Exercise-1

Marked Questions may have for Revision Questions.

OBJECTIVE QUESTIONS

Section (A) : Sine Rule

A-1 **Sol.** L.H.S. = a sin (B – C) + b sin (C – A) + c sin (A – B)

= k sin A sin (B – C) + k sin B sin (C – A) + k sin C sin (A – B)

= k $(\sin^2 B - \sin^2 C)$ + k $(\sin^2 C - \sin^2 A)$ + k $(\sin^2 A - \sin^2 B)$

= 0 = R.H.S.

A-2 **Sol.** L.H.S. = $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$

first term = $\frac{a^2 \sin(B-C)}{\sin A} = \frac{k^2 \sin^2 A \sin(B-C)}{\sin A}$

= $k^2 \sin(B + C) \sin(B - C)$

= $k^2 (\sin^2 B - \sin^2 C)$

Similarly $\frac{b^2 \sin(C-A)}{\sin B} = k^2 (\sin^2 C - \sin^2 A)$

and $\frac{c^2 \sin(A-B)}{\sin C} = k^2 (\sin^2 A - \sin^2 B)$

∴ L.H.S. = $k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$

= 0 = R.H.S.

A-3 **Sol.** ∵ 2B = A + C,

⇒ B = 60º

from Sine-rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ ⇒ $\frac{b}{c} = \frac{\sin B}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$

∴ sin C = $\frac{1}{\sqrt{2}}$ ⇒ C = 45º

∴ A = 75º

A-4. **Sol.** ∵ $\cos A + \cos B = 4 \sin^2 \frac{C}{2}$ ⇒ $2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) = 4 \sin^2 \frac{C}{2}$

⇒ $2 \cos\left(\frac{A-B}{2}\right) = 4 \sin \frac{C}{2}$ ⇒ $2 \cos \frac{C}{2} \cos\left(\frac{A-B}{2}\right) = 4 \sin \frac{C}{2} \cos \frac{C}{2}$

⇒ $2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) = 2 \sin C$ ⇒ sin A + sin B = 2 sin C

⇒ a + b = 2c ⇒ a, c, b are in A.P. a, c, b lekUrj Js<h esa gSA

A-5. **Sol.** ∵ $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$ ⇒ sin(B + C) sin(B – C) = sin(A + B) sin(A – B)

⇒ $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$ ⇒ $2 \sin^2 B = \sin^2 A + \sin^2 C$

⇒ $2b^2 = a^2 + c^2$ ⇒ a^2, b^2, c^2 are in A.P. ⇒ a^2, b^2, c^2

A-6. **Sol.** ∵ A : B : C = 3 : 5 : 4 ∴ A = 45° , B = 75° , C = 60°

∵ from Sine - rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \Rightarrow \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}}{2}} = k$ (∵ sin 75° = sin(45° + 30°))

$\therefore a = \frac{k}{\sqrt{2}}$, $b = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$ and $c = \frac{k\sqrt{3}}{2}$

$\therefore a + b + c\sqrt{2} = \frac{k}{\sqrt{2}} + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k + \left(\frac{k\sqrt{3}}{2}\right)\sqrt{2} = \frac{k}{2\sqrt{2}}\left[2 + (\sqrt{3}+1) + 2\sqrt{3}\right] = \frac{3 \; k(\sqrt{3}+1)}{2\sqrt{2}} = 3b$

A-7. Sol. given $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ ..... (i)

∵ a = k sin A, b = k sin B, c = k sin C ∴ (i) becomes

$\frac{\cot A}{k} = \frac{\cot B}{k} = \frac{\cot C}{k}$ ∴ A = B = C

∴ ΔABC is an equilateral triangle

A-8. Sol. ∵ $\frac{bc\sin^2 A}{\cos A + \cos B\cos C} = \frac{k^2 \sin B\sin C\sin^2 A}{-\cos(B+C) + \cos B\cos C} = \frac{k^2 \sin B\sin C\sin^2 A}{\sin B\sin C} = k^2 \sin^2 A = a^2$.

Section (B) : Cosine Rule, projection formula

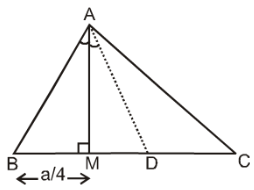
B-5. Sol. ∵ L.H.S. = b(cos A cos θ + sin A sin θ) + a(cos B cos θ – sin B sin θ)

= cos θ (b cos A + a cos B) + sin θ (b sin A – a sin B)

= c cos θ + 0 {∵ b sin A – a sin B = 0}

= c cos θ = R.H.S.

B-6. Sol. ∵ $\frac{a}{4} = c\cos B$



$\frac{a}{4} = c\left(\frac{a^2+c^2-b^2}{2ac}\right)$

$\therefore \frac{a^2}{2} = a^2 + c^2 - b^2$

$\therefore b^2 - c^2 = \frac{a^2}{2}$.

B-7.

Sol. L.H.S. () = $2a\sin^2\frac{C}{2} + 2c\sin^2\frac{A}{2}$

= a(1 – cos c) + c(1 – cos A)

= a + c – (a cos C + c cos A)

= a + c – b

= R.H.S.

B-8. **Sol.** $\because$ (a + b + c) (b + c – a) = kbc $\Rightarrow$ $(b + c)^2 - a^2 = kbc$

$b^2 + c^2 - a^2 = (k - 2)\, bc$ $\Rightarrow$ $\dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{k-2}{2} = \cos A$

$\because$ In a ΔABC $-1 < \cos A < 1$ $\therefore$ $-1 < \dfrac{k-2}{2} < 1$

$0 < k < 4$.

B-9. **Sol.** $\because$ a : b : c = 4 : 5 : 6

$\therefore$ a = 4k, b = 5k , c = 6k

$\because$ $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac} = \dfrac{9}{16}$

$\because$ $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{25 + 36 - 16}{2 \times 5 \times 6} = \dfrac{3}{4}$

$\because$ $\cos 3A = 4 \cos^3 A - 3 \cos A$

$= 4 \times \dfrac{27}{64} - 3 \times \dfrac{3}{4} = \dfrac{27}{16} - \dfrac{9}{4} = \dfrac{27 - 36}{16} = \dfrac{-9}{16}$

$\cos 3A = -\cos B = \cos (\pi - B)$

$\therefore$ $3A + B = \pi$

B-10. Sol. $\because$ $ED = \dfrac{a}{2} - c \cos B$

$= \dfrac{a}{2} - c\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right)$

$= \dfrac{a}{2} - \left(\dfrac{a^2 + c^2 - b^2}{2a}\right) = \dfrac{a^2 - a^2 - c^2 + b^2}{2a} = \dfrac{b^2 - c^2}{2a}$

Section (C) : Napiar formulae, Area of Triangle

C-1. **Sol.** L.H.S. = $4\Delta (\cot A + \cot B + \cot C)$

$= a^2 + b^2 + c^2$

$= 4\Delta \left(\dfrac{\cos A}{\sin A} + \dfrac{\cos B}{\sin B} + \dfrac{\cos C}{\sin C}\right) \left\{\because \ \Delta = \dfrac{1}{2} bc \sin A\right\}$

$= 2bc \cos A + 2\, ca \cos B + 2ab \cos C$

$= a^2 + b^2 + c^2$ = R.H.S.

C-2 **Sol.** $\because$ a = 6, b = 3 and cos(A – B) = 4/5

$\because$ $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$ .... (i)

$\because$ $\tan^2\left(\dfrac{A-B}{2}\right) = \dfrac{1 - \cos(A-B)}{1 + \cos(A-B)} = \dfrac{1}{9}$

$\therefore$ $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{1}{9}$

∴ from (i) we get $\frac{1}{3} = \frac{1}{3}\cot\frac{C}{2}$ ⇒ C = 90°

∴ Area = $\frac{1}{2}$ ab = 9 sq. unit

C-3. **Sol.** ∵ ∠A = 30° and $\Delta = \frac{\sqrt{3}a^2}{4}$

∵ $\frac{1}{2}$ bc sinA = $\frac{\sqrt{3}}{4}$ a^2 ⇒ $\frac{1}{2}$ bc sin30° = $\frac{\sqrt{3}}{4}$ a^2 ⇒ bc = $\sqrt{3}$ a^2

⇒ sinB sinC = $\sqrt{3}$ $\sin^2A$ ⇒ sinB sinC = $\frac{\sqrt{3}}{4}$ as sinA = $\frac{1}{2}$

⇒ cos(B – C) – cos (B + C) = $\frac{\sqrt{3}}{2}$ ⇒ cos (B – C) + cosA = $\frac{\sqrt{3}}{2}$

⇒ cos(B – C) = 0 (as ∠A = 30º ⇒ cos A = $\frac{\sqrt{3}}{2}$)

⇒ B – C = 90° or B – C = – 90°

But B + C = 150° as A = 30°

case (i) : if B – C = 90°

and B + C = 150°

⇒ B = 120° and C = 30° ⇒ B = 4C

case (ii) : if B – C = – 90°

and B + C = 150° ⇒ B = 30° and C = 120° ∴ C = 4B.

C-4. **Sol.** ∵ A = $\frac{2\pi}{3}$, b – c = $3\sqrt{3}$ and Area = $\frac{9\sqrt{3}}{2}$ cm^2

∵ Δ = $\frac{1}{2}$ bc sin A ⇒ $\frac{9\sqrt{3}}{2} = \frac{1}{2}$ bc sin $\frac{2\pi}{3}$ ⇒ bc = 18

∵ cos $\frac{2\pi}{3} = \frac{b^2+c^2-a^2}{2bc} = -\frac{1}{2}$ ⇒ $\frac{(b-c)^2+2bc-a^2}{2bc} = -\frac{1}{2}$ ⇒ a = 9

Section (D) : Half Angle formulae

D-1 **Sol.** ∵ L.H.S. () = $\frac{\cos^2\frac{A}{2}}{a} + \frac{\cos^2\frac{B}{2}}{b} + \frac{\cos^2\frac{C}{2}}{c}$

$= \frac{1}{a}.\frac{s(s-a)}{bc} + \frac{1}{b}.\frac{s(s-b)}{ca} + \frac{1}{c}.\frac{s(s-c)}{ab} = \frac{s\ \ (3s-(a+b+c))}{abc} = \frac{s^2}{abc}$.

