SIMPLE HARMONIC MOTION

1. PERIODIC MOTION

When a body or a moving particle repeats its motion along a definite path after regular intervals of time, its motion is said to be **Periodic Motion** and interval of time is called **time period** or harmonic motion period (T). The path of periodic motion may be linear, circular, elliptical or any other curve. For example, rotation of earth about the sun.

2. OSCILLATORY MOTION

'To and Fro' type of motion is called an **Oscillatory Motion**. It need not be periodic and need not have fixed extreme positions. For example, motion of pendulum of a wall clock.

The oscillatory motions in which energy is conserved are also periodic.

The force / torque (directed towards equilibrium point) acting in oscillatory motion is called restoring force / torque.

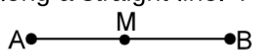
Damped oscillations are those in which energy is consumed due to some resistive forces and hence total mechanical energy decreases.

3. SIMPLE HARMONIC MOTION

If the restoring force/ torque acting on the body in oscillatory motion is directly proportional to the displacement of body/particle and is always directed towards equilibrium position then the motion is called simple Harmonic Motion (SHM). It is the simplest (easy to analyse) form of oscillatory motion.

3.1 TYPES OF SHM

(a) Linear SHM : When a particle moves to and fro about an equilibrium point, along a straight line. A and B are extreme positions. M is mean position. AM = MB = Amplitude



(b) Angular SHM : When body/particle is free to rotate about a given axis executing angular oscillations.

3.2 EQUATION OF SIMPLE HARMONIC MOTION (SHM) :

The necessary and sufficient condition for SHM is

$$F = -kx$$

where k = positive constant for a SHM = Force constant
x = displacement from mean position.

or $$m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{[differential equation of SHM]}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

It's solution is $x = A \sin(\omega t + \varphi)$

3.3 CHARACTERISTICS OF SHM

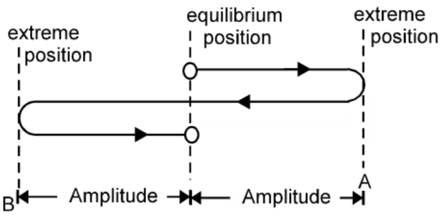
Note : In the figure shown, path of the particle is on a straight line.

(a) Displacement - It is defined as the distance of the particle from the mean position at that instant. Displacement in SHM at time t is given by $x = A \sin(\omega t + \varphi)$

(b) Amplitude - It is the maximum value of displacement of the particle from its equilibrium position.

Amplitude = $\frac{1}{2}$ [distance between extreme points or positions]

It depends on energy of the system.



(c) Angular Frequency (ω) : $\omega = \frac{2\pi}{T} = 2\pi f$ and its units is rad/sec.

(d) Frequency (f) : Number of oscillations completed in unit time interval is called frequency of oscillations, $f = \frac{1}{T} = \frac{\omega}{2\pi}$, its units is sec^{-1} or Hz.

(e) Time period (T) : Smallest time interval after which the oscillatory motion gets repeated is called time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Solved Examples

Example 1. For a particle performing SHM, equation of motion is given as $\frac{d^2x}{dt^2} + 4x = 0$. Find the time period.

Solution : $\frac{d^2x}{dt^2} = -4x$

$\omega^2 = 4$

$\omega = 2$

Time period; $T = \frac{2\pi}{\omega} = \pi$

(f) Phase : The physical quantity which represents the state of motion of particle (eg. its position and direction of motion at any instant).

The argument $(\omega t + \varphi)$ of sinusoidal function is called instantaneous phase of the motion.

(g) Phase constant (φ) : Constant φ in equation of SHM is called phase constant or initial phase.

It depends on initial position and direction of velocity.

(h) Velocity(v) : It is the rate of change of particle's displacement w.r.t time at that instant.

Let the displacement from mean position is given by

$$x = A \sin(\omega t + \varphi)$$

Velocity, $v = \frac{dx}{dt} = \frac{d}{dt}\left[A\sin(\omega t + \phi)\right]$

$v = A\omega \cos(\omega t + \varphi)$ or, $v = \omega\sqrt{A^2 - x^2}$

At mean position (x = 0), velocity is maximum.

$v_{max} = \omega A$

At extreme position (x = A), velocity is minimum.

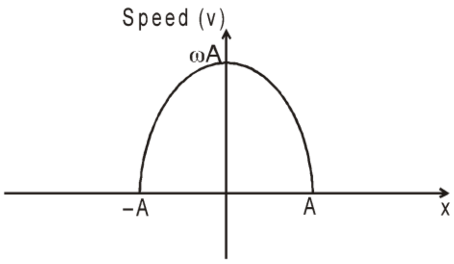
v_{min} = zero

GRAPH OF SPEED (v) VS DISPLACEMENT (x):

$v = \omega\sqrt{A^2 - x^2}$ $\qquad$ $v^2 = \omega^2\left(A^2 - x^2\right)$

$v^2 + \omega^2 x^2 = \omega^2 A^2$ $\qquad$ $\frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1$

GRAPH WOULD BE AN ELLIPSE



(i) Acceleration : It is the rate of change of particle's velocity w.r.t. time at that instant.

Acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}[A\omega\ \cos(\omega t + \phi)]$

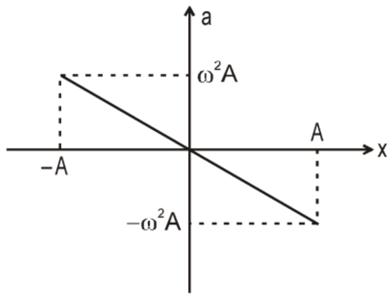
$a = -\omega^2 A \sin(\omega t + \varphi)$

$a = -\omega^2 x$

Note Negative sign shows that acceleration is always directed towards the mean postion.
At mean position (x = 0), acceleration is minimum.
a_{min} = zero
At extreme position (x = A), acceleration is maximum.
$a_{max} = \omega^2 A$

GRAPH OF ACCELERATION (A) VS DISPLACEMENT (x)

$a = -\omega^2 x$



Solved Examples

Example 2. The equation of particle executing simple harmonic motion is $x = (5\text{ m})\sin\left[(\pi\ \ s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at t = 1 s.

Solution : Comparing with equation $x = A\sin(\omega t + \delta)$, we see that
the amplitude = 5 m,

and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi\ \ s^{-1}} = 2s$.

The maximum speed $= A\omega = 5\text{ m} \times \pi\ s^{-1} = 5\pi$ m/s.

The velocity at time t = $\frac{dx}{dt}$ = A ω cos (ωt + δ)

At t = 1 s,

$$v = (5\text{ m})(\pi\text{ s}^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}\text{ m/s.}$$

Example 3. A particle executing simple harmonic motion has angular frequency 6.28 s^{-1} and ampitude 10 cm. Find (a) the time period, (b) the maximum speed, (c) the maximum acceleration, (d) the speed when the displacement is 6 cm from the mean position, (e) the speed at t = 1/6s assuming that the motion starts from rest at t = 0.

Solution : (a) Time period = $\frac{2\pi}{\omega} = \frac{2\pi}{6.28}$ s = 1 s.

(b) Maximum speed = Aω = (0.1 m) (6.28 s^{-1})
= 0.628 m/s.

(c) Maximum acceleration = $A\omega^2$
= (0.1 m) $(6.28\text{ s}^{-1})^2$
= 4 m/s^2.

(d) $v = \omega\sqrt{A^2 - x^2} = (6.28\text{ s}^{-1})\sqrt{(10\ \text{cm})^2 - (6\ \text{cm})^2}$
= 50.2 cm/s.

(e) At t = 0, the velocity is zero i.e., the particle is at an extreme. The equation for displacement may be written as

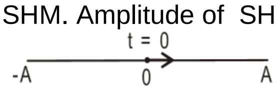
x = A cosωt.

The velocity is v = – A ω sin ωt.

At $t = \frac{1}{6}$ s, $v = -(0.1\text{ m})(6.28\text{ s}^{-1})\sin\left(\frac{6.28}{6}\right)$

$= (-0.628\text{ m/s})\sin\frac{\pi}{3}$ = 54.4 cm/s.

Example 4. A particle starts from mean position and moves towards positive extreme as shown. Find the equation of the SHM. Amplitude of SHM is A.



Solution : General equation of SHM can be written as x = A sin (ωt + φ)

At t = 0, x = 0

∴ 0 = A sinφ

∴ φ = 0, π $\phi \in [0, 2\pi)$

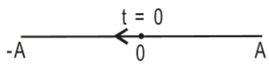
Also; at t = 0, v = +ve

∴ Aω cosφ = +ve

or, φ = 0

Hence, if the particle is at mean position at t = 0 and is moving towards +ve extreme, then the equation of SHM is given by x = A sinωt

Similarly



for

φ = π

∴ equation of SHM is x = A sin(ωt + π)

or, x = -A sinωt

Note : **If mean position is not at the origin, then we can replace x by x - x_0 and the equation. becomes x - x_0 = -A sinωt, where x_0 is the position co-ordinate of the mean position.**

Solved Examples

Example 5. A particle is performing SHM of amplitude "A" and time period "T". Find the time taken by the particle to go from 0 to A/2.

Solution : Let equation of SHM be $x = A \sin \omega t$

when $x = 0$, $t = 0$

when $x = A/2$; $A/2 = A \sin \omega t$

or $\sin \omega t = 1/2$ $\qquad \omega t = \pi/6$

$\frac{2\pi}{T} t = \pi/6$ $\qquad t = T/12$

Hence , time taken is T/12, where T is time period of SHM.

Example 6. A particle of mass 2 kg is moving on a straight line under the action force $F = (8 - 2x)$ N. It is released at rest from $x = 6$ m.

