Indefinite Integration

Exercise-3

PART - I : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. **Sol.** $I = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$ Let $x^n = t$ $n.\, x^{n-1}\, dx = dt$

$\Rightarrow$ $I = \frac{1}{n}\int \frac{1}{t(t+1)}dt = \frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$

$= \frac{1}{n}[\ell n\,|t| - \ell n\,|t+1|] + C = \frac{1}{n}\ell n\left|\frac{x^n}{x^n+1}\right| + C$

2. **Sol.** Put $x - \alpha = t$ $\Rightarrow$ $dx = dt$

$\Rightarrow$ $\int \frac{\sin x}{\sin(x+\alpha)}dx = \int \frac{\sin(\alpha+t)}{\sin t}dt = \sin\alpha \int \cot t\, dt + \cos\alpha \int dt$

$= \sin\alpha\,.\,\ell n\,|\sin t| + t\cos\alpha + C$

$= \sin\alpha\,.\ell n|\sin(x-\alpha)| + (x-\alpha)\cos\alpha + C$

$\Rightarrow$ $A = \cos\alpha\ \&\ B = \sin\alpha.$

3. **Sol.** $\int \frac{dx}{\cos x - \sin x} = \frac{1}{\sqrt{2}}\int \frac{dx}{\cos\left(x+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \sec\left(x+\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\,\ell n\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right| + C$

4. **Sol.** $I = \int\left\{\frac{(\ell n x - 1)}{1+(\ell n x)^2}\right\}^2 dx = \int \frac{1+(\ell n x)^2 - 2\ell n x}{(1+(\ell n x)^2)^2}dx$ ……………….(1)

$= \int \frac{1+(\ell n x)^2 - x\left(\frac{2\ell n x}{x}\right)}{(1+(\ell n x)^2)^2}dx = \left(\frac{x}{1+(\ell n x)^2}\right) + C$

OR

Put $\ell n x = t$ in (1), we have

$I = \int \frac{1+t^2-2t}{\left(1+t^2\right)^2}e^t\, dt = \int e^t\left(\frac{1}{1+t^2} - \frac{2t}{\left(1+t^2\right)^2}\right)dt = \frac{e^t}{1+t^2} + C = \frac{x}{1+(\ell n x)^2} + C$

5. **Sol.** $\int \frac{dx}{\cos x + \sqrt{3}\sin x} = \frac{1}{2}\int \frac{dx}{\cos\left(x-\frac{\pi}{3}\right)} = \frac{1}{2}\int \sec\left(x-\frac{\pi}{3}\right)dx$

$= \frac{1}{2}\,\ell n \tan\left(\frac{x}{2}+\frac{\pi}{12}\right) + C$

6. **Sol.** Put $x - \frac{\pi}{4} = t$

$dx = dt$

$I = \sqrt{2}\int \frac{\sin\left(t+\frac{\pi}{4}\right)}{\sin t}dt = \sqrt{2}\int\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cot t\right)dt = \int(1+\cot t)\,dt$

$= t + \ell n\,|\sin t| + C = x - \frac{\pi}{4} + \ell n\left|\sin\left(x-\frac{\pi}{4}\right)\right| + C = x + \ell n\left|\sin\left(x-\frac{\pi}{4}\right)\right| + C$

7. **Sol.** $\int \frac{5\tan x}{\tan x - 2}dx = \int \frac{5\sin x}{\sin x - 2\cos x}dx = \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)}dx$

$= \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx = x + 2\ln|(\sin x - 2\cos x)| + k \Rightarrow a = 2$

8. **Sol.** (3)

$\int f(x)dx = \psi(x)$

$I = \int x^5 f(x^3)dx$

put $x^3 = t \Rightarrow x^2dx = \frac{dt}{3}$

$= \frac{1}{3}\int tf(t)dt = \frac{1}{3}\left[t\psi(t) - \int \psi(t)dt\right] = \frac{1}{3}\left[x^3\psi(x^3) - 3\int x^2\psi(x^3)dx\right] + c = \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + c$

9. **Sol.** **Ans.** **(4)**

$\int \left(1 + x - \frac{1}{x}\right)e^{\left(x+\frac{1}{x}\right)}dx$

$\int e^{x+\frac{1}{x}}dx + \int\left(x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

Applying integration by parts in (I)

We get,

$\int e^{x+\frac{1}{x}}dx = x.\,e^{x+\frac{1}{x}} - \int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$

