GRAVITATION

1. INTRODUCTION

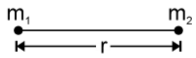
The motion of celestial bodies such as the sun, the moon, the earth and the planets etc. has been a subject of fascination since time immemorial. Indian astronomers of the ancient times have done brilliant work in this field, the most notable among them being Arya Bhatt the first person to assert that all planets including the earth revolve round the sun.

A millennium later the Danish astronomer Tycobrahe (1546-1601) conducted a detailed study of planetary motion which was interpreted by his pupil Johnaase Kepler (1571-1630), ironically after the master himself had passed away. Kepler formulated his important findings in three laws of planetary motion

2. UNIVERSAL LAW OF GRAVITATION : NEWTON'S LAW

According to this law "Each particle attracts every other particle. The force of attraction between them is directly proportional to the product of their masses and inversely proportional to square of the distance between them".

$F \propto \frac{m_1 \; m_2}{r^2}$ or $F = G\frac{m_1 \; m_2}{r^2}$



where $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} is the universal gravitational constant. This law holds good irrespective of the nature of two objects (size, shape, mass etc.) at all places and all times. That is why it is known as universal law of gravitation.

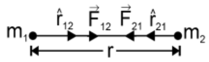
Dimensional formula of G :

$$G = \frac{Fr^2}{m_1 \; m_2} = \frac{[MLT^{-2}] \; [L^2]}{[M^2]} = [M^{-1} L^3 T^{-2}]$$

Newton's Law of gravitation in vector form :

$$\vec{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{r}_{12} \quad \& \quad \vec{F}_{21} = \frac{Gm_1m_2}{r^2}$$

Where $\vec{F}_{12}$ is the force on mass m_1 exerted by mass m_2 and vice-versa.



$$\vec{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{r}_{12} \quad \& \quad \vec{F}_{21} = \frac{Gm_1m_2}{r^2}\hat{r}_{21}$$

Now $\hat{r}_{12} = -\hat{r}_{21}$ Thus $\vec{F}_{21} = \frac{-\,G \;\; m_1 \;\; m_2}{r^2}\hat{r}_{12}$. Comparing above, we get $\vec{F}_{12} = -\vec{F}_{21}$

Important characteristics of gravitational force

(i) Gravitational force between two bodies form an action and reaction pair i.e. the forces are equal in magnitude but opposite in direction.

(ii) Gravitational force is a central force i.e. it acts along the line joining the centres of the two interacting bodies.

(iii) Gravitational force between two bodies is independent of the nature of the medium, in which they lie.

(iv) Gravitational force between two bodies does not depend upon the presence of other bodies.

(v) Gravitational force is negligible in case of light bodies but becomes appreciable in case of massive bodies like stars and planets.

(vi) Gravitational force is long range-force i.e., gravitational force between two bodies is effective even if their separation is very large. For example, gravitational force between the sun and the earth is of the order of 10^{27} N although distance between them is 1.5×10^7 km

Example 1. The centres of two identical spheres are at a distance 1.0 m apart. If the gravitational force between them is 1.0 N, then find the mass of each sphere. ($G = 6.67 \times 10^{-11}$ m^3 kg^{-1} sec^{-1})

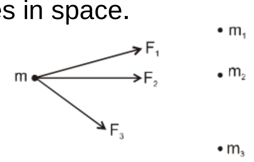
Solution Gravitational force $F = \frac{Gm.m}{r^2}$

on substituting F = 1.0 N , r = 1.0 m and $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-1}$

we get $m = 1.225 \times 10^5$ kg

Principle of superposition

The force exerted by a particle or other particle remains uneffected by the presence of other nearby particles in space.

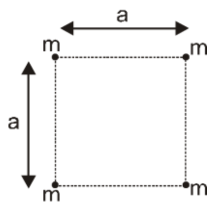


Total force acting on a particle is the vector sum of all the forces acted upon by the individual masses when they are taken alone.

$\vec{F}_1 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$

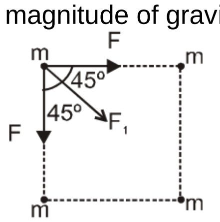
Solved Examples

Example 2.



Four point masses each of mass 'm' are placed on the corner of square of side 'a' . Calculate magnitude of gravitational force experienced by each particle.

Solution :



F_r = resultant force on each particle = 2F cos 45º + F_1

$$= \frac{2G.m^2}{a^2} \cdot \frac{1}{\sqrt{2}} + \frac{Gm^2}{(\sqrt{2}a)^2} = \frac{G.m^2}{2\ a^2}(2\sqrt{2}+1)$$

3. GRAVITATIONAL FIELD

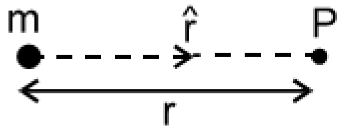
The space surrounding the body within which its gravitational force of attraction is experienced by other bodies is called gravitational field. Gravitational field is very similar to electric field in electrostatics where charge 'q' is replaced by mass 'm' and electric constant 'K' is replaced by gravitational constant 'G'. The intensity of gravitational field at a points is defined as the force experienced by a unit mass placed at that point.

$$\vec{E} = \frac{\vec{F}}{m}$$

The unit of the intensity of gravitational field is N kg^{-1}.

