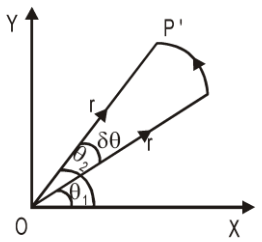
CIRCULAR MOTION

1. Fundamental parametre of circular motion

1.1 Radius vector : The vector joining the centre of the circle and the center of the particle performing circular motion is called radius vector.
It has constant magnitude and variable direction

1.2 Angular displacement (δθ or θ)
The angle described by radius vector is called angular displacement.



Infinitesimal angular displacement is a vector quantity. However, finite angular displacement is a scalar quantity.
S.I Unit Radian
Dimension : $M^0L^0T^0$

1 radian = $\frac{360}{2\pi}$

No. Of revolution = $\frac{\text{angular dispalcement}}{2\pi}$

In 1 revolution $\Delta\theta$ = 360° = 2π radian
In N revolution $\Delta\theta$ = $360^\circ \times N$ = $2\pi N$ radian
Clockwise rotation is taken as negative
Anticlockwise rotation is taken as positive

Solved Example

Example 1. If a particle complete one and half revolution along the circumference of a circle then its angular displacement is -
(1) 0 (2) π (3) 2π (4) 2π

Solution : (4) = 3π

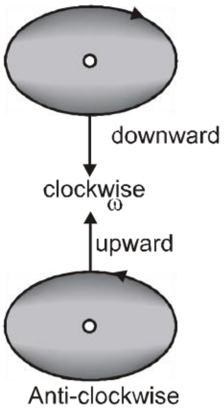
1.3 Angular velocity (ω)
The rate of change of angular displacement with time is called angular velocity.
It is a vector quantity.
The angle traced per unit time by the radius vector is called angular speed.

Instantaneous angular velocity = $\omega = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}$ or $\omega = \frac{d\theta}{dt}$

Average angular velocity $\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$

S.I. Unit **:** rad/sec
Dimension **:** $M^0L^0T^{-1}$
Direction : Infinitesimal angular displacement, angular velocity and angular acceleration are vector quantities whose direction is given by right hand rule.

Right hand Rule : Imagine the axis of rotation to be held in the right hand with fingers curled round the axis and the thumb stretched along the axis. If the curled fingers denote the sense of rotation, then the thumb denoted the direction of the angular velocity (or angular acceleration of infinitesimal angular displacement .



1.4 Angular acceleration (a)

The rate of change of angular velocity with time is called angular acceleration.

Average angular acceleration

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

It is a vector quantity, whose direction is along the change in direction of angular velocity.

S.I. Unit : radian/sec$_2$

Dimension : $M_0L_0T_{-2}$

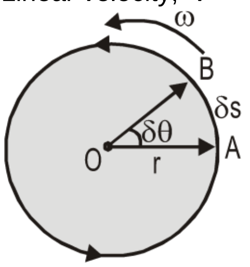
1.5 Relation between angular velocity and linear velocity

Suppose the particle moves along a circular path from point A to point B in infinitesimally small time δt.

As, $\delta t \to 0$, $\delta\theta \to 0$

∴ arc AB = chord AB i.e. displacement of the particle is along a straight line.

∴ Linear velocity, $v = \lim\limits_{\delta t \to 0} \frac{\delta s}{\delta t}$ But, $\delta s = r.\delta\theta$



∴ $v = \lim\limits_{\delta t \to 0} \frac{r.\delta s}{\delta t} = r \lim\limits_{\delta t \to 0} \frac{\delta\theta}{\delta t}$

But, $\lim\limits_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \omega$ = angular velocity

∴ $v = r . \omega$ [For circular motion only]

i.e. (linear velocity) = (Radian) × (angular velocity)

In vector notation, $\vec{v} = \vec{\omega} \times \vec{r}$ [in general]

Linear velocity of a particle performing circular motion is the vector product of its angular velocity and radius vector.

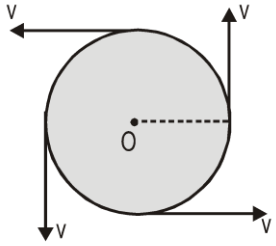
1.6 Relation between angular acceleration & linear acceleration

For perfect circular motion we know
$v = \omega r$
on differentiating with respect to time
we get $\frac{dv}{dt} = r\frac{d\omega}{dt}$
$a = r\,\alpha$
In vector form $\vec{a} = \vec{\alpha} \times \vec{r}$ (linear acc.) = (angular acc) × (radius)

2. Types of circular motion

2.1 uniform circular motion

Uniform circular motion : Motion of a particle along the circumference of a circle with a constant speed is called uniform circular motion. Uniform circular motion is an accelerated motion.
In case of uniform circular motion :
speed remains constant. v = constant

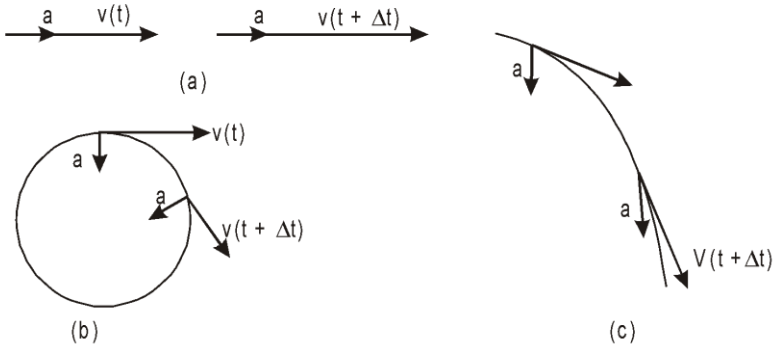


and $v = \omega r$
angular velocity ω = constant
motion will be periodic with time period = $T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$
Frequency of uniform circular motion : The number of revolutions performed per unit time by the particle performing uniform circular motion is called the frequency (n)
$\therefore$ $n = \frac{1}{T} = \frac{v}{2\pi r} = \frac{\omega}{2\pi}$
S.I. unit of frequency is Hz.
As ω = constant, from $\omega = \omega_0 + \alpha t$
angular acceleration $\alpha = 0$
As $a_t = \alpha r$, tangential acc. $a_t = 0$
As $a_t = 0$, $a = (a_r^2 + a_t^2)^{1/2}$ yields $a = a_r$, i.e. acceleration is not zero but along radius towards centre and has magnitude $a = a_r = (v^2/r) = r\omega^2$
Speed and magnitude of acceleration are constant. but their directions are always changing so velocity and acceleration are not constant. Direction of $\vec{v}$ is always along the tangent while that of $\vec{a}_r$ along the radius $\vec{v} \perp \vec{a}_r$

If the moving body comes to rest, i.e. $\vec{v} \to 0$, the body will move along the radius towards the centre and if radial acceleration a_r vanishes, the body will fly off along the tangent. So a tangential velocity and a radial acceleration (hence force) is a must for uniform circular motion.
As $\vec{F} = \frac{mv^2}{r} \neq 0$, so the body is not in equilibrium and linear momentum of the particle moving on the circle is not conserved. However, as the force is contral, i.e.,
$\vec{\tau} = 0$, so angular momentum is conserved, i.e., $\vec{p} \neq$ constant but $\vec{L}$ = constant
The work done by centripetal force is always zero as it is perpendicular to velocity and hence displacement. By work-energy theorem as work done = change in kinetic energy $\Delta K = 0$
So K (kinetic energy) remains constant
e.g. Planets revolving around the sun, motion of an electron around the nucleus in an atom

SPECIAL POINTS

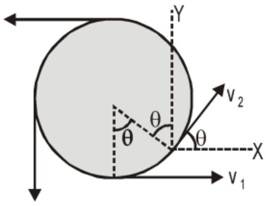
In one dimensional motion, acceleration is always parallel to velocity and changes only the magnitude of the velocity vector.



