(ii) Potential due to a uniform solid sphere

Potential at an external point

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of same mass placed at its centre. Thus,

$$V(r) = -\frac{GM}{r} \quad r \geq R$$

At the surface, $r = R$ and $V = -\frac{GM}{R}$

Potential at internal point

At some internal point, potential at a distance r from the centre is given by,

$$V(r) = -\frac{GM}{R^3}(1.5R^2 - 0.5r^2) \quad r \leq R$$

At $r = R$, $V = -\frac{GM}{R}$

while at $r = 0$, $V = -\frac{1.5\,GM}{R}$

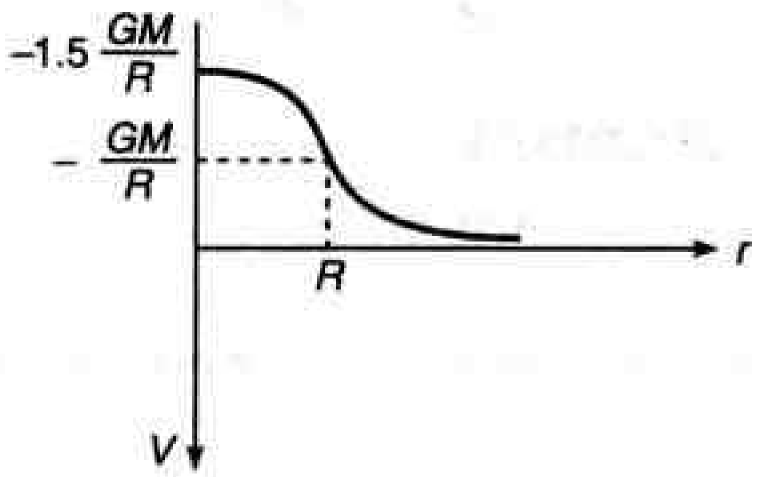


Fig. 10.13

i.e., at the centre of the sphere the potential is 1.5 times the potential at surface. The variation of V *versus* r graph is as shown in Fig. 10.13.

(iii) Potential due to a uniform thin spherical shell

Potential at an external point

To calculate the potential at an external point, a uniform spherical shell may be treated as a point mass of same magnitude at its centre. Thus, potential at a distance r is given by,

$$V(r) = -\frac{GM}{r} \quad r \geq R$$

at $r = R$, $V = -\frac{GM}{R}$

Potential at an internal point

The potential due to a uniform spherical shell is constant throughout at any point inside the shell and this is equal to $-\frac{GM}{R}$. Thus, V-r graph for a spherical shell is as shown in Fig. 10.14.

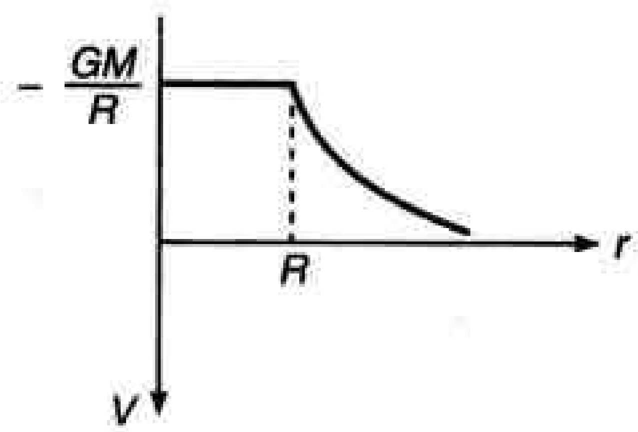


Fig. 10.14

(iv) Potential due to a uniform ring at a point on its axis

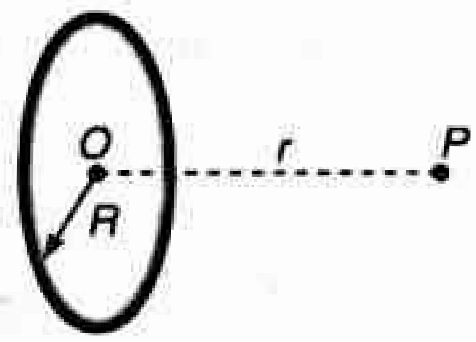


Fig. 10.15

The gravitational potential at a distance r from the centre on the axis of a ring of mass M and radius R is given by,

$$V(r) = -\frac{GM}{\sqrt{R^2 + r^2}} \qquad 0 \le r \le \infty$$

At $r = 0, V = -\frac{GM}{R}$, *i.e.*, at the centre of the ring gravitational potential is $-\frac{GM}{R}$.

The V-r graph is as shown in Fig. 10.16.

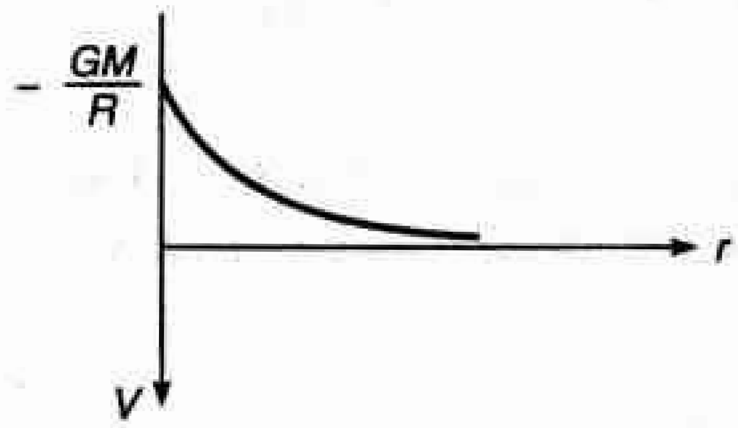


Fig. 10.16

10.6 Relation between Gravitational Field and Potential

Gravitational potential is a field function. It depends on the position of the point where potential is desired. Gravitational field and the gravitational potential are related by the following relation.

$$\vec{\mathbf{E}} = -\text{gradient } V = -\text{gradient } V$$

$$= -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$$

or

$$\vec{\mathbf{E}} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right] \qquad \text{...(i)}$$

Here, $\frac{\partial V}{\partial x}$ = Partial derivative of potential function V w.r.t. x, *i.e.*, differentiate V w.r.t. x assuming y and z to be constant.

Eq. (i) can be written in following different forms.

(i) $E = -\frac{dV}{dx}$, if gravitational field is along x-direction only

(ii) $dV = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$,

Here, $d\vec{\mathbf{r}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ and $\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$

Sample Example 10.4 *Two concentric spherical shells have masses m_1 and m_2 and radii r_1 and r_2 $(r_2 > r_1)$. What is the force exerted by this system on a particle of mass m_3 if it is placed at a distance $r(r_1 < r < r_2)$ from the centre?*

Solution The outer shell will have no contribution in the gravitational field at point P

$$\therefore \qquad E_P = \frac{Gm_1}{r^2}$$

Thus, force on mass m_3 placed at P is,

$$F = (m_3 E_P)$$

or
$$F = \frac{Gm_1 m_3}{r^2}$$

The field $\vec{\mathbf{E}}_P$ and the force $\vec{\mathbf{F}}$ both are towards centre O.

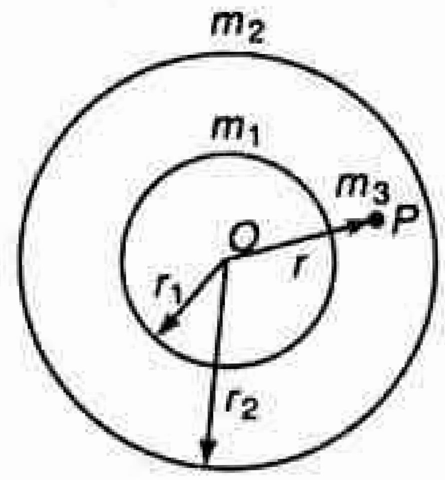


Fig. 10.17

Sample Example 10.5 *A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere.*

Solution Potential at the surface of sphere,

$$V = -\frac{GM}{R} = -\frac{(6.67 \times 10^{-11})(20)}{1} \text{ J/kg}$$

$$= -1.334 \times 10^{-9} \text{ J/kg}$$

i.e., 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus,

$$W = 1.334 \times 10^{-9} \text{ J}$$

Introductory Exercise 10.2

1. A particle of mass m is placed at the centre of a uniform spherical shell of same mass and radius R. Find the gravitational potential at a distance $\frac{R}{2}$ from the centre.
2. A particle of mass 20 g experiences a gravitational force of 4.0 N along positive x-direction. Find the gravitational field at that point.
3. The gravitational potential due to a mass distribution is $V = 3x^2y + y^3z$. Find the gravitational field.
4. Gravitational potential at $x = 2$ m is decreasing at a rate of 10 J/kg-m along the positive x-direction. It implies that the magnitude of gravitational field at $x = 2$ m is also 10 N/kg. Is this statement true or false?
5. The gravitational potential in a region is given by, $V = 20(x + y)$ J/kg. Find the magnitude of the gravitational force on a particle of mass 0.5 kg placed at the origin.
6. The gravitational field in a region is given by

$$\vec{\mathbf{E}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \text{ N/kg}.$$

Find the work done by the gravitational field when a particle of mass 1 kg is moved on the line $3y + 2x = 5$ from (1 m, 1 m) to (–2 m, 3 m).

10.7 Gravitational Potential Energy

The concept of potential energy has already been discussed in the chapter of work, energy and power. The word potential energy is defined only for a conservative force field. There we have discussed that the change in potential energy (dU) of a system corresponding to a conservative internal force is given by

$$dU = -\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

or

$$\int_i^f dU = -\int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

or

$$U_f - U_i = -\int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, *i.e.*, if we take $r_i = \infty$ (infinite) and $U_i = 0$ then we can write

$$U = -\int_{\infty}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -W$$

or potential energy of a body or system is negative of work done by the conservative forces in bringing it from infinity to the present position.

Gravitational Potential Energy of a two Particle System

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by,

Fig. 10.18

$$U = -\frac{Gm_1 m_2}{r}$$

This is actually the negative of work done in bringing those masses from infinity to a distance r by the gravitational forces between them.

Gravitational Potential Energy for a System of Particles

The gravitational potential energy for a system of particles (say m_1, m_2, m_3 and m_4) is given by

$$U = -G\left[\frac{m_4 m_3}{r_{43}} + \frac{m_4 m_2}{r_{42}} + \frac{m_4 m_1}{r_{41}} + \frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right]$$

Thus, for a n particle system there are $\frac{n(n-1)}{2}$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

Gravitational Potential Energy of a Body on Earth's Surface

The gravitational potential energy of mass m in the gravitational field of mass M at a distance r from it is,

$$U = -\frac{GMm}{r}$$

The earth behaves for all external points as if its mass M were concentrated at its centre. Therefore, a mass m near earth's surface may be considered at a distance R (the radius of earth) from M. Thus, the potential energy of m due to earth will be :

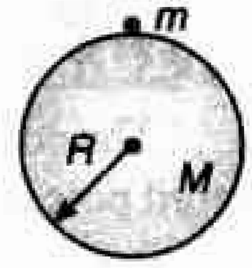


Fig. 10.19

$$U = -\frac{GMm}{R}$$

Difference in Potential Energy (ΔU)

Let us find the difference in potential energy of a mass m in two positions shown in figure. The potential energy of the mass on the surface of earth (at B) is,

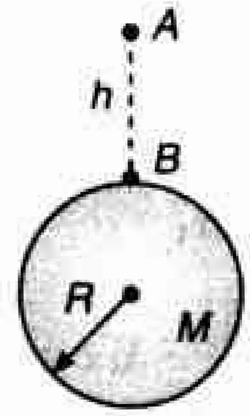


Fig. 10.20

$$U_B = -\frac{GMm}{R}$$

and potential energy of mass m at height h above the surface of earth (at A) is,

$$U_A = -\frac{GMm}{R+h}$$

$$(U_A > U_B)$$

$$\therefore \quad \Delta U = U_A - U_B = -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{GMmh}{R(R+h)}$$

$$= \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} \qquad \left(\frac{GM}{R^2} = g\right)$$

$$\therefore \quad \Delta U = \frac{mgh}{1+\frac{h}{R}}$$

For $\quad h << R, \ \Delta U \approx mgh$

Thus, what we read the mgh is actually the difference in potential energy (not the absolute potential energy), that too for $h << R$.

10.8 Binding Energy

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.

Suppose the mass m is placed on the surface of earth. The radius of the earth is R and its mass is M. Then, the kinetic energy of the particle $K = 0$ and potential energy of the particle is $U = -\frac{GMm}{R}$.

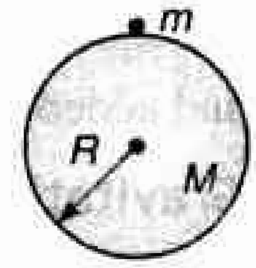


Fig. 10.21

Therefore, the total mechanical energy of the particle is,

$$E = K + U = 0 - \frac{GMm}{R}$$

or $$E = -\frac{GMm}{R}$$

$\therefore$ $$\text{Binding energy} = |E| = \frac{GMm}{R}$$

It is due to this energy, the particle is attached with the earth. If minimum this much energy is supplied to the particle in any form (normally kinetic) the particle no longer remains bound to the earth. It goes out of the gravitational field of earth.

Escape Velocity

As we discussed the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if v_e is the escape velocity of the particle, then

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

or $$v_e = \sqrt{\frac{2GM}{R}}$$

or $$v_e = \sqrt{2gR} \quad \text{as} \quad g = \frac{GM}{R^2}$$

Substituting the value of g (9.8 m/s^2) and R (6.4×10^6 m), we get

$$v_e \approx 11.2 \text{ km/s}$$

Thus, the minimum velocity needed to take a particle infinitely away from the earth is called the escape velocity. On the surface of earth its value is 11.2 km/s.

Sample Example 10.6 *Three masses of 1 kg, 2 kg and 3 kg are placed at the vertices of an equilateral triangle of side 1 m. Find the gravitational potential energy of this system.*
Take $G = 6.67 \times 10^{-11}$ *N-m²/kg²*

Solution $$U = -G\left(\frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right)$$

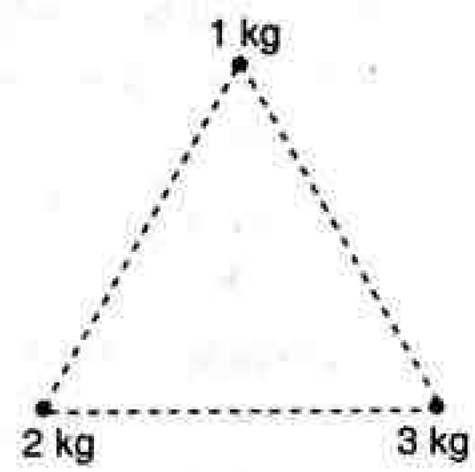


Fig. 10.22

Here, $r_{32} = r_{31} = r_{21} = 1.0$ m, $m_1 = 1$ kg,

$m_2 = 2$ kg and $m_3 = 3$ kg

Substituting in above, we get

$$U = -(6.67 \times 10^{-11})\left(\frac{3 \times 2}{1} + \frac{3 \times 1}{1} + \frac{2 \times 1}{1}\right)$$

or $$U = -7.337 \times 10^{-10} \text{ J}$$

Sample Example 10.7 *Calculate the escape velocity from the surface of moon. The mass of the moon is* 7.4×10^{22} *kg and radius* $= 1.74 \times 10^6$ *m*

Solution Escape velocity from the surface of moon is

$$v_e = \sqrt{\frac{2GM_m}{R_m}}$$

Substituting the values, we have

$$v_e = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^6}}$$

$$= 2.4 \times 10^3 \text{ m/s} \quad \text{or} \quad 2.4 \text{ km/s}$$

Sample Example 10.8 *A particle is projected from the surface of earth with an initial speed of 4.0 km/s. Find the maximum height attained by the particle. Radius of earth = 6400 km and* $g = 9.8$ *m/s*^{2}.

Solution The maximum height attained by the particle is,

$$h = \frac{v^2}{2g - \dfrac{v^2}{R}}$$

Substituting the values, we have

$$h = \frac{(4.0 \times 10^3)^2}{2 \times 9.8 - \dfrac{(4.0 \times 10^3)^2}{6.4 \times 10^6}} = 9.35 \times 10^5 \text{ m}$$

or

$$h \approx 935 \text{ km}$$

Introductory Exercise 10.3

1. The velocity of a particle is just equal to its escape velocity. Under such situation the total mechanical energy of the particle is zero. Is this statement true or false?
2. What is the kinetic energy needed to project a body of mass m from the surface of earth to infinity. Radius of earth is R and acceleration due to gravity on earth's surface is g.
3. Two particles of masses 20 kg and 10 kg are initially at a distance of 1.0 m. Find the speeds of the particles when the separation between them decreases to 0.5 m, if only gravitational forces are acting.
4. A particle is fired vertically upward with a speed of 15 km/s. Find the speed of the particle when it goes out of the earth's gravitational pull.
5. Show that if a body be projected vertically upward from the surface of the earth so as to reach a height nR above the surface :

   (i) the increase in its potential energy is $\left(\frac{n}{n+1}\right) mgR$,

   (ii) the velocity with which it must be projected is $\sqrt{\frac{2ngR}{n+1}}$, where R is the radius of the earth and m the mass of body.

10.9 Motion of Satellites

Just as the planets revolve around the sun, in the same way few celestial bodies revolve around these planets. These bodies are called 'Satellites'. For example moon is a satellite of earth. Artificial satellites are launched from the earth. Such satellites are used for telecommunication, weather forecast and other applications. The path of these satellites are elliptical with the centre of earth at a focus. However the difference in major and minor axes is so small that they can be treated as nearly circular for not too sophisticated calculations. Let us derive certain characteristics of the motion of satellites by assuming the orbit to be perfectly circular.

Orbital Speed

The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

$$\frac{mv_o^2}{r} = \frac{GMm}{r^2}$$

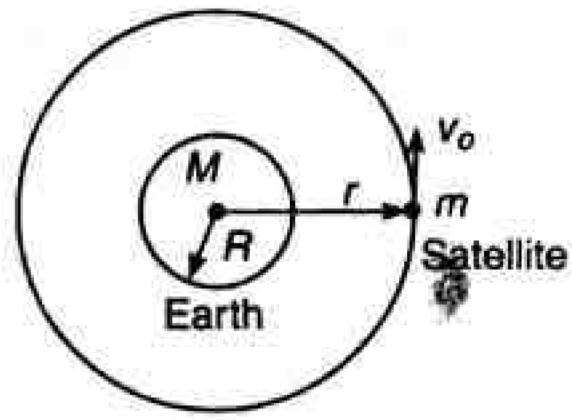


Fig. 10.23

$\therefore$ $$v_o = \sqrt{\frac{GM}{r}}$$

or $$v_o \propto \frac{1}{\sqrt{r}}$$

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface ($r \approx R$) is,

$$v_o = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}$$

Substituting $$v_e = 11.2 \text{ km/s}$$

$$v_o = 7.9 \text{ km/s}$$

Period of Revolution

The period of revolution (T) is given by

$$T = \frac{2\pi r}{v_o} \quad \text{or} \quad T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$

or $$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

or $$T = 2\pi\sqrt{\frac{r^3}{gR^2}} \qquad (\text{as } GM = gR^2)$$

From this expression of T, we can make the following conclusions

(i) $T \propto r^{3/2}$ or $T^2 \propto r^3$ (which is also the Kepler's third 'aw)

(ii) Time period of a satellite very close to earth's surface ($r \approx R$) is,

$$T = 2\pi \sqrt{\frac{R}{g}}$$

Substituting the values, we get $T \approx 84.6$ min

(iii) Suppose the height of a satellite is such that the time period of the satellite is 24 h and it moves in the same sense as the earth. The satellite will always be overhead a particular place on the equator. As seen from the earth, this satellite will appear to be stationary. Such a satellite is called a geostationary satellite. Putting $T = 24$ h in the expression of T, the radius of geostationary satellite comes out to be $r = 4.2 \times 10^4$ km. The height above the surface of earth is about 3.6×10^4 km.

Energy of Satellite

The potential energy of the system is

$$U = -\frac{GMm}{r}$$

The kinetic energy of the satellite is,

$$K = \frac{1}{2} mv_0^2 = \frac{1}{2} m\left(\frac{GM}{r}\right)$$

or

$$K = \frac{1}{2}\frac{GMm}{r}$$

The total energy is,

$$E = K + U = -\frac{GMm}{2r}$$

or

$$E = -\frac{GMm}{2r}$$

This energy is constant and negative, *i.e.,* the system is closed. The farther the satellite from the earth the greater its total energy.

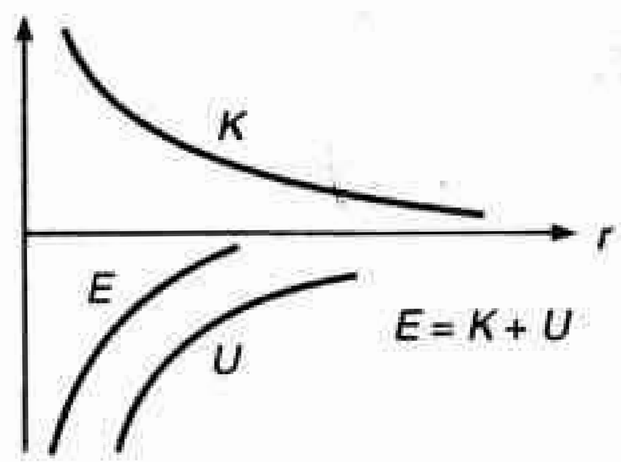


Fig. 10.24

Sample Example 10.9 *A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull. Radius of earth = 6400 km, g = 9.8 m/s².*

Solution The speed of the spaceship in a circular orbit close to the earth's surface is given by,

$$v_o = \sqrt{gR}$$

and escape velocity is given by,

$$v_e = \sqrt{2gR}$$

∴ Additional velocity required to escape

$$v_e - v_o = \sqrt{2gR} - \sqrt{gR}$$
$$= (\sqrt{2} - 1)\sqrt{gR}$$

Substituting the values of of g and R, we get

$$v_e - v_o = 3.278 \times 10^3 \text{ m/s}$$

10.10 Kepler's Laws

Kepler discovered three empirical laws that accurately described the motions of the planets. The three laws may be stated as,

(i) Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse. This law is also known as the law of elliptical orbits and obviously gives the shape of the orbits of the planets round the sun.

(ii) The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, *i.e.*, its areal velocity (or the area swept out by it per unit time) is constant. This is referred to as the law of areas and gives the relationship between the orbital speed of the planet and its distance from the sun.

(iii) The square of the planet's time period is proportional to the cube of the semi-major axis of its orbit. This is known as the harmonic law and gives the relationship between the size of the orbit of a planet and its time of revolution.

Kepler did not know why the planets move in this way. Three generations later when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be derived. They are consequences of Newton's law of motion and the law of gravitation.

Let us first consider the elliptical orbits described in Kepler's first law. Figure shows the geometry of the ellipse. The longest dimension is the major axis with half length a. This half length is called the semi-major axis.

$$SP + S'P = \text{constant}$$

Here, S and S' are the foci and P any point on the ellipse. The sun is at S and planet at P.

The distance of each focus from the centre of ellipse is ea, where e is the dimensionless number between 0 to 1 called the eccentricity. If $e = 0$, the ellipse is a circle. The actual orbits of the planets are nearly circular, their eccentricities range from 0.007 for Venus to 0.248 for Pluto. For earth $e = 0.017$. The point in the planet's orbit closest to the sun is the perihelion and the point most distant from the sun is aphelion.

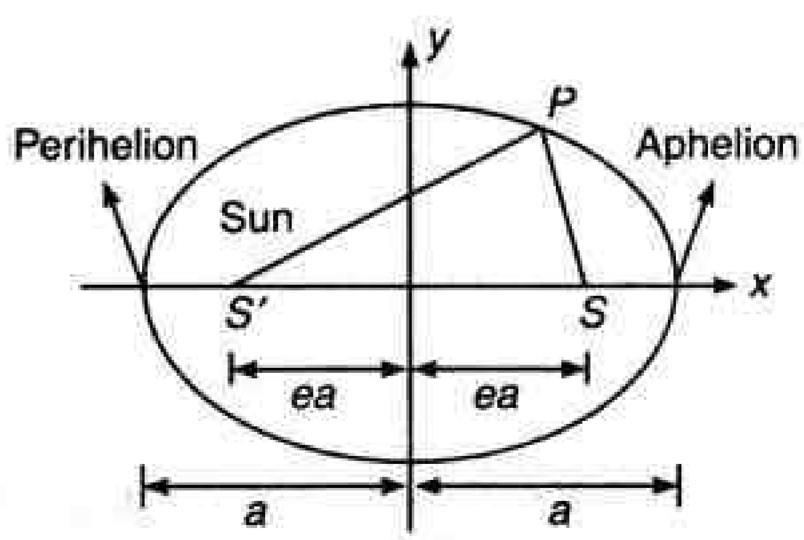


Fig. 10.25

Explanation of First Law

Newton was able to show that for a body acted on by an attractive force proportional to $\frac{1}{r^2}$, the only possible closed orbits are a circle or an ellipse. The open orbits must be parabolas or hyperbolas. He also showed that if total energy E is negative the orbit is an ellipse (or circle), if it is zero the orbit is a parabola and if E is positive the orbit is a hyperbola. Further, it was also shown that the orbits under the attractive force $F = \frac{K}{r^n}$, are stable for $n < 3$. Therefore, it follows that circular orbits will be stable for a force varying inversely as the distance or the square of the distance and will be unstable for the inverse cube (or a higher power) law.

Explanation of Second Law

$$PP' = v\,dt$$

$$P'M = (PP')\sin(180° - \theta) = PP'\sin\theta$$

$$= (v\sin\theta)\,dt$$

Kepler's second law is shown in figure. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out in this time interval is,

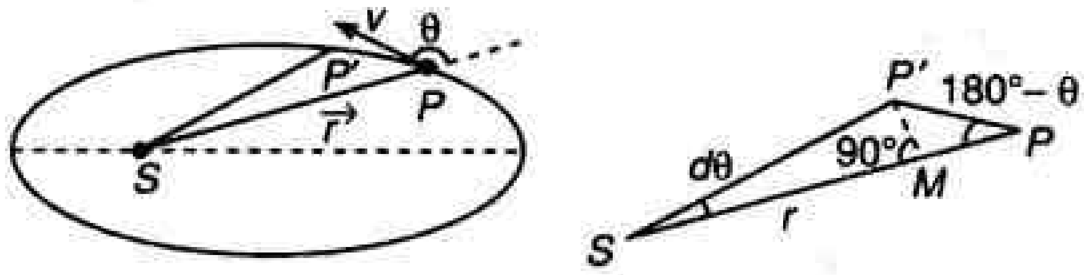


Fig. 10.26

$$dA = \text{area of triangle shown in figure}$$
$$= \frac{1}{2}(\text{base})(\text{height})$$
$$= \frac{1}{2}(SP)(P'M)$$
$$= \frac{1}{2}(r)(v\sin\theta)\,dt$$

∴ Areal velocity $$\frac{dA}{dt} = \frac{1}{2} rv\sin\theta \quad \text{...(i)}$$

Now, $rv\sin\theta$ is the magnitude of the vector product $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ which in turn is $\frac{1}{m}$ times the angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$ of the planet with respect to the sun. So we have,

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{\mathbf{r}} \times m\vec{\mathbf{v}}| = \frac{L}{2m} \quad \text{...(ii)}$$

or $$\frac{dA}{dt} = \frac{L}{2m}$$

Thus, Kepler's second law, that areal velocity is constant, means that angular momentum is constant. It is easy to see why the angular momentum of the planet must be constant. According to Newton's law the rate of change of $\vec{\mathbf{L}}$ equals the torque of the gravitational force $\vec{\mathbf{F}}$ acting on the planet,

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\tau}$$

Here, $\vec{\mathbf{r}}$ is the radius vector of planet from the sun and the force $\vec{\mathbf{F}}$ is directed from the planet to the sun. So, these vectors always lie along the same line and their vector product $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is zero. Hence, $\frac{d\vec{\mathbf{L}}}{dt} = 0$ or $\vec{\mathbf{L}}$ = constant. Thus, from Eq. (ii) we can see that $\frac{dA}{dt}$ = constant if $\vec{\mathbf{L}}$ = constant. Thus, second law is actually the law of conservation of angular momentum.

Explanation of Third Law

In Article 10.9 we have already derived Kepler's third law for the particular case of circular orbits ($T^2 \propto r^3$). Newton was able to show that the same relationship holds for an elliptical orbit, with the orbit radius r replaced by semimajor axis a. Thus,

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}} \quad \text{(elliptical orbit)}$$

Here, M_s is the mass of sun.

Important Points in Planetary Motion

- The areal velocity of a planet is constant (Kepler's second law) and is given by

$$\frac{dA}{dt} = \frac{L}{2m}$$

Here, L is the angular momentum of the planet about sun.

- Most of the problems of gravitation are solved by two conservation laws:
  (i) conservation of angular momentum about sun and
  (ii) conservation of mechanical (potential + kinetic) energy

Hence, the following two equations are used in most of the cases,

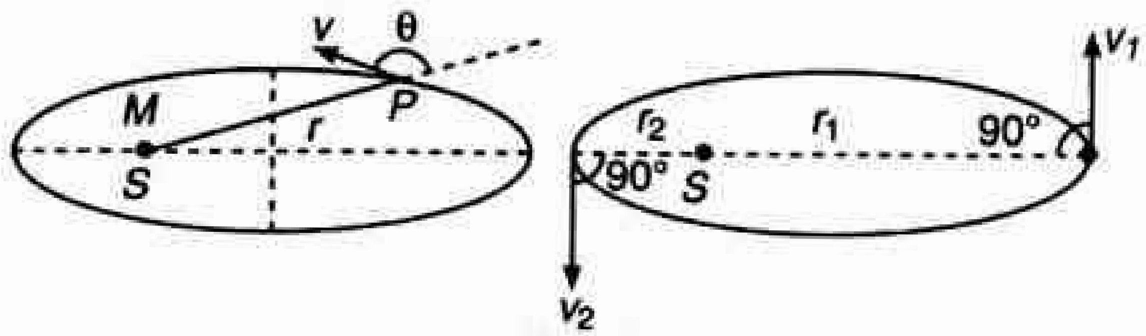


Fig. 10.27

$$mvr \sin\theta = \text{constant} \quad \text{...(i)}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant} \quad \text{...(ii)}$$

At aphelion and perihelion positions $\theta = 90°$

Hence, Eq. (i) can be written as,

$$mvr \sin 90° = \text{constant}$$

or

$$mvr = \text{constant} \quad \text{...(iii)}$$

Further, since mass of the planet (m) also remains constant, Eq. (i) can also be written as

$$vr \sin\theta = \text{constant} \quad \text{...(iv)}$$

or

$$v_1 r_1 = v_2 r_2 \quad (\theta = 90°)$$

$$r_1 > r_2$$

$$\therefore \quad v_1 < v_2$$

- Applying the above mentioned conservation laws in aphelion and perihelion positions with

$$r_1 = a(1+e) \quad \text{and} \quad r_2 = a(1-e)$$

We can show that

$$v_{min} = v_1 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

$$v_{max} = v_2 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

and total energy of the planet

$$E = -\frac{GMm}{2a}$$

Sample Example 10.10 *Find the speeds of a planet of mass m in its perihelion and aphelion positions. The semimajor axis of its orbit is a, eccentricity is e and the mass of the sun is M. Also find the total energy of the planet in terms of the given parameters.*

Solution Let v_1 and v_2 be the speeds of the planet at perihelion and aphelion positions.

$$r_1 = a(1-e)$$

and $$r_2 = a(1+e) \qquad \ldots\text{(i)}$$

Applying conservation of angular momentum of the planet at P (perihelion) and A (aphelion)

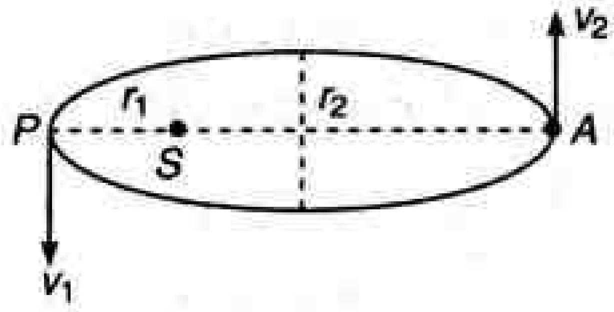


Fig. 10.28

$$mv_1 r_1 \sin 90° = mv_2 r_2 \sin 90°$$

or $$v_1 r_1 = v_2 r_2 \qquad \ldots\text{(ii)}$$

Applying conservation of mechanical energy in these two positions, we have

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$v_1 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

and $$v_2 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

Further, total energy of the planet

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1}$$

$$= \frac{1}{2}m\left[\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\right] - \frac{GMm}{a(1-e)}$$

$$= \frac{GMm}{a(1-e)}\left[\left(\frac{1+e}{2}\right) - 1\right]$$

$$= \frac{GMm}{a(1-e)}\left(\frac{e-1}{2}\right)$$

or $$E = -\frac{GMm}{2a}$$

Introductory Exercise 10.4

1. If a body is released from a great distance from the centre of the earth, find its velocity when it strikes the surface of the earth. Take $R = 6400$ km.
2. What quantities are constant in planetary motion?
3. Two satellites A and B of the same mass are orbiting the earth at altitudes R and $3R$ respectively, where R is the radius of the earth. Taking their orbits to be circular obtain the ratios of their kinetic and potential energies.
4. If a satellite is revolving close to a planet of density ρ with period T, show that the quantity ρT^2 is a universal constant.
5. A satellite is revolving around a planet in a circular orbit. What will happen, if its speed is increased from v_o to:
   (a) $\sqrt{1.5}\, v_o$ (b) $2v_o$

Extra Points

- Acceleration due to moon's gravity on moon's surface is $\frac{g_e}{6}$ because

$$\frac{M_m}{R_m^2} \approx \frac{1}{6}\frac{M_e}{R_e^2} \qquad \left(g = \frac{GM}{R^2}\right)$$

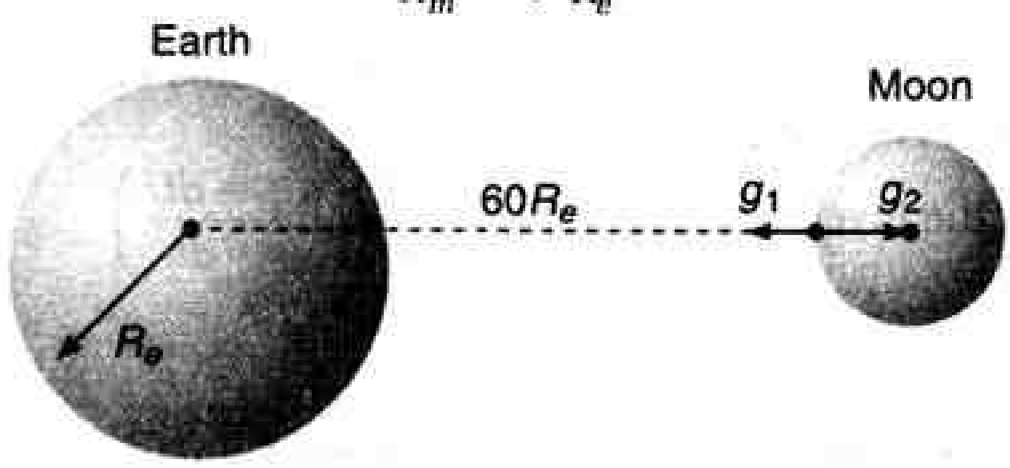


Fig. 10.29

While acceleration due to earth's gravity on moon's surface is approximately $\frac{g_e}{(60)^2}$ or $\frac{g_e}{3600}$. This is because distance of moon from the earth's centre is approximately equal to 60 times the radius of earth and $g \propto \frac{1}{r^2}$.

