RELATIVE MOTION

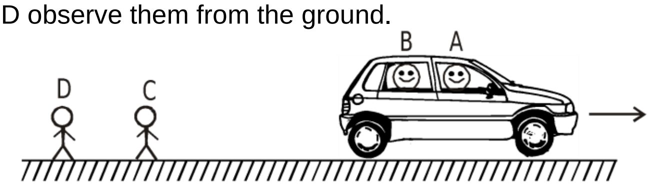
1 RELATIVE MOTION

Motion is a combined property of the object under study as well as the observer. It is always relative ; there is no such thing like absolute motion or absolute rest. Motion is always defined with respect to an observer or reference frame.

Reference frame :

Reference frame is an axis system from which motion is observed along with a clock attached to the axis, to measure time. Reference frame can be stationary or moving.

Suppose there are two persons A and B sitting in a car moving at constant speed. Two stationary persons C and D observe them from the ground.



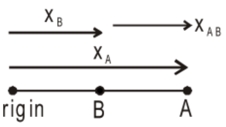
Here B appears to be moving for C and D, but at rest for A. Similarly C appears to be at rest for D but moving backward for A and B.

2 RELATIVE MOTION IN ONE DIMENSION :

2.1 Relative Position :

It is the position of a particle w.r.t. observer.

In general if position of A w.r.t. to origin is x_A and that of B w.r.t. origin is x_B then "Position of A w.r.t. B" x_{AB} is

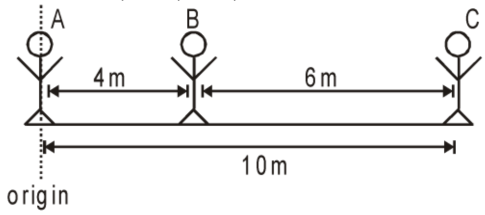


$x_{AB} = x_A - x_B$

Solved Examples

Example.1 See the figure (take +ve direction towards right and –ve towards left)

Find x_{BA} , x_{CA} , x_{CB} , x_{AB} and x_{AC} .



Solution : Here, Position of B w.r.t. A is 4 m towards right . (x_{BA} = +4m)

Position of C w.r.t. A is 10 m towards right . (x_{CA} = +10m)

Position of C w.r.t. B is 6 m towards right (x_{CB} = +6m)

Position of A w.r.t. B is 4 m towards left. (x_{AB} = –4 m)

Position of A w.r.t. C is 10 m towards left. (x_{AC} = –10m)

2.2 Relative Velocity

Definition **:** Relative velocity of a particle A with respect to B is defined as the velocity with which A appears to move if B is considered to be at rest. In other words, it is the velocity with which A appears to move as seen by B considering itself to be at rest.

or it is the rate at which relative position is changing.

NOTE 1 : All velocities are relative & have no significance unless observer is specified. However, when we say "velocity of A", what we mean is , velocity of A w.r.t. ground which is assumed to be at rest.

Relative velocity in one dimension -

If x_A is the position of A w.r.t. ground, x_B is position of B w.r.t. ground and x_{AB} is position of A w.r.t. B then we can say v_A = velocity of A w.r.t. ground = $\frac{dx_A}{dt}$

v_B = velocity of B w.r.t. ground = $\frac{dx_B}{dt}$

and v_{AB} = velocity of A w.r.t. B = $\frac{dx_{AB}}{dt} = \frac{d}{dt}(x_A - x_B)$

$= \frac{dx_A}{dt} - \frac{dx_B}{dt}$

Thus

$v_{AB} = v_A - v_B$

NOTE 2. : Velocity of an object w.r.t. itself is always zero.

