Exercise-1

* Marked Questions may have more than one correct option.

OBJECTIVE QUESTIONS

PARABOLA

Section (A) : Elementary Concepts of Parabola

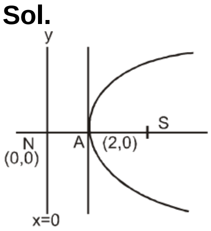
A-1. **Sol.** Eq. of the parabola is

$\sqrt{(x+3)^2+y^2} = |x + 5|$

$x^2 + 6x + 9 + y^2 = x^2 + 25 + 10x$

$y^2 = 4(x + 4)$

A-2. **Sol.**



A is the mid point of N & S, focus is (4, 0)

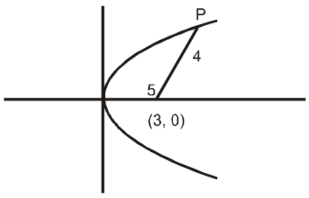
A-3. **Sol.** $(x-2)^2 + (y-3)^2 = \left|\frac{3x-4y+7}{5}\right|^2$

∴ focus is (2, 3) & diretrix is $3x - 4y + 7 = 0$

latus vectum = 2 × ⊥r distance from focus to

directrix = $2 \times \frac{1}{5}$ = 2/5

A-4. **Sol.**



Let the point P is $(3t^2, 6t)$

and PS = $3 + 3t^2 = 4$

$t^2 = 1/3$

$t = \pm\frac{1}{\sqrt{3}}$

∴Points are

$(1, 2\sqrt{3})$ & $(1, -2\sqrt{3})$

A-5. **Sol.** Latus rectum = 2 × distance of focus from directrix

$= 2 \times \left|\dfrac{3-4-2}{\sqrt{1^2+1^2}}\right| = 3\sqrt{2}$

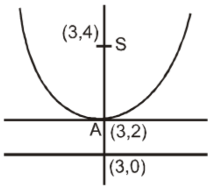
A-6. **Sol.** $y^2 - 12x - 4y + 4 = 0$

$y^2 - 4y = 12x - 4$

$(y - 2)^2 = 12x$

$Y^2 = 12X$

focus : X = A, Y = 0



x = 3, y = 2

A(3, 2)

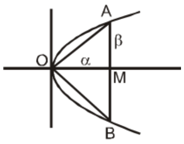
equ. of directrix is y = 0, PS = PM

$\sqrt{(x-3)^2+(y-4)^2} = |y|$

by squaring, we will get $(x-3)^2+(y-4)^2 = y^2$

$x^2 - 6x - 8y + 25 = 0$

A-7. **Sol.**



∠ AOM = 30° as angle ∠ AOB = 60º

$\tan 30° = \dfrac{\beta}{\alpha}$

$\alpha = \beta\sqrt{3}$

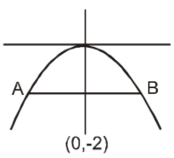
∴A is $(\beta\sqrt{3}, \beta)$

Now A will satisfy equation of parabola $y^2 = 4x$

$\beta^2 = 4\cdot\beta\sqrt{3}$ ⇒ $\beta = 4\sqrt{3}$ ⇒ $\beta \neq 0$

∴ AB = $8\sqrt{3}$

A-8. **Sol.**



$x^2 = -8y$

∴ a = 2

focus is (0, –2)

Clearly for A, B both, y = –2

∴ $x^2 = -8(-2) = 16$

∴ $x = \pm 4$

∴ A is $(-4, -2)$, B is $(4, -2)$

A-9. Sol. Parabola is $(x + 2)^2 = -2(y - 2)$

$$X^2 = -2Y$$

where $X = x + 2$, $Y = y - 2$

vertex is $(-2, 2)$

$4a = 2$

∴ $a = \frac{1}{2}$

latus rectum's equation is $Y = -a \Rightarrow Y = -\frac{1}{2}$

$\Rightarrow y - 2 = -\frac{1}{2}$ $\Rightarrow y = \frac{3}{2}$ $\Rightarrow 2y = 3$

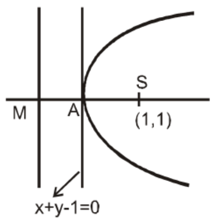
A-10. Sol. Parabola is $9\left(x^2 - \frac{2}{3}x\right) = -36y - 9$

$\Rightarrow 9\left(x - \frac{1}{3}\right)^2 = -36y - 8$

$\Rightarrow \left(x - \frac{1}{3}\right)^2 = -4\left(y + \frac{2}{9}\right)$

clearly, vertex is $\left(\frac{1}{3}, -\frac{2}{9}\right)$

A-11. Sol.



Point A is $\left(\frac{1}{2}, \frac{1}{2}\right)$

∴ M is $(0, 0)$

∴ Eq. of Directrix is $x + y = 0$

∴ Eq. of parabola is $(x - 1)^2 + (y - 1)^2 = \left(\frac{x + y}{\sqrt{2}}\right)^2$

Length of latus rectum = 2(⊥r distance from focus to the directrix) $= 2\left|\frac{1+1}{\sqrt{2}}\right| = 2\sqrt{2}$

A-12. Sol. $4a = 16$

∴ $a = 4$

as $y_1 = 2x_1$

∴ $y_1^2 = 16x_1$ gives $y_1 = 8$, $x_1 = 4$

∴ point is $(4, 8)$

focal distance = $x_1 + a = 4 + 4 = 8$

A-13. **Sol.** $x^2 - 2 = -2\cos t$, $y = 4\cos^2\frac{t}{2}$

$\cos t = \frac{x^2-2}{-2}$, $y = 4\cos^2\frac{t}{2}$

$y = 2\left(2\cos^2\frac{t}{2}\right)$

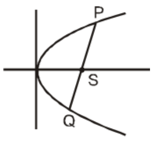
$y = 2(1 + \cos t)$

$y = 2\left(1+\frac{x^2-2}{-2}\right)$

$y = 2 + 2 - x^2$

$y = 4 - x^2$

A-14. **Sol.**



From the property $\frac{1}{PS} + \frac{1}{QS} = \frac{1}{a}$

$\frac{1}{3} + \frac{1}{2} = \frac{1}{a}$

$a = \frac{6}{5}$ $\therefore$ Latus rectum = $4a = \frac{24}{5}$

Section (B) : Position of point /Line, Chord

B-1. **Sol.** $(2)^2 - 4(3) < 0$

B-2. **Sol.** $(\alpha-1)^2 - 4\alpha + 4 < 0 \Rightarrow \alpha_2 - 6\alpha + 5 < 0$

