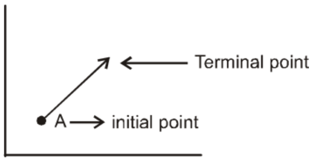
Vector

1. Vectors and their representation :



Vector quantities are specified by definite magnitude and definite direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\vec{AB}$ is expressed by $|\vec{AB}|$.

2. Types of Vectors :

(i) Zero vector :
A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O**. **The direction of zero vector is indeterminate.**

(ii) Unit vector :

A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$, symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

(iii) Equal vectors :

Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

(iv) Collinear vectors :

Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction they are named as **like vectors** otherwise unlike vectors.

Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (= \lambda)$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(v) Coplanar vectors :

A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"two vectors are always coplanar"**.

Example # 1 : Find unit vector of $\hat{i} - 2\hat{j} + 3\hat{k}$

Solution : $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore \quad |\vec{a}| = \sqrt{14} \Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

Example # 2 : Find values of x & y for which the vectors

$\vec{a} = (x + 2)\hat{i} - (x - y)\hat{j} + \hat{k}$

$\vec{b} = (x - 1)\hat{i} + (2x + y)\hat{j} + 2\hat{k}$ are parallel.

Solution : $\vec{a}$ and $\vec{b}$ are parallel if $\frac{x+2}{x-1} = \frac{y-x}{2x+y} = \frac{1}{2}$

$x = -5, y = -20$

3. Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then m is a vector parallel to $\vec{a}$ whose magnitude is |m| times that of $\vec{a}$. This multiplication is called **scalar multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then :

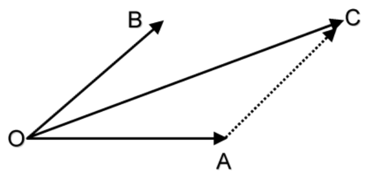
(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$

(ii) $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

(iv) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

4. Addition of vectors :



(i) If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

(ii) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ **(commutative)**

(iii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ **(associative)**

(iv) $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$

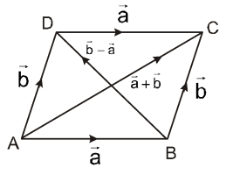
(v) $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$

(vi) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

(vii) $|\vec{a} - \vec{b}| \ge ||\vec{a}| - |\vec{b}||$

Example # 3: If $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Solution : Let ABCD be a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.



Then, $\vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow \vec{AC} = \vec{a} + \vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$|\vec{AC}| = \sqrt{9+16+25} = \sqrt{50}$

$\vec{AB} + \vec{BD} = \vec{AD}$

$\Rightarrow \quad \vec{BD} = \vec{AD} - \vec{AB} = \vec{b} - \vec{a} = -(\hat{i} + 2\hat{j} + 3\hat{k})$

$|\vec{BD}| = \sqrt{1+4+9} = \sqrt{14}$

$\therefore$ Unit vector along $\vec{AC} = \dfrac{\vec{AC}}{|\vec{AC}|} = \dfrac{1}{\sqrt{50}}\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right)$

and Unit vector along $\vec{BD} = \dfrac{\vec{BD}}{|\vec{BD}|} = -\dfrac{1}{\sqrt{14}}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

Example # 4 : ABCDE is a pentagon. Prove that the resultant of the forces $\vec{AB}$, $\vec{AE}$, $\vec{BC}$, $\vec{DC}$, $\vec{ED}$ and $\vec{AC}$ is 3. $\vec{AC}$

Solution : Let $\vec{R}$ be the resultant force

$\therefore \quad \vec{R} = \vec{AB} + \vec{AE} + \vec{BC} + \vec{DC} + \vec{ED} + \vec{AC}$

$\Rightarrow \quad \vec{R} = (\vec{AB} + \vec{BC}) + (\vec{AE} + \vec{ED} + \vec{DC}) + \vec{AC}$

$\Rightarrow \quad \vec{R} = \vec{AC} + \vec{AC} + \vec{AC}$

$\Rightarrow \quad \vec{R} = 3\vec{AC}$. Hence proved.

Self Practice Problems :

(1) Given a regular hexagon ABCDEF with centre O, show that
(i) $\vec{OB} - \vec{OA} = \vec{OC} - \vec{OD}$ (ii) $\vec{OD} + \vec{OA} = 2\,\vec{OB} + \vec{OF}$ (iii) $\vec{AD} + \vec{EB} + \vec{PC} = 4\,\vec{AB}$

(2) The vector $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$. Determine the unit vector along $\vec{c}$.

(3) The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

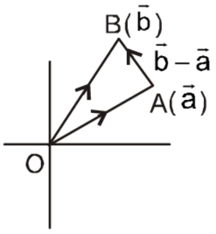
Ans. (2) $-\frac{1}{3}\hat{i} + \frac{2}{15}\hat{j} - \frac{14}{15}\hat{k}$

5. Position vector of a point :

Let O be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then

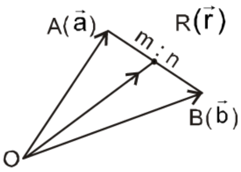
$\vec{AB} = \vec{b} - \vec{a}$ = position vector **(p.v.)** of B − position vector **(p.v.)** of A.

6. Distance formula :



Distance between the two points A $(\vec{a})$ and B $(\vec{b})$ is AB = $|\vec{a} - \vec{b}|$

7. Section formula :



If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the p.v. of a point which divides AB in the ratio m: n is given by $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$.

Note : Position vector of mid point of AB = $\dfrac{\vec{a} + \vec{b}}{2}$.

Example # 5 : The midpoint of two opposite sides of quadrilateral and the midpoint of the diagonals are vertices of a parallelogram. Prove using vectors.

Solution : Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of vertices A, B, C, D respectively.
Let E, F, G, H be midpoint of AB, CD, AC and BD resepectively

P.V of E = $\dfrac{\vec{a} + \vec{b}}{2}$

P.V of F = $\dfrac{\vec{c} + \vec{d}}{2}$

P.V. of G = $\dfrac{\vec{a} + \vec{c}}{2}$

P.V. of H = $\dfrac{\vec{b} + \vec{d}}{2}$

$\vec{EG} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{\vec{c}-\vec{b}}{2}$

$\vec{HF} = \frac{\vec{c}+\vec{d}}{2} - \left(\frac{\vec{b}+\vec{d}}{2}\right) = \frac{\vec{c}-\vec{b}}{2}$

$\vec{EG} = \vec{HF}$ $\Rightarrow$ $\vec{EG} \parallel \vec{HF}$ and EG = HF

hence EGHF is a parallelogram.

