Application of Derivative

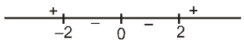
Exercise-3

✍ **Marked Questions may have for Revision Questions.**

* **Marked Questions may have more than one correct option.**

PART - I : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Sol. $f'(x) = \dfrac{(x+2)\ (x-2)}{2x^2}$



signs of f'(x)

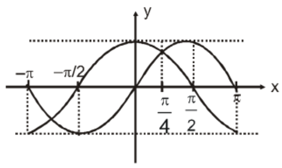
f' (x) changes sign as x crosses 2.

f(x) has minima at x = 2.

2. Sol. $\dfrac{f(3)-f(1)}{3-1} = f'(c)$, $1 < c < 3$

$\dfrac{\log_e^3 - 0}{2} = \dfrac{1}{c}$

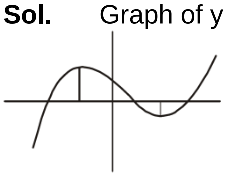
$c = \dfrac{2}{\log_e^3} = 2 \log_3 e$



Figure

3. Sol. $f'(x) = \dfrac{1}{1+(\sin x + \cos x)^2}$ $(\cos x - \sin x)$

4.✍ Sol. Graph of $y = x^3 - px + q$ cuts x-axis at three distinct pints



$\dfrac{dy}{dx} = 0$ $\Rightarrow$ $x = \pm\sqrt{\dfrac{p}{3}}$

Maxima at $x = -\sqrt{\dfrac{p}{3}}$, minima at $x = \sqrt{\dfrac{p}{3}}$

5. Sol. $P'(x) = 4x^3 + 3ax^2 + 2bx + c$ and $P'(0) = 0$

$\Rightarrow$ $c = 0$.

$P'(x) = x(4x^2 + 3ax + 2b)$

$D = 9a^2 - 32b < 0$ $\Rightarrow$ $b > \dfrac{9a^2}{32} > 0$ ($\because$ P' (x) = 0 has only one root x = 0)

$P(-1) < P(1)$

$\Rightarrow$ $a > 0$

P' (x) has only one change of sign. $\Rightarrow$ x = 0 is a point of minima.

$P(-1) = 1 - a + b + d$, $P(0) = d$

$P(1) = 1 + a + b + d$

$\Rightarrow$ $P(-1) < P(1)$, $P(0) < P(1)$, $P(-1) > P(0)$

$\Rightarrow$ P(–1) is not minimum but P(1) is maximum.

6. **Sol.** Condition for shortest distance is slope of tangent to $x = y^2$ must be same as slope of line $y = x + 1$.

$\Rightarrow \quad \frac{1}{2y} = 1 \quad \Rightarrow \quad y = \frac{1}{2}, \quad x = \frac{1}{4}$

$\left(\frac{1}{4}, \frac{1}{2}\right)$, $x - y + 1 = 0$.

Shortest distance = $\frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{2}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

7. **Ans.** **(3)**

Sol. $\lim_{x \to -1^+} f(x) = 1$

$f(-1) = k + 2$

$\lim_{x \to (-1)^-} f(x) = k + 2$

$\because$ f has a local minimum at $x = -1$

$x = -1$ ij f LFkkuh; fufEu"V j[krk gSA

$\therefore$ $f(-1_+) \geq f(-1) \leq f(-1_-)$

$1 \geq k + 2 \leq k + 2$

$\Rightarrow$ $k \leq -1$

possible value of k is -1 k dk laHkkfor eku -1 gSA

Hence correct option is (3)

vr% lgh fodYi (3) gSA

8. **Ans.** **(4)**

Sol. $e^x + 2e^{-x} \geq 2\sqrt{2}$ (AM $\geq$ GM)

$\frac{1}{e^x + 2e^{-x}} \leq \frac{1}{2\sqrt{2}}$

$\frac{1}{2\sqrt{2}} \geq f(x) > 0$ so statement- 2 is correct

As f(x) is continuous and $\frac{1}{3}$ belongs to range $\left(0, \frac{1}{2\sqrt{2}}\right]$ of f(x),

$\Rightarrow$ $f(c) = \frac{1}{3}$ for some C.

Hence correction option is (4).

9. **Ans.** **(3)**

Sol. $y = x + \frac{4}{x^2}$

$y' = 1 - \frac{8}{x^3} = 0 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2$

$y = 2 + \frac{4}{2^2} = 3$

(2, 3) is point of contact

Thus y = 3 is tangent

Hence correct option is (3)

10. **Sol.** **(2)**

$$f(x) = \begin{cases} \dfrac{\tan x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

In right neighbourhood of '0'
tan x > x
$\frac{\tan x}{x} > 1$
In left neighbourhood of '0'
tan x < x
$\frac{\tan x}{x} > 1$
at x = 0 ĳ , f(x) = 1
⇒ x = 0 is point of minima
so statement 1 is true.
statement 2 obvious

11. Sol. (2)

$y - x = 1$
$y^2 = x$
$2y \frac{dy}{dx} = 1$
$\frac{dy}{dx} = \frac{1}{2y} = 1$
$y = \frac{1}{2}$
$x = \frac{1}{4}$
tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$
$\frac{1}{2}$ $y = \frac{1}{2}\left(x + \frac{1}{4}\right)$
$y = x + \frac{1}{4}$
$y - x = \frac{1}{4}$
distance $= \left|\frac{1-\frac{1}{4}}{\sqrt{2}}\right| = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$ **Ans.**

12. Sol. Ans. (3)

$V = \frac{4}{3}\pi r^3$ $\qquad 4500\,\pi = \frac{4\pi r^3}{3}$
$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$ $\qquad 45 \times 25 \times 3 = r^3$
r = 15 m

after 49 min 49 min
= (4500 – 49.72)π
=972 π m^3

$972\pi = \frac{4}{3}\pi r^3$
$r^3 = 3\times243$
$= 3\times 3^5$
r = 9

$72\pi = 4\pi \times 9 \times 9 \left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \left(\frac{2}{9}\right)$

$r^3 = 3\times243$
$= 3\times 3^5$
r = 9

$72\pi = 4\pi \times 9 \times 9 \left(\frac{dr}{dt}\right) \Rightarrow \frac{dr}{dt} = \left(\frac{2}{9}\right)$