This can be understood from the Fig. 10.29.

$$g_1 = \frac{g_e}{(60)^2} \quad \text{while} \quad g_2 = \frac{g_e}{6}$$

- **Maximum height attained by a particle :**

Suppose a particle of mass m is projected vertically upwards with a speed v and we want to find the maximum height h attained by the particle. Then we can use conservation of mechanical energy, *i.e.*, Decrease in kinetic energy = increase in gravitational potential energy of particle.

$$\therefore \qquad \frac{1}{2}mv^2 = \Delta U \quad \text{or} \quad \frac{1}{2}mv^2 = \frac{mgh}{1 + \frac{h}{R}}$$

Solving this, we get

$$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

From this we can see that

(i) if $v = v_e$ or $v^2 = v_e^2 = 2gR$, $h = \infty$ and if

(ii) v is small $h = \dfrac{v^2}{2g}$

Both the results are quite obvious.

- **Time taken by the particle to reach a height h :**
Suppose a particle of mass m is projected vertically upwards with a speed v_0. We want to find the time taken by the particle to reach a height h. First of all we find the speed of the particle at a height x by applying conservation of mechanical energy, *i.e.*,

$$\frac{1}{2}m(v_0^2 - v^2) = \frac{mgx}{1 + \dfrac{x}{R}}$$

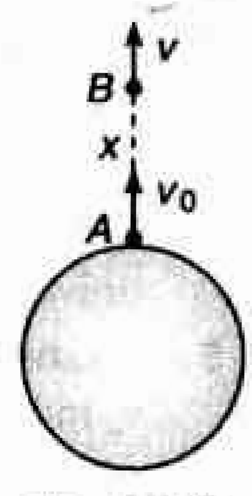


Fig. 10.30

or

$$v = \sqrt{v_0^2 - \frac{2gx}{1 + \dfrac{x}{R}}} \qquad \text{...(i)}$$

Now, v can be written as $-\dfrac{dx}{dt}$.

Hence, Eq. (i) reduces to

$$\frac{-dx}{\sqrt{v_0^2 - \dfrac{2gx}{1 + \dfrac{x}{R}}}} = dt$$

By integrating with limits from 0 to h on left hand side and from 0 to t on right hand side, we get the desired time.

- Total energy of a closed system is always negative. For example energy of planet-sun, satellite-earth or electron-nucleus system are always negative.

- If the law of force obeys the inverse square law $\left(F \propto \dfrac{1}{r^2},\ F = \dfrac{-dU}{dr}\right)$

$$K = \frac{|U|}{2} = |E|$$

The same is true for electron-nucleus system because there also, the electrostatic force $F_e \propto \dfrac{1}{r^2}$.

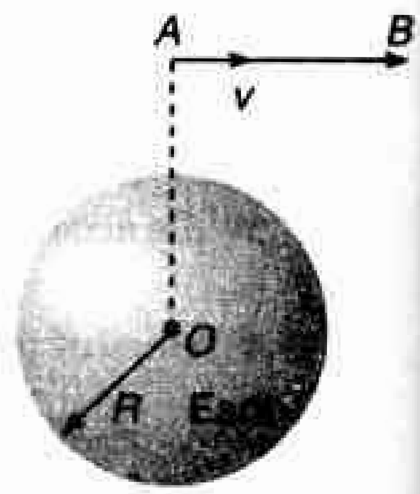


Fig. 10.31

- **Trajectory of a body projected from point A in the direction AB with different initial velocities :** Let a body be projected from point A with velocity v in the direction AB. For different values of v the paths are different. Here, are the possible cases.
  (i) If $v = 0$, path is a straight line from A to O.
  (ii) If $0 < v < v_o$, path is an ellipse with centre O of the earth as a focus.
  (iii) If $v = v_o$, path is a circle with O as the centre
  (iv) If $v_o < v < v_e$, path is again an ellipse with O as a focus.

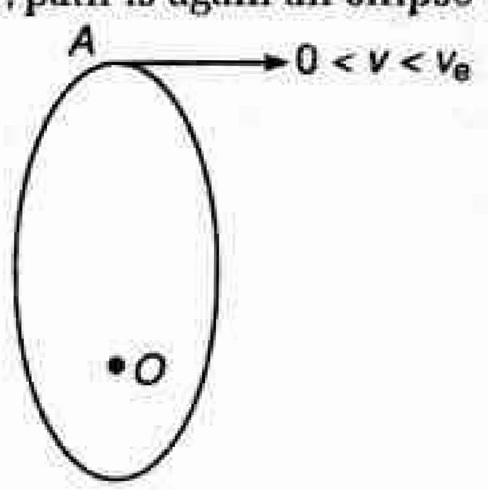


Fig. 10.32

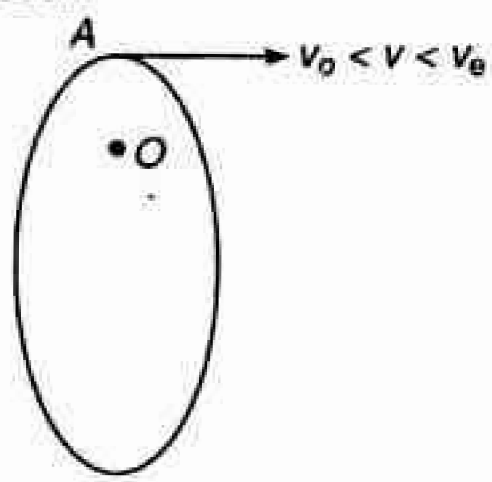


Fig. 10.33

(v) If $v = v_e$, body escapes from the gravitational pull of the earth and path is a parabola

(vi) If $v > v_e$, body again escapes but now the path is a hyperbola. Here, v_o = orbital speed $\left(\sqrt{\frac{GM}{r}}\right)$ at A and v_e = escape velocity at A.

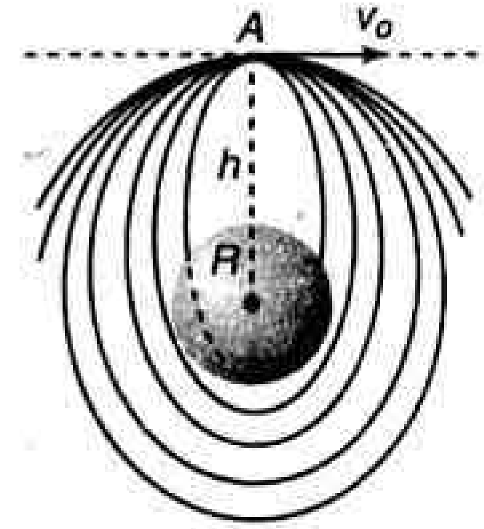


Fig. 10.34

Note
1. *From case (i) to (iv) total energy of the body is negative. Hence, these are the closed orbits. For case (v) total energy is zero and for case (vi) total energy is positive. In these two cases orbits are open*
2. *If v is not very large the elliptical orbit will intersect the earth and the body will fall back to earth.*

- If $F \propto r^n$

  then $T^2 \propto (r)^{1-n}$

  and if $U \propto r^m$

  then $T^2 \propto (r)^{2-m}$

  (Applicable only for circular orbits)

- In motion of a planet round the sun we have assumed the mass of the sun to be too large in comparison to the mass of the planet. Under such situation the sun remains stationary and the planet revolves round the sun. If however masses of sun and planet are comparable and motion of sun is also to be considered, then both of them revolve around their centre of mass with same angular velocity but different linear speeds in the circles of different radii. The centre of mass remains stationary.
  We use following equations under this condition.

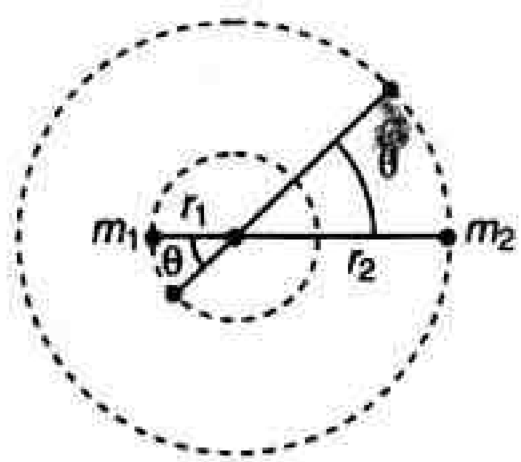


Fig. 10.35

$$m_1 r_1 = m_2 r_2 \quad \text{...(i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \quad \text{...(ii)}$$

Solving these two equations, we can find that

$$\omega = \sqrt{\frac{GM}{r^3}} \quad \text{or} \quad T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Here, $M = m_1 + m_2$

and $r = r_1 + r_2$

Further, angular momentum of the system about COM

$$L = (I_1 + I_2)\omega = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \omega = \mu r^2 \omega$$

Kinetic energy of system, $K = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 \omega^2 = \frac{1}{2}\mu r^2 \omega^2$

and moment of inertia of system,

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 = \mu r^2$$

Here, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass.

Thus, the two bodies can be replaced by a single body whose mass is equal to reduced mass. This single body revolve in a circular orbit whose radius is equal to the distance between two bodies and centripetal force of circular motion is equal to force of interaction between two bodies for actual separation.

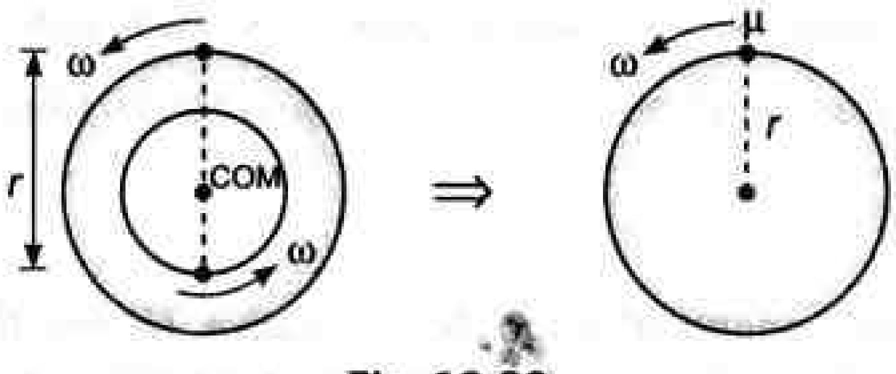


Fig. 10.36

- $T = 2\pi\sqrt{\frac{R}{g}} \approx$ min comes in following four places in whole physics.

(i) If time period of rotation of earth becomes 84.6 min, effective value of 'g' on equator becomes zero or we feel weightlessness on equator.

Exercise : Prove the above result.

(ii) Time period of a satellite close to earth's surface is 84.6 min.

(iii) Time pendulum of a pendulum of infinite length is 84.6 min.

(iv) If a tunnel is dug along any chord of the earth and a particle is released from the surface of earth along this tunnel, then motion of this particle is simple harmonic and time period of this is also 84.6 min.

Note *(a) Points (iii) and (iv) are part of chapter simple harmonic motion.*

(b) $T = 2\pi\sqrt{\frac{R}{g}}$ *is also the time period of small oscillations of a block inside a smooth spherical bowl of radius. R. But this is not 84.6 min because here R is the radius of bowl not the radius of earth.*

This expression can be compared with the time period of a pendulum $T = 2\pi\sqrt{\frac{l}{g}}$.

Fig. 10.37

Important Points Regarding Universe

- **There is no atmosphere on moon** The escape velocity on the moon's surface is only 2.5 km/s. The rms velocity of all gases is more than this value therefore they escape away from the gravitational field of moon. Hence, there is no atmosphere on moon.
- **Weight of a person in artificial satellite** A person feels weightlessness on a satellite. This is because the gravitational force exerted by the earth on the man is just equal to the necessary centripetal force there and the normal reaction between the foot of the man and the floor of the satellite becomes zero.
- **Exercise :** Show mathematically that $N = 0$ on a satellite.
- **Geostationary satellite or Parking satellite** If an artificial satellite revolves around the earth in an equatorial plane with a time period of 24 h in the same sense as that of earth, then it will appear stationary to the observer on earth. Such a satellite is known as Geostationary satellite or Parking satellite. The radius of Geostationary satellite is 42,400 km and its height above the surface of earth is 36,000 km. The orbital velocity of Geostationary satellite is 3.08 km/s.
- **A brief introduction of universe** Universe consists of all things such as stars, planets, satellites etc. There are two main theories regarding the universe :
  (a) Geocentric theory This theory was proposed by **Ptolemy.** According to this theory earth lies at the centre of universe. Earth is stationary and sun, moon etc. revolve round the earth.
  (b) Heliocentric theory This theory was proposed by **Copernicus.** According to this theory sun is at centre of the universe and planets revolve round the sun in elliptical orbits.
- **Galaxies** A large group of stars is called a galaxy. We are a member of galaxy called **Milky Way (Akash Ganga)**. It consists of 10^{11} stars (including the sun). **Andormeda galaxy** can be seen with naked eyes.
- **Stars** Heavenly bodies that shine like sun are called stars. They have the energy of their own to shine. After sun **Alpha Centauri** is nearest to earth.

  Birth of a Star Dust particles, hydrogen and helium gas molecules present in the intersteller space first come together (at about $-173°C$) to form a cloud. Then they start contracting. As a result of compression heating of cloud takes place. When temperature of the core becomes about 10^7 K, the fusion of hydrogen atoms is initiated with the release of energy. This energy keeps the star shining for millions of years.

  Death of a Star A star lasts until the hydrogen in the core of the star is exhausted. The core now starts contracting. It results in the rise in temperature of the star. As a result of rise in temperature, the outer layer of the star expand. Expansion of outer layer brings about cooling effect in them. The process continues till the temperature of outer layer falls enough to make the star appear red. It is then called **red giant.**

  At this stage, a violent explosion called **nova** or **supernova** occurs in the star. Due to the explosion its outer layer are thrown into intersteller space leaving behind the core of the star. The core of the star may further end up into one of the following three steller dead materials.

  (a) White dwarf The core dies as white dwarf if the original mass of the star was 1.4 solar masses. It was discovered by S. Chandrasekhar in 1930 and it is known as Chandrasekhar limit. The core is composed of protons and electrons. The core keeps on emitting heat and light for millions of years. As it cools steadily, its colour changes from white to yellow, then to red and finally it becomes black. It then becomes invisible forever as **black dwarf** and neither emits heat nor light.

  (b) Neutron star The core of the star finishes up as neutron star if the mass of the star was between 1.4 to 5 solar masses. The compressed core is made only of neutrons. Neutron stars produce very high magnetic fields. A spinning neutron star emitting electromagnetic waves is called **Pulsar.**

  (c) Black hole If the original mass of the star is greater than 5 solar masses the core dies into black holes. The mass of a black hole is greater than the mass of the sun but its size is very small, therefore the gravitational pull of a black hole is so strong that even the photon or radiation emitted by it can't escape from its surface. Since no radiation is received from a black hole, it cannot be seen through a telescope.

- **Solar System** The part of the universe in which the sun occupies the central position and nine planets revolving round it, is called solar system. The eight planets in the increasing order of their distance from the sun are : Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune. The important characteristics of the planets are as follows:
  - **(a) Mercury** Nearest, smallest and hottest planet of the solar system. Life is not possible on mercury.
  - **(b) Venus** Also called **morning star** and **the evening star** is brightest amongst all.
  - **(c) Earth** We all know about earth. It is the only planet which contain suitable conditions for evolution and survival of earth. It has only one natural satellite named moon.
  - **(d) Mars** Nearest planet to earth. It is reddish. It has traces of O_2 but percentage of O_2 is not sufficient for evolution and survival of life.
  - **(e) Jupiter** Largest planet of the solar system having the maximum number of satellites.
  - **(f) Saturn** It has ring around it.
  - **(g) Uranus** It is the only planet which rotates from east to west on its axis.
  - **(h) Neptune** It has no special characteristic.
- **Other Heavenly Objects in Solar System** In addition to the planets and their satellites asteroids, comets, meteors and meteoroids are other heavenly objects in solar system.
  - **(a) Asteroids** These are small pieces of planet like material revolving around the sun mostly between the orbits of **Mars** and **Jupiter** are called asteroids.
  - **(b) Comets** The small pieces of rock like material surrounded by a large amount of substances such as water, ammonia and methane possessing head and tail while moving past the sun are called comets. A comet does not have any tail, when it is far away from the sun. As it approaches the sun, it begins to get elongated in the direction away from the sun due to radiation pressure. When the comet is near the sun, substances like water are vaporised due to the heat of the sun and radiation pressure forces the vapour away in the shape of tail. Thus head of comet is made of heavy material like rock, while the tail is made of light material such as dust and gas. The comets have highly elliptical orbits. **Halley's comet** was discovered in 1757. It has a period of **76.2 years.**
  - **(c) Meteors and Meteoroids** It is also called **Shooting Star.** A small piece of planetary material (rock etc.) when enters the earth's atmosphere gets heated to very high temperature due to friction of air looking as bright lines of the fire and gets completely burnt before reaching the earth. Those meteors which are large enough to survive and manage to reach the earth are called **meteoroids.**
- **Big-Bang theory of Universe** According to this theory all the constituents of our universe were originally together as a single mass. A big explosion occurred and the single mass burst into a large number of fragments moving with different velocities.

  Hubble stated that the age of universe from the day, the universe has evolved to the present day is $t_0 = \frac{1}{H}$, where, H is Hubble's constant.

Solved Examples

For JEE Main

Example 1 *If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?*

Solution Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2\omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}MR^2\omega = \frac{1}{4}\times\frac{2}{5}MR^2\omega' \quad i.e., \quad \omega' = 4\omega$$

$$T' = \frac{1}{4}T = \frac{1}{4}\times 24 = 6\,\text{h}$$

Example 2 *The minimum and maximum distances of a satellite from the centre of the earth are 2R and 4R respectively, where R is the radius of earth and M is the mass of the earth. Find :*

(a) its minimum and maximum speeds,

(b) radius of curvature at the point of minimum distance.

Solution (a) Applying conservation of angular momentum

$$mv_1(2R) = mv_2(4R)$$

$$v_1 = 2v_2 \quad \text{... (i)}$$

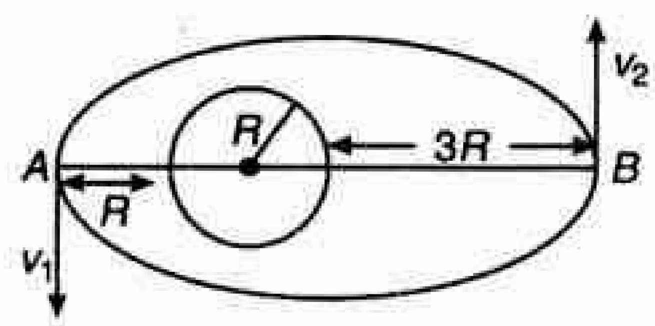


Fig. 10.38

From conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{2R} = \frac{1}{2}mv_2^2 - \frac{GMm}{4R} \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 = \sqrt{\frac{GM}{6R}}, \quad v_1 = \sqrt{\frac{2GM}{3R}}$$

(b) If r is the radius of curvature at point A

$$\frac{mv_1^2}{r} = \frac{GMm}{(2R)^2}$$

$$r = \frac{4v_1^2R^2}{GM} = \frac{8R}{3} \quad \text{(putting value of } v_1\text{)}$$

Example 3 *Three particles each of mass m, are located at the vertices of an equilateral triangle of side a. At what speed must they move if they all revolve under the influence of their gravitational force of attraction in a circular orbit circumscribing the triangle while still preserving the equilateral triangle ?*

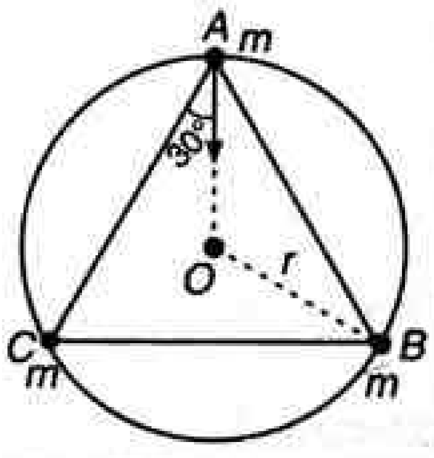


Fig. 10.39

Solution
$$\vec{F}_A = \vec{F}_{AB} + \vec{F}_{AC} = 2\left[\frac{GM^2}{a^2}\right]\cos 30° = \left[\frac{GM^2}{a^2}\cdot\sqrt{3}\right]$$

$$r = \frac{a}{\sqrt{3}},$$

Now
$$\frac{mv^2}{r} = F$$

or
$$\frac{mv^2\sqrt{3}}{a} = \frac{GM^2}{a^2}\sqrt{3}$$

$\therefore$
$$v = \sqrt{\frac{GM}{a}}$$

Example 4 *Two concentric shells of mass M_1 and M_2 are concentric as shown. Calculate the gravitational force on m due to M_1 and M_2 at points P, Q and R.*

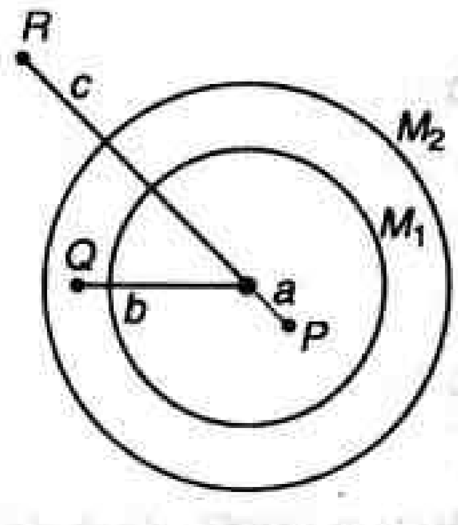


Fig. 10.40

Solution At P, $F = 0$

at Q,
$$F = \frac{GM_1 m}{b^2}$$

at R,
$$F = \frac{G(M_1 + M_2)m}{c^2}$$

Example 5 *What is the fractional decrease in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation h ? (h < < R)*

Solution
$$g = \frac{GM}{R^2}$$

$$\frac{dg}{dR} = \frac{-2GM}{R^3}$$

$\Rightarrow$ $$\frac{dg}{h} = \frac{-2GM}{R^2} \cdot \frac{1}{R}$$

$\Rightarrow$ $$\frac{dg}{g} = -2\left(\frac{h}{R}\right)$$

Example 6 *Three concentric shells of masses M_1, M_2 and M_3 having radii a, b and c respectively are situated as shown in figure. Find the force on a particle of mass m.*

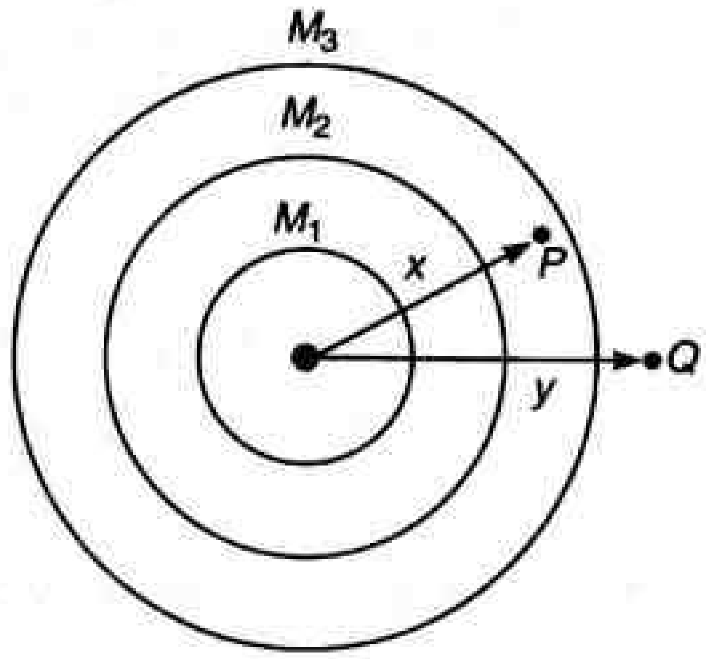


Fig. 10.41

(a) When the particle is located at Q.
(b) When the particle is located at P.

Solution Attraction at an external point due to spherical shell of mass M is $\left(\frac{GM}{r^2}\right)$ while at an internal point is zero.

(a) Point is external to shell M_1, M_2 and M_3,
So, force at Q will be

$$F_Q = \frac{GM_1m}{y^2} + \frac{GM_2m}{y^2} + \frac{GM_3m}{y^2}$$

$$= \frac{Gm}{y^2}(M_1 + M_2 + M_3)$$

(b) Force at P will be $$F_P = \frac{GM_1m}{x^2} + \frac{GM_2m}{x^2} + 0$$

$$= \frac{Gm}{x^2}(M_1 + M_2)$$

Example 7 *A planet of mass m revolves in elliptical orbit around the sun so that its maximum and minimum distances from the sun are equal to r_a and r_p respectively. Find the angular momentum of this planet relative to the sun.*

Solution Using conservation of angular momentum

$$mv_pr_p = mv_ar_a$$

As velocities are perpendicular to the radius vectors at apogee and perigee.

$\Rightarrow$ $$v_pr_p = v_ar_a$$

Using conservation of energy,

$$-\frac{GMm}{r_p}+\frac{1}{2}mv_p^2=\frac{-GMm}{r_a}+\frac{1}{2}mv_a^2$$

By solving, the above equations,

$$v_p=\sqrt{\frac{2GMr_a}{r_p(r_p+r_a)}}$$

$$L=mv_pr_p=m\sqrt{\frac{2GMr_pr_a}{(r_p+r_a)}}$$

For JEE Advanced

Example 1 *A planet of mass m_1 revolves round the sun of mass m_2. The distance between the sun and the planet is r. Considering the motion of the sun find the total energy of the system assuming the orbits to be circular.*

Solution Both the planet and the sun revolve around their centre of mass with same angular velocity (say ω)

$$r=r_1+r_2 \quad \ldots\text{(i)}$$

$$m_1r_1\omega^2=m_2r_2\omega^2=\frac{Gm_1m_2}{r^2} \quad \ldots\text{(ii)}$$

Fig. 10.42

Solving Eqs. (i) and (ii), we get

$$r_1=r\left(\frac{m_2}{m_1+m_2}\right)$$

$$r_2=r\left(\frac{m_1}{m_1+m_2}\right)$$

and

$$\omega^2=\frac{G(m_1+m_2)}{r^3}$$

Now, total energy of the system is

$$E=\text{PE}+\text{KE}$$

or

$$E=-\frac{Gm_1m_2}{r}+\frac{1}{2}m_1r_1^2\omega^2+\frac{1}{2}m_2r_2^2\omega^2$$

Substituting the values of r_1, r_2 and ω^2, we get

$$E=-\frac{Gm_1m_2}{2r}$$

Example 2 *Two masses m_1 and m_2 at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart.*

Solution Let v_r be their velocity of approach. From conservation of energy:

Increase in kinetic energy = decrease in gravitational potential energy

or

$$\frac{1}{2}\mu v_r^2=\frac{Gm_1m_2}{r} \quad \ldots\text{(i)}$$

Here, μ = reduced mass = $\dfrac{m_1 m_2}{m_1 + m_2}$

Substituting in Eq. (i), we get

$$v_r = \sqrt{\frac{2G(m_1 + m_2)}{r}}$$

Example 3 *If a planet was suddenly stopped in its orbit supposed to be circular, show that it would fall onto the sun in a time $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.*

Solution Consider an imaginary planet moving along a strongly extended flat ellipse, the extreme points of which are located on the planet's orbit and at the centre of the sun, The semi-major axis of the orbit of such a planet would apparently be half the semimajor axis of the planet's orbit. So the time period of the imaginary planet T' according to Kepler's law will be given by :

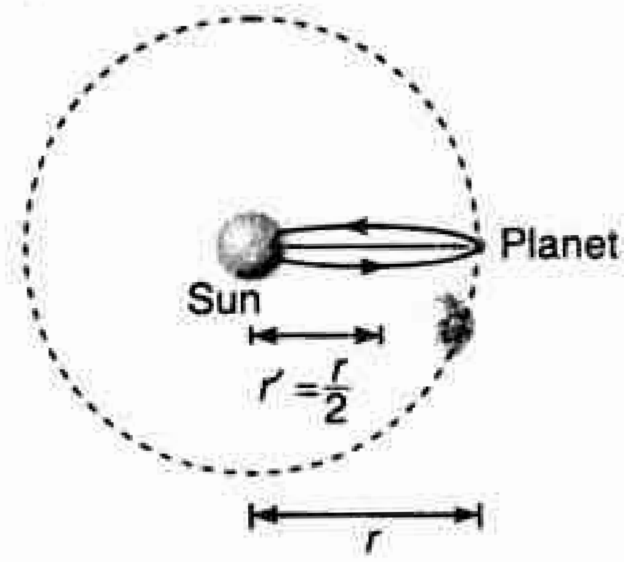


Fig. 10.43

$$\left(\frac{T'}{T}\right) = \left(\frac{r'}{r}\right)^{3/2}$$

or $$T' = T\left(\frac{1}{2}\right)^{3/2} \qquad \left(\text{as} \quad r' = \frac{r}{2}\right)$$

$\therefore$ Time taken by the planet to fall onto the sun is

$$t = \frac{T'}{2} = \frac{T}{2}\left(\frac{1}{2}\right)^{3/2}$$

$\Rightarrow$ $$t = \frac{\sqrt{2}}{8}\, T$$

Example 4 *A satellite is revolving round the earth in a circular orbit of radius r and velocity v_o. A particle is projected from the satellite in forward direction with relative velocity $v = (\sqrt{5/4} - 1)\, v_o$. Calculate its minimum and maximum distances from earth's centre during subsequent motion of the particle.*

Solution $$v_o = \sqrt{\frac{GM}{r}} = \text{orbital speed of satellite} \qquad \text{...(i)}$$

where M = mass of earth

Absolute velocity of particle would be:

$$v_p = v + v_o = \sqrt{\frac{5}{4}}\, v_o = \sqrt{1.25}v_o \qquad \text{...(ii)}$$

Since, v_p lies between orbital velocity and escape velocity, path of the particle would be an ellipse with r being the minimum distance.

Let r' be the maximum distance and v'_p its velocity at that moment.

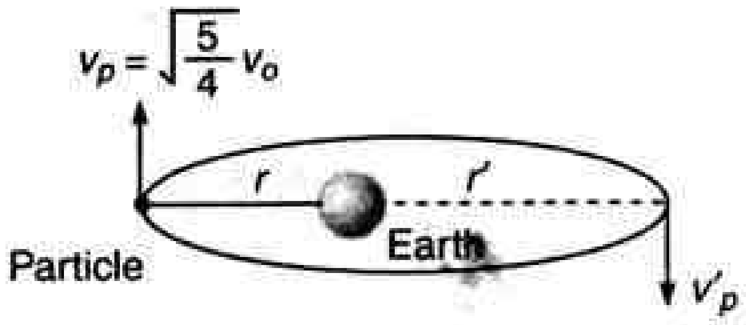


Fig. 10.44

Then from conservation of angular momentum and conservation of mechanical energy, we get

$$mv_p r = mv'_p r' \quad \text{...(iii)}$$

and
$$\frac{1}{2}mv_p^2 - \frac{GMm}{r} = \frac{1}{2}mv_p'^2 - \frac{GMm}{r'} \quad \text{...(iv)}$$

Solving the above Eqs. (i), (ii), (iii) and (iv), we get

$$r' = \frac{5r}{3} \quad \text{and} \quad r$$

Hence, the maximum and minimum distance are $\frac{5r}{3}$ and r respectively.