$= x\,e^{x+\frac{1}{x}} - \int\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

Thus $\int e^{x+\frac{1}{x}}dx + \int\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx = x\,e^{x+\frac{1}{x}} + C.$

10. **Ans.** **(4)**

Sol. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$

$\int \frac{dx}{x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

$1 + \frac{1}{x^4} = t^4$

$-4\frac{1}{x^5}\,dx = 4t^3\,dt$

$\frac{dx}{x^3} = -t^3\,dt$

$\int \frac{-t^3dt}{t^3} = -t + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

11. **Ans.** **(2)**

Sol. $I_n = \int \tan^n x dx = \int \tan^{n-2}(\sec^2 x - 1)dx$

$\int (\tan x)^{n-2}\sec^2 x dx - \int (\tan x)^{n-2}dx = \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$

$I_n + I_{n-2} = \frac{(\tan x)^{n-1}}{n-1}$

put n = 6

$I_4 + I_6 = \frac{1}{5}\tan^5 x = a\tan^5 x + bx^5 + c \quad \Rightarrow \quad a = \frac{1}{5} \quad b = 0 \quad c = 0$

$\therefore \quad (a, b) = \left(\frac{1}{5}, 0\right)$

12. **Sol.** **(4)**

$I = \frac{\tan^2 x.\sec^2 x}{(\tan^5 x + \tan^2 x + \tan^3 x + 1)^2}dx = \frac{\tan^2 x\sec^6 x}{(\tan^2 x+1)^2(\tan^3 x+1)^2}dx = \frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$

let $1 + \tan^3 x = t$

$3\tan^2 x \sec^2 x\, dx = dt = \frac{1}{3}\int \frac{1}{t^2}dt = -\frac{1}{3(1+\tan^3 x)} + C$

PART - II : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. **Solution :**

$\because \quad f(x) = \int e^x(x-1)(x-2)dx$

$\Rightarrow \quad f'(x) = e^x(x-1)(x-2) \le 0 \quad \Rightarrow \quad 1 \le x \le 2$

$\Rightarrow \quad x \in [1, 2]$

2. **Sol.** $I = \int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}} dx$ put $2 - \frac{2}{x^2} + \frac{1}{x^4} = z$ $\Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right) dx = dz$

$\Rightarrow I = \frac{1}{4}\int \frac{dz}{\sqrt{z}}$

$= \frac{1}{2}\sqrt{z} + c$ $\Rightarrow$ $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + C = \frac{\sqrt{2x^4-2x^2+1}}{2x^2} + C$

3. **Sol.** $f(f(x)) = \frac{f(x)}{(1+(f(x)^n)^{1/n}} = \frac{x/(1+x^n)^{1/n}}{\left(1+\frac{x^n}{1+x^n}\right)^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$

Also, $f(f(f(x))) = \frac{x/(1+2x^n)^{1/n}}{\left(1+\frac{x^n}{1+2x^n}\right)^{1/n}} = \frac{x}{\left(1+3x^n\right)^{1/n}}$ $\Rightarrow$ $g(x) = \frac{x}{(1+nx^n)^{1/n}}$

Hence $I = \int x^{n-2} \frac{x}{(1+nx^n)^{1/n}} dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}} dx$

Let $1 + n x_n = t$ $\Rightarrow x_{n-1} dx = \frac{1}{n^2} dt$

$\Rightarrow I = \frac{1}{n^2}\int \frac{dt}{t^{1/n}} = \frac{1}{n^2} \frac{t^{\frac{-1}{n}+1}}{1-\frac{1}{n}} = \frac{1}{n(n-1)} (1 + nx_n)_{1-1/n} + K$

4. **Solution**

Statement - 1 $F(x) = \int (\sin^2 x) dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

Which is aperiodic function. Hence statement is false.

Statement - 2 $\sin_2 x$ is periodic with the period π, hence statement is true.

