Complex Numbers

Exercise-1

Section (A) : Algebra , Modulus and Conjugate of complex number

A-1.

Sol. $\sqrt{-2}\,\sqrt{-3} = (i\sqrt{2})(i\sqrt{3}) = -\sqrt{6}$

A-2. **Sol.** $i^{4n} = 1$, $i^{4n-1} = -i$, $i^{4n+1} = i$, $i^{-4n} = 1$

A-3. **Sol.** $\sum_{n=1}^{200} i^n = i + i^2 + i^3 + ... + i^{200}$

$= \dfrac{i(1-i^{200})}{1-i} = \dfrac{i(1-1)}{1-i} = 0$

A-4. **Sol.** $i^{1+3+5+.....+(2n+1)} = i^{(n+1)^2} = \begin{cases} 1, \text{ if n is odd} \\ i, \text{ if n is even} \end{cases}$

A-5. **Sol.** $\left(\dfrac{1+i}{1-i}\right)^n = \left[\dfrac{(1+i)^2}{(1-i)^2}\right]^{\frac{n}{2}} = \left(\dfrac{2i}{-2i}\right)^{\frac{n}{2}} = (-1)^{n/2}$,

so n = 2 is mini. value of n ∈ N forwhich given expression is real.

A-6. **Sol.** $(1 - i)x + (1 + i)y = 1 - 3i$

comparing the real and imaginary part

$x + y = 1$ .... (1)

$-x + y = -3$ .... (2)

$\Rightarrow$ $x = 2, y = -1$

A-7. **Sol.** $\dfrac{(1-\cos\theta) - 2i\sin\theta}{(1-\cos\theta)^2 + 4\sin^2\theta} = \dfrac{1-\cos\theta - 2\,i\sin\theta}{(1-\cos\theta)(1-\cos\theta + 4\,(1+\cos\theta))}$

real part $= \dfrac{(1-\cos\theta)}{(1-\cos\theta)(5+3\cos\theta)} = \dfrac{1}{5+3\cos\theta}$

A-8. **Sol.** $\dfrac{(1+i)^2}{(2-i)} = \dfrac{2i}{2-i} \times \dfrac{2+i}{2+i} = \dfrac{-2+4i}{5}$

imaginary part $= \dfrac{4}{5}$

A-9. **Sol.** $z = 3 - 4i$

$(z - 3)^2 = (-4i)^2$

$\Rightarrow$ $z^2 - 6z + 25 = 0$

Now

$z^4 - 3z^3 + 3z^2 + 99z - 95$

$= (z^2 - 6z + 25)(z^2 + 3z - 4) + 5$

$= 5$

A-10. **Sol.** $3 + ix^2y = x^2 + y - 4i$

comparing real and imaginary part

$x^2 + y = 3$ .... (1)

$x^2y = -4$ .... (2)

from (1) and (2)
$x = \pm 2 \quad y = -1$

A-11. **Sol.** $z = \sqrt{-8-6i}$
$x + iy = \sqrt{-8-6i}$
squaring both sides
$x^2 - y^2 + 2ixy = -8 - 6i$
equating real and imaginary part on both sides
$x^2 - y^2 = -8$
$2xy = -6$
$(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2$
$64 + (36) = (x^2 + y^2)^2$
$x^2 + y^2 = \pm 10$
since x, y are real
$x^2 + y^2 = 10$
$x^2 - y^2 = -8$
$2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$
$2xy = -6$
$y = \frac{-3}{x}$
$x = 1,\ y = \frac{-3}{1} = -3$
$x = -1,\ y = \frac{-3}{-1} = 3$
$z = \pm(1 - 3i)$

A-12. **Sol.** $z = (-7 - 24i)^{1/2}$
$z^2 = -7 - 24i$
$z = x - iy$
$z^2 = x^2 - y^2 - 2ixy$
$x^2 - y^2 = -7$
$2xy = 24$
$xy = 12$
$(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2$
$49 + 4.144 = (x^2 + y^2)^2$
$(x^2 + y^2)^2 = 25^2$
$x^2 + y^2 = 25$

A-13. **Sol.** $\text{Re}(z^2) = 0,\ |z| = 2$
Let $z = x + iy$
$x^2 - y^2 = 0$
$x^2 + y^2 = 4$
Adding we get $2x^2 = 4$
$x = \pm\sqrt{2}$ Clearly $z = \sqrt{2}\ (\pm 1 \pm i)$
$y = \pm\sqrt{2}$ $\therefore$ 4 solutions

Section (B) : Representation of a complex number, Principal argument, Argument and its properties,

B-1. **Sol.** $z = 4e^{\frac{i5\pi}{6}} = 4\cos\frac{5\pi}{6} + i\,4\sin\frac{5\pi}{6} = 4 \times -\frac{\sqrt{3}}{2} + i4 \times \frac{1}{2} = -2\sqrt{3} + 2i$

