Fundamental of Mathematics

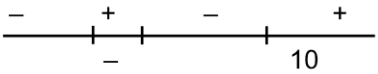
Exercise-1

PART - I : OBJECTIVE QUESTIONS

1. **Sol.** $x^2 \le 4$ $\Rightarrow$ $x \in [-2, 2]$
$x \ge -4$ $\quad$ $x = \{-2, -1, 0, 1, 2\}$

2. **Sol.** $\frac{x^2 - 4x + 3}{2x + 1} - 3 < 0$

$\frac{x^2 - 10x}{2x + 1} < 0$

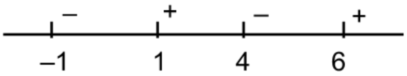


$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

3. **Sol.** $\frac{14x(x-4) - (9x-30)(x+1)}{(x+1)(x-4)} < 0$

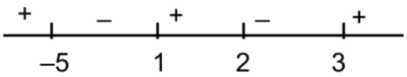
$\frac{14x^2 - 56x - (9x^2 - 21x - 30)}{(x+1)(x-4)} < 0$

$\frac{(x-1)(x-6)}{(x+1)(x-4)} < 0$



$\Rightarrow x \in \{0,5\}$

4. **Sol.** $\frac{7 + 9(x-2) + (x-2)(x-3)}{(x-2)(x-3)} < 0$ $\Rightarrow \frac{x^2 + 4x - 5}{(x-2)(x-3)} < 0$ $\Rightarrow \frac{(x-1)(x-5)}{(x-2)(x-3)} < 0$



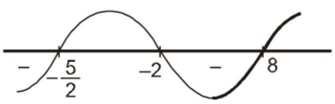
No positive Integers

5. **Sol.** $-5 \le x \le 10$ means $x = -5, -4, -3, ...9, 10$
and $0 \le x \le 15$ means $x = 0,1, 2, ...9, 10, 11, 12, ...15$
Required integers values of x are 0,1, 2, 3, 4,........9 $\Rightarrow$ Number of integers values of x = 10

6. **Sol.** $\frac{x^2 - 1}{2x + 5} - 3 < 0$ $\Rightarrow$ $\frac{x^2 - 1 - 6x - 15}{2x + 5} < 0$ $\Rightarrow$ $\frac{x^2 - 6x - 16}{2x + 5} < 0$ $\Rightarrow$

$\frac{x^2 - 8x + 2x - 16}{\left(x + \frac{5}{2}\right)} < 0$

$\Rightarrow \frac{x(x-8) + 2(x-8)}{x + \frac{5}{2}} < 0$ $\Rightarrow \frac{(x-8)(x-2)}{x + \frac{5}{2}} < 0$



$x \in (-\infty, -5/2) \cup (-2, 8)$

7. **Sol.** $5x + 2 < 3x + 8$ $\Rightarrow$ $2x < 6$ $\Rightarrow$ $x < 3$ ...(i)

$\frac{x+2}{x-1} < 4 \quad \Rightarrow \quad \frac{x+2}{x-1} - 4 < 0 \quad \Rightarrow \quad \frac{-3x+6}{x-1} < 0$

$\Rightarrow \quad \frac{x-2}{x-1} > 0 \quad \Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ ...(ii)

Taking intersection of (i) and (ii) $\quad x \in (-\infty, -1) \cup (2, 3)$

8. **Sol.** $x^2 + 9 < (x + 3)^2 < 8x + 25$

$\Rightarrow \quad (x + 3)^2 > x^2 + 9 \Rightarrow \quad x > 0$ ....(i)

and $\quad (x + 3)^2 < 8x + 25 \quad \Rightarrow \quad x^2 - 2x - 16 < 0$

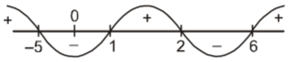
$\Rightarrow \quad x \in (1 - \sqrt{17}, 1 + \sqrt{17})$ .....(ii)

(i) $\cap$ (ii) $\Rightarrow x \in (0, 1 + \sqrt{17})$

Noumber of integers = 5

9. **Sol.** $\frac{x^2(x^2-3x+2)}{x^2-x-30} \geq 0 \Rightarrow \frac{x^2(x-1)(x-2)}{(x+5)(x-6)} \geq 0$