5. **Sol.** $J - I = \int \frac{(e^{3x}-e^x)\,dx}{e^{4x}+e^{2x}+1}$

Let $e_x = t$ $\Rightarrow$ $e_x\, dx = dt$

$\Rightarrow$ $J - I = \int \frac{t^2-1}{t^4+t^2+1} dt = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1} dt$ put $t + \frac{1}{t} = u \Rightarrow \left(1-\frac{1}{t^2}\right) dt = du$

$\Rightarrow$ $J - I = \int \frac{du}{u^2-1} = \frac{1}{2} \ell n \left|\frac{u-1}{u+1}\right| + C = \frac{1}{2} \ell n \left|\frac{t^2-t+1}{t^2+t+1}\right| + C = \frac{1}{2} \ell n \left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + C$

6. **Sol.** **(C)**

Put secx + tanx = t

(secx tanx + $\sec^2$x) dx = dt

secx . t dx = dt

secx – tanx = $\frac{1}{t}$

secx = $\frac{t+\frac{1}{t}}{2}$

$$\int \frac{\sec x.dt}{t^{9/2}.t} = \int \frac{1}{2}\frac{\left(t+\frac{1}{t}\right)}{t\ .\ t^{9/2}}dt = \frac{1}{2}\int\left(\frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\right)\ dt = -\frac{1}{2}\left[\frac{2}{7t^{7/2}}+\frac{2}{11t^{11/2}}\right]+k = -\frac{1}{t^{11/2}}\left[\frac{t^2}{7}+\frac{1}{11}\right]+k$$

Additional Problems For Self Practice (APSP)

PART - I : PRACTICE TEST PAPER

1. **Sol.** Differentiate both sides w.r.t. x

$$\frac{1}{(x+2)(x^2+1)} = \frac{2\alpha x}{1+x^2} + \frac{\beta}{1+x^2} + \frac{1}{x+2}\gamma$$

$$\Rightarrow \frac{1}{(x+2)(x^2+1)} = \frac{(2\alpha x+\beta)(x+2)+\gamma(1+x^2)}{(1+x^2)(2+x)}$$

$$\Rightarrow \frac{1}{(x+2)(x^2+1)} = \frac{(2\alpha+\gamma)x^2+(4\alpha+\beta)x+2\beta+\gamma}{(1+x^2)(x+2)}$$

$\Rightarrow$ $2\alpha + \gamma = 0$ and $4\alpha + \beta = 0$ and $2\beta + \gamma = 1$

$\Rightarrow$ $\beta - 2\gamma = 0$ and $2\beta + \gamma = 1$

$\Rightarrow$ $\gamma = \frac{1}{5}$ and $\beta = \frac{2}{5}$ and $\alpha = -\frac{1}{10}$

2. **Sol.** $I = \int \overset{II}{e^x}.\overset{I}{\ell n}(4x+1)dx + \int e^x.\frac{16}{(4x+1)^2}dx$

$= e^x \ell n(4x+1) - \int \frac{4}{4x+1}.e^x dx + 16\int \frac{e^x}{(4x+1)^2}dx$

$= e^x \ell n(4x+1) - \frac{4e^x}{4x+1} - 16\int \frac{1}{(4x+1)^2}.e^x dx + 16\int \frac{e^x}{(4x+1)^2}dx$

$= e^x\left(\ell n(4x+1) - \frac{4}{4x+1}\right) + C$

3. **Sol.** $I = \int \left(e^{\left(x^2-\frac{1}{x}\right)} + x(2x+\frac{1}{x^2}).e^{\left(x^2-\frac{1}{x}\right)}\right)dx$

$\therefore \int (f(x)+xf'(x))dx = x f(x) + C$

so , $I = xe^{\left(x^2-\frac{1}{x}\right)} + C$

4. **Sol.** Let , $\sin^2 x = t$

$2\sin x \cos x\, dx = dt$

$\Rightarrow I = \frac{1}{2}\int e^t(1+1-t)dt$

$= \frac{1}{2}.2\int e^t dt - \frac{1}{2}\int t.e^t dt = e^t - \frac{1}{2}(te^t - e^t) + C = \frac{3}{2}e^t - \frac{t}{2}e^t = \frac{1}{2}e^{\sin^2 x}(3-\sin^2 x) + C$

5. **Sol.** $I = \int \frac{\sec^4 x}{2\sqrt{\tan x}}dx$

put tanx = t

$\sec^2 x\, dx = dt$

$\Rightarrow I = \frac{1}{2}\int \frac{(1+t^2)}{\sqrt{t}}dt$ $\Rightarrow I = \frac{1}{2}\left[\frac{\sqrt{t}}{\frac{1}{2}} + \frac{t^{5/2}}{\frac{5}{2}}\right] + k$ $\Rightarrow I = (\tan x)^{1/2} + \frac{1}{5}(\tan x)^{5/2} + k$

so , $A + B + C = \frac{1}{2} + \frac{1}{5} + \frac{5}{2} = \frac{5+2+25}{10} = \frac{16}{5}$