D-2. **Sol.** L.H.S. () = 2bc(1 + cos A) + 2ca(1 + cos B) + 2ab(1 + cos C)

= 2bc + 2ca + 2ab + 2bc cos A + 2ca cos B + 2 ab cos C

= $2\sum ab$ + $a^2 + b^2 + c^2 = (a + b + c)^2$ = R.H.S.

D-3. **Sol.** L.H.S. = $\frac{2\ \ abc}{a+b+c}$ cos $\frac{A}{2}$. cos $\frac{B}{2}$. cos $\frac{C}{2}$

$= \frac{2\ \ abc}{2s}\sqrt{\frac{s(s-a)}{bc}.\frac{s(s-b)}{ca}.\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)\ \ (s-c)}$ = Δ = R.H.S. .

D-4. **Sol.** $\because\ 2b = a + c$ ...(i)

$\because\ \tan\frac{A}{2} + \tan\frac{C}{2} = \sqrt{\frac{(s-b)\ (s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)\ (s-b)}{s(s-c)}}$

$= \sqrt{\frac{s-b}{s}}\left[\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}}\right] = \frac{b}{s}\sqrt{\frac{s(s-b)}{(s-a)\ (s-c)}}$

$= \frac{2b}{2s}\sqrt{\frac{s(s-b)}{(s-a)\ (s-c)}} = \frac{2b}{3b}\cot\frac{B}{2} = \frac{2}{3}\cot\frac{B}{2}$.

D-5. **Sol.** $\because\ b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c.\ \Rightarrow\ b\frac{s(s-a)}{bc} + a\frac{s(s-b)}{ac} = \frac{3}{2}c.$

$\Rightarrow\ \frac{s}{c}[s-a+s-b] = \frac{3}{2}c \Rightarrow \frac{s}{c}\times c = \frac{3}{2}c\ \Rightarrow\ \frac{a+b+c}{2} = \frac{3c}{2}\ \Rightarrow\ a + b = 2c$

$\Rightarrow$ a, c, b are in A.P.

D-6. **Sol.** $\because\ \cot\frac{B}{2}\cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s}{s-a} = \frac{2s}{2s-2a}$

$= \frac{a+b+c}{b+c-a} = \frac{4a}{2a} = 2$ $(\because\ b + c = 3a)$

D-7. **Sol.** $\Delta = (a + b - c)\ (a - b + c)$

$\Delta = 4(s-c)\ (s-b)\ \Rightarrow\ \frac{\Delta}{(s-b)\ (s-c)} = \frac{1}{4}\ \therefore\ \tan\frac{A}{2} = \frac{1}{4}$

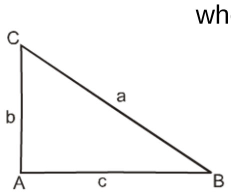
$\because\ \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}\ \Rightarrow\ \tan A = \frac{8}{15}$

D-8. **Sol.** $\because\ a^2 = b^2 + c^2$

$\because\ \tan C = \frac{c}{b}$

$\because\ \tan C = \frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}} = \frac{c}{b}$

$\frac{2t}{1-t^2} = \frac{c}{b}$ where $t = \tan\frac{C}{2}$



$t^2(c) + (2b)t - c = 0$

$\therefore\ t = \frac{-2b \pm \sqrt{4b^2 + 4\times c^2}}{2c}$

$t = \frac{-b \pm a}{c} \Rightarrow t = \frac{a-b}{c} = \tan\frac{C}{2}$

D-9. Sol. $\frac{2ab}{(a+b+c)\Delta} \cdot \cos^2\frac{C}{2} = \frac{2ab}{(2\ \ s).\Delta} \cdot \frac{s(s-c)}{ab} = \frac{s-c}{\Delta}$

Section (E) : Circumradius and Inradius

E-1. Sol. = Rr(sin A + sin B + sin C)

$= Rr\left(\frac{a+b+c}{2R}\right) \quad \because \quad r = \frac{\Delta}{s}$

$= \frac{r(2s)}{2} = rs = \Delta$

E-2. Sol. = a cos B cos C + cos A(b cos C + c cos B)

= a[cos B cos C + cos A]

= a[cos B cos C – cos (B + C)]

= a sin B sin C $= a.\frac{b}{2R}.\frac{c}{2R} = \frac{abc}{4R^2} = \frac{4R\Delta}{4R^2} = \frac{\Delta}{R}$

E-3. Sol. $= \frac{c+a+b}{abc} = \frac{2s}{4R\Delta} = \frac{1}{2R\frac{\Delta}{s}} = \frac{1}{2Rr}$

E-4. Sol. $= \frac{1}{2}$ (1+cosA +1 + cosB + 1 + cos C)

$= \frac{1}{2}\left(3+1+4\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 2 + \frac{1}{2}\frac{r}{R} = 2 + \frac{r}{2R}$

E-5. Sol. $= a\frac{\cos\ A}{\sin\ A} + b\frac{\cos\ B}{\sin\ B} + c\frac{\cos\ C}{\sin\ C}$

$= 2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 2R + 2r = 2(R + r)$

E-6. Sol. $\because \frac{b^2-c^2}{2aR} = \frac{4R^2\left(\sin^2 B - \sin^2 C\right)}{2.2R\sin A.R} = \frac{\sin(B+C).\sin(B-C)}{\sin A} = \sin(B-C)$

E-7. Sol. $\frac{a\cos A + b\cos B + c\cos C}{a+b+c} = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2R(\sin A + \sin B + \sin C)}$

$= \frac{4\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 4\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{R}.$

E-8. Sol. a = 3k ; b = 7k ; c = 8k

$\because$ s = 9 k.

$\because \quad \Delta = \sqrt{9k\cdot 6k.2k.k} = k^2\ 6\sqrt{3} \quad \therefore \quad R = \frac{abc}{4\Delta} = \frac{(3k)(7k)(8k)}{4\times k^2\times\ 6\sqrt{3}} = \frac{7k}{\sqrt{3}}$

$\because$ r = $\frac{\Delta}{s} = \frac{k^2 6\sqrt{3}}{9k} = \frac{2k}{\sqrt{3}}$ $\therefore$ R : r = 7 : 2

E-9. Sol. a = 1

$\because$ $2s = 6\left(\frac{\sin A + \sin B + \sin C}{3}\right)$

$2s = 2\left(\frac{a+b+c}{2R}\right)$

R = 1

$\because$ $\frac{a}{\sin\ A}$ = 2R ⇒ sin A = $\frac{1}{2}$

A = $\frac{\pi}{6}$

Section (F) : Length of Median, angle bisector, altitude

F-1. Sol. $\because$ $\alpha = \frac{2\Delta}{a}$, $\beta = \frac{2\Delta}{b}$, $\gamma = \frac{2\Delta}{c}$ ⇒ $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{a^2+b^2+c^2}{4\ \Delta^2}$

F-2. Sol. $\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$

F-3. Sol. $\because$ required distance = $\frac{2\Delta}{a}$

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

a = 13; b = 12; c = 5 ⇒ s = 15

$\therefore$ $\Delta = \sqrt{15 \times 2 \times 3 \times 10}$ = 5 × 3 × 2 = 30

$\therefore$ required distance = $\frac{2 \times 30}{13} = \frac{60}{13}$

F-4. Sol. $\because$ $AD^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$,

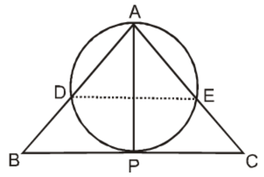
$BE^2 = \frac{1}{4}(2c^2 + 2a^2 - b^2)$ and

$CF^2 = (2a^2 + 2b^2 - c^2)$

$\therefore$ $\frac{AD^2 + BE^2 + CF^2}{BC^2 + CA^2 + AB^2} = \frac{3}{4}$

F-5. Sol. $\because$ $\frac{DE}{\sin A}$ = AP

⇒ DE = $\frac{2\Delta}{a}$ sin A



$= \frac{2\Delta \sin A}{2R \sin A} = \frac{\Delta}{R}$

F-6. **Sol.** $\because$ $\ell = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$\therefore$ $4\ell^2 = 2b^2 + 2c^2 - a^2$

$= a^2 + 2(b^2 + c^2 - a^2)$

$= a^2 + 2(2bc \cos A)$

$4\ell^2 = a^2 + 4bc \cos A$

Exercise-2

Marked Questions may have for Revision Questions.

PART - I : OBJECTIVE QUESTIONS

1. **Sol.** $\because$ $s - a = 3$ ...(1) and $s - c = 2$ ...(2)

by (1) – (2), we get

$c - a = 1$

(1) + (2), we get $2s - a - c = 5$ $\Rightarrow$ $b = 5$

$\because$ ΔABC is right angled at B

$\therefore$ $a^2 + c^2 = 25$ ...(3)

$\therefore$ $(c - a)^2 + 2ac = 25$

$ac = 12$ ...(4)

$\therefore$ $a(1 + a) = 12 \Rightarrow a^2 + a - 12 = 0$

$\Rightarrow (a + 4)(a - 3) = 0$

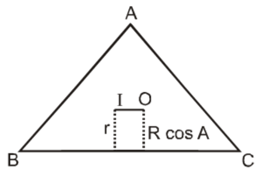
$\Rightarrow a = 3$ and $c = 4$.