(a) Is the particle moving simple harmonically.

(b) Find the equilibrium position of the particle.

(c) Write the equation of motion of the particle.

(d) Find the time period of SHM.

Solution : $F = 8 - 2x$

or $F = -2(x - 4)$

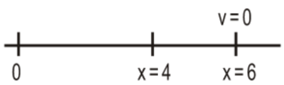
for equilibrium position $F = 0$ $\qquad \Rightarrow \qquad x = 4$ is equilibrium position

Hence the motion of particle is SHM with force constant 2 and equilibrium position $x = 4$.

(a) Yes, motion is SHM.

(b) Equilibrium position is $x = 4$

(c) At $x = 6$ m, particle is at rest i.e. it is one of the extreme position



Hence amplitude is A = 2 m and initially particle is at the extreme position.

$\therefore$ Equation of SHM can be written as

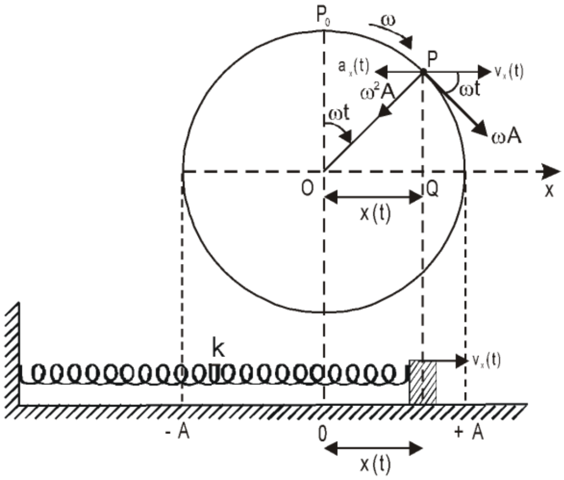
$x - 4 = 2 \cos \omega t$, where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$

i.e. $x = 4 + 2 \cos t$

(d) Time period, $T = \frac{2\pi}{\omega} = 2\pi$ sec.

4. SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION

Consider a particle moving on a circle of radius A with a constant angular speed ω as shown in figure.



Suppose the particle is on the top of the circle (Y-axis) at t = 0. The radius OP make an angle $\theta = \omega t$ with the Y-axis at time t. Drop a perpendicular PQ on X-axis. The components of position vector, velocity vector and acceleration vector at time t on the X-axis are

$x(t) = A \sin \omega t$

$v_x(t) = A\omega \cos \omega t$

$a_x(t) = - \omega^2 A \sin \omega t$

Above equations show that the foot of perpendicular Q executes a simple harmonic motion on the X-axis. The amplitude is A and angular frequency is ω. Similarly the foot of perpendicular on Y-axis will also executes SHM of amplitude A and angular frequency ω [$y(t) = A \cos \omega t$]. The phases of the two simple harmonic motions differ by $\pi/2$.

5. GRAPHICAL REPRESENTATION OF DISPLACEMENT, VELOCITY & ACCELERATION IN SHM

Displacement, $x = A \sin \omega t$

Velocity, $v = A\omega \cos \omega t = A\omega \sin (\omega t + \frac{\pi}{2})$ or $v = \omega \sqrt{A^2 - x^2}$

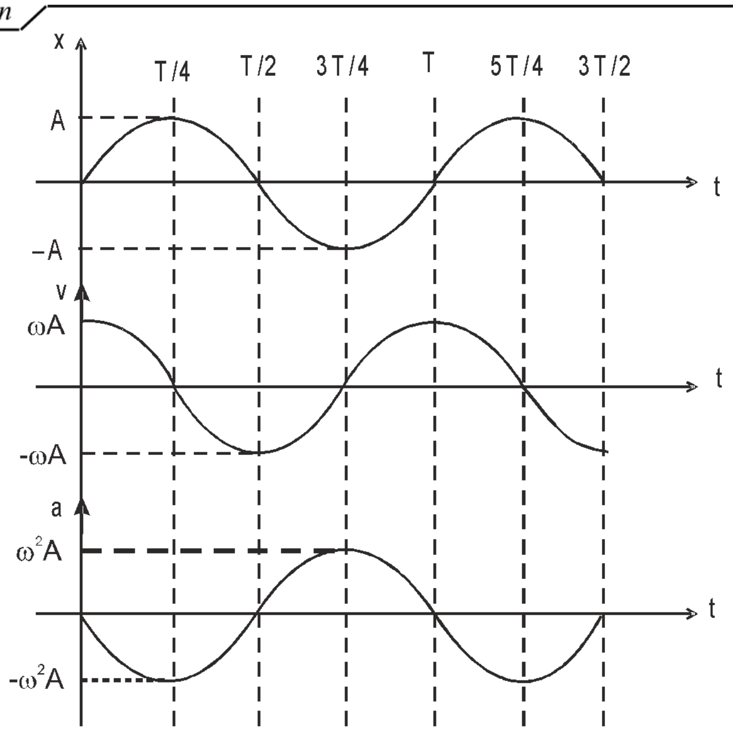
Acceleration, $a = - \omega^2 A \sin\omega t = \omega^2 A \sin (\omega t + \pi)$ or $a = -\omega^2 x$

Note : $v = \omega \sqrt{A^2 - x^2}$

$a = -\omega^2 x$

These relations are true for any equation of x.

time, t	0	T/4	T/2	3T/4	T
displacement, x	0	A	0	– A	0
Velocity, v	Aω	0	– Aω	0	Aω
acceleration, a	0	$-\omega^2A$	0	ω^2A	0



1. All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.
2. The velocity amplitude is ω times the displacement amplitude ($v_{max} = \omega A$).
3. The acceleration amplitude is ω^2 times the displacement amplitude ($a_{max} = \omega^2 A$).
4. In SHM, the velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$.
5. In SHM, the acceleration is ahead of velocity by a phase angle of $\frac{\pi}{2}$.

6. ENERGY OF SHM

6.1 Kinetic Energy (KE)

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2) \text{ (as a function of x)}$$

$$= \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \theta) = \frac{1}{2}KA^2\cos^2(\omega t + \theta) \text{ (as a function of t)}$$

$$KE_{max} = \frac{1}{2}kA^2 \quad ; \quad \langle KE \rangle_{0-T} = \frac{1}{4}kA^2 \quad ; \quad \langle KE \rangle_{0-A} = \frac{1}{3}kA^2$$

Frequency of KE = 2 × (frequency of SHM)

6.2 Potential Energy (PE)

$$\frac{1}{2}Kx^2 \text{ (as a function of x)} = \frac{1}{2}kA^2\sin^2(\omega t + \theta) \text{ (as a function of time)}$$

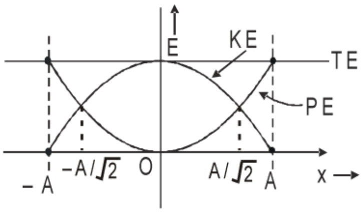
6.3 Total Mechanical Energy (TME)

Total mechanical energy = Kinetic energy + Potential energy

$$= \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}Kx^2 = \frac{1}{2}KA^2$$

Hence total mechanical energy is constant in SHM.

6.4 Graphical Variation of energy of SHM.



Solved Examples

Example 7. A particle of mass 0.50 kg executes a simple harmonic motion under a force F = – (50 N/m)x. If it crosses the centre of oscillation with a speed of 10 m/s, find the amplitude of the motion.

Solution : The kinetic energy of the particle when it is at the centre of oscillation is

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(0.50\ \text{kg})\ (10\ \text{m/s})^2 = 25\ \text{J}$$

The potential energy is zero here. At the maximum displacement x = A, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2}kA^2$. As there is no loss of energy,

$$\frac{1}{2}kA^2 = 25\ \text{J} \qquad ....(i)$$

The force on the particle is given by

$$F = -(50\ \text{N/m})x.$$

Thus, the spring constant is k = 50 N/m.

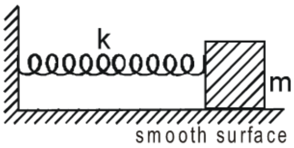
Equation (i) gives

$$\frac{1}{2}(50\ \text{N/m})\ A^2 = 25\ \text{J} \qquad \text{or,} \qquad A = 1\ \text{m}.$$

7. SPRING-MASS SYSTEM

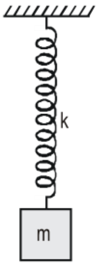
[Not in JEE, for other exams]

(1)



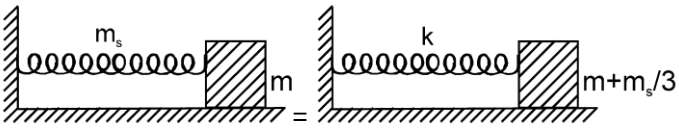
$$T = 2\pi\sqrt{\frac{m}{k}}$$

(2)



$$T = 2\pi\sqrt{\frac{m}{k}}$$

(3) If spring has mass m_s then



$$T = 2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$$

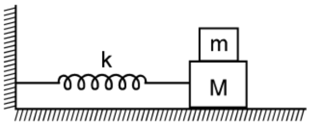
Solved Examples

Example 8. A particle of mass 200 g executes a simple harmonic motion. The restoring force is provided by a spring of spring cosntant 80 N/m. Find the time period.

Solution : The time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{200\times10^{-3}kg}{80\ \ N/m}} = 2\pi \times 0.05 \text{ s} = 0.31 \text{ s}.$$

Example 9. The friction coefficient between the two blocks shown in figure is μ and the horizontal plane is smooth. (a) If the system is slightly displaced and released, find the time period. (b) Find the magnitude of the frictional force between the blocks when the displacement from the meanposition is x. (c) What can be the maximum amplitude if the upper block does not slip relative to the lower block ?



