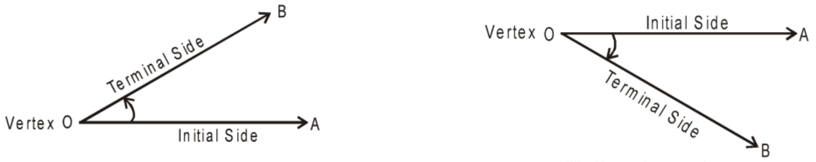
Trigonometry

Trigonometry

When writing about transcendental issues, be transcendentally clear .... Descartes, Rene

1. Angle :



(i) Positive angle (anticlockwise measurement)   (ii) Negative angle (clockwise measurement)

2. Systems For Measurement of Angles :

An angle can be measured in the following systems.

(i) Sexagesimal System (British System) : In this system $\frac{1}{360}$ of a complete circular turn is called a degree (°), $\frac{1}{60}$ of a degree is called a minute (') and $\frac{1}{60}$ of a minute is called a second ('').
One right angle = 90°, 1° = 60', 1' = 60''

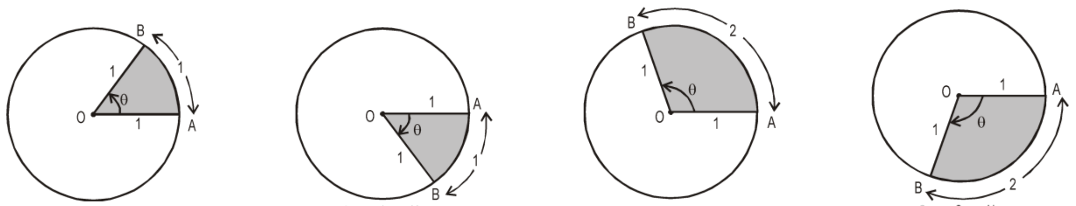
(ii) Centesimal System (French System) : In this system $\frac{1}{400}$ of a complete circular turn is called a grade ($_g$), $\frac{1}{100}$ of a grade is called a minute (`) and $\frac{1}{100}$ of a minute is called a second (``).

∴ One right angle = 100_g; 1_g = 100 ; 1' = 100''

Note : The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.

(iii) Circular System (Radian Measurement)
If in a circle of radius r, arc of length ℓ subtends an angle θ radian at the centre, we have

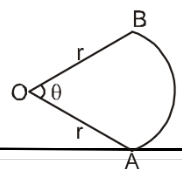
$\theta = \frac{\ell}{r}$ or $\ell = r\theta$.



θ = 1 radian (i)   θ = –1 radian (ii)   θ = 2 radian (iii)   θ = –2 radian (iv)

$\frac{\pi}{2}$ radian = 90° = 100_g

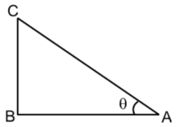
Notes: (i) If no symbol is mentioned while showing measurement of angle, then it is considered to be measured in radians.
e.g. θ = 15 implies 15 radian



(ii) Area of circular sector :

Area = $\frac{1}{2} r^2\theta$ sq. units

3. Trigonometric Ratios for Acute Angles :



$\sin\theta = \frac{BC}{AC}$, $\cos\theta = \frac{AB}{AC}$, $\tan\theta = \frac{BC}{AB}$,

$\text{cosec}\,\theta = \frac{AC}{BC}$ $\sec\theta = \frac{AC}{AB}$ $\cot\theta = \frac{AB}{BC}$

Trignometric ratios for some special acute angles

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}} = 1$
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.

Notes :

(i) $\sin^2 x + \cos^2 x = 1$

(ii) $1 + \tan^2 x = \sec^2 x$ $\{x \neq (2n + 1)\frac{\pi}{2}$; $n \in Z\}$

(iii) $1 + \cot^2 x = \text{cosec}^2 x$ $\{x \neq n\pi$; $n \in Z\}$

(iv) $\sec x = \frac{1}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer

(v) $\tan x = \frac{\sin x}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer.

(vi) $\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, where n is any integer.

4. Sign of The Trigonometric Functions :

2nd quadrant only sin θ and cosec θ are (+)ive	**1st quadrant** All trignometric ratios are (+)ive
3rd quadrant only tan θ and cot θ are (+)ive	**4th quadrant** only cos θ and sec θ are (+)ive

5. Trigonometric Ratios of allied angles :

If θ is any angle, then $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ etc. are called allied angles.

Trigonometry

(i) Trigonometric Ratios of $(-\theta)$:

(a) $\sin(-\theta) = -\sin\theta$

(b) $\cos(-\theta) = \cos\theta.$

(c) $\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$

(d) $\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta$

(e) $\sec(-\theta) = \frac{r}{x'} = \frac{r}{x} = \sec\theta.$

(f) $\text{cosec}(-\theta) = \frac{r}{y'} = \frac{r}{-y} = -\text{cosec}\,\theta.$