$x \neq -5, 6$



$x \in (-\infty, -5) \cup [1, 2] \cup (6, \infty) \cup \{0\}$

10. **Sol.** $x \in (-\infty, -9) \cup (-9, -3) \cup [-1, 0) \cup (0, 2) [4, 6)$

so +ve integral solution

11. **Sol.** $\frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^3+1} \geq 0$

$\frac{2x+2-x^2+x-1}{x^3+1} - \frac{(2x-1)}{x^2+1} \geq 0 = \frac{3x+1-x^2-2x+1}{x^3+1} \geq 0 = \frac{-x^2+x+2}{x^3+1} \geq 0 = \frac{x^2-x-2}{x^3+1} \leq 0$

$= \frac{(x+1)(x-2)}{(x+1)(x^2-x+1)} \leq 0 \Rightarrow \frac{(x-2)}{(x^2-x+1)} \leq 0$

required value of x, {0, 1, 2}

12. **Sol.** $a^4b^5 = 1 \Rightarrow$ log w.r.t. a $\rightarrow 4 + 5\log_a b = \log_a 1 \Rightarrow \quad \log_a b = -\frac{4}{5}$

Now $\log_a(a^5b^4) = 5 + 4\log_a b = 5 + 4\left(-\frac{4}{5}\right) = \frac{25-16}{5} = \frac{9}{5}$

13. **Sol.** $\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c} = \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$

$= \log_{abc} b + \log_{abc} c + \log_{abc} a = \log_{abc} abc = 1$

14. **Sol.** $\frac{1}{\log_{\sqrt{bc}} abc} + \frac{1}{\log_{\sqrt{ca}} abc} + \frac{1}{\log_{\sqrt{ab}} abc}$

$= \log_{abc}\sqrt{bc} + \log_{abc}\sqrt{ca} + \log_{abc}\sqrt{ab} = \log_{abc} abc = 1$

15. **Sol.** $\log_2 (5 \times 2).\log_2 (2^4 \times 5) - (\log_2 5)\log_2 (2^5 \times 5)$

$= (\log_2 5 + 1)(4 + \log_2 5) - \log_2 5(5 + \log_2 5)$, let $\log_2 5 = t$
$= (t + 1)(4 + t) - t(5 + t) = t^2 + 5t + 4 - 5t - t^2 = 4 = 4\log_2 2 = \log_2 16$

16. **Sol.** $y = \dfrac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$ $\Rightarrow$ $2^{\log_{2^{1/4}} a} = 2^{4\log_2 a} = a^4$

$3^{\log_{27}(a^2+1)^3} = 3^{\log_3(a^2+1)} = a^2 + 1$ $\Rightarrow$ $7^{4\log_{49} a} = 7^{2\log_7 a} = a^2$

$\therefore$ $y = \dfrac{a^4 - (a^2 + 1 + 2a)}{a^2 - a - 1} = \dfrac{a^4 - (a+1)^2}{a^2 - a - 1} = a^2 + a + 1$

17. **Sol.** $x = 2^{\log 3}$, $y = 3^{\log 2} = 2^{\log 3} = x$

18. **Sol.** $\log_a(ab) = x$ $\Rightarrow$ $1 + \log_a b = x$ $\Rightarrow$ $\log_a b = x - 1$ $\Rightarrow$ $\log_b a = \dfrac{1}{x-1}$

Now $\log_b (ab) = 1 + \log_b a = 1 + \dfrac{1}{x-1} = \dfrac{x-1+1}{x-1} = \dfrac{x}{x-1}$

19. **Sol.** $\log_{10}\pi$ is quantity lie between 0 to 1.

20. **Sol.** $\log_p \log_p (p)^{\frac{1}{p^n}} = \log_p\left(\dfrac{1}{p}\right)^n = -\log_p p^n = -n$ independent of p.

