Exercise-1

Section (A) : Basic Problems (Definition based, Substitution, By parts)

A-1. **Sol.** $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int_0^1 \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}dx$

$= \left[\frac{(x+1)^{3/2}}{3/2} - \frac{x^{3/2}}{3/2}\right]_0^1$

$= \frac{2}{3}[2\sqrt{2}-1-1+0]$

$= \frac{2}{3}(2\sqrt{2}-2) = \frac{4}{3}(\sqrt{2}-1)$

A-2. **Sol.** $\int_0^1 \underset{I}{x}\,\underset{II}{e^x}dx = \left[xe^x\right]_0^1 - \int_0^1 1.e^x dx$

= (e – 0) –

= e – 0 – e + 1

= 1

A-3. **Sol.** $\int_0^1 \frac{dx}{(x^2+1)(x^2+2)} = \int_0^1 \left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)dx$

$= \left[\tan^{-1}x - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^1$

$= \tan^{-1}1 - \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} - 0 + 0$

$= \frac{\pi}{4} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$

A-4. **Sol.** $I = \int_0^{\pi/4} \tan^2 x dx$

$\int_0^{\pi/4} (\sec^2 x - 1)dx = (\tan - x)_0^{\pi/4} = 1 - \frac{\pi}{4}$

A-5. **Sol.** $\int_0^2 \frac{3^{\sqrt{x}}}{\sqrt{x}} dx$

Put $\sqrt{x} = t \Rightarrow \frac{dx}{2\sqrt{x}} = dt$

$2\int_0^{\sqrt{2}} 3^t dt = 2\left[\frac{3^t}{\log 3}\right]_0^{\sqrt{2}} = \frac{2}{\ell n3}(3^{\sqrt{2}} - 1)$

A-6. **Sol.** $\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx = \int_0^{\pi/2} (\sin x + \cos x)dx$

$= \left[-\cos x + \sin x\right]_0^{\pi/2} = 2$

A-7. Sol. $\int_0^{\pi/4} x\tan x\sec^2 x\,dx = \left(x\frac{\tan^2 x}{2}\right)_0^{\pi/4} - \int_0^{\pi/4}\frac{\tan^2 x}{2}dx$

$= \frac{\pi}{4} - \frac{1}{2}$

A-8. Sol. $\int_{\ell n\pi - \ell n2}^{\ell n\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}dx = \frac{3}{2}\int_{\pi/3}^{2\pi/3}\frac{dt}{1-\cot} = \frac{3}{2}\int_{\pi/3}^{2\pi/3}\frac{dt}{1-(1-2\sin^2 t)}$

$= \frac{3}{2.2}\int_{\pi/3}^{2\pi/3}\operatorname{cosec}^2 t\;\;dt = \frac{3}{4}[-\cot\;\;t]_{\pi/3}^{2\pi/3}$

$= -\frac{3}{4}\left[-\sqrt{3}\;\;-\frac{1}{\sqrt{3}}\right] = \frac{3}{4}\left(\frac{4}{\sqrt{3}}\right) = \sqrt{3}$

A-9. Sol. $\int_1^2 e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\;dx$

$\therefore$ $\left.\frac{e^x}{x}\right|_1^2 = e\left(\frac{e}{2}-1\right)$

A-10. Sol. Let $I = \int_0^\infty e^{-ax^2}dx$

Put $\sqrt{a}\;\;x = t$ $\Rightarrow$ $dx = \frac{dt}{\sqrt{a}}$

then $I = \frac{1}{\sqrt{a}}\int_0^\infty e^{-t^2}dt = \frac{1}{\sqrt{a}}\int_0^\infty e^{-x^2}dx = \frac{1}{\sqrt{a}}\;\frac{\sqrt{\pi}}{2} = \frac{1}{2}\;\sqrt{\frac{\pi}{a}}$

A-11. Sol. $I_1 = \int_e^{e^2}\frac{dx}{\ell nx}$

Let $\ell nx = t$ $\Rightarrow$ $x = e^t$ $\Rightarrow$ $dx = e^t dt$

$I_1 = \int_1^2 \frac{e^t}{t}dt = I_2$

A-12. Sol. Given that

$\frac{d}{dx}\;f(x) = g(x)$

$I = \int_a^b f(x)g(x)\;\;dx = \left[\frac{f^2(x)}{2}\right]_a^b$

$= \frac{f^2(b)-f^2(a)}{2}$

A-13 Sol. $5 < x < 10$ $\Rightarrow$ $0 < x - 5 < 5$ $\Rightarrow$ $0 < \frac{x-5}{5} < 1$

$\therefore \quad \left[\frac{x-5}{5}\right] = 0$

A-14. **Sol.** $-\tan 1 < x < 0 \Rightarrow \tan(-1) < x < 0 \Rightarrow -1 < \tan^{-1} x < 0$

$\Rightarrow 0 < -\tan^{-1} x < 1$

$\therefore \quad [-\tan^{-1} x] = 0$

A-15. **Sol.** $\frac{\pi}{6} < x < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \sin x < \frac{\sqrt{3}}{2} \Rightarrow 1 < 2\sin x < \sqrt{3}$

$\therefore \quad [2\sin x] = 1$

Section (B) : Properties of definite integration

B-1. **Sol.** $\int_0^2 x^2 f(x)\ dx = \int_0^1 x^3\ dx + \int_1^2 (x^3 - x^2)\ dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{4} + \left[4 - \frac{8}{3} - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = 4 - \frac{8}{3} + \frac{1}{3} = 4 - \frac{7}{3} = \frac{5}{3}$

B-2. **Sol.** $\int_0^{\pi} |1 + 2\cos x|\ dx = \int_0^{2\pi/3} (1 + 2\cos x)dx - \int_{2\pi/3}^{\pi} (1 + 2\cos x)\ dx$

$= \frac{2\pi}{3} + 2\ .\ \left(\frac{\sqrt{3}}{2}\right) - \left[\pi - \left(\frac{2\pi}{3} + 2\ .\ \frac{\sqrt{3}}{2}\right)\right] = \frac{2\pi}{3} - \frac{\pi}{3} + \sqrt{3} + \sqrt{3} = \frac{\pi}{3} + 2\sqrt{3}$

B-3. **Sol.** $\int_0^1 |3x - 1|\, dx = \int_0^{1/3} (1 - 3x)dx + \int_{1/3}^1 (3x - 1)dx$

$= \left[x - \frac{3x^2}{2}\right]_0^{1/3} + \left[\frac{3x^2}{2} - x\right]_{1/3}^1$

$= \frac{5}{6}$

B-4. **Sol.** $\int_{1/e}^{e} |\ell nx|\, dx = \int_{1/e}^{1} -\ell nx dx + \int_1^e \ell nx\ dx$

$[x - x\ell nx]_{1/e}^1 + [x\ell nx - x]_1^e$

$= \left(1 - \frac{2}{e}\right) + 1 = 2\left(1 - \frac{1}{e}\right)$

B-5. **Sol.** $\int_0^{1.5} [x^2]\ dx = \int_0^1 0.dx + \int_1^{\sqrt{2}} 1.\ dx + \int_{\sqrt{2}}^{1.5} 2\ dx = 0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{1.5}$

$= 0 + \sqrt{2} - 1 + 2(1.5 - \sqrt{2}) = \sqrt{2} - 1 + 3 - 2\sqrt{2}$

$= 2 - \sqrt{2}$

B-6. **Sol.** $\int_{-1}^{3} |x-2| + [x]\ dx = \int_{-1}^{0} (2 - x - 1)\ dx + \int_0^1 (2 - x)\ dx + \int_1^2 (2 - x) + 1\ dx + \int_2^3 (x - 2 + 2)\ dx$

$= \left[x - \frac{x^2}{2}\right]_{-1}^0 + \left[2x - \frac{x^2}{2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 = -\left(-1 - \frac{1}{2}\right) + \left(2 - \frac{1}{2}\right) + \left(6 - 2\ \left(3 - \frac{1}{2}\right)\right) + \frac{9}{2} - 2$

$= \frac{3}{2}+\frac{3}{2}+4-\frac{5}{2}+\frac{5}{2}=7$

B-7. **Sol.** $\int_1^4 \log[x]dx = \int_1^2 \log[x]dx + \int_2^3 \log[x]dx + \int_3^4 \log[x]dx$

$= \int_1^2 \log 1 \ dx + \int_2^3 \log 2 dx + \int_3^4 \log 3 dy$

= log 2 + log 3 = log 6

B-8. **Sol.** $I = \int_{-2}^{-1} f(x) \ dx + \int_{-1}^{0} f(x) \ dx + \int_0^1 f(x) \ dx + \int_1^2 f(x) \ dx + \int_2^3 f(x) \ dx + \int_3^4 f(x) \ dx$

$= (-2)^2 + (-1)^2 + 0 + 1^2 + 2^2 + 3^2$

[using given relation]

= 19

B-9. **Sol.** $\int_0^{\pi} |\cos x| \, dx = \int_0^{\frac{\pi}{2}} \cos x \ dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \ dx = (\sin)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\pi}$

1 – 0 – (0 – 1) = 2

B-10. **Sol.** $I = \int_{-\pi/2}^{\pi/2} \tan x^3 \, dx$

$f(x) = \tan x^3$

$\therefore f(-x) = \tan(-x)^3 = -\tan x^3 = -f(x)$

$\therefore I = 0$

B-11. **Sol.** $I = \int_{-2}^{2} x^4 dx = 2\int_0^2 x^4 dx$

$= 2\left[\frac{x^5}{5}\right]_0^2 = 2\left(\frac{32}{5}-0\right) = \frac{64}{5}$

B-12.

Sol. $f(x) = x^{17} \cos^4 x$

f(x) is odd function

hence $\int_{-1}^{1} x^{17} \cos^4 x \ dx = 0$

B-13. **Sol.** $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)dx$

$= \int_{-\pi/2}^{\pi/2} 1. \ dx = \pi$ ($\because$ x^3, x cos x, $\tan^5 x$ are odd functions)

B-14. **Sol.** $I = \int_{\ell n\lambda}^{-\ell n\lambda} \frac{f\left(\frac{x^2}{4}\right) [f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right) [g \ (x)+g \ (-x)]} dx = 0$

Since $\dfrac{f\left(\frac{x^2}{4}\right)\ [f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)\ \left[g\ (x)+\int(-x)\right]}$ is an odd function

B-15. **Sol.** $I = \int_{-1}^{1}\frac{\sin x\ \ dx}{3-|x|} - \int_{-1}^{1}\frac{x^2\ \ dx}{3-|x|}$

$= 0 - \int_{-1}^{1}\frac{x^2\ \ dx}{3-|x|}$

$= \int_{0}^{1}\frac{-2x^2\ \ dx}{3-|x|}$ (∴ first is odd and second is even)

B-16. **Sol.** $I = \int_{-1}^{1}\frac{\cot^{-1}x}{\pi}\,dx \Rightarrow \int_{-1}^{1}\frac{\cot^{-1}(-x)}{\pi}\,dx \Rightarrow \int_{-1}^{1}\frac{\pi-\cot^{-1}x}{\pi}\,dx$

$\Rightarrow\ I = \int_{-1}^{1}1\ \ dx - \int_{-1}^{1}\frac{\cot^{-1}x}{\pi}\ \ dx$

$I = x|_{-1}^{1} - I$

$2I = 1 - (-1) = 2$

B.17. **Sol.** $I = \int_{1}^{2}\frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx = \int_{1}^{2}\frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}dx$

$\therefore\ 2I = \int_{1}^{2}\left(\frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}+\frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\right)dx$

$2\ I = \int_{1}^{2}dx = 2-1 \Rightarrow I = \frac{1}{2}$

B-18. **Sol.** $I = \int_{1}^{2}x\ \ f(x)dx$

$I = \int_{1}^{2}(3-x)f(3-x)dx$, using $\int_{a}^{b}f(x)dx = \int_{a}^{b}(a+b\ -\ x)dx$

Add

$2I = \int_{1}^{2}3f(3-x)dx = \int_{1}^{2}3f(x)dx$

$I = \frac{3}{2}\int_{1}^{2}f(x)dx$

B-19. **Sol.** $I = \int_{2-\log 3}^{3+\log 3}\frac{\log(4+x)}{\log(4+x)+\log(9-x)}\ dx$ ......(1)

using $\int_{a}^{b}f(x)\ \ dx = \int_{a}^{b}f(a+b-x)\ \ dx$

$I = \int_{2-\log 3}^{3+\log 3}\frac{\log(9-x)}{\log(9-x)+\log(4+x)}\ dx$ ......(2)

