Example # 33: Show that the vectors $\vec{a} = -4\hat{i} + 8\hat{j} - 4\hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{c} = -2\ \hat{i} - 2\ \hat{j} + 4\ \hat{k}$ are coplanar.

Solution : $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} -4 & 8 & -4 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix} = -4(-8-4) - 8(16-4) - 4(-8-4) = 0$

So vectors , , are coplanar

Self Practice Problems :

(42) Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\} . \vec{a} = 2 \left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(43) Show that $\vec{a}\ .\ (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$

(44) One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co-ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.

(45) Find the value of m such that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + m\hat{j} + 5\hat{k}$ are coplanar.

(46) Show that the vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$, $\vec{a} + \vec{b}$ are coplanar.

Ans. (44) 72 (45) – 4

27. Vector triple product :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a}\ \times\ (\vec{b}\ \times\ \vec{c})$ is a vector & is called a vector triple product.

28. Geometrical interpretation of $\vec{a}\ \times\ (\vec{b}\ \times\ \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors . $\vec{a}$ and $(\vec{b}\ \times\ \vec{c})$ Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $(\vec{b}\ \times\ \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which lies in the plane of $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\ \times (\vec{b}\ \times\ \vec{c})$ in terms of $\vec{b}$ and $\vec{c}$ i.e. $\vec{a}\ \times (\vec{b}\ \times\ \vec{c}) = x\vec{b} + y\vec{c}$, where x , y are scalars.

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} . \vec{c})\vec{b} - (\vec{b} . \vec{c})\vec{a}$
- In general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Example # 34 : If $\vec{b}$ and $\vec{c}$ are two non colinear vectors such that $\vec{a}\ ||\left(\vec{b} \times \vec{c}\right)$ then prove that $(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$ is equalto $|\vec{a}|^2\ (\vec{b}.\vec{c})$

Solution : Since $\vec{a} \parallel (\vec{b}\times\vec{c})$ $\quad \vec{a} = \lambda(\vec{b}\times\vec{c})$ and $\vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$

Now $\Rightarrow$ $(\vec{a}\times\vec{b}).(\vec{a}\times\vec{c}) = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{c} \end{vmatrix} = |\vec{a}|^2\,(\vec{b}.\vec{c})$

Example # 35 : Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = (\vec{b}\,.\,\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\,\vec{c})\,(\vec{a}\times\vec{d})$

Solution : We have, $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c} - (\vec{b}\,.\,\vec{c})\,\vec{d}\}$

$= \vec{a}\times\{(\vec{b}\,.\,\vec{d})\,\vec{c}\} - \vec{a}\times\{(\vec{b}\,.\,\vec{c})\,\vec{d}\}$ [by dist. law]

$= (\vec{b}\,.\,\vec{d})\,(\vec{a}\times\vec{c}) - (\vec{b}\,.\,\vec{c})\,(\vec{a}\times\vec{d})$.

Example # 36 : If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar non-zero vectors, then prove that

$(\vec{a}.\vec{a})\vec{b}\times\vec{c} + (\vec{a}.\vec{b})\vec{c}\times\vec{a} + (\vec{a}.\vec{c})\vec{a}\times\vec{b} = \left[\vec{b}\vec{c}\vec{a}\right]\vec{a}$

Sol. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar

then $\vec{a} = \lambda(\vec{b}\times\vec{c}) + \mu(\vec{c}\times\vec{a}) + \nu(\vec{a}\times\vec{b})$

$\therefore\ \vec{a}.\vec{a} = \lambda\left[\vec{b}\ \vec{c}\ \vec{a}\right]$; $\vec{a}.\vec{b} = \mu\left[\vec{c}\ \vec{a}\ \vec{b}\right]$; $\vec{a}.\vec{c} = \nu\left[\vec{a}\vec{b}\vec{c}\right]$

$\therefore\ a = \left[\dfrac{(\vec{a}.\vec{a})\vec{b}\times\vec{c}}{\left[\vec{b}\ \vec{c}\ \vec{a}\right]} + \dfrac{(\vec{a}.\vec{b})\vec{c}\times\vec{a}}{\left[\vec{c}\ \vec{a}\ \vec{b}\right]} + \dfrac{(\vec{a}.\vec{c})\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\right] \Rightarrow \vec{a}.\left[\vec{a}\vec{b}\vec{c}\right] = (\vec{a}.\vec{a})(\vec{b}\times\vec{c}) + (\vec{a}.\vec{b})(\vec{c}\times\vec{a}) + \vec{a}.\vec{c}(\vec{a}\times\vec{b})$

Example # 37 : If $\vec{A}+\vec{B}=\vec{a}$, $\vec{A}\,.\,\vec{a} = 1$ and $\vec{A}\times\vec{B} = \vec{b}$, then prove that $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$.

Solution : Given $\vec{A}+\vec{B}=\vec{a}$ .....(i)

$\Rightarrow\ \vec{a}\,.\,\left(\vec{A}+\vec{B}\right) = \vec{a}\,.\,\vec{a}$

$\Rightarrow\ \vec{a}\,.\,\vec{A}+\vec{a}\,.\,\vec{B} = \vec{a}\,.\,\vec{a} \quad\Rightarrow\quad 1+\vec{a}\,.\,\vec{B} = |\vec{a}|^2$

$\Rightarrow\ \vec{a}\,.\,\vec{B} = |\vec{a}|^2 - 1$ .....(ii)

Given $\vec{A}\times\vec{B}=\vec{b} \quad\Rightarrow\quad \vec{a}\times\left(\vec{A}\times\vec{B}\right) = \vec{a}\times\vec{b}$

$\Rightarrow\ \left(\vec{a}\,.\,\vec{B}\right)\vec{A} - \left(\vec{a}\,.\,\vec{A}\right)\vec{B} = \vec{a}\times\vec{b}$

$\Rightarrow\ \left(|\vec{a}|^2-1\right)\vec{A}-\vec{B} = \vec{a}\times\vec{b}$ .....(iii)

[Using equation (ii)]

solving equation (i) and (iii) simultaneously, we get

$\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$

Example # 38 : Solve for $\vec{r}$ satisfying the simultaneous equations $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$, $\vec{r}\ .\ \vec{a}=0$ provided $\vec{a}$ is not perpendicular to $\vec{b}$.

Solution : $(\vec{r}-\vec{c})\times\vec{b}=\vec{0}$ ⇒ $\vec{r}-\vec{c}$ and $\vec{b}$ are collinear

∴ $\vec{r}-\vec{c}=k\vec{b}$ ⇒ $\vec{r}=\vec{c}+k\vec{b}$ ....(i)

∵ $\vec{r}\ .\ \vec{a}=0$ ⇒ $(\vec{c}+k\vec{b})\ .\ =0$

⇒ $k=-\dfrac{\vec{a}\ .\ \vec{c}}{\vec{a}\ .\ \vec{b}}$ putting in (i) we get $\vec{r}=\vec{c}-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\vec{b}$

Example # 39 : If $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, where k is a scalar and $\vec{a},\ \vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a},\ \vec{b}$ and k.

Solution : $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$ ....(i)

Premultiply the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a})+k\ (\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$

⇒ $(\vec{a}\ .\ \vec{a})\ \vec{x}-(\vec{a}\ .\ \vec{x})\ \vec{a}+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$ ....(ii)

Premultiply (i) scalarly by

$[\vec{a}\ \ \vec{x}\ \ \vec{a}]+k\ (\vec{a}\ .\ \vec{x})=\vec{a}\ .\ \vec{b}$

$k(\vec{a}\ .\ \vec{x})=\vec{a}\ .\ \vec{b}$ ....(iii)

Substituting from (i) and from (iii) in (ii) we get

$$\vec{x}=\frac{1}{a^2+k^2}\left[k\vec{b}+(\vec{a}\times\vec{b})+\frac{(\vec{a}\ .\ \vec{b})}{k}\ \vec{a}\right]$$

Self Practice Problems :

(47) Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$.