21. **Sol.** $\log_{10} (\log_2 3) + \log_{10} (\log_3 4) + \log_{10} (\log_4 5) + \dots\dots + \log_{10}(\log_{1023} 1024) = \log_{10} (\log_2 1024)$
$= \log_{10}(\log_2 2^{10}) = \log_{10}(10) = 1$

22. **Sol.** Clearly Domain is $x > 0$ and $x \neq 1$

23. **Sol.** Let $\log_2 x = t$

$\dfrac{1 - t/2}{1+t} \dfrac{-1}{2} \leq 0$

$\dfrac{1-2t}{1+t} \leq 0$ $\Rightarrow$ $\dfrac{2t-1}{t+1} \geq 0$

$\log_2 x \leq -1$ or $\log_2 x \geq \dfrac{1}{2}$ $\Rightarrow x \leq \dfrac{1}{2}$ or $x \geq \sqrt{2}$

24. **Sol.** $\log_p(\log_q(\log_r x)) = 0$ $\Rightarrow$ $\log_q(\log_r x) = b$ $\Rightarrow$ $\log_r x = q \Rightarrow$ $x = r_q$ ......(i)
and $\log_q(\log_r(\log_p x)) = 0$
$\Rightarrow$ $\log_r(\log_p x) = 1$ $\Rightarrow$ $\log_p x = r$ $\Rightarrow$ $x = p_r$ ......(ii)
from (i) and (ii) $p_r = r_q$
$\Rightarrow$ $p = r_{q/r}$

25. **Sol.** $2\log 10x - \log 10(2x - 75) = 2$ $\Rightarrow$ $\dfrac{x^2}{2x-75} = 102 = 100$
$\Rightarrow x2 - 200x + 7500 = 0 \Rightarrow x = 50, x = 150$
sum = 200

26. **Sol.** $\log_x \log_{18}(\sqrt{2}+2\sqrt{2}) = \frac{1}{3}$ ⇒ $\log_x \log_{18}(\sqrt{18}) = \frac{1}{3}$ ⇒ $\log_x \frac{1}{2} = \frac{1}{3}$

⇒ $x^{1/3} = \frac{1}{2}$ ⇒ $x = \frac{1}{8}$ ⇒ 1000 x = 125.

27. **Sol.** $3^{2\log_3 x} - 2x - 3 = 0$ ⇒ $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$ ⇒ $x = 3, x = -1$ but $x \neq -1$

∴ $x = 3$.

28. **Sol.** $\sqrt{\log_{10}(-x)} = \log_{10}|x|$ ⇒ $-x > 0$ ⇒ $\sqrt{\log_{10}(-x)}$ $x < 0$

∴ $|x| = -x$ ⇒ $= \log_{10}(-x)$

$\log_{10}(-x)\,(\log_{10}(-x) - 1) = 0$

$\log_{10}(-x) = 0$ $\log_{10}(-x) = 1$

⇒ $-x = 1$ $-x = 10$

$x = -1$ $x = -10$.

29. **Sol.** $\log_{\sin\frac{\pi}{3}}(x^2 - 3x + 2) \geq 2$ ⇒ $x^2 - 3x + 2 \leq \left(\frac{\sqrt{3}}{2}\right)^2$ ⇒ $4x^2 - 12x + 8 \leq 3$

⇒ $4x^2 - 12x + 5 \leq 0$ ⇒ $(2x - 5)(2x - 1) \leq 0$ ⇒ $x \in \left[\frac{1}{2}, \frac{5}{2}\right]$

But domain $x^2 - 3x + 2 > 0$ ⇒ $(x - 1)(x - 2) > 0$ ⇒ $x \in (-\infty, 1) \cup (2, \infty)$

Hence $x \in \left[\frac{1}{2}, 1\right) \cup \left(2, \frac{5}{2}\right]$

30. **Sol.** $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$; $\log_{0.3}(x - 1) < \frac{\log_{0.3}(x-1)}{2}$

⇒ $\log_{0.3}(x - 1) < 0$ ⇒ $x - 1 > 1$ ⇒ $x > 2$

31. **Sol.** $2 - \log_2(x^2 + 3x) \geq 0$ ⇒ $\log_2(x^2 + 3x) \leq 2$

$x^2 + 3x > 0$ ⇒ $x \in (-\infty, -3) \cup (0, \infty)$ .....(i)

and $x^2 + 3x \leq 4$

⇒ $(x - 1)(x + 4) \leq 0$ ⇒ $x \in [-4, 1]$ .....(ii)

(i) ∩ (ii) ⇒ $x \in [-4, -3) \cup (0, 1]$

32. **Sol.** $\log_{0.5}\log_5(x^2 - 4) > \log_{0.5} 1$; $\log_{0.5}\log_5(x^2 - 4) > 0$