Self Practice Problems

(4) Express vectors $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ in terms of the vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$

(5) If $\vec{a}, \vec{b}$ are position vectors of the points(1,–1),(–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

(6) The position vectors of the points A, B, C, D are $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$, $\hat{i}-6\hat{j}-\hat{k}$ respectively. Show that the lines AB and CD are parallel and find the ratio of their lengths.

(7) The vertices P, Q and S of a ΔPQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) If M is the mid point of PQ, then find position vector of M in terms of $\vec{p}$ and $\vec{q}$

(ii) Find $\vec{t}$, the position vector of T on SM such that ST:TM = 2 : 1, in terms of $\vec{p}, \vec{q}$ and $\vec{s}$.

(iii) If the parallelogram PQRS is now completed. Express $\vec{r}$, the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

(8) D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle.

Show $\vec{FE} = \frac{1}{2}\vec{BC}$ and that the sum of the vectors $\vec{AD}$, $\vec{BE}$, $\vec{CF}$ is zero.

(9) The median AD of a ΔABC is bisected at E and BE is produced to meet the side AC in F. Show that AF = $\frac{1}{3}$ AC and EF = $\frac{1}{4}$ BF.

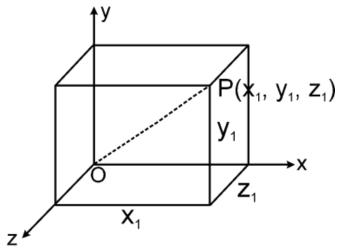
(10) Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\vec{AL}$ + $\vec{BM}$ + $\vec{CN}$ is a vector parallel to $\vec{CK}$, when K divides AB in the ratio 1 : 3.

Ans. (4) $\vec{BC} = \vec{OC} - \vec{OB}$, $\vec{CA} = \vec{OA} - \vec{OC}$, $\vec{AB} = \vec{OB} - \vec{OA}$

(5) m = 2

(6) 1 : 2 (7) $\vec{m} = \frac{1}{2}(\vec{p}+\vec{q})$, $\vec{t} = \frac{1}{2}(\vec{p}+\vec{q}+\vec{s})$, $\vec{r} = \frac{1}{2}(\vec{q}+\vec{p}-\vec{s})$

8. Coordinate of a point in space :



x-coordinate of point P = distance of P from y-z plane
y-coordinate of point P = distance of P from x-z plane
z-coordinate of point P = distance of P from x-y plane

9. Vector representation of a point in space :

If coordinate of a point P in space is (x, y, z), then the position vector of the point P with respect to the same origin is $x\hat{i} + y\hat{j} + z\hat{k}$.

10. Distance formula :

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Vector method

We know that if position vector of two points A and B are given as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then

$AB = |\overrightarrow{OB} - \overrightarrow{OA}|$

$\Rightarrow$ $AB = |(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})|$

$\Rightarrow$ $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

11. Distance of a point from coordinate axes :

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then

$PA = \sqrt{y^2 + z^2}$, $PB = \sqrt{z^2 + x^2}$, $PC = \sqrt{x^2 + y^2}$

Example # 6 : Show by using distance formula that the points (4, 5, –5), (0, –11, 3) and (2, –3,–1) are collinear.

Solution : Let A ≡ (4, 5, –5), B ≡ (0, –11, 3), C ≡ (2, –3, –1).

$AB = \sqrt{(4-0)^2 + (5+11)^2 + (-5-3)^2} = \sqrt{336} = \sqrt{4 \times 84} = 2\sqrt{84}$

$BC = \sqrt{(0-2)^2 + (-11+3)^2 + (3+1)^2} = \sqrt{84}$

$AC = \sqrt{(4-2)^2 + (5+3)^2 + (-5+1)^2} = \sqrt{84}$

BC + AC = AB

Hence points A, B, C are collinear and C lies between A and B.

Example # 7 : Find the locus of a point which moves such that the sum of its distances from points A(0, 0, – α) and B(0, 0, α) is constant.

Solution : Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$\therefore \quad \sqrt{(x-0)^2+(y-0)^2+(z+\alpha)^2} + \sqrt{(x-0)^2+(y-0)^2+(z-\alpha)^2} = 2a$

$\Rightarrow \quad \sqrt{x^2+y^2+(z+\alpha)^2} = 2a - \sqrt{x^2+y^2+(z-\alpha)^2}$

$\Rightarrow \quad x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2+y^2+(z-\alpha)^2}$

$\Rightarrow \quad 4z\alpha - 4a^2 = -4a\sqrt{x^2+y^2+(z-\alpha)^2} \quad \Rightarrow \quad \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$

or, $\quad x^2 + y^2 + z^2\left(1-\frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2 \quad \Rightarrow \quad \frac{x^2}{a^2-\alpha^2} + \frac{y^2}{a^2-\alpha^2} + \frac{z^2}{a^2} = 1$

This is the required locus.

Self Practice problems :

(11) One of the vertices of a cuboid is (1, 2, 3) and the edges from this vertex are along the +ve x-axis, +ve y-axis and +ve z-axis respectively and are of lengths 2, 3, 2 respectively find out the vertices.

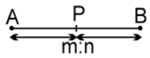
(12) Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

(13) Find the locus of point P if $AP^2 - BP^2 = 18$,where A ≡ (1, 2,– 3) and B ≡ (3, – 2, 1).

Ans. (11) (1, 2, 5), (3, 2, 5), (3, 2, 3), (1, 5, 5), (1, 5, 3), (3, 5, 3), (3, 5, 5).
(13) 2x – 4y + 4z – 9 = 0

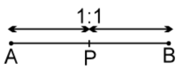
12. Section formula :

If point P divides the distance between the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio of m : n, internally then coordinates of P are given as $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$



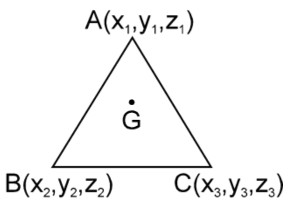
Note :- Mid point

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$



13. Co-ordinates of special points of a triangle :

(i) Centroid of a triangle :



$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

(ii) Incentre of triangle ABC :

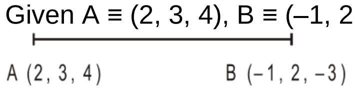
$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c},\frac{az_1+bz_2+cz_3}{a+b+c}\right)$$ Where AB = c, BC = a, CA = b

(iii) **Centroid of a tetrahedron** :

A (x_1, y_1, z_1) B (x_2, y_2, z_2) C (x_3, y_3, z_3) and D (x_4, y_4, z_4) are the vertices of a tetrahedron, then coordinate of its centroid (G) is given as $\left(\frac{\sum_{i=1}^{4}x_i}{4},\frac{\sum_{i=1}^{4}y_i}{4},\frac{\sum_{i=1}^{4}z_i}{4}\right)$

Example # 8 : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution : Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).



