Understanding-Physics
For-Jee-Main-And-Advanced

Mehanics

Part-2

Contents

Syllabus *of* JEE Mains

Rotational Motion

Basic concepts of rotational motion; Moment of a force, Torque, Angular momentum, conservation of angular momentum and its applications; Moment of inertia, Radius of gyration. Values of moments of inertia for simple geometrical objects, Parallel and perpendicular axes theorems and their applications. Rigid body rotation, Equations of rotational motion.

Gravitation

The universal law of gravitation. Acceleration due to gravity and its variation with altitude and depth. Kepler's laws of planetary motion. Gravitational potential energy; Gravitational potential, Escape velocity. Orbital, velocity of a satellite, Geo-stationary satellites.

Properties of Solids and Liquids

Elastic behaviour, Stress-strain relationship, Hooke's Law, Young's modulus, Bulk modulus, modulus of rigidity. Pressure due to a fluid column; Pascal's law and its applications. Viscosity, Stokes' law, Terminal velocity, Streamline and turbulent flow, Reynolds number. Bernoulli's principle and its applications. Surface energy and surface tension, Angle of contact, Application of surface tension – drops, bubbles and capillary rise.

Oscillations

Periodic motion – period, frequency, displacement as a function of time. Periodic functions. Simple harmonic motion (S.H.M.) and its equation; Phase; Oscillations of a spring -restoring force and force constant; energy in S.H.M. – Kinetic and potential energies; Simple pendulum – derivation of expression for its time period; Free, forced and damped oscillations, resonance.

Experimental Skills

- Simple Pendulum-dissipation of energy by plotting a graph between square of amplitude and time.
- Metre Scale – mass of a given object by principle of moments.
- Young's modulus of elasticity of the material of a metallic wire.
- Surface tension of water by capillary rise and effect of detergents.
- Co-efficient of Viscosity of a given viscous liquid by measuring terminal velocity of a given spherical body.

JEE *Advanced*

General

Determination of g using simple pendulum. Young's modulus by Searle's method.

Gravitation

Law of gravitation, Gravitational potential and field, Acceleration due to gravity, Motion of planets and satellites in circular orbits, Escape velocity.

Rotational Motion

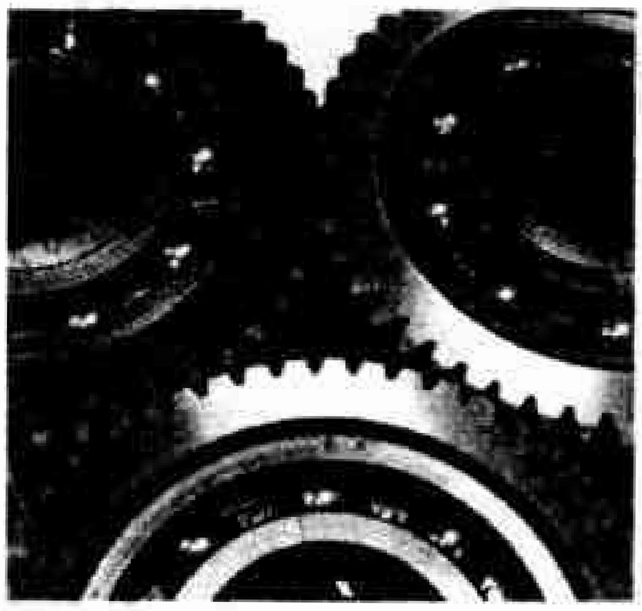
Rigid body, moment of inertia, Parallel and perpendicular axes theorems, Moment of inertia of uniform bodies with simple geometrical shapes, Angular momentum, Torque, Conservation of angular momentum, Dynamics of rigid bodies with fixed axis of rotation, Rolling without slipping of rings, cylinders and spheres, Equilibrium of rigid bodies, Collision of point masses with rigid bodies.

Oscillations

Linear and angular simple harmonic motions.

Properties of solids and liquids

Hooke's law, Young's modulus. Pressure in a fluid, Pascal's law, Buoyancy, Surface energy and surface tension, capillary rise, Viscosity (Poiseuille's equation excluded), Stoke's law, Terminal velocity, Streamline flow, Equation of continuity, Bernoulli's theorem and its applications.



9 MECHANICS of ROTATIONAL MOTION

Chapter Contents

9.1 Moment of Inertia

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating . The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion. If a body is at rest, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult it is to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis) and it depends on the mass as well as its distribution about that axis.

Moment of Inertia of a Single Particle

For a very simple case the moment of inertia of a single particle about an axis is given by,

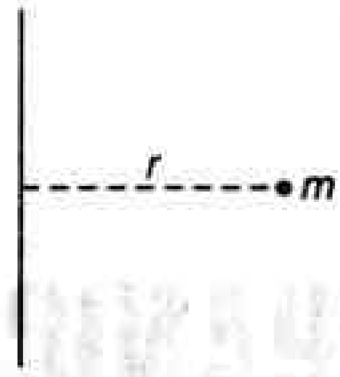


Fig. 9.1

$$I = mr^2 \qquad \ldots(i)$$

Here, m is the mass of the particle and r its distance from the axis under consideration.

Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by,

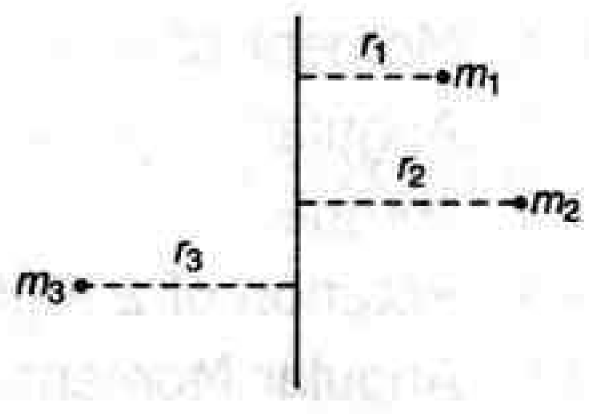


Fig. 9.2

$$I = \sum_i m_i r_i^2 \qquad \ldots(ii)$$

where r_i is the perpendicular distance from the axis to the ith particle, which has a mass m_i.

Moment of Inertia of Rigid Bodies

For a continuous mass distribution such as found in a rigid body, we replace the summation of Eq. (ii) by an integral. If the system is divided into infinitesimal elements of mass dm and if r is the distance from a mass element to the axis of rotation, the moment of inertia is,

$$I = \int r^2 \, dm$$

where the integral is taken over the system.

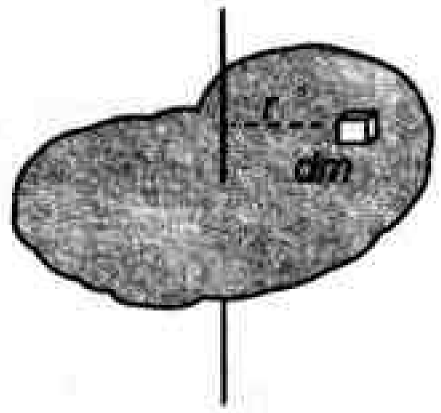


Fig. 9.3

Moment of Inertia of a Uniform Cylinder

Let us find the moment of inertia of a uniform cylinder about an axis through its centre of mass and perpendicular to its base. Mass of the cylinder is M and radius is R.

We first divide the cylinder into annular shells of width dr and length l as shown in figure. The moment of inertia of one of these shells is

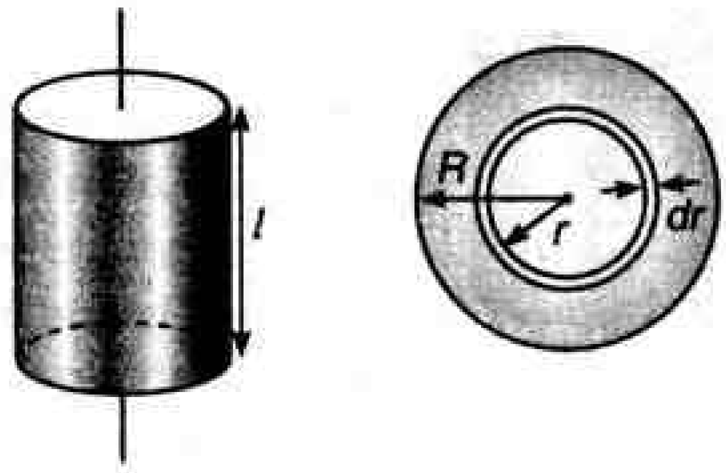


Fig. 9.4

$$dI = r^2 dm = r^2 (\rho \cdot dV)$$

Here, ρ = density of cylinder

and dV = volume of shell = $2\pi rl\, dr$

$\therefore$ $dI = 2\pi\rho l\, r^3\, dr$

The cylinder's moment of inertia is found by integrating this expression between 0 and R,

So, $$I = 2\pi\rho l \int_0^R r^3\, dr = \frac{\pi\rho l}{2} R^4 \quad \text{...(iii)}$$

The density ρ of the cylinder is the mass divided by the volume.

$\therefore$ $$\rho = \frac{M}{\pi R^2 l} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we have

$$I = \frac{1}{2} MR^2$$

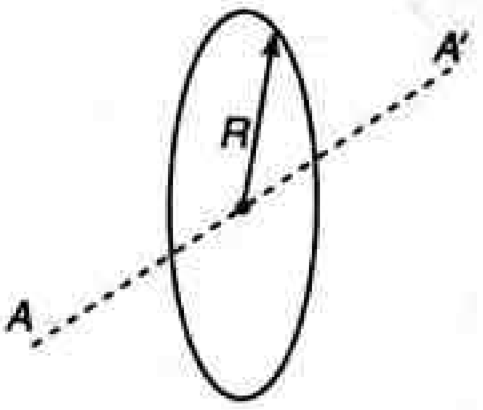
Proceeding in the similar manner we can find the moment of inertia of certain rigid bodies about some given axis. Moments of inertia of several rigid bodies with symmetry are listed in Table. 9.1.

In all cases except (f) the rotational axis AA' passes through the centre of mass.

Table 9.1

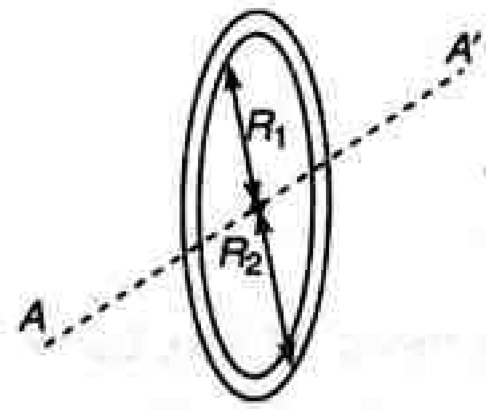
(a) Very thin circular hoop

$$I = MR^2$$



(b) Uniform circular hoop

$$I = M\frac{R_1^2 + R_2^2}{2}$$

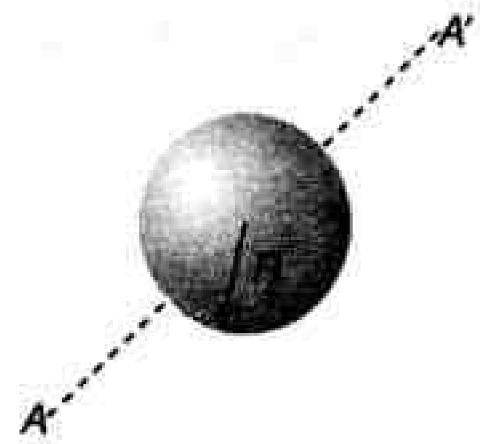


(c) Uniform solid cylinder

$$I = \frac{1}{2}MR^2$$

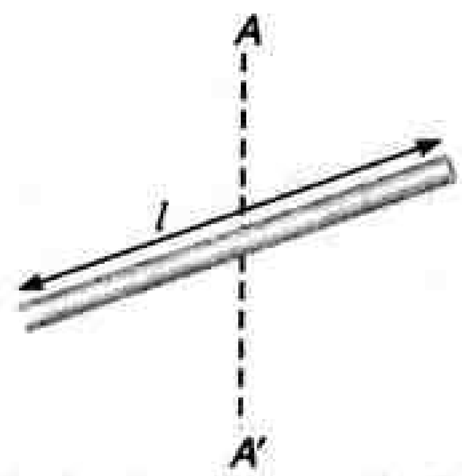
(d) Uniform solid sphere

$$I = \frac{2}{5}MR^2$$



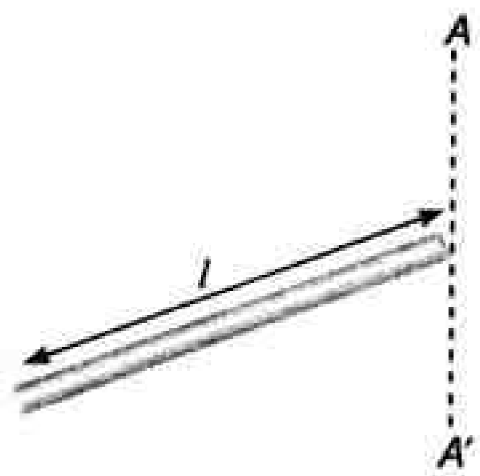
(e) Uniform thin rod

$$I = \frac{1}{12}Ml^2$$



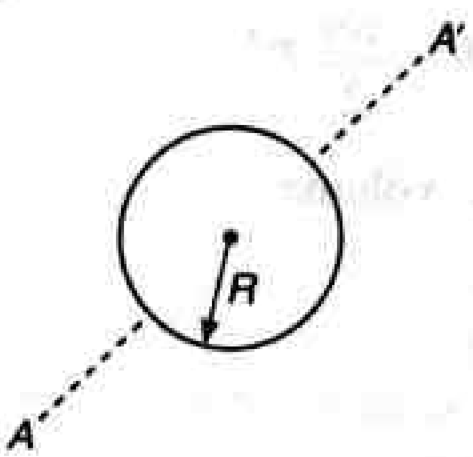
(f) Uniform thin rod

$$I = \frac{1}{3}Ml^2$$



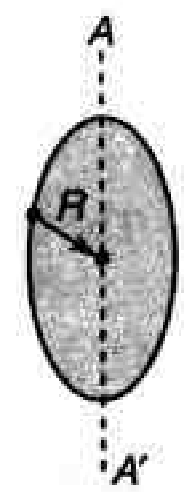
(g) Very thin spherical shell

$$I = \frac{2}{3}MR^2$$



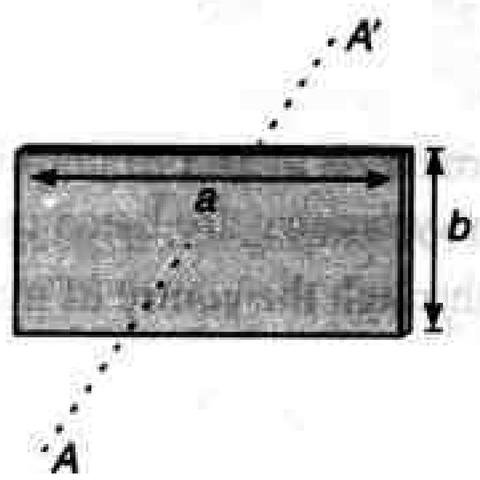
(h) Thin circular sheet

$$I = \frac{1}{4}MR^2$$



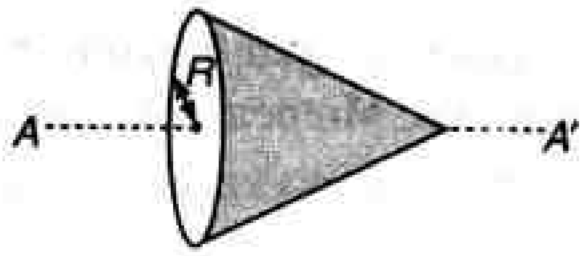
(i) Thin rectangular sheet

$$I = M\frac{a^2 + b^2}{12}$$



(j) Uniform right cone

$$I = \frac{3}{10}MR^2$$



Theorems on Moment of Inertia

There are two important theorems on moment of inertia, which, in some cases, enable the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

(i) Theorem of parallel axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.

Two such axes are shown in figure for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them, these moments are related by,

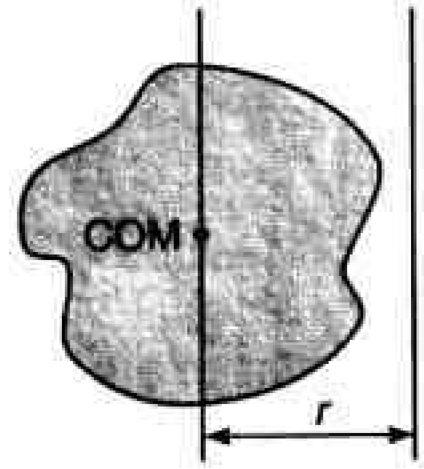


Fig. 9.5

$$I = I_{COM} + Mr^2$$

We now present a proof of the above theorem.

Proof : A cross-section view of a rigid body is shown in Fig. 9.6. The body is oriented so that the two axes, one through COM (the centre of mass) and the other through A, are perpendicular to the plane of the pages. The coordinates of these two points are $(x_{COM}, y_{COM}, 0)$ and $(x_{COM} + a, y_{COM} + b, 0)$ respectively. The distance between the two axes is

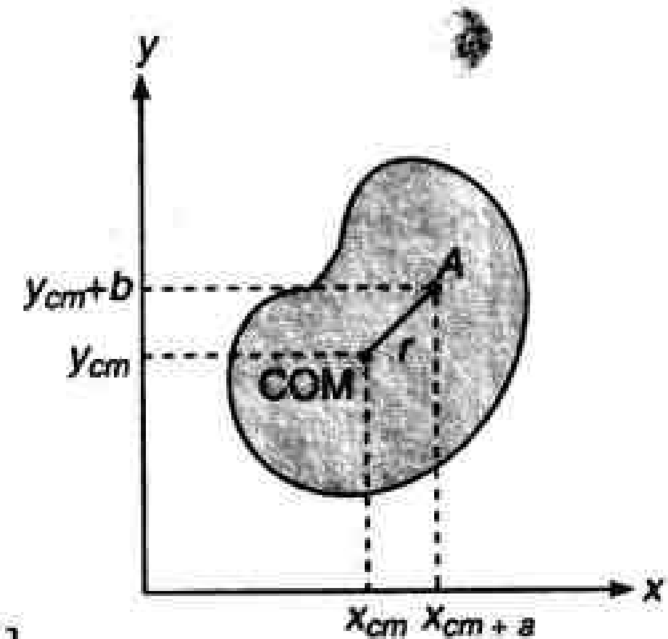


Fig. 9.6

$$r^2 = a^2 + b^2$$

The moment of inertia around the axis through CM is

$$I_{COM} = \sum_i m_i [(x_i - x_{COM})^2 + (y_i - y_{COM})^2]$$

and the moment of inertia around the parallel axis through A is,

$$I_A = \sum_i m_i [(x_i - x_{COM} - a)^2 + (y_i - y_{COM} - b)^2]$$

Which after some rearrangement, can be written as

$$I_A = \sum_i m_i [(x_i - x_{COM})^2 + (y_i - y_{COM})^2]$$

$$- 2a \sum_i m_i (x_i - x_{COM}) - 2b \sum_i m_i (y_i - y_{COM}) + (a^2 + b^2) \sum_i m_i$$

The first of these terms is I_{COM}. The second and third terms are zero from the definition of the centre of mass.

$$\sum_i m_i x_i = x_{COM} \sum_i m_i$$

and

$$\sum_i m_i y_i = y_{COM} \sum_i m_i$$

Since, $r^2 = a^2 + b^2$ and $M = \sum_i m_i$,

The fourth term is Mr^2. Thus,

$$I_A = I_{cm} + 0 + 0 + Mr^2 \quad \text{or} \quad I_A = I_{cm} + Mr^2$$

which is the parallel axis theorem.

Note *From the above theorem we can see that among too many parallel axes moment of inertia is least about an axis which passes through centre of mass. e.g.,* I_2 *is least among* I_1, I_2 *and* I_3. *Similarly,* I_5 *is least among* I_4, I_5 *and* I_6.

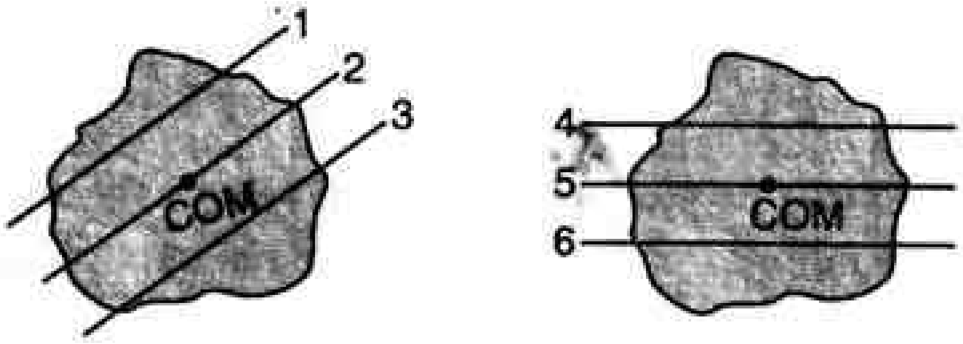


Fig. 9.7

(ii) Theorem of perpendicular axes

This theorem is applicable only to the plane bodies (two dimensional). The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it. Let x and y axes be chosen in the plane of the body and z-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that

$$I_z = I_x + I_y$$

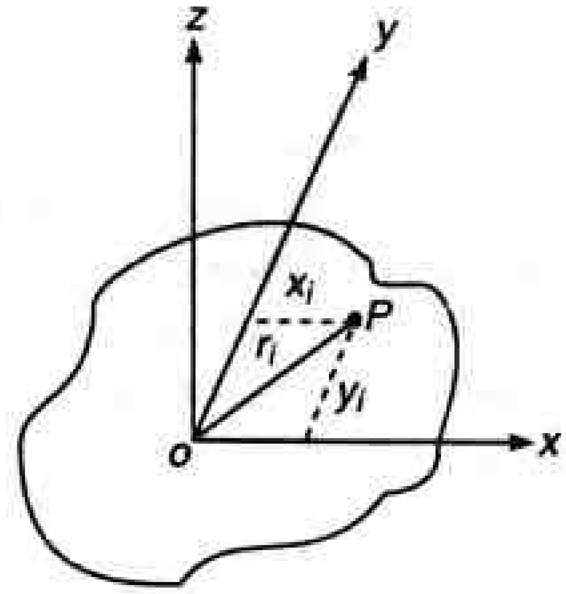


Fig. 9.8

Proof : Consider an arbitrary particle P of mass m_i, distant r_i from o and x_i and y_i are the perpendicular distances of point P from the axes oy and ox respectively, we have

$$I_z = \sum_i m_i r_i^2, \quad I_x = \sum_i m_i y_i^2 \quad \text{and} \quad I_y = \sum_i m_i x_i^2$$

So that,

$$I_x + I_y = \sum_i m_i y_i^2 + \sum_i m_i x_i^2$$

$$= \sum_i m_i (y_i^2 + x_i^2)$$

$$= \sum_i m_i r_i^2$$

$$= I_z$$

i.e.,

$$I_z = I_x + I_y$$

Radius of Gyration

Radius of gyration (K) of a body about an axis is the effective distance from this axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$I = MK^2$$

or

$$K = \sqrt{\frac{I}{M}}$$

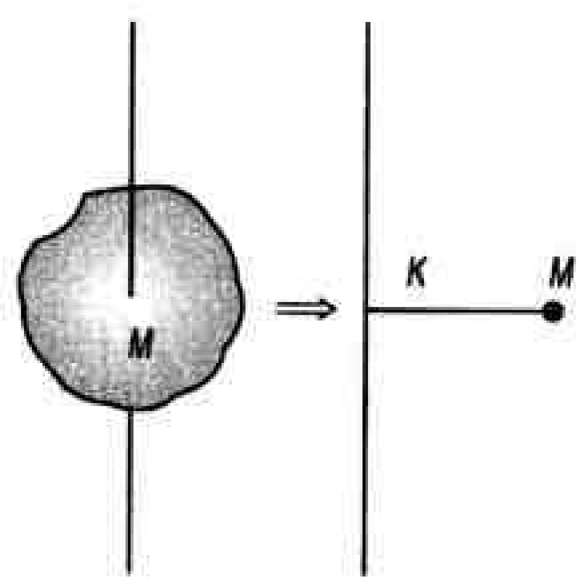


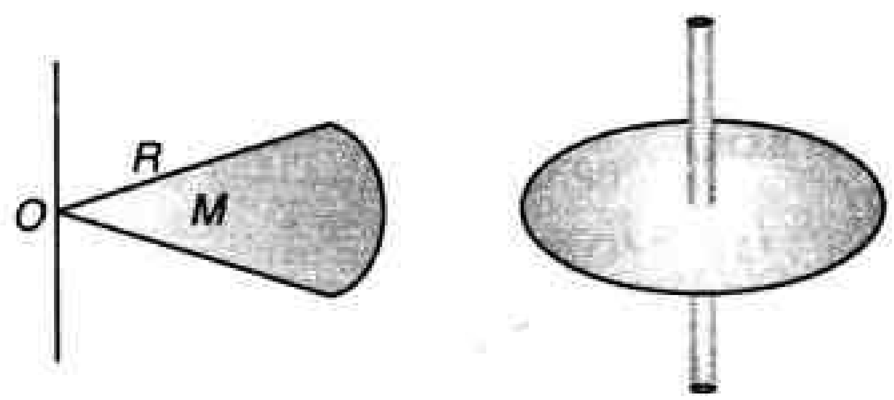
Fig. 9.9

e.g., radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

$$K = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

Important points in Moment of Inertia

- Theorem of parallel axes is applicable for any type of rigid body whether it is a two dimensional or three dimensional, while the theorem of perpendicular axes is applicable for laminar type or two dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes (x, y and z) may be any point on the plane of body (it may even lie outside the body). This point may or may not be the centre of mass of the body.
- Moment of inertia of a part of a rigid body (symmetrically cut from the whole mass) is the same as that of the whole body. *e.g.*, in figure (a) moment of inertia of the section shown (a part of a circular disc) about an axis perpendicular to its plane and passing through point O is $\frac{1}{2}MR^2$ as the moment of inertia of the complete disc is also $\frac{1}{2}MR^2$. This can be shown as in Fig. 9.10.



(a) (b)

Fig. 9.10

Suppose the given section is $\frac{1}{n}$ th part of the disc, then mass of the disc will be nM.

$$I_{\text{disc}} = \frac{1}{2}(nM)R^2$$

$$\therefore \quad I_{\text{section}} = \frac{1}{n} I_{\text{disc}} = \frac{1}{2}MR^2$$

Sample Example 9.1 *Three rods each of mass m and length l are joined together to form an equilateral triangle as shown in figure. Find the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of the triangle.*

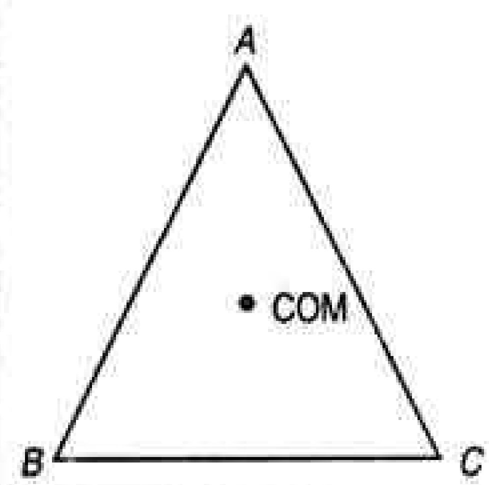


Fig. 9.11

Solution Moment of inertia of rod BC about an axis perpendicular to plane of triangle ABC and passing through the mid-point of rod BC ($i.e.$, D) is

$$I_1 = \frac{ml^2}{12}$$

From theorem of parallel axes, moment of inertia of this rod about the asked axis is

$$I_2 = I_1 + mr^2 = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}$$

∴ Moment of inertia of all the three rods is

$$I = 3I_2 = 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}$$

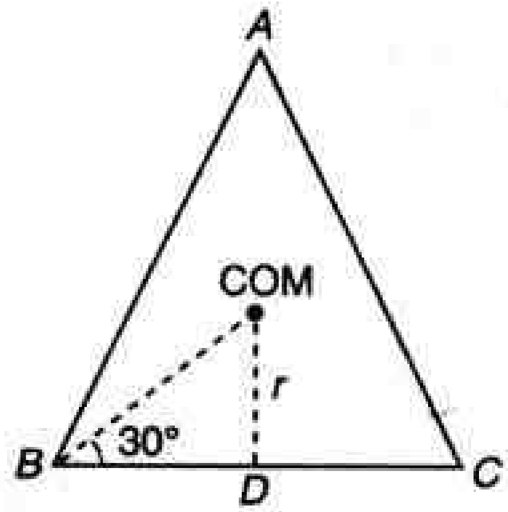


Fig. 9.12

Sample Example 9.2 *Find the moment of inertia of a solid sphere of mass M and radius R about an axis XX shown in figure.*

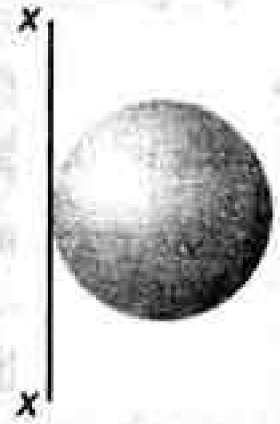


Fig. 9.13

Solution From theorem of parallel axis,

$$I_{XX} = I_{COM} + Mr^2$$
$$= \frac{2}{5}MR^2 + MR^2$$
$$= \frac{7}{5}MR^2$$

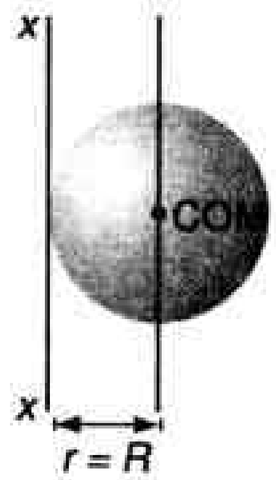


Fig. 9.14

Sample Example 9.3 *Consider a uniform rod of mass m and length 2l with two particles of mass m each at its ends. Let AB be a line perpendicular to the length of the rod and passing through its centre. Find the moment of inertia of the system about AB.*

Solution
$$I_{AB} = I_{\text{rod}} + I_{\text{both particles}}$$
$$= \frac{m(2l)^2}{12} + 2(ml^2)$$
$$= \frac{7}{3}ml^2$$

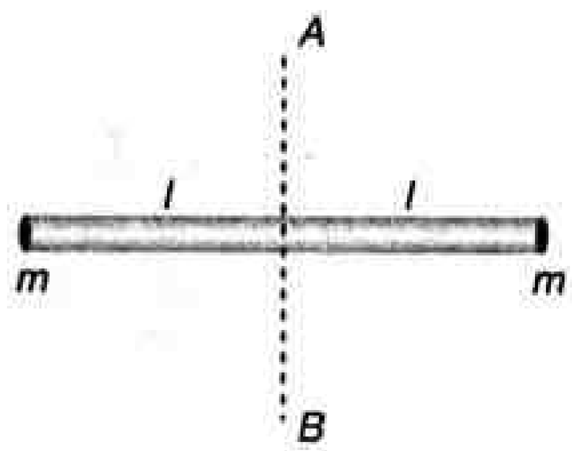


Fig. 9.15

Sample Example 9.4 *Find the moment of inertia of the rod AB about an axis YY as shown in figure. Mass of the rod is m and length is l.*

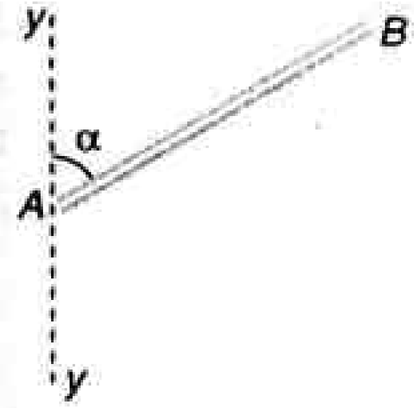


Fig. 9.16

Solution Mass per unit length of the rod $= \frac{m}{l}$

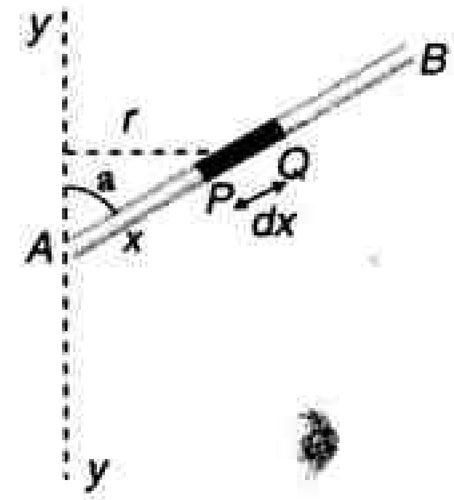


Fig. 9.17

Mass of an element PQ of the rod is, $dm = \left(\frac{m}{l}\right) dx$

Perpendicular distance of this elemental mass about yy is $r = x \sin \alpha$

$\therefore$ Moment of inertia of this small element of the rod (can be assumed as a point mass) about yy is,

$$dI = (dm)r^2 = \left(\frac{m}{l} dx\right)(x \sin \alpha)^2$$

$$= \frac{m}{l} \sin^2 \alpha \, x^2 \, dx$$

$\therefore$ Moment of inertia of the complete rod,

$$I = \int_{x=0}^{x=l} dI = \frac{m}{l} \sin^2 \alpha \int_0^l x^2 \, dx = \frac{ml^2}{3} \sin^2 \alpha$$ **Ans.**

Note *(i)* $I = 0$ *if* $\alpha = 0$ *(ii)* $I = \frac{ml^2}{3}$ *if* $\alpha = \frac{\pi}{2}$ *or* $90°$

Introductory Exercise 9.1

1. About what axis would a uniform cube have its minimum moment of inertia?
2. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 the moment of inertia of the ring formed by the same rod about an axis passing through the centre of mass of the ring and perpendicular to the plane of the ring. Then find the ratio $\frac{I_1}{I_2}$.
3. Find the radius of gyration of a rod of mass m and length $2l$ about an axis passing through one of its ends and perpendicular to its length.
4. There are four solid balls with their centres at the four corners of a square of side a. The mass of each sphere is m and radius is r. Find the moment of inertia of the system about (i) one of the sides of the square (ii) one of the diagonals of the square.
5. A non-uniform rod AB has a mass M and length $2l$. The mass per unit length of the rod is mx at a point of the rod distant x from A. Find the moment of inertia of this rod about an axis perpendicular to the rod (a) through A (b) through the mid-point of AB.
6. A circular lamina of radius a and centre O has a mass per unit area of kx^2, where x is the distance from O and k is a constant. If the mass of the lamina is M, find in terms of M and a, the moment of inertia of the lamina about an axis through O and perpendicular to the lamina.

7. The uniform disc shown in the figure has a moment of inertia of 0.6 kg-m^2 around the axis that passes through O and is perpendicular to the plane of the page. If a segment is cut out from the disc as shown, what is the moment of inertia of the remaining disc?

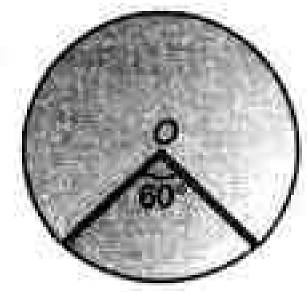


Fig. 9.18

8. Particles of masses 1 g, 2 g, 3 g, ..., 100 g are kept at the marks 1 cm, 2 cm, 3 cm, ..., 100 cm respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.
9. The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius R. Find the distance of the line from the centre.
10. If two circular disks of the same weight and thickness are made from metals having different densities. Which disk, if either will have the larger moment of inertia about its central axis.

9.2 Angular Velocity

The term angular velocity $(\vec{\omega})$ is defined for a particle about a point.

Suppose a particle P is moving with a velocity $\vec{\mathbf{v}}$, its position vector at some moment of time $t = t$, is $\vec{\mathbf{r}}$ with respect to a fixed point O. At time $t = t + dt$ the radius vector becomes $\vec{\mathbf{r}} + d\vec{\mathbf{r}}$. It has been rotated an angle $d\theta$ in time dt. Then the angular speed of particle P about point O, *i.e.*,

$$\omega = \frac{d\theta}{dt}$$

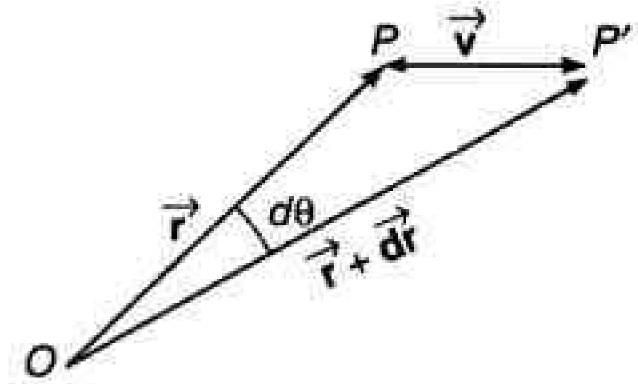


Fig. 9.19

This is also equal to the component of velocity perpendicular to $\vec{\mathbf{r}}$ divided by the distance of particle P from point O at that instant or,

$$\omega = \frac{v_{\perp}}{r}$$

In vector form the linear velocity, the angular velocity and the radius vector are related by,

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

Thus, angular velocity may be represented as a vector quantity whose direction is perpendicular to the plane of motion given by the right hand rule.

Important Points in Angular Velocity

- If a particle is moving in a circle it is in pure rotational motion about the centre of the circle, while for a moment it may be in pure translational motion about some other point.

  If a particle P is moving in a circle, its angular velocity about centre of the circle (ω_C) is two times the angular velocity about any point on the circumference of the circle (ω_O)

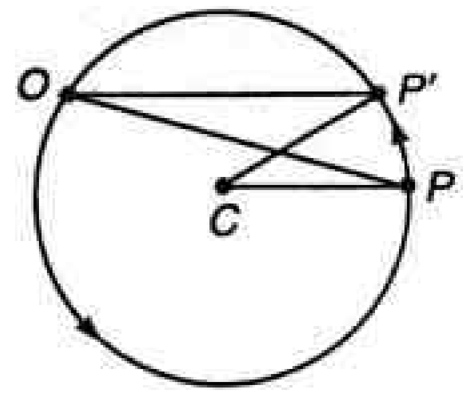


Fig. 9.20

or $$\omega_C = 2\omega_O$$

This is because $\angle P'CP = 2\angle P'OP$ (by property of a circle)

$$\omega_C = \frac{\angle P'CP}{t_{pp'}}, \quad \omega_O = \frac{\angle P'OP}{t_{pp'}}$$

From these relations we can see that $\omega_C = 2\omega_O$.

- If a rigid body is rotating about a fixed axis with angular speed ω, all the particles in rigid body rotate same angle in same interval of time, *i.e.*, their angular speed is same (ω). They rotate in different circles of different radii. The planes of these circles are perpendicular to the rotational axis. Linear speeds of different particles are different. Linear speed of a particle situated at a distance r from the rotational axis is

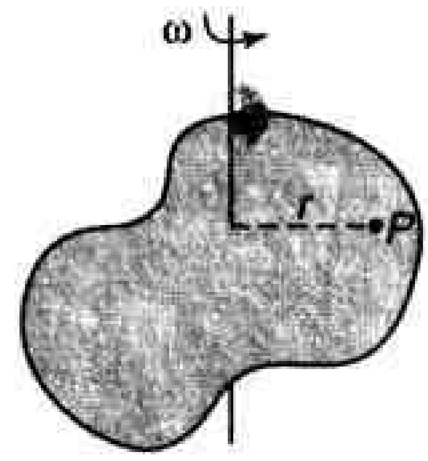


Fig. 9.21

$$v = r\omega$$

or $$v \propto r$$

- Angular velocity of a rigid body (ω) is $\frac{d\theta}{dt}$. Here θ is the angle between the line joining any two points (say A and B) on the rigid body and any reference line (dotted) as shown in figure.