B-2. **Sol.** $z = \frac{1+i}{1-i}$

$z = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-1+2i}{2} = i = 1.\ e^{i\frac{\pi}{2}}$

$\arg(z) = \frac{\pi}{2}$, modulus = 1

B-3. **Sol.** $z = \frac{1-i\sqrt{3}}{1+i\sqrt{3}} = \frac{2e^{-i\frac{\pi}{3}}}{2e^{i\frac{\pi}{3}}} = e^{-i\frac{2\pi}{3}}$

$-\frac{2\pi}{3} = -120° \approx 240°$

B-4. **Sol.** $z = (1 + i\sqrt{3})^8$

$z = (2e^{i\frac{\pi}{3}})^8$

$= 256\, e^{i\frac{8\pi}{3}}$

$= 256\, e^{i\frac{2\pi}{3}}$

B-6. **Sol.** $z = \frac{1+i\sqrt{3}}{\sqrt{3}+i} = \frac{2e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}} = e^{i\frac{\pi}{6}}$

$\bar{z} = e^{-i\frac{\pi}{6}}$

$\bar{z}_{100} = e^{-i\frac{100\pi}{6}} = e^{-i\frac{50\pi}{3}} = e^{-\frac{2i\pi}{3}}$

which is in III quadrant

B-7. **Solution:**

$-\theta = \arg(z) < 0$

$\arg(-z) = \pi - \theta$

$\Rightarrow \arg(-z) - \arg(z) = \pi - \theta - (-\theta) = \pi$

B-8. **Sol.** $z = 1 + \cos\frac{11\pi}{9} + i\sin\frac{11\pi}{9}$

$z = 1 - \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}$

$z = 2\sin^2\frac{\pi}{9} - 2i\sin\frac{\pi}{9}\cos\frac{\pi}{9}$

$z = 2\sin\frac{\pi}{9}\left(\sin\frac{\pi}{9} - i\cos\frac{\pi}{9}\right)$

$= 2\cos\left(\frac{7\pi}{18}\right)\left(\cos\frac{7\pi}{18} - i\sin\frac{7\pi}{18}\right)$

$= 2\cos\left(\frac{7\pi}{18}\right)\left(\cos\left(-\frac{7\pi}{18}\right) + i\sin\left(\frac{-7\pi}{18}\right)\right)$

arg (z) = $\frac{-7\pi}{18}$

| z | = 2cos $\frac{7\pi}{18}$

B-9. **Sol.** $z = \sin\frac{\pi}{5} + i\left(1-\cos\frac{\pi}{5}\right)$

$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

$= 2\sin\frac{\pi}{10}\,.\,e^{i\frac{\pi}{10}}$

Arg(z) = $\frac{\pi}{10}$

B-10. Sol. $\sin\frac{6\pi}{5} + i\left(1+\cos\frac{6\pi}{5}\right)$

$= -\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$

$=- 2\sin\frac{\pi}{10}\cos\frac{\pi}{10}+2i\sin^2\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\left(-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

$= 2\sin\frac{\pi}{10}\left(\cos\frac{9\pi}{10}+i\sin\frac{9\pi}{10}\right)$

Arg (z) = $\left(\frac{9\pi}{10}\right)$

Section (C) : Properties of conjugate and modulus and Triangle inequality

C-1. **Sol.** $z.\bar{z} = 0$, let z = x + iy ⇒ $x^2 + y^2 = 0$ ⇒ x = y = 0

C-2. **Sol.** (2 + i) (2 + 2i) (2 + 3i) ..... (2 + 9i) = x + iy
5.8.13......... 85 = $(x^2 + y^2)$

C-3. **Sol.** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 + |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 = z\,(|z_1|^2 + |z_2|^2)$

C-4. **Sol.** $|z_1 + z_2| = |z_1 - z_2|$

$z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

$z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

$\frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$

$\frac{z_1}{z_2}$ is purely imaginary $\left|\arg\left(\frac{z_1}{z_2}\right)\right| = \frac{\pi}{2}$

C-5. **Solution :**
$|z_1| = |z_2| = |z_3| = 1$

$z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$

Given

$1 = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = |\overline{z_1 + z_2 + z_3}| = 1$

$1 = |z_1 + z_2 + z_3|$

C-6. **Sol.** $\dfrac{z-i}{z+i} = \lambda i$

$\dfrac{z-i}{z+i} + \dfrac{\bar{z}+i}{\bar{z}-i} = 0$

$\dfrac{(z-i)(\bar{z}-i)+(\bar{z}+i)(z+i)}{(z+i)(\bar{z}-i)} = 0$

$z\bar{z} - i\bar{z} - iz - 1 + z\bar{z} + iz + i\bar{z} - 1 = 0$

$2z\bar{z} - 2 = 0$

$z\bar{z} = 1$

C-7. **Sol.** $\dfrac{z-1}{z+1}$ is purely imaginary

So $\dfrac{z-1}{z+1} = -\dfrac{\bar{z}-1}{\bar{z}+1}$

$z\bar{z} + z - \bar{z} - 1 = -z\bar{z} + z - \bar{z} + 1$

$2z\bar{z} = 2 \qquad \Rightarrow |z|^2 = 1$

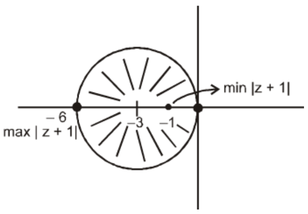
C-8. **Sol.** $|z_1 + z_2| \le |z_1| + |z_2|$