$(\alpha-1)(\alpha-5) < 0 \Rightarrow \alpha \in (1, 5)$

B-3. **Sol.** $(2 - 2x)^2 = 4x \Rightarrow x^2 - 2x + 1 = x$

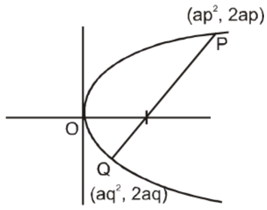
$x^2 - 3x + 1 = 0$ (roots x_1, x_2)

$(x_1 - x_2)^2 = 9 - 4 = 5$

similarly $(y_1 - y_2)^2 = 20$

Length of chord = $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{5+20} = 5$

B-4. **Sol.**



slope of PQ = $\frac{2a(p-q)}{a(p-q)(p+q)}$ = 1

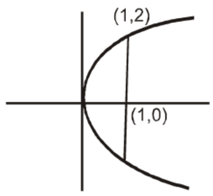
$\therefore$ p + q = 2

B-5. **Sol.** Length of chord = $\frac{4}{m^2}\sqrt{a(a-mc)(1+m^2)}$

m = tan 60° = $\sqrt{3}$

Length of chord = $\frac{4}{3}\sqrt{3(3-\sqrt{3}\times 0)(1+3)} = \frac{4}{3}\sqrt{36} = 8$

B-6. **Sol.**



$y^2 = 4x$, the other end of focal chord will be (1, –2)

B-7. **Sol.** $4a = 6 \Rightarrow a = \frac{3}{2}$

Negative end of latus rectum is (a, –2a), i.e. $\left(\frac{3}{2}, -3\right)$, vertex is (0, 0)

Line through these two points is $\frac{y-0}{x-0} = \left(\frac{-3-0}{\frac{3}{2}-0}\right)$ or 2x + y = 0

Section-(C) : Tangent of Parabola

C-1. **Sol.** $\lambda = c = \frac{a}{m} = \frac{2}{1} = 2$

C-2. **Sol.** $y = 2x - 3$, $y^2 = 4a\left(x - \frac{1}{3}\right)$

$(2x-3)^2 = 4a\left(x - \frac{1}{3}\right)$

$4x^2 - 12x + 9 = 4ax - \frac{4a}{3}$

$4x^2 - (12 + 4a)x + \left(9 + \frac{4a}{3}\right) = 0$

$\because$ D = 0

$(12 + 4a)^2 - 4.4.\left(9 + \frac{4a}{3}\right) = 0$

$\therefore$ a = $\frac{-14}{3}$

C-3. **Sol.** Slope of tangent = $\frac{1-0}{4-3}$ = 1

also $\frac{dy}{dx} = 2(x - 3)$

$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x, - 3) = 1$ $\Rightarrow$ $x_1 - 3 = \frac{1}{2}$

$x_1 = \frac{7}{2}$

$\therefore y_1 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$

Equation of tangent is

$y - \frac{1}{4} = 1\left(x - \frac{7}{2}\right)$

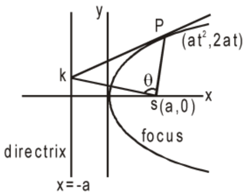
4y – 1 = 2(2x – 7)

4x – 4y = 13

C-4. **Sol.** $\Rightarrow c = \frac{a}{m} \Rightarrow -(a + 3) = \frac{3a}{2(a+3)}$

$\Rightarrow 2a^2 + 15a + 18 \Rightarrow a = -\frac{3}{2}$ or – 6

C-5. **Sol.**



Let P be $(at^2, 2at)$, S be (a, 0) equation of tangent at P is

$ty = x + at^2$

at K, x = –a,

$\therefore$ y = $\frac{-a + at^2}{t} = \frac{a(t^2 - 1)}{t}$

$\therefore$ K is $\left(-a, \frac{a(t^2 - 1)}{t}\right)$

slope KS = $\frac{a(t^2-1)}{t} \times \frac{1}{-2a} = -\frac{1}{2}\frac{t^2-1}{t}$

slope PS = $\frac{2at-0}{at^2-a} = \frac{2t}{t^2-1}$

clearly, slope of KS × slope of PS = –1

∴ θ = 90º

C-6. **Sol.** Let the equation of tangent is $y = mx + \frac{a}{m}$

$y = mx + \frac{3}{m}$ ...(1)

$\tan 45° = \left|\frac{m-3}{1+3m}\right|$

⇒ $\frac{m-3}{1+3m} = \pm 1$

⇒ 4m – 2 = 0 & 2m + 4 = 0

⇒ $m = \frac{1}{2}$ & – 2

∴ equation of tangents

$y = -2x - \frac{3}{2}$ & $y = \frac{1}{2}x + 6$

2y = –4x – 3 & 2y = x + 12

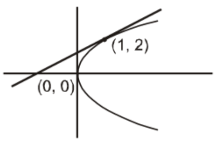
C-7. **Sol.** $y = mx + \frac{a}{m}$

$y = \frac{1}{4}x + \frac{7.4}{4.1}$

$y = \frac{x}{4} + 7 \Rightarrow x - 4y + 28 = 0$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ = (28, 14)

C-8. **Sol.**



Equation of tangent at(1, 2) is

2y = 2(x + 1)

x – y + 1 = 0 .... (i)

Image of (0, 0) in the line (i) is (– 1, 1)

∴ Vertex of required parabola will be (– 1, 1)