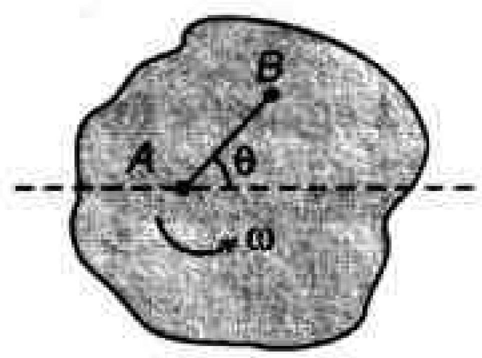


Fig. 9.22

  For example AB is a rod of length 4 m. End A is resting against a vertical wall OY and B is moving towards right with constant speed $v_B = 10$ m/s To find the angular speed of rod at $\theta = 30°$, we can proceed as under.

$$OB = x = AB\cos\theta$$

$\therefore$ $$x = 4\cos\theta$$

or $$\frac{dx}{dt} = -4\sin\theta\left(\frac{d\theta}{dt}\right)$$

$\therefore$ $$\left(\frac{d\theta}{dt}\right) = -\frac{(dx/dt)}{4\sin\theta} \qquad \left(\frac{dx}{dt} = v_B = 10\text{ m/s}\right)$$

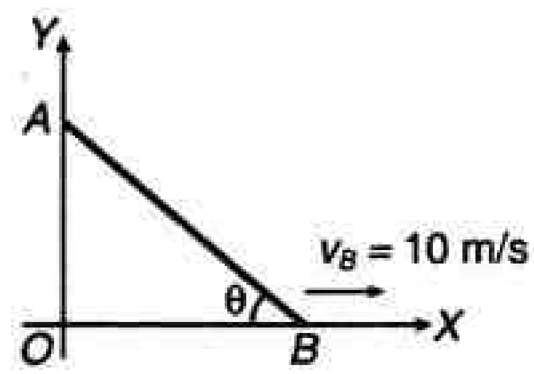


Fig. 9.23

or $$\omega = -\frac{10}{4\sin 30°} = -5\text{ rad / s}$$

Here, negative sign implies that θ decreases as t increases $\left(\frac{d\theta}{dt} < 0\right)$.

Sample Example 9.5 *A particle A moves along a circle of radius $R = 10$ cm so that its radius vector $\vec{r}$ relative to O rotates with constant angular velocity $\omega = 0.2$ rad/s. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.*

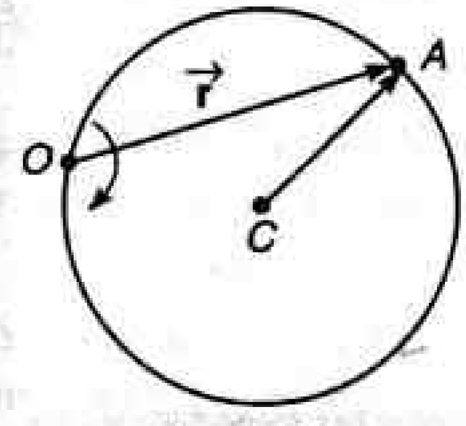


Fig. 9.24

Solution Given that $\omega_O = 0.2$ rad/s, $R = 10\text{cm} = 0.1\text{m}$

$$\therefore \qquad \omega_C = 2\omega_O = 0.4 \text{ rad/s}$$

(i) Modulus of velocity $|\vec{v}| = R\omega_C = (0.1)(0.4)$

or $$|\vec{v}| = 0.04 \text{ m/s} \quad \text{or} \quad 4 \text{ cm/s}$$

(ii) Modulus of total acceleration $|\vec{a}| = R\omega_C^2$

or $$|\vec{a}| = (0.1)(0.4)^2 = 0.016 \text{ m/s}^2$$

or $$|\vec{a}| = 1.6 \text{ cm/s}^2$$

(iii) The direction of its total acceleration (centripetal acceleration) will be towards centre C.

9.3 Torque

Suppose a force $\vec{F}$ is acting on a particle P and let $\vec{r}$ be the position vector of this particle about some reference point O. The torque of this force $\vec{F}$, about O is defined as,

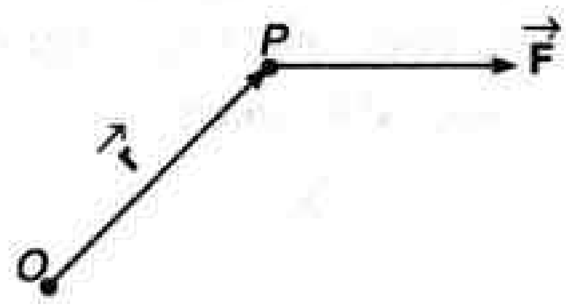


Fig. 9.25

$$\vec{\tau} = \vec{r} \times \vec{F}$$

This is a vector quantity having its direction perpendicular to both $\vec{r}$ and $\vec{F}$ according to the rule of cross product.

Note *Here,* $\vec{r} = \vec{r}_F - \vec{r}_O$

$\vec{r}_F$ = *position vector of point, where force is acting and*

$\vec{r}_O$ = *position vector of point about which torque is required. See Sample Example 9.8.*

Torque of a force about a line

Consider a rigid body rotating about a fixed axis *AB*. Let $\vec{F}$ be a force acting on the body at point *P*. Take the origin *O* somewhere on the axis of rotation. The torque of $\vec{F}$ about *O* is

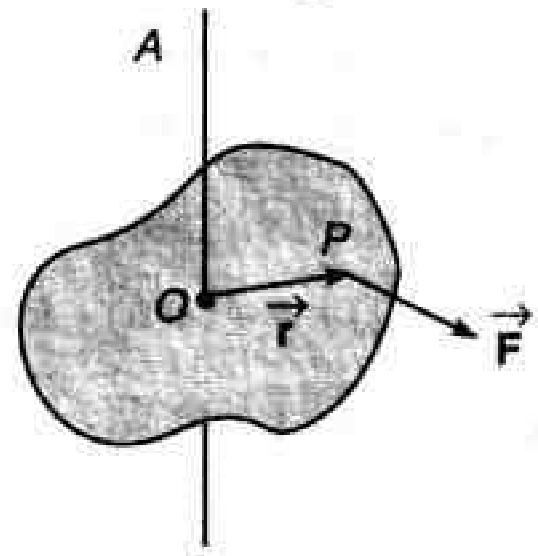


Fig. 9.26

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Its component along *AB* is called the torque of $\vec{F}$ about *AB*.

Important Points in Torque

- When a rigid body is rotating about a fixed axis and a force is applied on it at some point then we are concerned with the component of torque of this force about the axis of rotation not with the net torque.
- The component of torque about axis of rotation is independent of the choice of the origin *O*, so long as it is chosen on the axis of rotation, *i.e.*, we may choose point *O* anywhere on the line *AB*.
- Component of torque along axis of rotation *AB* is zero if

  (a) $\vec{F} \,||\, AB$

  (b) $\vec{F}$ intersects *AB* at some point
- If $\vec{F}$ is perpendicular to *AB*, but does not intersect it, then component of torque about line *AB* = magnitude of force $\vec{F}$ × perpendicular distance of $\vec{F}$ from the line *AB* (called the lever arm or moment arm) of this torque.
- If there are more than one force $\vec{F}_1, \vec{F}_2, \ldots$ acting on a body, the total torque will be

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$$

  But if the forces act on the same particle, one can add the forces and then take the torque of the resultant force, or

$$\vec{\tau} = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \ldots)$$

Sample Example 9.6 *Find the torque of a force* $\vec{\mathbf{F}} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ *N about a point O. The position vector of point of application of force about O is* $\vec{\mathbf{r}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ *m.*

Solution Torque

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-9+2) + \hat{\mathbf{j}}(-1+6) + \hat{\mathbf{k}}(4-3)$$

or

$$\vec{\tau} = (-7\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) \text{ N-m}$$

Sample Example 9.7 *A small ball of mass* 1.0 *kg is attached to one end of a* 1.0 *m long massless string and the other end of the string is hung from a point. When the resulting pendulum is* 30° *from the vertical, what is the magnitude of torque about the point of suspension. [Take* $g = 10\, m/s^2$*]*

Solution Two forces are acting on the ball :

(i) tension (T)

(ii) weight (mg)

Torque of tension about point O is zero, as it passes through O.

$$\tau_{mg} = F \times r_{\perp}$$

Here,

$$r_{\perp} = OP = 1.0 \sin 30° = 0.5 \text{ m}$$

$\therefore$

$$\tau_{mg} = (mg)(0.5) = (1)(10)(0.5) = 5 \text{ N-m}$$

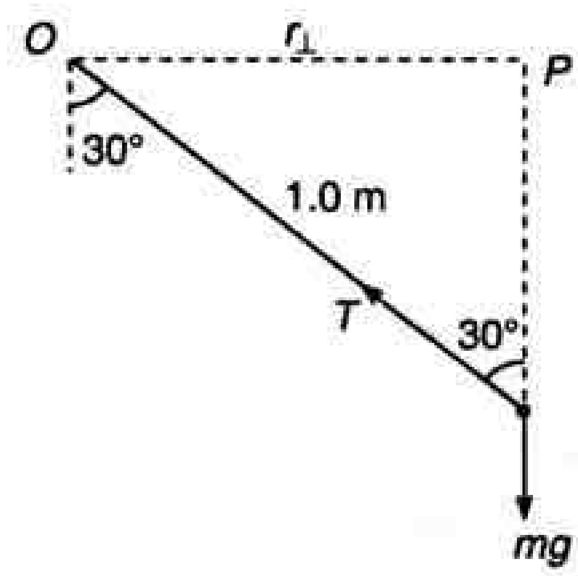


Fig. 9.27

Sample Example 9.8 *A force* $\vec{\mathbf{F}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *N is acting at point* $(2\,m, -3\,m, 6\,m)$. *Find torque of this force about a point whose position vector is* $(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ *m.*

Solution

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Here,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_F - \vec{\mathbf{r}}_0$$

$$= (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$= (2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \text{ m}$$

Now,

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = (-17\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \text{ N-m}$$

9.4 Rotation of a Rigid Body about a Fixed Axis

When a body is rotating about a fixed axis, any point P located in the body travels along a circular path. Before, analysing the circular motion of point P, we will first study the angular motion properties of a rigid body.

Angular motion

Since, a point is without dimension, it has no angular motion. Only lines or bodies undergo angular motion. Let us consider the angular motion of a radial line r located with the shaded plane.

Angular position

The angular position of r is defined by the angle θ, measured between a fixed reference line OA and r.

Angular displacement

The change in the angular position, often measured as a differential $d\vec{\theta}$ is called the angular displacement. (Finite angular displacements are not vector quantities, although differential rotations $d\vec{\theta}$ are vectors). This vector has a magnitude $d\theta$ and the direction of $d\vec{\theta}$ is along the axis.

Specifically, the direction of $d\vec{\theta}$ is determined by right hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb or $d\vec{\theta}$ points upward.

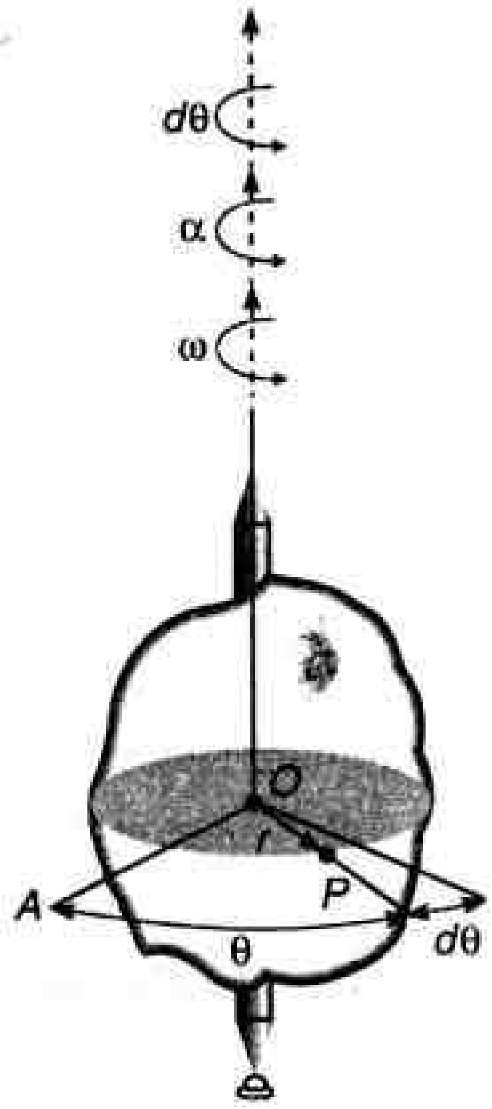


Fig. 9.28

Angular velocity

The time rate of change in the angular position is called the angular velocity $\vec{\omega}$. Thus,

$$\omega = \frac{d\theta}{dt} \quad \text{...(i)}$$

It is expressed here in scalar form, since its direction is always along the axis of rotation, *i.e.*, in the same direction as $d\vec{\theta}$.

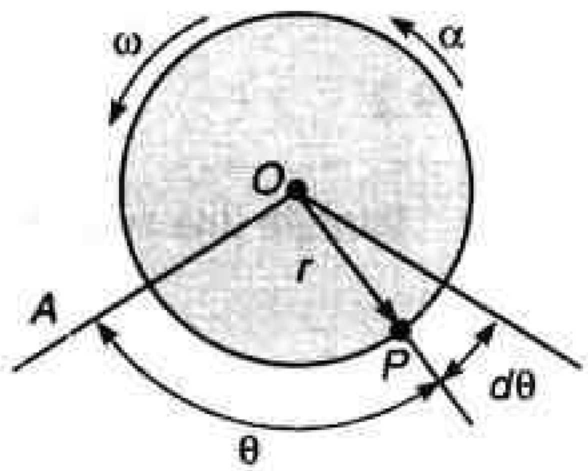


Fig. 9.29

Angular acceleration

The angular acceleration $\vec{\alpha}$ measures the time rate of change of the angular velocity. Hence, the magnitude of this vector may be written as,

$$\alpha = \frac{d\omega}{dt} \quad \text{...(ii)}$$

It is also possible to express α as,

$$\alpha = \frac{d^2\theta}{dt^2}$$

The line of action of $\vec{\alpha}$ is the same as that for $\vec{\omega}$, however its sense of direction depends on whether ω is increasing or decreasing with time. In particular, if ω is decreasing, $\vec{\alpha}$ is called an angular deceleration and therefore, has a sense of direction which is opposite to $\vec{\omega}$.

Torque and angular acceleration for a rigid body

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}$$

Thus, for a rigid body we have the rotational analog of Newton's second law :

$$\Sigma\tau = I\alpha \qquad \text{...(iii)}$$

Following two points are important regarding the above equation.

(i) The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration α is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

Rotation with constant angular acceleration

If the angular acceleration of the body is constant then Eqs. (i) and (ii) when integrated yield a set of formulae which relate the body's angular velocity, angular position and time. These equations are similar to equations used for rectilinear motion. Table given ahead compares the linear and angular motion with constant acceleration.

Table 9.2

Straight line motion with constant linear acceleration	Fixed axis rotation with constant angular acceleration
a = constant	α = constant
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = s_0 + ut + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2a(s - s_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity respectively.

Kinetic Energy of a rigid body rotating about a fixed axis

Suppose a rigid body is rotating about a fixed axis with angular speed ω. Then, kinetic energy of the rigid body will be :

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\omega r_i)^2$$

$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2$$

$$= \frac{1}{2} I\omega^2 \qquad (\text{as } \sum_i m_i r_i^2 = I)$$

Thus, $\quad KE = \frac{1}{2} I\omega^2$

Sometimes it is called the rotational kinetic energy.

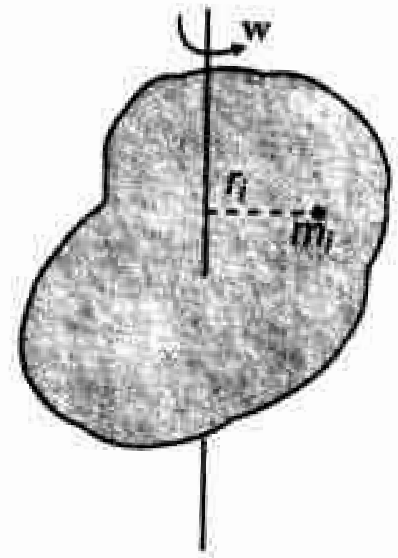


Fig. 9.30

Sample Example 9.9 *A solid sphere of mass 2 kg and radius 1 m is free to rotate about an axis passing through its centre. Find a constant tangential force F required to rotate the sphere with 10 rad/s in 2 s. Also find the number of rotations made by the sphere in that time interval.*

Solution Since, the force is constant, the torque produced by it and the angular acceleration α will be constant. Hence, we can apply

$$\omega = \omega_0 + \alpha t \quad \text{etc.}$$

$$10 = 0 + (\alpha)(2)$$

$$\therefore \quad \alpha = 5 \text{ rad/s}^2$$

Further, the force is tangential. Therefore, the perpendicular distance from the axis of rotation will be equal to the radius of the sphere.

$$\therefore \quad \alpha = \frac{\tau}{I} = \frac{F \cdot R}{\frac{2}{5} mR^2} = \frac{5F}{2mR}$$

or

$$F = \frac{2mR\alpha}{5}$$

Substituting the value, we have

$$F = \frac{(2)(2)(1)(5)}{(5)} = 4 \text{ N}$$

Further, $\quad$ Angle rotated $\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}(5)(2)^2$

$$= 10 \text{ rad}$$

$\therefore \quad$ Number of rotations $n = \frac{\theta}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi}$

Sample Example 9.10 *The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where θ is in radians and t is in seconds. What are the angular velocities at*

(a) $t = 2.0$ s and

(b) $t = 4.0$ s?

(c) What is the average angular acceleration for the time interval that begins at $t = 2.0$ s and ends at $t = 4.0$ s?

(d) What are the instantaneous angular acceleration at the beginning and the end of this time interval?

Solution Angular velocity $\omega = \frac{d\theta}{dt} = \frac{d}{dt}(4t - 3t^2 + t^3)$

or $\omega = 4 - 6t + 3t^2$

(a) At $t = 2.0\,s$, $\omega = 4 - 6 \times 2 + 3(2)^2$

or $\omega = 4$ rad/s

(b) At $t = 4.0\,s$, $\omega = 4 - 6 \times 4 + 3(4)^2$

or $\omega = 28$ rad/s

(c) Average angular acceleration

$$\alpha_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{28 - 4}{4 - 2}$$

or $\alpha_{av} = 12\ \text{rad/s}^2$

(d) Instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(4 - 6t + 3t^2)$$

or $\alpha = -6 + 6t$

At $t = 2.0\,s$, $\alpha = -6 + 6 \times 2 = 6\ \text{rad/s}^2$

At $t = 4.0\,s$, $\alpha = -6 + 6 \times 4 = 18\ \text{rad/s}^2$

Introductory Exercise 9.2

1. A body rotates about a fixed axis with an angular acceleration 1 rad/s^2. Through what angle does it rotates during the time in which its angular velocity increases from 5 rad/s to 15 rad/s.

2. A wheel starting from rest is uniformly accelerated at 4 rad/s^2 for 10 s. It is allowed to rotate uniformly for the next 10 s and is finally brought to rest in the next 10 s. Find the total angle rotated by the wheel.

3. A flywheel of moment of inertia 5.0 kg m^2 is rotated at a speed of 10 rad/s. Because of the friction at the axis it comes to rest in 10 s. Find the average torque of the friction.

4. A wheel of mass 10 kg and radius 0.2 m is rotating at an angular speed of 100 rpm, when the motion is turned off. Neglecting the friction at the axis, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 rev. Assume wheel to be a disc.

5. A solid body rotates about a stationary axis according to the law $\theta = 6t - 2t^3$. Here, θ is in radian and t in seconds. Find :

(a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop,

(b) the angular acceleration at the moment when the body stops.

Hint *If $y = y(t)$, then mean/average value of y between t_1 and t_2 is* $\langle y \rangle = \frac{\int_{t_1}^{t_2} y(t)\,dt}{t_2 - t_1}$.

6. A solid body starts rotating about a stationary axis with an angular acceleration $\alpha = (2.0 \times 10^{-2})t$ rad/s^2, here, t is in seconds. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\theta = 60°$ with its velocity vector?

9.5 Angular Momentum

A mass moving in a straight line has linear momentum $(\vec{\mathbf{P}})$. When a mass rotates about some point/axis, there is momentum associated with rotational motion called the angular momentum $(\vec{\mathbf{L}})$. Just as net external force is required to change the linear momentum of an object a net external torque is required to change the angular momentum of an object. Keeping in view the problems asked in JEE, the angular momentum is classified in following three types.

(i) Angular momentum of a particle about some point

Suppose a particle A of mass m is moving with linear momentum $\vec{\mathbf{P}} = m\vec{\mathbf{v}}$. Its angular momentum $\vec{\mathbf{L}}$ about point O is defined as:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{P}} = \vec{\mathbf{r}} \times (m\vec{\mathbf{v}}) = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})$$

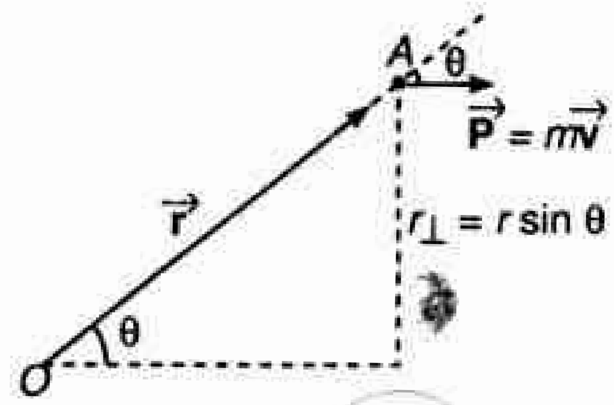


Fig. 9.31

Here, $\vec{\mathbf{r}}$ is the radius vector of particle A about O at that instant of time. The magnitude of $\vec{\mathbf{L}}$ is

$$L = mvr\sin\theta = mvr_\perp$$

Here, $r_\perp = r\sin\theta$ is the perpendicular distance of line of action of velocity $\vec{\mathbf{v}}$ from point O. The direction of $\vec{\mathbf{L}}$ is same as that of $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$.

Note *The angular momentum of a particle about a line (say AB) is the component along AB of the angular momentum of the particle about any point (say O) on the line AB. This component is independent of the choice of point O, so far as it is chosen on the line AB.*

Sample Example 9.11 *A particle of mass m is moving along the line $y = b, z = 0$ with constant speed v. State whether the angular momentum of particle about origin is increasing, decreasing or constant.*

Solution

$$|\vec{\mathbf{L}}| = mvr\sin\theta$$
$$= mvr_\perp$$
$$= mvb$$

$\therefore \quad |\vec{\mathbf{L}}| = \text{constant} \quad$ as $\quad m, v$ and b all are constants.

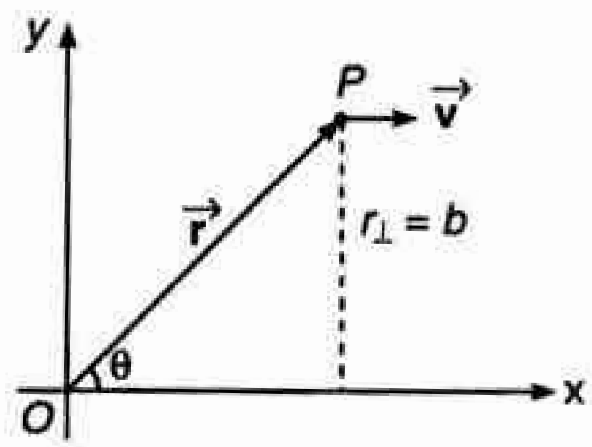


Fig. 9.32

Direction of $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ also remains the same. Therefore, angular momentum of particle about origin remains constant with due course of time.

Note *In this problem $|\vec{\mathbf{r}}|$ is increasing, θ is decreasing but $r\sin\theta$, i.e., b remains constant. Hence, the angular momentum remains constant.*

Sample Example 9.12 *A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of particle at any time t about O before the particle strikes the ground again.*

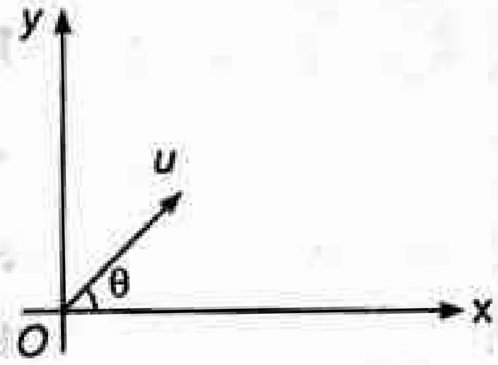


Fig. 9.33

Solution $$\vec{L} = m(\vec{r} \times \vec{v})$$

Here, $$\vec{r}(t) = x\hat{i} + y\hat{j} = (u\cos\theta)t\hat{i} + (ut\sin\theta - \frac{1}{2}gt^2)\hat{j}$$

and $$\vec{v}(t) = v_x\hat{i} + v_y\hat{j} = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$$

∴ $$\vec{L} = m(\vec{r} \times \vec{v}) = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (u\cos\theta)t & (u\sin\theta)t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

$$= m\left[(u^2\sin\theta\cos\theta)t - (u\cos\theta)gt^2 - (u^2\sin\theta\cos\theta)t + \frac{1}{2}(u\cos\theta)gt^2\right]\hat{k}$$

$$= -\frac{1}{2}m(u\cos\theta)gt^2\,\hat{k}$$

(ii) Angular Momentum of a rigid body rotating about a fixed axis

Suppose a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case $\vec{r}$ and $\vec{P}$ are perpendicular to each other and $\vec{r} \times \vec{P}$ is along the axis. Thus, component of $\vec{r} \times \vec{P}$ along the axis is mvr itself. The angular momentum of the whole rigid body about AB is the sum of components of all particles, *i.e.*,

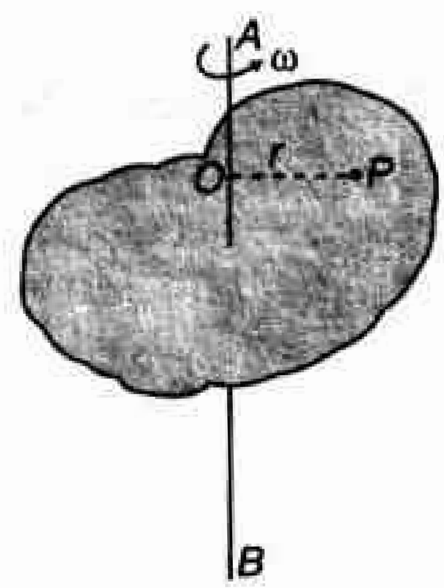


Fig. 9.34

$$L = \sum_i m_i r_i v_i$$

Here, $$v_i = r_i\omega$$

∴ $$L = \sum_i m_i r_i^2\omega \quad \text{or} \quad L = \omega\sum_i m_i r_i^2$$

or $$\mathbf{L = I\omega} \qquad (\text{as } \sum_i m_i r_i^2 = I)$$

Here, I is the moment of inertia of the rigid body about AB.

Note *The vector relation $\vec{L} = I\vec{\omega}$ is not correct in the above case because $\vec{L}$ and $\vec{\omega}$ do not point in the same direction, but we could write $L_{AB} = I\omega$. If however the body is symmetric about the axis of rotation $\vec{L}$ and $\vec{\omega}$ are parallel and we can write ($L = I\omega$) in vector form as $\vec{L} = I\vec{\omega}$.*

By symmetric we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation.

Thus, remember that $\vec{L} = I\vec{\omega}$ applies only to bodies that have symmetry about the (fixed) rotational axis. Here, $\vec{L}$ stands for total angular momentum. However the relation $L_{AB} = I\omega$ holds for any rigid body symmetrical or not that is rotating about a fixed axis.

(iii) Angular momentum of a rigid body in combined rotation and translation

Let O be a fixed point in an inertial frame of reference. Angular momentum of the body about O is

$$\vec{L} = \sum_i m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum_i m_i (\vec{r}_{i,cm} + \vec{r}_0) \times (\vec{v}_{i,cm} + \vec{v}_0)$$

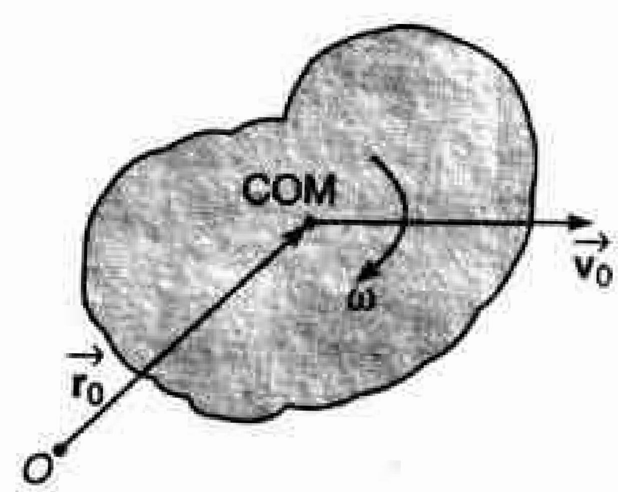


Fig. 9.35

Here, $\vec{r}_0$ is the position vector of the centre of mass and $\vec{v}_0$ its velocity.

Thus, $\vec{L} = \sum_i m_i (\vec{r}_{i\,cm} \times \vec{v}_{i,cm}) + \{\sum_i m_i \vec{r}_{i,cm}\} \times \vec{v}_0 + \vec{r}_0 \times \{\sum_i m_i \vec{v}_{i,cm}\} + \{\sum_i m_i\} \vec{r}_0 \times \vec{v}_0$

Now, $\sum_i m_i \vec{r}_{i,cm} = M\vec{R}_{cm,cm} = 0$

Similarly, $\sum_i m_i \vec{v}_{i,cm} = M\vec{v}_{cm,cm} = 0$

Thus, $\vec{L} = \sum_i m_i (\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + M\vec{r}_0 \times \vec{v}_0$

or $$\vec{L} = \vec{L}_{cm} + M(\vec{r}_0 \times \vec{v}_0)$$

The first term $\vec{L}_{cm}$ represents the angular momentum of the body as seen from the centre of mass frame. The second term $M(\vec{r}_0 \times \vec{v}_0)$ equals the angular momentum of centre of mass about point O.

Sample Example 9.13 *A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = \frac{v}{R}$ in clockwise direction as shown in figure. Find the magnitude of the total angular momentum of the disc about bottommost point O of the disc.*

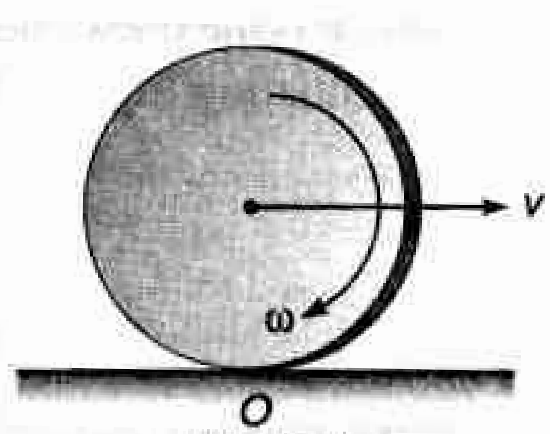


Fig. 9.36

Solution $$\vec{L} = \vec{L}_{cm} + m(\vec{r}_0 \times \vec{v}_0) \quad \text{...(i)}$$

Here, $\vec{L}_{cm} = I\omega$ (perpendicular to paper inwards)

$$= \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)$$

$$= \frac{1}{2} mvR$$

and $m(\vec{r}_0 \times \vec{v}_0) = mRv$ (perpendicular to paper inwards)

Since, both the terms of right hand side of Eq. (i) are in the same direction.

$$\therefore \quad |\vec{L}| = \frac{1}{2} mvR + mvR$$

or $$|\vec{L}| = \frac{3}{2} mvR$$

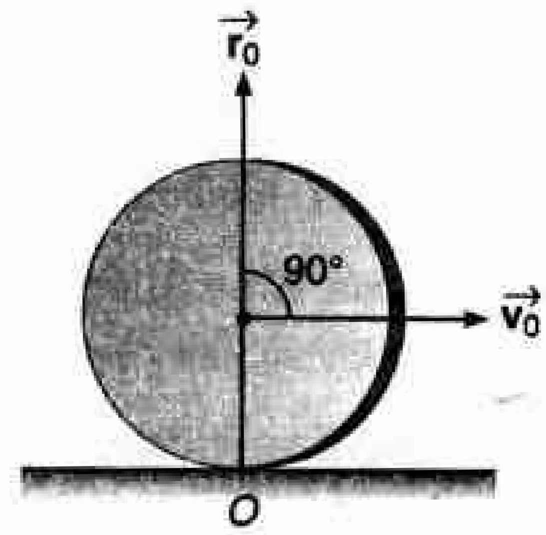


Fig. 9.37

Introductory Exercise 9.3

1. Two particles each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is the same about any point taken as origin.
2. In sample example number 9.13 suppose the disc starts rotating anticlockwise with the same angular velocity $\omega = \frac{v}{R}$, then what will be the angular momentum of the disc about bottommost point in this new situation?
3. A particle of mass m moves in xy plane along the line $y = x - 4$, with constant speed v. Find the angular momentum of particle about origin at any instant of time t.
4. A particle of mass m is projected from the ground with an initial speed u at an angle α. Find the magnitude of its angular momentum at the highest point of its trajectory about the point of projection.
5. If the angular momentum of a body is zero about some point. Is it necessary that it will be zero about a different point?

9.6 Conservation of Angular Momentum

As we have seen in Article 9.5, the angular momentum of a particle about some reference point O is defined as,

$$\vec{L} = \vec{r} \times \vec{p} \qquad \text{...(i)}$$

Here, $\vec{P}$ is the linear momentum of the particle and $\vec{r}$ its position vector with respect to the reference point O. Differentiating Eq. (i) with respect to time, we get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} \qquad \text{...(ii)}$$

Here, $$\frac{d\vec{p}}{dt} = \vec{F}$$

and $$\frac{d\vec{r}}{dt} = \vec{v} \text{ (velocity of particle)}$$

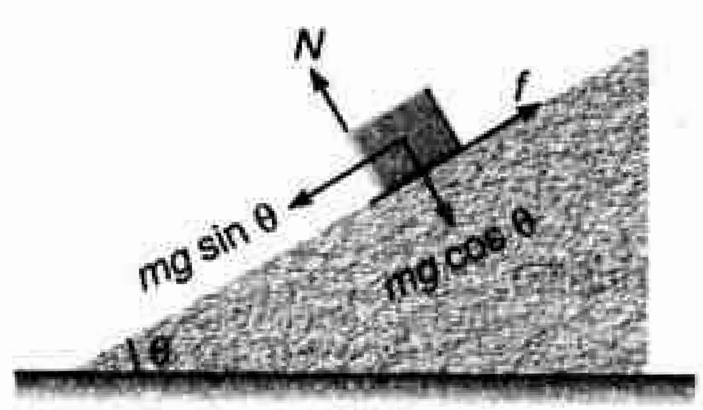


Fig. 9.38

Hence, Eq. (ii) can be rewritten as,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

Now, $\vec{v} \times \vec{p} = 0$, because $\vec{v}$ and $\vec{p}$ are parallel to each other and the cross product of two parallel vectors is zero. Thus,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

or

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \text{...(iii)}$$

Which states that the time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it. This result is rotational analog of the equation $\vec{F} = \frac{d\vec{P}}{dt}$, which states that the time rate of change of the linear momentum of a particle is equal to the force acting on it. Eq. (iii) like all vector equations, is equivalent to three scalar equations, namely

$$\tau_x = \left(\frac{dL}{dt}\right)_x, \qquad \tau_y = \left(\frac{dL}{dt}\right)_y$$

and

$$\tau_z = \left(\frac{dL}{dt}\right)_z$$

The same equation can be generalised for a system of particles as, $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$. According to which the time rate of change of the total angular momentum of a system of particles about some reference point of an inertial frame of reference is equal to the sum of all external torques (of course the vector sum) acting on the system about the same reference point.

Now, suppose that $\vec{\tau}_{ext} = 0$, then $\frac{d\vec{L}}{dt} = 0$, so that $\vec{L} = \text{constant}$.

"When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega$$

It is possible for the moment of inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then L_z must remain constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as

$$I\omega = \text{constant} \qquad \text{...(iv)}$$

Sample Example 9.14 *A wheel of moment of inertia I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary particle of mass m at its edge. Find the new angular speed of the wheel.*

Solution Net external torque on the system is zero. Therefore, angular momentum will remain conserved. Thus,

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad \omega_2 = \frac{I_1\omega_1}{I_2}$$

Here, $I_1 = I$, $\omega_1 = \omega_0$, $I_2 = I + mR^2$

$$\therefore \qquad \omega_2 = \frac{I\omega_0}{I + mR^2}$$

Introductory Exercise 9.4

1. A thin circular ring of mass M and radius R is rotating about its axis with an angular speed ω_0. Two particles each of mass m are now attached at diametrically opposite points. Find the new angular speed of the ring.
2. If the ice at the poles melts and flows towards the equator, how will it affect the duration of day-night?
3. When tall buildings are constructed on earth, the duration of day night slightly increases. Is this statement true or false?

9.7 Combined Translational and Rotational Motion of a Rigid Body

Up until now we have considered only bodies rotating about some fixed axis. In JEE however questions are frequently asked on combined translational and rotational motion of a rigid body.

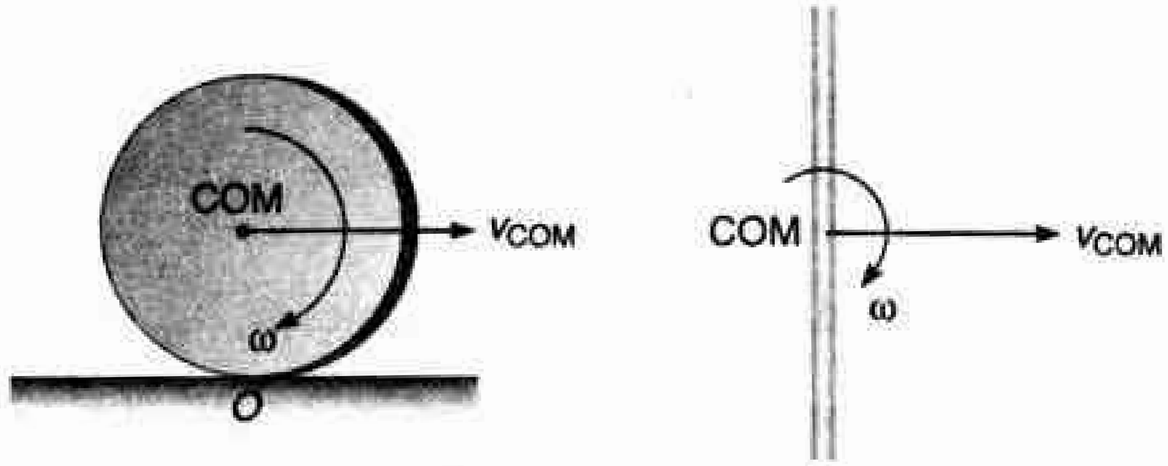


Fig. 9.39

In such problems if two things are known:
(i) velocity of centre of mass (v_{COM}) (ii) angular velocity of the rigid body (ω).

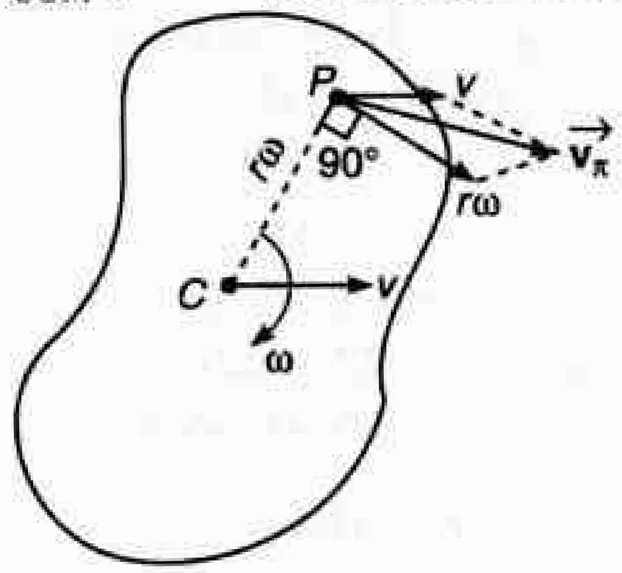


Fig. 9.40

The motion of whole rigid body can be described.

For example, let the velocity of centre of mass of a rigid body shown in figure is v and angular velocity of the rigid body is ω. Then velocity of any point P on the rigid body can be obtained as,

$$\vec{v}_p = \vec{v}_{COM} + \vec{v}_{p,COM}$$

Here, $$|\vec{v}_{COM}| = v$$

and $\vec{v}_{p,COM} = r\omega$ in a direction perpendicular to line CP.

Thus, the velocity of point P is the vector sum of $\vec{v}_{COM}$ and $\vec{v}_{p,COM}$ as shown in figure.

Kinetic energy of rigid body in combined translational and rotational motion

Here, two energies are associated with the rigid body. One is translational $\left(=\frac{1}{2} m v_{COM}^2\right)$ and another is rotational $\left(=\frac{1}{2} I_{COM}\, \omega^2\right)$. Thus, total kinetic energy of the rigid body is

$$K = \frac{1}{2} m v_{COM}^2 + \frac{1}{2} I_{COM}\, \omega^2$$

For better understanding of this article let us take an example based on the above theory asked in JEE-2000.