In uniform circular motion, acceleration is always perpendicular to velocity and changes only the direction of the velocity vector.
In the more general case, like projectile motion, acceleration is neither parallel nor perpendicular to figure summarizes these three cases.
If a particle moving with uniform speed v on a circle of radius r suffers angular displacement θ in time Δt then change in its velocity.

$\Delta\vec{v} = \Delta\vec{v}_2 - \Delta\vec{v}_1$ $\qquad$ $\vec{v}_1 = \vec{v}_1\,\hat{i}$ $\qquad$ $\vec{v}_2 = \vec{v}_2\cos\theta\,\hat{i} + \vec{v}_2\sin\theta\,\hat{j}$

$\Delta\vec{v} = (\vec{v}_2\cos\theta - \vec{v}_1)\,\hat{i} + \vec{v}_2\sin^2\,\hat{j}$



$|\Delta\vec{v}| = \sqrt{(\vec{v}_2\cos\theta - \vec{v}_1)^2 + \vec{v}_2\sin^2}$

$|\Delta\vec{v}| = \sqrt{2v^2 - 2v^2\cos\theta} = \sqrt{2v^2(1-\cos\theta)} = \sqrt{2v^2\left(2\sin^2\frac{\theta}{2}\right)}$

($\because$ $v_1 = v_2 = v$)

$|\Delta\vec{v}| = 2v\sin\frac{\theta}{2}$

Solved Example

Example 2. A particle is moving in a circle of radius r centrad at O with constant speed v. What is the change in velocity in moving from A to B ? Given ∠AOB = 40º .

Solution $|\Delta\vec{v}|$ = 2v sin40º/2 = 2 v sin 20º

2.2 Non uniform circular motion

A circular motion in which both the direction and magnitude of the velocity changes is called nonuniform circular motion.
A merry-go-round spinning up from rest to full speed, or a ball whirling around in a vertical circle.
The acceleration is neither parallel nor perpendicular to the velocity.
We can resolve the acceleration vector into two components :
Radial Acceleration : a_r perpendicular to the velocity ⇒ changes only the directions of velocity Acts just like the acceleration in uniform circular motion.

a_c = or $a_r = \frac{v^2}{r}$

Centripetal force : $F_c = \frac{mv^2}{r} = m\omega^2 r$

Tagential acceleration : a_r parallel to the velocity (since it is tangent to the path)
⇒ changes magnitude of the velocity acts just like one-dimensional acceleration

$\Rightarrow \quad a_t = \dfrac{dv}{dt}$

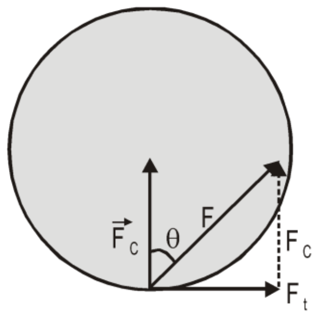
Tangential acceleration : $a_t = \dfrac{dv}{dt}$, where $v = \dfrac{ds}{dt}$ and s = length of arc
Tangential force : $F_t = ma_t$
The net acceleration vector is obtained by vector addition of these two components.

$$\boxed{a = \sqrt{a_r^2 + a_t^2}}$$

(a) In non-uniform circular motion :

speed $|\vec{v}| \neq$ constant angular velocity $\omega \neq$ constant
i.e. speed ≠ constant i.e. angular velocity ≠ constant
(b) In at any instant ⇒ v = magnitude of velocity of particle
⇒ r = radius of circular path
⇒ ω = angular velocity of a particle
then, at that instant $v = r\,\omega$

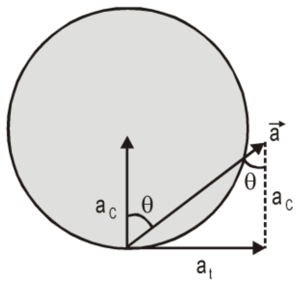
Net force on the particle



$\vec{F} = \vec{F}_c + \vec{F}_t \quad \Rightarrow \quad F = \sqrt{F_c^2 + F_t^2}$
If θ is the angle made by $F = F_c$,

then $\tan\theta = \dfrac{F_t}{F_c} \quad \Rightarrow \quad \theta = \tan^{-1}\left[\dfrac{F_t}{F_c}\right]$
[Note angle between F_c and F_t is 90º]
Angle between F and F_t is (90º – θ)

Net acceleration : $a = \sqrt{a_c^2 + a_t^2} = \dfrac{F_{net}}{m}$



The angle made by 'a' with a_c, $\tan\theta = \dfrac{a_t}{a_c} = \dfrac{F_t}{F_c}$

Special note :
In both uniform and non-uniform circular motion F_c is perpendicular to velocity.
So work done by centripetal force will be zero in both the cases.
In uniform circular motion $F_t = 0$, as $= a_t = 0$, so work done will be zero by tangential force.
But in non-uniform circular motion $F_t \neq 0$, so work done by tangential force is non zero.
Rate of work done by net force in non-uniform circular motion = rate of work done by tangential force

$\Rightarrow \quad P = \frac{dW}{dt} = \vec{F}_t . \vec{v} = \vec{F}_t . \frac{d\vec{x}}{dt}$

In a circle as tangent and radius are always normal to each other, so $\vec{a}_t \perp \vec{a}_r$.

Net acceleration in case of circular motion a = $a_r^2 = a_t^2$

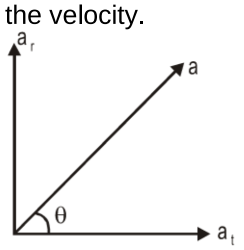
Here is must be noted that a_t governs the magnitude of $\vec{v}$ while a_r its direction of motion so that

if $a_r = 0$ and $a_t = 0$ $\quad$ $a \rightarrow 0 \Rightarrow$ motion is uniform translatory
if $a_r = 0$ and $a_t \neq 0$ $\quad$ $a \rightarrow a_t \Rightarrow$ motion is accelerated translatory
if $a_r \neq 0$ and $a_t = 0$ $\quad$ $a \rightarrow a_r \Rightarrow$ motion is uniform circular
if $a_r \neq 0$ and $a_t \neq 0$ $\quad$ $a \rightarrow \sqrt{a_r^2 + a_t^2} \Rightarrow$ motion is non-uniform circular.

Solved Examples

Example 3. A road makes a 90º bend with a radius of 190 m. A car enters the bend moving at 20 m/s. Finding this too fast, the driver decelerates at 0.92 m/s2. Determine the acceleration of the car when its speed rounding the bend has dropped to 15 m/s.

Solution Since it is rounding a curve, the car has a radial acceleration associated with its changing direction, in addition to the tangential deceleration that changes its speed. We are given that a_t = 0.92 m/s2; since the car is slowing down, the tangential acceleration is directed opposite the velocity.



The radial acceleration is $a_r = \frac{v^2}{r} = \frac{(15m/s)^2}{190m} = 1.2$ m/s2

Magnitude of net acceleration,

$a = \sqrt{a_r^2 + a_t^2}$ = [(1.2 m/s)2 + (0.92 m/s)2]1/2 = 1.5 m/s2

and points at an angle $\theta = \tan^{-1}\left(\frac{a_r}{a_t}\right) = \tan^{-1}\left(\frac{1.2m/s^2}{0.92m/s^2}\right) = 53°$

relative to the tangent line to the circle.

Example 4. A particle lis constrained to move in a circular path of radius r = 6m. Its velocity varies with time according to the relation v = 2t (m/s). Determine its
(i) centripetal acceleration,
(ii) tangential acceleration,
(iii) instantaneous acceleration at (a) t = 0 sec. and (b) t = 3 sec.