∴ Mirror image of parabola is $(x + 1)^2 = 4(y - 1)$

C-9. **Sol.** Let the equation of tangent is

$y = mx + \frac{9}{4m}$ .... (i)
Above equation pases through (4, 10)

$10 = 4m + \frac{9}{4m}$
$16m^2 - 40m + 9 = 0$
$(4m - 9)(4m - 1) = 0$

$m = \frac{9}{4}, \frac{1}{4}$
put in (i), we get

$y = \frac{9}{4}x + 1$ & $y = \frac{1}{4}x + 9$
$4y = 9x + 4$ & $4y = x + 36$

Section-(D) : Normal, Pair of tangents, Director circle, Chord of conctact,chord with given mid point

D-1. **Sol.** Director circle of parabola is direcctrix
so $X = -a$

$\Rightarrow x - 1 = -\frac{5}{2} \Rightarrow 2x + 3 = 0$

D-2. **Sol.** Equation of normal to the parabola $y^2 = 4ax$ at its points $(am^2, 2am)$ is
$y = -mx + 2am + am^3$

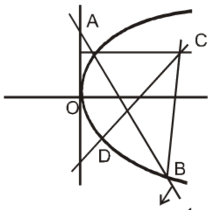
D-3. **Sol.** Point is $(am^2, -2am)$, where $m = \pm 1$
$\therefore$ point is (1, 2)

D-4. **Sol.** Line : $y = -2x - \lambda$
Parabola : $y^2 = -8x$
$c = -2am - am^3$ (condition for line to be normal to the parabola)
$-\lambda = -2 \times -2 \times -2 - (-2)(-8)$
$-\lambda = -8 - 16$
$\lambda = 24$

D-5. **Sol.** Equation of tangent is $y = x + A$ ...(1)
And the equation of normal is
$y = mx - 2Am - Am^3$
Where $m = 1$
$y = x - 2A - A$
$y = x - 3A$ ...(2)

Distance b/w (1) & (2) is $\left|\frac{3A + A}{\sqrt{2}}\right| = 2\sqrt{2}\,A$.

D-6. **Sol.**



c

A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)

$y_1 + y_2 + y_3 = 0$ ...(1)

$y^2 = 4ax$

$y^2 = 4a(1 - y)$

$y^2 + 4ay - 4a = 0$

$y_1 + y_2 = -4a$ ...(2)

Using (2) in (1)

$y_3 = 4a$

D(4a, 4a)

D-7.

Sol. Let the parabola be $y^2 = 4ax$

Equation of any normal is

$y = - tx + 2at + at^3$

It passes through the focus (a, 0)

$0 = - at + 2at + at^3 = at(1 + t^2)$

$\Rightarrow t = 0$

$\therefore$ There is only 1 real normal.

D-8. **Sol.** Equation of any normal to $y^2 = 4x$ is $y = -tx + 2t + t^3$

comparing it with $y = 2x + c - 4$

$t = -2$ and $2t + t^3 = c - 4$

$\therefore$ $c = 4 - 4 - 8 = - 8$

$\therefore$ $c = -8$

D-9. **Sol.** Pair of tangent $SS_1 = T^2$

$\Rightarrow (y^2-4x)(9-8) = [3y-2(x+2)]^2$

$y^2-4x = (3y- 2x-4)^2$ $\Rightarrow x^2+2y^2- 3xy+5x- 6y+4 = 0$

$x- y+1 = 0$, $x-2y+4 = 0$

D-10. **Sol.** $\because$ (–a, 2a) lies on the directrix of the parabola.

$y^2 = 4ax$

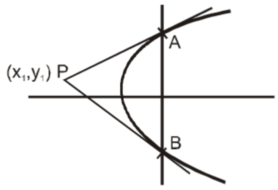
$\therefore$ Angle = $\frac{\pi}{2}$

D-11. **Sol.** $\because$ (–a, 2a) lies on directrix of the parabola.

$y^2 = 4ax$

$\therefore$ Angle = $\frac{\pi}{2}$

D-12. **Sol.**



Eq. of AB is :
T = 0
$yy_1 = 2(x + x_1)$
$2x - yy_1 + 2x_1 = 0$ ...(1)
$4x - 7y + 10 = 0$ .... (2)
equ. (1) & (2) are identical

$\therefore \frac{2}{4} = \frac{y_1}{7} = \frac{2x_1}{10}$

$y_1 = \frac{7}{2}$ & $x_1 = \frac{5}{2}$

D-13. Sol. From the property : the feet of the ⊥r will lie on the tangent at vertex of the parabola.
$y = (x - 1)^2 - 3 - 1$
$(x - 1)^2 = (y + 4)$
Tangent at vertex of above parabola is y + 4 = 0.

D-14. Sol. Eq. of chord is $T = S_1$
$ky - 2(x + h) = k^2 - 4h$ ...(1)
$\because$ Above eq. passes through focus (1, 0)
$\therefore$ $0.k - 2(1 + h) = k^2 - 4h$
$-2 - 2x = y^2 - 4x$
$y^2 = 2(x - 1)$

D-15. Sol. $T = S_1$
$2y - 2(x + 2) = 4 - 8$
$2y - 2x = 0 \Rightarrow y = x$

ELLIPSE

Section (A) : Standard

A-1. Sol. $e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

A-2. Sol. $e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$
focii = (±ae, 0) = (±4,0)

A-3. **Sol.** $\sqrt{(x-1)^2+(y+1)^2} = \frac{1}{2}\left|\frac{x-y-3}{\sqrt{1^2+1^2}}\right|$

Squaring, we have

$7x^2 + 7y^2 + 7 - 10x + 10y + 2xy = 0$

A-4. **Sol.** $4x^2 + 9y^2 + 8x + 36y + 4 = 0$

$\Rightarrow 4(x^2 + 2x + 1) + 9[y^2 + 4y + 4] = 36$

þ $4(x+1)^2 + 9(y+2)^2 = 36$

$\Rightarrow \frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$

$\Rightarrow e = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$

A-5. **Sol.** $2 \times \frac{a}{e} = 3 \times 2ae \Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$

