i) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.

(ii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2 : 1.



$$\frac{OG}{GC} = \frac{2}{1}$$

(iii)

Type of triangle	Location of special point(s)
Isosceles	G, O, I & C lie on the same line
Equilateral	G, O, I & C coincide
Right angled	Orthocentre is at right angled vertex and circumcentre is mid point of hypotenuse
Obtuse angled	Circumcentre and orthocentre both are out side the triangle.

Example # 27 : Find the co-ordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 0), (6, 0) and (0, 8).

Solution : (i) We know that the co-ordinates of the centroid of a triangle whose angular points are $(x_1, y_1), (x_2, y_2)\ (x_3, y_3)$ are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

So the co-ordinates of the centroid of a triangle whose vertices are (0, 0), (6, 0) and (0, 8) are $\left(\frac{0+0+6}{3}, \frac{0+0+8}{3}\right)$ or $\left(2, \frac{8}{3}\right)$.

(ii) Let A (0, 0), B (6, 0) and C(0, 8) be the vertices of triangle ABC.

Then c = AB = $\sqrt{(0-8)^2+(6-0)^2}$ = 10, b = CA = 8

and a = BC = 6

The co-ordinates of the in-centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

or $\left(\frac{10\times0+6\times8+0\times6}{10+8+6}, \frac{10\times0+6\times0+8\times6}{10+8+6}\right)$ or (2, 2)

Self practice problems :

(21) Find the co-ordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also find its circumradius.

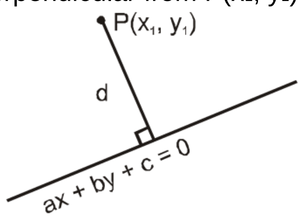
(22) Two vertices of a triangle are (3, –5) and (–7, 4). If the centroid is (2, –1), find the third vertex.

(23) Find the co-ordinates of the centre of the circle inscribed in a triangle whose vertices are (– 36, 7), (20, 7) and (0, – 8)

Ans. (21) (5, 2), 5 (22) (10, – 2) (23) (–1, 0)

17. Length of perpendicular from a point to a line :

The length of perpendicular from $P(x_1, y_1)$ on ax + by + c = 0 is $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$.



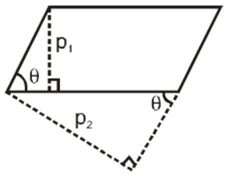
18. Distance between parallel lines :

The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is = $\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$.

Note that coefficients of x & y in both the equations must be same.

19. Area of the parallelogram :

The area of the parallelogram = $\frac{p_1p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.



Example # 28 : Find the distance between the line 12x – 5y + 9 = 0 and the point (0, 0)

Solution : The required distance = $\left|\dfrac{12\times 0 - 5\times 0 + 9}{\sqrt{12^2 + (-5)^2}}\right| = \dfrac{9}{13}$

Example # 29 : Two sides of a square lie on the lines x + y = 1 and x + y + 7 = 0. What is its area ?

Solution : Clearly the length of the side of the square is equal to the distance between the parallel lines

x + y – 1 = 0 ....(i) and x + y + 2 = 0 ....(ii)

Putting x = 0 in (i), we get y = 1. So (0, 1) is a point on line (i).

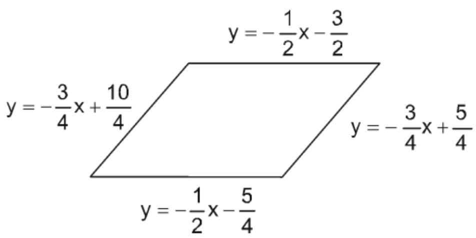
Now, Distance between the parallel lines

= length of the ⊥ from (0, 1) to x + y + 2 = 0 = $\dfrac{|0+1+7|}{\sqrt{1^2+1^2}} = \dfrac{8}{\sqrt{2}}$

Thus, the length of the side of the square is $\dfrac{8}{\sqrt{2}}$ and hence its area = $\left(\dfrac{8}{\sqrt{2}}\right)^2$ = 32