13. **Sol.** **Ans. (2)**

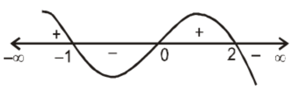
$f'(x) = \frac{1}{x} + 2bx + a$
at x = – 1 $-1 - 2b + a = 0$
$a - 2b = 1$ ...(i)

at x = 2 $\frac{1}{2} + 4b + a = 0$

$a + 4b = -\frac{1}{2}$ ...(ii)

On solving (i) and (ii) $a = \frac{1}{2}$, $b = -\frac{1}{4}$

$f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$



So maxima at x = – 1, 2

14. **Sol.** **(4)**
$f(x) = 2x^3 + 3x + k$
$f'(x) = 6x^2 + 3 > 0 \quad \forall x \in R$
⇒ f(x) is strictly increasing function
⇒ f(x) = 0 has only one real root, so two roots are not possible

15. **Sol.** **Ans. (2)**
Consider f(x) – 2g(x) = h(x)
Then, h(x) is continuous and differentiable in [0, 1]
Also h(0) = 2 & h(1) = 2
Hence h(x) satisfies conditions of Rolles Theorem in (0, 1)
Thus, There exist a 'c' such that h'(c) = 0 where c ∈ (0, 1)
⇒ f'(c) = 2g'(c)

16. **Sol.** **Ans. (1)**
$f(x) = \alpha \ell n|x| + \beta x^2 + x$

(1). $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$.

$= \frac{2\beta x^2 + x + \ell}{x}$

Since x = –1, 2 are extreme points ⇒ f '(x) = 0 at these points.

Hence $2\beta - 1 + \alpha = 0$

$8\beta + 2 + \alpha = 0$

$-6\beta - 3 = 0 \Rightarrow \beta = -\frac{1}{2}$ & $\alpha = 2$.

17. Ans. (2)

Sol. $4x + 2\pi r = 2$ ….(i)

$x^2 + \pi r^2$ = minimum ⇒ So $f(r) = \left(\frac{1-\pi r}{2}\right)^2 + \pi r^2$

$\frac{df}{dr} = \pi^2 \frac{r}{2} - \frac{\pi}{2} + 2\pi r = 0$ $\Rightarrow r = \frac{1}{\pi + 4}$

using equation (i) $x = \frac{(1-\pi r)}{2}$

$\Rightarrow x = 2r$

18. Ans. (1)

Sol. at $x = \frac{\pi}{6}$ ⇒ $y = \frac{\pi}{3}$

$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right)$ $x \in \left(0, \frac{\pi}{2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$f(x) = \frac{\pi}{4} + \frac{x}{2}$ $f'(x) = \frac{1}{2}$

slope of normal = –2

equation of normal $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

$y = -2x + \frac{2\pi}{3}$

19. Ans. (3) **AOD**

Sol. $2r + \ell = 20 \Rightarrow 2r + r\theta = 20 \Rightarrow \theta = \frac{20 - 2r}{r}$

$A = \frac{\pi r^2 \theta}{360} = \frac{r^2}{2} \cdot \frac{20 - 2r}{r} = r(10 - r)$

$A = 10r - r^2$

$\frac{dA}{dr} = 10 - 2r = 0 \Rightarrow r = 5$

$\therefore \theta = \frac{10}{5} = 2$

$\therefore$ Maximum area $= \frac{1}{2} \times 25 \times 2 = 25$ sq. m.

20. Ans. (2)

Sol. $y(x-2)(x-3) = x + 6$

Intersection with y-axis ;

Put $x = 0 \Rightarrow y = 1$

$\Rightarrow$ Point of Intersection is (0, 1)

Now , $y = \frac{x+6}{x^2-5x+6}$

$y' = \frac{(x^2-5x+6)-(x+6)(2x-5)}{(x^2-5x+6)^2}$

$y' = \frac{6-(-30)}{36} = 1$ at (0,1)

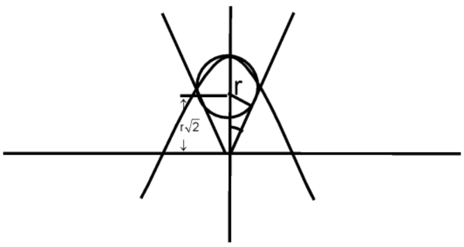
$\therefore$ Equation of normal is given by

$(y-1) = -1(x-0)$

$x + y - 1 = 0$

21. Ans. (3)

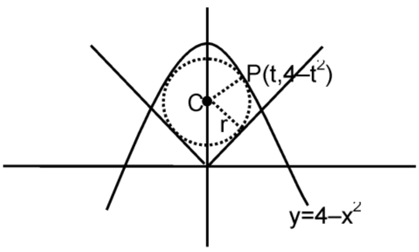
Sol.



let radius of circle be r, its center lies on y-axis as y-axis bisects the 2 rays of y = |x|

Now $4 - r\sqrt{2} = r \Rightarrow r = \frac{4}{\sqrt{2}+1} = 4(\sqrt{2}-1)$

NOTE : The correct solution should be,



due to symmetry center of the circle must be on y-axis

let center be (0, k)

Length of perpendicular from (0, k) to y = x,

i.e. $r = \left|\frac{k}{\sqrt{2}}\right|$

$\therefore$ Equation of circle : $x^2 + (y - k)^2 = \frac{k^2}{2}$

solving circle and parabola,

$4 - y + y^2 - 2ky + \frac{k^2}{2} = 0$

$y^2 - (2k + 1)\, y + \left(\frac{k^2}{2}+4\right) = 0$

Because circle touches the parabola

$\therefore D = 0$

$(2k + 1)^2 = 4\left(\frac{k^2}{2}+4\right)$

$4k^2 + 4k + 1 = 2k^2 + 16$

On solving we get $k = \frac{-4+\sqrt{136}}{4}$

Therefore radius = $k/\sqrt{2} \approx 1.3546$

However among the given choices the following method will yield one of the choice.