Intensity of gravitational field due to point mass :

The force due to mass m on test mass m_0 placed at point P is given by :



$$F = \frac{GMm_0}{r^2}$$

Hence $E = \frac{F}{m_0}$ $\Rightarrow$ $E = \frac{Gm}{r^2}$

In vector form $\vec{E} = -\frac{GM}{r^2}\hat{r}$

Dimensional formula of intensity of gravitational field = $\frac{F}{m} = \frac{[\ MLT^{-2}]}{[\ M\ ]} = [\ M^0\ \ LT^{-2}\ ]$

Solved Examples

Example 3. Find the relation between the gravitational field on the surface of two planets A & B of masses m_A, m_B & radius R_A & R_B respectively if

(i) they have equal mass

(ii) they have equal (uniform) density

Solution : Let E_A & E_B be the gravitational field intensities on the surface of planets A & B.

then, $E_A = \frac{Gm_A}{R_A^2} = \frac{G\frac{4}{3}\pi R_A^3 \rho_A}{R_A^2} = \frac{4G\pi}{3}\rho_A R_A$

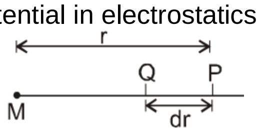
Similarly, $E_B = \frac{Gm_B}{R_{B^2}} = \frac{4G}{3}\pi\ \rho_B R_B$

(i) for $m_A = m_B$ $\frac{E_A}{E_B} = \frac{R_B^2}{R_A^2}$

(ii) For & $\rho_A = \rho_B$ $\frac{E_A}{E_B} = \frac{R_A}{R_B}$

4. GRAVITATIONAL POTENTIAL

The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy). Gravitational potential is very similar to electric potential in electrostatics.



Gravitational potential due to a point mass :

Let the unit mass be displaced through a distance dr towards mass M, then work done is given by

$dW = F\,dr = \frac{Gm}{r^2}dr$

Total work done in displacing the particle from infinity to point P is

$$W = \int dW = \int_\infty^r \frac{GM}{r^2}dr = \frac{-GM}{r}.$$

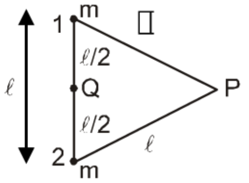
Thus gravitational potential, $\boxed{V = -\frac{GM}{r}}$.

The unit of gravitational potential is J kg^{-1}. Dimensional Formula of gravitational potential

$= \frac{\text{work}}{\text{mass}} = \frac{[\ ML^2T^{-2}]}{[\ M\ ]} = [M^\circ L^2 T^{-2}].$

Solved Examples

Example 4.



Find out potential at P and Q due to the two point mass system. Find out work done by external agent in bringing unit mass from P to Q. Also find work done by gravitational force.

Solution : (i) V_{P1} = potential at P due to mass 'm' at '1' = $-\frac{Gm}{\ell}$

$V_{P2} = -\frac{Gm}{\ell}$ ∴ $V_P = V_{P1} + V_{P2} = -\frac{2Gm}{\ell}$

(ii) $V_{Q1} = -\frac{GM}{\ell/2}$

$V_{Q2} = -\frac{Gm}{\ell/2}$ ∴ $V_Q = V_{Q1} + V_{Q2} = -\frac{Gm}{\ell/2} - \frac{Gm}{\ell/2} = -\frac{4Gm}{\ell}$

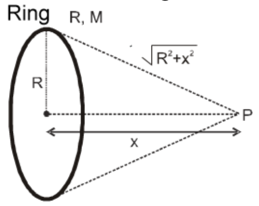
Force at point Q = 0

(iii) work done by external agent = $(V_Q - V_P) \times 1 = -\frac{2GM}{\ell}$

(iv) work done by gravitational force = $V_P - V_Q = \frac{2GM}{\ell}$

Example 5. Find potential at a point 'P' at a distance 'x' on the axis away from centre of a uniform ring of mass M and radius R.

Solution : Ring can be considered to be made of large number of point masses (m_1 , m_2 ....etc)



$V_P = -\frac{Gm_1}{\sqrt{R^2+x^2}} - \frac{Gm_2}{\sqrt{R^2+x^2}} - ....$

$= -\frac{G}{\sqrt{R^2+x^2}}(m_1 + m_2.....) = -\frac{GM}{\sqrt{R^2+x^2}}$, where $M = m_1 + m_2 + m_3 + ...$

Potential at centre of ring $= -\frac{G.M}{R}$

5. RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

The work done by an external agent to move unit mass from a point to another point in the direction of the field E, slowly through an infinitesimal distance dr = Force by external agent × distance moved = –Edr.

Thus $dV = -Edr$ ⇒ $E = -\frac{dV}{dr}$.

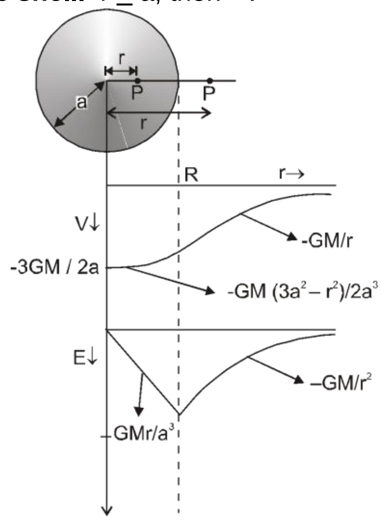
Therefore, gravitational field at any point is equal to the negative gradient at that point.