Example 5 *An earth satellite is revolving in a circular orbit of radius a with velocity v_o. A gun is in the satellite and is aimed directly towards the earth. A bullet is fired from the gun with muzzle velocity $\frac{v_o}{2}$. Neglecting resistance offered by cosmic dust and recoil of gun, calculate maximum and minimum distance of bullet from the centre of earth during its subsequent motion.*

Solution Orbital speed of satellite is

$$v_o = \sqrt{\frac{GM}{a}} \quad \text{...(i)}$$

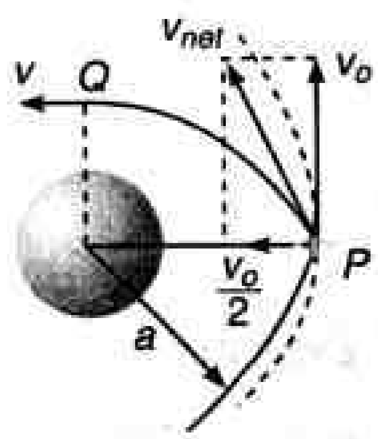


Fig. 10.45

From conservation of angular momentum at P and Q, we have

$$mav_o = mvr$$

or
$$v = \frac{av_o}{r} \quad \text{...(ii)}$$

From conservation of mechanical energy at P and Q, we have

$$\frac{1}{2}m\left(v_o^2 + \frac{v_o^2}{4}\right) - \frac{GMm}{a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

or
$$\frac{5}{8}v_o^2 - \frac{GM}{a} = \frac{v^2}{2} - \frac{GM}{r}$$

Substituting values of v and v_o from Eqs. (i) and (ii), we get

$$\frac{5}{8}\frac{GM}{a} - \frac{GM}{a} = \frac{a^2}{r^2}\cdot\left(\frac{GM}{2a}\right) - \frac{GM}{r}$$

or
$$-\frac{3}{8a} = \frac{a}{2r^2} - \frac{1}{r} \quad \text{or} \quad -3r^2 = 4a^2 - 8ar$$

or $$3r^2 - 8ar + 4a^2 = 0$$

or $$r = \frac{8a \pm \sqrt{64a^2 - 48a^2}}{6}$$

or $$r = \frac{8a \pm 4a}{6} \quad \text{or} \quad r = 2a \quad \text{and} \quad \frac{2a}{3}$$

Hence, the maximum and minimum distances are $2a$ and $\frac{2a}{3}$ respectively.

Example 6 *Binary stars of comparable masses m_1 and m_2 rotate under the influence of each other's gravity with a time period T. If they are stopped suddenly in their motions, find their relative velocity when they collide with each other. The radii of the stars are R_1 and R_2 respectively. G is the universal constant of gravitation.*

Solution Both the stars rotate about their centre of mass (COM).

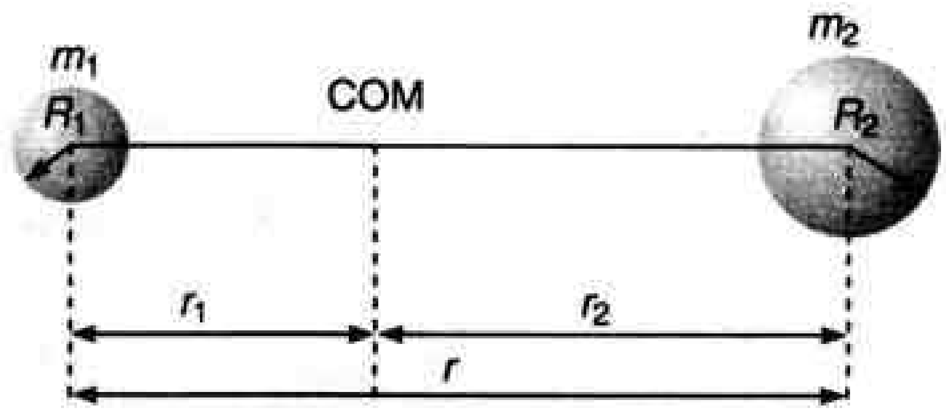


Fig. 10.46

For the position of COM $$\frac{r_1}{m_2} = \frac{r_2}{m_1} = \frac{r_1 + r_2}{m_1 + m_2} = \frac{r}{m_1 + m_2} \qquad (r = r_1 + r_2)$$

Also, $$m_1 r_1 \omega^2 = \frac{Gm_1 m_2}{r^2} \quad \text{or} \quad \omega^2 = \frac{Gm_2}{r_1 r^2} \qquad \left(\omega = \frac{2\pi}{T}\right)$$

But, $$r_1 = \frac{m_2 r}{m_1 + m_2}$$

$\therefore$ $$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

or $$r = \left\{\frac{G(m_1 + m_2)}{\omega^2}\right\}^{1/3} \qquad \ldots\text{(i)}$$

Applying conservation of mechanical energy we have

$$-\frac{Gm_1 m_2}{r} = -\frac{Gm_1 m_2}{(R_1 + R_2)} + \frac{1}{2}\mu v_r^2 \qquad \ldots\text{(ii)}$$

Here, $$\mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

and v_r = relative velocity between the two stars.

From Eq. (ii), we find that

$$v_r^2 = \frac{2Gm_1 m_2}{\mu}\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$

$$= \frac{2Gm_1 m_2}{\frac{m_1 m_2}{m_1 + m_2}} \left(\frac{1}{R_1 + R_2} - \frac{1}{r} \right)$$

$$= 2G(m_1 + m_2) \left(\frac{1}{R_1 + R_2} - \frac{1}{r} \right)$$

Substituting the value of r from Eq. (i), we get

$$v_r = \sqrt{2G(m_1 + m_2) \left[\frac{1}{R_1 + R_2} - \left\{ \frac{4\pi^2}{G(m_1 + m_2)T^2} \right\}^{1/3} \right]}$$

Example 7 *Find the maximum and minimum distances of the planet A from the sun S, if at a certain moment of time it was at a distance r_0 and travelling with the velocity v_0, with the angle between the radius vector and velocity vector being equal to ϕ.*

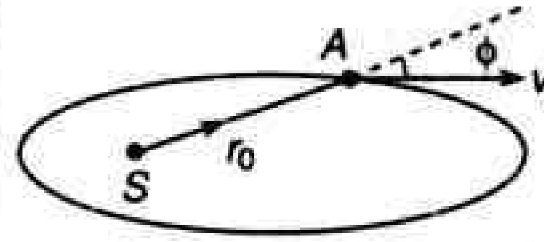


Fig. 10.47

Solution At minimum and maximum distances velocity vector ($\vec{v}$) makes an angle of 90° with radius vector. Hence, from conservation of angular momentum,

$$mv_0 r_0 \sin \phi = mrv \qquad \text{...(i)}$$

Here, m is the mass of the planet.

From energy conservation law it follows that.

$$\frac{mv_0^2}{2} - \frac{GMm}{r_0} = \frac{mv^2}{2} - \frac{GMm}{r} \qquad \text{...(ii)}$$

Here, M is the mass of the sun.

Solving Eqs. (i) and (ii) for r, we get two values of r, one is r_{max} and another is r_{min}. So,

$$r_{max} = \frac{r_0}{2-K} (1 + \sqrt{1 - K(2-K)\sin^2 \phi})$$

and

$$r_{min} = \frac{r_0}{2-K} (1 - \sqrt{1 - K(2-K)\sin^2 \phi})$$

Here,

$$K = \frac{r_0^2 v_0^2}{GM}$$

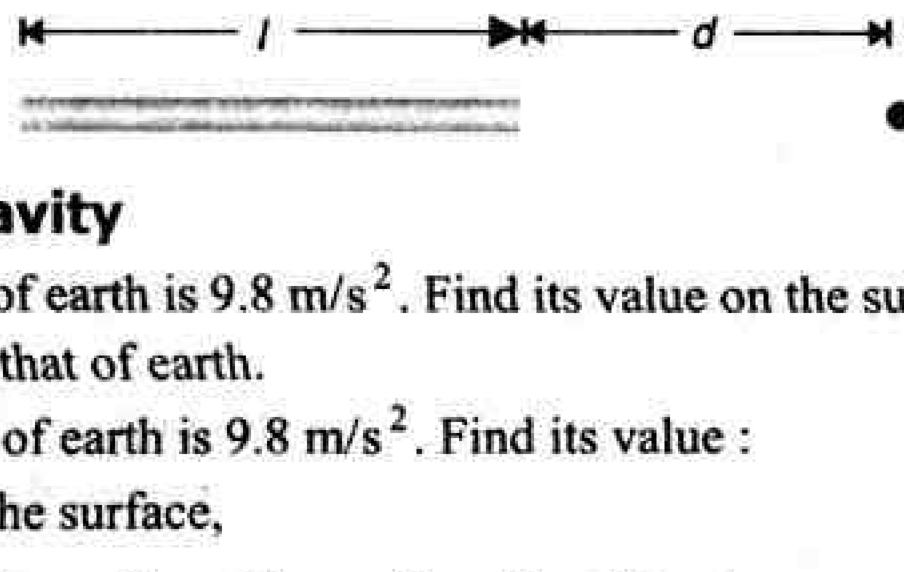
EXERCISES

For JEE Main

Subjective Questions

Newton's Law of Gravitation

1. Two particles of masses 1.0 kg and 2.0 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.
2. Three particles A, B and C, each of mass m, are placed in a line with $AB = BC = d$. Find the gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.
3. Four particles having masses m, $2m$, $3m$ and $4m$ are placed at the four corners of a square of edge a. Find the gravitational force acting on a particle of mass m placed at the centre.
4. Three uniform spheres each having a mass M and radius a are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any of the spheres due to the other two.
5. The figure shows a uniform rod of length l whose mass per unit length is λ. What is the gravitational force of the rod on a particle of mass m located a distance d from one end of the rod?



Acceleration due to Gravity

6. Value of g on the surface of earth is $9.8\ m/s^2$. Find its value on the surface of a planet whose mass and radius both are two times that of earth.
7. Value of g on the surface of earth is $9.8\ m/s^2$. Find its value :
   (a) at height $h = R$ from the surface,
   (b) at depth $h = \frac{R}{2}$ from the surface. (R = radius of earth)
8. Calculate the distance from the surface of the earth at which the acceleration due to gravity is the same below and above the surface of the earth.
9. A body is weighed by a spring balance to be 1000 N at the north pole. How much will it weight at the equator. Account for the earth's rotation only.
10. At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation ?
11. Assuming earth to be spherical, at what height above the north pole, value of g is same as that on the earth's surface at equator ?

Gravitational Field Strength and Potential

12. Two concentric spherical shells have masses m_1, m_2 and radii $R_1, R_2 (R_1 < R_2)$. Calculate the force exerted by this system on a particle of mass m, if it is placed at a distance $\frac{(R_1 + R_2)}{2}$ from the centre.

13. Two spheres one of mass M has radius R. Another sphere has mass $4M$ and radius $2R$. The centre to centre distance between them is $12R$. Find the distance from the centre of smaller sphere where :
(a) net gravitational field is zero,
(b) net gravitational potential is half the potential on the surface of larger sphere.

14. A semicircular wire has a length L and mass M. Find the gravitational field at the centre of the circle.

15. A uniform solid sphere of mass M and radius a is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance (a) $\frac{3}{2}a$ from the centre, (b) $\frac{5}{2}a$ from the centre.

16. The density inside a solid sphere of radius a is given by $\rho = \rho_0 a/r$, where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centre.

17. A particle of mass m is placed at the centre of a uniform spherical shell of same mass and radius R. Find the gravitational potential at a distance $\frac{R}{2}$ from the centre.

Gravitational Potential Energy

18. A rocket is accelerated to speed $v = 2\sqrt{gR}$ near earth's surface (R = radius of earth). Show that very far from earth its speed will be $v = \sqrt{2gR}$.

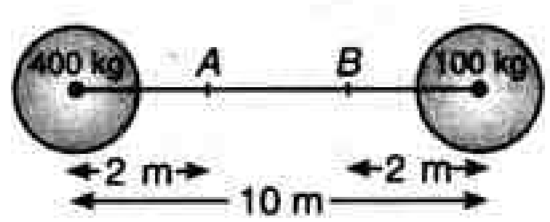
19. Two neutron stars are separated by a distance of 10^{10} m. They each have a mass of 10^{30} kg and a radius of 10^5 m. They are initially at rest with respect to each other.

As measured from the rest frame, how fast are they moving when :
(a) their separation has decreased to one-half its initial value,
(b) they are about to collide.

20. A projectile is fired vertically from earth's surface with an initial speed of 10 km/s. Neglecting air drag, how far above the surface of earth will it go ?

21. A mass m is taken to a height R from the surface of the earth and then is given a vertical velocity v. Find the minimum value of v, so that mass never returns to the surface of the earth.
(Radius of earth is R and mass of the earth M).

22. In the figure masses 400 kg and 100 kg are fixed.



(a) How much work must be done to move a 1 kg mass from point A to point B ?
(b) What is the minimum kinetic energy with which the 1 kg mass must be projected from A to the right to reach the point B ?

Planets and Satellites : Kepler's Law

23. A sky lab of mass 2×10^3 kg is first launched from the surface of earth in a circular orbit of radius $2R$ and then it is shifted from this circular orbit to another circular orbit of radius $3R$. Calculate the energy required :

(a) to place the lab in the first orbit,

(b) to shift the lab from first orbit to the second orbit. ($R = 6400\,\text{km}$, $g = 10\,\text{m/s}^2$)

24. Two identical stars of mass M orbit around their centre of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.

(a) Find the gravitational force of one star on the other.

(b) Find the orbital speed of each star and the period of the orbit.

(c) What minimum energy would be required to separate the two stars to infinity ?

25. Consider two satellites A and B of equal mass, moving in the same circular orbit of radius r around the earth but in the opposite sense and therefore a collision occurs.

(a) Find the total mechanical energy $E_A + E_B$ of the two satellite-plus-earth system before collision.

(b) If the collision is completely inelastic, find the total mechanical energy immediately after collision. Describe the subsequent motion of the combined satellite.

26. Two satellites A and B revolve around a planet in two coplanar circular orbits in the same sense with radii 10^4 km and 2×10^4 km respectively. Time period of A is 28 hours. What is time period of another satellite.

27. A satellite of mass 1000 kg is supposed to orbit the earth at a height of 2000 km above the earth's surface. Find (a) its speed in the orbit, (b) its kinetic energy, (c) the potential energy of the earth-satellite system and (d) its time period. Mass of the earth = 6×10^{24} kg.

28. In a certain binary star system, each star has the same mass as our sun. They revolve about their centre of mass. The distance between them is the same as the distance between earth and the sun. What is their period of revolution in years ?

29. (a) Does it take more energy to get a satellite upto 1500 km above earth than to put it in circular orbit once it is there.

(b) What about 3185 km?

(c) What about 4500 km? (Take $R_e = 6370\,\text{km}$)

Objective Questions

Single Correct Option

1. A satellite orbiting close to the surface of earth does not fall down because the gravitational pull of earth

(a) is balanced by the gravitational pull of moon

(b) is balanced by the gravitational pull of sun

(c) provides the necessary acceleration for its motion along the circular path

(d) makes it weightless

2. For the planet-sun system identify the correct statement

(a) the angular momentum of the planet is conserved

(b) the total energy of the system is conserved

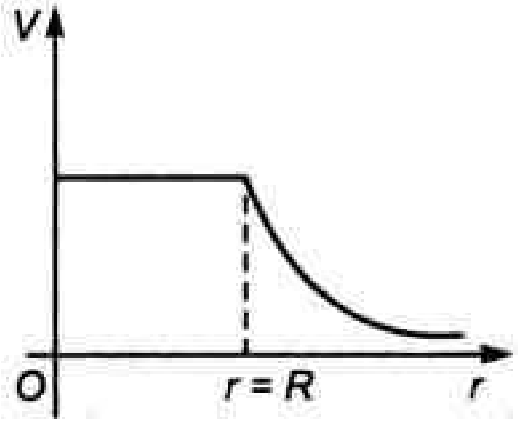
(c) the momentum of the planet is conserved

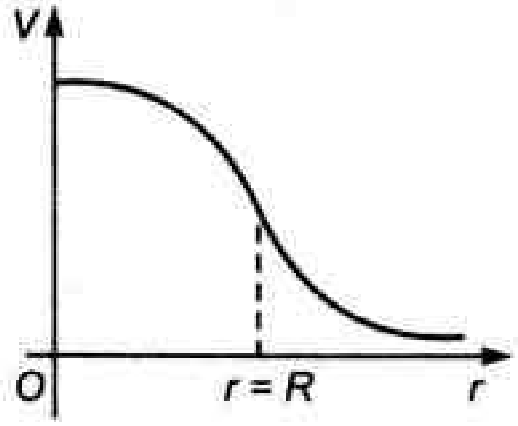
(d) All of the above

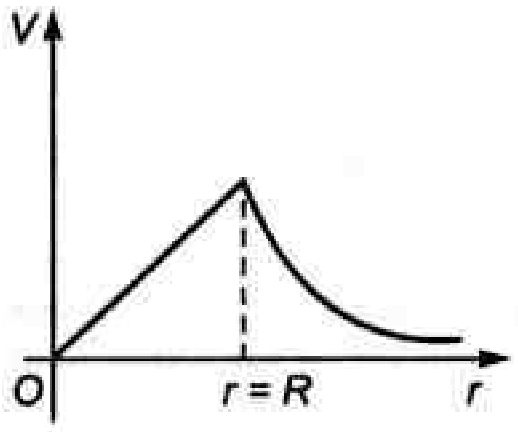
3. If the earth stops rotating about its axis, then the magnitude of gravity
   (a) increases everywhere on the surface of earth
   (b) will increase only at the poles
   (c) will not change at the poles
   (d) All of the above

4. For a body to escape from earth, angle from horizontal at which it should be fired is
   (a) $45°$ (b) $0°$ (c) $90°$ (d) any angle

5. The correct variation of gravitational potential V with radius r measured from the centre of earth of radius R is given by

(a)


(b)


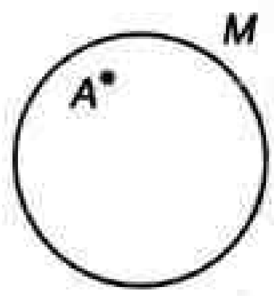
(c)


(d) None of the above

6. The Gauss' theorem for gravitational field may be written as
   (a) $\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{S}} = \frac{m}{G}$ (b) $-\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{S}} = 4\pi mG$ (c) $\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{S}} = \frac{m}{4\pi G}$ (d) $-\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{S}} = \frac{m}{G}$

7. In the earth-moon system, if T_1 and T_2 are period of revolution of earth and moon respectively about the centre of mass of the system then
   (a) $T_1 > T_2$ (b) $T_1 = T_2$ (c) $T_1 < T_2$ (d) Insufficient data

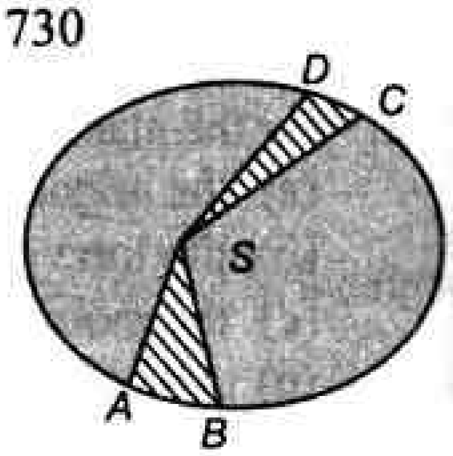
8. The figure shows a spherical shell of mass M. The point A is not at the centre but away from the centre of the shell. If a particle of mass m is placed at A, then



   (a) it remains at rest
   (b) it experiences a net force towards the centre
   (c) it experiences a net force away from the centre
   (d) None of the above

9. If the distance between the earth and the sun were reduced to half its present value, then the number of days in one year would have been
   (a) 65 (b) 129 (c) 183 (d) 730

10. The figure represents an elliptical orbit of a planet around sun. The planet takes time T_1 to travel from A to B and it takes time T_2 to travel from C to D. If the area CSD is double that of area ASB, then



   (a) $T_1 = T_2$ (b) $T_1 = 2T_2$
   (c) $T_1 = 0.5T_2$ (d) Data insufficient

11. At what depth from the surface of earth the time period of a simple pendulum is 0.5% more than that on the surface of the Earth? (Radius of earth is 6400 km)
(a) 32 km (b) 64 km (c) 96 km (d) 128 km

12. If M is the mass of the earth and R its radius, the ratio of the gravitational acceleration and the gravitational constant is
(a) $\frac{R^2}{M}$ (b) $\frac{M}{R^2}$ (c) MR^2 (d) $\frac{M}{R}$

13. The height above the surface of earth at which the gravitational field intensity is reduced to 1% of its value on the surface of earth is
(a) $100R_e$ (b) $10R_e$ (c) $99R_e$ (d) $9R_e$

14. For a satellite orbiting close to the surface of earth the period of revolution is 84 minute. The time period of another satellite orbiting at a height three times the radius of earth from its surface will be
(a) $(84)\,2\sqrt{2}$ minute (b) (84) 8 minute (c) $(84)\,3\sqrt{3}$ minute (d) (84) 3 minute

15. The angular speed of rotation of earth about its axis at which the weight of man standing on the equator becomes half of its weight at the poles is given by
(a) 0.034 rad s^{-1} (b) $8.75\times10^{-4} \text{ rad s}^{-1}$
(c) $1.23\times10^{-2} \text{ rad s}^{-1}$ (d) $7.65\times10^{-7} \text{ rad s}^{-1}$

16. The height from the surface of earth at which the gravitational potential energy of a ball of mass m is half of that at the centre of earth is (where R is the radius of earth)
(a) $\frac{R}{4}$ (b) $\frac{R}{3}$ (c) $\frac{3R}{4}$ (d) $\frac{4R}{3}$

17. A body of mass m is lifted up from the surface of earth to a height three times the radius of the earth R. The change in potential energy of the body is
(a) $3mgR$ (b) $\frac{5}{4}mgR$ (c) $\frac{3}{4}mgR$ (d) $2mgR$

18. A satellite is revolving around earth in its equatorial plane with a period T. If the radius of earth suddenly shrinks to half its radius without change in the mass. Then, the new period of revolution will be
(a) $8T$ (b) $2\sqrt{2}\,T$ (c) $2T$ (d) T

19. If the radius of moon is 1.7×10^6 m and its mass is 7.34×10^{22} kg. Then its escape velocity is
(a) $2.4\times10^3 \text{ ms}^{-1}$ (b) $2.4\times10^2 \text{ ms}^{-1}$ (c) $3.4\times10^3 \text{ ms}^{-1}$ (d) $3.4\times10^2 \text{ ms}^{-1}$

20. A planet has twice the density of earth but the acceleration due to gravity on its surface is exactly the same as on the surface of earth. Its radius in terms of earth's radius R will be
(a) $R/4$ (b) $R/2$ (c) $R/3$ (d) $R/8$

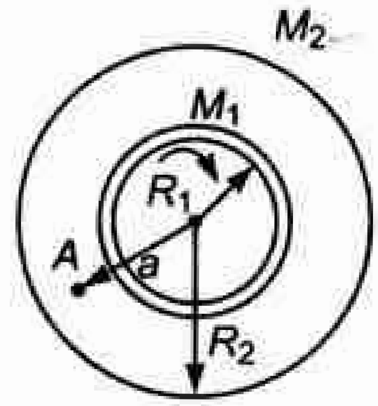
21. The speed of earth's rotation about its axis is ω. Its speed is increased to x times to make the effective acceleration due to gravity equal to zero at the equator, then x is around ($g = 10 \text{ ms}^{-2}$; $R = 6400$ km)
(a) 1 (b) 8.5 (c) 17 (d) 34

22. A satellite is seen every 6 hours over the equator. It is known that it rotates opposite to that of earth's direction. Then the angular velocity (in radian per hour) of satellite about the centre of earth will be
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

23. For a planet revolving around sun, if a and b are the respective semi-major and semi-minor axes, then the square of its time period is proportional to

(a) $\left(\frac{a+b}{2}\right)^3$ (b) $\left(\frac{a-b}{2}\right)^3$ (c) b^3 (d) a^3

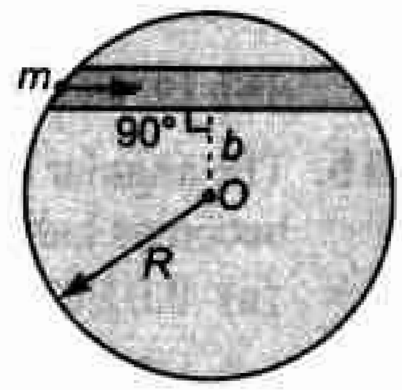
24. The figure represents two concentric shells of radii R_1 and R_2 and masses M_1 and M_2 respectively. The gravitational field intensity at the point A at distance a ($R_1 < a < R_2$) is



(a) $\frac{G(M_1+M_2)}{a^2}$ (b) $\frac{GM_1}{a^2}+\frac{GM_2}{R_2^2}$

(c) $\frac{GM_1}{a^2}$ (d) Zero

25. A straight tunnel is dug into the earth as shown in figure at a distance b from its centre. A ball of mass m is dopped from one of its ends. The time it takes to reach the other end is approximately



(a) 42 min (b) 84 min

(c) $84\left(\frac{b}{R}\right)$ min (d) $42\left(\frac{b}{R}\right)$ min

26. Three identical particles each of mass m are placed at the corners of an equilateral triangle of side l. The work done by external force to increase the side of triangle from l to $2l$ is

(a) $-\frac{3}{2}\frac{GM^2}{l}$ (b) $\frac{-3GM^2}{l}$ (c) $\frac{3}{2}\frac{GM^2}{l}$ (d) $\frac{3GM^2}{l}$

27. A particle is thrown vertically upwards from the surface of earth and it reaches to a maximum height equal to the radius of earth. The ratio of the velocity of projection to the escape velocity on the surface of earth is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2\sqrt{2}}$

28. The gravitational potential energy of a body at a distance r from the centre of earth is U. Its weight at a distance $2r$ from the centre of earth is

(a) $\frac{U}{r}$ (b) $\frac{U}{2r}$ (c) $\frac{U}{4r}$ (d) $\frac{U}{\sqrt{2}r}$

For **JEE Advanced**

Assertion and Reason

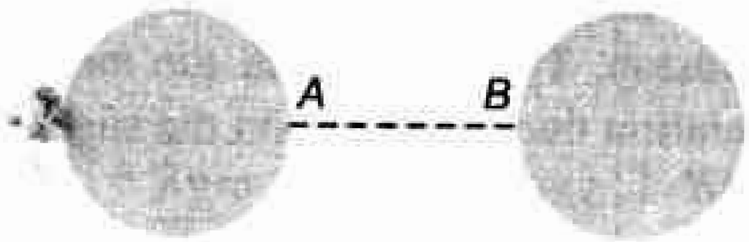
Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***
(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***
(*c*) *If **Assertion** is true, but the **Reason** is false.*
(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : When two masses come closer, their gravitational potential energy decreases.
Reason : Two masses attract each other.

2. **Assertion :** In moving from centre of a solid sphere to its surface, gravitational potential increases.
Reason : Gravitational field strength increases.

3. **Assertion :** There are two identical spherical bodies fixed in two positions as shown. While moving from A to B gravitational potential first increases then decreases.



Reason : At centre of A and B field strength will be zero.

4. **Assertion :** If we plot potential versus x-coordinate graph along the x-axis, then field strength is zero where slope of V-x graph is zero.
Reason : If potential is function of x-only then
$$E = -\frac{dV}{dx}$$

5. **Assertion :** A particle is projected upwards with speed v and it goes to a height h. If we double the speed then it will move to height $4h$.
Reason : In case of earth, acceleration due to gravity g varies as
$$g \propto \frac{1}{r^2} \qquad (\text{for } r \geq R)$$

6. **Assertion :** In planetary motion angular momentum of system remains constant. But linear momentum does not remain constant.
Reason : Net torque on this system about any point is zero.

7. **Assertion :** Plane of space satellite is always equitorial plane.
Reason : On the equator value of g is minimum.

8. **Assertion :** On satellites we feel weightlessness. Moon is also a satellite of earth. But we do not feel weightlessness on moon.
Reason : Mass of moon is considerable.

9. **Assertion :** Plane of geostationary satellites always passes through equator.
Reason : Geostationary satellites always lies above Moscow.

10. **Assertion :** It we double the circular radius of a satellite, then its potential energy, kinetic energy and total mechanical energy will become half.
Reason : Orbital speed of a satellite.
$$v \propto \frac{1}{\sqrt{r}}$$
where r is its radius of orbit.

11. **Assertion :** If the radius of earth is decreased keeping its mass constant, effective value of g may increase or decrease at pole.
Reason : Value of g on the surface of earth is given by $g = \frac{GM}{R^2}$.

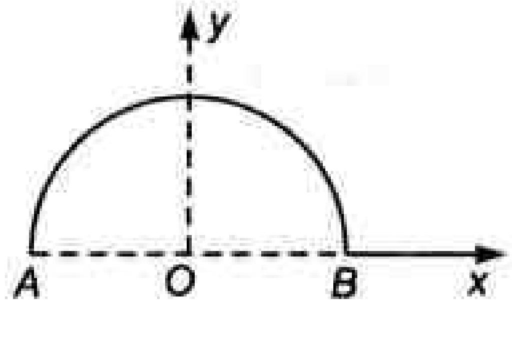
Objective Questions

Single Correct Option

1. An artificial satellite of mass m is moving in a circular orbit at a height equal to the radius R of the earth. Suddenly due to internal explosion the satellite breaks into two parts of equal pieces. One part of the satellite stops just after the explosion. The increase in the mechanical energy of the system due to explosion will be (Given : acceleration due to gravity on the surface of earth is g)

   (a) mgR (b) $\frac{mgR}{2}$ (c) $\frac{mgR}{4}$ (d) $\frac{3mgR}{4}$

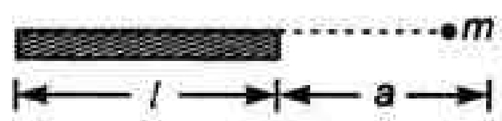
2. Gravitational field at the centre of a semicircle formed by a thin wire AB of mass M and length l is



   (a) $\frac{GM}{l^2}$ along x-axis (b) $\frac{GM}{\pi l^2}$ along y-axis

   (c) $\frac{2\pi GM}{l^2}$ along x-axis (d) $\frac{2\pi GM}{l^2}$ along y-axis

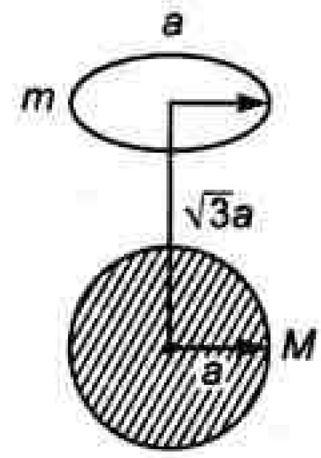
3. A mass m is at a distance a from one end of a uniform rod of length l and mass M. The gravitational force on the mass due to the rod is



   (a) $\frac{GMm}{(a+l)^2}$ (b) $\frac{GmM}{a\,(l+a)}$

   (c) $\frac{GMm}{a^2}$ (d) $\frac{GmM}{2\,(l+a)^2}$

4. A uniform ring of mass m is lying at a distance $\sqrt{3}\,a$ from the centre of a sphere of mass M just over the sphere (where a is the radius of the ring as well as that of the sphere). Then magnitude of gravitational force between them is



   (a) $\frac{GMm}{8a^2}$ (b) $\frac{GMm}{\sqrt{3}a^2}$

   (c) $\sqrt{3}\frac{GMm}{a^2}$ (d) $\sqrt{3}\frac{GMm}{8a^2}$

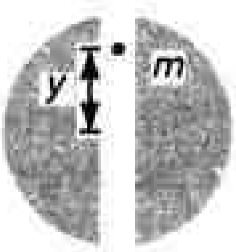
5. Four particles, each of mass M, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

   (a) $\frac{GM}{R}$ (b) $\sqrt{2\sqrt{2}\frac{GM}{R}}$ (c) $\sqrt{\frac{GM}{R}(2\sqrt{2}+1)}$ (d) $\sqrt{\frac{GM}{R}\frac{2\sqrt{2}+1}{4}}$

6. A projectile is fired from the surface of earth of radius R with a velocity kv_e (where v_e is the escape velocity from surface of earth and $k < 1$). Neglecting air resistance, the maximum height of rise from the centre of earth is

   (a) $\frac{R}{k^2-1}$ (b) k^2R (c) $\frac{R}{1-k^2}$ (d) kR

7. Suppose a vertical tunnel is along the diameter of earth, assumed to be a sphere of uniform mass density ρ. If a body of mass m is thrown in this tunnel, its acceleration at a distance y from the centre is given by



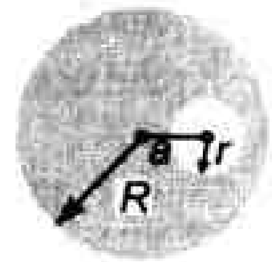
   (a) $\frac{4\pi}{3}G\rho ym$ (b) $\frac{3}{4}\pi\rho y$ (c) $\frac{4}{3}\pi\rho y$ (d) $\frac{4}{3}\pi G\rho y$

8. A train of mass m moves with a velocity v on the equator from east to west. If ω is the angular speed of earth about its axis and R is the radius of the earth then the normal reaction acting on the train is

(a) $mg\left[1-\frac{(\omega R-2v)\omega}{g}-\frac{v^2}{Rg}\right]$ (b) $mg\left[1-2\frac{(\omega R-v)\omega}{g}-\frac{v^2}{Rg}\right]$

(c) $mg\left[1-\frac{(\omega R+2v)\omega}{g}-\frac{v^2}{Rg}\right]$ (d) $mg\left[1-2\frac{(\omega R-v)\omega}{g}-\frac{v^2}{Rg}\right]$

9. The figure represents a solid uniform sphere of mass M and radius R. A spherical cavity of radius r is at a distance a from the centre of the sphere. The gravitational field inside the cavity is



(a) non-uniform
(b) towards the centre of the cavity
(c) directly proportional to a
(d) All of the above

10. If v_e is the escape velocity for earth when a projectile is fired from the surface of earth. Then the escape velocity if the same projectile is fired from its centre is

(a) $\sqrt{\frac{3}{2}}\,v_e$ (b) $\frac{3}{2}v_e$ (c) $\sqrt{\frac{2}{3}}v_e$ (d) $\frac{2}{3}v_e$

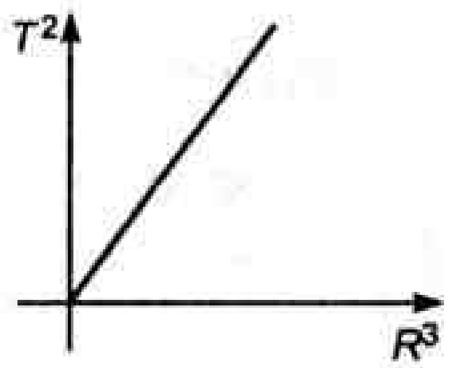
11. If the gravitational field intensity at a point is given by $g=\frac{GM}{r^{2.5}}$. Then the potential at distance r is

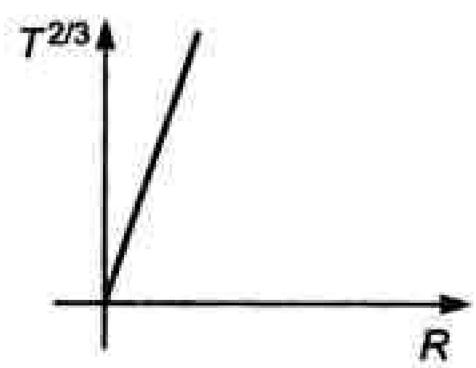
(a) $\frac{-2GM}{3r^{1.5}}$ (b) $\frac{-GM}{r^{3.5}}$ (c) $\frac{2GM}{3r^{1.5}}$ (d) $\frac{GM}{r^{3.5}}$

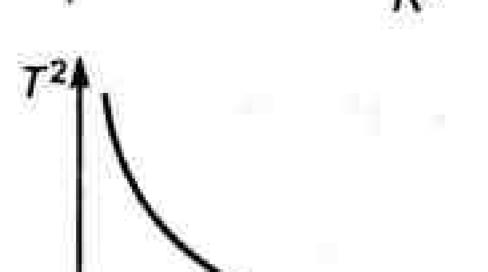
12. Three identical particles each of mass M move along a common circular path of radius R under the mutual interaction of each other. The velocity of each particle is

(a) $\sqrt{\frac{GM}{R}\sqrt{\frac{2}{3}}}$ (b) $\sqrt{\frac{GM}{\sqrt{3}R}}$ (c) $\sqrt{\frac{GM}{3R}}$ (d) $\sqrt{\frac{2}{3}\frac{GM}{R}}$

13. If T be the period of revolution of a planet revolving around sun in an orbit of mean radius R, then identify the **incorrect** graph.