Solution : (a) For small amplitude, the two blocks oscillate together. The angular frequency is

$$\omega = \sqrt{\frac{k}{M+m}} \quad \text{and so the time period } T = 2\pi \cdot \sqrt{\frac{M+m}{k}}$$

(b) The acceleration of the blocks at displacement x from the mean position is

$$a = -\,\omega^2 x = \left(\frac{-kx}{M+m}\right)$$

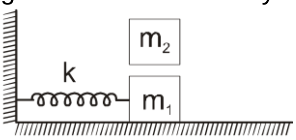
The resultant force on the upper block is, therefore, $ma = \left(\frac{-mkx}{M+m}\right)$

This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\left(\frac{mk\,|\,x\,|}{M+m}\right)$

(c) Maximum force of friction required for simple harmonic motion of the upper block is $\frac{mk\ \ A}{M+m}$ at the extreme positions. But the maximum frictional force can only be μ mg. Hence

$$\frac{mkA}{M+m} = \mu\, mg \qquad \text{or,} \qquad A = \frac{\mu(M+m)g}{k}$$

Example 10 A block of mass 4kg attached with spring of spring constant 100 N/m executing SHM of amplitude 0.1m on smooth horizontal surface as shown in figure. If another block of mass 5 kg is gently placed on it, at the instant it passes through the mean position then find the frequency and amplitude of the motion assuming that two blocks always move together.



Solution : Frequency $n_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m_1+m_2}}$

$$= \frac{1}{2\pi}\sqrt{\frac{100}{4+5}}$$

$$= \frac{1}{2\pi}\times\frac{10}{3} = \frac{5}{3\pi}$$

By conservation of linear momentum at mean position,

$P_i = P_f$

$\Rightarrow \quad m_1\omega_1A_1 = (m_1 + m_2)\,\omega_2A_2$

$\Rightarrow \quad m_1\sqrt{\frac{k}{m_1}}A_1 = (m_1 + m_2)\sqrt{\frac{k}{m_1+m_2}}A_2$

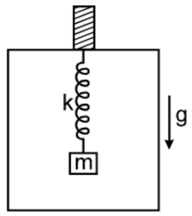
$\Rightarrow \quad \sqrt{km_1}A_1 = \sqrt{k(m_1+m_2)}\;A_2$

$\Rightarrow \quad A_2 = \frac{2}{30}$ m.

Ans. $\frac{5}{3\pi}$ Hz, $\frac{2}{30}$ m

Example 11. A block of mass m is suspended from the ceiling of a stationary elevator through a spring of spring constant k and suddenly, the cable breaks and the elevator starts falling freely. Show that block now executes a simple harmonic motion of amplitude mg/k in the elevator.

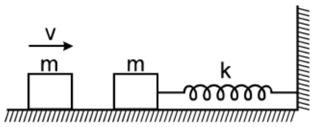
Solution: When the elevator is stationary, the spring is stretched to support the block. If the extension is x, the tension is kx which should balance the weight of the block.



Thus, x = mg/k. As the cable breaks, the elevator starts falling with acceleration 'g'. We shall work in the frame of reference of the elevator. Then we have to use a psuedo force mg upward on the block. This force will 'balance' the weight.

Thus, the block is subjected to a net force kx by the spring when it is at a distance x from the position of unstretched spring. Hence, its motion in the elevator is simple harmonic with its mean position corresponding to the unstretched spring. Initially, the spring is stretched by x = mg/k, where the velocity of the block (with respect to the elevator) is zero. Thus, the amplitude of the resulting simple harmonic motion is mg/k.

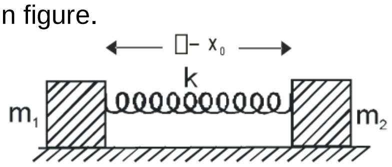
Example 12. The left block in figure collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.



Solution : Assuming the collision to last for a small interval only, we can apply the principle of conservation of momentum. The common velocity after the collision is $\frac{v}{2}$.

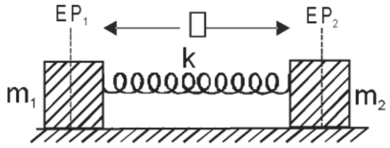
The kinetic energy = $\frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is A, the total energy can also be written as $\frac{1}{2}kA^2$. Thus, $\frac{1}{2}kA^2 = \frac{1}{4}mv^2$, giving A = $\sqrt{\frac{m}{2k}}\,v$.

Example 13. Two blocks of mass m_1 and m_2 are connected with a spring of natural length l and spring constant k. The system is lying on a smooth horizontal surface. Initially spring is compressed by x_0 as shown in figure.



Show that the two blocks will perform SHM about their equilibrium position.Also find the time period.

Solution : Here both the blocks will be in equilibrium at the same time when spring is in its natural length. Let EP_1 and EP_2 be equilibrium positions of block A and B as shown in figure.



Let at any time during oscillations, blocks are at a distance of x_1 and x_2 from their euilibrium positions.

As no external force is acting on the spring block system

$\therefore \quad (m_1 + m_2)\Delta x_{cm} = m_1x_1 - m_2x_2 = 0 \quad$ or $\quad m_1x_1 = m_2x_2$

For 1st particle, force equation can be written as

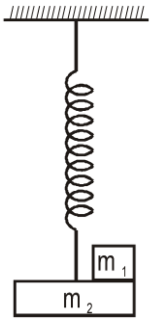
$$k(x_1 + x_2) = - m_1 \frac{d^2x_1}{dt^2} \qquad \text{or,} \qquad k\left(x_1 + \frac{m_1}{m_2} x_1\right) = - m_1a_1$$

or, $\quad a_1 = - \dfrac{k(m_1+m_2)}{m_1m_2} x_1 \qquad \therefore \quad \omega^2 = \dfrac{k(m_1+m_2)}{m_1m_2}$

Hence, $T = 2\pi\sqrt{\dfrac{m_1m_2}{k(m_1+m_2)}} = 2\pi\sqrt{\dfrac{\mu}{K}}$ where $\mu = \dfrac{m_1m_2}{(m_1+m_2)}$ which is known as reduced mass

Similarly time period of 2nd particle can be found. Both will be having the same time period.

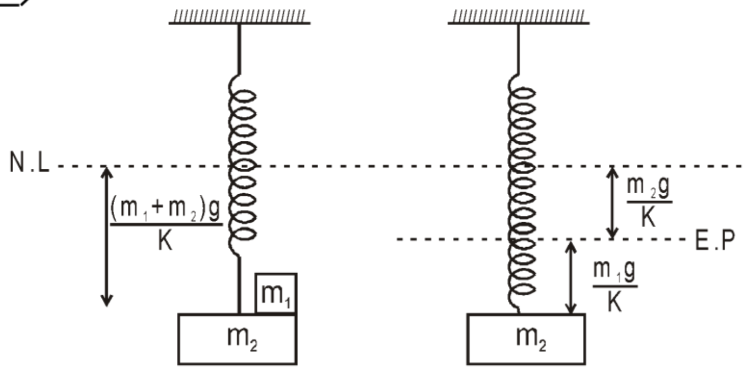
Example 14. The system is in equilibrium and at rest. Now mass m_1 is removed from m_2. Find the time period and amplitude of resultant motion. Spring constant is K.



Solution : Initial extension in the spring

$$x = \frac{(m_1+m_2)g}{K}$$

Now, if we remove m_1, equilibrium position(E.P.) of m_2 will be $\dfrac{m_2g}{K}$ below natural length of spring.



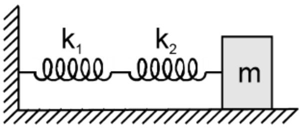
At the initial position, since velocity is zero i.e. it is the extreme position.

Hence Amplitude = $\frac{m_1 g}{K}$

Time period = $2\pi\sqrt{\frac{m_2}{K}}$

8. COMBINATION OF SPRINGS

8.1 Series Combination :



Total displacement $x = x_1 + x_2$

Tension in both springs $= k_1 x_1 = k_2 x_2$

∴ Equivalent constant in series combination K_{eq} is given by :

$1/k_{eq} = 1/k_1 + 1/k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$

Note :

- In series combination, tension is same in all the springs & extension will be different. (If k is same then deformation is also same)
- In series combination , extension of springs will be reciprocal of its spring constant.
- Spring constant of spring is reciprocal of its natural length

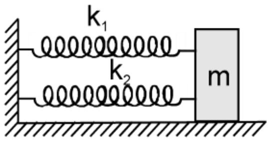
  $\because\ k \propto 1/\ell$

  $\therefore k_1 \ell_1 = k_2 \ell_2 = k_3 \ell_3$
- If a spring is cut in 'n' pieces then spring constant of one piece will be nk.

8.2 Parallel combination :

Extension is same for both springs but force acting will be different.

Force acting on the system = F

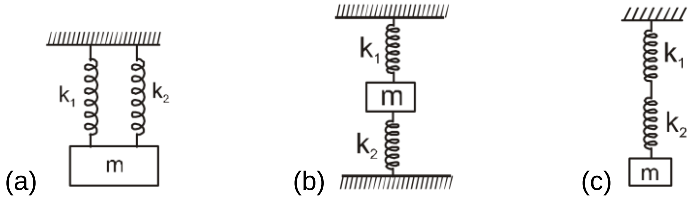


∴ $F = -(k_1 x + k_2 x) \quad \Rightarrow \quad F = -(k_1 + k_2)\,x \quad \Rightarrow \quad F = -k_{eq}x$

∴ $k_{eq} = k_1 + k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$

Solved Examples

Example 15 Three spring mass systems are shown in figure. Assuming gravity free space, find the time period of oscillations in each case. What should be the answer if space is not gravity free ?