$\le 2 + 1$

≤ 3

C-9. **Sol.** $|z_1 - z_2| \Rightarrow$ minimum distance b/w z_1 & $z_2 = 1$

C-10 **Sol.**



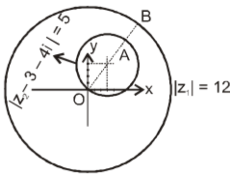
C-11. **Sol.** By using property $|z_1 - z_2| \ge |z_1| - |z_2|$

$|z_1 - z_2| = |z_1 - (z_2 - 3 - 4i) - (3 + 4i)| \ge |z_1| - |z_2 - 3 - 4i| - |3 + 4i| = 12 - 5 - 5 = 2$

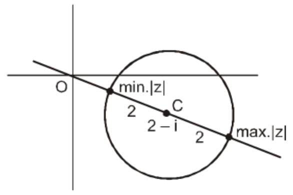
$|z_1 - z_2| \ge 2.$

Clearly from the figure : minimum value of

$|z_1 - z_2| = AB = OB - OA = 12 - 10 = 2$



C-12. **Sol.**

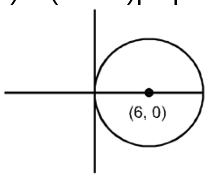


OC = $\sqrt{5}$

min.|z| = $\sqrt{5} - 2$

max. |z| = $\sqrt{5} + 2$

C-13. **Sol.** $|(z_1 - 1) + (z_2 - 2) + (z_3 - 3)| \le |z_1 - 1| + |z_2 - z| + |z_3 - 3|$



⇒ $|z_1 + z_2 + z_3 - 6| < 6$

Let $z = z_1 + z_2 + z_3$

then $|z - 6| < 6$ is circular disc

Clearly $0 < |z| < 12$

Section (D) : Geometry of complex number and Rotation theorem

D-1. **Sol.** Length of segment = $|-1 - i - 2 - 3i|$

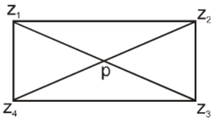
$= |-3 - 4i| = 5$

D-2. **Sol.** $z = -4 + 5i$

$z_{new} = 1.5\,(-4 + 5i)\; e^{i\pi}$

$= \frac{3}{2}(4 - 5i) \quad = 6 - \frac{15i}{2}$

D-3. **Sol.**



$p(z) = \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$

D-4. **Sol.** ∴ $|z - (2 - i)| = |z - (3 + i)|$

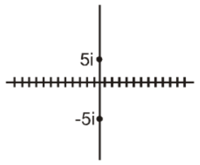
Locus of z is the perpendicular bisector of

(2, –1) and (3, 1)

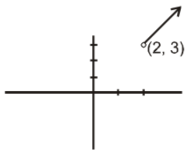
i.e. $2x + 4y = 5$

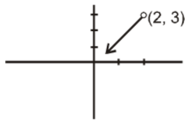
D-5. **Sol.** $|z - 5i| = |z + 5i| \Rightarrow |z - 5i| = |z - (-5i)|$

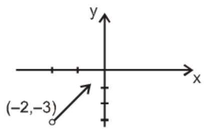
z is equidistant from 5i and –5i

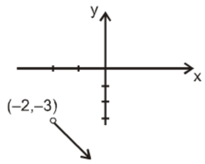


D-6. If Arg $(z - 2 - 3i) = \frac{\pi}{4}$, then the locus of z is

(1*) 

(2) 

(3) 

(4) 

Sol. Ray joining, z to 2 + 3i should make on angle of $\frac{\pi}{4}$ with +ve direction of real axis.