Example # 30 : Find the area of the parallelogram whose sides are x + 2y + 3 = 0, 3x + 4y – 5 = 0, 2x + 4y + 5 = 0 and 3x + 4y – 10 = 0

Solution :



Here, $c_1 = -\dfrac{3}{2}$, $c_2 = -\dfrac{5}{4}$, $d_1 = \dfrac{10}{4}$, $d_2 = \dfrac{5}{4}$, $m_1 = -\dfrac{1}{2}$, $m_2 = -\dfrac{3}{4}$

∴ Area = $\left|\dfrac{\left(-\frac{3}{2}+\frac{5}{4}\right)\left(\frac{10}{4}-\frac{5}{4}\right)}{\left(-\frac{1}{2}+\frac{3}{4}\right)}\right| = \dfrac{5}{4}$ sq. units

Self practice problem :

(24) Find the length of the altitudes from the vertices of the triangle with vertices :(–1, 1), (5, 2) and (3, –1).

(25) Find the area of parallelogram whose sides are given by 4x – 5y + 1 = 0, x – 3y – 6 = 0, 4x – 5y – 2 = 0 and 2x – 6y + 5 = 0

Ans. (24) $\dfrac{16}{\sqrt{13}}, \dfrac{8}{\sqrt{5}}, \dfrac{16}{\sqrt{37}}$ (25) $\dfrac{51}{14}$ sq. units

20. Foot of perpendicular, reflection and image of a point about a line :

(i) Foot of the perpendicular from a point (x_1, y_1) on the line ax + by + c = 0 is

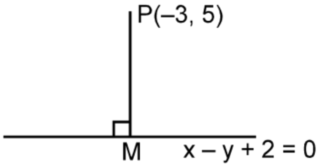
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

(ii) The image of a point (x_1, y_1) about the line ax + by + c = 0 is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right).$$

Example # 31 : Find the foot of perpendicular of the line drawn from P (– 3, 5) on the line x – y + 2 = 0.

Solution :



Slope of PM = – 1

∴ Equation of PM is

x + y – 2 = 0 ....(i)

solving equation (i) with x – y + 2 = 0, we get co-ordinates of M (0, 2)

Aliter Here, $\frac{x+3}{1} = \frac{y-5}{-1} = -\frac{(1\times(-3)+(-1)\times 5+2)}{(1)^2+(-1)^2}$

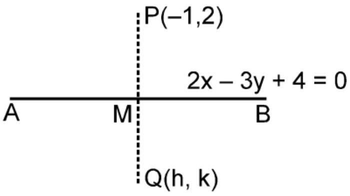
⇒ $\frac{x+3}{1} = \frac{y-5}{-1} = 3$ ⇒ x + 3 = 3 ⇒ x = 0 and y – 5 = – 3

⇒ y = 2

∴ M is (0, 2)

Example # 32 : Find the image of the point P(–1, 2) in the line mirror 2x – 3y + 4 = 0.

Solution :



Let image of P is Q.

∴ PM = MQ & PQ ⊥ AB

Let Q is (h, k) ∴ M is $\left(\frac{h-1}{2}, \frac{k+2}{2}\right)$

It lies on 2x – 3y + 4 = 0.

∴ $2\left(\frac{h-1}{2}\right) - 3\left(\frac{k+2}{2}\right) + 4 = 0.$ or 2h – 3k = 0 ....(i) slope of PQ = $\frac{k-2}{h+1}$

PQ $\perp$ AB

$\therefore$ $\frac{k-2}{h+1} \times \frac{2}{3} = -1.$ $\Rightarrow$ $3h + 2k - 1 = 0$. ....(ii)

soving (i) & (ii), we get $h = \frac{3}{13}$, $k = \frac{2}{13}$

$\therefore$ Image of P(– 1, 2) is Q$\left(\frac{3}{13}, \frac{2}{13}\right)$

Aliter The image of P (– 1, 2) about the line 2x – 3y + 4 = 0 is $\frac{x+1}{2} = \frac{y-2}{-3} = -2\,\frac{[2(-1)-3(2)+4]}{2^2+(-3)^2}$

$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13} \Rightarrow 13x + 13 = 16 \Rightarrow x = \frac{3}{13}$ & $13y - 26 = -24$

$\Rightarrow y = \frac{2}{13}$ $\therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

Self practice problems :

(26) Find the foot of perpendicular of the line drawn from (– 2, – 3) on the line 3x – 2y – 1 = 0.

