## **Exercise-3**

\* Marked Questions may have more than one correct option.

PART-1: JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)  
1. Sol. 
$$ax + bx + c = 0$$
  
 $a(x - c) (x - \beta) = 0$   
 $\lim_{x \to a} \frac{1 - \cos(ax^2 + bx + c)}{(x - a)^2} = \lim_{x \to a} \frac{2\sin^2\left(\frac{ax^2 + bx + c}{2}\right)}{(x - a)^2} = \lim_{x \to a} \frac{2\sin^2\left(\frac{a}{2}(x - a) (x - \beta)\right)}{(x - a)^2}$   
 $= \lim_{x \to a} \left(\frac{\sin\left(\frac{a(x - a)}{2}(x - \beta)\right)}{\frac{a(x - a)}{2}}\right)^2 \cdot \frac{2a^2(x - \beta)^2}{4} = \lim_{x \to a} \frac{2\sin^2\left(\frac{a}{2}(x - \beta)^2\right)}{4} = \frac{a^2(a - \beta)^2}{2}$   
2. Sol. Since  $|f(x) - f(y)| \le (x - y)$ ?  
 $\lim_{x \to y} \frac{|f(x)| \le 0}{|x - y| \le x - y|}$   
 $\Rightarrow \quad |f'(y)| \le 0$   
 $\Rightarrow \quad f(y) = 0$   
 $\Rightarrow \quad f(y) = 0$  [ $\because f(0) = 0$  given]  
 $\Rightarrow \quad f(y) = 0$  [ $\because f(0) = 0$  given]  
 $\Rightarrow \quad f(1) = 0$   
3. Sol.  $f'(1) = \lim_{x \to 0} \frac{f(1 + h) - f(1)}{h}$   
 $= \lim_{x \to 0} \frac{f(1 + h) - \lim_{x \to 0} \frac{f(1)}{h}}{h} = 5$ , so  $\lim_{x \to 0} \frac{f(1)}{h}$  must be  
finite as  $f'(1)$  exist and  $\lim_{x \to 0} \frac{f(1)}{h} = 0$   
 $\therefore \quad f'(1) = \lim_{x \to 0} \frac{f(1 + h)}{h} = 5$ .  
4. Sol. Since,  $f(x) = \frac{x}{1 + |x|}$   
Let  $f(x) = f(x) = 1 + |x|$  are differentiable on  $(-\infty, \infty)$  and  $(-\infty, 0) \cup (0, \infty)$  respectively  
Thus,  $f(x)$  is differentiable on  $(-\infty, 0) \cup (0, \infty)$ .

$$\therefore \qquad \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x}{1 + |x|} - 0}{x}$$

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 $= \lim_{x \to 0} \frac{1}{1 + |x|} = 1$ Hence, f (x) is differentiable on  $(-\infty, \infty)$  $\lim_{x \to 0} \left( \frac{1}{x} - \frac{2}{e^{2x} - 1} \right)_{=} \lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x(e^{2x} - 1)}$ Now. 5. Sol.  $2e^{2x} - 2$  $= \lim_{x \to 0} \frac{2e^{-2x}}{(e^{2x} - 1) + 2xe^{2x}}$ (using L'Hospital's rule) 4e<sup>2x</sup>  $\lim_{x \to 0} \frac{12}{4e^{2x} + 4xe^{2x}} = 1$  $4e^{2x} + 4xe^{2x} = 1$  (using L'Hospital's rule) f(x) is continuous at x = 0, then = *:*..  $\lim_{x \to 0} f(x) = f(0) \implies 1 = f(0)$ 6.  $f(x) = \min \{x + 1, |x| + 1\}$ Sol.  $f(x) = x + 1, \forall x \in R.$ (0, 1) 0 x It is clear from the figure that  $f(x) \ge 1, \forall x \in \mathbb{R}.$ Now, f' (1) =  $\lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{f(1-h-1)\sin\left(\frac{1}{1-h-1}\right) - 0}{-h} = \lim_{h \to 0} \sin\left(-\frac{1}{h}\right) = --$ Sol. 7.  $\lim_{h \to 0} \sin \frac{1}{h}$ and f' (1+)=  $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h-1)\sin\left(\frac{1}{1+h-1}\right) - 0}{h} - \lim_{h \to 0} \sin \frac{1}{h}$ **.**.  $f'(1_{-}) \neq f'(1_{+})$ f is not differentiable at x = 1. Again, now  $\frac{(0+h+1)sin\left(\frac{1}{0+h+1}\right)-sin1}{-h}$  $f'(0) = \lim_{h \to 0}$  $\left[-\left\{(h+1)\cos\left(\frac{1}{h+1}\right)\times\left(\frac{1}{(h+1)^2}\right)\right\}+\sin\left(\frac{1}{(h+1)}\right)\right]$ lim (using L' hospital's rule) =  $= \cos 1 - \sin 1$  $\lim_{h \to 0} \frac{(0+h-1) \sin \left(\frac{1}{0+h-1}\right) - \sin 1}{h}$  $f'(0_+) =$ and (using L'Hospital's rule) =  $= \cos 1 - \sin 1 \Rightarrow f'(0_{-}) = f'(0_{+})$ *.*.. f is differentiable at x = 0. 8. Sol. gof (x) = sin (x |x|)