2. **Sol.** $\because$ $R \cos A = r$

$R \cos A = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\cos A = \cos A + \cos B + \cos C - 1$

$\cos B + \cos C = 1$



3. **Sol.** MINA is a cyclic quadrilatral

$\therefore \frac{MN}{\sin A} = AI \Rightarrow MN = r \operatorname{cosec} \frac{A}{2} \sin A = 2r \cos \frac{A}{2}$

IM = IN = r

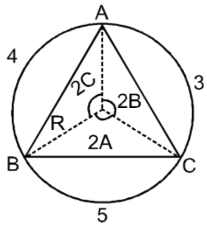
$\therefore x = \frac{\left(2r\cos\frac{A}{2}\right)(r)(r)}{4 \times \frac{1}{2} r \times r \sin A} = \frac{2r^3 \cos \frac{A}{2}}{2r^2 \sin A}$

$= \frac{r\cos\frac{A}{2}}{\sin A} = \frac{r}{2\sin\frac{A}{2}}$

similarly $y = \dfrac{r}{2\sin\frac{B}{2}}$ and $z = \dfrac{r}{2\sin\frac{C}{2}}$

$\therefore\ xyz = \dfrac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \dfrac{r^3}{2\frac{r}{R}} = \dfrac{1}{2} r^2 R$

4. Sol.



angle = $\dfrac{\text{arc}}{\text{radius}}$ ....(1)

$\because\ 4 + 5 + 3 = 2\pi R \quad \Rightarrow \quad R = 6/\pi$

$\because\ 2A = \dfrac{5}{R} = \dfrac{5\pi}{6}$,

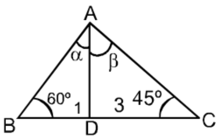
$2B = \dfrac{3}{R} = \dfrac{\pi}{2}$ and

$2C = \dfrac{4}{R} = \dfrac{2\pi}{3}$

Area of Δ ABC = $\dfrac{1}{2} R^2\left[\sin\dfrac{2\pi}{3} + \sin\dfrac{5\pi}{6} + \sin\dfrac{\pi}{2}\right]$

$= \dfrac{R^2}{2}\left[\dfrac{\sqrt{3}}{2} + \dfrac{1}{2} + 1\right] = \dfrac{R^2}{2}\left[\dfrac{\sqrt{3}+3}{2}\right] = \dfrac{\sqrt{3}(\sqrt{3}+1)}{4} \times \dfrac{36}{\pi^2} = \dfrac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$

5. Sol.



if we apply Sine-Rule in Δ BAD, we get

$\dfrac{BD}{\sin\ \alpha} = \dfrac{AD}{\text{Sin}\ 60^\circ}$ ...(1)

if we apply Sine-Rule in Δ CAD, we get.

$\dfrac{CD}{\sin\ \beta} = \dfrac{AD}{\sin\ 45^\circ}$ ...(2)

divide (2) by (1)

$\dfrac{\sin\ \alpha}{\sin\ \beta} \times \dfrac{CD}{BD} = \dfrac{\sin\ 60^\circ}{\sin\ 45^\circ}$

$\dfrac{\sin\ \alpha}{\sin\ \beta} \times \dfrac{3}{1} = \dfrac{\sqrt{3}}{2\times\frac{1}{\sqrt{2}}}$

$\dfrac{\sin\ \alpha}{\sin\ \beta} = \dfrac{1}{\sqrt{6}}$

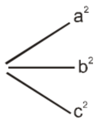
6. Sol. $\because \dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = k \quad \therefore \left.\begin{matrix} b+c = 11\ k \\ c+a = 12\ k \\ a+b = 13\ k \end{matrix}\right\}$ $\Rightarrow$ a = 7k, b = 6k, c = 5k

$\because \quad \cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{36+25-49}{2\times6\times5} = \dfrac{1}{5}$

$\cos B = \dfrac{c^2+a^2-b^2}{2ca} = \dfrac{25+49-36}{2\times5\times7} = \dfrac{19}{35}$

$\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{49+36-25}{2\times7\times6} = \dfrac{5}{7} \qquad \therefore \dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$

7. Sol. $\because \quad x^3 - Px^2 + Qx - R = 0$



$\therefore \quad a^2 + b^2 + c^2 = P$

$a^2b^2 + b^2c^2 + c^2a^2 = Q$

$a^2b^2c^2 = R \Rightarrow abc = \sqrt{R}$

$\because \quad \dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{1}{2abc}[a^2 + b^2 + c^2] = \dfrac{P}{2\sqrt{R}}$

8. Sol. $\because$ L.H.S. $= (b-c)\cot\dfrac{A}{2} + (c-a)\cot\dfrac{B}{2} + (a-b)\cot\dfrac{C}{2}$

$\because \ (b-c)\cot\dfrac{A}{2} = k(\sin B - \sin C)\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = 2k\cos\left(\dfrac{B+C}{2}\right)\sin\left(\dfrac{B-C}{2}\right)\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$

$= 2k\sin\left(\dfrac{B+C}{2}\right)\sin\left(\dfrac{B-C}{2}\right) = k[\cos C - \cos B]$

similarly $(c-a)\cot\dfrac{B}{2} = k[\cos A - \cos C]$

and () $\qquad (a-b)\cot\dfrac{C}{2} = k[\cos B - \cos A]$

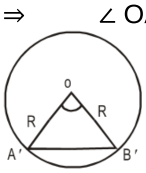
$\therefore$ L.H.S. $= k[\cos C - \cos B + \cos A - \cos C + \cos B - \cos A]$

$= 0$

= R.H.S.

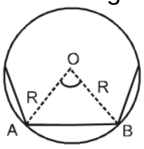
9. Sol. For dodecagon $\angle A'OB' = \dfrac{2\pi}{12} = 30°$

$\Rightarrow \quad \angle OA'B' = \angle OB'A' = 75° \Rightarrow \dfrac{R}{\sin 75°} = \dfrac{\sqrt{3}-1}{\sin 30°}$



⇒ R = $\dfrac{(\sqrt{3}-1)(\sqrt{3}+1)}{2\sqrt{2}\times\dfrac{1}{2}}$ ⇒ R = $\sqrt{2}$

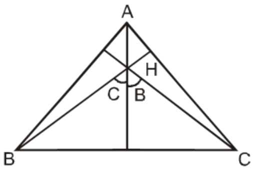
For hexagon ∠ AOB = $\dfrac{2\pi}{6}$ = 60°



⇒ Δ AOB is equilatecal ⇒ AB = R = $\sqrt{2}$

10. **Sol.** In ΔHBC is we apply Sine-rule, then we get

$\dfrac{BC}{\sin(B+C)}$ = 2R'



$\dfrac{a}{\sin\ A}$ = 2R' ⇒ 2R = 2R' ⇒ R = R'

∴ circumradius of ΔHBC (i.e. R') = R

Similarly we can prove for ΔHCA and ΔHAB.

11 **Sol.** f = RcosA , g = R cos B, h = R cosC.

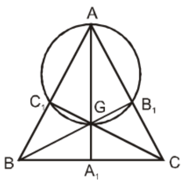
∴ $\dfrac{a}{f}+\dfrac{b}{g}+\dfrac{c}{h} = \dfrac{2R\sin A}{R\cos A}+\dfrac{2R\sin B}{R\cos B}+\dfrac{2R\sin C}{R\cos C}$

= $2\left(\sum \tan A\right)$

∴ $\dfrac{abc}{fgh} = 8\left(\prod \tan A\right)$

$\dfrac{a}{f}+\dfrac{b}{g}+\dfrac{c}{h} = \lambda\dfrac{abc}{fgh}$ ⇒ $2\sum \tan A = \lambda.8\left(\prod \tan A\right)$ ⇒ $\lambda = \dfrac{1}{4}$

12. **Sol.** ∵ A, C_1 , G and () B_1 are cyclic



∴ $BC_1 . BA = BG . BB_1$

$\dfrac{c}{2} . c = \left(\dfrac{2}{3}\ BB_1\right) . BB_1$

$\dfrac{c^2}{2} = \dfrac{2}{3}\times\dfrac{1}{4}(2c^2 + 2a^2 - b^2)$

⇒ $c^2 + b^2 = 2a^2$

PART - II : MISCELLANEOUS QUESTIONS

A-1. Sol. Statement-1 :

$\because$ H.M. of the three ex-radii = $\dfrac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} = \dfrac{3\Delta}{s-a+s-b+s-c} = \dfrac{3\Delta}{s} = 3r$

= 3 times the inradius

$\therefore$ statement-1 is true

Statement-2 : $\because$ L.H.S. = $r_1 + r_2 + r_3$

$= \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c}$

$= \Delta\left[\dfrac{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right]$

$= \Delta\left[\dfrac{3s^2-2s(a+b+c)+ab+bc+ca}{\Delta^2}\right]$

$= \dfrac{s\Delta\ \left[ab+bc+ca-s^2\right]}{\Delta^2}$

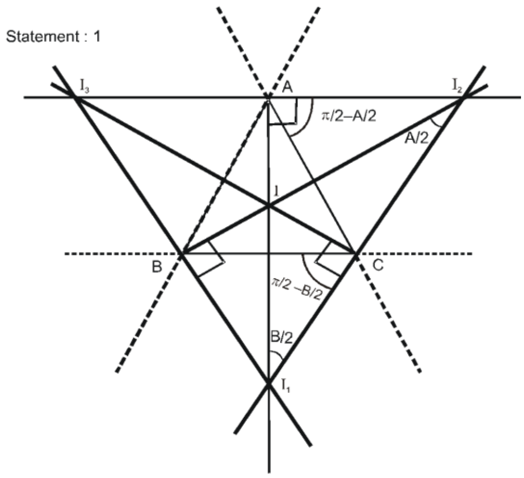
$= \dfrac{s\ \left(ab+bc+ca-s^2\right)}{\Delta}$

$\because$ R.H.S. = 4R = $\dfrac{abc}{\Delta}$

$\therefore$ L.H.S. ≠ R.H.S.

$\therefore$ Statement 2 is false.

A-2. Sol.