Sample Example 9.15 *A disc of radius R has linear velocity v and angular velocity ω as shown in the figure. Given v = rω. Find velocity of points A, B, C and D on the disc.*

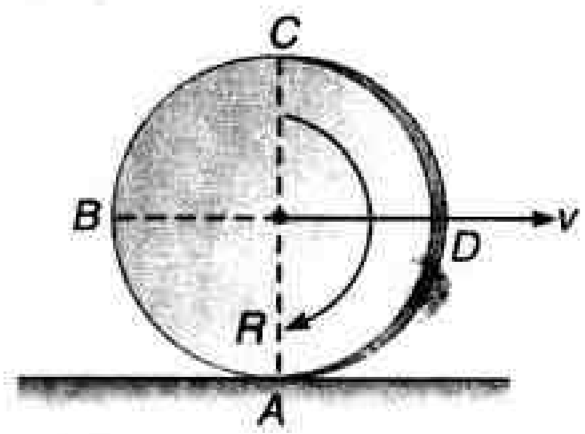


Fig. 9.41

Solution As stated in above article, velocity of any point of the rigid body in rotation plus translation is the vector sum of v (the velocity of centre of mass) and $r\omega$. Here, r is the distance of the point under consideration from the centre of mass of the body. Direction of this $r\omega$ is perpendicular to the line joining the point with centre of mass in the sense of rotation. Based on this, velocities of points A, B, C and D are as shown below:

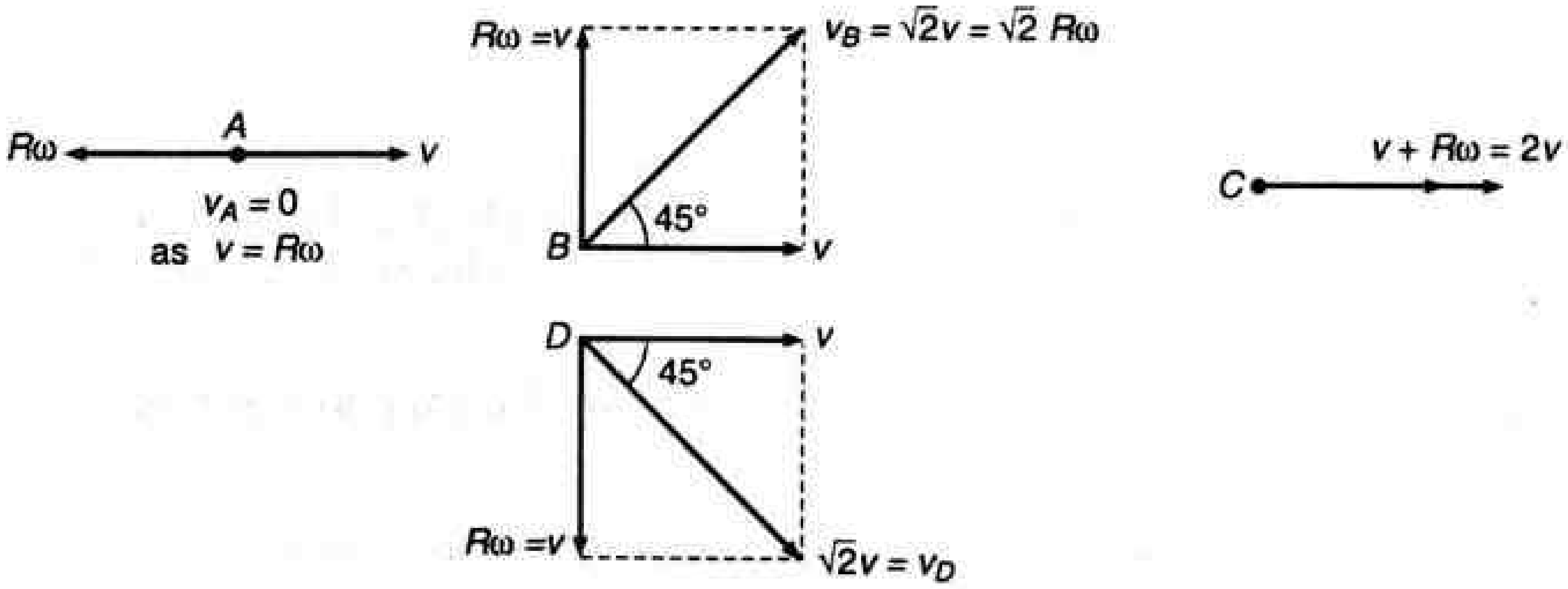


Fig. 9.42

Thus, v_A is zero, velocity of B and D is $\sqrt{2}v$ or $\sqrt{2}R\omega$ and velocity of C is $2v$ or $2R\omega$ in the directions shown in figure.

9.8 Instantaneous Axis of Rotation

The combined effects of translation of the centre of mass and rotation about an axis through the centre of mass are equivalent to a pure rotation with the same angular speed about an axis passing through a point of zero velocity. Such an axis is called the instantaneous axis of rotation. (*IAOR*). This axis is always perpendicular to the plane used to represent the motion and the intersection of the axis with this plane defines the location of instantaneous centre of zero velocity (*IC*).

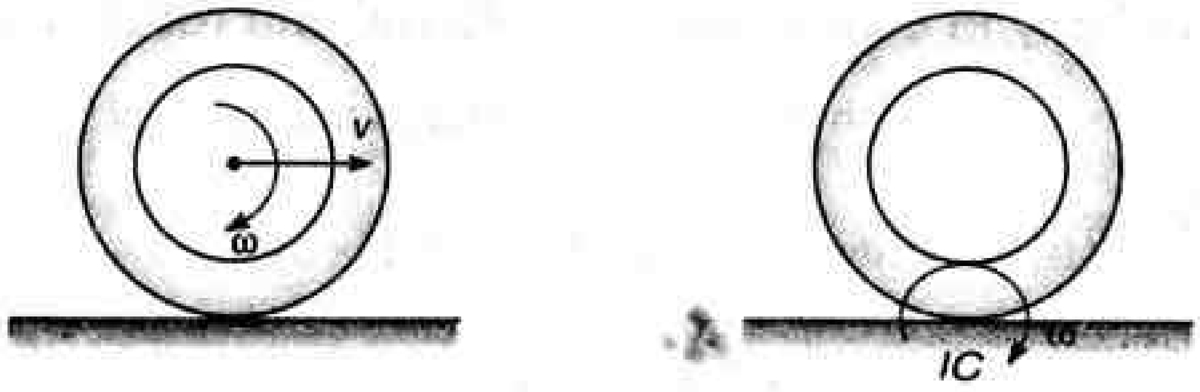


Fig. 9.43

For example consider a wheel which rolls without slipping. In this case the point of contact with the ground has zero velocity. Hence, this point represents the *IC* for the wheel. If it is imagined that the wheel is momentarily pinned at this point, the velocity of any point on the wheel can be found using $v = r\omega$. Here r is the distance of the point from *IC*. Similarly, the kinetic energy of the body can be assumed to be pure rotational about *IAOR* or,

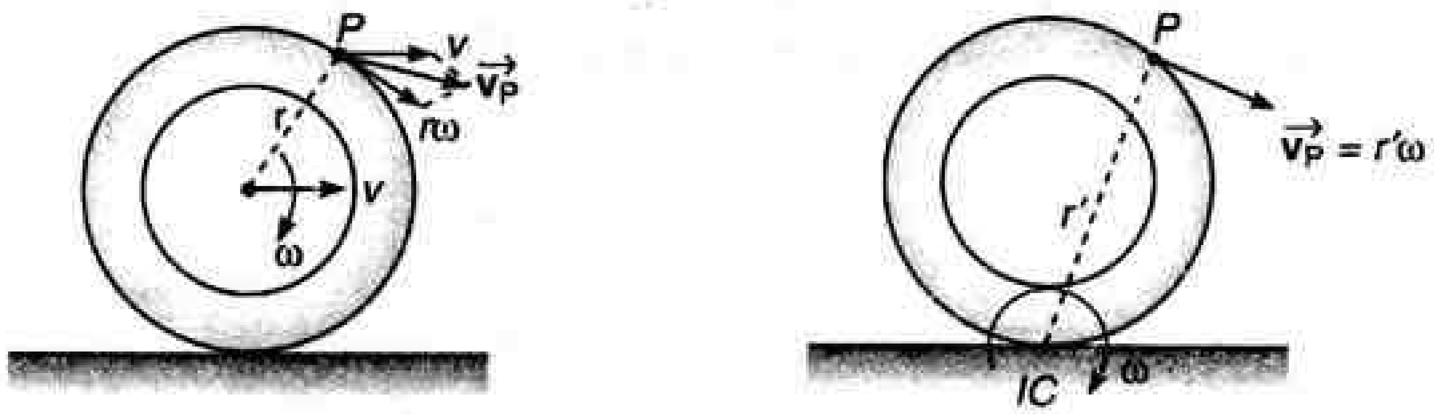


Fig. 9.44

$$K = \frac{1}{2} I_{IAOR}\ \omega^2$$

Rotation + Translation $\Rightarrow$ Pure rotation about *IAOR* passing through *IC*

$$\text{KE} = \frac{1}{2} m v_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}} \omega^2 \quad \Rightarrow \quad \text{KE} = \frac{1}{2} I_{IAOR}\ \omega^2$$

Location of the IC

If the location of the *IC* is unknown, it may be determined by using the fact that the relative position vector extending from the *IC* to a point is always perpendicular to the velocity of the point. Following three possibilities exist.

(i) Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body

If v and ω are known, the *IC* is located along the line drawn perpendicular to $\vec{v}$ at *P*, such that the distance from *P* to *IC* is, $r = \frac{v}{\omega}$. Note that *IC* lie on that side of *P* which causes rotation about the *IC*, which is consistent with the direction of motion caused by $\vec{\omega}$ and $\vec{v}$.

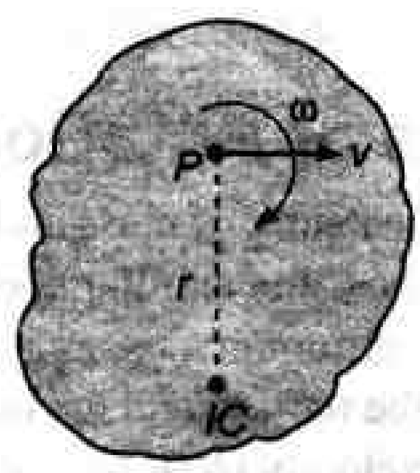


Fig. 9.45

Sample Example 9.16 *A rotating disc moves in the positive direction of the x-axis. Find the equation y(x) describing the position of the instantaneous axis of rotation if at the initial moment the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counterclockwise with a constant angular acceleration α. The initial angular velocity is equal to zero.*

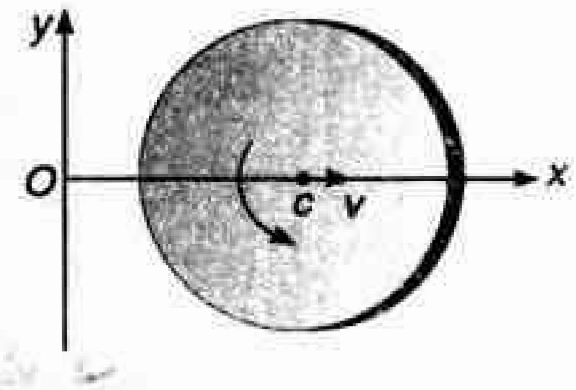


Fig. 9.46

Solution

$$t = \frac{x}{v}$$

and

$$\omega = \alpha t = \frac{\alpha x}{v}$$

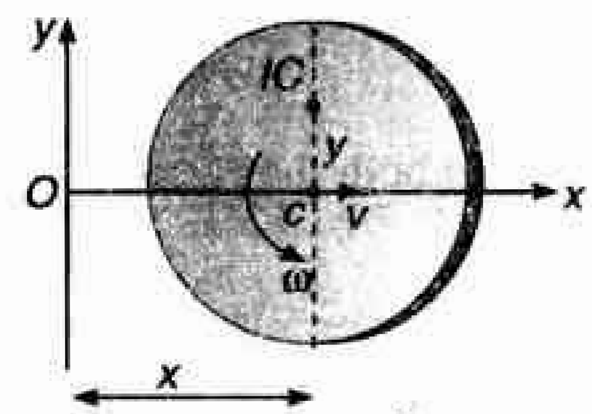


Fig. 9.47

The position of *IAOR* will be at a distance

$$y = \frac{v}{\omega}$$

or

$$y = \frac{v}{\frac{\alpha x}{v}}$$

or

$$y = \frac{v^2}{\alpha x}$$

or

$$xy = \frac{v^2}{\alpha} = \text{constant}$$

This is the desired *x-y* equation. This equation represents a rectangular hyperbola.

(ii) Given the lines of action of two non-parallel velocities

Consider the body shown in figure where the line of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known. Draw perpendiculars at *A* and *B* to these lines of action. The point of intersection of these perpendiculars as shown locates the *IC* at the instant considered.

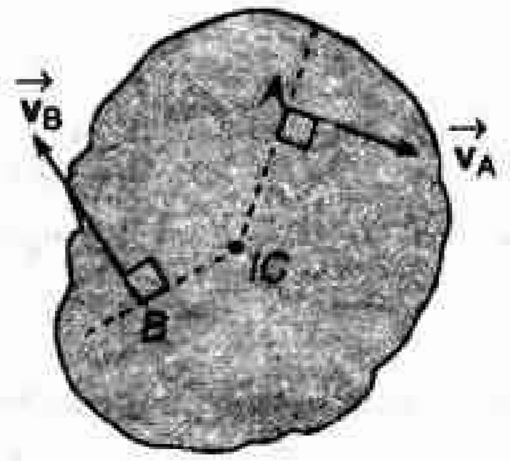


Fig. 9.48

(iii) Given the magnitude and direction of two parallel velocities

When the velocities of points *A* and *B* are parallel and have known magnitudes v_A and v_B then the location of the *IC* is determined by proportional triangles as shown in figure.

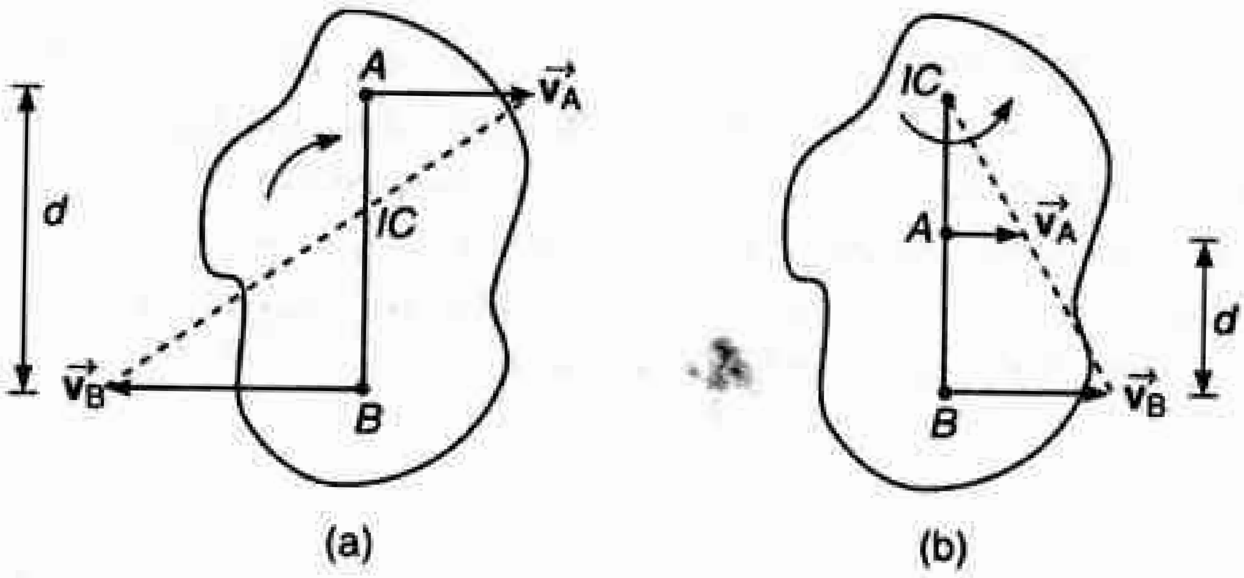


Fig. 9.49

In both the cases,

$$r_{A,IC} = \frac{v_A}{\omega}$$

and

$$r_{B,IC} = \frac{v_B}{\omega}$$

In Fig. (a) $$r_{A,IC} + r_{B,IC} = d$$

and in Fig. (b) $$r_{B,IC} - r_{A,IC} = d$$

As a special case, if the body is translating, $v_A = v_B$ and the *IC* would be located at infinity, in which case $\omega = 0$.

Sample Example 9.17 *A uniform thin rod of mass m and length l is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with horizontal.*

Solution As the floor is smooth, mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod, hence the centre of mass moves vertically downwards in a straight line. Thus velocities of *COM* and the lower end *B* are in the directions shown in figure. The location of *IC* at this instant can be found by drawing perpendiculars to $\vec{v}_C$ and $\vec{v}_B$ at respective points. Now, the rod may be assumed to be in pure rotational motion about *IAOR* passing through *IC* with angular speed ω.

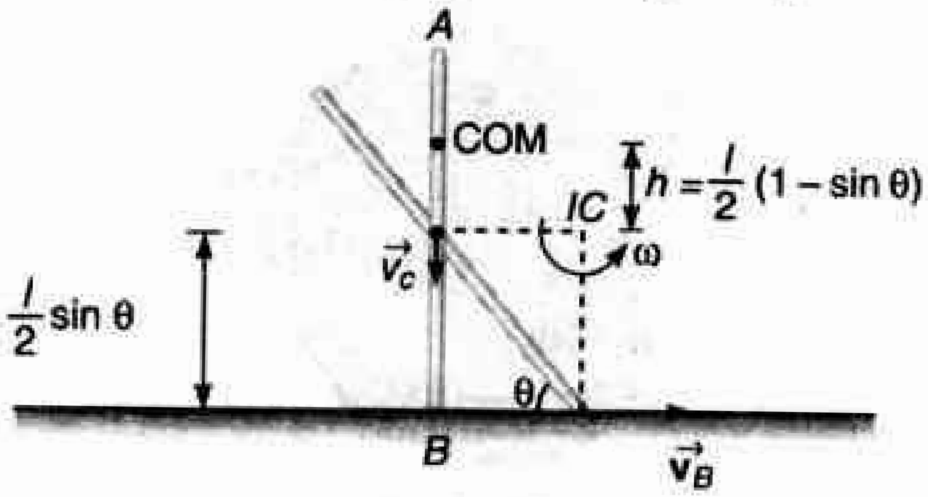


Fig. 9.50

Applying conservation of mechanical energy. Decrease in gravitational potential energy of the rod = increase in rotational kinetic energy about *IAOR*

$$\therefore \qquad mgh = \frac{1}{2} I_{IAOR}\, \omega^2$$

or

$$mg\frac{l}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega = \sqrt{\frac{12g(1-\sin\theta)}{l(1+3\cos^2\theta)}}$$

Now,

$$|\vec{v}_C| = \left(\frac{l}{2}\cos\theta\right)\omega$$

$$= \sqrt{\frac{3gl(1-\sin\theta)\cos^2\theta}{(1+3\cos^2\theta)}}$$

Introductory Exercise 9.5

1. In sample example 9.16, find the equation $y(x)$ if at the initial moment the axis c of the disc was located at the point O after which it moved with a constant linear acceleration a_0 (and the zero initial velocity) while the disc rotates counter clockwise with a constant angular velocity ω.
2. A uniform bar of length l stands vertically touching a wall OA. When slightly displaced, its lower end begins to slide along the floor. Obtain an expression for the angular velocity ω of the bar as a function of θ. Neglect friction everywhere.

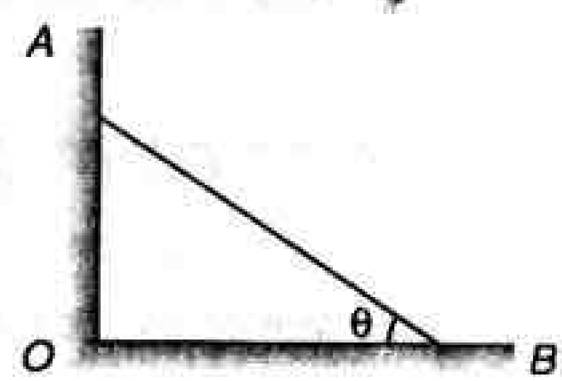


Fig. 9.51

9.9 Uniform Pure Rolling

Pure rolling means no relative motion (or no slipping) at point of contact between two bodies.

For example, consider a disc of radius R moving with linear velocity v and angular velocity ω on a horizontal ground. The disc is said to be moving without slipping if velocities of points P and Q (shown in figure b) are equal, *i.e.*,

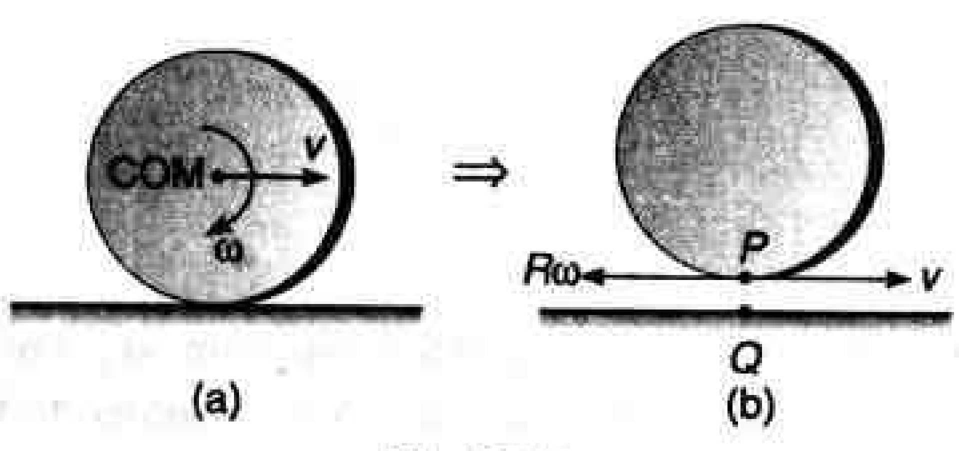


Fig. 9.52

$$v_P = v_Q$$

or

$$v - R\omega = 0$$

or

$$v = R\omega$$

If $v_P > v_Q$ or $v > R\omega$, the motion is said to be forward slipping and if $v_P < v_Q$ or $v < R\omega$, the motion is said the backward slipping (or sometimes called forward english).

Thus, $v = R\omega$ is the condition of pure rolling on a stationary ground. Sometimes it is simply said rolling. Suppose the base over which the disc in rolling, is also moving with some velocity (say v_0) then in that case condition of pure rolling is different.

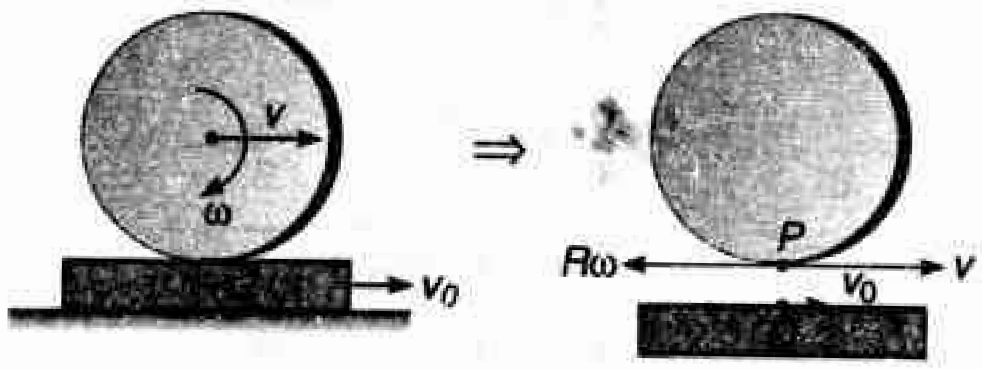


Fig. 9.53

For example, in the above figure,

$$v_P = v_Q$$

or

$$v - R\omega = v_0$$

Thus, in this case $v - R\omega \neq 0$, but $v - R\omega = v_0$. By uniform pure rolling we mean that v and ω are constant. They are neither increasing nor decreasing.

Important points in Uniform Pure Rolling on Ground

In case of pure rolling on a stationary horizontal ground, following points are important to note:

- Distance moved by the centre of mass of the rigid body in one full rotation is $2\pi R$.

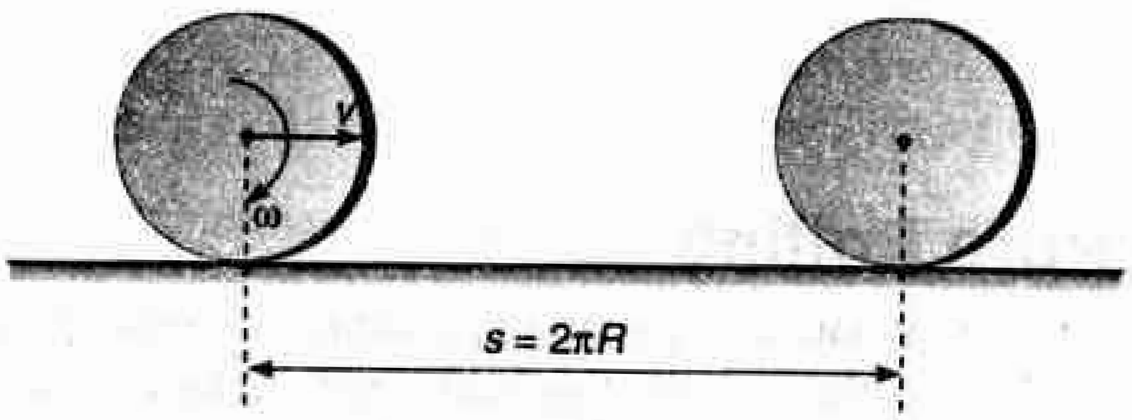


Fig. 9.54

This can be shown as under:

$$s = v \cdot T = (\omega R)\left(\frac{2\pi}{\omega}\right) = 2\pi R$$

In forward slipping $\quad s > 2\pi R \quad$ (as $v > \omega R$)
and in backward slipping $\quad s < 2\pi R \quad$ (as $v < \omega R$)

- Instantaneous axis of rotation ($IAOR$) passes through the bottommost point, as it is a point of zero velocity. Thus, the combined motion of rotation and translation can be assumed to be pure rotational motion about bottommost point with same angular speed ω.

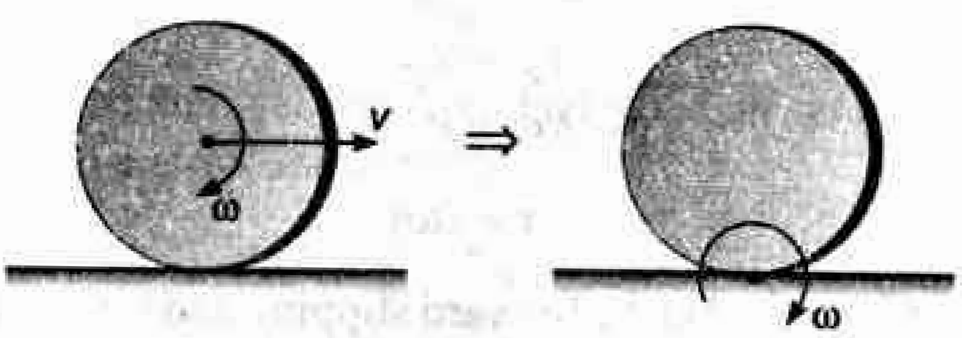


Fig. 9.55

- The speed of a point on the circumference of the body at the instant shown in figure is $2v\sin\frac{\theta}{2}$ or $2R\omega\sin\frac{\theta}{2}$. *i.e.*,

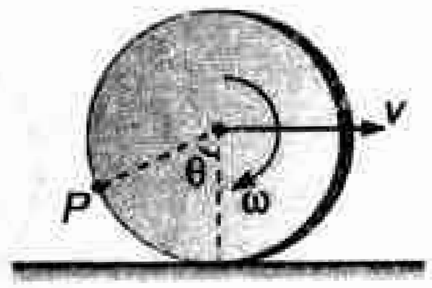


Fig. 9.56

$$|\vec{\mathbf{v}}_P| = v_p = 2v\sin\frac{\theta}{2} = 2R\omega\sin\frac{\theta}{2}$$

This can be shown by following two methods.

Method 1:

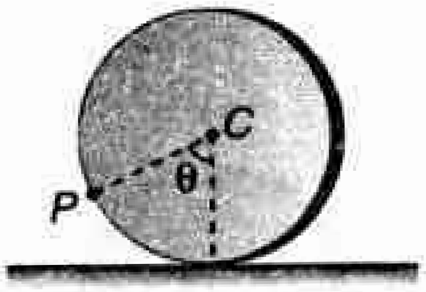


Fig. 9.57

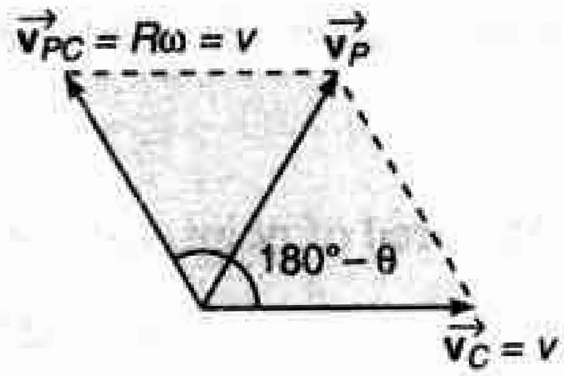


Fig. 9.58

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_C + \vec{\mathbf{v}}_{PC}$$

$$\therefore \quad |\vec{\mathbf{v}}_P| = \sqrt{v^2 + v^2 + 2v\cdot v\cos(180^\circ - \theta)}$$

$$= 2v\sin\frac{\theta}{2}$$

Method 2:

$$|\vec{\mathbf{v}}_P| = (OP)\omega$$

Here, $$OP = 2R\sin\frac{\theta}{2}$$

$$\therefore \quad |\vec{\mathbf{v}}_P| = \left(2R\sin\frac{\theta}{2}\right)\omega$$

$$= 2R\omega\sin\left(\frac{\theta}{2}\right)$$

$$= 2v\sin\frac{\theta}{2}$$

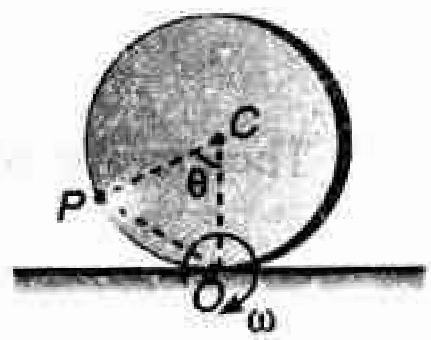


Fig. 9.59

- From point number (3) we can see that

$v_A = 0$ as $\theta = 0^\circ$

$v_B = \sqrt{2}v$ as $\theta = 90^\circ$

and $v_C = 2v$ as $\theta = 180^\circ$

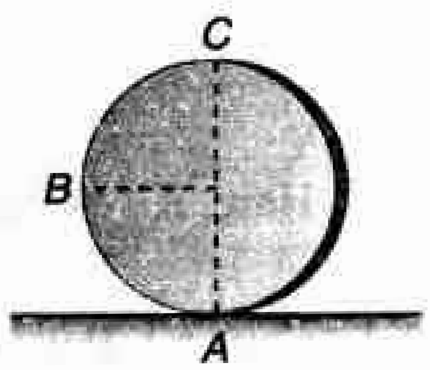


Fig. 9.60

- The path of a point on circumference is a cycloid and the distance moved by this point in one full rotation is $8R$.

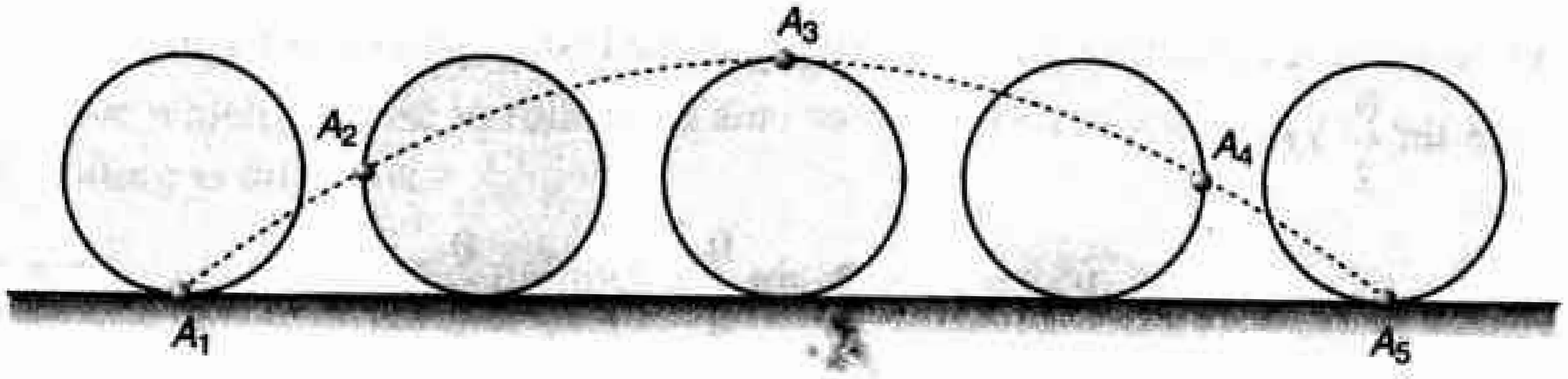


Fig. 9.61

In the figure, the dotted line is a cycloid and the distance $A_1 A_2 \dots A_5$ is $8R$. This can be proved as under.

In figure 9.61 $$\theta = \omega t$$

According to point (3), speed of point A at this moment is,

$$v_A = 2R\omega \sin\left(\frac{\omega t}{2}\right)$$

Distance moved by it in time dt is,

$$ds = v_A \; dt = 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

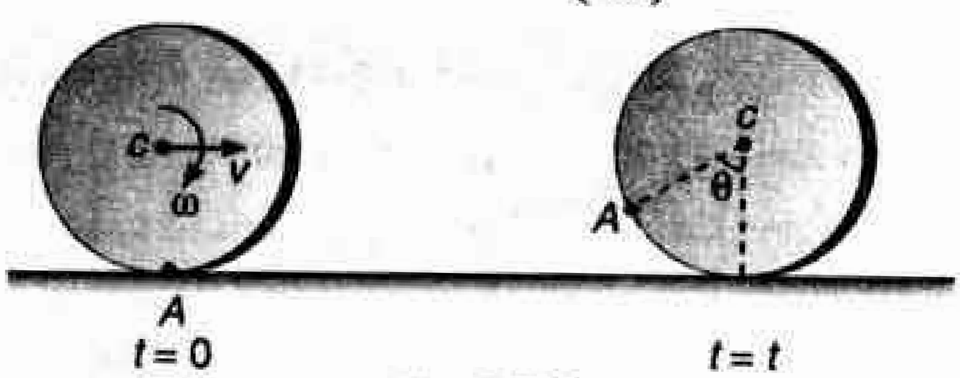


Fig. 9.62

Therefore, total distance moved in one full rotation is,

$$s = \int_0^{T = 2\pi/\omega} ds$$

or $$s = \int_0^{T=2\pi/\omega} 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

On integration we get, $$s = 8R.$$

- $$\frac{K_R}{K_T} = 1 \text{ for a ring} = \frac{1}{2} \text{ for a disc}$$
$$= \frac{2}{5} \text{ for a solid sphere}$$
$$= \frac{2}{3} \text{ for a hollow sphere etc.}$$

Here, K_R stands for rotational kinetic energy $\left(= \frac{1}{2} I\omega^2\right)$ and K_T for translational kinetic energy $\left(= \frac{1}{2} mv^2\right)$. For example, for a disc :

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4} mv^2 \quad \text{and} \quad K_T = \frac{1}{2} mv^2$$

$$\therefore \quad \frac{K_R}{K_T} = \frac{1}{2}$$

Sample Example 9.18 *A disc of radius R start at time $t = 0$ moving along the positive x axis with linear speed v and angular speed ω. Find the x and y coordinates of the bottommost point at any time t.*

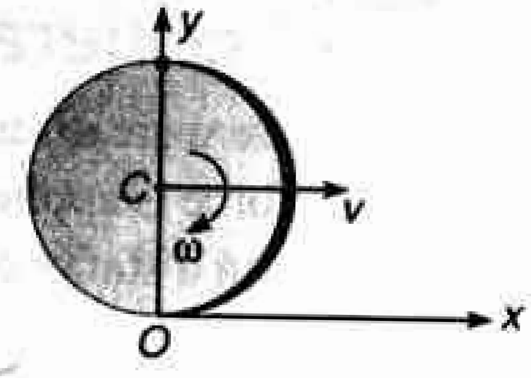


Fig. 9.63

Solution At time t the bottommost point will rotate an angle $\theta = \omega t$ with respect to the centre of the disc C. The centre C will travel a distance $s = vt$.

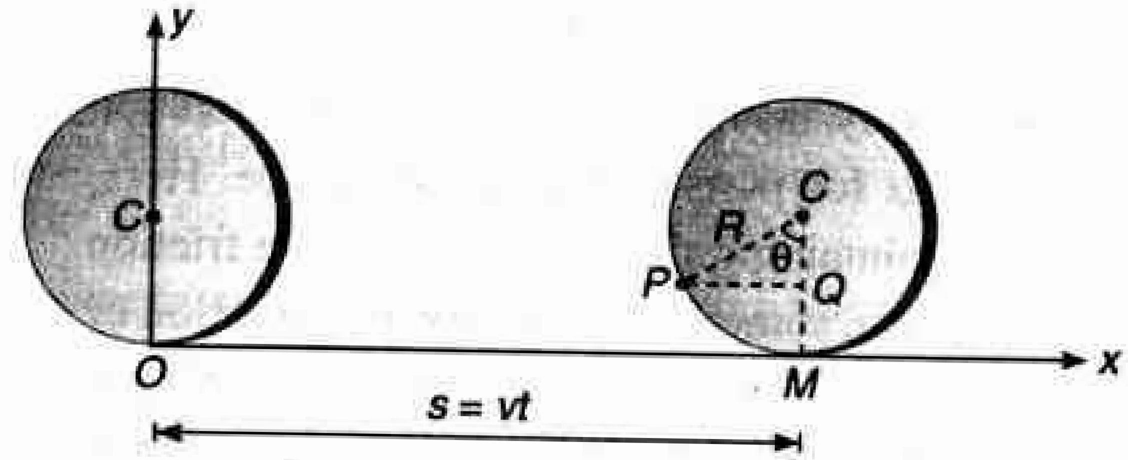


Fig. 9.64

In the figure, $PQ = R\sin\theta = R\sin\omega t$

and $CQ = R\cos\theta = R\cos\omega t$

Coordinates of point P at time t are,

$$x = OM - PQ = vt - R\sin\omega t$$

and $$y = CM - CQ = R - R\cos\omega t$$

$\therefore$ $$(x, y) \equiv (vt - R\sin\omega t, R - R\cos\omega t)$$

Introductory Exercise 9.6

1. A solid sphere of mass m rolls down an inclined plane a height h. Find rotational kinetic energy of the sphere.

   [**Hint :** Mechanical energy will remain conserved]

2. A ring of radius R rolls on a horizontal ground with linear speed v and angular speed ω. For what value of θ the velocity of point P is in vertical direction. ($v < R\omega$)

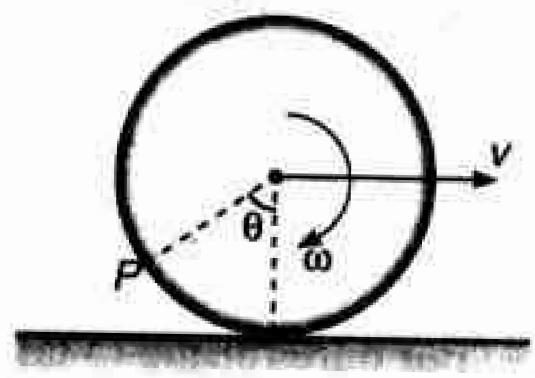


Fig. 9.65

3. The topmost and bottommost velocities of a disc are v_1 and v_2 ($< v_1$) in the same direction. The radius is R. Find the value of angular velocity ω.

9.10 Accelerated Pure Rolling

So, far we were discussing the uniform pure rolling in which v and ω were constants. Now, suppose an external force is applied to the rigid body, the motion will no longer remain uniform. The condition of pure rolling on a stationary ground is,

$$v = R\omega$$

Differentiating this equation with respect to time, we have

$$\frac{dv}{dt} = R \cdot \frac{d\omega}{dt}$$

or

$$a = R\alpha$$

Thus, in addition to $v = R\omega$ at every instant of time, linear acceleration $= R \times$ angular acceleration or $a = R\alpha$ for pure rolling to take place. Here, friction plays an important role in maintaining the pure rolling. The friction may sometimes act in forward direction, sometimes in backward direction or under certain conditions it may be zero. Here, we should not forget the basic nature of friction, which is a self adjusting force (upto a certain maximum limit) and which has a tendency to stop the relative motion between two bodies in contact. Let us take an example illustrating the above theory.