(ii) Trigonometric Ratios of $\pi - \theta$

(a) $\sin(\pi-\theta) = \sin\theta.$

(b) $\cos(\pi-\theta) = -\cos\theta.$

(c) $\tan(\pi-\theta) = -\tan\theta.$

(d) $\cot(\pi-\theta) = -\cot\theta.$

(e) $\sec(\pi-\theta) = -\sec\theta.$

(f) $\text{cosec}(\pi-\theta) = \text{cosec}\,\theta.$

(iii) Trigonometric Ratios of $\left(\frac{\pi}{2}-\theta\right)$ **:**

(a) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$,

(b) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta,$

(c) $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta,$

(d) $\text{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta,$

(e) $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta,$

(f) $\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\,\theta$

(iv) Trigonometric Ratios of $\left(\frac{\pi}{2}+\theta\right)$

(a) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta,$

(b) $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta,$

(c) $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta,$

(d) $\text{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta,$

(e) $\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta,$

(f) $\sec\left(\frac{\pi}{2}+\theta\right) = -\text{cosec}\,\theta$

(v) Trigonometric Ratios of $(\pi+\theta)$

(a) $\sin(\pi+\theta) = -\sin\theta,$

(b) $\cos(\pi+\theta) = -\cos\theta,$

(c) $\tan(\pi+\theta) = \tan\theta,$

(d) $\text{cosec}(\pi+\theta) = -\text{cosec}\,\theta,$

(e) $\cot(\pi+\theta) = \cot\theta,$

(f) $\sec(\pi+\theta) = -\sec\theta$

(vi) Trigonometric Ratios of $\left(\frac{3\pi}{2}-\theta\right)$

(a) $\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta,$

(b) $\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta,$

(c) $\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta,$

(d) $\text{cosec}\left(\frac{3\pi}{2}-\theta\right) = -\sec\theta$

(e) $\cot\left(\frac{3\pi}{2}-\theta\right) = \tan\theta,$

(f) $\sec\left(\frac{3\pi}{2}-\theta\right) = -\text{cosec}\,\theta,$

(vii) **Trigonometric Ratios of $\left(\frac{3\pi}{2}+\theta\right)$**

(a) $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, (b) $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$,

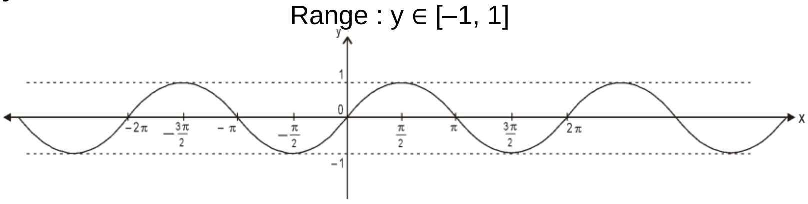
(c) $\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$, (d) $\cot\left(\frac{3\pi}{2}+\theta\right) = -\tan\theta$,

(e) $\sec\left(\frac{3\pi}{2}+\theta\right) = \text{cosec}\,\theta$, (f) $\text{cosec}\left(\frac{3\pi}{2}+\theta\right) = -\sec\theta$

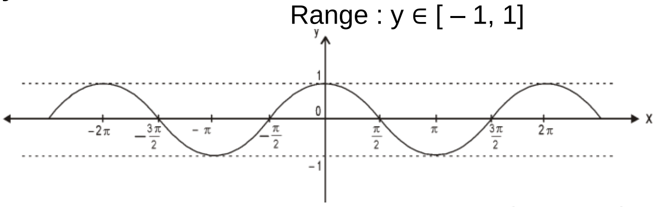
Note ; (i) $\sin\pi = \cos\frac{3\pi}{2} = 0$ (ii) $\cos\pi = \sin\frac{3\pi}{2} = -1$

6. <u>Trigonometric functions</u> :

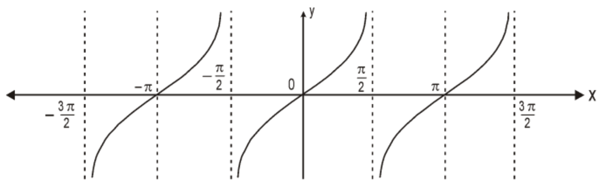
(i) **y = sin x** Domain : $x \in R$
Range : $y \in [-1, 1]$



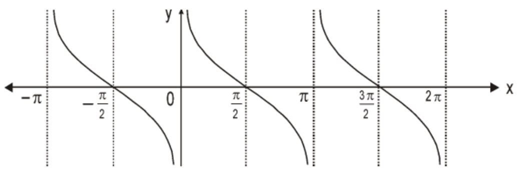
(ii) **y = cos x** Domain : $x \in R$
Range : $y \in [-1, 1]$



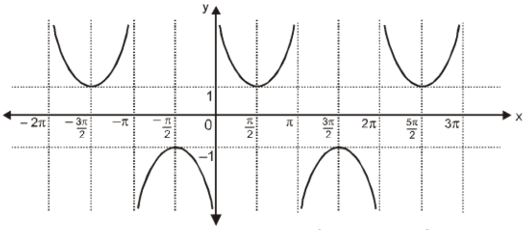
(iii) **y = tan x** Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$
Range : $y \in R$



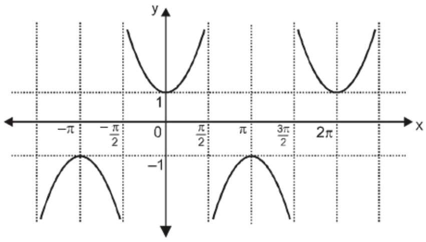
(iv) **y = cot x** Domain : $x \in R - \{n\pi\}$, $n \in I$
Range : $y \in R$



(v) **y = cosec x** Domain : $x \in R - \{n\pi\}$, $n \in I$
Range : $y \in (-\infty, -1] \cup [1, \infty)$



