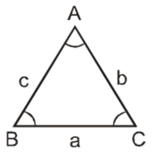
Solution of Triangle

According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect....................Proclus

1. Sine rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides



i.e. . $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example # 1 : How many triangles can be constructed with the data :
$a = 5$, $b = 7$, $\sin A = 3/4$

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{5}{3/4} = \frac{7}{\sin B}$

$\Rightarrow \quad \sin B = \frac{21}{20} > 1$ not possible

$\therefore$ no triangle can be constructed.

Example # 2 : If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution : We have $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

$\Rightarrow \sin(B + C)\sin(B–C) = \sin(A + B)\sin(A–B)$

$\Rightarrow \sin^2 B – \sin^2 C = \sin^2 A – \sin^2 B$

$\Rightarrow b^2 – c^2 = a^2 – b^2$

$\Rightarrow a^2, b^2, c^2$ are in A.P.

Self Practice Problems :

(1) In a ΔABC, the sides a, b and c are in A.P. , then prove that $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) : \cot\frac{B}{2} = 2 : 3$

(2) If the angles of ΔABC are in the ratio 1 : 2 : 3, then find the ratio of their corresponding sides

(3) In a ΔABC prove that $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

Ans. (2) $1 : \sqrt{3} : 2$

2. Cosine formula :

In any ΔABC

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A = b_2 + c_2 + 2bc \cos (B + C)$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Example # 3 : In a triangle ABC, A, B, C are in A.P. Show that

$$2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2 - ac + c^2}}.$$

Solution : $A + C = 2B \Rightarrow A + B + C = 3B \Rightarrow B = 60^o$

$\therefore \cos 60^o = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow a_2 - ac + c_2 = b_2$

$\Rightarrow \frac{a+c}{\sqrt{a^2 - ac + c^2}} = \frac{a+c}{b} = \left[\frac{\sin A + \sin C}{\sin B}\right]$

$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B}$

$= 2\cos\frac{A-C}{2}$ $\quad (\because A + C = 2B)$

Example # 4 : In a ΔABC , prove that $a (b\cos C - c \cos B) = b_2 - c_2$

Solution : Since $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\therefore$ L.H.S. $= a\left\{b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)\right\}$

$= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = (b_2 - c_2) =$ R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Example # 5 : The sides of ΔABC are AB = $\sqrt{13}$ cm, BC = $4\sqrt{3}$ cm and CA = 7 cm. Then find the value of $\sin\theta$ where θ is the smallest angle of the triangle.

Solution : Angle opposite to AB is smallest . Therefore,

$\cos\theta = \frac{49 + 48 - 13}{2.7.4\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{1}{2}$

Self Practice Problems :

(4) If in a triangle ABC, 3 sinA = 6 sinB = $2\sqrt{3}$ sinC, Then find the angle A.

(5) If two sides a, b and angle A be such that two triangles are formed, then find the sum of two values of the third side.

Ans. (4) 90º (5) 2b cosA

3. Projection formula :

In any ΔABC

(i) a = b cosC + c cosB **(ii)** b = c cosA + a cosC **(iii)** c = a cosB + b cosA

Example # 6 : If in a ΔABC, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show that a, b, c are in A.P.

Solution : c (1 + cosA) + a (1 + cosC) = 3b ⇒ a + c + (c cosA + acosC) = 3b
⇒ a + c + b = 3b ⇒ a + c = 2b

Example # 7 : In a ΔABC, prove that (b + c) cos A + (c + a) cos B + (a + b) cos C = a + b + c.

Solution : ∵ L.H.S. = (b + c) cos A + (c + a) cos B + (a + b) cos C
= b cos A + c cos A + c cos B + a cos B + a cos C + b cos C
= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)
= a + b + c
= R.H.S. Hence L.H.S. = R.H.S. **Proved**

Self Practice Problems :

(6) The roots of $x^2 - 2\sqrt{3}x + 2 = 0$ represent two sides of a triangle. If the angle between them is $\frac{\pi}{3}$, then find the perimeter of triangle.

(7) In a triangle ABC, if cos A + cosB + cos C = 3/2, then show that the triangle is an equilateral triangle.

(8) In a ΔABC, prove that $\frac{\cos A}{c\cos B + b\cos C} + \frac{\cos B}{a\cos C + c\cos A} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2+b^2+c^2}{2abc}$.

Ans. (6) $2\sqrt{3} + \sqrt{6}$

4. Napier's Analogy - tangent rule :

In any ΔABC

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ **(ii)** $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Example # 8 : Find the unknown elements of the ΔABC in which a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 90°.

