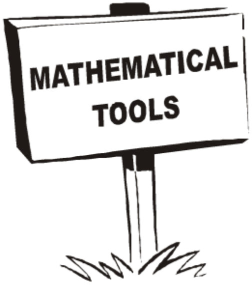
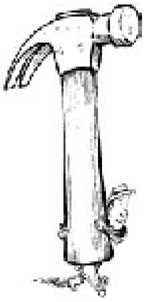
MATHEMATICAL TOOLS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics.



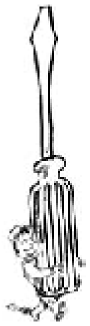
Tools are required to do physical work easily and mathematical tools are required to solve numerical problems easily.

MATHEMATICAL TOOLS



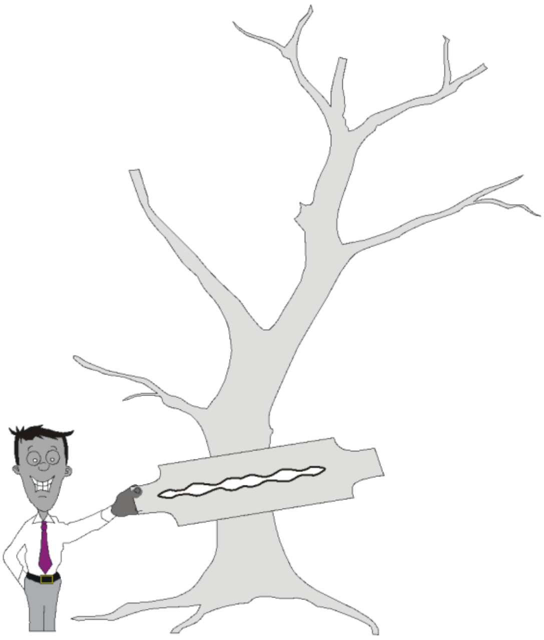
Differentiation

Integration

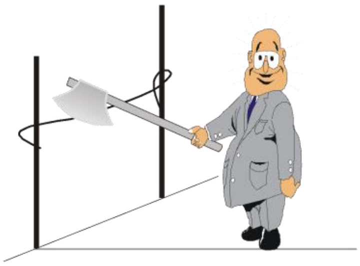


Vectors

To solve the problems of physics Newton made significant contributions to Mathematics by inventing differentiation and integration.



Cutting a tree with a blade



Cutting a string with an axe

APPROPRIATE CHOICE OF TOOL IS VERY IMPORTANT

What is mathematical tools

This chapter includes all the necessary mathematics knowledge which we require to possess to study physics efficiently **Importance of mathematical tools**. This is the most important chapter of physics as it will be repeatedly used as in all the upcoming chapters. **Importance of Mathematical Tools in Physics**: This chapter is the foundation of Physics to be studied in Class 11 & 12. Here we will learn about the mathematics that will be involved in Physics.

1. FUNCTION

Function is a rule of relationship between two variables in which one is assumed to be dependent and the other independent variable, for example :

e.g. The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent & temperature is the dependent variable

e.g. The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each of the above example, value of one variable quantity (dependent variable) , which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as y = f(x).

Here f represents the function, x the independent variable & y is the dependent variable.

All possible values of independent variables (x) are called *domain* of function.

All possible values of dependent variable (y) are called *range* of function.

Think of a function f as a kind of machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usually call the area A and the radius r. Since area depends on radius, we say that A is a function of r, A = f(r) . The equation $A = \pi r_2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(r) = \pi r_2$. (Here the rule of relationship which describes the function may be described as square & multiply by π).

If r = 1 A = π ; if r = 2A = 4π ; if r = 3 A = 9π

The set of all possible input values for the radius is called the domain of the function. The set of all output values of the area is the range of the function.

We usually denote functions in one of the two ways :

1. By giving a formula such as $y = x_2$ that uses a dependent variable y to denote the value of the function.
2. By giving a formula such as $f(x) = x_2$ that defines a function symbol f to name the function.

   Strictly speaking, we should call the function f and not f(x),

   y = sin x. Here the function is sine, x is the independent variable.

Solved Examples

Example 1. The volume V of a ball (solid sphere) of radius r is given by the function $V(r) = (4/3)\pi\,(r)^3$
The volume of a ball of radius 3m is ?

Solution : $V(3) = 4/3\pi(3)^3 = 36\pi\ m_3$.

Function of a function :

Suppose we are given 2 functions, $f(x) = x^2$ and $g(x) = x + 1$
If we are required to find f(g(x))
i.e., value of f (x) at x = g(x)
$f(g(x)) = (x + 1)^2$ [put g (x) in place of x in f(x)]
$g(f(x)) = x^2 + 1$ [put f (x) in place of x in g (x)]

Example 2. Suppose that the function F is defined for all real numbers r by the formula F(r) = 2(r – 1) + 3.
Evaluate F at the input values 0, 2, x + 2, and F(2).

Solution : In each case we substitute the given input value for r into the formula for F :
F(0) = 2(0 – 1) + 3 = – 2 + 3 = 1 ;
F(2) = 2(2 – 1) + 3 = 2 + 3 = 5
F(x + 2) = 2(x +2 – 1) + 3 = 2x + 5 ;
F(F(2)) = F(5) = 2(5 – 1) + 3 = 11.

Example 3. A function ƒ(x) is defined as ƒ(x) = x_2 + 3, Find ƒ(0) , ƒ(1), ƒ(x_2), ƒ(x+1) and ƒ(ƒ(1)).

Solution : ƒ(0) = 0_2 + 3 = 3 ;
ƒ(1) = 1_2 + 3 = 4 ;
ƒ(x_2) = $(x_2)_2$+3 = x_4+3
ƒ(x+1) = $(x + 1)_2$ + 3 = x_2 + 2x + 4 ;
ƒ(ƒ(1)) = ƒ(4) = 4_2+3 = 19

Example 4. If function F is defined for all real numbers x by the formula F(x) = x_2.
Evaluate F at the input values 0, 2, x + 2 and F(2)

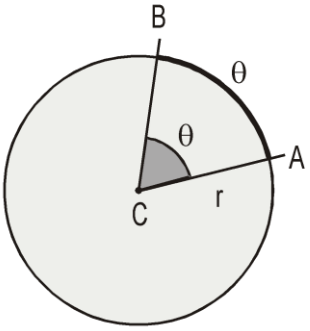
Solution : F(0) = 0 ;
F(2) = 2_2 = 4 ;
F(x+2) = $(x+2)_2$;
F(F(2)) = F(4) = 4_2 = 16

2. TRIGONOMETRY

2.1 MEASUREMENT OF ANGLE AND RELATIONSHIP BETWEEN DEGREES AND RADIAN

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations.

Let ACB be a central angle in a **circle** of radius r , as in figure.
Then the angle ACB or θ is defined in radius as **-**



$\theta = \frac{\text{Arc length}}{\text{Radius}} \Rightarrow \theta = \frac{\overset{\frown}{AB}}{r}$

If r = 1 then θ = AB

The **radian measure** for a circle of unit radius of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360º, the relation between radians and degrees is given by : π radians = 180º

<u>Angle Conversion formulas</u>

1 degree = $\frac{\pi}{180}$ (≈ 0.02) radian

1 radian ≈ 57 degrees

Degrees to radians : multiply by $\frac{\pi}{180}$

Radians to degrees : multiply by $\frac{180}{\pi}$

Solved Examples

Example 5. (i) Convert 45º to radians.

(ii) Convert $\frac{\pi}{6}$ rad to degrees.

Solution : (i) $45 \cdot \frac{\pi}{180} = \frac{\pi}{4}$ rad

(ii) $\frac{\pi}{6} \cdot \frac{180}{\pi} = 30^o$

Example 6. Convert 30º to radians.

Solution : $30^o \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

Example 7. Convert $\frac{\pi}{3}$ rad to degrees.

Solution : $\frac{\pi}{3} \times \frac{180}{\pi} = 60^o$

Standard values

(1) $30_o = \frac{\pi}{6}$ rad

(2) $45_o = \frac{\pi}{4}$ rad

(3) $60_o = \frac{\pi}{3}$ rad

(4) $90_o = \frac{\pi}{2}$ rad

(5) $120_o = \frac{2\pi}{3}$ rad

(6) $135_o = \frac{3\pi}{4}$ rad

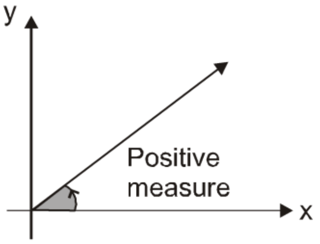
(7) $150_o = \frac{5\pi}{6}$ rad

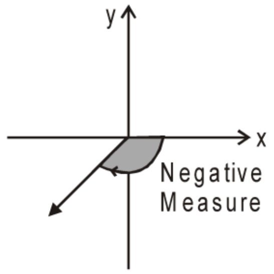
(8) $180_o = \pi$ rad

(9) $360_o = 2\pi$ rad

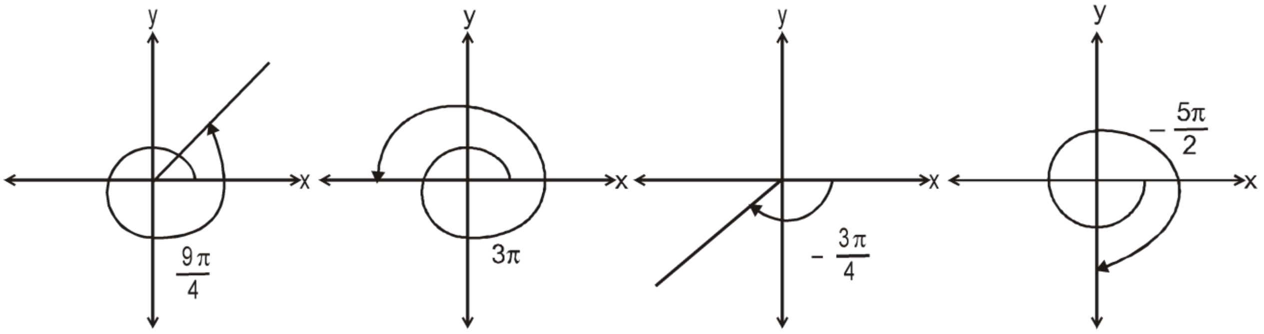
(Check these values yourself to see that the satisfy the conversion formulae)

2.2. MEASUREMENT OF POSITIVE AND NEGATIVE ANGLES





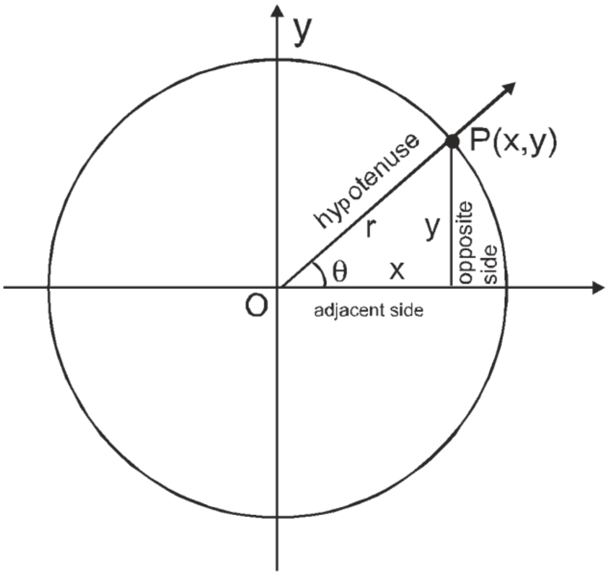
An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig.). Angles measured counterclockwise from the positive x-axis are assigned positive measures ; angles measured clockwise are assigned negative measures.