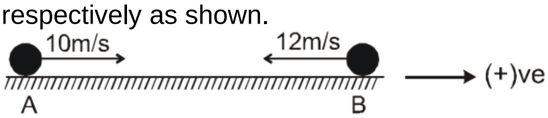
Solved Examples

Example.2 An object A is moving with 5 m/s and B is moving with 20 m/s in the same direction. (Positive x-axis)
(i) Find velocity of B with respect to A.
(ii) Find velocity of A with respect to B

Solution : (i) $v_B = +20$ m/s $v_A = +5$ m/s $\mathbf{v_{BA}} =$ $v_B - v_A = +15$ m/s
(ii) $v_B = +20$ m/s, $v_A = +\ 5$ m/s ; $v_{AB} = v_A - v_B = -15$ m/s

Note : $\mathbf{v_{BA} = -v_{AB}}$

Example.3 Two objects A and B are moving towards each other with velocities 10 m/s and 12 m/s respectively as shown.



(i) Find the velocity of A with respect to B.
(ii) Find the velocity of B with respect to A

Solution : $v_A = +10$, $v_B = -12$
(i) $v_{AB} = v_A - v_B = (10) - (-12) = 22$ m/s.
(ii) $v_{BA} = v_B - v_A = (-12) - (10) = -22$ m/s.

2.3 Relative Acceleration

It is the rate at which relative velocity is changing.

$a_{AB} = \frac{d\ v_{AB}}{dt} = \frac{dv_A}{dt} - \frac{dv_B}{dt} = a_A - a_B$

Equations of motion when relative acceleration is constant.

$v_{rel} = u_{rel} + a_{rel}\, t$

$s_{rel} = u_{rel}\, t + \frac{1}{2}\, a_{rel}\, t^2$

$v^2_{rel} = u^2_{rel} + 2a_{rel}\, s_{rel}$

2.4 Velocity of Approach / Separation

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach / separation is simply equal to magnitude of relative velocity of A w.r.t. B.

Solved Examples

Example.4 A particle A is moving with a speed of 10 m/s towards right and another particle B is moving at speed of 12 m/s towards left. Find their velocity of approach.

Solution : $V_A = +10$, $V_B = -12 \Rightarrow V_{AB} = V_A - V_B \Rightarrow 10 - (-12) = 22$ m/s

since separation is decreasing hence $V_{app} = |V_{AB}| = 22$ m/s

Ans. : 22 m/s

Example.5 A particle A is moving with a speed of 20 m/s towards right and another particle B is moving at a speed of 5 m/s towards right. Find their velocity of approach.

Solution : $V_A = +20$, $V_B = +5$

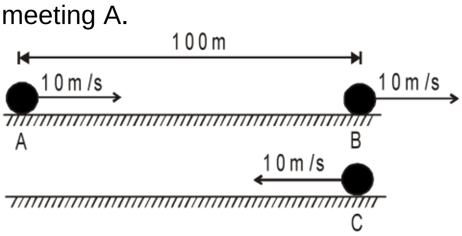
$V_{AB} = V_A - V_B$

$20 - (+5) = 15$ m/s

since separation is decreasing hence $V_{app} = |V_{AB}| = 15$ m/s

Ans. : 15 m/s

Example.6 A particle A is moving with a speed of 10 m/s towards right, particle B is moving at a speed of 10 m/s towards right and another particle C is moving at speed of 10 m/s towards left. The separation between A and B is 100 m. Find the time interval between C meeting B and C meeting A.



Solution ; $t = \dfrac{\text{seperation between A and C}}{V_{app} \text{ of A and C}} = \dfrac{100}{10-(-10)} = 5$ sec.

Ans. : 5 sec.