IV. Uniform Solid Sphere

(a) Point P inside the shell. $r \le a$, then

$V = -\frac{Gm}{2a^3}(3a^2 - r^2)$ & $E = -\frac{GM\ r}{a^3}$, and at the centre $V = -\frac{3GM}{2a}$ and E = 0

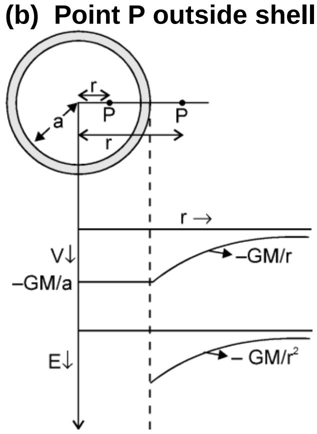
(b) Point P outside the shell. $r \ge a$, then $V = -\frac{GM}{r}$ & $E = -\frac{GM}{r^2}$



V. Uniform Thin Spherical Shell

(a) Point P Inside the shell. $r \le a$, then $V = \frac{-\ GM}{a}$ & E = 0

(b) Point P outside shell. $r \ge a$, then $V = \frac{-\ GM}{r}$ & $E = -\frac{GM}{r^2}$



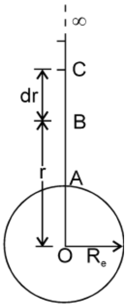
6. GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy of two mass system is equal to the work done by an external agent in assembling them, while their initial separation was infinity. Consider a body of mass m placed at a distance r from another body of mass M. The gravitational force of attraction between them is given by,

$F = \frac{GMm}{r^2}$.

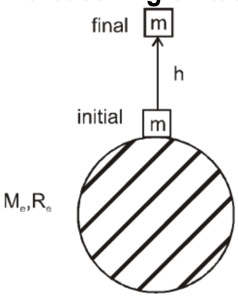
Now, Let the body of mass m is displaced from point. C to B through a istance 'dr' towards the mass M, then work done by internal conservative force (gravitational) is given by,

dW = F dr = $\frac{GM\ m}{r^2}$ dr ⇒ $\int dW = \int_{\infty}^{r} \frac{GM\ m}{r^2}$ dr



∴ Gravitational potential energy, $U = -\frac{GMm}{r}$

Increase in gravitational potential energy :



Suppose a block of mass m on the surface of the earth. We want to lift this block by 'h' height.
Work required in this process = increase in P.E. = $U_f - U_i = m(V_f - V_i)$

$W_{ext} = \Delta U = (m)\left[-\left(\frac{GM_e}{R_e+h}\right) - \left(-\frac{GM_e}{R_e}\right)\right]$

$W_{ext} = \Delta U = GM_em\left(\frac{1}{R_e} - \frac{1}{R_e+h}\right) = \frac{GM_em}{R_e}\left(1-\left(1+\frac{h}{R_e}\right)^{-1}\right)$

(as h << R_e , we can apply Binomical theorem)

$W_{ext} = \Delta U = \frac{GM_em}{R_e}\left(1-\left(1-\frac{h}{R_e}\right)\right) = (m)\left(\frac{GM_e}{R_e^{\ 2}}\right)h$

$W_{ext} = \Delta U = mgh$

* This formula is valid only when h << R_e

Example 6. A body of mass m is placed on the surface of earth. Find work required to lift this body by a height

(i) h = $\frac{R_e}{1000}$ (ii) h = R_e

Solution : (i) h = $\frac{R_e}{1000}$, as h << Re , so we can apply
W_{ext} = U↑ = mgh

$W_{ext} = \left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{1000}\right)(m) = \frac{GM_e m}{1000\ R_e}$

(ii) $h = R_e$, in this case h is not very less than R_e, so we cannot apply $\Delta U = mgh$ so we cannot apply $\Delta U = mgh$

$W_{ext} = U\uparrow = U_f - U_i = m(V_f - V_i)$

$W_{ext} = m\left[\left(-\frac{GM_e}{R_e + R_e}\right) - \left(-\frac{GM_e}{R_e}\right)\right] \Rightarrow W_{ext} = \frac{GM_e m}{2R_e}$

7. ACCELERATION DUE TO GRAVITY :

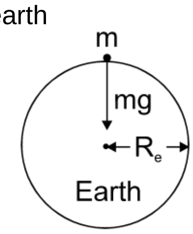
It is the acceleration, a freely falling body near the earth's surface acquires due to the earth's gravitational pull. The property by virtue of which a body experiences or exerts a gravitational pull on another body is called **gravitational mass m_G**, and the property by virtue of which a body opposes any change in its state of rest or uniform motion is called its **inertial mass m_I** thus if $\vec{E}$ is the gravitational field intensity due to the earth at a point P, and g is acceleration due to gravity at the same point, then $m_I\vec{g} = m_G\vec{E}$

Now the value of inertial & gravitational mass happen to be exactly same to a great degree of accuracy for all bodies. Hence, $\vec{g} = \vec{E}$

The gravitational field intensity on the surface of earth is therefore numerically equal to the acceleration due to gravity (g), there. Thus we get,

$$g = \frac{GM_e}{R_e^2}$$

where , M_e = Mass of earth



R_e = Radius of earth

Note :

- Here the distribution of mass in the earth is taken to be spherical symmetrical so that its entire mass can be assumed to be concentrated at its center for the purpose of calculation of g.