(a)


(b)


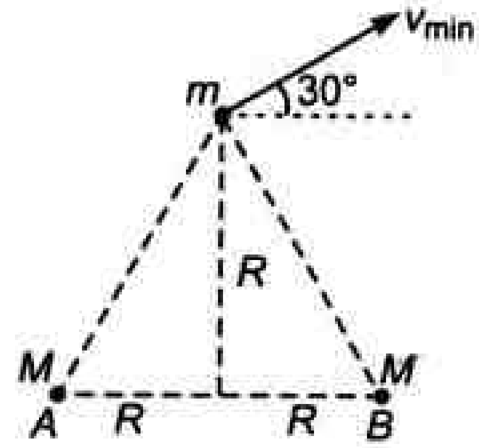
(c)


(d) None of these

14. A person brings a mass of 1 kg from infinity to a point A. Initially, the mass was at rest but it moves at a speed of 3 m/s as it reaches A. The work done by the person on the mass is – 5.5 J. The gravitational potential at A is

(a) – 1 J/kg (b) – 4.5 J/kg (c) – 5.5 J/kg (d) – 10 J/kg

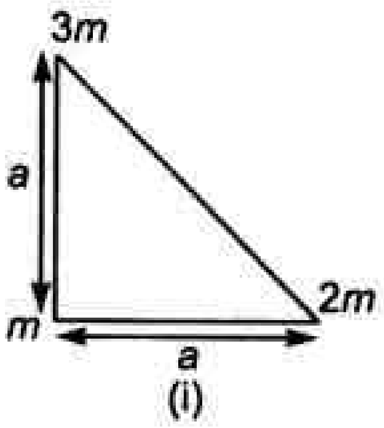
15. With what minimum speed should m be projected from point C in presence of two fixed masses M each at A and B as shown in the figure such that mass m should escape the gravitational attraction of A and B?

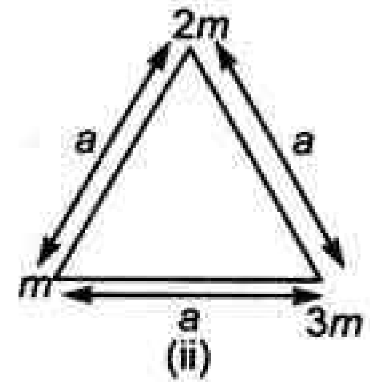


(a) $\sqrt{\frac{2GM}{R}}$ (b) $\sqrt{\frac{2\sqrt{2}GM}{R}}$

(c) $2\sqrt{\frac{GM}{R}}$ (d) $2\sqrt{2}\sqrt{\frac{GM}{R}}$

16. Consider two configurations of a system of three particles of masses m, $2m$ and $3m$. The work done by gravity in changing the configuration of the system from figure (i) to figure (ii) is





(a) zero (b) $\frac{6Gm^2}{a}\left\{1+\frac{1}{\sqrt{2}}\right\}$

(c) $\frac{6Gm^2}{a}\left\{1-\frac{1}{\sqrt{2}}\right\}$ (d) $\frac{6Gm^2}{a}\left\{2-\frac{1}{\sqrt{2}}\right\}$

17. A tunnel is dug along the diameter of the earth. There is a particle of mass m at the centre of the tunnel. Find the minimum velocity given to the particle so that is just reaches to the surface of the earth. (R = radius of earth)

(a) $\sqrt{\frac{GM}{R}}$

(b) $\sqrt{\frac{GM}{2R}}$

(c) $\sqrt{\frac{2GM}{R}}$

(d) it will reach with the help of negligible velocity

18. A body is projected horizontally from the surface of the Earth (radius = R) with a velocity equal to n times the escape velocity. Neglect rotational effects of the earth. The maximum height attained by the body from the earth's surface is $R/2$. Then n must be

(a) $\sqrt{0.6}$ (b) $(\sqrt{3})/2$ (c) $\sqrt{0.4}$ (d) $1/2$

19. A tunnel is dug in the earth across one of its diameter. Two masses m and $2m$ are dropped from the two ends of the tunnel. The masses collide and stick each other. They perform SHM, the amplitude of which is (R = radius of earth)

(a) R (b) $R/2$ (c) $R/3$ (d) $2R/3$

20. There are two planets. The ratio of radius of the two planets is k but ratio of acceleration due to gravity of both planets is g. What will be the ratio of their escape velocity?

(a) $(kg)^{1/2}$ (b) $(kg)^{-1/2}$ (c) $(kg)^2$ (d) $(kg)^{-2}$

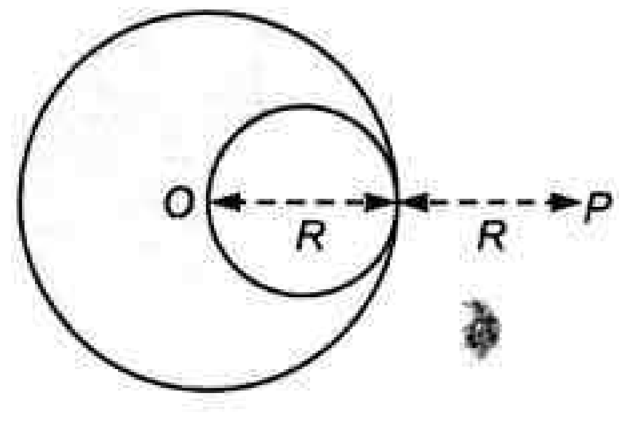
21. A body of mass 2 kg is moving under the influence of a central force whose potential energy is given by $U = 2r^3$ J. If the body is moving in a circular orbit of 5 m, its energy will be

(a) 625 J (b) 250 J (c) 500 J (d) 125 J

22. A research satellite of mass 200 kg circles the earth in an orbit of average radius $3R/2$, where R is the radius of the earth. Assuming the gravitational pull on the mass of 1 kg on the earth's surface to be 10 N, the pull on the satellite will be
(a) 1212 N (b) 889 N (c) 1280 N (d) 960 N

23. A satellite of mass m revolves around the earth of radius R at a height x from its surface. If g is the acceleration due to gravity on the surface of the earth, the orbital speed of the satellite is
(a) $\sqrt{gx}$ (b) $\sqrt{\dfrac{gR}{R-x}}$ (c) $\sqrt{\dfrac{gR^2}{R-x}}$ (d) $\sqrt{\dfrac{gR^2}{R+x}}$

24. A solid sphere of uniform density and radius R applies a gravitational force of attraction equal to F_1 on a particle placed at P, distance $2R$ from the centre O of the sphere. A spherical cavity of radius $R/2$ is now made in the sphere as shown in figure. The sphere with cavity now applies a gravitational force F_2 on same particle placed at P. The ratio F_2/F_1 will be



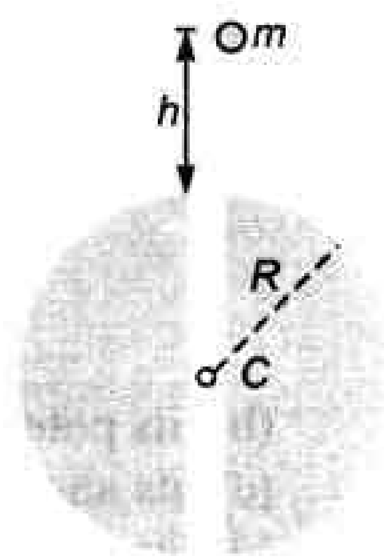
(a) 1/2 (b) 7/9
(c) 3 (d) 7

More than One Correct Options

1. Three planets of same density have radii R_1, R_2 and R_3 such that $R_1 = 2R_2 = 3R_3$. The gravitational field at their respective surfaces are g_1, g_2 and g_3 and escape velocities from their surfaces are v_1, v_2 and v_3, then
(a) $g_1/g_2 = 2$ (b) $g_1/g_3 = 3$ (c) $v_1/v_2 = 1/4$ (d) $v_1/v_3 = 3$

2. For a geostationary satellite orbiting around the earth identify the necessary condition.
(a) it must lie in the equatorial plane of earth
(b) its height from the surface of earth must be 36000 km
(c) it period of revolution must be $2\pi\sqrt{\dfrac{R}{g}}$ where R is the radius of earth
(d) its period of revolution must be 24 hrs

3. A ball of mass m is dropped from a height h equal to the radius of the earth above the tunnel dug through the earth as shown in the figure. Choose the correct options.

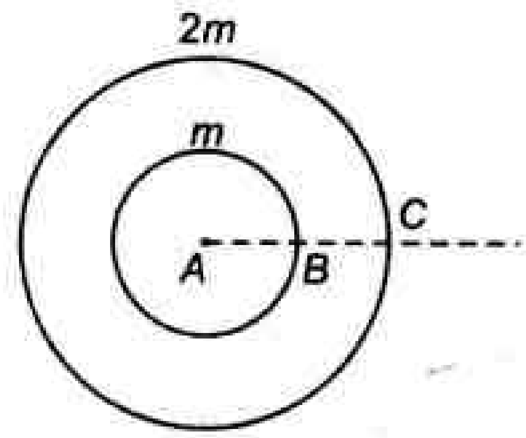


(a) Particle will oscillate through the earth to a height h on both sides
(b) Particle will execute simple harmonic motion
(c) Motion of the particle is periodic
(d) Particle passes the centre of earth with a speed $v = \sqrt{\dfrac{2GM}{R}}$

4. Two point masses m and $2m$ are kept at points A and B as shown. E represents magnitude of gravitational field strength and V the gravitational potential. As we move from A to B

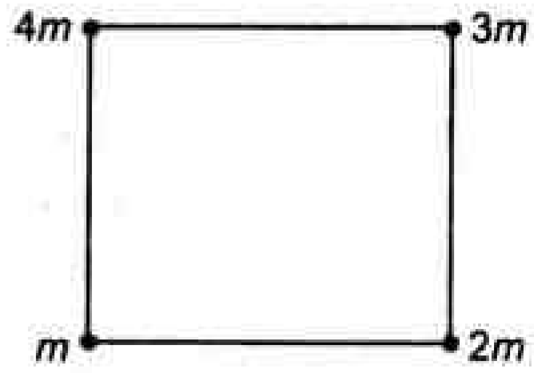
(a) E will first decrease then increases
(b) E will first increase then decrease
(c) V will first decrease then increase
(d) V will first increase then decrease

5. Two spherical shells have mass m and $2m$ as shown. Choose the correct options.



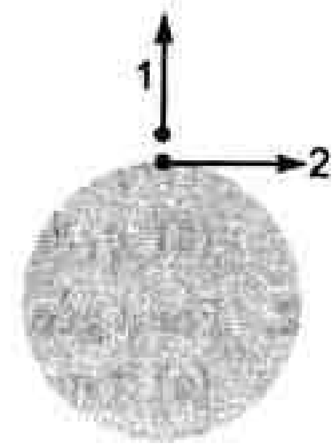
(a) Between A and B gravitational field strength is zero
(b) Between A and B gravitational potential is constant
(c) There will be two points one lying between B and C and other lying between C and infinity where gravitational field strength are same
(d) There will be a point between B and C where gravitational potential will be zero

6. Four point masses are placed at four corners of a square as shown. When positions of m and $2m$ are interchanged



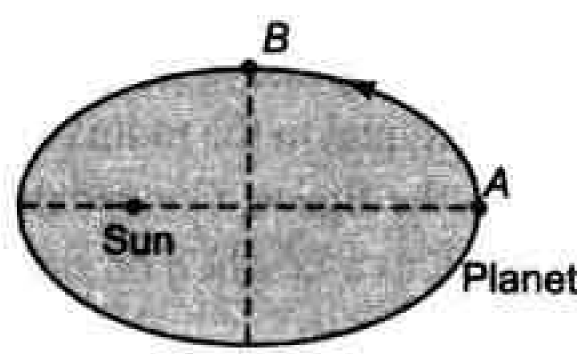
(a) gravitational field strength at centre will increase
(b) gravitational field strength at centre will decrease
(c) gravitational potential at centre will remain unchanged
(d) gravitational potential at centre will decrease

7. Two identical particles 1 and 2 are projected from surface of earth with same velocities in the directions shown in figure.



(a) Both the particles will stop momentarily (before striking with ground) at different times
(b) Particle-2 will rise upto lesser height compared to particle-2
(c) Minimum speed of particle-2 is more than that of particle-1
(d) Particle-1 will strike the ground earlier

8. A planet is moving round the sun in an elliptical orbit as shown. As the planet moves from A to B
(a) its kinetic energy will decrease



(b) its potential energy will remain unchanged
(c) its angular momentum about centre of sun will remain unchanged
(d) its speed is minimum at A

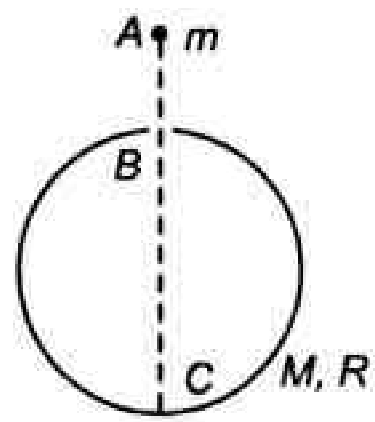
9. A satellite of mass m is just placed over the surface of earth. In this position mechanical energy of statellite is E_1. Now it starts orbiting round the earth in a circular path at height h = radius of earth. In this position, kinetic energy, potential energy and total mechanical energy of satellite are K_2, U_2 and E_2 respectively. Then

(a) $U_2 = \frac{E_1}{2}$ (b) $E_2 = \frac{E_1}{4}$ (c) $K_2 = -E_2$ (d) $K_2 = -\frac{U_2}{2}$

10. A satellite is revolving round the earth in circular orbit
   (a) if mass of earth is made four times, keeping other factors constant, orbital speed of satellite will become two times
   (b) corresponding to change in part (a), times period of satellite will remain half
   (c) when value of G is made two times orbital speed increases and time period decreases
   (d) G has no effect on orbital speed and time period

Match the Columns

1. There is a small hole in a spherical shell of mass M and radius R. A particle of mass m is dropped from point A as shown. Match the two columns for the situation shown in figure.



Column I	Column II
(a) Potential energy from A to B	(p) Continuously increases
(b) Potential energy from B to C	(q) Continuously decreases
(c) Speed of particle from B to C	(r) First increases then remains constant
(d) Acceleration of particle from A to C	(s) None of these

2. Five point masses m each are placed at five corners of a regular pentagon. Distance of any corner from centre is r. Match the following two columns.

Column I	Column II
(a) Gravitational field strength at centre	(p) Gm/r^2
(b) Gravitational potential at centre	(q) $4Gm/r$
(c) When one mass is removed gravitational field strength at centre	(r) zero
(d) When one mass is removed gravitational potential at centre	(s) None of these

3. | Potential | on the surface of a solid sphere is x and radius is y. Match the following two columns.

Column I	Column II
(a) Field strength at distance $2y$ from centre	(p) $\frac{x}{2y}$
(b) | Potential | at distance $\frac{y}{2}$ from centre	(q) $\frac{x}{2}$
(c) Field strength at distance $y/2$ from centre	(r) $\frac{x}{4y}$
(d) | Potential | at distance $2y$ from centre	(s) None

4. Match the following two columns.

Column I	Column II
(a) Work done is raising a mass m to a height $h = R$	(p) $\frac{1}{4}mgR$
(b) Kinetic energy of a satellite of mass m at height $h = R$	(q) mgR
(c) Difference in energies of two satellites each of mass m but one at height $h_1 = R$ and another of height $h_2 = 2R$	(r) $\frac{1}{2}mgR$
(d) Kinetic energy required to raise a particle of mass m to a height $h = R$ if projected vertically from surface of earth.	(s) None

5. Match the following two columns.

Column I	Column II
(a) Gravitational field strength is maximum at	(p) $r = 0$
(b) Gravitational field strength is zero at	(q) $r = R$
(c) Gravitational potential is minimum at	(r) $r = \frac{R}{\sqrt{2}}$
(d) Gravitational potential is zero at	(s) None of these

Here r is distance from centre of a solid sphere or distance from centre of a ring along its axis.

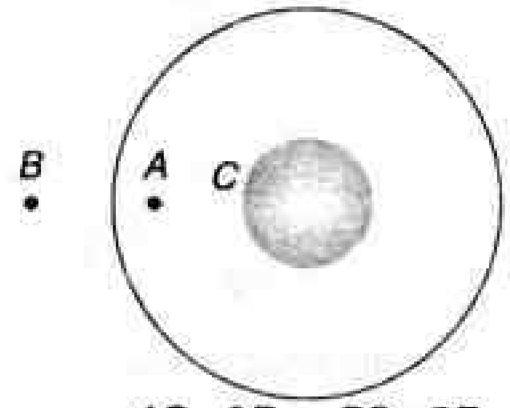
Subjective Questions

1. Three particles of mass m each are placed at the three corners of an equilateral triangle of side a. Find the work which should be done on this system to increase the side of the triangle to $2a$.

2. A man can jump vertically to a height of 1.5 m on the earth. Calculate the radius of a planet of the same mean density as that of the earth from whose gravitational field he could escape by jumping. Radius of earth is 6.41×10^6 m.

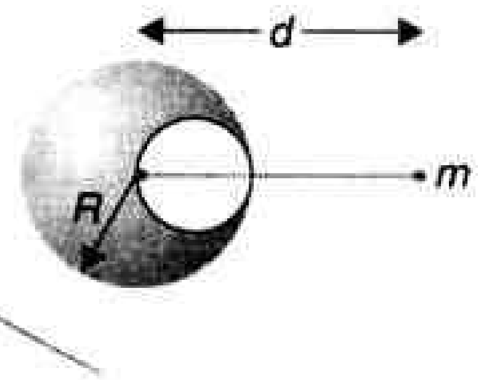
3. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the surface of earth. (Radius of earth = 6400 km)
   (a) Determine the height of the satellite above the earth's surface.
   (b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the earth, find the speed with which it hits the surface of earth.

4. A uniform metal sphere of radius R and mass m is surrounded by a thin uniform spherical shell of same mass and radius $4R$. The centre of the shell C falls on the surface of the inner sphere. Find the gravitational fields at points A and B.

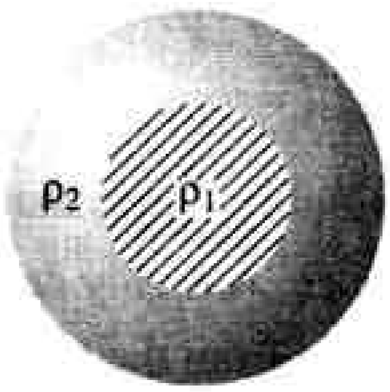


$AC = 3R$, $BC = 5R$

5. Figure shows a spherical cavity inside a lead sphere. The surface of the cavity passes through the centre of the sphere and touches the right side of the sphere. The mass of the sphere before hollowing was M. With what gravitational force does the hollowed out lead sphere attract a particle of mass m that lies at a distance d from the centre of the lead sphere on the straight line connecting the centres of the spheres and of the cavity.



6. The density of the core of a planet is ρ_1 and that of the outer shell is ρ_2, the radii of the core and that of the planet are R and $2R$ respectively. The acceleration due to gravity at the surface of the planet is same as at a depth R. Find the ratio of $\frac{\rho_1}{\rho_2}$.



7. If a satellite is revolving around a planet of mass M in an elliptical orbit of semi-major axis a. Show that the orbital speed of the satellite when it is at a distance r from the focus will be given by

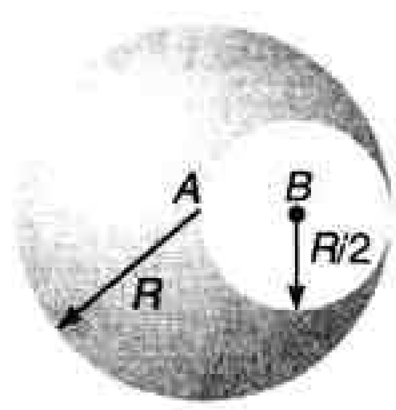
$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

8. A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance $\sqrt{3}a$ from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.

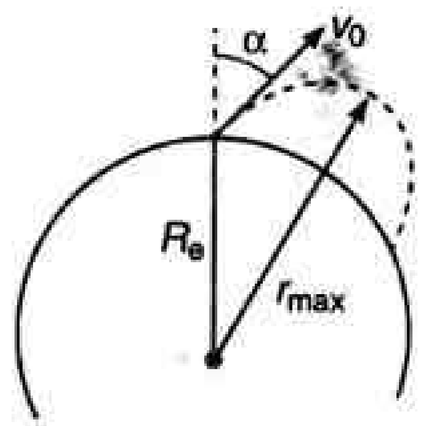
9. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

10. A smooth tunnel is dug along the radius of earth that ends at centre. A ball is released from the surface of earth along tunnel. Coefficient of restitution for collision between soil at centre and ball is 0.5. Calculate the distance travelled by ball just before second collision at centre. Given mass of the earth is M and radius of the earth is R.

11. Inside a fixed sphere of radius R and uniform density ρ, there is spherical cavity of radius $\frac{R}{2}$ such that surface of the cavity passes through the centre of the sphere as shown in figure. A particle of mass m_0 is released from rest at centre B of the cavity. Calculate velocity with which particle strikes the centre A of the sphere. Neglect earth's gravity. Initially sphere and particle are at rest.



12. A projectile of mass m is fired from the surface of the earth at an angle $\alpha = 60°$ from the vertical. The initial speed v_0 is equal to $\sqrt{\frac{GM_e}{R_e}}$. How high does the projectile rise ? Neglect air resistance and the earth's rotation.



13. A ring of radius $R = 4$ m is made of a highly dense material. Mass of the ring is $m_1 = 5.4 \times 10^9$ kg distributed uniformly over its circumference. A highly dense particle of mass $m_2 = 6 \times 10^8$ kg is placed on the axis of the ring at a distance $x_0 = 3$ m from the centre. Neglecting all other forces, except mutual gravitational interaction of the two. Calculate :

(i) displacement of the ring when particle is at the centre of ring, and

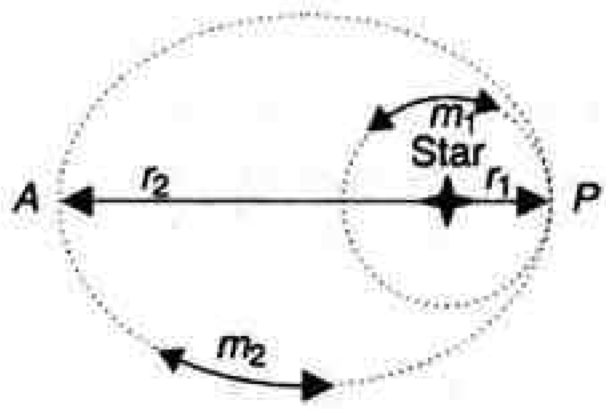
(ii) speed of the particle at that instant.

14. Two planets of equal mass orbit a much more massive star (figure). Planet m_1 moves in a circular orbit of radius 1×10^8 km with period 2 yr. Planet m_2 moves in an elliptical orbit with closest distance $r_1 = 1 \times 10^8$ km and farthest distance $r_2 = 1.8 \times 10^8$ km, as shown.

(a) Using the fact that the mean radius of an elliptical orbit is the length of the semi-major axis, find the period of m_2's orbit.

(b) Which planet has the greater speed at point P ? Which has the greater total energy ?

(c) Compare the speed of planet m_2 at P with that at A.



15. In a double star, two stars one of mass m_1 and another of mass m_2, with a separation d, rotate about their common centre of mass. Find :

(a) an expression for their time period of revolution,

(b) the ratio of their kinetic energies,

(c) the ratio of their angular momenta about the centre of mass, and

(d) the total angular momentum of the system,

(e) the kinetic energy of the system

ANSWERS

Introductory Exercise 10.1

1. -0.0168 ms^{-2} **2.** 7.8×10^{-4} rad/s **3.** 1600 km **4.** $\frac{a}{r^2}$ **5.** $\frac{2\pi GMm}{L^2}$

Introductory Exercise 10.2

1. $-\frac{3Gm}{R}$ **2.** 200 N/kg along positive x-direction **3.** $-[6xy\hat{\mathbf{i}} + (3x^2 + 3y^2z)\hat{\mathbf{j}} + y^3\hat{\mathbf{k}}]$ **4.** False

5. $10\sqrt{2}$ N **6.** Zero

Introductory Exercise 10.3

1. True **2.** mgR **3.** $2.1 \times 10^{-5} \text{ ms}^{-1}$, $4.2 \times 10^{-5} \text{ ms}^{-1}$ **4.** 10 kms^{-1}

Introductory Exercise 10.4

1. 11.2 kms^{-1} **2.** Angular momentum, mechanical energy **3.** 2 : 1, 2 : 1

5. (a) Orbit will become elliptical (b) The satellite will escape

For JEE Main

Subjective Questions

1. $a_1 = 5.3 \times 10^{-10} \text{ ms}^{-2}$, $a_2 = 2.65 \times 10^{-10} \text{ ms}^{-2}$ **2.** $\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)\frac{Gm^2}{d^2}$ (along PB) **3.** $\frac{4\sqrt{2}Gm^2}{a^2}$

4. $\frac{\sqrt{3}GM^2}{4a^2}$ **5.** $\frac{Gm\lambda l}{d(l+d)}$ **6.** 4.9 ms^{-2} **7.** (a) 2.45 ms^{-2} (b) 4.9 ms^{-2}

8. $\frac{(\sqrt{5}-1)R}{2}$, where R is the radius of earth **9.** 997 N **10.** $1.237 \times 10^{-3} \text{ rads}^{-1}$, 84.6 min

11. approximately 10 km **12.** $F = \frac{4Gm_1m}{(R_1+R_2)^2}$ **13.** (a) $4R$ (b) 7.65 R and 1.49 R **14.** $E = \frac{2\pi GM}{L^2}$

15. (a) $\frac{4GM}{9a^2}$ (towards the centre) (b) $\frac{8GM}{25a^2}$ (towards the centre) **16.** $\frac{\pi G\rho_0 a}{2}$ **17.** $\frac{-3Gm}{R}$

19. (a) 82 km/s (b) $1.8 \times 10^4 \text{ kms}^{-1}$ **20.** 2.5×10^4 km **21.** $v = \sqrt{\frac{GM}{R}}$

22. (a) 7.5×10^{-9} J (b) 8.17×10^{-9} J **23.** (a) 9.6×10^{10} J (b) 1.07×10^{10} J

24. (a) $F = \frac{GM^2}{4R^2}$ (b) $v = \sqrt{\frac{GM}{4R}}$, $T = \frac{4\pi R^{3/2}}{\sqrt{GM}}$ (c) $\frac{GM^2}{4R}$ **25.** (a) $\frac{-GMm}{r}$ (b) $\frac{-2GMm}{r}$

26. $56\sqrt{2}\ h$

27. (a) 6.90 kms^{-1} (b) 2.38×10^{10} J (c) -4.76×10^{10} J with usual reference (d) 2.1 h

28. 0.71 yr **29.** (a) No (b) Same (c) Yes

Objective Questions

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (a)	**9.** (b)	**10.** (c)
11. (b)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (c)	**27.** (a)	**28.** (c)		

For JEE Advanced

Assertion and Reason

1. (b) 2. (b) 3. (b) 4. (d) 5. (d) 6. (d) 7. (d) 8. (a) 9. (c) 10. (b)
11. (d)

Objective Questions

1. (c) 2. (d) 3. (b) 4. (d) 5. (d) 6. (c) 7. (d) 8. (a) 9. (c) 10. (a)
11. (a) 12. (b) 13. (d) 14. (d) 15. (b) 16. (c) 17. (a) 18. (a) 19. (c) 20. (a)
21. (a) 22. (b) 23. (d) 24. (b)

More than One Correct Options

1. (a,b,d) 2. (a,b,d) 3. (a,c,d) 4. (a,d) 5. (a,b,c) 6. (a,c) 7. (b,c,d)
8. (c,d) 9. (all) 10. (a,b,c)

Match the Columns

1. (a) → q (b) → s (c) → s (d) → r
2. (a) → r (b) → s (c) → p (d) → s
3. (a) → r (b) → s (c) → p (d) → q
4. (a) → r (b) → p (c) → s (d) → r
5. (a) → q,r (b) → p (c) → p (d) → s

Subjective Questions

1. $\frac{3Gm^2}{2a}$ 2. 3.1×10^3 m 3. (a) 6400 km (b) 7.92 kms^{-1} 4. $\frac{Gm}{16R^2}, \frac{61Gm}{900R^2}$

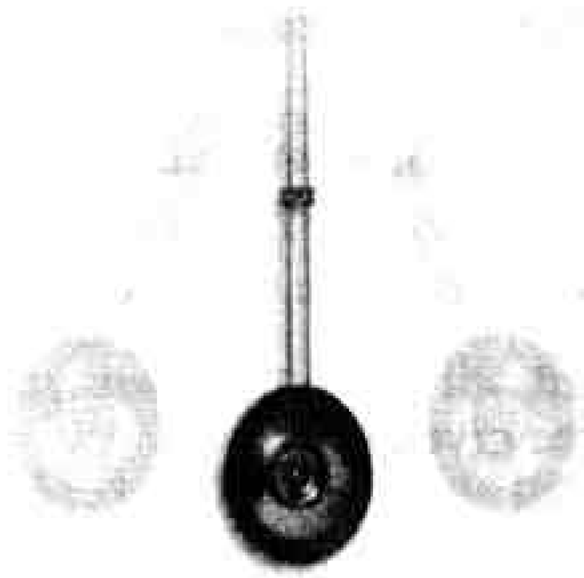
5. $\frac{GMm}{d^2}\left[1 - \frac{1}{8(1-R/2d)^2}\right]$ 6. 7/3 8. $\frac{\sqrt{3}GMm}{8a^2}$ 9. $\frac{3}{2}\sqrt{\frac{5GM}{a}}$ 10. $d = 2R$ 11. $\sqrt{\frac{2}{3}\pi G\rho R^2}$

12. $\frac{R_e}{2}$ 13. (i) 0.3 m (ii) 18 cms^{-1}

14. (a) 3.31 yr (b) m_2 has greater speed and greater total energy (c) $v_P = 1.8v_A$

15. (a) $2\pi\sqrt{\frac{d^3}{G(m_1 + m_2)}}$ (b) $\frac{m_2}{m_1}$ (c) $\frac{m_2}{m_1}$ (d) $\mu\omega d^2$

(e) $\frac{1}{2}\mu\omega^2 d^2$, where μ is the reduced mass and ω the angular velocity.