(48) Find the unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$.

(49) Prove that $\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}=(\vec{a}\ .\ \vec{a})\ (\vec{b}\times\vec{a})$.

(50) Given that $\vec{x}+\dfrac{1}{p^2}(\vec{p}\ .\ \vec{x})\ \vec{p}=\vec{q}$, show that $\vec{p}\ .\ \vec{x}=\dfrac{1}{2}\vec{p}\ .\ \vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

(51) If $\vec{x}\ .\ \vec{a}=0$, $\vec{x}\ .\ \vec{b}=0$ and $\vec{x}\ .\ \vec{c}=0$ for some non-zero vector , then show that $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$

(52) Prove that $\vec{r}=\dfrac{(\vec{r}\ .\ \vec{a})\ (\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}+\dfrac{(\vec{r}\ .\ \vec{b})\ (\vec{c}\times\vec{a})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}+\dfrac{(\vec{r}\ .\ \vec{c})\ (\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}$ where $\vec{a},\ \vec{b},\ \vec{c}$ are three non-coplanar vectors

Ans. (48) $\pm\dfrac{1}{\sqrt{2}}\ (-\hat{j}+\hat{k})$ (50) $\vec{x}=\vec{q}-\left(\dfrac{\vec{p}\ .\ \vec{q}}{2|\vec{p}|^2}\right)\vec{p}$

29. <u>Linear combinations</u> :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots..$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$ for any x, y, z..... ∈ R. We have the following results :

(i) If $\vec{a}, \vec{b}$ are non zero, non−collinear vectors, then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'\ ,\quad y = y'$

(ii) **Fundamental Theorem in plane :** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$
i.e. there exist some unique x, y ∈ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(iii) If $\vec{a}, \vec{b}, \vec{c}$ are non−zero, non−coplanar vectors, then
$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x'\ ,\ y = y'\ ,\ z = z'$

(iv) **Fundamental theorem in space:** Let $\vec{a}, \vec{b}, \vec{c}$ be non−zero, non−coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. there exist some unique x,y, z ∈ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(v) If $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are n non zero vectors and $k_1, k_2, \ldots.., k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots. + k_n\vec{x}_n = \vec{0} \ \Rightarrow\ k_1 = 0,\ k_2 = 0\ ,\ \ldots..,\ k_n = 0$, then we say that vectors $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are **linearly independent vectors**.

(vi) If $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 \ldots\ldots + k_r\vec{x}_r + \ldots\ldots + k_n\vec{x}_n = \vec{0}$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are said to be **linearly dependent vectors.**
If $k_r \neq 0$ then $\vec{x}_r$ is expressed as a linear combination of vectors
$\vec{x}_1, \vec{x}_2, \ldots\ldots.., \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots\ldots...., \vec{x}_n$

Note 1: If $k_r \neq 0$; $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 \ldots\ldots + k_r\vec{x}_r + \ldots\ldots + k_n\vec{x}_n = \vec{0}$

$\Rightarrow\ -k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots. + k_{r-1}.\vec{x}_{r-1} + k_{r+1}.\vec{x}_{r+1} + \ldots\ldots + k_n\vec{x}_n$

$\Rightarrow\ -k_r\dfrac{1}{k_r}\vec{x}_r = k_1\dfrac{1}{k_r}\vec{x}_1 + k_2\dfrac{1}{k_r}\vec{x}_2 + \ldots.. + k_{r-1}.\dfrac{1}{k_r}\vec{x}_{r-1} + \ldots.. + k_n\dfrac{1}{k_r}\vec{x}_n$

$\Rightarrow\ \vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots\ldots + c_{r-1}\vec{x}_{r-1} + c_{r+1}\vec{x}_{r+1} + \ldots\ldots. + c_n\vec{x}_n$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1, \vec{x}_2, \ldots\ldots.., \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots\ldots...., \vec{x}_n$
Hence $\vec{x}_r$ with $\vec{x}_1, \vec{x}_2, \ldots\ldots.., \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots\ldots., \vec{x}_n$ forms a linearly dependent set of vectors.

Note 2 :

☞ If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a **Linear Combination** of vectors $\hat{i}, \hat{j}, \hat{k}$. Also $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, in 3 dimensional space every set of four vectors is a linearly dependent system.

☞ $\hat{i}, \hat{j}, \hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = \vec{0}$
$\Rightarrow\ K_1 = K_2 = K_3 = 0$

☞ Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = \vec{0} \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq \vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

☞ If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. Conversely if $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ then the vectors are linearly independent.

Self Practice Problems :

(53) Given that $\hat{i} - \hat{j}$, $\hat{i} - 2\hat{j}$ are two vectors. Find a unit vector coplanar with these vectors and perpendicular to the first vector $\hat{i} - \hat{j}$. Find also the unit vector which is perpendicular to the plane of the two given vectors.

(54) If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$. Express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$ where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

(55) Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ in the form

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Ans. (53) $\pm(\hat{i} + \hat{j})\dfrac{1}{\sqrt{2}}$; $\hat{k}$ (54) $\vec{\beta}_1 = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$, $\vec{\beta}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$

30. Test of collinearity :

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0 = \vec{0}$, where x + y + z = 0.

31. Test of coplanarity :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where x + y + z + w = 0.

Example # 40 : Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors.

Solution : Let, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a} - \vec{b} + 3\vec{c} = x\left(\vec{a} + \vec{b} - 2\vec{c}\right) + y\left(\vec{a} + \vec{b} - 3\vec{c}\right)$, for some scalars x and y.

$\Rightarrow$ $2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\vec{a} + (x + y)\vec{b} + (-2x - 3y)\vec{c}$

$\Rightarrow$ 2 = x + y, –1 = x + y and 3 = – 2x – 3y.

Solving first and third of these equations, we get x = 9 and y = –7.

Clearly these values do not satisfy the second equation. Hence the given vectors are not coplanar.

Example # 41 : Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Solution : Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\overrightarrow{PR} + y\overrightarrow{PS}$

⇒ $-\vec{a}-5\vec{b}+4\vec{c} = x(\vec{a}+\vec{b}-\vec{c}) + y(-\vec{a}-9\vec{b}+7\vec{c})$

⇒ $-\vec{a}-5\vec{b}+4\vec{c} = (x-y)\vec{a} + (x-9y)\vec{b} + (-x+7y)\vec{c}$

⇒ $x-y=-1,\ x-9y=-5,\ -x+7y=4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.
These values also satisfy the third equation. Hence the given four points are coplanar.

Self Practice Problems :

(56) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=\vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point (P) at which AC and BD meet. Also find the ratio in which P divides AC and BD.

(57) Show that the vector $\vec{a}-\vec{b}+\vec{c}$, $\vec{b}-\vec{c}-\vec{a}$ and $2\vec{a}-3\vec{b}-4\vec{c}$ are non-coplanar, where $\vec{a}, \vec{b}, \bar{c}$ are any non-coplanar vectors.

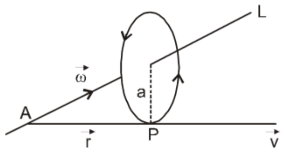
(58) Find the value of λ for which the four points with position vectors $-\hat{j}-\hat{k}$, $4\hat{i}+5\hat{j}+\lambda\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $-4\hat{i}+4\hat{j}+4\hat{k}$ are coplanar.