D-7. **Sol.** $|z - 1|^2 + |z + 1|^2 = 2$

$z\bar{z} + 1 - (z + \bar{z}) + z\bar{z} + 1 + (z + \bar{z}) = 2$

$\Rightarrow$ $z\bar{z} = 0$

$\Rightarrow$ $z = 0$

D-8. Sol. Put $z = x + iy \Rightarrow$ $(x + 1)^2 + y^2 + x^2 + y^2 = 4$ $\Rightarrow$ $2x^2 + 2y^2 + 2x - 3 = 0$

$x^2 + y^2 + x - \frac{3}{2} = 0$

D-9. **Sol.** $x^2 + y^2 = 4$

$(x - 3)^2 + y^2 = 4$

On solving

$x = \frac{3}{2}$ $y = \pm \frac{\sqrt{7}}{2}$

$z = \frac{1}{2}(3 \pm i\sqrt{7})$

D-10. **Sol.** $\frac{|z-2|}{|z-3|} = 2.$

End points of diameters are $\left(\frac{8}{3}, 0\right)$ and (4, 0)

radius = $\frac{1}{2}\sqrt{\left(\frac{8}{3} - 4\right)^2} = \frac{2}{3}$

D-11. **Sol.** $|z + 1| = \sqrt{2}\,|z - 1| \Rightarrow\ = \sqrt{2}$

$\sqrt{2} \neq 1$, so locus of z is a circle.

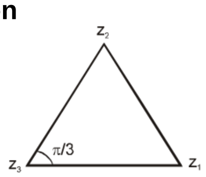
D-12. **Sol.** G $\rightarrow$ Centroid of $\Delta = \frac{z_1 + z_2 + z_3}{3}$

H $\rightarrow$ Orthocentre = z say, O $\rightarrow$ Circum centre = 0

$\because$ G divides HO in ratio 2 : 1

$\frac{z_1 + z_2 + z_3}{3} = \frac{2\cdot 0 + 1\cdot z}{2 + 1}$ $\Rightarrow$ $z = z_1 + z_2 + z_3$

D-15. **Solution**



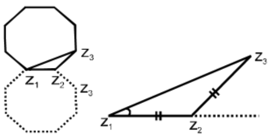
∴ $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{\left(1-\sqrt{3}\,i\right)\left(1+\sqrt{3}\,i\right)}{2\left(1+\sqrt{3}\,i\right)} = \dfrac{4}{2\left(1+\sqrt{3}\,i\right)} = \dfrac{2}{1+\sqrt{3}\,i}$

$\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$ ⇒ $\left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$ and arg $\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$

Hence triangle is equilatral

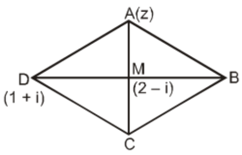
D-16. **Sol.** $\dfrac{z_3 - z_2}{z_2 - z_1} = e^{\pm i\frac{\pi}{4}}$

∵ $|z_1 - z_2| = |z_2 - z_3|$



⇒ $z_3 = z_2 + \left(\dfrac{1+i}{\sqrt{2}}\right)(z_2 - z_1)$

D-17. **Sol.**

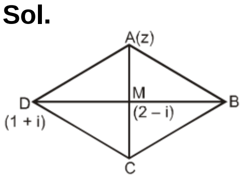


$\dfrac{z-(2-i)}{-1+2i} = \dfrac{1}{2}\, e^{\pm\frac{i\pi}{2}} = \pm\dfrac{i}{2}$

⇒ $z = 1 - \dfrac{3i}{2}$ or (;k½ $3 - \dfrac{i}{2}$

D-18. **Sol.** $e^{iz} = e^{i(r\cos\theta + ir\sin\theta)} = e^{-r\sin\theta}\,(\cos(r\cos\theta) + i\sin(r\cos\theta))$

D-17. **Sol.**



$\dfrac{z-(2-i)}{-1+2i} = \dfrac{1}{2}\, e^{\pm\frac{i\pi}{2}} = \pm\dfrac{i}{2}$ ⇒ $z = 1 - \dfrac{3i}{2}$ or (;k½ $3 - \dfrac{i}{2}$

D-18 **Sol.** $e^{iz} = e^{i(r\cos\theta + ir\sin\theta)} = e^{-r\sin\theta}\,(\cos(r\cos\theta) + i\sin(r\cos\theta))$

Section (E) : De moivre's theorem, cube roots and nth roots of unity

E-1-. **Sol.** $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \dfrac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta - i\sin\theta)^5} = -i(\cos 9\theta + i\sin 9\theta) = \sin 9\theta - i\cos 9\theta$

E-2. **Sol.** $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$

$= e^{(-8\theta - 20\theta + 6\theta - 27\theta)i}$

$= e^{(-49\theta)i} = \cos 49\theta - i\sin 49\theta$

E-3. **Sol.** $\left[\frac{1+\cos(\pi/8)+i\sin(\pi/8)}{1+\cos(\pi/8)-i\sin(\pi/8)}\right]^8 = \left(\frac{2\cos^2\frac{\pi}{16}+i.2\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{2\cos^2\frac{\pi}{16}-i2\cos\frac{\pi}{16}\sin\frac{\pi}{16}}\right)^8$

$= \left(\frac{\cos\frac{\pi}{16}+i.\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}-i\sin\frac{\pi}{16}}\right)^8 = \cos\pi + i\sin\pi = -1$