11

SIMPLE HARMONIC MOTION

Chapter Contents

11.1 Introduction

A particle has oscillatory (vibrational) motion when it moves periodically about stable equilibrium position. The motion of a pendulum is oscillatory. A weight attached to a stretched spring, once it is released, starts oscillating.

When the particle is moved away from the equilibrium position and released, a force comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side and is again pulled back toward equilibrium.

Of all the oscillatory motions, the most important is called **simple harmonic motion (SHM).** In this type of oscillatory motion, displacement velocity, acceleration and force all vary (w.r.t. time) in a way that can be described by either the sine or the cosine function collectively called **sinusoids.** Any oscillatory motion that cannot be described so simply is called **anharmonic oscillation.**

Besides being the simplest motion to describe and analyze, it constitutes a rather accurate description of many oscillations found in nature.

Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents and light.

11.2 The Causes of Oscillation

Consider a particle free to move on x-axis, is being acted upon by a force given by,

$$F = -kx^n$$

Here, k is a positive constant.

Now, following cases are possible depending on the value of n.

(i) If n is an even integer (0, 2, 4, ... etc), force is always along negative x-axis, whether x is positive or negative. Hence, the motion of the particle is not oscillatory. If the particle is released from any position on the x-axis (except at $x = 0$) a force in negative direction of x-axis acts on it and it moves rectilinearly along negative x-axis.

(ii) If n is an odd integer (1, 3, 5, ... etc), force is along negative x-axis for $x > 0$, along positive x-axis for $x < 0$ and zero for $x = 0$. Thus, the particle will oscillate about stable equilibrium position, $x = 0$. The force in this case is called the restoring force. Of these, if $n = 1$, *i.e.*, $\boldsymbol{F = -kx}$ the motion is said to be SHM.

Sample Example 11.1 *Describe the motion of a particle acted upon by a force*

(i) $F = -2(x-2)^3$

(ii) $F = -2(x-2)^2$

(iii) $F = -2(x-2)$

Solution (i) $F = -2(x-2)^3$

$$F = 0 \quad \text{at} \quad x = 2$$

Force is along negative x-direction for $x > 2$

and it is along positive x-direction for $x < 2$. Thus, the motion of the particle is oscillatory (but not simple harmonic) about $x = 2$.

(ii) $F = 0$ for $x = 2$, but force is always along negative x-direction for any value of x except at $x = 2$. Thus, the motion of the particle is rectilinear along negative x-direction.

(iii) Let, us take $x-2=X$, then the given force can be written as,

$$F=-2X$$

This is the equation of SHM. Hence, the particle oscillates simple harmonically about $X=0$ or $x=2$.

11.3 Kinematics of Simple Harmonic Motion

A particle has simple harmonic motion along an axis OX when its displacement x relative to the origin of the co-ordinate system is given as a function of time by the relation,

$$x=A\cos(\omega t+\phi)$$

The quantity $(\omega t+\phi)$ is called the **phase angle** or simply the phase of the SHM and ϕ is the initial phase, *i.e.*, the phase at $t=0$. Although, we have defined SHM in terms of a cosine function, it may just as well be expressed in terms of a sine function. The only difference between the two forms is an initial phase difference of $\frac{\pi}{2}$. Since, the cosine (or sine) function varies between a value of -1 and $+1$, the displacement of the particle varies between $x=-A$ and $x=+A$. The maximum displacement from the origin A, is the amplitude of the SHM. The cosine or sine function repeats itself every time the angle ωt increases by 2π. Thus, the displacement of the particle repeats itself after a time interval of $\frac{2\pi}{\omega}$. Therefore, SHM is periodic, and its **period** is,

$$T=\frac{2\pi}{\omega}$$

The **frequency** ν of a SHM is equal to the number of complete oscillations per unit time. Thus,

$$\nu=\frac{1}{T}=\frac{\omega}{2\pi}$$

and is measured in hertz. The quantity ω, called the angular frequency of the oscillating particle is related to the frequency by the relation similar to the equation for a circular motion, namely

$$\omega=\frac{2\pi}{T}=2\pi\nu$$

The velocity of the particle is

$$v=\frac{dx}{dt}=-\omega A\sin(\omega t+\phi)$$

which varies periodically between the values $+\omega A$ and $-\omega A$. Similarly, the acceleration is given by,

$$a=\frac{dv}{dt}=-\omega^2 A\cos(\omega t+\phi)=-\omega^2 x$$

and therefore, varies periodically between the values $+\omega^2 A$ and $-\omega^2 A$. This expression also indicates that,

In SHM the acceleration is proportional and opposite to the displacement.

In figure x, v and a as functions of time are illustrated.

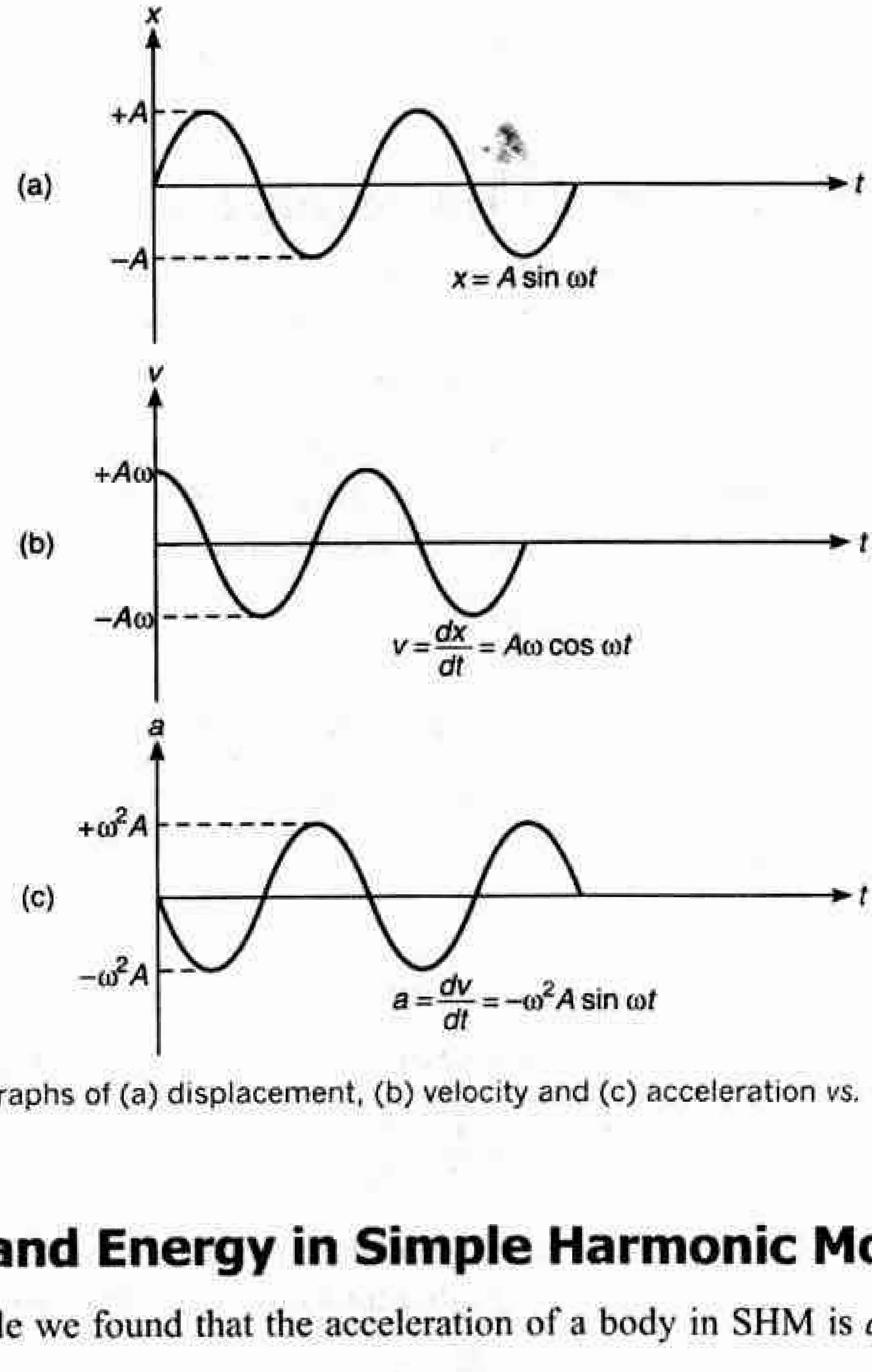


Fig. 11.1 Graphs of (a) displacement, (b) velocity and (c) acceleration *vs.* time in SHM.

11.4 Force and Energy in Simple Harmonic Motion

In the above article we found that the acceleration of a body in SHM is $a = -\omega^2 x$. Applying the equation of motion $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, we have

$$F = -m\omega^2 x = -kx$$

where,

$$\omega = \sqrt{\frac{k}{m}}$$

Thus, **in SHM the force is proportional and opposite to the displacement.**

That is when the displacement is to the right (positive) the force points to the left, and when the displacement is to the left (negative) the force points to the right. Thus, the force is always pointing toward the origin O. Such type of force appears when an elastic body such as a spring is deformed. That is why the constant $k = m\omega^2$ is sometimes called the **elastic constant.** Further, since

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and,} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Kinetic Energy

The kinetic energy of the particle is,

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m A^2 \sin^2(\omega t + \phi)$$

Since, $\sin^2\theta = 1 - \cos^2\theta$

and using $x = A\cos(\omega t + \phi)$ for the displacement, we can also express the kinetic energy as,

$$K = \frac{1}{2}m\omega^2 A^2 [1 - \cos^2(\omega t + \phi)]$$

which can be written as,

$$K = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

From this expression we can see that, the kinetic is maximum at the centre ($x = 0$) and zero at the extremes of oscillation ($x = \pm A$).

Potential Energy

To obtain the potential energy we use the relation,

$$F = -\frac{dU}{dx} \quad \text{or} \quad \frac{dU}{dx} = kx \qquad (\text{as } F = -kx)$$

$$\therefore \qquad \int_0^U dU = \int_0^x kx\,dx$$

$$\therefore \qquad U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

Thus, the potential energy has a minimum value at the centre ($x = 0$) and increases as the particle approaches either extreme of the oscillation ($x = \pm A$).

Total Energy

Total energy can be obtained by adding potential and kinetic energies. Therefore,

$$E = K + U = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$$

$$= \frac{1}{2}m\omega^2 A^2$$

or $$E = \frac{1}{2} kA^2 \qquad (\text{as } m\omega^2 = k)$$

Which is a **constant** quantity. This was to be expected since the force is conservative.

Therefore, we may conclude that, during an oscillation, there is a continuous exchange of kinetic and potential energies. While moving away from the equilibirium position, the potential energy increases at the expense of the kinetic energy. When the particle moves towards the equilibrium position, the reverse happens.

Figure shows the variation of total energy (E), potential energy (U) and kinetic energy (K) with displacement (x).

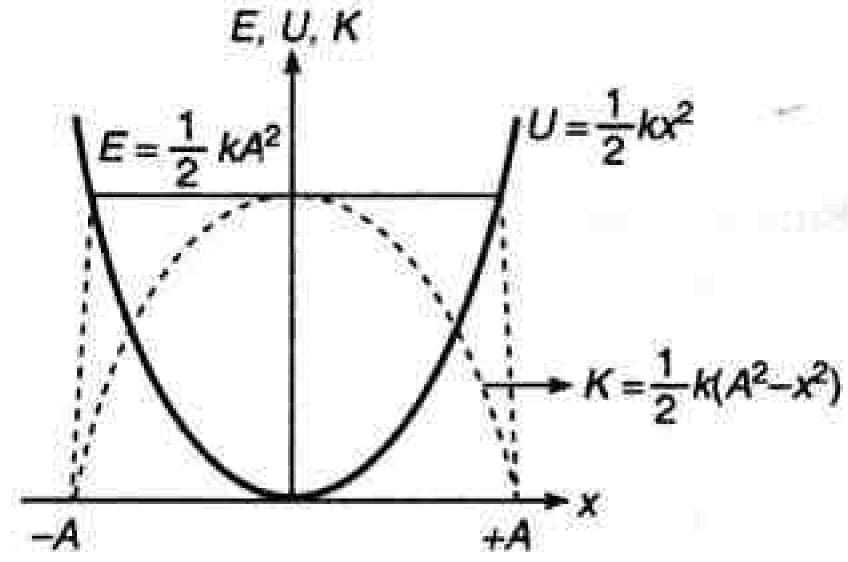


Fig. 11.2

• *Important Points* in SHM

1. In SHM, $F = -kx$ or $a = -\omega^2 x$, *i.e.*, F–x graph or a–x graph is a straight line passing through origin with negative slope. The corresponding graphs are shown below.

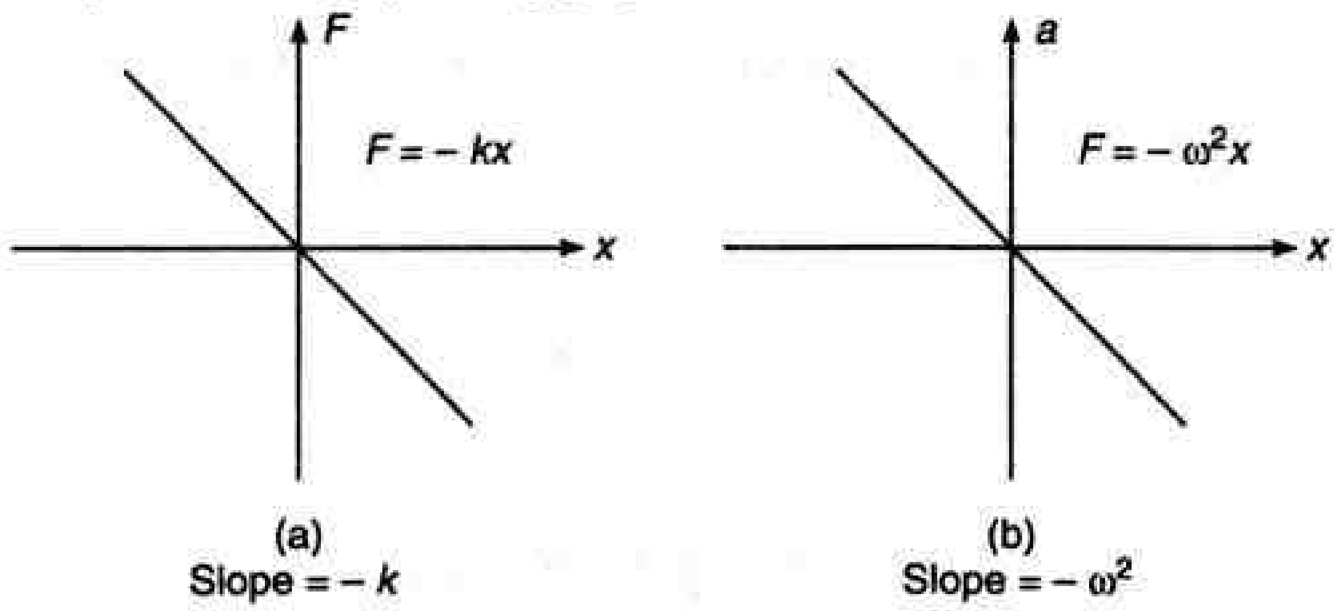


Fig. 11.3

2. Any function of t, say $y = y(t)$ oscillates simple harmonically if

$$\frac{d^2y}{dt^2} \propto -y$$

or we can say if above condition is satisfied, y will oscillate simple harmonically.

3. All sine and cosine functions of t are simple harmonic in nature. *i.e.*, for the function

$$y = A \sin(\omega t \pm \phi)$$

or $$y = A \cos(\omega t \pm \phi)$$

$\frac{d^2y}{dt^2}$ is directly proportional to $-y$. Hence, they are simple harmonic in nature.

4. How the different physical quantities (*e.g.*, displacement, velocity, acceleration, kinetic energy, etc.) vary with time or displacement are listed below in tabular form.

Table 11.1

S. No.	Name of the equation	Expression of the equation	Remarks
1.	Displacement–time	$x = A\cos(\omega t + \phi)$	x varies between $+A$ and $-A$
2.	Velocity - time $\left(v = \frac{dx}{dt}\right)$	$v = -A\omega\sin(\omega t + \phi)$	v varies between $+A\omega$ and $-A\omega$
3.	Acceleration - time $\left(a = \frac{dv}{dt}\right)$	$a = -A\omega^2\cos(\omega t + \phi)$	a varies between $+A\omega^2$ and $-A\omega^2$
4.	Kinetic energy - time $\left(K = \frac{1}{2}mv^2\right)$	$K = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$	K varies between 0 and $\frac{1}{2}mA^2\omega^2$
5.	Potential energy - time $\left(U = \frac{1}{2}m\omega^2x^2\right)$	$K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$	U varies between $\frac{1}{2}mA^2\omega^2$ and 0
6.	Total energy - time $(E = K + U)$	$E = \frac{1}{2}m\omega^2A^2$	E is constant
7.	Velocity - displacement	$v = \omega\sqrt{A^2 - x^2}$	$v = 0$ at $x = \pm A$ and at $x = 0$ $v = \pm A\omega$
8.	Acceleration - displacement	$a = -\omega^2x$	$a = 0$ at $x = 0$ $a = \pm\omega^2A$ at $x = \mp A$
9.	Kinetic energy - displacement	$K = \frac{1}{2}m\omega^2(A^2 - x^2)$	$K = 0$ at $x = \pm A$ $K = \frac{1}{2}m\omega^2A^2$ at $x = 0$
10.	Potential energy - displacement	$U = \frac{1}{2}m\omega^2x^2$	$U = 0$ at $x = 0$ $U = \frac{1}{2}m\omega^2A^2$ at $x = \pm A$
11.	Total energy - displacement	$E = \frac{1}{2}m\omega^2A^2$	E is constant

5. From the above table we see that x, v and a are sine or cosine functions of time. So, they all oscillate simple harmonically with same angular frequency ω. Phase difference between x and a is π and between any other two is $\frac{\pi}{2}$.

6. Kinetic energy versus time equation can also be written as K

$$= \frac{1}{4}mA^2\omega^2[1 - \cos 2(\omega t + \phi)]$$

This function is also periodic with angular frequency 2ω. Thus, kinetic energy in SHM is also periodic with double the frequency then that of x, v and a. But these oscillations are not simple harmonic in nature, because $\frac{d^2K}{dt^2}$ is not proportional to $-K$. But,

$$K - \frac{1}{4}mA^2\omega^2 = -\frac{1}{4}mA^2\omega^2\cos 2(\omega t + \phi)$$

$$= K_0 \text{ (say)}$$

Here, K_0 is simply a cosine function of time. So, K_0 will oscillate simple hormonically with angular frequency 2ω.

Same is the case with potential energy function. U also oscillate with angular frequency 2ω but the oscillations are not simple harmonic in nature. Total energy does not oscillate. It is constant.

Thus,

$x \rightarrow$ oscillate simple harmonically with angular frequency ω
$v \rightarrow$ oscillate simple harmonically with angular frequency ω
$a \rightarrow$ oscillate simple harmonically with angular frequency ω
$K \rightarrow$ oscillate with angular frequency 2ω but not simple harmonically
$U \rightarrow$ oscillate with angular frequency 2ω but not simple harmonically
$E \rightarrow$ does not oscillate.

7. In the above discussion we have read that potential energy is zero at mean position and maximum at extreme positions and kinetic energy is maximum at mean position and zero at extreme positions. But the correct statement is like this,

At mean position $\rightarrow$ K is maximum and U is minimum (it may be zero also, but it is not necessarily zero).

at extreme positions $\rightarrow$ K is zero and U is maximum.

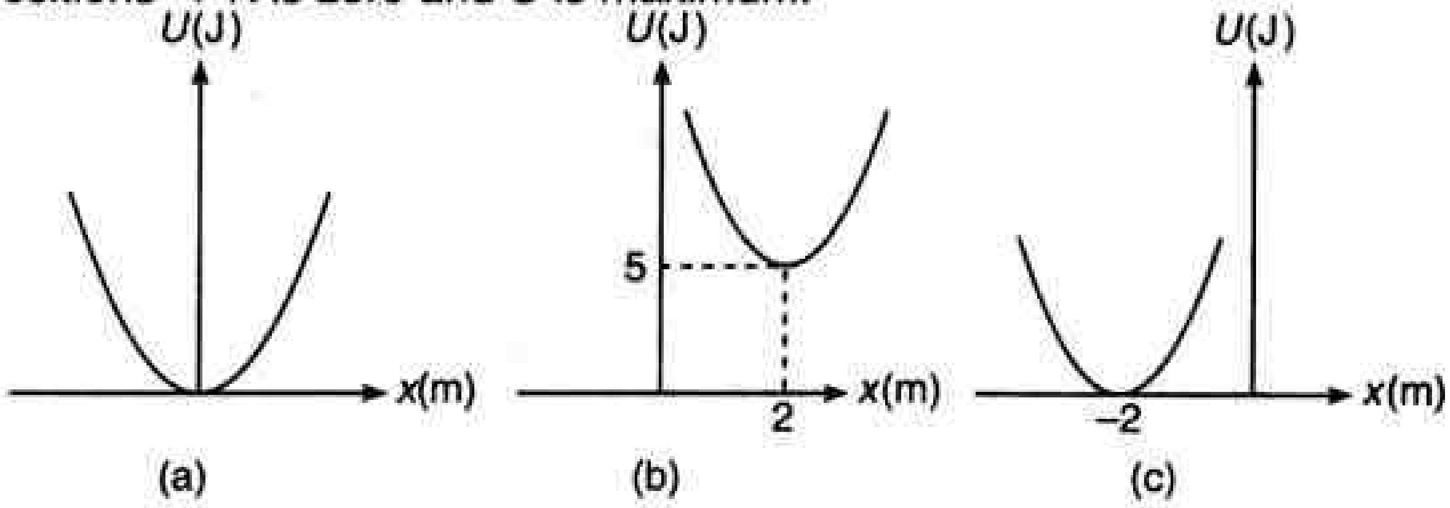


Fig. 11.4

Thus, in figure (a), oscillations will take place about the mean position $x = 0$ and minimum potential energy at mean position is zero.

In figure (b) mean position is at $x = 2$ m and the minimum potential energy in this position is 5 J.

In figure (c) mean position is at $x = -2$ m and the minimum potential energy in this position is again zero.

8. A function $f(t)$ is said to be periodic of time period T if

$$f(t + T) = f(t)$$

All sine or cosine functions of time are periodic. Thus,

$Y = A \sin \omega t$ or $A \cos \omega t$ is periodic, of time period $T = \dfrac{2\pi}{\omega}$.

Sample Example 11.2 *Find the period of the function,*

$$y = \sin \omega t + \sin 2\omega t + \sin 3\omega t$$

Solution The given function can be written as,

$$y = y_1 + y_2 + y_3$$

Here, $$y_1 = \sin \omega t, \quad T_1 = \frac{2\pi}{\omega}$$

$$y_2 = \sin 2\omega t, \qquad T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

and $$y_3 = \sin 3\omega t, \qquad T_3 = \frac{2\pi}{3\omega}$$

$\therefore$ $$T_1 = 2T_2 \quad \text{and} \quad T_1 = 3T_3$$

So, the time period of the given function is T_1 or $\frac{2\pi}{\omega}$. **Ans.**

Because in time $T = \frac{2\pi}{\omega}$, first function completes one oscillation, the second function two oscillations and the third, three.

Sample Example 11.3 *A linear harmonic oscillator has a total mechanical energy of 200 J. Potential energy of it at mean position is 50 J. Find :*

(i) the maximum kinetic energy,
(ii) the minimum potential energy,
(iii) the potential energy at extreme positions.

Solution At mean position, potential energy is minimum and kinetic energy is maximum. Hence,

$$U_{min} = 50 \text{ J} \qquad \text{(at mean position)}$$

and $$K_{max} = E - U_{min} = 200 - 50$$
$$= 150 \text{ J} \qquad \text{(at mean position)}$$

At extreme positions, kinetic energy is zero and potential energy is maximum

$\therefore$ $$U_{max} = E$$
$$= 200 \text{ J} \qquad \text{(at extreme position)}$$

Sample Example 11.4 *The potential energy of a particle oscillating on x-axis is given as*

$$U = 20 + (x-2)^2$$

Here, U is in joules and x in metres. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.
(b) Find the mean position.
(c) Find the maximum kinetic energy of the particle.

Solution (a) $$F = -\frac{dU}{dx} = -2(x-2)$$

By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$

The motion of the particle is simple harmonic

(b) The mean position of the particle is $X = 0$ or $x - 2 = 0$, which gives $x = 2$ m

(c) Maximum kinetic energy of the particle is,

$$K_{max} = E - U_{min} = 36 - 20$$
$$= 16 \text{ J}$$

Note *U_{min} is 20 J at mean position or at $x = 2$ m.*

Introductory Exercise 11.1

1. A particle moves under the force $F(x) = (x^2 - 6x)$ N, where x is in metres. For small displacements from the origin what is the force constant in the simple harmonic motion approximation?
2. At $x = \frac{A}{2}$, what fraction of the mechanical energy is potential? What fraction is kinetic? Assume potential energy to be zero at mean position.
3. The initial position and velocity of a body moving in SHM with period $T = 0.25$ s are $x = 5.0$ cm and $v = 218$ cm/s. What are the amplitude and phase constant of the motion?
4. A cart of mass 2.00 kg is attached to the end of a horizontal spring with force constant $k = 150$ N/m. The cart is displaced 15.0 cm from its equilibrium position and released. What are
   (a) the amplitude (b) the period (c) the frequency (d) the mechanical energy (e) the maximum velocity of the cart? Neglect friction.
5. A body of mass 0.10 kg is attached to a vertical massless spring with force constant 4.0×10^3 N/m. The body is displaced 10.0 cm from its equilibrium position and released. How much time elapses as the body moves from a point 8.0 cm on one side of the equilibrium position to a point 6.0 cm on the same side of the equilibrium position?
6. A 2.0 kg particle undergoes SHM according to $x = 1.5 \sin\left(\frac{\pi t}{4} + \frac{\pi}{6}\right)$ (in SI units)
   (a) What is the total mechanical energy of the particle?
   (b) What is the shortest time required for the particle to move from $x = 0.5$ m to $x = -0.75$ m?
7. A body of mass 200 g is in equilibrium at $x = 0$ under the influence of a force $F(x) = (-100x + 10x^2)$ N
   (a) If the body is displaced a small distance from equilibrium, what is the period of its oscillations?
   (b) If the amplitude is 4.0 cm, by how much do we error in assuming that $F(x) = -kx$ at the end points of the motion.

11.5 Relation Between Simple Harmonic Motion and Uniform Circular Motion

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P oscillates betwen B and C. The angle that the radius OQ makes with the x-axis is, $\theta = \omega t + \phi$. Here, ϕ is the angle made by the radius OQ with the x-axis at time $t = 0$. Further,

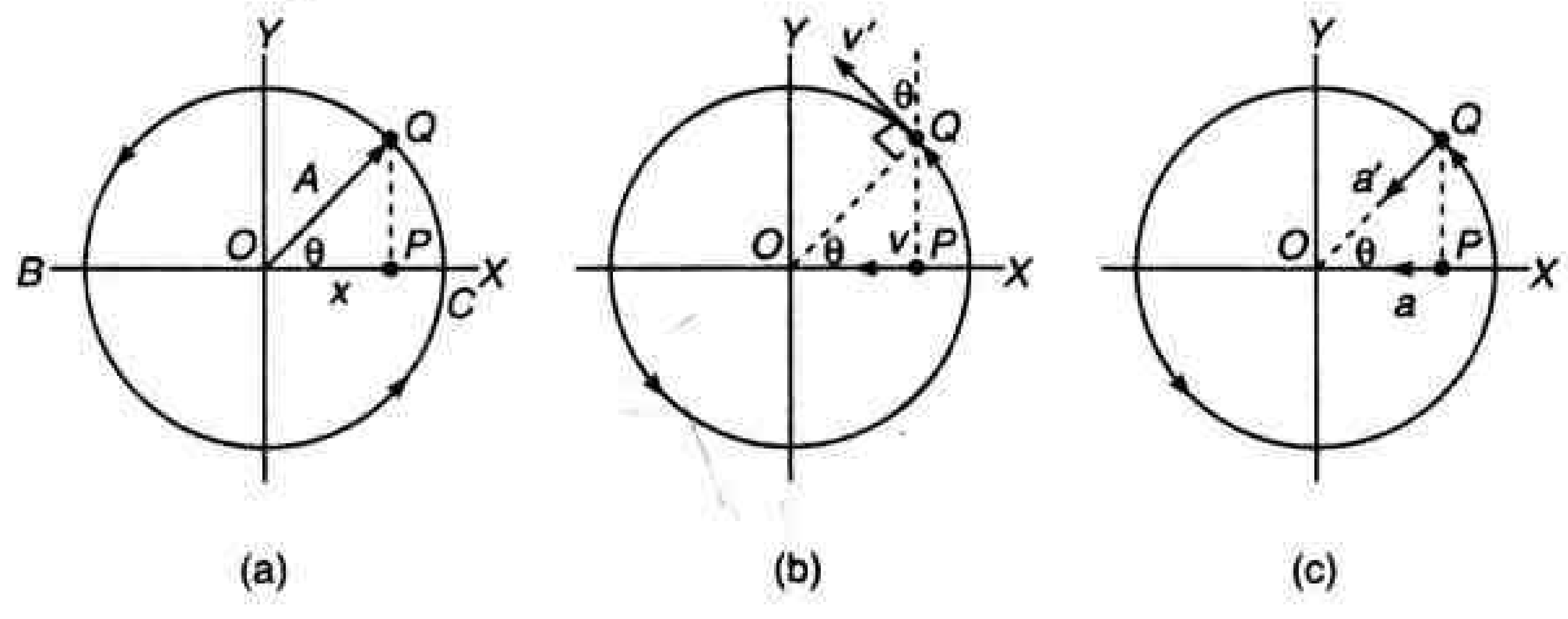


Fig. 11.5 Relation between SHM and uniform circular motion. (a) Position, (b) velocity and (c) acceleration

$$OP = OQ \cos\theta$$

or $$x = A\cos(\omega t + \phi)$$

In other words, P moves with SHM. That is :

When a particle moves with uniform circular motion, its projection on a diameter moves with SHM.

The velocity of Q is perpendicular to OQ and has a magnitude of velocity $v' = \omega A$. The component of v' along the x-axis is,

$$v = -v' \sin\theta$$

or $$v = -\omega A \sin(\omega t + \phi)$$

which is also the velocity of P. The acceleration of Q is centripetal and has a magnitude, $a' = \omega^2 A$.

The component of a' along the x-axis is

$$a = -a' \cos\theta$$

or $$a = -\omega^2 A \cos(\omega t + \phi)$$

Which again coincides with the acceleration of P.

11.6 Basic Differential Equation of SHM

The force required to produce SHM is of nature,

$$F = -kx$$

From the equation of motion $F = ma$ and remembering that in rectilinear motion $a = \dfrac{d^2x}{dt^2}$ we may write the above equation as,

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0$$

Setting, $$\omega^2 = \frac{k}{m}, \quad \text{we have}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{...(i)}$$

This is an equation which relates displacement with acceleration. The solution of this equation is cosine or sine functions of ωt. For instance, substituting $x = A\cos(\omega t + \phi)$ and its second time derivative in the above equation, the equation is satisfied, regardless of the values of A and ϕ. Thus,

$$x = A\cos(\omega t + \phi)$$

is a general solution of Eq. (i). Similarly, by the same process, we can see that

$$x = A\sin(\omega t + \phi) \quad \text{and} \quad x = A\sin\omega t + B\cos\omega t$$

are also general solutions of Eq. (i).

Note *Eq. (i) appears in many different situations in physics. Whenever it is found, it indicates that the corresponding phenomenon is ocillatory according to the law,*

$$x = A\cos(\omega t + \phi)$$

or one of the other general solutions. The phenomenon is oscillatory whether x describes a linear or angular displacement of a particle, a current in an electric circuit or any other physical situations.

11.7 Method of Finding Time Period of a Simple Harmonic Motion

There are basically following two methods of finding time period of a SHM. These are the restoring force or torque method and the energy method.

Restoring Force or Torque Method

The following steps are usually followed in this method.

Step 1

Find the stable equilibrium position which usually is also known as the mean position. Net force or torque on the particle in this position is zero. Potential energy is minimum.

Step 2

Displace the particle from its mean position by a small displacement x (in case of a linear SHM) or θ (in case of an angular SHM).

Step 3

Find net force or torque in this displaced position.

Step 4

Show that this force or torque has a tendency to bring the particle back to its mean position and magnitude of force or torque is proportional to displacement, *i.e.*,

$$F \propto -x \quad \text{or} \quad F = -kx \qquad \text{...(i)}$$

or

$$\tau \propto -\theta \quad \text{or} \quad \tau = -k\theta \qquad \text{...(ii)}$$

This force or torque is also known as restoring force or restoring torque.