(27) Find the image of the point (1, 2) in y-axis.

Ans. (26) $\left(\frac{-23}{13}, \frac{-41}{13}\right)$ (27) (– 1, 2)

21. Bisectors of the angles between two lines :

Equations of the bisectors of angles between the lines ax + by + c = 0 & a'x + b'y + c' = 0 (ab' ≠ a'b) are : $\frac{ax + by + c}{\sqrt{a^2+b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$

Note : Equation of straight lines passing through P(x_1, y_1) & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Example # 33 : Find the equations of the bisectors of the angle between the straight lines 3x – 4y + 7 = 0 and 12 x – 5y – 8 = 0.

Solution : The equations of the bisectors of the angles between 3x – 4y + 7 = 0 and 12 x – 5y – 8 = 0 are

$\frac{3x-4y+7}{\sqrt{3^2+(-4)^2}} = \pm \frac{12x-5y-8}{\sqrt{12^2+(-5)^2}}$ or $\frac{3x-4y+7}{5} = \pm \frac{12x-5y-8}{13}$

or 39x – 52y + 91 = ± (60 x – 25 y – 40)

Taking the positive sign, we get 21 x + 27 y – 131 = 0 as one bisector

Taking the negative sign, we get 99 x – 77 y + 51 = 0 as the other bisector.

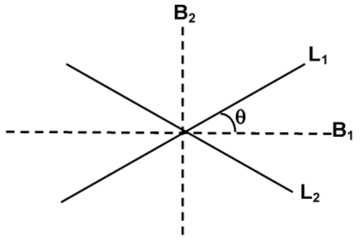
Self practice problem :

(28) Find the equations of the bisectors of the angles between the following pairs of straight lines 3x + 4y + 13 = 0 and 12x – 5y + 32 = 0

Ans. 21x – 77y – 9 = 0 and 99x + 27y + 329 = 0

22. Methods to discriminate between the acute angle bisector & the obtuse angle bisector :

(i)

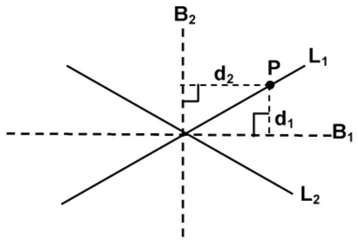


L_1 and L_2 are given lines B_1 and B_2 are found angle bisectors

If $|\tan \theta| < 1$, then $2\theta < 90° \Rightarrow$ B_1 will be acute angle bisector.

If $|\tan \theta| > 1$, then $2\theta > 90° \Rightarrow$ B_1 will be obtuse angle bisector.

(ii)



Let P is a general point on L_1 (other then intersecting point of L_1 and L_2)

If $d_1 < d_2$, B_1 will be acute angle bisector .

If $d_1 > d_2$, B_1 will be obtuse angle bisector .

(iii) If $aa' + bb' < 0$, then the equation of the bisector of this acute angle is

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

If, however, $aa' + bb' > 0$, the equation of the bisector of the obtuse angle is :

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

Example # 34 : For the straight lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$, find the equation of the

(i) bisector of the obtuse angle between them;

(ii) bisector of the acute angle between them;

Solution : The equations of the bisectors of the angles between $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ are

$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = \pm \frac{12x - 5y - 8}{\sqrt{12^2 + (-5)^2}} \quad \text{or} \quad \frac{3x - 4y + 7}{5} = \pm \frac{12x - 5y - 8}{13}$$

or $39x - 52y + 91 = \pm (60x - 25y - 40)$

Taking the positive sign, we get $21x + 27y - 131 = 0$ as one bisector

Taking the negative sign, we get $99x - 77y + 51 = 0$ as the other bisector.