(vi) **y = sec x** Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$
Range : $y \in (-\infty, -1] \cup [1, \infty)$



7. Trigonometric functions of sum or difference of two angles :

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \sin A \sin B$

(iii) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(iv) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(v) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(vi) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(vii) $\sin(A + B + C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$

(viii) $\cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(ix) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

(x) $\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \dfrac{S_1 - S_3 + S_5 - ......}{1 - S_2 + S_4 - ....}$

where S_i denotes sum of product of tangent of angles taken i at a time

Example # 1 : Prove that

(i) $\sin(45^o + A)\cos(45^o - B) + \cos(45^o + A)\sin(45^o - B) = \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right) \tan\left(\frac{3\pi}{4}+\theta\right) = -1$

Solution : (i) Clearly $\sin(45^o + A)\cos(45^o - B) + \cos(45^o + A)\sin(45^o - B)$
$= \sin(45^o + A + 45^o - B) = \sin(90^o + A - B) = \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right) \times \tan\left(\frac{3\pi}{4}+\theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta} \times \dfrac{-1+\tan\theta}{1+\tan\theta} = -1$

Self Practice Problems :

(1) If $\sin\alpha = \frac{3}{5}$, $\cos\beta = \frac{5}{13}$, then find $\sin(\alpha+\beta)$

(2) Find the value of $\sin 105^o$

(3) Prove that $1 + \tan A \tan\frac{A}{2} = \tan A \cot\frac{A}{2} - 1 = \sec A$

Ans. (1) $-\frac{33}{65}, \frac{63}{65}$ (2) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

8. Transformation formulae :

(i) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

(ii) $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

(iii) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

(iv) $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

Example # 2 : Prove that $\sin 5A + \sin 3A = 2\sin 4A\cos A$

Solution : L.H.S. $\sin 5A + \sin 3A = 2\sin 4A\cos A$ = R.H.S. $\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

Example # 3 : Find the value of $2\sin 3\theta\cos\theta - \sin 4\theta - \sin 2\theta$

Solution : $2\sin 3\theta\cos\theta - \sin 4\theta - \sin 2\theta = 2\sin 3\theta\cos\theta - [2\sin 3\theta\cos\theta] = 0$

Example # 4 : Prove that

(i) $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$ (ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta\cos 4\theta$

Solution : (i) $\frac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta}$

$= \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta} = \frac{2\sin 2\theta\cos 5\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta$

(ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin 5\theta\cos 3\theta + \sin 3\theta\cos 5\theta}{\sin 5\theta\cos 3\theta - \sin 3\theta\cos 5\theta} = \frac{\sin 8\theta}{\sin 2\theta} = 4\cos 2\theta\cos 4\theta$

Self Practice Problems :

(4) Prove that

(i) $\cos 8x - \cos 5x = -2\sin\frac{13x}{2}\sin\frac{3x}{2}$

(ii) $\dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot \dfrac{A}{2}$

(iii) $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

(iv) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(v) $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$

(5) Prove that $\sin\dfrac{\theta}{2} + \sin\dfrac{7\theta}{2}\sin\dfrac{3\theta}{2}\sin\dfrac{11\theta}{2} = \sin 2\theta \sin 5\theta$

(6) Prove that cos A sin (B – C) + cos B sin (C – A) + cos C sin (A – B) = 0

(7) Prove that $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13} = 0$

9. Multiple and sub-multiple angles :

(i) $\sin 2A = 2 \sin A \cos A$

Note : $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ etc.

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Note : $2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$.

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Note : $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(iv) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(v) $\sin 3A = 3\sin A - 4\sin^3 A$

(vi) $\cos 3A = 4\cos^3 A - 3\cos A$

(vii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(viii) $\sin A \cdot \sin(60° - A)\sin(60° + A) = \dfrac{1}{4}\sin 3A$

(ix) $\cos A \cdot \cos(60° - A)\cos(60° + A) = \dfrac{1}{4}\cos 3A$

(x) $\tan A \cdot \tan(60° - A)\tan(60° + A) = \tan 3A$

Example # 5 : Prove that

(i) $\frac{\sin 2A}{1+\cos 2A}$ = tan A

(ii) tan A + cot A = 2 cosec 2 A

(iii) $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\frac{A}{2}\cot\frac{B}{2}$

Solution : (i) L.H.S. $\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A}$ = tan A

(ii) L.H.S. tan A + cot A = $\frac{1+\tan^2 A}{\tan A} = 2\left(\frac{1+\tan^2 A}{2\tan A}\right) = \frac{2}{\sin 2A}$ = 2 cosec 2 A

(iii) L.H.S. $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \frac{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\sin\left(\frac{A}{2}+B\right)}{2\cos^2\frac{A}{2}-2\cos\frac{A}{2}\cos\left(\frac{A}{2}+B\right)}$

$= \tan\frac{A}{2}\left[\frac{\sin\frac{A}{2}+\sin\left(\frac{A}{2}+B\right)}{\cos\frac{A}{2}-\cos\left(\frac{A}{2}+B\right)}\right] = \tan\frac{A}{2}\left[\frac{2\sin\frac{A+B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A+B}{2}\sin\left(\frac{B}{2}\right)}\right] = \tan\frac{A}{2}\cot\frac{B}{2}$

Self Practice Problems :