Step 5

Find linear acceleration by dividing Eq. (i) by mass m or angular acceleration by dividing Eq. (ii) by moment of inertia I. Hence,

$$a = -\frac{k}{m}.x = -\omega^2 x \quad \text{or} \quad \alpha = -\frac{k}{I}\theta = -\omega^2\theta$$

Step 6

Finally,

$$\omega = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

or

$$\frac{2\pi}{T} = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$\therefore$

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

Energy Method

Repeat step 1 and step 2 as in method 1. Find the total mechanical energy (E) in the displaced position. Since, mechanical energy in SHM remains constant.

$$\frac{dE}{dt}=0$$

By differentiating the energy equation with respect to time and substituting $\frac{dx}{dt}=v, \frac{d\theta}{dt}=\omega, \frac{dv}{dt}=a,$ and $\frac{d\omega}{dt}=\alpha$ we come to step 5. The remaining procedure is same.

Note (*i*) *E usually consists of following terms :*
(a) Gravitational PE (b) Elastic PE (c) Electrostatic PE (d) Rotational KE and (e) Translational KE
(*ii*) *For gravitational PE, choose the reference point* ($h=0$) *at mean position.*

Now, let us take few examples of finding time period (T) of certain simple harmonic motions.

The simple pendulum

An example of SHM is the motion of a pendulum. A simple pendulum is defined as a particle of mass m suspended from a point O by a string of length l and of negligible mass.

When the particle is pulled aside to position B, so that the string makes an angle θ_0 with the vertical OC and then released, the pendulum will oscillate between B and the symmetric position B'. The oscillatory motion is due to the tangential component F_T of the weight mg of the particle. This force F_T is maximum at B and B', and zero at C. Thus, we can write,

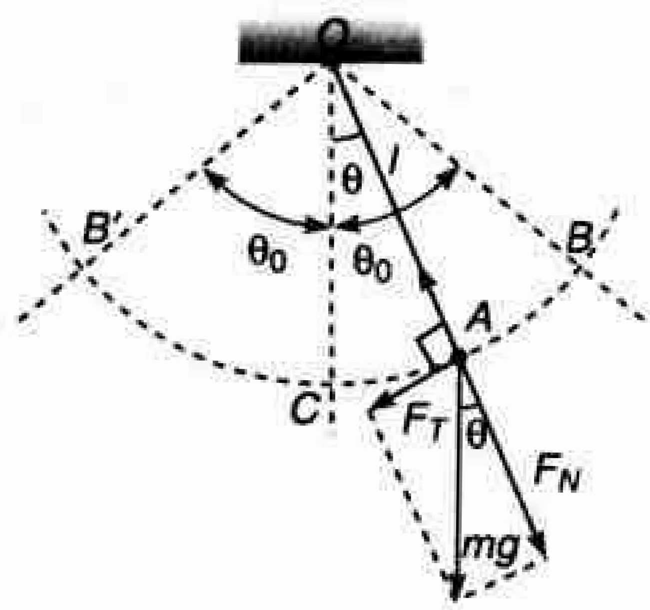


Fig. 11.6

$$F_T=-mg\sin\theta$$

Here, minus sign appears because it is opposite to the displacement.

$$x=CA$$

$\therefore$ $$ma_T=-mg\sin\theta \quad \text{...(i)}$$

Here, $$a_T=l\alpha \quad \left(\text{where, } \alpha=\frac{d^2\theta}{dt^2}\right)$$

and $$\sin\theta\approx\theta \quad \text{for small oscillations}$$

$\therefore$ $$ml\alpha=-mg\theta$$

or $$\alpha=-\left(\frac{g}{l}\right)\theta$$

or $$\left|\frac{\theta}{\alpha}\right|=\frac{l}{g}$$

$\therefore$ $$T=2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

or $$T = 2\pi\sqrt{\frac{l}{g}}$$

Note that the period is independent of the mass of the pendulum.

Energy Method

Let us derive the same expression by energy method. Suppose ω be the angular velocity of particle at angular displacement θ about point O. Then, total mechanical energy of particle in position A is,

$$E = \frac{1}{2} I\omega^2 + mg(h_A - h_C)$$

or $$E = \frac{1}{2}(ml^2)\omega^2 + mgl(1 - \cos\theta)$$

E is constant, therefore, $$\frac{dE}{dt} = 0$$

or $$0 = ml^2\omega\left(\frac{d\omega}{dt}\right) + mgl\sin\theta\left(\frac{d\theta}{dt}\right)$$

Putting $\frac{d\theta}{dt} = \omega$, $\frac{d\omega}{dt} = \alpha$ and $\sin\theta \approx \theta$, we get the same expression viz.

$$\alpha = -\left(\frac{g}{l}\right)\theta$$

$\therefore$ $$T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

or $$T = 2\pi\sqrt{\frac{l}{g}}$$

Following points should be remembered in case of a simple pendulum.

(1) For large amplitudes the approximation $\sin\theta \approx \theta$ is not valid and the calculation of the period is more complex.
The time period in this case depends on the amplitude θ_0 and is given by,

$$T = 2\pi\sqrt{\frac{l}{g}\left(1 + \frac{\theta_0^2}{16}\right)}$$

Here the amplitude θ_0 must be expressed in radians. This is sufficient approximation for most practical situations.

(2) If the time period of a simple pendulum is 2 seconds, it is called **seconds pendulum.**

(3) If length of the pendulum is large, g no longer remain vertical but will be directed towards the centre of the earth and expression for time period is given by,

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l} + \frac{1}{R}\right)}}$$

Here, R is the radius of earth. From this expression we can see that,

(a) if $l << R, \frac{1}{l} >> \frac{1}{R}$ and $T = 2\pi\sqrt{\frac{l}{g}}$

(b) as $l \to \infty, \frac{1}{l} \to 0$ and $T = 2\pi\sqrt{\frac{R}{g}}$ and substituting the value of R and g, we get $T = 84.6$ minutes.

(4) Time period of a simple pendulum depends on acceleration due to gravity 'g' $\left(\text{as } T \propto \frac{1}{\sqrt{g}}\right)$ so take $|\vec{g}_{eff}|$ in $T = 2\pi\sqrt{\frac{l}{g}}$. Following two cases are possible :

(i) If a simple pendulum is in a carriage which is accelerating with acceleration $\vec{a}$, then

$$\vec{g}_{\text{eff}} = \vec{g} - \vec{a}$$

e.g., if the acceleration $\vec{a}$ is upwards, then

$$|\vec{g}_{\text{eff}}| = g + a \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{l}{g+a}}$$

If the acceleration $\vec{a}$ is downwards, then $(g > a)$

$$|\vec{g}_{\text{eff}}| = g - a \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{l}{g-a}}$$

If the acceleration $\vec{a}$ is in horozontal direction, then

$$|\vec{g}_{\text{eff}}| = \sqrt{a^2 + g^2}$$

In a freely falling lift $g_{eff} = 0$ and $T = \infty$, *i.e.*, the pendulum will not oscillate.

(ii) If in addition to gravity one additional force $\vec{F}$, (*e.g.*, electrostatic force $\vec{F}_e$) is also acting on the bob, then in that case,

$$\vec{g}_{\text{eff}} = \vec{g} + \frac{\vec{F}}{m}$$

Here, m is the mass of the bob.

(5) Due to change in temperature, length of pendulum and so the time period will change. If $\Delta\theta$ is the increase in temperature then,

$$l' = l(1 + \alpha\Delta\theta) \qquad \text{or} \qquad \frac{l'}{l} = 1 + \alpha\Delta\theta$$

$$\therefore \qquad \frac{T'}{T} = \sqrt{\frac{l'}{l}} = (1 + \alpha\Delta\theta)^{1/2}$$

$$\approx \left(1 + \frac{1}{2}\alpha\Delta\theta\right)$$

$$\therefore \qquad \frac{T'}{T} - 1 \approx \frac{1}{2}\alpha\Delta\theta$$

or
$$\frac{T' - T}{T} = \frac{1}{2}\alpha\Delta\theta$$

or
$$\Delta T = \frac{1}{2} T\alpha\Delta\theta$$

Note *In case of a pendulum clock, time is lost if T increases and gained if T decreases. Time lost or gained in time t is given by,*

$$\Delta t = \frac{\Delta T}{T'} . t$$

e.g., if $T = 2$ s, $T' = 3$ s, *then* $\Delta T = 1$ s

$\therefore$ *Time lost by the clock in 1 hr.* $\Delta t = \frac{1}{3} \times 3600 = 1200$ s

Sample Example 11.5 *A simple pendulum of length l is suspended from the ceiling of a cart which is sliding without friction on an inclined plane of inclination θ. What will be the time period of the pendulum?*

Solution Here, point of suspension has an acceleration. $\vec{a} = g \sin\theta$ (down the plane). Further, $\vec{g}$ can be resolved into two components $g \sin\theta$ (along the plane) and $g \cos\theta$ (perpendicular to plane).

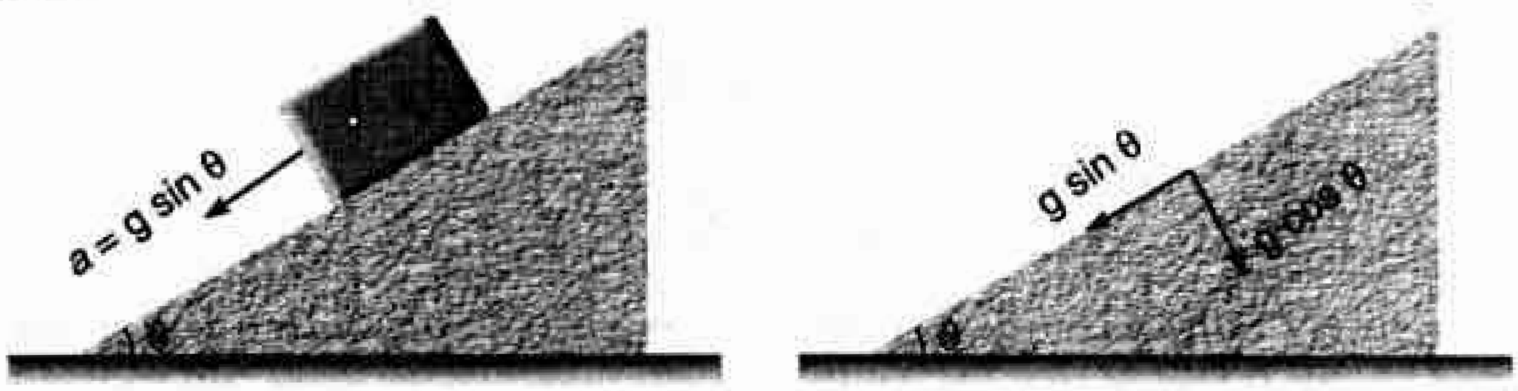


Fig. 11.7

$$\therefore \qquad \vec{g}_{eff} = \vec{g} - \vec{a}$$

$$= g\cos\theta \qquad \text{(perpendicular to plane)}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{l}{|\vec{g}_{eff}|}}$$

$$= 2\pi\sqrt{\frac{l}{g\cos\theta}} \qquad \textbf{Ans.}$$

Note *If* $\theta = 0°$, $T = 2\pi\sqrt{\frac{l}{g}}$ *which is quiet obvious.*

Sample Example 11. 6 *A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will pendulum oscillate if the electrostatic force acting on the sphere is less than the grapvitational force?*

Solution The two forces acting on the bob are shown in figure

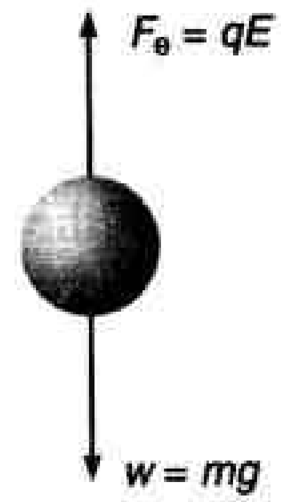


Fig. 11.8

g_{eff} in this case will be $\dfrac{w - F_e}{m}$

or
$$g_{\text{eff}} = \frac{mg - qE}{m} = g - \frac{qE}{m}$$

$\therefore$
$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$
$$= 2\pi\sqrt{\frac{l}{g - \dfrac{qE}{m}}} \qquad \textbf{Ans.}$$

Introductory Exercise 11.2

1. A simple pendulum of length l and mass m is suspended in a car that is moving with constant speed v around a circle of radius r. Find the period of oscillation and equilibrium position of the pendulum.
2. Find the period of oscillation of a pendulum of length l if its point of suspension is.
   (a) moving vertically up with acceleration a.
   (b) moving vertically down with acceleration a ($< g$).
   (c) falling freely under gravity
   (d) moving horizontally with acceleration a.
3. A clock with an iron pendulum keeps correct time at 20°C. How much time will it lose or gain in a day if the termprature changes to 40°C. Thermal coefficient of linear expansion $\alpha = 0.000012$ per °C.
4. A simple pendulum with a solid metal bob has a period T. What will be the period of the same pendulum if it is made to oscillate in a non-viscous liquid of density one-tenth of the metal of the bob?

Spring-block System

Suppose a mass m is attached to the free end of a massless spring of spring constant k, with its other end fixed to a rigid support.

If the mass be displaced through a distance x, as shown, a linear restoring force,
$$F = -kx \qquad \text{...(i)}$$
starts acting on the mass, tending to bring it back into its original position. The negative sign simply indicates that it is directed oppositely to the displacement of the mass.

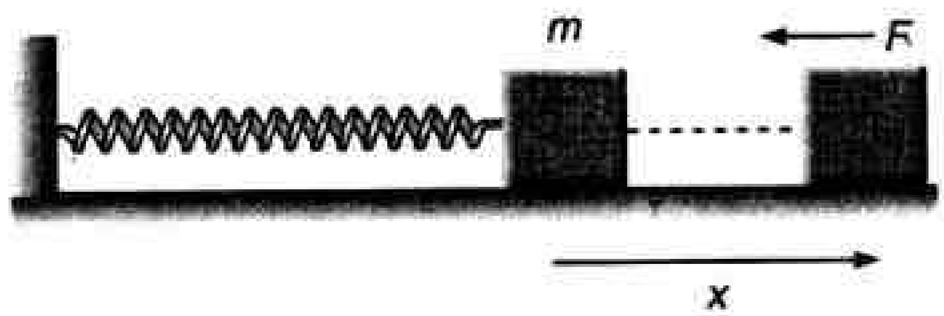


Fig. 11.9

Eq. (i) can be written as,
$$ma = -kx \qquad \text{...(ii)}$$
or
$$\left|\frac{x}{a}\right| = \frac{m}{k}$$

$\therefore$ $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or $$T = 2\pi\sqrt{\frac{m}{k}}$$

Energy Method

The time period of the spring-block system can also be obtained by the energy method. Let v be the speed of the mass in displaced position. Then total mechanical energy of the spring-block system is.

$$E = \text{ kinetic energy of the block + elastic potential energy}$$

or $$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Since, E is constant

$$\frac{dE}{dt} = 0 \quad \text{or,} \quad 0 = mv\left(\frac{dv}{dt}\right) + kx\left(\frac{dx}{dt}\right)$$

Substituting, $$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = a$$

We have, $$ma = -kx$$

$\therefore$ $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\frac{m}{k}}$$

Following points are important for a spring block system.

(i) Although we have considered a horizontal system. But the same is true for vertical system also.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

(ii) In case of a vertical spring-block system, time period can also be written as,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

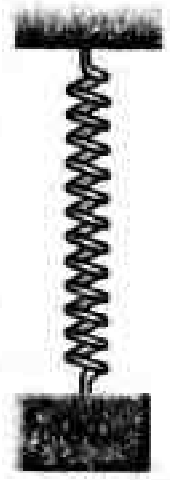


Fig. 11.10

Here, l = extension in the spring when the mass m is suspended from the spring.

This can be seen as under :

$$kl = mg \quad \text{(in equilibrium position)}$$

$\therefore$ $$\frac{m}{k} = \frac{l}{g}$$

or $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$

(iii) Equivalent force constant (k) : If a spring pendulum is constructed by using two springs and a mass, the following three situations are possible.

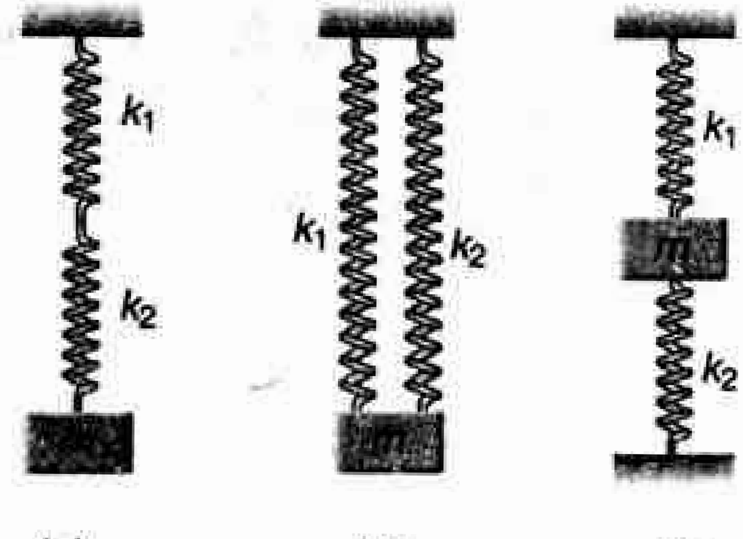


Fig. 11.11

Refer figure (a)

In this case,

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$$

or $$k=\frac{k_1 k_2}{k_1+k_2}$$

$\therefore$ $$T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$$

Refer figures (b) and (c)

In both the cases,

$$k=k_1+k_2$$

or $$T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m}{k_1+k_2}}$$

(iv) If spring has a mass m_s and a mass m is suspended from it, then time period is given by,

$$T=2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$$

Note *The derivation of it can be seen in miscellaneous examples.*

(v) If two masses m_1 and m_2 are connected by a spring and made to oscillate on horizontal surface. Then time period is given by,

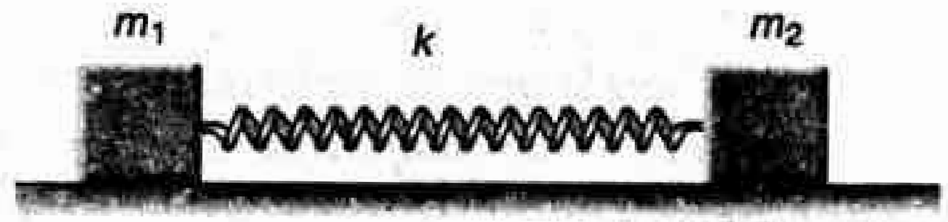


Fig. 11.12

$$T=2\pi\sqrt{\frac{\mu}{k}}$$

Here, $\mu=\text{reduced mass}=\dfrac{m_1 m_2}{m_1+m_2}$

(vi) The force constant (k) of a spring is inversely proportional to the length of the spring. *i.e.*,

$$k\propto\frac{1}{\text{length of spring}}$$

This can be visualized as under :

A spring of length l and spring constant k can be supposed to be made up by two springs in series, of length $\frac{l}{2}$ and force constant $2k$. In series,

$$k_{\text{eff}} = \frac{(2k)(2k)}{2k + 2k} = k$$

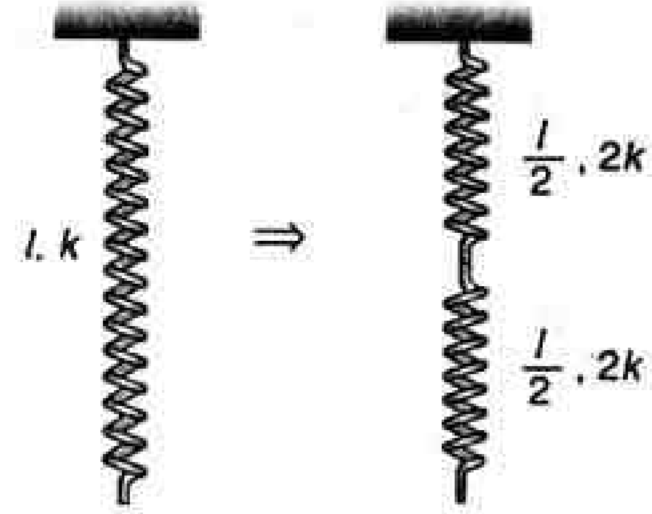


Fig. 11.13

Sample Example 11.7 *A block with a mass of 3.00 kg is suspended from an ideal spring having negligible mass and stretches the spring 0.2 m.*

(a) What is the force constant of the spring?

(b) What is the period of oscillation of the block if it is pulled down and released?

Solution (a) In equilibrium, $kl = mg$

$$\therefore \qquad k = \frac{mg}{l}$$

Substituting the proper values, we have $k = \frac{(3.00)(9.8)}{0.2} = 147$ N/m **Ans.**

(b)
$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$= 2\pi\sqrt{\frac{0.2}{9.8}} = 0.897\text{ s}$$ **Ans.**

Sample Example 11.8 *A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position a lump of putty with mass m is drropped vertically on the block from a very small height and sticks to it.*

(a) Find the new amplitude and period.

(b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

Solution (a) Before the lump of putty is dropped the total mechanical energy of the block and spring is

$$E_1 = \frac{1}{2}kA_1^2.$$

Since, the block is at the equilibrium position, $U = 0$, and the energy is purely kinetic. Let v_1 be the speed of the block at the equilibrium position, we have

$$E_1 = \frac{1}{2}Mv_1^2 = \frac{1}{2}kA_1^2$$

$$\therefore \qquad v_1 = \sqrt{\frac{k}{M}}A_1$$

During the process momentum of the system in horizontal direction is conserved. Let v_2 be the speed of the combined mass, then

$$(M+m)v_2 = Mv_1$$

$$\therefore \qquad v_2 = \frac{M}{M+m} v_1$$

Now, let A_2 be the amplitude afterwards. Then,

$$E_2 = \frac{1}{2} kA_2^2 = \frac{1}{2}(M+m)v_2^2$$

Substituting the proper values, we have

$$A_2 = A_1 \sqrt{\frac{M}{M+m}}$$ **Ans.**

Note *$E_2 < E_1$, as some energy is lost into heating up the block and putty.*

Further,

$$T_2 = 2\pi \sqrt{\frac{M+m}{k}}$$ **Ans.**

(b) When the putty drops on the block, the block is instantaneously at rest. All the mechanical energy is stored in the spring as potential energy. Again the momentum in horizontal direction is conserved during the process. But now it is zero just before and after putty is dropped. So, in this case, adding the extra mass of the putty has no effect on the mechanical energy, *i.e.*,

$$E_2 = E_1 = \frac{1}{2} kA_1^2$$

and the amplitude is still A_1. Thus, $\qquad A_2 = A_1$

and

$$T_2 = 2\pi \sqrt{\frac{M+m}{k}}$$ **Ans.**

Introductory Exercise 11.3

1. Find the period of oscillation of the system shown in Fig. 11.14.

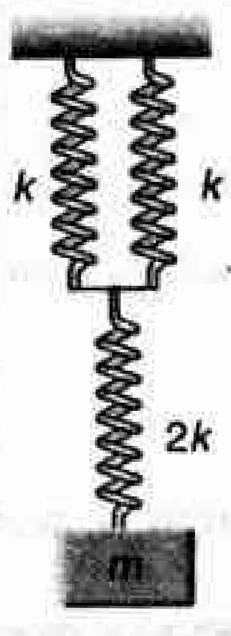


Fig. 11.14

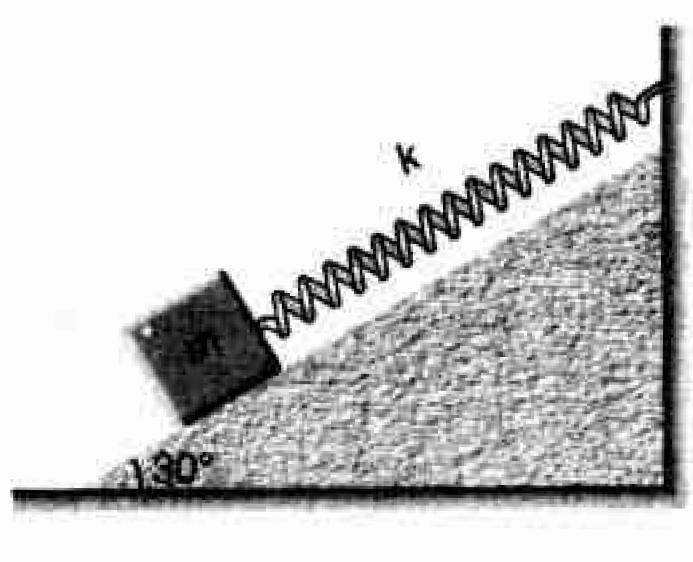


Fig. 11.15

2. A block of mass 0.2 kg is attached to a massless spring of force constant 80 N/m as shown in Fig. 11.15. Find the period of oscillation. Take $g = 10 \text{ m/s}^2$. Neglect friction.

3. A bullet of mass m strikes a block of mass M. The bullet remains embedded in the block. Find the amplitude of the resulting SHM.

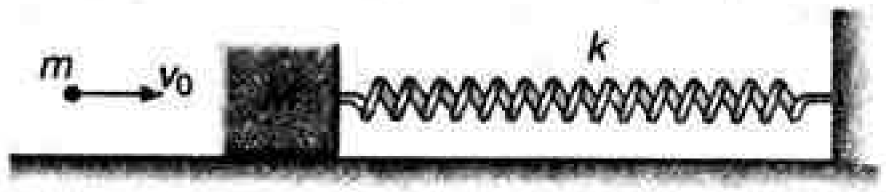


Fig. 11.16

4. A spring is cut into three equal pieces and connected as shown in problem number (1) of the same exercise. By what factor will the time period of oscillation change if a block is attached before and after?

The Physical Pendulum

The physical pendulum is just a rigid body, of whatever shape, capable of oscillating about a horizontal axis passing through it. For small oscillations the motion of a physical pendulum is almost as easy as for a simple pendulum. Figure shows a body of irregular shape pivoted at O so that it can oscillate without friction about an axis passing through O. In equilibrium the centre of gravity (G) in directly below O.

In the position shown in figure, the body is displaced from equilibrium by an angle θ. The distance from O to the centre of gravity is l. The moment of inertia of the body about the axis of rotation through O is I and the total mass is m. In the displaced position, the weight mg causes a restoring torque,

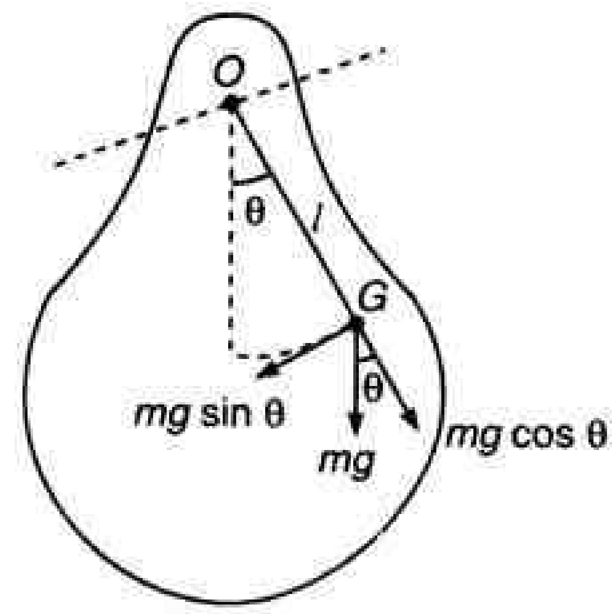


Fig. 11.17

$$\tau = -(mg)(l \sin \theta)$$

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise and *vice-versa*.

For small oscillations, $\sin \theta \approx \theta$ and $\Sigma\tau = I\alpha$

$\therefore$ $$-(mgl)\theta = I\alpha$$

As α is proportional to $-\theta$, the motion is simple harmonic, the time period of which is,

$$T = 2\pi \sqrt{\left|\frac{\theta}{\alpha}\right|}$$

or $$T = 2\pi \sqrt{\frac{I}{mgl}}$$

Following points are important regarding a physical pendulum.

(i) If I_G is the moment of inertia of the pendulum about an axis through G, parallel to the axis through O, we have, from the theorem of parallel axes,

$$I = I_G + ml^2$$

and if K is the radius of gyration of the pendulum about this axis through G, we have

$$I_G = mK^2$$

$\therefore$ $$I = mK^2 + ml^2 = m(K^2 + l^2)$$

Thus, the time period of the pendulum can be written as,

$$T = 2\pi\sqrt{\frac{\frac{(K^2+l^2)}{l}}{g}} = 2\pi\sqrt{\frac{l_{eff}}{g}}$$

Here, $l_{eff} = \frac{K^2+l^2}{l}$ is called the length of an equivalent simple pendulum.

Thus, the time period of a physical pendulum is the same as that of a simple pendulum of length,

$$l_{eff} = \frac{K^2+l^2}{l}$$

Since, K^2 is always greater than or equal to zero, the length of equivalent simple pendulum is always greater than or equal to l, the length of the compound pendulum.

(ii) A point O' on the other side of the centre of gravity G of the pendulum in a line with OG and at a distance $\frac{K^2}{l}$ from G is called the centre of oscillation of the pendulum. The time period of the pendulum is the same as that about O. Thus,

$$T_0 = T_0'$$

(iii) Time period T will be a maximum or minimum when $\frac{dT}{dl} = 0$, *i.e.*, when $l^2 = K^2$ or $l = K$.

Since, $\frac{d^2T}{dl^2}$ comes out to be positive hence, T is a minimum at $l = K$.

$$\therefore \qquad T_{min} = 2\pi\sqrt{\frac{K^2+K^2}{Kg}} = 2\pi\sqrt{\frac{2K}{g}}$$

On the other hand, we see from the expression for T that if $l = 0$, $T = \infty$ or a maximum. Thus, the time period of a compound pendulum is the maximum when its length is zero, *i.e.*, when the axis of suspension passes through its centre of gravity or the centre of gravity itself is the point of suspension. This is obvious because the pendulum is then in a state of neutral equilibrium, with no restoring torque due to gravity on it.

Sample Example 11.9 *A uniform circular disc of radius R oscillates in a vertical plane about a horizontal axis. Find the distance of the axis of rotation from the centre for which the period is minimum. What is the value of this period?*

Solution The time period of a compound pendulum is the minimum when its length is equal to the radius of gyration about its centre of gravity, *i.e.*, $l = K$.

Since, the moment of inertia of a disc about an axis perpendicular to its plane and passing through its centre is equal to,

$$I = MK^2 = \frac{1}{2}MR^2$$

$$\therefore \qquad K = \frac{R}{\sqrt{2}}$$

Thus, the disc will oscillate with the minimum time period when the distance of the axis of rotation from the centre is $\frac{R}{\sqrt{2}}$. **Ans.**

And the value of this minimum time period will be,

$$T_{min} = 2\pi \sqrt{\frac{\frac{2R}{\sqrt{2}}}{g}} = 2\pi \sqrt{\frac{\sqrt{2}R}{g}}$$

or

$$T_{min} \approx 2\pi \sqrt{\frac{1.414R}{g}}$$ **Ans.**

Sample Example 11.10 *Find the period of small oscillations of a uniform rod with length l, pivoted at one end.*

SOLUTION

$$T = 2\pi \sqrt{\frac{I_0}{mg(OG)}}$$

Here, $I_0 = \frac{1}{3} ml^2$ and $OG = \frac{l}{2}$

$\therefore$

$$T = 2\pi \sqrt{\frac{\left(\frac{1}{3} ml^2\right)}{(m)(g)\left(\frac{l}{2}\right)}}$$

or

$$T = 2\pi \sqrt{\frac{2l}{3g}}$$ **Ans.**

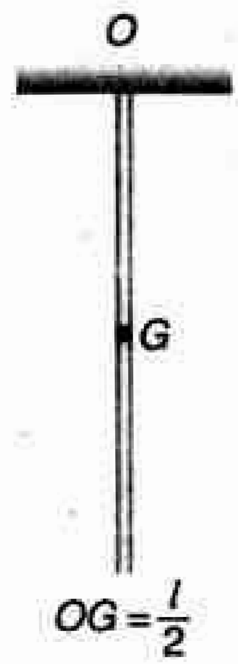


Fig. 11.18

Introductory Exercise 11.4

1. An annular ring of internal and outer radii r and R respectively oscillates in a vertical plane about a horizontal axis perpendicular to its plane and passing through a point on its outer edge. Calculate its time period and show that the length of an equivalent simple pendulum is $\frac{3R}{2}$ as $r \to 0$ and $2R$ as $r \to R$.
2. A body of mass 200 g oscillates about a horizontal axis at a distance of 20 cm from its centre of gravity. If the length of the equivalent simple pendulum is 35 cm, find its moment of inertia about the point of suspension.

Oscillations of a Fluid Column

Initially the level of liquid in both the columns is same. The area of cross-section of the tube is uniform. If the liquid is depressed by x in one limb, it will rise by x along the length of the tube is the other limb. Here, the restoring force is provided by the hydrostatic pressure difference.