Writing the equation of the lines so that constants become positive we have

$3x - 4y + 7 = 0$ ....(1)

and $12x - 5y - 8 = 0$ ....(2)

Here $a_1 = 3$, $a_2 = 12$, $b_1 = -4$, $b_2 = -5$

Now $a_1a_2 + b_1b_2 = 3 \times 12 + 20 = 56 > 0$

so we get the obtuse angle bisector by taking positive sign so obtuse angle bisector is

$21x + 27y - 131 = 0$

and acute angle bisector is $99x - 77y + 51 = 0$

Self practice problem :

(29) Find the equations of the bisectors of the angles between the lines $x + y - 3 = 0$ and $7x - y + 5 = 0$ and state which of them bisects the acute angle between the lines.

Ans. $x - 3y + 10 = 0$ (bisector of the obtuse angle);
$6x + 2y - 5 = 0$ (bisector of the acute angle)

23. To discriminate between the bisector of the angle containing a point :

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then ; $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point (α, β) is,

$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ or $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ according as

$a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

Example # 35 : For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the bisector of the angle which contains the origin.

Solution : For point O(0, 0), $4x + 3y - 6 = -6 < 0$ and $5x + 12y + 9 = 9 > 0$

Hence for point O(0, 0) $4x + 3y - 6$ and $5x + 12y + 9$ are of opposite signs.

Hence equation of the bisector of the angle between the given lines containing the origin will be

$$\frac{4x + 3y - 6}{\sqrt{(4)^2 + (3)^2}} = - \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

or $\frac{4x + 3y - 6}{5} = - \frac{5x + 12y + 9}{13}$ or $52x + 39y - 78 = -25x - 60y - 45.$

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$

Self practice problem :

(30) Find the equation of the bisector of the angle between the lines $x + 2y - 11 = 0$ and $3x - 6y - 5 = 0$ which contains the point $(1, -3)$.

Ans. $3x - 19 = 0$

24. Condition of Concurrency :

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Alternatively : If three constants A, B & C (not all zero) can be found such that

$A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

Example # 36 : Prove that the straight lines 4x + 7y = 9, 5x – 8y + 15 = 0 and 9x – y + 6 = 0 are concurrent.

Solution : Given lines are

$4x + 7y - 9 = 0$ ....(1)

$5x - 8y + 15 = 0$........(2) and $9x - y + 6 = 0$ ....(3)

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix} = 4(-48 + 15) - 7(30 - 135) - 9(-5 + 72) = -132 + 735 - 603 = 0$$

Hence lines (1), (2) and (3) are concurrent.

Self practice problem :

(31) Find the value of m so that the lines 3x + y + 2 = 0, 2x – y + 3 = 0 and x + my – 3 = 0 may be concurrent.

Ans. 4

25. Family of Straight Lines :

The equation of a family of straight lines passing through the point of intersection of the lines, $L_1 \equiv a_1x + b_1y + c_1 = 0$ & $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $\lambda L_1 + k L_2 = 0$ i.e.

$\lambda(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where λ and k are arbitrary real numbers.

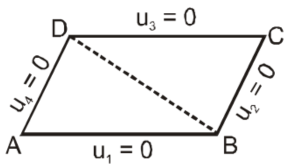
Note : (i) For parallelogram

If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$,

$u_4 = a'x + b'y + d'$

then $u_1 = 0; u_2 = 0; u_3 = 0; u_4 = 0$ form a parallelogram.

The diagonal BD can be given by $u_2u_3 - u_1u_4 = 0$.



(ii) For any quadrilateral

The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and

$u_2 + \mu u_3 = 0$, if the two equations are identical for some real λ and μ.

[For getting the values of λ & μ compare the coefficients of x, y & the constant terms].

Example # 37 : Find the equation of the straight line which passes through the origin and the point of intersection of the lines x + y + 4 = 0 and 3x – y – 8 = 0.