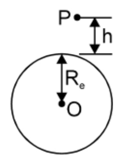
8. VARIATION OF ACCELERATION DUE TO GRAVITY

(a) Effect of Altitude

Acceleration due to gravity on the surface of the earth is given by, $g = \frac{GM_e}{R_e^2}$

Now, consider the body at a height 'h' above the surface of the earth, then the acceleration due to gravity at height 'h' given by

$$g_h = \frac{GM_e}{(R_e + h)^2} = g\left(1 + \frac{h}{R_e}\right)^{-2} \simeq g\left(1 - \frac{2\ h}{R_e}\right) \text{ when } h << R.$$

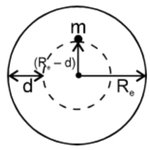


The decrease in the value of 'g' with height h = $g - g_h = \frac{2gh}{R_e}$.

Then percentage decrease in the value of 'g' = $\dfrac{g - g_h}{g} \times 100 = \dfrac{2\ h}{R_e} \times 100\%$

(b) Effect of depth

The gravitational pull on the surface is equal to its weight i.e. mg = $\dfrac{GM_e\ \ m}{R_e^2}$



∴ mg = $\dfrac{G \times \frac{4}{3}\pi R_e^3\ \ \rho\ \ m}{R_e^2}$ or g = $\frac{4}{3}$ π G R$_e$ ρ ....(1)

When the body is taken to a depth d, the mass of the sphere of radius (R$_e$ – d) will only be effective for the gravitational pull and the outward shall will have no resultant effect on the mass. If the acceleration due to gravity on the surface of the solid sphere is g_d, then

g_d = $\frac{4}{3}$ π G (R$_e$ – d) ρ ....(2)

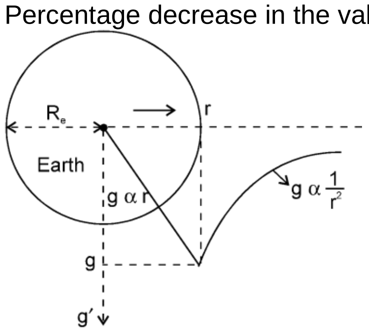
By dividing equation (2) by equation (1)

⇒ $g_d = g\left(1 - \dfrac{d}{R_e}\right)$

IMPORTANT POINTS

(i) At the center of the earth, d = R$_e$, so $g_{centre} = g\left(1 - \dfrac{R_e}{R_e}\right) = 0$.
Thus weight (mg) of the body at the centre of the earth is zero.

(ii) Percentage decrease in the value of 'g' with the depth

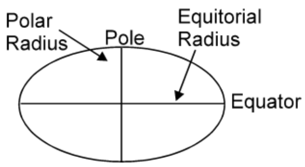


= $\left(\dfrac{g - g_d}{g}\right) \times 100$ = $\dfrac{d}{R_e} \times 100$.

(c) Effect of the surface of Earth

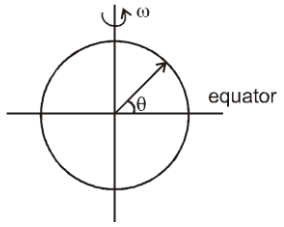
The equatorial radius is about 21 km longer than its polar radius.

We know, g = $\dfrac{GM_e}{R_e^2}$ Hence $g_{pole} > g_{equator}$. The weight of the body increase as the body taken from the equator to the pole.



(d) Effect of rotation of the Earth

The earth rotates around its axis with angular velocity ω. Consider a particle of mass m at latitude θ. The angular velocity of the particle is also ω.



g' =g $\left[1 - \frac{R_e \ \omega^2}{g}\cos^2\theta\right]$.

At pole θ = 90° ⇒ g_{pole} = g ,

At equator θ = 0 ⇒ $g_{equator}$ = g $\left[1 - \frac{R_e \ \omega^2}{g}\right]$.

Hence $g_{pole} > g_{equator}$

If the body is taken from pole to the equator, then g′ = g $\left(1 - \frac{R_e\omega^2}{g}\right)$.

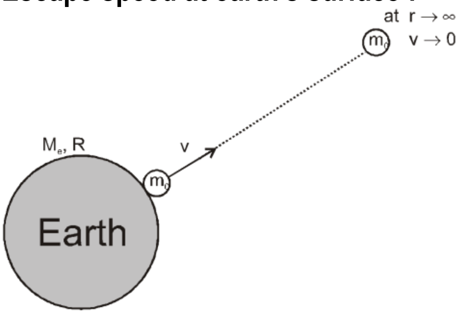
Hence % change in weight =

$$\frac{mg - mg\left(1 - \frac{R_e\omega^2}{g}\right)}{mg}\times 100 = \frac{m \ R_e \ \omega^2}{mg}\times 100 = \frac{R_e \ \omega^2}{g}\times 100$$

9. ESCAPE SPEED

The minimum speed required to send a body out of the gravity field of a planet (send it to r → ∞)

9.1 Escape speed at earth's surface :



Suppose a particle of mass m is on earth's surface

We project it with a velocity V from the earth's surface, so that it just reaches r → ∞ (at r → ∞, its velocity become zero)

Applying energy conservation between initial position (when the particle was at earth's surface) and find positions (when the particle just reaches to r → ∞)

Ki + Ui = Kf + Uf

$\frac{1}{2}$ mv2 + m0 $\left(-\frac{GM_e}{R}\right)$ = 0 + m0 $\left(-\frac{GM_e}{(r\to\infty)}\right)$ ⇒ v = $\sqrt{\frac{2GM_0}{R}}$

Escape speed from earth is surface $V_e = \sqrt{\frac{2GM_e}{R}}$

If we put the values of G, M_e, R the we get

Ve = 11.2 km/s.