2.3 SIX BASIC TRIGONOMETRIC FUNCTIONS

The trigonometric function of a general angle θ are defined in terms of x, y, and r.

Sine : $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$ Cosecant : $\text{cosec}\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y}$



Cosine : $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$ Secant : $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x}$

Tangent : $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$ Cotangent : $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$

VALUES OF TRIGONOMETRIC FUNCTIONS

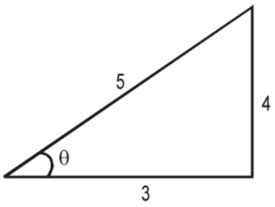
If the circle in (Fig. above) has radius r = 1, the equations defining sinθ and cos θ become

$\text{Cos}\ \theta = x,\qquad \sin\theta = y$

We can then calculate the values of the cosine and sine directly from the coordinates of P.

Solved Examples

Example 8. Find the six trigonometric ratios from given figure



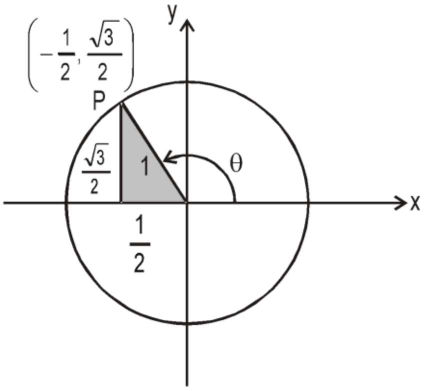
Solution : $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$; $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$;

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$; $\text{cosec}\ \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$;

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$; $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

Example 9 Find the sine and cosine of angle θ shown in the unit circle if coordinate of point p are as shown.

Solution :



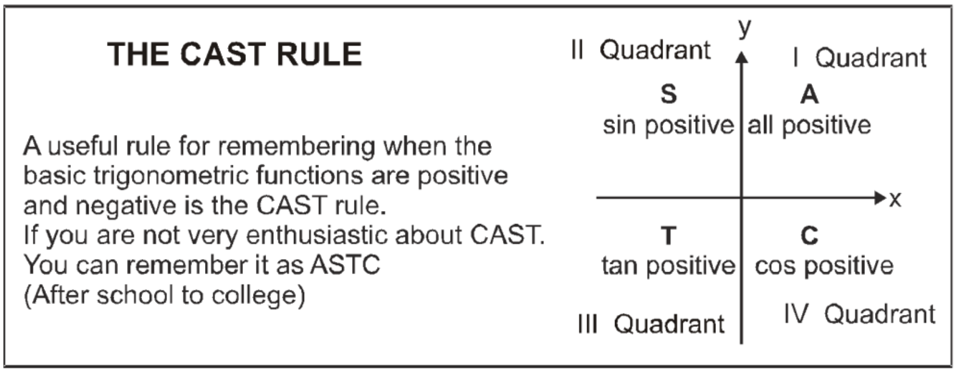
cos θ = x-coordinate of P = $-\frac{1}{2}$ sin θ = y-coordinate of P = $\frac{\sqrt{3}}{2}$.

2.4 RULES FOR FINDING TRIGONOMETRIC RATIO OF ANGLES GREATER THAN 90°

Step 1 → Identify the quadrant in which angle lies.

Step 2 →

(a) If angle = (nπ ± θ) where n is an integer. Then trigonometric function of (nπ ± θ) = same trigonometric function of θ and sign will be decided by CAST Rule.



A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule.
If you are not very enthusiastic about CAST.
You can remember it as ASTC
(After school to college)

(b) If angle = $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ where n is an integer. Then trigonometric function of $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ = complimentary trigonometric function of θ and sign will be decided by CAST Rule.

Values of sin θ, cos θ and tan θ for some standard angles.

Degree	0	30	37	45	53	60	90	120	135	180
Radians	0	$\pi/6$	$37\pi/180$	$\pi/4$	$53\pi/180$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$\sin\theta$	0	$1/2$	$3/5$	$1/\sqrt{2}$	$4/5$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	0
$\cos\theta$	1	$\sqrt{3}/2$	$4/5$	$1/\sqrt{2}$	$3/5$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	-1
$\tan\theta$	0	$1/\sqrt{3}$	$3/4$	1	$4/3$	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	0

Solved Examples

Example 10 Evaluate sin 120°

Solution : sin 120° = sin (90° + 30°) = cos 30° = $\frac{\sqrt{3}}{2}$

Aliter sin 120° = sin (180° – 60°) = sin 60° = $\frac{\sqrt{3}}{2}$

Example 11 Evaluate cos 135°

Solution cos 135° = cos (90° + 45°) = – sin 45° = – $\frac{1}{\sqrt{2}}$

Example 12 Evaluate cos 210°

Solution : cos 210° = cos (180° + 30°) = – cos30° = – $\frac{\sqrt{3}}{2}$

Example 13 Evaluate tan 210°

Solution : tan 210° = tan (180° + 30°) = tan 30° = $\frac{1}{\sqrt{3}}$

2.5 GENERAL TRIGONOMETRIC FORMULAS :

1.

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \mathrm{cosec}^2\theta.$$

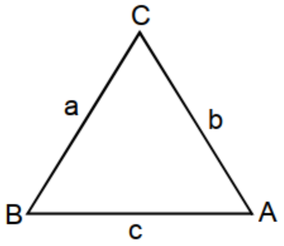
2.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

3. sin 2θ = 2 sin θ cos θ ; cos 2θ = $\cos_2\theta - \sin_2\theta$ = $2\cos_2\theta - 1$ = $1 - 2\sin_2\theta$

$\cos_2\theta = \frac{1+\cos 2\theta}{2}$; $\sin_2\theta = \frac{1-\cos 2\theta}{2}$

4.



In ⊗ ABC, sine rule

ΔABC need not be right angled, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

5. Cosine rule :

$\cos A = \frac{b^2 + C^2 - a^2}{2bc}$

$\cos B = \frac{a^2 + C^2 - b^2}{2ac}$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

3. COORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

Origin

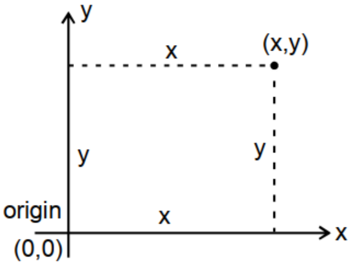
This is any fixed point which is convenient to you. All measurement are taken w.r.t. this fixed point.

• Axis or Axes

Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

3.1 Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure.



Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.

3.2 Distance Formula

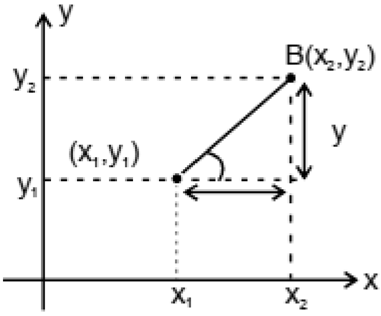
The distance between two points (x_1, y_1) and (x_2, y_2) is given by d = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note: In space d = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3.3 Slope of a Line

The slope of a line joining two points A(x_1, y_1) and B(x_2, y_2) is denoted by m and is given by

m = $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$ [If both axes have identical scales]



Here ⎝ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.

Example 14. For point (2, 14) find abscissa and ordinate. Also find distance from y and x-axis.

Solution: Abscissa = x-coordinate = 2 = distance from y-axis.
Ordinate = y-coordinate = 14 = distance from x-axis.

Example 15. Find value of a if distances between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Solution: By using distance formula d= $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ ® 13 $\sqrt{[3-(-9)]^2+[3-a]^2}$
$\Rightarrow 13^2 = 12^2 + (3 - a)^2 \Rightarrow (3 - a)^2 = 13^2 - 12^2 = 5^2 \Rightarrow (3 - a) = \pm 5 \Rightarrow a = 2$ cm or 8 cm

Example 16. A dog wants to catch a cat. The dog follows the path whose equation is y-x = 0 while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are.

(1) (2, – 2) (2) (2, 2) (3) (–2, 2) (4) (–2, 2)

Ans. (2, 4)

Solution : Let catching point be (x^1, y^1) then, $y^1 - x^1 = 0$ and $x^{12} + y^{12} = 8$ Therefore, $2x^{12} = 8 \Rightarrow x^{12} = 4$
$\Rightarrow x^1 = \pm 2$; so possible ae (2, 2) and (–2, –2).

4. ALGEBRA

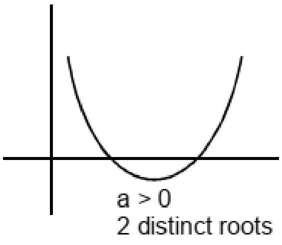
4.1 Quadratic equation and its solution :

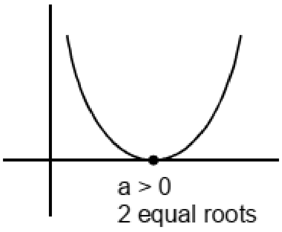
An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. Equation $ax^2 + bx + c = 0$ is the general quadratic equation.
The general solution of the above quadratic equation or value of variable x

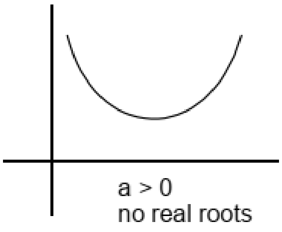
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

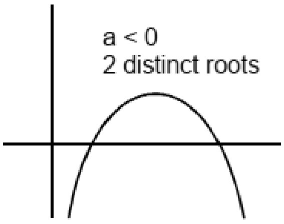
Sum of roots = $x_1 + x_2 = -\frac{b}{a}$ and product of roots = $x_1 + x_2 = \frac{c}{a}$
For real roots $b^2 - 4ac \geq 0$ and for imaginary roots $b^2 - 4ac < 0$.

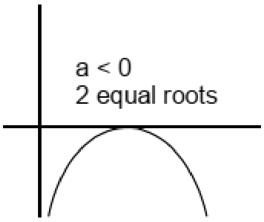
Example 17. find roots of $f(x) = x^2 - 4x + 3$, $f(x) = -x^2 + 3x - 5$ Every quadratic equation has 2 roots (x_1 x_2) such that f (x_1) and f (x_2) is zero.
Graph of quadratic Equation : $ax^2 + bx + c$

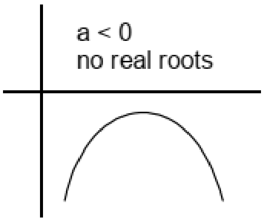












- ® Graph of quadratic equation is of parabolic nature.
- ® Points where graph cuts x-axis are roots of quadratic equation.

Example 18. Draw graphs of
$f(x) = x^2 + 4x + 3$
$f(x) = x^2 + 8x + 16$
$f(x) = -x^2 + 7x - 13$

4.2 BINOMIAL EXPRESSION

An algebraic expression containing two terms is called binomial expression.