Solution (a) At = 0, v = 0, Thus $a_r = 0$

but $\frac{dv}{dt}$ =2 thus a_t = 2 m/s2 and a = $\sqrt{a_t^2 + a_r^2}$ = 2 m/s2

(b) At t = 3 sec. v = 6 m/s so $a_r = \frac{v^2}{r} = \frac{(6)^2}{6} = 6$ m/s2

and $a_t = \frac{dv}{dt} = 2$ m/s2 Therefore, a = $a = \sqrt{2^2 + 6^2}$ = $\sqrt{40}$ m/s2

Example 5. The kinetic energy of a particle moving along a circle of radius r depends on distance covereds as K = As2 where A is a const. Find the force acting on the particle as a function of s.

Solution According to given problem

$\frac{1}{2}$ mv2 = As2 or $v = s\sqrt{\frac{2A}{m}}$ ....(1)

So $a_r = \frac{v^2}{r} = \frac{2As^2}{mr}$ ....(2)

Further more as $a_t = \frac{dv}{dt} = \frac{dv}{ds}.\frac{ds}{dt} = v\frac{dv}{ds}$ ....(3)

from eqn. (1), ⇒ ....(4)

Substitute values from eqn. (1) & eqn. (4) in eqn. (3)

$$a_t = \left[s\sqrt{\frac{2A}{m}}\right]\left[\sqrt{\frac{2A}{m}}\right] = \frac{2As}{m}$$

so $$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left[\frac{2As^2}{mr}\right]^2 + \left[\frac{2As}{m}\right]^2}$$

i.e. $$a = \frac{2As}{m}\sqrt{1+[s/r]^2}$$

so $F = ma = 2As\sqrt{1+[s/r]^2}$

Example 6. A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration as is varying with time t as $a_c = k_2 rt_2$, where k is a constant. Determine the power delivered to particle by the forces acting on it.

Solution If v is instantaneous velocity, centripetal acceleration $a_c = \frac{v^2}{r}$ ⇒ $\frac{v^2}{r} = k_2 rt_2$ ⇒ v = krt

In circular motion work done by centripetally force is always zero & work is done only by tangential force.

∵ Tangent acceleration $a_t = \frac{dv}{dt} = \frac{d}{dt}(krt) = kr$

∴ Tangential force $F_t = ma_t = mkr$

Power $P = F_t v = (mkr)(krt) = mk_2 r_2 t$

3. Centripetal and centrifugal force

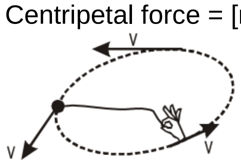
3.1 Centripetal force : In uniform circular motion the force acting on the particle along the radius and towards the centre keeps the body moving along the circular path . This force is called centripetal force.

Explanation :

(i) Centripetal force in necessary for uniform circular motion.

(ii) It is along the radius and towards the centre.

(iii) Centripetal force = [mass] × [centripetal acceleration] = $\frac{mv^2}{r}$ = $mr\omega_2$



(iv) Centripetal force is due to known interaction. Therefore it is a real force. If an object tied to a string its revolved uniformly in a horizontal circle, the centripetal force is due to the tension imparted to the string by the hand.

When a satellite is revolving in circular orbit round the earth, the centripetal force is due to the gravitational force of attraction between the satellite and the earth.

In an atom, an electron revolves in a circular orbit round the nucleus. The centripetal force is due to the electrostatic force of attraction between the positively charged nucleus and negatively charged.

Solved Examples

Example 7. Stone of mass 1kg is whirled in a circular path of radius 1 m. Find out the tension in the string if the linear velocity is 10 m/s ?

Solution Tension $\frac{mv^2}{R} = \frac{1\times(10)^2}{1}$ = 100 N

Example 8. A satellite of mass 10_7 kg is revolving around the earth with a time period of 30 days at a height of 1600 km. Find out the force of attraction on satellite by earth?

Solution Force = $m\omega^2R$ and $\omega = \frac{2\pi}{T} = \frac{2\times3.4}{30\times86400} = \frac{6.28}{2.59\times10^6}$

Force = $m\omega^2 r = \left(\frac{6.28}{2.59\times10^6}\right)^2 \times 10^7 \times (6400 + 1600) \times 10^3 = 2.34 \times 10^6$ N

3.2 Centrifugal force

The pseudo force experienced by a particle performing uniform circular motion due to accelerated frame of reference which is along the radius and directed way from the centre is called centrifugal force.

Explanation :

(i) Centrifugal force is a pseudo force as it is experienced due to accelerated frame of reference. The interaction of origin and away from the centre.

(ii) It is along the radius and away from the centre.

(iii) The centrifugal force in having the same magnitude as that of centripetal force. But, its direction is opposite to that of centripetal force . It is not due to reaction of centripetal force because without action, reaction not possible, but centrifugal force can exists without centripetal force.

(iv) Magnitude of the centrifugal force is $\frac{mv^2}{r}$ or $mr\omega^2$.

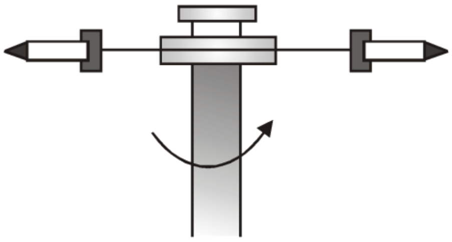
Note : Pseudo force acts in non inertial frame i.e. accelerated frame of reference in which Neutron's law's of motion do not hold good.
When a car moving along a horizontal curve takes a turn, the person in the car experiences a push in the outward direction.
The coin placed slightly away from the centre of a rotating gramophone disc slips towards the edge of the disc.
A cyclist moving fast along a curved road has to lean inwards to keep his balance.

3.3 Difference between centripetal force and centrifugal force

Centripetal force	Centrifugal force
Centripetal force is directed along the radius, towards the centre of the circle	Centrifugal force is directed along the radius, away from the centre of the circle.
It is a real force.	It is a pseudo force.
This force produces uniform motion.	This force is the effect of uniform circular motion.
It arises in both inertial and non-inertial frames of reference.	It arises only in the non-inertial frame of reference of in a rotating frame of reference.
e.g. when a satellite is revolving in circular orbit round the earth, the centripetal force is due to the gravitational force of attraction	e.g. along a curved road the passenger in the vehicle has a feeling of push in the outward direction. This push is due to centrifugal force

3.4 Applications of centrifugal force

The centrifugal pump used to lift the water works on the principle of centrifugal force.



A cream-separator used in the diary work, works on the principle of centrifugal force. Centrifuge used for the separation of suspended particle from the liquid, works on the principle of centrifugal force. Centrifugal drier.

3.5 A Centrifuge

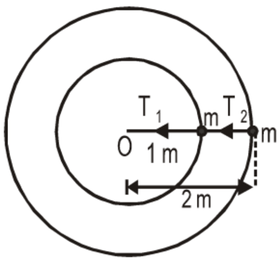
A centrifuge works on the principle of centrifugal force.
The centrifuge consists of two steel tubes suspended from the ends of a horizontal bar which can be rotated at high speed in a horizontal plane by an electric motor.
The tubes are filled with the liquid and the bar is set into rotation.
Due to rotational motion, the tubes get tied and finally be come horizontal.
Due to heavy mass, the heavier particles experience more centrifugal force than that of the liquid particles. Therefore, is then stopped so that the tubes becomes vertical.

Solved Example

Example 9. Two balls of equal masses are attached to a string at distance 1 m and 2 m from one end as shown in fig. The string with masses is then moved in a horizontal circle with constant speed. Find the ratio of the tension T_1 and T_2?