Ans. (56) $\vec{p} = \frac{3\vec{a}+\vec{c}}{4}$ divides AC in 1 : 3 and BD in 1 : 1 ratio (58) $\lambda = 1$

32. Application of vectors :

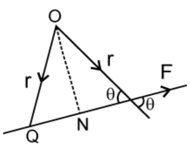
(i) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $W = \vec{F}.\vec{s}$

(ii) The tangential velocity $\vec{V}$ of a body moving in a circle is given by, $\vec{V} = \vec{\omega}$ x $\vec{r}$ where $\vec{r}$ is the position vector of the point P.



(iii) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r}$ x $\vec{F}$, where $\vec{r}$ is the position vector of P w.r.t. 'O'. The direction $\vec{M}$ of is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ & $\vec{M}$ form a right handed system.

(iv) Moment of the couple = $(\vec{r}_1-\vec{r}_2)$ x $\vec{F}$, where $\vec{r}_1$ and $\vec{r}_2$ are position vectors of the point of the application of the forces $\vec{F}$ and $-\vec{F}$.



Example # 42 : Forces of magnitudes 5 and 3 units acting in the directions $6\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+6\hat{k}$ respectively act on a particle which is displaced from the point (2, 2, –1) to (4, 3, 1). Find the work done by the forces.

Solution : Let $\vec{F}$ be the resultant force and $\vec{d}$ be the displacement vector. Then,

$$\vec{F} = 5\frac{(6\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{36+4+9}} + 3\frac{(3\hat{i}-2\hat{j}+6\hat{k})}{\sqrt{9+4+36}} = \frac{1}{7}(39\hat{i}+4\hat{j}+33\hat{k})$$

and $\vec{d} = (4\hat{i}+3\hat{j}+\hat{k}) - (2\hat{i}+2\hat{j}-\hat{k}) = 2\hat{i}+\hat{j}+2\hat{k}$

$\therefore$ Total work done $= \vec{F}.\vec{d} = \frac{1}{7}(39\hat{i}+4\hat{j}+33\hat{k}).(2\hat{i}+\hat{j}+2\hat{k})$

$= \frac{1}{7}(78+4+66) = \frac{148}{7}$ units.

Self Practice Problems :

(59) A point describes a circle uniformly in the $\hat{i}$, $\hat{j}$ plane taking 12 seconds to complete one revolution. If its initial position vector relative to the centre is $\hat{i}$ and the rotation is from $\hat{i}$ to $\hat{j}$, find the position vector at the end of 7 seconds. Also find the velocity vector.

(60) The force represented $3\hat{i}+2\hat{k}$ by is acting through the point $5\hat{i}+4\hat{j}-3\hat{k}$. Find its moment about the point $\hat{i}+3\hat{j}+\hat{k}$.

(61) Find the moment of the couple formed by the forces $5\hat{i}+\hat{k}$ and $-5\hat{i}-\hat{k}$ acting at the points (9, –1, 2) and (3, –2, 1) respectively

Ans. (59) $-\frac{1}{2}\left(\sqrt{3}\ \hat{i}+\hat{j}\right)$, $\frac{\pi}{12}\left(\hat{i}-\sqrt{3}\ \hat{j}\right)$ (60) $2\hat{i}-20\hat{j}-3\hat{k}$ (61) $\hat{i}-\hat{j}-5\hat{k}$

33. Miscellaneous solved examples :

Example # 43 : Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

Solution : We have,

$\overrightarrow{AB} = (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{BC} = (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

and, $\overrightarrow{CA} = (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

Since $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = (-\hat{i}-2\hat{j}-6\hat{k}) + (2\hat{i}-\hat{j}+\hat{k}) + (-\hat{i}+3\hat{j}+5\hat{k}) = \vec{0}$

So A, B and C are the vertices of a triangle.

Now, $\overrightarrow{BC}.\overrightarrow{CA} = (2\hat{i}-\hat{j}+\hat{k}).(-\hat{i}+3\hat{j}+5\hat{k}) = -2-3+5 = 0$

$\Rightarrow \overrightarrow{BC} \perp \overrightarrow{CA} \Rightarrow \angle BCA = \frac{\pi}{2}$ $\Rightarrow$ ABC is a right angled triangle.

Since A is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore

$$\cos A = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(-\hat{i}-2\hat{j}-6\hat{k}).(\hat{i}-3\hat{j}-5\hat{k})}{\sqrt{(-1)^2+(-2)^2+(-6)^2}\ \sqrt{1^2+(-3)^2+(-5)^2}}$$

$= \dfrac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \dfrac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\dfrac{35}{41}} \Rightarrow A = \cos^{-1}\sqrt{\dfrac{35}{41}}$

$\cos B = \dfrac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}||\vec{BC}|} = \dfrac{(\hat{i}+2\hat{j}+6\hat{k}) \cdot (2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+2^2+6^2}\sqrt{2^2+(-1)^2+(1)^2}}$

$\Rightarrow \quad \cos B = \dfrac{2-2+6}{\sqrt{41}\sqrt{6}} = \sqrt{\dfrac{6}{41}} \Rightarrow B = \cos^{-1}\sqrt{\dfrac{6}{41}}$

Example # 44 : If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Solution : Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$ ....(i)

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2 \quad [\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ....(ii)

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$\cos\theta_1 = \dfrac{\vec{a} \cdot (\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \dfrac{\vec{a} \cdot \vec{a}+\vec{a} \cdot \vec{b}+\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \dfrac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$

$= \dfrac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \dfrac{\lambda}{\sqrt{3}\lambda} = \dfrac{1}{\sqrt{3}}$ [Using (ii)] $\Rightarrow \quad \theta_1 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\therefore$ Similarly, $\theta_2 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Because $\theta_1 = \theta_2 = \theta_3$, so $\vec{a}+\vec{b}+\vec{c}$ is equally inclineded with $\vec{a}, \vec{b}$ and $\vec{c}$

Example # 45 : Using vectors : Prove that cos (A + B) = cos A cos B – sin A sin B

Solution : Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX andOY respectively. Let ∠XOP = A and ∠XOQ = B. Drawn PL ⊥ OX and QM ⊥ OX.Clearlyangle between $\vec{OP}$ and $\vec{OQ}$ is A + B

In ΔOLP, OL = OP cos A and LP = OP sin A. Therefore $\vec{OL}$ = (OP cos A) and

$\vec{LP}$ = (OP sin A) $(-\hat{j})$

Now, $\vec{OL} + \vec{LP} = \vec{OP}$

$\Rightarrow \vec{OP}$ = OP [(cos A) $\hat{i}$ – (sin A) $\hat{j}$] ......(i)

In ΔOMQ, OM = OQ cos B and MQ = OQ sin B.

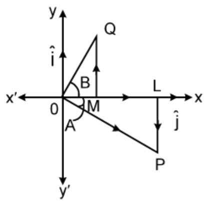
Therefore, $\vec{OM}$ = (OQ cos B) $\hat{i}$, $\vec{MQ}$ = (OQ sin B) $\hat{j}$

Now, $\vec{OM} + \vec{MQ} = \vec{OQ}$

$\Rightarrow \quad \vec{OQ} = OQ \; [(\cos B)\,\hat{i} + (\sin B)\hat{j}]$ ......(ii)

From (i) and (ii), we get

$\vec{OP} \,.\, \vec{OQ}$ = OP [(cos A) $\hat{i}$ – (sin A) $\hat{j}$] . OQ [(cos B) $\hat{i}$ + (sin B) $\hat{j}$]



= OP . OQ [cos A cos B – sin A sin B]

But, $\vec{OP} \,.\, \vec{OQ} = |\vec{OP}|\,|\vec{OQ}|$ cos (A + B) = OP . OQ cos (A + B)

$\therefore$ OP . OQ cos (A + B) = OP . OQ [cos A cos B – sin A sin B]

$\Rightarrow$ cos (A + B) = cos A cos B – sin A sin B

Example # 46 : Prove that in any triangle ABC

(i) $c^2 = a^2 + b^2 - 2ab \cos C$

(ii) c = bcosA + acosB.