 $f(0_{*}) = \lim_{x \to 0^{+}} \frac{(1+x)^{1/2} - 1}{x} = \lim_{x \to 0^{+}} \frac{1 + \frac{1}{2}x + \dots - 1}{x} = \frac{1}{2}$  $\lim_{x\to 0^-} \frac{\sin (p+1)x + \sin x}{x}$  $f(0_{-}) = x \to 0^{-}$  $f(0_{-}) = \lim_{x \to 0^{-}} \frac{(\cos(p+1)x)(p+1) + (\cos x)}{1} = (p+1) + 1 = p + 2$  $\therefore \quad p+2=q=\frac{1}{2} \quad \Rightarrow \quad p=-\frac{3}{2}, q=\frac{1}{2} \text{ Ans.}$  $\int x \sin(1/x)$ ,  $x \neq 0$  $f(x) = \begin{cases} 0 & x = 0 \\ 0 & x = 0 \end{cases}$  at x = 0 ij 14. Sol. LHL =  $\lim_{h \to 0^{-}} \left\{ -h \sin \left( -\frac{1}{h} \right) \right\} = 0 \times a$  finite quantity between - 1 and 1  $\lim_{h \to 0^{-}} h \sin \frac{h}{h} = 0$ f(0) = 0 $\therefore$  f(x) is continuous on R.  $f_2(x)$  is not continuous at x = 0Sol. (3) 15.  $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a} = \lim_{x \to a} \frac{2x f(a) - a^2 f'(x)}{1} = 2a f(a) - a_2 f'(a)$  $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a} = \lim_{x \to a} \frac{x^2 f(a) - a^2 f(a) + a^2 f(a) - a^2 f(x)}{x - a}$ Alter  $= \lim_{x \to a} \frac{(x^2 - a^2)f(a) - a^2(f(x) - f(a))}{x - a} = \lim_{x \to a} (x + a) f(a) - a_2$  $= 2af(a) - a_2f'(a)$ 16. Sol. Ans. (1) Doubtful points are  $x = n, n \in I$  $\lim_{L.H.L = x \to n^{-}} [x] \cos^{\left(\frac{2x-1}{2}\right)} \pi = (n-1) \cos^{\left(\frac{2n-1}{2}\right)} \pi = 0$  $\lim_{R.H.L. = x \to n^+} [x] \cos \left(\frac{2n-1}{2}\right)_{\pi} = n \cos \left(\frac{2n-1}{2}\right)_{\pi} = 0$ f(n) = 0Hence continuous 17. Sol. Ans. (3)  $f(x) = 3 \qquad 2 \le x \le 5$ f'(x) = 02 < x < 5 f'(4) = 0