(8) Prove that $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$ = tan θ

(9) Prove that sin 20º sin 40º sin 60º sin 80º = $\frac{3}{16}$

(10) Prove that tan 3A tan 2A tan A = tan 3A – tan 2A – tan A

(11) Prove that $\tan\left(45^\circ+\frac{A}{2}\right)$ = sec A + tan A

10. Important trigonometric ratios of standard angles :

(i) sin nπ = 0 ; cos nπ = $(-1)^n$; tan nπ = 0, where n ∈ I

(ii) sin 15° or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ = cos 75° or $\cos\frac{5\pi}{12}$;

cos 15° or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ = sin 75° or $\sin\frac{5\pi}{12}$;

tan 15° = $\frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$ = cot 75° ; tan 75° = $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ = cot 15°

(iii) $\sin\frac{\pi}{10}$ or sin 18° = $\frac{\sqrt{5}-1}{4}$ = cos 72°

$\cos\frac{\pi}{5}$ or cos 36° = $\frac{\sqrt{5}+1}{4}$ = sin 54°

$\sin\frac{\pi}{5}$ or $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$

$\cos\frac{\pi}{10}$ or $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$

(iv) $\sin\frac{\pi}{8}$ or $\sin\left(22\frac{1}{2}\right)^\circ = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \cos\left(67\frac{1}{2}\right)^\circ$

$\cos\frac{\pi}{8}$ or $\cos\left(22\frac{1}{2}\right)^\circ = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sin\left(67\frac{1}{2}\right)^\circ$

$\tan\frac{\pi}{8}$ or $\tan\left(22\frac{1}{2}\right)^\circ = \sqrt{2}-1 = \cot\left(67\frac{1}{2}\right)^\circ$

$\cot\frac{\pi}{8}$ or $\cot\left(22\frac{1}{2}\right)^\circ = \sqrt{2}+1 = \tan\left(67\frac{1}{2}\right)^\circ$

11. Conditional identities :

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

Example # 6 : If $A + B + C = 180°$, Prove that, $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$.

Solution : Let $S = \sin^2 A + \sin^2 B + \sin^2 C$

so that $2S = 2\sin^2 A + 1 - \cos 2B + 1 - \cos 2C$

$= 2\sin^2 A + 2 - 2\cos(B + C)\cos(B - C)$

$= 2 - 2\cos^2 A + 2 - 2\cos(B + C)\cos(B - C)$

$\therefore$ $S = 2 + \cos A [\cos(B - C) + \cos(B + C)]$

since $\cos A = -\cos(B+C)$

$\therefore$ $S = 2 + 2\cos A \cos B \cos C$

Example # 7 : If $x + y + z = xyz$, Prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} . \frac{2y}{1-y^2} . \frac{2z}{1-z^2}$.

Solution : Put $x = \tan A$, $y = \tan B$ and $z = \tan C$,

so that we have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Rightarrow A + B + C = n\pi$, where $n \in I$

Hence L.H.S.

$\therefore \quad \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C}$.

= tan2A + tan2B + tan2C [$\because$ A + B + C = nπ]

= tan2A tan2B tan2C = $\frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$

Self Practice Problems :

(12) If A + B + C = 180°, prove that

(i) $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$

(ii) $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

(13) If A + B + C = 2S, prove that

(i) sin(S – A) sin(S – B) + sinS sin (S – C) = sinA sinB.

(ii) $\sin(S - A) + \sin(S - B) + \sin(S - C) - \sin S = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

12. Sine and Cosine series :

(i) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots + \sin\{\alpha + (n-1)\beta\} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$

(ii) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots. + \cos\{\alpha + (n-1)\beta\} = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$

where : $\beta \neq 2m\pi$, $m \in I$

Example # 8 : Find the summation of the following series

(i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(ii) $\sum_{i=1}^{13}\sin\left(\frac{\pi}{7}i\right)$

(iil) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

Solution : (i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{\cos\frac{\left(\frac{2\pi}{7}+\frac{6\pi}{7}\right)}{2}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$

$= \frac{\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = \frac{-\cos\frac{3\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = -\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$

(ii) $\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} \ldots\ldots\ldots\ldots + \sin\frac{13\pi}{7}$

$$\Rightarrow \frac{\sin\dfrac{\left(\dfrac{\pi}{7}+\dfrac{13\pi}{7}\right)}{2}\sin\left(\dfrac{13\pi}{14}\right)}{\sin\dfrac{\pi}{14}} \quad \Rightarrow \frac{\sin\pi.\sin\dfrac{13\pi}{14}}{\sin\dfrac{\pi}{14}} = 0$$

(iii) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

$$= \frac{\cos\dfrac{10\pi}{22}\sin\dfrac{5\pi}{11}}{\sin\dfrac{\pi}{11}} = \frac{\sin\dfrac{10\pi}{11}}{2\sin\dfrac{\pi}{11}} = \frac{1}{2}$$

Self Practice Problems :

Find sum of the following series :

(14) $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots\ldots$ up to n terms.