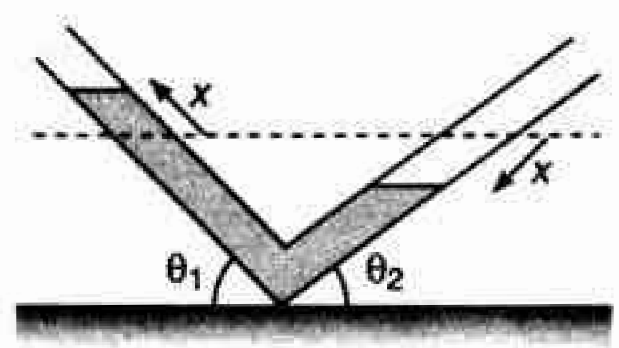


Fig. 11.19

$\therefore$

$$F = -(\Delta P)A = -(h_1 + h_2)\rho g A$$
$$= -\rho g A (\sin\theta_1 + \sin\theta_2)\, x$$

Let, m be the mass of the liquid in the tube. Then,

$$ma = -\rho g A(\sin\theta_1 + \sin\theta_2)x$$

Since, F or a is proportional to $-x$, the motion of the liquid column is simple harmonic in nature, time period of which is given by,

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or

$$T = 2\pi\sqrt{\frac{m}{\rho g A(\sin\theta_1 + \sin\theta_2)}}$$

Note *For a U-tube if the liquid is filled to a height l, $\theta_1 = 90° = \theta_2$ and $m = 2(lA\rho)$*

So,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Thus, we see that the expression $T = 2\pi\sqrt{\frac{l}{g}}$ comes in picture at three places.

(i) Time period of a simple pendulum for small oscillations.
(ii) Time period of a spring-block system in vertical position.
(iii) Time period of a liquid column in a U-tube filled to a height l.
But l has different meanings at different places.

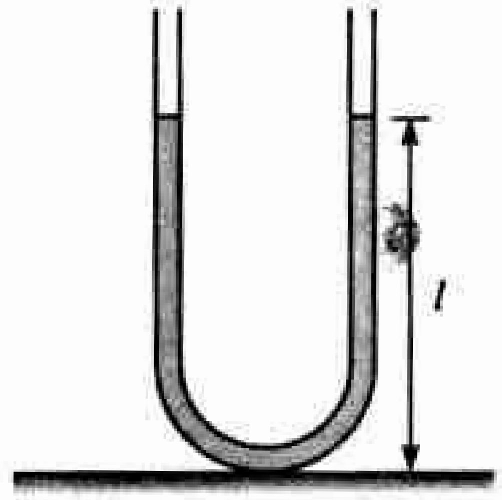


Fig. 11.20

11.8 Vector Method of Combining Two or More Simple Harmonic Motions in Same Direction

A simple harmonic motion is produced when a force (called restoring force) proportional to the displacement acts on a particle. If a particle is acted upon by two such forces the resultant motion of the particle is a combination of two simple harmonic motions. Suppose the two individual motions are represented by,

$$x_1 = A_1 \sin\omega t \qquad \text{and} \qquad x_2 = A_2\sin(\omega t + \phi)$$

Both the simple harmonic motions have same angular frequency ω.

The resultant displacement of the particle is given by,

$$x = x_1 + x_2 = A_1\sin\omega t + A_2\sin(\omega t + \phi)$$

$$= A\sin(\omega t + \alpha)$$

Here,

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

and

$$\tan\alpha = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

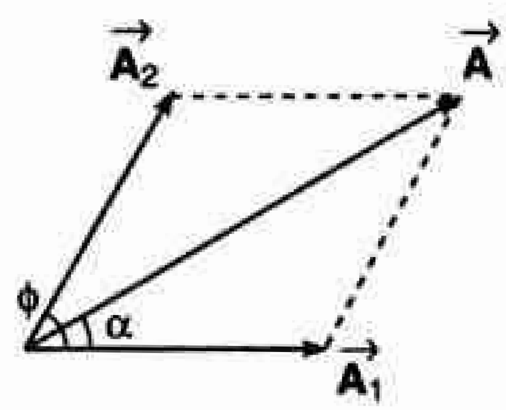


Fig. 11.21

Thus, we can see that this is similar to the vector addition. The same method of vector addition can be applied to the combination of more than two simple harmonic motions.

Sample Example 11.11 *Find the displacement equation of the simple harmonic motion obtained by combining the motions.*

$$x_1 = 2\sin\omega t,\quad x_2 = 4\sin\left(\omega t + \frac{\pi}{6}\right)\quad and\quad x_3 = 6\sin\left(\omega t + \frac{\pi}{3}\right)$$

Solution The resultant equation is,

$$x = A\sin(\omega t + \phi)$$

$$\Sigma A_x = 2 + 4\cos 30° + 6\cos 60° = 8.46$$

and $$\Sigma A_y = 4\sin 30° + 6\cos 30° = 7.2$$

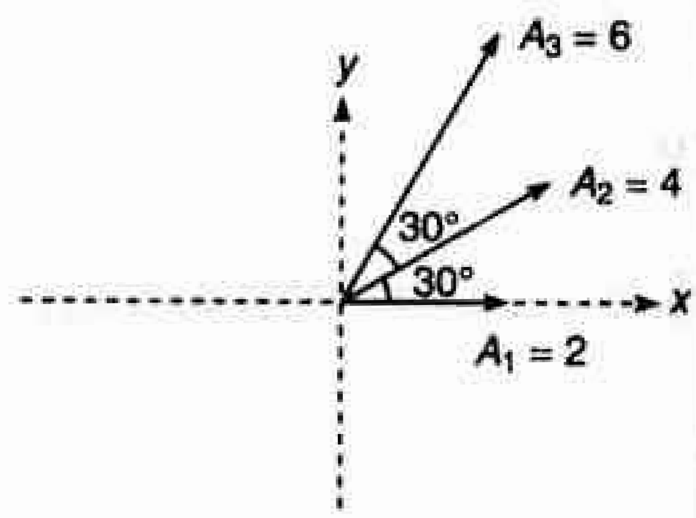


Fig. 11.22

$\therefore$ $$A = \sqrt{(\Sigma A_x)^2 + (\Sigma A_y)^2}$$

$$= \sqrt{(8.46)^2 + (7.2)^2}$$

$$= 11.25$$

and $$\tan\phi = \frac{\Sigma A_y}{\Sigma A_x} = \frac{7.2}{8.46} = 0.85$$

or $$\phi = \tan^{-1}(0.85) = 40.4°$$

Thus, the displacement equation of the combined motion is,

$$x = 11.25\sin(\omega t + \phi)$$

where $$\phi = 40.4°$$ **Ans.**

Introductory Exercise 11.5

1. A particle is subjected to two simple harmonic motions of the same frequency and direction. The amplitude of the first motion is 4.0 cm and that of the second is 3.0 cm. Find the resultant amplitude if the phase difference between the two motions is:
   (a) 0° (b) 60° (c) 90° (d) 180°
2. A particle is subjected to two simple harmonic motions.

$$x_1 = 4.0\sin(100\pi t)\quad \text{and}\quad x_2 = 3.0\sin\left(100\pi t + \frac{\pi}{3}\right)$$

Find :
(a) the displacement at $t = 0$
(b) the maximum speed of the particle and
(c) the maximum acceleration of the particle.

11.9 Free, Forced and Damped Oscillations, Resonance

Free And Damped Oscillations

We know that in reality, a spring won't oscillate forever with constant amplitude. These constant amplitude oscillations (which will really occur in vacuum) are called **"free oscillations"**. Frictional forces will diminish the amplitude of oscillation until eventually the system comes to rest.

We will now add frictional forces to the mass and spring. Imagine that the mass was put in a viscous liquid. To incorporate friction, we can just say that there is a frictional force that's proportional to the velocity of the mass. This is a pretty good approximation for a body moving at a low velocity in air, or in a liquid. So we say the frictional force $f_r = -bv$. The constant b depends on the kind of liquid and the shape of the mass. The negative sign, just says that the force is in the opposite direction to the body's motion. Let's add this frictional force in to the equation $f_{net} = ma$

$$-kx - bv = ma \qquad ...(i)$$

In terms of derivatives

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad ...(ii)$$

This is a differential equation. In the solution of this type of differential equation instead of the amplitude being constant, it's decaying with time.

$$A(t) = A_0 e^{-\text{const } t}$$

So,

$$x(t) = A(t)\cos(\omega t + \delta) = A_0 e^{-\text{const } t}\cos(\omega t + \delta)$$

Here's a plot of an example of such a function.

These types of oscillations which eventually come to end are called **"damped oscillations"**. There are further three types of damped oscillations, namely

(i) Under damped
(ii) Critically damped and
(iii) Over damped oscillations

But their detailed discussion is out of our syllabus at this stage.

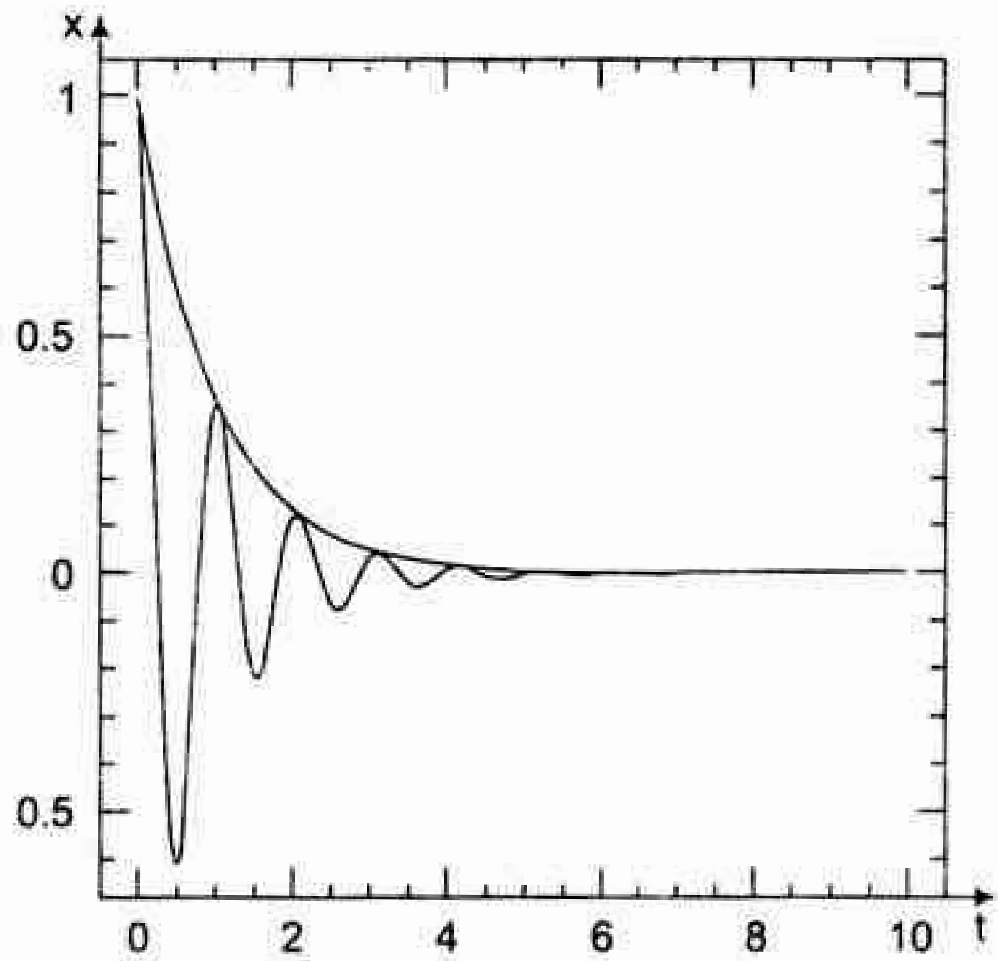


Fig. 11.23

Natural frequency (or Characteristic Frequency)

That is the frequency at which a system would oscillate by itself if displaced. The natural frequency of a spring-mass system is $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is the spring constant and m is the mass of the object attached to the spring.

Forced Oscillations

A periodic force at a given frequency (called driving frequency f_d) is applied to an oscillating system of natural frequency f_0. At the begining (transient stage), there is a mixture of two kinds of oscillations, one has the frequency f_0 and the other has f_d. The former will gradually die out because of the damping effect. Eventually (at the steady state) the system settles down with oscillation at the frequency of the driving force (f_d).

Resonance

When the driving frequency is at the same frequency as the natural frequency of the oscillator, the amplitude of oscillation is at its greatest. When this happens the energy of the oscillator becomes a maximum. This is called a condition of resonance.

Resonance and its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance ; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. Inexpensive loudspeakers often have an unwanted boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone. Resonance also occurs in electric circuits ; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step ; the frequency of their steps was close to the natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge.

Nearly everyone has seen the film of the collapse of the Tacoma Narrows suspension bridge in 1940.

Extra Points

- **Lissajous figures :** Suppose two forces act on a particle, the first alone would produce a simple harmonic motion in x-direction given by,

$$x = a \sin \omega t$$

and the second would produce a simple harmonic motion in y-direction given by,

$$y = b \sin (\omega t + \phi)$$

The amplitudes a and b may be different and their phases differ by ϕ. The frequencies of the two simple harmonic motions are assumed to be equal. The resultant motion of the particle is a combination of the two simple harmonic motions.

Depending on the value of ϕ and relation between a and b, the particle follows different paths. Given below are few special cases :

Case 1 : (When $\phi = 0°$) When the phase difference between two simple harmonic motions is 0°, *i.e.*,

$$x = a \sin \omega t \Rightarrow \sin \omega t = \frac{x}{a} \quad \text{...(i)}$$

$$y = b \sin \omega t \Rightarrow \sin \omega t = \frac{y}{b} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x}{a} = \frac{y}{b} \quad \text{or} \quad y = \left(\frac{b}{a}\right) x$$

which is equation of a straight line with slope $\frac{b}{a}$. Thus, the path of the particle is a straight line. As a special case $y = x$ if $a = b$ or slope is 1.

Case 2 : $\left(\text{When } \phi = \frac{\pi}{2}\right)$ When the phase difference is $\frac{\pi}{2}$ *i.e.*,

$$x = a \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{a} \qquad \text{...(iii)}$$

$$y = b \sin\left(\omega t + \frac{\pi}{2}\right) = b \cos \omega t$$

$$\Rightarrow \qquad \cos \omega t = \frac{y}{b} \qquad \text{...(iv)}$$

Squaring and adding, Eqs. (iii) and (iv), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an **ellipse.** Again as a special case, the above equation reduces to,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad \text{or} \quad x^2 + y^2 = a^2 \qquad \text{for } a = b$$

This is an equation of a **circle.**

Table 11.1 given below shows some of the figures (called **Lissajous figures**) depending upon the phase difference ϕ and ratio b/a.

- **Two Body Oscillator**

A system of two bodies connected by a spring so that both are free to oscillate simple harmonically along the length of the spring constitutes a two body harmonic oscillator.

Suppose, two masses m_1 and m_2 are connected by a horizontal massless spring of force constant k, so as to be free to oscillate along the length of the spring on a frictionless horizontal surface.

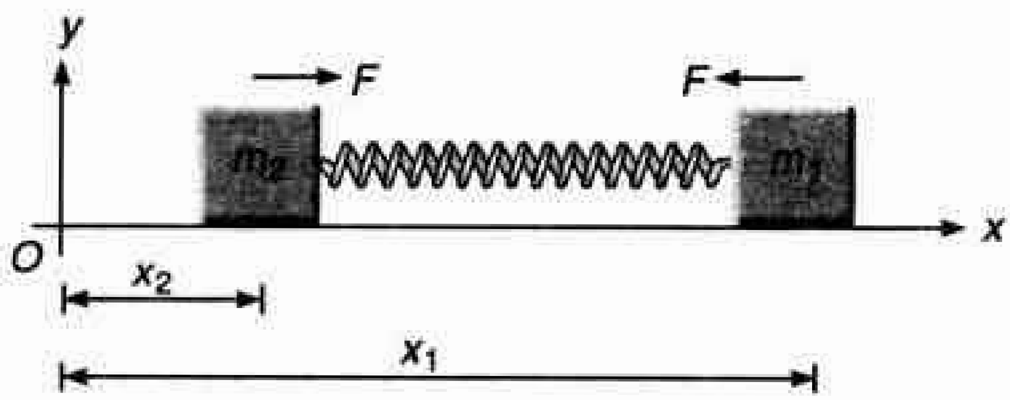


Fig. 11.24

Let l_0 be the natural length of the spring and let x_1 and x_2 be the co-ordinates of the two masses at any instant of time. Then,

Extension of the spring $\qquad x = (x_1 - x_2) - l_0 \qquad$ ...(i)

For $x > 0$, the spring force $F = kx$ acts on the two masses in the directions shown in figure.

Thus, we can write

$$m_1 \frac{d^2 x_1}{dt^2} = -kx \qquad \text{...(ii)}$$

$$m_2 \frac{d^2 x_2}{dt^2} = kx \qquad \text{...(iii)}$$

Multiplying Eq. (ii) by m_2 and Eq. (iii) by m_1 and subtracting the latter from the former, we have

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -(m_2 + m_1)\, kx$$

or $$m_1 m_2 \frac{d^2(x_1 - x_2)}{dt^2} = -kx(m_1 + m_2)$$

or $$\left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2}{dt^2}(x_1 - x_2) = -kx \qquad \text{...(iv)}$$

Differentiating Eq. (i), twice with respect to time, we have

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_1 - x_2)$$

Also, $$\frac{m_1 m_2}{m_1 + m_2} = \mu = \text{reduced mass of the two blocks}$$

Substituting these values in Eq. (iv), we have

$$\mu \frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \mu a_r = -kx$$

(Here $a_r = \frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2}$ = Relative acceleration)

This, is the standard differential equation of SHM. Time period of which is

$$T = 2\pi\sqrt{\left|\frac{x}{a_r}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{\mu}{k}}$$

The above equation is identical with the equation of motion obtained for a single body oscillator, with the difference that μ, here, is the reduced mass of the system instead of mass m of the single body there and x here, is the relative displacement of the two masses from their equilibrium positions instead of the displacement of mass m alone from its equilibrium position there.

Further, if $m_2 >> m_1$ then $\frac{m_1}{m_2} \approx 0$ and the reduced mass $\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} \approx m_1$

and $$T = 2\pi\sqrt{\frac{m_1}{k}}$$

Thus, if a mass m is attached to the free end of a spring whose other end is fixed to a rigid support like a wall, constitutes a harnonic oscillator. This too is in fact a two body oscillator, only one of the bodies *viz*., the wall is rigidly connected to the earth and has thus, effectively infinite mass and the reduced mass is m.

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k}}$$

- **Concept of Reduced Mass :** If net external force on a system of two bodies is zero and only internal conservative forces are acting on them, mechanical energy of the system will remain conserved. The relative speed between them can be found by putting reduced mass in energy equation. Let us take two examples in support of the theory.

Sample Example 11.1 Suppose two masses m_1 and m_2 are initially at infinite separation and we want to find relative speed (v_r) between them when they are at a separation r.

Then this can be found by energy conservation principle.

decrease in gravitational potential energy = increase in kinetic energy

or, $$\frac{Gm_1 m_2}{r} = \frac{1}{2}\mu v_r^2$$

Here, $$\mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

$\therefore$ $$v_r = \sqrt{\frac{2G(m_1 + m_2)}{r}}$$

Sample Example 11.2 Two blocks of masses m_1 and m_2 are connected by a spring of force constant k. They are stretched by x_0 and released. The relative speed between them when the spring is unstretched can also be found by using energy conservation principle, *i.e.*,

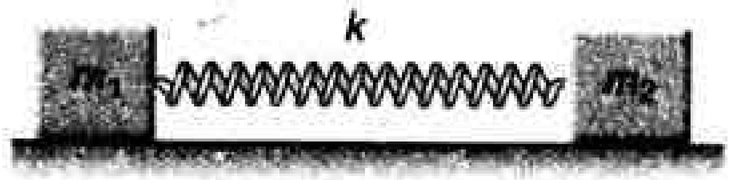


Fig. 11.25

Decrease in elastic potential energy = increase in kinetic energy

$\therefore$ $$\frac{1}{2} k x_0^2 = \frac{1}{2} \mu v_r^2$$

$\therefore$ $$v_r = \sqrt{\frac{k}{\mu}} \cdot x_0$$

Here, $$\mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

Energy of vibration : Suppose two blocks of masses m_1 and m_2 are connected by an ideal spring of stiffness k as shown in figure. They are lying on smooth horizontal surface. Block m_1 is given a velocity v_0 towards right. Velocity of centre of mass will be,

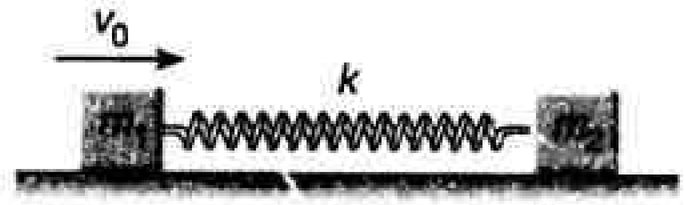


Fig. 11.26

$$v_{cm} = \frac{m_1 v_0}{m_1 + m_2}$$

Kinetic energy of centre of mass is,

$$K_{CM} = \frac{1}{2}(m_1 + m_2)\, v_{cm}^2 = \frac{m_1^2 v_0^2}{2(m_1 + m_2)} \quad \text{...(i)}$$

Energy of two blocks is, $$E = \frac{1}{2} m_1 v_0^2 \quad \text{...(ii)}$$

Mathematically we can see that $E > K_{CM}$. Their difference is the energy of vibration *i.e.*,

$$E_{vib} = E - K_{CM} \quad \text{...(iii)}$$

At maximum compression of spring the whole of vibrational energy is the elastic potential energy of spring. Or,

$$\frac{1}{2} k x_m^2 = E_{vib} \quad \text{or} \quad x_m = \sqrt{\frac{2E_{vib}}{k}}$$

Motion of two blocks connected by spring as seen from centre of mass frame : Two blocks m_1 and m_2 connected by an ideal spring are lying on a smooth horizontal surface as shown. A constant horizontal force F is applied on m_2. We are interested in finding the maximum extension in the spring. The problem becomes simple when solved from centre of mass frame (also sometimes called C-frame).

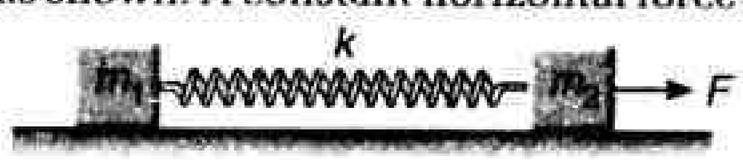


Fig. 11.27

Acceleration of centre of mass will be,

$$a_{cm} = \frac{F}{m_1 + m_2} = a \text{ (say)}$$

Now let us draw free body diagrams of m_1 and m_2 from C-frame (accelerated).

Fig. 11.28

In the figure $m_1 a$ = pseudo force on m_1 and

$m_2 a$ = pseudo force on m_2

Further free body diagrams have been shown with the spring unstretched (at $t = 0$).

Net force on $$m_2 = F - m_2 a = F - \frac{m_2 F}{m_1 + m_2} = \frac{m_1 F}{m_1 + m_2}$$

and on $$m_1 = m_1 a = \frac{m_1 F}{m_1 + m_2}$$

i.e., force on m_1 and on m_2 comes out to be equal and opposite say it is F_0. The situation is as shown in Fig. 11.29.

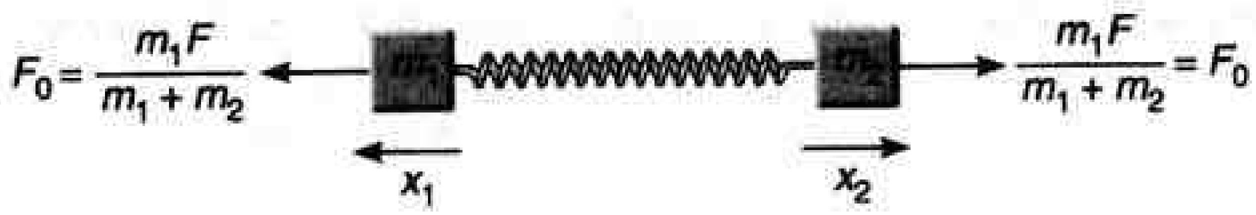


Fig. 11.29

At maximum extension (x_m) both the blocks instantaneously come at rest. Suppose the left block is displaced through a distance x_1 and the right block through a distance x_2 from their initial positions. Total work done by the external forces is equal to the increase in potential energy of spring. Hence,

$$F_0 x_1 + F_0 x_2 = \frac{1}{2} k x_m^2 \quad \text{or} \quad F_0 (x_1 + x_2) = \frac{1}{2} k x_m^2$$

or $$F_0 x_m = \frac{1}{2} k x_m^2 \qquad (\text{as } x_1 + x_2 = x_m)$$

Substituting $F_0 = \frac{m_1 F}{m_1 + m_2}$, we have $$x_m = \frac{2F_0}{k} = \frac{2m_1 F}{k(m_1 + m_2)}$$

Solved Examples

For **JEE Main**

Example 1 *If a SHM is represented by the equation* $x = 10 \sin\left(\pi t + \frac{\pi}{6}\right)$ *in SI units determine its amplitude, time period and maximum velocity* v_{max} *?*

Solution Comparing the above equation with

$$x = A \sin(\omega t + \phi), \text{ we get}$$

$$A = 10 \text{ m} \qquad \textbf{Ans.}$$

$$\omega = \pi \text{ s}^{-1} \quad \text{and} \quad \phi = \frac{\pi}{6}$$

$$\because \qquad T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = 2\text{s} \qquad \textbf{Ans.}$$

$$v_{max} = \omega A = 10\pi \text{ m/s} \qquad \textbf{Ans.}$$

Example 2 *A particle executes SHM with a time period of 4 s. Find the time taken by the particle to go directly from its mean position to half of its amplitude.*

Solution $x = A \sin(\omega t + \phi)$

At $t = 0, \quad x = 0 \quad \Rightarrow \quad A \sin\phi = 0 \quad \text{or} \quad \phi = 0$

Hence, $x = A \sin(\omega t)$

or $\frac{A}{2} = A \sin(\omega t)$

or $\frac{1}{2} = \sin(\omega t)$

$$\omega t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$t = \frac{\pi}{6\omega} = \frac{\pi T}{6(2\pi)}$$

as $\omega = \frac{2\pi}{T} \quad \Rightarrow \quad t = \frac{T}{12} = \frac{1}{3}\text{s}$ **Ans.**

Example 3 *A particle executes SHM.*

(a) *What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude?*

(b) *At what value of displacement are the kinetic and potential energies equal?*

Solution We know that

$$E_{\text{total}} = \frac{1}{2} m\omega^2 A^2$$

$$\text{KE} = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

and

$$U = \frac{1}{2} m\omega^2 x^2$$

(a) When

$$x = \frac{A}{2}$$

$$\text{KE} = \frac{1}{2} m\omega^2 \frac{3A^2}{4} \quad \Rightarrow \quad \frac{\text{KE}}{E_{\text{total}}} = \frac{3}{4}$$ **Ans.**

At $x = \frac{A}{2}$,

$$U = \frac{1}{2} m\omega^2 \frac{A^2}{4}$$

$\Rightarrow$

$$\frac{\text{PE}}{E_{\text{total}}} = \frac{1}{4}$$ **Ans.**

(b) Since,

$$K = U$$

$$\frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} m\omega^2 x^2$$

or

$$2x^2 = A^2 \quad \text{or} \quad x = \frac{A}{\sqrt{2}} = 0.707A$$ **Ans.**

Example 4 *Two particles move parallel to x-axis about the origin with the same amplitude and frequency. At a certain instant they are found at distance $\frac{A}{3}$ from the origin on opposite sides but their velocities are found to be in the same direction. What is the phase difference between the two ?*

Solution Let equations of two SHM be

$$x_1 = A \sin \omega t \quad \text{... (i)}$$

$$x_2 = A \sin (\omega t + \phi) \quad \text{... (ii)}$$

Give that

$$\frac{A}{3} = A \sin \omega t$$

and

$$-\frac{A}{3} = A \sin (\omega t + \phi)$$

Which gives

$$\sin \omega t = \frac{1}{3} \quad \text{... (iii)}$$

$$\sin (\omega t + \phi) = -\frac{1}{3} \quad \text{... (iv)}$$

From Eq. (iv),

$$\sin \omega t \cos \phi + \cos \omega t \sin \phi = -\frac{1}{3}$$

$$\Rightarrow \qquad \frac{1}{3}\cos\phi + \sqrt{1-\frac{1}{9}}\sin\phi = -\frac{1}{3}$$

Solving this equation, we get

$$\text{or} \qquad \cos\phi = -1, \frac{7}{9}$$

$$\Rightarrow \qquad \phi = \pi \quad \text{or} \quad \cos^{-1}\left(\frac{7}{9}\right)$$

Differentiating Eqs. (i) and (ii), we obtain

$$v_1 = A\omega\cos\omega t \quad \text{and} \quad v_2 = A\omega\cos(\omega t + \phi)$$

If we put $\phi = \pi$, we find v_1 and v_2 are of opposite signs. Hence, $\phi = \pi$ is not acceptable.

$$\therefore \qquad \phi = \cos^{-1}\left(\frac{7}{9}\right)$$ **Ans.**

Example 5 *A particle executes simple harmonic motion about the point $x = 0$. At time $t = 0$ it has displacement $x = 2$ cm and zero velocity. If the frequency of motion is $0.25\ s^{-1}$, find (a) the period, (b) angular frequency, (c) the amplitude, (d) maximum speed, (e) the displacement at $t = 3$ s and (f) the velocity at $t = 3$ s.*

Solution (a) Period $\qquad T = \frac{1}{f} = \frac{1}{0.25\ s^{-1}} = 4\ s$ **Ans.**

(b) Angular frequency $\qquad \omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}\ \text{rad/s} = 1.57\ \text{rad/s}$ **Ans.**

(c) Amplitude is the maximum displacement from mean position. Hence, $A = 2 - 0 = 2$ cm.

(d) Maximum speed $\qquad v_{max} = A\omega = 2 \cdot \frac{\pi}{2} = \pi\ \text{cm/s} = 3.14\ \text{cm/s}$ **Ans.**

(e) The displacement is given by $\qquad x = A\sin(\omega t + \phi)$

Initially at $t = 0$, $\qquad x = 2$ cm, then

$$2 = 2\sin\phi$$

$$\text{or} \qquad \sin\phi = 1 = \sin 90^\circ$$

$$\text{or} \qquad \phi = 90^\circ$$

Now, at $t = 3$ s $\qquad x = 2\sin\left(\frac{\pi}{2}\times 3 + \frac{\pi}{2}\right) = 0$ **Ans.**

(f) Velocity at $x = 0$ is v_{max} *i.e.*, 3.14 cm/s. **Ans.**

Example 6 *Show that the period of oscillation of simple pendulum at depth h below earth's surface is inversely proportional to $\sqrt{R-h}$, where R is the radius of earth. Find out the time period of a second pendulum at a depth R/2 from the earth's surface ?*

Solution At earth's surface the value of time period is given by

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{or} \quad T \propto \frac{1}{\sqrt{g}}$$

At a depth h below the surface, $$g' = g\left(1 - \frac{h}{R}\right)$$

$\therefore$ $$\frac{T'}{T} = \sqrt{\frac{g}{g'}} = \sqrt{\frac{1}{\left(1 - \frac{h}{R}\right)}} = \sqrt{\frac{R}{R-h}}$$

$\therefore$ $$T' = T\sqrt{\frac{R}{R-h}}$$

or $$T' \propto \frac{1}{\sqrt{R-h}}$$ **Hence proved.**

Further, $$T_{R/2} = 2\sqrt{\frac{R}{R - R/2}} = 2\sqrt{2} \text{ sec}$$ **Ans.**

Example 7 *A spring mass system is hanging from the ceiling of an elevator in equilibrium. The elevator suddenly starts accelerating upwards with acceleration a. Find :*

(a) the frequency and

(b) the amplitude of the resulting SHM.

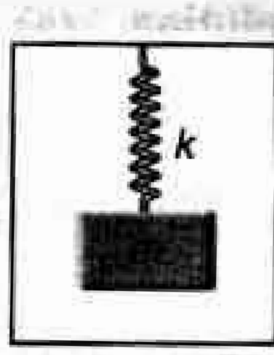


Fig. 11.30

Solution (a) Frequency $= 2\pi\sqrt{\frac{m}{k}}$ (Frequency is independent of g in spring)

(b) Extension in spring in equilibrium

$$\text{initial} = \frac{mg}{k}$$

Extension in spring in equilibrium in accelerating lift $= \frac{m(g+a)}{k}$

$\therefore$ $$\text{Amplitude} = \frac{m(g+a)}{k} - \frac{mg}{k} = \frac{ma}{k}$$ **Ans.**

Example 8 *A ring of radius r is suspended from a point on its circumference. Determine its angular frequency of small oscillations.*

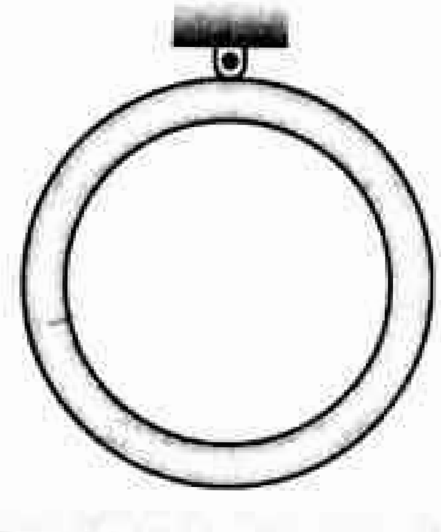


Fig. 11.31

Solution It is a physical pendulum, the time period of which is,

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Here, I = moment of inertia of the ring about point of suspension

$= mr^2 + mr^2$

$= 2mr^2$

and l = distance of point of suspension from centre of gravity

$= r$

$$T = 2\pi\sqrt{\frac{2mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}$$

∴ Angular frequency $\omega = \frac{2\pi}{T}$

or $\omega = \sqrt{\frac{g}{2r}}$ **Ans.**

For **JEE Advanced**

Example 1 *For the arrangement shown in figure, the spring is initially compressed by 3 cm. When the spring is released the block collides with the wall and rebounds to compress the spring again.*

(a) If the coefficient of restitution is $\frac{1}{\sqrt{2}}$, find the maximum compression in the spring after collision.