22. **Ans. (2)**

Sol. $y^2 = 6x$ and $9x^2 + by^2 = 16$

$2y\frac{dy}{dx} = 6 \Rightarrow \frac{dy}{dx} = \frac{3}{y}$

$18x + 2by\frac{dy}{dx} = 0$

$9x + by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-9x}{by}$

$\frac{3}{y}\times\frac{-9x}{by} = -1$

(b) $6x = 27x$

$b = \frac{27}{6} \Rightarrow b = \frac{9}{2}$

23. **Sol. (2)**

$f(x)\ x^2 + \frac{1}{x^2}$, $g(x) = x = \frac{1}{x}$

$h(x) = \frac{f(x)}{g(x)} = \frac{x^2+\frac{1}{x^2}}{x-\frac{1}{x}} = \frac{\left(x-\frac{1}{x}\right)^2+2}{\left(x-\frac{1}{x}\right)}$

$x - \frac{1}{x} = t$

$h(t) = \frac{t^2+2}{t} = t+\frac{2}{t}$ $\quad |t| \geq 2$

AM $\geq$ GM . $\frac{t+\frac{2}{t}}{2} \geq \sqrt{t.\frac{2}{t}}$

$t + \frac{2}{t} \geq 2\sqrt{2}$

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Sol. Given $y^3 + 3x^2 = 12y$ ....(1)

Differentiating w.r.t. x

$3y^2 \frac{dy}{dx} + 6x = 12\frac{dy}{dx}$

$\frac{dy}{dx} \to \infty$ ($\because$ tangent is vertical)

$12 - 3y^2 = 0$

$y = \pm 2$

Put y = 2 in equation (1) $8 + 3x^2 = 24 \Rightarrow x = \pm\frac{4}{\sqrt{3}}$

Put y = –2 then $3x^2 = -16$ no real solution

hence point $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

2. Sol. Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

The longest interval in which sinx is increasing, is of length $\frac{\pi}{3}$

3. Sol. In case of function given in (A), f is continuous on [0, 1] but not differentiable at $x = 1/2 \in (0, 1)$.
Note that Lf' (1/2) = – 1 and Rf' (1/2) = 0
Thus, the Lagrange's mean value theorem is not applicable. The function in (B) is continuous on [0, 1] and differentiable on]0, 1[and hence the Lagrange's mean value theorem is applicable.
The function in (C) is $f(x) = x|x| = x \cdot x = x^2$ and in (D) is $f(x) = |x| = x$ on [0, 1]. As both are polynomial function, the Lagrange's mean value theorem is applicable.

4. Sol. $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$. Put x = 0 and we get $\frac{0}{0}$ form. Also because 'f' is strictly increasing and differentiable. Apply L-Hospital rule, weget

$\lim_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)} = -1$. Since 'f' is strictly increasing f¢(x) ¹ 0 in an internal

5. Sol. $f(x) = \begin{cases} x^\alpha \ell nx & , \ 0 < x \le 1 \\ 0 & , \ x = 0 \end{cases}$

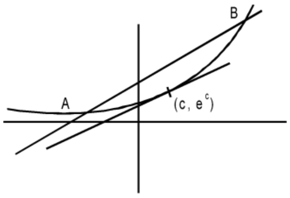
For Rolles theorem to be applicable in [0, 1], f(x) should be continuous in [0, 1], differentiable in (0, 1) and f(0) = f(1).

Last two conditions hold and $\lim_{x\to 0^+} x^a \ell n\, x = 0$

Put $x = e^{-t}$, then as $x \to 0$, $t \to \infty$

$\lim_{t\to\infty}\frac{-t}{e^{t\alpha}} = 0$ which is true for all a > 0

6. Sol. $f'(x) = 3x^2 + 2bx + c$ whose discriminant is $4(b^2 - 3c)$ which is negative as $0 < b^2 < c$.
Thus f¢(x) is always positive and f(x) is strictly increasing.



7. **Sol.** Slope of the line joining the points $(c - 1, e^{c-1})$

and $(c + 1, e^{c+1})$ is equal to $\dfrac{e^{c+1} - e^{c-1}}{2} > e^c$

tangent to the curve $y = e^x$ will intersect line to the left of the line $x = c$.

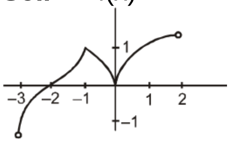
8. **Sol.** $g(u) = 2\tan^{-1}(e^u) - \dfrac{\pi}{2}$

$g(u) = \tan^{-1}(e^u) - \cot^{-1}(e^u)$

$g(u) = \tan^{-1}(e^u) - \tan^{-1}(e^{-u})$ odd function.

$g'(u) = \dfrac{e^u}{1+e^{2u}} + \dfrac{e^{-u}}{1+e^{-2u}} > 0$ Strictly increasing function

9. **Sol.** $f(x) = \begin{cases} (2+x)^2 & , \ -3 < x \le -1 \\ x^{2/3} & , \ -1 < x < 2 \end{cases}$



one point of local maxima at $x = -1$

one point of local minima at $x = 0$

$\therefore$ (C) is the correct option

10. **Sol.** **(C)**

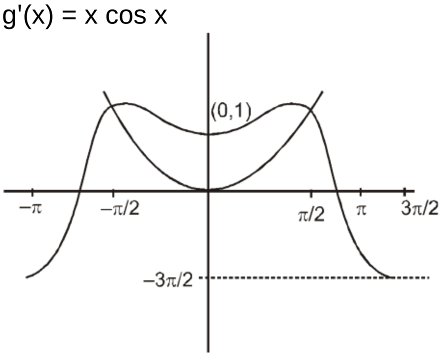
$x^2 = x\sin x + \cos x$

$f(x) = x^2$

$g(x) = x\sin x + \cos x$

$g'(x) = \sin x + x\cos x - \sin x$

$g'(x) = x\cos x$



Only two solution.