(15) $\sin2\alpha + \sin3\alpha + \sin4\alpha + \ldots.. + \sin n\alpha$, where $(n + 2)\alpha = 2\pi$

Ans. (14) $\frac{1}{2}$ (15) 0

13. Product series of cosine angles :

$$\cos\theta \cdot \cos2\theta \cdot \cos2^2\theta \cdot \cos2^3\theta \ldots\ldots \cos2^{n-1}\theta = \frac{\sin2^n\theta}{2^n\sin\theta}$$

14. Range of trigonometric expression :

$E = a\sin\theta + b\cos\theta$

$$\Rightarrow E = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$

Let $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ & $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$

$\Rightarrow E = \sqrt{a^2+b^2}\sin(\theta+\alpha)$, where $\tan\alpha = \frac{b}{a}$

Hence for any real value of θ,

$$-\sqrt{\mathbf{a^2+b^2}} \le \mathbf{E} \le \sqrt{\mathbf{a^2+b^2}}$$

Example # 9 : Find maximum and minimum values of following :

(i) $3\sin x + 4\cos x$ (ii) $1 + 2\sin x + 3\cos^2 x$

Solution : (i) We know

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le \sqrt{3^2+4^2}$ $\quad -5 \le 3\sin x + 4\cos x \le 5$

(ii) $1 + 2\sin x + 3\cos^2 x = -3\sin^2 x + 2\sin x + 4$

$$= -3\left(\sin^2 x - \frac{2\sin x}{3}\right) + 4 = -3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3}$$

Now $0 \le \left(\sin x - \frac{1}{3}\right)^2 \le \frac{16}{9} \Rightarrow -\frac{16}{3} \le -3\left(\sin x - \frac{1}{3}\right)^2 \le 0$

$-1 \le -3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3} \le \frac{13}{3}$

Self Practice Problem :

(16) Find maximum and minimum values of following

(i) $3 + (\sin x - 2)^2$ (ii) $10\cos^2 x - 6\sin x \cos x + 2\sin^2 x$

(iii) $\cos\theta + 3\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) + 6$

Ans. (i) max = 12, min = 4. (ii) max = 11, min = 1. (iii) max = 11, min = 1

15. Trigonometric Equations

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

(i) Solution of Trigonometric Equation :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

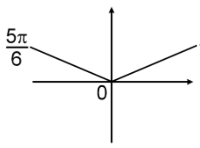
e.g. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$, ...

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :

(a) Principal solution (b) General solution.

(a) Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions. e.g. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.



$\because$ $\sin x = \frac{1}{2}$ $\because$ there exists two values

i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

(b) General Solution :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution.

General solution of some standard trigonometric equations are given below.

General Solution of Some Standard Trigonometric Equations :

- If $\sin\theta = \sin\alpha$ $\Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.
- If $\cos\theta = \cos\alpha$ $\Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

- If $\tan\theta = \tan\alpha$ $\Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.
- If $\sin^2\theta = \sin^2\alpha$ $\Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.
- If $\cos^2\theta = \cos^2\alpha$ $\Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.
- If $\tan^2\theta = \tan^2\alpha$ $\Rightarrow \theta = n\pi \pm \alpha$, $n \in I$. [**Note:** α is called the principal angle]

Some Important deductions :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2},\ n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2},\ n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2},\ n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi,\ n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi,\ n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in I$

Example # 10 : Solve $\sin\theta = \frac{1}{2}$.

Solution : $\because$ $\sin\theta = \frac{1}{2}$ $\Rightarrow$ $\sin\theta = \sin\frac{\pi}{6}$ $\therefore$ $\theta = n\pi + (-1)^n\frac{\pi}{6},\ n \in I$

Example # 11 : Solve : $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Solution : $\because$ $\sec 2\theta = -\frac{2}{\sqrt{3}}$ $\Rightarrow$ $\cos 2\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow$ $\cos 2\theta = \cos\frac{5\pi}{6}$

$\Rightarrow$ $2\theta = 2n\pi \pm \frac{5\pi}{6},\ n \in I$ $\Rightarrow$ $\theta = n\pi \pm \frac{5\pi}{12},\ n \in I$

Example # 12 : Solve $\tan\theta = 1$

Solution : $\because$ $\tan\theta = 1$ .... (i)

Let $1 = \tan\alpha \Rightarrow \tan\theta = \tan\frac{\pi}{4} \Rightarrow \theta = n\pi + \frac{\pi}{4},\ n \in I$

Self Practice Problems :

(17) Solve $\cot\theta = -1$ (18) Solve $\cos 3\theta = -\frac{1}{2}$

Ans. (17) $\theta = n\pi - \frac{\pi}{4},\ n \in I$ (18) $\frac{2n\pi}{3} \pm \frac{2\pi}{9},\ n \in I$

Example # 13 : Solve $\sin^2\theta = \frac{3}{4}$

Solution : $\because \sin^2\theta = \frac{3}{4} \Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2\theta = \sin^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$

Example # 14 : Solve $4\cot 2\theta = \cot^2\theta - \tan^2\theta$

Solution : This equation is not defined for $\theta = \frac{n\pi}{2}$ where $n \in I$

$\frac{4}{\tan 2\theta} = \frac{1}{\tan^2\theta} - \tan^2\theta \Rightarrow \frac{4(1-\tan^2\theta)}{2\tan\theta} = \frac{1-\tan^4\theta}{\tan^2\theta}$

or $(1 - \tan^2\theta)\,[2\tan\theta - (1 + \tan^2\theta)] \Rightarrow (1 - \tan^2\theta)\,[\tan\theta - 1]^2 = 0$

$\tan^2\theta = 1$

$\tan^2\theta = \tan^2\frac{\pi}{4}$ $\theta = n\pi \pm \frac{\pi}{4}, n \in I$

Self Practice Problem :

(19) Solve $7\cos^2\theta + 3\sin^2\theta = 4$. (20) Solve $2\sin^2x + \sin^2 2x = 2$

Ans. (19) $n\pi \pm \frac{\pi}{3}, n \in I$ (20) $(2n + 1)\frac{\pi}{2}, n \in I$ or $n\pi \pm \frac{\pi}{4}, n \in I$

(ii) Types of Trigonometric Equations :

Type -1

Trigonometric equations which can be solved by use of factorization.