18.	Sol. (4)
	$\lim_{x \to 1} \frac{(1 - \cos 2x)(3 + \cos x)}{x^2} \cdot \frac{x}{\tan 4x} = \lim_{x \to 1} \frac{2\sin^2 x}{x^2} \cdot \frac{3 + \cos x}{4} \cdot \frac{x}{\tan 4x} = \frac{1}{4}$
	$I = x \to 0$ $X^ I$ $tan 4x = x \to 0$ $X^ I$ $tan 4x = 2.4$ . $4 = 2$
19.	Sol. Ans. (2)
	$\operatorname{Lim} \frac{\sin(\pi \cos^2 x)}{2} \qquad \left(\frac{0}{0} \operatorname{form}\right)$
	$x \to 0$ $X^ (0^-)$ $\sin(\pi \cos^2 x) = \sin(\pi \sin^2 x) = \pi \sin^2 x$
	$-\lim_{x \to 0} \frac{\sin(\pi \cos x)}{x^2} - \frac{\sin(\pi \sin x)}{\pi \sin^2 x} + \frac{\pi \sin^2 x}{x^2} - \pi$
20.	Ans. (3)
	$(1 - \cos 2x)(3 + \cos x)$ $2\sin^2 x \frac{(3 + \cos x)}{\sin 4x} \cos 4x$
Sol.	$\lim_{x \to 0} \frac{1}{x \tan 4x} = \lim_{x \to 0} \frac{1}{x^2} \frac{1}{4x} = 2$
21.	Ans. (1) $[1, 1]$
	$g(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ mx+2 & 3 \le x \le 5 \end{cases}$
Sol.	
	$\lim \frac{k\sqrt{x+1}-2k}{k} = \lim k \left\{ \frac{(x+1-4)}{k} \right\}$
	$x \to 3^{-}$ $x - 3$ $x + 3^{-}$ $\left  (x - 3)(\sqrt{x + 1} + 2) \right  = \frac{\kappa}{4}$
	L(g(3)) = mx + 2 - 2k
	$\lim_{x \to 3^+} \frac{1}{x - 3}$
	Since this limit exists $3m + 2 - 2k = 0 \implies 2k = 3m + 2$ (i)
	So $R(g'(3)) = m$ by L-Hospital rule
	Solving (i) & (ii) $(1 - 1)^{-1}$
	$\frac{2}{8}$ $\frac{8}{8}$
	$m = {}^{5}$ , $k = {}^{5}$ $\Rightarrow$ $k + m = 2$ (ii)
22.	Ans. (2)
	$\lim_{n \to \infty} (1 - 2 - 7)^{\frac{1}{n}}$
Sol.	$P = x \to 0^+ (1 + \tan^2 \sqrt{x})^{2x}$ then log p =
	$\lim_{x \to 1} (1 + \tan^2 \sqrt{x} - 1) \frac{1}{2x} = \lim_{x \to 1} \frac{(\tan \sqrt{x})^2}{2(\sqrt{x})^2} = \frac{1}{2}$
	$P = e^{x \to 0^{-}} \qquad \qquad e^{x} = e^{x \to 0^{-} z(v^{x})} = e^{z}$
	$\log \mathbf{P} = \log \mathbf{P}^{\frac{1}{2}} = \frac{1}{2}$
	$\log r = \log 3 = 2$
23	Ans (1)
20	
Sol.	$f(x) = \frac{ f(x) - f(x) }{ f(x) - f(x) }$
	$f(f(x)) = \frac{ (n_2 - \sin n_1) (n_2 - \sin n_2) }{ (n_1 - \sin n_2) (n_2 - \sin n_2) }$
	In the vicinity of $x = 0$ q(x) = ln2 = sin(ln2 - sinx)
	$g(x) = a_1 z^2 - a_1 (a_1 z^2 - a_1 x)$ Hence $g(x)$ is differentiable at $x = 0$ as it is sum and composite of differentiable function
	vr%g(x), $x = 0$ ij vodyuh; g $S$ t $Slk$ fd ; $g$ vodyuh; Qyuksa dk ; $ksx$ r $Fkk$ la; $kstu$ g $SA$
	$g'(x) = \cos(\ell n 2 - \sin x). \cos x$
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 $g'(0) = \cos(\ell n 2)$ 

24. Ans. (2)

Sol. 
$$x \rightarrow \frac{\pi}{2} + t$$
  

$$\lim_{t \rightarrow 0} \frac{-\tan t + \sin t}{(2t)^3}$$

$$\lim_{t \rightarrow 0} \frac{-\sin t(1 - \cos t)}{-8t^3 \cos t} = \frac{1}{16}$$