11. **Ans.** **(B)**

Sol. $f''(x) > 0$ for all $x \in R$, $f(1/2) = 1/2, f(1) = 1$

$\Rightarrow f'(x)$ increases

Let $g(x) = f(x) - x$, $x \in [1/2,1]$

Then $g'(x) = 0$ has atleast one real root in $(1/2,1)$

$f'(x) = 1$ has atleast one real root in $(1/2,1)$

Hence $f'(x)$ increases $\Rightarrow f'(1) > 1$

Additional Problems For Self Practice (APSP)

PART - I : PRACTICE TEST PAPER

1. **Sol.** Let at $P(\alpha,\beta)$ tangent passes through O

$\therefore$ $\left(\frac{dy}{dx}\right)_{at\,\alpha} = 12\alpha^2 - 10\alpha^4$

$\therefore$ tangent is $y - \beta = (12\alpha_2 - 10\alpha_4)(x - \alpha)$

passes through (0,0)

$\therefore$ $-\beta = -\alpha(12\alpha_2 - 10\alpha_4)$

$\Rightarrow$ $4\alpha_3 - 2\alpha_5 = 12\alpha_3 - 10\alpha_5$

$\Rightarrow$ $8\alpha_5 - 8\alpha_3 = 0 \Rightarrow \alpha = 0, 1, -1$: 3 tangents are possible

2. **Sol.** $y = e_{px} + px \therefore \frac{dy}{dx} = pe^{px} + p$

Given $\left|\frac{y}{dy/dx}\right| = \left|y\frac{dy}{dx}\right| \Rightarrow \left(\frac{dy}{dx}\right)^2 = 1 \Rightarrow \frac{dy}{dx} = \pm 1$

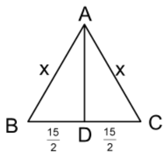
$\Rightarrow$ $\pm 1 = pe_{px} + p \Rightarrow \pm 1 = 2p$ at (0,1)

$\Rightarrow$ $p = \pm \frac{1}{2}$

3. **Sol.** $\frac{x^2}{a^2} + \frac{y^2}{4} = 1 \Rightarrow \frac{dy}{dx} = \frac{-4x}{a^2y}$

and $y_3 = 16x \Rightarrow \frac{dy}{dx} = \frac{16}{3y^2}$

$\therefore$ $\frac{-4x}{a^2y}\cdot\frac{16}{3y^2} = -1 \Rightarrow \frac{64x}{3} = a^2y^3 = a^2 16x \Rightarrow a_2 = \frac{4}{3}$

4. **Sol.** 

AD = $\sqrt{x^2 - \frac{225}{4}}$

Area A = BC.AD = $\frac{1}{2}.15\sqrt{x^2 - \frac{225}{4}}$

$\therefore$ $\frac{dA}{dt} = \frac{15}{2}\cdot\frac{2}{2\sqrt{4x^2-225}}.2x\frac{dx}{dt}$

$= .\frac{15.15}{\backslash\sqrt{4x^2-225}} = \frac{15.15}{15\sqrt{3}}$ (–2)

$= -10\sqrt{3}$ cm^2/s

5. **Sol.** $x + 4y = 14$, $y_2 = \alpha x_3 - \beta$ .....(1)

$\therefore \quad 9 = 8\alpha - \beta \quad .....(2)$

differentiate (1) $2y\dfrac{dy}{dx} = 3\alpha x^2$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{at(2,3)} = \dfrac{3\alpha.4}{2.3} = 2\alpha$

$\therefore \quad$ slope of normal $= -\dfrac{1}{2\alpha} = -\dfrac{1}{4}$

$\Rightarrow \alpha = 2 \ \therefore 9 = 16 - \beta$

$\Rightarrow \beta = 7$

$\therefore \quad \alpha^2 + \beta^2 = 4 + 49 = 53$

6. **Sol.** Let h(x) = f(x) – 2g (x) is differentiable in (0,1) and continuous in [0,1]

h(x) = f(x) – 2g (x), (0,1)

h(0) = f(0) – 2g(0) = 0

h(1) = f(1) – 2g(1) = 6–2g(1)

$\therefore$ $c\in(0,1)$ s. t. h'(x) = 0

$\therefore$ h(1) = h(0) $\Rightarrow$ 6–2g(1) = 0 $\Rightarrow$ g(1) = 3

7. **Sol.** For increasing function, $f'(x) \geq 0 \ \forall \ x\in[1,2]$

$\Rightarrow \quad 3x^2 + 2x + k \geq 0 \ \forall \ x\in[1,2]$

$\Rightarrow \quad k \geq -3x^2 - 2x \ \forall \ x\in[1,2]$

Now if ;fn $x\in[1,2]$ then $-3x^2-2x\in[-16,-5]$

$\therefore$ min k = – 5

8. **Sol.** Here $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

$\Rightarrow \quad g(\alpha^2 - 2\alpha) > g(3\alpha - 4) \quad \therefore$ f is increasing

$\Rightarrow \quad \alpha^2 - 2\alpha < 3\alpha - 4 \quad \therefore$ g is decreasing

$\Rightarrow \quad \alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow \quad \alpha\in(1, 4)$

9. **Sol.** $g(x) = f(x)\sqrt{1-2(f(x))^2}$

$\Rightarrow g'(x) = f'(x)\sqrt{1-2(f(x))^2} + \dfrac{f(x).(-4f(x).f'(x)}{2\sqrt{1-2(f(x))^2}}$

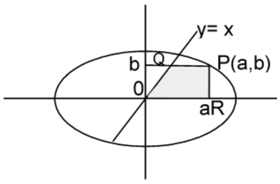
$= \dfrac{1-4(f(x))^2}{\sqrt{1-2(f(x))^2}}f'(x)$

f(x) & g(x) are strictly increasing $\therefore$ f'(x) > 0, g'(x) > 0

$\therefore \quad 1 - 4(f(x))^2 > 0$

$\Rightarrow \quad 4(f(x))^2 - 1 < 0$

$\Rightarrow \quad f(x)\in\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$ & $\sqrt{1-2(f(x))^2}$ is also defined



10. **Sol.**

P(a,b) = p ($\sqrt{3}$ cosθ, 2sinθ)

Area trap = $\frac{1}{2}$ (PQ + OR) PR

$= \frac{1}{2}\left(\left(\sqrt{3}\cos\theta - 2\sin\theta\right) + \sqrt{3}\cos\theta\right) 2\sin\theta$

$= 2\sqrt{3}\sin\theta\cos\theta - 2\sin^2\theta$

A $= \sqrt{3}\sin\theta\cos\theta - (1 - \cos2\theta)$

$= \sqrt{3}\sin2\theta + \cos2\theta - 1$

$= 2\left(\frac{\sqrt{3}}{2}\sin2\theta + \frac{1}{2}\cos2\theta\right) - 1$

$= 2\cos\left(2\theta - \frac{\pi}{3}\right) - 1$

A will be maximum when $2\theta - \frac{\pi}{3} = 0 \Rightarrow \theta = \frac{\pi}{6}$