6. **Sol.** $I = \int \frac{\sec^2\left(\frac{x}{2}\right)}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}dx$

$= \int \frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)}dx$

put $\tan\left(\frac{x}{2}\right) = t$

$\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$ $\Rightarrow I = 2\int \frac{1}{(3)^2+t^2}dt$

$= 2\left[\frac{1}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right)\right] + C$

so , $a = \frac{2}{3}$, $b = \frac{1}{3}$

7. $I = -\frac{2}{5}\int \frac{1}{1+t}dt = -\frac{2}{5}\ell n\left(1+\frac{1}{x^{5/2}}\right)+C = \frac{2}{5}\ell n\left(\frac{x^{5/2}}{x^{5/2}+1}\right)+C$

so , $\alpha = \frac{2}{5}$ and $\beta = \frac{5}{2}$

8. **Sol.** Let $\tan^{-1}x = t$

$\frac{1}{1+x^2}dx = dt$ $\Rightarrow I = \int x^3.\frac{1}{1+x^2}dx = \int \frac{x(x^2+1-1)}{1+x^2}dx$

$= \int x dx - \int \frac{x}{1+x^2}dx = \frac{x^2}{2} - \frac{1}{2}\ell n(1+x^2)+C$

9. **Sol.** $I = -e^{-x}\ln(e^x+1) - \int \frac{1}{e^x+1}.e^x(-e^{-x})\,dx$

$= -e^{-x}\ln(e^x+1) + \int \frac{e^{-x}}{1+e^{-x}}\,dx$

put $1 + e^{-x} = t$

$= -e^{-x}dx = dt$

$\Rightarrow I = -e^{-x}\ln(e^x+1) - \ln(1+e^x) + \ln e^x + C$

$= x - (e^{-x}+1)\ln(e^x+1) + C$

10. **Sol.** put $\ln(g(x)) - \ln(f(x) = t$ $\Rightarrow \left(\frac{g'(x)}{g(x)} - \frac{f'(x)}{f(x)}\right)dx = dt$

$\Rightarrow I = \int t\,dt$ $= \frac{t^2}{2} + C$ $= \frac{1}{2}\left(\ln(g(x)) - \ln(f(x))\right)^2 + C$

11. **Sol.** $f(x) = \int \frac{(1+x^2)-(1-\sin^2 x)}{1+x^2}.\sec^2 x\,dx$

$= \int \sec^2 x\,dx - \int \frac{1}{1+x^2}\,dx$

$f(x) = \tan x - \tan^{-1}x + C$

 $f(0) = 0 \Rightarrow c = 0$

so , $f(1) = \tan 1 - \frac{\pi}{4}$

12. **Sol.** Let $\cos x = t$

so , $I = \int \frac{1}{1+t^2}\,dt$

$= \tan^{-1}(\cos x) + C$

$\Rightarrow I = -\cot^{-1}(\cos x) + C$

13. **Sol.** put $\cot^{-1}x = t$

$-\frac{1}{1+x^2}\,dx = dt$

$\Rightarrow I = \int e^{\tan^{-1}x}(1+x+x^2).\left(-\frac{1}{1+x^2}dx\right)$ $I = -\int e^{\tan^{-1}(x)}\left(1+\frac{x}{1+x^2}\right)dx$

$= -\int\left(e^{\tan^{-1}x} + x.\frac{e^{\tan^{-1}x}}{1+x^2}\right)dx$ $= -xe^{\tan^{-1}x} + C$

14. **Sol.** $I_n = x(\ln x)^n - \int n\frac{(\ln x)^{n-1}}{x}x\,dx$

$= x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$

$\Rightarrow$ $I_n + nI_{n-1} = x^{(\ell nx)^n}$

put n = 5

= n = 5

15. **Sol.** $I = \int \frac{\cos x}{\sin^2 x(1+\sin^5 x)^{4/5}} dx$ $= \int \frac{\cos x}{\sin^6 x(1+\sin^{-5} x)^{4/5}} dx$

put $1 + \sin^{-5}x = t^5$ $\Rightarrow$ $-5\sin^{-6}x \cos x\, dx = 5t^4\, dt$

$\Rightarrow$ $I = -\int \frac{1}{(t^5)^{4/5}}.t^4 dt$ $= -(1 + \sin^{-5}x)^{1/5} + C$