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25. Sol. (1)  $\lim_{x\to 0^+} x\left(\left\lceil \frac{1}{x}\right\rceil + \left\lceil \frac{2}{x}\right\rceil + \dots + \left\lceil \frac{15}{x}\right\rceil\right)\right)$  $\lim_{x \to 0^+} \left( x \left[ \frac{1}{x} \right] + x \left[ \frac{2}{x} \right] + \dots + x \left[ \frac{15}{x} \right] \right)$ = 1 + 2 + 3 + .....+ 15  $\frac{15}{2}(15+1) = 120$ 26. Sol. (3)  $f(x) = |x - \pi|$ .  $(e^{|x|} - 1) \sin|x|$ According to given options we have to check only at x = 0 and  $\pi$ at x = 0, f(0) = 0 $\lim \frac{f(0-h)-f(0)}{f(0-h)-f(0)}$ -h LHD =  $h \rightarrow 0^{-1}$  $\lim \frac{(\pi+h).(e^h-1)\sinh}{h}$ \_h  $= \operatorname{RHD}_{h \to 0^{-}} \lim_{h \to 0^{-}} \frac{(\pi - h)(e^{h} - 1)\sinh}{h}$ = 0 \Rightarrow e<sup>lin</sup> h→0<sup>-</sup>  $= 0 \Rightarrow$  diff. at x = 0 Now at  $x = \pi$  $f(\pi) = 0$  $\lim_{h \to 0^{-}} \frac{f(\pi - h) - f(\pi)}{-h} = \lim_{h \to 0^{-}} \frac{k.(e^{\pi - h} - 1).\sinh}{h}$  $LHD = h \rightarrow 0$ = 0  $\lim_{n \to \infty} \frac{-h(e^{\pi + h} - 1)}{\sin h}$  $\lim \frac{f(\pi+h)-f(\pi)}{2}$ h =  $h \rightarrow 0$ h  $\mathsf{RHD} = {}^{\mathsf{h} \to \mathsf{0}}$ = 0 differential at  $x = \pi$  also, hence answer is (3)

## PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Sol.  

$$\begin{aligned}
& \lim_{x \to 0} \frac{\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 1\right] \left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots\right)\right]}{x^n} \\
& \lim_{x \to 0} \frac{x^3 \left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots\right) \left(-1 - x \left(\frac{1}{2!} + \frac{1}{2!}\right) - \dots\right)}{x^n} \\
& \lim_{x \to 0} \frac{\left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots\right) \left(-1 - x \left(\frac{1}{2!} + \frac{1}{2!}\right) - \dots\right)}{x^{n-3}}
\end{aligned}$$

is finite and non-zero if n = 3

## 2. Solution

$$\begin{cases} -\frac{1}{2}(x+1) , & x < -1 \\ \tan^{-1}x , & -1 \le x \le 1 \\ \frac{1}{2}(x-1) , & x > 1 \\ at & x = -1 \text{ LHL} = 0 \\ RHL = -\frac{\pi}{4} & \text{so LHL} \neq RHL \\ \text{therefore function is discontinuous at } x = -1 \\ \text{therefore function is not differentiable at } x = -1 \\ \pi \end{cases}$$

At x = 1, LHL = 
$$\frac{4}{4}$$
  
RHL = 0  
therefore LHL  $\neq$  RHL so function

therefore function is not differentiable at x = 1 f(x) is discontinuous at x = -1 and x = 1.

3. Solution 
$$\lim_{x \to 0} \left( \frac{f(1+x)}{f(1)} \right)^{1/x} (1_{\infty} \text{ form}) (1_{\infty} : \mathbf{i}) = e^{\lim_{x \to 0} \frac{1}{x} \left( \frac{f(1+x)-f(1)}{f(1)} \right)} = e^{\frac{f'(1)}{f(1)}} = e_{6/3} = e_2.$$

= 0

is discontinuous at x = 1

**ISol.** 
$$\lim_{x \to 0} \frac{i((a-n)nx - tanx)sinnx}{x^2}$$

$$\lim_{x \to 0} \left( \frac{\sin nx}{nx} \right) (n) \left[ (a-n) \quad n - \frac{\tan x}{x} \right] = 0$$

$$\Rightarrow \qquad (1) (n) \left[ (a-n) (n) - 1 \right] = 0$$

$$\Rightarrow \qquad n(a-n) - 1 = 0 \qquad [\because n \neq 0]$$

$$\Rightarrow \qquad a - n = \frac{1}{n}$$

4.