$\therefore P\left(\frac{3}{2}, 1\right) \therefore a = \frac{3}{2}$, b = 1 $\therefore$ 2a + b = 4

11. **Sol.** $g(x) = f(\tan^2 x - 2\tan x + 4)$

$= f(t^2 - 2t + 4),\ t \in (0, \infty)$

$g'(x) = f'(t^2 - 2t + 4),\ (2t - 2) > 0\ \forall t \in (0, \infty)$ as $f''(x) > 0$ & $f'(3) = 0$

$\Rightarrow t > 1$

$\Rightarrow \tan x > 1 \Rightarrow x > \frac{\pi}{4}$

$\Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

12. **Sol.** $4x^4 + 9y^4 = c^6$

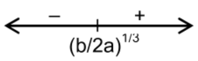
$\therefore$ Let $4x^4 = c^6\cos^2\theta$ & $9y^4 = c^6\sin^2\theta$

$\Rightarrow$ $x = c^{\frac{3}{2}}\sqrt{\frac{\cos\theta}{2}},\ y = c^{\frac{3}{2}}\sqrt{\frac{\sin\theta}{3}}$

$\therefore$ $xy = c^3\sqrt{\frac{\sin\theta\cos\theta}{6}} = c^3\sqrt{\frac{\sin2\theta}{12}}$ $\therefore$ maximum $xy = \frac{c^3}{2\sqrt{3}}$

13. **Sol.** Let $f(x) = ax^2 + \frac{b}{x} \therefore f'(x) = 2ax - \frac{b}{x^2} = 0$

$\Rightarrow$ $x^3 = \frac{b}{2a} \Rightarrow x = \left(\frac{b}{2a}\right)^{\frac{1}{3}}$



$\Rightarrow$ f(x) has a minimum at $x = \left(\frac{b}{2a}\right)^{\frac{1}{3}}$ f(x), $x = \left(\frac{b}{2a}\right)^{\frac{1}{3}}$

$\therefore$ $f\left(\left(\frac{b}{2a}\right)^{\frac{1}{3}}\right) \geq c$ as given

$\Rightarrow a\left(\frac{b}{2a}\right)^{\frac{2}{3}} + \frac{b}{(b/2a)^{1/3}} \geq c \Rightarrow 27ab^2 \geq 4c^3$

14. **Sol.** Here $y = ke^{kx}$ $\therefore \frac{dy}{dx} = k^2e^{kx}$

Curve intersect y-axis at (0,k) $\therefore \left(\frac{dy}{dx}\right)_{at(0,k)} = k^2$

$\therefore \tan\theta = k^2$ $\therefore$ angle made by y-axis is $\frac{\pi}{2} - \theta$

$\Rightarrow \cot\theta = k^2$ $\Rightarrow \theta = \cot^{-1}(k^2)$

15. **Sol.** Let $y = \tan A \tan B = \tan A \tan\left(\frac{\pi}{3} - A\right)$

$= \tan A.\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A} = \frac{\sqrt{3}x - x^2}{1 + \sqrt{3}x}$ (Let x = tanA)

$\frac{dy}{dx} = \frac{(1+\sqrt{3}x)\ \ (\sqrt{3} - 2x)\ \ -(\sqrt{3}x - x^2)\sqrt{3}}{(1+\sqrt{3}x)^2}$

$= \frac{-\left(\sqrt{3}x - 1\right)\left(x + \sqrt{3}\right)}{\left(1 + \sqrt{3}x\right)^2} = 0$ at $x = \frac{1}{\sqrt{3}} = \tan A$

$\Rightarrow$ y has maximum at $x = \frac{1}{\sqrt{3}}$ or $A = \frac{\pi}{6}$

$x = \frac{1}{\sqrt{3}}$ $A = \frac{\pi}{6}$

$\therefore y_{maximum} = \tan\frac{\pi}{6}\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}} = \frac{1}{3}$

16. **Sol.** tangent at $(6\cos^3\theta, 6\sin^3\theta)$ is

$x\sin\theta + y\cos\theta = 6\sin\theta\cos\theta$

& normal at $(6\cos^3\theta, 6\sin^3\theta)$ is

$x\cos\theta - y\sin\theta - 6\cos2\theta = 0$

$\therefore$ $\alpha = 6\sin\theta\cos\theta \Rightarrow 2\alpha = 6\sin2\theta$

$\beta = 6\cos2\theta$ $\therefore$ $4\alpha^2 + \beta^2 = 36(\sin^2 2\theta + \cos^2 2\theta) = 36$

17. **Sol.** Let P (α,β) on $y = x^4 + 3x^2 + 2x$ $\therefore \beta = \alpha^4 + 3\alpha^2 + 2\alpha$

$\frac{dy}{dx} = 4x^3 + 6x + 2 \Rightarrow 4\alpha^3 + 6\alpha + 2 = 2$

$\Rightarrow 4\alpha^3 + 6\alpha = 0$

$\Rightarrow 2\alpha(2\alpha^2 + 3) = 0, \alpha = 0$

$\therefore \beta = 0$ $\therefore P(0,0)$

$\therefore$min distance $= \frac{0 + 0 + 1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}$

18. **Sol.** $x + y = 8$ Let $P = x^3 + y^3$

$\Rightarrow$ $P = x^3 + (8-x)^3$

$\frac{dp}{dx} = 3x^2 + 3(8-x)^2(-1)$
$= 3(x^2 - (8-x)^2)$
$= 3(x + x - 8)(x - x + 8)$
$= 24(2x-8) = 0$
$= x = 4$

P has a minima at x = 4 x = 4

$\therefore x = 4, y = 4 \therefore xy = 16$

19. Sol. $f(1) = m+b$, $\lim_{x\to1^-} f(x) = 2 + a$

$\therefore m+b = 2 + a$ ...(1)