Solution : ∵ a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 90° ∵ A + B + C = 180°
∴ A + B = 90° ....(i)
∵ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$= \frac{(\sqrt{3}+1) - (\sqrt{3}-1)}{(\sqrt{3}+1) + (\sqrt{3}-1)} \cot 45° = \frac{2}{2\sqrt{3}} \cot 45° \Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{\sqrt{3}}$$

$\therefore \quad \frac{A-B}{2} = \frac{\pi}{6} \quad \Rightarrow A - B = \frac{\pi}{3}$ ....(ii)

From equation (i) and (ii), we get

$A = \frac{5\pi}{12}$ and $B = \frac{\pi}{12}$

Now,

$c = \sqrt{a^2+b^2} = 2\sqrt{2} \quad \therefore \quad c = 2\sqrt{2}$, $A = \frac{5\pi}{12}$, $B = \frac{\pi}{12}$ Ans.

Self Practice Problems :

(9) In a ΔABC if b = 3, c = 5 and cos (B – C) = $\frac{7}{25}$, then find the value of sin $\frac{A}{2}$.

(10) If in a ΔABC, we define x = $\tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, y = $\tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and z = $\tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$, then show that x + y + z = – xyz.

Ans. (9) $\frac{1}{\sqrt{10}}$

5. Trigonometric functions of Half Angles :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$, where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

6. Area of triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

Example # 9 : If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then show that $p_1^{-1} + p_2^{-1} - p_3^{-1} = \frac{s-c}{\Delta}$

Solution : We have

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta} = \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$$

Example # 10 : In a ΔABC if b sinC(b cosC + c cosB) = 64, then find the area of the ΔABC.

Solution : $\because$ b sinC (b cosC + c cosB) = 64 ....(i) given

$\because$ From **projection rule**, we know that

a = b cosC + c cosB put in (i), we get

ab sinC = 64 ....(ii)

$\because$ $\Delta = \frac{1}{2}$ ab sinC $\therefore$ from equation (ii), we get

$\therefore$ Δ = 32 sq. unit

Example # 11 : If A,B,C are the angle of a triangle, then prove that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$

Solution : $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$= \frac{\sqrt{s}(s-a+s-b+s-c)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s}{\Delta}$ (3s – 2s) $= \frac{s^2}{\Delta}$

7. Radius of circumcirlce :

If R be the circumradius of ΔABC, then $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

Example # 12 : In a ΔABC , prove that sin2A + sin2B + sin2C = 2Δ/R₂

Solution : In a ΔABC, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

and sin2A + sin2B + sin2C = 4sinAsinBsinC

$= \frac{4abc}{8R^3} = \frac{16\Delta R}{8R^3} = \frac{2\Delta}{R^2}$

Example # 13 : In a ΔABC if a = 22 cm, b = 28 cm and c = 36 cm, then find its circumradius.

Solution : $\because$ $R = \frac{abc}{4\Delta}$ ....(i)

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because$ $s = \frac{a+b+c}{2}$ = 43 cm

$\therefore$ $\Delta = \sqrt{43 \times 21 \times 15 \times 7} = 21\sqrt{215}$

$\therefore$ $R = \frac{22 \times 28 \times 36}{4 \times 21\sqrt{215}} = \frac{264}{\sqrt{215}}$ cm

Example # 14 : In a ΔABC, if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

Solution : We have : $8R^2 = a^2 + b^2 + c^2$

$\Rightarrow 8R^2 = [4R^2 \sin^2A + 4R^2 \sin^2B + 4R^2 \sin C]$ [$\because$ a = 2R sin A etc.]

$\Rightarrow 2 = \sin^2A + \sin^2B + \sin^2C \Rightarrow (1 - \sin^2A) - \sin^2 B + (1 - \sin^2C) = 0$

$\Rightarrow (\cos^2 A - \sin^2B) + \cos^2 C = 0 \Rightarrow \cos (A + B) \cos (A - B) + \cos^2C = 0$

$\Rightarrow -\cos C \cos (A - B) + \cos^2 C = 0 \Rightarrow -\cos C \{\cos (A - B) - \cos C\} = 0$

$\Rightarrow -\cos C[\cos (A - B) + \cos(A + B)] = 0 \Rightarrow -2\cos A \cos B \cos C = 0$

$\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$

$\Rightarrow \Delta ABC$ is a right angled triangle.

Example # 15 : $\frac{b^2 - c^2}{2a} = R \sin (B - C)$

Solution : $\frac{b^2 - c^2}{2a} = \frac{4R^2(\sin^2 B - \sin^2 C)}{4R \sin A} = \frac{R \sin(B+C)\sin(B-C)}{\sin A} = R \sin (B - C)$

Self Practice Problems :

(11) In a ΔABC, prove that $(a + b) = 4R \cos \left(\frac{A-B}{2}\right) \cos \frac{C}{2}$

(12) In a ΔABC , if $b = 15$ cm and $\cos B = \frac{4}{5}$, find R.