Solution : Any line through the intersection of the lines x + y + 4 = 0 and 3x – y – 8 =0 has the equation

$(x + y + 4) + \lambda(3x - y - 8) = 0$ ....(i)

This will pass through (0, 0) if

$4 - 8\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

Putting the value of λ in (i), the required line is $(x + y + 4) + (½)(3x - y - 8) = 0$

$\frac{5x}{2} + \frac{y}{2} = 0$ or $5x + y = 0$

Example # 38 : Obtain the equations of the lines passing through the intersection of lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and equally inclined to the axes.

Solution : The equation of any line through the intersection of the given lines is

$(4x - 3y - 1) + \lambda (2x - 5y + 3) = 0$

or $x(2\lambda + 4) - y(5\lambda + 3) + 3\lambda - 1 = 0$ ....(i)

Let m be the slope of this line. Then $m = \frac{2\lambda + 4}{5\lambda + 3}$

As the line is equally inclined with the axes, therefore

$m = \tan 45^o$ or $m = \tan 135^o$ ⇒ $m = \pm 1$, $\frac{2\lambda + 4}{5\lambda + 3} = \pm 1$

⇒ $\lambda = -1$ or $\frac{1}{3}$, putting the values of λ in (i), we get $2x + 2y - 4 = 0$ and $14x - 14y = 0$

i.e. $x + y - 2 = 0$ and $x = y$ as the equations of the required lines.

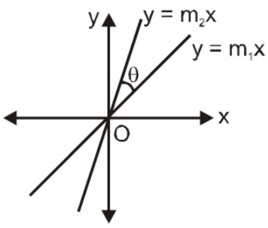
Self practice problem :

(32) Find the equation of the lines through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and whose distance from the origin is $\sqrt{5}$

Ans. $2x + y - 5 = 0$

26. A Pair of straight lines through origin :

(i)



A homogeneous equation of degree two,

"$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab$ ⇒ lines are real & distinct .

(b) $h^2 = ab$ ⇒ lines are coincident .

(c) $h^2 < ab$ ⇒ lines are imaginary with real point of intersection i.e. (0, 0)

This equation is obtained by multiplying the two equations of lines $(m_1x - y)(m_2x - y) = 0$

⇒ $m_1m_2x_2 - (m_1 + m_2)xy + y_2 = 0$

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(iv) The condition that these lines are :

(a) at right angles to each other is $a + b = 0$. i.e. co−efficient of x^2 + co−efficient of $y^2 = 0$.

(b) coincident is $h^2 = ab$.
(c) equally inclined to the axis of x is h = 0.i.e. coeff. of xy = 0 .

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angles between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Example # 39 : Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 3y^2 = 0$.

Solution : We have $2x^2 - 7xy + 3y^2 = 0$.

$\Rightarrow \quad 2x^2 - 6xy - xy + 3y^2 = 0 \Rightarrow \quad 2x(x - 3y) - y(x - 3y) = 0$

$\Rightarrow \quad (x - 3y)(2x - y) = 0 \quad \Rightarrow \quad x - 3y = 0$ or $2x - y = 0$

Thus the given equation represents the lines x – 3y = 0 and 2x – y = 0. The equations of the lines passing through the origin and perpendicular to the given lines are y – 0 = –3 (x – 0)

and $y - 0 = -\frac{1}{2}(x - 0)$ [$\because$ (Slope of x – 3 y = 0) is 1/3 and (Slope of 2x – y = 0) is 2]

$\Rightarrow \quad y + 3x = 0$ and $2y + x = 0$

Example # 40 : Find the angle between the pair of straight lines $5x^2 + 26xy + 5y^2 = 0$

Solution : Given equation is $5x^2 + 26xy + 5y^2 = 0$

Here a = coeff. of $x^2 = 5$, b = coeff. of $y^2 = 5$

and 2h = coeff. of xy = 26 $\quad \therefore \quad$ h = 13

Now $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{169 - 25}}{5 + 5}\right| = \frac{2 \times 12}{10} = \frac{12}{5}$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}\frac{12}{5}$ and obtuse angle between them is $\pi - \tan^{-1}\frac{12}{5}$