Now $f'(x) = \begin{cases} -2x+3, 0<x<1 \\ m, 1<x<2 \end{cases}$

$f'(1^-) = f'(1^+) \Rightarrow 1 = m$

$\therefore 1 + b = 2 + a$

$\lim_{x\to0^+} f(x) = f(0)$ $\Rightarrow a = 3$

$\therefore b = 4$

$\therefore (a + b)\,m = (3 + 4).1 = 7$

20. Sol. Let $y = f(x) = ax^2 + bx + c$

it passes through $(-1,0) \therefore a - b + c = 0$ ....(1)

$y = x$ touches at $x = 1 \therefore f'(1) = 1$

$\Rightarrow 2a + b = 1$ ....(2)

and (1,1) lies on curve $\therefore a + b + c = 1$ ....(3)

solving (1), (2), (3) we get $y = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4} = f(x)$

$\therefore f(2) = 1 + 1 + \frac{1}{4} = \frac{9}{4}$

21. Sol. $\because x^3 + y^3 = c^3$

$\because \left(\frac{dy}{dx}\right)$ at $(\alpha,\beta) = \frac{-\alpha^2}{\beta^2}$ & $\alpha^3 + \beta^3 = c^3$, $\alpha_1^3 + \beta_1^3 = c^3$

slope of tangent = slope of line joining (α,β) & (α_1,β_1)

$\Rightarrow -\frac{\alpha^2}{\beta^2} = \frac{\beta_1 - \beta}{\alpha_1 - \alpha}$ & $\alpha^3 + \beta^3 = \alpha_1^3 + \beta_1^3$

$\Rightarrow \beta_1^3 - \beta^3 = \alpha^3 - \alpha_1^3$

$\Rightarrow (\beta_1 - \beta)(\beta^2 + \beta\beta_1 + \beta_1^2) = -(\alpha_1 - \alpha)(\alpha^2 + \alpha\alpha_1 + \alpha_1^2)$

$\therefore + \frac{\alpha^2}{\beta^2} = + \frac{\alpha^2 + \alpha\alpha_1 + \alpha_1^2}{\beta^2 + \beta\beta_1 + \beta_1^2}$

$\Rightarrow \frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta} = -1$

22. Sol. $f'(x) = \frac{\ell nx.1 - x.\frac{1}{x}}{(\ell nx)^2} = \frac{\ell nx - 1}{(\ell nx)^2}$

for decreasing $f'(x) < 0$ $\therefore \ell nx - 1 < 0$

$\Rightarrow \ell nx < 1$

$\Rightarrow x < e$

$\Rightarrow x \in (0,e)$

23. **Sol.** Here $f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9) = 0$

$\Rightarrow x = 7 - a \pm \sqrt{58-14a}$, $x_1 = 7-a+\sqrt{58-14a}$, $x_2 = 7-a-\sqrt{58-14a}$

$58 - 14a > 0 \Rightarrow 14a < 58 \Rightarrow a < \frac{29}{7}$

$f''(x) = 6x + 6(a-7)$, $f''(x_2) < 0 \Rightarrow$ at $x = a_2$, $f(x)$ has a maxima

$\therefore x_2 > 0 \Rightarrow 7 - a - \sqrt{58-14a} > 0$, $\sqrt{58-14a} < 7 - a$

$\Rightarrow \quad 58 - 14a < (7-a)^2 \Rightarrow a^2 - 9 > 0$

$\Rightarrow \quad a \in (-\infty,-3) \cup (3,\infty)$ & $a < \frac{29}{7}$

$\therefore a \in (-\infty,-3) \cup \left(3,\frac{29}{7}\right)$

24. **Sol.** $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r \Rightarrow x = r\cos\theta,\ y = r\sin\theta$

$\therefore 5r^2\cos^2\theta + 5r^2\sin^2\theta - 6r^2\sin\theta\cos\theta = 4$

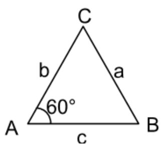
$5r^2 - 6r^2\sin\theta\cos\theta = 4$

$\Rightarrow r^2 = \frac{4}{5-3\sin 2\theta}$

for minimum distance Let $\sin 2\theta = -1$

$\Rightarrow \quad r = \frac{1}{\sqrt{2}}$

25. **Sol.**



$a + b + c = \ell \qquad \ldots(1)$

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{2}$

$\Rightarrow (b+c)^2 - a^2 = 3bc$

$\Rightarrow \ell^2 - 2a\ell = 3bc \qquad \ldots(2)$

Now $bc = \frac{1}{3}\ell(\ell-2a)$ and $b + c = \ell - a$ & $(b-c)^2 = (b+c)^2 - 4bc \geq 0$

$\Rightarrow \frac{1}{3}(-\ell^2 + 2a\ell + 3a^2) \geq 0$

$\Rightarrow (\ell - 3a)\ (\ell + a) \leq 0$

$\Rightarrow \ell \leq 3a \qquad \Rightarrow a \geq \frac{\ell}{3}$

$S = \text{Area} = \frac{1}{2}\,bc\sin 60 = \frac{\sqrt{3}}{4}\,bc = \frac{1}{4\sqrt{3}}\left(\ell^2 - 2a\ell\right)$

$\frac{ds}{da} = \frac{1}{4\sqrt{3}}(-2\ell) < 0$ S is decreasing

S will be maximum for a = $\frac{\ell}{3}$

maximum Area = $\frac{\sqrt{3}}{4}\frac{\ell^2}{9} = \frac{\ell^2}{12\sqrt{3}}$

26. **Sol.** We have $\frac{-\pi}{2} < \ell n\theta < \ell n\frac{\pi}{2}$

$\Rightarrow \cos\frac{-\pi}{2} < \cos(\ell n\theta) < \cos(\ell n\frac{\pi}{2})$

$\Rightarrow \cos(\ell n\theta) > 0$ ......(1)

Also $-1 \le \cos\theta \le 1 \forall\theta$

$\Rightarrow -\infty < \ell n(\cos\theta) \le 0 \forall 0 < \cos\theta \le 1$

$\Rightarrow \ell n(\cos\theta) < 0$ ....(2)