$= -\frac{1}{\sin x}(\sin^5 x + 1)^{1/5} + C$

16. **Sol.** $I = \int \frac{1}{x^{13}(1+x^{-4})^{1/2}} dx$

$1 + x^{-4} = t^2$

$-4x^{-5}dx = 2tdt$

$\Rightarrow$ $I = -\frac{1}{4}\int \frac{2t}{x^8 t} dt$ $\Rightarrow$ $I = -\frac{1}{2}\int (t^2 - 1)^2 dt$

$\Rightarrow$ $I = -\frac{1}{2}\int (t^4 - 2t^2 + 1) dt$ $\Rightarrow$ $I = \frac{t^5}{-10} - \frac{t^3}{-3} - \frac{t}{2} + k$

so , c – b – a = 2 – (–3) – (–10) = 15

17. **Sol.** $f'(x) = \frac{\sqrt{x^2+1}+x}{(x^2+1)-x^2}$

integrating both sides w.r.t. x

$\int f'(x)dx = \int (\sqrt{x^2+1} + x)dx$ $\Rightarrow$ $f(x) = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ell n(x + \sqrt{x^2+1}) + C$

$f(0) = -\frac{\sqrt{2}+1}{2}$

Now (1) = $\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2}\ell n\ (1+\sqrt{2})\ -\frac{\sqrt{2}+1}{2}$ $= \frac{1}{2}\ell n(\sqrt{2}+1)$

$= \ell n\ (\sqrt{\sqrt{2}+1})$

18. **Sol.** $\therefore \vec{a}.\vec{b} = 0$

$\Rightarrow f(x).\ f''(x) - (f'(x))^2 = 0 \Rightarrow$ $\ell n(f'(x)) = \ell n(f(x)) + \ell nC$

Integrating both sides w.r.t.

$\ell n(f'(x)) = \ell n(f(x)) + \ell nC$

$f'(x) = Cf(x)$

$f(0) = 1,\ f'(0) = 2$

$\Rightarrow$ C = 2 $\Rightarrow \frac{f'(x)}{f(x)} = 2$ $\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int 2dx$

$\ell n(f(x)) = 2x + C_1$

put x = 0 $\Rightarrow C_1 = 0$ $\Rightarrow f(x) = e^{2x}$

19. **Sol.** $g(x) = \int \frac{3x+2}{\sqrt{x-9}}\,dx$

Let $\sqrt{x-9} = t \Rightarrow x - 9 = t^2$

$\Rightarrow$ dx = 2tdt

$\Rightarrow g(x) = \int \frac{3(t^2+9)+2}{t}.2tdt$

$g(x) = 2\int (29+3t^2)dt$

$= 2[29t + t^3] + C$

$g(x) = 2\left[29\sqrt{x-9} + (x-9)^{3/2}\right] + C$

$\therefore g(13) = 132$

$\Rightarrow$ 132 = 2(58+8) + C

$\Rightarrow$ C = 0

so , g(10) = 60

20. **Sol.** $I = \int\left(\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right)dx$

$= \int\left(\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right| + \left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|\right)\,dx$

$\therefore \frac{x}{2} \in \left(\frac{\pi}{8}, \frac{\pi}{4}\right)$ $\Rightarrow I = \int 2\cos\frac{x}{2}dx$ $= 4\sin\frac{x}{2} + C$

21. **Sol.** $I = \int \overset{\text{I}}{(\cos x)^{-2017}}.\overset{\text{II}}{\operatorname{cosec}^2 x}\,dx - 2017\int (\cos x)^{-2017}\,dx$

$= (-\cot x).(\cos x)^{-2017} - \int (-2017).(\cos x)^{-2018}.(-\sin x)(-\cot x)dx - 2017\int (\cos x)^{-2017}dx$

$I = \frac{-\cot x}{(\cos x)^{2017}} + 2017\int (\cos x)^{-2017}dx - \int (2017)(\cos x)^{-2017}dx$

$I = -\frac{\operatorname{cosec} x}{(\cos x)^{2016}} + C$

22. **Sol.** f(x) + xf '(x) = 6f(x).f '(x) $\Rightarrow$ f(x) = (6f(x) – x) f '(x)