E-4. **Sol.** $e^{i\theta}.e^{2i\theta}.e^{3i\theta}........e^{ni\theta} = 1$

$= e^{i\theta(1+2+........+n)} = 1$

$= e^{\frac{in(n+1)\theta}{2}} = 1$

$\Rightarrow \frac{n(n+1)\theta}{2} = 2k\pi \qquad k \in I$

$\Rightarrow \theta = \frac{4k\pi}{n(n+1)} \qquad k \in I$ or $\theta = \frac{4m\pi}{n(n+1)}$, $\quad m \in I$

E-5. **Sol.** $\frac{\left(-1+i\sqrt{3}\right)^{15}}{(1-i)^{20}} + \frac{\left(-1-i\sqrt{3}\right)^{15}}{(1+i)^{20}}$

$= \frac{2^{15}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{15}}{2^{10}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{20}} + \frac{2^{15}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)^{15}}{2^{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{20}}$

$= 2\times2^5$ [cos 15π + i sin 15 π] $= 2^6(-1) = -64$

E-6. **Sol.** $\left(\frac{-1+i\sqrt{3}}{2}\right)^{20} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{20} = \omega^{20} + \omega^{40} = \omega^2 + \omega = -1$

E-7. **Sol.** $(3 + 5\omega + 3\omega^2)^2 + (3 + 3\omega + 5\omega^2)^2 = (2\omega)^2 + (2\omega^2)^2 \qquad = 4\omega^2 + 4\omega^4 = -4$

E-8. **Sol.** $(1 + \omega^2)^n = (1 + \omega^4)^n \quad \Rightarrow \quad (-\omega)^n = (-\omega^2)^n$ which is true for n = 3 for least positive integer

E-9. **Sol.** $(1 - \omega + \omega^2)(1 + \omega^4 - \omega^2) = (-2\omega)(-2\omega^2) = 4$

similarly total we have and terms and each equal to 4

∴ Ans. $= 4^n$

E-10. **Sol.** Given , $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} = 1(\omega^{3n} - 1) - \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$

$= 1(1 - 1) - 0 + \omega^{2n}(\omega^n - \omega^n) = 0$

E-11. **Sol.** $4 + 5\omega^{334} + 3\omega^{365} = 4 + 5\omega + 3\omega^2 \qquad = 1 + 2\omega = 1 + 2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = i\sqrt{3}$

E-12. **Sol.** $x = a + b + c$, $y = a\omega + b\omega^2 + c$, $z = a\omega^2 + b\omega + c$

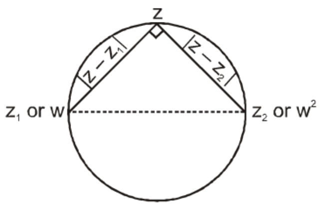
$yz = (a\omega + b\omega^2 + c)(a\omega^2 + b\omega + c)$
$= a^2 + b^2 + c^2 + ab(\omega^4 + \omega^2) + bc(\omega^2 + \omega) + ca(\omega^2 + \omega)$
$= a^2 + b^2 + c^2 + ab(\omega^2 + \omega) + bc(\omega^2 + \omega) + ca(\omega^2 + \omega)$ $\{\omega^4 = \omega\}$
$= a^2 + b^2 + c^2 - ab - bc - ca$ $\{\omega^2 + \omega + 1 = 0\}$
$xyz = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$= a^3 + b^3 + c^3 - 3abc$

E-13. Sol. $\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega^2+\frac{1}{\omega^2}\right)^2+\left(\omega^3+\frac{1}{\omega^3}\right)^2 \ldots\ldots \left(\omega^{27}+\frac{1}{\omega^{17}}\right)^2$
there are 9 term which have ω^{3p}.
so sum 9 × 4 = 36
there are 18 term which not have ω^{3p}
so sum is = 18
total sum = 18 + 36 = 54

NEW

E-13_ Sol. Obvious

E-14. Sol. ∵ Circle



so by pythogorous theorem
$\lambda = |w - w^2|^2 = \left|\sqrt{3}\right|^2 = 3$

E-15. Sol. $\alpha = 2, \beta = 2\omega, \gamma = 2\omega^2 \Rightarrow \frac{a\alpha+b\beta+c\gamma}{a\beta+b\gamma+c\alpha} = \frac{2a+2\omega b+2\omega^2 c}{2\omega a+2\omega^2 b+2c} = \frac{1}{\omega} = \omega^2$

E-16. Sol. $\frac{z-1}{2} = (1)^{\frac{1}{4}}$
$\frac{z-1}{2} = 1, -1, i, -i$
$z = 3, -1, 1 + 2i, 1 - 2i$
sum of roots = 4

E-17. Sol. Product of roots of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}$
$= (-1)^3 (\cos\pi + i\sin\pi)$
$= 1$

E-18. Sol. $z_1z_2z_3z_4z_5 = e^{i\frac{2\pi}{5}[1+2......5]} = e^{\frac{2\pi}{5}\cdot\frac{5}{2}\cdot 6} = e^{i6\pi} = 1$