For example $(a + b)$, $(a + b)^3$, $(2x - 3y)^{-1}$, $\left(x + \frac{1}{y}\right)$, etc are binomial theorem

Binomial Theorem

$(a + b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2 \times 1} a^{n-2}b^2 + \ldots\ldots\ldots\ldots$, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1} x^2 + \ldots\ldots\ldots$

Binomial Approximation

If x is very small, then terms containing higher powers of x can be neglected so

(1 + x)n = 1 + nx

4.3 LOGARITHM :

Definition : Every positive real number N can be expressed in exponential form as

$N = a^x$ ......(1) e.g. $49 = 7^2$

where ‘a’ is also a positive real different than unity and is called the base and ‘x’ is called the exponent.

We can write the relation (1) in logarithmic form as

$\log_a N = x$ ....(2)

Hence the two relations

and $\log_a \left.\begin{matrix} a^x = N \\ N = x \end{matrix}\right]$

are identical where N > 0, a > 0, a ≠ 1

Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number. Logarithm of zero does not exist and logarithm of (–) ve reals are not defined in the system of real numbers.

a is raised what power to get N

Example 19.

Find value of

(i)$\text{Log}_{81} 27$ (ii) $\log_{10}100$ (iii) $\log 9\sqrt{3}$

Sol. (i) let $\log_{81}27$ ⇒ $3^3 = 3^{4x}$ gives x = 3/4

(ii) $\log_{10}100$ ⇒ $10^2 = 10^x$ gives x = 2

(iii) $\log 9\sqrt{3}$ ⇒ $9\sqrt{3} \left(\frac{1}{3}\right)^x$ ⇒ $3^{5/2} = 3^{-x}$ gives x = –5/2

Note that :

(a) Unity has been excluded from the base of the logarithm as in this case $\log_1 N$ will not be possible and if N = 1 then $\log_1 1$ will have infinitely many solutions and will not be unique which is necessary in the functional notation.

(b) a N log N a = is an identify for all N > 0 and a > 0, a ≠ 1 e.g. $2^{\log_2 5} = 5$

(c) the number N in (2) is called the antilog of ‘x’ to the base ‘a’. Hence If $\log_2 512$ is 9 then $\text{antilog}_2 9$ is equal to $2^2 = 512$

(d) Using the basic definition of log we have 3 important deductions :

(i) $\log_N N = 1$ i.e. logarithm of number to the same base is 1.

(ii) $\log_{\frac{1}{N}} N = -1$ i.e. logarithm of a number to its reciprocal is –1.

(iii) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(basic constraints on number and base must be observed.)

(vi) $\log^{\log_a n}$ = n is an identify for all N > 0 and a > 0; a ≠ 1 e.g. $\log^{\log_a n}$ = 5

(e) Whenever the number and base are on the same side of unity then logarithm of that number to the base is (+ve), however if the number and base are located an different side of unity then logarithm of that number to the base is (–ve)

e.g. (i) $\log_{10} 100 = 2$

(ii) $\log_{1/10} 100 = -2$

(f) For a non negative number ‘a’ & n ≥ 2, n ∈ N $\sqrt[n]{a} = a^{1/n}$

Example 20.

(i) $\log_{\sin 30^o} \cos 60^o = 1$

(ii) $\log_{3/4} 1.\overline{3}$ –1

(iii) $\log_{2-\sqrt{3}} 2+\sqrt{3} = -1$

(iv) $\log_5 \sqrt{5\sqrt{5\sqrt{5........\infty}}} = 1$

Sol. Let $\sqrt{5\sqrt{5\sqrt{5........\infty}}} = x$

® $\sqrt{5x} = x$ ® $x^2 = 5x$

® x = 5x ® $\log_2 5 = 1$

(v) (log tan 1º) (log tan 2º) (log tan 3º) ....(log tan 89º) = 0

Sol. since tan 45º = 1 thus log tan 45º = 0

(vi) $7^{\log 7x} + 2x + 9 = 0$

Sol. 3x + 9 = 0 ⇒ (x = – 3) as it makes initial problem undefined

x = f

(vii) $2^{\log 2(x-3)} + 2(x-3) - 12 = 0$

Sol. x – 3 + 2x – 6 – 12 = 0

3x = 21 ® x = 7

(viii) $\log_2 (x-3) = 4$

Sol. x–3 = 24

x = 19

4.4 Componendo and Dividendo Rule

If $\frac{p}{q} = \frac{a}{b}$ then $\frac{p+q}{p-q} = \frac{a+b}{a-b}$

4.5 Arithmetic progression (AP)

General from : a, a + d, a + 2d, ...a + (n – 1)d

Here a = first term, d = common difference

Sum of n terms $S_n = \frac{n}{2}[a + a + (n-1)d] = \frac{n}{2}$ [Ist term + nth term]

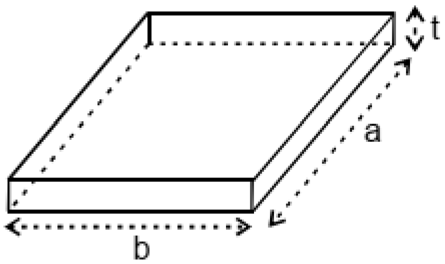
5. GEOMETRY

5.1 Formulae for determination of area :

1. Area of a square = (side)2
2. Area of rectangle = length × breadth
3. Area of a triangle = $\frac{1}{2}$ (base × height)
4. Area of trapezoid = $\frac{1}{2}$ (distance between parallel side) × (sum of parallel side)
5. Area enclosed by a circle = πr^2 (r = radius)

6. Surface area of a sphere = $4\pi r^2$ (r = radius)
7. Area of a parallelogram = base × height
8. Area of curved surface of cylinder = $2\pi r\ \ell$(r = radius and ℓ = length)
9. Area of ellipse = π ab (a and b are semi major and semi minor axes respectively)
10. Surface area of a cube = $6(\text{side})^2$
11. Total surface area of cone = $\pi r^2 + \pi r\ \ell$ where $\pi r\ell = \pi r\sqrt{r^2+h^2}$ = lateral area

5.2 Formulae for determination of volume :



1. Volume of rectangular slab = length × breadth × height = abt
2. Volume of a cube = $(\text{side})^3$
3. Volume of a sphere = $\frac{4}{3}\pi r^3\ell$ (r = radius)
4. Volume of a cylinder = $\pi r^2\lambda\ell$ (r = radius and ℓ is length)
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$ (r = radius and h is height)

Note : $\pi = \frac{22}{7} = 3.14$, $\pi^2 = 9.8776 \approx 10$ and $\frac{1}{\pi} = 0.3182 \approx 0.3$

6. DIFFERENTIATION

6.1 FINITE DIFFERENCE

The finite difference between two values of a physical quantity is represented by Δ notation.

For example :

Difference in two values of y is written as Δy as given in the table below.

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

INFINITELY SMALL DIFFERENCE :

The infinitely small difference means very-very small difference. And this difference is represented by ‘d’ notation instead of ‘Δ’.

For example infinitely small difference in the values of y is written as ‘dy’

if $y_2 = 100$ and $y_1 = 99.99999999$........

then dy = 0.000000...................00001

6.2 DEFINITION OF DIFFERENTIATION

Another name for differentiation is derivative. Suppose y is a function of x or y = f(x)

Differentiation of y with respect to x is denoted by symbol $f'(x)$

where $f'(x) = \frac{dy}{dx}$

dx is very small change in x and dy is corresponding very small change in y.

NOTATION : There are many ways to denote the derivative of a function y = f(x). Besides $f'(x)$, the most common notations are these :

y'	**"y prime" or "y dash"**	***Nice and brief but does not name the independent variable.***
$\frac{dy}{dx}$	"dy by dx"	*Names the variables and uses d for derivative.*
$\frac{df}{dx}$	"df by dx"	*Emphasizes the function's name.*
$\frac{d}{dx}f(x)$	"d by dx of f"	*Emphasizes the idea that differentiation is an operation performed on f.*
$D_x f$	"dx of f"	*A common operator notation.*
$\dot{y}$	"y dot"	*One of Newton's notations, now common for time derivatives i.e.* $\frac{dy}{dt}$.
f′(x)	*f dash x*	*Most common notation, it names the independent variable and Emphasize the function's name.*

6.3 SLOPE OF A LINE

It is the tan of angle made by a line with the positive direction of x-axis, measured in anticlockwise direction.

Slope = tan θ (In 1_{st} quadrant tan θ is +ve & 2nd quadrant tan θ is –ve)

In Figure - 1 slope is positive

θ < 90° (1st quadrant)

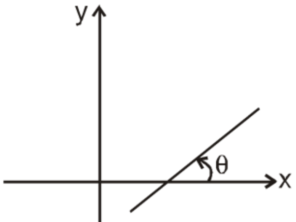


Figure - 1

In Figure - 2 slope is negative

θ > 90° (2nd quadrant)

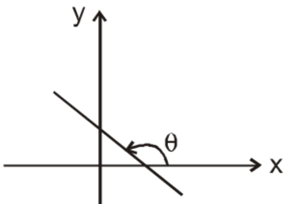


Figure - 2

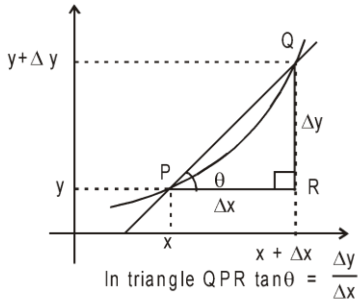
6.4 AVERAGE RATES OF CHANGE :

Given an arbitrary function y = f(x) we calculate the average rate of change of y with respect to x over the interval (x , x + Δx) by dividing the change in value of y, i.e. Δy = f(x + Δx) – f(x), by length of interval Δx over which the change occurred.

The average rate of change of y with respect to x over the interval [x, x + Δx] = $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Geometrically, $\frac{\Delta y}{\Delta x} = \frac{QR}{PR}$ = tan θ = Slope of the line PQ

therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P & Q.



6.5 THE DERIVATIVE OF A FUNCTION

We know that, average rate of change of y w.r.t. x is $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

If the limit of this ratio exists as Δx → 0, then it is called the derivative of given function f(x) and is denoted as

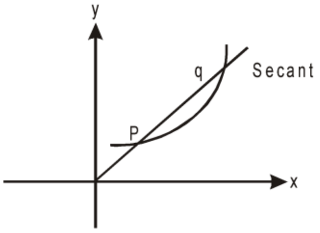
$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

6.6 GEOMETRICAL MEANING OF DIFFERENTIATION

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve

Secant and tangent to a curve

Secant : A secant to a curve is a straight line, which intersects the curve at any two points.



Tangent:-

A tangent is a straight line, which touches the curve at a particular point. Tangent is a limiting case of secant

which intersects the curve at two overlapping points.

In the figure-1 shown, if value of Δx is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (Figure - 2) value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P .

Therefore $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx}$ = tan θ

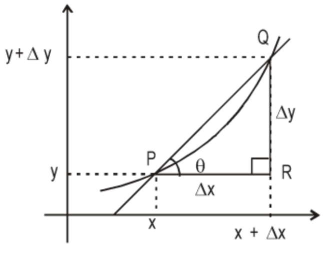


Figure - 1

we can say that differentiation of y with respect to x, i.e. $\left(\frac{dy}{dx}\right)$ is equal to slope of the tangent at point P (x, y) or tanθ = $\frac{dy}{dx}$

(From fig. 1, the average rate of change of y from x to x + Δx is identical with the slope of secant PQ.)