Let C divide AB internally in the ratio k : 1, then

C ≡ $\left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right)$ ∴ $\frac{-k+2}{k+1}$ = – 4 ⇒ 3k = – 6 ⇒ k = –2

For this value of k, $\frac{2k+3}{k+1}$ = 1, and $\frac{-3k+4}{k+1}$ = –10

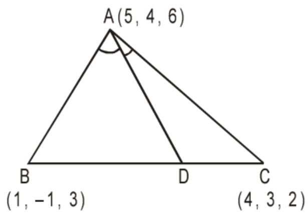
Since k < 0, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Example # 9 : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of ∠BAC meets BC in D. Find AD.

Solutio n : AB = $\sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

AC = $\sqrt{1^2+1^2+4^2} = 3\sqrt{2}$

Since AD is the internal bisector of BAC



∴ $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$

∴ D divides BC internally in the ratio 5 : 3

∴ D ≡ $\left(\frac{5\times4+3\times1}{5+3}, \frac{5\times3+3(-1)}{5+3}, \frac{5\times2+3\times3}{5+3}\right)$ or, D = $\left(\frac{23}{8},\frac{12}{8},\frac{19}{8}\right)$

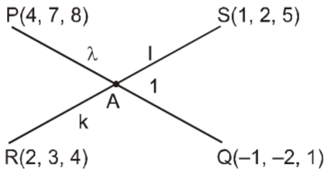
∴ AD = $\sqrt{\left(5-\frac{23}{8}\right)^2+\left(4-\frac{12}{8}\right)^2+\left(6-\frac{19}{8}\right)^2}$ = $\frac{\sqrt{1530}}{8}$ unit

Example # 10 : If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution : Let the lines PQ and RS intersect at point A.

Let A divide PQ in the ratio λ : 1, (λ ≠ –1) then A ≡ $\left(\frac{-\lambda+4}{\lambda+1}, \frac{-2\lambda+7}{\lambda+1}, \frac{\lambda+8}{\lambda+1}\right)$. .... (1)

Let A divide RS in the ratio k : 1, then A ≡ $\left(\frac{k+2}{k+1}, \frac{2k+3}{k+1}, \frac{5k+4}{k+1}\right)$ ..... (2)



From (1) and (2), we have,

$\frac{-\lambda+4}{\lambda+1} = \frac{k+2}{k+1}$ ⇒ – λk – λ + 4k + 4 = λk + 2λ + k + 2 ⇒ 2λk + 3λ – 3k – 2 = 0 .....(3)

$\frac{-2\lambda+7}{\lambda+1} = \frac{2k+3}{k+1}$ ⇒ –2λk – 2λ + 7k + 7 = 2λk + 3λ + 2k + 3 ⇒ 4λk + 5λ – 5k – 4 = 0 .....(4)

$\frac{\lambda+8}{\lambda+1} = \frac{5k+4}{k+1}$ ..... (5)

Multiplying equation (3) by 2, and subtracting from equation (4), we get – λ + k = 0 or, λ = k

Putting λ = k in equation (3), we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0$ ⇒ λ = 1 = k

Clearly λ = k = 1 satisfies eqn. (5), hence our assumption is correct.

∴ A ≡ $\left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right)$ or, A ≡ $\left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right)$.

Self Practice problems :

(14) Find the ratio in which xy plane divides the line joining the points A (1, 2, 3) and B (2, 3, 6).

(15) Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).

(16) Two vertices of a triangle are (4,–6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3}, -1, 2\right)$. Find the third vertex.

(17) If centroid of the tetrahedron OABC, where co-ordinates of A, B, C are (a, 2, 3), (1, b, 2) and (2, 1, c) respectively be (1, 2, 3), then find the distance of point (a, b, c) from the origin, where O is the origin.

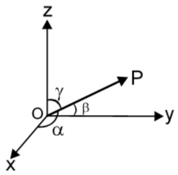
(18) Show that $\left(-\frac{1}{2}, 2, 0\right)$ is the circumcentre of the triangle whose vertices are A(1,1,0), B (1,2,1) and C (– 2, 2, –1) and hence find its orthocentre.

Ans. (14) 1 : 2 Externally (15) (3, 4, 5) (16) (2, 5, 2)
(17) $\sqrt{107}$ (18) (1, 1, 0)

14. Direction cosines and direction ratios :

(i) Direction cosines :

Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.



Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell_2 + m_2 + n_2 = 1$

(iii) **Direction ratios :**

Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\left(\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

or $$\left(\ell = \frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, n = \frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$$

(v) If OP = r, when O is the origin and the direction cosines of OP are ℓ, m, n then the coordinates of P are (ℓr, mr, nr).

If direction cosines of the line AB are ℓ, m,n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1) then the coordinates of B is given as $(x_1 + r\ell, y_1 + rm, z_1 + rn)$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

(vii) **Direction cosines of axes :**

Since the positive x–axis makes angles 0º, 90º, 90º with axes of x, y and z respectively. Therefore

Direction cosines of x–axis are (1, 0, 0)

Direction cosines of y–axis are (0, 1, 0)

Direction cosines of z–axis are (0, 0, 1)

Example # 11 : Find the direction cosines of a line which is connected by $\ell + m + n = 0$, $2\ell_2 + 2m_2 - n_2 = 0$

Solution : $\ell_2 + m_2 + n_2 = 1$

$\ell + m + n = 0$

$2\ell^2 + 2m^2 - n^2 = 0$

$2(1 - n^2) - n^2 = 0$

$\Rightarrow \quad 3n^2 = 2 \quad$ or $\quad n = \pm\sqrt{\frac{2}{3}}$ and $2(\ell^2 + m^2) = n^2 = (-(\ell + m))^2$

or $\quad \ell = m \quad \ell + m = \pm\sqrt{\frac{2}{3}} \quad$ or $\quad 2\ell = \pm\sqrt{\frac{2}{3}}$

$\ell = \pm\frac{1}{\sqrt{6}}$, $m = \pm\frac{1}{\sqrt{6}}$

direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$ and $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$

or $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$

Self practice problems:

(19) Find the direction cosines of a line lying in the xy plane and making angle 30° with x-axis.

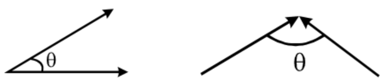
(20) A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis.

(21) A plane intersects the co-ordinates axes at point A(a, 0, 0), B(0, b, 0), C(0, 0, c) ; O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face ABC.

Ans. (19) $\ell = \frac{\sqrt{3}}{2}$, $m = \pm\frac{1}{2}$, $n = 0$ (20) 45° (21) $\frac{a}{3}$, $-b$, $\frac{c}{3}$,

15. <u>Angle between two vectors</u> :

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. Note that $0^\circ \le \theta \le 180^\circ$.