⇒ $x^2 - 4 > 0$ ⇒ $x \in (-\infty, -2) \cup (2, \infty)$ ....(i)

$\log_5(x^2 - 4) > 0$ ⇒ $x^2 - 5 > 0$

⇒ $x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$ ....(ii)

$\log_5(x^2 - 4) < 1$

⇒ $x^2 - 9 < 0$ ⇒ $x \in (-3, 3)$ ....(iii)

(i) ∩ (ii) ∩ (iii) ⇒ $x \in (\sqrt{5} -3,) \cup (\sqrt{5}, 3)$

33. **Sol.** $\left(\frac{1}{2}\right)^{x^2-2x} < \left(\frac{1}{2}\right)^2$ ⇒ $x^2 - 2x > 2$

⇒ $x^2 - 2x - 2 > 0$ ⇒ $x \in (-\infty, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, \infty)$

34. Sol. $|x^2-4x+3| \Rightarrow x^2-4x+3$ $\quad (x-1)(x-3) \geq 0$

$-(x^2-4x+3)$ $\quad (x-1)(x-3)<0$

$\therefore$ $y = x^2-4x+3+x=7$ $\quad x \in (-\infty, 1] \cup [3, \infty)$

$x^2-4x+3+7-x=0$ $\quad x \in (1,3)$

35. Sol. $y =$

$-1-2x$	$-\infty < x < -3$
5	$-3 \leq x < 0$
$5-2x$	$0 \leq x < 2$
1	$2 \leq x < 3$
$2x-5$	$x \geq 3$

Solve for $y = 7x + 1 \Rightarrow x = 4/9$

36. Sol. If p is solution, then –p also will be another sol.

37. Sol. Let $|x-1| = t$; $t^2 - 3t + 2 = 0 \Rightarrow t = 1,2$

$|x-1| = 1$ $\quad |x-1| = 2$

$x = 0,2$ $\quad x = -1, 3$

38. Sol. Minimum will be at $x = 1$; $f(1) = 2$

39. Sol. $|x-4| = 5$ or $|x-4| = 3$

$x = 9, -1$ $\quad x = 7, 1$

40. Sol. $|x^3 - 9x^2 + 26x - 24| = |x-2|\,|x-3|\,|x-4|$

When $x \in I$; above 3 are consecutive positive integers, hence multiplication can never be a prime number

41. Sol. $y =$

$-2x+1$	$x \in (-\infty, -1)$
3	$x \in [-1, 2)$
$2x-1$	$x \in [2, \infty)$

$y \geq 3 \quad \forall x \in R$

42. Sol. $-1 \leq |x-2|-1 \leq 1$

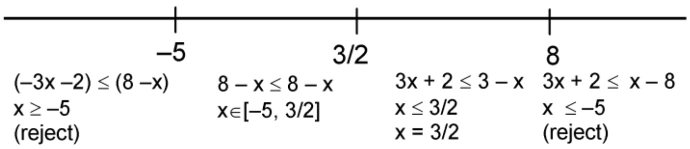
$0 \leq |x-2| \leq 2$

$|x-2| \geq 0$ and $|x-2| \leq 2$

$x \in R$ $\quad 0 \leq x \leq 4$

43. Ans. $x \in [-5, 3/2]$

Sol.



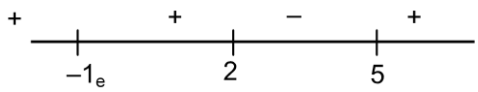
44. Ans. $x \in [2, 5] \cup \{-1\}$

Sol. $|-x^2 + 4x + 5| + |x^2 - x - 2| = |3x + 3|$

$|a| + |b| = |a + b| \Rightarrow ab \geq 0$

$(x^2 - 4x - 5)(x^2 - x - 2) \leq 0$

$(x+1)^2 (x-5)(x-2) \leq 0$



45. **Sol.**

Case- I
$\log_2 x \ge 0$
$x \ge 1$
$\log_2 x \ge 2/3$
$x \ge 2^{\frac{2}{3}}$

Case-II
$\log_2 x < 0$
$0 < x < 1$
$-\log_2 x \ge 2$
$x \le \frac{1}{4}$ $\quad x \in \left(0, \frac{1}{4}\right]$