1 a = n + n ⇒ 5. Solution  $f(x^2)-f(x)$ 0  $\lim_{x \to 0} \overline{f(x) - f(0)}$ . Put x = 0 and we get  $\overline{0}$  form. Also because 'f' is strictly increasing and differentiable. Apply L-Hospital rule, weget  $\lim_{x \to 0} \frac{2xf'(x^2) - f'(x)}{f'(x)} = -1.$  Since 'f' is strictly increasing f'(x) ≠0 in an internal 6. Sol. for x > 0x > 0 ds fy,  $\lim_{x\to 0} (\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}$  $\lim_{x \to 0} (\sin x)^{\frac{1}{x}} + \lim_{x \to 0} \left(\frac{1}{x}\right)^{\sin x} = 0 + \lim_{x \to 0} \left(\frac{1}{x}\right)^{\sin x} = e^{\lim_{x \to 0} (\sin x) \cdot \ln\left(\frac{1}{x}\right)} = e^{\lim_{x \to 0} \frac{\ln(x)}{-\csc x}} \left(\frac{\infty}{\infty} \text{ form}\right)$  $= e^{\lim_{x \to 0} \frac{1}{x}} = e^{\lim_{x \to 0} \frac{\sin^2 x}{x \cos x}} = e^0 = 1$ Clearly 'p' =  $\lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h} = \frac{|h| - 0}{-h} = -1$ 7. Sol.  $\lim_{x \to 1^{n}} \frac{(x-1)^{n}}{\log \cos^{m}(x-1)} = -1$ Now  $\lim_{x \to 1^{n}} \frac{(x-1)^{n}}{m \frac{\log [1 + \cos(x-1) - 1]}{\cos(x-1) - 1} \times [\cos(x-1) - 1]} = -1$  $\lim_{x \to T} \frac{(x-1)^{n}}{m} \frac{1}{-2\sin^{2}\frac{(x-1)}{2}}$ or. Hence n = 2, m = 2. (C) is correct answer. Aliter : LHD of |x - 1| at 1 is  $-1 \equiv p$  $\lim_{x \to 1^{+}} g(x) = \lim_{t \to 0^{-}} \frac{t^{n}}{\log \cos^{m} t} = \frac{1}{m} \lim_{t \to 0^{-}} \frac{\cos t - 1}{\tan t} \frac{t^{2}}{\cos t - 1}$  $= - \frac{2}{m} \lim_{t \to 0^{-}} t_{n-2} \qquad (\because \lim_{t \to 0} \frac{1 - \cos t}{t^2} = \frac{1}{2})$  $\therefore \qquad -\frac{2}{m}\lim_{t\to 0^-}t_{n-2}=-1 \quad \Rightarrow \qquad n=2, m=2$ 8. Sol. Ans  $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)$  $\lim \left( \frac{x^2(1-a) + x(1-a-b) + (1-b)}{x+1} \right) - 4$ Limit is finite It exists when  $1 - a = 0 \implies a = 1$ 

then  

$$\lim_{x \to \infty} \left( \frac{1 - a - b + \frac{1 - b}{x}}{1 + \frac{1}{x}} \right) = 4$$

$$\therefore \quad 1 - a - b = 4 \quad \Rightarrow \qquad b = -4$$

## **9. Sol. Ans (B)** (I) for derivability at x = 0

L.H.D. = f'(0\_-) = 
$$\lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0^-} \frac{h^2 \cdot \left| \cos\left(-\frac{\pi}{h}\right) \right| - 0}{-h} = \lim_{h \to 0^-} -h \cdot \left| \cos\frac{\pi}{h} \right| = 0$$
  
RHD f'(0\_+) = 
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 \cdot \left| \cos\left(\frac{\pi}{h}\right) \right| - 0}{h} = 0$$
  
So f(x) is derivable at x = 0  
(ii) check for derivability at x = 2

$$RHD = f'(2_{*}) = \lim_{h \to 0^{+}} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^{+}} \frac{(2+h)^{2}}{h} \left| \frac{\cos\left(\frac{\pi}{2+h}\right)}{h} \right| - 0}{h} = \lim_{h \to 0^{+}} \frac{(2+h)^{2} \cdot \cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h \to 0^{+}} \frac{(2+h)^{2} \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2+h}\right)}{h} = \lim_{h \to 0^{+}} \frac{(2+h)^{2} \cdot \sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi}{2(2+h)}\right)h} \cdot \frac{\pi}{2(2+h)} = (2)_{2} \cdot \frac{\pi}{2(2)} = \pi$$

$$LHD = \lim_{h \to 0^{+}} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0^{+}} \frac{(2-h)^{2}}{-h} \left| \frac{\cos\left(\frac{\pi}{2-h}\right)}{-h} \right| - 0}{-h}$$

$$= \lim_{h \to 0^{+}} \frac{(2-h)^{2} \cdot \left(-\cos\left(\frac{\pi}{2-h}\right)\right) - 0}{-h} = \lim_{h \to 0^{+}} \frac{e_{1}}{h} \frac{(2-h)^{2} \cos\left(\frac{\pi}{2-h}\right)}{h}$$

$$= \lim_{h \to 0^{+}} \frac{e_{1}}{h} \frac{(2-h)^{2} \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2-h}\right)}{h} = \lim_{h \to 0^{+}} \frac{e_{1}}{h} \frac{(2-h)^{2} \cdot \sin\left(-\frac{\pi h}{2(2-h)}\right)}{h} \cdot \frac{-\pi}{2(2-h)}$$

So f(x) is not derivable at x = 2