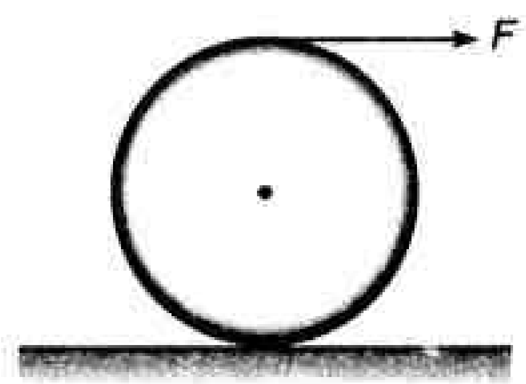


Fig. 9.66

Suppose a force F is applied at the topmost point of a rigid body of radius R, mass M and moment of inertia I about an axis passing through the centre of mass. Now, the applied force F can produce by itself:

(i) a linear acceleration a and

(ii) an angular acceleration α.

If $a = R\alpha$, then there is no need of friction and force of friction $f = 0$. If $a < R\alpha$, then to support the linear motion the force of friction f will act in forward direction. Similarly, if $a > R\alpha$, then to support the angular motion the force of friction will act in backward direction. So, in this case force of friction will be either backward, forward or even zero also. It all depends on M, I and R. For calculation you can choose any direction of friction. Let we assume it in forward direction,

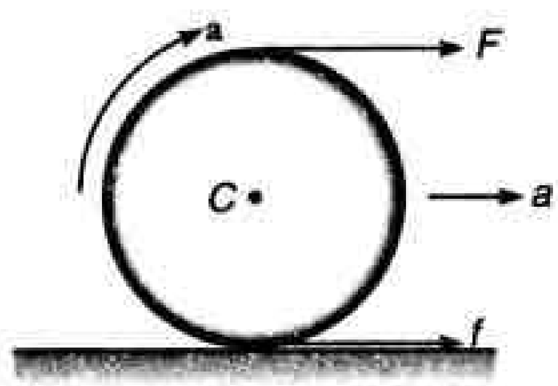


Fig. 9.67

Let, a = linear acceleration, α = angular acceleration

then,

$$a = \frac{F_{net}}{M} = \frac{F + f}{M} \quad \text{...(i)}$$

$$\alpha = \frac{\tau_c}{I} = \frac{(F - f)R}{I} \quad \text{...(ii)}$$

For pure rolling to take place,

$$a = R\alpha \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{(MR^2 - I)}{(MR^2 + I)} \cdot F \quad \text{...(iv)}$$

From Eq. (iv) following conclusions can be drawn

(i) If $I = MR^2$ (*e.g.*, in case of a ring)

$$f = 0$$

i.e., if a force F is applied on the top of a ring, the force of friction will be zero and the ring will roll without slipping.

(ii) If $I < MR^2$, (*e.g.*, in case of a solid sphere or a hollow sphere), f is positive, *i.e.*, force of friction will be forward.

(iii) If $I > MR^2$, f is negative, *i.e.*, force of friction will be backwards. Although under no condition $I > MR^2$. (Think why?). So force of friction is either in forward direction or zero.

Here, it should be noted that the force of friction f obtained in Eq. (iv) should be less than the limiting friction (μMg), for pure rolling to take place. Further, we saw that if $I < MR^2$ force of friction acts in forward direction. This is because α is more if I is small $\left(\alpha = \frac{\tau}{I}\right)$ *i.e.*, to support the linear motion force of friction is in forward direction.

Note *It is often said that rolling friction is less than the sliding friction. This is because the force of friction calculated by equation number (iv) normally comes less than the sliding friction ($\mu_k N$) and even sometimes it is in forward direction, i.e., it supports the motion.*

There are certain situations in which the direction of friction is fixed. For example in the following situations the force of friction is backward.

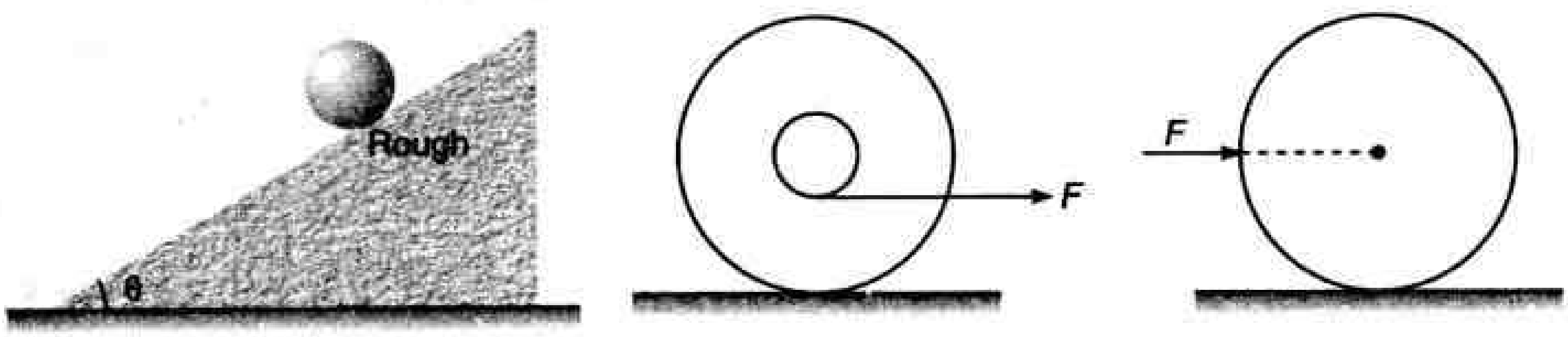


Fig. 9.68

Rolling on Rough Inclined Plane

As we said earlier also, force of friction in this case will be backward. Equations of motion are :

$$a = \frac{Mg \sin \theta - f}{M} \quad \text{...(i)}$$

$$\alpha = \frac{fR}{I} \quad \text{...(ii)}$$

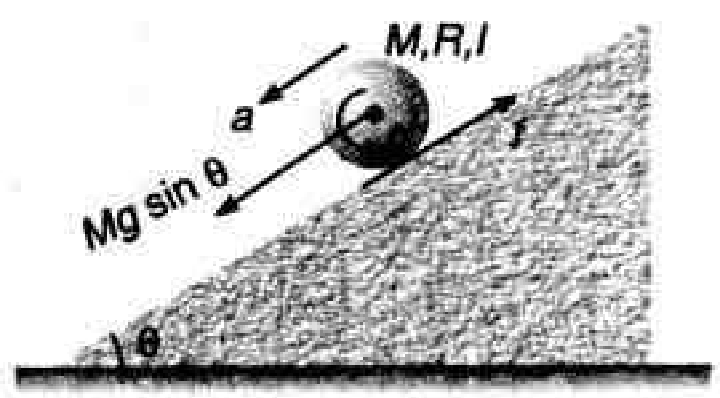


Fig. 9.69

For pure rolling to take place,

$$a = R\alpha \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{Mg \sin \theta}{1 + \frac{MR^2}{I}} \quad \text{...(iv)}$$

and

$$a = \frac{g \sin \theta}{1 + \frac{I}{MR^2}} \quad \text{...(v)}$$

From Eq. (v), we can see that if a solid sphere and a hollow sphere of same mass and radius are released from a rough inclined plane the solid sphere reaches the bottom first because :

$$I_{solid} < I_{hollow} \quad \text{or} \quad a_{solid} > a_{hollow}$$

$\therefore$ $$t_{solid} < t_{hollow}$$

Further, the force of friction calculated in Eq. (iv) for pure rolling to take place should be less than or equal to the maximum friction $\mu Mg \cos\theta$.

or $$\frac{Mg\sin\theta}{1+\frac{MR^2}{I}} \le \mu\, Mg\cos\theta$$

or $$\mu \ge \frac{\tan\theta}{1+\frac{MR^2}{I}}$$

Sample Example 9.19 *In the arrangement shown in figure the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions $\frac{T_1}{T_2}$ of the vertical sections of the thread in the process of motion.*

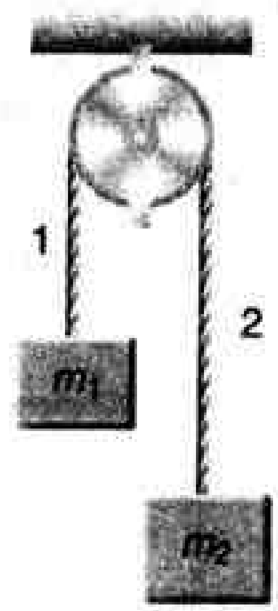


Fig. 9.70

Solution Let α = angular acceleration of the cylinder

and a = linear acceleration of two bodies

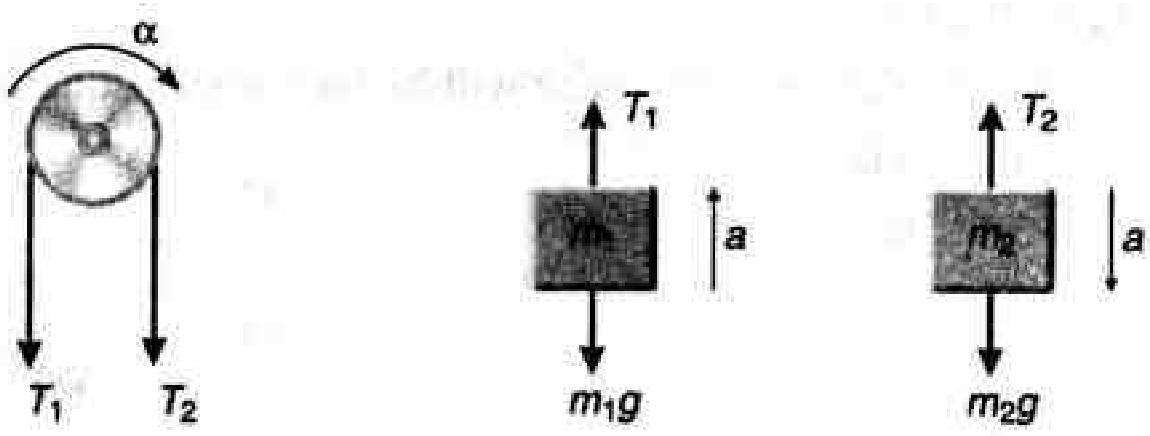


Fig. 9.71

Equations of motion are:

For mass m_1, $$T_1 - m_1g = m_1a \quad \text{...(i)}$$

For mass m_2, $$m_2g - T_2 = m_2a \quad \text{...(ii)}$$

For cylinder, $$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2}mR^2} \quad \text{...(iii)}$$

For no slipping condition $$a = R\alpha \quad \text{...(iv)}$$

Solving these equations, we get

$$\alpha = \frac{2(m_2 - m_1)g}{(2m_1 + 2m_2 + m)R}$$

and

$$\frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

Sample Example 9.20 *Consider the arrangement shown in figure. The string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a massless, frictionless pulley to a falling weight. Determine the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.*

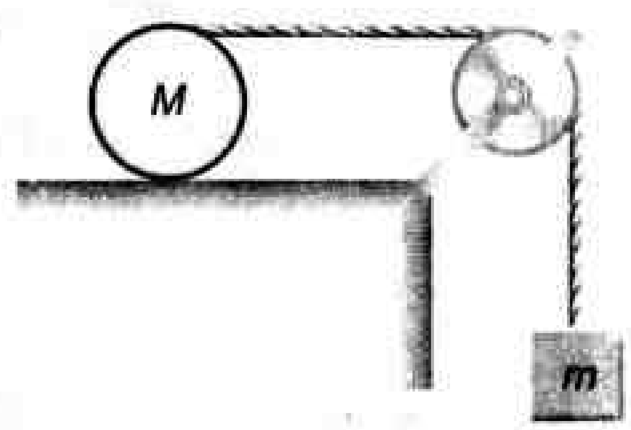


Fig. 9.72

Solution Let T be the tension in the string and f the force of (static) friction, between the cylinder and the surface

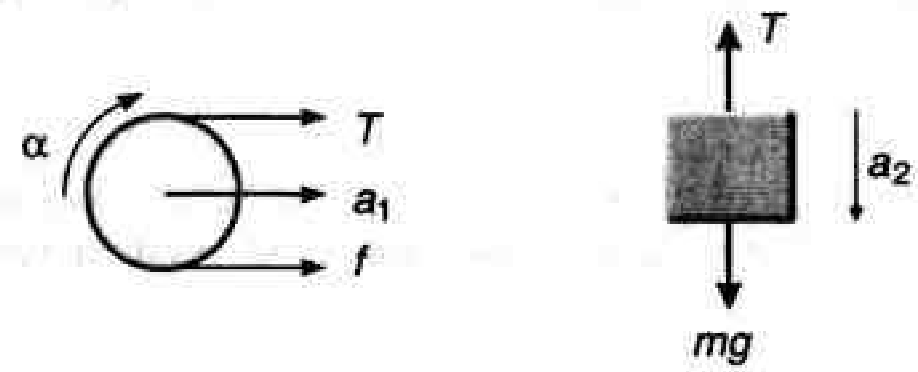


Fig. 9.73

a_1 = acceleration of centre of mass of cylinder towards right

a_2 = downward acceleration of block m

α = angular acceleration of cylinder (clockwise)

Equations of motion are:

For block, $$mg - T = ma_2 \quad \text{...(i)}$$

For cylinder, $$T + f = Ma_1 \quad \text{...(ii)}$$

$$\alpha = \frac{(T - f)R}{\frac{1}{2}MR^2} \quad \text{...(iii)}$$

The string attaches the mass m to the highest point of the cylinder, hence

$$v_m = v_{COM} + R\omega$$

Differentiating, we get $$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

We also have (for rolling without slipping)

$$a_1 = R\alpha \quad \text{...(v)}$$

Solving these equations, we get $$a_2 = \frac{8mg}{3M + 8m}$$

Alternate Solution (Energy Method)

Since, there is no slipping at all contacts mechanical energy of the system will remain conserved.

∴ Decrease in gravitational potential energy of block m in time t = increase in translational kinetic energy of block + increase in rotational as well as translational kinetic energy of cylinder.

$$\therefore \quad mgh = \frac{1}{2} mv_2^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} Mv_1^2$$

or

$$mg\left(\frac{1}{2} a_2 t^2\right) = \frac{1}{2} m (a_2 t)^2 + \frac{1}{2}\left(\frac{1}{2} MR^2\right)(\alpha t)^2 + \frac{1}{2} M(a_1 t)^2 \quad \text{...(vi)}$$

Solving Eqs. (iv), (v) and (vi), we get the same result.

Introductory Exercise 9.7

1. A ball of mass M and radius R is released on a rough inclined plane of inclination θ. Friction is not sufficient to prevent slipping. The coefficient of friction between the ball and the plane is μ. Find:
   (a) the linear acceleration of the ball down the plane,
   (b) the angular acceleration of the ball about its centre of mass.
2. Work done by friction in pure rolling is always zero. Is this statement true or false?
3. A spool is pulled by a force in vertical direction as shown in figure. What is the direction of friction in this case? The spool does not loose contact with the ground.

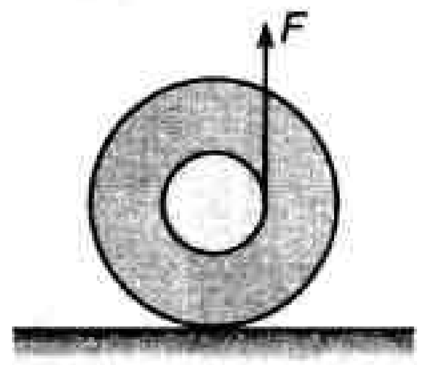


Fig. 9.74

4. A cylinder is rolling down a rough inclined plane. Its angular momentum about the point of contact remains constant. Is this statement true or false?
5. Two forces F_1 and F_2 are applied on a spool of mass M and moment of inertia I about an axis passing through its centre of mass. Find the ratio $\frac{F_1}{F_2}$, so that the force of friction is zero. Given that $I < 2Mr^2$.

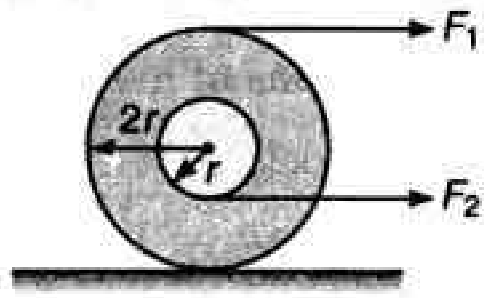


Fig. 9.75

6. A disc is placed on the ground. Friction coefficient is μ. What is the minimum force required to move the disc if it is applied at the topmost point?
7. When a body rolls, on a stationary ground, the acceleration of the point of contact is always zero. Is this statement true or false?

9.11 Angular Impulse

The angular impulse of a torque in a given time interval is defined as $\int_{t_1}^{t_2} \vec{\tau}\, dt$

Here, $\vec{\tau}$ is the resultant torque acting on the body. Further, since

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} \quad \therefore \quad \vec{\tau}\,dt = d\vec{\mathbf{L}}$$

or
$$\int_{t_1}^{t_2} \vec{\tau}dt = \text{angular impulse} = \vec{\mathbf{L}}_2 - \vec{\mathbf{L}}_1$$

Thus, the angular impulse of the resultant torque is equal to the change in angular momentum. Let us take few examples based on the angular impulse.

Sample Example 9.21 *A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the result be affected in the case of a perfectly smooth inclined plane? The angle of inclination of the plane is θ.*

Solution Applying the equation.

Angular impulse = change in angular momentum about point of contact we have,

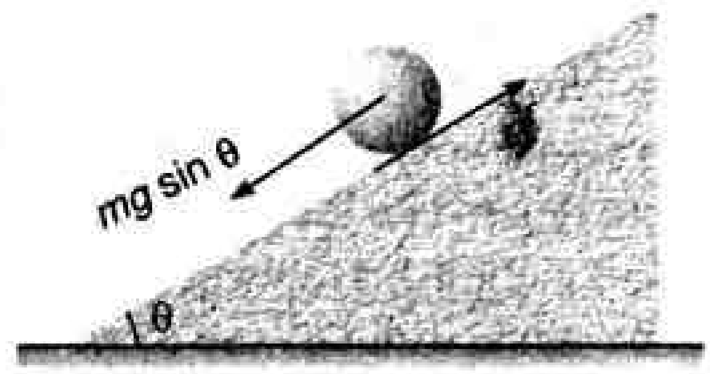


Fig. 9.76

$$\int \vec{\tau}\,dt = \Delta\vec{\mathbf{L}}$$

or
$$L = (mg\sin\theta)Rt$$

There will be no change in the result, as the torque of force of friction in the first case is zero about point of contact. So, it hardly matters whether the surface is rough or smooth.

9.12 Toppling

You might have seen in your practical life that if a force F is applied to a block A of smaller width it is more likely to topple down, before sliding while if the same force F is applied to an another block B of broader base, chances of its sliding are more compared to its toppling. Have you ever throught why it happens sc. To understand it better let us take an example.

Fig. 9.77

Suppose a force F is applied at a height b above the base AE of the block. Further, suppose the friction f is sufficient to prevent sliding. In this case, if the normal reaction N also passes through C, then despite the fact that the block is in translational equilibrium ($F = f$ and $N = mg$), an unbalanced torque (due to the couple of forces F and f) is there.

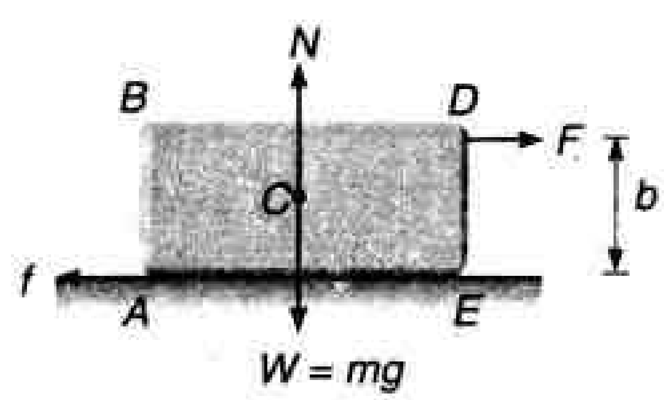


Fig. 9.78

This torque has a tendency to topple the block about point E. To cancel the effect of this unbalanced torque the normal reaction N is shifted towards right a distance 'a' such that, net anticlockwise torque is equal to the net clockwise torque or

$$Fb = (mg)a$$

or

$$a = \frac{Fb}{mg}$$

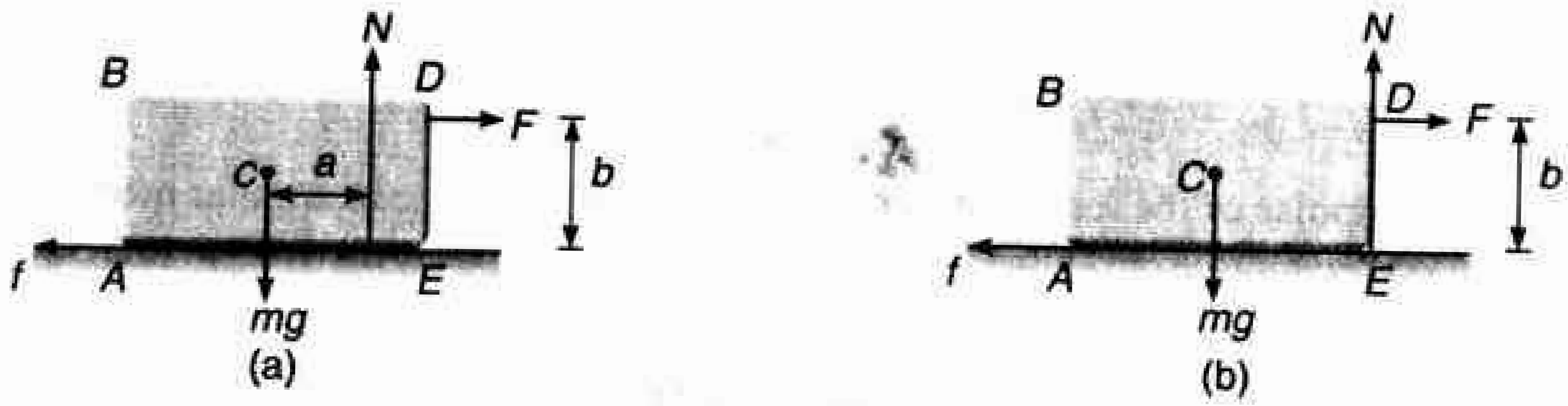


Fig. 9.79

Now, as F or b (or both) are increased, distance a also increases. But it can not go beyond the right edge of the block. So, in extreme case (beyond which the block will topple down), the normal reaction passes through E as shown in Fig. 9.79(b).

Now, if F or b are further increased, the block will topple down. This is why the block having the broader base has less chances of toppling in comparison to a block of smaller base. Because the block of larger base has more margin for the normal reaction to shift. On the similar ground we can see why the rolling is so easy.

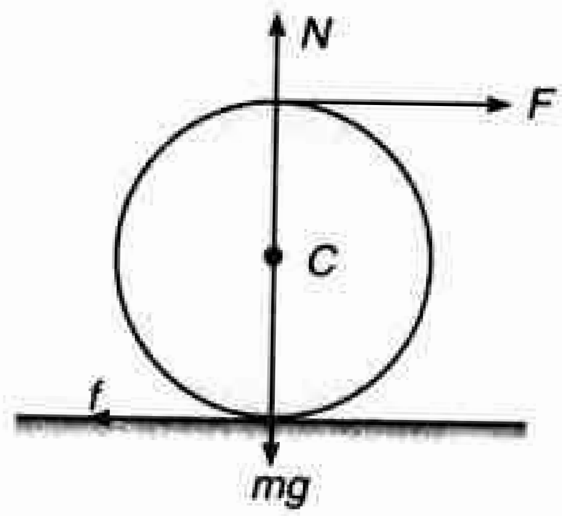


Fig. 9.80

Because in this case the normal reaction has zero margin to shift. So even if the body is in translational equilibrium ($F = f$, $N = mg$) an unbalanced torque is left behind and the body starts rolling clockwise. As soon as the body starts rolling the force of friction is so adjusted (both in magnitude and direction) that either the pure rolling starts (if friction is sufficient enough) or the body starts sliding. Let us take few examples related to toppling.

Sample Example 9.22 *A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $\frac{3a}{4}$ above the base. What is the minimum value of F for which the cube begins to tip about an edge?*

Solution In the limiting case normal reaction will pass through O. The cube will tip about O if torque of F exceeds the torque of mg.

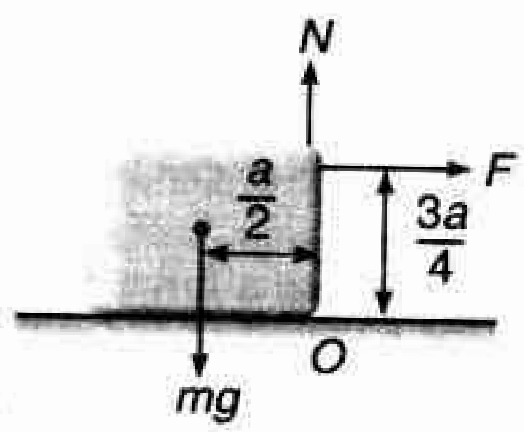


Fig. 9.81

Hence,

$$F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

or

$$F > \frac{2}{3}mg$$

Therefore, minimum value of F is $\frac{2}{3}mg$.

Sample Example 9.23 *A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will (a) slide before toppling (b) topple before sliding.*

Solution (a) The cylinder will slide if

$$mg\sin\theta > \mu mg\cos\theta$$

or $$\tan\theta > \mu \quad \text{...(i)}$$

The cylinder will topple if $(mg\sin\theta)\dfrac{h}{2} > (mg\cos\theta)r$

or $$\tan\theta > \frac{2r}{h} \quad \text{...(ii)}$$

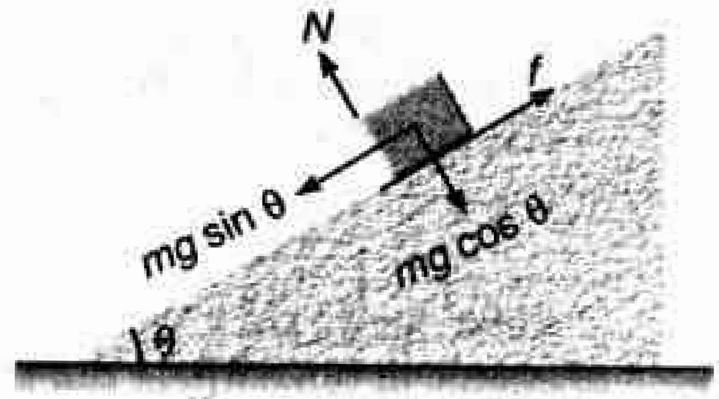


Fig. 9.82

Thus, the condition of sliding is $\tan\theta > \mu$ and condition of toppling is $\tan\theta > \dfrac{2r}{h}$. Hence, the cylinder will slide before toppling if

$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if $\mu > \dfrac{2r}{h}$

Introductory Exercise 9.8

1. A cube is resting on an inclined plane. If the angle of inclination is gradually increased, what must be the coefficient of friction between the cube and plane so that,
   (a) cube slides before toppling?
   (b) cube topples before sliding?

2. A solid sphere of mass M and radius R is hit by a cue at a height h above the centre C. For what value of h the sphere will roll without slipping?

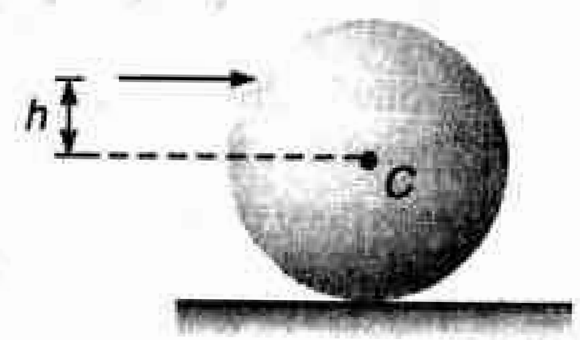


Fig. 9.83

Extra Points

- Whether a particle is in translational motion, rotational motion or in both it merely depends on the reference point with respect to which the motion of the particle is described.

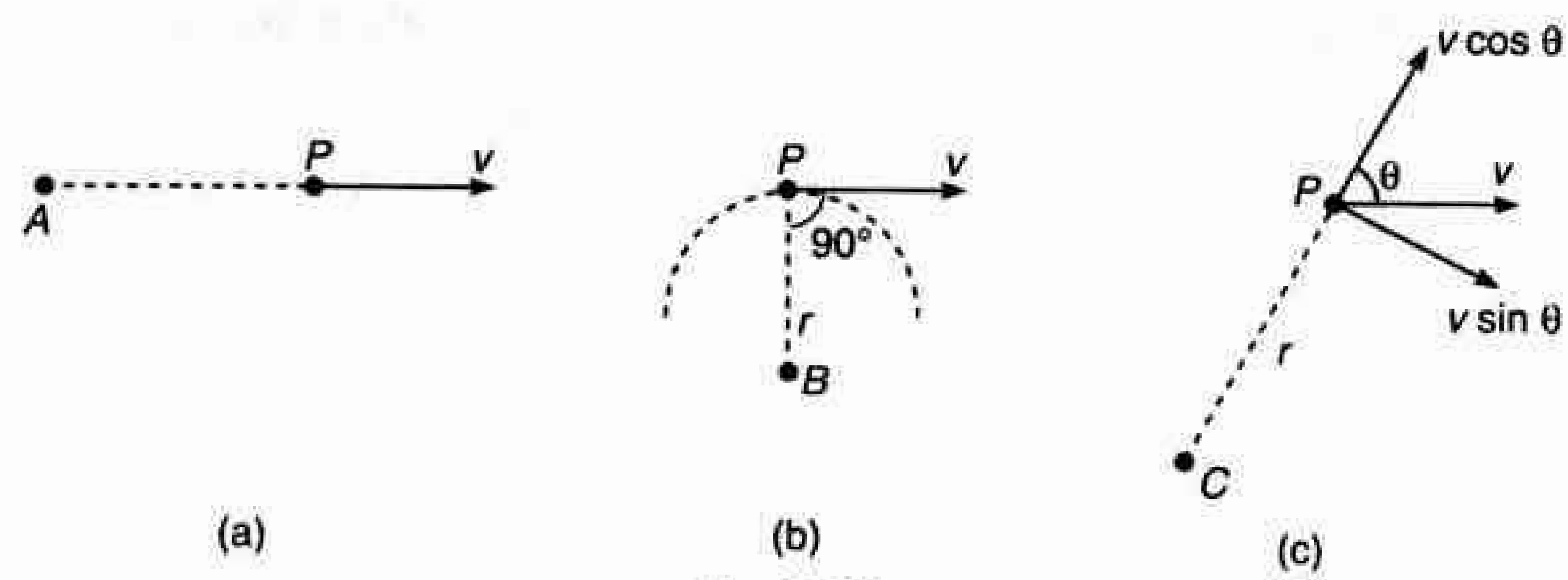


Fig. 9.84

For example : Suppose a particle P of mass m is moving in a straight line as shown in figures (a), (b) and (c).

Refer figure (a) : With respect to point A, the particle is in pure translational motion. Hence, kinetic energy of the particle can be written as

$$KE = \frac{1}{2} mv^2$$

Refer figure (b) : With respect to point B, the particle is in pure rotational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} I\omega^2 = \frac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2$$
$$= \frac{1}{2} mv^2$$

Refer figure (c) : With respect to point C, the particle can be assumed to be in rotational as well as translational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} m (v \cos \theta)^2 + \frac{1}{2} I\omega^2$$
$$= \frac{1}{2} m (v \cos \theta)^2 + \frac{1}{2}(mr^2)\left(\frac{v \sin \theta}{r}\right)^2$$
$$= \frac{1}{2} mv^2$$

Thus, in all the three cases, the kinetic energy of the particle comes out to be the same.

- If $\vec{F}$ is perpendicular to AB, but does not intersect it, then component of torque about line AB = magnitude of force $\vec{F}$ × perpendicular distance of $\vec{F}$ from the line AB (called the lever arm or moment arm) of this torque.
- Work done by friction in pure rolling on a stationary ground in zero as the point of application of the force is at rest. Therefore, mechanical energy can be conserved if all other dissipative forces are ignored.
- In cases where pulley is having some mass and friction is sufficient enough to prevent slipping, the tension on two sides of the pulley will be different and rotational motion of the pulley is also to be considered.
-

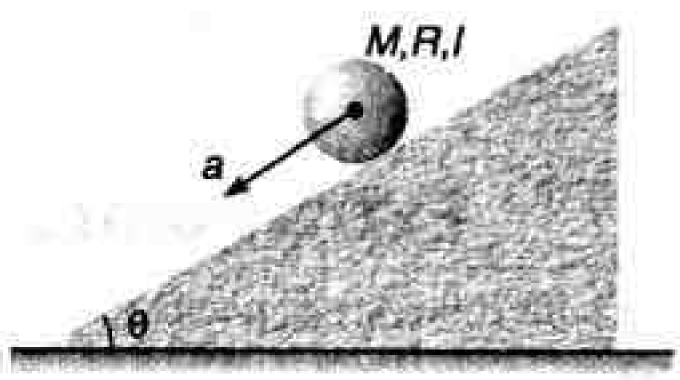


Fig. 9.85

$a = g \sin \theta$	if surface is smooth
$a = g \sin \theta - \mu g \cos \theta$	if surface is rough but friction is insufficient to prevent slipping. (forward slipping will take place)
$a = \dfrac{g \sin \theta}{1 + I/MR^2}$	if pure rolling is taking place, *i.e.*, friction is sufficient to prevent slipping.

Solved Examples

For JEE Main

Example 1 *If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?*

Solution Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2\omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}MR^2\omega = \frac{1}{4}\times\frac{2}{5}MR^2\omega'$$

i.e.,

$$\omega' = 4\omega$$

$$T' = \frac{1}{4}T = \frac{1}{4}\times 24 = 6\,\text{h}$$

Example 2 *A particle of mass m is projected with velocity v at an angle θ with the horizontal. Find its angular momentum about the point of projection when it is at the highest point of its trajectory.*

Solution At the highest point it has only horizontal velocity $v_x = v\cos\theta$

Length of the perpendicular to the horizontal velocity from 'O' is the maximum height, where

$$H_{max} = \frac{v^2\sin^2\theta}{2g}$$

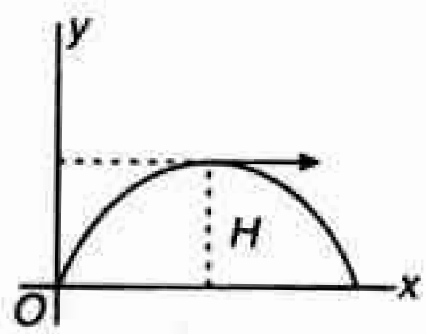


Fig. 9.86

$\Rightarrow$ Angular momentum $L = \dfrac{mv^3\sin^2\theta\cos\theta}{2g}$

Example 3 *A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is μ. What is maximum value of F, for which there is no slipping ?*

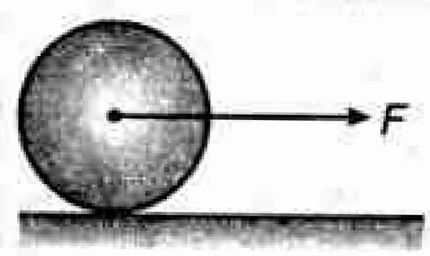


Fig. 9.87

Solution

$$F - f = Ma$$

$$f \cdot R = \frac{2}{5} MR^2 \frac{a}{R} \Rightarrow f = \frac{2}{5} Ma$$

$\Rightarrow$
$$f = \frac{2}{7} F$$

$$\frac{2}{7} F \leq \mu mg \Rightarrow F \leq \frac{7}{2} \mu mg$$

Example 4 *A tangential force F acts at the top of a thin spherical shell of mass m and radius R. Find the acceleration of the shell if it rolls without slipping.*

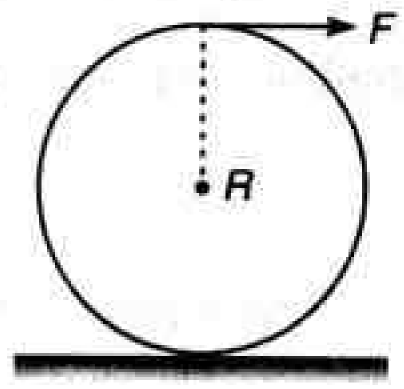


Fig. 9.88

Solution Let f be the force of friction between the shell and the horizontal surface.

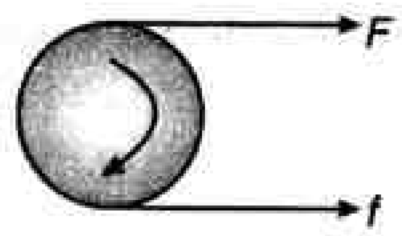


Fig. 9.89

For translational motion,

$$F + f = ma \qquad \ldots\text{(i)}$$

For rotational motion,

$$FR - fR = I\alpha = I\frac{a}{R} \qquad [\because a = R\alpha \text{ for pure rolling}]$$

$\Rightarrow$
$$F - f = I\frac{a}{R^2} \qquad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2F = \left(m + \frac{I}{R^2}\right)a = \left(m + \frac{2}{3}m\right)a = \frac{5}{3}ma$$

or
$$F = \frac{5}{6}ma \qquad \left[\because I_{\text{shell}} = \frac{2}{3}mR^2\right]$$

$\Rightarrow$
$$a = \frac{6F}{5m}$$

Example 5 *A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its centre of mass when its centre of mass has fallen a height h.*

Solution Considering the two shown positions of the cylinder. As it does not slip hence total mechanical energy will be conserved.

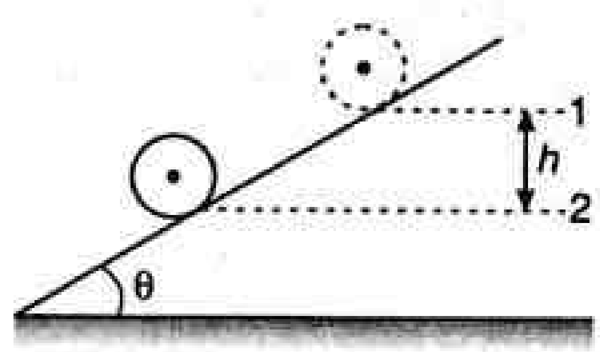


Fig. 9.90

Energy at postion 1 is $\quad E_1 = mgh$

Energy at position 2 is $\quad E_2 = \frac{1}{2} mv_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}}\, \omega^2$

$\because \quad \frac{v_{\text{COM}}}{r} = \omega$ and $I_{\text{COM}} = \frac{mr^2}{2}$

$$\Rightarrow \qquad E_2 = \frac{3}{4} m v_{COM}^2$$

From COE, $E_1 = E_2$

$$\Rightarrow \qquad v_{COM} = \sqrt{\frac{4}{3} gh}$$

Example 6 *A disc starts rotating with constant angular acceleration of* π *rad/s²* *about a fixed axis perpendicular to its plane and through its centre. Find :*

(a) the angular velocity of the disc after 4 s.

(b) the angular displacement of the disc after 4 s and

Solution Here $\alpha = \pi \text{ rad/s}^2, \quad \omega_0 = 0, \quad t = 4\text{ s}$

(a) $\omega_{(4\text{ s})} = 0 + (\pi \text{ rad/s}^2) \times 4\text{ s} = 4\pi \text{ rad/s}.$

(b) $\theta_{(4\text{ s})} = 0 + \frac{1}{2}(\pi \text{ rad/s}^2) \times (16\text{s}^2) = 8\pi \text{ rad}$

Example 7 *A small solid cylinder of radius r is released coaxially from point A inside the fixed large cylindrical bowl of radius R as shown in figure. If the friction between the small and the large cylinder is sufficient enough to prevent any slipping, then find :*

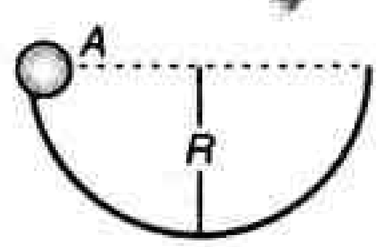


Fig. 9.91

(a) What fractions of the total energy are translational and rotational, when the small cylinder reaches the bottom of the larger one?

(b) The normal force exerted by the small cylinder on the larger one when it is at the bottom.