Ans. (a) $2\pi\sqrt{\dfrac{m}{k_1+k_2}}$, $k_{eq.} = k_1 + k_2$; (b) $2\pi\sqrt{\dfrac{m}{k_1+k_2}}$, $k_{eq.} = k_1 + k_2$; (c) $2\pi\sqrt{\dfrac{m\ (k_1+k_2)}{k_1k_2}}$, $k_{eq.} = \dfrac{k_1k_2}{k_1+k_2}$

Answers will remain same

Solution : (a) When spring is stretched by x then restoring force.

$F = K_1 x + K_2 x$

$F = K_{eq} x$

$K_{eq} x = K_1 x + K_2 x$

$K_{eq} = K_1 + K_2$

$$T = 2\pi\sqrt{\frac{m}{K_{eq}}} = 2\pi\sqrt{\frac{m}{K_1+K_2}}$$

(b) When bock is displaced by x from mean position then restoring force.

$F = K_1 x + K_2 x$

$K_{eq} x = K_1 x + K_2 x$

$K_{eq} = K_1 + K_2$

$$T = 2\pi\sqrt{\frac{m}{K_{eq}}} = 2\pi\sqrt{\frac{m}{K_1+K_2}}$$

(c) When block is displaced by x and extension in upper spring is x_1, extension in lower spring is x_2

then $F = K_1 x_1 \Rightarrow x_1 = \dfrac{F}{K_1}$

$F = K_2 x_2 \Rightarrow x_2 = \dfrac{F}{K_2}$

$F = K_{eq} x \Rightarrow x = \dfrac{F}{K_{eq}}$

$x = x_1 + x_2 \Rightarrow \dfrac{F}{K_{eq}} = \dfrac{F}{K_1} + \dfrac{F}{K_2}$

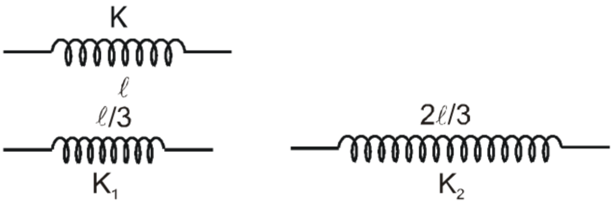
$\Rightarrow K_{eq} = \dfrac{K_1K_2}{K_1+K_2}$

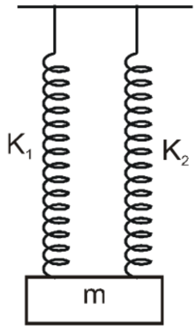
$$T = 2\pi\sqrt{\frac{m}{K_{eq}}} = 2\pi\sqrt{\frac{m(K_1+K_2)}{K_1K_2}}$$

When space is not gravity free then answers do not change as time period of spring mass system is independent of gravity.

Example 16 : A spring of force constant 'k' is cut into two parts whose lengths are in the ratio 1 : 2. The two parts are now connected in parallel and a block of mass 'm' is suspended at the end of the combined spring. Find the period of oscillation performed by the block.

Solution :





We have $\frac{K_1K_2}{K_1+K_2} = K$

and $K_1 \times \frac{\ell}{3} = K \times \ell = K_2 \propto \frac{2\ell}{3}$

$\Rightarrow \quad \frac{K_1}{K_2} = \frac{1}{2} \quad \Rightarrow \quad K_2 = 2K_1$

So $\quad \frac{K_1 \times 2K_1}{K_1 + 2K_1} = K$

$\Rightarrow \quad K_1 = \frac{3}{2}K \quad \& \quad K_2 = 3K$

When connected in parallel

$K_{eq} = K_1 + K_2$

$= \frac{3}{2}K + 3K = \frac{9}{2}K$

So time period $\quad T = 2\pi\sqrt{\frac{m}{K}} \quad = 2\pi\sqrt{\frac{2m}{9K}}$

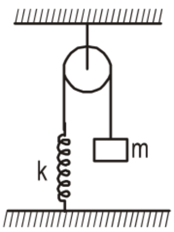
9. METHOD'S TO DETERMINE TIME PERIOD, ANGULAR FREQUENCY IN S.H.M.

(a) Force / torque method

(b) Energy method

Solved Examples

Example 17. The string, the spring and the pulley shown in figure are light.
Find the time period of the mass m.



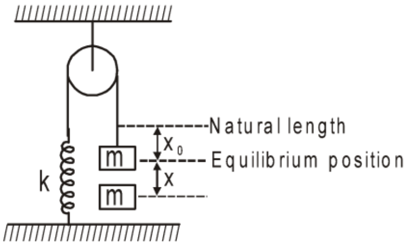
Solution (a) **Force Method**

Let in equilibrium position of the block, extension in spring is x_0.

$\therefore \quad kx_0 = mg \quad$ -- (1)

Now if we displace the block by x in the downward direction, net force on the block towards mean position is

$F = k(x + x_0) - mg \quad = kx$ using (1)



Hence the net force is acting towards mean position and is also proportional to x.So, the particle will perform S.H.M. and its time period would be

$$T = 2\pi\sqrt{\frac{m}{k}}$$

(b) Energy Method

Let gravitational potential energy to be zero at the level of the block when spring is in its natural length.

Now at a distance x below that level, let speed of the block be v.

Since total mechanical energy is conserved in S.H.M.

$\therefore$ $- mgx + \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant}$

Differentiating w.r.t. time, we get

$- mgv + kxv + mva = 0$

where a is acceleration.

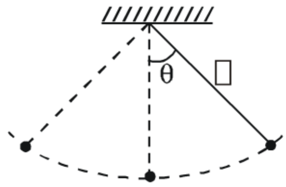
$\therefore$ $F = ma = - kx + mg$ or $F = - k(x - \frac{mg}{k})$

This shows that for the motion, force constant is k and equilibrium position is $x = \frac{mg}{k}$.

So, the particle will perform S.H.M. and its time period would be $T = 2\pi\sqrt{\frac{m}{k}}$

10. SIMPLE PENDULUM

If a heavy point-mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum



Time period of a simple pendulum $T = 2\pi\sqrt{\frac{\ell}{g}}$

(some times we can take $g = \pi^2$ for making calculation simple)

Note :

- If angular amplitude of simple pendulum is more, then time period
  $T = 2\pi\sqrt{\frac{\ell}{g}}\left(1+\frac{\theta_0^2}{16}\right)$ **(Not in JEE, For other exams)**
  where θ_0 is in radians.
- General formula for time period of simple pendulum for pendulum oscillating near the surface of earth.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{R}+\frac{1}{\ell}\right)}}$$ where, R = Radius of the earth

- Time period of simple pendulum of infinite length is maximum and is given by:

  $T = 2\pi\sqrt{\frac{R}{g}}$ = 84.6 min (Where R is radius of earth)
- Time period of seconds pendulum is 2 sec and ℓ = 0.993 m.
- Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear S.H.M.
- If time period of clock based on simple pendulum increases then clock will be slow but if time period decrease then clock will be fast.
- If g remains constant & $\Delta\ell$ is change in length, then $\frac{\Delta T}{T}\times 100 = \frac{1}{2}\frac{\Delta\ell}{\ell}\times 100$
- If ℓ remain constant & Δg is change in acceleration then, $\frac{\Delta T}{T}\times 100 = -\frac{1}{2}\frac{\Delta g}{g}\times 100$
- If $\Delta\ell$ is change in length & Δg is change in acceleration due to gravity then,

  $$\frac{\Delta T}{T}\times 100 = \left[\frac{1}{2}\frac{\Delta\ell}{\ell} - \frac{1}{2}\frac{\Delta g}{g}\right]\times 100$$

Solved Examples

Example 18 A simple pendulum of length 40 cm oscillates with an angular amplitude of 0.04 rad. Find (a) the time period, (b) the linear amplitude of the bob, (c) the speed of the bob when the string makes 0.02 rad with the vertical and (d) the angular acceleration when the bob is in momentary rest. Take g = 10 m/s^2.

Solution : (a) The angular frequency is $\omega = \sqrt{g/\ell} = \sqrt{\frac{10\,m/s^2}{0.4m}} = 5\ s^{-1}$

the time period is $\frac{2\pi}{\omega} = \frac{2\pi}{5s^{-1}} = 1.26$ s.

(b) Linear amplitude = 40 cm × 0.04 = 1.6 cm

(c) Angular speed at displacement 0.02 rad is

$\Omega = (5\ s^{-1})\sqrt{(0.04)^2 - (0.02)^2}$ rad = 0.17 rad/s.

where speed of the bob at this instant

= (40 cm) × 0.175^{-1} = 6.8 cm/s.

(d) At momentary rest, the bob is in extreme position.

Thus, the angular acceleration α = (0.04 rad) (25 s^{-2}) = 1 rad/s^2.

10.1 Time Period of Simple Pendulum in accelerating Reference Frame :

$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$ where

$g_{eff.}$ = Effective acceleration due to gravity in reference system = $|\vec{g} - \vec{a}|$
= acceleration of the point of suspension w.r.t. ground.

Condition for applying this formula: $|\vec{g} - \vec{a}|$ = constant

Solved Examples

Example 19. A simple pendulum is suspended from the ceiling of a car accelerating uniformly on a horizontal road. If the acceleration is a_0 and the length of the pendulum is ℓ, find the time period of small oscillations about the mean position.