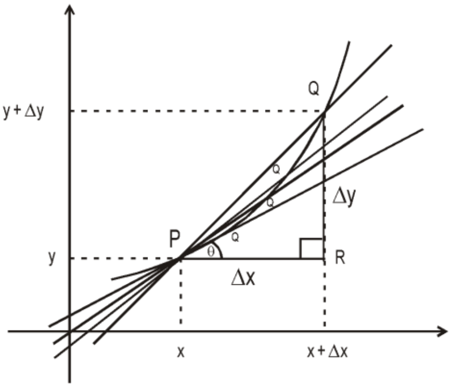


Figure - 2

6.7 RULES FOR DIFFERENTIATION

RULE NO. 1 : DERIVATIVE OF A CONSTANT

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\frac{d}{dx}c$ = 0.

Example 21 $\frac{d}{dx}(8) = 0$, $\frac{d}{dx}\left(-\frac{1}{2}\right) = 0$,

RULE NO. 2 : POWER RULE

If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$.

To apply the power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

Example 22

f	x	x^2	x^3	x^4	....
f'	1	2x	$3x^2$	$4x^3$	....

Example 23 (i) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}$ $(x_{-1}) = (-1)x_{-2} = -\frac{1}{x^2}$

(ii) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}$ $(x_{-3}) = 4(-3)x_{-4} = -\frac{12}{x^4}$.

Example 24 (a) $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Function defined for $x \geq 0$ derivative defined only for $x > 0$

(b) $\frac{d}{dx}(x^{1/5}) = \frac{1}{5}x^{-4/5}$

| |

Function defined for $x \geq 0$ derivative not defined at $x = 0$

RULE NO. 3 : THE CONSTANT MULTIPLE RULE

If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$

In particular, if n is a positive integer, then $\frac{d}{dx}(cx^n) = cn\, x_{n-1}$

Example 25 The derivative formula

$$\frac{d}{dx}(3x^2) = 3(2x) = 6x$$

says that if we rescale the graph of $y = x_2$ by multiplying each y–coordinate by 3, then we multiply the slope at each point by 3.

Example 26 A useful special case

The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with c = –1 gives.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\frac{d}{dx}(u) = -\frac{d}{dx}(u)$$

RULE NO. 4 : THE SUM RULE

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point where u and v are both differentiable functions is their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx}+(-1)\frac{dv}{dx} = \frac{du}{dx}-\frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If u_1, u_2,.........u_n are differentiable at x, then so is $u_1 + u_2 + \text{........} + u_n$, and

$$\frac{d}{dx}(u_1+u_2+.....+u_n) = \frac{du_1}{dx}+\frac{du_2}{dx}+.......+\frac{du_n}{dx}.$$

Example 27 **(a)** $y = x_4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4)+\frac{d}{dx}(12x)$$

(b) $y = x_3 + \frac{4}{3} x_2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(\frac{4}{3}x^2\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)$$

$= 4x_3 + 12$ $\quad = 3x_2 + \frac{4}{3}.2x - 5 + 0 = 3x_2 + \frac{8}{3}x - 5.$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in above example.

RULE NO. 5 : THE PRODUCT RULE

If u and v are differentiable at x, then so is their product uv, and $\frac{d}{dx}$ (uv) = $u\frac{dv}{dx} + v\frac{du}{dx}$.
The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In prime notation (uv)′ = uv′ + vu′.

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$\frac{d}{dx}$ **(x.x) =** $\frac{d}{dx}$ **(x_2) = 2x,** **while** $\frac{d}{dx}$ **(x)** $\frac{d}{dx}$ **.(x) = 1.1 = 1.**

Example 28 Find the derivatives of $y = (x_2 + 1)(x_3 + 3)$.

Solution : From the product Rule with $u = x_2 + 1$ and $v = x_3 + 3$, we find

$$\frac{d}{dx}[(x^2+1)(x^3+3)] = (x_2 + 1)(3x_2) + (x_3 + 3)(2x)$$
$$= 3x_4 + 3x_2 + 2x_4 + 6x = 5x_4 + 3x_2 + 6x.$$

Example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check : $y=(x_2 + 1)(x_3 + 3)=x_5 +x_3+3x_2 + 3$

$$\frac{dy}{dx} = 5x_4 + 3x_2 + 6x.$$

This is in agreement with our first calculation.
There are times, however, when the product Rule must be used. In the following examples. We have only numerical values to work with.

Example 29 Let y = uv be the product of the functions u and v. Find y′(2) if u′(2) = 3, u′(2) = –4, v(2) = 1, and v′(2) = 2.

Solution : From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have $y'(2) = u(2)\,v'(2) + v(2)\,u'(2) = (3)(2) + (1)(-4) = 6 - 4 = 2.$

RULE NO. 6 : THE QUOTIENT RULE

If u and v are differentiable at x, and v(x) ≠ 0, then the quotient u/v is differentiable at x,

and $\frac{d}{dx}\left(\frac{u}{v}\right)$ **=** $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

Example 30 Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$

Solution : We apply the Quotient Rule with u = $t_2 - 1$ and v = $t_2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1).2t-(t^2-1).2t}{(t^2+1)^2} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt)-u(dv/dt)}{v^2}$$

$$= \frac{2t^3+2t-2t^3+2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$$

RULE NO. 7 : DERIVATIVE OF SINE FUNCTION

$$\frac{d}{dx}(\sin x) = \cos x$$

Example 31 (a) y = $x_2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule

$= 2x - \cos x$

(b) y = $x_2 \sin x$: $\frac{dy}{dx} = x_2 \frac{d}{dx}(\sin x) + 2x \sin x$ Product Rule

$= x_2 \cos x + 2x \sin x$

(c) y = $\frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2}$ Quotient Rule

$= \frac{x\cos x - \sin x}{x^2}$.

RULE NO. 8 : DERIVATIVE OF COSINE FUNCTION

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 32 (a) y = 5x + cos x

$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$ Sum Rule

$= 5 - \sin x$

(b) y = sinx cosx

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$ Product Rule

$= \sin x(-\sin x) + \cos x(\cos x)$

$= \cos_2 x - \sin_2 x$

RULE NO. 9 : DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

Because sin x and cos x are differentiable functions of x , the related functions

$\tan x = \frac{\sin x}{\cos x}$; $\sec x = \frac{1}{\cos x}$

$\cot x = \frac{\cos x}{\sin x}$; $\text{cosec } x = \frac{1}{\sin x}$

are differentiable at every value of x at which they are defined. There derivatives. Calculated from the Quotient Rule, are given by the following formulas.

$\frac{d}{dx}$ **(tan x) = sec$_2$ x ;** $\frac{d}{dx}$ **(sec x) = sec x tan x**

$\frac{d}{dx}$ **(cot x) = – cosec^2 x ;** $\frac{d}{dx}$ **(cosec x) = – cosec x cot x**

Example 33 Find dy / dx if y = tan x .

Solution : $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Example 34 (a) $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \text{cosec}^2 x$

(b) $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\text{cosec } x) = 2\frac{d}{dx}(\text{cosec } x)$

$= 2(-\text{cosec } x \cot x) = -2 \text{ cosec } x \cot x$

RULE NO. 10 : DERIVATIVE OF LOGARITHM AND EXPONENTIAL FUNCTIONS

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$ ⇒ $\frac{d}{dx}(e^x) = e^x$

Example 35. $y = e^x . \log_e(x)$

$\frac{dy}{dx} = \frac{d}{dx}(e^x) . \log(x) +. \frac{d}{dx}[\log_e(x)] e^x$ ⇒ $\frac{dy}{dx} = e^x . \log_e(x) + \frac{e^x}{x}$

Example 36. $\frac{d}{dt}(\sin \omega t)$

Answer : $\omega \cos \omega t$

Example 37. $\frac{d}{dt}(\cos \omega t)$

Answer : $-\omega \sin \omega t$

Example 38 (a) $\frac{d}{dx}\cos 3x = -\sin 3x \frac{d}{dx} 3x = -3 \sin 3x$

(b) $\frac{d}{dx}\sin 2x = \cos 2x \frac{d}{dx}(2x) = \cos 2x . 2$

$= 2 \cos 2x$

(c) $\frac{d}{dt}(A \sin(\omega t + \phi))$

$= A \cos(\omega t + \phi) \frac{d}{dt}(\omega t + \phi)$

$= A \cos(\omega t + \phi). \omega.$

$= A \omega \cos(\omega t + \phi)$

Example 39 $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} \quad = -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$

$= -1(3x-2)^{-2}(3) = -\frac{3}{(3x-2)^2}$

Example 40 $\frac{d}{dt}\left[A\cos(\omega t+\varphi)\right]$

$= -A\omega \sin(\omega t + \phi)$

RULE NO. 11 : CHAIN RULE :

If f (x) is given as function of g(x) i.e., y = f(g(x)) and we are required to find $\frac{dy}{dx}$, assume g(x)= u

⇒ y = f(4) ⇒ $\frac{dy}{du} = f'(4)$, $\frac{du}{dx} = g'(x)$

Example 41 $y = \sin(x^2)$
$y = \log(x^2 + 5x)$
$y = \sin(\cos x)$
$y = A\sin(\omega t + \phi)$, A,ω,ϕ, are constant

RULE NO. 12 : RADIAN VS. DEGREES

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x^\circ) .$$

6.8 DOUBLE DIFFERENTIATION

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f . Using Leibniz notation, we write the second derivative of y = f(x) as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D^2 f(x) = D^2 f(x)$

INTERPRETATION OF DOUBLE DERIVATIVE

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve y = f (x) .

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If s = s(t) is the position function of an object that moves in a straight line, we known that its first derivative represents the velocity v(t) of the object as a function of time :

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the acceleration a(t) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function :

$$a(t) = v'(t) = s''(t)$$

or in Leibniz notation, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Solved Examples

Example 42 : If f (x) = x cos x , find f ″ (x).

Solution : Using the Product Rule, we have

$$f'(x) = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)$$

$$= -x \sin x + \cos x$$

To find f ″ (x) we differentiate f' (x) :

$$f''(x) = \frac{d}{dx}(-x \sin x + \cos x)$$

$$= -x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(-x) + \frac{d}{dx}(\cos x)$$

$$= -x \cos x - \sin x - \sin x \quad = -x \cos x - 2 \sin x$$

Example 43 : The position of a particle is given by the equation

$$s = f(t) = t_3 - 6t_2 + 9t$$

where t is measured in seconds and s in meters.

Find the acceleration at time t. What is the acceleration after 4 s ?

Solution : The velocity function is the derivative of the position function :

$$s = f(t) = t_3 - 6t_2 + 9t \quad \Rightarrow \quad v(t) = \frac{ds}{dt} = 3t_2 - 12t + 9$$

The acceleration is the derivative of the velocity function :

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12 \quad \Rightarrow \quad a(4) = 6(4) - 12 = 12 \text{ m/s}_2$$

6.9 APPLICATION OF DERIVATIVES

6.9.1 DIFFERENTIATION AS A RATE OF CHANGE

$\frac{dy}{dx}$ is rate of change of ‘y’ with respect to ‘x’ :

For examples :

(i) $v = \frac{dx}{dt}$ this means velocity ‘v’ is rate of change of displacement ‘x’ with respect to time ‘t’

(ii) $a = \frac{dv}{dt}$ this means acceleration ‘a’ is rate of change of velocity ‘v’ with respect to time ‘t’ .