Solution (a)

$$K_{trans} = \frac{1}{2} m v^2$$

$$K_{rot} = \frac{1}{2} I\omega^2 = \frac{1}{4} m v^2$$

$$K = K_{trans} + K_{rot} = \frac{3}{4} m v^2$$

$$\therefore \qquad \frac{K_{trans}}{K} = \frac{2}{3}$$

$$\frac{K_{rot}}{K} = \frac{1}{3}$$

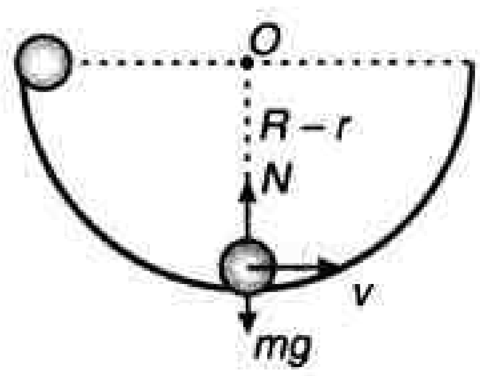


Fig. 9.92

(b) From conservation of energy,

$$mg(R - r) = \frac{3}{4} m v^2$$

$$\therefore \qquad \frac{mv^2}{R - r} = \frac{4}{3} mg$$

Now, $$N - mg = \frac{mv^2}{R - r}$$

$$\therefore \qquad N = \frac{7}{3} mg$$

Example 8 *A wheel rotates around a stationary axis so that the rotation angle θ varies with time as $\theta = at^2$, where $a = 0.2 rad/s^2$. Find the magnitude of net acceleration of the point A at the rim at the moment $t = 2.5$ s if the linear velocity of the point A at this moment is $v = 0.65$ m/s.*

Solution Instantaneous angular velocity at time t is

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(at^2)$$

or $$\omega = 2at = 0.4t \qquad \text{(as } a = 0.2 \text{ rad/s}^2\text{)}$$

Further, instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.4t)$$

or $$\alpha = 0.4 \text{ rad/s}^2$$

Angular velocity at $t = 2.5 \text{ s}$ is

$$\omega = 0.4 \times 2.5 = 1.0 \text{ rad/s}$$

Further, radius of the wheel $R = \frac{v}{\omega}$ or $R = \frac{0.65}{1.0} = 0.65 \text{ m}$

Now, magnitude of total acceleration is,

$$a = \sqrt{a_n^2 + a_t^2}$$

Here, $$a_n = R\omega^2 = (0.65)(1.0)^2 = 0.65 \text{ m/s}^2$$

and $$a_t = R\alpha = (0.65)(0.4) = 0.26 \text{ m/s}^2$$

∴ $$a = \sqrt{(0.65)^2 + (0.26)^2}$$

or $$a = 0.7 \text{ m/s}^2$$

Example 9 *A solid ball of radius 0.2 m and mass 1 kg is given an instantaneous impulse of 50 N-s at point P as shown. Find the number of rotations made by the ball about its diameter before hitting the ground. The ball is kept on smooth surface initially.*

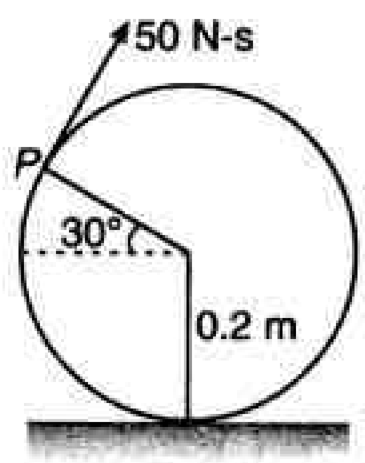


Fig. 9.93

Solution Impulse gives translational velocity

$$u = \frac{\text{Impulse}}{\text{Mass}} \text{ along impulse} = 50 \text{ m/s}$$

$$T = \text{time of flight of projectile}$$

$$= \frac{2u \sin\theta}{g} = \frac{2 \times 50 \times \sin 60°}{10} = 5\sqrt{3} \text{ sec}$$

Impulse give angular impulse also

$$\omega = \frac{\text{Impulse} \times R}{I}$$

or

$$\omega = \frac{\text{Impulse} \times R}{\frac{2}{5} mR^2}$$

Number of rotations,

$$n = \frac{\omega T}{2\pi} = \frac{3125\sqrt{3}}{2\pi}$$

For **JEE Advanced**

Example 1 *A solid ball rolls down a parabolic path ABC from a height h as shown in figure. Portion AB of the path is rough while BC is smooth. How high will the ball climb in BC?*

Hint *In case of pure rolling mechanical energy is conserved.*

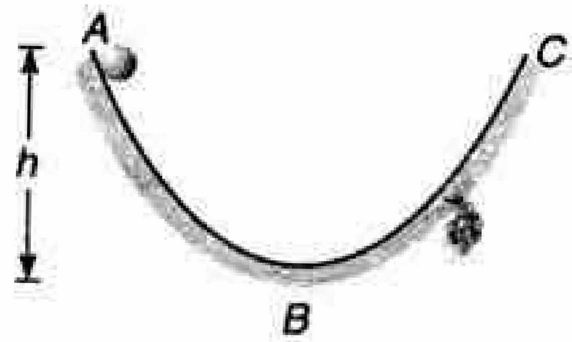


Fig. 9.94

Solution At B, total kinetic energy $= mgh$

Here, $m =$ mass of ball

The ratio of rotational to translational kinetic energy would be,

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$\therefore$

$$K_R = \frac{2}{7} mgh \quad \text{and} \quad K_T = \frac{5}{7} mgh$$

In portion BC, friction is absent. Therefore, rotational kinetic energy will remain constant and translational kinetic energy will convert into potential energy. Hence, if H be the height to which ball climbs in BC, then

$$mgH = K_T$$

or

$$mgH = \frac{5}{7} mgh \quad \text{or} \quad H = \frac{5}{7} h$$

Example 2 *A thread is wound around two discs on either sides. The pulley and the two discs have the same mass and radius. There is no slipping at the pulley and no friction at the hinge. Find out the accelerations of the two discs and the angular acceleration of the pulley.*

Fig. 9.95

Solution Let R be the radius of the discs and T_1 and T_2 be the tensions in the left and right segments of the rope.

Fig. 9.96

Acceleration of disc 1,

$$a_1 = \frac{mg - T_1}{m} \quad ...(i)$$

Acceleration of disc 2,

$$a_2 = \frac{mg - T_2}{m} \quad ...(ii)$$

Angular acceleration of disc 1, $$\alpha_1 = \frac{\tau}{I} = \frac{T_1 R}{\frac{1}{2} mR^2} = \frac{2T_1}{mR} \quad ...(iii)$$

Similarly, angular acceleration of disc 2, $$\alpha_2 = \frac{2T_2}{mR} \quad ...(iv)$$

Both α_1 and α_2 are clockwise.

Angular acceleration of pulley,

$$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2} mR^2} = \frac{2(T_2 - T_1)}{mR} \quad ...(v)$$

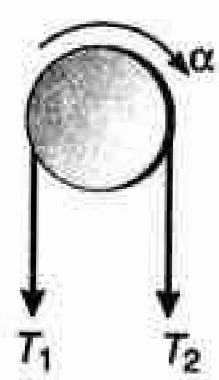


Fig. 9.97

For no slipping, $$R\alpha_1 - a_1 = a_2 - R\alpha_2 = R\alpha \quad ...(vi)$$

Solving these equations, we get

$$\alpha = 0 \quad \text{and} \quad a_1 = a_2 = \frac{2g}{3}$$

Alternate Solution

As both the discs are in identical situation, $T_1 = T_2$ and $\alpha = 0$. *i.e.*, each of the discs falls independently and identically. Therefore, this is exactly similar to the problem shown in figure.

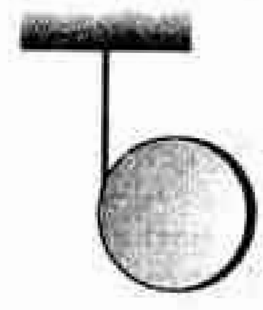


Fig. 9.98

Example 3 *A thin massless thread is wound on a reel of mass* 3 *kg and moment of inertia* 0.6 *kg-m*^{2}. *The hub radius is* $R = 10$ *cm and peripheral radius is* $2R = 20$ *cm. The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of* 1 *kg.*

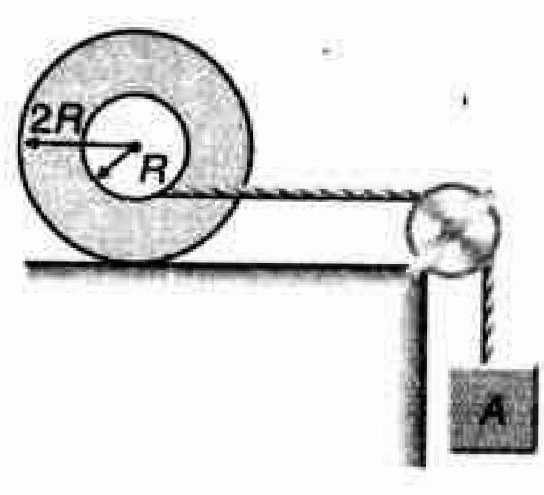


Fig. 9.99

Solution Let, a_1 = acceleration of centre of mass of reel

a_2 = acceleration of 1 kg block

α = angular acceleration of reel (clockwise)

T = tension in the string

and f = force of friction

Free body diagram of reel is as shown below: (only horizontal forces are shown).

Equations of motion are :

$$T - f = 3a_1 \quad \text{...(i)}$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T.R}{I} = \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \quad \text{...(ii)}$$

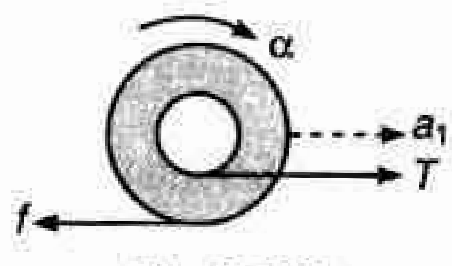


Fig. 9.100

Free body diagram of mass is,

Equation of motion is,

$$10 - T = a_2 \quad \text{...(iii)}$$

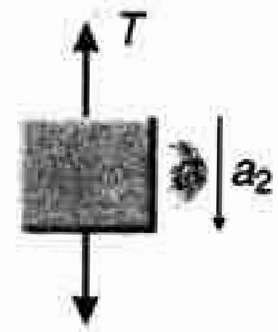


Fig. 9.101

For no slipping condition,

$$a_1 = 2R\alpha \quad \text{or} \quad a_1 = 0.2\alpha \quad \text{...(iv)}$$

and $$a_2 = a_1 - R\alpha \quad \text{or} \quad a_2 = a_1 - 0.1\alpha \quad \text{...(v)}$$

Solving the above five equations, we get

$$a_1 = 0.27 \text{ m/s}^2$$

and $$a_2 = 0.135 \text{ m/s}^2$$

Example 4 *A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the time t when the slipping stops. In addition, state the linear velocity v and angular velocity ω at the end of slipping.*

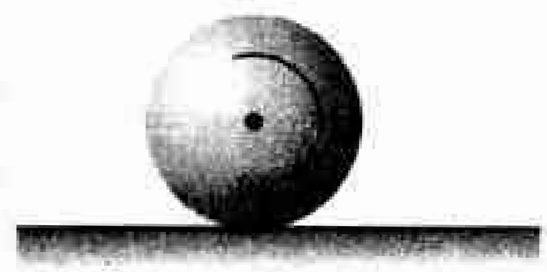


Fig. 9.102

Solution Let m be the mass of the sphere.

Since, it is a case of backward slipping, force of friction is in forward direction. Limiting friction will act in this case.

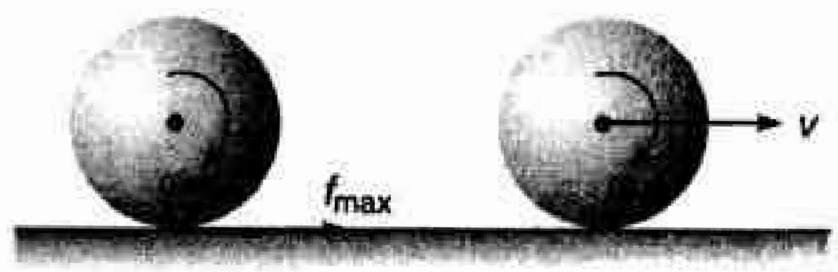


Fig. 9.103

Linear acceleration $$a = \frac{f}{m} = \frac{\mu mg}{m} = \mu g$$

Angular retardation $$\alpha = \frac{\tau}{I} = \frac{f \cdot r}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$$

Slipping is ceased when. $$v = r\omega$$

or $$(at) = r(\omega_0 - \alpha t)$$

or $$\mu gt = r\left(\omega_0 - \frac{5}{2}\frac{\mu gt}{r}\right)$$

or
$$\frac{7}{2}\mu g t = r\omega_0$$

$\therefore$
$$t = \frac{2}{7}\frac{r\omega_0}{\mu g}$$

$$v = at = \mu g t = \frac{2}{7} r\omega_0$$

and
$$\omega = \frac{v}{r} = \frac{2}{7}\omega_0$$

Alternate Solution

Net torque on the sphere about the bottommost point is zero. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

$$L_i = L_f$$

$\therefore$
$$I\omega_0 = I\omega + mrv$$

or
$$\frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

$\therefore$
$$\omega = \frac{2}{7}\omega_0 \quad \text{and} \quad v = r\omega = \frac{2}{7}r\omega_0$$

Example 5 *A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in figure. The ball leaves the cue with a speed v_0 and because of its forward english (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$. Show that*

$$h = \frac{4}{5}R$$

where R is the radius of the ball.

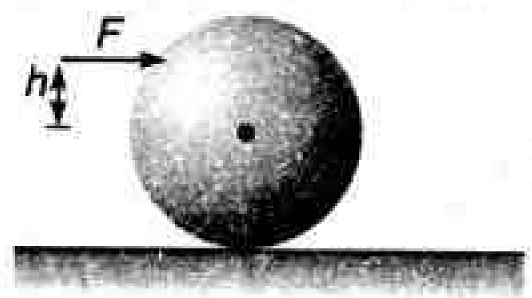


Fig. 9.104

Solution Let ω_0 be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let v be the linear speed and ω the angular speed when slipping is ceased.

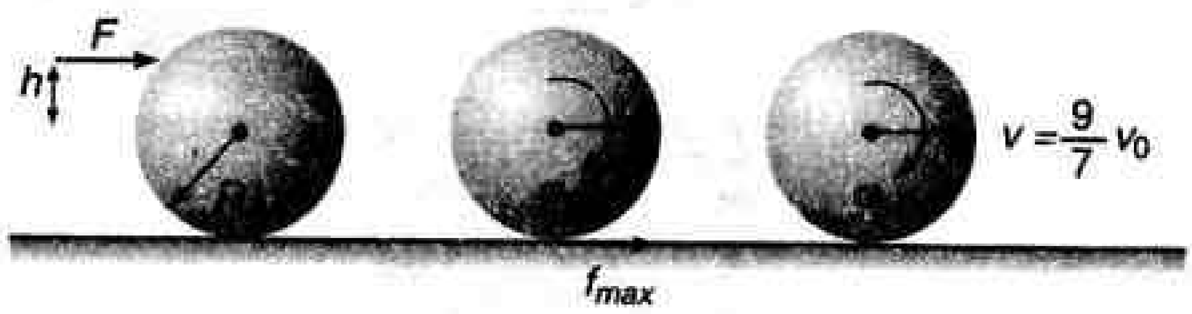


Fig. 9.105

$\therefore$
$$v = R\omega \quad \text{or} \quad \omega = \frac{v}{R}$$

Given,
$$v = \frac{9}{7}v_0 \qquad \text{...(i)}$$

$\therefore$
$$\omega = \frac{9}{7}\frac{v_0}{R} \qquad \text{...(ii)}$$

Applying, Linear impulse = change in linear momentum

$$\therefore \quad F\,dt = mv_0 \quad \text{...(iii)}$$

Angular impulse = change in angular momentum

$$\therefore \quad \tau\,dt = I\omega_0$$

or
$$Fh\,dt = \frac{2}{5}mR^2\,\omega_0 \quad \text{...(iv)}$$

Angular momentum about bottommost point will remain conserved.

i.e.,
$$L_i = L_f$$

or
$$I\omega_0 + mRv_0 = I\omega + mRv$$

$$\therefore \quad \frac{2}{5}mR^2\,\omega_0 + mRv_0 = \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \quad \text{...(v)}$$

Solving Eqs. (iii), (iv) and (v), we get

$$h = \frac{4}{5}R$$ **Proved.**

Example 6 *Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are μ_s and μ_k respectively.*

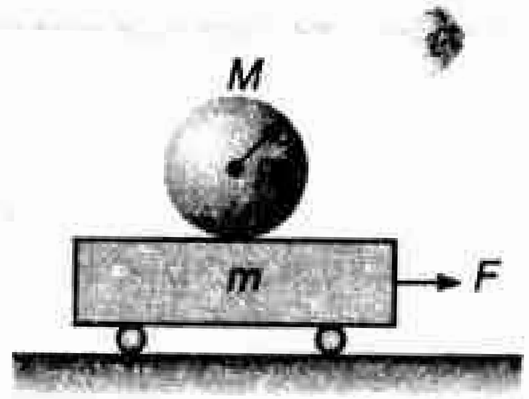


Fig. 9.106

Solution The free body diagrams of the sphere and the plank are as shown below:

Fig. 9.107

Writing equations of motion :

For sphere : Linear acceleration
$$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \quad \text{...(i)}$$

Angular acceleration
$$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2} = \frac{5}{2}\frac{\mu_s g}{R} \quad \text{...(ii)}$$

For plank : Linear acceleration
$$a_2 = \frac{F - \mu_s Mg}{m} \quad \text{...(iii)}$$

For no slipping :
$$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

Solving the above four equations, we get $F = \mu_s g\left(M + \frac{7}{2}m\right)$

Thus, maximum value of F can be $\mu_s g\left(M + \frac{7}{2}m\right)$

Example 7 *A uniform disc of radius r_0 lies on a smooth horizontal plane. A similar disc spinning with the angular velocity ω_0 is carefully lowered onto the first disc. How soon do both discs spin with the same angular-velocity if the friction coefficient between them is equal to μ ?*

Solution From the law of conservation of angular momentum.

$$I\omega_0 = 2I\omega$$

Here, I = moment of inertia of each disc relative to common rotation axis

$\therefore$ $$\omega = \frac{\omega_0}{2} = \text{steady state angular velocity}$$

The angular velocity of each disc varies due to the torque τ of the friction forces. To calculate τ, let us take an elementary ring with radii r and $r + dr$. The torque of the friction forces acting on the given ring is equal to.

$$d\tau = \mu r\left(\frac{mg}{\pi r_0^2}\right)2\pi r\, dr = \left(\frac{2\mu mg}{r_0^2}\right)r^2\, dr$$

where m is the mass of each disc. Integrating this with respect to r between 0 and r_0, we get

$$\tau = \frac{2}{3}\mu mgr_0$$

The angular velocity of the lower disc increases by $d\omega$ over the time interval

$$dt = \left(\frac{I}{\tau}\right)d\omega = \left(\frac{3r_0}{4\mu g}\right)d\omega$$

Integrating this equation with respect to ω between 0 and $\frac{\omega_0}{2}$, we find the desired time

$$t = \frac{3r_0\,\omega_0}{8\mu g}$$

EXERCISES

For **JEE Main**

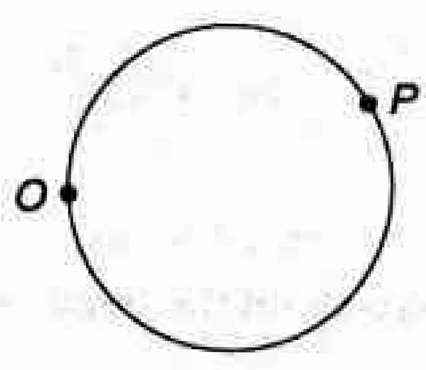
Subjective Questions

Moment of Inertia

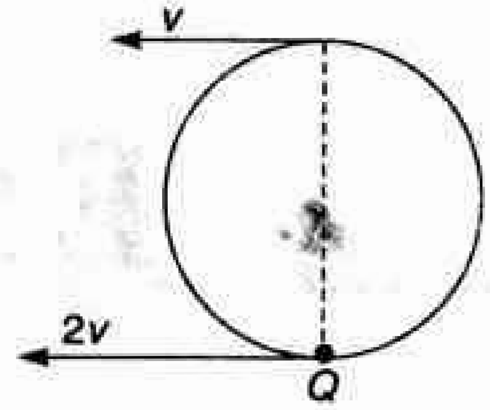
1. Four thin rods each of mass m and length l are joined to make a square. Find moment of inertia of all the four rods about any side of the square.
2. A mass of 1 kg is placed at (1 m, 2 m, 0). Another mass of 2 kg is placed at (3 m, 4 m, 0). Find moment of inertia of both the masses about z-axis.
3. Moment of inertia of a uniform rod of mass m and length l is $\frac{7}{12}ml^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.
4. Find the moment of inertia of a uniform square plate of mass M and edge a about one of its diagonals.
5. Radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass.
6. Two point masses m_1 and m_2 are joined by a weightless rod of length r. Calculate the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the rod.
7. Linear mass density (mass/length) of a rod depends on the distance from one end (say A) as $\lambda_x = (\alpha x + \beta)$. Here, α and β are constants. Find the moment of inertia of this rod about an axis passing through A and perpendicular to the rod. Length of the rod is l.

Angular Velocity

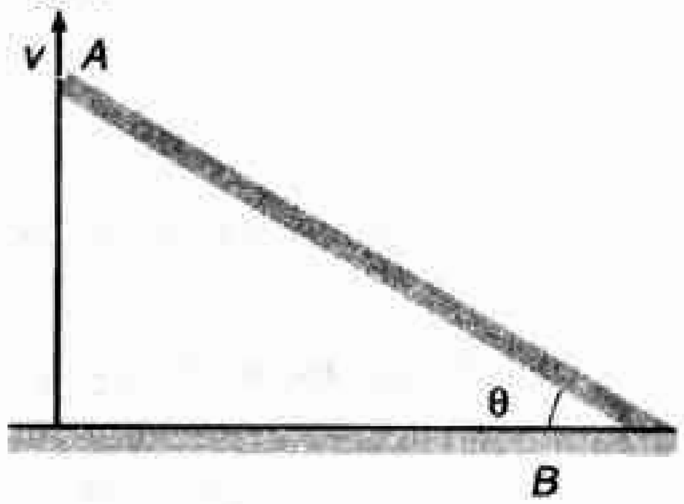
8. Find angular speed of second's clock.
9. A particle is located at (3 m, 4 m) and moving with $\vec{v} = (4\hat{i} - 3\hat{j})$ m/s. Find its angular velocity about origin at this instant.
10. Particle P shown in figure is moving in a circle of radius $R = 10$ cm with linear speed $v = 2$ m/s. Find the angular speed of particle about point O.



11. Two points P and Q, diametrically opposite on a disc of radius R have linear velocities v and $2v$ as shown in figure. Find the angular speed of the disc.



12. Point A of rod AB ($l = 2\text{m}$) is moved upwards against a wall with velocity $v = 2\,\text{m/s}$. Find angular speed of the rod at an instant when $\theta = 60°$.

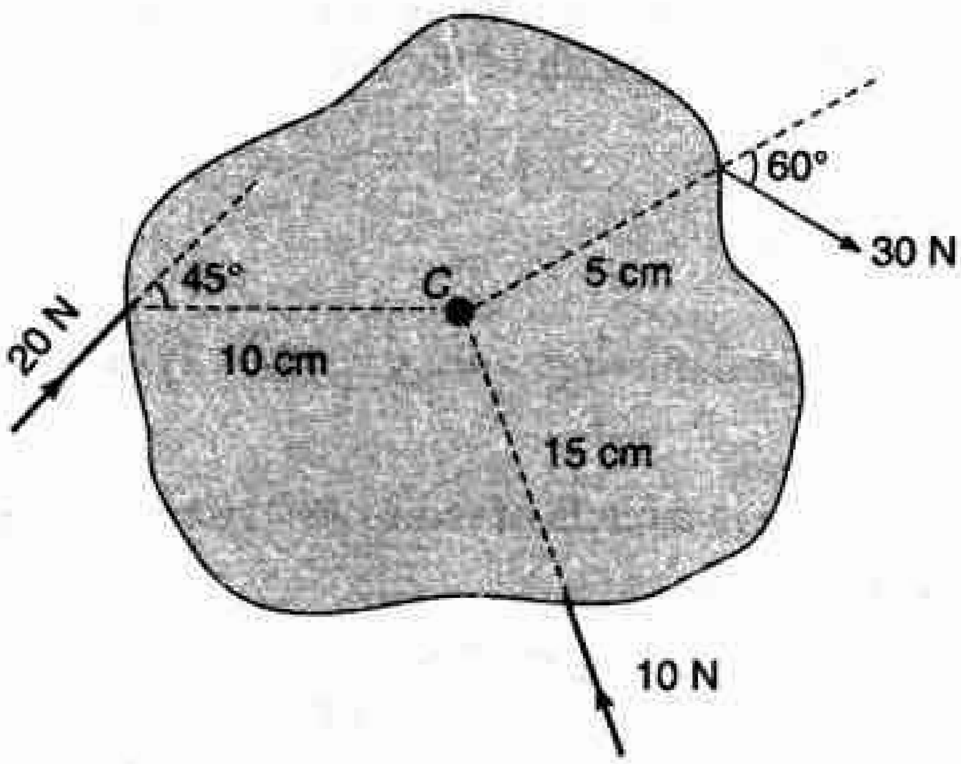


Torque

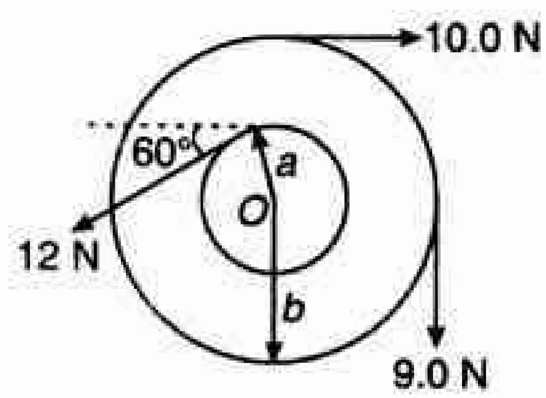
13. A force $\vec{\mathbf{F}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ N is acting on a body at point (2 m, 4 m, –2 m). Find torque of this force about origin.

14. A particle of mass $m = 1\,\text{kg}$ is projected with speed $u = 20\sqrt{2}$ m/s at angle $\theta = 45°$ with horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.

15. Point C is the centre of mass of the rigid body shown in figure. Find the total torque acting on the body about point C.



16. Find the net torque on the wheel in figure about the point O if $a = 10\,\text{cm}$ and $b = 25\,\text{cm}$.



Rotation of a Rigid Body About a Fixed Axis

Uniform angular acceleration

17. A wheel rotating with uniform angular acceleration covers 50 rev in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.

18. A wheel starting from rest is uniformly accelerated with $\alpha = 2\ \text{rad/s}^2$ for 5 s. It is then allowed to rotate uniformly for the next two seconds and is finally brought to rest in the next 5 s. Find the total angle rotated by the wheel.

19. A wheel whose moment of inertia is $0.03\ \text{kg m}^2$, is accelerated from rest to 20 rad/s in 5 s. When the external torque is removed, the wheel stops in 1 min. Find :

(a) the frictional torque, (b) the external torque.

20. A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of $2\ \text{rad/s}^2$. At what time will the body have kinetic energy same as the initial value if the torque continues to act ?

21. A uniform disc of mass 20 kg and radius 0.5 m can turn about a smooth axis through its centre and perpendicular to the disc. A constant torque is applied to the disc for 3 s from rest and the angular velocity at the end of that time is $\frac{240}{\pi}$ rev/min. Find the magnitude of the torque. If the torque is then removed and the disc is brought to rest in t seconds by a constant force of 10 N applied tangentially at a point on the rim of the disc, find t.

22. A uniform disc of mass m and radius R is rotated about an axis passing through its centre and perpendicular to its plane with an angular velocity ω_0. It is placed on a rough horizontal plane with the axis of the disc keeping vertical. Coefficient of friction between the disc and the surface is μ. Find :

(a) the time when disc stops rotating,

(b) the angle rotated by the disc before stopping.

Non-uniform angular acceleration

23. A flywheel whose moment of inertia about its axis of rotation is $16\ \text{kg-m}^2$ is rotating freely in its own plane about a smooth axis through its centre. Its angular velocity is $9\ \text{rad s}^{-1}$ when a torque is applied to bring it to rest in t_0 seconds. Find t_0 if :

(a) the torque is constant and of magnitude of 4 Nm,

(b) the magnitude of the torque after t seconds is given by kt.

24. A shaft is turning at 65 rad/s at time zero. Thereafter, angular acceleration is given by $\alpha = -10\ \text{rad/s}^2 - 5t\ \text{rad/s}^2$

where t is the elapsed time.

(a) Find its angular speed at $t = 3.0$ s.

(b) How far does it turn in these 3s ?

25. The angular velocity of a gear is controlled according to $\omega = 12 - 3t^2$ where ω, in radian per second, is positive in the clockwise sense and t is the time in seconds. Find the net angular displacement $\Delta\theta$ from the time $t = 0$ to $t = 3$ s. Also, find the number of revolutions N through which the gear turns during the 3s.

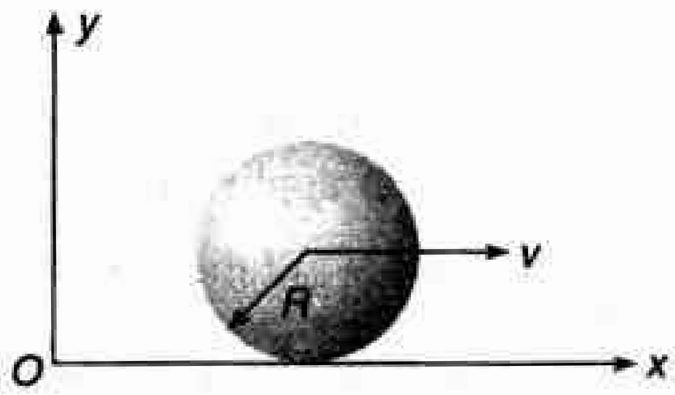
26. A solid body rotates about a stationary axis according to the law $\theta = at - bt^3$, where $a = 6$ rad/s and $b = 2\ \text{rad/s}^3$. Find the mean values of the angular velocity and acceleration over the time interval between $t = 0$ and the time, when the body comes to rest.

Angular Momentum

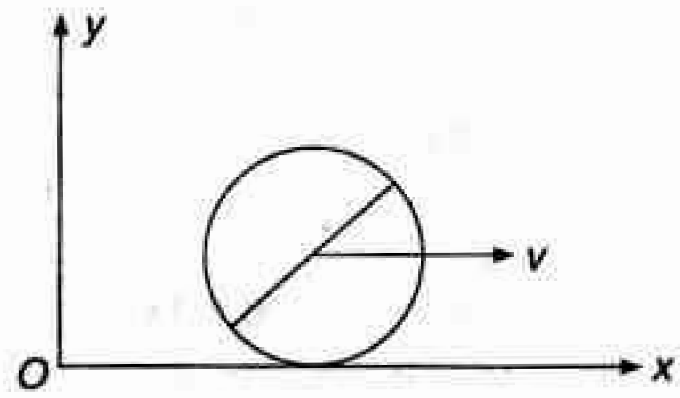
27. A particle of mass 1 kg is moving along a straight line $y = x + 4$. Both x and y are in metres. Velocity of the particle is 2 m/s. Find magnitude of angular momentum of the particle about origin.

28. A uniform rod of mass m is rotated about an axis passing through point O as shown. Find angular momentum of the rod about rotational axis.

29. A solid sphere of mass m and radius R is rolling without slipping as shown in figure. Find angular momentum of the sphere about z-axis.



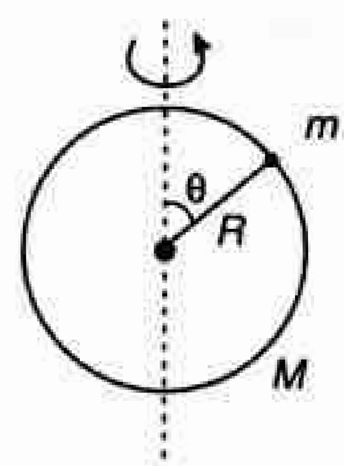
30. A rod of mass m and length $2R$ is fixed along the diameter of a ring of same mass m and radius R as shown in figure. The combined body is rolling without slipping along x-axis. Find the angular momentum about z-axis.



Conservation of Angular Momentum

31. If radius of earth is increased, without change in its mass, will the length of day increase, decrease or remain same?

32. The figure shows a thin ring of mass $M = 1\,\text{kg}$ and radius $R = 0.4$ m spinning about a vertical diameter. (Take $I = \frac{1}{2}MR^2$). A small bead of mass $m = 0.2$ kg can slide without friction along the ring. When the bead is at the top of the ring, the angular velocity is 5 rad/s. What is the angular velocity when the bead slips halfway to $\theta = 45°$.



33. A horizontal disc rotating freely about a vertical axis makes 100 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90. Calculate the moment of inertia of disc.

34. A man stands at the centre of a circular platform holding his arms extended horizontally with 4 kg block in each hand. He is set rotating about a vertical axis at 0.5 rev/s. The moment of inertia of the man plus platform is 1.6 kg-m^2, assumed constant. The blocks are 90 cm from the axis of rotation. He now pulls the blocks in toward his body until they are 15 cm from the axis of rotation. Find (a) his new angular velocity and (b) the initial and final kinetic energy of the man and platform. (c) how much work must the man do to pull in the blocks?

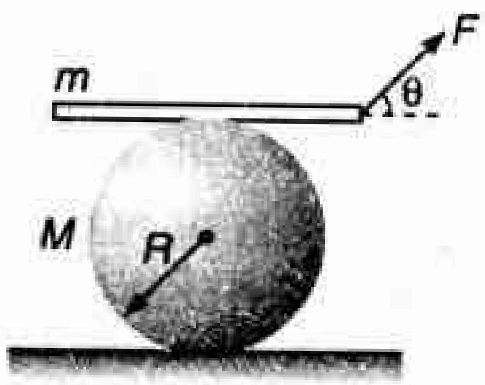
35. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0. Then, by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find :

(a) the angular velocity of the system in its final state,

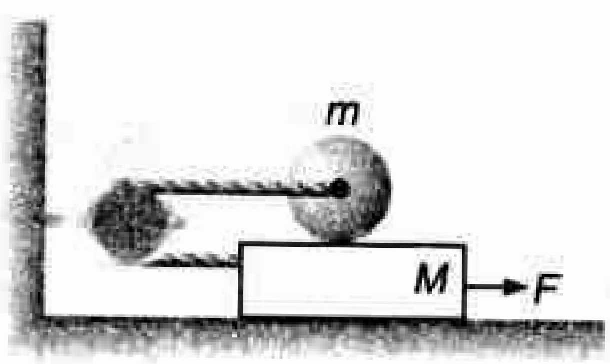
(b) the work performed by the force F.

Pure Rolling

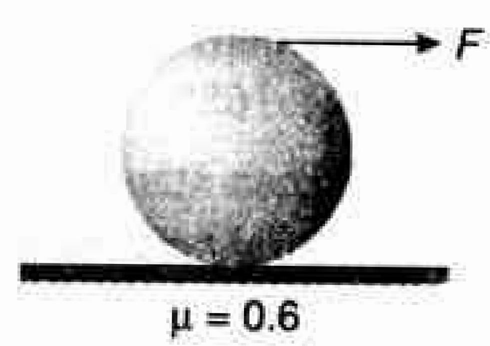
36. Consider a cylinder of mass M and radius R lying on a rough horizontal plane. It has a plank lying on its top as shown in figure. A force F is applied on the plank such that the plank moves and causes the cylinder to roll. The plank always remains horizontal. There is no slipping at any point of contact. Calculate the acceleration of the cylinder and the frictional forces at the two contacts.



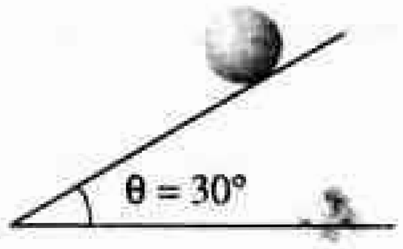
37. Find the acceleration of the cylinder of mass m and radius R and that of plank of mass M placed on smooth surface if pulled with a force F as shown in figure. Given that sufficient friction is present between cylinder and the plank surface to prevent sliding of cylinder.



38. In the figure shown a force F is applied at the top of a disc of mass 4 kg and radius 0.25 m. Find maximum value of F for no slipping



39. In the figure shown a solid sphere of mass 4 kg and radius 0.25 m is placed on a rough surface. Find : $(g = 10\,m/s^2)$



(a) minimum coefficient of friction for pure rolling to take place.
(b) If $\mu > \mu_{min}$, find linear acceleration of sphere.
(c) If $\mu = \frac{\mu_{min}}{2}$, find linear acceleration of cylinder.

Here, μ_{min} is the value obtained in part (a).

Angular Impulse

40. A uniform rod AB of length $2l$ and mass m is rotating in a horizontal plane about a vertical axis through A, with angular velocity ω, when the mid-point of the rod strikes a fixed nail and is brought immediately to rest. Find the impulse exerted by the nail.

41. A uniform rod of length L rests on a frictionless horizontal surface. The rod is pivoted about a fixed frictionless axis at one end. The rod is initially at rest. A bullet travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. The mass of the bullet is one-sixth the mass of the rod.
(a) What is the final angular velocity of the rod ?
(b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision ?

42. A uniform rod AB of mass $3m$ and length $2l$ is lying at rest on a smooth horizontal table with a smooth vertical axis through the end A. A particle of mass $2m$ moves with speed $2u$ across the table and strikes the rod at its mid-point C. If the impact is perfectly elastic. Find the speed of the particle after impact if :
(a) it strikes the rod normally,
(b) its path before impact was inclined at 60° to AC.

Objective Questions

Single Correct Option

1. The moment of inertia of a body does not depend on
(a) mass of the body
(b) the distribution of the mass in the body
(c) the axis of rotation of the body
(d) None of these

2. The radius of gyration of a disc of radius 25 cm is
(a) 18 cm (b) 12.5 cm (c) 36 cm (d) 50 cm

3. A shaft initially rotating at 1725 rpm is brought to rest uniformly in 20s. The number of revolutions that the shaft will make during this time is
(a) 1680 (b) 575 (c) 287 (d) 627

4. A man standing on a platform holds weights in his outstretched arms. The system is rotated about a central vertical axis. If the man now pulls the weights inwards close to his body, then
(a) the angular velocity of the system will increase
(b) the angular momentum of the system will remain constant

(c) the kinetic energy of the system will increase
(d) All of the above

5. The moment of inertia of a uniform semicircular disc of mass M and radius r about a line perpendicular to the plane of the disc through the centre is

(a) Mr^2 (b) $\frac{1}{2}Mr^2$ (c) $\frac{1}{4}Mr^2$ (d) $\frac{2}{5}Mr^2$

6. Two bodies A and B made of same material have the moment of inertial in the ratio $I_A : I_B = 16:18$. The ratio of the masses $m_A : m_B$ is given by

(a) cannot be obtained (b) 2 : 3
(c) 1 : 1 (d) 4 : 9

7. When a sphere rolls down an inclined plane, then identity the correct statement related to the work done by friction force

(a) The friction force does positive translational work
(b) The friction force does negative rotational work
(c) The net work done by friction is zero
(d) All of the above

8. A circular table rotates about a vertical axis with a constant angular speed ω. A circular pan rests on the turn table (with the centre coinciding with centre of table) and rotates with the table. The bottom of the pan is covered with a uniform thick layer of ice which also rotates with the pan. The ice starts melting. The angular speed of the turn table

(a) remains the same
(b) decreases
(c) increases
(d) may increase or decrease depending on the thickness of ice layer

9. If R is the radius of gyration of a body of mass M and radius r, then the ratio of its rotational to translational kinetic energy in the rolling condition is

(a) $\frac{R^2}{R^2+r^2}$ (b) $\frac{R^2}{r^2}$ (c) $\frac{r^2}{R^2}$ (d) 1

10. A solid sphere rolls down two different inclined planes of the same height but of different inclinations

(a) in both cases the speeds and time of descend will be same
(b) the speeds will be same but time of descend will be different
(c) the speeds will be different but time of descend will be same
(d) speeds and time of descend both will be different

11. For the same total mass which of the following will have the largest moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the body

(a) a disc of radius R (b) a ring of radius R
(c) a square lamina of side $2R$ (d) four rods forming a square of side $2R$

12. A disc and a solid sphere of same mass and radius roll down an inclined plane. The ratio of the friction force acting on the disc and sphere is

(a) $\frac{7}{6}$ (b) $\frac{5}{4}$
(c) $\frac{3}{2}$ (d) depends on angle of inclination

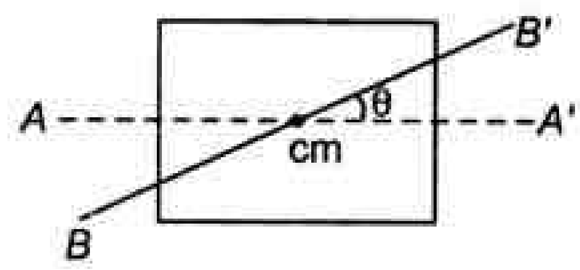
13. A horizontal disc rotates freely with angular velocity ω about a vertical axes through its centre. A ring, having the same mass and radius as the disc, is now gently placed coaxially on the disc. After some time, the two rotate with a common angular velocity. Then
(a) no friction exists between the disc and the ring
(b) the angular momentum of the system is conserved
(c) the final common angular velocity is $\frac{1}{2}\omega$
(d) All of the above

14. A solid homogeneous sphere is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere
(a) total kinetic energy of the sphere is conserved
(b) angular momentum of the sphere about any point on the horizontal surface is conserved
(c) only the rotational kinetic energy about the centre of mass is conserved
(d) None of the above

15. A particle of mass $m = 3$ kg moves along a straight line $4y - 3x = 2$ where x and y are in metre, with constant velocity $v = 5\ \text{ms}^{-1}$. The magnitude of angular momentum about the origin is
(a) $12\ \text{kg m}^2\text{s}^{-1}$ (b) $6.0\ \text{kg m}^2\text{s}^{-1}$ (c) $4.5\ \text{kg m}^2\text{s}^{-1}$ (d) $8.0\ \text{kg m}^2\text{s}^{-1}$

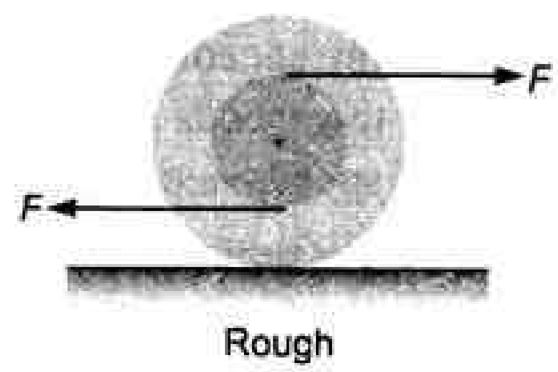
16. A solid sphere rolls without slipping on a rough horizontal floor, moving with a speed v. It makes an elastic collision with a smooth vertical wall. After impact,
(a) it will move with a speed v initially
(b) its motion will be rolling with slipping initially and its rotational motion will stop momentarily at some instant
(c) its motion will be rolling without slipping only after some time
(d) All of the above

17. The figure shows a square plate of uniform mass distribution. AA' and BB' are the two axes lying in the plane of the plate and passing through its centre of mass. It I_o is the moment of inertia of the plate about AA' then its moment of inertia about the BB' axis is



(a) I_o (b) $I_o \cos\theta$
(c) $I_o \cos^2\theta$ (d) None of these

18. A spool is pulled horizontally on rough surface by two equal and opposite forces as shown in the figure. Which of the following statements are correct?