Now $I = \int \frac{2x(x-6f(x))+f(x)}{(6f(x)-x)(x^2-f(x))^2}\,dx = -\int \frac{2x(6f(x)-x)-(6f(x)-x).f'(x)}{(6f(x)-x)(x^2-f(x))^2}\,dx$

$= -\int \frac{2x - f'(x)}{(x^2-f(x))^2}\,dx$

put $x^2 - f(x) = t$

$(2x - f'(x))\,dx = dt$

so , I = $\frac{1}{x^2 - f(x)} + C$

23. **Sol.** $I_{4,3} = \int \cos^4 x.\sin 3x\,dx$

$= -\frac{1}{3}\cos 3x.\cos^4 x - \int 4\cos^3 x(-\sin x)\frac{(-\cos 3x)}{3}dx$

$= -\frac{1}{3}\cos 3x.\cos^4 x - \frac{4}{3}\int \cos^3 x\sin x\cos 3x\,dx$

$= -\frac{\cos 3x.\cos^4 x}{3} - \frac{4}{3}\int \cos^3 x.\sin x(4\cos^3 x - 3\cos x)dx$

$= -\frac{\cos 3x.\cos^4 x}{3} - \frac{4}{3}\int \cos^4 x.\sin x(4(1-\sin^2 x)-3)dx$

$= -\frac{\cos 3x.\cos^4 x}{3} - \frac{4}{3}\int \cos^4 x.\sin x\,(1 - 4\sin^2 x)\,dx$

$= -\frac{\cos 3x.\cos^4 x}{3} - \frac{4}{3}\int \cos^4 x.(\sin 3x - 2\sin x)\,dx$

$= -\frac{1}{3}\cos 3x.\cos^4 x - \frac{4}{3}I_{4,3} + \frac{4}{3}\int \cos^3 x.\sin 2x)\,dx$

so $\frac{7}{3}I_{4,3} - \frac{4}{3}I_{3,2} = -\frac{\cos 3x.\cos^4 x}{3} + C$

24. **Sol.** $I = \int \overset{II}{1}.\overset{I}{f^{-1}}(x)dx = xf^{-1}(x) - \int \frac{d}{dx}(f^{-1}(x)).xdx$

Let $f^{-1}(x) = t \Rightarrow x = f(t) \Rightarrow \left(\frac{d}{dx}(f^{-1}(x))\right)dx = dt$

$I = xf^{-1}(x) - \int f(t)dt = xf^{-1}(x) - g(t) + C = xf^{-1}(x) - g(f^{-1}(x)) + C$

25. **Sol.** $I = \int \sin(2016x + x).(\sin x)^{2015}\,dx$

$= \int ((\sin x)^{2015}.\cos x).\sin(2016x)\;\;dx + \int (\sin x)^{2016}.\cos(2016x)\;\;dx$

$= \sin(2016x)\frac{(\sin x)^{2016}}{2016} - \int (2016).\cos(2016x).\frac{(\sin x)^{2016}}{2016}dx + \int (\sin x)^{2016}.(\cos(2016x))dx$

$I = \frac{\sin(2016x)(\sin x)^{2016}}{2016} + C$

26. **Sol.** $I = \int \frac{1 + x^2 + 2x}{x(x^2+1)}dx = \int \left(\frac{1}{x} + \frac{2}{x^2+1}\right)dx$

I = $\ell n|x| + 2\tan^{-1}x + C$

27. **Sol.** I = $\int (6x^5 + 5x^4 + 4x^3)(x^4 + x^5 + x^6)^6\,dx$

Let $x^6 + x^5 + x^4 = t$

$(6x^5 + 5x^4 + 4x^2)\,dx = dt$

I = $\int t^6 dt = \frac{(x^4 + x^5 + x^6)^7}{7} + C = \frac{x^{28}(1 + x + x^2)^7}{7} + C$

28. **Sol.** I = $\int \frac{\sqrt{x}}{\sqrt{x}(1+\sqrt{x})^{2017}}\,dx$

$1 + \sqrt{x} = t$

$\frac{1}{\sqrt{x}}dx = 2dt \quad \Rightarrow \quad I = 2\int \frac{t-1}{t^{2017}}dt = 2\int (t^{-2016} - t^{-2017})dt$