Example # 41 : Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$

Solution : Given equation is $3x^2 - 5xy + 4y^2 = 0$ ....(1)

comparing it with the equation $ax^2 + 2hxy + by^2 = 0$ ....(2)

we have a = 3, 2h = –5; and b = 4

Now the equation of the bisectors of the angle between the pair of lines (1) is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

or $\frac{x^2 - y^2}{3 - 4} = \frac{xy}{-\frac{5}{2}}$; or $\frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$ or $5x^2 - 2xy - 5y^2 = 0$

Self practice problems :

(33) Find the area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and y = 9.

(34) If the pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that pq = –1.

27. General equation of second degree representing a pair of Straight lines :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Such an equation is obtained again by multiplying the two equation of lines $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Example # 42 : Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection.

Solution : Given equation is $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$

Writing the equation (1) as a quadratic equation in x we have

$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$

$$\therefore \quad x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - 4.2(3y^2+7y+4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^2+60y+36-24y^2-56y-32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^2+4y+4}}{4} = \frac{-(5y+6) \pm (y+2)}{4}$$

$$\therefore \quad x = \frac{-5y-6+y+2}{4}, \frac{-5y-6-y-2}{4}$$

or $4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$ or $x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence equation (1) represents a pair of straight lines whose equation are

$x + y + 1 = 0$ .....(1) and $2x + 3y + 4 = 0$ .....(2)

Solving these two equations, the required point of intersection is $(1, -2)$.

Self practice problem :

(35) Find the combined equation of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ and find the angle between them.

Ans. $x^2 - 5xy + 4y^2 + 3x - 3y = 0$, $\tan^{-1}\left(\frac{3}{5}\right)$

28. Homogenization :

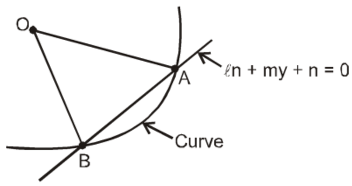
This method is used to write the joint equation of two lines connecting origin to the points of intersection of a given line and a given second degree curve.

The equation of a pair of straight lines joining origin to the points of intersection of the line $L \equiv \ell x + my + n = 0$ and a second degree curve

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$.

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



Notes : (i) Here we have written 1 as $\frac{\ell x + my}{-n}$ and converted all terms of the curve to second degree expressions

(ii) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$, where λ & μ are parameters.

Example # 43 : Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$.

Solution : Equation of the given curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$

and equation of the given straight line is $y - 3x = 2$; $\therefore$ $\frac{y-3x}{2} = 1$

Making equation (1) homogeneous equation of the second degree in x any y with the help of (1), we have $x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right) - 11\left(\frac{y-3x}{2}\right)^2 = 0$

or $x^2 + 2xy + 3y^2 + \frac{1}{2}(4xy + 8y^2 - 12x^2 - 24xy) - \frac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$

or $-119x^2 + 34xy + 17y^2 = 0$ or $119x^2 - 34xy - 17y^2 = 0$

or $7x^2 - 2xy - y^2 = 0$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$

we have $a = 7$, $b = -1$ and $2h = -2$

i.e. $h = -1$

If θ be the acute angle between pair of lines (3), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3} \qquad \therefore \qquad \theta = \tan^{-1}\frac{2\sqrt{2}}{3}.$$

Self practice problems :

(36) Find the equation of the straight lines joining the origin to the points of intersection of the line $3x + 4y - 5 = 0$ and the curve $2x^2 + 3y^2 = 5$.

(37) Find the equation of the straight lines joining the origin to the points of intersection of the line $lx + my + n = 0$ and the curve $y^2 = 4ax$. Also, find the condition of their perpendicularity.

Ans. (36) $x^2 - y^2 - 24xy = 0$ (37) $4alx^2 + 4amxy + ny^2 = 0; 4al + n = 0$