(iii) $F = \frac{dp}{dt}$ this means force ‘F’ is rate of change of momentum ‘p’ with respect to time ‘t’ .

(iv) $\tau = \frac{dL}{dt}$ this means torque 'τ' is rate of change of angular momentum 'L' with respect to time 't'

(v) Power = $\frac{dW}{dt}$ this means power 'P' is rate of change of work 'W' with respect to time 't'

(vi) $I = \frac{dq}{dt}$ this means current 'I' is rate of flow of charge 'q' with respect to time 't'

Solved Examples

Example 44. The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4} D^2$.
How fast is the area changing with respect to the diameter when the diameter is 10 m?

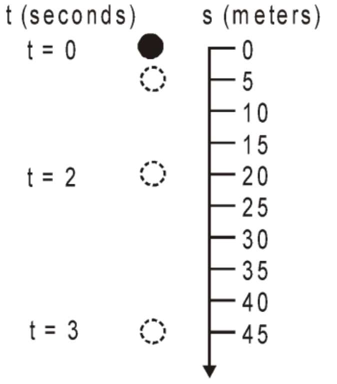
Solution : The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} 2D = \frac{\pi D}{2}$$

When D = 10 m, the area is changing at rate (π/2) 10 = 5π m^2/m. This means that a small change
ΔD m in the diameter would result in a change of about 5π ΔD m^2 in the area of the circle.

Example 45. Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2} gt^2 ,$$



A ball bearing falling from rest

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects in air. Figure shows the free fall of a heavy ball bearing released from rest at time t = 0 sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

Solution : (a) The free–fall equation is $s = 4.9\, t^2$.
During the first 2 sec. the ball falls
$s(2) = 4.9(2)^2 = 19.6$ m,

(b) At any time t, velocity is derivative of displacement :

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8\, t.$$

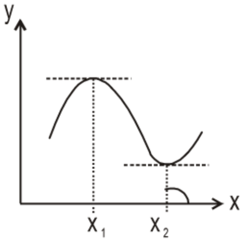
At t = 2, the velocity is v(2) = 19.6 m/sec
in the downward (increasing s) direction. The speed at t = 2 is

speed = |v(2)| = 19.6 m/sec. $a = \frac{d^2s}{dt^2}$ = 9.8 m/s2

6.9.2 MAXIMA AND MINIMA

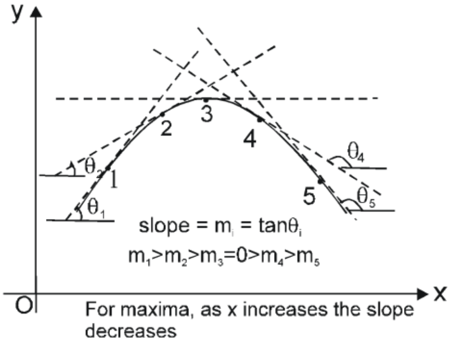
Suppose a quantity y depends on another quantity x in a manner shown in the figure. It becomes maximum at x_1 and minimum at x_2. At these points the tangent to the curve is parallel to the x–axis and hence its slope is tan θ = 0. Thus, at a maximum or a minimum,

slope = $\frac{dy}{dx}$ = 0.



MAXIMA

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ < 0 at maximum.



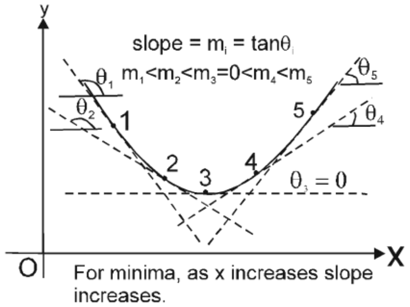
The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$.

Conditions for maxima are:– (a) $\frac{dy}{dx}$ = 0 (b) $\frac{d^2y}{dx^2}$ < 0

MINIMA

Similarly, at a minimum the slope changes from negative to positive.
Hence with the increases of x. the slope is increasing that means
the rate of change of slope with respect to x is positive

hence $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ > 0.



For minima, as x increases slope increases.

Conditions for minima are:– (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} > 0$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

Solved Examples

Example 46. Find minimum value of y = 1 + x_2 – 2x

$\frac{dy}{dx} = 2x - 2$ for minima $\frac{dy}{dx} = 0$

$2x - 2 = 0$

$x = 1$

$\frac{d^2y}{dx^2} = 2$ $\Rightarrow$ $\frac{d^2y}{dx^2} > 0$

at x = 1 there is minima

for minimum value of y

$y_{minimum} = 1 + 1 - 2 = 0$

7. INTEGRATION

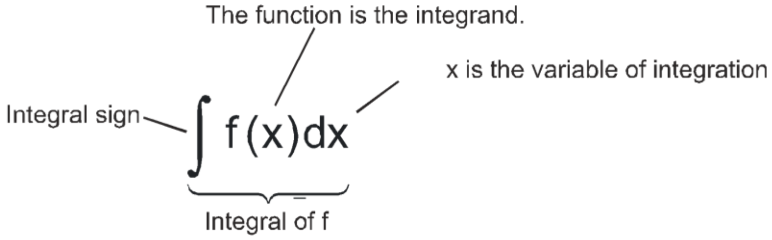
In mathematics, for each mathematical operation, there has been defined an inverse operation.

For example- Inverse operation of addition is subtraction, inverse operation of multiplication is division and inverse operation of square is square root. Similarly there is a inverse operation for differentiation which is known as integration

7.1 ANTIDERIVATIVES OR INDEFINITE INTEGRALS

Definitions :

A function F(x) is an antiderivative of a function f(x) if F´(x) = f(x) for all x in the domain of f. The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by



The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of

integration.

For example f(x) = x_3 then f′(x) = $3x_2$

So the integral of $3x_2$ is x_3

Similarly if f(x) = x_3 + 4 then f′(x) = $3x_2$

So the integral of $3x_2$ is x_3 + 4

there for general integral of $3x_2$ is x_3 + c where c is a constant

One antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way :

$$\int f(x)dx = F(x) + C. \qquad \text{............(i)}$$

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, “The indefinite integral of f with respect to x is F(x) + C.” When we find F(x)+ C, we say that we have integrated f and evaluated the integral.

Solved Examples

Example 47. Evaluate $\int$ 2x dx.

Solution :

$$\int 2x\,dx = \overset{\text{an antiderivative of }2x}{x^2} + \underset{\text{the arbitrary constant}}{C}$$

The formula x_2 + C generates all the antiderivatives of the function 2x. The function x_2 + 1, x_2 – π, and x_2 + $\sqrt{2}$ are all antiderivatives of the function 2x, as you can check by differentiation.

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

7.2 INTEGRAL FORMULAS

	Indefinite Integral	**Reversed derivative formula**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$,n ≠ –1, n rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)$ = x_n
	$\int dx$ = $\int 1dx = x + C$ (special case)	$\frac{d}{dx}(x)$ = 1
2.	$\int \sin(Ax+B)dx = \frac{-\cos(Ax+B)}{A} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right)$ = sin kx
3.	$\int \cos kx\,dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right)$ = cos kx

Solved Examples

Example 48. Examples based on above formulas :

(a) $\int x^5 dx = \frac{x^6}{6} + C$ Formula 1 with n = 5

(b) $\int \frac{1}{\sqrt{x}}dx$ = $\int x^{-1/2}dx = 2x^{1/2} + C = 2\sqrt{x} + C$ Formula 1 with n = –1/2

(c) $\int \sin 2x\,dx = \frac{-\cos 2x}{2} + C$ Formula 2 with k = 2

(d) $\int \cos\frac{x}{2}dx = \int \cos\frac{1}{2}xdx = \frac{\sin(1/2)x}{1/2} + C = 2\sin\frac{x}{2} + C$ Formula 3 with k = 1/2

Example 49. Right : $\int$ x cosx dx = x sin x + cos x + C

Reason : The derivative of the right-hand side is the integrand:

Check : $\frac{d}{dx}$ (x sin x + cos x + C) = x cos x + sin x – sin x + 0 = x cos x.

Wrong : $\int$ x cosx dx = x sin x + C

Reason : The derivative of the right-hand side is not the integrand:

Check : $\frac{d}{dx}$ (x sin x + C) = x cos x + sin x + 0 x cos x.

7.3 RULES FOR INTEGRATION

RULE NO. 1 : CONSTANT MULTIPLE RULE

A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.

$\int k\,f(x)dx = k\int f(x)dx$ **; where k is a constant**

Example 50. $\int 5x^2dx = \frac{5x^3}{3} + C$

Example 51. $\int \frac{7}{x^2}dx = \int 7x^{-2}dx = -\frac{7x^{-1}}{1} + C = \frac{-7}{x} + C$

Example 52. $\int \frac{t}{\sqrt{t}}\,dt = \int t^{1/2}dt = \frac{t^{3/2}}{3/2} + C = \frac{2}{3}t^{3/2} + C$

RULE NO. 2 : SUM AND DIFFERENCE RULE

A function is an antiderivative of a sum or difference f ± g if and only if it is the sum or difference of an antiderivative of f an antiderivative of g.

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Solved Examples

Example 53. Term–by–term integration

Evaluate : $\int$ (x_2 – 2x + 5) dx.

Solution. If we recognize that (x_3 /3) – x_2 + 5x is an antiderivative of x_2 – 2x + 5, we can evaluate the integral as

$$(x^2 - 2x + 5)dx = \overbrace{\frac{x^3}{3} \ -x^2 + 5x}^{\text{antiderivative}} + C \nearrow \text{arbitrary constant}$$

If we do not recognize the antiderivative right away, we can generate it term by term with the sum and difference Rule:

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x dx + \int 5dx = \frac{x^3}{3} + C_1 - x_2 + C_2 + 5x + C_3.$$

This formula is more complicated than it needs to be. If we combine C_1,C_2 and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x_2 + 5x + C$$

and still gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x dx + \int 5dx = \frac{x^3}{3} - x_2 + 5x + C.$$

Find the simplest antiderivative you can for each part add the constant at the end.

Example 54. Find a body velocity from its acceleration and initial velocity. The acceleration of gravity near the surface of the earth is 9.8 m/sec$_2$. This means that the velocity v of a body falling freely in a vacuum changes at the rate of $\frac{dv}{dt}$ = 9.8 m/sec$_2$. If the body is dropped from rest, what will its velocity be t seconds after it is released?

Solution. In mathematical terms, we want to solve the initial value problem that consists of

The differential condition : $\frac{dv}{dt} = 9.8$

The initial condition: $v = 0$ when $t = 0$ (abbreviated as $v(0) = 0$)

We first solve the differential equation by integrating both sides with respect to t:

$\frac{dv}{dt} = 9.8$ The differential equation

$\int \frac{dv}{dt} dt = \int 9.8 dt$ Integrate with respect to t.

$v + C_1 = 9.8t + C_2$ Integrals evaluated

$v = 9.8t + C.$ Constants combined as one

This last equation tells us that the body’s velocity t seconds into the fall is 9.8t + C m/sec.