Example # 15 : Solve $(2\cos x - \sin x)(1 + \sin x) = \cos^2x$.

Solution : $\because$ $(2\cos x - \sin x)(1 + \sin x) = \cos^2x$

$\Rightarrow$ $(2\cos x - \sin x)(1 + \sin x) - \cos^2x = 0$

$\Rightarrow$ $(2\cos x - \sin x)(1 + \sin x) - (1 - \sin x)(1 + \sin x) = 0$

$\Rightarrow$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $1 + \sin x = 0$ or $2\cos x - 1 = 0$

$\Rightarrow$ $\sin x = -1$ or $\cos x = \frac{1}{2}$

$\Rightarrow$ $x = (4n - 1)\frac{\pi}{2}, n \in I$ or $\cos x = \cos\frac{\pi}{3}$ $\Rightarrow x = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ Solution of given equation is

$(4n - 1)\frac{\pi}{2}, n \in I$ or $2n\pi \pm \frac{\pi}{3}$

Self Practice Problems :

(21) Solve $\cos^3x + \cos^2x - 4\cos^2\frac{x}{2} = 0$

(22) Solve $\cot^2\theta + 3\mathrm{cosec}\theta + 3 = 0$

Ans. (21) $(2n + 1)\pi,\ n \in I$ (22) $2n\pi - \frac{\pi}{2},\ n \in I$ or $n\pi + (-1)^{n+1}\frac{\pi}{6},\ n \in I$

Type - 2

Trigonometric equations which can be solved by reducing them in quadratic equations.

Example # 16 : Solve $2\cos^2\theta + 3\sin\theta = 0$

Solution : $2(1 - \sin^2\theta) + 3\sin\theta = 0 \Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$

$\Rightarrow (\sin\theta - 2)(2\sin\theta + 1) = 0 \Rightarrow \sin\theta \neq 2$

so $2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

$\sin\theta = \sin\left(-\frac{\pi}{6}\right)$ $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ or $\theta = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right)$

Self Practice Problems :

(23) Solve $\cos 2\theta - (\sqrt{2} + 1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right) = 0$ (24) Solve $4\cos\theta - 3\sec\theta = \tan\theta$

Ans. (23) $2n\pi \pm \frac{\pi}{3},\ n \in I$ or $2n\pi \pm \frac{\pi}{4},\ n \in I$

(24) $n\pi + (-1)^n\alpha$ where $\alpha = \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right),\ n \in I$

or $n\pi + (-1)^n\beta$ where $\beta = \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right),\ n \in I$

Type - 3

Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product.

Example # 17 : Solve $\cos x + \cos 3x = 2\cos 2x$

Solution : $\cos x + \cos 3x = 2\cos 2x$

$\Rightarrow 2\cos 2x \cos x - 2\cos 2x = 0 \Rightarrow 2\cos 2x\,[\cos x - 1] = 0$

$\cos 2x = 0$ or $\cos x = 1$

$2x = (2n + 1)\frac{\pi}{2}$ or $x = 2m\pi,\ n, m \in Z$

$x = (2n + 1)\frac{\pi}{4}$ or $x = 2m\pi,\ n, m \in Z$

Self Practice Problems :

(25) Solve $\sin 7\theta = \sin 3\theta + \sin\theta$

(26) Solve $5\sin x + 6\sin 2x + 5\sin 3x + \sin 4x = 0$

(27) Solve $\cos\theta - \sin 3\theta = \cos 2\theta$

Ans. (25) $\frac{n\pi}{3},\ n \in I$ or $\frac{n\pi}{2} \pm \frac{\pi}{12},\ n \in I$

(26) $\frac{n\pi}{2}$, $n \in I$ or $2n\pi \pm \frac{2\pi}{3}$, $n \in I$

(27) $\frac{2n\pi}{3}$, $n \in I$ or $2n\pi - \frac{\pi}{2}$, $n \in I$ or $n\pi + \frac{\pi}{4}$, $n \in I$

Type - 4

Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference.

Example # 18 : Solve $\sin 9x.\cos 7x = \sin 10x.\cos 6x$

Solution : $\because$ $\sin 9x.\cos 7x = \sin 10x.\cos 6x$

$\Rightarrow$ $2\sin 9x.\cos 7x = 2\sin 10x.\cos 6x$

$\Rightarrow$ $\sin 16x + \sin 2x = \sin 16x + \sin 4x$

$\Rightarrow$ $\sin 4x - \sin 2x = 0$

$\Rightarrow$ $2\sin 2x.\cos 2x - \sin 2x = 0$

$\Rightarrow$ $\sin 2x (2\cos 2x - 1) = 0$

$\Rightarrow$ $\sin 2x = 0$ or $2\cos 2x - 1 = 0$

$\Rightarrow$ $2x = n\pi, n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow$ $x = \frac{n\pi}{2}$, $n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3}$, $n \in I$ $\Rightarrow$ $x = n\pi \pm \frac{\pi}{6}$, $n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2}$, $n \in I$ or $n\pi \pm \frac{\pi}{6}$, $n \in I$

Type - 5

Trigonometric Equations of the form $a \sin x + b \cos x = c$, where $a, b, c \in R$, can be solved by dividing both sides of the equation by $\sqrt{a^2 + b^2}$.