$= 2\left[\frac{1}{(-2015)(1+\sqrt{x})^{2015}} - \frac{1}{(-2016)(1+\sqrt{x})^{2016}}\right] + C = 2\left[\frac{1}{2016(1+\sqrt{x})^{2016}} - \frac{1}{2015(1+\sqrt{x})^{2015}}\right] + C$

so, α – β = 1

29. **Sol.** I = $\frac{1}{\sqrt{2}}\int \frac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}\,dx$

put sinx + cosx = t

(cosx – sinx) dx = dt

I = $\frac{1}{\sqrt{2}}\int \frac{1}{1+t^2}\,dt$

$\frac{1}{\sqrt{2}}\tan^{-1}(\sin x + \cos x) + C$

so, A.f(x) = $\frac{1}{\sqrt{2}}(\sin x + \cos x) + C$

Range ∈ [–1, 1]

30. **Sol.** I = $\int \frac{e^x(x-2)}{x^3\left(1 + \frac{e^x}{x^2}\right)}\,dx$

put $1 + \frac{e^x}{x^2} = t$

$\frac{x^2e^x - e^x 2x}{x^4}\,dx = dt \quad \Rightarrow \quad \frac{e^x}{x^3}(x-2)dx = dt$

so, I = $\int \frac{1}{t}dt = \ell n\left(1 + \frac{e^x}{x^2}\right) + C$

Practice Test (JEE-Main Pattern)

OBJECTIVE RESPONSE SHEET (ORS)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.										
Que.	11	12	13	14	15	16	17	18	19	20
Ans.										
Que.	21	22	23	24	25	26	27	28	29	30
Ans.										

PART - II : PRACTICE QUESTIONS

1. **Sol.** $I = \int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}}dx$ Put $1 + \ell n|x| = t^2$, then $\frac{1}{x}dx = 2t\,dt$

$\Rightarrow I = \int \frac{t^2-1}{t}.2t\,dt = 2\left[\frac{t^3}{3}-t\right] + C = \frac{2}{3}\sqrt{1+\ell n|x|}\,(\ell n|x| - 2) + C$

2. **Sol.** $I = \int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}dx$ $0 < \alpha < x < \pi$

$= \int \frac{\sqrt{2}\sin\frac{x}{2}dx}{\sqrt{2\cos^2\frac{\alpha}{2} - 1 - 2\cos^2\frac{x}{2} + 1}} = \int \frac{\sin\frac{x}{2}dx}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{x}{2}}}$ put $\cos\frac{x}{2} = t \Rightarrow -\frac{1}{2}\sin\frac{x}{2}dx = dt$

$\Rightarrow I = \int \frac{-2\,dt}{\sqrt{\cos^2\frac{\alpha}{2} - t^2}} = -2\sin^{-1}\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right) + C$

3. **Sol.** $I = \int \frac{\cos^3 x}{\sin^2 x + \sin x}dx = \int \frac{\cos x.(1-\sin^2 x)}{\sin x(1+\sin x)}dx$ Put sin x = t , then cosx dx = dt

$\Rightarrow$ $I = \int \frac{(1-t)(1+t)\,dt}{t(1+t)} = \ell n\,|t| - t + C = \ell n\,|\sin x| - \sin x + C$

4. **Sol.** $I = 2\int \sin x\,.\,\mathrm{cosec}\,4x\,dx = 2\int \frac{\sin x\,dx}{4\sin x\cos x\cos 2x}$

$= \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)}$ Put sinx = t, then cos x dx = dt

$\Rightarrow$ $I = \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)}$

$= \frac{1}{2}\left[-\int \frac{1}{1-t^2}dt + \int \frac{2}{1-2t^2}dt\right]$ [By partial fraction]

$= \frac{1}{2}\left[-\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}t}{1-\sqrt{2}\,t}\right)\right|\right]$

$= \frac{1}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right)\right| - \frac{1}{4}\ell n\left|\left(\frac{1+\sin x}{1-\sin x}\right)\right| + C$

5. **Sol.** $\int \{\sec^2 x + \tan^2 x + 2\tan x\sec x\}^{1/2}dx = \int (\sec x + \tan x)\,dx$

$= \ell n\,|(\sec x + \tan x)| + \ell n\,|\sec x| + C = \ell n\,|\sec x\,(\sec x + \tan x)| + C$

6. **Sol.** $\int \frac{\sin^8 x - \cos^8 x}{\sin^4 x + \cos^4 x} dx = \int (\sin^4 x - \cos^4 x)\, dx$