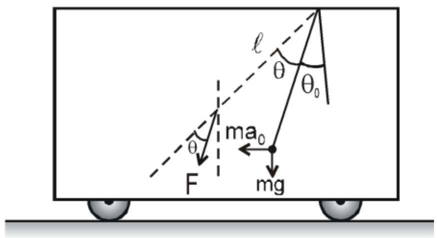
Solution : We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a psuedo force ma_0 on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If θ_0 be the angle made by the string with the vertical, the tension, weight and the psuedo force will add to zero in this position.

Hence, resultant of mg and ma_0 (say F = $m\sqrt{g^2 + a_0^2}$) has to be along the string.

$\therefore \quad \tan\theta_0 = \frac{ma_0}{mg} = \frac{a_0}{g}$

Now, suppose the string is further deflected by an angle θ as shown in figure.



Now, restoring torque can be given by

$(F \sin\theta)\,\ell = -\,m\,\ell^2\alpha$

Substituting F and using $\sin\theta = \theta$, for small θ.

$\left(m\sqrt{g^2 + a_0^2}\right)\ell\theta = -\,m\,\ell^2\alpha$

or, $\quad \alpha = -\frac{m\sqrt{g^2 + a_0^2}}{}\,\theta \qquad$ so; $\quad \omega^2 = \frac{\sqrt{g^2 + a_0^2}}{\ell}$

This is an equation of simple harmonic motion with time period $T = \frac{2\pi}{\omega} = 2\pi \frac{\sqrt{\ell}}{(g^2 + a_0^2)^{1/4}}$

10.2 If forces other then $m\vec{g}$ acts then :

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}} \quad \text{where} \quad g_{eff.} = \left|\vec{g} + \frac{\vec{F}}{m}\right|$$

$\vec{F}$ = constant force acting on 'm'.

Solved Examples

Example 20. A simple pendulum of length 'ℓ' and having bob of mass 'm' is doing angular SHM inside water.

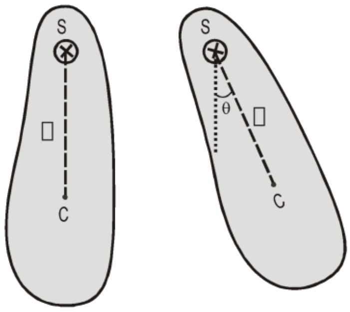
A constant buoyant force equal to half the weight of the bob is acting on the ball. Find the time period of oscillations?

Solution : Here $g_{eff.} = g - \frac{mg/2}{m} = g/2$.

Hence $T = 2\pi\sqrt{\frac{2\ell}{g}}$

11. COMPOUND PENDULUM / PHYSICAL PENDULUM

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.



C = Position of center of mass
S = Point of suspension
ℓ = Distance between point of suspension and center of mass (it remains constant during motion)

For small angular displacement "θ" from mean position
The restoring torque is given by

$\tau = -mg\ell\sin\theta$

$\tau = -mg\ell\theta$ $\quad\because$ for small θ, $\sin\theta = \theta$

or, $I\alpha = -mg\ell\theta$ where, I = Moment of inertia about point of suspension.

or, $\alpha = -\frac{mg\ell}{I}\theta$ or, $\omega^2 = \frac{mg\ell}{I}$

Time period, $T = 2\pi\sqrt{\frac{I}{mg\ell}}$ $\quad I = I_{CM} + m\ell^2$

Where I_{CM} = moment of inertia relative to the axis which passes from the center of mass & parallel to the axis of oscillation.

$$T = 2\pi\sqrt{\frac{I_{CM} + m\ell^2}{mg\ell}}$$

where $I_{CM} = mk^2$

k = gyration radius (about axis passing from centre of mass)

$T = 2\pi\sqrt{\frac{mk^2 + m\ell^2}{mg\ell}}$ $\quad T = 2\pi\sqrt{\frac{k^2 + \ell^2}{\ell g}} = 2\pi\sqrt{\frac{L_{eq}}{g}}$

$L_{eq} = \dfrac{k^2}{\ell} + \ell$ = equivalent length of simple pendulum ;

T is minimum when ℓ = k.

$T_{min} = 2\pi\sqrt{\dfrac{2k}{g}}$

Graph of T vs ℓ

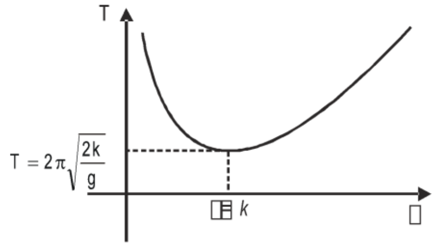
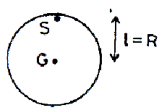
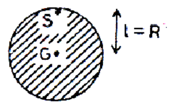
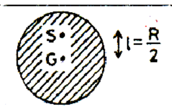
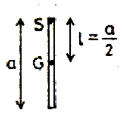
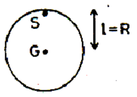
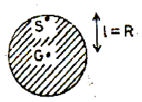


Table showing time periods for some rigid body pendulums

	Shape	Figure showing position of axis	l	k^2	$L = \dfrac{k^2}{l} + l$	$T = 2\pi\sqrt{\dfrac{L}{g}}$	For second pendulum T = 2 s, L = 1 m
	Circular ring		R	R^2	2R	$T = 2\pi\sqrt{\dfrac{2R}{g}}$	2R = 1m or R = 0.5m
	Circular disc		R	$\dfrac{R^2}{2}$	$\dfrac{3R}{2}$	$T = 2\pi\sqrt{\dfrac{3R}{2g}}$	$\dfrac{3}{2}R = 1$ m or $R = \dfrac{2}{3}$ m
3.	Circular disc		$\dfrac{R}{2}$	$\dfrac{R^2}{2}$	$\dfrac{3R}{2}$	$T = 2\pi\sqrt{\dfrac{3R}{2g}}$	same as above
4.	Light Rod		$\dfrac{a}{2}$	$\dfrac{a^2}{12}$	$\dfrac{2a}{3}$	$T = 2\pi\sqrt{\dfrac{2a}{3g}}$	$\dfrac{2}{3}a = 1$ m or $a = \dfrac{3}{2}$ m
5.	Spherical shell		R	$\dfrac{2}{3}R^2$	$\dfrac{5}{3}R$	$T = 2\pi\sqrt{\dfrac{5R}{3g}}$	$\dfrac{5}{3}R = 1$ m or $R = \dfrac{3}{5}$ m
6.	Solid sphere		R	$\dfrac{2}{5}R^2$	$\dfrac{7}{5}R$	$T = 2\pi\sqrt{\dfrac{7R}{5g}}$	$\dfrac{7}{5}R = 1$ m or $R = \dfrac{5}{7}$ m

Solved Examples

Example 21. A uniform rod of length 1.00 m is suspended through an end and is set into oscillation with small amplitude under gravity. Find the time period of oscillation. (g = 10 m/s²)

Solution : For small amplitude the angular motion is nearly simple harmonic and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mg(\ell/2)}} = 2\pi\sqrt{\frac{(m\ell^2/3)}{mg(\ell/2)}}$$

$$= 2\pi\sqrt{\frac{2\ell}{3g}} = 2\pi$$

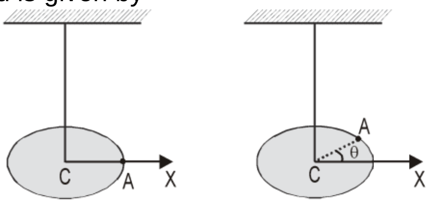
$$= \sqrt{\frac{2\times 1.00\ \ m}{3\times 10\ \ m/s^2}}\ s.$$

----

12. TORSIONAL PENDULUM

In torsional pendulum, an extended object is suspended at the centre by a light torsion wire. A torsion wire is essentially inextensible, but is free to twist about its axis. When the lower end of the wire is rotated by a slight amount, the wire applies a restoring torque causing the body to oscillate rotationally when released.

The restoring torque produced is given by



$\tau = -C\theta$ where, C = Torsional constant

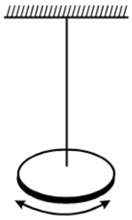
or, $I\alpha = -C\theta$ where, I = Moment of inertia about the vertical axis.

or, $\alpha = -\frac{C}{I}\theta$ ∴ Time Period, $T = 2\pi\sqrt{\frac{I}{C}}$

Solved Examples

Example 22. A uniform disc of radius 5.0 cm and mass 200 g is fixed at its centre to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

Solution : The situation is shown in figure. The moment of inertia of the disc about the wire is



$$I = \frac{mr^2}{2} = \frac{(0.200\ \ kg)(5.0\times10^{-2}m)^2}{2} = 2.5 \times 10^{-4}\ kg\text{ - }m^2.$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{or,} \quad C = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2(2.5\times10^{-4}kg-m^2)}{(0.20\ \ s)^2} = 0.25\ \frac{kg-m^2}{s^2}.$$

----

13. SUPERPOSITION OF TWO SHM'S

13.1 In same direction and of same frequency.

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin (\omega t + \theta)$, then resultant displacement

$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin (\omega t + \theta) = A \sin (\omega t + \varphi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$ and $\varphi = \tan^{-1}\left[\frac{A_2\sin\theta}{A_1 + A_2\cos\theta}\right]$

If $\theta = 0$, both SHM's are in phase and $A = A_1 + A_2$

If $\theta = \pi$, both SHM's are out of phase and $A = | A_1 - A_2 |$

The resultant amplitude due to superposition of two or more than two SHM's of this case can also be found by phasor diagram also.

13.2 In same direction but are of different frequencies.

$x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$ This resultant motion is not SHM.