46. **Sol.** $\frac{(x+1)(x-1)(e^x-1)}{(x-1)} \le 0$

Sign scheme: $+$ -1 $-$ 0 $+$ 1 $+$ $\quad [-1, 0]$

47. **Sol.** $\frac{6x^2-5x-3}{x^2-2x+6} - 4 \le 0 \Rightarrow \frac{2x^2+3x-27}{x^2-2x+6} \le 0$

denominator $x^2 - 2x + 6 > 0 \quad \forall \; x \in R \quad (\because D < 0)$

then $2x^2 + 3x - 27 \le 0 \Rightarrow (2x + 9)(x - 3) \le 0$

$-\frac{9}{2} \le x \le 3 \Rightarrow 0 \le x^2 \le \frac{81}{4}$

$(4x^2)_{max} = 4\left(-\frac{9}{2}\right)^2 = 81 \Rightarrow (4x^2)_{min} = 4(0) = 0$

48. **Sol.** $\frac{4\alpha}{\alpha^2+1} \ge 1 \Rightarrow \alpha^2 - 4\alpha + 1 \le 0 \Rightarrow \alpha + \frac{1}{\alpha} \le 4$

Now $\alpha + \frac{1}{\alpha} = 1$. But $\alpha + \frac{1}{\alpha} \ge 2$
so reject

and $\alpha + = 3 \Rightarrow \alpha^2 - 3\alpha + 1 = 0 \Rightarrow \alpha = \frac{3 \pm \sqrt{9-4}}{2}$

$\Rightarrow \alpha = \frac{3 \pm \sqrt{5}}{2}$
two real values

49. $y^2 = 6 + y \Rightarrow y = 3$
$\log_2 \log_9 3 = 1$

50. **Sol.** $\log_a 625 = I \Rightarrow 5^4 = a^I$
$a = 5, 25, 625$

51. **Sol.** $xy = 15$; $\frac{y-x}{xy} = \frac{2}{15} \Rightarrow y - x = 2$
$y = 5$; $x = 3$

52. **Sol.** $b = a^2$, $c = b^2$, $\frac{c}{a} = 3^3 \Rightarrow c = 27a \Rightarrow b^2 = 27a \Rightarrow a^4 = 27a$
$\Rightarrow a = 3, a > 0$
$c = 81, b = 9$
$\therefore a + b + c = 3 + 9 + 81 = 93$

Fundamental of Mathematics

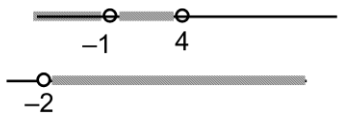
53. **Sol.** $a^{(\log_3 7)^2} = \left(a^{\log_3 7}\right)^{\log_3 7} = 27^{\log_3 7} = 27^{\log_3 7} = 7_3 = 343$

$b^{(\log_7 11)^2} = \left(b^{\log_7 11}\right)^{\log_7 11} = 49^{\log_7 11} = 11^{\log_7 49} = 121$

$c^{(\log_{11} 25)^2} = \left(c^{\log_{11} 25}\right)^{\log_{11} 25} = \left(\sqrt{11}\right)^{\log_{11} 25} = 25^{\log_{11}\sqrt{11}} = 5$

hence the sum is 343 + 121 + 5 = 469

54. **Sol.** $\frac{\ell n(x+2)}{x^2 - 3x - 4} \le 0 \Rightarrow$



55. **Sol.** $\log_{(0.2)^2}(x-1) \ge \log_{0.2}(x-1)$

$\log_{0.2}(x-1) \ge 2\log_{0.2}(x-1)$

$(x-1) \le (x-1)^2$

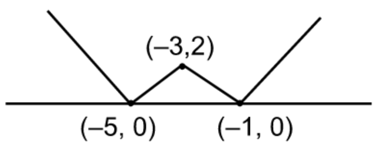
$\Rightarrow x \ge 2$

56. **Sol.** Domain $\frac{x-1}{x+2} > 0 \Rightarrow x < -2$ or $x > 1$

Since $\log_9 10 > 1 \Rightarrow \log_2\left(\frac{x-1}{x+2}\right) > 1 \Rightarrow \left(\frac{x-1}{x+2}\right) > 2$

$\Rightarrow \frac{x+5}{x+2} < 0 \Rightarrow x \in (-5, -2) \Rightarrow$ integers = {– 4,–3}