16. <u>Scalar product (Dot Product) of two vectors</u> :

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta, \ (0 \le \theta \le \pi)$$

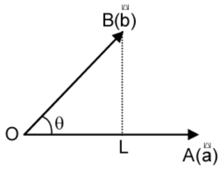
Note: (i) If θ is acute, then $\vec{a}.\vec{b} > 0$ and if θ is obtuse, then $\vec{a}.\vec{b} < 0$.

(ii) $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0, \ \vec{b} \ne 0)$

(iii) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}||\vec{b}|$

(iv) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}||\vec{b}|$

17. <u>Geometrical interpretation of scalar product</u> :



Let $\vec{a}$ and $\vec{b}$ be vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw BL ⊥ OA and AM ⊥ OB

From ΔOBL and ΔOAM, we have OL = OB cos θ and OM = OA cosθ

Here OL are known as projections of $\vec{b}$ on $\vec{a}$.

and $\vec{a}$ on $\vec{b}$ respectively.

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|(|\vec{b}|\cos\theta) = |\vec{a}|(OB\cos\theta) = |\vec{a}|(OL)$

= (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

(i) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

(ii) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ **(commutative)**

(iii) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ **(distributive)**

(iv) $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$, where m is a scalar.

(v) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$

(vii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(viii) $|\vec{a} \pm \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$, where θ is the angle between the vectors

(ix) Any vector $\vec{a}$ can be written as $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$.

Example # 12 : Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are
(i) perpendicular (ii) parallel

Solution : (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow (3\hat{i} + 2\hat{j} + 9\hat{k}).(\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow$ 3 + 2p + 27 = 0 $\Rightarrow$ p = – 15

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$\dfrac{3}{1} = \dfrac{2}{p} = \dfrac{9}{3} \Rightarrow 3 = \dfrac{2}{p} \Rightarrow p = \dfrac{2}{3}$

Example # 13 : If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 6$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution : We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{c}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \cos\theta = |\vec{c}|^2$

$\Rightarrow \quad 9 + 25 + 2 (3) (5) \cos\theta = 36 \quad \Rightarrow \quad \cos\theta = \frac{2}{30} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{1}{15}$

Example # 14 : Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.

Solution : The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

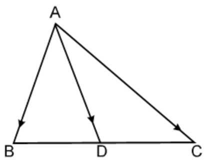
Now, θ is obtuse $\Rightarrow \cos\theta < 0 \Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} < 0 \Rightarrow \vec{a} \cdot \vec{b} < 0 \quad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow 14x^2 - 8x + x < 0 \Rightarrow 7x(2x - 1) < 0 \Rightarrow x(2x - 1) < 0 \Rightarrow 0 < x < \frac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Example # 15 : D is the mid point of the side BC of a ΔABC, show that $AB^2 + AC^2 = 2(AD^2 + BD^2)$

Solution : We have $\vec{AB} = \vec{AD} + \vec{DB} \Rightarrow AB^2 = (\vec{AD} + \vec{DB})^2 \Rightarrow AB^2 = AD^2 + DB^2 + 2\vec{AD} \cdot \vec{DB}$ .....(i)



Also we have $\vec{AC} = \vec{AD} + \vec{DC} \Rightarrow AC^2 = (\vec{AD} + \vec{DC})^2 = AD^2 + DC^2 + 2\vec{AD} \cdot \vec{DC}$ .....(ii)

Adding (i) and (ii), we get $AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\vec{AD} \cdot (\vec{DB} + \vec{DC})$

$\Rightarrow AB^2 + AC^2 = 2(AD^2 + BD^2) \quad \because \vec{DB} + \vec{DC} = \vec{0}$

Example # 16 : If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $= 2\vec{a} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but ⊥ to $\vec{a}$.

Solution : (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}$; Here $\vec{a} \cdot \vec{b} = 2 - 1 + 3 = 4$ and $|\vec{a}|^2 = 3$

Hence $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{4}{3}\vec{a} = \frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but ⊥ to $\vec{a}$ is $\vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{1}{3}(2\hat{i} - 7\hat{j} + 5\hat{k})$

Self Practice Problems :

(22) If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\frac{\theta}{2} = \frac{|\vec{a} - \vec{b}|}{|\vec{a} + \vec{b}|}$.

(23) Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

(24) Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

(25) The points O, A, B, C, D are such that $\overrightarrow{OA} = \vec{a}$ $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = 2\vec{a} + 3\vec{b}$, $\overrightarrow{OD} = \vec{a} + 2\vec{b}$ Given that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$. Show that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular.

Ans. (23) $x = -\frac{31}{12}$, $y = \frac{41}{12}$ (24) (– 3, –2) ∪ (2, 3)

18. Projection of a line segment on a line :

(i) If the coordinates of P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $\left| \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \right|$

(ii) **Vector form :**

projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a} \cdot \hat{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ In the above case we can consider $\overrightarrow{PQ}$ as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ in place of $\vec{a}$ and $\ell\hat{i} + m\hat{j} + n\hat{k}$ in place of $\vec{b}$.

(iii) $\ell|\vec{r}|$, $m|\vec{r}|$ & $n|\vec{r}|$ are the projection of $\vec{r}$ on OX, OY & OZ axes.

(iv) $\vec{r} = |\vec{r}|(\ell\hat{i} + m\hat{j} + n\hat{k})$

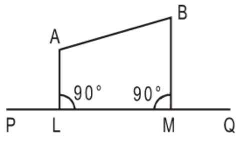
Example # 17 : Find the projection of the line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios 2, 3, – 6.

Solution : Let A ≡ (1, 2, 3), B ≡ (–1, 4, 2)

Direction ratios of the given line PQ are 2, 3, – 6

$\sqrt{2^2 + 3^2 + (-6)^2} = 7$

∴ direction cosines of PQ are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$



Projection of AB on PQ

$= |\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

$= \left|\frac{2}{7}(-1-1)+\frac{3}{7}(4-2)-\frac{6}{7}\ (2-3)\right| = \left|\frac{-4+6+6}{7}\right| = \frac{8}{7}$

Self Practice problems :

(26) A (6, 3, 2), B (5, 1, 1,), C(3, –1, 3) D (0, 2, 5). Find the projection of line segment AB on CD line.

(27) The projections of a directed line segment on co-ordinate axes are – 2, 3, – 6. Find its length and direction cosines.

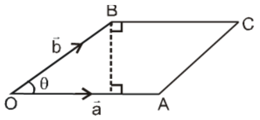
(28) Find the projection of the line segment joining (2, – 1, 3) and (4, 2, 5) on a line which makes equal acute angles with co-ordinate axes.