A-6. **Sol.** $9x^2 + 4y^2 = 1$

$\Rightarrow \frac{x}{1/9} + \frac{y^2}{1/4} = 1 \Rightarrow$ Length of latus rectum $= \frac{2a^2}{b} = \frac{4}{9}$

A-7. **Sol.** Line $\frac{x}{7} + \frac{y}{2} = 1$ meet x-axis at, $y = 0$

$\Rightarrow x = 7$

line $\frac{x}{3} - \frac{y}{5} = 1$ meet y-axis at $x = 0 \Rightarrow y = -5$

$\therefore$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through (7, 0) and (0, – 5)

hence $\frac{49}{a^2} + 0 = 1 \Rightarrow a^2 = 49$

$0 + \frac{25}{b^2} = 1 \Rightarrow b^2 = 25$

$\therefore \frac{x^2}{49} + \frac{y^2}{25} = 1 \Rightarrow e = \sqrt{1-\frac{25}{49}}$

A-8. **Sol.** Max. area $= \frac{1}{2} \times 2ae \times b = \frac{1}{2} \times 2 \times 3 \times 4 = 12$

A-9. **Sol.** Distance between focii $= 2ae = \sqrt{(2-4)^2+(2-2)^2} = 2$

$2a = 10$

$\therefore e = \frac{2}{10} = \frac{1}{5}$

$\therefore b^2 = a^2(1 - e^2)$

$\therefore$ $b^2 = 25\left(1-\frac{1}{25}\right) = 24$

Centre = $\left(\frac{2+4}{2}, \frac{2+2}{2}\right) = (3, 2)$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{24} = 1$$

A-10. Sol. $(3x)^2 - 2.3x + (1)^2 + (2y)^2 + 2.2y + (1)^2 = 1$

$\Rightarrow (3x-1)^2 + (2y+1)^2 = 1$ $\Rightarrow 9\left(x-\frac{1}{3}\right)^2 + 4\left(y+\frac{1}{2}\right)^2 = 1$

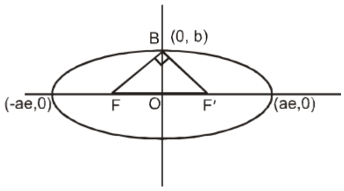
$\therefore$ $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} + \frac{\left(y+\frac{1}{2}\right)^2}{\frac{1}{4}} = 1$

Length of axes $2a = 2 \times \frac{1}{3} = \frac{2}{3}$

$2b = 2 \times \frac{1}{2} = 1$

A-11. Sol. $\frac{x^2}{r-2} + \frac{y^2}{5-r} = 1$ For ellipse $r - 2 > 0$ and $5 - r > 0 \Rightarrow 2 < r < 5$

A-12. Sol.



Use the relation between a, b and e

Given $\angle FBF' = 90^o$

$\therefore$ $\frac{b}{-ae} \times \frac{b}{ae} = -1$

$\Rightarrow$ $b^2 = a^2e^2$

A-13. Sol. $2ae = 10$ $\frac{a}{e} - ae = 15$

$ae = 5$ $\frac{5}{e^2} - 5 = 15$ $\Rightarrow \frac{5}{e^2} = 20$ $\Rightarrow e = \frac{1}{2}$

A-14. Sol. Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$e = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

foci = ($\pm$ ae,0) = (5 × 3/5,0) = ($\pm$ 3, 0)

$\Rightarrow F_1$ & F_2 are foci.

$PF_1 + PF_2 = 2a = 10$

A-15. Sol. Equation of axis : $x - y + k = 0,1$

$3 - 4 + k = 0$

$k = 1$

$x - y + 1 = 0$

point of intersetions of axis and directrix

$x - y + 1 = 0$

$x + y - 1 = 0$

$x = 0, y = 1$

Now, $\frac{AS}{AM} = \frac{1}{2}$

For internal division

$\therefore$ $x = \frac{1\times 0 + 2\times 3}{3} = 2$

$y = \frac{1\times 1 + 2\times 4}{3} = 3$

For external division

$x = \frac{1\times 0 - 2\times 3}{1-2} = 6$

$y = \frac{1\times 1 - 2\times 4}{1-2} = 7$

A-16. Sol. $3(x-3)^2 + 4(y+2)^2 = C$

if $C = 0$ a point

if $C > 0$ ellipse

if $C < 0$ no locus.

A-17. Sol. Eccentricity of ellipse = $\sqrt{1-\frac{b^2}{a^2}} = \sqrt{\frac{a^2-b^2}{a}} = e$

$\therefore$ $P(\theta) = (a\cos\theta, b\sin\theta)$

$\therefore$ $PS = \sqrt{(a\cos\theta \pm ae)^2 + b^2\sin^2\theta}$

$= \sqrt{a^2\cos^2\theta \pm 2ae\cos\theta + a^2e^2 + b^2\sin^2\theta}$

$= \sqrt{a^2\cos^2\theta \pm 2ae\cos\theta + a^2e^2 + a^2(1-e^2)\sin^2\theta}$

$= \sqrt{a^2 \pm 2ae\cos\theta + a^2e^2\cos^2\theta}$

$= a \pm ae\cos\theta$

$PS = a(1 \pm e\cos\theta)$

A-18. Sol. $\frac{x}{3} = (\cos t + \sin t)$, $\frac{y}{4} = (\cos t - \sin t)$.

Squaring and adding

$\frac{x^2}{9} + \frac{y^2}{16} = 2$

Section (B) : Position of Point, Chord/Tangent of Ellipse

B-1. Sol. $S_1 = \frac{16}{8} + \frac{9}{9} - 1 = 2 > 0$ outside ellipse

B-2. **Sol.** $4(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = 36$

$4(x-2)^2 + 9(y-3)^2 = 36$

$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1.$

Equation of major axis y = 3.