From equation (1) and (2)

2I= $\int_{2-\log 3}^{3+\log 3} dx = 1+2\log 3$

∴ I = $\frac{1}{2}+\log 3$

B-20. **Sol.** I = $\int_0^{\pi/2} \frac{a \sin x + b\cos x}{\sin x + \cos x} dx$ ..... (1)

I = $\int_0^{\pi/2} \frac{a \cos x + b \sin x}{\cos x + \sin x}$ ..... (2)

by using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Add (1) and (2)

2I = $\int_0^{\pi/2} (a+b) dx$ = $(a+b)\frac{\pi}{2}$

∴ I = (a + b) $\frac{\pi}{4}$

B-21. **Sol.** I = $\int_0^1 x(1-x)^n dx$ ....(1)

I = $\int_0^1 (1-x)(x)^x dx$ ....(2)

= $\int_0^1 (x^n - x^{n+1})^x dx$ = $\left[\frac{x^{n+1}}{x+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1$

= $\frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)\ (n+2)}$

B-22. **Sol.** I = $\int_0^a f(a-x)\ g(a-x)\ dx$ = $\int_0^a f(x)\ [2-g(x)]\ dx$

I = $2\int_0^a f(x)\ dx - \int_0^a f(x)\ g(x)\ dx$ ⇒ I = $\int_0^a f(x)dx$

B-23. **Sol.** I = $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ ..... (1)

= $\int_0^{\pi/2} \frac{dx}{1+\tan^3\left(\frac{\pi}{2}-x\right)}$

= $\int_0^{\pi/2} \frac{dx}{1+\cot^3 x}$ = $\int_0^{\pi/2} \frac{\tan^3 x dx}{1+\tan^3 x}$ ..... (2)

Add (1) and (2), 2I = $\int_0^{\pi/2} \frac{1+\tan^3 x}{1+\tan^3 x} dx$ = $\frac{\pi}{2}$

I = $\frac{\pi}{4}$

B-24. Sol. $I=\int_0^{2\pi}|\sin x|\,dx=2\int_0^{\pi}|\sin x|\,dx \quad \because f(2\pi-x)=f(x)$

$=4\int_0^{\pi/2}|\sin x|\,dx=4\int_0^{\pi/2}\sin x\,dx=4\left[-\cos x\right]_0^{\pi/2}$

= 4 (– 0 + 1) = 4

B-25. Sol. $I=\int_0^{\pi} x\,\ell n\ \sin x\ dx$

$I=\int_0^{\pi}(\pi-x)\ell n\ \sin(\pi-x)\ dx$

$=\int_0^{\pi}(\pi-x)\ell n\ \sin x\ dx$

Add $2I=\pi\int_0^{\pi}\ell n\ \sin x\ dx$

$I=\frac{\pi}{2}\int_0^{\pi}\ell n\ \sin x\ dx$

$=-\frac{\pi^2}{2}\ell n2$

B-26. Sol. $\tan x+\cot x=\frac{1}{\sin x\ \cos x}$

$I=-\int_0^{\pi/2}\ell n(\sin x\cos x)dx$

$=-\int_0^{\pi/2}\ell n(\sin x)dx-\int_0^{\pi/2}\ell n(\cos x)\ dx$

Put $x\to\frac{\pi}{2}-x$

$I=-2\int_0^{\pi/2}\ell n\sin dx$

$=-2\left(-\frac{\pi}{2}\ell n2\right)=\pi\ell n2$

B-27. Sol. $I=2\int_0^{\pi/2}\ell n\sin x\ dx-\int_0^{\pi/2}\ell n\sin 2x\ dx$

let $I_1=\int_0^{\pi/2}\ell n\sin 2x\ dx$, put 2x = t

$I_1=\frac{1}{2}\int_0^{\pi}\ell n\sin t dt=\frac{2}{2}\int_0^{\pi/2}\ell n\sin t\ dt=\int_0^{\pi/2}\ell n\sin x dx$

$I=2\int_0^{\pi/2}\ell n\sin x dx-\int_0^{\pi/2}\ell n\ \sin\ xdx=\int_0^{\pi/2}\ell n\sin x dx=\frac{\pi}{2}\ell n\frac{1}{2}$

Section (C) : Integration of periodic functions

C-1. **Sol.** C-1_. $\int_0^{2\pi} |\sin 3x|$

$= \int_0^{6.\frac{\pi}{3}} |\sin 3x|\,dx$ $= 6\int_0^{\pi/3} \sin 3x \; dx$

$= 6\left[\frac{-\cos 3x}{3}\right]_0^{\pi/3}$

$= -\frac{6}{3}(\cos\pi - \cos 0) = -2\times -2$

= 4

C-2. **Sol.** $I = \int_0^{11} 11^{\{x\}}dx$ $= \int_0^{11\times 1} 11^{\{x\}}dx = 11\int_0^1 11^{\{x\}}dx$ {∵ {x} is periodic with period 1 }

$= 11\int_0^1 11^x dx = 11\left[\frac{11^x}{\ell n 11}\right]_0^1$ $= 11\left[\frac{11}{\ell n 11} - \frac{1}{\ell n 11}\right]$

$= \frac{110}{\ell n 11} = \frac{k}{\ell n 11} \Rightarrow$ k = 110

C-3. Sol. $\int_{-2}^{10} \text{sgn}\left(\frac{x}{2} - \left[\frac{x}{2}\right]\right)$

$= 6\int_0^2 \text{sgn}\left\{\frac{x}{2}\right\} \; dx$ ∴ period of {x} is 1

= 6.2 = 12

C-4. **Sol.** We know that x – [x] is periodic with period 1

$\int_0^{[x]} (x - [x])dx$ $= [x]\int_0^1 (x - [x]) \; dx$

$= [x]\int_0^1 (x - 0) \; dx$

$= \frac{[x]}{2}$

C-5. **Sol.** $I = \int_0^{20\frac{\pi}{2}} (|\sin x| + |\cos x|)dx$

$= 20\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$ $\left[∵ \text{ prd of } |\sin x| + |\cos x| = \frac{\pi}{2}\right]$ $\left[∵ |\sin x| + |\cos x| = \frac{\pi}{2}\right]$

= 40

C-6. **Sol.** ∵ $0 \le \left|\frac{\sin x}{2}\right| \le$ ∴ $\left[\left|\frac{\sin x}{2}\right|\right] = 0$

$I = \int_0^{2n\pi} |\sin x|\, dx = 2n \int_0^{\pi} (\sin x)\ dx = 4n \int_0^{\pi/2} \cos x\ dx = 4n$

C-7. **Sol.** $I = 2\int_0^{\pi} \sin^2 x dx = 4 \int_0^{\pi/2} \sin^2 x dx$

Section (D) : Leibinitz theorem, Estimation of definite integrals, Definite integral as limit of sum

D-1. **Sol.** $I = \lim_{h \to 0} \dfrac{\int_a^x \ell n^2 t\ dt + \int_x^{x+h} \ell n^2 t\ dx - \int_a^x \ell n^2 t\ dt}{h}$

$I = \dfrac{\int_x^{x+h} \ell n^2 t\ dx}{h}$

Using L hospital we get

$I = \lim_{h \to 0} \ell n^2 (x+h) = \ell n^2 x$

D-2. **Sol.** $\int_a^y \cos\ t^2\ dt = \int_a^{x^2} \dfrac{\sin t}{t} dt$

differentiating both sides w.r.t x we get

$\dfrac{d}{dx} \int_a^y \cot\ t^2\ dt = \dfrac{d}{dx} \int_a^{x^2} \dfrac{\sin t}{t} dt$

RHS = $\dfrac{\sin\ [x^2]}{x^2}\ \dfrac{dx^2}{dx} = 2x \dfrac{\sin x^2}{x^2}$

L.H.S. = $\dfrac{d}{dy}\left(\int_a^y \cos\ t^2\ dt\right) \dfrac{dy}{dx} = \cos\ y^2 \dfrac{dy}{dx}$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{2\sin x^2}{x\cos y^2}$

D-3. **Sol.** According to leibnitz theorem

$\dfrac{d}{dx} \int_{f(x)}^{g(x)} \phi(t) dt$ = g'(x) φ (g(x)) – f' (x) φ(f(x)).

D-4. **Sol.** f(x) = 1 + x + $\int_1^x (\ell n^2 t + 2\ell nt)$ dt

Differentiate both sides w.r.t. x

by using Leibinitz theorem

$f'(x) = 1 + \ell n_2 x + 2\ell nx = 0$

$(1 + \ell nx)_2 = 0$ ∴ $\ell nx = -1$

∴ $x = \dfrac{1}{e}$

D-5. **Sol.** $x = \int_2^{\sin t} \sin^{-1} z\ dz$, $y = \int_n^{\sqrt{t}} \dfrac{\sin^{-1} z^2}{z} dz$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(\sin t/\sqrt{t}).\frac{1}{2\sqrt{t}}}{\sin^{-1}(\sin t).\cos t}$$

$$= \frac{\sin t}{2t.t.\cos t} = \frac{\tan\ t}{2t^2}$$

D-6. **Sol.** $\ell im_{x\to 0} \frac{\sqrt{\cos x^3}.3x^2}{1-\sqrt{\cos x}}$

$$= \ell im_{x\to 0} \frac{\sqrt{\cos x^3}.3x^2}{1-\sqrt{1-\frac{x^2}{2\ !}+.....}}$$

= 12

D-7. **Sol.** f(x) = $\frac{\tan x}{x}, x \in (0,\frac{\pi}{4})$

f'(x)= $\frac{x\sec^2 x - \tan x}{x^2} > 0 \forall x \in (0,\frac{\pi}{4})$

∴ f(x) is increasing is $(0,\frac{\pi}{4})$

∴ f(x), $(0,\frac{\pi}{4})$

∴ $\lim_{x\to 0^+}$ f(x) < f(x) < f($\frac{\pi}{4}$) ⇒ 1 < f(x) < $\frac{\pi}{4}$

⇒ $I = \int_0^{\pi/4} dx < I < \frac{4}{\pi}\int_0^{\pi/4} dx$

⇒ $\frac{\pi}{4}$ < I < 1

D-8. Sol. We know that tanx > x $\forall$ x∈ (0,1)

⇒ $\frac{\tan x}{\sqrt{x}} > \sqrt{x}$

⇒ $\int_0^1 \frac{\tan x}{\sqrt{x}} dx > \int_0^1 \sqrt{x} dx$

⇒ I > $\left[\frac{x^{3/2}}{3/2}\right]_0^1$

⇒ I > $\frac{2}{3}$

D-9. **Sol.** Let f(x) = $\sqrt{1+x^4}$

∴ f(1) = $\sqrt{2}$ and f(2) = $\sqrt{17}$

∴ $\sqrt{2} \le \sqrt{1+x^4} \le \sqrt{17}$ ⇒ $\frac{1}{\sqrt{2}} \ge \frac{1}{\sqrt{1+x^4}} \ge \frac{1}{\sqrt{17}}$

⇒ ∴ $\frac{1}{\sqrt{17}} \le I \le \frac{1}{\sqrt{2}}$

D-10. Sol. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{r\sqrt{r}}{n^{5/2}} = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{r}{n}\sqrt{\frac{r}{n}}.\frac{1}{n} = \int_0^1 x\sqrt{x}\ \ dx$

D-11. Sol. $I = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left[\frac{\frac{1}{n}\left(\frac{r^3}{n^3}\right)}{\frac{r^4}{n^4}+1}\right] = \int_0^1 \frac{x^3}{1+x^4}dx = \frac{1}{4}\ell n2$

D-12. Sol. $A = \left[\lim\limits_{n\to\infty}\left(\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}.......\sin\frac{(n-1)}{2\ \ n}\right)\right]^{1/n}$

$\Rightarrow$ $\ell nA = \frac{1}{n}\sum\limits_{r=1}^{2(n-1)}\ell n\sin\frac{r\pi}{2n} = \int_0^2 \ell n\sin\left(\frac{\pi x}{2}\right)\ dx$

put $\frac{\pi x}{2} = t$

$\Rightarrow$ $\ell nA = \frac{2}{\pi}\int_0^{\pi}\ell n(\sin t)\ dt = \frac{4}{\pi}\int_0^{\pi/2}\ell n(\sin t)dt$