E-19. Sol. $\alpha = 1^{1/5}$
consider $x^5 - 1 = 0$
so $2^{\left|1+\alpha+\alpha^2+\frac{\alpha^3}{\alpha^5}-\frac{\alpha^4}{\alpha^5}\right|} = 2^{|1+\alpha+\alpha^2+\alpha^3-\alpha^4|}$

$= 2^{|-\alpha^4-\alpha^4|} = 4^{|\alpha^4|} = 4$

E-20. Sol. $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$\Rightarrow \sum_{k=1}^{6} -i\left[\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right]$

$= -i(-1) = i$

E-21. Sol. $k \cdot \frac{2\pi}{n} = \frac{\pi}{2} \Rightarrow n = 4k$

E-22. Sol. $-1(-1)^{n-1} = -(-1)^{3-1} = -1$

Exercise-2

PART - I : OBJECTIVE QUESTIONS

1. Sol. $z = 2e^{i\pi}e^{i\pi/6} = 2e^{-i5\pi/6}$

$|z| = 2$, $\text{Arg } z = -\frac{5\pi}{6}$.

2. Sol. $z = 1 + e^{i\frac{18\pi}{25}} = e^{i\frac{9\pi}{25}}\left[e^{i\frac{9\pi}{25}} + e^{-i\frac{9\pi}{25}}\right]$

$z = 2\cos\left(\frac{9\pi}{25}\right) e^{i\frac{9\pi}{25}}$

$|z| = 2\cos\left(\frac{9\pi}{25}\right)$ $\quad$ $\text{Arg } z = \frac{9\pi}{25}$

3. Sol. $(a + ib)^5 = \alpha + i\beta$

$i^5 (b - ia)^5 = \alpha + i\beta$

$(b - ia)^5 = -i\alpha + \beta$

$(b + ia)^5 = \beta + i\alpha$

4. Sol. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

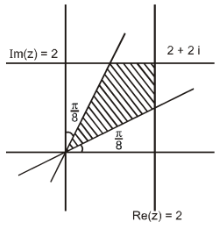
$z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

$\Rightarrow \frac{z_1}{z_2} = -\frac{\bar{z}_1}{\bar{z}_2}$

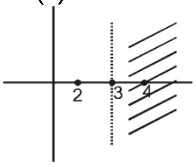
$\frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)} = 0 \Rightarrow \frac{z_1}{z_2}$ is purely imaginary

so amp $\left(\frac{z_1}{z_2}\right)$ may be $\frac{\pi}{2}$

5. Sol.

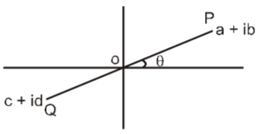


6. **Sol.** $|z - 4| < |z - 2|$
In the shaded region
$Re(z) > 3$



7. **Sol.** Obvious

8. **Sol.**

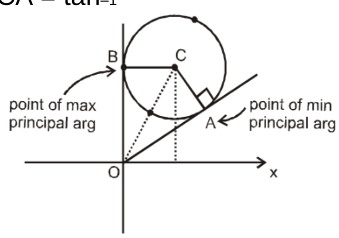


$|a + ib| = |(c + id)|$
and $a + c = b + d$

9. **Sol.** max (arg z) = $\frac{\pi}{2}$
max | z | = d + r
min | z | = d – r
d = OC = $\sqrt{5}$
r = 1

$\theta = \angle OCX = \tan^{-1}\frac{2}{1}$

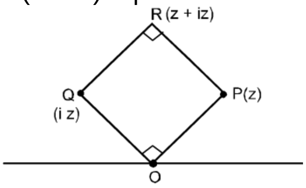
$\alpha = \angle OCA = \tan^{-1}$ $\left(\because \ \sin\alpha = \frac{1}{\sqrt{5}}\right)$



So principal Arg of A = θ – α $= \tan^{-1} 2 - \tan^{-1}\frac{1}{2}$

$= \tan^{-1}\frac{2-\frac{1}{2}}{1+1} = \tan^{-1}\frac{3}{4}$

10. **Sol.** $iz = ze^{i\frac{\pi}{2}}$, Q is obtained by rotating P about origin through an angle $\frac{\pi}{2}$
R(z + iz) represents vertex of parallelogram(square) OPRQ.