13.3 In two perpendicular directions.

$x = A \sin \omega t$

$y = B \sin (\omega t + \theta)$

Case (i) : If $\theta = 0$ or π then $y = \pm (B/A)\, x$. So path will be straight line & resultant displacement will be $r = (x^2 + y^2)^{1/2} = (A^2 + B^2)^{1/2} \sin\omega t$

which is equation of SHM having amplitude $\sqrt{A^2 + B^2}$

Case (ii) : If $\theta = \frac{\pi}{2}$ then. $x = A \sin \omega t$
$y = B \sin (\omega t + \pi/2) = B \cos \omega t$

so, resultant will be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$. i.e. equation of an ellipse and if A = B, then superposition will be an equation of circle. This resultant motion is not SHM.

13.4 Superposition of SHM's along the same direction (using phasor diagram)

If two or more SHM's are along the same line, their resultant can be obtained by vector addition by making phasor diagram.

1. Amplitude of SHM is taken as length(magnitude) of vector.

2. Phase difference between the vectors is taken as the angle between these vectors. The magnitude of resultant of vector's give resultant amplitude of SHM and angle of resultant vector gives phase constant of resultant SHM.

For example;

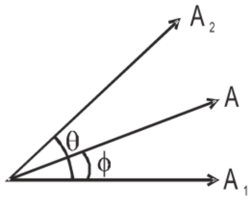
$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin (\omega t + \theta)$

If equation of resultant SHM is taken as $x = A \sin (\omega t + \varphi)$

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$

$\tan \varphi = \frac{A_2 \sin\theta}{A_1 + A_2 \cos\theta}$



Phasor Diagram

Solved Examples

Example 23. Find the amplitude of the simple harmonic motion obtained by combining the motions

$x_1 = (2.0 \text{ cm}) \sin\omega t$

and $x_2 = (2.0 \text{ cm}) \sin (\omega t + \pi/3)$.

Solution : The two equations given represent simple harmonic motions along X-axis with amplitudes $A_1 = 2.0$ cm and $A_2 = 2.0$ cm. The phase differnce between the two simple harmonic motions is $\pi/3$. The resultant simple harmonic motion will have an amplitude A given by

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta} = \sqrt{(2.0\text{cm})^2 + (2.0\ \text{cm})^2 + 2(2.0\ \text{cm})^2 \cos\frac{\pi}{3}} = 3.5$ cm

Example 24. $x_1 = 3 \sin \omega t$; $x_2 = 4 \cos \omega t$

Find (i) amplitude of resultant SHM. (ii) equation of the resultant SHM.

Solution : First write all SHM's in terms of sine functions with positive amplitude. Keep "ωt" with positive sign.

$\therefore$ $x_1 = 3 \sin \omega t$

$x_2 = 4 \sin (\omega t + \pi/2)$

$A = \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \cos\frac{\pi}{2}} = \sqrt{9+16} = \sqrt{25} = 5$

$\tan \varphi = \dfrac{4\sin\frac{\pi}{2}}{3 + 4\cos\frac{\pi}{2}} = \dfrac{4}{3}$, $\varphi = 53^o$

equation $x = 5 \sin (\omega t + 53^o)$

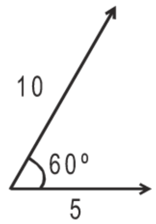
Example 25 $x_1 = 5 \sin (\omega t + 30^o)$

$x_2 = 10 \cos (\omega t)$

Find amplitude of resultant SHM.

Solution : $x_1 = 5 \sin (\omega t + 30^o)$

$x_2 = 10 \sin (\omega t + \frac{\pi}{2})$



Phasor Diagram

$A = \sqrt{5^2 + 10^2 + 2 \times 5 \times 10 \cos 60^o} = \sqrt{25 + 100 + 50}$

$= \sqrt{175} = 5\sqrt{7}$

Example 26 A particle is subjected to two simple harmonic motions

$x_1 = A_1 \sin \omega t$

and $x_2 = A_2 \sin (\omega t + \pi/3)$.

Find (a) the displacement at t = 0, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle.

Solution : (a) At $t = 0$, $x_1 = A_1 \sin \omega t = 0$

and $x_2 = A_2 \sin (\omega t + \pi/3) = A_2 \sin (\pi/3) = \dfrac{A_2\sqrt{3}}{2}$.

Thus, the resultant displacement at t = 0 is

$x = x_1 + x_2 = A_2 \dfrac{\sqrt{3}}{2}$

(b) The resultant of the two motions is a simple harmonic motion of the same angular frequency ω. The amplitude of the resultant motion is

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\pi/3)} = \sqrt{A_1^2 + A_2^2 + A_1A_2}$.

The maximum speed is $u_{max} = A\,\omega = \omega\sqrt{A_1^2 + A_2^2 + A_1A_2}$

(c) The maximum acceleration is $a_{max} = A\,\omega^2 = \omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}$.

Example 27. A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.

Solution : Let the amplitudes of the individual motions be A each. The resultant amplitude is also A. If the phase difference between the two motions is δ,

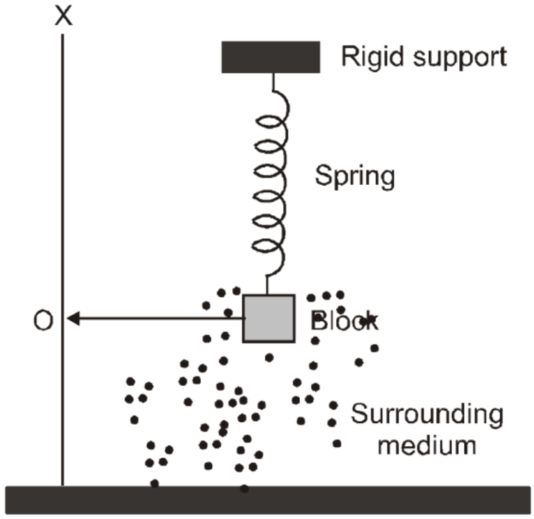
$A = \sqrt{A^2 + A^2 + 2A.\ A.\cos\delta}$

$A = A\sqrt{2(1+\cos\delta)} = 2A\cos\frac{\delta}{2} \Rightarrow \cos\frac{\delta}{2} = \frac{1}{2} \Rightarrow \delta = 2\pi/3.$

14. DAMPED HARMONIC OSCILLATOR

We know that the motion of a simple pendulum, swinging in air, dies out eventually. This is because the air drag and the friction at the support oppose the motion of the pendulum and dissipate its energy gradually. The pendulum is said to execute damped oscillations. In damped oscillations, although the energy of the system is continuously dissipated, but for small damping,the oscillations remain approximately periodic. The dissipating forces are generally the frictional forces. To understand the effecet of such external forces on the motion of an ocillator, let us consider a system a shwon in the fig. A block mass m connected to an elastic spring of spring constant k oscillates vertically. If the block is pushed down a little and relesed , its angular frequency of oscillation is $\omega = \sqrt{\frac{k}{m}}$

However, in practice, the surrounding medium (air) will exert a damping force on the motion of the block and mechanical energy of block spring system will decrease .The energy loss will appear as heat of the surrounding medium (and the block also)



The Damping force depends on the nature of the surrounding medium . The damping force is generally proportional to velocity of bob . If the damping force is denoted F_d. We have

$F_d = -bV$

Where the positive constant b depends on characteristics of the medium (viscosity, for example) and the size and shape of the block. etc. Eq. is uaually valid only for small velocity.

Thus the total force acting on the mass at any time t, is

$F = -kx - bV$.

$m\,a(t) = -kx(t) - bV(t)$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The soultion of equation is

$x(t) = A\,e^{-bt/2m}\cos(\omega' t + \varphi)$

amplitude at time t is $A\ e^{\frac{-bt}{2m}}$ and ω' is the angular frequency of the damped oscillator given by,

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Now the mechanical energy of the undamped oscillator is $1/2\ kA^2$ for small damped oscillator

$$E(t) = \frac{1}{2}\ kA^2 e^{-bt/m}$$

Equation shows that the total energy of the sysetm decreases exponentially with time. Note that small damping mean that the dimensionless ratio $\left(\frac{b}{\sqrt{km}}\right)$ is much less then one

FORCED OSCILATONS AND RESONANCE

When a system (such as a simple pendulum or a block attached to a spring) is displaced from tis equlibrium position and released, it oscillates with its nature frequency ω, and the oscillations are called freee oscillations. All free oscillations eventually die out because of the ever present damping forces. However, and external agency can maintain these oscillations. These are called force or driven oscillations.

Suppose an external force F(t) of amplitude F_0 that varies periodically with time is applied to a damped oscillator .Such a force can be represented as.

$$F(t) = F_0 \cos \omega_d t$$

The motion of a particle under the combined action of a linear restoring force, damping force and a time dependent driving force represented by Eq. is given by ,

$m\,a(t) = -kx(t) - bv(t) + F_0 \cos\omega_d t'$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos \omega_d t$$

The solution of equation is $x(t) = A \cos (\omega_0 t + \varphi)$

The amplitude A is function of the forced frequency ω_d and the natural frequency ω. Analysis shows that it is given by

$$A = \frac{F_0}{\left\{m^2\left(\omega^2 - \omega_d^2\right)^2 + \omega_d^2 b^2\right\}^{1/2}}$$

and

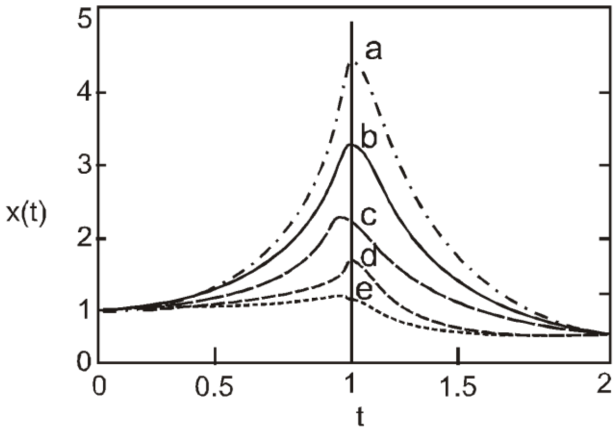
$$\tan\phi = \frac{-v_0}{\omega_d x_0}$$

where m is the mass of the particle and v_0 and x_0 are the velocity and the displacement of the particle at time t = 0 , which is the moment when we apply the periodic force. Equation shows that the amplitude of the forced oscillator depends on the (angular) frequecny of the driving force. We can see a different behaviour of the oscillator when ω_d is far from ω and when It is close to ω. We consider these two cases.