Ans. (26) $\frac{5}{\sqrt{22}}$ (27) 7, $\frac{-2}{7},\frac{3}{7},\frac{-6}{7}$ (28) $\frac{7}{\sqrt{3}}$

19. Vector product (Cross Product) of two vectors :

(i) If $\vec{a}$, $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iv) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})\ m(\vec{a} \times \vec{b})$, where m is a scalar.

(v) $\vec{a} \times (\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)

(vi) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0\ ,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(ix) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\frac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(x) If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

(xi) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of ΔABC = $\frac{1}{2}(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$. The points A, B and C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

(xii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

(xiii) **Lagrange's Identity :**

For any two vectors $\vec{a}$ and $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

Example # 18 : Find a vector of magnitude 9, which is perpendicular to both the vectors $\hat{i} - 7\hat{j} + 7\hat{k}$ and $3\hat{i} - 2\hat{j} + 2\hat{k}$.

Solution : Let $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$. Then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = (-14 + 14) - (2 - 21) + (-2 + 21)\ \hat{k} = 19\hat{j} + 19\hat{k}$$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = 19\sqrt{2}$

$\therefore$ Required vector $= \pm 9\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right) = \pm\frac{9}{\sqrt{2}}(\hat{j} + \hat{k})$

Example # 19 : For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$.

Solution : We have, $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$ [Using distributive law]

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{b}\times\vec{c} = \vec{0}$ [$\because \vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$ etc]

Example # 20 : For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$

Solution : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then

$\vec{a}\times\hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i} = a_1(\hat{i}\times\hat{i}) + a_2(\hat{j}\times\hat{i}) + a_3(\hat{k}\times\hat{i}) = -a_2\hat{k} + a_3\hat{j}$

$\Rightarrow \quad |\vec{a}\times\hat{i}|^2 = a_{22} + a_{32}$

$\vec{a}\times\hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{j} = a_1\hat{k} - a_3\hat{i}$

$\Rightarrow \quad |\vec{a}\times\hat{j}|^2 = a_{21} + a_{32}$

$\vec{a}\times\hat{k} = (a_i\ \hat{i} + a_2\hat{j} + a_3\hat{k}) \times \vec{k} = -\ a_i\hat{j} + a_2\hat{i}$

$\Rightarrow \quad |\vec{a}\times\hat{k}|^2 = a_{12} + a_{22}$

$\therefore \quad |\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = a_{22} + a_{33} + a_{12} + a_{32} + a_{12} + a_{22}$

$= 2(a_{12} + a_{22} + a_{32}) = 2|\vec{a}|^2$

Self Practice Problems :

(29) If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º. Find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

(30) Prove that the normal to the plane containing the three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}$

Ans. (29) 3/4 sq. units

20. A line :

(i) Equation of a line

(a) A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non−symmetrical form.

(b) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

(c) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(d) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(e) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(f) Reduction of cartesion form of equation of a line to vector form & vice versa
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

Note: Straight lines parallel to co-ordinate axes:

	Straight lines	**Equation**
(i)	Through origin	y = mx and z = nx
(ii)	x−axis	y = 0 and z = 0
(iii)	y−axis	x = 0 and z = 0
(iv)	z−axis	x = 0 and y = 0
(v)	Parallel to x−axis	y = p, z = q
(vi)	Parallel to y−axis	x = h, z = q
(vii)	Parallel to z−axis	x = h, y = p

(ii) **Equation of angle bisector of two lines :**

The equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are : $\vec{r} = \vec{a} + t\left(\hat{b}+\hat{c}\right)$ and $\vec{r} = \vec{a} + p\left(\hat{c}-\hat{b}\right)$.

Note : A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is . $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is, $\lambda\left(\hat{a}+\hat{b}\right)$ where $\lambda \in R^+$. Bisector of the exterior angle between and is , $\lambda \in R^+$.

Bisector of the exterior angle between $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a}-\hat{b}\right)$, $\lambda \in R^+$

Example # 21 : Find the equation of the line through the points (1, 2, 4) and (2, 4, 6) in vector form as well as in cartesian form.

Solution : Let A ≡ (1, 2, 4), B ≡ (2, 4, 6)

Now $\vec{a} = \vec{OA} = \hat{i} + 2\hat{j} + 4\hat{k} \Rightarrow \vec{b} = \vec{OB} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = \hat{i} + 2\hat{j} + 4\hat{k} + t(\hat{i} + 2\hat{j} + 2\hat{k})$ ..... (1)

Equation in cartesian form :

Equation of AB is $\frac{x-1}{2-1} = \frac{y-2}{4-2} = \frac{z-4}{6-4}$ or, $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-4}{2}$

Example # 22 : Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 10 units from point (1, – 2, 3).

Solution : Given line is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ ..... (1)

Let P ≡ (1, –2, 3)

Direction ratios of line (1) are 2, 3, 6

∴ Direction cosines of line (1) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Equation of line (1) may be written as $\frac{x-1}{\frac{2}{7}} = \frac{y+2}{\frac{3}{7}} = \frac{z-3}{\frac{6}{7}}$ ..... (2)

Co-ordinates of any point on line (2) may be taken as $\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Let Q ≡ $\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Distance of Q from P = | r |

According to question | r | = 10 ∴ r = ±10

Putting the value of r, we have

Q ≡ $\left(\frac{27}{7}, \frac{16}{7}, \frac{81}{7}\right)$ or Q ≡ $\left(-\frac{13}{7}, -\frac{44}{7}, -\frac{39}{7}\right)$

Example # 23: Find the equation of the line drawn through point (3, 0, 1) to meet at right angles the line $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$

Solution : Given line is $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$ ..... (1)

Let P ≡ (3, 0, 1)

Co-ordinates of any point on line (1) may be taken as

Q ≡ (3r – 1, – 2r + 2, – r – 1)

Direction ratios of PQ are 3r – 4, – 2r + 2, – r – 2

Direction ratios of line AB are 3, – 2, – 1

Since PQ AB

∴ 3 (3r – 4) – 2 (– 2r + 2) – 1 (– r – 2) = 0

⇒ 9r – 12 + 4r – 4 + r + 2 = 0 ⇒ 14r = 14 ⇒ r = 1

Therefore, direction ratios of PQ are 1, 0, 3 or, –1, 0, –3

Equation of line PQ is $\frac{x-3}{1} = \frac{y-0}{0} = \frac{z-1}{3}$ or, $\frac{x-3}{-1} = \frac{y-0}{0} = \frac{z-1}{-3}$

Example # 24 : Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2}$ = z intersect. Find also the point of intersection of these lines.

Solution : Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ..... (1)

and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ ..... (2)

Any point on line (1) is P (2r + 1, 3r + 2, 4r +3)

and any point on line (2) is Q (5λ + 4, 2λ + 1, λ)

Lines (1) and (2) will intersect if P and Q coincide for some value of λ and r.