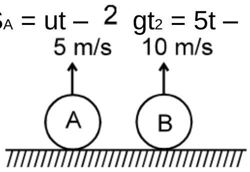
Note : $a_{app} = \left(\dfrac{d}{dt}\right)v_{app}$, $a_{sep} = \dfrac{d}{dt}v_{sep}$

$v_{app} = \int a_{app}dt$, $v_{sep} = \int a_{sep}dt$

Example.7 A and B are thrown vertically upward with velocity, 5 m/s and 10 m/s respectively ($g = 10\ m/s^2$. Find

separation between them after one second

Solution : $S_A = ut - \dfrac{1}{2}gt^2 = 5t - \dfrac{1}{2} \times 10 \times t^2$



$= 5 \times 1 - 5 \times 1^2$

$= 5 - 5 \quad = 0$

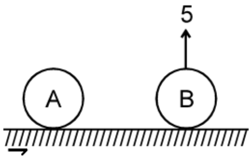
$S_B = ut - \dfrac{1}{2}gt^2 . = 10 \times 1 - \dfrac{1}{2} \times 10 \times 1^2 = 10 - 5 = 5$

$\therefore$ $S_B - S_A$ = separation = 5m.

Aliter :

$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = (-10) - (-10) = 0$

Also $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 10 - 5 = 5$ m/s



∴ $\vec{s}_{BA}$ (in 1 sec) $= \vec{v}_{BA} \times t$

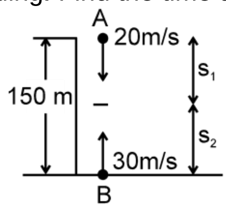
$= 5 \times 1$

$= 5$ m

∴ Distance between A and B after 1 sec = 5 m.

Solved Examples

Example.8 A ball is thrown downwards with a speed of 20 m/s from the top of a building 150 m high and simultaneously another ball is thrown vertically upwards with a speed of 30 m/s from the foot of the building. Find the time after which both the balls will meet. ($g = 10$ m/s^2)



Solution : $S_1 = 20t + 5t^2$

$S_2 = 30t - 5t^2$

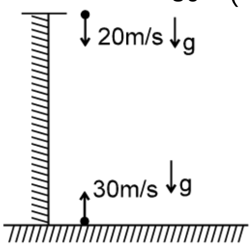
$S_1 + S_2 = 150 \Rightarrow 150 = 50t \Rightarrow t = 3$ s

Aliter :

Relative acceleration of both is zero since both have same acceleration in downward direction

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = g - g = 0$$

$$\vec{v}_{BA} = 30 - (-20) = 50$$



$s_{BA} = v_{BA} \times t$

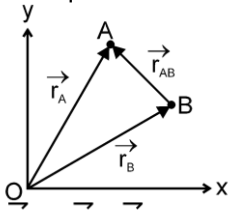
$$t = \frac{s_{BA}}{v_{BA}} = \frac{150}{50} = 3 \text{ s}$$

3. RELATIVE MOTION IN TWO DIMENSION

$\vec{r}_A$ = position of A with respect to O

$\vec{r}_B$ = position of B with respect to O

$\vec{r}_{AB}$ = position of A with respect to B.



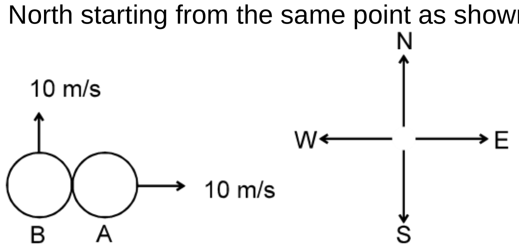
$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$ (The vector sum $\vec{r}_A - \vec{r}_B$ can be done by Δ law of addition or resolution method)

∴ $\dfrac{d\,(\vec{r}_{AB})}{dt} = \dfrac{d\,(\vec{r}_A)}{dt} - \dfrac{d\,(\vec{r}_B)}{dt}$. $\Rightarrow$ $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$

$$\frac{d\,(\vec{v}_{AB})}{dt} = \frac{d\,(\vec{v}_{A})}{dt} - \frac{d\,(\vec{v}_{B})}{dt} \Rightarrow \vec{a}_{AB} = \vec{a}_{A} - \vec{a}_{B}$$