$\Rightarrow$ $\ell n A = -2\ \ell n\ 2$ $\Rightarrow$ $A = \frac{1}{4}$

D-13. Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\ \ dx$

$\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n} f\left(\frac{r+n}{n}\right) = \int_0^2 f(1+x)\ \ dx = \int_1^3 f(t)\ \ dt = \int_1^3 f(x)\ \ dx$

$\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{n+1}^{2n} f\left(\frac{r}{n}\right) = \int_1^2 f(x)\ \ dx$

$\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)\ \ dx$

Section (E) : Reduction Formulae, Walli's formula

E-1. Sol. $U_{10} = \left(-x^{10}\cos x\right)_0^{\pi/2} + \int_0^{\pi/2} 10x^9\cos x\ \ dx$

$= 10\ \left[x^9\sin x\right]_0^{\pi/2} - 10\times 9\int_0^{\pi/2} x^8\sin x\ \ dx$

$U_{10} + 90U_8 = \frac{10.\pi^9}{2^9}$

E-2. Sol. $I_n = \left(-x^n\ \ e^{-x}\right)_0^{\infty} + n\int_0^{\infty} x^{n-1}\ \ e^{-x}\ \ dx = 0 + n\ I_{n-1} = ... = n! = \int_0^{\infty} e^{-x}dx = n!$

E-3. Sol. $I = \int_0^{\pi/2}\sin^6 x\ \ dx$

$= \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ by wallis formula

$= \frac{5\pi}{32}$

E-4. **Sol.** $I = \int_0^{\pi} \sin^7 x \cos^6 x dx = 2\int_0^{\pi/2} \sin^7 x \cos^6 x dx$ ∵ $f(\pi - x) = f(x)$

$= 2.\frac{6.4.2.5.3.1}{13.11.9.7.5.3.1}$

$= \frac{32}{3003}$

E-5. **Sol.** Let $I = \int_0^3 x^{5/2}(3-x)^{3/2} dx$

Put $x = 3\sin^2\theta$ ∴ $dx = 6\sin\theta\cos\theta d\theta$

$I = 6\int_0^{\pi/2} (3\sin^2\theta)^{5/2}(3-3\sin^2\theta)^{3/2} \sin\theta\cos\theta d\theta$

$= 6.3^4 \int_0^{\pi/2} \sin^6\theta\cos^4\theta d\theta = \frac{6.81.5.3.1.3.1}{10.8.6.4.2}\frac{\pi}{2} = \frac{3^6\pi}{2^8}$

E-6. Sol. $I_n = \int_0^{\pi/4} \tan^n x \, dx =$

$\Rightarrow I_n = \frac{1}{n-1} - I_{n-2}$

∴ $I_6 = \frac{1}{5} - I_4 = \frac{1}{5} - \frac{1}{3} + I_2 = \frac{1}{5} - \frac{1}{3} + 1 - I_0$

$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{3-5+15}{15} - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}$

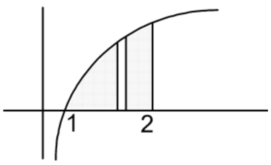
Section (F) : Area Under the Curves

F-1. **Sol.** $A = \int_1^2 \ln x dx$

$= [\ln x.x - x]_1^2$

$= 2\ln 2 - 1$

$= \ln \frac{4}{e}$



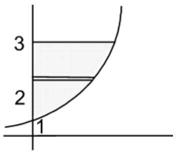
F-2. **Sol.** $A = \int_1^3 x dy$

$$A = \int_1^3 \ell n y \, dy$$

$= [y\ell ny - y]_1^3$

$= 3\ell n3 - 3 - 0 + 1$

$= 3\ell n3 - 2$



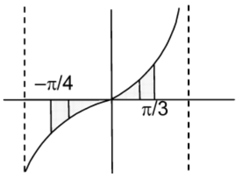
F-3. **Sol.** $A = \int_{-\pi/4}^{0} (-\tan x)dx + \int_0^{\pi/3} \tan x dx$

$= -[\ell n \sec x]_{-\pi/4}^{0} + [\ell n \sec x]_0^{\pi/3}$

$= -(\ell n \sec 0 - \ell n \sec \frac{\pi}{4}) + \ell n \sec \frac{\pi}{3} - \ell n \sec 0$

$= -0 + \ell n\sqrt{2} + \ell n2 - 0$

$= \frac{3}{2}\ell n2$

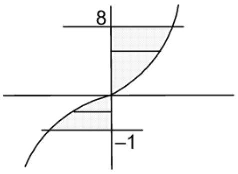


F-4. **Sol.** $A = \int_{-1}^{0} \left(0 - y^{\frac{1}{3}}\right)dy + \int_0^8 y^{\frac{1}{3}}dy$

$= -\left[\frac{y^{4/3}}{4/3}\right]_{-1}^{0} + \left[\frac{y^{4/3}}{4/3}\right]_0^8$

$-\frac{3}{4}(0-1) + \frac{3}{4}(16-0)$

$= \frac{3}{4}(16+1) = \frac{51}{4}$



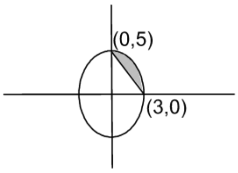
F-5. **Sol.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$

& $\frac{x}{3} + \frac{y}{5} = 1$

$$A = \int_0^3 \left(5\sqrt{1-\frac{x^2}{9}} - \frac{15-5x}{3}\right)dx$$

$$= \frac{5}{3}\int_0^3 \sqrt{9-x^2}\,dx - 5\int_0^3 dx + \frac{5}{3}\int_0^3 x\,dx$$

$$= \frac{15}{2}\left(\frac{\pi}{2}-1\right)$$



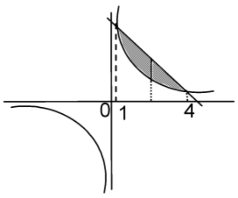
F-6. Sol. $A = \int_1^4 \left(5 - x - \frac{4}{x}\right)dx$

$$= \left[5x - \frac{x^2}{2} - 4\ell nx\right]_1^4$$

$$= 20 - 8 - 4\ell n4 - 5 + \frac{1}{2} + 0$$

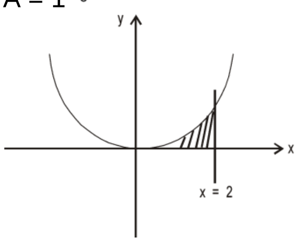
$$= 7 + \frac{1}{2} - 4\ell n4$$

$$= \frac{15}{2} - 4\ell n4$$



F-7. Sol. $y = \frac{x^2}{4}$

$$A = 1\int_0^2 \frac{x^2}{4}\,dx$$

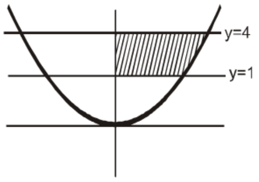


$$A = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$A = \frac{1}{4}\left(\frac{8}{3}\right)$$

$$A = \frac{2}{3}$$

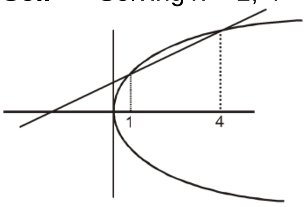
F-8.



Sol. Graph of $y = 4x^2$

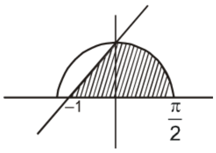
Area = $\int_1^4 \frac{1}{2}\sqrt{y}dy$ = $\frac{7}{3}$

F-9. **Sol.** Solving x = 1, 4



From graph it is clear that required

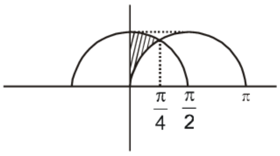
area = $\int_1^4 \left(2\sqrt{x} - \frac{1}{3}(2x+4)\right)dx$ = $\frac{1}{3}$



F-10. **Sol.**

From figure it is clear that required

area = $\frac{1}{2} + \int_0^{\pi/2} \cos x \; dx = \frac{3}{2}$



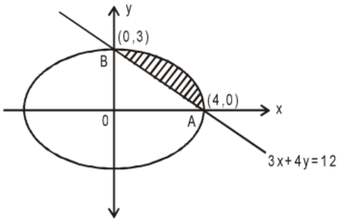
F-11. **Sol.**

From figure

Area = $\int_0^{\pi/4} (\cos x - \sin x)dx$ = $\sqrt{2} - 1$

F-12. **Sol.** Area = $\int_0^4 \left(3\sqrt{1-\frac{x^2}{16}} - \frac{1}{4}(12-3x)\right) dx$

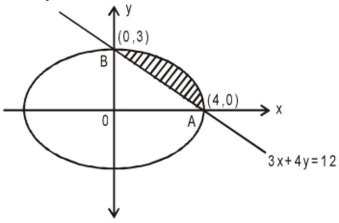
= $\frac{3}{4}\left[\frac{x}{4}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 - \frac{1}{4}\left[12x - \frac{3x^2}{2}\right]_0^4$



$= \frac{3}{4}\left[(0+4\pi)-(0+0)\right]-\frac{1}{4}[48-24]$

= 3π – 6 = 3(π – 2)

Alter Required area = $\frac{1}{4}$ area of ellipse – ΔOAB



$= \frac{1}{4}$ (π.4.3) – (4.3)

3π – 6

Exercise-2

1. Sol. $I = \int_0^1 \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha} = \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1 = \frac{1}{\sin\alpha}\left(\alpha-\frac{\alpha}{2}\right) = \frac{\alpha}{2\sin\alpha}$

2. Sol. Let $\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$\therefore \quad I = \int_0^{\pi/4} \frac{t \;\; \tan \;\; t \cdot dt}{\sqrt{1+\tan^2 t}} = \int_0^{\pi/4} t \cdot \sin t \;\; dt$

$= -\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{4-\pi}{4\sqrt{2}}$

3. Sol. $I = \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)} dt$

put $t = \frac{1}{x}$

$I = \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{e}^{\tan x} \frac{t}{\frac{1}{x}\left(1+\frac{1}{x^2}\right)} dx \cdot \left(-\frac{1}{x^2}\right) dx$

$I = \int_{1/e}^{\tan x} \frac{t}{1+t^2} + \int_{\tan x}^{e} \frac{x}{1+x^2} dx = \int_{1/e}^{e} \frac{t}{1+t^2} dt = \left(\frac{1}{2}\ell n \;\; (1+t^2)\right)_{1/e}^{e} = 1$

4. $= \left(\frac{1}{2}\right)+\left(\frac{2}{3}-\frac{1}{2}\right)+\left(\frac{3}{4}-\frac{2}{3}\right)+....+\left(\frac{n}{n+1}-\frac{n-1}{n}\right)+....+1=\frac{n}{n+1}+....+1$ as $n \to \infty$

taking limit $n \to \infty$

we get $\int_0^2 f(x)\,dx = 1 + 1 = 2$

5. **Sol.** $\sum_{r=1}^{100} \left(\int_0^1 f(r-1+x)\,dx \right)$

$= \int_0^1 f(x)\,dx + \int_0^1 f(1+x)\,dx + \int_0^1 f(2+x)\,dx + ... + \int_0^1 f(99+x)\,dx$

$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + ... + \int_{99}^{100} f(x)\,dx$

{using shifting property}

$= \int_0^{100} f(x)\,dx = a$

6. **Sol.** $I = \int_0^{\infty} \left[2e^{-x}\right] dx$

∴ $2e^{-x}$ decreases for $x \in [0, \infty)$

⇒ $0 < 2e^{-x} \le 2 \ \forall\ x \in [0, \infty)$

for $x > \ell n2$, $[2e^{-x}] = 0$

⇒ $I = \int_0^{\ell n2} [2e^{-x}]\,dx + \int_{\ell n 2}^{\infty} [2e^{-x}]\,dx$

$= \int_0^{\ell n2} 1 \,.\, dx + \int_{\ell n2}^{\infty} 0.dx = \ell n2$

7. **Sol.** $\int_{-\frac{\pi}{2}}^{\pi/2} \frac{|x|\,dx}{8\cos^2 2x + 1}$

$= 2 \int_0^{\pi/2} \frac{x\,dx}{8\cos^2 2x + 1} = 2I$

$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) dx}{8\cos^2(\pi - 2x) + 1}$