$I_1 I_2 = 4R \cos\dfrac{C}{2}$ if we apply Sine-Rule in $\Delta I_1 I_2 I_3$, then $\Delta I_1 I_2 I_3$

$2R_{ex} = \dfrac{I_1 \ I_2}{\sin\left(\frac{A}{2}+\frac{B}{2}\right)} = \dfrac{4R\cos\frac{C}{2}}{\sin\left(\frac{A+B}{2}\right)}$

$= \dfrac{4R\cos\frac{C}{2}}{\cos\frac{C}{2}}$

$2R_{ex}$ = 4R R_{ex} = 2R

∴ Δ ABC is pedal triangle of Δ $I_1 I_2 I_3$

B-1. Ans. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

Sol. (A) ∵ $2B = A + C \Rightarrow B = \frac{\pi}{3}$ and $A + C = \frac{2\pi}{3}$

∵ $b^2 = ac$

$\Rightarrow \sin^2 B = \sin A.\sin C$

$\Rightarrow \sin A \sin C = \frac{3}{4}$

$\Rightarrow \cos(A - C) - \cos(A + C) = \frac{3}{2}$ ∵ $A + C = \frac{2\pi}{3}$

$\Rightarrow \cos(A - C) = \frac{\pi}{3}$ 1

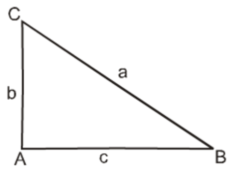
$\Rightarrow A = C = \ = B$

$\Rightarrow a = b = c$

∴ $\frac{a^2(a+b+c)}{3abc} = 1$

(B) ∵ $a^2 = b^2 + c^2$ and $2R = a$

∴ $\frac{a^2+b^2+c^2}{R^2} = \frac{2a^2}{R^2} = 8$



(C) ∵ $\Delta = \frac{1}{2}$ bc sin A $\Rightarrow \Delta = \frac{1}{2}$.9. sin A = $\frac{9}{2} \times \frac{a}{2R}$ ∵ a = 2

∴ $2R\Delta = 9$

(D) ∵ a = 5, b = 3 and c = 7

and because we know that

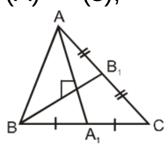
b cos C + c cos B = a

∴ 3 cos C + 7 cos B = 5

B-2 Ans. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

Ans. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

Sol.



Match the column

(A) AA_1 and BB_1 are perpendicular

∴ $a^2 + b^2 = 5c^2$

∴ $c^2 = \frac{a^2+b^2}{5} = 5$ ⇒ $c = \sqrt{5}$

∴ $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{16+9-5}{2\times4\times3} = \frac{5}{6}$

∵ $\sin C = \frac{\sqrt{11}}{6}$

∴ $\Delta = \frac{1}{2}ab\sin C = \sqrt{11}$

∴ $\Delta^2 = 11$

(B) ∵ G.M. ≥ H.M.

$(r_1 r_2 r_3)^{1/3} \ge \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$

⇒ $(r_1 r_2 r_3)^{1/3} \ge 3r$

⇒ $\frac{r_1 r_2 r_3}{r^3} \ge 27$

(C) ∵ $\tan^2\frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)}$ a = 5, b = 4 2s = 9 + c

$= \frac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \frac{c^2-1}{81-c^2}$

⇒ $\frac{7}{9} = \frac{c^2-1}{81-c^2}$ ⇒ $c^2 = 36$ ⇒ c = 6

(D) $2a^2 + 4b^2 + c^2 = 4ab + 2ac.$

⇒ $(a-2b)^2 + (a-c)^2 = 0$

⇒ a = 2b = c

∵ $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{7}{8}$

∴ 8 cos B = 7

Section (C) : ONE OR MORE THAN ONE OPTIONS CORRECT

C-1. **Sol.** (1) ∵ $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ ....(i)

∵ $\tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$

∴ $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}}$ ∵ a = 5 and = 4

∴ from equation (i), we get

$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right)\cot\frac{C}{2}$ ⇒ $\frac{1}{3\sqrt{7}} = \frac{1}{9}\cot\frac{C}{2}$ ⇒ $\cot\frac{C}{2} = \frac{3}{\sqrt{7}}$

$\because$ cos C = $\frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$

$\because$ cos C = $\frac{b^2+a^2-c^2}{2ab}$ $\Rightarrow$ $c^2 = a^2 + b^2 - 2ab\cos C$ $\Rightarrow$ c = 6

(2), (3) $\because$ Area $= \frac{1}{2}$ ab sinC $\because$ cosC = $\frac{1}{8}$ $\Rightarrow$ sinC = $\sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$

Area = $\frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$

Area = $\frac{15\sqrt{7}}{4}$ sq. unit.

$\because$ From Sine rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $\Rightarrow$ sinA = $\frac{a\sin C}{c} = \frac{5\times 3\sqrt{7}}{6\times 8}$ $\therefore$ sinA = $\frac{5\sqrt{7}}{16}$

C-2. **Sol.** (1) $\because$ $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$

$= \frac{a^2+b^2+c^2}{2abc}$

(2) $\because$ $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{a}{2R.a} + \frac{b}{2R.b} + \frac{c}{2R.c} = \frac{3}{2R}$

(3) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ $\Rightarrow$ cot A = cot B = cot C $\Rightarrow$ A = B = C

true for equilateral triangle only

(4) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

$\Rightarrow$ $\frac{2\sin A\cos A}{k^2\sin^2 A} = \frac{2\sin B\cos B}{k^2\sin^2 B} = \frac{2\sin C\cos C}{k^2\sin^2 C}$

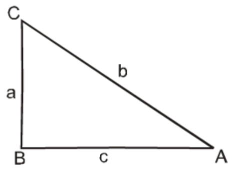
$\Rightarrow$ cot A = cot B = cot C $\Rightarrow$ A = B = C $\Rightarrow$ true for equilateral triangle only

C-3. **Sol.** $\because$ r = (s – b) tan $\frac{B}{2}$

r = s – b ($\because$ B = 90º)

$\therefore$ r = $\frac{2s-2b}{2} = \frac{AB+BC+CA-2CA}{2}$

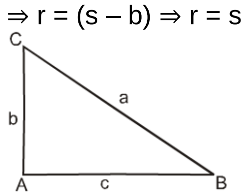
$\therefore$ r = $\frac{AB+BC-CA}{2}$.



Again. $\because$ R = $\frac{b}{2}$

$\because$ r = (s – b) tan $\frac{B}{2}$

$\Rightarrow$ r = (s – b) $\Rightarrow$ r = s – 2R $\Rightarrow$ R = $\frac{s-r}{2}$



C-4. **Sol.** $\because$ $\sin C = \frac{1-\cos A\cos B}{\sin A\sin B} \le 1$ $\Rightarrow$ cos (A – B) $\ge$ 1 $\Rightarrow$ cos (A – B) = 1

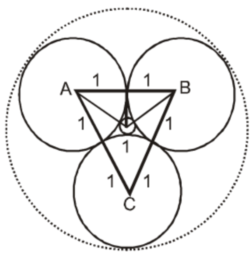
$\Rightarrow$ A – B = 0 $\Rightarrow$ A = B $\therefore$ $\sin C = \frac{1-\cos^2 A}{\sin^2 A} = 1$ $\Rightarrow$ C = 90º

C-5. **Sol.** Product of distances of incenter from angular points

$= \frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r^3}{r/4R} = 4Rr^2 = \frac{abc}{\Delta}r^2 = \frac{\frac{(abc)(r)}{\Delta}}{r} = \frac{(abc)\ \ (r)}{s}$.

C-6. **Sol.** Let the radius of the inner circle be x

$\because$ $\cos 30° = \frac{1}{x+1} = \frac{\sqrt{3}}{2}$



$x + 1 = \frac{2}{\sqrt{3}}$

$x = \frac{2-\sqrt{3}}{\sqrt{3}}$

$\therefore$ radius of other (shaded) circle

= 2 + x = 2 + $\frac{2-\sqrt{3}}{\sqrt{3}} = \frac{2+\sqrt{3}}{\sqrt{3}}$

C-7 **Sol.** $\because$ $\beta_a = \frac{2bc}{b+c}\cos\frac{A}{2}$

(A) correct
(B) incorrect

(C) $\frac{abc\,cosec\frac{A}{2}}{2R(b+c)} = \frac{abc\,cosec\frac{A}{2}}{\frac{a}{\sin A}.(b+c)} = \frac{bc\quad 2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}.(b+c)} = \frac{2bc}{(b+c)}\cos\frac{A}{2}$

(D) $\because \frac{2\Delta}{(b+c)}\,cosec\,\frac{A}{2} = \frac{bc\sin A}{(b+c)}.\frac{1}{\sin\frac{A}{2}} = \frac{2bc\sin\frac{A}{2}\cos\frac{A}{2}}{(b+c)}.\frac{1}{\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}$

Exercise-3

1. **Sol.** $\tan\left(\frac{\pi}{n}\right) = \frac{a}{2r}$; $\sin\left(\frac{\pi}{n}\right) = \frac{a}{2R}$

$r + R = \frac{a}{2}\left[\cot\frac{\pi}{n} + cosec\frac{\pi}{n}\right] \Rightarrow r + R = \frac{a}{2}.\cot\left(\frac{\pi}{2n}\right)$

2. **Sol.** $a = \frac{s(s-c)}{a\ b} + c.\frac{s(s-a)}{bc}$

$\Rightarrow \frac{s}{b}(s-c+s-a) = \frac{3b}{2}$

$\Rightarrow$ a + b + c = 3b.

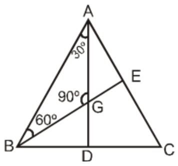
$\Rightarrow$ a + c = 2b

$\Rightarrow$ a, b, c are in A.P.

3. **Sol.** AD = 4

$\because$ AG = $\frac{2}{3} \times 4 = \frac{8}{3}$

$\therefore$ Area of Δ ABG = $\frac{1}{2}$ × AB × AG sin 30º



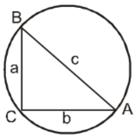
$\because = \frac{1}{2} \times \frac{16}{3\sqrt{3}} \times \frac{8}{3} \times \frac{1}{2} = \frac{32}{9\sqrt{3}}$ $\because$ Sin 60º = $\frac{AG}{AB}$ $\Rightarrow$ AB = $\frac{2AG}{\sqrt{3}} = \frac{16}{3\sqrt{3}}$

$\therefore$ Area of Δ ABC = 3(Area of Δ ABG) $= \frac{32}{3\sqrt{3}}$

4. **Sol.** $\cos\beta = \frac{\sin^2\alpha + \cos^2\alpha \quad -1-\sin\alpha\cos\alpha}{2\sin\alpha.\cos\alpha} = -\frac{1}{2}$

$\Rightarrow$ β = 120º

5. **Sol.** ∠C = π/2

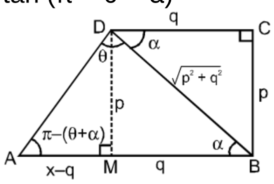


$r = (s - c)\tan\frac{C}{2}$ ∴ $C = 90^o$
$r = s - 2R$
∵ $2r + 2R = 2(s - 2R) + 2R.$
$= 2s - 2R$
$= (a + b + c) - \frac{c}{\sin C}$ ∵ $C = 90^o$
$= a + b + c - c$
$= a + b$

6. **Sol.** $\frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c}$ are in H.P.
$\frac{a}{2\Delta}, \frac{b}{2\Delta}, \frac{c}{2\Delta}$ are in A.P.
⇒ a,b,c are in A.P.