9.2 Escape speed depends on :

(i) Mass (Me) and size (R) of the planet

(ii) Position from where the particle is projected.

9.3 Escape speed does not depend on :

(i) Mass of the body which is projected (m0)

(ii) Angle of projection.

If a body is thrown from Earth's surface with escape speed, it goes out of earth's gravitational field and never returns to the earth's surface. But it starts revolving around the sun.

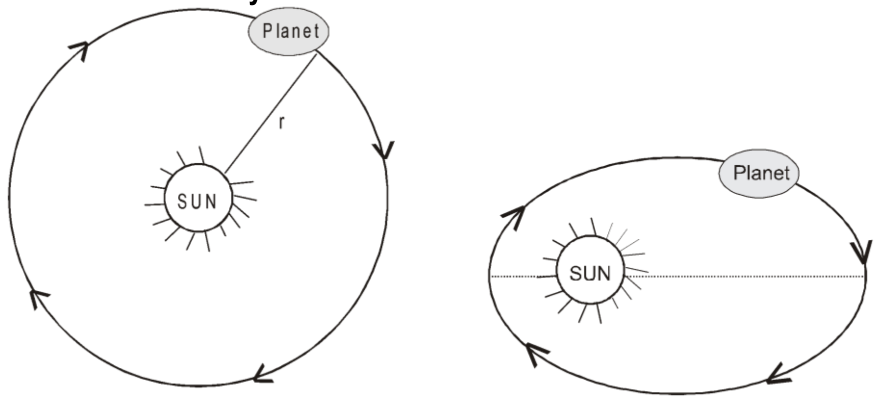
10. KEPLER'S LAW FOR PLANETARY MOTION

Suppose a planet is revolving around the sun, or a satelite is revolving around the earth, then the planetary motion can be studied with help of Kepler's three laws.

10.1 Kepler's Law of orbit

Each planet moves around the sun in a circular path or elliptical path with the sun at its focus. (Infact circular path is a subset of elliptical path)

10.2 Law of areal velocity :

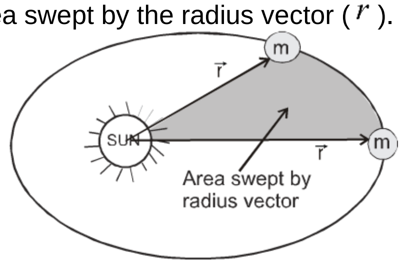


To understand this law, lets understand the angular momentum conservation for the planet.
If a planet moves in an elliptical orbit, the gravitation force acting on it always passes through the centre of the sun. So torque of this gravitation force about the centre of the sun will be zero. Hence we can say that angular momentum of the planet about the centre of the sun will remain conserved (constant)
τ about the sun = 0

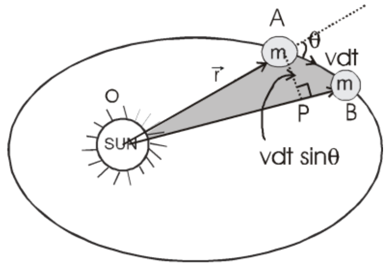
$\Rightarrow \quad \frac{dJ}{dt} = 0 \quad \Rightarrow \quad J_{planet}$ / sun = constant $\quad \Rightarrow \quad$ mvr sinθ = constant

Now we can easly study the Kapler's law of areal velocity.

If a planet moves around the sun, the radius vector ($\vec{r}$) also rotates are sweeps area as shown in figure. Now lets find rate of area swept by the radius vector ($\vec{r}$).



Suppose a planet is revolving around the sun and at any instant its velocity is v, and angle between radius vector ($\vec{r}$) and velocity ($\vec{v}$). In dt time, it moves by a distance vdt, during this dt time, area swept by the radius vector will be OAB which can be assumed to be a triangle



dA = 1/2 (Base) (Perpendicular height)
dA = 1/2 (r) (vdtsinθ)

so rate of area swept $\frac{dA}{dt} = \frac{1}{2}$ vr sinθ

we can write $\frac{dA}{dt} = \frac{1}{2}\frac{mvr \sin\theta}{m}$

where mvr sinθ = angular momentum of the planet about the sun, which remains conserved (constant)

⇒ $\frac{dA}{dt} = \frac{L_{planet/sun}}{2m}$ = constant

so **Rate of area swept by the radius vector is constant**

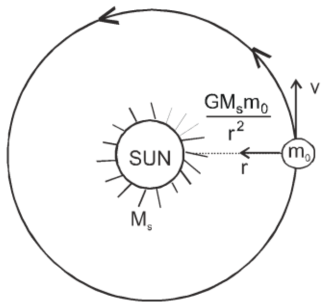
10.3 Kepler's law of time period :