$I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{8\cos^2 2x + 1} - I = \pi \int_0^{\pi/4} \frac{dx}{8\cos^2 2x + 1}$ ∴ $I = \frac{\pi}{2} \int_0^{\pi/4} \frac{dx}{8\cos^2 2x + 1}$

$= \frac{\pi}{4} \int_0^{\pi/4} \frac{2\sec^2 2x\,dx}{9 + \tan^2 2x} = \frac{\pi}{4} \cdot \frac{1}{3} \tan^{-1} \left(\frac{1}{3}\tan 2x\right)_0^{\pi/4} = \frac{\pi^2}{24}$

∴ given integral $= \frac{\pi^2}{12}$

8. **Sol.** $I = \int_4^{10} \frac{[(x-14)^2]}{[x^2] + [(x-14)^2]}\,dx$

⇒ $2I = \int_4^{10} dx = 6$ ⇒ $I = 3$

9. **Sol.** $x = \tan\theta$

$\int_0^{\pi/2} 2\ell n \sec\theta \, d\theta$

$= -2 \int_0^{\pi/2} \ell n \cos\theta \, d\theta = -2\left(-\frac{\pi}{2}\ell n 2\right)$

$= \pi \ell n 2$

10. Sol. Using properties ...... $\int_{-1}^{1} \frac{e^x+1}{e^x-1} dx = 0$

11. Sol. $I_1 = \int_0^{\pi} x \; f(\sin^3 x + \cos^2 x)dx$ ....(1)

$= \int_0^{\pi} (\pi - x) \; f(\sin^3(\pi - x) + \cos^2(\pi - x)dx$ ....(2)

(1) + (2)

$2I_1 = \pi \int_0^{\pi} f(\sin^3 x + \cos^2 x)dx$

$2I_1 = 2\pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)dx$

$\therefore \quad I_1 = \pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)dx$

12. Sol. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$

$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x \sin x}{\cos^4 x + \sin^4 x} dx$

$\therefore 2I = \int_0^{\pi/2} \frac{\left(x + \frac{\pi}{2} - x\right)\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$= \frac{\pi}{4}\int_0^{\pi/2} \frac{2\sin x \cos x \; dx}{\sin^4 x + \cos^4 x} dx$

$= \frac{\pi}{4}\int_0^{\pi/2} \frac{2\tan x \sec^2 x dx}{1 + \tan^4 x} dx$

$= \frac{\pi}{4}\int_0^{\infty} \frac{dt}{1+t^2}$ Put t = $\tan_2 x$

$= \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{8}$

$\therefore I = \frac{\pi^2}{16}$

13. Sol. Let $I = \int_0^{\pi/2} \frac{\sin x \quad dx}{1+\sin x+\cos x}$

$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx = \int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x} dx$

$\therefore 2I = \int_0^{\pi/2} \frac{\sin x \ +\cos x}{1+\sin x+\cos x} dx = \int_0^{\pi/2} \left(1-\frac{1}{1+\sin x+\cos x}\right) dx$

$= \frac{\pi}{2} - \int_0^1 - \frac{2dt}{2+2t}$ Put $\tan\frac{x}{2}=t$

$= \frac{\pi}{2} - \ell n2$ $\therefore I = \frac{\pi}{4}-\frac{1}{2}\ell n2$

a = 4, b = 2 ∴ a+b = 6

14. Sol. $I = \int_0^{\pi} \frac{x^2\cos^4 x\sin x}{2\pi x-\pi^2} dx$

$= \int_0^{\pi} \frac{(\pi-x)^2\cos^4(\pi-x)\sin(\pi-x)}{2\pi(\pi-x)-\pi^2} dx$

$= \int_0^{\pi} \frac{(\pi-x)^2\cos^4 x\sin x}{\pi^2-2\pi x} dx$

$\therefore 2I = \int_0^{\pi} \cos^4 x\sin x.\left(\frac{x^2}{2\pi x-\pi^2}+\frac{(\pi-x)^2}{\pi^2-2\pi x}\right)dx$

$= \int_0^{\pi} \cos^4 x\sin x.\frac{x^2-\pi^2-x^2+2\pi x}{2\pi x-\pi^2} dx$

$= \int_0^{\pi} \cos^4 x\sin x dx = 2\int_0^{\pi/2} \cos^4 x\sin x \ dx$

$= 2.\frac{3.1}{5.3.1} = \frac{2}{5}$ $\therefore I = \frac{1}{5}$

15. Sol. $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}} dx = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}} dx$

$= \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}} dx$

$\therefore 2I = \int_0^{\pi} \frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}} dx = \int_0^{\pi} dx = \pi$

$I = \frac{\pi}{2}$

16. **Sol.** $\int_{-1}^{-1+6(1/2)} \{2x\}\,dx = 6\int_0^{1/2} \{2x\}\,dx = 6\int_0^{1/2} 2x\,dx$

$= 12\left[\frac{x^2}{2}\right]_0^{1/2} = \frac{12}{8} = \frac{3}{2}$

17. **Sol.** $I = \int_0^{400\pi} \sqrt{1-\cos 2x}\,dx$

$= \int_0^{400\pi} \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int_0^{400\pi} |\sin x|\,dx$

$= \sqrt{2}\times 400\int_0^{\pi} \sin x\,dx$

$= 800\sqrt{2}\int_0^{\pi/2} \sin x\,dx = 800\sqrt{2}$

18. **Sol.** $f(x+\pi) = \int_0^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)dt$

$= \int_0^{x} (2\cos^2 3t + 3\sin^2 3t)dt + \int_x^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)dt$

$= f(x) + g(x)$

let $g(x) = \int_x^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)dt$

$= \int_0^{\pi} (2\cos^2 3t + 3\sin^2 3t)dt$

$= 2\int_0^{\pi/2} (2\cos^2 3t + 3\sin^2 3t)dt$

$= 2f\left(\frac{\pi}{2}\right)$

$\therefore \quad f(x+\pi) = f(x) + 2f\left(\frac{\pi}{2}\right)$

19. **Sol.** Let $f(x) = (\tan^{-1} x)^2$ $\therefore f'(x) = \frac{2\tan^{-1} x}{1+x^2} > 0$ then $\forall x \in (0,1)$

$\therefore f(0) < f(x) < f(1) \Rightarrow 0 < f(x) < \frac{\pi^2}{16}$

$\therefore 0 < \int_0^1 f(x)dx < \frac{\pi^2}{16}$

20. **Sol.** Let $f(x) = e^{\cos^{-1} x}$ $\therefore f'(x) = e^{\cos^{-1} x}\left(\frac{-1}{\sqrt{1-x^2}}\right) < 0 \; \forall x \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$\therefore f(1) < f(x) < f\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow 1 < f(x) < e^{\pi/4}$

$\therefore\ 1-\frac{1}{\sqrt{2}} < I < \left(1-\frac{1}{\sqrt{2}}\right)e^{\pi/4}$

21. **Sol.** Let $y = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

$\ell n\, y = \lim_{n\to\infty}\frac{1}{n}\ell n\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty}\frac{1}{n}\ell n\left(\frac{1\ .\ 2\ .\ 3\ ....n}{n^n}\right)$

$= \lim_{n\to\infty}\frac{1}{n}\left[\ell n\left(\frac{1}{n}\right)+\ell n\left(\frac{2}{n}\right)+\ell n\left(\frac{3}{n}\right)+.....+\ell n\left(\frac{n}{n}\right)\right]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(\frac{r}{n}\right) = \int_0^1 \ell nx\ dx = x\ \ \ell n\ \ x - x\Big]_0^1$

$= (0-1) - \underset{x\to0^+}{\text{Lt}}\ x\ \ell n\ x + 0$

$= -1 - 0 = -1$

$\Rightarrow\quad y = \frac{1}{e}$

22.. **Sol.** $f'(x) = \frac{1}{\ell nx^3}.\ \ 3x^2 - \frac{1}{\ell n(x^2)}.2x$

$= \frac{x^2}{\ell nx} - \frac{x}{\ell nx} = \frac{x^2-x}{\ell nx}$

for increasing, $f'(x) > 0$

$\Rightarrow\quad \frac{x^2-x}{\ell nx} > 0$

23. **Sol.** $I = \int_0^1 x^5(1-x^2)^4 dx$

Put $x = \sin\theta\ \ \therefore dx = \cos\theta d\theta$

$I = \int_0^{\pi/2} \sin^5\theta\cos^8\theta\cos\theta\ d\theta$

$= \int_0^{\pi/2} \sin^5\theta\cos^9\theta\ d\theta$

$= \frac{4.2.8.6.4.2}{14.12.10.8.6.4.2}$

$= \frac{1}{210}$

24. **Sol.** $I_n = \int_0^1 x^n\tan^{-1}x dx = \left[\tan^{-1}x.\frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}.\frac{x^{n+1}}{n+1}dx$

$= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx$

$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$

$$(n-2+1)I_{n-2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n-1}}{1+x^2}dx$$

$$\therefore \quad (n+1)\ I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}+x^{n-1}}{1+x^2}dx$$

$$= \frac{\pi}{2} - \int_0^1 x^{n-1}dx = \frac{\pi}{2} - \frac{1}{n}$$

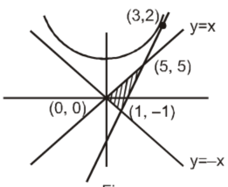
25. Sol. $I_n = \int_0^1 a^x x^n dx$

$$= \left[x^n \frac{a^x}{\ell na}\right]_0^1 - \int_0^1 nx^{n-1}\frac{a^x}{\ell na}dx$$

$$= \frac{a}{\ell na} - 0 - \frac{n}{\ell na}I_{n-1} = \frac{a}{\ell na} - \frac{n}{\ell na}I_{n-1}$$

26. Sol. $\left.\frac{dy}{dx}\right|_{(3,2)} = \frac{3}{2}$. Tangent $y = \frac{3x}{2} - \frac{5}{2}$

$y = x,\quad y = \frac{3x}{2} - \frac{5}{2} \quad \Rightarrow \quad (5, 5)$

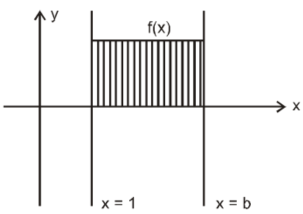


Figure

$y = -x,\quad y = \frac{3x}{2} - \frac{5}{2} \quad \Rightarrow \quad (1, -1)$

closed figure formed is right angled triangle. Its area is $\frac{1}{2}(\sqrt{2})\ (5\sqrt{2}) = 5$

27. Sol. Required area $(b-1)\sin(3b+4) = \int_0^b f(x)\ dx$

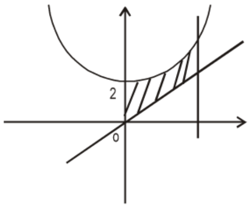


diff. w.r.t. b

$3(b-1)\cos(3b+4) + \sin(3b+4) = f(b)$

$\Rightarrow \quad f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$

28. Sol. $A = \int_0^3 (x^2+2-x)\ dx$



A = $\left[\frac{x^3}{3}+2x-\frac{x^2}{2}\right]_0^3$

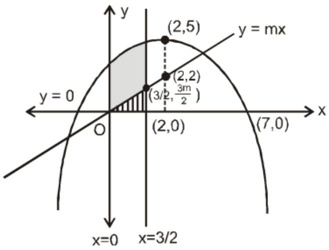
A = $\left[\frac{27}{3}+6-\frac{9}{2}\right]$

A = $\frac{21}{2}$

29. **Sol.** $y = 1 + 4x - x^2 \Rightarrow (x-2)^2 = -(y-5)$
vertex (2, 5)

A = $\int_0^{3/2}(1+4x-x^2)\ dx = \left(x+\frac{4x^2}{2}-\frac{x^3}{3}\right)_0^{3/2} = \frac{39}{8}$ ....(1)

$\Delta OAB = \frac{1}{2}\times\frac{3}{2}\times\frac{3}{2}\times m = \frac{9}{8}m$ ....(2)



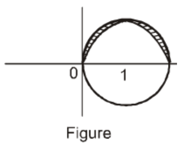
From (1) & (2) $\frac{9}{8}m = \frac{1}{2}\left(\frac{39}{8}\right)$