7. **Ans.** **(2)**
Sol. $\frac{r}{R} = \cos\left(\frac{\pi}{n}\right)$
Let $\cos\frac{\pi}{n} = \frac{2}{3}$ for some $n \geq 3, n \in N$
As $\frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}}$ ⇒ $\cos\frac{\pi}{3} < \cos\frac{\pi}{n} < \cos\frac{\pi}{4}$ ⇒ $\frac{\pi}{3} > \frac{\pi}{n} > \frac{\pi}{4}$
⇒ $3 < n < 4$, which is not possible
so option (2) is the false statement
so it will be the right choice
Hence correct option is (2)

8. **Sol.** **(1)**
Let AB = x
$\tan(\pi - \theta - \alpha) = \frac{p}{x - q}$ ⇒ $\tan(\theta + \alpha) = \frac{p}{q - x}$



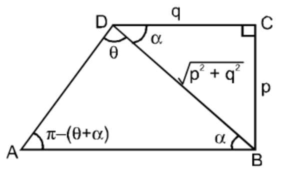
⇒ $q - x = p\cot(\theta + \alpha)$
⇒ $x = q - p\cot(\theta + \alpha)$
$= q - p\left(\frac{\cot\theta\ \cot\alpha - 1}{\cot\alpha + \cot\theta}\right)$
$= q - p\left(\frac{\frac{q}{p}\cot\theta\ - 1}{\frac{q}{p} + \cot\theta}\right) = q - p\left(\frac{q\cot\theta\ - p}{q + p\cot\theta}\right) = q - p\left(\frac{q\cos\theta\ - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$

$\Rightarrow \quad x = \dfrac{q^2 \sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta + q\sin\theta} \quad \Rightarrow AB = \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

Alternative

From Sine Rule

$\dfrac{AB}{\sin\theta} = \dfrac{\sqrt{p^2+q^2}}{\sin(\pi-(\theta+\alpha))}$



$AB = \dfrac{\sqrt{p^2+q^2}\ \sin\theta}{\sin\theta\ \cos\alpha + \cos\theta\ \sin\alpha}$

$= \dfrac{(p^2+q^2)\ \sin\theta}{q\sin\ \theta + p\cos\theta} \qquad \left(\because \cos\alpha = \dfrac{q}{\sqrt{p^2+q^2}} \right)$

$= \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. **Sol.** $I_n = 2n \times$ area of $\Delta\, OA_1I_1$

$\Rightarrow \quad I_n = 2n \times \dfrac{1}{2} \times A_1I_1 \times OI_1$

$\Rightarrow \quad I_n = n \times \sin\dfrac{\pi}{n} \times \cos\dfrac{\pi}{n}$

$\Rightarrow \quad I_n = \dfrac{n}{2}\sin\dfrac{2\pi}{n}$. ....(1)

$O_n = 2n \times$ area of $\Delta\, OB_1O_1$

$\Rightarrow \quad O_n = 2n \times \dfrac{1}{2} \times B_1O_1 \times O_1O = n \times \tan\dfrac{\pi}{n} \times 1 = n\tan\dfrac{\pi}{n}$

$\Rightarrow O_n = n\tan\dfrac{\pi}{n}$ .....(2)

Now R.H.S. $= \dfrac{O_n}{2}\left[1+\sqrt{1-\left(\dfrac{2\ I_n}{n}\right)^2}\right] = \dfrac{O_n}{2}\left[1+\cos\dfrac{2\pi}{n}\right]$

$= \dfrac{O_n}{2} \times 2\cos^2\dfrac{\pi}{n} \qquad = O_n.\cos^2\dfrac{\pi}{n}$

$= n\tan\dfrac{\pi}{n}.\cos^2\dfrac{\pi}{n} \qquad = \dfrac{n}{2}\sin\dfrac{2\pi}{n} = I_n$ = L.H.S

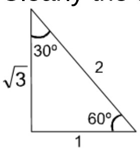
2. **Sol.** Let angle of the triangle be 4x, x and x .
Then $4x + x + x = 180°$ $\Rightarrow$ $x = 30°$
Longest side is opposite to the largest angle.
Using the law of sines

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

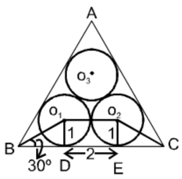
$\therefore$ a = R, b = R, c = $\sqrt{3}$ R $\therefore$ 2S = $(2+\sqrt{3})$ R $\therefore$ = $(2+\sqrt{3})$ R

3. **Solution**

Clearly the triangle is right angled. Hence angles are 30º, 60º and 90º are in ratio 1 : 2 : 3



4. **Sol.** ΔO_1BD, $\dfrac{BD}{O_1D}$ = cot 30º ⇒ BD = $\sqrt{3}$



⇒ BC = AB = AC = 2 + 2$\sqrt{3}$

area of ΔABC = $\dfrac{\sqrt{3}}{4}(2+2\sqrt{3})^2$

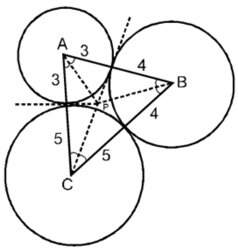
= $\dfrac{\sqrt{3}}{4}(1+3+2\sqrt{3})$ 4 = 6 + $4\sqrt{3}$ sq. unit

5. **Sol.** Consider $\dfrac{b-c}{a} = \dfrac{k(\sin B-\sin C)}{k\sin A}$

$= \dfrac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \dfrac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \dfrac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$

6. **Ans.** $\sqrt{5}$

Solution :



r = $\dfrac{\Delta}{s} = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

2s = 7 + 8 + 9 ⇒ s = 12

r = $\sqrt{\dfrac{5.4.3}{12}} = \sqrt{5}$

7. **Sol.** $\Delta = \dfrac{1}{2}$. b . b . sin 120º = $\dfrac{\sqrt{3}}{4}$ b^2 ....(1)

Also $\frac{\sin 120^\circ}{a} = \frac{\sin 30^\circ}{b}$ ⇒ $a = \sqrt{3}b$ ....(2)

and $\Delta = \sqrt{3}s$ and $s = \frac{1}{2}(a + 2b)$

⇒ $\Delta = \frac{\sqrt{3}}{2}(a + 2b)$ ....(3)

From (1), (2) and (3), we get $\Delta = \left(12 + 7\sqrt{3}\right)$

8.* **Sol.** We have ΔABC = ΔABD + ΔACD

⇒ $\frac{1}{2}$ bc sin A = $\frac{1}{2}$ c AD sin $\frac{A}{2}$ + $\frac{1}{2}$ b × AD sin $\frac{A}{2}$

⇒ AD = $\frac{2bc}{b+c}\cos\frac{A}{2}$

Again AE = AD sec $\frac{A}{2}$

= $\frac{2bc}{b+c}$ ⇒ AE is HM of b and c.

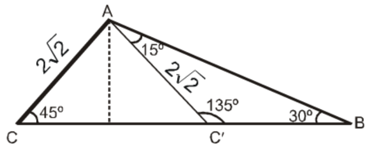
EF = ED + DF = 2DE = 2 × AD tan $\frac{A}{2}$ = $\frac{2 \times 2bc}{b+c}$ × cos $\frac{A}{2}$ × tan $\frac{A}{2}$

= $\frac{4bc}{b+c}$ ⇒ sin $\frac{A}{2}$

As $AD \perp EF$ and DE = DF and AD is bisector ⇒ Δ AEF is isosceles.
Hence A, B, C and D are correct answers.

9. **Ans.** 4
Sol. In Δ ABC, by sine rule

$\frac{a}{\sin A} = \frac{2\sqrt{2}}{\sin 30^\circ} = \frac{4}{\sin C}$ ⇒ C = 45º, C' = 135º
When C = 45º ⇒ A = 180º – (45º + 30º) = 105º
When C' = 135º ⇒ A = 180º – (135º + 30º) = 15º

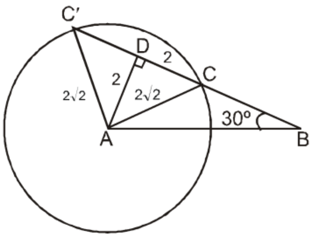


Area of Δ ABC' = $\frac{1}{2}$ AB . AC'.sin ∠ BAC' = $\frac{1}{2}$ × 4 × $2\sqrt{2}$ sin (15º) = $4\sqrt{2}$ × $\frac{\sqrt{3}-1}{2\sqrt{2}}$ = $2\left(\sqrt{3}-1\right)$

Area of Δ ABC = $\frac{1}{2}$ AB . AC .sinA = $\frac{1}{2}$ × 4 × $2\sqrt{2}$ sin (105º) = $2\left(\sqrt{3}+1\right)$

Absolute difference of areas of triangles = $\left| 2\left(\sqrt{3}+1\right) - 2\left(\sqrt{3}-1\right) \right| = 4$

Aliter



AD = 2 , DC = 2

Difference of Areas of triangle ABC and ABC' = Area of triangle ACC'

$= \frac{1}{2}$ AD × CC' $= \frac{1}{2} \times 2 \times 4 = 4$

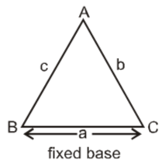
10*. **Sol.** $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$

$\Rightarrow \quad 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin^2\frac{A}{2} \quad \Rightarrow \quad 2\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right] = 0$

$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) - 2\cos\left(\frac{B+C}{2}\right) = 0 \qquad$ as $\sin\frac{A}{2} \neq 0$

$\Rightarrow \quad -\cos\frac{B}{2}\cos\frac{C}{2} + 3\sin\frac{B}{2}\sin\frac{C}{2} = 0$

$\Rightarrow \quad \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$



$\Rightarrow \quad \sqrt{\frac{(s-a)(s-c)}{s(s-b)} \cdot \frac{(s-b)(s-a)}{s(s-c)}} =$

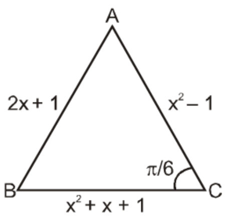
$\Rightarrow \quad \frac{s-a}{s} = \frac{1}{3} \quad \Rightarrow \quad 2s = 3a \quad \Rightarrow \quad b + c = 2a$

∴ Locus of A is an ellipse

11. **Sol.** $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \frac{2}{2R}(a\cos C + c\cos A) = \frac{b}{R} = 2\sin B = 2\sin 60^\circ = \sqrt{3}$

12. **Sol.** $\cos\frac{\pi}{6} = \frac{(x^2-1)^2 + (x^2+x+1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$



$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow \sqrt{3} = \dfrac{x^2-1+x(x+2)}{x^2+x+1}$ $\Rightarrow$ $\sqrt{3}(x^2 + x + 1) = 2x^2 + 2x - 1$

$\Rightarrow (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) = 0$

on solving

$x^2 + x - (3\sqrt{3}+5) = 0$ we get

$x = \sqrt{3} + 1, -(2+\sqrt{3})$

$\because$ At $x = -(2+\sqrt{3})$, Side c becomes negative.