For value of C : What value? We find out from the initial condition :

$v = 9.8t + C$

$0 = 9.8(0) + C \quad v(0) = 0$

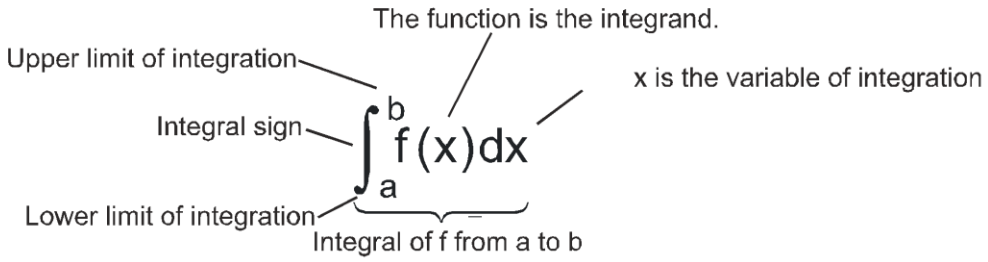
$C = 0.$

Conclusion : The body’s velocity t seconds into the fall is

$v = 9.8t + 0 = 9.8t$ m/sec.

The indefinite integral F(x) + C of the function f(x) gives the general solution y = F(x) + C of the differential equation dy/dx = f(x). The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We solve the differential equation by finding its general solution. We then solve the initial value problem by finding the particular solution that satisfies the initial condition $y(x_o) = y_o$ (y has the value y_o when $x = x_o$.).

7.4 DEFINITE INTEGRATION OR INTEGRATION WITH LIMITS



$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a)$$

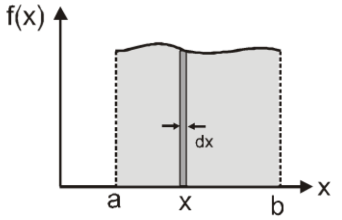
where g(x) is the antiderivative of f(x) i.e. g´(x) = f(x)

Solved Examples

Example 55. $\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3\left[x\right]_{-1}^{4}$ = 3[4 – (–1)] = (3) (5) = 15

$\int_0^{\pi/2} \sin x dx = \left[-\cos x\right]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0)$ = –0 + 1 = 1

7.5 APPLICATION OF DEFINITE INTEGRAL : CALCULATION OF AREA OF A CURVE



From graph shown in figure if we divide whole area in infinitely small strips of dx width.

We take a strip at x position of dx width.

Small area of this strip dA = f(x) dx

So, the total area between the curve and x–axis = sum of area of all strips = $\int_a^b f(x)dx$

Let f(x) ≥ 0 be continuous on [a,b]. The area of the region between the graph of f and the x-axis is

A = $\int_a^b f(x)dx$

Solved Examples

Example 56. Find area under the curve of y = x from x = 0 to x = a

Answer : $\int_0^a ydx = \left.\frac{x^2}{2}\right|_0^a = \frac{a^2}{2}$

8. VECTOR

In physics we deal with two type of physical quantity one is scalar and other is vector. Each scalar quantities has magnitude.

Magnitude of a physical quantity means product of numerical value and unit of that physical quantity.

For example mass = 4 kg

Magnitude of mass = 4 kg

and unit of mass = kg

Example of scalar quantities : mass, speed, distance etc.

Scalar quantities can be added, subtracted and multiplied by simple laws of algebra.

8.1 DEFINITION OF VECTOR

If a physical quantity in addition to magnitude -

(a) has a specified direction.

(b) It should obey commutative law of additions $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

(c) obeys the law of parallelogram of addition, then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be a vector.

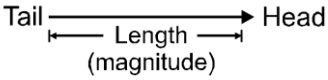
If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not a be vector. e.g. time, pressure, surface tension or current etc. have directions but are not vectors because they do not obey parallelogram law of addition.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.

Example of vector quantity : Displacement, velocity, acceleration, force etc.

Representation of vector :

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as

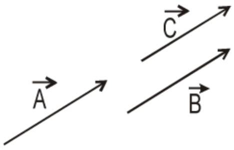


Mathematically, vector is represented by $\vec{A}$

Sometimes it is represented by bold letter **A** .

IMPORTANT POINTS :

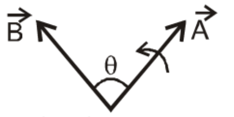
If a vector is displaced parallel to itself it does not change (see Figure)



$\vec{A}=\vec{B}=\vec{C}$

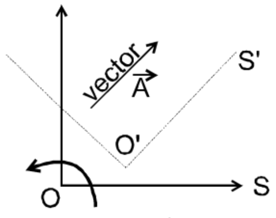
Transition of a vector parallel to itself

If a vector is rotated through an angle other than multiple of 2π (or 360º) it changes (see Figure).



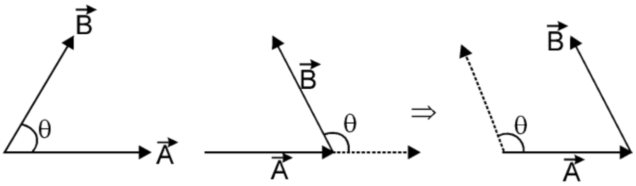
$\vec{A} \neq \vec{B}$ Rotation of a vector

- If the frame of reference is translated or rotated the vector does not change (though its components may change). (see Figure).



moving frame

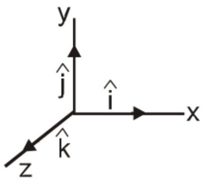
- Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.
- Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e. $0 \le \theta \le \pi$).



8.2 UNIT VECTOR

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector ($\vec{A}$) can be written as the product of unit vector ($\hat{A}$) in that direction and magnitude of the given vector.

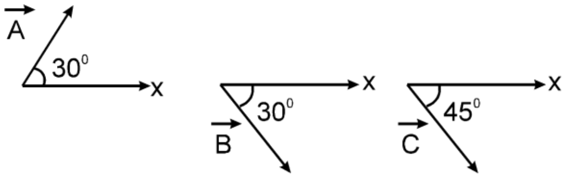
$$\vec{A} = A\hat{A} \quad \text{or} \quad \hat{A} = \frac{\vec{A}}{A}$$



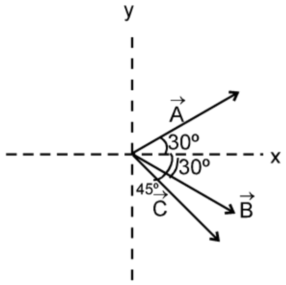
A unit vector has no dimensions and unit. Unit vectors along the positive x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}|$ = 1.

Solved Examples

Example 57. Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are shown in the figure. Find angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.



Solution. To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ such that tails of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are connected as shown in figure.



Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60º, $\vec{B}$ and $\vec{C}$ is 15º and between $\vec{A}$ and $\vec{C}$ is 75º.

Example 58. A unit vector along East is defined as $\hat{i}$. A force of 10_5 dynes acts west wards. Represent the force in terms of $\hat{i}$.

Solution. $\vec{F} = -10_5\ \hat{i}$ dynes

8.3 MULTIPLICATION OF A VECTOR BY A SCALAR

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}$ (= λ$\vec{A}$) whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is – λ times $|\vec{A}|$.

Solved Examples

Example 59. A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a}$ = 3m/s_2 East wards

(ii) $\vec{a}$ = –4m/s_2 North wards

Solution. (i) $\vec{F} = m\vec{a}$ = 3 × 3 ms_{-2} East wards = 9 N East wards

(ii) $\vec{F} = m\vec{a}$ = 3 × (–4) N North wards

= – 12N North wards = 12 N South wards

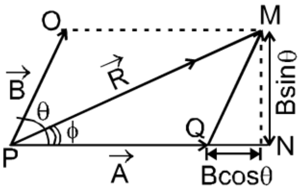
8.4 ADDITION OF VECTORS

Addition of vectors is done by parallelogram law or the triangle law :

(a) Parallelogram law of addition of vectors : If two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram both pointing outwards (and their tails coinciding) as shown. Then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e., vector sum of $\vec{A}$ and $\vec{B}$).

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

The direction of resultant vector $\vec{R}$ from $\vec{A}$ is given by

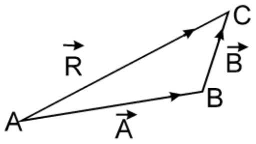


$$\tan\phi = \frac{MN}{PN} = \frac{MN}{PQ+QN} = \frac{B\sin\theta}{A+B\cos\theta}$$

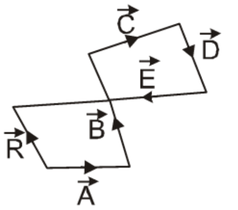
$$\varphi = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$$

(b) **Triangle law of addition of vectors :** To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$. The sum $\vec{A}$ + $\vec{B}$ is a vector $\vec{R}$ drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A} + \vec{B} = \vec{R}$. As the figure formed is a triangle, this method is called ‘ triangle method’ of addition of vectors.

If the ‘triangle method’ is extended to add any number of vectors in one operation as shown . Then the figure formed is a polygon and hence the name Polygon Law of addition of vectors is given to such type of addition.



$\vec{A} + \vec{B} = \vec{R}$



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$

IMPORTANT POINTS :

☞ To a vector only a vector of same type can be added that represents the same physical quantity and the resultant is a vector of the same type.

☞ As R = $[A_2 + B_2 + 2AB\cos\theta]_{1/2}$ so R will be maximum when, cos θ = max = 1, i.e., θ = 0^o, i.e. vectors are like or parallel and R_{max} = A + B.

☞ The resultant will be minimum if, cos θ = min = -1, i.e., θ = 180^o , i.e. vectors are antiparallel and R_{min} = A ~ B.

☞ If the vectors A and B are orthogonal, i.e., θ = 90^o, R = $\sqrt{A^2+B^2}$

☞ As previously mentioned that the resultant of two vectors can have any value from (A ~ B) to (A + B) depending on the angle between them and the magnitude of resultant decreases as θ increases 0^o to 180^o

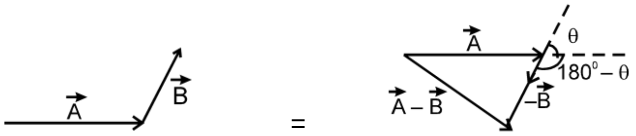
☞ Minimum number of unequal coplanar vectors whose sum can be zero is three.

☞ The resultant of three non-coplanar vectors can never be zero, or minimum number of non coplanar vectors whose sum can be zero is four.

☞ Subtraction of a vector from a vector is the addition of negative vector, i.e.,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

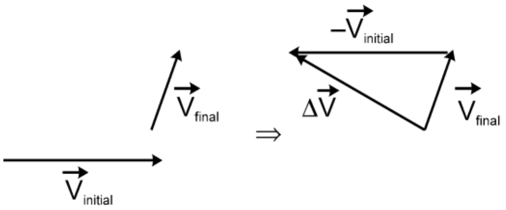
(a) From figure it is clear that $\vec{A} - \vec{B}$ is equal to addition of $\vec{A}$ with reverse of $\vec{B}$



$$|\vec{A} - \vec{B}| = [(A)_2 + (B_2) + 2AB\cos(180^\circ - \theta)]_{1/2}$$

$$|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

(b) Change in a vector physical quantity means subtraction of initial vector from the final vector.