$\Rightarrow$ m = $\frac{39}{2\times 9} = \frac{13}{6}$

30. **Sol.** A = $\int_0^{\pi/4}\sec^2 x\ \ dx$

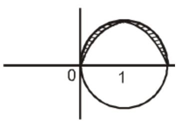
A = $[\tan x]_0^{\pi/4}$

A = 1



Figure

31. **Sol.** $(x-1)^2 + y^2 = 1$
Area of cile is πr^2



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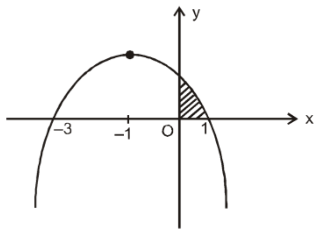
Required area = $\frac{\pi}{2}-\int_0^2\sin\frac{\pi x}{2}dx$

$= \frac{\pi}{2}-\frac{4}{\pi}$

32.. **Sol.** $0 < y < 3 - 2x - x^2$, $x > 0$

$y = 3 - 2x - x^2 \Rightarrow \quad (x + 1)^2 = -(y - 2)$

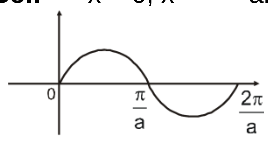
vertex $(-1, 2)$ at $y = 0$, $x = -3$, $x = 1$



Area = $\int_0^1 (3 - 2x - x^2)\ dx$ **Ans.**

33. **Sol.** Area = $\int_0^1 (x - x^2)dx = \frac{1}{6}$

34. **Sol.** $x = 0$, $x = \frac{\pi}{a}$ are successive points of inflection

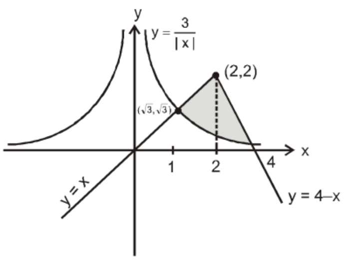


Graph of y = sin ax

Area = $\int_0^{\pi/a} \sin ax\ dx = \frac{2}{a}$

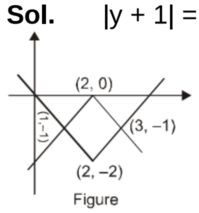
35. **Sol.** $A = \int_{\sqrt{3}}^{2}\left(x - \frac{3}{x}\right) dx + \int_{2}^{3}\left(4 - x - \frac{3}{x}\right) dx$

$= \left(\frac{x^2}{2} - 3\ell nx\right)_{\sqrt{3}}^{2} + \left(4x - \frac{x^2}{2} - 3\ell nx\right)_{2}^{3}$



$= \frac{4 - 3\ell n3}{2}$

36. **Sol.** $|y + 1| = 1 - |x - 2|$



Figure

$y = -1 \pm (1 - |x - 2|)$

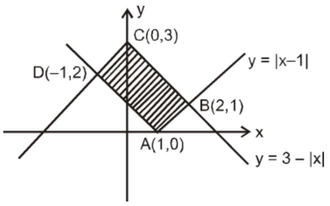
$y = -|x - 2|\ (y \geq -1)$

$y = -2 - |x - 2|$ $(y \le -1)$

Area = $\frac{1}{2}\begin{pmatrix} 2 & 3 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 & -2 \end{pmatrix}$ (formula from coordinate geometry)

$= \frac{1}{2}(4) = 2$

37.



Sol.

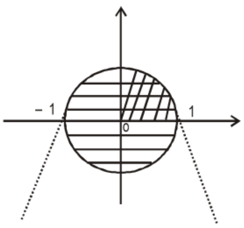
$AB = \sqrt{2}$, BC = $2\sqrt{2}$

Area of rectangle ABCD

$= \sqrt{2} \times 2\sqrt{2} = 4$

38. **Sol.** $A = 4\int_0^1 (1 - x^2)\ dx$

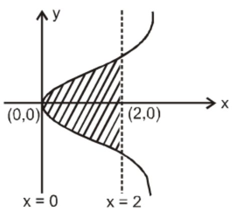
$= 4\left[x - \frac{x^3}{3}\right]_0^1$



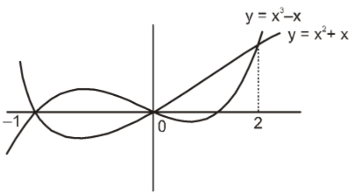
$= 4\left[1 - \frac{1}{3}\right]$

$= 4 \cdot \frac{2}{3} = \frac{8}{3}$

39. **Sol.**



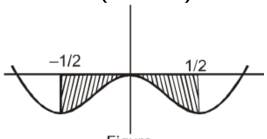
$A = 2\int_0^2 x^3 dx = 8$



40. **Sol.** Figure

Area = $\int_{-1}^{0} (x^3 - x) - (x^2 + x)\ dx + \int_0^2 (x^2 + x - (x^3 - x))\ dx = \frac{37}{12}$

41. **Sol.** $\frac{dy}{dx} = 8x^3 - 2x$, $\frac{dy}{dx} = 0$ $\Rightarrow$ $(4x^2 - 1)\,x = 0$



Figure

$\Rightarrow$ $x = -\frac{1}{2}, 0, \frac{1}{2}$

Required area = $-2\int_0^{1/2} (2x^4 - x^2)\,dx = \frac{7}{120}$

42. **Sol.** $y = xe^x$, $y = xe^{-x}$

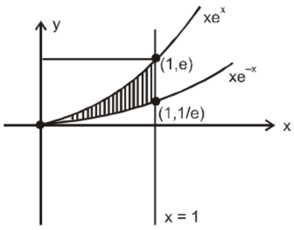
$xe^x = x\,e^{-x} \Rightarrow x(e^x - e^{-x}) = 0$

x = 0, Intersection point (0,0)

x = 0, (0,0)

Area = $\int_0^1 (xe^x - xe^{-x})\,dx$

$= x\left(e^x + e^{-x}\right)_0^1 - \int_0^1 1.(e^x + e^{-x})\,dx$



$= \left(e + \frac{1}{e} - 0\right) - \left[e^x - e^{-x}\right]_0^1 = \frac{2}{e}$

PART - II : MISCELLANEOUS QUESTIONS

A-1. **Ans.** **(3)**

Sol. Statement-2 : Let α(x) be differentiable and periodic with period T. Then $\alpha(x + T) = \alpha(x)\ \forall\ x \in D_f$

$\alpha'(x + T) = \lim_{h \to 0} \frac{\alpha(x+T+h) - \alpha(x+T)}{h} = \lim_{h \to 0} \frac{\alpha(x+h) - \alpha(x)}{h}$

$\therefore$ α'(x) is periodic with period T.

Statement-1 : $g(x + T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt$

$= \int_a^x f(t)\,dt + \int_0^T f(t)\,dt = g(x) + \int_0^T f(t)\,dt$

which is not true if $f(x) > 0\ \forall\ x \in D_f$

A-2.

Ans. **(1)**

Sol. $\int_0^t \{x\}\,dx = \int_0^{[t]} \{x\}\,dx + \int_{[t]}^t \{x\}\,dx = [t]\int_0^1 x\,dx + \int_0^{\{t\}} x\,dx = \frac{[t]}{2} + \frac{\{t\}^2}{2}$

$\therefore$ statement-2 is true.

$\int_0^{5.5} \{x\}\,dx = \frac{5}{2} + \frac{(.5)^2}{2} = \frac{21}{8}$

∴ statement-1 is true and is explained by statement-2.

A-3. Ans. (2)

Sol. **Statement -1 :** $10\int_0^{\pi}|\cos x|\,dx = 10\left[\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi}-\cos x\,dx\right] = 10 . 2 = 20$

Statement - 2 : $\int_0^{3\pi/4}\cos x\,dx = \sin x\Big|_0^{3\pi/4} = \frac{1}{\sqrt{2}}$

but $\cos x < 0,\ \forall x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

∴ statement-2 is false.

A-4. Ans. (1)

Sol. $\int_0^{2\pi}\tan^2 x\,dx = 2\int_0^{\pi}\tan^2 x\,dx$

$= 2\left[\int_0^{\pi/2}\tan^2 x\,dx + \int_0^{\pi/2}\tan^2(\pi - x)\,dx\right] = 4\int_0^{\pi/2}\tan^2 x\,dx$

∴ Statement 1 is true

statement-2 $\int_0^{nT} f(x)dx = \int_0^{T} f(x)dx + \int_T^{2T} f(x)\,dx + ... + \int_{(n-1)T}^{nT} f(x)\,dx$

$= \int_0^T f(x)\,dx + \int_0^T f(x+T)\,dx + ... + \int_0^T f(x+(n-1)T)\,dx$

$= \int_0^T f(x)\,dx + \int_0^T f(x)\,dx + ... + \int_0^T f(x)\,dx$ (∵ f has a period T)

$= n\int_0^T f(x)\,dx$

Section (B) : MATCH THE COLUMN

<u>Note : Only one answer type (1 × 1)</u>

B-1.

Ans. (A) → (s), (B) → (s), (C) → (p), (D) → (r)

Sol. (A) $\int_{-1}^{1}\frac{dx}{1+x^2} = \left(\tan^{-1}x\right)_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$ → (s)

(B) $\int_0^1\frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1}x\right)_0^1 = \frac{\pi}{2}$ → (s)

(C) $\int_2^3\frac{dx}{1-x^2} = \frac{1}{2}\left(\ell n\left|\frac{1+x}{1-x}\right|\right)_2^3 = \frac{1}{2}\ \ell n\left(\frac{2}{3}\right)$ → (p)

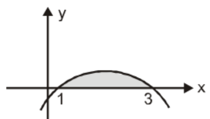
(D) $\int_1^2\frac{dx}{x\sqrt{x^2-1}} = \left(\sec^{-1}x\right)_1^2 = \sec^{-1}2 - \sec^{-1}(1) = \frac{\pi}{3}$ → (r)

B-2. Ans. (A) → (s), (B) → (s), (C) → (q), (D) → (p)

Sol. (A) $0 \le y \le 4x - x^2 - 3$

at $y = 0$, $x^2 - 4x + 3 = 0 \Rightarrow x = 1, x = 3$

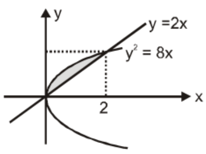
$A = \int_1^3 (4x - x^2 - 3)dx = \frac{4}{3}$



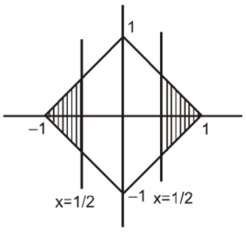
Figure

(B) $8x = 4x^2$
intersection point (0,0), (2,4)

$A = \int_0^4 \left(\frac{y}{2} - \frac{y^2}{8}\right) dy = \left(\frac{y^2}{4} - \frac{y^3}{24}\right)_0^4 = 4 - \frac{8}{3} = \frac{4}{3}$



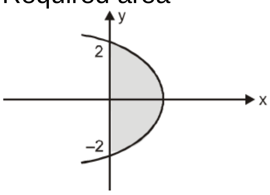
(C)



Figure

Required area = $4\left(\frac{1}{2} . \frac{1}{2} . \frac{1}{2}\right) = \frac{1}{2}$

(D)



$A = \int_{-2}^{2} (4 - y^2)\ dy = \left(4y - \frac{y^3}{3}\right)_{-2}^{2} = \frac{32}{3}$

Section (C) : ONE OR MORE THAN ONE OPTIONS CORRECT

C-1. **Sol.** given $2 = 2a + 4b \Rightarrow a + 2b = 1$ .....(1)

and $\int_0^4 (a\sqrt{x} + bx)\ dx = 16$ as $y \geq 0$

$\Rightarrow \left[\frac{2a}{3}x^{3/2} + \frac{bx^2}{2}\right]_0^4 = 16 \Rightarrow \frac{2a}{3} . 8 + 8b = 16 \Rightarrow 2a + 3b = 6$ ......(2)

from (1) and (2) $a = 9, b = -4$

C-2. **Sol.** $\varphi'(x) = (3x + 4)\int_x^3 f(u)\ du$

$\varphi''(x) = 3\int_x^3 f(u)\ du + (3x + 4)(0 - f(x))$

so $\varphi'(0) = (0 + 4)\int_x^3 f(u)\ du = 12$

$\varphi''(3) = -13f(3)$

C-3. **Sol.** $f'(x) = \sin x . \frac{1}{2\sqrt{x}} - \left(\sin\left(\frac{1}{x^2}\right)\right)\left(-\frac{1}{x^2}\right) = \frac{\sin x}{2\sqrt{x}} + \frac{1}{x^2}\sin\left(\frac{1}{x^2}\right)$

$f'(1) = \frac{3}{2}\sin 1$, $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{\sin x}{2\sqrt{x}} + \frac{\sin(1/x^2)}{x^2}\right)$

= 0 as sinx is bounded.