Solution : (i) In ΔABC, $\vec{AB} + \vec{BC} + \vec{CA} = 0$

$\Rightarrow \quad \vec{BC} + \vec{CA} = -\vec{AB}$ ......(i)

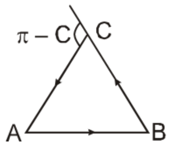
Squaring both sides

$(\vec{BC})^2 + (\vec{CA})^2 + 2\,(\vec{BC}).\vec{CA} = (\vec{AB})^2$

$\Rightarrow \quad a^2 + b^2 + 2\,(\vec{BC}.\vec{CA}) = c^2 \qquad \Rightarrow \qquad c^2 = a^2 + b^2 + 2\,ab \cos(\pi - C)$

$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab \cos C$

(ii) $(\vec{BC} + \vec{CA}).\vec{AB} = -\,\vec{AB}.\vec{AB}$



$\vec{BC}\,.\,\vec{AB} + \vec{CA}\,.\,\vec{AB} = -\,c^2$

$-\,ac \cos B - bc \cos A = -\,c^2$

acosB + bcosA = c.

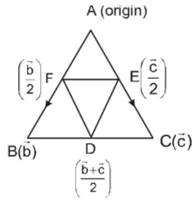
Example # 47 : If D, E, F are the mid-points of the sides of a triangle ABC, prove by vector method that area of ΔDEF = $\frac{1}{4}$ (area of ΔABC)

Solution : Taking A as the origin, let the position vectors of B and C be and respectively. Then the position vectors of D, E and F are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

Now, $\vec{DE} = \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{b}}{2}$ and $\vec{DF} = \frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{c}}{2}$

$\therefore$ Vector area of ΔDEF = $\frac{1}{2}(\vec{DE}\times\vec{DF}) = \frac{1}{2}\left(\frac{-\vec{b}}{2}\times\frac{-\vec{c}}{2}\right)$

$= \frac{1}{8}(\vec{b}\times\vec{c}) = \frac{1}{4}\left\{\frac{1}{2}(\vec{AB}\times\vec{AC})\right\} = \frac{1}{4}$ (vector area of ΔABC)

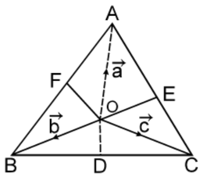


Hence area of ΔDEF $\frac{1}{4}$ = area of ΔABC.

Example # 48 : In any triangle, show that the perpendicular bisectors of the sides are concurrent.

Solution : Let ABC be the triangle and D, E and F are respectively middle points of sides BC, CA and AB. Let the perpendicular bisectors of BC and CA meet at O. Join OF. We are required to prove that OF is ⊥ to AB. Let the position vectors of A, B, C with O as origin of reference be $\vec a$, $\vec b$ and $\vec c$ respectively.

$\therefore \quad \overrightarrow{OD} = \frac{1}{2}(\vec b + \vec c)$, $\overrightarrow{OE} = \frac{1}{2}(\vec c + \vec a)$ and $\overrightarrow{OF} = \frac{1}{2}(\vec a + \vec b)$



Also $\overrightarrow{BC} = \vec c - \vec b$, $\overrightarrow{CA} = \vec a - \vec c$ and $\overrightarrow{AB} = \vec b - \vec a$

Since OD ⊥ BC

$\Rightarrow \frac{1}{2}(\vec b + \vec c)\,.\,(\vec c - \vec b) = 0 \qquad \Rightarrow b^2 = c^2$ ....(i)

Similarly OE ⊥ CA $\quad \Rightarrow \frac{1}{2}(\vec c + \vec a)\,.\,(\vec a - \vec c) = 0 \quad \Rightarrow \quad a^2 = c^2$ ....(ii)

from (i) and (ii) we have $a^2 - b^2 = 0$

$\Rightarrow \quad (\vec a + \vec b)\,.\,(\vec b - \vec a) = 0 \quad \Rightarrow \frac{1}{2}(\vec b + \vec a)\,.\,(\vec b - \vec a) = 0 \quad \Rightarrow \overrightarrow{OF} \perp \overrightarrow{AB}$.

Example # 49 : A, B, C, D are four points in space. using vector methods, prove that $AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2$ what is the implication of the sign of equality.

Solution : Let the position vector of A, B, C, D be $\vec a$, $\vec b$, $\vec c$ and $\vec d$ respectively then

$AC^2+BD^2 + AD^2 +BC^2 = (\vec c - \vec a).(\vec c - \vec a) + (\vec d - \vec b).(\vec d - \vec b) + (\vec d - \vec a).(\vec d - \vec a) + (\vec c - \vec b).(\vec c - \vec b) = |\vec c|^2$
$+ |\vec a|^2 - 2\vec a\,.\,\vec c + |\vec d|^2 + |\vec b|^2 - 2\vec d.\vec b + |\vec d|^2 + |\vec a|^2 - 2\vec a.\vec d + |\vec c|^2 + |\vec b|^2 - 2\vec b\,.\,\vec c$
$= |\vec a|^2 + |\vec b|^2 - 2\vec a.\vec b + |\vec c|^2 + |\vec d|^2 - 2\vec c.\vec d + |\vec a|^2 + |\vec b|^2 + |\vec c|^2 + |\vec d|^2$
$+ 2\vec a.\vec b + 2\vec c.\vec d - 2\vec a.\vec c - 2\vec b.\vec d - 2\vec a.\vec d - 2\vec b.\vec c$
$= (\vec a - \vec b).(\vec a - \vec b) + (\vec c - \vec d).(\vec c - \vec d) + (\vec a + \vec b - \vec c - \vec d)^2$
$= AB^2 + CD^2 + (\vec a + \vec b - \vec c - \vec d).(\vec a + \vec b - \vec c - \vec d) \ge AB^2 + CD^2$
$\Rightarrow \quad AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2$

for the sign of equality to hold, $\vec a + \vec b - \vec c - \vec d = 0$ or $\vec a - \vec c = \vec d - \vec b$

$\Rightarrow \quad \overrightarrow{AC}$ and $\overrightarrow{BD}$ are collinear, the four points A, B, C, D are collinear

34. <u>A plane and its equation</u> :

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

(i) The equation $(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$ represents a plane containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane. The above equation can also be written as $\vec{r}\cdot\vec{n}=d$, where d = $\vec{r}_0\cdot\vec{n}$

(ii) Angle between two planes is the angle between two normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

(iii) The length of perpendicular (p) from a point having position vector to the plane is given by p = $\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$

(iv) If $(\vec{r}-\vec{a})\cdot\vec{n}_1=0$ and $(\vec{r}-\vec{a})\cdot\vec{n}_2=0$ are the equations of two planes, then the equation of line of intersection of these planes is given by $\vec{r}=\vec{a}+\lambda\,(\vec{n}_1\times\vec{n}_2)$.

(v) Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m, n are the directioncosines of the normal to the plane and p is the distance of the plane from the origin.