∴ 2r + 1 = 5λ + 4 ⇒ 2r – 5λ = 3 ..... (3)

3r + 2 = 2λ + 1 ⇒ 3r – 2λ = – 1 ..... (4)

4r + 3 = λ ⇒ 4r – λ = – 3 ..... (5)

Solving (3) and (4), we get r = – 1, λ = – 1

Clearly these values of r and λ satisfy eqn. (5)

Now P ≡ (– 1, – 1, – 1)

Hence lines (1) and (2) intersect at (– 1, – 1, – 1).

Self Practice problems:

(31) Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, – 1, 5. Prove that this line passes through (4, – 1, 7).

(32) Find the equation of the line parallel to line $\frac{x-2}{3} = \frac{y+1}{1} = \frac{z-7}{9}$ and passing through the point (3, 0, 5).

(33) Find the coordinates of the point when the line through (3, 4, 1) and (5, 1, 6) crosses the xy plane.

Ans. (31) $\frac{x-1}{3} = \frac{y}{-1} = \frac{z-2}{5}$ (32) $\frac{x-3}{3} = \frac{y}{1} = \frac{z-5}{9}$ (33) $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

21. <u>Foot, length and equation of perpendicular from a point to a line</u> :

(i) Cartesian form :

Let equation of the line be $\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n}$ = r (say) ….......(i)

and A (α, β, γ) be the point. Any point on line (i) is P (ℓr + a, mr + b, nr + c) .... (ii)

If it is the foot of the perpendicular from A on the line, then AP is perpendicular to the line. So $\ell(\ell r + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$ i.e. $r = (\alpha - a)\ell + (\beta - b)m + (\gamma - c)n$ since $\ell^2 + m^2 + n^2 = 1$. Putting this value of r in (ii), we get the foot of perpendicular from point A on the given line. Since foot of perpendicular P is known, then the length of perpendicular is given by AP $\sqrt{(\ell r + a - \alpha)^2 + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$ = the equation of perpendicular is given by $\frac{x - \alpha}{\ell r + a - \alpha} = \frac{y - \beta}{mr + b - \beta} = \frac{z - \gamma}{nr + c - \gamma}$

(ii) Vector Form :

Equation of a line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is parallel to $\vec{b}_1 \times \vec{b}_2$. So the vector equation of such a line is $\vec{r} = \vec{\alpha} + \lambda(\vec{b}_1 \times \vec{b}_2)$. Position vector $\vec{\beta}$ of the image of a point $\vec{\alpha}$ in a straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ is given by $\vec{\beta} = 2\vec{a} - \left[\frac{2(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right]\vec{b} - \vec{\alpha}$. Position vector of the foot of the perpendicular on line is $\vec{f} = \vec{a} - \left[\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right]\vec{b}$. The equation of the perpendicular is $\vec{r} = \vec{\alpha} + \mu\left[(\vec{a} - \vec{\alpha}) - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}\right]$.

22. <u>To find image of a point with respect to a line</u> :

Let L ≡ $\frac{x - x_2}{a} = \frac{y - y_2}{b} = \frac{z - z_2}{c}$ is a given line

Let (x', y', z') is the image of the point P (x_1, y_1, z_1) with respect to the line L. Then

(i) $a(x_1 - x') + b(y_1 - y') + c(z_1 - z') = 0$

(ii) $\frac{\frac{x_1 + x'}{2} - x_2}{a} = \frac{\frac{y_1 + y'}{2} - y_2}{b} = \frac{\frac{z_1 + z'}{2} - z_2}{c} = \lambda$

from (ii) get the value of x', y', z' in terms of λ as $x' = 2a\lambda + 2x_2 - x_1$,
$y' = 2b\lambda + 2y_2 - y_1$, $z' = 2c\lambda + 2z_2 - z_1$
now put the values of x', y', z' in (i) get λ and resubtitute the value of λ to to get (x' y' z').

Example # 25 : Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Solution : Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ ..... (1)

P ≡ (2, – 3, 1)

Co-ordinates of any point on line (1) may be taken as Q ≡ (2r – 1, 3r + 3, – r – 2)

Direction ratios of PQ are 2r – 3, 3r + 6, – r – 3

Direction ratios of AB are 2, 3, – 1

Since PQ AB

∴ 2 (2r – 3) + 3 (3r + 6) – 1 (– r – 3) = 0

or, 14r + 15 = 0 ∴ $r = \frac{-15}{14}$

$\therefore$ $Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$ $\therefore$ $PQ = \sqrt{\frac{531}{14}}$ units.

Second method :

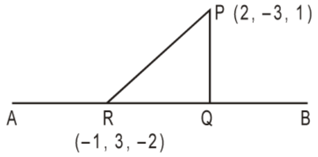
Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

$P \equiv (2, -3, 1)$

Direction ratios of line (1) are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$

RQ = length of projection of RP on AB

$= \left|\frac{2}{\sqrt{14}}(2+1) + \frac{3}{\sqrt{14}}(-3-3) - \frac{1}{\sqrt{14}}(1+2)\right| = \frac{15}{\sqrt{14}}$



$PR^2 = 3^2 + 6^2 + 3^2 = 54$

$\therefore$ $PQ = \sqrt{PR^2 - RQ^2} = \sqrt{54 - \frac{225}{14}} = \sqrt{\frac{531}{14}}$ units.

Self Practice problems :

(34) Find the length and foot of perpendicular drawn from point (2, –1, 5) to the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. Also find the image of the point in the line.

(35) Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

(36) Find the foot and hence the length of perpendicular from (5, 7, 3) to the line $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. Find also the equation of the perpendicular.

Ans. (34) $\sqrt{14}$, N ≡ (1, 2, 3), I ≡ (0, 5, 1) (35) (1, 0 , 7)

(36) (9, 13, 15) ; 14 ; $\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

23. Angle between two line :

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Two parallel lines have same direction cosines i.e. $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$

Example # 26 : What is the angle between the lines whose direction cosines are

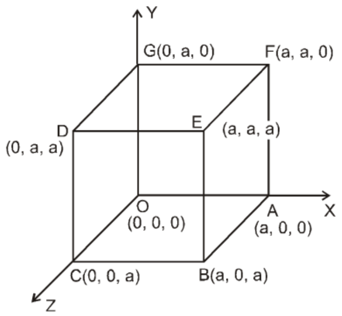
$$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$$

Solution : Let θ be the required angle, then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2} \Rightarrow \theta = 120°,$$

Example # 27: Find the angle between any two diagonals of a cube.

Solution : The cube has four diagonals
OE, AD, CF and GB
The direction ratios of OE are
a, a, a or 1, 1, 1



∴its direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Direction ratios of AD are – a, a, a or – 1, 1, 1.