(13) In a triangle ABC if α, β, γ are the distances of the vertices of triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = r^2$

Ans. (12) 12.5

8. Radius of the incircle :

If 'r' be the inradius of ΔABC, then

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}$

(iii) $r = \frac{a \sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Example # 16 : $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$

Solution : LHS = $\cos A + \cos B + \cos C$

$= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2 \frac{C}{2}$

$= 2 \sin \frac{C}{2} \left\{\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right\} + 1 \quad = 2\sin \frac{C}{2} \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\} + 1$

$= 2\sin \frac{C}{2} \left\{2\sin\frac{A}{2}\sin\frac{B}{2}\right\} + 1 = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$= 1 + \frac{1}{R}\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 1 + \frac{r}{R}$ = RHS

Self Practice Problems :

(14) In a triangle ABC, a : b : c = 4 : 5 : 6 . Find the ratio of the radius of the circumcircle to that of the incircle.

(15) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC, then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

Ans. (14) 6, 8 , 10 (15) 16 : 7

9. Length of angle bisectors, medians & altitudes :

(i) Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$;

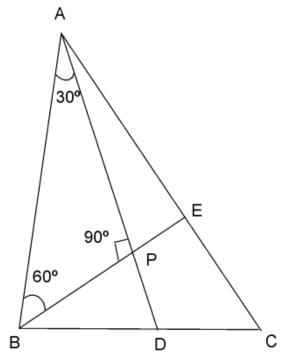
(ii) Length of median from the angle A = $m_a = \dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$

(iii) Length of altitude from the angle A = $A_a = \dfrac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

Example # 17 : In ΔABC, AD & BE are its two median . If AD = 4 , $\angle DAB = \dfrac{\pi}{6}$ and $\angle ABE = \dfrac{\pi}{3}$ then find the length of BE and area of ΔABC.

Solution : $AP = \dfrac{2}{3}AD = \dfrac{8}{3}$; $PD = \dfrac{4}{3}$; Let PB = x



$\tan 60° = \dfrac{8/3}{x}$ or $x = \dfrac{8}{3\sqrt{3}}$

Area of $\Delta ABP = \dfrac{1}{2} \times \dfrac{8}{3} \times \dfrac{8}{3\sqrt{3}} = \dfrac{32}{9\sqrt{3}}$

$\therefore$ Area of $\Delta ABC = 3 \times \dfrac{32}{9\sqrt{3}} = \dfrac{32}{3\sqrt{3}}$

Also, $BE = \dfrac{3}{2}x = \dfrac{4}{\sqrt{3}}$

Self Practice Problem :

(16) In a ΔABC if ∠A = 90°, b = 5 cm, c = 12 cm. If 'G' is the centroid of triangle, then find circumradius of ΔGAB.

Ans. (16) $\dfrac{13\sqrt{601}}{30}$ cm

10. The distances of the special points from vertices and sides of triangle :

(i) Circumcentre (O) : OA = R and O_a = R cos A

(ii) Incentre (I) : IA = r cosec $\dfrac{A}{2}$ and I_a = r

(iii) Excentre (I_1) : $I_1A = r_1 \operatorname{cosec}\dfrac{A}{2}$ and $I_{1a} = r_1$

(iv) Orthocentre (H) : HA = 2R cos A and H_a = 2R cos B cos C

(v) Centroid (G) : $GA = \dfrac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and $G_a = \dfrac{2\Delta}{3a}$

Example # 18 : If p_1, p_2, p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, prove that :

(i) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$ (ii) $\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R}$

Solution : (i) use $\frac{1}{p_1} = \frac{a}{2\Delta}, \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$

$\therefore$ LHS = $\frac{1}{2\Delta}$ (a cosA + b cosB + c cosC)

= $\frac{R}{2\Delta}$ (sin 2A + sin 2B + sin 2C) = $\frac{4R\sin A \;\; \sin B \;\; \sin C}{2\Delta}$

= $\frac{4R}{2\Delta} \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{1}{4\Delta R^2} abc = \frac{1}{4\Delta R^2} . (4R\Delta) = \frac{1}{R}$ = RHS

(ii) LHS = $\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b}$ = $\frac{a^2+b^2+c^2}{2R}$ = $\frac{2b\Delta}{ac} + \frac{2c\Delta}{ab} + \frac{2a\Delta}{bc}$ = $\frac{2\Delta(a^2+b^2+c^2)}{abc}$

= $\frac{2\Delta(a^2+b^2+c^2)}{4\Delta R}$ = $\frac{a^2+b^2+c^2}{2R}$

Self Practice Problems :

(17) If I be the incentre of ΔABC, then prove that IA . IB . IC = abc tan $\frac{A}{2}$ tan $\frac{B}{2}$ tan $\frac{C}{2}$.

(18) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.