Equation of minor axis x = 2

B-3. **Sol.** $c = \pm\sqrt{8 \times 4 + 4} = \pm 6$

B-4. **Sol.** $3x^2 + 4y^2 = 1$

$3xx_1 + 4yy_1 = 1$

given $3x + 4y = -\sqrt{7}$

comparing

$\therefore \frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{-\sqrt{7}}$

Reuqired point $\left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$

B-5 **Sol.** $y + 4 = m(x - 15)$

$y = mx - (15m + 4)$ .... (i)

for tangent

$(15m + 4)^2 = 50m^2 + 32$

$175m^2 + 120m - 16 = 0$

$(35m - 4)(5m + 4) = 0$

$m = \frac{4}{35}, \frac{-4}{5}$

putting in (i)

$4x + 5y = 40$

and $4x - 35y = 200$

B-6. **Sol.** $\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\phi = 1$ ......(1)

$x^2 + y^2 = a^2$

$ax\cos\varphi + ay\sin\varphi = a^2$

$x\cos\varphi + y\sin\varphi = a$

$\frac{x}{a}\cos\varphi + \frac{y}{a}\sin\varphi = 1$ ......(2)

Solving (1) and (2) y = 0

B-7. **Sol.** Let eccentric angle be θ, then equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ...(1)

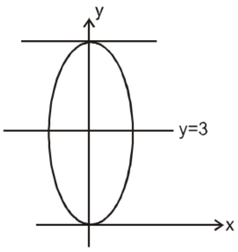
given equation is

$\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ ...(2)

comparing (1) and (2)

$\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $\theta = 45°$

B-8. **Sol.** $9x^2 + 5(y-3)^2 = 45$



$\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$

Ends of major axis

(0, 0) and (0, 6)

Equation of tangent y = 0, y = 6

B-9. **Sol.** $\therefore$ eq^n of tangent at any point $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$P(a\sec\theta, 0)$ & $Q(0, b\,\text{cosec}\,\theta)$

$\therefore \Delta\ OPQ = \frac{1}{2}\ ab\sec\theta\ \text{cosec}\,\theta$

$\Delta = \frac{ab}{\sin 2\theta}$

Now Δ will be minimum if $\sin 2\theta = 1$

$\therefore\ \Delta = ab$

Section (C) : Normal, Pair of Tangents, Director circle, Chord of conctact, chord with given mid point of ellipse

C-1. **Sol.** $y = \left(\frac{3}{4\lambda}\right)x + \left(\frac{9}{8\lambda}\right)$

$m = \frac{3}{4\lambda}$, $c = \frac{9}{8\lambda}$

condition of normal

$c = \frac{-(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$

$\frac{9}{8\lambda} = -\frac{[-3].m}{\sqrt{1+4m^2}}$ but $m = \frac{3}{4}\lambda$

solving

$\lambda = \pm\frac{\sqrt{3}}{2}$

C-2. **Sol.** Normal at a point having eccentric angle φ is

$ax.\sec\varphi - by\ \text{cosec}\ \varphi = a^2 - b^2$

for given ellipse a = 5, b = 3

$5x \sec\varphi - 3y\ \text{cosec}\ \varphi = 16$

comparing with given line

$5x - 3y = 8\sqrt{2}$

$\Rightarrow \frac{\sec\phi}{1} = \frac{\text{cosec}\phi}{1} = \frac{16}{8\sqrt{2}} = \sqrt{2}$

$\varphi = \frac{\pi}{4}$ hence statement is false.

C-3. **Sol.** Equation of normal at $\left(ae, \frac{b^2}{a}\right)$

$\frac{a^2x}{ae} - \frac{b^2ya}{b^2} = a^2 - b^2$

$\frac{ax}{e} - ay = a^2 - b^2$ $\qquad x - ey = ae^3$

C-4. **Sol.** Equation of normal at P(θ) is

$\frac{\sqrt{14} \times x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 9$

it passes through Q(2θ) so

$\frac{14\cos 2\theta}{\cos\theta} - \frac{5\sin 2\theta}{\sin\theta} = 9$

$\Rightarrow \frac{14\cos 2\theta}{\cos\theta} = 10\cos\theta + 9$

$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow \cos\theta = -2/3$ and $\cos\theta = 7/6$ (reject)

C-5. **Sol.** Equation of normal at $\left(ae, \frac{b^2}{a}\right)$

$\frac{a^2x}{ae} - \frac{b^2y}{b^2} \times a = a^2 - b^2$

$\frac{ax}{e} - ay = a^2 - b^2 = a^2e^2$

$x - ey = ae^3$

passes through (0, – b)

$+ be = ae^3$

$b = ae^2$

$a^2(1 - e^2) = a^2e^4$

$e^4 + e^2 - 1 = 0$

C-6. Sol. $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right)(-1) = \left[\frac{x}{4}+\frac{y}{3}-1\right]^2$

$\frac{x^2}{16}+\frac{y^2}{9}-1 = \frac{x^2}{16}+\frac{y^2}{9}+1 + \frac{xy}{6}-\frac{2y}{3} - \frac{x}{2}$

$\frac{xy}{6}-\frac{2y}{3}-\frac{x}{2}+2=0 \Rightarrow xy - 4y - 3x + R = 0$

$3x + 4y - xy - 12 = 0$

C-7. Sol. $x^2 + y^2 = a^2 + b^2$

$\Rightarrow x^2 + y^2 = 25 + 16 = 41$

C-8. Sol. Locus of point 'A' will be director circle at given ellipse

hence $x^2 + y^2 = a^2 + b^2$

C-9. Sol. Let point of intersection is R(h, k)

PQ is chord of contact

$\frac{hx}{25}+\frac{ky}{16}=1$ ...... (i)

equation $\frac{x}{4}+\frac{y}{3}=1$ ...... (ii)

Comparing (i) and (ii)

$\Rightarrow$ equation $\frac{h}{25}=\frac{1}{4}$, $k=\frac{16}{3}$

C-10. Sol. $3x^2 + 2y^2 = 5$

Let m be the slope of tangent to the ellipse from the point (1, 2).