(a) The centre of mass moves towards left
(b) The centre of mass moves towards right
(c) The centre of mass remains stationary
(d) The net torque about the centre of mass of the spool is zero

19. Two identical discs are positioned on a vertical axis as shown in the figure. The bottom disc is rotating at angular velocity ω_0 and has rotational kinetic energy K_0. The top disc is initially at rest. It then falls and sticks to the bottom disc. The change in the rotational kinetic energy of the system is

(a) $K_0/2$ (b) $-K_0/2$
(c) $-K_0/4$ (d) $K_0/4$

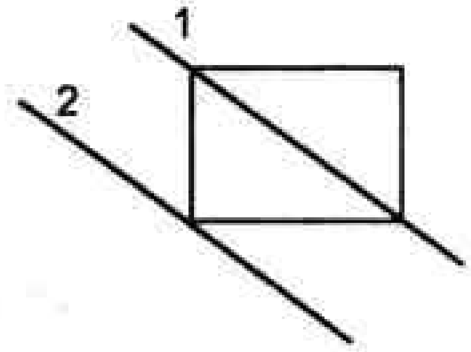
20. The moment of inertia of hollow sphere (mass M) of inner radius R and outer radius $2R$, having material of uniform density, about a diametric axis is

(a) $31MR^2/70$ (b) $43MR^2/90$ (c) $19MR^2/80$ (d) None of these

21. A rod of uniform cross-section of mass M and length L is hinged about an end to swing freely in a vertical plane. However, its density is non uniform and varies linearly from hinged end to the free end doubling its value. The moment of inertia of the rod, about the rotation axis passing through the hinge point is

(a) $\frac{2ML^2}{9}$ (b) $\frac{3ML^2}{16}$ (c) $\frac{7ML^2}{18}$ (d) None of these

22. Let I_1 and I_2 be the moment of inertia of a uniform square plate about axes shown in the figure. Then the ratio $I_1 : I_2$ is



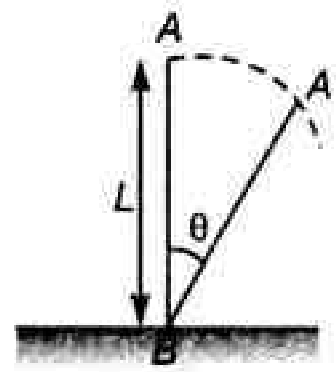
(a) $1:\frac{1}{7}$ (b) $1:\frac{12}{7}$

(c) $1:\frac{7}{12}$ (d) $1:7$

23. Moment of inertia of a uniform rod of length L and mass M, about an axis passing through $L/4$ from one end and perpendicular to its length is

(a) $\frac{7}{36}ML^2$ (b) $\frac{7}{48}ML^2$ (c) $\frac{11}{48}ML^2$ (d) $\frac{ML^2}{12}$

24. A uniform rod of length L is free to rorate in a vertical plane about a fixed horizontal axis through B. The rod begins rotating from rest. The angular velocity ω at angle θ is given as



(a) $\sqrt{\left(\frac{6g}{L}\right)}\sin\frac{\theta}{2}$ (b) $\sqrt{\left(\frac{6g}{L}\right)}\cos\frac{\theta}{2}$

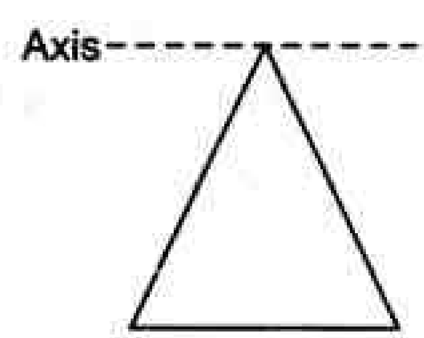
(c) $\sqrt{\left(\frac{6g}{L}\right)}\sin\theta$ (d) $\sqrt{\left(\frac{6g}{L}\right)}\cos\theta$

25. Two paricles of masses 1 kg and 2 kg are placed at a distance at a distance of 3m. Moment of inertia of the particles about an axis passing through their centre of mass and perpendicular to the line joining them is (in kg-m^2).

(a) 6 (b) 9

(c) 8 (d) 12

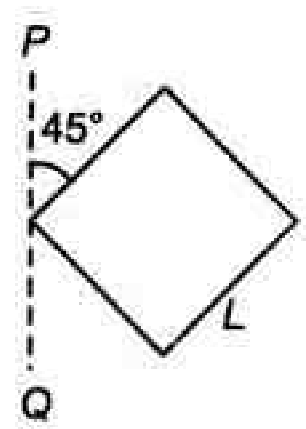
26. Find moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis as shown in figure. The length of each side is L



(a) $ML^2/8$ (b) $3ML^2/8$

(c) $7ML^2/8$ (d) None of these

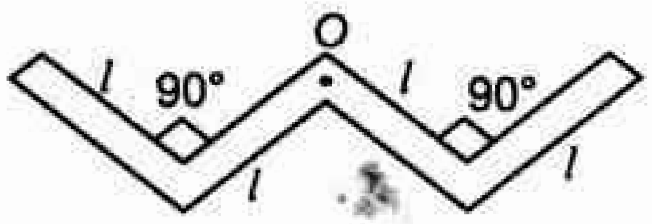
27. A square is made by joining four rods each of mass M and length L. Its moment of inertia about an axis PQ, in its plane and passing through one of its corner is



(a) $6ML^2$ (b) $\frac{4}{3}ML^2$

(c) $\frac{8}{3}ML^2$ (d) $\frac{10}{3}ML^2$

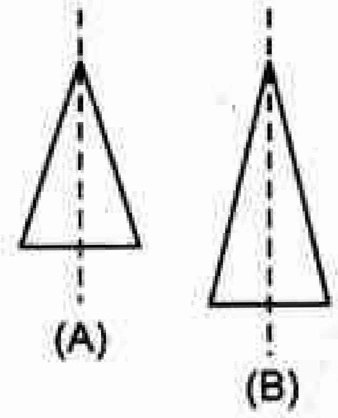
28. A thin rod of length $4l$, mass $4m$ is bent at the points as shown in the figure. What is the moment of inertia of the rod about the axis passing through O and perpendicular to the plane of the paper?



(a) $\frac{ml^2}{3}$ (b) $\frac{10ml^2}{3}$ (c) $\frac{ml^2}{12}$ (d) $\frac{ml^2}{24}$

29. The figure shows two cones A and B with the conditions : $h_A < h_B$; $\rho_A > \rho_B$; $R_A = R_B$ $m_A = m_B$. Identify the correct statement about their axis of symmetry.

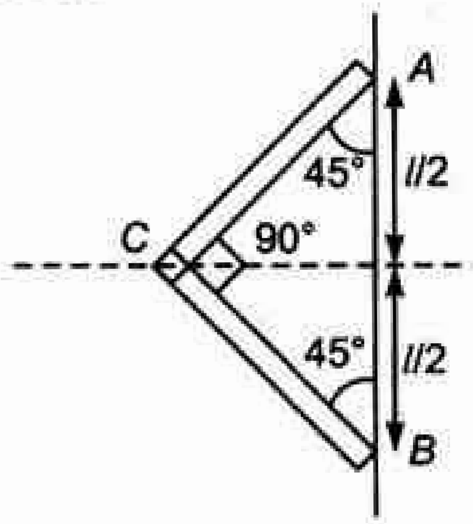
(a) Both have same moment of inertia
(b) A has greater moment of inertia
(c) B has greater moment of inertia
(d) Nothing can be said



30. Linear mass density of the two rods system, AC and CB is x. Moment of inertia of two rods about an axis passing through AB is

(a) $\frac{xl^3}{4\sqrt{3}}$ (b) $\frac{xl^3}{\sqrt{2}}$

(c) $\frac{xl^3}{4}$ (d) $\frac{xl^3}{6\sqrt{2}}$



For JEE Advanced

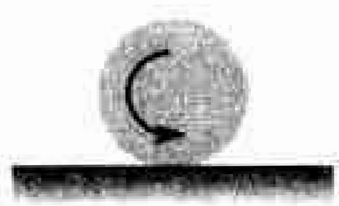
Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***
(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***
(*c*) *If **Assertion** is true, but the **Reason** is false.*
(*d*) *If **Assertion** is false but the **Reason** is true.*

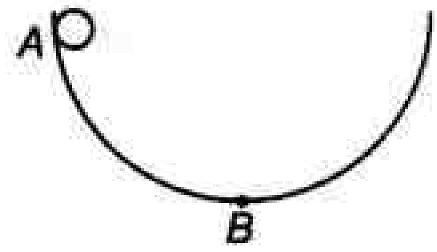
1. Assertion : Moment of inertia of a rigid body about any axis passing through its centre of mass is minimum.
Reason : From theorem of parallel axis,
$$I = I_{cm} + Mr^2$$

2. Assertion : A ball is released on a rough ground in the condition shown in figure. It will start pure rolling after some time towards left side.
Reason : Friction will convert the pure rotational motion of the ball into pure rolling.

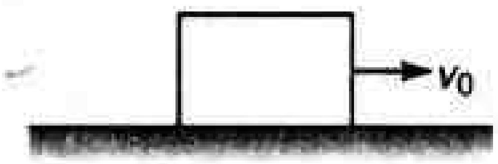


3. Assertion : A solid sphere and a hollow sphere are rolling on ground with same total kinetic energies. If translational kinetic energy of solid sphere is K, then translational kinetic energy of hollow sphere should be greater than K.
Reason : In case of hollow sphere rotational kinetic energy is less than its translational kinetic energy.

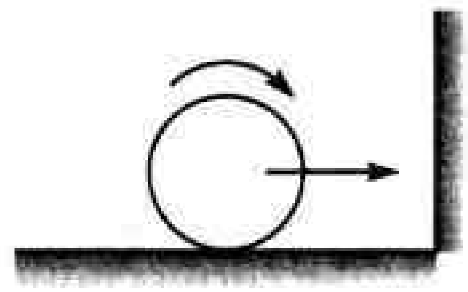
4. **Assertion :** A small ball is released from rest from point A as shown. If bowl is smooth than ball will exert more pressure at point B, compared to the situation if bowl is rough.
Reason : Linear velocity and hence, centripetal force in smooth situation is more.



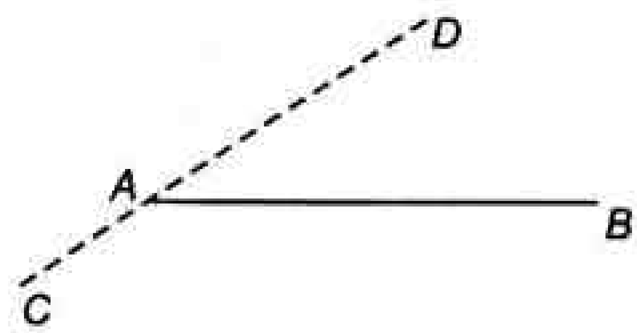
5. **Assertion :** A cubical block is moving on a rough ground with velocity v_0. During motion net normal reaction on the block from ground will not pass through centre of cube. It will shift towards right.
Reason : It is to keep the block in rotational equilibrium.



6. **Assertion :** A ring is rolling without slipping on a rough ground. It strikes elastically with a smooth wall as shown in figure. Ring will stop after some time while travelling in opposite direction.
Reason : Net angular momentum about an axis passing through bottommost point and perpendicular to plane of paper is zero.



7. **Assertion :** There is a thin rod AB and a dotted line CD. All the axes we are talking about are perpendicular to plane of paper. As we take different axes moving from A to D, moment of inertia of the rod may first decrease then increase.



Reason : Theorem of perpendicular axis cannot be applied here.

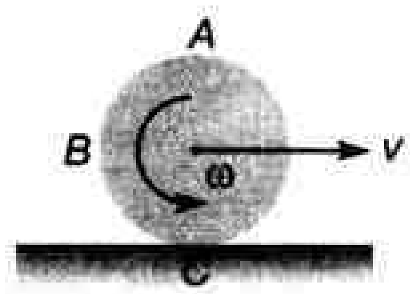
8. **Assertion :** If linear momentum of a particle is constant, then its angular momentum about any axis will also remain constant.

Reason : Linear momentum remain constant, if $\vec{F}_{net} = 0$ and angular momentum remains constant if $\vec{\tau}_{net} = 0$

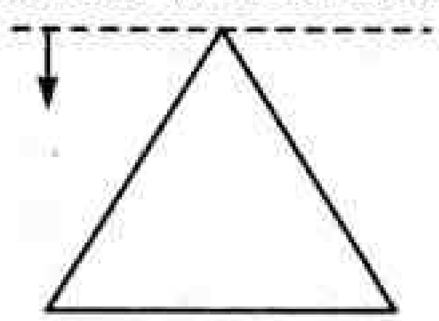
9. **Assertion :** In the figure shown, A, B and C are three points on the circumference of a disc. Let v_A, v_B and v_C are speeds of these three points, then

$$v_C > v_B > v_A$$

Reason : In case of rotational plus translational motion of a rigid body, net speed of any point (other than centre of mass) is greater than, less than or equal to the speed of centre of mass.

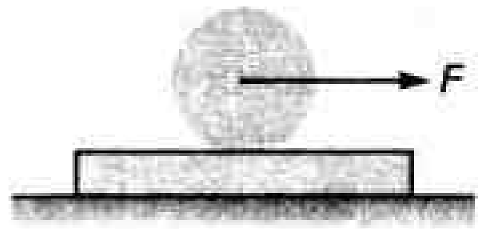


10. **Assertion :** There is a triangular plate as shown. A dotted axis is lying in the plane of slab. As the axis is moved downwards, moment of inertia of slab will first decrease then increase.



Reason : Axis is first moving towards its centre of mass and then it is receding from it.

11. **Assertion :** A horizontal force F is applied at the centre of solid sphere placed over a plank. The minimum coefficient of friction between plank and sphere required for pure rolling is μ_1 when plank is kept at rest and μ_2 when plank can move, then $\mu_2 < \mu_1$.

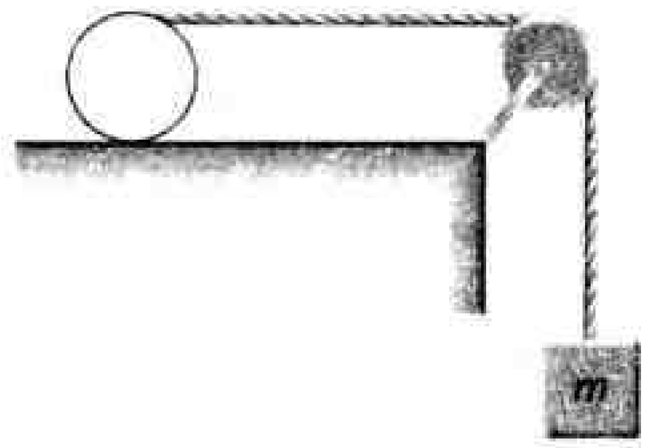


Reason : Work done by frictional force on the sphere in both cases is zero.

Objective Questions

Single Correct Option

1. In the given figure a ring of mass m is kept on a horizontal surface while a body of equal mass m is attached through a string, which is wounded on the ring. When the system is released, the ring rolls without slipping. Consider the following statement and choose the correct option.



(i) acceleration of the centre of mass of ring is $\frac{2g}{3}$

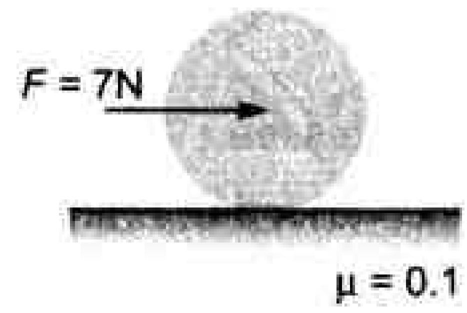
(ii) acceleration of hanging particle is $\frac{4g}{3}$

(iii) frictional force (on the ring) acts in forward direction

(iv) frictional force (on the ring) acts in backward direction

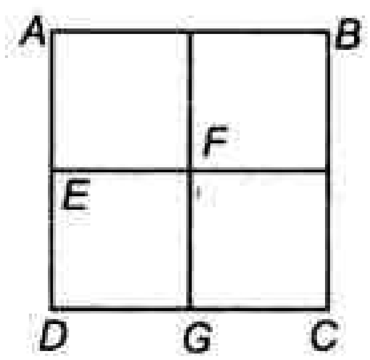
(a) statement (i) and (ii) only are correct
(b) statement (ii) and (iii) only are correct
(c) statement (iii) and (iv) only are correct
(d) None of these

2. A solid sphere of mass 10 kg is placed on rough surface having coefficient of friction $\mu = 0.1$. A constant force $F = 7$ N is applied along a line passing through the centre of the sphere as shown in the figure. The value of frictional force on the sphere is



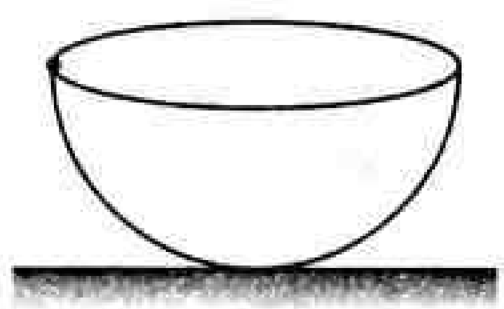
(a) 1 N
(b) 2 N
(c) 3 N
(d) 7 N

3. From a uniform square plate of side a and mass m, a square portion $DEFG$ of side $\frac{a}{2}$ is removed. Then, the moment of inertia of remaining portion about the axis AB is



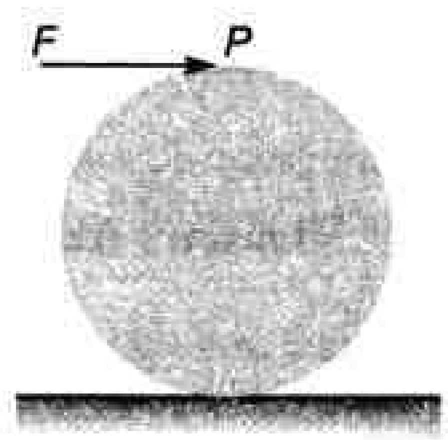
(a) $\frac{7ma^2}{16}$
(b) $\frac{3ma^2}{16}$
(c) $\frac{3ma^2}{4}$
(d) $\frac{9ma^2}{16}$

4. A small solid sphere of mass m and radius r starting from rest from the rim of a fixed hemispherical bowl of radius $R (>> r)$ rolls inside it without sliding. The normal reaction exerted by the sphere on the hemisphere when it reaches the bottom of hemisphere is



(a) $(3/7)\,mg$
(b) $(9/7)\,mg$
(c) $(13/7)\,mg$
(d) $(17/7)\,mg$

5. A uniform solid cylinder of mass m and radius R is placed on a rough horizontal surface. A horizontal constant force F is applied at the top point P of the cylinder so that it starts pure rolling. The acceleration of the cylinder is



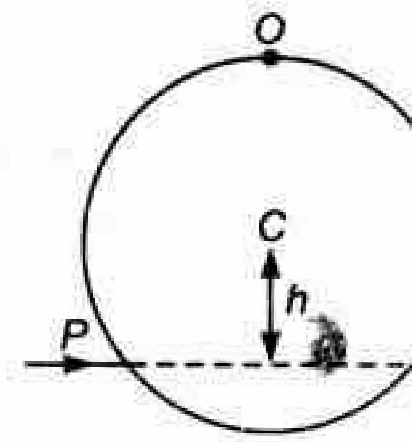
(a) $F/3m$
(b) $2F/3m$
(c) $4F/3m$
(d) $5F/3m$

6. In the above question, the frictional force on the cylinder is
(a) $F/3$ towards right
(b) $F/3$ towards left
(c) $2F/3$ towards right
(d) $2F/3$ towards left

7. A small pulley of radius 20 cm and moment of inertia $0.32\ \text{kg-m}^2$ is used to hang a 2 kg mass with the help of massless string. If the block is released, for no slipping condition acceleration of the block will be

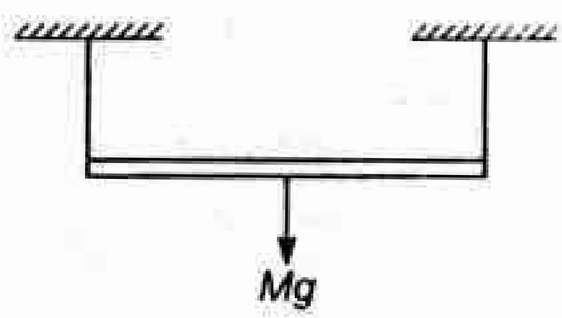
(a) $2\ \text{m/s}^2$
(b) $4\ \text{m/s}^2$
(c) $1\ \text{m/s}^2$
(d) $3\ \text{m/s}^2$

8. A uniform circular disc of radius R is placed on a smooth horizontal surface with its plane horizontal and hinged at circumference through point O as shown. An impulse P is applied at a perpendicular distance h from its centre C. The value of h so that the impulse due to hinge is zero, is



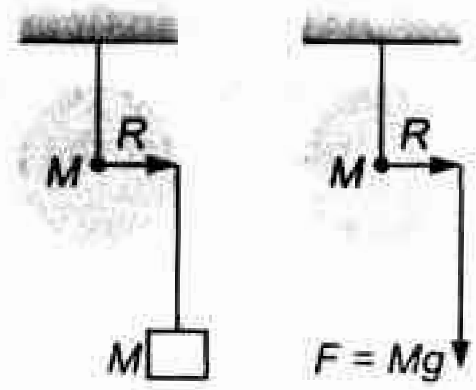
(a) R
(b) $R/2$
(c) $R/3$
(d) $R/4$

9. A rod is supported horizontally by means of two strings of equal length as shown in figure. If one of the string is cut. Then tension in other string at the same instant will



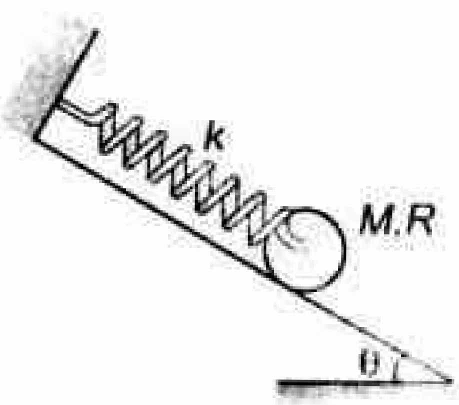
(a) remain unaffected
(b) increase
(c) decrease
(d) become equal to weight of the rod

10. The figure represents two cases. In first case a block of mass M is attached to a string which is tightly wound on a disc of mass M and radius R. In second case $F = Mg$. Initially, the disc is stationary in each case. If the same length of string is unwound from the disc, then



(a) same amount of work is done on both discs
(b) angular velocities of both the discs are equal
(c) both the discs have unequal angular accelerations
(d) All of the above

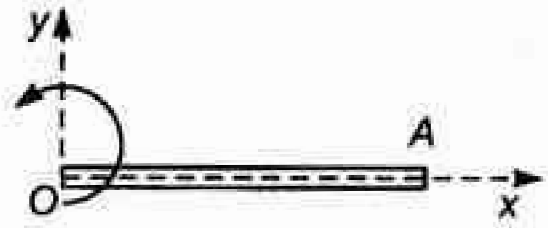
11. A uniform cylinder of mass M and radius R is released from rest on a rough inclined surface of inclination θ with the horizontal as shown in figure. As the cylinder rolls down the inclined surface, the maximum elongation in the spring of stiffness k is



(a) $\dfrac{3}{4}\dfrac{Mg\sin\theta}{k}$
(b) $\dfrac{2\,Mg\sin\theta}{k}$
(c) $\dfrac{Mg\sin\theta}{k}$
(d) None of these

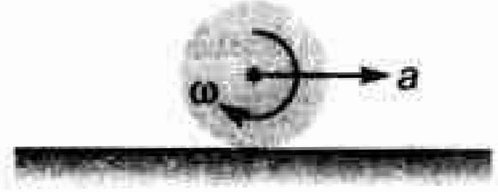
12. A uniform rod of mass m and length l rotates in a horizontal plane with an angular velocity ω about a vertical axis passing through one end. The tension in the rod at a distance x from the axis is
(a) $\frac{1}{2}m\omega^2 x$
(b) $\frac{1}{2}m\omega^2\left(1-\frac{x^2}{l}\right)$
(c) $\frac{1}{2}m\omega^2 l\left(1-\frac{x^2}{l^2}\right)$
(d) $\frac{1}{2}m\omega^2 l\left[1-\frac{x}{l}\right]$

13. A rod of length 1 m rotates in the xy plane about the fixed point O in the anticlockwise sense, as shown in figure with velocity $\omega = a + bt$ where $a = 10 \text{rad s}^{-1}$ and $b = 5 \text{ rad s}^{-2}$. The velocity and acceleration of the point A at $t = 0$ is



(a) $+10\hat{i} \text{ ms}^{-1}$ and $+5\hat{i} \text{ ms}^{-2}$
(b) $+10\hat{j} \text{ ms}^{-1}$ and $(-100\hat{i} + 5\hat{j}) \text{ ms}^{-2}$
(c) $-10\hat{j} \text{ ms}^{-1}$ and $(100\hat{i} + 5\hat{j}) \text{ ms}^{-2}$
(d) $-10\hat{j} \text{ ms}^{-1}$ and $-5\hat{j} \text{ ms}^{-2}$

14. A ring of radius R rolls on a horizontal surface with constant acceleration a of the centre of mass as shown in figure. If ω is the instantaneous angular velocity of the ring, then the net acceleration of the point of contact of the ring with ground is

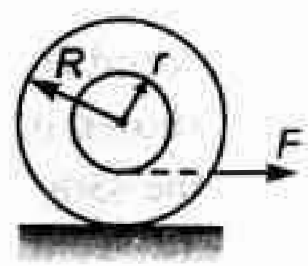


(a) zero
(b) $\omega^2 R$
(c) a
(d) $\sqrt{a^2 + (\omega^2 R)^2}$

15. The density of a rod AB increases linearly from A to B. Its midpoint is O and its centre of mass is at C. Four axes pass through A, B, O and C, all perpendicular to the length of the rod. The moments of inertia of the rod about these axes are I_A, I_B, I_O and I_C respectively. Then

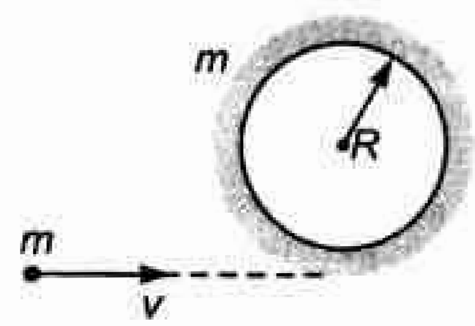
(a) $I_A > I_B$
(b) $I_C < I_B$
(c) $I_O > I_C$
(d) All of these

16. The figure shows a spool placed at rest on a horizontal rough surface. A tightly would string on the inner cylinder is pulled horizontally with a force F. Identity the correct alternative related to the friction force f acting on the spool



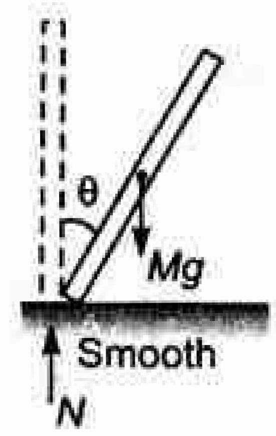
(a) f acts leftwards with $f < F$
(b) f acts leftwards but nothing can be said about its magnitude
(c) $f < F$ but nothing can be said about its magnitude
(d) None of the above

17. A circular ring of mass m and radius R rests flat on a horizontal smooth surface as shown in figure. A particle of mass m, and moving with a velocity v, collides inelastically ($e = 0$) with the ring. The angular velocity with which the system rotates after the particle strikes the ring is



(a) $\dfrac{v}{2R}$
(b) $\dfrac{v}{3R}$
(c) $\dfrac{2v}{3R}$
(d) $\dfrac{3v}{4R}$

18. A stationary uniform rod in the upright position is allowed to fall on a smooth horizontal surface. The figure shows the instantaneous position of the rod. Identity the correct statement.

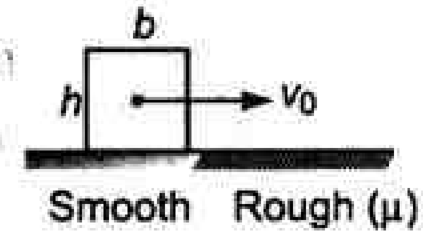


(a) normal reaction N is equal to Mg
(b) N does positive rotational work about the centre of mass
(c) a couple of equal and opposite forces acts on the rod
(d) All of the above

19. A thin uniform rod of mass m and length l is free to rotate about its upper end. When it is at rest. It receives an impulse J at its lowest point, normal to its length. Immediately after impact

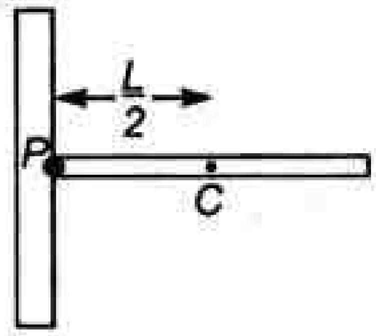
(a) the angular momentum of the rod is Jl
(b) the angular velocity of the rod is $3J/ml$
(c) the kinetic energy of the rod is $3J^2/2m$
(d) All of the above

20. A rectangular block of size $(b \times h)$ moving with velocity v_o enters on a rough surface where the coefficient of friction is μ as shown in figure. Identify the correct statement.



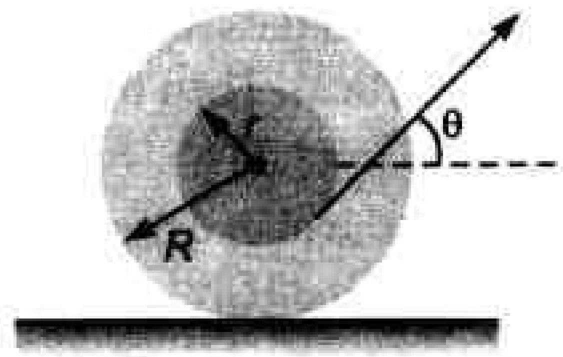
(a) The net torque acting on the block about its COM is $\mu\, mg \frac{h}{2}$ (clockwise)
(b) The net torque acting on the block about its COM is zero
(c) The net torque acting on the block about its COM is in the anticlockwise sense
(d) None of the above

21. A uniform rod of length L and mass m is free to rotate about a frictionless pivot at one end as shown in figure. The rod is held at rest in the horizontal position and a coin of mass m is placed at the free end. Now the rod is released. The reaction on the coin immediately after the rod starts falling is



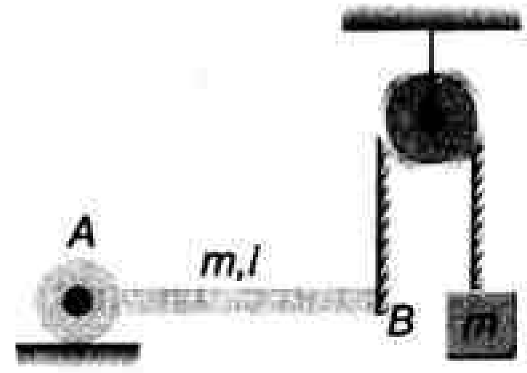
(a) $\frac{3mg}{2}$ (b) $2mg$
(c) zero (d) $\frac{mg}{2}$

22. A spool is pulled at an angle θ with the horizontal on a rough horizontal surface as shown in the figure. If the spool remains at rest, the angle θ is equal to



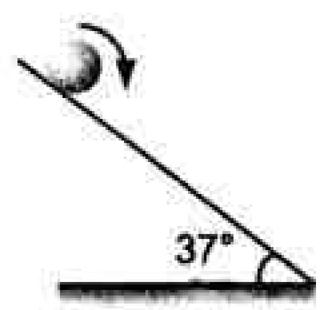
(a) $\cos^{-1}\left(\frac{R}{r}\right)$ (b) $\sin^{-1}\left(\sqrt{1-\frac{r^2}{R^2}}\right)$
(c) $\pi - \cos^{-1}\left(\frac{r}{R}\right)$ (d) $\sin^{-1}\left(\frac{r}{R}\right)$

23. Uniform rod AB is hinged at end A in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Mass of block and rod is same and is equal to m. Then acceleration of block just after release from this position is



(a) $6g/13$ (b) $g/4$
(c) $3g/8$ (d) None of these

24. A cylinder having radius 0.4 m, initially rotating (at $t = 0$) with $\omega_0 = 54$ rad/sec is placed on a rough inclined plane with $\theta = 37°$ having friction coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is ($g = 10\,\text{m/s}^2$)

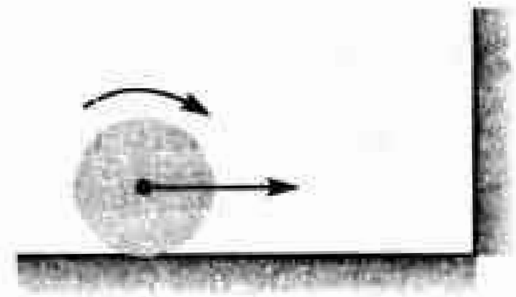


(a) 5.4 s (b) 1.2 s
(c) 1.4 s (d) 1.8 s

25. A disc of mass M and radius R is rolling purely with center's velocity v_0 on a flat horizontal floor when it hits a step in the floor of height $R/4$. The corner of the step is sufficiently rough to prevent any slipping of the disc against itself. What is the velocity of the centre of the disc just after impact?