(a) **Small Damping, Driving Frequency far from natural Fequency :** In this case, $\omega_d b$ will be much smaller then $m\left(\omega^2 - \omega_d^2\right)$ and we can neglect tern, Then Eq. reduces to

$$A = \frac{F_0}{m\left(\omega^2 - \omega_d^2\right)}$$

Fig. Shows the dependence of the displacement amplitude of an oscillator on the angular frequency of the driving force for different amounts of damping present in the system . It may be noted that in all the case the amplitude is greatest when $\omega_d/\omega = 1$. The curves in this figure show that samller the damping the taller and narrower is the resonance peak.



(b) **DRIVING FREQUENCY CLOSE TO NATURAL FREQUENCY :** If ω_d is very close to ω, $m\left(\omega^2 - \omega_d^2\right)$ whould be much less than ω_d b, for any reasonable value of b the eq. reduces to $A = \frac{F_0}{\omega_d b}$

This makes it clear that the maximum possible amplitude for a given driving frequency is governed by the driving frequency and the damping and is never infinity. The phenomenon of increase in amplitude when the driving force is closed to the natural frequency of the oscillator is called resonance .

MAINTAINED OSCILLATION :

The oscillations, in which the energy loss of the oscillator is compensated by supplying energy from an external source are known as maintained oscillations.

The world is full of object that vibrate; banjo strings, electrons in a television antenna, our eardrums, the atoms in molecules, and ducks on a rough sea. Vibrations are not always desirable. Tall buildings must be constructed so that they do not sway excessively, and shock absorbers in your car dampen its up-and-down vibrations while traveling on a bumpy road. At other times much effort is made to produce vibrations. The hills and valleys on a record groove initiate mechanical and electrical vibrations in a stereo system, and when amplified, the electrical vibrations cause the diaphragm of a speaker to vibrate, The sound waves produced by the speaker cause our eardrums to vibrate, and we hear sound. Sound, like most forms of communication, is created and detected by objects that vibrate.

Solved Examples

Example 28. A damped oscillator attains the first amplitude of 500 mm after starting from rest. After 100 oscillations the amplitude remains 50 mm. The time period of motion is 2.3 sec. Find the damping coefficient.

Solution : The amplitude of damped oscillations is $a = a_0e^{-kt}$

The first amplitude is attained at t = T/4

After 100 oscillations, 201^{st} amplitude will be attained at

t = (100 T + T/4)

Therefore

$a_1 = a_0e^{-kT/4}$ and $a_{201} = a_0e^{-k(100\,T\,+\,T/4)}$

or $\frac{a_{201}}{a_1} = e^{-100kT}$

or $\frac{50}{500} = e^{-100\times k\times 2.3}$ or $e^{230k} = 100$ or $k = 0.01\ sec^{-1}$

Example 29. The relaxations time in the above question will be -

Solution : From above question $k = 0.01\ sec^{-1}$

$\therefore\ \tau = \frac{1}{2k} = \frac{1}{2\times 0.01} = 50$ sec.

Example 30. The frequency of a turning fork is 300 Hz. Its quantity factor $Q = 5 \times 10^4$. After what time its energy will become $1/10^{th}$ of the initial value -

Solution : Energy $E = E_0e^{-2kt} = E_0e^{-t/\tau}$

or $\frac{E_0}{10} = E_0e^{-t/\tau}$ or $e^{t/\tau} = 10$

or $t = 2.3\tau = 2.3\left(\frac{Q}{\omega}\right) = \frac{2.3\times 5\times 10^4}{2\pi\times 300} = 60$ sec.

Example 31. The amplitude of a damped harmonic oscillator become halved in 1 minute. After three minutes the amplitude will become 1/x of initial amplitude where x is -

Solution : The variation in amplitude of a damped harmonic oscillator with time is given by

$A = A_0e^{-bt}$;

A_0 = initial amplitude, b = damping factor

It is given that after 1 minute $A_1 = A_0/2 = A_0e^{-bt}$

$\therefore\ 2 = e^b$

$\therefore$ after 3 minutes $A_3 = A_0/x = A_0e^{-3b}$ $\therefore\ x = e^{3b} = 2^3$

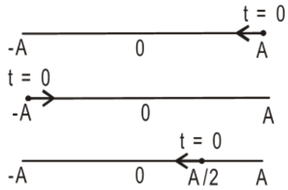
Solved Miscellaneous Problems

Problem 1. Write the equation of SHM for the situations shown below:

(a)

(b)

(c)



Solution :

(a) At t = 0 , x = +A

$x = A\sin(\omega t + \varphi)$

$A = A\sin(\varphi)$

$\varphi = \pi/2$

$x = A\sin\left(\omega t + \frac{\pi}{2}\right) = A\cos(\omega t)$

(b) At t = 0 , x = – A

$x = A\sin(\omega t + \varphi)$

$-A = A\sin\varphi$

$\varphi = \frac{3\pi}{2}$

$x = A\sin\left(\omega t + \frac{3\pi}{2}\right)$

$x = -A\cos(\omega t)$

(c) At t = 0 , $x = \frac{A}{2}$

$x = A\sin(\omega t + \varphi)$

$\frac{A}{2} = A\sin(\omega t + \varphi)$

$\frac{1}{2} = \sin\varphi \Rightarrow f = 30^\circ$, 150

Particular is moving towards the mean position and in negative direction.

velocity $v = A\omega\cos(\omega t + \varphi)$

At t = 0 , v = –ve

$v = A\omega\cos\varphi$

hence $\varphi = 150^\circ$

$x = A\sin(\omega t + 150^\circ)$

Ans. **(a)** $x = A\cos\omega t$; **(b)** $x = -A\cos\omega t$; **(c)** $x = A\sin(\omega t + 150^\circ)$

Problem 2. Block A of mass m is performing SHM of amplitude a. Another block B of mass m is gently placed on A when it passes through mean position and B sticks to A. Find the time period and amplitude of new SHM.

Solution :

Time period of mass m = $2\pi\sqrt{\frac{m}{K}}$

Time period of mass 2m = $2\pi\sqrt{\frac{2m}{K}}$

At mean position Kinetic energy = Total Energy

For mass m : $\frac{1}{2}mu^2 = \frac{1}{2}m\omega^2a^2$ .... (1)

For mass 2m : $\frac{1}{2}2mv^2 = \frac{1}{2}2m\left(\frac{\omega}{\sqrt{2}}\right)^2 A^2$ .... (2)

By Conservation of momentum

$mu = 2mv$

$v = \frac{u}{2}$

$\therefore \frac{1}{2}2m\left(\frac{u}{2}\right)^2 = \frac{1}{2}2m\left(\frac{\omega}{\sqrt{2}}\right)^2 A^2$ .... (3)

Divide equation (1) & (3)

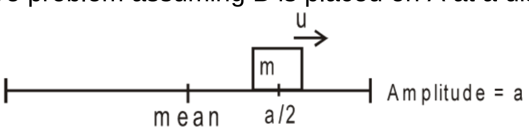
$4 = \frac{2a^2}{A^2}$

New Amplitude $A = \frac{a}{\sqrt{2}}$

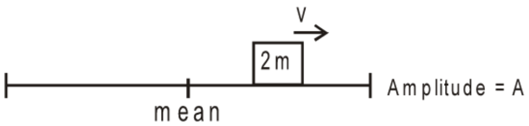
Ans. $T = 2\pi\sqrt{\frac{2m}{K}}$ Amplitude = $\frac{a}{\sqrt{2}}$

Problem 3. Repeat the above problem assuming B is placed on A at a distance $\frac{a}{2}$ from mean position.

Solution : $T = 2\pi\sqrt{\frac{m}{K}}$



$T = 2\pi\sqrt{\frac{2m}{K}}$



By conservation of momentum

$mu = 2mv \Rightarrow v = \frac{u}{2}$

Kinetic Energies at $\frac{a}{2}$

For mass m : $\frac{1}{2}mu^2 = \frac{1}{2}m\omega^2\left[a^2 - \left(\frac{a}{2}\right)^2\right]$ .... (1)

For mass 2m : $\frac{1}{2}2mv^2 = \frac{1}{2}2m\left(\frac{\omega}{\sqrt{2}}\right)^2\left[A^2 - \left(\frac{a}{2}\right)^2\right]$ .... (2)

$\frac{1}{2}2m\frac{u^2}{4} = \frac{1}{2}2m\frac{\omega^2}{2}\left[A^2 - \frac{a^2}{4}\right]$ .... (3)

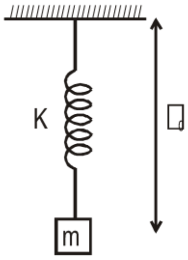
Dividing equation (1) & (3)

$$2 = \frac{a^2 - \frac{a^2}{4}}{A^2 - \frac{a^2}{4}}$$

New Amplitude $A = \sqrt{\frac{5}{8}}\,a$

Ans. $T = 2\pi\sqrt{\frac{2m}{K}}$, Amplitude = $a\sqrt{\frac{5}{8}}$

Problem 4. The block is allowed to fall, slowly from the position where spring is in its natural length. Find, maximum extension in the string.