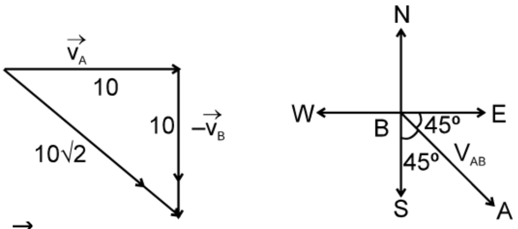
Solved Examples

Example.9 Object A and B both have speed of 10 m/s. A is moving towards East while B is moving towards North starting from the same point as shown. Find velocity of A relative to B ($\vec{v}_{AB}$)



Solution : **Method 1**

$\vec{v}_{AB} = \vec{v}_{A} - \vec{v}_{B}$



∴ $|\vec{v}_{AB}| = 10\sqrt{2}$

Method 2

$\vec{v}_{A} = 10\hat{i}$, $\vec{v}_{B} = 10\hat{j}$

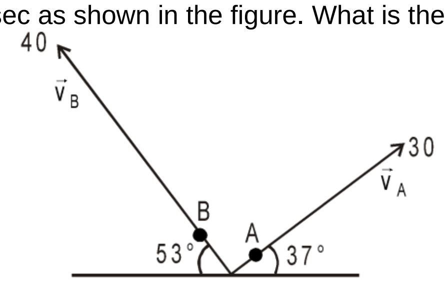
$\vec{v}_{AB} = \vec{v}_{A} - \vec{v}_{B} = 10\hat{i} - 10\hat{j}$

∴ $|\vec{v}_{AB}| = 10\sqrt{2}$

NOTE : $|\vec{v}_{A} - \vec{v}_{B}| = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$, where θ is angle between $\vec{v}_A$ and $\vec{v}_B$

Self Practice Problems

1. Two particles A and B are projected in air. A is thrown with a speed of 30 m/sec and B with a speed of 40 m/sec as shown in the figure. What is the separation between them after 1 sec.



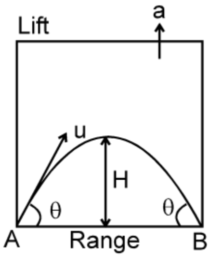
Answer Key : 1. (50 m)

3.1 Relative Motion in Lift
Projectile motion in a lift moving with acceleration a upwards

(1) In the reference frame of lift, acceleration of a freely falling object is g + a

(2) Velocity at maximum height = u cos θ

(3) $T = \frac{2u\sin\theta}{g+a}$



(4) Maximum height (H) = $\dfrac{u^2 \sin^2\theta}{2(g+a)}$

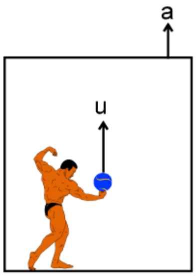
(5) Range = $\dfrac{u^2 \sin 2\theta}{g+a}$

Solved Examples

Example.10 A lift is moving up with acceleration a. A person inside the lift throws the ball upwards with a velocity u relative to hand.

(a) What is the time of flight of the ball?

(b) What is the maximum height reached by the ball in the lift?



Solution :

(a) $\vec{a}_{BL} = \vec{a}_B - \vec{a}_L = g + a$

$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}_{BL}t^2$

$0 = uT - \frac{1}{2}(g + a)T^2$ $\therefore$ $T = \dfrac{2u}{(g+a)}$

(b) $v^2 - u^2 = 2as$

$0 - u^2 = -2(g + a)H$

$H = \dfrac{u^2}{2(g+a)}$

4. RELATIVE MOTION IN RIVER FLOW

If a man can swim relative to water with velocity $\vec{v}_{mR}$ and water is flowing relative to ground with velocity $\vec{v}_R$, velocity of man relative to ground $\vec{v}_m$ will be given by :

$\vec{v}_{mR} = \vec{v}_m - \vec{v}_R$ or $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R$

If $\vec{v}_R = 0$, then $\vec{v}_m = \vec{v}_{mR}$

in words, velocity of man in still water = velocity of man w.r.t. river

4.1 River Problem in One Dimension :

Velocity of river is u & velocity of man in still water is v.