$\therefore$ by (1) & (2) Is $\ell n(\cos\theta) < \cos(\ell n\theta)$

27. **Sol.** h(x) = f(g(x) h'(x) = f' (g(x)) g'(x) < 0

$\therefore$ h(0) = 0 $\therefore$h(x) ≤ 0 ⇒ h(x) = 0 $\forall x \in (0,\infty)$

⇒ h(x) – h(1) = 0

28. **Sol.** $\frac{dy}{dx} = 2(x-2) = \frac{4-0}{4-2} = 2$

x = 3 $\therefore$ y = 1

$\therefore$ (a+b) = 3+1 = 4

29. **Sol.** Let A (4cosθ, 3sinθ)

area S of ΔABC = $\frac{1}{2}$ AB.CD

= $\frac{1}{2}$ 6sinθ.(4–4cosθ)

S= 12 sinθ (1–cosθ)

$\therefore \frac{ds}{d\theta}$ = 12 cosθ (1–cosθ) + 12 sinθ(0+sinθ) = 0

⇒ cosθ – $\cos^2\theta$ + $\sin^2\theta$ = 0 ⇒ cos2θ = cosθ

2θ = 2nπ± θ, $n \in I$

$\therefore$ θ = $\frac{2\pi}{3}$ as $\theta \in (0,\pi)$

$\therefore$ maximum area = $\frac{3\sqrt{3}}{4} \times 12 = 9\sqrt{3}$

30. **Sol.** Let x = 5, h = 0.001

f(x+h) = f(x) + hf'(x), f'(x) = $3x^2$ – 14x

f(5.001) = f(5) + 0.001.f'(5)

= (5^3–7.25+15) + (3.25–14.5) × (0.001)

= –34.995

Practice Test (JEE-Main Pattern)

OBJECTIVE RESPONSE SHEET (ORS)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.										
Que.	11	12	13	14	15	16	17	18	19	20
Ans.										
Que.	21	22	23	24	25	26	27	28	29	30
Ans.										

PART - II : PRACTICE QUESTIONS

1. **Sol.** $y = e^{\{x\}} = e^{x-a}$ in $x \in [a, a + 1)$

$\frac{dy}{dx} = e^{x-a} = e^{\{x\}}$

equation of tangent

$(Y - y) = \frac{dy}{dx}(X - x)$

passing through (–1/2, 0)

$(0 - y) = e^{\{x\}}(-1/2 - x)$

$\Rightarrow -1 = -\frac{1}{2} - x \Rightarrow x = \frac{1}{2}$

$\therefore$ point $\left(\frac{1}{2}, e^{1/2}\right)$

Number of tangent = 1

2. **Sol.** $f'(x) = e^{x(1-x)} + xe^{x(1-x)}(1 - 2x)$

$= e^{x(1-x)}(1 + x(1 - 2x))$

$= -e^{x(1-x)}(x - 1)(2x + 1)$

$f'(x) \geq 0$ when $x \in \left[-\frac{1}{2}, 1\right]$

$\Rightarrow$ f(x) is increasing in $\left[-\frac{1}{2}, 1\right]$

3. **Sol.** Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

The longest interval in which sinx is increasing, is of length $\frac{\pi}{3}$

4. **Sol.** Since coefficient of x^2 is (+ve)

$\Rightarrow m(b) = -\frac{D}{4a}$

$\Rightarrow m(b) = -\frac{(4b^2 - 4(1+b^2))}{4(1+b^2)}$

$\Rightarrow m(b) = \frac{1}{1+b^2}$

$\Rightarrow b^2 \geq 0$

$\Rightarrow 1 + b^2 \geq 1$

$\Rightarrow \quad 0 < \dfrac{1}{1+b^2} \le 1$

$\Rightarrow \quad m(b) \in (0, 1]$

5. Sol. Let $g(x) = f(x) - x^2$

$\Rightarrow$ g(x) has atleast 3 real roots which are x = 1,2,3

By langrange mean value theorem (LMVT)

$\Rightarrow$ g'(x) has atleast 2 real roots in $x \in (1, 3)$

$\Rightarrow$ g''(x) has atleast 1 real roots in $x \in (1, 3)$

$\Rightarrow$ f''(x) – 2 = 0 for atleast 1 real roots in $x \in (1, 3)$

$\Rightarrow$ f''(x) = 2 for atleast one $x \in (1, 3)$

6. Sol. Clearly $f(x) = e^{x^2} + e^{-x^2}$

$f'(x) = 2x\,(e^{x^2} - e^{-x^2}) \ge 0$ increasing $\Rightarrow$ $f_{max} = f(1) = e + \dfrac{1}{e}$

$g(x) = x\,e^{x^2} + e^{-x^2}$ $\Rightarrow$ $g'(x) = e^{x^2} + 2x^2 e^{x^2} - 2x\,e^{-x^2} > 0$ increasing

$\Rightarrow \quad g_{max} = g(1) = e + \dfrac{1}{e}$

$h(x) = x^2 e^{x^2} + e^{-x^2}$ $\Rightarrow$ $h'(x) = 2x + 2x^3 e^{x^2} - 2x\,e^{-x^2} = 2x\left(e^{x^2} + x^2 e^{x^2} - e^{-x^2}\right) > 0$

$\Rightarrow \quad h_{max} = h(1) = e + \dfrac{1}{e}$,

so, a = b = c

7. Sol. $f(x) = \begin{cases} x^{\alpha}\ell nx & , \quad 0 < x \le 1 \\ 0 & , \quad x = 0 \end{cases}$

For Rolles theorem to be applicable in [0, 1], f(x) should be continuous in [0, 1], differentiable in (0, 1) and f(0) = f(1).