57. **Sol.**



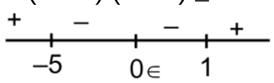
P = 2

no. of values = 1

58. **Sol.** $|a| + |b| = |a + b| \Rightarrow a.b. \ge 0$

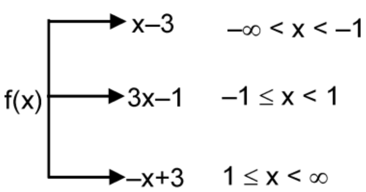
$x(x + 5)(x)(1 - x) \ge 0$

$x^2(x + 5)(x - 1) \le 0$



$x \in [-5,1] \Rightarrow \{-5, -4, -3, -2, -1, 0, 1\}$

59. **Sol.**



max at $x = 1$ $f(1) = 2 - 2|1 - 1| = 2$

60. **Sol.** $x = 2 + |y|$

$|x + 1| + y = 5$

$|2 + |y| + 1| + y = 5 \Rightarrow 3 + |y| + y = 5$

$y + |y| = 2$

$y > 0 \Rightarrow y = 1$; $x = 3$

PART - II : MISCELLANEOUS QUESTIONS

Section (A) : ASSERTION/REASONING

A-1 **Ans. (1)**

Sol. Since $0 < \sqrt{13} - \sqrt{12} < 1$ $\therefore$ $\log_{10} (\sqrt{13} - \sqrt{12}) < 0$

Since $0 < \sqrt{14} - \sqrt{13} < 1$ $\therefore$ $\log_{0.1} (\sqrt{14} - \sqrt{13}) > 0$

A-2 **Ans. (1)**

Sol. Statement-1 : $y = \log_{1/3}(x^2 - 4x + 5)$ is max.
when $x^2 - 4x + 5$ is min.

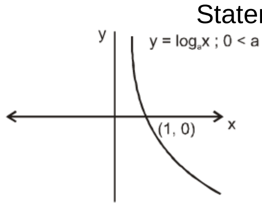
Let $f(x) = x^2 - 4x + 5$

$\Rightarrow$ $(x - 2)^2 + 1$

$f(x)_{min} = 1$

$y_{max} = \log_{1/3}1 = 0$

Statement-1 is true



Statement-2 : $\log_a x \leq 0$ for $x \geq 1$, $0 < a < 1$

$\therefore$ Statement-2 is true and
correct explanation for statement-1

A-3. **Ans. (1)**

Sol. If a & b are of same sign then $|a + b| = |a| + |b|$ $ab \geq 0$

$\therefore$ $(x - 2)(x - 7) \geq 0$ $\Rightarrow$ $x \leq 2$ or $x \geq 7$

$\because$ $2x - 9 = (x - 2) + (x - 7)$

Section (B) : MATCH THE COLUMN

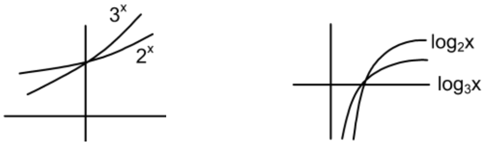
B-1. **Ans.** (A) → (p), (B) → (s), (C) → (r), (D) → (q)

Section (C) : ONE OR MORE THAN ONE OPTIONS CORRECT

C-1. **Sol.** $N = \dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3} = (\log_3 27 + \log_3 5)(\log_3 15) - \log_3 5.\log_3 405$

$= (3 + \log_3 5)(1 + \log_3 5) - \log_3 5 \log_3(81 \times 5) = (3 + \log_3 5)(1 + \log_3 5) - \log_3 5(4 + \log_3 5) = 3$

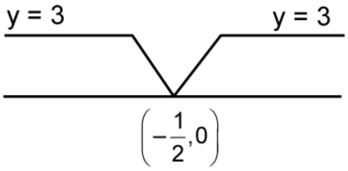
C-2. **Sol.**



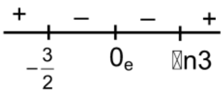
C-3. **Sol.** $-\infty < x < -2 \Rightarrow f(x) = 3$

$-2 \le x < 1 \quad f(x) = |2x + 1|$

$1 \le x < \infty \quad f(x) = 3$



C-4. **Sol.**



$x \in \left(-\infty, -\frac{3}{2}\right] \cup (\ell n3, \infty) \cup \{0\}$