C-4. **Sol.** $f'(x) = \frac{d}{dx}\int_{-1}^{x}|t|\,dt = |x|$

$\because$ $f'(x) = 1$ $\Rightarrow$ $x = \pm 1$

so for x = 1 $\Rightarrow$ $f(1) = \int_{-1}^{1}|t|\,dt = 2\int_{0}^{1}|t|\,dt = 2\int_{0}^{1}t\,dt = 1$

hence point of contact is (1, 1) $\Rightarrow$ tangent is y = x

for x = –1 $\Rightarrow$ $f(-1) = \int_{-1}^{-1}|t|\,dt = 0$

point of contact is (–1, 0)

so tangent is y = x + 1

$f'(x) = \frac{d}{dx}\int_{-1}^{x}|t|\,dt = |x|$

$f'(x) = 1$ $\Rightarrow$ $x = \pm 1$

C-5. **Sol.** $x \in (0, 1)$

so, $1 + x_8 > 1 + x_9$

and $1 + x_3 > 1 + x_4$

$\Rightarrow$ $(1 + x_8)(1 + x_3) > (1 + x_9)(x + x_4)$ $\Rightarrow$ $\frac{1+x^8}{1+x^4} > \frac{1+x^9}{1+x^3}$

$\Rightarrow$ $\int_0^1 \frac{1+x^8}{1+x^4}\,dx > \int_0^1 \frac{1+x^9}{1+x^3}\,dx$ $\Rightarrow$ $I_1 > I_2$

Now again

$\frac{1+x^8}{1+x^4} < 1$ $\because$ $1 + x_4 > 1 + x_8$

$\Rightarrow$ $\int_0^1 \frac{1+x^8}{1+x^4}\,dx < \int_0^1 dx$ $\Rightarrow$ $I_1 < 1$

C-6. **Sol.** (1) $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$

$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\,dx$ (by property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$)

$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}\,dx$

$2I_n = 2\int_{0}^{\pi} \frac{\sin nx}{\sin x}\,dx$

$I_n = \int_{0}^{\pi} \frac{\sin nx}{\sin x}\,dx$

$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)\ x - \sin nx}{\sin x}\ dx = \int_0^{\pi} \frac{2\cos(n+1)\ x \sin x}{\sin x}\ dx = 2\left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = 0$

$\Rightarrow \quad I_{n+2} = I_n$

(2) $I_3 = I_5 = \ldots.. = I_{21}$

$\therefore \quad \sum_{m=1}^{10} I_{2m+1} = 10I_3 = 10\int_0^{\pi} \frac{\sin 3x}{\sin x}\ dx = 10\int_0^{\pi} (3 - 4\sin^2 x)\ dx$

$= 10\left[3x - 2x + 2\sin 2x\right]_0^{\pi} = 10\pi$

(3) $I_2 = I_4 = \ldots\ldots.. = I_{20}$

$\sum_{m=1}^{10} I_{2m} = 10\int_0^{\pi} \frac{\sin 2x}{\sin x}\ dx = 20\left[\sin x\right]_0^{\pi} = 0$

Exercise-3

1. Sol. Let $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$ ...(i)

$= \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-9+x}+\sqrt{9-x}}dx$

$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}dx$

$\Rightarrow 2I = \int_3^6 \frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}dx$

$= \int_3^6 1 \; dx$

$\Rightarrow I = \frac{3}{2}$

2. Sol. Let $I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)] \; dx$ ...(i)

and $I = \int_{-3\pi/2}^{-\pi/2}\left[\left(-\frac{\pi}{2}-\frac{3\pi}{2}-x+\pi\right)^3+\cos^2\left(-\frac{\pi}{2}-\frac{3\pi}{2}-x+3\pi\right)\right]dx$

$\Rightarrow I = \int_{-3\pi/2}^{-\pi/2} [-(x+\pi)^3 + \cos^2(\pi-x)] \; dx$ ...(ii)

(i) + (ii)

$2I = \int_{-3\pi/2}^{-\pi/2} 2\cos^2 x \; dx$

$= \int_{-3\pi/2}^{-\pi/2} (1+\cos 2x) \; dx$

$\Rightarrow I = \frac{\pi}{2}$

3. Sol. Let I = xf(sin x) dx ...(i)

$\Rightarrow I = \int_0^{\pi} (\pi - x) \; f \; [\sin \; (\pi - x)] \; dx$

$\Rightarrow I = \int_0^{\pi} (\pi - x) \; f \; (\sin x) \; dx$ ...(ii)

(i) + (ii)

$2I = \int_0^{\pi} \pi \; f \; (\sin x) \; dx$

$\Rightarrow I = \pi\int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2}-x\right)\right] dx = \pi\int_0^{\pi/2} f \; (\cos x) \; dx$

4. Sol. Let a = k + h, where [a] = k and 0 ≤ h < 1

$\therefore \int_1^a [x] \; f'(x) \; dx = \int_1^2 1f'(x) \; dx + \int_2^3 2f'(x) \; dx + ... \int_{k-1}^{k} (k-1)f'(x)dx + \int_k^{k+h} kf'(x) \; dx$

$\{f(2) - f(1)\} + 2\{f(3) - f(2)\} + 3\{f(4) - f(3)\} + ... + (k - 1)\{f(k) - f(k - 1)\} + k\{f(k + h) - f(k)\}$

$= -f(1) - f(2) - f(3) .... -f(k) + k\, f(k + h)$

$= [a]\, f(a) - \{f(1) + f(2) + f(3) + ... + f([a])\}$

5. **Sol.** Since, $f(x) = \int_1^x \frac{\log t}{1+t}\, dt$

and $F(e) = f(e) + f\left(\frac{1}{e}\right)$

$\Rightarrow F(e) = \int_1^e \frac{\log\ t}{1+t} dt + \int_1^e \frac{\log\ t}{1+t} dt$

By putting $t = 1/x \Rightarrow dt = -1/x^2\, dx$

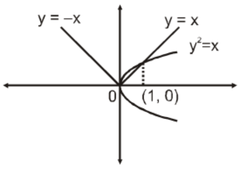
$= \int_1^e \frac{\log\ t}{1+t} dt + \int_1^e \frac{\log\ t}{(1+t)t} dt$

$= \int_1^e \frac{\log\ t}{t} dt = \left[\frac{(\log t)^2}{2}\right]_1^e$

$= \frac{1}{2}$

6. **Sol.** $\sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{2}$ $\Rightarrow$ $\sec^{-1} x = \frac{3\pi}{4}$ $\Rightarrow$ $x = -\sqrt{2}$

7.



Figure

Sol. Required area, $A = \int_0^1 (\sqrt{x} - x)dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1$

$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ sq unit

8. **Sol.** Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx$,

because $x \in (0, 1)$, $x > \sin x$

$I < \int_0^1 \sqrt{x}\ \ dx = \frac{2}{3}\left[x^{3/2}\right]_0^1$

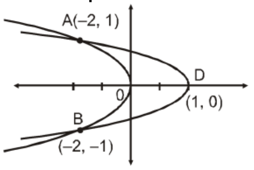
$\Rightarrow I < \frac{2}{3}$

and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 x^{-\frac{1}{2}} dx = 2$

$J < 2$

9. **Sol.** $x + 2y^2 = 0$ $\Rightarrow$ $y^2 = -\frac{1}{3}(x - 1)$

[Left handed parabola with vertex at (1, 0)]
Solving the two equations we get the points of intersection as (–2, 1), (–2, –1)
The required area is AOBDA, given by



Figure

$= \left|\int_{-1}^{1}(1-y^2)dy\right| = \left|\left[y-\frac{y^3}{3}\right]_{-1}^{1}\right|$

$= 2 \times \frac{2}{3} = \frac{4}{3}$ sq. units.

10. Sol. Let $I = \int_0^{\pi}[\cot x]dx$ ...(i)

$\Rightarrow I = \int_0^{\pi}[\cot(\pi - x)]dx = \int_0^{\pi}[-\cot x]dx$ ...(ii)

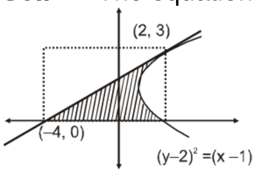
(i) + (ii)

$2I = \int_0^{\pi}[\cot x]dx + \int_0^{\pi}[-\cot x]dx$

$= \int_0^{\pi}(-1)dx = -\pi$

$I = -\frac{\pi}{2}$

11. Sol. The equation of tangent at (2, 3) to the given parabola is x = 2y – 4



Figure

∴ Required area $= \int_0^3 \{(y-2)^2+1-2y+4\}\ dy = \left[\frac{(y-2)^3}{3} - y^2 + 5y\right]_0^3$

$= \frac{1}{3} - 9 + 15 + \frac{8}{3}$

= 9 sq. unit

12. Ans.

Sol. p'(x) = p'(1 – x)

⇒ p(x) = –p(1 – x) + c

put x = 0

p(0) = – p(1) + c

⇒ c = 42

$I = \int_0^1 p(x)dx$ $I = \int_0^1 p(1-x)dx$

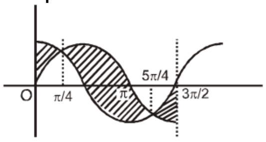
$2I = \int_0^1 (p(x)+p(1-x))\ dx = \int_0^1 c\,dx = \int_0^1 42dx$

$2I = 42 \quad \Rightarrow \quad I = 21$

Hence correct option is (1)

13. Ans. (4)

Sol. Required area = $\int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx + \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)dx$



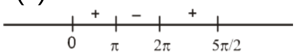
$= 2\left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4} = 4\sqrt{2} - 2$

Hence correct option is (4)

14. Sol. (4)

$f(x) = \int_0^x \sqrt{t}\sin t\,dt$

$f'(x) = \sqrt{x}\ \sin x$



local maximum at π

and local minimum at 2π **Ans.**

15. 2

Sol. (1)

$x = \tan\theta$

$dx = \sec^2\theta\ d\theta$

$I = \int_0^{\pi/4} \frac{8\ell n(1+\tan\theta)}{\sec^2\theta}\sec^2 d\theta = 8\int_0^{\pi/4} \ell n(1+\tan\theta)d\theta$

$\Rightarrow \quad I = 8\int_0^{\pi/4} \ell n(1+\tan(\pi/4-\theta))d\theta$

$= 8\int_0^{\pi/4} \ell n\left(\frac{2}{1+\tan\theta}\right)\ d\theta \quad \Rightarrow \quad 2I = 8\int_0^{\pi/4} \ell n(2)\ d\theta$

$= 8\ \frac{\pi}{4}\ \ell n2 = 2\pi\ell n2 \quad \Rightarrow \quad I = \pi\ell n2$

16. Sol. (3)

$\int_0^1 x\ [x^2]\ dx + \int_1^{\sqrt{2}} x\ [x^2]\ dx + \int_{\sqrt{2}}^{1.5} x\ [x^2]\ dx$

$\int_0^1 x.\ 0\ dx + \int_1^{\sqrt{2}} x\ dx + \int_{\sqrt{2}}^{1.5} 2x\ dx$

$0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[x^2\right]_{\sqrt{2}}^{1.5}$

$\frac{1}{2}$ (2 – 1) + (2.25 – 2)

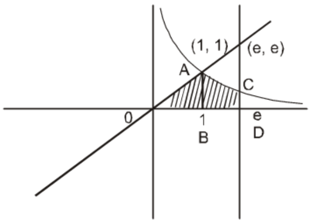
$\frac{1}{2} + .25$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

17. **Sol.** **(3)**

Required area

= OAB + ACDB

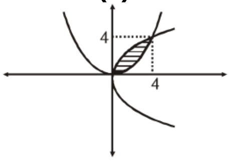


$= \frac{1}{2} \times 1 \times 1 + \int_1^e \frac{1}{x} dx$

$= \frac{1}{2} + (\ell n x)_1^e$

$= \frac{3}{2}$ square unit **Ans.**

18. **Sol.** **(2)**



Area $= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \left(2\left(\frac{x^{3/2}}{3/2}\right) - \frac{x^3}{12}\right)_0^4$

$= \frac{4}{3} \times 8 - \frac{64}{12}$

$= \frac{32}{3} - \frac{16}{3}$

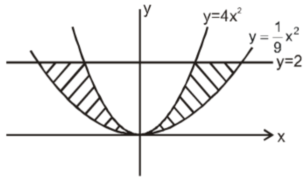
$= \frac{16}{3}$

19*. **Sol.** $g(x + \pi) = \int_0^{x+\pi} \cos 4t \quad dt = g(x) + \int_0^{\pi} \cos 4t \quad dt$

$= g(x) + g(\pi)$

Here $g(\pi) = \int_0^{\pi} \cos 4t \quad dt = 0$

so answers are (2) or (3)



20. Sol.

$y = 4x^2$

$y = \frac{1}{9}x^2$

Area = $2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$

$= 2\left[\frac{5}{2}\frac{y\sqrt{y}}{3/2}\right]_0^2 = 2.\ \frac{5}{3}\ 2\sqrt{2} = \frac{20\sqrt{2}}{3}$

21. Sol. (1)

$\frac{dy}{dx} = |x| = 2$

$x = \pm 2$

points $y = \int_0^{\pm 2} |t|\,dt = \pm 2$

∴ equation of tangent is

$y - 2 = 2(x - 2)$ or $y + 2 = 2(x + 2)$

⇒ x - intercept = ± 1.