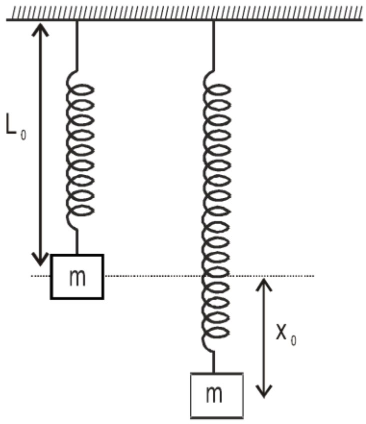
Solution : Since the block falls slowly from rest the maximum extension occurs when

$mg = Kx_0$

$x_0 = \frac{mg}{K}$ is maximum extension **Ans.** $\frac{mg}{K}$

Problem 5. In the above problem if block is released from there, what would be maximum extension.

Solution :



Let x_0 = maximum extension

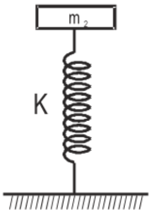
applying conservation of energy

$mgx_o = \frac{1}{2}Kx_0^2$

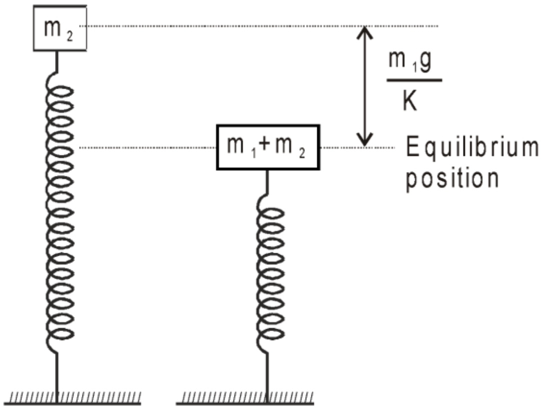
The velocity at the point of maximum extension is zero.

$x_0 = \frac{2mg}{K}$ is maximum extension **Ans.** $\frac{2mg}{K}$

Problem 6. Block of mass m_2 is in equilibrium as shown in figure. Another block of mass m_1 is kept gently on m_2. Find the time period of oscillation and amplitude.



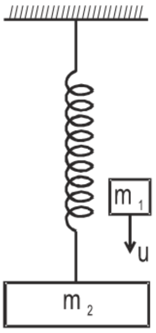
Solution : Time period $T = 2\pi\sqrt{\dfrac{m_1+m_2}{K}}$



At initial position since velocity is zero it is the extreme position.

Amplitude $A = \dfrac{m_1 g}{K}$ **Ans.** $T = 2\pi\sqrt{\dfrac{m_1+m_2}{K}}$ Amplitude $= \dfrac{m_1 g}{K}$

Problem 7. Block of mass m_2 is in equilibrium and at rest. The mass m_1 moving with velocity u vertically downwards collides with m_2 and sticks to it. Find the energy of oscillation.



Solution : Let the velocity of m_1 & m_2 be v after collision By conservation of momentum

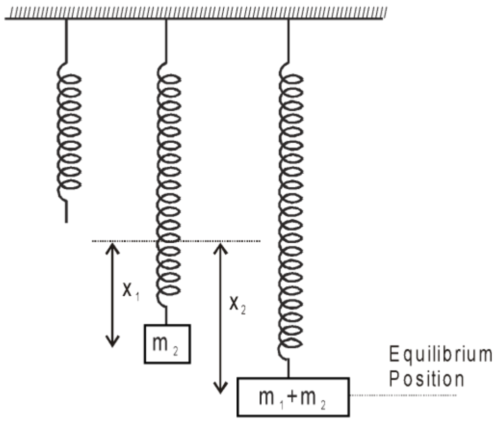
$m_1u = (m_1 + m_2)v$

$$v = \frac{m_1 u}{m_1+m_2}$$

Hence, $KE = \dfrac{1}{2}(m_1+m_2)v^2 = \dfrac{1}{2}\dfrac{m_1^2}{m_1+m_2}u^2$

$m_2g = Kx_1$

$(m_1 + m_2)g = Kx_2$



$$PE = \frac{1}{2}K(x_2 - x_1)^2 = \frac{1}{2}K\left(\frac{m_1 g}{K}\right)^2$$

$$\Rightarrow \quad PE = \frac{1}{2}\frac{m_1^2 g^2}{K}$$

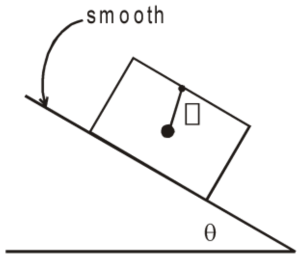
Therefore energy of oscillation is -

E = KE + PE

$$E = \frac{1}{2}\frac{m^2u^2}{m_1+m_2} + \frac{m_1{}^2g^2}{2K} \Rightarrow \frac{1}{2}\left[\frac{m_1{}^2u^2}{m_1+m_2} + \frac{m_1{}^2g^2}{K}\right]$$ **Ans.**

Problem 8. A block is placed on a smooth inclined plane and it is free to move.

A simple pendulum is attached in the block. Find its time period.



Solution : Let F_P = Pseudo force

For equilibrium position

$T\sin\alpha + F_P = mg\sin\theta$

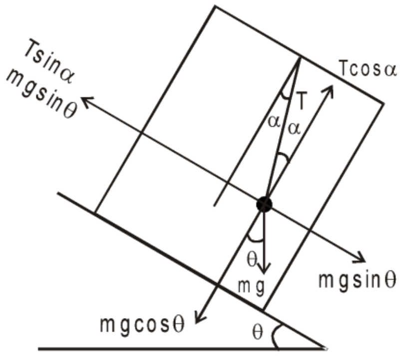
$T\sin\alpha + ma = mg\sin\theta$

since $a = g\sin\theta$

$T\sin\alpha + mg\sin\theta = mg\sin\theta$

$T\sin\alpha = 0$

$\therefore \alpha = 0$



Hence in equilibrium position the string is perpendicular to incline plane.

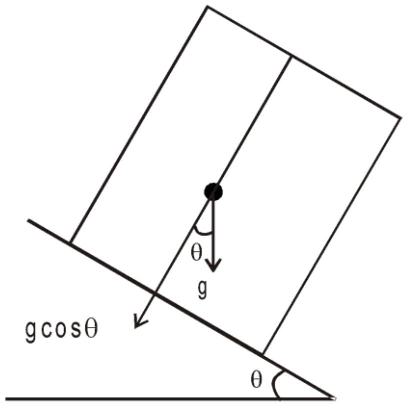
There fore effective 'g' is

$g_{eff} = g\cos\theta$

Time period

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

$$T = 2\pi\sqrt{\frac{\ell}{g\cos\theta}}$$ **Ans.**



Problem 9. $x_1 = 5\sin\omega t$

$x_2 = 5\sin(\omega t + 53^o)$

$x_3 = -10\cos\omega t$

Find amplitude of resultant SHM

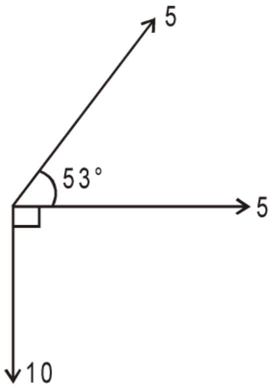
Solution : $x_1 = 5\sin\omega t$

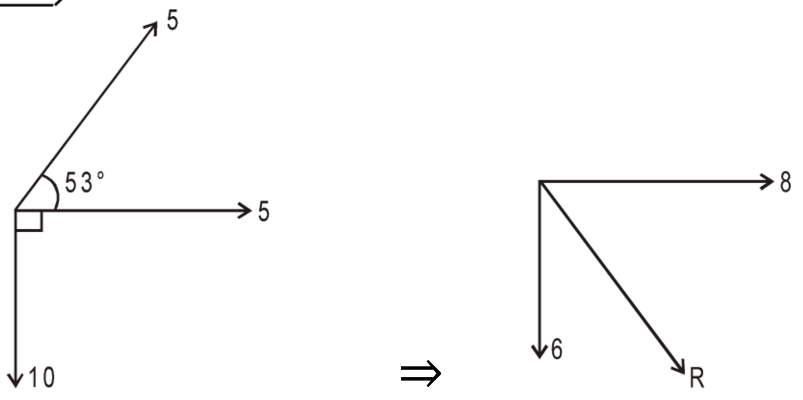
$x_2 = 5\sin(\omega t + 53^o)$

$x_3 = -10\cos\omega t$

we can write $x_3 = 10\sin(\omega t + 270°)$

Finding the resultant amplitude by vector notation.





Resultant Amplitude $|R| = \sqrt{8^2 + 6^2} = 10$ **Ans.**

Problem 10. Equation of two S.H.M. $x_1 = 5 \sin (2\pi t + \pi/4)$, $x_2 = 5\sqrt{2} (\sin 2\pi t + \cos 2\pi t)$ ratio of amplitude & phase difference will be

Solution : $x_1 = 5\sin (2\pi t + \pi/4)$

$x_2 = 5\sqrt{2}$ $(\sin 2\pi t + \cos 2\pi t)$

$x_2 = 10 (\sin 2\pi t \cos \pi/4 + \cos 2\pi t \sin \pi/4)$

$x_2 = 10 \sin (2\pi t + \pi/4)$

Ratio of amplitude = 5 : 10 = 1 : 2

Phase difference = $(2\pi t + \pi/4) - (2\pi t + \pi/4) = 0$