Example # 19 : Solve $\sqrt{3} \sin x + \cos x = \sqrt{3}$

Solution : $\because$ $\sqrt{3} \sin x + \cos x = \sqrt{3}$ .... (i)

Here $a = \sqrt{3}$, $b = 1$.

$\therefore$ divide both sides of equation (i) by 2, we get

$\sin x . \frac{\sqrt{3}}{2} + \cos x. \frac{1}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\sin x.\sin \frac{\pi}{3} + \cos x.\cos \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}, n \in I \qquad \Rightarrow \quad x = 2n\pi \pm \frac{\pi}{6} + \frac{\pi}{3}, n \in I$

$\therefore$ Solution of given equation is $x = 2n\pi + \frac{\pi}{2}$ or $2n\pi + \frac{\pi}{6}$

Note : Trigonometric equation of the form a sinx + b cosx = c can also be solved by changing sinx and cosx into their corresponding tangent of half the angle.

Example # 20 : Solve $\cos x + \sqrt{3}\sin x = \sqrt{3}$ by changing sinx and cosx to there half angles

Solution : $\because \quad \cos x + \sqrt{3}\sin x = \sqrt{3}$ .... (i)

$\because \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$\Rightarrow \quad \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + \sqrt{3}\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = \sqrt{3}$ .... (ii)

Let $\tan\frac{x}{2} = t$

$\therefore$ equation (ii) becomes $\left(\frac{1-t^2}{1+t^2}\right) + \sqrt{3}\left(\frac{2t}{1+t^2}\right) = \sqrt{3}$

$\Rightarrow (\sqrt{3}+1)t^2 - 2\sqrt{3}\,t + \sqrt{3} - 1 = 0 \Rightarrow (\sqrt{3}+1)\,t^2 - (\sqrt{3}+1)t + (1-\sqrt{3})t + \sqrt{3} - 1 = 0$

$\Rightarrow [(\sqrt{3}+1)t + (1-\sqrt{3})]\,(t-1) = 0 \Rightarrow \tan\frac{x}{2} = 1$ or $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

so $x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi + \frac{\pi}{6}$

Self Practice Problems :

(28) Solve $\sqrt{3}\cos x + \sin x = 2$ (29) Solve $\sin x + \tan\frac{x}{2} = 0$

Ans. (28) $2n\pi + \frac{\pi}{6}$, $n \in I$ (29) $x = 2n\pi$, $n \in I$

Type - 6

Trigonometric equations of the form P(sinx ± cosx, sinx cosx) = 0, where p(y, z) is a polynomial, can be solved by using the substitution sinx ± cosx = t.

Example # 21 : Solve 5(sinx + cosx) = 5 + 2sinx.cosx

Solution : $\because$ 5(sinx + cosx) = 5 + 2sinx.cosx ....(i)

Let sinx + cosx = t

$\Rightarrow \quad \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow \quad \sin x.\cos x = \frac{t^2-1}{2}$

Now put sinx + cosx = t and sinx.cosx = $\frac{t^2 - 1}{2}$ in (i)

⇒ $t^2 - 5t + 4 = 0$ ⇒ t = 1 or t = 4

⇒ t ≠ 4 because $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

so t = 1

⇒ sinx + cosx = 1 .... (ii)

divide both sides of equation (ii) by $\sqrt{2}$, we get

⇒ $\sin x . \frac{1}{\sqrt{2}} + \cos x . \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

⇒ $\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ ⇒ $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(i) if we take positive sign, we get $x = 2n\pi + \frac{\pi}{2}$, n ∈ I

(ii) if we take negative sign, we get

x = 2nπ, n ∈ I

Self Practice Problems :

(30) Solve sin2x + 5sinx + 1 + 5cosx = 0

(31) Solve 3cosx + 3sinx + sin3x – cos3x = 0

(32) Solve $(1 - \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.

Ans. (30) $n\pi - \frac{\pi}{4}$, n ∈ I (31) $n\pi - \frac{\pi}{4}$, n ∈ I

(32) $2n\pi + \frac{\pi}{2}$, n ∈ I or 2nπ, n ∈ I or $n\pi + \frac{\pi}{4}$, n ∈ I

Type - 7

Trigonometric equations which can be solved by the use of boundness of the trigonometric ratios sinx and cosx.

Example # 22 : Solve $\cos^{50}x - \sin^{50}x = 1$

Solution : $\cos^{50}x - \sin^{50}x = 1$ ⇒ $\cos^{50}x = \sin^{50}x + 1$ ⇒ L.H.S. ≤ 1 & R.H.S. ≥ 1

so, both sides should be equal to 1

$\cos^{50}x = 1$ & $\sin^{50}x = 0$

or sinx = 0 or x = nπ where n ∈ I

Self Practice Problems :

(33) Solve sin3x + cos2x = – 2

(34) Solve $\sqrt{3\sin 5x - \cos^2 x - 3} = 1 - \sin x$

Ans (33) $(4p - 3)\frac{\pi}{2}$, p ∈ I (34) $2m\pi + \frac{\pi}{2}$, m ∈ I

Important points :

- Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method is wrong and those obtained by another method is correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of $n = ...-2, -1, 0, 1, 2, 3.......$ etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

- While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For Example, suppose we have the equation $\tan x = 2 \sin x$. Here by dividing both sides by $\sin x$, we get $\cos x = \frac{1}{2}$. This is not equivalent to the original equation. Here the roots obtained by $\sin x = 0$, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.
- While equating one of the factors to zero, take care of the other factor that it should not become infinite. For Example, if we have the equation $\sin x = 0$, which can be written as $\cos x \tan x = 0$. Here we cannot put $\cos x = 0$, since for $\cos x = 0$, $\tan x = \sin x / \cos x$ is infinite.
- Avoid squaring : When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.