$y - 2 = m(x - 1)$

$y = mx + 2 - m$

from condition of tangency in ellipse

$c^2 = a^2m^2 + b^2$

$\Rightarrow (2 - m)^2 = \frac{5}{3} m^2 + \frac{5}{2}$

$\Rightarrow 4 - 4m + m^2 = \frac{5m^2}{3} + \frac{5}{2}$

$\Rightarrow \frac{2m^2}{3} + 4m + \frac{5}{2} - 4 = 0$

$\Rightarrow \frac{2m^2}{3} + 4m - \frac{3}{2} = 0$

$4m^2 + 24m - 9 = 0$

$\therefore m_1 + m_2 = -6$

$m_1m_2 = \frac{-9}{4}$

$m_1 - m_2 = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

tan θ = $\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3\sqrt{5}}{1 - \frac{9}{4}}\right| = \frac{12}{\sqrt{5}}$

θ = $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$

C-11. **Sol.** $(S_1 F_1) \cdot (S_2 F_2) = b^2 = 3$

C-12. **Sol.** (6, 2) is a focus

∴the reflected ray passes through the other focus i.e. the point (–4, 2)

∴equation of the reflected ray $y - 6 = \frac{4}{8}(x - 4)$ i.e. $x - 2y + 8 = 0$

C-13. **Sol.** $T = S_1 \Rightarrow \frac{x}{2} + \frac{y}{2} - 1 = \frac{1}{4} + \frac{1}{2} - 1$

⇒ $\frac{x}{2} + \frac{y}{2} = \frac{3}{4}$

⇒ $2x+2y=3$

SECTION HYPERBOLA

Section- (A) : Elementary Concepts of Hyperbola/Conjugate/ Rectangular Hyperbola(xy= c2)

A-1.

Sol. Given hyperbola

$(x - 2)^2 - (y - 2)^2 = -16$

Rectangular hyperbola

∴ $e = \sqrt{2}$.

A-2. **Sol.** If e_1 & e_2 are eccentircities of two conjugate hyperbolas

then $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

∴ $e_1 = \sec\alpha$ $e_2 = \mathrm{cosec}\,\alpha$

A-3. **Sol.** Length of latus rectum $\frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$

A-4.

Sol. $\frac{2b^2}{a} = 8$ ... (1)

and $2b = \frac{2ae}{2}$ ...(2)

and $e^2 = 1 + \frac{b^2}{a^2}$ ... (3)

by (1), (2), (3) $e = \frac{2}{\sqrt{3}}$ Ans.

A-5. **Sol.** $2b = 5$

$\Rightarrow b = \frac{5}{2}$

$2ae = 13$

$\therefore ae = \frac{13}{2}$

$\therefore b^2 = a^2e^2 - a^2$

$\Rightarrow \frac{25}{4} = \frac{169}{4} - a^2$

$\Rightarrow a^2 = \frac{144}{4} = 36$

$a = 6$

$\therefore$ Equation of Hyperbola is

$\frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$

A-6. **Sol.** $2a = 7$

$\therefore a = \frac{7}{2}$

$\therefore$ Let equation of hyperbola is

$\frac{x^2}{\frac{49}{4}} - \frac{y^2}{b^2} = 1$

It passes through (5, –2)

$\Rightarrow \frac{25 \times 4}{49} - \frac{4}{b^2} = 1$ $\Rightarrow \frac{100 - 49}{49} = \frac{4}{b^2}$ $\Rightarrow \frac{51}{49} = \frac{4}{b^2}$

$\therefore b^2 = \frac{196}{51}$ $\Rightarrow \frac{4x^2}{49} - \frac{51y^2}{196} = 1$

A-7. **Sol.** $PS = ePM$

$\Rightarrow PS^2 = e^2PM^2$

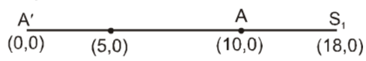
$\Rightarrow (x - 2)^2 + (y - 1)^2 = 4 \frac{(x + 2y - 1)^2}{5}$

$\Rightarrow x^2 + y^2 - 4x - 2y + 5 = \frac{4}{5}(x^2 + 4y^2 + 1 + 4xy - 4y - 2x)$

$\Rightarrow 5x^2 + 5y^2 - 20x - 10y + 25 = 4x^2 + 16y^2 + 4 + 16xy - 16y - 8x$

$x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$

A-8. **Sol.** Centre of hyperbola ≡ (5, 0)



$\therefore$ 2a = 10

$\therefore$ a = 5

$\therefore$ ae = 13

$b^2 = a^2e^2 - a^2$

$b^2 = 169 - 25$

$\therefore$ $b^2 = 144$

$\therefore \dfrac{(x-5)^2}{25} - \dfrac{y^2}{144} = 1$

A-9. Sol. $(x + 1)^2 - y^2 - 1 + 5 = 0$

$y^2 - (x + 1)^2 = 4$

$\dfrac{y^2}{4} - \dfrac{(x+1)^2}{4} = 1$

Equation of directrices

$\therefore$ $y = \pm \dfrac{2}{\sqrt{2}}$ $\Rightarrow$ $y = \pm \sqrt{2}$

A-10. Sol. $e = \sqrt{1-\dfrac{5}{9}}$, $e' = \sqrt{1+\dfrac{45/4}{45/5}}$

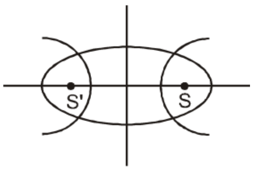
$e = \dfrac{2}{3}$, $e' = \dfrac{3}{2}$

$\therefore$ $e\,e' = 1$

A-11. Sol. Equation of auxilary circle $x^2 + y^2 = a^2$

$\Rightarrow x^2 + y^2 = 9$

A-12. Sol.



ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Hyperbola, $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$

$\therefore e_1^2 = 1 - \dfrac{b^2}{a^2}$, $e_2^2 = 1 + \dfrac{B^2}{A^2}$

and $2ae_1 = 2Ae_2$

Also, $b = B$

So, $\dfrac{b}{ae_1} = \dfrac{B}{Ae_2}$

$\therefore$ $e_1^2 = 1 - \dfrac{B^2}{A^2}\,\dfrac{e_1^2}{e_2^2}$

$= 1 - \frac{\left(e_2^2 - 1\right)e_1^2}{e_2^2}$

$e_1^2\ e_2^2 = e_2^2 - e_1^2\ e_2^2 + e_1^2$

$\Rightarrow e_1^{-2} + e_2^{-2} = 2$

A-13. Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2} \Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $(e')^2 = 1 + \frac{a^2}{b^2} = \frac{b^2+a^2}{b^2}$

$\frac{1}{e^2} + \frac{1}{(e')^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{b^2+a^2} = \frac{a^2+b^2}{a^2+b^2} = 1$

So the point lie on $x^2 + y^2 = 1$

A-14. Sol. $\sqrt{2}^2 \sec^2\theta + \sqrt{2}^2 \tan^2\theta = 6$

$\Rightarrow 1 + 2\tan^2\theta = 3$

$\therefore \theta = \pi/4$ for first quadrant

A-15. Sol. Centre of hyperbola = (0, 2)

focii of hyperbola = (±ae, 2)

$\therefore e = \frac{5}{4}$

focii ≡ (±5, 2)

A-16. Sol. Centre of ellipse = (0, 0)

Centre of hyperbola = (0, 0)

$\therefore$ Ecentricity of ellipse = $\sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

$\therefore$ focii of ellipse ≡ (±ae, 0) = (±3, 0)

$\therefore$ Ecentricity of hyperbola = $\sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5}$

focii of hyperbola = (±ae, 0) = $(\pm\sqrt{41}, 0)$

Vertex of ellipse and hyperbola = (±5, 0)

A-17. Sol. Curve $xy = c^2$

Point P $\left(ct, \frac{c}{t}\right)$ Point Q $\left(ct', \frac{c}{t'}\right)$

Equation of normal $xt^3 - yt = c\,(t^4 - 1)$

Point Q satisfy the equation $ct't^3 - \frac{c}{t'}\,t = c\,(t^4 - 1)$

$t't^3 - \frac{t}{t'} = t^4 - 1$

$(t')^2 t^3 - t = t'(t^4 - 1)$

$t'^2t^4 + t' - t - t't^4 = 0$

$\Rightarrow t'(t't^3 + 1) - t(1 + t't^3) = 0$

$t' = t$ or $t' = -\frac{1}{t^3}$

so only possibility $t' = -\frac{1}{t^3}$

A-18. Sol. By property, orthocentre always lie on rect. hyperbola

$\therefore$ $\lambda \times 4 = 16$

$\therefore$ $\lambda = 4$

A-19. Sol. Distance between foci = $\sqrt{19^2 + 5^2} = \sqrt{386}$

Now by PS + S'P = 2a (for ellipse) (take point P at origin) we get a = 19

$\therefore$ $2ae = \sqrt{386} \Rightarrow e = \frac{\sqrt{386}}{38}$

If conic is hyperbola

$|PS - PS'| = 2a \Rightarrow a = 6$

by $2ae' = \sqrt{386}$ $e' = \frac{\sqrt{386}}{12}$

Section-(B) : Position of Point/Line, Tangent, Chord of hyperbola

B-1. Sol. Since x + y = a touches the hyperbola

$x^2 - 2y^2 = 18$

$\therefore$ $x^2 - 2(a - x)^2 = 18$ has equal roots

i.e. $x^2 - 4ax + 18 + 2a^2 = 0$ has equal roots

$\therefore$ $16a^2 - 4(18 + 2a^2) = 0$

$8a^2 - 72 = 0$ $a = \pm 3$

$\therefore$ $|b| = 3$

B-2. Sol. Let tangent given by

$y = mx + \sqrt{m^2 - 5}$

$\because$ it passes through (2, 8)

$(8 - 2m)^2 = m^2 - 5$

$3m^2 - 32m + 69 = 0$

$\Rightarrow m = 3$ or $23/3$

$\therefore$ tangent can be

$3x - y + 2 = 0$

or $23x - 3y - 22 = 0$ Ans.

B-3. Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

tangent at point P $(a \sec\theta, b \tan\theta)$

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ or $\frac{x}{a\cos\theta} + \frac{y}{(-b\cot\theta)} = 1$

Point A(acosθ, 0), B(0, – b cotθ)

Cordinate of point P is

(h, k) ≡ (a cosθ, – b cotθ)

$\cos\theta = \frac{h}{a}$, $\cot\theta = -\frac{k}{b}$

$\cot\theta = \frac{h}{\sqrt{a^2 - h^2}} = -\frac{k}{b}$

$\frac{h^2}{a^2 - h^2} = \frac{k^2}{b^2}$

$\frac{a^2}{h^2} - 1 = \frac{b^2}{k^2}$

So locus is

$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

B-4. **Sol.** $4x^2 - 9y^2 = 36 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$

$5x + 2y - 10 = 0$

$m = \frac{-5}{2}$ $m^1 = \frac{2}{5}$

Equation of tangent $y = m\pi \pm \sqrt{a^2m^2 - b^2}$

$y = \frac{2}{5}x \pm \sqrt{9 \times \frac{4}{25} - 16}$

$y = \frac{2}{5}x \pm \sqrt{-ve}$ so not possible

B-5. **Sol.** $\frac{x^2}{18} - \frac{y^2}{9} = 1$

given line is

y = x

∴ slope of tangent

∴ equation is

$y = mx \pm \sqrt{a^2m^2 - b^2} \Rightarrow y = -x \pm 3$

B-6. **Sol.** $y = mx \pm \sqrt{a^2m^2 - b^2}$

$y = x \pm \sqrt{5 - b^2}$

∴b = 0, ±1, ±2

b can not be zero

∴four values are possible

B-7. **Sol.** Equation of the hyperbola can be written as $\frac{X^2}{5^2} - \frac{Y^2}{4^2} = 1$
where X = x – 3 and Y = y – 2.