(vi) General form: ax + by + cz + d = 0 is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

(vii) The equation of a plane passing through the point (x_1, y_1, z_1) is given by a $(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(viii) Plane through three points: The equation of the plane through three non−collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

(ix) Intercept Form: The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(x) Vector form: The equation of a plane passing through a point having position vector & normal to vector $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n} = 0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

Note: **(i)** Vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\vec{n} = d$

(ii) **Coordinate planes**

(a) Equation of yz−plane is x = 0

(b) Equation of xz−plane is $y = 0$

(c) Equation of xy−plane is $z = 0$

(iii) **Planes parallel to the axes:**

If $a = 0$, the plane is parallel to x−axis i.e. equation of the plane parallel to the x−axis is $by + cz + d = 0$.

Similarly, equation of planes parallel to y−axis and parallel to z−axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

(iv) **Plane through origin :** Equation of plane passing through origin is $ax + by + cz = 0$.

(v) **Transformation of the equation of a plane to the normal form:**

To reduce any equation $ax + by + cz - d = 0$ to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2+b^2+c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}} + \frac{by}{\pm\sqrt{a^2+b^2+c^2}} + \frac{cz}{\pm\sqrt{a^2+b^2+c^2}} = \frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

Where (+) sign is to be taken if $d > 0$ and (−) sign is to be taken if $d < 0$.

(vi) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$. Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(vii) **Equation of a plane passing through a given point & parallel to the given vectors:**

The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ & $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars. or $\vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$ (non parametric form)

(viii) A plane $ax + by + cz + d = 0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

(ix) The xy−plane divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\frac{z_1}{z_2}$. Similarly yz−plane in $-\frac{x_1}{x_2}$ and zx−plane in $-\frac{y_1}{y_2}$

(x) Coplanarity of four points the points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplaner then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$$

very similar in vector method the points A ($\vec{r}_1$), B($\vec{r}_2$), C($\vec{r}_3$) and D($\vec{r}_4$) are coplanar if $[\vec{r}_4 - \vec{r}_1,\ \vec{r}_4 - \vec{r}_2,\ \vec{r}_4 - \vec{r}_3] = 0$

Example # 50 : Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Solution : If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = p$ ..... (1)

Here p = 10

Direction ratios of normal to the plane are 3, 2, 6

$\sqrt{3^2 + 2^2 + 6^2} = 7$

$\therefore$ Direction cosines of normal to the required plane are $\ell = \frac{3}{7}$, $m = \frac{2}{7}$, $n = \frac{6}{7}$

Putting the values of ℓ, m, n, p in (1), equation of required plane is

$\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$ or, $3x + 2y + 6z = 70$

Example # 51 : Show that the points (0, – 1, 0), (2, 1, – 1), (1, 1, 1), (3, 1, –3) are coplanar.

Solution : Let A ≡ (0, – 1, 0), B ≡ (2, 1, – 1), C ≡ (1, 1, 1) and D ≡ (3, 1, –3)

Equation of a plane through A (0, – 1, 0) is

$a(x - 0) + b(y + 1) + c(z - 0) = 0$

or, $ax + by + cz + b = 0$ ..... (1)

If plane (1) passes through B (2, 1, – 1) and C (1, 1, 1)

Then $2a + 2b - c = 0$ ..... (2)

and $a + 2b + c = 0$ ..... (3)

From (2) and (3), we have $\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2}$

or, $\frac{a}{4} = \frac{b}{-3} = \frac{c}{2} = k$ (say)

Putting the value of a, b, c, in (1), equation of required plane is

$4kx - 3k(y + 1) + 2kz = 0$

or, $4x - 3y + 2z - 3 = 0$ ..... (2)

Clearly point D (3, 1, –3) lies on plane (2)

Thus point D lies on the plane passing through A, B, C and hence points A, B, C and D are coplanar.

Example # 52 : If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that OP . OQ = p_2, show that the locus of the point Q is $p(\ell x + my + nz) = x_2 + y_2 + z_2$.

Solution : Let P ≡ (α, β, γ), Q ≡ (x_1, y_1, z_1)

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have

$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k$ (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(\ell x_1 + my_1 + nz_1) = p$ ..... (2)

Given OP . OQ = p_2

$\therefore$ $\sqrt{\alpha^2 + \beta^2 + \gamma^2}\ \sqrt{x_1^2 + y_1^2 + z_1^2} = p_2$

or, $\sqrt{k^2(x_1^2 + y_1^2 + z_1^2)}\ \sqrt{x_1^2 + y_1^2 + z_1^2} = p_2$

or, $k(x_1^2 + y_1^2 + z_1^2) = p_2$ ..... (3)

On dividing (2) by (3), we get $\frac{\ell x_1 + my_1 + nz_1}{x_1^2 + y_1^2 + z_1^2} = \frac{1}{p}$

or, $p(\ell x_1 + my_1 + nz_1) = x_1^2 + y_1^2 + z_1^2$

Hence the locus of point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Example # 53 : A point P moves on a plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the co-ordinate axes in A, B and C. If the planes through A, B and C parallel to the planes x = 0, y = 0, z = 0 intersect in Q, find the locus of Q.

Solution : Given plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ..... (1)

Let P ≡ (h, k, ℓ)

Then $\frac{h}{a}+\frac{k}{b}+\frac{\ell}{c} = 1$ ..... (2)

OP = $\sqrt{h^2+k^2+\ell^2}$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

∴ Equation of the plane through P and normal to OP is

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = (h^2 + k^2 + \ell^2)$

∴ $A \equiv \left(\frac{h^2+k^2+\ell^2}{h}, 0, 0\right)$, $B \equiv \left(0, \frac{h^2+k^2+\ell^2}{k}, 0\right)$, $C \equiv \left(0, 0, \frac{h^2+k^2+\ell^2}{\ell}\right)$

Let Q ≡ (α, β, γ), then $\alpha = \frac{h^2+k^2+\ell^2}{h}$, $\beta = \frac{h^2+k^2+\ell^2}{k}$, $\gamma = \frac{h^2+k^2+\ell^2}{\ell}$ ..... (3)

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{h^2+k^2+\ell^2}{(h^2+k^2+\ell^2)^2} = \frac{1}{(h^2+k^2+\ell^2)}$ ..... (4)

From (3), h = $\frac{h^2+k^2+\ell^2}{\alpha}$

$$\therefore \frac{h}{a} = \frac{h^2+k^2+\ell^2}{a\alpha}$$

Similarly $\frac{k}{b} = \frac{h^2+k^2+\ell^2}{b\beta}$ and $\frac{\ell}{c} = \frac{h^2+k^2+\ell^2}{c\gamma}$

∴ $\frac{h^2+k^2+\ell^2}{a\alpha}+\frac{h^2+k^2+\ell^2}{b\beta}+\frac{h^2+k^2+\ell^2}{c\gamma} = \frac{h}{a}+\frac{k}{b}+\frac{\ell}{c} = 1$ [from (2)] or, $\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+\ell^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ [from (4)]

$\therefore$ Required locus of Q (α, β, γ) is $\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$.

Self practice problems :

(62) Check whether given points are coplanar if yes find the equation of plane containing them A ≡ (1, 1, 1), B ≡ (0, – 1, 0), C ≡ (2, 1, –1), D ≡ (3, 3, 0)

(63) Find the plane passing through point (– 3, – 3, 1) and perpendicular to the line joining the points (2, 6, 1) and (1, 3, 0).

(64) Find the equation of plane parallel to plane x + 5y – 4z + 5 = 0 and cutting intercepts on the axes whose sum is 150.

(65) Find the equation of plane passing through (2, 2, 1) and (9, 3, 6) and perpendicular to the plane x + 3y + 3z = 8.

(66) Find the equation of the plane parallel to $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}$ and passing through (1, 1, 2).

(67) Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes x + 2y + 3z – 7 = 0 and 2x – 3y + 4z = 0.