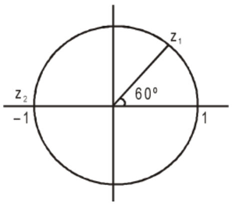
⇒ ΔPQR = 200

⇒ $\frac{1}{2}$ |z| |iz| = 200

$|z|^2 = 400$ ⇒ |z| = 20

11. Sol. $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$
⇒ $(a-1)^2 + (1-b)^2 = a^2 + 1 = b^2 + 1$
⇒ $a = b$ and $a^2 - 2a + 1 + b^2 - 2b + 1 = a^2 + 1$
⇒ $a^2 - 4a + 1 = 0$
⇒ $a = 2 - \sqrt{3} = b$ ∴ $0 < a, b < 1$

12. **Sol.**



13. **Sol.** All the three vertices lies on circle |z| = 1

so there centroid at O i.e. $\frac{z_1 + z_2 + z_3}{3} = 0$

14. $2z_2 = z_1 + z_3$

⇒ $z_2 = \frac{z_1 + z_3}{2}$
⇒ straight line

15. **Sol.** $z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + k = 0$ is equation of circle
centre = – α
= – 4 – 3i
radius = $\sqrt{\alpha\bar{\alpha} - k}$
= $\sqrt{25 - 5}$ = $2\sqrt{5}$

16. **Sol.** Given = $e^{\frac{\pi}{2}i + \frac{\pi}{2^2}i + ... + \infty}$ = $e^{\frac{\pi i/2}{1 - \frac{1}{2}}}$ = $e^{\pi i}$ = –1

17. **Sol.** $\left(\frac{e^{i\alpha}}{e^{-i\alpha}}\right)^n - \left[\frac{e^{in\alpha}}{e^{-in\alpha}}\right]$
⇒ $e^{i2n\alpha} - e^{i2n\alpha} = 0$

18. **Sol.** $x = e^{i\theta}$ $y = e^{i\varphi}$ $\Rightarrow$ $x^n + \frac{1}{x^n} = 2\cos n\theta$ $\Rightarrow$ $x^n - \frac{1}{x^n} = 2i \sin n\theta$

19. **Sol.** $h(\omega) = \omega f(\omega^3) + \omega^2(\omega^3) = 0$
and $h(\omega^2) = \omega^2 f(\omega^6) + \omega^4 g(\omega^6) = 0$
$\Rightarrow$ $\omega f(1) + \omega^2 g(1) = 0$ and $\omega^2 f(1) + \omega g(1) = 0$
$\Rightarrow$ $f(1) = 0$ and $g(1) = 0$ $\Rightarrow$ $h(1) = 0$

20. **Sol.** $x^n - 1 = (x - 1)(x - \omega)(x - \omega^2) \ldots\ldots. (x - \omega^{n-1})$
put x = 5
$\frac{5^n - 1}{4} = (5 - \omega)(5 - \omega^2) \ldots. (5 - \omega^{n-1})$

21. **Sol.** $(z - 1)(z - \alpha_1) \ldots\ldots\ldots(z - \alpha_4) = z^5 - 1$
Put $z = \omega$, $z = \omega^2$ and divide
$$\frac{(\omega-1)(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-1)(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)} = \frac{\omega^5-1}{\omega^{10}-1}$$
$$\frac{(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)}$$
$= \frac{(\omega^2-1)^2}{(\omega-1)^2} = (\omega + 1)^2 = \omega^4 = \omega$

22. **Sol.** Real $(1 + \alpha + \alpha^2 + \ldots\ldots\ldots + \alpha^{10}) = 0$
$\Rightarrow$ $1 + \text{Real}(\alpha + \alpha^2 + \ldots\ldots \alpha^5) + \text{Real}(\alpha^6 + \alpha^7 + \ldots\ldots + \alpha^{10}) = 0$
$\Rightarrow$ $2\,\text{Real}(\alpha + \alpha^2 + \ldots\ldots\ldots \alpha^5) = -1$ $\Rightarrow$ $(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = -\frac{1}{2}$

23. **Sol.** Sum of root $= a + a^2 + a^3 + a^4 + a^5 + a^6 = -1$
product of root $= 3a^7 + (a + a^2 + a^3 + a^4 + a^5 + a^6) = 3 - 1 = 2 \Rightarrow$ quadratic equation is $x^2 + x + 2 = 0$

PART - II : MISCELLANEOUS QUESTIONS

Section (A) : ASSERTION/REASONING

A-1. **Ans. (2)**

Sol. Statement-1 Arg (2 + 3i) is $\tan^{-1}\frac{3}{2}$
Arg (2 – 3i) is $\tan^{-1}\left(-\frac{3}{2}\right)$
Arg (2 + 3i) + Arg (2 – 3i) = 0
Statement-2 Let $z = -2 + 0i$, then $\bar{z} = -2 - 0i$
$\therefore$ Arg (z) + Arg $(\bar{z}) = 2\pi \neq 0$
$\therefore$ statement is wrong.