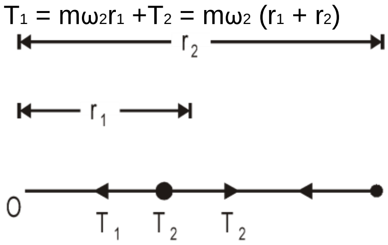
Solution Let the balls of the two circles are r_1 and r_2. The linear speed of the two masses are $v_1 = \omega r_1$, $v_2 = \omega r_2$
where ω is the angular speed of the circular motion. The tension in the strings are such that

$$T_2 = \frac{mv_2^2}{r_2} = m\omega^2 r_2$$



$$T_1 - T_2 = \frac{mv_1^2}{r_1} = m\omega^2 r_1$$

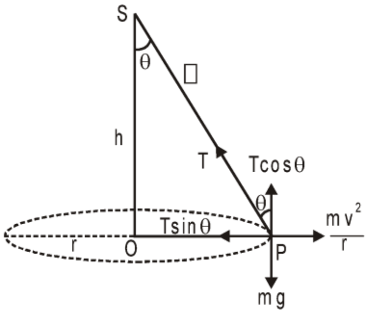
∴ $T_1 = m\omega^2 r_1 + T_2 = m\omega^2 (r_1 + r_2)$



∴ $$\frac{T_1}{T_2} = \frac{r_1 + r_2}{r_2} = \frac{1+2}{2} = \frac{3}{2}$$

4. Conical pendulum

(This the best example of uniform circular motion)
A conical pendulum consists of a body attached to a string, such that it can revolve in a horizontal circle with uniform speed. The string traces out a cone in the space.
The force acting on the bob are



(a) Tension T (b) weight mg

The horizontal component T sinθ of the tension T provides the centripetal force and the vertical component T cos θ balances the weight to bob

∴ T sinθ = and T cosθ = mg

From these equations ....(i)

and tan θ = ...(ii)

If h = height of conical pendulum tanθ = = ...(iii)

From (ii) & (iii)

The time period of revolution =

4.1 Hints to solve numerical problem (UCM)

(i) First show all force acting on a particle

(ii) Resolve these forces along radius and tangent.

(iii) Resultant force along radial direction provides necessary centripetal force.

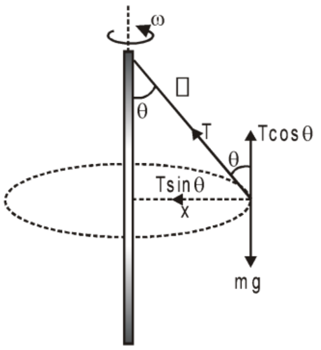
(iv) Resultant force along tangent = $Ma_T = 0$ (a_T = tangential acceleration)

Solved Examples

Example 10. A vertical rod is rotating about its axis with a uniform angular speed ω. A simple pendulum of length ℓ is attached to its upper end what is its inclination with the rod ?

Solution Let the radius of the circle in which the bob is rotating is, the tension in the string is T, weight of the bob mg, and inclination of the string θ. Then T cos θ balances the weight mg and T sin θ provides the centripetal force necessary for circular motion.

That is –



$T \cos\theta = mg$ and $T \sin\theta = m\omega^2 x$

but $x = \ell \sin\theta$

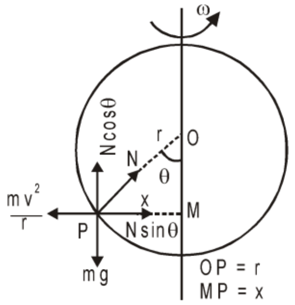
∴ $T = m\omega^2 \ell$

and $\cos\theta = \dfrac{mg}{T} = \dfrac{mg}{m\omega^2\ell}$ or $\theta = \cos^{-1}\left(\dfrac{g}{\omega^2\ell}\right)$

Example 11. A circular loop has a small bead which can slide on it without friction. The radius of the loop is r. Keeping the loop vertically it is rotated about a vertical diameter at a constant angular speed ω. What is the value of angle θ, when the bead is in dynamic equilibrium ?

Solution Centripetal force is provided by the horizontal component of the normal reaction N.

The vertical component balances the weight. Thus



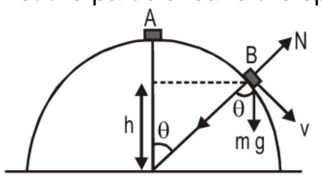
$N \sin\theta = m\omega_2 x$ and $N\cos\theta = mg$

Also $x = r\sin\theta$ $\Rightarrow$ $N = m\omega_2 r$

$\cos\theta = \dfrac{g}{\omega^2 r}$ or $\theta = \cos^{-1}\left(\dfrac{g}{r\omega^2}\right)$

Example 12. A particle of mass m slides down from the vertex of semihemisphere, without any initial velocity. At what height from horizontal will the particle leave the sphere.

Solution Let the particle leave the sphere at height h, $\dfrac{mv^2}{R} = mg\cos\theta - N$



When the particle leaves the sphere N = 0,

$\dfrac{mv^2}{R} = mg\cos\theta \Rightarrow v_2 = gR\cos\theta$

According to law of conservation of energy

(K . E.+ P. E.) at A =(K . E.+ P. E.) at B

$\Rightarrow$ $o + mgR = \dfrac{1}{2}mv^2 + mgh \Rightarrow v^2 = 2g(R-h)$.......(2)

From (1) & (2) $h = \dfrac{2R}{3}$, Also $\cos\theta = 2/3$

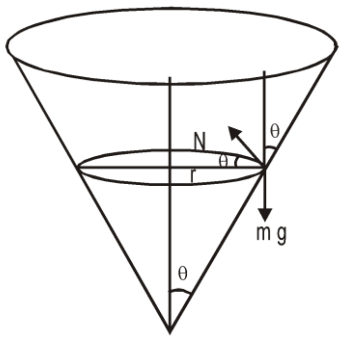
Example 13. A particle describes a horizontal circle of radius r in a funnel type vessel of frictionless surface with half one angle θ (as shown in figure). If mass of the particle is m, then in dynamical equilibrium the speed of the particle must be -

Solution The normal reaction N and weight mg are the only forces acting on the particle (inertial frame view), the N is making an angle $\left(\dfrac{\pi}{2}-\theta\right)$ with the vertical. The vertical component of N balances the weight mg and the horizontal component provides the centripetal force required for circular motion.

Thus

$N\cos\left(\dfrac{\pi}{2}-\theta\right) = mg$

$N\sin\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{mv^2}{r}$

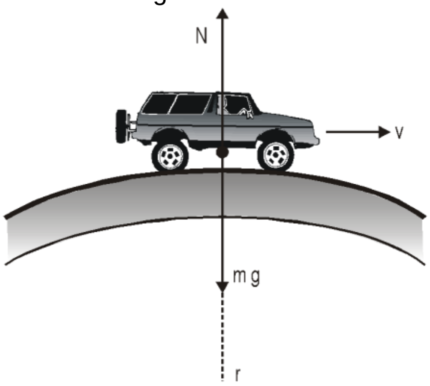


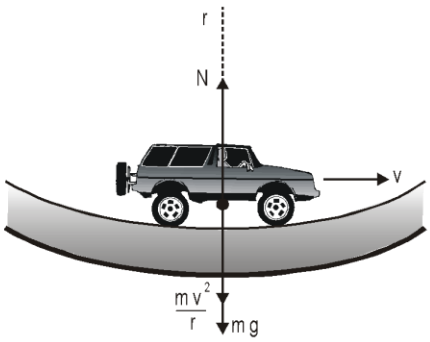
or N sin θ = $\frac{mv^2}{r}$ mg.

N cos θ = $\tan\theta = \frac{rg}{v^2}$, on dividing we get , so $v = \sqrt{\frac{rg}{\tan\theta}}$

Example 14 Prove that a motor car moving over a (i) Convex bridge is lighter than the same car resting on the same bridge.
(ii) Concave bridge is heavier than the same car resting on the same bridge.

Solution Apparent weight of car = N (normal reaction)
(i) Convex bridge
The motion of the motor car over a convex bridge is the motion along the segment of a circle. The centripetal force is provided by the difference of weight mg of the car and the normal reaction N of the bridge.