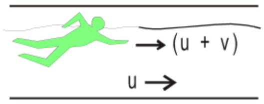
Case - 1

Man swimming downstream (along the direction of river flow)

In this case velocity of river $v_R = +u$

velocity of man w.r.t. river $v_{mR} = +v$

now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = u + v$



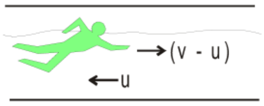
Case - 2

Man swimming upstream (opposite to the direction of river flow)

In this case velocity of river $\vec{v}_R = -u$

velocity of man w.r.t. river $\vec{v}_{mR} = +v$

now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = (v - u)$



Solved Examples

Example.11 A swimmer capable of swimming with velocity 'v' relative to water jumps in a flowing river having velocity 'u' . The man swims a distance d (with respect to ground) down stream and returns back to the original position. Find out the time taken in complete motion.

Solution : Total time = time of swimming downstream + time of swimming upstream

$$t = t_{down} + t_{up} = \frac{d}{v+u} + \frac{d}{v-u} = \frac{2dv}{v^2 - u^2}$$ **Ans.**

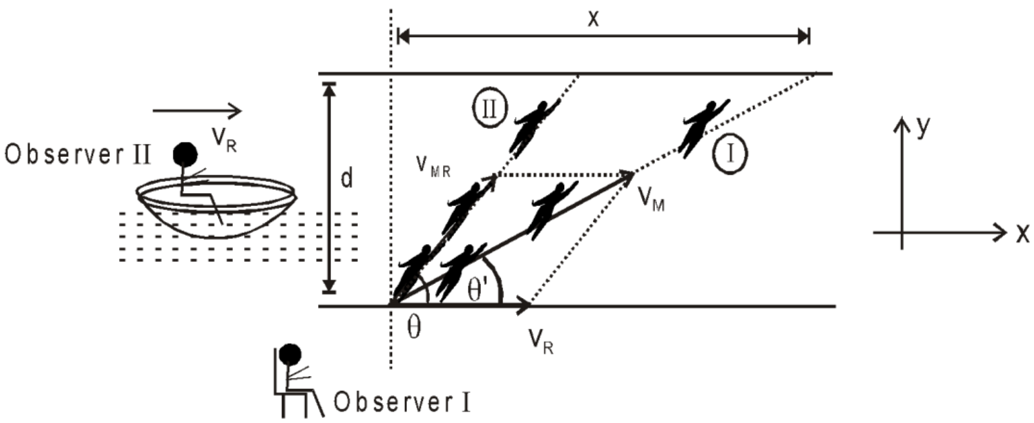
4.2 Motion of Man Swimming in a River

Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow The velocity of river is $\vec{v}_R$.

Let there be two observers I and II , observer I is on ground and observer II is on a raft floating along with the river and hence moving with the same velocity as the river. Hence motion w.r.t. observer II is same as motion w.r.t. river. i.e. the man will appear to swim at an angle θ with the river flow for observer II .

For observer I the velocity of swimmer will be $\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$,

Hence the swimmer will appear to move at an angle θ′ with the river flow.



I : Motion of swimmer for observer I

II : Motion of swimmer for observer II

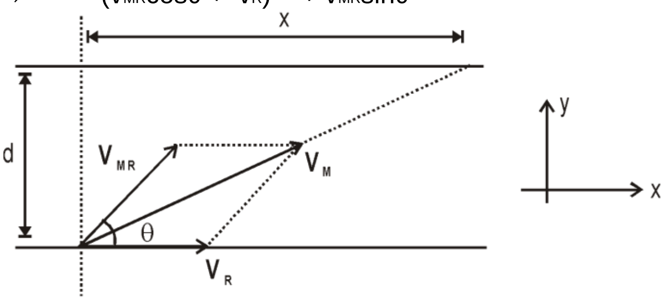
4.3 River problem in two dimension (crossing river) :

Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow . The velocity of river is V_R and the width of the river is d