$\therefore$ $x = \sqrt{3}+1$

13. **Ans.** 3

Sol. Area of triangle = $\frac{1}{2}$ ab sin C = $15\sqrt{3}$

$\Rightarrow$ $\frac{1}{2}$. 6 . 10 sin C = $15\sqrt{3}$ $\Rightarrow$ sin C = $\frac{\sqrt{3}}{2}$

$\Rightarrow$ C = $\frac{2\pi}{3}$ (C is obtuse angle)

Now cos C = $\Rightarrow$ $-\frac{1}{2} = \dfrac{36+100-c^2}{2.6.10}$ $\Rightarrow$ c = 14

$\therefore$ $r = \dfrac{\Delta}{s} = \dfrac{\frac{15\sqrt{3}}{6+10+14}}{2} = \sqrt{3}$ $\Rightarrow$ $r^2 = 3$

14. **Sol.** **Ans.** **(C)**

a = 2 = QR

b = $\frac{7}{2}$ = PR

c = $\frac{5}{2}$ = PQ

s = $\dfrac{a+b+c}{2} = \dfrac{8}{4} = 4$

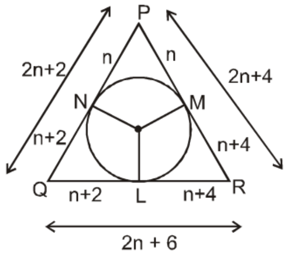
$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)} = \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

15.* **Sol.** **(B, D)**

$$\cos P = \frac{(2n+2)^2 + (2n+4)^2 - (2n+6)^2}{2\ (2n+2)\ (2n+4)} = \frac{1}{3}$$

$$\Rightarrow \frac{4n^2-16}{8\ (n+1)\ (n+2)} = \frac{1}{3}$$



$= \frac{n^2-4}{2(n+1)(n+2)} = \frac{1}{3}$ $\Rightarrow \frac{n-2}{2\ (n+1)} = \frac{1}{3}$

= 3n – 6 = 2n + 2

⇒ n = 8

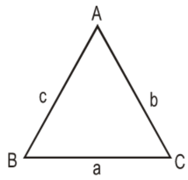
⇒ 2n + 2 = 18

⇒ 2n + 4 = 720

⇒ 2n + 6 = 22

16. Ans. (B)

Sol. a + b = x



ab = y

$x^2 - c^2 = y$

$(a + b)^2 - c^2 = ab$

$a^2 + b^2 + ab = c^2$ $a^2 + b^2 - c^2 = -ab$

$\frac{a^2+b^2-c^2}{2ab} = \frac{7}{2}$

cosC = $\frac{-1}{2}$

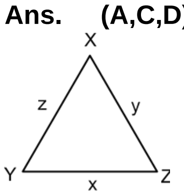
C = $\frac{2\pi}{3}$

$\frac{r}{R} = \frac{\Delta \times 4\Delta}{s \times abc} = \frac{4 \times \frac{1}{4}a^2b^2\sin^2 C}{(a+b+c)abc} = \frac{3ab}{4c(x+c)}$

$= \frac{3y}{4c(x+c)}$

17. Ans. (A,C,D)

Sol.



2S = x + y + z ⇒ $\frac{S-x}{4} = \frac{S-y}{3} = \frac{S-z}{2} = \lambda$

S – x = 4λ

S – y = 3λ

S –z = 2λ

S = 9λ

Adding all we get

S = 9λ, x = 5λ, y = 6λ, = z = 7λ

$\pi r^2 = \frac{8\pi}{3} \quad \Rightarrow \quad r^2 = \frac{8}{3}$

$\Delta = \sqrt{S(S-x)(S-y)(S-z)} \quad \Rightarrow \quad \Delta = \sqrt{9\lambda.4\lambda.3\lambda.2\lambda.} = 6\sqrt{6}\,\lambda^2$

$R = \frac{xyz}{4\Delta} = \frac{5\lambda.6\lambda.7\lambda}{4.6\sqrt{6}\lambda^2} = \frac{35}{4\sqrt{6}}\lambda \quad \Rightarrow \quad r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2} = \frac{24}{9}\lambda^2 = \frac{8}{3}\lambda^2 = \frac{8}{3}$

we get $\lambda = 1$

(A) $\Delta = 6\sqrt{6}$ (B) $R = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$

(C) $r = 4R \sin\frac{X}{2} \sin\frac{Y}{2} \sin\frac{Z}{2} \quad \Rightarrow \quad \frac{2\sqrt{2}}{\sqrt{3}} = 4.\frac{35}{4\sqrt{6}}.\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$

$\frac{4}{35} = \sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{S(S-z)}{xy} = \frac{9.2}{5.6} = \frac{3}{5}$

Additional Problems For Self Practice (APSP)

PART - I : PRACTICE TEST PAPER

1. Sol. $a + b = 2\sqrt{3}$, $ab = 2$

$(a - b)^2 = (a + b)^2 - 4ab = 12 - 8 = 4$

$\Rightarrow a - b = 2$

$a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 - 4\cos\left(\frac{\pi}{3}\right) = 2(3 + 1) - 2$

$c = \sqrt{6}$

Perimeter $= a + b + c = 2\sqrt{3} + \sqrt{6}$

2. Sol. Here $\angle c = 90°$ $\quad (\because c^2 = a^2 + b^2)$

Now, $4s(s - a)(s - b)(s - c)$

$= 4\Delta^2$

$= 4\left(\frac{1}{2}ab\right)^2 = a^2b^2$

3. Sol. $\Delta = (a - b + c)(a + b - c)$

$\Delta = (2s - 2b)(2s - 2c)$

$\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}4(s - b)(s - c)$

$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{4}$

$\tan\frac{A}{2} = \frac{1}{4}$

$\tan A = \frac{2(1/4)}{1-(1/4)} = \frac{8}{15}$

4. Sol. $a^2 \sin 2B + b^2 \sin 2A$

$= a^2(2\sin B\cos B) + b^2(2\sin A\cos A)$

$= 2a^2(kb)\cos B + 2b^2(ka)\cos A$

$= 2abk(a\cos B + b\cos A)$

$= 2ab(ck)$

$= 4\left(\frac{1}{2}ab\sin C\right)$

$= 4\Delta$

5. Sol. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$\frac{1}{3} \times \frac{2}{3} + \frac{2}{3}\tan\frac{C}{2} + \left(\tan\frac{C}{2}\right)\left(\frac{1}{3}\right) = 1$

$\tan\frac{C}{2} = 1 - \frac{2}{9} = \frac{7}{9}$

6. **Sol.** $\cos A = \frac{kb}{2(kc)} = \frac{b}{2c}$

$\frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2c}$

$\Rightarrow b^2 + c^2 - a^2 = b^2$

$c^2 - a^2 = 0$

$a = c$

isosceles,

7. **Sol.** $\frac{\sin A}{1/3} = \frac{\sin B}{1/6} = \frac{\sin C}{1/2\sqrt{3}}$

$\Rightarrow \frac{\sin A}{1} = \frac{\sin B}{1/2} = \frac{\sin C}{\sqrt{3}/2}$

$\angle A = 90°$

$\angle B = 30°$

$\angle C = 60°$

8. **Sol.** $r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

$s = \frac{3p + 5p + 6p}{2} = 7p$

$\therefore 4 = \sqrt{\frac{(7p-3p)(7p-5p)(7p-6p)}{7p}}$

$p = \sqrt{14}$

9. **Sol.** $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-a)(s-c)} \cdot \frac{s(s-c)}{(s-a)(s-b)}}$

$= \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$

$= \frac{s^2}{\Delta} = \frac{s}{(\Delta/s)}$

10. **Sol.** $c^2 + a^2 - 2ac + ac = b^2$

$c^2 + a^2 - b^2 = ac$

$\frac{c^2 + a^2 - b^2}{2ac} = \frac{1}{2} \Rightarrow \cos B = \frac{1}{2}$

$B = 60°$

11. **Sol.** Let $a = 4k$

$b = 5k$

$c = 7k$

$S = \frac{(4+5+7)k}{2}$

$= 8k$

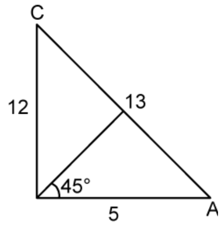
$\Delta = \sqrt{s(s-a)(s-b)(s-C)} = \sqrt{8k(4k)(3k)(k)} = 4\sqrt{6}\,k^2$

R = $\frac{abc}{4\Delta}$ = $\frac{(4k)(5k)(7k)}{4(4\sqrt{6}k^2)}$ = $\frac{35}{4\sqrt{6}}$ k

r = $\frac{\Delta}{S}$ = $\frac{4\sqrt{6}k^2}{8k}$ = $\frac{4\sqrt{6}}{8}$

$\therefore \frac{R}{r} = \frac{\frac{35k}{4\sqrt{6}}}{\left(\frac{4\sqrt{6}}{8}k\right)} = \frac{35\times 8}{16\times 6} = \frac{35}{12}$