(a) $4v_0/5$ (b) $4v_0/7$
(c) $5v_0/6$ (d) None of these

26. A solid sphere is rolling purely on a rough horizontal surface (coefficient of kinetic friction $= \mu$) with speed of centre $= u$. It collides inelastically with a smooth vertical wall at a certain moment, the coefficient of restitution being $\frac{1}{2}$. The sphere will begin pure rolling after a time



(a) $\frac{3u}{7\mu g}$ (b) $\frac{2u}{7\mu g}$ (c) $\frac{3u}{5\mu g}$ (d) $\frac{2u}{5\mu g}$

27. A thin hollow sphere of mass m is completely filled with non viscous liquid of mass m. When the sphere roll-on horizontal ground such that centre moves with velocity v, kinetic energy of the system is equal to

(a) mv^2 (b) $\frac{4}{3}mv^2$ (c) $\frac{4}{5}mv^2$ (d) None of these

28. A solid uniform disc of mass m rolls without slipping down a fixed inclined plank with an acceleration a. The frictional force on the disc due to surface of the plane is

(a) $\frac{1}{4}ma$ (b) $\frac{3}{2}ma$ (c) ma (d) $\frac{1}{2}ma$

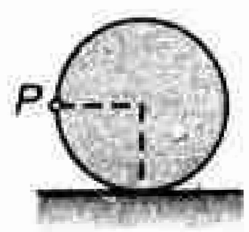
29. A uniform slender rod of mass m and length L is released from rest, with its lower end touching a frictionless horizontal floor. At the initial moment, the rod is inclined at an angle $\theta = 30^\circ$ with the vertical. Then the value of normal reaction from the floor just after release will be

(a) $4mg/7$ (b) $5mg/9$ (c) $2mg/5$ (d) None of these

30. In the above problem, the initial acceleration of the lower end of the rod will be

(a) $g\sqrt{3}/4$ (b) $g\sqrt{3}/5$ (c) $3g\sqrt{3}/7$ (d) None of these

31. A disc of radius R is rolling purely on a flat horizontal surface, with a constant angular velocity. The angle between the velocity and acceleration vectors of point P is

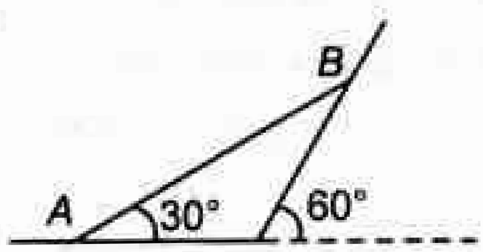


(a) zero (b) 45°
(c) $\tan^{-1}(2)$ (d) $\tan^{-1}(1/2)$

32. A straight rod AB of mass M and length L is placed on a frictionless horizontal surface. A force having constant magnitude F and a fixed direction starts acting at the end A. The rod is initially perpendicular to the force. The initial acceleration of end B is

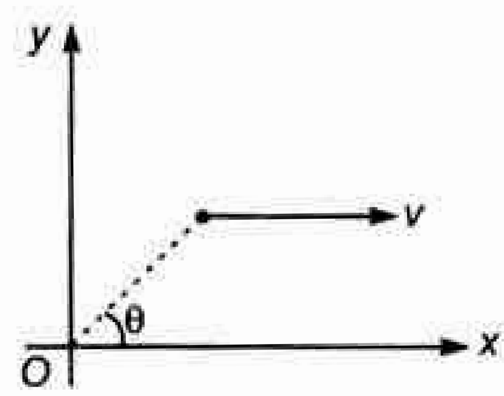
(a) zero (b) $2F/M$ (c) $4F/M$ (d) None of these

33. In the figure shown, the instantaneous speed of end A of the rod is v to the left. The angular velocity of the rod of length L, must be



(a) $v/2L$ (b) v/L (c) $\frac{\sqrt{3}v}{2L}$ (d) $\frac{2v}{L}$

34. A particle moves parallel to x-axis with constant velocity v as shown in the figure. The angular velocity of the particle about the origin O

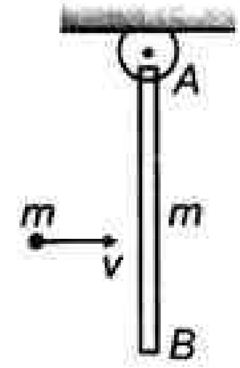


(a) remains constant
(b) continuously increases
(c) continuously decreases
(d) oscillates

35. A thin uniform rod of mass M and length L is hinged at its upper end, and released from rest from a horizontal position. The tension at a point located at a distance $L/3$ from the hinge point, when the rod becomes vertical, will be

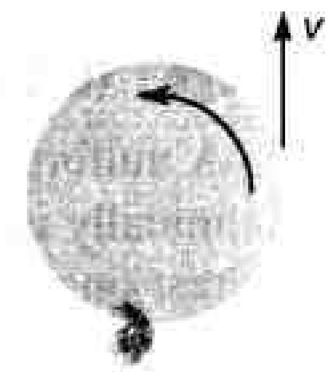
(a) $22\,Mg/27$ (b) $11\,Mg/13$ (c) $6\,Mg/11$ (d) $2\,Mg$

36. A uniform rod AB of length L and mass m is suspended freely at A and hangs vertically at rest when a particle of same mass m is fired horizontally with speed v to strike the rod at its mid point. If the particle is brought to rest after the impact. Then the impulsive reaction at A in horizontal direction is



(a) $mv/4$ (b) $mv/2$
(c) mv (d) $2\,mv$

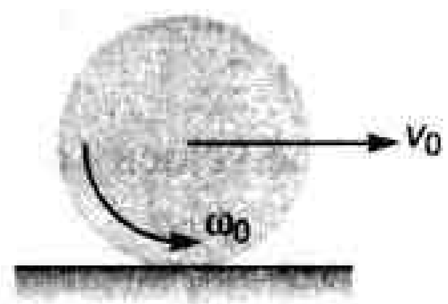
37. A child with mass m is standing at the edge of a merry go round having moment of inertia I, radius R and initial angular velocity ω as shown in the figure. The child jumps off the edge of the merry go round with tangential velocity v with respect to the ground. The new angular velocity of the merry go round is



(a) $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$ (b) $\sqrt{\dfrac{(I + mR^2)\omega^2 - mv^2}{I}}$

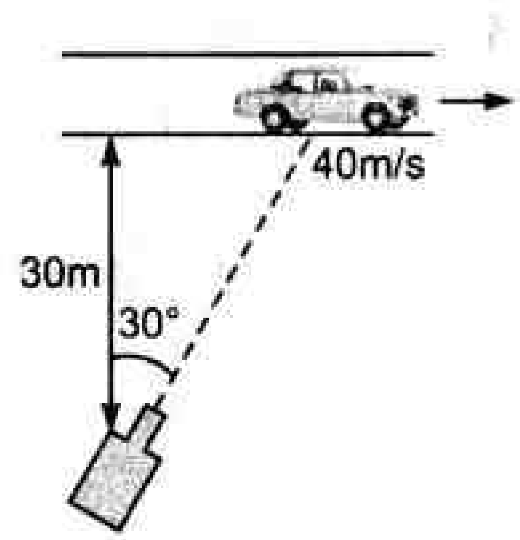
(c) $\dfrac{I\omega - mvR}{I}$ (d) $\dfrac{(I + mR^2)\omega - mvR}{I}$

38. A disc of radius R is spun to an angular speed ω_0 about its axis and then imparted a horizontal velocity of magnitude $\dfrac{\omega_0 R}{4}$. The coefficient of friction is μ. The sense of rotation and direction of linear velocity are shown in the figure. The disc will return to its initial position



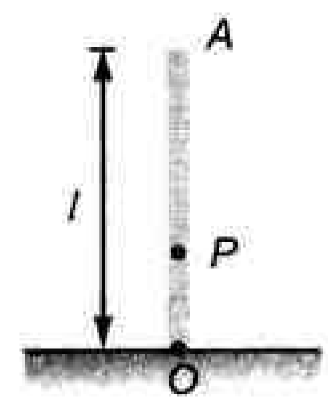
(a) if the value of $\mu < 0.5$ (b) irrespective of the value of μ
(c) if the value of $0.5 < \mu < 1$ (d) if $\mu > 1$

39. A racing car is travelling along a straight track at a constant velocity of 40 m/s. A fixed TV camera is recording the event as shown in figure. In order to keep the car in view, in the position shown, the angular velocity of camera should be



(a) 3 rad/s
(b) 2 rad/s
(c) 4 rad/s
(d) 1 rad/s

40. A uniform rod OA of length l, resting on smooth surface is slightly distributed from its vertical position. P is a point on the rod whose locus is a circle during the subsequent motion of the rod. Then the distance OP is equal to



(a) $l/2$
(b) $l/3$
(c) $l/4$
(d) there is no such point

41. In the above question, the velocity of end O when end A hits the ground is

(a) zero
(b) along the horizontal
(c) along the vertical
(d) at some inclination to the ground ($\neq 90°$)

42. In the above question, the velocity of end A at the instant it hits the ground is

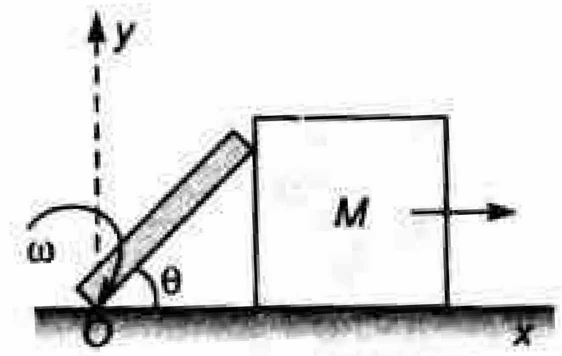
(a) $\sqrt{3gl}$ (b) $\sqrt{12gl}$ (c) $\sqrt{6gl}$ (d) None of these

43. A solid sphere of mass m and radius R is gently placed on a conveyer belt moving with constant velocity v_0. If coefficient of friction between belt and sphere is 2/7, the distance traveled by the centre of the sphere before it starts pure rolling is

(a) $\dfrac{v_0^2}{7g}$ (b) $\dfrac{2v_0^2}{49g}$ (c) $\dfrac{2v_0^2}{5g}$ (d) $\dfrac{2v_0^2}{7g}$

Passage : (Q. No. 44 to 47)

A uniform rod of mass m and length l is pivoted at point O. The rod is initially in vertical position and touching a block of mass M which is at rest on a horizontal surface. The rod is given a slight jerk and it starts rotating about point O. This causes the block to move forward as shown. The rod loses contact with the block at $\theta = 30°$. All surfaces are smooth. Now answer the following questions.



44. The value of ratio M/m is

(a) 2 : 3 (b) 3 : 2 (c) 4 : 3 (d) 3 : 4

45. The velocity of block when the rod loses contact with the block is

(a) $\dfrac{\sqrt{3gl}}{4}$ (b) $\dfrac{\sqrt{5gl}}{4}$ (c) $\dfrac{\sqrt{6gl}}{4}$ (d) $\dfrac{\sqrt{7gl}}{4}$

46. The acceleration of centre of mass of rod, when it loses contact with the block is

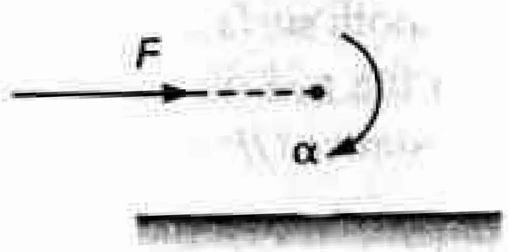
(a) $5g/4$ (b) $5g/2$ (c) $3g/2$ (d) $3g/4$

47. The hinge reaction at O on the rod when it loses contact with the block is

(a) $\dfrac{3\,mg}{4}(\hat{i}+\hat{j})$ (b) $\left(\dfrac{mg}{4}\right)\hat{j}$ (c) $\left(\dfrac{mg}{4}\right)\hat{i}$ (d) $\dfrac{mg}{4}(\hat{i}+\hat{j})$

Passage : (Q. No. 48 to 50)

Consider a uniform disc of mass m, radius r, rolling without slipping on a rough surface with linear acceleration a and angular acceleration α due to an external force F as shown in the figure. Coefficient of friction is μ



48. The work done by the frictional force at the instant of pure rolling is

(a) $\dfrac{\mu mgat^2}{2}$ (b) $\mu mgat^2$ (c) $\mu mg\dfrac{at^2}{\alpha}$ (d) zero

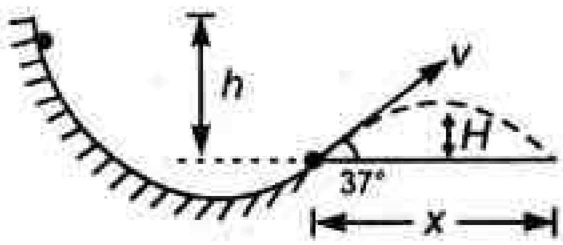
49. The magnitude of frictional force acting on the disc is

(a) ma (b) μmg (c) $\dfrac{ma}{2}$ (d) zero

50. Angular momentum of the disc will be conserved about
(a) centre of mass
(b) point of contact
(c) a point at a distance $3R/2$ vertically above the point of contact
(d) a point at a distance $4R/3$ vertically above the point of contact

Passage : (Q. No. 51 to 53)

A tennis ball, starting from rest, rolls down the hill in the drawing. At the end of the hill the ball becomes airborne, leaving at an angle of 37° with respect to the ground. Treat the ball as a thin-walled spherical shell.



51. The velocity of projection v is
(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{10}{7}gh}$ (c) $\sqrt{\frac{5}{7}gh}$ (d) $\sqrt{\frac{6}{5}gh}$

52. Maximum height reached by ball H above ground is
(a) $\frac{9h}{35}$ (b) $\frac{18h}{35}$ (c) $\frac{18h}{25}$ (d) $\frac{27h}{125}$

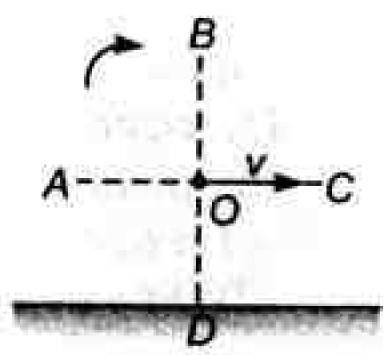
53. Range x of the ball is
(a) $\frac{144}{125}h$ (b) $\frac{48}{25}h$ (c) $\frac{48}{35}h$ (d) $\frac{24}{7}h$

More than One Correct Options

1. A mass m of radius r is rolling horizontally without any slip with a linear speed v. It then rolls up to a height given by $\frac{3}{4}\frac{v^2}{g}$
(a) the body is identified to be a disc or a solid cylinder
(b) the body is a solid sphere
(c) moment of inertia of the body about instantaneous axis of rotation is $\frac{3}{2}mr^2$
(d) moment of inertia of the body about instantaneous axis of rotation is $\frac{7}{5}mr^2$

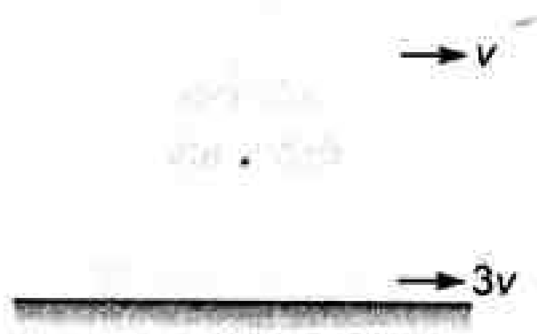
2. Four identical rods each of mass m and length l are joined to form a rigid square frame. The frame lies in the xy plane, with its centre at the origin and the sides parallel to the x and y axes. Its moment of inertia about
(a) the x-axis is $\frac{2}{3}ml^2$
(b) the z-axis is $\frac{4}{3}ml^2$
(c) an axis parallel to the z-axis and passing through a corner is $\frac{10}{3}ml^2$
(d) one side is $\frac{5}{3}ml^2$

3. A uniform circular ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure. Then



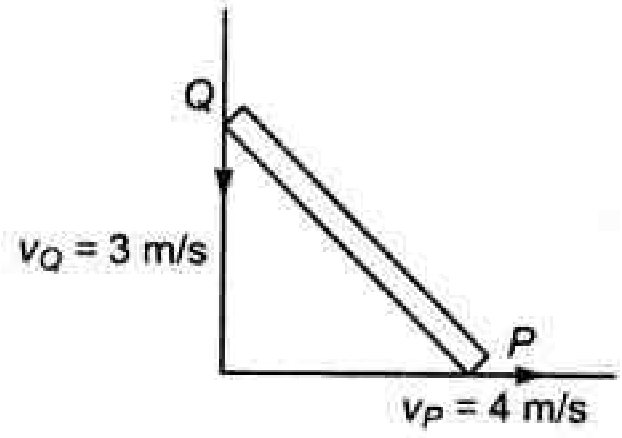
(a) section ABC has greater kinetic energy than section ADC
(b) section BC has greater kinetic energy than section CD
(c) section BC has the same kinetic energy as section DA
(d) the sections CD and DA have the same kinetic energy

4. A cylinder of radius R is to roll without slipping between two planks as shown in the figure. Then



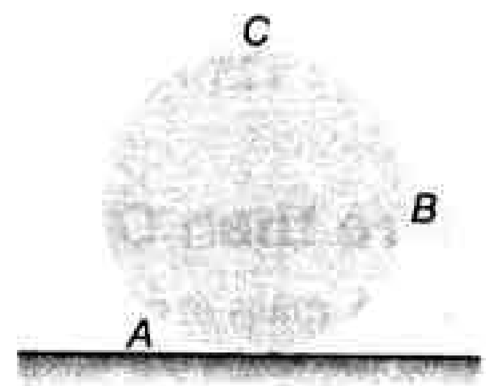
(a) angular velocity of the cylinder is $\frac{v}{R}$ counter clockwise
(b) angular velocity of the cylinder is $\frac{2v}{R}$ clockwise
(c) velocity of centre of mass of the cylinder is v towards left
(d) velocity of centre of mass of the cylinder is $2v$ towards right

5. A uniform rod of mass $m = 2$ kg and length $l = 0.5$ m is sliding along two mutually perpendicular smooth walls with the two ends P and Q having velocities $v_P = 4$ m/s and $v_Q = 3$ m/s as shown. Then



(a) The angular velocity of rod, $\omega = 10$ rad/s, counter clockwise
(b) The angular velocity of rod, $\omega = 5.0$ rad/s, counter clockwise
(c) The velocity of centre of mass of rod, $v_{cm} = 2.5$ m/s
(d) The total kinetic energy of rod, $K = \frac{25}{3}$ joule

6. A wheel is rolling without slipping on a horizontal plane with velocity v and acceleration a of centre of mass as shown in figure. Acceleration at



(a) A is vertically upwards
(b) B may be vertically downwards
(c) C cannot be horizontal
(d) a point on the rim may be horizontal leftwards

7. A uniform rod of length l and mass $2m$ rests on a smooth horizontal table. A point mass m moving horizontally at right angles to the rod with velocity v collides with one end of the rod and sticks it. Then

(a) angular velocity of the system after collision is $\frac{v}{l}$
(b) angular velocity of the system after collision is $\frac{v}{2l}$
(c) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{mv^2}{6}$
(d) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{7mv^2}{24}$

8. A nonuniform ball of radius R and radius of gyration about geometric centre $= R/2$, is kept on a frictionless surface. The geometric centre coincides with the centre of mass. The ball is struck horizontally with a sharp impulse $= J$. The point of application of the impulse is at a height h above the surface. Then

(a) the ball with slip on surface for all cases
(b) the ball will roll purely if $h = 5R/4$
(c) the ball will roll purely if $h = 3R/2$
(d) there will be no rotation if $h = R$

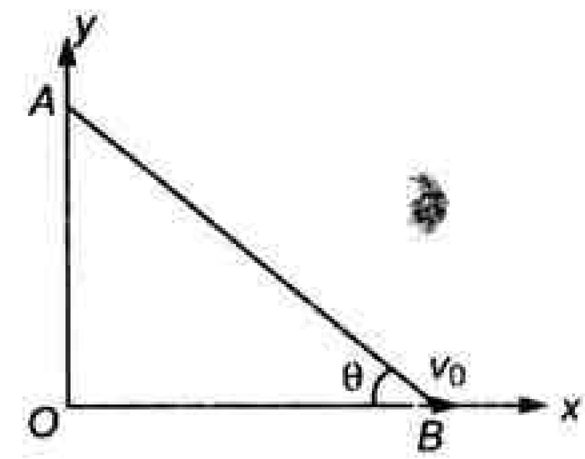
9. A hollow spherical ball is given an initial push up an incline of inclination angle α. The ball rolls purely. Coefficient of static friction between ball and incline $= \mu$. During its upwards journey

(a) friction acts up along the incline
(b) $(2\tan\alpha)/5$
(c) friction acts down along the incline
(d) $(2\tan\alpha)/7$

10. A uniform disc of mass m and radius R rotates about a fixed vertical axis passing through its centre with angular velocity ω. A particle of same mass m and having velocity of $2\omega R$ towards centre of the disc collides with the disc moving horizontally and sticks to its rim.

(a) The angular velocity of the disc will become $\omega/3$
(b) The angular velocity of the disc will become $5\omega/3$
(c) The impulse on the particle due to disc is $\frac{\sqrt{37}}{3} m\omega R$
(d) The impulse on the particle due to disc is $2m\omega R$

11. The end B of the rod AB which makes angle θ with the floor is being pulled with a constant velocity v_0 as shown. The length of the rod is l.



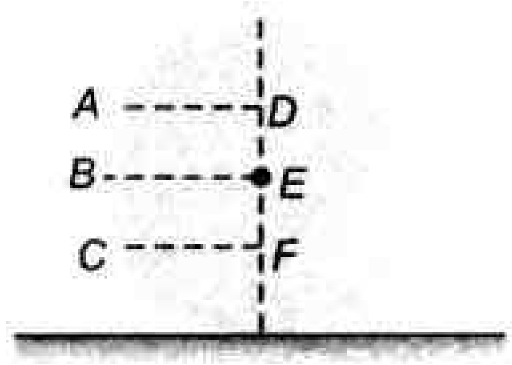
(a) At $\theta = 37^\circ$ velocity of end A is $\frac{4}{3} v_0$ downwards
(b) At $\theta = 37^\circ$ angular velocity of rod is $\frac{5v_0}{3l}$
(c) Angular velocity of rod is constant
(d) Velocity of end A is constant

Match the Columns

1. A solid sphere, a hollow sphere and a disc of same mass and same radius are released from a rough inclined plane. All of them rolls down without slipping. On reaching the bottom of the plane, match the two columns.

Column I	Column II
(a) time taken to reach the bottom	(p) maximum for solid sphere
(b) total kinetic energy	(q) maximum for hollow sphere
(c) rotational kinetic energy	(r) maximum for disc
(d) translational kinetic energy	(s) same for all

2. A solid sphere is placed on a rough ground as shown. E is the centre of sphere and $DE > EF$. We have to apply a linear impulse either at point A, B or C. Match the following two columns.

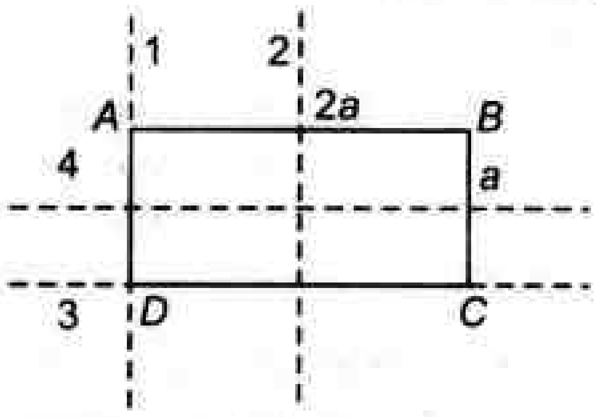


Column I	Column II
(a) Sphere will acquire maximum angular speed it impulse is applied at	(p) A
(b) Sphere will acquire maximum linear speed it impulse is applied at	(q) B
(c) Sphere can roll without slipping if impulse is applied at	(r) C
(d) Sphere can roll with forward slipping if impulse is applied at	(s) at any point A, B or C

3. The inclined surfaces shown in column I are sufficiently rough. In column II direction and magnitudes of frictional forces are mentioned. Match the two columns.

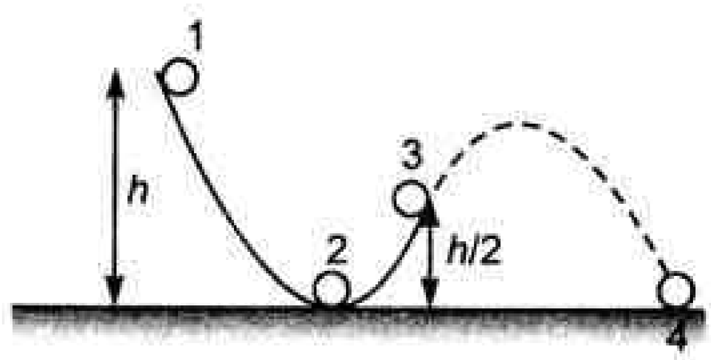
Column I	Column II
(a)  Rolling upwards	(p) upwards
(b)  Kept in rotating position	(q) downwards
(c)  Kept in translational position	(r) maximum friction will act
(d)  Kept in translational position	(s) required value of friction will act

4. A rectangular slab $ABCD$ have dimensions $a \times 2a$ as shown in figure. Match the following two columns.



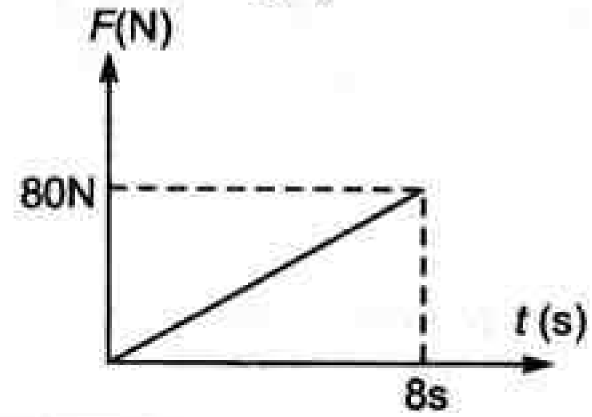
Column I	Column II
(a) Radius of gyration about axis-1	(p) $\frac{a}{\sqrt{12}}$
(b) Radius of gyration about axis-2	(q) $\frac{2a}{\sqrt{3}}$
(c) Radius of gyration about axis-3	(r) $\frac{a}{\sqrt{3}}$
(d) Radius of gyration about axis-4	(s) None

5. A small solid ball rolls down along sufficiently rough surface from 1 to 3 as shown in figure. From point-3 onwards it moves under gravity. Match the following two columns.



Column I	Column II
(a) Rotational kinetic energy of ball at point-2	(p) $\frac{1}{7}mgh$
(b) Translational kinetic energy of ball at point-3	(q) $\frac{2}{7}mgh$
(c) Rotational kinetic energy of ball at point-4	(r) $\frac{5}{7}mgh$
(d) Translational kinetic energy of ball at point-4	(s) None

6. A uniform disc of mass 10 kg, radius 1 m is placed on a rough horizontal surface. The co-efficient of friction between the disc and the surface is 0.2. A horizontal time varying force is applied on the centre of the disc whose variation with time is shown in graph.



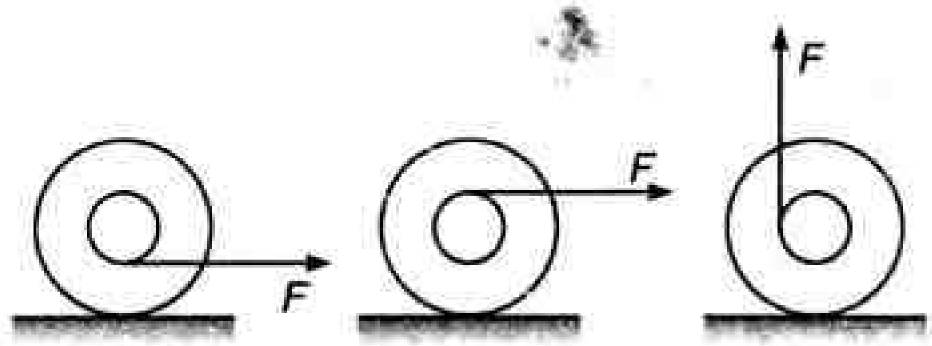
Column I	Column II
(a) Disc rolls without slipping	(p) at $t = 7$s
(b) Disc rolls with slipping	(q) at $t = 3$ s
(c) Disc starts slipping at	(r) at $t = 4$ s
(d) Friction force is 10 N at	(s) None

7. Match the columns.

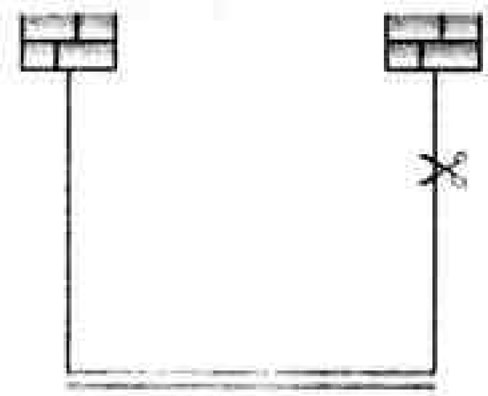
Column I	Column II
(a) Moment of inertia of a circular disc of mass M and radius R about a tangent parallel to plane of disc	(p) $\frac{MR^2}{2}$
(b) Moment of inertia of a solid sphere of mass M and radius R about a tangent	(q) $\frac{7}{5}MR^2$
(c) Moment of inertia of a circular disc of mass M and radius R about a tangent perpendicular to plane of disc	(r) $\frac{5}{4}MR^2$
(d) Moment of inertia of a cylinder of mass M and radius R about its axis	(s) $\frac{3}{2}MR^2$

Subjective Questions

1. Figure shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo the string is pulled in the direction shown. In each case there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate ?



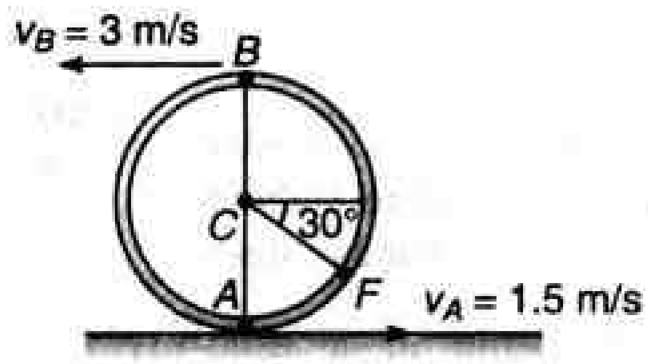
2. A uniform rod of mass m and length l is held horizontally by two vertical strings of negligible mass, as shown in the figure.
   (a) Immediately after the right string is cut, what is the linear acceleration of the free end of the rod ?
   (b) Of the middle of the rod ?
   (c) Determine the tension in the left string immediately after the right string is cut.



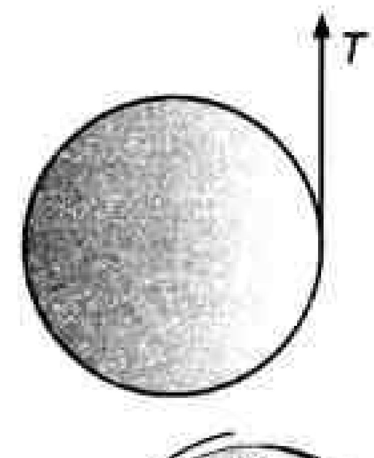
3. A solid disk is rolling without slipping on a level surface at a constant speed of 2.00 m/s. How far can it roll up a 30° ramp before it stops? (Take $g = 9.8\,\text{m/s}^2$)

4. A lawn roller in the form of a thin-walled hollow cylinder of mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

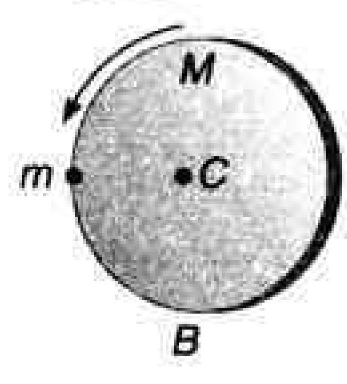
5. Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the centre point C and point F at this instant.



6. A uniform cylinder of mass M and radius R has a string wrapped around it. The string is held fixed and the cylinder falls vertically, as in figure.
   (a) Show that the acceleration of the cylinder is downward with magnitude $a = \frac{2g}{3}$.
   (b) Find the tension in the string.

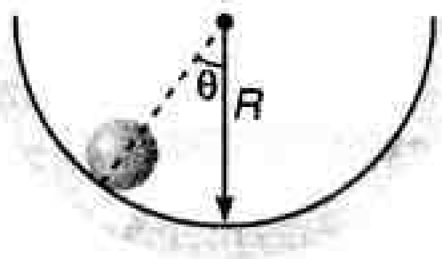


7. A uniform disc of mass M and radius R is pivoted about the horizontal axis through its centre C. A point mass m is glued to the disc at its rim, as shown in figure. If the system is released from rest, find the angular velocity of the disc when m reaches the bottom point B.

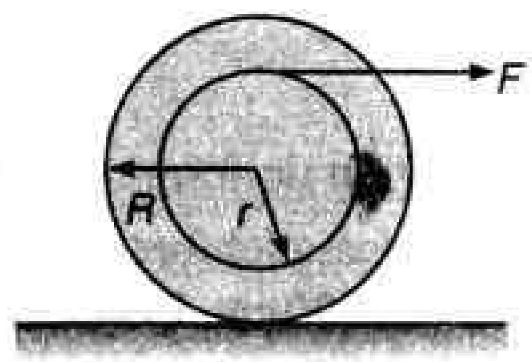


8. A disc of radius R and mass m is projected on to a horizontal floor with a backward spin such that its centre of mass speed is v_0 and angular velocity is ω_0. What must be the minimum value of ω_0 so that the disc eventually returns back?

9. A ball of mass m and radius r rolls along a circular path of radius R. Its speed at the bottom ($\theta = 0°$) of the path is v_0. Find the force of the path on the ball as a function of θ.

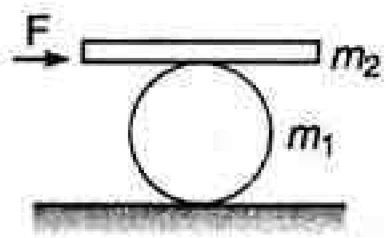


10. A heavy homogeneous cylinder has mass m and radius R. It is accelerated by a force F, which is applied through a rope wound around a light drum of radius r attached to the cylinder (figure). The coefficient of static friction is sufficient for the cylinder to roll without slipping.



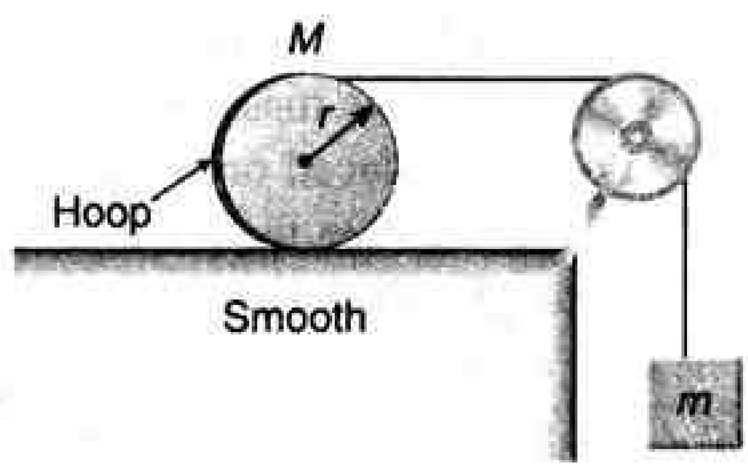
(a) Find the friction force.
(b) Find the acceleration a of the centre of the cylinder.
(c) Is it possible to choose r, so that a is greater than $\frac{F}{m}$? How ?
(d) What is the direction of the friction force in the circumstances of part (c) ?

11. A man pushes a cylinder of mass m_1 with the help of a plank of mass m_2 as shown. There is no slipping at any contact. The horizontal component of the force applied by the man is F. Find :



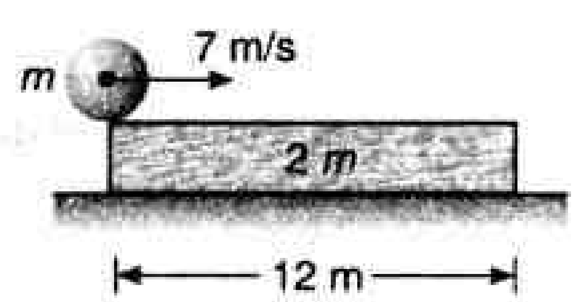
(a) the acceleration of the plank and the centre of mass of the cylinder and
(b) the magnitudes and directions of frictional forces at contact points.

12. For the system shown in figure, $M = 1\,\text{kg}$, $m = 0.2\,\text{kg}$, $r = 0.2\,\text{m}$.
Calculate : ($g = 10\,\text{m/s}^2$)
(a) the linear acceleration of hoop,
(b) the angular acceleration of the hoop of mass M and
(c) the tension in the rope.

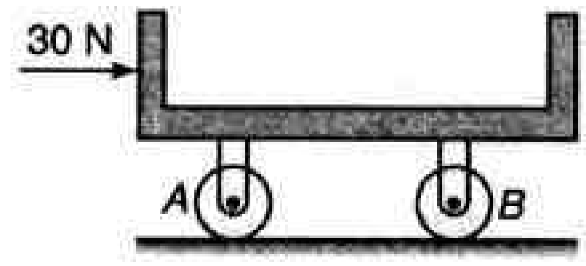


Note *Treat hoop as the ring. Assume no slipping between string and hoop.*

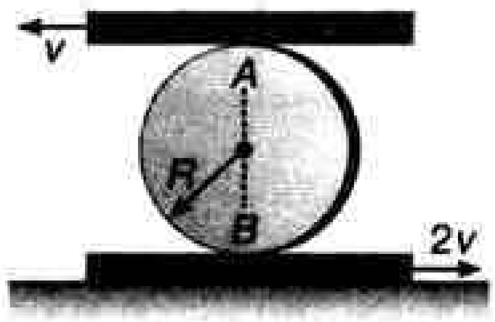
13. A cylinder of mass m is kept on the edge of a plank of mass $2m$ and length 12 m, which in turn is kept on smooth ground. Coefficient of friction between the plank and the cylinder is 0.1. The cylinder is given an impulse, which imparts it a velocity 7 m/s but no angular velocity. Find the time after which the cylinder falls off the plank. ($g = 10\,\text{m/s}^2$)



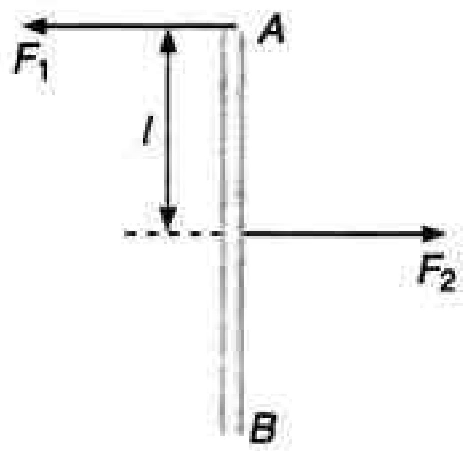
14. The 9 kg cradle is supported as shown by two uniform disks that roll without sliding at all surfaces of contact. The mass of each disk is $m = 6\,\text{kg}$ and the radius of each disk is $r = 80\,\text{mm}$. Knowing that the system is initially at rest, determine the velocity of the cradle after it has moved 250 mm.



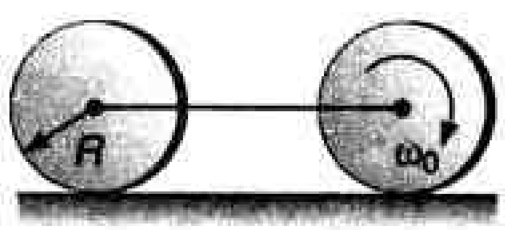
15. The disc of the radius R is confined to roll without slipping at A and B. If the plates have the velocities shown, determine the angular velocity of the disc.



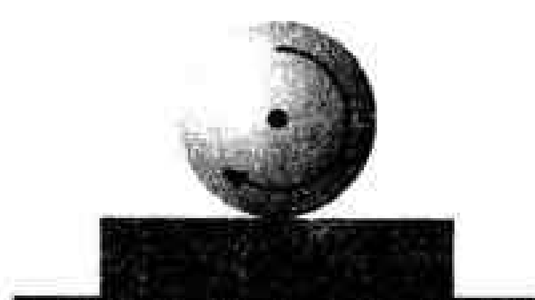
16. A thin uniform rod AB of mass $m = 1\,\text{kg}$ moves translationally with acceleration $a = 2\,\text{m/s}^2$ due to two antiparallel forces F_1 and F_2. The distance between the points at which these forces are applied is equal to $l = 20\,\text{cm}$. Besides, it is known that $F_2 = 5\,\text{N}$. Find the length of the rod.



17. The assembly of two discs as shown in figure is placed on a rough horizontal surface and the front disc is given an initial angular velocity ω_0. Determine the final linear and angular velocity when both the discs start rolling. It is given that friction is sufficient to sustain rolling in the rear wheel from the starting of motion.



18. A horizontal plank having mass m lies on a smooth horizontal surface. A sphere of same mass and radius r is spined to an angular frequency ω_0 and gently placed on the plank as shown in the figure. If coefficient of friction between the plank and the sphere is μ. Find the distance moved by the plank till the sphere starts pure rolling on the plank. The plank is long enough.



19. A ball rolls without sliding over a rough horizontal floor with velocity $v_0 = 7\,\text{m/s}$ towards a smooth vertical wall. If coefficient of restitution between the wall and the ball is $e = 07$. Calculate velocity v of the ball long after the collision.

20. A uniform rod of mass m and length l rests on a smooth horizontal surface. One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result the rod obtains velocity v_0. Find the force with which one-half of the rod will act on the other in the process of motion.

21. A sphere, a disk and a hoop made of homogeneous materials have the same radius (10 cm) and mass (3 kg). They are released from rest at the top of a 30° incline and roll down without slipping through a vertical distance of 2 m. ($g = 9.8\,\text{m/s}^2$)

(a) What are their speeds at the bottom ?
(b) Find the frictional force f in each case
(c) If they start together at $t = 0$, at what time does each reach the bottom ?

22. ABC is a triangular framework of three uniform rods each of mass m and length $2l$. It is free to rotate in its own plane about a smooth horizontal axis through A which is perpendicular to ABC. If it is released from rest when AB is horizontal and C is above AB. Find the maximum velocity of C in the subsequent motion.

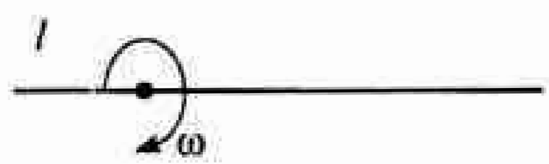
23. A uniform stick of length L and mass M hinged at one end is released from rest at an angle θ_0 with the vertical. Show that when the angle with the vertical is θ, the hinge exerts a force F_r along the stick and F_t perpendicular to the stick given by

$$F_r = \frac{1}{2} Mg\,(5\cos\theta - 3\cos\theta_0) \quad \text{and} \quad F_t = \frac{1}{4} Mg\sin\theta$$

24. A uniform rod AB of mass $3m$ and length $4l$, which is free to turn in a vertical plane about a smooth horizontal axis through A, is released from rest when horizontal. When the rod first becomes vertical, a point C of the rod, where $AC = 3l$, strikes a fixed peg. Find the linear impulse exerted by the peg on the rod if :

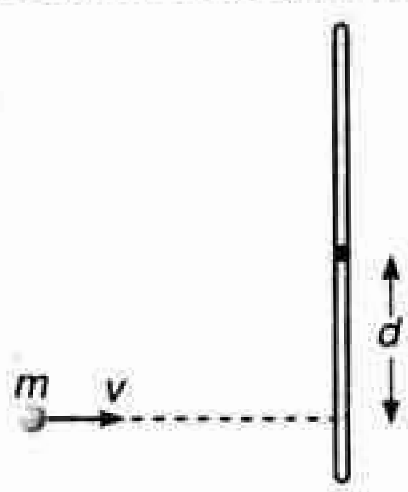
(a) the rod is brought to rest by the peg,
(b) the rod rebounds and next comes to instantaneous rest inclined to the downward vertical at an angle $\frac{\pi}{3}$ radian.

25. A uniform rod of length $4l$ and mass m is free to rotate about a horizontal axis passing through a point distant l from its one end. When the rod is horizontal, its angular velocity is ω as shown in figure. Calculate :

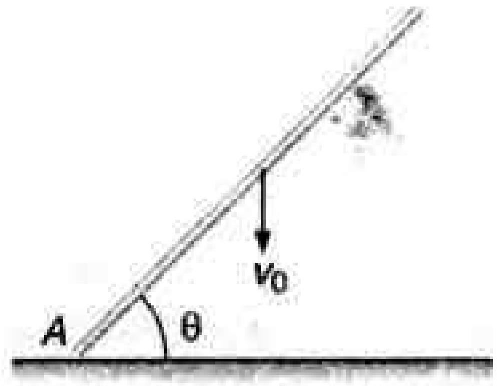


(a) reaction of axis at this instant,
(b) acceleration of centre of mass of the rod at this instant,
(c) reaction of axis and acceleration of centre mass of the rod when rod becomes vertical for the first time,
(d) minimum value of ω, so that centre of rod can complete circular motion.

26. A stick of length l lies on horizontal table. It has a mass M and is free to move in any way on the table. A ball of mass m, moving perpendicularly to the stick at a distance d from its centre with speed v collides elastically with it as shown in figure. What quantities are conserved in the collision ? What must be the mass of the ball, so that it remains at rest immediately after collision?