Ans. (62) yes, 4x – 3y + 2z = 3 (63) x + 3y + z + 11 = 0
(64) $x + 5y - 4z = \frac{3000}{19}$ (65) 3x + 4y – 5z = 9
(66) x + y – 2z + 2 = 0 (67) 17x + 2y – 7z = 26

35. Sides of a plane :

A plane divides the three dimensional space in two equal parts. Two points A (x_1 y_1 z_1) and B (x_2 y_2 z_2) are on the same side of the plane ax + by + cz + d = 0 if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Example # 54 : Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane x + 4y + z – 3 = 0.

Solution : Since the numbers 1+ 4 × 2 + 3 – 3 = 9 and 2 – 4 + 4 – 3 = – 1 are of opposite sign, then points are on opposite sides of the plane.

36. A plane & a point :

(i) Distance of the point (x', y', z') from the plane ax + by + cz+ d = 0 is given by $\left|\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}\right|$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}.\vec{n} = d$ is $p = \frac{|\vec{a}\ .\ \vec{n} - d|}{|\vec{n}|}$.

(iii) The coordinates of the foot of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(iv) **To find image of a point w.r.t. a plane.**

Let P (x_1, y_1, z_1) is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image of the point, then

(a) $x' - x_1 = \lambda a$, $y' - y_1 = \lambda b$, $z' - z_1 = \lambda c$

$\Rightarrow$ $x' = \lambda a + x_1$, $y' = \lambda b + y_1$, $z' = \lambda c + z_1$ ...... (i)

(b) $a\left(\frac{x' + x_1}{2}\right) + b\left(\frac{y' + y_1}{2}\right) + c\left(\frac{z' + z_1}{2}\right) = 0$ ...... (ii)

from (i) put the values of x', y', z' in (ii) and get the values of λ and subtitute in (i) to get (x' y' z').

The coordinate of the image of point (x_1, y_1, z_1) w.r.t the plane $ax + by + cz + d = 0$ are given by $\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = -2\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(v) The distance between two parallel planes $ax + by + cx + d = 0$ and $ax + by + cx + d' = 0$ is $\frac{|d - d'|}{\sqrt{a^2 + b^2 + c^2}}$

Example # 55 : Find the image of the point P (3, 5, 7) in the plane $2x + y + z = 0$.

Solution : Given plane is $2x + y + z = 0$ ..... (1)

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R then PQ $\perp$ plane (1)

Let R ≡ (2r + 3, r + 5, r + 7)

Since R lies on plane (1)

$\therefore$ $2(2r + 3) + r + 5 + r + 7 = 0$ or, $6r + 18 = 0$ $\therefore$ $r = -3$

$\therefore$ R ≡ (– 3, 2, 4)

Let Q ≡ (α, β, γ)

Since R is the middle point of PQ

$\therefore$ $-3 = \frac{\alpha + 3}{2}$ $\Rightarrow$ $\alpha = -9$ and $2 = \frac{\beta + 5}{2}$ $\Rightarrow$ $\beta = -1$ and $4 = \frac{\gamma + 7}{2}$ $\Rightarrow$ $\gamma = 1$

$\therefore$ Q = (– 9, – 1, 1).

Example # 56 : Find the distance between the planes $2x - y + 2z = 4$ and $6x - 3y + 6z = 9$.

Solution : Given planes are $2x - y + 2z - 4 = 0$ .....(1)

and $6x - 3y + 6z - 9 = 0$ .....(2)

We find that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence planes (1) and (2) are parallel.

Plane (2) may be written as $2x - y + 2z - 3 = 0$ ..... (3)

$\therefore$ Required distance between the planes $= \frac{|4-3|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{1}{3}$

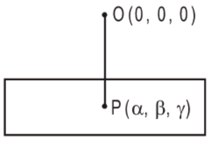
Example # 57 : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$

Solution : Let the equation of the variable plane be $\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c) $\therefore$ $\ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are



$\alpha - 0, \beta - 0, \gamma - 0$ i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n;

α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$\therefore$ $\frac{\alpha}{\ell} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{1}{k}$ (say) $\therefore$ $\ell = k\alpha, m = k\beta, n = k\gamma$ ..... (3)

Putting the values of ℓ, m, n in equation (2), we get $ka\alpha + kb\beta + kc\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1) $\therefore$ $\ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$ [putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Self practice problems :

(68) Find the intercepts of the plane $3x + 4y - 7z = 84$ on the axes. Also find the length of perpendicular from origin to this plane and direction cosines of this normal.

(69) Find : (i) perpendicular distance
(ii) foot of perpendicular
(iii) image of (1, 0, 2) in the plane $2x + y + z = 5$

Ans. (68) a = 28, b = 21, c = – 12, p = $\frac{84}{\sqrt{74}}$; $\frac{3}{\sqrt{74}}, \frac{4}{\sqrt{74}}, \frac{-7}{\sqrt{74}}$

(69) (i) $\frac{1}{\sqrt{6}}$ (ii) $\left(\frac{4}{3}, \frac{1}{6}, \frac{13}{6}\right)$ (iii) $\left(\frac{5}{3}, \frac{1}{3}, \frac{7}{3}\right)$

37. Angle between two planes :

(i) Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a', b', c') respectively, hence θ, the angle between them, is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(ii) The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\ |\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$.

38. Angle bisectors :

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(ii) Equation of bisector of the angle containing origin: First make both the constant terms positive. Then the positive sign in $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ gives the bisector of the angle which contains the origin.

(iii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

39. Family of planes :

(i) Any plane passing through the line of intersection of non−parallel planes or equation of the plane through the given line in non symmetric form.
$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is
$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is. $\vec{r}\cdot(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Example # 58 : The plane $x + y + z = 4$ is rotated through 90° about its line of intersection with the plane $x + y + 2z = 4$. Find its equation in the new position.

Solution : Given planes are $x + y + z = 4$ ..... (1)
and $x + y + 2z = 4$ ..... (2)

Since the required plane passes through the line of intersection of planes (1) and (2)
∴ its equation may be taken as
$x + y + 2z - 4 + k(x + y + z - 4) = 0$
or $(1 + k)x + (1 + k)y + (2 + k)z - 4 - 4k = 0$ ..... (3)
Since planes (1) and (3) are mutually perpendicular,
∴ $(1 + k) + (1 + k) + (2 + k) = 0$

or, $4 + 3k = 0$ or $k = -\frac{4}{3}$

Putting $k = -\frac{4}{3}$ in equation (3), we get $-x - y + 2z - 28 = 0$
This is the equation of the required plane.

Example # 59 : Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Solution : Given planes are $x + y + z - 6 = 0$ ..... (1)
and $2x + 3y + 4z + 5 = 0$ ..... (2)
Given point is P (1, 1, 1).
Equation of any plane through the line of intersection of planes (1) and (2) is
$x + y + z - 6 + k (2x + 3y + 4z + 5) = 0$ ..... (3)
If plane (3) passes through point P, then
$1 + 1 + 1 - 6 + k (2 + 3 + 4 + 5) = 0$ or, $k = \frac{3}{14}$
From (3) required plane is $20x + 23y + 26z - 69 = 0$