12. **Sol.**



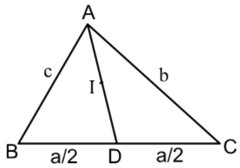
a = 12, b = 13, c = 5

length of angle bisector of ∠B

$= \frac{2ac}{a+c}\cos\left(\frac{B}{2}\right) = \frac{2(12)(5)}{12+5}\cos(45°) = \frac{2(60)}{17} \times \frac{1}{\sqrt{2}} = \frac{120}{17\sqrt{2}}$

13. **Sol.** $= \frac{\Delta^2}{s(s-a)(s-b)(s-c)}.s^2 \qquad = s^2$

14. **Sol.**



$\frac{BD}{AB} = \frac{DI}{AI}$

$\Rightarrow \frac{\frac{a}{2}}{c} = \frac{1}{1}$

$\therefore$ a = 2c

b = 2c

cosA = $\frac{b^2+c^2-a^2}{2bc}$ f= $\frac{4c^2+c^2-4c^2}{2(2c)(c)} = \frac{1}{4}$

sin A= $\sqrt{1-\left(\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$

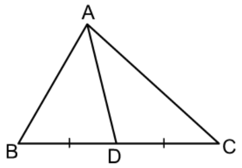
15. **Sol.** $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}\tan\left(\frac{A+B}{2}\right)$

$= \frac{1}{3}\left(\cot\left(\frac{C}{2}\right)\right) \Rightarrow \frac{a-b}{a+b} = \frac{1}{3} \quad \Rightarrow 3a - 3b = a + b$

2a = 4b

b : a = 1 : 2

16. **Sol.**



BD = $\frac{a}{2}$ AND $\angle BAD = 90°$

$\cos B = \frac{c}{(a/2)} = \frac{2c}{a} \Rightarrow \frac{c^2+a^2-b^2}{2ca} = \frac{2c}{a} \Rightarrow a^2 - b^2 = 3c^2$

17. **Sol.** $2R = 4R\sin\frac{A}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right) = 4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) \Rightarrow 1 = 2\sin\frac{A}{2}\left(\sin\frac{A}{2}\right)$

$\sin\frac{A}{2} = \frac{1}{\sqrt{2}} \Rightarrow \angle A = 90°$

18. **Sol.** $2\left(\frac{b^2+c^2-a^2}{2abc}\right) + \left(\frac{c^2+a^2-b^2}{2abc}\right) + 2\left(\frac{a^2+b^2-c^2}{2abc}\right) = \frac{a^2+b^2}{abc}$

$\Rightarrow b^2 + c^2 = a^2 \Rightarrow \angle A = 90°$

20. **Sol.** $\frac{s-a}{s} = \frac{s-c}{s-b} \Rightarrow \frac{(s-a)(s-b)}{s(s-c)} = 1$

$\tan^2\frac{C}{2} = 1 \Rightarrow \tan\frac{C}{2} = 1 \Rightarrow \frac{C}{2} = 45°$

$\angle C = 90°$

21. **Sol.** $2a^2 + 9b^2 + c^2 = 6ab + 2ac$

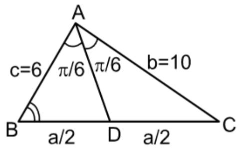
$\Rightarrow a^2 + 9b^2 - 6ba + a^2 + c^2 - 2ac = 0 \Rightarrow (a-3b)^2 + (a-c)^2 = 0$

$\Rightarrow a = 3b, a = c \Rightarrow a = 3b = c$

$\cos B = \frac{c^2+a^2-b^2}{2ac}$ $\frac{a}{1} = \frac{b}{1/3} = \frac{c}{1}$

$= \frac{1+1-\left(\frac{1}{3}\right)^2}{2\times1\times1} = \frac{2-\frac{1}{9}}{2} = \frac{17}{18}$

22. **Sol.**



$\cos A = \cos\frac{\pi}{3} = \frac{b^2+c^2-a^2}{2bc}$

$\frac{1}{2} = \frac{100+36-a^2}{2\times10\times6}$

$\Rightarrow a^2 = 136 - 60$

a2 = 76

length of median AD is

$= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$= \frac{1}{2}\sqrt{200 + 72 - 76}$

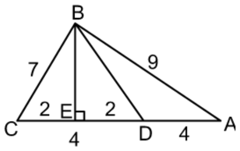
$= \frac{1}{2}\sqrt{196}$

$= \frac{1}{2}\sqrt{2 \times 2 \times 49}$

$= \frac{1}{2} \times 2 \times 7$

= 7

23. **Sol.**



median = $\frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

$= \frac{1}{2}\sqrt{162 + 98 - 64}$

$= \frac{1}{2}\sqrt{196}$

$= \frac{1}{2} \times 14 = 7$

BE = $\sqrt{7^2 - 2^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$

24. **Sol.** Angles are 15°, 75°, 90°

$\frac{a}{\sin 15} = \frac{b}{\sin 75^\circ} = \frac{c}{\sin 90^\circ}$

$\frac{a}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{1}$

$\frac{a}{\sqrt{3}-1} = \frac{b}{\sqrt{3}+1} = \frac{c}{2\sqrt{2}}$

S = $\frac{a+b+c}{2} = \frac{1}{2}(2\sqrt{3} + 2\sqrt{2}) = (\sqrt{3} + \sqrt{2})$

25. **Sol.** $\frac{\sin\alpha}{a-2} = \frac{\sin 2\alpha}{a+2}$

$\Rightarrow 2\cos\alpha = \frac{a+2}{a-2}$

Now, $\cos\alpha = \frac{(a+2)^2 + a^2 - (a-2)^2}{2(a+2)a}$

$= \dfrac{a^2+4a+4+a^2-a^2+4a-4}{2a(a+2)}$

$= \dfrac{a(a+8)}{2a(a+2)}$

$\Rightarrow \dfrac{a+8}{2(a+2)}\ \dfrac{a+2}{2(a-2)} \Rightarrow a^2 + 6a - 16 = a^2 + 4a + 4$

$2a = 20$

$a = 10$

26. **Sol.** $\cos(A - B) = \dfrac{4}{5} \Rightarrow \tan(A - B) = \dfrac{3}{4}$

$\dfrac{2\tan\left(\dfrac{A-B}{2}\right)}{1-\tan^2\left(\dfrac{A-B}{2}\right)} = \dfrac{3}{4}$

$\Rightarrow 8\tan\left(\dfrac{A-B}{2}\right) = 3 - 3\tan^2\left(\dfrac{A-B}{2}\right)$

$\tan = \left(\dfrac{A-B}{2}\right) - 3, \dfrac{1}{3}$

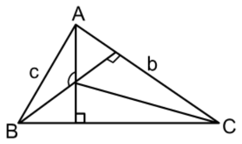
$\because \dfrac{A-B}{2}$ is acute, $\therefore \tan\left(\dfrac{A-B}{2}\right) = \dfrac{1}{3}$

Now, $\dfrac{a-b}{a+b}\cot\dfrac{C}{2} = \dfrac{1}{3}$

$\dfrac{8-4}{8+4}\cot\dfrac{C}{2} = \dfrac{1}{3}$

$\Rightarrow \cot\dfrac{C}{2} = 1 \Rightarrow \angle C = 90°$

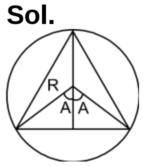
27. **Sol.**



$\dfrac{AH}{\sin(90-A)} = \dfrac{c}{\sin(90-A)}$

$AH = \dfrac{c\cos A}{\sin(180-C)} = \dfrac{c\cos A}{\sin C}$

28. **Sol.**



$\cos A = \dfrac{\ell}{R}$

$\ell = R\cos A$

PART - II : PRACTICE QUESTIONS

Practice Questions: 20-50 depending on chapter length.

1. **Sol.** $\because$ a tan A + b tan B = (a + b) tan $\left(\frac{A+B}{2}\right)$

$\Rightarrow a\left[\tan A - \tan\left(\frac{A+B}{2}\right)\right] = b\left[\tan\left(\frac{A+B}{2}\right) - \tan B\right]$

$\Rightarrow a\left[\frac{\sin A\cos\left(\frac{A+B}{2}\right) - \sin\left(\frac{A+B}{2}\right)\cos A}{\cos A\cos\left(\frac{A+B}{2}\right)}\right] = b\,\frac{\sin\left(\frac{A+B}{2} - B\right)}{\cos B\cos\left(\frac{A+B}{2}\right)}$

$\Rightarrow \frac{a\ \sin\left(A - \frac{A+B}{2}\right)}{\cos A\cos\left(\frac{A+B}{2}\right)} = \frac{b\ \sin\left(\frac{A-B}{2}\right)}{\cos B\cos\left(\frac{A+B}{2}\right)}$

$\Rightarrow \sin\left(\frac{A-B}{2}\right)\left(\frac{a}{\cos A} - \frac{b}{\cos B}\right) = 0$

$\Rightarrow \sin\left(\frac{A-B}{2}\right) = 0$ or ;k $\frac{a}{\cos A} - \frac{b}{\cos B} = 0$

$\Rightarrow$ A = B or ;k 2R (tan A – tan B) = 0

$\Rightarrow$ tan A = tan B

$\Rightarrow$ A = B

2. **Sol.** $\because$ cos A cot $\frac{A}{2}$ = $\left(1 - 2\sin^2\frac{A}{2}\right)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$ = cot $\frac{A}{2}$ – sin A

Similarly cos B cot $\frac{B}{2}$ = cot $\frac{B}{2}$ – sin B

and cos C cot $\frac{C}{2}$ = cot $\frac{C}{2}$ – sin C

$\because$ a, b, c are in A.P.

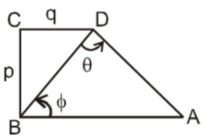
$\therefore$ sin A, sin B, sin C are also in A.P.

$\because$ a, b, c are in A.P.

$\therefore$ cot $\frac{A}{2}$, cot $\frac{B}{2}$, cot $\frac{C}{2}$ are also in A.P.

$\therefore$ cot $\frac{A}{2}$ – sin A, cot $\frac{B}{2}$ – sin B, cot $\frac{C}{2}$ – sin C are also in A.P.