∴ $mg - N = \frac{mv^2}{r}$ or $N = mg - \frac{mv^2}{r}$

Clearly N < mg, i.e., the apparent weight of the moving car is less than the weight of the stationary car.

(ii) Concave bridge $N - mg = \frac{mv^2}{r}$

Apparent weight $N = mg + \frac{mv^2}{r}$

5. Motion in vertical circle

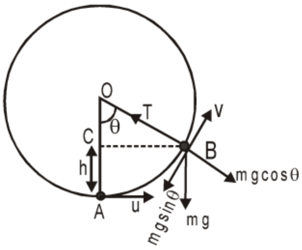
Motion of a body suspended by string :
This is the best example of non-uniform circular motion.
Suppose a particle of mass m is attached to an inexcusable light string of length r. The particle is moving in a vertical circle of radius r, about a fixed point O.

At lost point A velocity of particle = u (in horizontal direction)
After covering ∠θ velocity of particle = v (at point B)
Resolve weight (mg) into two components
(i) mg cos θ (along radial direction)
(ii) mg sin θ (tangential direction)

Then force T – mg cos θ provides necessary centripetal force

$T - mg\cos\theta = \dfrac{mv^2}{r}$ ......(i)

Δ OCB $\cos\theta = \dfrac{r-h}{r}$ ......(ii)



or $h = r(1 - \cos\theta)$

By conservation of energy at point A & B

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$ or $u^2 = v^2 + 2gh$

or $\boxed{v^2 = u^2 - 2gh}$ ....(iii)

Substitute value of cos θ and v^2 in equn. (i)

$T - mg\left[\frac{r-h}{r}\right] = \frac{m}{r}(u^2 - 2gh)$ or $T = \frac{m}{r}[u^2 - 2gh + gr - gh)$ or $T = \frac{m}{r}[u^2 + gr - 3gh]$ ....(iv)

(i) If velocity becomes zero at height h_1

$O = u^2 - 2gh$, or $h_1 = \dfrac{u^2}{2g}$ .... (v)

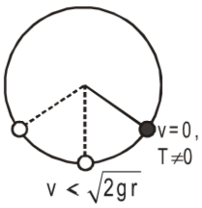
(ii) If tension becomes zero at height h_2

$O = \frac{m}{r}[u^2 + gr - 3gh_2]$

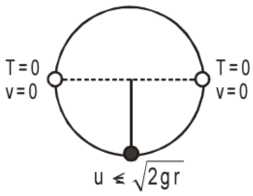
or $u^2 + gr - 3gh_2 = 0$ or $h_2 = \dfrac{u^2 + gr}{3g}$ ....(vi)

(iii) Case of oscillation

It v = 0, T ≠ 0 then $h_1 < h_2$



Case of oscillation



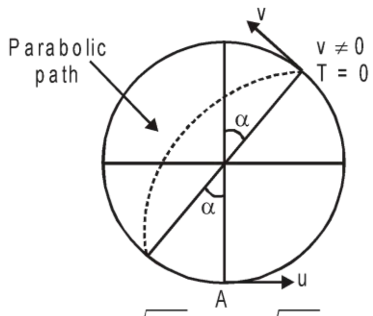
Case of oscillation
semi-circular path

$\frac{u^2}{2g} < \frac{u^2 + gr}{3g}$ ⇒ $3u^2 < 2u^2 + 2gr$ ⇒ $u^2 < 2gr$ ⇒ $u < \sqrt{2gr}$

(iv) Case of leaving the circle

If v ≠ 0, T = 0

then $h_1 > h_2$ $\frac{u^2}{2g} > \frac{u^2 + gr}{3g}$



$\sqrt{5gr} > u > \sqrt{2gr}$

Case of leaving the circle

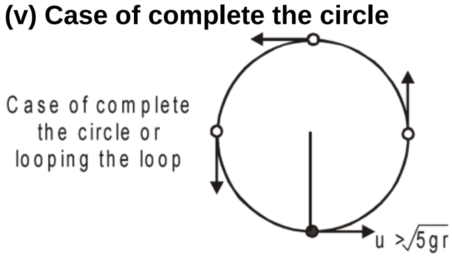
$3u^2 > 2u^2 + 2gr$

$u^2 > 2gr$

$u > \sqrt{2gr}$

$\sqrt{5gr} > u > \sqrt{2gr}$

(v) Case of complete the circle



$u \geq \sqrt{5gr}$

$T \geq 0$

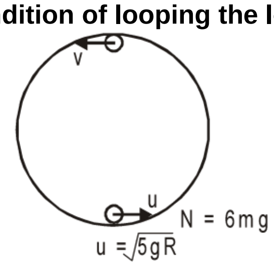
$v \neq 0$

SPECIAL NOTE :

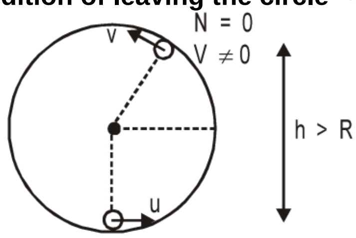
The same conditions apply if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.

This is shown in the figure given below $v = \sqrt{gR}$ $N = 0$

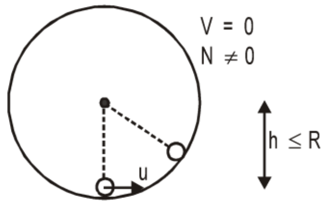
(i) Condition of looping the loop is u ≥ $\sqrt{5gR}$



(ii) Condition of leaving the circle $\sqrt{2gR} < u < \sqrt{5gR}$



(iii) Condition of oscillation is 0 < u ≥ $\sqrt{2gR}$



Solved Examples

Example 15. A ball is released from height h as shown in fig. Find the condition for the particle to complete the circular path.

Solution According to law of conservation of energy (K.E. + P.E) at A = (K.E. + P.E) at B

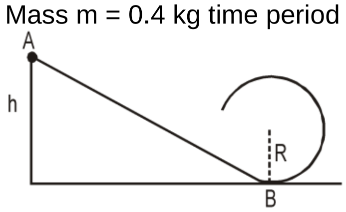
$\Rightarrow \quad 0 + mgh = \frac{1}{2} mv^2 + 0 \quad \Rightarrow \quad v = \sqrt{2gh}$

But velocity at the lowest point of circle,

$$v \geq \sqrt{5gR} \Rightarrow \sqrt{2gh} \geq \sqrt{5gR} \Rightarrow h \geq \frac{5R}{2}$$

Example 16. A body weighing 0.4 kg is whirled in a vertical circle making 2 revolutions per second. If the radius of the circle is 1.2 m, find the tension in the string, when the body is (a) at the top of the circle (b) at the bottom of the circle. Given : g = 9.8 ms^{-2} and π = 1.2 m

Solution Mass m = 0.4 kg time period = $\frac{1}{2}$ second and radius, r =1.2 m



Angular velocity, $\omega = \frac{2\pi}{1/2}$ = 4π rad s^{-1} = 12.56 rad s^{-1}

(a) At the top of the circle, $T = \frac{mv^2}{r} - mg = mr\omega^2 - mg = m\,(r\omega^2 - g)$

= 0.4 (1.2 × 12.56 × 12.56 – 9.8) N = 71.8 N

(b) At the lowest point, T = m($r\omega^2$ + g) = 79.64 m

Example 17. In a circus a motorcyclist moves in vertical loop inside a 'death well' (a hollow spherical chamber with holes, so that the spectators can watch from outside). Explain clearly why the motorcyclist does not drop down when the is at the uppermost point, with no support from below. What is the minimum speed required to perform a vertical loop is the radius of the chamber is 25 m.

Solution When the motorcyclist is at the highest point of te death-well, the normal reaction R on the motorcyclist by the ceiling of the chamber acts downwards. His weight mg also acts downwards. These two forces are balanced by the outward centrifugal force acting on him.