$\therefore$ tangent $Y = X \pm \sqrt{25-16}$

$\Rightarrow$ y = x + 2 or y = x – 4

B-8. **Sol.** Equation of chord joining given points

$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$

If (ae,0) satisfies it

$\frac{1}{e} = \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}$

Now by componendo dividendo

$\frac{1-e}{1+e} = \tan\frac{\theta}{2}\tan\frac{\phi}{2}$ .... (B)

again if (–ae, 0) satisfies it

$-e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$

$\Rightarrow \frac{1+e}{1-e} = \tan\frac{\theta}{2}\tan\frac{\phi}{2}$ .... (D)

B-9. **Sol.** by $T = S_1$

3xh – 2yk + 2(x + h) – 3(y + k)

= $3h^2 - 2k^2 + 4h - 6k$

$\Rightarrow x(3h + 2) + y(-2k - 3) = 3h^2 - 2k^2 + 2h - 3k$

If is parallel to y = 2x

$\therefore \frac{(3h+2)}{(2k+3)} = 2$

$\Rightarrow$ 3x – 4y = 4 Ans.

B-10. **Sol.** Locus of R will be

T = 0

$\frac{x.2}{16} - \frac{y.1}{9} - 1 = 0$

9x – 8y – 72 = 0

Section- (C): Normal/Director Circle / Chord of Contact/Chord with given midpoint/ pair of tangent of hyperbola

C-1. **Sol.** $(1, 2\sqrt{2})$ lies on director circle of $\frac{x^2}{25} - \frac{y^2}{16} = 1$ i.e. $x^2 + y^2 = 9$

$\therefore$ Required angle $\pi/2$

C-2. **Sol.** $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

locus of perpendicular tangents

(Director circle) $x^2 + y^2 = a^2 - b^2$

$x^2 + y^2 = \cos^2\alpha - \sin^2\alpha = \cos2\alpha$

But $0 < \alpha < \frac{\pi}{4}$ $\Rightarrow$ $0 < \cos2\alpha < 1$

$0 < x^2 + y^2 < 1$

So there are infinite points.

C-3. **Sol.** The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola

$\frac{x^2}{25} - \frac{y^2}{3} = 1$ is 3

$\therefore 3 = \sqrt{k}$, hence k = 9

C-4. **Sol.** True by property that locus of feet of perpendicular from the focus of hyperbola upon any of its tangent is its auxillary circle.

C-5. **Sol.** $2x - 3y - 2(x+2) + 2(y+3) = 0$

$-y + 2 = 0 \Rightarrow y = 2$

C-6. **Sol.** $T = S_1 \Rightarrow \frac{3x}{2} - y = \frac{9}{2} - 1$

$\Rightarrow 3x - 2y = 7$

Section-(D): Problems involving more than one conic

D-1. **Sol.** $2y = x + 4$

$y = \frac{x}{2} + 2$ $\Rightarrow$ $M = \frac{1}{2}$

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$2 = \pm\sqrt{4m^2 + b^2}$

$\Rightarrow$ $b^2 = 3$ $\Rightarrow$ $b = \pm\sqrt{3}$

$\Rightarrow$ $\frac{1}{m} = \pm\sqrt{4m^2 + 3}$

$\Rightarrow$ $\frac{1}{m^2} = 4m^2 + 3$

$\Rightarrow \quad 4m^4 + 3m^2 - 1 = 0$

$\Rightarrow \quad m = \pm \frac{1}{2}$

Hence $y = -\frac{1}{2}x - 2,\ 2y = -x - 4$

D-2. **Sol.** $c = \frac{a}{m} = \pm b\sqrt{1+m^2}$

$\frac{2}{m} = \sqrt{2}\sqrt{1+m^2}$

$2 = m^2(1 + m^2)$

$\Rightarrow m^4 + m^2 - 2 = 0$

$m^4 + 2m^2 - m^2 - 2 = 0$

$(m^2 - 1) = 0$

$m = \pm 1$

$y = \pm x \pm 2$

$y = x + 2$

or $y = -x - 2$

D-3. **Sol.** $\frac{16x^2}{225} = 1 \Rightarrow x = \pm\frac{15}{4}$

Hence intersection points are $P\left(\frac{15}{4},\frac{15}{4}\right)$

$Q\left(-\frac{15}{4},-\frac{15}{4}\right)$

$2a = PQ = 2\sqrt{2} \times \frac{15}{4} = \frac{15}{\sqrt{2}} \Rightarrow a = \frac{15}{2\sqrt{2}}$

given $b = \frac{5}{2\sqrt{2}}$

$e = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$

D-4. **Sol.** $y = mx - 2am - am^3$

$y = mx - 2m - m^3$

Passes through centre of circle (6,0)

$= 6m - 2m - m^3 \Rightarrow 4m - m^3 = 0$

$\Rightarrow m = 0, m = 2, m = -2$

D-5. **Sol.** $c^2 = a^2m^2 + b^2 = a^2c_1 + m^2$

$\Rightarrow 9m^2 + 1 = 5 + 5m^2$

$\Rightarrow 4m^2 = 4 \Rightarrow m = \pm 1$

$y = x + 10$

$y = -x + 10$

$y = -x - 10$

D-6. **Sol.** $c^2 = a^2m^2 + b^2 = a^2m - b^2$

$2m^2 + 5 = 3m^2 - 11$

$\Rightarrow m^2 = 16 = 3x^2 - 11$

$\Rightarrow y = \pm 4x \pm 37$

D-7. **Sol.**

$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

End points of latus rectum $\left(3, \frac{16}{5}\right)$

put in $y^2 = 4x \Rightarrow \frac{256}{25} = 4a(3)$

$\Rightarrow a = \frac{64}{75}$

D-8. **Sol.** $c = \frac{a}{m} = \sqrt{a^2m^2 - b^2}$

$\frac{2}{m} = \sqrt{m^2 - 3}$

$\Rightarrow 4 = m^2(m^2 - 3) \Rightarrow m^4 - 3m^2 - 4 = 0$

$(m^2 - 4)(m^2 + 1) = 0$

$\Rightarrow m = \pm 2$

$y = \pm 2x \pm 1$

$\Rightarrow \pm y = 2x + 1$