$\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$

$\Rightarrow \vec{v}_M = (v_{MR}\cos\theta\,\hat{i} + v_{MR}\sin\theta\,\hat{j}) + v_R\,\hat{i}$

$\Rightarrow \vec{v}_M = (v_{MR}\cos\theta + v_R)\,\hat{i} + v_{MR}\sin\theta\,\hat{j}$



Here $v_{MR}\sin\theta$ is the component of velocity of man in the direction perpendicular to the river flow.
This component of velocity is responsible for the man crossing the river. Hence if the time to cross the river is t , then

$$t = \frac{d}{v_y} = \frac{d}{v\ \sin\theta}$$

DRIFT : It is defined as the displacement of man in the direction of river flow. (see the figure).
It is simply the displacement along x-axis, during the period the man crosses the river. $(v_{MR}\cos\theta + v_R)$ is the component of velocity of man in the direction of river flow and this component of velocity is responsible for drift along the river flow. If drift is x then,
Drift = $v_x \times t$

$$x = (v_{MR}\cos\theta + v_R) \times \frac{d}{v_{MR}\sin\theta}$$

4.4 Crossing the river in shortest time

As we know that $t = \frac{d}{v_{MR}\sin\theta}$. Clearly t will be minimum when θ = 90° i.e. time to cross the river will be minimum if man swims perpendicular to the river flow. Which is equal to $\frac{d}{v_{MR}}$.

4.5 Crossing the river in shortest path, Minimum Drift

The minimum possible drift is zero. In this case the man swims in the direction perpendicular to the river flow as seen from the ground. This path is known as ***shortest path***
here $x_{min} = 0 \Rightarrow (v_{MR}\cos\theta + v_R) = 0$

or $\cos\theta = -\frac{v_R}{v_{MR}}$

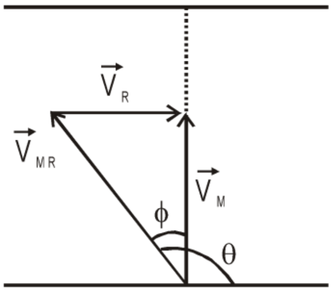
since cos θ is – ve, ∴ θ > 90° , i.e. for minimum drift the man must swim at some angle ϕ with the perpendicular in backward direction.

Where $\sin\phi = \frac{v_R}{v_{MR}}$

$\theta = \cos^{-1}\left(\frac{-v_R}{v_{MR}}\right)$ ∴ $\left|\frac{v_R}{v_{MR}}\right| \le 1$ i.e. $v_R \le v_{MR}$

i.e. minimum drift is zero if and only if velocity of man in still water is greater than or equal to the velocity of river.

Time to cross the river along the shortest path



$$t = \frac{d}{v_{MR}\sin\theta} = \frac{d}{\sqrt{v_{MR}^2 - V_R^2}}$$

NOTE :

If $v_R > v_{MR}$ then it is not possible to have zero drift. In this case the minimum drift (corresponding to shortest possible path is non zero and the condition for minimum drift can be proved to be

$\cos\theta = -\frac{v_{MR}}{v_R}$ or $\sin\phi = \frac{v_R}{v_{MR}}$ **for minimum but non zero drift.**

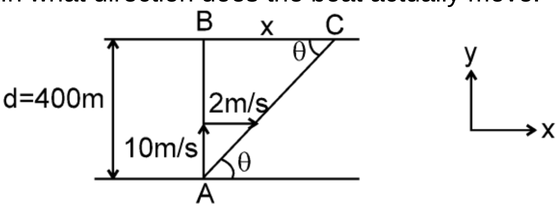
Solved Examples

Example.12 A 400 m wide river is flowing at a rate of 2.0 m/s.A boat is sailing with a velocity of 10 m/s with respect to the water, in a direction perpendicular to the river.

(a) Find the time taken by the boat to reach the opposite bank.

(b) How far from the point directly opposite to the starting point does the boat reach the opposite bank.