22. Sol. (4)

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$

⇒ $2I = \int_{\pi/6}^{\pi/3} dx$

⇒ $I = \frac{1}{2}\ ,\ \left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12}$ statement -1 is false

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ it is property

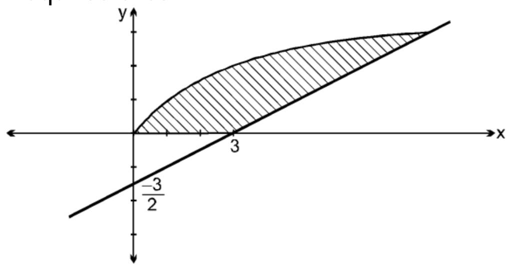
23. Sol. (1)

$y = \sqrt{x}$ ....(1)

and $2y - x + 3 = 0$ ....(2)

On solving both y = – 1, 3

Required area = $\int_0^3 \left\{(2y+3) - y^2\right\} dy$



$= \left. y^2 + 3y - \frac{y^3}{3} \right|_0^3$

= 9 + 9 – 9

= 9.

24. Sol. Ans. (2)

$I = \int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx = \int_0^{\pi} |1 - 2\sin x/2|\, dx$

$= \int_0^{\pi/3} |1 - 2\sin x/2| + \int_{\pi/3}^{\pi} |1 - 2\sin x/2|\, dx = \int_0^{\pi/3} (1 - 2\sin x/2) dx + \int_{\pi/3}^{\pi} (2\sin x/2 - 1) dx$

$= \left(x + 2\frac{\cos\frac{x}{2}}{\frac{1}{2}}\right)_0^{\pi/3} + \left(-2\frac{\cos\frac{x}{2}}{\frac{1}{2}} - x\right)_{\pi/3}^{\pi} = \left(\frac{\pi}{3} + 4\frac{\sqrt{3}}{2}\right)$ – (4) + (0 – π) – $\left(\pi - 4 \times \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)$

$= \frac{\pi}{3} + 2\sqrt{3} - 4 - \pi + 2\sqrt{3} + \pi/3 = -4 - \pi/3 + 4\sqrt{3}$

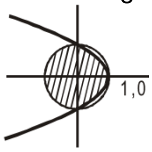
25. Sol. Ans. (3)

Intersection of $x^2 + y^2 = 1$ & $y^2 = 1 - x$

is x = 0, 1

The required portion is shaded as shown.

Area of region is area of semi-circle plus area bounded by parabola & y-axis.



Area of semi-circle is . $\frac{\pi}{2}$

Area bounded by parabola = $\frac{2}{3}$ of corresponding rectangle

$= \frac{2}{3} \times 1 \times 2 = \frac{4}{3}$

Hence total area $= \frac{\pi}{2} + \frac{4}{3}$.

Method - 1

Required area = area of semi circle + area bounded by parabola

$= \frac{\pi}{2} + \int_0^1 (1 - y^2)\, dy = \frac{\pi}{2} + 2\left(y - \frac{y^3}{3}\right)_0^1$

$= \frac{\pi}{2} + 2\left(1-\frac{1}{3}\right) \Rightarrow \frac{\pi}{2} + \frac{4}{3}$

26. **Ans.** **(3)**

Sol. $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(x^2 - 12x + 36)} dx$

$I = \frac{2}{2}\int_2^4 \frac{\log x}{\log x + \log(6-x)} dx$ ...(i)

$I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x} dx \quad \left\{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right\}$ ...(ii)

Equation (i) & (ii) gives

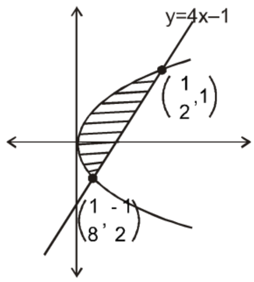
$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)} dx = \int_2^4 dx = 2$

Hence I = 1

27. **Ans.** **(4)**

Sol. $\int_{-1/2}^{2} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy$

$\Rightarrow \frac{1}{4}\left\{\frac{y^2}{2} + y\right\}_{-1/2}^{1} - \frac{1}{6}\{y^3\}_{-1/2}^{1}$



$\Rightarrow \frac{1}{4}\left\{\left(\frac{1}{2}+1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$

$\Rightarrow \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\} \Rightarrow \frac{15}{32} - \frac{6}{32} = \frac{9}{32}$

28. **Ans.** **(1)**

Sol. $p = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)......(n+2n)}{n^{2n}}\right)$

$\log p = \frac{1}{n}\left(\lim_{n\to\infty}\sum_{r=1}^{2n} \log\left(1 + \frac{r}{n}\right)\right)$

$\log p = \int_0^2 \log(1+x) dx$

$\log p = (x\log(1+x))_0^2 - \int_0^2 \frac{x}{1+x} dx$

$\log p = 2\log 3 - \int_0^2 \left(1 - \frac{1}{1+x}\right) dx$

$\log p = 2\log 3 - (x - \log(1 + x))\Big|_0^2$

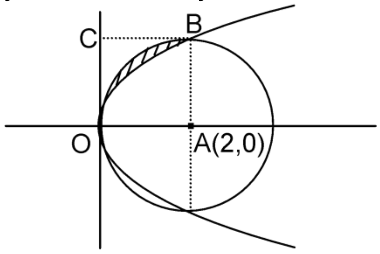
$\log p = 2\log 3 - (2 - \log 3)$

$\log p = 3\log 3 - 2 = \log \frac{27}{e^2}$

$p = \frac{27}{e^2}$

29. Ans. (1)

Sol. $y^2 = 2x$ and $x^2 + y^2 = 4x$ meet at O(0, 0) and B(2, 2) {(2, –2) is not considered as x, y ≥ 0}

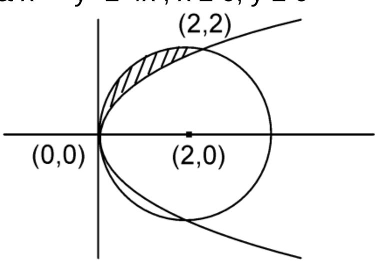


Now required area = (Area of quadrant of circle) – $\frac{2}{3}$ (Area of rectangle OABC)

$= \pi - \frac{2}{3}.(2.2) = \pi - \frac{8}{3}$

Alter :

$y^2 \ge 2x$ & $x^2 + y^2 \le 4x$; $x \ge 0, y \ge 0$



$x^2 + 2x = 4x$

$x^2 - 2x = 0$

$x = 0, 2$

$\int_0^2 (\sqrt{4x - x^2} - \sqrt{2x})\, dx = \pi - \frac{8}{3}$

$\int_0^2 (\sqrt{4 - (x-2)^2} - \sqrt{2}\sqrt{x})\, dx$

$\left(\left(\frac{(x-2)}{2}\sqrt{4x - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right) - \frac{\sqrt{2}(x^{3/2})}{3}(2)\right)_0^2$

$\left(\frac{-2\sqrt{2}}{3}(2^{3/2}) - \left(2\sin^{-1}(-1)\right)\right)$

$\frac{-2\sqrt{2}}{3}(2\sqrt{2}) - 2\left(\frac{-\pi}{2}\right) = \pi - \frac{8}{3}$

30. **Ans. (1)**

Sol. $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

$\int \frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$

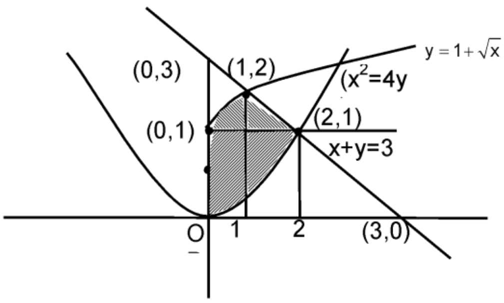
Let ¼½ $1+\frac{1}{x^2}+\frac{1}{x^5}=t$

$\frac{dt}{dx}=\frac{-2}{x^3}-\frac{5}{x^6}$

$\int \frac{-dt}{t^3}=\frac{1}{2t^2}+C=\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+C=\frac{x^{10}}{2(x^5+x^3+1)^2}+C$

31. **Ans. (4)**

Sol.



$y=1+\sqrt{x}$

$(y-1)^2 \le x$

Required area $=\int_0^1\left(1+\sqrt{x}\right)dx+\int_1^2(3-x)dx-\int_0^2\frac{x^2}{4}dx$

$=\left(x+\frac{2x^{3/2}}{3}\right)\Big|_0^1+\left(3x-\frac{x^2}{7}\right)\Big|_1^2-\left(\frac{x^3}{12}\right)_0^2=1+\frac{2}{3}+\left\{(6-2)-\frac{5}{2}\right\}-\frac{8}{12}$

$=1+\frac{2}{3}+\left(4-\frac{5}{2}\right)-\frac{2}{3}=1+\frac{3}{2}=\frac{5}{2}$

32. **Ans. (2)**

Sol. $I=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{1+\cos x}dx$

Using property , $\int_a^b f(x).dx = \int_a^b f(a+b-x).dx$

I = $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1-\cos x}dx$

On adding we get,

2I = $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1-\cos^2 x}dx$

2I = $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\operatorname{cosec}^2 x.dx$

I = $(-\cot x)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ = 2

33. Sol. (2)

I = $\int_{+\pi/2}^{+\pi/2} \frac{\sin^2 x}{1+2^x}dx$

I = $\int_0^{+\pi/2} \left(\frac{\sin^2 x}{1+2^x} + \frac{\sin^2 x}{1+2^{-x}}\right)dx$

property $\int_{-9}^{+9} f(x)dx = \int_0^9 (f(x)+(-x))dx$

I = $\int_0^{\pi/2} \sin^2 x \quad dx = \int_0^{\pi/2} \frac{(1-\cos(2x))}{2}dx = \left(\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)_0^{\pi/2}\right) = \frac{1}{2}[\pi/2 - 0] = \pi/4$

PART - II : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Solution

L = $\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\ dt}{x^2 - \frac{\pi^2}{16}} \quad \frac{0}{0}$ form

By L. Hospital rule

L = $\lim_{x\to\frac{\pi}{4}} \frac{(f(\sec^2 x))2\sec^2 x\tan x - 0}{2x}$

L = $\frac{8f(2)}{\pi}$

L = $\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\ dt}{x^2 - \frac{\pi^2}{16}} \quad \frac{0}{0}$:i

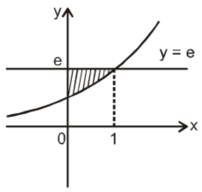
$L = \dfrac{\lim\limits_{x\to\frac{\pi}{4}} (f(\sec^2 x))2\sec^2 x \tan x - 0}{2x}$

$L = \dfrac{8f(2)}{\pi}$

2. **Sol.** Shaded area = $\mathbf{e - \left(\int_0^1 e^x \ dx\right) = 1}$

Also $\int_1^e \boldsymbol{\ell n\,(e + 1 - y)\ dy}$

put $e + 1 - y = t \quad \Rightarrow \quad - dy = dt$



$= \int_e^1 \ell n\ t\ (-dt) = \int_1^e \ell n\ t\ dt = \int_1^e \boldsymbol{\ell n\ y\ dy = 1}$