Example # 60 : Find the planes bisecting the angles between planes $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Solution : Given planes are $-2x - y - 2z + 9 = 0$ ..... (1)
and $3x - 4y + 12z + 13 = 0$ ..... (2)
Equations of bisecting planes are $\dfrac{-2x-y-2z+9}{\sqrt{(-2)^2+(-1)^2+(-2)^2}} = \pm \dfrac{3x-4y+12z+13}{\sqrt{3^2+(-4)^2+(12)^2}}$
or, $13 [-2x - y - 2z + 9] = \pm 3 (3x - 4y + 12z + 13)$
or, $35x + y + 62z = 78$, ... (3) [Taking +ve sign]
and $17x + 25y - 10z = 156$ ..... (4) [Taking – ve sign]
Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)(3) + (-1)(-4) + (-2)(12) = -6 + 4 - 24 = -26 < 0$
$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$
$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example # 61 : If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Solution : Given planes are $x - cy - bz = 0$ ..... (1)
$cx - y + az = 0$ ..... (2)
$bx + ay - z = 0$ ..... (3)
Equation of any plane passing through the line of intersection of planes
(1) and (2) may be
taken as $x - cy - bz + \lambda (cx - y + az) = 0$
or, $x (1 + \lambda c) - y (c + \lambda) + z (-b + a\lambda) = 0$ ..... (4)
If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$\therefore$ $\dfrac{1+c\lambda}{b} = \dfrac{-(c+\lambda)}{a} = \dfrac{-b+a\lambda}{-1}$
(i) (ii) (iii)

From (i) and (ii), $a + ac\lambda = -bc - b\lambda$
or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... (5)
From (ii) and (iii),
or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... (5)
From (ii) and (iii),

$c + \lambda = -ab + a^2\lambda$ or $\lambda = \dfrac{-(ab+c)}{1-a^2}$ ..... (6)

From (5) and (6), we have $\dfrac{-(a+bc)}{ac+b} = \dfrac{-(ab+c)}{(1-a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$ or $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$

or, $a^2 + b^2 + c^2 + 2abc = 1$.

Self practice problems :

(70) A tetrahedron has vertices at O(0, 0, 0), A(1, 2, 1), B(2, 1, 3) and C(–1, 1, 2). Prove that the angle between the faces OAB and ABC will be $\cos^{-1}\left(\dfrac{19}{35}\right)$.

(71) Find the equation of plane passing through the line of intersection of the planes $4x - 5y - 4z = 1$ and $2x + y + 2z = 8$ and the point (2, 1, 3).

(72) Find the equations of the planes bisecting the angles between the planes $x + 2y + 2z - 3 = 0$, $3x + 4y + 12z + 1 = 0$ and sepecify the plane which bisects the acute angle between them.

(73) Prove that the planes $12x - 15y + 16z - 28 = 0$, $6x + 6y - 7z - 8 = 0$ and $2x + 35y - 39z + 12 = 0$ have a common line of intersection.

Ans. (71) $32x - 5y + 8z - 83 = 0$

(73) $2x + 7y - 5z = 21$, $11x + 19y + 31z = 18$; $2x + 7y - 5z = 21$

40. Area of a triangle :

Let A (x_1, y_1, z_1), B (x_2, y_2, z_2), C (x_3, y_3, z_3) be the vertices of a triangle, then $\Delta = \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)}$

where $\Delta_x = \dfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y \dfrac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$ = and $\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Vector Method – From two vector $\vec{AB}$ and $\vec{AC}$. Then area is given by

$$\frac{1}{2}\,|\vec{AB} \times \vec{AC}| = \frac{1}{2}\begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Example # 62 : Through a point P (h, k, ℓ) a plane is drawn at right angles to OP to meet the co-ordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\left|\dfrac{p^5}{2hk\ell}\right|$, where O is the origin.

Solution : $OP = \sqrt{h^2 + k^2 + \ell^2} = p$

Direction cosines of OP are $\dfrac{h}{\sqrt{h^2+k^2+\ell^2}}, \dfrac{k}{\sqrt{h^2+k^2+\ell^2}}, \dfrac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$ ..... (1)

$\therefore$ $A \equiv \left(\frac{p^2}{h}, 0, 0\right)$, $B \equiv \left(0, \frac{p^2}{k}, 0\right)$, $C \equiv \left(0, 0, \frac{p^2}{\ell}\right)$

Now area of ΔABC, $\Delta = \sqrt{A^2_{xy} + A^2_{yz} + A^2_{zx}}$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|} \text{ Similarly, } A_{yz} = \frac{1}{2}\frac{p^4}{|k\ell|} \text{ and } A_{zx} = \frac{1}{2}\frac{p^4}{|\ell h|}$$

$$\therefore \quad \Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2\ell^2} + \frac{1}{4}\frac{p^8}{\ell^2h^2} = \frac{p^8}{4h^2k^2\ell^2}(\ell^2 + k^2 + h^2) = \frac{p^{10}}{4h^2k^2\ell^2} \text{ or } \Delta = \left|\frac{p^5}{2hk\ell}\right|.$$

41. Volume of a tetrahedron :

Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) is V

$$= \text{modulus of } \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

42. Angle between a plane and a line :

(i) If θ is the angle between line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\ell^2 + m^2 + n^2}}\right].$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}\right]$.

(iii)	Condition for perpendicularity	$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$	$\vec{b} \times \vec{n} = 0$
(iv)	Condition for parallel	$a\ell + bm + cn = 0$	$\vec{b}\cdot\vec{n} = 0$

43. Condition for a line to lie in a plane :

(i) Cartesian form:

Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form:

Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$

44. Coplanar lines :

(i) If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection/coplanarity is $\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ & equation of plane containing the above two lines is $\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ or $\begin{vmatrix} x-\alpha' & y-\beta' & z-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$

(ii) Condition of coplanarity if both the lines are in general form Let the lines be

$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ &

$\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Alternative method

get vector along the line of shortest distance as

$$\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}$$

Now get unit vector along $\vec{u}$, let it be $\hat{u}$

Let $\vec{v} = (\alpha - \alpha')\hat{i} + (\beta - \beta')\hat{j} + (\gamma - \gamma')\hat{k}$

S. D. $= \hat{u}.\vec{v}$

Example # 63 : Find the distance of the point (1, 0, – 3) from the plane x – y – z = 9 measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Solution : Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point Q(1, 0, – 3) and parallel to line (2) is

$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r.$ ..... (3)

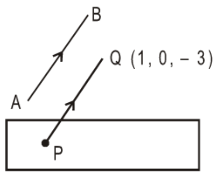
Co-ordinates of any point on line (3) may be taken as

P (2r + 1, 3r, – 6r – 3)

If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

$\therefore$ $(2r + 1) - (3r) - (-6r - 3) = 9$, $r = 1$ or $P \equiv (3, 3, -9)$

Distance between points Q (1, 0, – 3) and P (3, 3, – 9)



$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-3))^2} = \sqrt{4+9+36} = 7.$

Example # 64 : Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Solution : Equation of any plane passing through (1, 2, 0) may be taken as

$a(x-1) + b(y-2) + c(z-0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ $a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13} = k$ (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6(x-1) + 2(y-2) + 13(z-0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example # 65 : Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 9$.

Solution : Let the given line AB be $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ ..... (1)

Given plane is $x + 2y + z = 9$ ..... (2)

Let DC be the projection of AB on plane (2)

Clearly plane ABCD is perpendicular to plane (2).

Equation of any plane through AB may be taken as (this plane passes through the point (1, – 1, 3) on line AB)

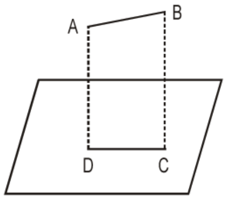
$a(x-1) + b(y+1) + c(z-3) = 0$ ..... (3)

where $2a - b + 4c = 0$ ..... (4)

[$\because$ normal to plane (3) is perpendicular to line (1)]

Since plane (3) is perpendicular to plane (2),

$\therefore$ $a + 2b + c = 0$ ..... (5)



Solving equations (4) & (5), we get $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$.