$= \int (\sin^2 x - \cos^2 x)\, dx = -\frac{1}{2}\sin 2x + C$

7. **Sol.** $\int \frac{x^3 - 1}{x^3 + x} dx = \int \left(1 - \frac{1}{1+x^2} - \frac{1}{x} + \frac{x}{x^2+1}\right) dx = x - \tan^{-1}x - \ell n|x| + \frac{1}{2}\ell n(x^2 + 1) + C$

8. **Sol.** $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ put $\sqrt{x} = \cos 2t \Rightarrow dx = -4 \sin 2t \cos 2t\, dt$

$= \int \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}\,(-4 \sin 2t \cos 2t)\, dt \qquad = -4\int \frac{\sin t}{\cos t}.2\sin t \cos t \cos 2t\, .dt$

$= -4\int (1-\cos 2t) \cos 2t\, dt \qquad = -4\int \cos 2t\, dt + 4\int \cos^2 2t\, dt$

$= -\frac{4}{2}\sin 2t + 2\int (\cos 4t + 1)\, dt \qquad = -2 \sin 2t + 2 \times \frac{\sin 4t}{4} + 2t + C$

$= -2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} + \cos^{-1}\sqrt{x} + C$

9. **Sol.** $\int \frac{1}{[(x-1)^3 (x+2)^5]^{1/4}} dx$

$\int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4} (x+2)^2} dx$

Let $\frac{x-1}{x+2} = t \Rightarrow \frac{(x+2)-(x-1)}{(x+2)^2} dx = dt$

$\Rightarrow \frac{3}{(x+2)^2} dx = dt \qquad = \frac{1}{3}\int t^{-3/4} dt = \frac{1}{3}\frac{t^{1/4}}{(1/4)} + c = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

10. **Sol.** $\therefore \int \frac{\frac{x^2-1}{x^2}}{\left(\frac{1}{x^4}+\frac{1}{x^2}\right)\sqrt{x^8 + x^6 + x^4}} dx = \sec^{-1} f(x) + c$

$\Rightarrow \int \frac{\left(1-\frac{1}{x^2}\right)}{\frac{1}{x^3}\left(x+\frac{1}{x}\right)x^3\sqrt{x^2+1+\frac{1}{x^2}}} dx = \sec^{-1}(f(x)) + C$

$\Rightarrow \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2 - 1}} = \sec^{-1} f(x) + C$

⇒ $\sec^{-1}\left(x+\frac{1}{x}\right)$ + C = $\sec^{-1}$ (f(x)) + C

⇒ $\sec^{-1}\left(x+\frac{1}{x}\right)$ + C = $\sec^{-1}$(f(x)) + C

⇒ $f'(x) = \frac{(x+1)(x-1)}{x^2} = 0$

⇒ $x = \pm 1$

Maximum occur at x = –1 and minimum occur at x = 1

11. Sol. $\int \frac{3x^4-1}{(x^4+x+1)^2}dx = \int \frac{x(4x^3+1)-(x^4+x+1)}{(x^4+x+1)^2}dx$

$= \int \frac{d}{dx}\left(\frac{-x}{(x^4+x+1)}\right)dx = -\frac{x}{x^4+x+1}$

12. Sol. $I = \int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$ put $2-\frac{2}{x^2}+\frac{1}{x^4} = z$ ⇒ $\left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx = dz$

$I = \frac{1}{4}\int \frac{dz}{\sqrt{z}}$

$= \frac{1}{2}\sqrt{z} + c$ ⇒ $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + C = \frac{\sqrt{2x^4-2x^2+1}}{2x^2} + C$

13. Sol. Let $I = \int \frac{1+x^4}{\left(1-x^4\right)^{3/2}}dx$

$= \int \frac{1+x^4}{x^3\left(\frac{1}{x^2}-x^2\right)^{3/2}}dx = \int \frac{x+\frac{1}{x^3}}{\left(\frac{1}{x^2}-x^2\right)^{3/2}}dx$ Put $\frac{1}{x^2} - x^2 = t$ ⇒ $\left(\frac{-2}{x^3}-2x\right)dx = dt$

⇒ $I = -\frac{1}{2}\int \frac{dt}{t^{3/2}} = t^{-1/2} + C = \frac{1}{\sqrt{\frac{1}{x^2}-x^2}} + C$