∴its direction cosines are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Similarly, direction cosines of CF and GB respectively are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

We take any two diagonals, say OE and AD

Let θ be the acute angle between them, then

$$\cos\theta = \left| \left(\frac{1}{\sqrt{3}}\right) \left(\frac{-1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right).\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right).\left(\frac{1}{\sqrt{3}}\right) \right| = \frac{1}{3}$$

or, $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Self Practice problems:

(37) Find the angle between the lines whose direction cosines are given by
$\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$

(38) Let P (6, 3, 2), Q (5, 1, 4), R (3, 3, 5) are vertices of a Δ find ∠Q.

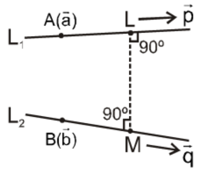
(39) Show that the direction cosines of a line which is perpendicular to the lines having directions cosines $\ell_1\ m_1\ n_1$ and $\ell_2\ m_2\ n_2$ respectively are proportional to

$m_1n_2 - m_2n_1$, $n_1\ell_2 - n_2\ell_1$, $\ell_1m_2 - \ell_2m_1$

Ans. (37) 60° (38) 90°

24. Skew lines and shortest distance between two lines :

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect and are also not parallel are called **skew line**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines.



Let $\overrightarrow{LM}$ be the shortest distance vector between the lines L_1 and L_2 . Then $\overrightarrow{LM}$ is perpendicular to both $\vec{p}$ and $\vec{q}$ i.e. $\overrightarrow{LM}$ is parallel to $\vec{p} \times \vec{q}$. Therefore the magnitude of the shortest distance vector (i.e. $|\overrightarrow{LM}|$) would be equal to that of $\overrightarrow{AB}$ the projection of along the direction of the line of shortest distance.

∴ $|\overrightarrow{LM}| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}\right| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p} \times \vec{q}\right|$

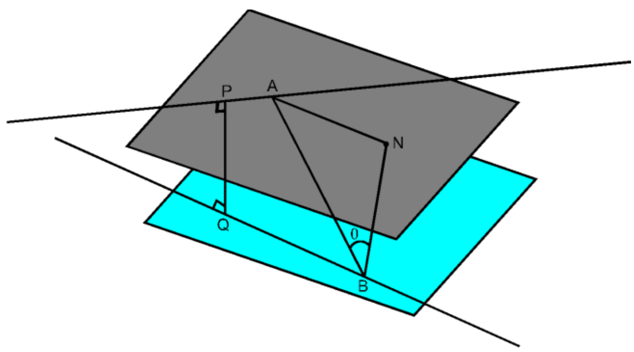
$= \left|\dfrac{\overrightarrow{AB}\cdot(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right| = \left|\dfrac{(\vec{b}-\vec{a})\cdot(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

(i) The two lines directed along $\vec{p}$ and $\vec{q}$ will intersect only if shortest distance = 0 i.e. $(\vec{b}-\vec{a})\cdot(\vec{p} \times \vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ and $\vec{q}$. ⇒ d $(\vec{b}-\vec{a})\cdot\vec{p} \times \vec{q} = 0$

(ii) If two parallel lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, then distance (d) between them is given by $d = \left|\dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

(iii) The straight lines which are not parallel and non−coplanar i.e. non−intersecting are calledskew lines. If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ & $\dfrac{x-\alpha'}{\ell'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$ are skew lines.

(iv) Shortest distance: Suppose the equation of the lines are $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and $\dfrac{x-\alpha'}{\ell'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$ are



S.D. = $\left|\dfrac{(\alpha-\alpha')\ \ (mn'-m'n)\ \ +(\beta-\beta')\ \ (n\ell'-n'\ell)+(\gamma-\gamma')\ \ (\ell m'-\ell' m)}{\sqrt{\sum(mn'-m'n)^2}}\right|$

$= \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum(mn' - m'n)^2}$

(v) Vector Form: For lines $\vec r = \vec a_1 + \lambda_1 \vec b_1$ & $\vec r = \vec a_2 + \lambda_2 \vec b_2$ to be skew ($\vec b_1 \times \vec b_2$). ($\vec a_2 - \vec a_1$) 0 or $[\vec b_1\ \vec b_2\ (\vec a_2 - \vec a_1)] \neq 0$.

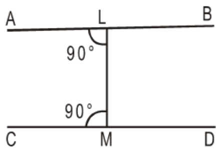
(vi) Shortest distance between the two parallel lines $\vec r = \vec a_1 + \lambda \vec b$ & $\vec r = \vec a_2 + \mu \vec b$ is d = $\left|\dfrac{(\vec a_2 - \vec a_1)\ \times\ \vec b}{|\vec b|}\right|$.

Example # 28 : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec r = 3\hat i + 8\hat j + 3\hat k + \lambda\ (3\hat i - \hat j + \hat k)$ and $\vec r = -3\hat i - 7\hat j + 6\hat k + \mu\ (-3\hat i + 2\hat j + 4\hat k)$

Solution : Given lines are $\vec r = 3\hat i + 8\hat j + 3\hat k + \lambda\ (3\hat i - \hat j + \hat k)$ ..... (1)

and $\vec r = -3\hat i - 7\hat j + 6\hat k + \mu\ (-3\hat i + 2\hat j + 4\hat k)$ ..... (2)

Equation of lines (1) and (2) in cartesian form is



AB : $\dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1} = \lambda$

and CD : $\dfrac{x+3}{-3} = \dfrac{y+7}{2} = \dfrac{z-6}{4} = \mu$

Let L ≡ (3λ + 3, – λ + 8, λ + 3)

and M ≡ (– 3μ – 3, 2μ – 7, 4μ + 6)

Direction ratios of LM are

3λ + 3μ + 6, – λ – 2μ + 15, λ – 4μ – 3.

Since LM ⊥ AB

∴ 3 (3λ + 3μ + 6) – 1 (– λ – 2μ + 15) + 1 (λ – 4μ – 3) = 0

or, 11λ + 7μ = 0 ..... (5)

Again LM ⊥ CD

∴ – 3 (3λ + 3μ + 6) + 2 (– λ – 2μ + 15) + 4 (λ – 4μ – 3) = 0

or, – 7λ – 29μ = 0 ..... (6)

Solving (5) and (6), we get $\lambda = 0, \mu = 0$

$\therefore$ L ≡ (3, 8, 3), M ≡ (– 3, – 7, 6)

Hence shortest distance LM = $\sqrt{(3+3)^2+(8+7)^2+(3-6)^2}$ = $\sqrt{270}$ = 3 $\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\hat{i}+8\hat{j}+3\hat{k}+t\ (6\hat{i}+15\hat{j}-3\hat{k})$

Note : Cartesian equation of LM is $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$.

Example # 29 : Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes y+z=0, x+z=0, x+y=0, x+y+z= $\sqrt{3}$ a is $\sqrt{2}$ a.