Substituting these values of a, b and c in equation (3), we get

9 (x – 1) – 2 (y + 1) – 5 (z – 3) = 0

or, 9x – 2y – 5z + 4 = 0 ...... (6)

Since projection DC of AB on plane (2) is the line of intersection of plane ABCD and plane (2), therefore equation of DC will be

$$\left.\begin{aligned} &9x - 2y - 5z + 4 = 0 \quad .....(i) \\ \text{and } &x + 2y + z - 9 = 0 \quad .....(ii) \end{aligned}\right\} \quad ..... (7)$$

Let ℓ, m, n be the direction ratios of the line of intersection of planes (i) and (ii)

∴ $9\ell - 2m - 5n = 0$ ..... (8) and $\ell + 2m + n = 0$ ..... (9)

∴ $\frac{\ell}{-2+10} = \frac{m}{-5-9} = \frac{n}{18+2}$ ⇒ $\frac{\ell}{4} = \frac{m}{-7} = \frac{n}{10}$

Let any point on line (7) is (α, β, 0) ⇒ 9α – 2β + 4 = 0

α + 2β – 9 = 0 ⇒ α = $\frac{1}{2}$, β = $\frac{17}{4}$

So equation of line is $\frac{x-\frac{1}{2}}{4} = \frac{y-\frac{17}{4}}{-7} = \frac{z-0}{10}$

Example # 66 : Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.

Solution : Given lines are $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ = r (say)..... (1) and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ = R (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.

Any point on line (1) may be taken as (2r + 3, – 3r – 1, r – 2) = P (let).

Any point on line (2) may be taken as (– 3R + 7, R, 2R – 7) = P (let).

∴ 2r + 3 = – 3R + 7 or, 2r + 3R = 4 ..... (3)

Also – 3r – 1 = R or, – 3r – R = 1 ..... (4)

and r – 2 = 2R – 7 or, r – 2R = – 5. ..... (5)

Solving equations (3) and (4), we get, r = – 1, R = 2

Clearly r = – 1, R = 2 satisfies equation (5).

Hence lines (1) and (2) intersect.

∴ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is $\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$

or, $(x-3)(-6-1)-(y+1)(4+3)+(z+2)(2-9)=0$

or, $-7(x-3)-7(y+1)-7(z+2)=0$ or, $x-3+y+1+z+2=0$ or, $x+y+z=0$.

Self practice problems:

(74) Find the values of a and b for which the line $\frac{x-2}{a}=\frac{y+3}{4}=\frac{z-6}{-2}$ is perpendicular to the plane $3x-2y+bz+10=0$.

(75) Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar. Also find the equation of the plane in which they lie.

(76) Find the plane containing the line $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{5}$ and parallel to the line $\frac{x+1}{1}=\frac{y-1}{-2}=\frac{-z+1}{1}$

(77) Show that the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ & $\frac{x-4}{5}=\frac{y-1}{2}=z$ are intersecting each other. Find their intersection point and the plane containing the line.

(78) Show that the lines $\vec{r}=(\hat{i}+3\hat{j}+5\hat{k})+\lambda(3\hat{i}+5\hat{j}+7\hat{k})$ & $\vec{r}=(2\hat{i}+4\hat{j}-6\hat{k})+\mu(\hat{i}-4\hat{j}+7\hat{k})$ are coplanar and find the plane containing the line.

Ans. (74) $a=-6,\ b=1$ (75) $3x-y-z+2=0$

(76) $13x+3y-7z-7=0$ (77) $(-1,-1,-1)$ & $5x-18y+11z-2=0$

(78) $\bar{r}.(\hat{i}-2\hat{j}+\hat{k})=0$

45. Reduction of non-symmetrical form to symmetrical form :

Let equation of the line in non–symmetrical form be $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$. To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

(i) Direction ratios:

Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes.

So $a_1\ell+b_1m+c_1n=0$, $a_2\ell+b_2m+c_2n=0$. From these equations, proportional values of ℓ, m, n can be found by cross–multiplication as $\frac{\ell}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$

Alternative method

The vector $\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ = i ($b_1c_2 - b_2c_1$) + j ($c_1a_2 - c_2a_1$) + k ($a_1b_2 - a_2b_1$) will be parallel to the line of intersection of the two given planes. hence ℓ : m: n = ($b_1c_2 - b_2c_1$): ($c_1a_2 - c_2a_1$): ($a_1b_2 - a_2b_1$)

(ii) Point on the line :

Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non−zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it. Let it intersect xy−plane in (x_1, y_1, 0). Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$.Solving these,we get a point on the line. Then its equation becomes.

$$\frac{x-x_1}{b_1c_2-b_2c_1} = \frac{y-y_1}{c_1a_2-c_2a_1} = \frac{z-0}{a_1b_2-a_2b_1} \text{ or } \frac{x-\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1} = \frac{y-\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1} = \frac{z-0}{a_1b_2-a_2b_1}$$

Note: If $\ell \neq 0$, take a point on yz−plane as (0, y_1, z_1) and if m ≠ 0, take a point on xz−plane as (x_1, 0, z_1).

Alternative method

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ Put z = 0 in both the equations and solve the equations $a_1x + b_1y + d_1 = 0$,$a_2x + b_2y + d_2 = 0$ otherwise Put y = 0 and solve the equations $a_1x + c_1z + d_1 = 0$ and $a_2x + c_2z + d_2 = 0$

Example # 67 : Find the eqatuion of line x + y – z – 3 = 0 = 2x + 3y + z + 4 in symmetric from?

Solution : Let ℓ, m, n be direction ratio of line

then $2\ell + 3m + n = 0$ and $\ell + m - n = 0$

$$\therefore \quad \frac{\ell}{-3-1} = \frac{m}{1+2} = \frac{n}{2-3} \text{ or } \frac{\ell}{-4} = \frac{m}{3} = \frac{n}{-1} \text{ or } \frac{\ell}{4} = \frac{m}{-3} = \frac{n}{1}$$

let x = 0 then $y - z = 3$

$3y + z = -4$

solving that we get y = $\frac{-1}{4}$; z = $\frac{-13}{4}$

line is passing through point $\left(0, -\frac{1}{4}, -\frac{13}{4}\right)$ and having direction ratios 4, –3 and 1. $\frac{x-0}{4} = \frac{y+\frac{1}{4}}{-3} = \frac{z+\frac{13}{4}}{1}$

Example # 68 : Find the angle between the lines x – 3y – 4 = 0, 4y – z + 5 = 0 and x + 3y – 11 = 0, 2y – z + 6 = 0.

Solution : Given lines are $\left.\begin{matrix} x-3y-4=0 \\ 4y-z+5=0 \end{matrix}\right\}$ ..... (1)

and $\left.\begin{matrix} x+3y-11=0 \\ 2y-z+6=0 \end{matrix}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes

x – 3y – 4 = 0 and 4y – z + 5 = 0

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3)

and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$

or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4}$ = k (let).

Since line (2) is perpendicular to the normals of each of the planes

x + 3y – 11 = 0 and 2y – z + 6 = 0,

$\therefore$ $\ell_2 + 3m_2 = 0$ ..... (5)

and $2m_2 - n_2 = 0$ ..... (6)

$\therefore$ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore$ $\frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2}$ = t (let).

If θ be the angle between lines (1) and (2), then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

= (3k) (– 3t) + (k) (t) + (4k) (2t) = – 9kt + kt + 8kt = 0

$\therefore$ θ = 90°.

Self Practice problems :

(79) Find the equation of the line of intersection of the plane
4x + 4y – 5z = 12, 8x + 12y – 13z = 32.

(80) Prove that the three planes 2x + y – 4z – 17 = 0, 3x + 2y – 2z – 25 = 0, 2x – 4y + 3z + 25 = 0 intersect at a point and find its co-ordinates.

Ans. (79) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-0}{4}$ (80) (3, 7, – 1)