Solved Examples

Example 60. Find the resultant of two forces each having magnitude F_0, and angle between them is θ.

Solution. $F^2_{Resultant} = F_0^2 + F_0^2 + 2F_0^2\cos\theta$

$$= 2F_0^2(1 + \cos\theta) = 2F_0^2\left(1 + 2\cos_2\frac{\theta}{2} - 1\right) = 2 \times 2F_0^2\cos_2\frac{\theta}{2}$$

$$F_{resultant} = 2F_0\cos\frac{\theta}{2}$$

Example 61. Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Find angle between $\vec{A}$ and $\vec{B}$?

Solution. $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$ ⇒ $A_2 + B_2 + 2AB\cos\theta = A_2 + B_2 - 2AB\cos\theta$

⇒ $4AB\cos\theta = 0$ ⇒ $\cos\theta = 0$ ⇒ $\theta = \frac{\pi}{2}$

Example 62. If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference?

Solution. Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then

$$|\hat{R}| = |\hat{A} + \hat{B}|$$

$$1 = \sqrt{(\hat{A})^2 + (\hat{B})^2 + 2|\hat{A}||\hat{B}|\cos\theta}$$

$$1 = 1 + 1 + 2\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

$$|\hat{A} - \hat{B}| = \sqrt{(\hat{A})^2 + (\hat{B})^2 - 2|\hat{A}||\hat{B}|\cos\theta} = \sqrt{1 + 1 - 2 \times 1 \times 1(-\frac{1}{2})} = \sqrt{3}$$

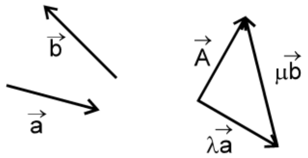
8.5 RESOLUTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two nonzero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors - one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number .

$\vec{A} = \lambda \vec{a} + \mu \vec{b}$ **(**where λ and μ are real numbers)

We say that $\vec{A}$ has been resolved into two component vectors namely

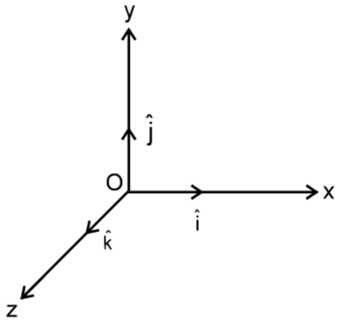
$\lambda\vec{a}$ and $\mu\vec{b}$



$\lambda \vec{a}$ and $\mu \vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane.

Resolution along rectangular component :

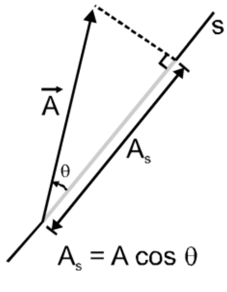
It is convenient to resolve a general vector along axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x,y and z-axis as shown in figure below:



Resolution in two Dimension

Consider a vector $\vec{A}$ that lies in xy plane as shown in figure,

$\vec{A} = \vec{A_1} + \vec{A_2}$



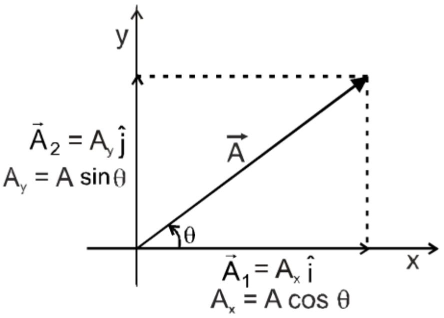
$\vec{A_1} = A_x\hat{i}$, $\vec{A_2} = A_y\hat{j}$

$\Rightarrow \quad \vec{A} = A_x\hat{i} + A_y\hat{j}$

The quantities A_x and A_y are called x- and y- components of the vector $\vec{A}$.

A_x is itself not a vector but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$.

$A_x = A \cos \theta$ and $A_y = A \sin \theta$



Its clear from above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways :

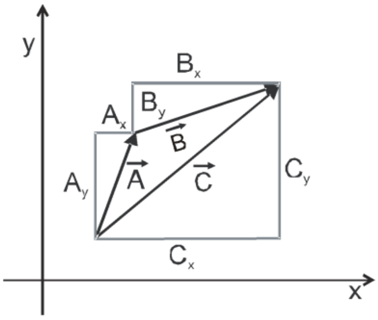
(a) its magnitude A and the direction θ it makes with the x-axis; or

(b) its components A_x and A_y. $A = \sqrt{A_x^2 + A_y^2}$, $\theta = \tan^{-1}\frac{A_y}{A_x}$

Note : If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e.
components of a vector perpendicular to itself is always zero.
The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in figure. We saw that
$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both

$C_x = A_x + B_x$

and $C_y = A_y + B_y$



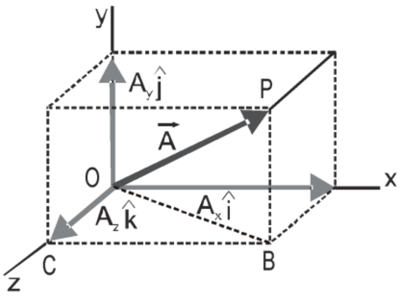
Resolution in three dimensions. A vector $\vec{A}$ in components along x-, y- and z-axis can be written as :

$$\vec{OP} = \vec{OB} + \vec{BP} = \vec{OC} + \vec{CB} + \vec{BP}$$

$$\Rightarrow \quad \vec{A} = \vec{A}_Z + \vec{A}_x + \vec{A}_y = \vec{A}_x + \vec{A}_y + \vec{A}_Z$$

$$= A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



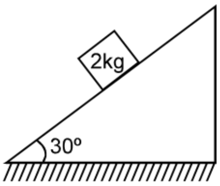
$A_x = A \cos \alpha, \; A_y = A \cos \beta, \; A_z = A \cos \gamma$

where cos α, cos β and cos γ are termed as **Direction Cosines** of a given vector $\vec{A}$.

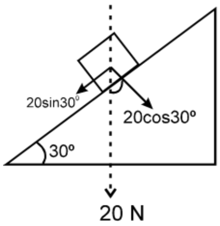
$$\cos_2 \alpha + \cos_2 \beta + \cos_2 \gamma = 1$$

Solved Examples

Example 63. A mass of 2 kg lies on an inclined plane as shown in figure.Resolve its weight along and perpendicular to the plane.(Assumeg=10m/s$_2$)

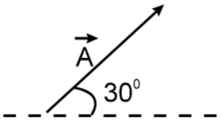


Solution. Component along the plane = 20 sin 30 = 10 N



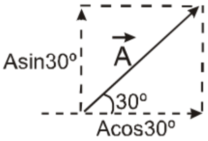
component perpendicular to the plane = 20 cos 30 = 10 $\sqrt{3}$ N

Example 64. A vector makes an angle of 30º with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component?



Solution. Let vector is $\vec{A}$

$A_x = A \cos 30_0 = 250 = \frac{A\sqrt{3}}{2} \quad \Rightarrow \quad A = \frac{500}{\sqrt{3}}$



$A_y = A \sin 30_0 = \frac{500}{\sqrt{3}} \times \frac{1}{2} = \frac{250}{\sqrt{3}}$

Example 65. $\vec{A} = \hat{i} + 2\hat{j} - 3\hat{k}$, when a vector $\vec{B}$ is added to $\vec{A}$, we get a unit vector along x-axis. Find the value of $\vec{B}$? Also find its magnitude

Solution. $\vec{A} + \vec{B} = \hat{i}$

$\vec{B} = \hat{i} - \vec{A} = \hat{i} - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{j} + 3\hat{k}$

$\Rightarrow \quad |\vec{B}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

Example 66. In the above question find a unit vector along $\vec{B}$?

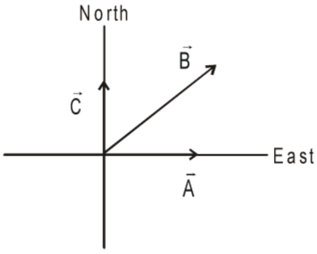
Solution. $\vec{B} = \frac{\vec{B}}{B} = \frac{-2\hat{j} + 3\hat{k}}{\sqrt{13}}$

Example 67. Vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitude 5, $5\sqrt{2}$ and 5 respectively, direction of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are towards east, North-East and North respectively. If $\hat{i}$ and $\hat{j}$ are unit vectors along East and North respectively. Express the sum $\vec{A} + \vec{B} + \vec{C}$ in terms of $\hat{i}$, $\hat{j}$. Also Find magnitude and direction of the resultant.

Solution. $\vec{A} = 5\hat{i}$ $\qquad$ $\vec{C} = 5\hat{j}$

$\vec{B} = 5\sqrt{2}\cos 45\hat{i} + 5\sqrt{2}\sin 45\hat{j} \qquad = 5\hat{i} + 5\hat{j}$

$\vec{A} + \vec{B} + \vec{C} = 5\hat{i} + 5\hat{i} + 5\hat{j} + 5\hat{j} \qquad = 10\hat{i} + 10\hat{j}$



$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{(10)^2 + (10)^2} = 10\sqrt{2}$

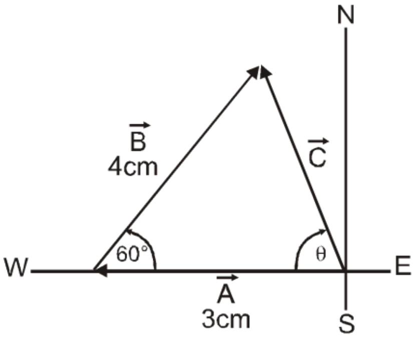
$\tan\theta = \frac{10}{10} = 1 \qquad \Rightarrow \qquad \theta = 45^\circ$ from East

Example 68. You walk 3 Km west and then 4 Km headed 60° north of east. Find your resultant displacement

(a) graphically and

(b) using vector components.

Solution. Picture the Problem : The triangle formed by the three vectors is not a right triangle, so the magnitudes of the vectors are not related by the Pythagoras theorem. We find the resultant graphically by drawing each of the displacements to scale and measuring the resultant displacement.



(a) If we draw the first displacement vector 3 cm long and the second one 4 cm long, we find the resultant vector to be about 3.5 cm long. Thus the magnitude of the resultant displacement is 3.5 Km. The angle θ made between the resultant displacement and the west direction can then be measured with a protractor. It is about 75°.

(b) 1. Let $\vec{A}$ be the first displacement and choose the x-axis to be in the easterly direction. Compute A_x and A_y, $A_x = -3$, $A_y = 0$

2. Similarly, compute the components of the second displacement $\vec{B}$, B_x = 4cos 60° = 2 , B_y = 4 sin 60° = $2\sqrt{3}$

3. The components of the resultant displacement = $\vec{A} + \vec{B}$ are found by addition,

   $\vec{C} = (-3+2)\hat{i} + (2\sqrt{3})\hat{j} = -\ \hat{i} + 2\sqrt{3}\ \hat{j}$

4. The Pythagorean theorem gives the magnitude of $\vec{C}$.

   $C = \sqrt{1^2 + \left(2\sqrt{3}\right)^2} = \sqrt{13} = 3.6$

5. The ratio of C_y to C_x gives the tangent of the angle θ between $\vec{C}$ and the x axis.

   $\tan\theta = \dfrac{2\sqrt{3}}{-1} \Rightarrow \theta = -74°$

Remark : Since the displacement (which is a vector) was asked for, the answer must include either the magnitude and direction, or both components. in (b) we could have stopped at step 3 because the x and y components completely define the displacement vector. We converted to the magnitude and direction to compare with the answer to part (a). Note that in step 5 of (b), a calculator gives the angle as –74°. But the calculator can't distinguish whether the x or y components is negative. We noted on the figure that the resultant displacement makes an angle of about 75° with the negative x axis and an angle of about 105° with the positive x axis. This agrees with the results in (a) within the accuracy of our measurement.