(c) In what direction does the boat actually move.



Solution :

(a) time taken to cross the river

$$t = \frac{d}{v_y} = \frac{400\ \text{m}}{10\ \text{m/s}} = 40\ \text{s}$$ **Ans.**

(b) drift (x) = $(v_x)(t)$ = (2m/s) (40s) = 80 m **Ans.**

(c) Actual direction of boat,

$$\theta = \tan^{-1}\left(\frac{10}{2}\right) = \tan^{-1} 5,$$ (downstream) with the river flow.

5. WIND AIRPLANE PROBLEMS

This is very similar to boat river flow problems. The only difference is that boat is replaced by aeroplane and river is replaced by wind.

Thus,

velocity of aeroplane with respect to wind

$$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w$$

or $$\vec{v}_a = \vec{v}_{aw} + \vec{v}_w$$

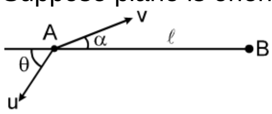
where, $\vec{v}_a$ = velocity of aeroplane w.r.t. ground

and, $\vec{v}_w$ = velocity of wind.

Solved Examples

Example.13 An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is ℓ and the aeroplane maintains the constant speed v w.r.t. wind. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.

Solution : Suppose plane is oriented at an angle α w.r.t. line AB while the plane is moving from A to B :

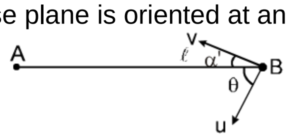


Velocity of plane along AB = $v \cos\alpha - u\cos\theta$,
and for no-drift from line AB

$v \sin\alpha = u\sin\theta \quad \Rightarrow \quad \sin\alpha = \dfrac{u\sin\theta}{v}$

time taken from A to B : $t_{AB} = \dfrac{\ell}{v\cos\alpha - u\cos\theta}$

Suppose plane is oriented at an angle α′ w.r.t. line AB while the plane is moving from B to A :



velocity of plane along BA = $v\cos\alpha + u\cos\theta$ and for no drift from line AB

$v\sin\alpha = u\sin\theta$

$\Rightarrow \quad \sin\alpha = \dfrac{u\sin\theta}{v} \quad \Rightarrow \quad \alpha = \alpha'$

time taken from B to A : $t_{BA} = \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

total time taken = $t_{AB} + t_{BA} = \dfrac{\ell}{v\cos\alpha - u\cos\theta} + \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

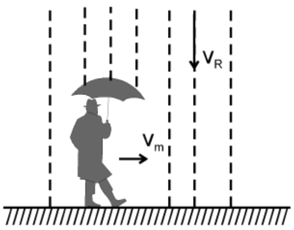
$$= \frac{2v\ell\cos\alpha}{v^2\cos^2\alpha + u^2\cos^2\theta} = \frac{2v\ell\sqrt{1-\dfrac{u^2\sin^2\theta}{v^2}}}{v^2 - u^2}.$$

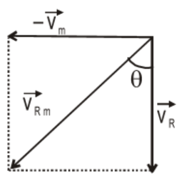
6. RAIN PROBLEM

If rain is falling vertically with a velocity $\vec{v}_R$ and an observer is moving horizontally with velocity $\vec{v}_m$, the velocity of rain relative to observer will be :

$$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_m \qquad \text{or} \qquad v_{Rm} = \sqrt{v_R^2 + v_m^2}$$

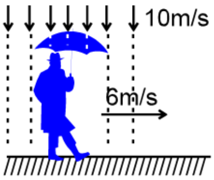
and direction $\theta = \tan^{-1}\left(\dfrac{v_m}{v_R}\right)$ with the vertical as shown in figure.





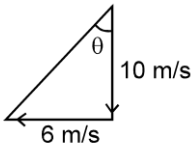
Solved Examples

Example.14 Rain is falling vertically and a man is moving with velocity 6 m/s. Find the angle at which the man should hold his umbrella to avoid getting wet.