A-2. **Ans. (1)**
Sol. Statement -1 $(1 + z)^6 = -z^6$
take modulus
$|1 + z|^6 = |z|^6$
$\left|\frac{1+z}{z}\right| = 1$
which is straight line

Statement -2 $\quad z_2 = \dfrac{z_1 + z_3}{2}$

$\therefore z_2$ is mid point of line joining z_1 & z_3. Hence z_1, z_2, z_3 are collinear

A-3. **Ans. (1)**

Sol. For statement-1

$$\frac{1}{z_1 - z_2} + \frac{1}{(z_2 - z_3)} + \frac{1}{(z_3 - z_1)} = 0$$

$\Rightarrow \quad \dfrac{1}{z_1 - z_2} + \dfrac{1}{(z_2 - z_3)} + \dfrac{1}{(z_3 - z_1)} = 0$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$\Rightarrow \quad z_1\ z_2\ z_3$ are vertices of equilateral triangle

For statement-2

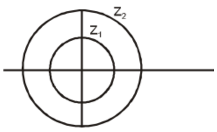
$|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$

$\therefore \quad z_0$ is circum-centre

Section (B) : MATCH THE COLUMN

B-1. **Ans.** (A) → (p), (B) → (q), (C) → (r), (D) → (s)

Sol.



(A) $|z_1 + z_2| \le |z_1| + |z_2| \le 2 + 1 \le 3$

(A) → P

(B) $|z_1 - z_2| \Rightarrow$ minimum distance b/w z_1 & $z_2 = 1$

$|z_1 - z_2| \Rightarrow z_1 = 1$

B → q

(C) $|2z_1 + 3z_2|$ minimum is = 6 – 2 = 4

(C) → r

(D) $|z_1 - 2z_2| \le |z_1| + |-2z_2|$

$1 + 4 \le 5$

(D) → s

Section (C) : ONE OR MORE THAN ONE OPTIONS CORRECT

Section (C) : ,d ;k ,d ls vf/kd lgh fodYi çdkj ¼ONE OR MORE THAN ONE OPTIONS CORRECT½

C-1. **Sol.** $\displaystyle\sum_{r=1}^{6} \cos\frac{(2r-1)\pi}{13} = \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \ldots\ldots + \cos\frac{11\pi}{13}$

$$= \frac{\sin\frac{6\pi}{13}}{\sin\left(\frac{2\pi}{2\times 13}\right)} \cos\left(\frac{\pi}{13} + 5\frac{\pi}{13}\right)$$

$$= \frac{\sin\frac{6\pi}{13}\cos\frac{6\pi}{13}}{\sin\left(\frac{\pi}{13}\right)} = \frac{2\sin\frac{6\pi}{13}\cos\frac{6\pi}{13}}{2\sin\frac{\pi}{13}}$$

Re(z) = $\dfrac{\sin\frac{12\pi}{13}}{2\sin\frac{\pi}{13}} = \dfrac{1}{2}$

($\because$ cos α + cos (α + β) + cos (α + 2β) + ... + cos(α + (n – 1)β))

= $\dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$ cos(α + (n – 1)β/2)

Im(z) = cos $\frac{\pi}{19}$ + cos $\frac{3\pi}{19}$ + ... + cos $\frac{17\pi}{19}$

= $\dfrac{\sin\left(\frac{2\pi}{19}\right)}{\sin\frac{\pi}{19}}$ $\cos\left(\frac{\pi}{19}+\frac{9\pi}{19}\right)$

= $\dfrac{\sin\left(\frac{2\pi}{19}\right)\cos\left(\frac{\pi}{19}+\frac{9\pi}{19}\right)}{\sin\frac{\pi}{19}}$

Im(z) = $\dfrac{-2\sin\frac{9\pi}{19}\cos\frac{9\pi}{19}}{2\sin\frac{\pi}{19}}$ = $\dfrac{-\sin\frac{18\pi}{19}}{2\sin\frac{\pi}{19}} = -\dfrac{1}{2}$

$\therefore$ z = $\frac{1}{2}-\frac{i}{2}$

arg(z) = – π/4

|z| = $\sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$

$\left|z+\frac{1}{2}-\frac{i}{2}\right| = |1-i| = \sqrt{2}$

C-2. **Sol.** amp (z_1z_2) = 0 ⇒ amp z_1 + amp z_2 = 0

$\therefore$ amp z_1 = –amp z_2 = amp $\bar{z}_2$

Since $|z_1| = |z_2|$, we get $|z_1| = |\bar{z}_2|$ ij. So, $z_1 = \bar{z}_2$.

Also z_1z_2 =

$\bar{z}_2 z_2 = |z_2|^2 = 1$ because $|z_2| = 1$.

C-3. **Sol.** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

⇒ $\dfrac{z_1}{z_2} = -\dfrac{\bar{z}_1}{\bar{z}_2}$

$\dfrac{z_1}{z_2} + \overline{\left(\dfrac{z_1}{z_2}\right)} = 0$ ⇒ $\dfrac{z_1}{z_2}$ is purely imaginary

so amp $\left(\dfrac{z_1}{z_2}\right)$ is may be $\frac{\pi}{2}$ or $-\frac{\pi}{2}$

C-4. **Sol.** $z = i^n$

Now principal argument of z can be 0, π, π/2, –π/2

C-5. Sol.