8.6 MULTIPLICATION OF VECTORS

8.6.1 THE SCALAR PRODUCT

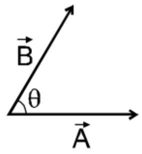
The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus, $\vec{A}.\vec{B} = AB\cos\theta$ {here θ is the angle between the vectors}

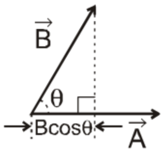
PROPERTIES :

- It is always a scalar which is positive if angle between the vectors is acute (i.e. < 90º) and negative if angle between them is obtuse (i.e. 90º < θ ≤ 180º)
- It is commutative, i.e., $\vec{A}.\vec{B} = \vec{A}.\vec{B}$
- It is distributive, i.e. $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

- As by definition $\vec{A}.\vec{B}$ = AB cos θ. The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}.\vec{B}}{AB}\right]$
- $\vec{A}.\vec{B}$ = A (B cos θ) = B (A cos θ)

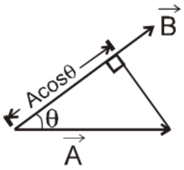
  Geometrically, B cos θ is the projection of $\vec{B}$ onto $\vec{A}$ and A cos θ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}.\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.





Component of $\vec{B}$ along $\vec{A}$ = B cosθ = $\frac{\vec{A}.\vec{B}}{A}$ = $\vec{A}.\vec{B}$

Component of $\vec{A}$ along $\vec{B}$ = A cosθ = $\frac{\vec{A}.\vec{B}}{B}$ = $\vec{A}.\vec{B}$



- Scalar product of two vectors will be maximum when cos θ = max = 1, i.e., θ = 0°, i.e., vectors are parallel ⇒ $(\vec{A}.\vec{B})_{max}$ = AB
- If the scalar product of two nonzero vectors vanishes then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by

  $(\vec{A})^2 = \vec{A}.\vec{A}$ = AA cosθ = AAcos0° = A^2 ⇒ A = $\sqrt{\vec{A}.\vec{A}}$
- In case of unit vector $\hat{n}$,

  $\hat{n}.\hat{n}$ = 1 x 1 x cos 0° = 1 ⇒ $\hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$
- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
- $\vec{A}. = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

Solved Examples

Example 69. If the Vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?

Solution. If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

⇒ $\vec{P}.\vec{Q} = 0$ ⇒ $(a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0$

⇒ $a^2 - 2a - 3 = 0$ ⇒ $a^2 - 3a + a - 3 = 0$

⇒ $a(a-3)+1(a-3)=0$ ⇒ $a=-1, 3$

Example 70. Find the component of $3\hat{i} + 4\hat{j}$ along $\hat{i} + \hat{j}$?

Solution. Component of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}\cdot\vec{B}}{B}$ hence required component

$$= \frac{(3\hat{i}+4\hat{j})\cdot(\hat{i}+\hat{j})}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

Example 71. Find angle between $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 12\hat{i} + 5\hat{j}$?

Solution. We have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(3\hat{i}+4\hat{j})\cdot(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$$\cos\theta = \frac{36+20}{5\times13} = \frac{56}{65} \qquad \theta = \cos^{-1}\frac{56}{65}$$

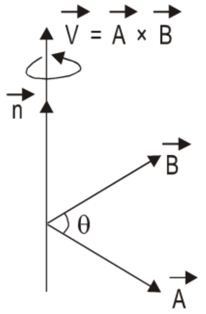
8.6.2 VECTOR PRODUCT

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \times \vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as : $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand-thumb rule.

Right-Hand-Thumb Rule:

To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding . Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$. The direction of the thumb gives the direction of $\hat{n}$.



PROPERTIES :

☞ Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

☞ Vector product of two vectors is not commutative i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$.

But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}|$ = AB sin θ

☞ The vector product is distributive when the order of the vectors is strictly maintained i.e.

$$A\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}.$$

☞ The magnitude of vector product of two vectors will be maximum when sinθ = max = 1, i.e,, θ = 90º

$$|\vec{A}\times\vec{B}|_{max} = AB$$

i.e., magnitude of vector product is maximum if the vectors are orthogonal.

☞ The magnitude of vector product of two non–zero vectors will be minimum when |sinθ| = minimum = 0,i.e., θ = 0º or 180º and $|\vec{A}\times\vec{B}|_{min} = 0$ i.e., if the vector product of two non–zero vectors vanishes, the vectors are collinear.

Note : When θ = 0º then vectors may be called as like vector or parallel vectors and when θ = 180º then vectors may be called as unlike vectors or antiparallel vectors.

☞ The self cross product i.e. product of a vector by itself vanishes i.e. is a null vector.

Note : Null vector or zero vector : A vector of zero magnitude is called zero vector. The direction of a zero vector is in determinate (unspecified).

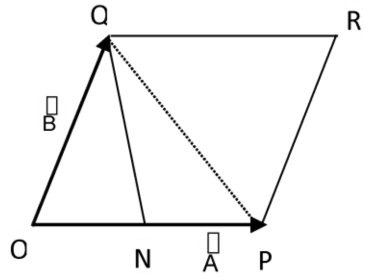
$$\vec{A}\times\vec{A} = AA \sin 0^{o}\, \hat{n} = \vec{0}.$$

Note : Geometrical meaning of vector product of two vectors

(i) Consider two vectors $\vec{A}$ and $\vec{B}$ which are represented by $\overrightarrow{OP}$ and $\overrightarrow{QP}$ and ∠POQ = θ

(ii) Complete the parallelogram OPRQ. Join P with Q. Here OP = A and OQ = B. Draw QN $\perp$ OP

(iii) Magnitude of cross product of $\vec{A}$ and $\vec{B}$



$|\vec{A}\times\vec{B}|$ = AB sin θ

= (OP)(OQ sin θ)

= (OP)(NQ) (∵ NQ = OQ sin θ)

= base × height

= Area of parallelogram OPRQ

Area of ΔPOQ = ΔPOQ = $\frac{\text{base}\times\text{height}}{2} = \frac{(OP)(NQ)}{2} = \frac{1}{2}|\vec{A}\times\vec{B}|$

∴ Area of parallelogram OPRQ = 2 [area of Δ OPQ] = $|\vec{A}\times\vec{B}|$

Formulae to find Area

If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a triangle, then its area = $\frac{1}{2}\,|\vec{A}\times\vec{B}|$

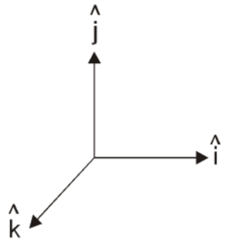
If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a parallelogram, then its area = $|\vec{A}\times\vec{B}|$

If $\vec{A}$ and $\vec{B}$ are diagonals of a parallelogram then its area = $\frac{1}{2}\,|\vec{A}\times\vec{B}|$

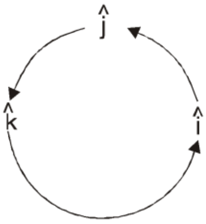
☞ In case of unit vector $\hat{n}$, $\hat{n}\times\hat{n}=\vec{0}$ $\Rightarrow$ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$

☞ In case of orthogonal unit vectors $\hat{i},\hat{j}$ and $\hat{k}$
in accordance with right-hand-thumb-rule,

$\hat{i}\times\hat{j}=\hat{k}$ $\qquad$ $\hat{j}\times\hat{k}=\hat{i}$ $\qquad$ $\hat{k}\times\hat{i}=\hat{j}$



(A)



(B)

In terms of components, $\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\hat{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\hat{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\hat{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$

$$\vec{A}\times\vec{B}=\hat{i}\,(A_yB_z-A_zB_y)+\hat{j}\,(A_zB_x-A_xB_z)+\hat{k}\,(A_xB_y-A_yB_x)$$

The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A}$ and $\vec{B}$ equal to

Example 72. $\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A}\times\vec{B}$?

Solution. Applying right hand thumb rule we find that $\vec{A}\times\vec{B}$ is along North.

Example 73. If $\vec{A}\cdot\vec{B}=|\vec{A}\times\vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$

Solution. $\vec{A}\cdot\vec{B}=|\vec{A}\times\vec{B}|$ $\quad$ AB cos θ = AB sin θ $\quad$ tan θ = 1 $\quad$ ⇒ $\quad$ θ = 45°

Example 74. Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Find a unit vector which is perpendicular to both $\vec{A}$ and $\vec{B}$

Solution. $\vec{A}\times\vec{B}$ = AB sin θ $\hat{n}$

⇒ $\hat{n}=\dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Example 75. Find $\vec{A} \times \vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$.

Solution. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}\,(-4 - \hat{j}\,(-4)) - (2 - 12) + \hat{k}\,(-1-(-6)) = 10\hat{j} + 5\hat{k}$

Miscellaneous Problems

Problem 1. Find the value of

(a) $\sin(-\theta)$ **(b)** $\cos(-\theta)$ **(c)** $\tan(-\theta)$

(d) $\cos\left(\frac{\pi}{2} - \theta\right)$ **(e)** $\sin\left(\frac{\pi}{2} + \theta\right)$ **(f)** $\cos\left(\frac{\pi}{2} + \theta\right)$

(g) $\sin(\pi - \theta)$ **(h)** $\cos(\pi - \theta)$ **(i)** $\sin\left(\frac{3\pi}{2} - \theta\right)$

(j) $\cos\left(\frac{3\pi}{2} - \theta\right)$ **(k)** $\sin\left(\frac{3\pi}{2} + \theta\right)$ **(l)** $\cos\left(\frac{3\pi}{2} + \theta\right)$

(m) $\tan\left(\frac{\pi}{2} - \theta\right)$ **(n)** $\cot\left(\frac{\pi}{2} - \theta\right)$

Answers : **(a)** – sin θ **(b)** cos θ **(c)** – tan θ **(d)** sin θ **(e)** cos θ **(f)** – sin θ

(g) sin θ **(h)** – cos θ **(i) – cos** θ **(j)** – sin θ **(k)** – cos θ **(l)** sin θ

(m) cot θ **(n)** tan θ

Problem 2. (i) For what value of m the vector $\vec{A} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ is perpendicular to $\vec{B} = 3\hat{i} - m\hat{j} + 6\hat{k}$

(ii) Find the components of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the direction of $\hat{i} + \hat{j}$?

Answers : (i) m = –10 (ii) $\frac{5}{\sqrt{2}}$.

Problem 3. (i) $\vec{A}$ is North–East and $\vec{B}$ is down wards, find the direction of $\vec{A} \times \vec{B}$.

(ii) Find $\vec{B} \times \vec{A}$ if $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$.

Answers : (i) North - West.

(ii) $-4\hat{i} - 3\hat{j} + \hat{k}$