$\therefore \quad R + mg = \frac{mv^2}{r}$ ....(i) r = radius of the circle

Here v is the speed of the motorcyclist and m is the mass of the motorcyclist (including the mass the motor cycle). Because of the balancing of the forces, the motorcyclist does not fall down.

The minimum speed required to perform a vertical loop is given by equation (i), when R = 0

$\therefore \quad mg = \frac{mv^2_{min}}{r}$ or $v^2_{min} = gr$

or $v_{min} = \sqrt{gr} = \sqrt{9.8 \times 25}$ ms^{-1} = 15.65 ms^{-1}

So, the minimum speed at the top required to perform a vertical loop is 15.65 ms^{-1}.

Example 18. A 4kg ball in swung in a vertical circle at the end of a cord 1 m long. What is the maximum speed which is can swing if the cord can sustain maximum tension of 163.6 N ?

Solution Maximum tension = $T = \frac{mv^2}{r} + mg$ (at lowest point)

$\therefore$ $\frac{mv^2}{r} = T - mg$

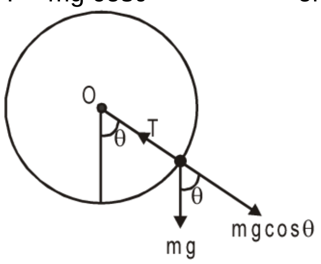
or $\frac{4v^2}{1} = 163.6 - 4 \times 9.8$

solving we get v = 6 m/sec

Example 19. A small body of mass m = 0.1 kg swings in a vertical circle at the end of a chord of length 1 m. Its speed is 2 m/s when the chord makes an angle θ = 30º with the vertical. Find the tension in the chord.

Solution The equation of motion is

$T - mg\cos\theta = \frac{mv^2}{r}$ or $T = mg\cos\theta + \frac{mv^2}{r}$



Substituting the given values, we get

$T = 0.1 \times 9.8 \times \cos 30 + \frac{0.1 \times (2)^2}{1} = 0.98 \times \left(\frac{\sqrt{3}}{2}\right) + 0.4$

= 0.85 + 0.4 = 1.25 N

Important Point :

If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottom most points in not . Because in this case, velocity of the particle at the topmost point can be zero also. Using conservation of mechanical energy between points A and B as shown in fig. (a) we get

$$\frac{1}{2} m (u_2 - v_2) = mgh$$

or $\frac{1}{2} mu_2 = mg (2R)$

$\therefore$ $u = 2\sqrt{gR}$

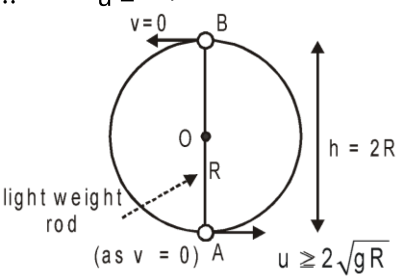


fig (a)

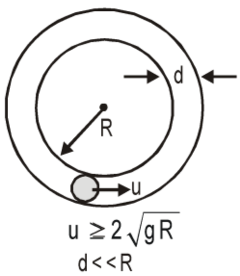


fig (b)

Therefore, the minimum values of u in this case is $2\sqrt{gR}$

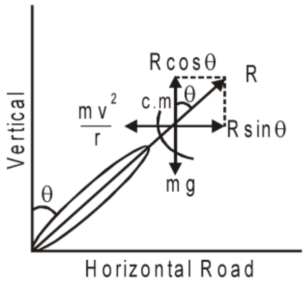
Same in the case when a particle is compelled to move inside a smooth vertical tube as shown in fig. (b)

6. Particle application of circular motion

6.1 A Cyclist making A turn

Let a cyclist moving on a circular path of radius r bend away from the vertical by an angle θ. If R is the reaction of the ground, then R may be resolved into two components horizontal and vertical. The vertical component R cos θ balances the weight mg of the cyclist and the horizontal component R sin θ provides the necessary centripetal force for circular motion.

$R \sin\theta = \frac{mv^2}{r}$ ....(1)



and R cos θ = mg ....(2)

Dividing (1) by (2), we get

$\tan\theta = \frac{v^2}{rg}$ ....(3)

For less beding of cyclist, his speed v should be smaller and radius r of circular path should be greater.

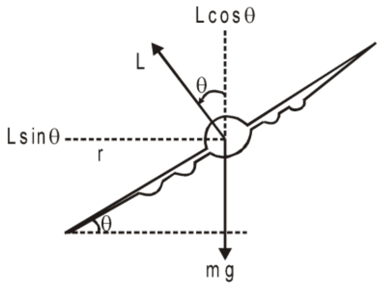
If μ is coefficient of friction, then for no skidding of cycle (or overturning of cyclist)

$\mu \geq \tan\theta$ ......(4) $\mu \geq \frac{v^2}{rg}$

6.2 An aeroplane making a turn

In order to makes a circular turn, a plane must roll at some angle θ in such a manner that the horizontal component of the lift force L provides the necessary centripetal force for circular motion. The vertical component of the lift force balances the weight of the plane.

$L \sin\theta = \frac{mv^2}{r}$

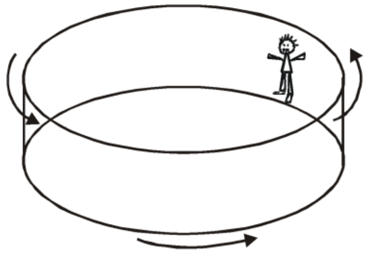


and L cos θ = mg

or the angle θ should be such that $\tan\theta = \frac{v^2}{rg}$

6.3 Death well and rotor

Example of uniform circular motion In 'death well' a person drives a bicycle on a vertical surface of a large wooden well.



(a) A passenger on a 'rotor ride'

In 'death well' walls are at rest while person revolves.

In a rotor at a certain angular speed of rotor a person hangs resting against the wall without any floor.

In rotor person is at rest and the walls rotate.

In both these cases friction balances the weight of person while reaction provides the centripetal force necessary for circular motion i.e.

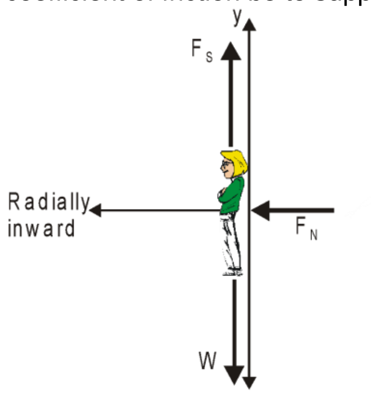
Force of fiction F_S = mg and Normal reaction $F_N = \frac{mv^2}{r}$

so $\frac{F_N}{F_S} = \frac{v^2}{rg}$, i.e., $v = \sqrt{\frac{rgF_N}{F_S}}$

Now for v to be minimum F_S must be maximum, i.e., $v_{min} = \sqrt{\frac{rg}{\mu}}$
[as $F_{S\,max} = \mu F_N$]

Solved Examples

Example 20. A 62 kg woman is a passenger in a "rotor ride" at an amuse ment park. A drum of radius 5.0 m is spun with an angular velocity of 25 rpm. The woman is pressed against the wall of the rotating drum as shown in figure (a) Calculate the normal force of the drum of the woman (the centripetal force that prevents her from leaving her circular path). (b) While the drum rotates, the floor is lowered. A vertical static friction force supports the woman's weight. What must the coefficient of friction be to support her weight ? (ω = 25 rev/min, r = 5m)



(b) A force diagram for the person

Solution Normal force exerted by the drum on woman towards the centre

$F_N = ma_c = m\omega^2 r = 62\text{ kg} \times \left(25\frac{\text{rev}}{\text{min}} \times \frac{2\pi\text{rad}}{1\text{rev}} \times \frac{1\text{min}}{60\text{ s}}\right)^2 \times 5\text{m} = 2100\text{ N}$

(b) $\mu F_N = F = mg$ ....(2)
dividing eqn. (2) be eq. (1)

$\mu = \frac{g}{\omega^2 r} = \left(\frac{60}{2\pi \times 25}\right)^2 \times \frac{10}{5} = 0.292$

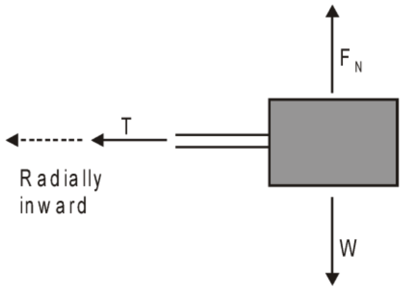
Example 21. A 1.1 kg block slides on a horizontal frictionless surface in a circular path at the end of a 0.50 m long string. (a) Calculate the block's speed if the tension in the string in 86 N. (b) By what percent does the tension change if the block speed decreases by 10 percent?