Suppose a planet is revolving around the sun in circular orbit

then $\frac{m_0 v^2}{r} = \frac{GM_s m_0}{r^2}$ v = $\sqrt{\frac{GM_s}{r}}$

Time period of revolution is

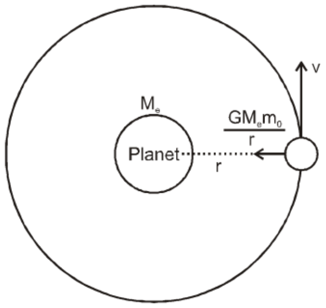
T = $\frac{2\pi r}{v}$ = 2πr $\sqrt{\frac{r}{GM_s}}$



$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3$ ⇒ $T^2 \propto r^3$

For all the planet of a sun , $T^2 \propto r^3$

11. CIRCULAR MOTION OF A SATELLITE AROUND A PLANET



Suppose at satellite of mass m0 is at a distance r from a planet. If the satellite does not revolve, then due to the gravitational attraction, it may collide to the planet.

To avoid the collision, the satellite revolve around the planet, for circular motion of satellite.

⇒ $\frac{GM_e m_0}{r^2} = \frac{m_0 v^2}{r}$ ....(1)

⇒ v = $\sqrt{\frac{GM_e}{r}}$ this velocity is called orbital velocity (v0) ⇒ v0 = $\sqrt{\frac{GM_e}{r}}$

11.1 Total energy of the satellite moving in circular orbit :

(i) KE = $\frac{1}{2}$ m0v2 and from equation .....(1)

$\frac{m_0v^2}{r} = \frac{GM_em_0}{r^2}$ ⇒ m0v2 = $\frac{GM_em_0}{r}$ ⇒ KE = $\frac{1}{2}m_0v^2 = \frac{GM_em_0}{2r}$

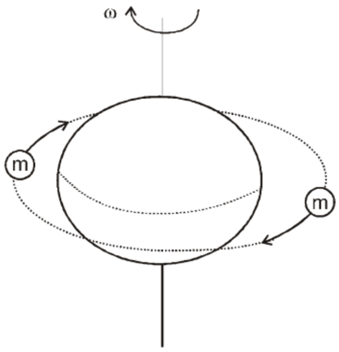
(ii) Potential energy

U = – $\frac{GM_em_0}{r}$

Total energy = KE + PE = $\left(\frac{GM_em_0}{2r}\right) + \left(\frac{-GM_em_0}{r}\right)$ ⇒ TE = – $\frac{GM_em_0}{2r}$

Total energy is –ve. It shows that the satellite is still bounded with the planet.

12. GEO - STATIONERY SATELITE :



We know that the earth rotates about its axis with angular velocity ω_{earth} and time period T_{earth} = 24 hours. Suppose a satellite is set in an orbit which is in the plane of the equator, whose ω is equal to ω_{earth}, (or its T is equal to T_{earth} = 24 hours) and direction is also same as that of earth. Then as seen from earth, it will appear to be stationery. This type of satellite is called geo- stationery satellite. For a geo-stationery satelite,

$w_{satelite} = w_{earth}$

⇒ $T_{satelite} = T_{earth}$ = 24 hr.

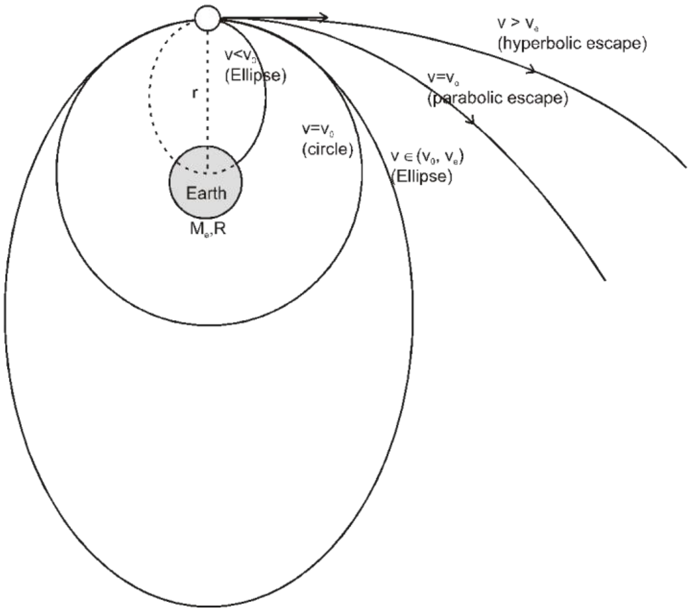
So time period of a geo-stationery satelite must be 24 hours. To achieve T = 24 hour, the orbital radius geo-stationery satelite :

$T_2 = \left(\frac{4\pi^2}{GM_e}\right) r_3$

Putting the values, we get orbital radius of geo stationery satelite r = 6.6 R_e (here Re = radius of the earth)

height from the surface h = 5.6 R_e.

13. PATH OF A SATELLITE ACCORDING TO DIFFERENT SPEED OF PROJECTION



Suppose a stallite is at a distance r from the centre of the earth. If we give different velocities (v) to the satellite, its path will be different

(i) If v < v0 $\left(\text{or } v < \sqrt{\frac{GM_e}{r}}\right)$ then the satellite will move is an elliptical path and strike the earth's surface.