3. **Sol.** $\lim\limits_{x\to 0} \dfrac{x\ell n\ (1+x)}{(x^4+4)\ \times 3x^2} = \lim\limits_{x\to 0} \dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{12}$

4. **Sol.** $\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \dfrac{x^4[(1+x^2)-2x]^2}{1+x^2} dx = \int_0^1 \dfrac{x^4[(1+x^2)^2 - 4x(1+x^2)+4x^2]}{1+x^2} dx$

$= \int_0^1 x^4\left[(1+x^2) - 4x + \dfrac{4x^2}{1+x^2}\right] dx = \int_0^1 \left[x^6 + x^4 - 4x^5 + \dfrac{4x^6}{1+x^2}\right] dx$

Now on polynomial division of x^6 by $1 + x^2$, we obtain

$= \int_0^1 \left[x^6 + x^4 - 4x^5 + 4\left[(x^4 - x^2 + 1) - \dfrac{1}{1+x^2}\right]\right] dx = \int_0^1 \left[\left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right) - \dfrac{4}{1+x^2}\right] dx$

$= \left[\dfrac{x^7}{7} - \dfrac{4x^6}{6} + \dfrac{5\ .\ x^5}{5} - \dfrac{4x^3}{3} + 4x\right]_0^1 - 4\left[\tan^{-x} x\right]_0^1 = \left(\dfrac{1}{7} - \dfrac{4}{6} + 1 - \dfrac{4}{3} + 4\right) - 4\left(\dfrac{\pi}{4}\right) = \left[\dfrac{1}{7} - \dfrac{12}{6} + 5\right] - \pi$

$= \left(\dfrac{1}{7} + 3\right) - \pi = \dfrac{22}{7} - \pi$

5. **Sol.** $f(x) = e^{\left(2 + \int_0^x \sqrt{t^4+1}\ dt\right)}$

Let $g(x) = f^{-1}(x) \quad \Rightarrow \quad g(f(x)) = x$

$\Rightarrow \quad g'(f(x))\ f'(x) = 1$

$\Rightarrow \quad g'(2) = \dfrac{1}{f'(0)} \quad (\because f(0) = 2)$

Now $f'(x) = e^{\left(2 + \int_0^x \sqrt{t^4+1}\ dt\right)} + e^x \sqrt{x^4+1}$ (Applying Leibinitz Rule)

$\Rightarrow \quad f'(0) = 2 + 1 = 3$

$\Rightarrow \quad g'(2) = \dfrac{1}{3}$

⇒ $(f^{-1})'(2) = \frac{1}{3}$

6. **Ans.** **(A)**

Sol. Put $x^2 = t$

$x\,dx = \frac{dt}{2}$

$I = \int_{\ell n2}^{\ell n3} \frac{\sin t}{\sin t + \sin\ (\ell n6 - t)} \cdot \frac{dt}{2}$ ......(1)

apply $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin(\ell n6 - t)}{\sin(\ell n6 - t) + \sin t}\,dt$ ....(2)

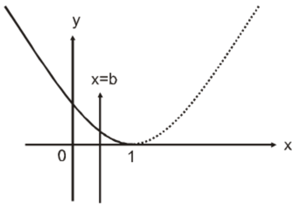
adding (1) and (2)

$2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} 1.dt$

⇒ $I = \frac{1}{4}\ell n\frac{3}{2}$

7. **Sol.** $R_1 = \int_0^b (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_0^b = \frac{(b-1)^3 + 1}{3}$

also $R_2 = \int_b^1 (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_b^1 = -\frac{(b-1)^3}{3}$



⇒ $R_1 - R_2 = \frac{2(b-1)^3}{3} + \frac{1}{3}$

⇒ $\frac{1}{4} = \frac{2(b-1)^3}{3} + \frac{1}{3}$ ⇒ $(b-1)^3 = -\frac{1}{8}$

⇒ $b = \frac{1}{2}$

8. **Sol.** $R_2 = \int_{-1}^{2} f(x)\ dx$ and $R_1 = \int_{-1}^{2} xf(x)\ dx$

⇒ $= \int_{-1}^{2} (1-x)f(1-x)\ dx$

$= \int_{-1}^{2} (1-x)f(x)\ dx$

$R_1 = R_2 - R_1$

⇒ $2R_1 = R_2$

9. **Sol.** **Ans. (B)**

$\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\left(\frac{\pi+x}{\pi-x}\right)\right)$ cosx dx = $2\int_0^{\pi/2} x^2\cos x dx + 0$ $\left(\because \ell n\left(\frac{\pi+x}{\pi-x}\right) \text{is an odd function}\right)$

$= 2\left[\left(x^2\sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x\sin x\ dx\right]$

$= 2\left(\frac{\pi^2}{4}-0\right) - 4\int_0^{\pi/2} x\sin x\ dx$

$= \frac{\pi^2}{2} - 4\left[\left(-x\cos x\right)_0^{\pi/2} + \int_0^{\pi/2}\cos x\ dx\right]$

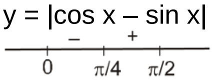
$= \frac{\pi^2}{2} - 4$

10. **Sol.** **(B)**

Given y = sin x + cos x $\qquad x \in [0, \pi/2]$

$\frac{dy}{dx}$ = cos x – sin x

y = |cos x – sin x| = $\begin{cases} \cos x - \sin x & x \in [0, \pi/4] \\ \sin x - \cos x & x \in [\pi/4, \pi/2] \end{cases}$



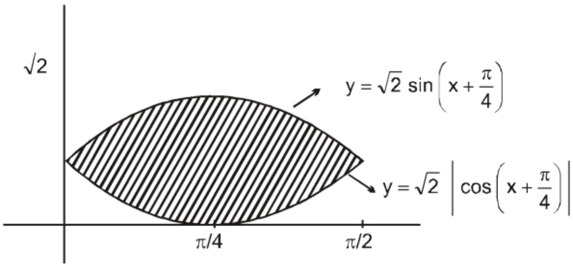
required area = $\int_0^{\pi/4} \left|(\sin x + \cos x) - (\cos x - \sin x)\right| dx + \int_{\pi/4}^{\pi/2} |2\cos x|\ dx$

$= \int_0^{\pi/4} |2\sin x|\ dx + \int_{\pi/4}^{\pi/2} |2\cos x|\ dx = 2\left(-\cos x\right)_0^{\pi/4} + 2\left(\sin x\right)_{\pi/4}^{\pi/2} = 2\left[-\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right]$

$= 2\left(2 - \frac{2}{\sqrt{2}}\right)$

$= 2\left(2 - \sqrt{2}\right)$

$= 4 - 2\sqrt{2}$



$= 2\sqrt{2}\left(\sqrt{2} - 1\right)$

11. **Sol.** **(D)**

f'(x) – 2f(x) < 0

$\frac{d}{dx}\left(e^{-2x} f(x)\right) < 0$

⇒ $e^{-2x} f(x)$ is decreasing

⇒ x > 1/2

$e^{-2x} f(x) < 1/e$

⇒ $f(x) < e^{2x-1}$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)\ dx < \int_{1/2}^{1} \left(e^{2x-1}\right)\ dx \qquad \Rightarrow \qquad 0 < \int_{1/2}^{1} f(x)\ dx < \frac{e-1}{2}$

12.* **Sol.** **(B) (JEE given B, D answer)**

$$\frac{2\sum r^a}{(n+1)^{a-1}\ \left(2n^2a+n^2+n\right)} \Rightarrow \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{(1+1/n)^{a-1}\ \left(2n^2a+n^2+n\right)} \Rightarrow \frac{2\int_0^1 x^a dx}{2a+1}$$

$$\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$$

120 = (2a + 1) (a + 1)

a = 7,–17/2 **(–17/2 reject)**

13. **Ans.** **(A)**

Sol. $I = \int_{\pi/4}^{\pi/2} (2\operatorname{cosec}x)^{17} dx$

Put $\ell n \tan x/2 = t$ $\qquad \tan\frac{x}{2} = e^t$

$\Rightarrow \qquad \sin x = \frac{2e^t}{1+e^{2t}}$

$\operatorname{cosec} x = \frac{e^t+e^{-t}}{2}$

$$I = 2\int_{\ell n(\sqrt{2}-1)}^{0} (e^t+e^{-t})^{16}.dt$$

$$= 2\int_{-\ell n(\sqrt{2}+1)}^{0} (e^t+e^{-t})^{16}.dt$$

since$(e^t + e^{-t})^{16}$ is an even function

$$\int_{-a}^{0} = \int_{0}^{a}$$

Hence $I = \int_{0}^{\ell n(\sqrt{2}+1)} 2(e^t+e^{-t})^{16} dt$

14. **Ans.** **(B)**

Sol. f'(x) = 2x f(x)

$\frac{f'(x)}{f(x)} = 2x$

$\ell n(f(x)) = x^2 + c$

x = 0, f(0) = 1

c = 0

∴ $\ell n(f(x)) = x^2$

$f(x) = e^{x^2}$

∴ F(x) = f(x) + c

$F(x) = e^{x^2} + c$

$F(0) = 0$

∴ $c = -1$

∴ $f(x) = e^{x^2} - 1$

$f(2) = e^4 - 1$

15. **Sol.** $\alpha = \int_0^1 e^{9x+3\tan^{-1}x}\cdot\left(\frac{12+9x^2}{1+x^2}\right) dx$

⇒ $\alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$

⇒ $\alpha = e^{9+\frac{3\pi}{4}} - 1$

⇒ $\ell n(1 + \alpha) = 9 + \frac{3\pi}{4}$

Aliter :

$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t$

⇒ $\left(9+\frac{3}{1+x^2}\right)dx = dt$ ⇒ $\left(\frac{12+9x^2}{1+x^2}\right)dx = dt$

⇒ $\alpha = \int_0^{9+3\pi/4} e^t dt = \left(e^t\right)_0^{9+3\pi/4}$ $= e^{9+3\pi/4} - 1$

Now $\log_e|1 + \alpha| - 3\pi/4 = \log_e e^{(9+3\pi/4)} - 3\pi/4 = 9$

16. **Sol.** $I = \int_{-1}^2 \frac{x[x^2]}{2+[x+1]}dx = \int_{-1}^2 \frac{x[x^2]}{3+[x+1]}dx = \int_{-1}^0 \frac{0}{3-1}dx + \int_0^1 \frac{0}{3+0}dx + \int_1^{\sqrt{2}} \frac{x.1}{3+1}dx$

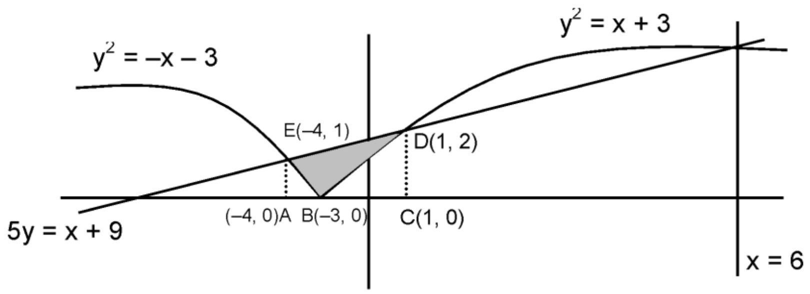
$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} = \frac{2-1}{8} = \frac{1}{4}$ ∴ $4I - 1 = 0$

17. **Ans.** **(A)**

Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{x^2\cos x}{(1+e^x)}dx$ ⇒ $I = \int_0^{\pi/2}\left(\frac{x^2\cos x}{1+e^x} + \frac{x^2\cos x}{1+e^{-x}}\right)dx$

$I = \int_0^{\pi/2} x^2\cos x dx = (x^2\sin x - 2x(-\cos x) + (2)(-\sin x))_0^{\pi/2}$ $= \left(\frac{\pi^2}{4} - 2\right) - (0) = \frac{\pi^2}{4} - 2$

18. **Ans.** **(C)**



Area ABE (under parabola) = $\int_{-4}^{-3}\sqrt{-x-3}\,dx = \frac{2}{3}$

Area BCD (under parabola) = $\int_{-3}^{1}\sqrt{x+3}\,dx = \frac{16}{3}$

Area of trapezium ACDE = $\frac{1}{2}(1 + 2)5 = \frac{15}{2}$

Required area = $\frac{15}{2}-\frac{16}{3}-\frac{2}{3}=\frac{3}{2}$