Solution :

$\vec{v}_{rain} = -10\hat{j}$ $\Rightarrow$ $\vec{v}_{man} = 6\hat{j}$

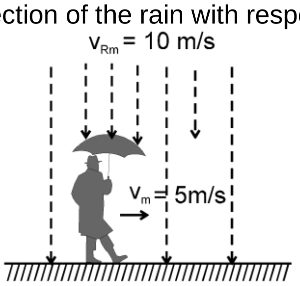


$\vec{v}_{r.w.r.t.\ man} = -10\hat{j} - 6\hat{i}$

$\tan\theta = \frac{6}{10}$ $\Rightarrow$ $\theta = \tan^{-1}\left(\frac{3}{5}\right)$

Where θ is angle with vertical

Example.15 A man moving with 5m/s observes rain falling vertically at the rate of 10 m/s. Find the speed and direction of the rain with respect to ground.

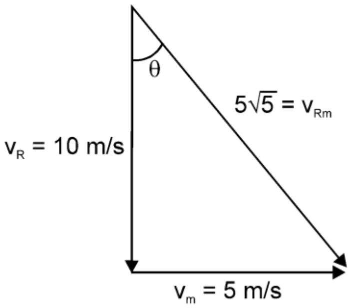


Solution : $v_{RM} = 10$ m/s, $v_M = 5$ m/s

$\vec{v}_{RM} = \vec{v}_{Ru} - \vec{v}_M$

$\Rightarrow$ $\vec{v}_{Ru} = \vec{v}_{RM} + \vec{v}_M$

$\Rightarrow$ $\vec{v}_R = 5\sqrt{5}$



$\tan\theta = \frac{1}{2}$, $\theta = \tan^{-1}\frac{1}{2}$.

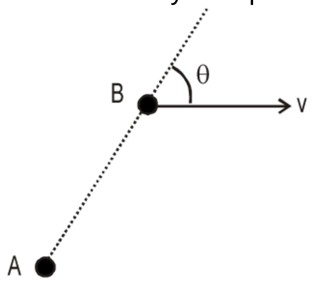
7. VELOCITY OF APPROACH / SEPARATION IN TWO DIMENSION

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

Solved Examples

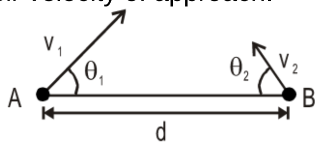
Example.16 Particle A is at rest and particle B is moving with constant velocity v as shown in the diagram at t = 0. Find their velocity of separation



Solution : $v_{BA} = v_B - v_A = v$

v_{sep} = component of v_{BA} along line AB = $v\cos\theta$

Example.17 Two particles A and B are moving with constant velocities v_1 and v_2 . At t = 0, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B. Find their velocity of approach.



Solution : Velocity of approach is relative velocity along line AB

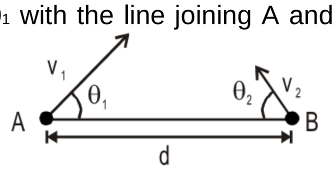
$v_{APP} = v_1\cos\theta_1 + v_2\cos\theta_2$

7.1 Condition for uniformly moving particles to collide

If two particles are moving with uniform velocities and the relative velocity of one particle w.r.t. other particle is directed towards each other then they will collide.

Solved Examples

Example.18 Two particles A and B are moving with constant velocities v_1 and v_2 . At t = 0, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B.



(i) Find the condition for A and B to collide.

(ii) Find the time after which A and B will collide if separation between them is d at t = 0

Solution :

(i) For A and B to collide, their relative velocity must be directed along the line joining them. Therefore their relative velocity along the perpendicular to this line must be zero.
Thus $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

(ii) $v_{APP} = v_1\cos\theta_1 + v_2\cos\theta_2$

$$t = \frac{d}{v_{app}} = \frac{d}{v_1\cos\theta_1 + v_2\cos\theta_2}$$