Solution (a) Force diagram for the block is shown in fig. The upward normal force balances the block's weight. The tension force of the string on the block provides the centripetal force that keeps the block moving in a circle. Newton's second law for forces along the radial direction is $\sum F$ (in radial direction) = T = $\frac{mv^2}{r}$,

or $v = \sqrt{\frac{Tr}{m}} = \sqrt{\frac{(80\text{N})(0.50\text{m})}{1.2\text{ kg}}}$ = 5.0 m/s

(b) A 10 percent reduction in the speed results in a speed v' = 5.4 m/s. The new tension is

$T' = \frac{mv'^2}{r} - \frac{(1.2\text{kg})\,(5.4\text{m}/\text{s})^2}{0.50\text{m}} = 70\text{N}$



Thus, $\frac{T'}{T} = \frac{70N}{86N} = 0.81$

percentage reduction in the tension is about 19%.
The same result is obtained using a proportionality method.

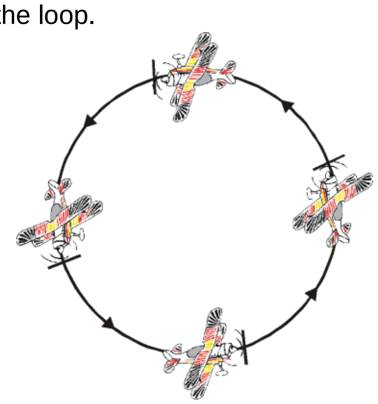
$$\frac{T'}{T} = \frac{(mv'^2/r)}{(mv^2/r)} = \left(\frac{v'}{v}\right)^2 = \left(\frac{0.90}{v}\right)^2 = 0.81$$

6.4 Looping the loop

This is the best example of non uniform circular motion in vertical plane.
For looping the pilot of the plane puts off the engine at lowest point and traverses a vertical loop. (with variable velocity).

Solved Example

Example 22. An aeroplane moves at 64 m/s in a vertical loop of radius 120 m, as shown in figure. Calculate the force of the plane's seat on 72 kg pilot while passing through the
bottom part of the loop.



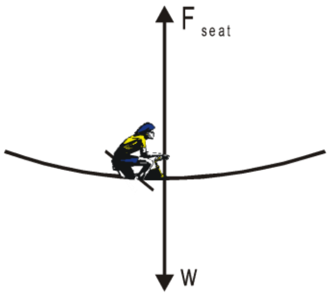
Solution Two forces acts on the pilot his downward weigth force w and the upward force of the aeroplane's seat F_{seat}. Because the pilot moves in a circular path, these forces along the radial direction must, according to Newton's seconds law (∑ F = ma), equal the pilot's mass times his centripetal acceleration, where

$a_c = v^2/r$. We find that ∑ F (in radial direction) = $F_{seat} - w = \frac{mv^2}{r}$

Remember that force pointing towards the center of the circle (F_{seat}) are positive & those pointing away from the center (w) are negative.

Substituting ω = mg and rearranging, we find that the force of the aeroplane seat on the pilot is

$$F_{seat} = m\left(\frac{v^2}{r} + g\right) = 72\ kg\left[\frac{64(m/s)^2}{120m} + 9.8m/s^2\right]$$



= 72 kg (34.1
m/s2 + 9.8m/s2) = 3160.8 N

The pilot in this example feels very heavy. To keep him in the circular path, the seat must push the pilot upwards with a force of 3160 N, 4.5 times his normal weight. He experiences an acceleration of 4.5 g, that is, 4.5 times the acceleration of gravity.

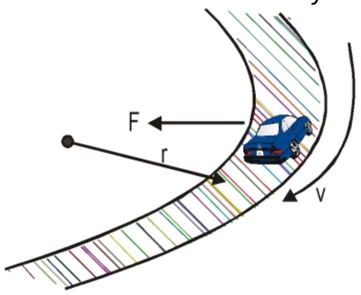
6.5 A car taking a turn on a level road

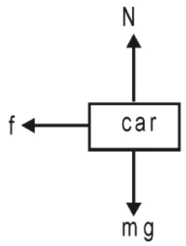
When a car takes a turn on a level road, the portion of the turn can be approximated by an arc of a circle of radius r (see fig.) If the car makes the turn at a constant speed v, then there must be some centripetal force acting on the car. This force is generated by the friction between the tyre and the road. (car has a tendency to slip radially outward, so frictional force acts inwards)

μ_s is coefficient of static friction

N = mg is the normal reaction of the surface

The maximum safe velocity v is –





free body diagram of car

$$\frac{mv^2}{r} - \mu_S N = \mu_S mg$$

or $$\mu_S = \frac{v^2}{rg}$$

or $$v = \sqrt{\mu_S rg}$$

It is independent of the mass of the car. The safe velocity is same for all vehicles of larger and smaller mass.

Solved Example

Example 23. A car is travelling at 30 km/h in a circle of radius 60 m. What is the minimum value of μ_s for the car to make the turn without skidding ?

Solution The minimum μ_s should be that

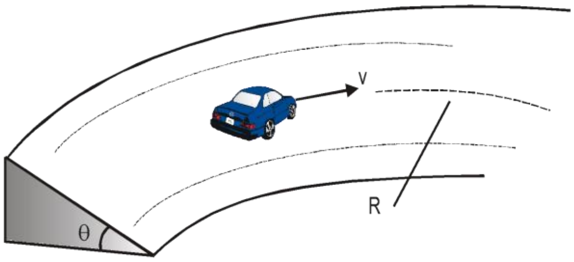
$$\mu_S mg = \frac{mv^2}{r} \quad \text{or} \quad \mu_S = \frac{v^2}{rg}$$

Hare v = 30 $\frac{km}{h} = \frac{30 \times 1000}{3600} = \frac{25}{3}$ m/s $\Rightarrow$ $\mu_S = \frac{25}{3} \times \frac{25}{3} \times \frac{1}{60 \times 10}$ = 0.115

For all values of μ_s greater than or equal to the above value, the car can make the turn without skidding. If the speed of the car is high so that minimum μ_s is greater than the standard values (rubber tyre on dry concrete μ_s = 1 and on wet concrete μ_s = 0.7), then the car will skid.

6.6 Banking of road

If a cyclist takes a turn, he can bend from his vertical position. This is not possible in the case of car, truck of train.



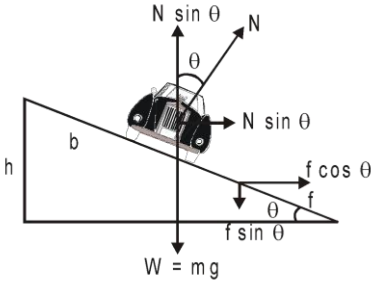
The tilting of the vehicle is achieved by raising the outer edge of the circular track, slightly above the inner edge. This is known as banking of curved track. The angle of inclination with the horizontal is called the angle of banking. If driver moves with slow velocity that friction does not play any role in negotiating the turn. The various forces acting on the vehicle are :
(i) Weight of the vehicle (mg) in the downward direction.
(ii) Normal reaction (N) perpendicular to the inclined surface of the road.
Resolve N in two components.
N cosθ, vertically upwards which balances weight of the vehicle.