  For Example : Consider the equation,

  $\sin\theta + \cos\theta = 1$ .....(1)

  Squaring we get

  $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .....(2)

  i.e. $2\theta = n\pi$ or $\theta = n\pi/2$,

  This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$

  Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as

  $\sin\pi + \cos\pi = -1, \neq 1$ and $\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1, \neq 1$.

  The reason for this is simple.

  The equation (2) is not equivalent to (1) and (2) contains two equations : $\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore we get extra solutions.

  Thus if squaring is must, verify each of the solution.
- Some necessary restrictions :

  If the equation involves $\tan x$, $\sec x$, take $\cos x \neq 0$. If $\cot x$ or $\text{cosec}\, x$ appear, take $\sin x \neq 0$.

  If log appear in the equation, i.e. $\log [f(\theta)]$ appear in the equation, use $f(\theta) > 0$ and base of $\log > 0, \neq 1$.

  Also note that $\sqrt{[f(\theta)]}$ is always positive, for Example $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm \sin\theta$.

- Verification : Student are adviced to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

16. Trigonometric InEquations :

Solutions of elementary trigonometric inequalities are obtained from graphs.

Inequality	**Set of solutions of inequality (n∈z)**
$\sin x > a$ ($\|a\| < 1$)	$x \in (\sin^{-1}a + 2\pi n, \pi - \sin^{-1}a + 2\pi n)$
$\sin x < a$ ($\|a\| < 1$)	$x \in (-\pi - \sin^{-1}a + 2\pi n, \sin^{-1}a + 2\pi n)$
$\cos x > a$ ($\|a\| < 1$)	$x \in (-\cos^{-1} a + 2\pi n, \cos^{-1} a + 2\pi n)$
$\cos x < a$ ($\|a\| < 1$)	$x \in (\cos^{-1} a + 2\pi n, 2\pi - \cos^{-1} a + 2\pi n)$
$\tan x > a$	$x \in (\tan^{-1} a + \pi n, \pi/2 + \pi n)$
$\tan x < a$	$x \in \left(-\frac{\pi}{2} + \pi n,\ \tan^{-1} a + \pi n\right)$

Inequalities of the form R(y) > 0, R(y) < 0, where R is a certain rational function and y is a trigonometric function (sine, cosine or tangent), are usually solved in two stages: first the rational inequality is solved for the unknown y and then follows the solution of an elementary trigonometric inequality.

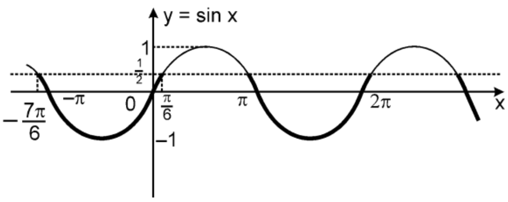
Example # 23 : Solve the inequality $2 \sin^2 x - 7 \sin x + 3 > 0$

Solution : Designating $\sin x = y$, we get an inequality

$2y^2 - 7y + 3 > 0$,

whose set of solutions is $y < 1/2$, $y > 3$. Returning to the initial unknown, we find that the given inequality is equivalent to two inequalities :

$\sin x < 1/2$ and $\sin x > 3$.



The second inequality has no solution and the solution of the first is

$x \in \left(-\frac{7\pi}{6} + 2n\pi,\ \frac{\pi}{6} + 2n\pi\right)$, $n \in Z$

Example # 24 : Solve the inequality $\cos x + \cos 2x + \cos 3x > 0$

Solution : Transforming the sum of the end terms into a product, we get an inequality

$$\cos 2x + 2 \cos 2x \cos x > 0$$

Substituting cos 2x before the brackets, we obtain

$$\cos 2x (2 \cos x + 1) > 0$$

This inequality is equivalent to two systems of elementary inequalities :

cos 2x < 0, and cos 2x > 0

cos x < –1/2, cos x > –1/2

Combining the solutions of these systems, we get a solution of the initial inequality

$$\left(\frac{2\pi}{3}+2\pi n,\ \frac{3\pi}{4}+2\pi n\right) \cup \left(\frac{5\pi}{4}+2\pi n,\ \frac{4\pi}{3}+2\pi n\right) \cup \left(-\frac{\pi}{4}+2\pi n,\ \frac{\pi}{4}+2\pi n\right) \quad (n \in Z)$$

17. Heights and distances

Angle of elevation and depression :

Let OX be a horizontal line and P be a point which is above point O. If an observer (eye of observer) is at point O and an object is lying at point P then ∠XOP is called angle of elevation as shown in figure. If an observer (eye of observer) is at point P and object is at point O then ∠QPO is called angle of depression.