3. **Sol.**



If we apply Sine-Rule in Δ ABD , we get

$\frac{AB}{\sin\ \theta} = \frac{BD}{\sin(\pi-(\theta+\phi))}$ $\Rightarrow$ $AB = \frac{BD\sin\theta}{\sin(\theta+\phi)} = \frac{BD\ \sin\theta}{\sin\theta\ \cos\phi+\cos\theta\sin\phi}$ ...(i)

$\sin\varphi = \frac{p}{\sqrt{p^2+q^2}}$ and $\cos\varphi = \frac{p}{\sqrt{p^2+q^2}}$

$\therefore$ from equation (i), we get

$$AB = \frac{\left(\sqrt{p^2+q^2}\right)\sin\theta}{\frac{q\sin\theta}{\sqrt{p^2+q^2}}+\frac{p\cos\theta}{\sqrt{p^2+q^2}}}$$

$$\therefore\ AB = \frac{(p^2+q^2)\sin\theta}{q\sin\theta+p\cos\theta}$$

4*. **Sol.** cos A(sin B – sin C) + (sin 2B – sin 2C) = 0

$\Rightarrow$ cos A.(sin B – sin C) + 2 cos(B + C) sin (B – C) = 0 $\because$ B + C = π – A

$\Rightarrow$ cos A.(sin B – sin C) – 2 cos A.sin (B – C) = 0

$\Rightarrow$ cos A[(sin B – sin C) – 2(sin B cos C– cos Bsin C)] = 0

$\Rightarrow$ either cos A = 0 $\Rightarrow$ A = 90° $\Rightarrow$ right angled

or (sin B – sin C) – 2(sin B cos C – cos B sin C) = 0

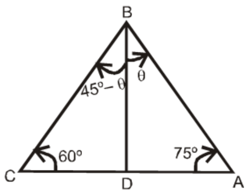
$$\Rightarrow (b-c) - 2\left(b.\frac{a^2+b^2-c^2}{2ab} - c.\frac{a^2+c^2-b^2}{2ac}\right) = 0$$

$\Rightarrow a(b-c) - 2(b^2-c^2) = 0$

$(b-c)\,[a-2(b+c)] = 0$

$\therefore$ b – c = 0 $\Rightarrow$ b = c $\Rightarrow$ isosceles

5. **Sol.**



Area of ΔBAD = $\sqrt{3}$ × Area of ΔBCD ΔBAD = $\sqrt{3}$ × ΔBCD

$\Rightarrow$ $\frac{1}{2}$ BD × BA sin θ = $\sqrt{3}$ × $\frac{1}{2}$ BC × BD sin (45° –θ)

$\frac{BA}{BC} = \sqrt{3}\ \frac{\sin(45^o-\theta)}{\sin\theta}$ ....(1)

$\because$ From Sine-Rule

$\frac{BC}{\sin75^o} = \frac{AB}{\sin60^o}$

$\therefore$ $\frac{BA}{BC} = \frac{\sin60^o}{\sin75^o} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}+1}$

$\therefore$ From equation (1) lehdj.k (1) ls

$$\frac{\sqrt{3}.\sqrt{2}}{(\sqrt{3}+1)} = \sqrt{3}\left[\frac{1}{\sqrt{2}}\cot\theta-\frac{1}{\sqrt{2}}\right]$$

$\Rightarrow \quad \dfrac{2}{(\sqrt{3}+1)}$ = cot θ –1 $\quad \Rightarrow \quad \dfrac{2\left(\sqrt{3}-1\right)}{2}$ = cot θ – 1

$\Rightarrow$ cot θ = $\sqrt{3}$ $\quad \Rightarrow$ θ = 30º $\quad \Rightarrow$ ∠ ABD = 30º

6. **Sol.** $\because \dfrac{\text{Area of incircle}}{\text{Area of } \Delta ABC} = \dfrac{\pi r^2}{\frac{1}{2}bc \quad \sin A}$

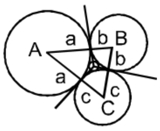
$= \dfrac{\pi \times 16R^2 \times \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\frac{1}{2}(2R\sin B)(2R\sin C)\left(2\sin\frac{A}{2} \quad \cos\frac{A}{2}\right)}$

$= \dfrac{4\pi\sin\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)\cos\frac{A}{2}}$

$= \dfrac{\pi\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \dfrac{\pi}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$

$= \pi : \cot\frac{A}{2} \cot\frac{B}{2} \cot\frac{C}{2}$

7. **Sol.**



required distance = inradius of Δ ABC

$\because$ 2s = a + b + b + c + c + a

= 2 (a + b + c)

s = a + b + c

 Δ = $\sqrt{s(s-(a+b))(s-(b+c))(s-(c+a))}$

= $\sqrt{(a+b+c)(abc)}$

∴ required distance

8. **Sol.** $\because$ AD = $\dfrac{abc}{b^2-c^2}$

$\Rightarrow \quad \dfrac{2\Delta}{a} = \dfrac{abc}{b^2-c^2} \Rightarrow \dfrac{bc\sin A}{a} = \dfrac{abc}{b^2-c^2}$

$\Rightarrow$ $(b^2 - c^2)$ sin A = a^2

$\Rightarrow$ sin A $\left(\sin^2 B - \sin^2 C\right)$ = $\sin^2$ A

$\Rightarrow$ sin(B + C) sin (B – C) = sin (B + C)

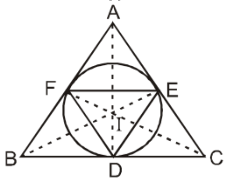
$\Rightarrow$ sin (B – C) = 1

$\Rightarrow$ B – C = 90º

$\Rightarrow$ B = 90° + C = 90° + 23°

∴ B = 113°

9. Sol. (i) EIFA is a cyclic quadrilateral



∴ $\frac{EF}{\sin A} = AI$

∵ AI = r cosec A/2

∴ EF = r cosec A/2.sin A

= 2 r cos A/2

similarly DF = 2 r cos B/2

and DE = 2r cos C/2.

(ii) IECD is a cyclic quadrilateral

∴ $\angle ICE = \angle IDE = \frac{C}{2}$

similarly $\angle IDF = \angle IBF = \frac{B}{2}$

∴ $\angle FDE = \frac{B}{2} + \frac{C}{2} = \frac{\pi - A}{2}$

$= \frac{\pi}{2} - \frac{A}{2}$

(iii) area of $\Delta DEF = \frac{1}{2}$ FD. DE sin ∠FDE

$= \frac{1}{2}\left(2r\cos\frac{B}{2}\right)\left(2r\cos\frac{C}{2}\right)\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)$

$= 2r^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$= 2r^2 \left(\frac{\sin A + \sin B + \sin C}{4}\right)$

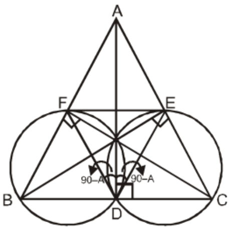
$= \frac{r^2}{2}\left(\frac{2\Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab}\right)$

$= \frac{r^2}{2}\left[\frac{2\Delta \ (a+b+c)}{abc}\right] = \frac{r^2\Delta.2s}{abc}$

$= \frac{2r^2.\Delta s^2}{(abc)s} = \frac{2\Delta(rs)^2}{(abc)s}$

$= \frac{2\Delta^3}{(abc)s} = \frac{1}{2}\frac{r\Delta}{R}$.

10. Sol.



$\angle$EDF = 90 – A + 90 – A
= 180 – A

11*. **Sol.** ΔAEF : AF = bcosA, AE = ccosA

∴ $\cos A = \dfrac{b^2\cos^2 A + C^2\cos^2 A - EF^2}{2b\cos A.c\cos A}$

⇒ $(EF)^2 = (b^2 + c^2 - 2bc\cos A)\cos^2 A$

$(EF)^2 = a^2\cos^2 A$

EF = a.cosA

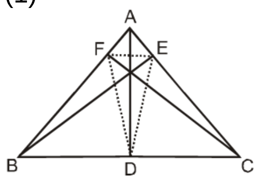
12. **Sol.** Circumraii of the triangle PBC = $\dfrac{BC}{2.\sin(B+C)} = \dfrac{a}{2\sin(\pi - A)} = \dfrac{a}{2\sin A} = R$

13*. **Sol.** ∵ FE = a cos A = R sin 2A

DE = c cos C = R sin 2C

FD = b cos B = R sin 2B

(1) = $\dfrac{R\ \left(\sum \sin 2A\right)}{a+b+c}$



$= \dfrac{R(4\sin A\sin B\sin C)}{2R\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)} = \dfrac{8\left(\prod \sin\frac{A}{2}\right)\left(\prod \cos\frac{A}{2}\right)}{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \dfrac{r}{R}$

(2) ∵ Area of ΔDEF

$= \frac{1}{2}$ FD × DE sin (π – 2A) $= \frac{1}{2}$ b cos B.c cos C.sin 2A

$= \frac{1}{2}$ bc cos B.cos C.2 sin A.cos A = $2\left(\frac{1}{2}bc\sin A\right)$ cos A.cos B.cos C

= 2Δ cos A.cos B. cos C

(3) Area of ΔAEF $= \frac{1}{2}$ AE ×AF sin A

$= \frac{1}{2}$ (c cos A) (b cos A) sin A = $\left(\frac{1}{2}\ bc\ \sin\ A\right)$ $\cos^2 A = \Delta\cos^2 A$

(4) $R_{DEF} = \frac{FE \times DE \times FD}{4\Delta_{DEF}} = \frac{abc\cos A\cos B\cos C}{4 \times 2\Delta\cos A\cos B\cos C} = \frac{abc}{8\Delta} = \frac{4R\Delta}{8\Delta} = \frac{R}{2}$ ss

14. **Sol.** Clearly

15. Angles of the $\Delta I_1 I_2 I_3$ are

Sol. Let $\angle I_3 I_1 I_2 = \theta$

Then angle of pedal trinagle = $\pi - 2\theta = A$

$\theta = \frac{\pi}{2} - \frac{A}{2}$

16. **Sol.** Side of pedal triangle = $I_2I_3\cos\theta = BC$

$I_2I_3 = \frac{a}{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)}$

$I_2I_3 = 4R\cos\left(\frac{A}{2}\right)$

17. **Sol** $I I_1 = 4R\sin\frac{A}{2}$

$I_2I_3 = 4R\cos\frac{A}{2}$

$\therefore$ $II_1^2 + I_2I_3^2 = 16R^2$