$\therefore \quad N\cos\theta = mg \quad$ ....(i)

N sin θ, in horizontal direction which provides necessary centripetal force.

$\therefore \quad N\sin\theta = \frac{mv^2}{r} \quad$ ....(ii)

on dividing eqn. (ii) by eqn. (i)



$$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{r}}{mg} \quad \text{or} \quad \tan\theta = \frac{v^2}{rg} \quad \therefore \quad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

Where m is the mass of the vehicle, r is radius of curvature of the road, v is speed of the vehicle and θ is the banking angle (sinθ =h/b) .
Factors that decide the value of angle of banking are as follows :
Thus, there is no need of mass of the vehicle to express the value of angle of banking
i.e. angle of banking ⇒ does not dependent on the mass of the vehicle.

$\because \quad v_2 = gr\tan\theta \quad \therefore \quad v = \sqrt{gr\tan\theta}$ (maximum safe speed)

This gives the maximum safe speed of the vehicle. In actual practice, some frictional forces are always present. So, the maximum safe velocity is always much greater than that given by the above equation. While construction the curved track, the value of θ is calculated for fixed values of v_{max} and r. This explains why along the curved roads, the speed limit at which the curve is to be negotiated is clearly incited on sign boards.
The outer side of the road is raised by h = b × θ.

When θ i small, then $\tan\theta \approx \sin\theta = \frac{h}{b}$;

Also $\tan\theta = \frac{v^2}{rg} \quad \therefore \quad \frac{v^2}{rg} = \frac{h}{b}$ or $h = \frac{v^2}{rg} \times b$

Solved Example

Example 24 At what should a highway be banked for cars travelling at a speed of 100 km/h if the radius of the road is 400 m and no frictional forces are involved?

Solution The banking should be done at an angle θ such that

$\tan\theta = \frac{v^2}{rg} = \frac{\frac{250}{9}\times\frac{250}{9}}{400\times10}$ or $\tan\theta = \frac{625}{81\times40} = 0.19$ or $\theta = \tan^{-1} 0.19 \approx 0.19$ radian

$\approx 0.19 \times 57.3^o \approx 11^o$

Example 25 The radius of curvature of a railway line at a place when the train is moving with a speed of 36 kmh $^{-1}$ is 1000 m, the distance between the two rails being 1.5 metre. Calculate the elevation of the outer rail above the inner rail so that there may be no side pressure on the rails.

Solution Velocity, $v = 36$ km $h^{-1} = \frac{36\times1000}{3600}$ $ms^{-1} = 10$ ms^{-1}

radius, $r = 1000$ m ; $\tan\theta = \frac{v^2}{rg} = \frac{}{1000\times9.8} = \frac{1}{9.8}$

Let h be the height through which outer rail is raised. Let ℓ be the distance between the two rails.

Then, $\tan\theta = \frac{h}{\ell}$ [∵ θ is very small] or $h = \ell \tan\theta$

$h = 1.5 \times \frac{1}{98}$ m = 0.0153 m [∵ $\ell = 1.5$ m]

Example 26. An aircraft executes a horizontal loop at a speed of 720 km h^{-1} with its wing banked at 15º. Calculate the radius of the loop.

Solution Speed , $v = 720$ km $h^{-1} = \frac{720\times1000}{3600}$ $ms^{-1} = 200$ ms^{-1} and $\tan\theta = \tan 15^o = 0.2679$

$\tan\theta = \frac{v^2}{rg}$ or $r = \frac{v^2}{g\tan\theta} = \frac{200\times200}{9.8\times0.2679}$ m = 1523.7m = 15.24 km,

Example 27. A train rounds an unbanked circular bend of radius 30 m at a speed of 54 km h^{-1}. The mass of the train is 10^6 kg. What provides the centripetal force required for this propose ? The engine or the rails? The outer of inner rails ? Which rail will wear out faster, the outer or the inner rail ? What is the angle of banking required to prevent wearing out of the rails?

Solution $r = 30$ m, $v = 54$ km $h^{-1} = \frac{54\times5}{18}$ $ms^{-1} = 15$ ms^{-1} $m = 10^6$ kg, θ = ?

(i) The centripetal force is provided by the lateral thrust by the outer rail on the flanges of the wheel of the train. The train causes an equal and opposite thrust on the outer rail (Newton’s third law of motion).

Thus, the outer rails wears out faster.

(ii) $\tan\theta = \frac{v^2}{rg} = \frac{15\times15}{60\times9.8} = 0.7653$ ∴ $\theta = \tan^{-1}(0.7653) = 37.43^o$

7. Special points about circular motion

Centripetal force does not increase the kinetic energy of the particle moving in circular path, hence the work done by the force is zero.

Centrifuges are the apparatuses used to separate small and big particles from a liquid.

The physical quantities which remain constant for a particle moving in circular path are speed, kinetic energy and angular momentum.

If a body is moving on a curved road with speed greater than the speed limit, the reaction at the inner wheel disappears and it will leave the ground first.

On unbanked curved roads the minimum radius of curvature of the curve for safe driving is $r = v^2/\mu g$, where v is the speed of the vehicle and μ is the coefficient of friction.

The skidding of a vehicle will occur if $v^2/r > \mu g$ i.e., skidding will take place if the speed is large, the curve is sharp and μ is small.

If r is the radius of curvature of the speed breaker, then the maximum speed with which the vehicle can run on it s without leaving contact with the ground is $v = \sqrt{(gr)}$,

While taking a turn on the level road sometimes vehicles overturn due to centrifugal force.

8. Points to be remember

- **Uniform motion in a circle -**

Angular velocity $\omega = \frac{d\theta}{dt} = 2\pi n = \frac{2\pi}{T}$ Linear velocity $v = \vec{\omega} \times \vec{r}$

$v = \omega r$ when $\vec{\omega}$ and $\vec{r}$ are perpendicular to each other.

Centripetal acceleration $a = \frac{v^2}{r} = \omega^2 r = \omega v = 4\pi^2 n^2 r$

- **Equations of motion -**

For constant angular acceleration -

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha\, t^2$ (iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

- **Motion of a car on a plane circular road -**

For motion without skidding $\frac{Mv_{max}^2}{r} = \mu M_g, v_{max}\sqrt{\mu rg}$

- **Motion on a banked road -** Angle of banking = θ

$\tan\theta = \frac{h}{b}$

Maximum safe speed at the bend $v_{max} = \left[\frac{rg(\mu + \tan\theta)}{1-(\mu\tan\theta)}\right]^{1/2}$

If friction is negligible $v_{max} = \sqrt{rg\tan\theta} = \sqrt{\frac{rhg}{b}}$ and $\tan\theta = \frac{v^2 max}{rg}$

- **Motion of cyclist on a curve -**

In equilibrium angle with vertical is θ then $\tan\theta = \frac{v^2}{rg}$

Maximum safe speed = $v_{max} = \sqrt{\mu rg}$

- **Motion in a vertical circle (particle tied to string) -**

At the top position - Tension $T_A = m\left(\frac{v_A^2}{r} - g\right)$

For $T_A = 0$, critical speed $= \sqrt{gr}$

At the bottom - Tension $T_B = m\left(\frac{v_B^2}{r} + g\right)$

For completing the circular motion minimum speed at the bottom $v_B = \sqrt{5gr}$,

Tension $T_B = 6mg$

- **Conical pendulum (Motion in a horizontal circle)**

Tension is string $= \frac{mg\ell}{(\ell^2 - r^2)^{1/2}}$

Angular velocity $= \sqrt{\frac{g}{\ell\cos\theta}}$ Periodic time $= 2\pi\sqrt{\frac{\ell\cos\theta}{g}} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$