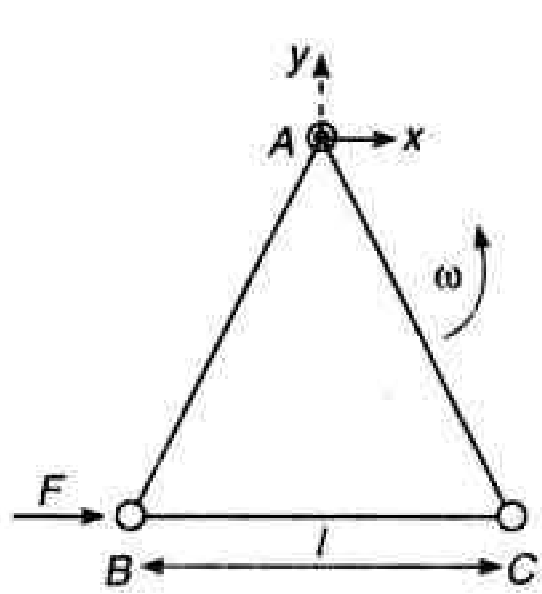
27. A rod of length l forming an angle θ with the horizontal strikes a frictionless floor at A with its centre of mass velocity v_0 and no angular velocity. Assuming that the impact at A is perfectly elastic. Find the angular velocity of the rod immediately after the impact.



28. Three particles A, B and C, each of mass m, are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side l. This body is placed on a horizontal frictionless table (x-y plane) and is hinged to it at the point A, so that it can move without friction about the vertical axis through A (see figure). The body is set into rotational motion on the table about A with a constant angular velocity ω.

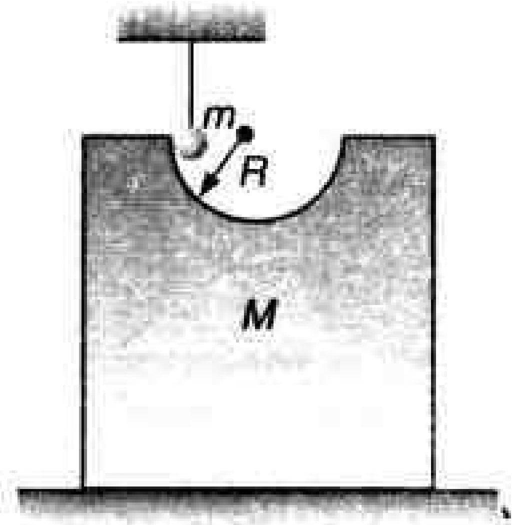
(a) Find the magnitude of the horizontal force exerted by the hinge on the body.

(b) At time T, when the side BC is parallel to the x-axis, a force F is applied on B along BC (as shown). Obtain the x-component and the y-component of the force exerted by the hinge on the body, immediately after time T.



29. A semicircular track of radius $R = 62.5\,\text{cm}$ is cut in a block. Mass of block, having track, is $M = 1$ kg and rests over a smooth horizontal floor. A cylinder of radius $r = 10\,\text{cm}$ and mass $m = 0.5\,\text{kg}$ is hanging by thread such that axes of cylinder and track are in same level and surface of cylinder is in contact with the track as shown in figure. When the thread is burnt, cylinder starts to move down the track. Sufficient friction exists between surface of cylinder and track, so that cylinder does not slip.

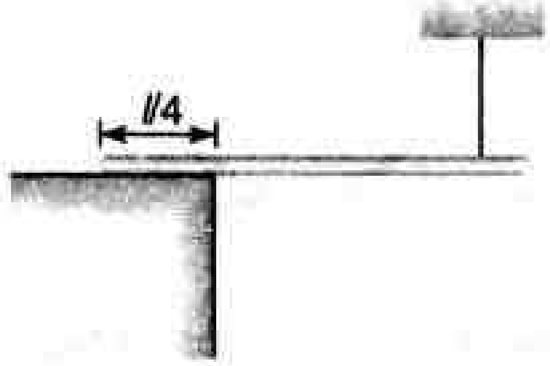
Calculate velocity of axis of cylinder and velocity of the block when it reaches bottom of the track. Also find force applied by block on the floor at that moment. ($g = 10\,\text{m/s}^2$)



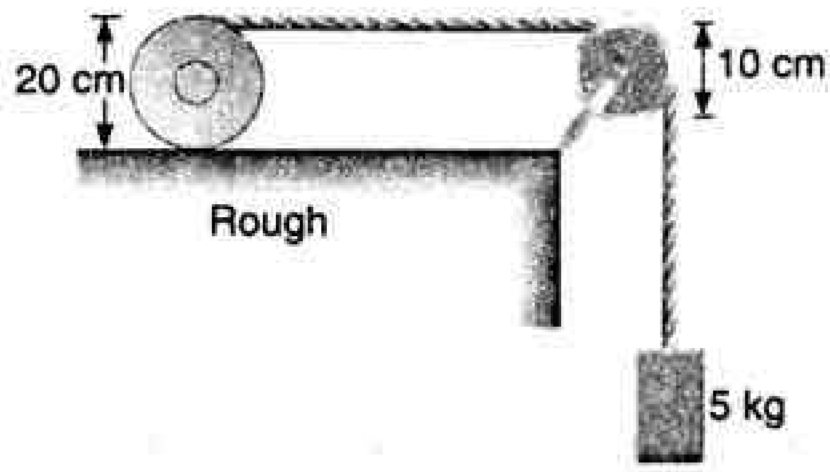
30. A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity. It is placed in contact with a plane inclined at an angle α to the horizontal. If there is a coefficient of friction μ for sliding between the cylinder and plane. Find the distance the cylinder moves up before sliding stops. Also, calculate the maximum distance it travels up the plane. Assume $\mu > \tan\alpha$.

31. Show that if a rod held at angle θ to the horizontal and released, its lower end will not slip if the friction coefficient between rod and ground is greater than $\dfrac{3\sin\theta\cos\theta}{1+3\sin^2\theta}$.

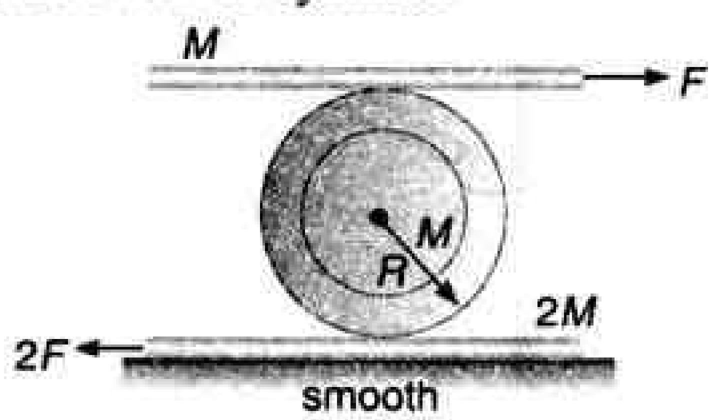
32. One-fourth length of a uniform rod of mass m and length l is placed on a rough horizontal surface and it is held stationary in horizontal position by means of a light thread as shown in the figure. The thread is then burnt and the rod starts rotating about the edge. Find the angle between the rod and the horizontal when it is about to slide on the edge. The coefficient of friction between the rod and the surface is μ.



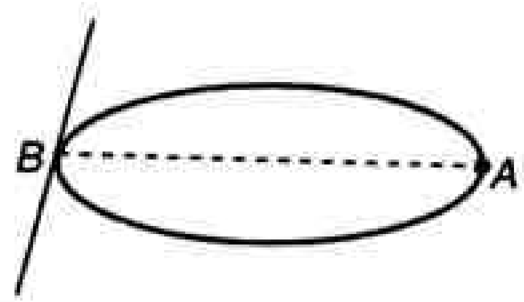
33. In figure the cylinder of mass 10 kg and radius 10 cm has a tape wrapped round it. The pulley weighs 100 N and has a radius 5 cm. When the system is released, the 5 kg mass comes down and the cylinder rolls without slipping. Calculate the acceleration and velocity of the mass as a function of time.



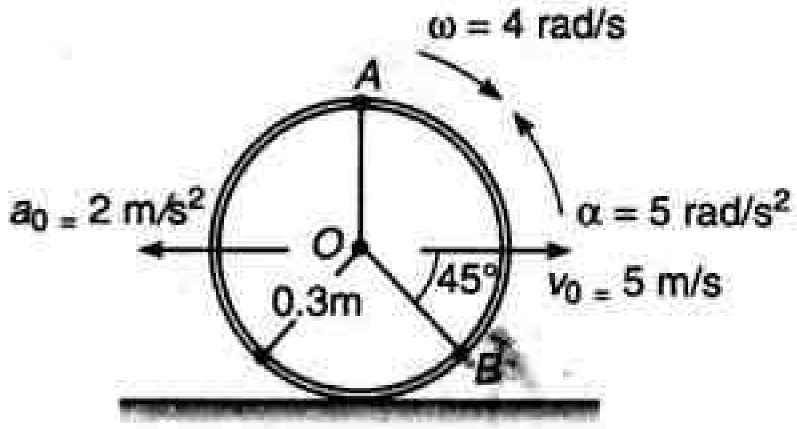
34. A cylinder is sandwiched between two planks. Two constant horizontal forces F and $2F$ are applied on the planks as shown. Determine the acceleration of the centre of mass of cylinder and the top plank, if there is no slipping at the top and bottom of cylinder.



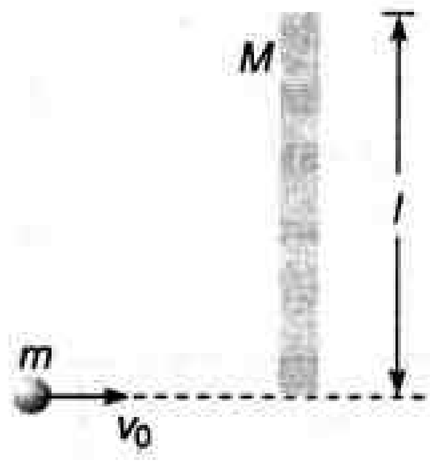
35. A ring of mass m and radius r has a particle of mass m attached to it at a point A. The ring can rotate about a smooth horizontal axis which is tangential to the ring at a point B diametrically opposite to A. The ring is released from rest when AB is horizontal. Find the angular velocity and the angular acceleration of the body when AB has turned through an angle $\frac{\pi}{3}$.



36. A hoop is placed on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular deceleration $\alpha = 5$ rad/s^2. Also, its centre has a velocity of $v_0 = 5$ m/s and a deceleration $a_0 = 2$ m/s^2. Determine the magnitude of acceleration of point B at this instant.



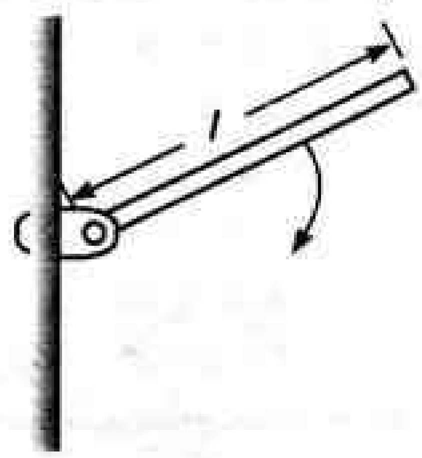
37. A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.



(a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

(b) One point on the plank is at rest immediately after the collision. Where is it?

38. A thin plank of mass M and length l is pivoted at one end. The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



ANSWERS

Introductory Exercise 9.1

1. About a diagonal, because the mass is more concentrated about a diagonal **2.** $\frac{\pi^2}{3}$ **3.** $\frac{2}{\sqrt{3}} l$

4. (i) $\frac{8}{5} mr^2 + 2ma^2$ (ii) $\frac{8}{5} mr^2 + ma^2$ **5.** (a) $2Ml^2$ (b) $\frac{1}{3} Ml^2$ **6.** $\frac{2}{3} Ma^2$

7. 0.5 kg-m^2 **8.** 0.43 kg-m^2 **9.** $\frac{R}{\sqrt{2}}$ **10.** The one having the smaller density

Introductory Exercise 9.2

1. 100 rad **2.** 800 rad **3.** 5 N-m **4.** 0.87 N **5.** (a) 4 rad-s^{-1}, – 6 rad-s^{-2} (b) – 12 rad-s^{-2}

6. 7 s

Introductory Exercise 9.3

2. $\frac{1}{2} mRv$ **3.** $2\sqrt{2}\, mv$ **4.** $\frac{mu^3 \cos\alpha \sin^2\alpha}{2g}$ **5.** No

Introductory Exercise 9.4

1. $\frac{\omega_0 M}{M + 2m}$ **2.** Duration of day-night incerease **3.** True

Introductory Exercise 9.5

1. $y^2 = \left(\frac{2a_0}{\omega^2}\right) x$ **2.** $\sqrt{\frac{3g}{l}(1 - \sin\theta)}$

Introductory Exercise 9.6

1. $\frac{2}{7} mgh$ **2.** $\pm \cos^{-1}\left(\frac{v}{R\omega}\right)$ **3.** $\frac{v_1 - v_2}{2R}$

Introductory Exercise 9.7

1. (a) $g \sin\theta - \mu g \cos\theta$ (b) $\frac{5}{2} \frac{\mu g \cos\theta}{R}$ **2.** False **3.** Leftwards **4.** False **5.** $\frac{I + 2Mr^2}{4Mr^2 - I}$

6. $\lim_{F \to 0}$ can make the body move **7.** False

Introductory Exercise 9.8

1. (a) $\mu < 1$ (b) $\mu > 1$ **2.** $\frac{2}{5} R$

For JEE Main

Subjective Questions

1. $\frac{5}{3} ml^2$ **2.** 55 kg-m^2 **3.** $\frac{l}{\sqrt{2}}$ **4.** $\frac{Ma^2}{12}$ **5.** 8 cm

6. $I = \mu r^2$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass of two masses.

7. $I = \left(\frac{\alpha l^4}{4} + \frac{\beta l^3}{3}\right)$ **8.** $\frac{\pi}{30}$ rad-s^{-1} **9.** $(-\hat{k})$ rad-s^{-1} **10.** 10 rad-s^{-1} **11.** $\omega = \frac{v}{2R}$ **12.** 2 rad/s

13. $(-2\hat{i} - 2\hat{k})$ N-m **14.** 400 N-m (perpendicular to the plane of motion) **15.** 2.71 N-m **16.** $\frac{83}{20}$ N-m

17. (8π)rad-s^{-2}, (40π)rad-s^{-1} **18.** 70 rad **19.** (a) 0.01 N-m (b) 0.13 N-m **20.** 20 s

21. $\frac{20}{3}$ N-m, 4 s **22.** (a) $\frac{3\omega_0 R}{4\mu g}$ (b) $\frac{3\omega_0^2 R}{8\mu g}$ **23.** (a) 36 s (b) $12\sqrt{\frac{2}{k}}$

24. (a) $\omega = 12.5$ rad-s^{-1} (b) 127.5 rad **25.** 9 rad, 1.43 **26.** $\omega_{av} = 4$ rad/s, $\alpha_{av} = -6.0$ rad-s^{-2}

27. $4\sqrt{2}$ kg-m^{2}s^{-1} **28.** $ml^2\omega$ **29.** $-\left(\frac{7}{5}mRv\right)\hat{k}$ **30.** $-\left(\frac{10}{3}mRv\right)\hat{k}$ **31.** Increase **32.** $\frac{25}{6}$ rad-s^{-1}

33. 7.29×10^{-4} kg-m^2 **34.** (a) 14.3 rad-s^{-1} = 2.27 rev-s^{-1} (b) $E_i = 39.9$ J, $E_f = 181$ J (c) 141.1 J

35. (a) $\omega = \left(1 + \frac{2m}{M}\right)\omega_0$ (b) $\frac{1}{2}m\omega_0^2 R^2\left(1 + \frac{2m}{M}\right)$ **36.** $\frac{4F\cos\theta}{3M + 8m}, \frac{3MF\cos\theta}{3M + 8m}, \frac{MF\cos\theta}{3M + 8m}$ **37.** $\frac{F}{M + 3m}$

38. 72 N **39.** (a) $\frac{2}{7\sqrt{3}}$ (b) $\frac{25}{7}$ ms^{-2} (c) $\frac{30}{7}$ ms^{-2} **40.** $\frac{4}{3}ml\omega$ **41.** (a) $\frac{2v}{9L}$ (b) $\frac{1}{9}$

42. (a) $\frac{2u}{3}$ (b) $\frac{2u}{\sqrt{3}}$

Objective Questions

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (b)
11. (d)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (d)
21. (c)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (d)

For JEE Advanced

Assertion and Reason

1. (d)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (a)
11. (c)									

Objective Questions

1. (d)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (c)
11. (b)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (d)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (d)	**20.** (b)
21. (c)	**22.** (b)	**23.** (c)	**24.** (d)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (a)	**30.** (c)
31. (b)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (d)	**36.** (a)	**37.** (d)	**38.** (b)	**39.** (d)	**40.** (c)
41. (a)	**42.** (a)	**43.** (a)	**44.** (c)	**45.** (a)	**46.** (d)	**47.** (b)	**48.** (d)	**49.** (c)	**50.** (c)
51. (d)	**52.** (d)	**53.** (a)							

More than One Correct Options

1. (a,c)	**2.** (all)	**3.** (a,b,d)	**4.** (a,b)	**5.** (a,c,d)	**6.** (all)	**7.** (a,c)
8. (b,d)	**9.** (a,b)	**10.** (a,c)	**11.** (a,b)			

Match the Columns

1. (a) → q (b) → s (c) → q (d) → p
2. (a) → p (b) → s (c) → p (d) → r
3. (a) → p,s (b) → p,r (c) → q,r (d) → p,r
4. (a) → q (b) → r (c) → r (d) → p
5. (a) → q (b) → s (c) → p (d) → s
6. (a) → q,r (b) → p (c) → s (d) → q
7. (a) → r (b) → q (c) → s (d) → p

Subjective Questions

1. In each case in clockwise direction **2.** (a) $3g/2$, (b) $3g/4$ (c) $mg/4$ **3.** 0.612 m **4.** $\frac{F}{2M}, \frac{F}{2}$

5. 0.75 ms^{-1}, 1.98 ms^{-1} **6.** (b) $\frac{1}{3} Mg$ **7.** $\sqrt{\frac{4mg}{(2m+M)R}}$ **8.** $\frac{2v_0}{R}$

9. $f = \frac{2}{7} mg \sin\theta$, $N = \frac{mg}{7}(17 \cos\theta - 10) + \frac{mv_0^2}{(R-r)}$

10. (a) $f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right) F$ assuming f opposite to F (b) $a = \left(\frac{2F}{3mR}\right)(R + r)$ (c) yes, if r is greater than $\frac{1}{2} R$.

(d) f in same direction as F.

11. (a) $\frac{8F}{3m_1 + 8m_2}$, $\frac{4F}{3m_1 + 8m_2}$

(b) $\frac{3m_1 F}{3m_1 + 8m_2}$ (between plank and cylinder) $\frac{m_1 F}{3m_1 + 8m_2}$ (between cylinder and ground)

12. (a) 1.43 ms^{-2} (b) 7.15 rad·s^{-2} (c) 1.43 N **13.** 2.25 s **14.** 0.745 ms^{-1} (rightwards)

15. $\frac{3}{2}\frac{v}{r}$ (anticlockwise) **16.** 1 m **17.** $\frac{\omega_0 R}{6}; \frac{\omega_0}{6}$

18. $S = \frac{2\omega_0^2 r^2}{81\mu g}$ **19.** $v = 1.5$ ms^{-1} **20.** $\frac{9}{2}\frac{mv_0^2}{l}$

21. (a) Sphere, 5.29 ms^{-1}, disk 5.11 ms^{-1}, hoop 4.43 ms^{-1} (b) Sphere 4.2 N, disk 4.9 N, hoop 7.36 N

(c) Sphere, 1.51 s disk 1.56 s hoop 1.81 s

22. $2l\sqrt{\frac{g\sqrt{3}}{l}}$ **24.** (a) $\left(\frac{8}{3} m\right)\sqrt{3gl}$, (b) $\frac{4}{3} m\sqrt{6gl}(\sqrt{2} + 1)$

25. (a) $\frac{4}{7} mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$ (b) $\sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$ (c) $\left(\frac{13}{7} mg + ml\omega^2\right), \left(\frac{6g}{7} + l\omega^2\right)$ (d) $\sqrt{\frac{6g}{7l}}$

26. $\frac{Ml^2}{12d^2 + l^2}$ **27.** $\omega = \frac{6v_0}{l}\left(\frac{\cos\theta}{1 + 3\cos^2\theta}\right)$

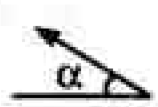
28. (a) $\sqrt{3}\, ml\omega^2$ (b) $F_x = -\frac{F}{4}$, $F_y = \sqrt{3}\, ml\omega^2$

29. 2.0 ms^{-1}, 1.5 ms^{-1}, 16.67 N

30. $d_1 = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$, $d_{max} = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha(3\mu\cos\alpha - \sin\alpha)}$

32. $\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$ **33.** 3.6 ms^{-2}, $\frac{4gt}{11}$ **34.** $\frac{F}{26M}, \frac{21F}{26M}$

35. $\sqrt{\frac{6g\sqrt{3}}{11r}}, \frac{3g}{11r}$ **36.** 6.21 ms^{-2}

37. (b) $\frac{2l}{3}$ from the boy **38.** $\frac{\sqrt{10}}{4} Mg$, 

$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

10

GRAVITATION

Chapter Contents

10.1 Introduction

Why are planets, moon and the sun all nearly spherical? Why do some earth satellites circle the earth in 90 minutes, while the moon takes 27 days for the trip? And why don't satellites fall back to earth? The study of gravitation provides the answers for these and many related questions.

Gravitation is one of the four classes of interactions found in nature. These are :

(i) the gravitational force
(ii) the electromagnetic force
(iii) the strong nuclear force (also called the hadronic forces)
(iv) the weak nuclear forces.

Although, of negligible importance in the interactions of elementary particles, gravity is of primary importance in the interactions of large objects. It is gravity that holds the universe together.

In this chapter we will learn the basic laws that govern gravitational interactions.

10.2 Newton's Law of Gravitation

Along with his three laws of motion, Newton published the law of gravitation in 1687. According to him; *"every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them."*

Thus, the magnitude of the gravitational force F between two particles m_1 and m_2 placed at a distance r is,

$$F \propto \frac{m_1 m_2}{r^2}$$

or

$$F = G\frac{m_1 m_2}{r^2}$$

Here, G is a universal constant called gravitational constant whose magnitude is,

$$G = 6.67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2$$

The direction of the force F is along the line joining the two particles.

Following three points are important regarding the gravitational force :

(i) Unlike the electrostatic force, it is independent of the medium between the particles.
(ii) It is conservative in nature.
(iii) It expresses the force between two point masses (of negligible volume). However, for external points of spherical bodies the whole mass can be assumed to be concentrated at its centre of mass.

Gravity

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is earth then the gravitation is called 'gravity'. Hence, gravity is the force by which earth attracts a body towards its centre. It is a special case of gravitation.

Sample Example 10.1 *Spheres of the same material and same radius r are touching each other. Show that gravitational force between them is directly proportional to* r^4.

Solution

$$m_1 = m_2 = (\text{volume})(\text{density})$$

$$= \left(\frac{4}{3}\pi r^3\right)\rho$$

$\therefore$
$$F = \frac{Gm_1 m_2}{r^2}$$

$$= \frac{G\left(\frac{4}{3}\pi r^3\right)\left(\frac{4}{3}\pi r^3\right)\rho^2}{r^2}$$

or
$$F \propto r^4$$

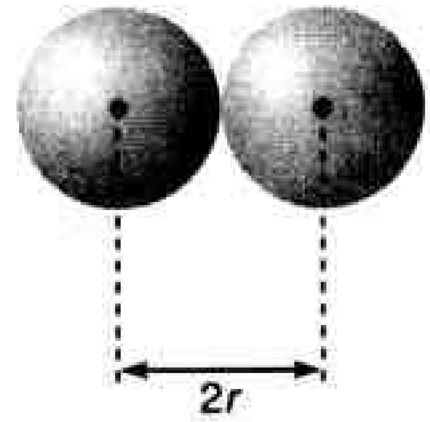


Fig. 10.1

Hence proved.

10.3 Acceleration due to Gravity

When a body is dropped from a certain height above the ground it begins to fall towards the earth under gravity. The acceleration produced in the body due to gravity is called the acceleration due to gravity. It is denoted by g. Its value close to the earth's surface is 9.8 m/s^2.

Suppose that the mass of the earth is M, its radius is R, then the force of attraction acting on a body of mass m close to the surface of earth is

$$F = \frac{GMm}{R^2}$$

According to Newton's second law, the acceleration due to gravity

$$g = \frac{F}{m} = \frac{GM}{R^2}$$

This expression is free from m. If two bodies of different masses are allowed to fall freely they will have the same acceleration, *i.e.*, if they are allowed to fall from the same height, they will reach the earth simultaneously.

Variation in the value of g

The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of earth. Thus, value of g depends on following factors.

(i) Shape of the earth

The earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius. And since

$$g = \frac{GM}{R^2}$$

or
$$g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles.

(ii) Height above the surface of the earth

The force of gravity on an object of mass m at a height h above the surface of earth is,

$$F = \frac{GMm}{(R+h)^2}$$

$\therefore$ Acceleration due to gravity at this height will be,

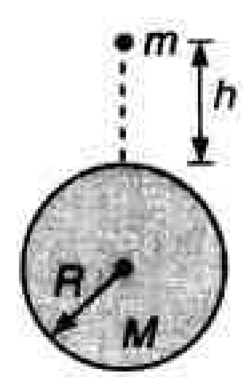


Fig. 10.2

$$g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

This can also be written as,
$$g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

or
$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \quad \text{as} \quad \frac{GM}{R^2} = g$$

Thus,
$$g' < g$$

i.e., the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1+\frac{h}{R}\right)^{-2}$$

or
$$g' \approx g\left(1-\frac{2h}{R}\right) \quad \text{if} \quad h << R$$

(iii) Depth below the surface of the earth

Let an object of mass m is situated at a depth h below the earth's surface. Its distance from the centre of earth is $(R-h)$. This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. The gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.

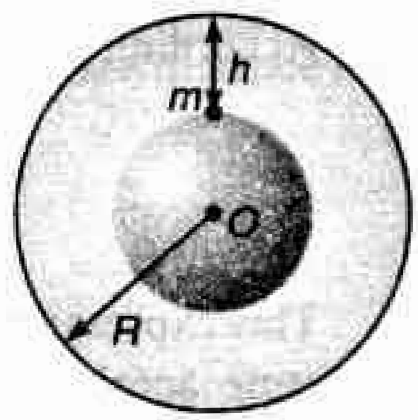


Fig. 10.3

The mass of this sphere is,

$$M' = \left(\frac{M}{4/3\pi R^3}\right)\frac{4}{3}\pi (R-h)^3$$

or
$$M' = \frac{(R-h)^3}{R^3} . M$$

$$F = \frac{GM'm}{(R-h)^2} = \frac{GMm(R-h)}{R^3}$$

and
$$g' = \frac{F}{m}$$

Substituting the values, we get
$$g' = g\left(1-\frac{h}{R}\right)$$

i.e.,
$$g' < g$$

Note *We can see from this equation that $g' = 0$ at $h = R$, i.e., acceleration due to gravity is zero at the centre of the earth.*

Thus, the variation in the value of g with r (the distance from the centre of earth) is as follows :

For $r \le R$,
$$g' = g\left(1-\frac{h}{R}\right) = \frac{gr}{R}$$

as
$$R - h = r$$

or $$g' \propto r$$

For $r > R$,
$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} = \frac{gR^2}{r^2}$$

or $$g' \propto \frac{1}{r^2}$$

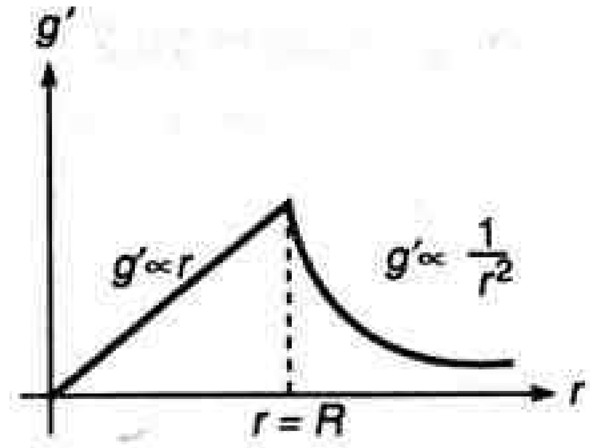


Fig. 10.4

(iv) Axial rotation of the earth

Let us consider a particle P at rest on the surface of the earth, in latitude ϕ. Then the pseudo force acting on the particle is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the centre O of the earth. Thus, the effective acceleration g' is the resultant of g and $r\omega^2$, or

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180-\phi)}$$

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2 \cos\phi} \qquad \text{...(i)}$$

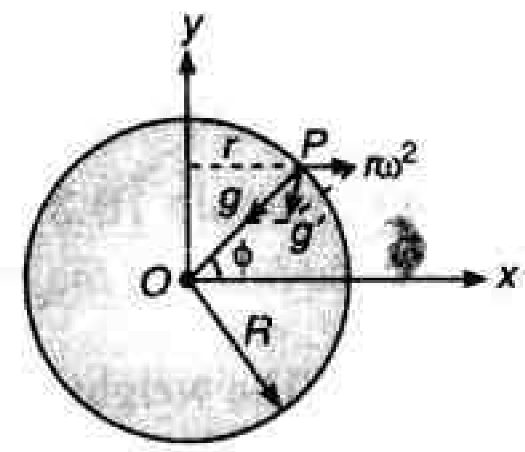


Fig. 10.5

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$ rad/s is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq. (i) can be written as

$$g' = (g^2 - 2gR\omega^2\cos^2\phi)^{1/2}$$

$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2}$$

$$\approx g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $$g' = g - R\omega^2\cos^2\phi$$

Following conclusions can be drawn from the above discussion :

(i) The effective value of g is not truely vertical.

(ii) The effect of centrifugal force due to rotation of earth is to reduce the effective value of g.

(iii) At equators $\phi = 0°$.

Therefore, $g' = g - R\omega^2$

and at poles $\phi = 90°$,

Therefore, $g' = g$

Thus, at equator g' is minimum while at poles g' is maximum.

Sample Example 10.2 *Assuming earth to be a sphere of uniform mass density, how much would a body weigh half way down the centre of the earth, if it weighed* 100 *N on the surface?*

Solution Given,

$$mg = 100\text{ N}$$

$$g' = g\left(1 - \frac{h}{R}\right)$$

$$\frac{h}{R} = \frac{1}{2}$$

$$\therefore \quad g' = g\left(1 - \frac{1}{2}\right) = \frac{g}{2}$$

$$\therefore \quad mg' = \frac{mg}{2} = \frac{100}{2} = 50\text{ N}$$

Sample Example 10.3 *Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero? Take* $g = 10\ m/s^2$ *and radius of earth* $R = 6400\ km$.

Solution The weight will become zero, when

$$g' = 0$$

or

$$g - R\omega^2 = 0 \qquad \text{(on the equator } g' = g - R\omega^2\text{)}$$

or

$$\omega = \sqrt{\frac{g}{R}}$$

$$\therefore \quad \frac{2\pi}{T} = \sqrt{\frac{g}{R}} \text{ or } T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values,

$$T = \frac{2\pi\sqrt{\dfrac{6400 \times 10^3}{10}}}{3600}\text{ h}$$

or

$$T = 1.4\text{ h}$$

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Introductory Exercise 10.1

1. Calculate the change in the value of g at altitude 45°. Take radius of earth $= 6.37 \times 10^3$ km
2. Determine the speed with which the earth would have to rotate on its axis, so that a person on the equator would weigh $\frac{3}{5}$th as much as at present. Take $R = 6400$ km
3. At what height from the surface of earth will the value of g be reduced by 36% from the value at the surface? $R = 6400$ km
4. The distance between two bodies A and B is r. Taking the gravitational force according to the law of inverse square of r, the acceleration of the body A is a. If the gravitational force follows an inverse fourth power law, then what will be the acceleration of the body A?
5. Find the force of attraction on a particle of mass m placed at the centre of a semicircular wire of length L and mass M.

10.4 Gravitational Field

The space around a body in which any other body experiences a force of attraction is called the gravitational field of the first body.

The force experienced (both in magnitude and direction) by a unit mass placed at a point in a gravitational field is called the **gravitational field strength** or **intensity of gravitational field** at that point. Usually it is denoted by $\vec{\mathbf{E}}$. Thus,

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{m}$$

In Article 10.3 we have seen that acceleration due to gravity $\vec{\mathbf{g}}$ is also $\frac{\vec{\mathbf{F}}}{m}$. Hence, for the earth's gravitational field $\vec{\mathbf{g}}$ and $\vec{\mathbf{E}}$ are same. The E *versus* r (the distance from the centre of earth) graph are same as that of g' *versus* r graph.

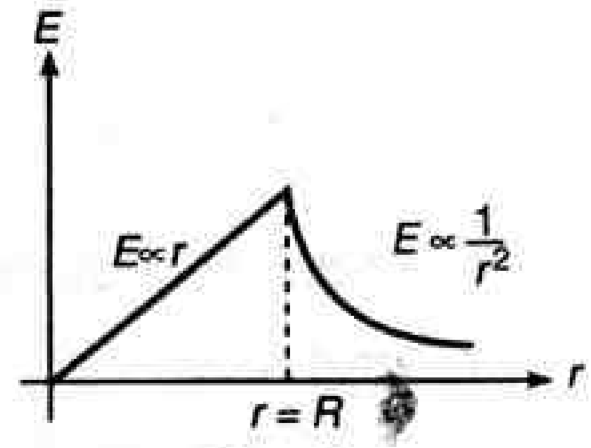


Fig. 10.6

(i) Field due to a point mass

Suppose, a point mass M is placed at point O. We want to find the intensity of gravitational field $\vec{\mathbf{E}}$ at a point P, a distance r from O. Magnitude of force F acting on a particle of mass m placed at P is,

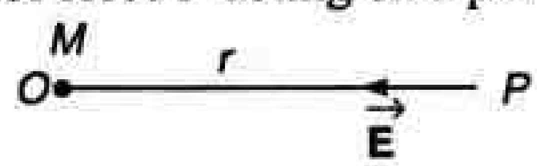


Fig. 10.7

$$F = \frac{GMm}{r^2}$$

$\therefore$
$$E = \frac{F}{m} = \frac{GM}{r^2}$$

or
$$E = \frac{GM}{r^2}$$

The direction of the force F and hence of E is from P to O as shown in Fig. 10.7.

(ii) Gravitational field due to a uniform solid sphere

Field at an external point

A uniform sphere may be treated as a single particle of same mass placed at its centre for calculating the gravitational field at an external point. Thus,

$$E(r) = \frac{GM}{r^2} \quad \text{for} \quad r \geq R$$

or
$$E(r) \propto \frac{1}{r^2}$$

Here, r is the distance of the point from the centre of the sphere and R the radius of sphere.

Field at an internal point

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre itself, it is zero and at surface it is $\frac{GM}{R^2}$, where R is the radius of the sphere. Thus,

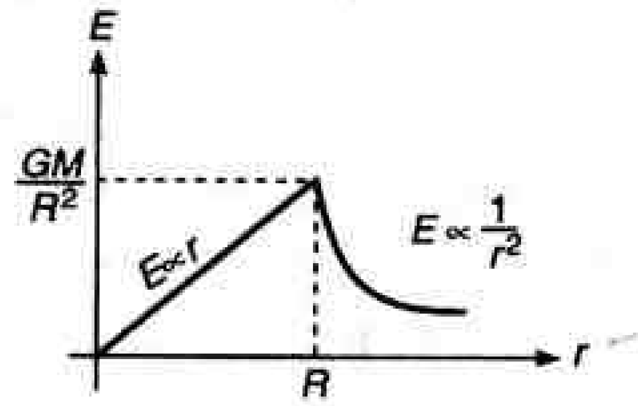


Fig. 10.8

$$E(r) = \frac{GM}{R^3} r \quad \text{for} \quad r \le R$$

or $$E(r) \propto r$$

Hence, E *versus* r graph is as shown in Fig. 10.8.

(iii) Field due to a uniform spherical shell

At an external point

For an external point the shell may be treated as a single particle of same mass placed at its centre. Thus, at an external point the gravitational field is given by,

$$E(r) = \frac{GM}{r^2} \qquad \text{For} \quad r \ge R$$

at $$r = R \qquad \text{(the surface of shell)}$$

$$E = \frac{GM}{R^2}$$

and otherwise $$E \propto \frac{1}{r^2}$$

At an internal point

The field inside a uniform spherical shell is zero.

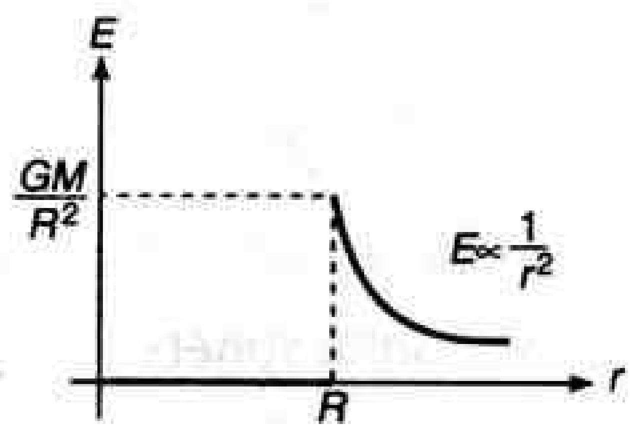


Fig. 10.9

Thus, E *versus* r graph is as shown in Fig. 10.9.

(iv) Field due to a uniform circular ring at a point on its axis

Field strength at a point P on the axis of a circular ring of radius R and mass M is given by,

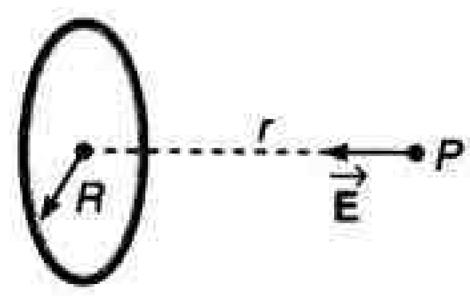


Fig. 10.10

$$E(r) = \frac{GMr}{(R^2 + r^2)^{3/2}}$$

This is directed towards the centre of the ring. It is zero at the centre of the ring and maximum at $r = \frac{R}{\sqrt{2}}$ (can be obtained by putting $\frac{dE}{dr} = 0$). Thus, E-r graph is as shown in Fig. 10.11.

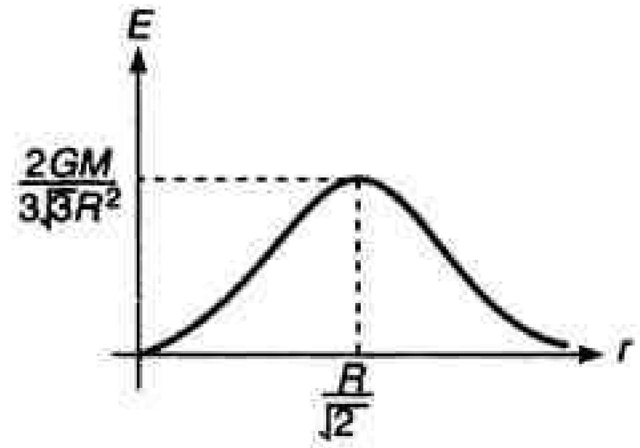


Fig. 10.11

The, maximum value is $E_{max} = \frac{2GM}{3\sqrt{3}R^2}$.

Note *Equivalent **Gauss theorem** for gravitational field is as follows:*

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = -4\pi G(m)$$

Here, m = enclosed mass

10.5 Gravitational Potential

If a body is moved in a gravitational field from one place to the other either work is done against the gravitational attraction or it is obtained.

The work done in bringing a unit mass from infinity to a point in the gravitational field is called the 'gravitational potential' at that point.

This work is obtained (not done) by the agent in bringing the mass. The gravitational potential is denoted by V. So, let W joule of work is obtained in bringing a test mass m from infinity to some point then gravitational potential at that point will be

$$V = \frac{W}{m}$$

Since, work is obtained, it is negative. Hence, gravitational potential is always negative.

(i) Potential due to a point mass

Suppose a point mass M is situated at a point O. We want to find the gravitational potential due to this mass at a point P a distance r from O. For this let us find work done in taking the unit mass from P to infinity. This will be,

Fig. 10.12

$$W = \int_r^\infty F\,dr = \int_r^\infty \frac{GM}{r^2}\,.\,dr = \frac{GM}{r}$$

Hence, the work done in bringing unit mass from infinity to P will be $-\frac{GM}{r}$. Thus, the gravitational potential at P will be,

$$V = -\frac{GM}{r}$$