But if size of earth were small, the satellite would complete the elliptical orbit, and the centre of the earth will be at is farther focus.

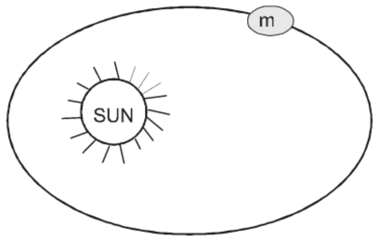
(ii) If v = v0 $\left(or \quad v = \sqrt{\frac{GM_e}{r}}\right)$, then the satellite will revolve in a circular orbit.

(iii) If v0 > V > v0 $\left(\text{or} \quad \sqrt{\frac{2GM_e}{r}} > v > \sqrt{\frac{GM_e}{r}}\right)$, then the satellite will revolve in a elliptical orbital, and the centre of the earth will be at its nearer focus.

(iv) If v = ve $\left(\text{or} \quad v = \sqrt{\frac{2GM_e}{r}}\right)$, then the satellite will just escape with parabolic path.

Solved Miscellaneous Problems

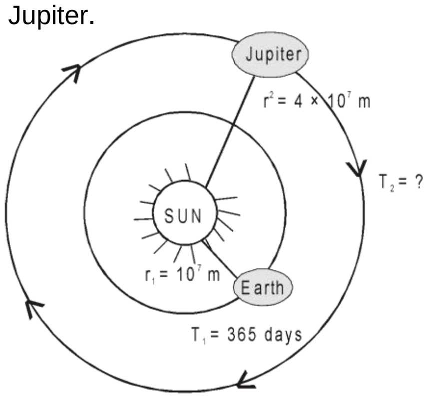
Problem 1. Suppose a planet is revolving around the sun in an elliptical path given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
Find time period of revolution. Angular momentum of the planet about the sun is L.



Solution : Rate of area swept = $\frac{dA}{dt}=\frac{L}{2m}$ constant

$\Rightarrow$ dA = $\frac{L}{2m}dt$ $\Rightarrow$ $\int_{A=0}^{A=\pi ab} dA = \int_{t=0}^{t=T}\frac{L}{2m}dt$ $\Rightarrow$ $\pi ab = \frac{L}{2m}T$ $\Rightarrow$ $T = \frac{2\pi mab}{L}$

Problem 2. The Earth and Jupiter are two planets of the sun. The orbital radius of the earth is 10^7 m and that of Jupiter is 4×10^7 m. If the time period of revolution of earth is T = 365 days, find time period of revolution of the Jupiter.

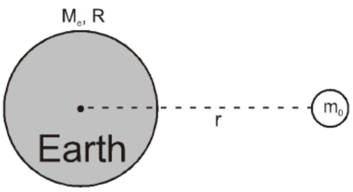


Solution : For both the planets $T^2 \propto r^3$

$\left(\frac{T_{jupiter}}{T_{earth}}\right)^2 = \left(\frac{T_{jupiter}}{r_{earth}}\right)^3$ $\Rightarrow$ $\left(\frac{T_{jupiter}}{365\ \ days}\right)^2 = \left(\frac{4\ \times 10^7}{10^7}\right)^3$

$T_{jupiter}$ = 8 × 365 days

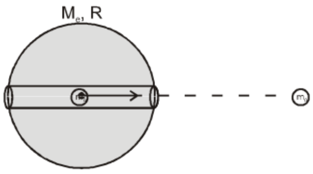
Problem 3.



Suppose earth has radius R and mass M. A point mass m_o is at a distance r from the centre. Find the gravitational potential energy of the mass due to earth.

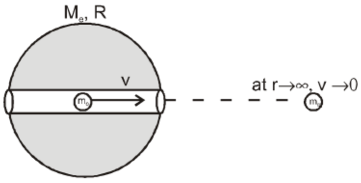
Solution : $U_g = (m_o)\ (V_{earth})$

$U_g = (m_o)\left(-\frac{GM_e}{r}\right) = \left(-\frac{GM_e m_0}{r}\right)$



Problem 4.

Suppose the earth has mass and radius R. A small groove in made and point mass m_0 is placed at the centre of the sphere. With what minimum velocity should we project the particle so that it may escape out of the gravity field (reaches to r → ∞)



Solution :

Suppose the particle projected with speed v, and to send it to infinity, its velocity should be zero at r → ∞. Applying energy conservation between its initial position (centre) and final position (r → ∞)

$K_i + U_i = k_f + U_f$

$\frac{1}{2} m_o v^2 + (m_o)(V_{earth}) = 0 + (m_o)(V_{earth\ at\ infinitely})$ ⇒ $\frac{1}{2} m_o v^2 + (m_o)\left(-\frac{3GM_e}{2\ R}\right) = 0 + m_o\,(0)$

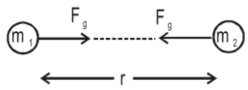
$v = \left(\sqrt{\frac{3GM_e}{R}}\right)$

SUMMARY

☞ Newton's law of gravitation :

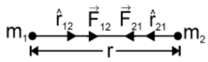
Gravitational attraction force between two point masses

$F_g = \frac{G\ m_1\ m_2}{r^2}$ and its direction will be attractive.