Last two conditions hold and $\lim\limits_{x \to 0^+} x^{\alpha}\,\ell n\, x = 0$

Put $x = e^{-t}$, then as $x \to 0$, $t \to \infty$

$\lim\limits_{t \to \infty} \dfrac{-t}{e^{t\alpha}} = 0$ which is true for all $\alpha > 0$

8. Sol. $\int P(x)\, dx = \dfrac{x^{102}}{2} - 23\,x^{101} - 45\dfrac{x^2}{2} + 1035x + c.$

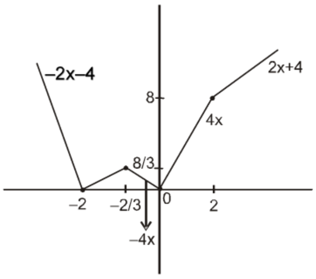
Now consider a function

$f(x) = \dfrac{x^2}{2}(x^{100} - 45) - 23x\,(x^{100} - 45) = \dfrac{x}{2}(x - 46)(x^{100} - 45)$

It satisfies all conditions of Rolle's theorem in $[45^{1/100}, 46]$. Hence there exist at least one $c \in \left(45^{1/100}, 46\right)$ for which f'(c) = 0

$\Rightarrow$ P(c) = 0 which proves that there exist a root of P(x) = 0 in $\left(45^{1/100}, 46\right)$

10*. Sol. (1, 2)



$f(x) = 2|x| + |x + 2| - \big|\, |x + 2| - 2\,|x| \,\big|$

$$= \begin{cases} -2x-4 & x \le -2 \\ 2x+4 & -2 < x \le -2/3 \\ -4x & -2/3 < x \le 0 \\ 4x & 0 < x \le 2 \\ 2x+4 & x > 2 \end{cases}$$

Graph of y = f(x) is

minima at x = –2, 0 ; maxima at x = –2/3

y = f(x)

x = –2, 0 ; x = –2/3

11*. **Ans.** **(1,3,4)**

Sol. $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$

$f'(x) = \dfrac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \dfrac{x}{x^2} e^{-\left(x+\frac{1}{x}\right)}$

$f'(x) = \dfrac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$

(1) For $x \in [1, \infty)$ $f'(x) > 0$ so (1) is correct.

(2) Obvious wrong.

(3) $f(x) + f(1/x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \underbrace{\int_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t}}_{\text{put } t=\frac{1}{p}} dt$

$\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} - \int_{1/x}^{x} e^{-\left(p+\frac{1}{p}\right)} \frac{dp}{p}$

= 0

(3) is correct

(4) Since $f(x) = -f\left(\frac{1}{x}\right)$

$f(2^x) = -f\left(\frac{1}{2^x}\right)$

$f(2^x) = -f(2^{-x})$

odd.

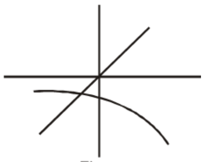
(4) is correct

1,3,4 is answer

COMPREHENSION

(COMPREHENSION)
Comprehension # 1 (Q.12 to 14)

If a continuous function f defined on the real line **R**, assumes positive and negative values in **R** then the equation f(x) = 0 has a root in **R**. For example, if it is known that a continuous function f on **R** is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in **R**.
Consider $f(x) = ke^x - x$ for all real x where k is a real constant.



14. **Sol.** **12.** Figure

13.
Consider $y = ke^x$ and $y = x$
Let (α, ke^α) be a point on $y = ke^x$
if it lies on y = x also then $\alpha = ke^\alpha$

$\frac{dy}{dx} = ke^x$

$\therefore \left.\frac{dy}{dx}\right|_{x=\alpha} = ke^\alpha = \alpha = 1$ $\{\because$ y = x is tangent to $y = ke^x$ at one point$\}$

$\therefore$ 1 = ke i.e. k = 1/e

14.
Consider $y = ke^x$ and $y = x$

from above question $e^x = \frac{x}{k}$ if we decrease the value of k from $\frac{1}{e}$, then slope of y = increases

$\therefore$ $y = e^x$ and $y = \frac{x}{k}$ intersect at two distinct points

$\therefore$ $k \in \left(0, \frac{1}{e}\right)$

Comprehension # 1 (Q.15 to 16)

Let f : [0, 1] → R (the set of all real numbers) be a function. Suppose the function f is twice differentiable, f(0) = f(1) = 0 and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.
f(0) = f(1) = 0 $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$

15. **Sol.** **(4)**
$f''(x) - 2f'(x) + f(x) \geq e^x$
$f''(x)\cdot e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \geq 1$

$\frac{d}{dx}(f'(x)e^{-x}) - \frac{d}{dx}(f(x)\cdot e^{-x}) \geq 1$

$\frac{d}{dx}(f'(x)e^{-x} - f(x)e^{-x}) \geq 1$

$\Rightarrow \frac{d^2}{dx^2}(e^{-x}f(x)) \geq 1 \quad \forall x \in [0, 1]$

Let $\varphi(x) = e^{-x}f(x)$

$\Rightarrow \varphi(x)$ is concave upward
$f(0) = f(1) = 0$
$\Rightarrow \varphi(0) = 0 = \varphi(1) \quad \Rightarrow \quad \varphi(x) < 0$
$\Rightarrow f(x) < 0$

16. Sol. (3)
$\varphi'(x) < 0,\ x \in (0, 1/4)$
and
$\varphi'(x) > 0,\ x \in (1/4, 1) \quad \Rightarrow e^{-x} f'(x) - e^{-x} f(x) < 0,\ x \in (0, 1/4)$
$f'(x) < f(x),\ 0 < x < 1/4$

DIRECTIONS : (Q. 17)

Each question has 4 choices (1), (2), (3) and (4) out of which ONLY ONE is correct.
(1) Both the statements are true.
(2) Statement-I is true, but Statement-II is false.
(3) Statement-I is false, but Statement-II is true.
(4) Both the statements are false.

17. Ans. (1)

Sol. $ax^2 + by^2 = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{-ax}{by}$

$Ax^2 + By^2 = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{-Ax}{By}$

Product of slopes $= -1 \quad \Rightarrow \quad aAx^2 + bBy^2 = 0$

Eliminating x^2, y^2. $\Rightarrow \begin{vmatrix} a & b & 1 \\ A & B & 1 \\ aA & bB & 0 \end{vmatrix} = 0$

$\Rightarrow \quad (AbB - aAB) - (abB - abA) = 0$
$AB(b - a) - ab(B - A) = 0 \quad \Rightarrow \quad ab(A - B) = AB(a - b)$
Statement-2 is true.
Using statement-2 we have in statement-1

$$\frac{1}{a^2}\left(\frac{-1}{b^2}\right)\left(\frac{1}{1+a^2} - \frac{1}{1-b^2}\right) = \frac{1}{(1+a^2)(1-b^2)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

$$\frac{-(-b^2 - a^2)}{a^2b^2(1+a^2)(1-b^2)} = \frac{(b^2 + a^2)}{(1+a^2)(1-b^2)a^2b^2}.$$

Statement-1 is true.