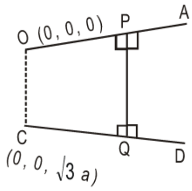
Solution : Given planes are $y + z = 0$ ..... (i)

$x + z = 0$ ..... (ii)

$x + y = 0$ ..... (iii)

$x + y + z = \sqrt{3}\ a$ ..... (iv)

Clearly planes (i), (ii) and (iii) meet at O(0, 0, 0)
Let the tetrahedron be OABC



Let the equation to one of the pair of opposite edges OA and BC be

$y + z = 0, x + z = 0$ ..... (1)

$x + y = 0, x + y + z = \sqrt{3}\ a$ ..... (2)

equation (1) and (2) can be expressed in symmetrical form as

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{-1} \quad ..... (3)$$

and, $$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-\sqrt{3}\ a}{0} \quad ..... (4)$$

d. r. of OA and BC are respectively (1, 1, – 1) and (1, – 1, 0).

Let PQ be the shortest distance between OA and BC having direction cosines (ℓ, m, n)

$\therefore$ PQ is perpendicular to both OA and BC.

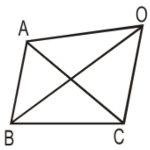
$\therefore$ $\ell + m - n = 0$

and $\ell - m = 0$

Solving (5) and (6), we get, $\frac{\ell}{1} = \frac{m}{1} = \frac{n}{2}$ = k (say)

also, $\ell^2 + m^2 + n^2 = 1$

$\therefore$ $k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm \frac{1}{\sqrt{6}}$

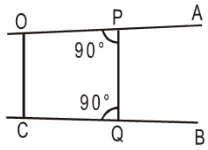


∴ $\ell = \pm\frac{1}{\sqrt{6}}$, $m = \pm\frac{1}{\sqrt{6}}$, $n = \pm\frac{2}{\sqrt{6}}$

Shortest distance between OA and BC

i.e. PQ = The length of projection of OC on PQ

$= |\,(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)\,n\,|$



$= \left|0\ .\ \frac{1}{\sqrt{6}} + 0\ .\ \frac{1}{\sqrt{6}} + \sqrt{3}\ a\ .\ \frac{2}{\sqrt{6}}\right| = \sqrt{2}$ a.

Self practice problems:

(40) Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Find also its equation.

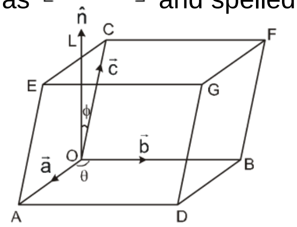
(41) Prove that the shortest distance between the diagonals of a rectangular parallelopiped whose coterminous sides are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

Ans. (40) $\frac{1}{\sqrt{6}}$, 6x – y = 10 – 3y = 6z – 25

25. <u>Scalar triple product (Box Product) (S.T.P.)</u> :

(i) The scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$, $\sin\theta\ .\ \cos\phi$ where θ is the angle between $\vec{a}, \vec{b}$ **(i.e.** $(\vec{a} \wedge \vec{b}) = \theta$ **)** and φ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$; $(\vec{a} \times \vec{b}) \wedge \vec{c} = \varphi$**)**. It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$



(iii) In a scalar triple product the position of dot and cross can be interchanged i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c} \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

(iv) $\vec{a}.(\vec{b} \times \vec{c}) = -\vec{a}.(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{\ell}\,\vec{m}\,\vec{n}]$, where $\vec{\ell}$, $\vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(vi) If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

(vii) Scalar product of three vectors, two of which are equal or parallel is 0 $\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

(viii) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a}\,\vec{b}\,\vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(ix) (i) $[\hat{i}\ \ \hat{j}\ \ \hat{k}] = 1$ (ii) $[K\vec{a}\,\vec{b}\vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ (iii) $[(\vec{a}+\vec{b})\ \vec{c}\ \ \vec{d}] = [\vec{a}\vec{c}\vec{d}] + [\vec{b}\vec{c}\ \vec{d}]$

(x) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ and $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\vec{b}\ \vec{c}]$

(xi) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

26. Tetrahedron and their properties :

(i) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

(ii) If the position vectors of the vertices of tetrahedron are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$.

Note : that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Example # 30 : Find the volume of a parallelopiped whose sides are given by $-6\hat{i}+14\hat{j}+10\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$

Solution : Let $\vec{a} = -6\hat{i}+14\hat{j}+10\hat{k}$, $\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is $\left|[\vec{a}\ \ \vec{b}\ \ \vec{c}]\right|$.

Now $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} -6 & 14 & 10 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$ = –6(–21 – 15) –14(15 + 21) + 10(25 – 49) = –528

So required volume of the parallelopiped = $\left|[\vec{a}\ \ \vec{b}\ \ \vec{c}]\right|$ = | –528 | = 528 cubic units.

Example # 31 : Simplity $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

Solution : $\left[\vec a+\vec b \quad \vec b+\vec c \quad \vec c+\vec a\right] = (\vec a+\vec b)\,.\,\left[(\vec b+\vec c)\times(\vec c+\vec a)\right] = (\vec a+\vec b)\,.\,\left[\vec b\times\vec c+\vec b\times\vec a+\vec c\times\vec a\right]$.

$= [\vec a \ \vec b \ \vec c] + [\vec a \ \vec b \ \vec c] = 2[\vec a \ \vec b \ \vec c]$

Example # 32 : Find the volume of the tetrahedron whose four vertices have position vectors $\vec a$, $\vec b$, $\vec c$ and $\vec d$.

Solution : Let four vertices be A, B, C, D with position vectors $\vec a$, $\vec b$, $\vec c$ and $\vec d$ respectively.

$\therefore \quad \overrightarrow{DA} = (\vec a - \vec d)$

$\overrightarrow{DB} = (\vec b - \vec d)$

$\overrightarrow{DC} = (\vec c - \vec d)$

Hence volume $V = \frac{1}{6}\,[\vec a - \vec d \quad \vec b - \vec d \quad \vec c - \vec d]$

$= \frac{1}{6}\,(\vec a - \vec d)\,.\,[(\vec b - \vec d)\times(\vec c - \vec d)] = \frac{1}{6}\,(\vec a - \vec d)\,.\,[\vec b\times\vec c - \vec b\times\vec d + \vec c\times\vec d]$

$= \frac{1}{6}\,\{[\vec a\ \vec b\ \vec c] - [\vec a\ \vec b\ \vec d] + [\vec a\ \vec c\ \vec d] - [\vec d\ \vec b\ \vec c]\} = \frac{1}{6}\,\{[\vec a\ \vec b\ \vec c] - [\vec a\ \vec b\ \vec d] + [\vec a\ \vec c\ \vec d] - [\vec b\ \vec c\ \vec d]\}$