☞ Gravitational force on (1) due to (2) in vector form

$\vec{F}_{12} = \frac{G\ m_1\ m_2}{r^2}$



☞ Gravitational Field :- Gravitational force acting on unit mass.

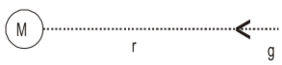
$g = \frac{F}{m}$

☞ Gravitational Potential : - Gravitation potential energy of unit mass

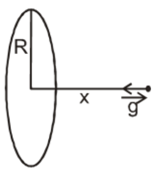
$V_g = \frac{U}{m}$ ⇒ $g = -\frac{dV_g}{dr}$ and $V_B - V_A = -\int_A^B \vec{g} \cdot \vec{dr}$

(i) For point mass :

$g = \frac{GM}{r^2}$, $V = -\frac{GM}{r}$



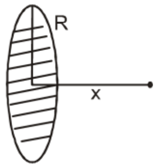
(ii) For circular ring $g = \frac{GMx}{(R^2 + x^2)^{3/2}}$



$V = -\frac{GM}{\sqrt{R^2 + x^2}}$

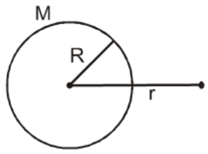
(iii) For thin circular disc

g = $\frac{2GM}{R^2}\left(1-\frac{1}{\sqrt{1+\left(\frac{R}{x}\right)^2}}\right)$



V = $\frac{-2GM}{R^2}\left(\sqrt{R^2+x^2}-x\right)$

(iv) Uniform thin spherical shell : -



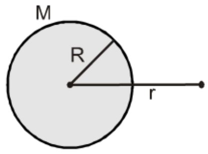
$g_{out} = \frac{GM}{r^2}$ ⇒ $g_{surface} = \frac{GM}{R^2}$

$g_{in} = 0$

☞ **Potential :**

$V_{out} = -\frac{GM}{r}$ ⇒ $V_{surface} = -\frac{GM}{R}$ ⇒ $V_{in} = -\frac{GM}{R}$

(v) Uniform solid sphere :- (Most Important)



$g_{out} = \frac{GM}{r^2}$ ⇒ $g_{surface} = \frac{GM}{R^2}$

$g_{in} = \frac{GM}{R^3}r$ ⇒ $g_{centre} = 0$

☞ **Potential :** $V_{out} = -\frac{GM}{r}$ ⇒ $V_{in} = -\frac{GM}{2R^3}(3R^2 - r^2)$

$V_{surface} = -\frac{GM}{R}$ ⇒ $V_{centre} = -\frac{3}{2}\frac{GM}{R}$

☞ **Self energy :**

Self energy of hollow sphere = $U_{self} = -\frac{1}{2}\frac{GM^2}{R}$

Gravitational Self energy of a Uniform Sphere = $U_{self} = -\frac{3}{5}\frac{GM^2}{R}$

Escape speed from earth's surface

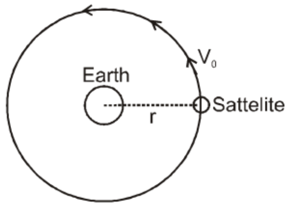
$V_e = \sqrt{\frac{2GM_e}{R}}$ = 11.2 km/sec.

If a satellite is moving around the earth is circular orbit, then its orbital speed is

$V_0 = \sqrt{\frac{GM_e}{r}}$

where r is distance of satellite from the centre of earth.

PE . of the satellite = $-\frac{GM_e \; m}{r}$



KE of the satellite = $\frac{1}{2}mv_0^2$ = $\frac{GM_e m}{2 \; r}$

TE of the satellite = $-\frac{GM_e m}{2 \; r}$

☞ **Time period of Geo - stationary satellite = 24 hours**

Kapler's laws : -

(i) Law of Orbit :- If a planet is revolving around a sun, its path is either elliptical (or circular)

(ii) Law of Area : -

View (i) If a planet is revolving around a sun, the angular momentum of the planet about the sun remains conserved

View (ii) The radius vector from the sum to the planet sweeps area at constant rate

Areal velocity = $\frac{dA}{dt}$ = $\frac{L}{2m}$ = constant

(iii) For all the planets of a sun

$T^2 \propto R^3$ ⇒ $T^2 = \left(\frac{4\pi^2}{GM_s}\right) R^3$

☞ **Factors Affecting Acceleration Due to Gravity :**

1. **Effect of Altitude :** $g_h = \frac{GM_e}{(R_e+h)^2} = g\left(1+\frac{h}{R_e}\right)^{-2} \simeq g\left(1-\frac{2\;h}{R_e}\right)$ when h << R.

2. **Effect of depth :** $g_d = g\left(1-\frac{d}{R_e}\right)$

3. **Effect of the surface of Earth**

The equatorial radius is about 21 km longer than its polar radius.

We know, $g = \frac{GM_e}{R_e^2}$ Hence $g_{pole} > g_{equator}$.

4. **Effect of rotation of the Earth**

Consider a particle of mass m at latitude θ. $g' = g - \omega^2 R_e \cos^2\theta$

At pole θ = 90°

⇒ $g_{pole} = g$, At equator θ = 0

⇒ $g_{equator} = g - \omega^2 R_e$. Hence $g_{pole} > g_{equator}$