(b) If the time starts at the instant when spring is released, find the minimum time after which the block becomes stationary.

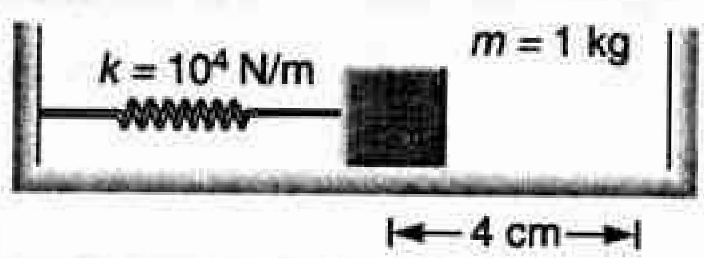


Fig. 11.32

Solution (a) Velocity of the block just before collision,

$$\frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2$$

or $$v_0 = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

Here, $x_0 = 0.03$ m, $x = 0.01$ m, $k = 10^4$ N/m, $m = 1$ kg

$\therefore$ $$v_0 = 2\sqrt{2}\ \text{m/s}$$

After collision, $$v = ev_0 = \frac{1}{\sqrt{2}} 2\sqrt{2} = 2\ \text{m/s}$$

Maximum compression in the spring is

$$\frac{1}{2}kx_m^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

or $$x_m = \sqrt{x^2 + \frac{m}{k}v^2} = \sqrt{(0.01)^2 + \frac{1(2)^2}{10^4}}\ \text{m}$$

$$= 2.23\ \text{cm}$$ **Ans.**

(b) In the case of spring-mass system, since the time period is independent of the amplitude of oscillation.

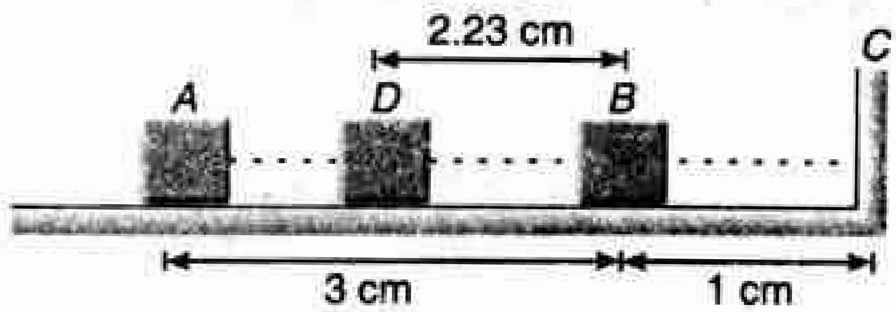


Fig. 11.33

$$\text{Time} = t_{AB} + 2t_{BC} + t_{BD}$$

$$= \frac{T_0}{4} + 2\left(\frac{T_0}{2\pi}\right)\sin^{-1}\left(\frac{1}{3}\right) + \frac{T_0}{4}$$

Here, $$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

Substituting the values, we get

$$\text{Total time} = \sqrt{\frac{m}{k}}\left[\pi + 2\sin^{-1}\left(\frac{1}{3}\right)\right]$$ **Ans.**

Example 2 *With the assumption of no slipping, determine the mass m of the block which must be placed on the top of a 6 kg cart in order that the system period is 0.75 s. What is the minimum coefficient of static friction μ_s for which the block will not slip relative to the cart if the cart is displaced 50 mm from the equilibrium position and released? Take ($g = 9.8\ m/s^2$).*

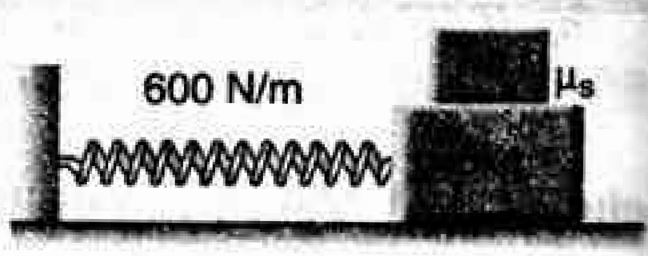


Fig. 11.34

Solution (a) $$T = 2\pi\sqrt{\frac{m+6}{600}} \qquad \left(T = 2\pi\sqrt{\frac{m}{k}}\right)$$

or $$0.75 = 2\pi\sqrt{\frac{m+6}{600}}$$

$\therefore$ $$m = \frac{(0.75)^2 \times 600}{(2\pi)^2} - 6$$

$$= 2.55 \text{ kg}$$ **Ans.**

(b) Maximum acceleration of SHM is,

$$a_{max} = \omega^2 A \qquad (A = \text{amplitude})$$

i.e., maximum force on mass 'm' is $m\omega^2 A$ which is being provided by the force of friction between the mass and the cart. Therefore,

$$\mu_s mg \geq m\omega^2 A$$

or $$\mu_s \geq \frac{\omega^2 A}{g}$$

or $$\mu_s \geq \left(\frac{2\pi}{T}\right)^2 \cdot \frac{A}{g}$$

or $$\mu_s \geq \left(\frac{2\pi}{0.75}\right)^2 \left(\frac{0.05}{9.8}\right) \qquad (A = 50 \text{ mm})$$

or $$\mu_s \geq 0.358$$

Thus, the minimum value of μ_s should be 0.358. **Ans.**

Example 3 *A long uniform rod of length L and mass M is free to rotate in a vertical plane about a horizontal axis through its one end 'O'. A spring of force constant k is connected vertically between one end of the rod and ground. When the rod is in equilibrium it is parallel to the ground.*

(a) What is the period of small oscillation that result when the rod is rotated slightly and released?

(b) What will be the maximum speed of the displaced end of the rod, if the amplitude of motion is θ_0?

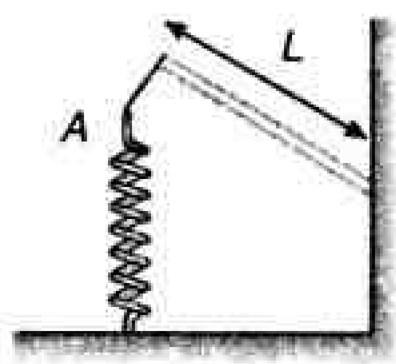


Fig. 11.35

Solution (a) Restoring torque about 'O' due to elastic force of the spring

$$\tau = -FL = -kyL \qquad (F = ky)$$

$$\tau = -kL^2\theta \qquad (\text{as } y = L\theta)$$

$$\tau = I\alpha = \frac{1}{3}ML^2\frac{d^2\theta}{dt^2}$$

$$\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} = -kL^2\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$\omega = \sqrt{\frac{3k}{M}} \Rightarrow T = 2\pi\sqrt{\frac{M}{3k}}$$ **Ans.**

(b) In angular SHM, maximum angular velocity

$$\left(\frac{d\theta}{dt}\right)_{max} = \theta_0\,\omega = \theta_0\sqrt{\frac{3k}{M}}$$

$$v = r\left(\frac{d\theta}{dt}\right)$$

So,
$$v_{max} = L\left(\frac{d\theta}{dt}\right)_{max} = L\theta_0\sqrt{\frac{3k}{M}}$$ **Ans.**

Example 4 *A block with a mass of 2 kg hangs without vibrating at the end of a spring of spring constant 500 N/m, which is attached to the ceiling of an elevator. The elevator is moving upwards with an acceleration $\frac{g}{3}$. At time $t = 0$, the acceleration suddenly ceases.*

(a) What is the angular frequency of oscillation of the block after the acceleration ceases?
(b) By what amount is the spring stretched during the time when the elevator is accelerating?
(c) What is the amplitude of oscillation and initial phase angle observed by a rider in the elevator? Take the upward direction to be positive. Take $g = 10.0\ m/s^2$.

Solution (a) Angular frequency

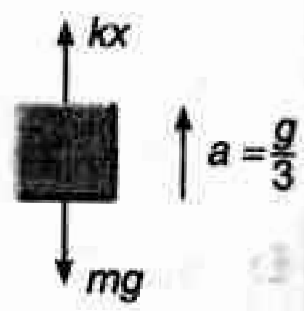


Fig. 11.36

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = \sqrt{\frac{500}{2}}$$

or
$$\omega = 15.81 \text{ rad/s}$$

(b) Equation of motion of the block (while elevator is accelerating) is,

$$kx - mg = ma = m\frac{g}{3}$$

$\therefore$
$$x = \frac{4mg}{3k} = \frac{(4)(2)(10)}{(3)(500)} = 0.053 \text{ m}$$

or
$$x = 5.3 \text{ cm}$$

(c) (i) In equilibrium when the elevator has zero acceleration, the equation of motion is,

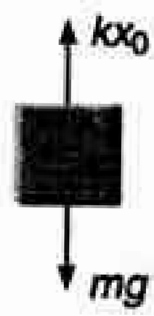


Fig. 11.37

$$kx_0 = mg$$

or
$$x_0 = \frac{mg}{k} = \frac{(2)(10)}{500} = 0.04 \text{ m}$$

$$= 4 \text{ cm}$$

$\therefore$ Amplitude
$$A = x - x_0 = 5.3 - 4.0$$
$$= 1.3 \text{ cm}$$ **Ans.**

(ii) At time $t = 0$, block is at $x = -A$. Therefore, substituting $x = -A$ and $t = 0$ in equation,

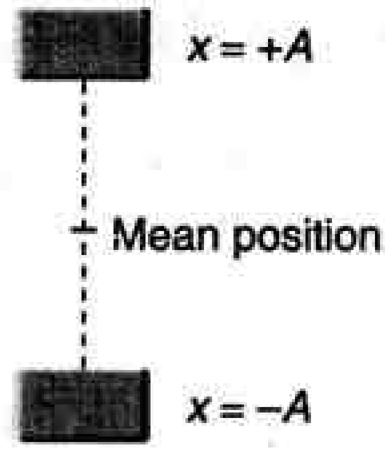


Fig. 11.38

$$x = A \sin(\omega t + \phi)$$

We get initial phase $\phi = \frac{3\pi}{2}$ **Ans.**

Example 5 *Figure shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down from its equilibrium position and released, find the period of its vertical oscillation in cases (a), (b) and (c).*

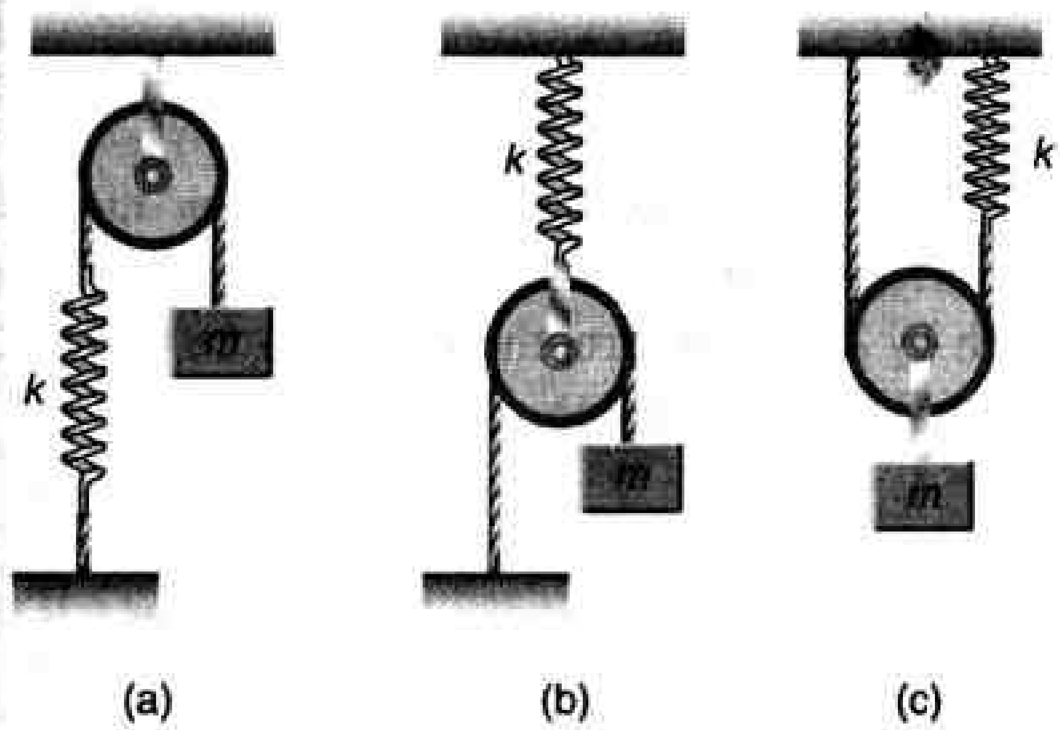


Fig. 11.39

Solution (a) In equilibrium, $kx_0 = mg$ ...(i)

When further depressed by an amount x, net restoring force (upwards) is,

$$F = -\{k(x + x_0) - mg\}$$

$$F = -kx \quad (\text{as } kx_0 = mg)$$

or $$a = -\frac{k}{m}x$$

$\therefore$ $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or $$T = 2\pi\sqrt{\frac{m}{k}}$$ **Ans.**

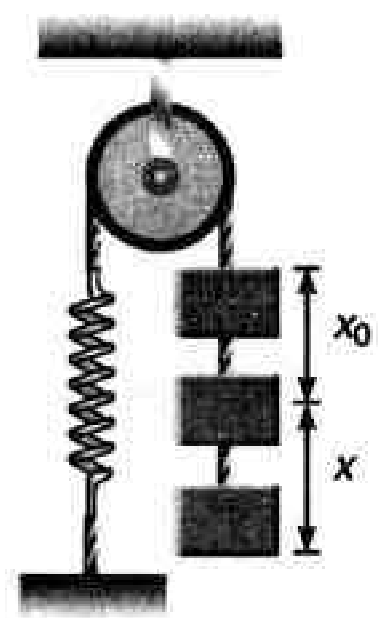


Fig. 11.40

(b) In this case if the mass m moves down a distance x from its equilibrium position, then pulley will move down by $\frac{x}{2}$. So, the extra force in spring will be $k\frac{x}{2}$. Now, as the pulley is massless, this force $\frac{kx}{2}$ is equal to extra $2T$ or $T = \frac{kx}{4}$. This is also the restoring force of the mass. Hence,

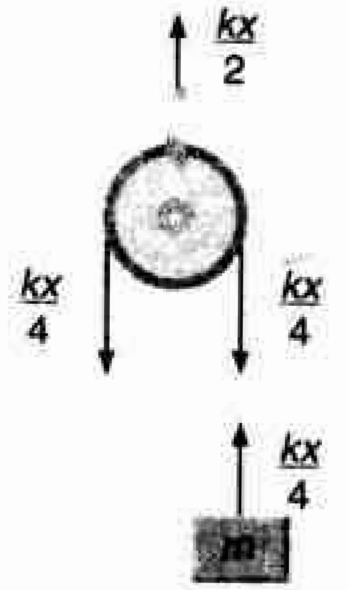


Fig. 11.41

$$F = -\frac{kx}{4}$$

or $$a = -\frac{k}{4m}x$$

or $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or $$T = 2\pi\sqrt{\frac{4m}{k}}$$ **Ans.**

(c) In this situation if the mass m moves down a distance x from its equilibrium position, the pulley will also move by x and so the spring will stretch by $2x$. Therefore, the spring force will be $2kx$. The restoring force on the block will be $4kx$. Hence,

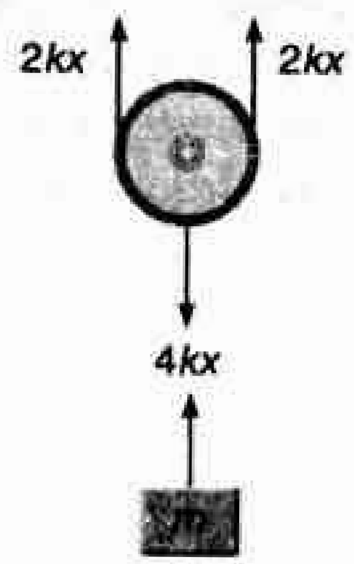


Fig. 11.42

$$F = -4kx$$

or $$a = -\frac{4k}{m}.x$$

$\therefore$ $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or $$T = 2\pi\sqrt{\frac{m}{4k}}$$ **Ans.**

Example 6 *Calculate the angular frequency of the system shown in figure. Friction is absent everywhere and the threads, spring and pulleys are massless. Given that $m_A = m_B = m$.*

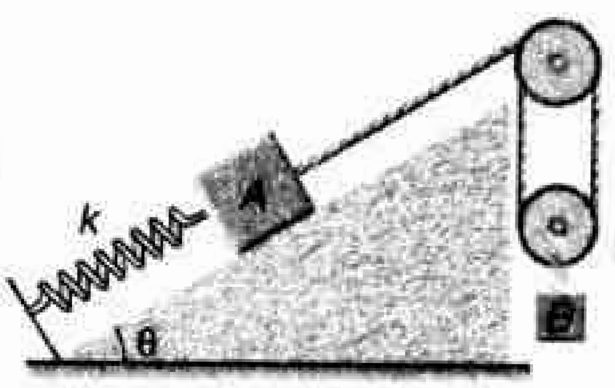


Fig. 11.43

Solution Let x_0 be the extension in the spring in equilibrium. Then equilibrium of A and B give,

$$T = kx_0 + mg\sin\theta \quad \text{...(i)}$$

and $$2T = mg \quad \text{...(ii)}$$

Here, T is the tension in the string. Now, suppose A is further displaced by a distance x from its mean position and v be its speed at this moment. Then B lowers by $\frac{x}{2}$ and speed of B at this instant will be $\frac{v}{2}$. Total energy of the system in this position will be,

$$E = \frac{1}{2} k (x + x_0)^2 + \frac{1}{2} m_A v^2 + \frac{1}{2} m_B \left(\frac{v}{2}\right)^2 + m_A g h_A - m_B g\, h_B$$

or $$E = \frac{1}{2} k (x + x_0)^2 + \frac{1}{2} mv^2 + \frac{1}{8} mv^2 + mgx \sin\theta - mg \frac{x}{2}$$

or $$E = \frac{1}{2} k(x + x_0)^2 + \frac{5}{8} mv^2 + mgx \sin\theta - mg \frac{x}{2}$$

Since, E is constant,

$$\frac{dE}{dt} = 0$$

or $$0 = k(x + x_0) \frac{dx}{dt} + \frac{5}{4} mv \left(\frac{dv}{dt}\right) + mg (\sin\theta) \left(\frac{dx}{dt}\right) - \frac{mg}{2} \left(\frac{dx}{dt}\right)$$

Substituting, $$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = a$$

and $$kx_0 + mg \sin\theta = \frac{mg}{2}$$ [From Eqs. (i) and (ii)]

We get, $$\frac{5}{4} m\, a = - kx$$

Since, $$a \propto - x$$

Motion is simple harmonic, time period of which is,

$$T = 2\pi \sqrt{\left|\frac{x}{a}\right|} = 2\pi \sqrt{\frac{5m}{4k}}$$

$\therefore$ $$\omega = \frac{2\pi}{T} = \sqrt{\frac{4k}{5m}}$$ **Ans.**

Example 7 *A solid sphere (radius = R) rolls without slipping in a cylindrical through (radius = 5R). Find the time period of small oscillations.*

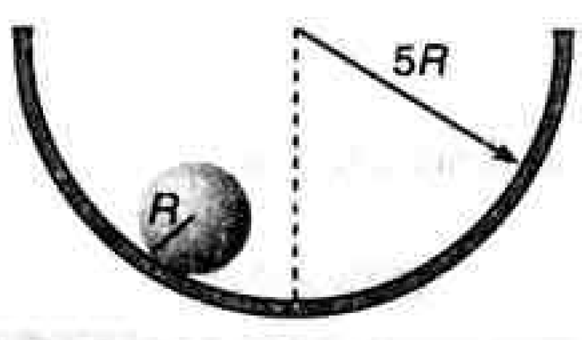


Fig. 11.44

Solution For pure rolling to take place,

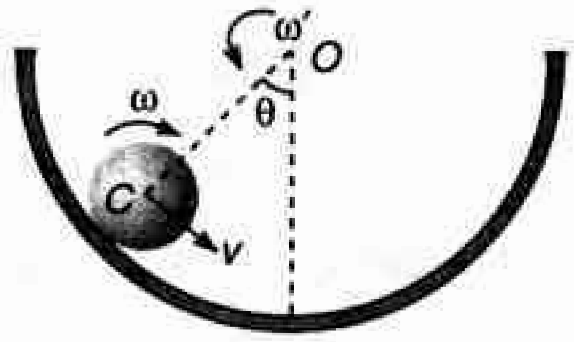


Fig. 11.45

$$v = R\omega$$

ω' = angular velocity of COM of sphere C about O

$$= \frac{v}{4R} = \frac{R\omega}{4R} = \frac{\omega}{4}$$

$$\therefore \quad \frac{d\omega'}{dt} = \frac{1}{4}\frac{d\omega}{dt}$$

or $$\alpha' = \frac{\alpha}{4}$$

$$\alpha = \frac{a}{R} \text{ for pure rolling}$$

where, $$a = \frac{g \sin\theta}{1 + \frac{I}{mR^2}} = \frac{5g\sin\theta}{7}$$

as, $$I = \frac{2}{5}mR^2$$

$$\therefore \quad \alpha' = \frac{5g\sin\theta}{28R}$$

For small θ, $\sin\theta \approx \theta$, being restoring in nature,

$$\alpha' = -\frac{5g}{28R}\theta$$

$$\therefore \quad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha'}\right|} = 2\pi\sqrt{\frac{28R}{5g}}$$ **Ans.**

Example 8 *Consider the earth as a uniform sphere of mass M and radius R. Imagine a straight smooth tunnel made through the earth which connects any two points on its surface. Show that the motion of a particle of mass m along this tunnel under the action of gravitation would be simple harmonic. Hence, determine the time that a particle would take to go from one end to the other through the tunnel.*

Solution Suppose at some instant the particle is at radial distance r from centre of earth O. Since, the particle is constrained to move along the tunnel, we define its position as distance x from C. Hence, equation of motion of the particle is,

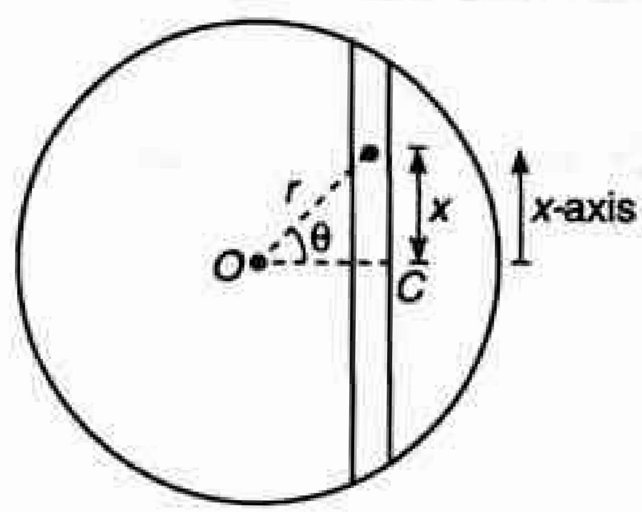


Fig. 11.46

$$ma_x = F_x$$

The gravitational force on mass m at distance r is,

$$F = \frac{GMmr}{R^3} \qquad \text{(towards } O\text{)}$$

Therefore,
$$F_x = -F\sin\theta = -\frac{GMmr}{R^3}\left(\frac{x}{r}\right)$$
$$= -\frac{GMm}{R^3}\cdot x$$

Since, $F_x \propto -x$, motion is simple harmonic in nature. Further,
$$ma_x = -\frac{GMm}{R^3}\cdot x \quad \text{or} \quad a_x = -\frac{GM}{R^3}\cdot x$$

$\therefore$ Time period of oscillation is,
$$T = 2\pi\sqrt{\left|\frac{x}{a_x}\right|} = 2\pi\sqrt{\frac{R^3}{GM}}$$

The time taken by particle to go from one end to the other is $\frac{T}{2}$.

$\therefore$
$$t = \frac{T}{2} = \pi\sqrt{\frac{R^3}{GM}}$$
Ans.

EXERCISES

For **JEE Main**

Subjective Questions

Kinematics of SHM

1. A 50 g mass hangs at the end of a massless spring. When 20 g more are added to the end of the spring, it stretches 7.0 cm more.
(a) Find the spring constant.
(b) If the 20 g are now removed, what will be the period of the motion ?

2. A body of weight 27 N hangs on a long spring of such stiffness that an extra force of 9 N stretches the spring by 0.05 m. If the body is pulled downward and released, what is the period ?

3. A 0.5 kg body performs simple harmonic motion with a frequency of 2 Hz and an amplitude of 8 mm. Find the maximum velocity of the body, its maximum acceleration and the maximum restoring force to which the body is subjected.

4. A body describing SHM has a maximum acceleration of $8\pi \text{ m/s}^2$ and a maximum speed of 1.6 m/s. Find the period T and the amplitude A.

5. Given that the equation of motion of a mass is $x = 0.20 \sin(3.0t)$ m. Find the velocity and acceleration of the mass when the object is 5 cm from its equilibrium position. Repeat for $x = 0$.

6. A particle executes simple harmonic motion of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant δ, if the equation is written as $x = A\sin(\omega t + \delta)$.

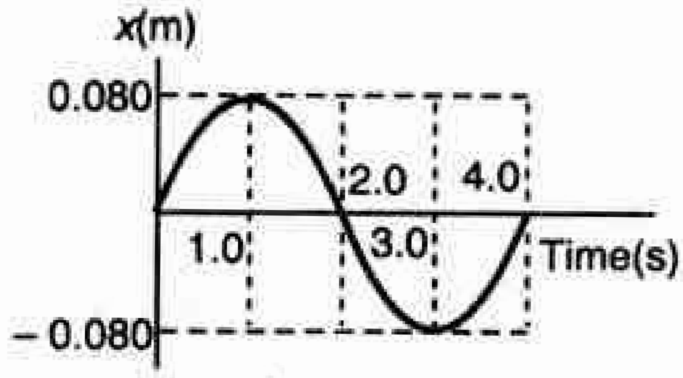
7. A body makes angular simple harmonic motion of amplitude $\pi/10$ rad and time period 0.05 s. If the body is at a displacement $\theta = \pi/10$ rad at $t = 0$, write the equation giving angular displacement as a function of time.

8. The equation of motion of a particle started at $t = 0$ is given by $x = 5\sin\left(20t + \frac{\pi}{3}\right)$, where x is in cm and t in sec. When does the particle
(a) first come to rest,
(b) first have zero acceleration,
(c) first have maximum speed.

9. A particle executes simple harmonic motion of period 16 s. Two seconds later after it passes through the centre of oscillation its velocity is found to be 2 m/s. Find the amplitude.

10. The period of a particle in SHM is 8 s. At $t = 0$ it is in its equilibrium position.
(a) Compare the distance travelled in the first 4 s and the second 4 s.
(b) Compare the distance travelled in the first 2 s and the second 2 s.

11. An object of mass 0.8 kg is attached to one end of a spring and the system is set into simple harmonic motion. The displacement x of the object as a function of time is shown in the figure. With the aid of the data, determine



(a) the amplitude A of the motion,
(b) the angular frequency ω,
(c) the spring constant k,
(d) the speed of the object at $t = 1.0$ s and
(e) the magnitude of the object's acceleration at $t = 1.0$ s.

12. (a) The motion of the particle in simple harmonic motion is given by $x = a \sin \omega t$.

If its speed is u, when the displacement is x_1 and speed is v, when the displacement is x_2, show that the amplitude of the motion is

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{1/2}$$

(b) A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values x_1 and x_2, the corresponding values of velocity are u_1 and u_2, show that the period is

$$T = 2\pi \left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$$

Energy in SHM

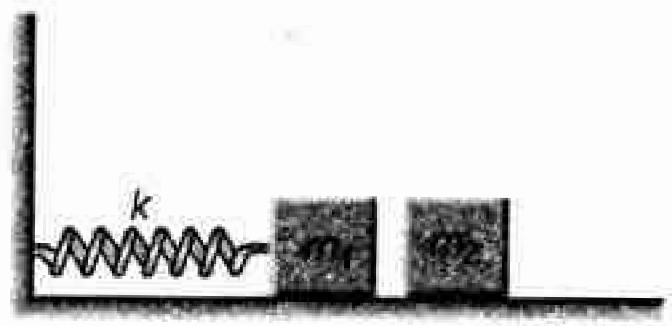
13. A spring-mass oscillator has a total energy E_0 and an amplitude x_0.

(a) How large will K and U be for it when $x = \frac{1}{2} x_0$?

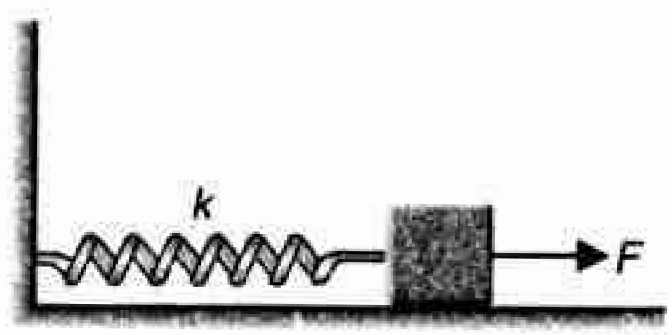
(b) For what value of x will $K = U$?

14. Show that the combined spring energy and gravitational energy for a mass m hanging from a light spring of force constant k can be expressed as $\frac{1}{2} ky^2$, where y is the distance above or below the equilibrium position.

15. The masses in figure slide on a frictionless table. m_1 but not m_2, is fastened to the spring. If now m_1 and m_2 are pushed to the left, so that the spring is compressed a distance d, what will be the amplitude of the oscillation of m_1 after the spring system is released ?

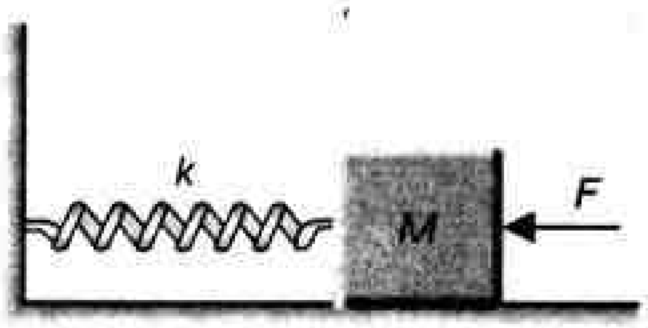


16. The spring shown in figure is unstretched when a man starts pulling on the cord. The mass of the block is M. If the man exerts a constant force F, find



(a) the amplitude and the time period of the motion of the block,
(b) the energy stored in the spring when the block passes through the equilibrium position and
(c) the kinetic energy of the block at this position.

17. In figure, $k = 100$ N/m, $M = 1$ kg and $F = 10$ N.



(a) Find the compression of the spring in the equilibrium position.
(b) A sharp blow by some external agent imparts a speed of 2 m/s to the block towards left. Find the sum of the potential energy of the spring and the kinetic energy of the block at this instant.
(c) Find the time period of the resulting simple harmonic motion.
(d) Find the amplitude.
(e) Write the potential energy of the spring when the block is at the left extreme.
(f) Write the potential energy of the spring when the block is at the right extreme.

The answers of (b), (e) and (f) are different. Explain why this does not violate the principle of conservation of energy?

18. A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Write down the equation of motion of this particle when the initial phase of oscillation is 45°.

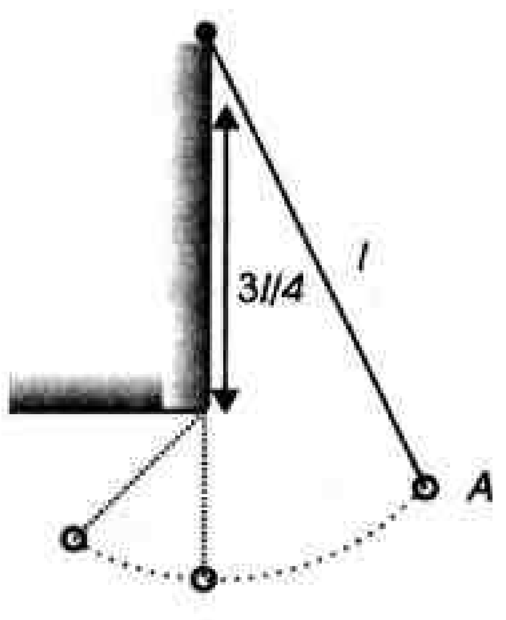
19. Potential energy of a particle in SHM along x-axis is given by :

$$U = 10 + (x - 2)^2$$

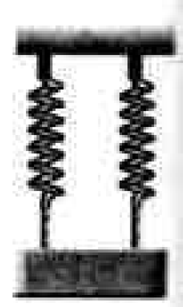
Here, U is in joule and x in metre. Total mechanical energy of the particle is 26 J. Mass of the particle is 2 kg. Find :
(a) angular frequency of SHM,
(b) potential energy and kinetic energy at mean position and extreme position,
(c) amplitude of oscillation,
(d) x-coordinates between which particle oscillates.

Time period of SHM

20. A pendulum has a period T for small oscillations. An obstacle is placed directly beneath the pivot, so that only the lowest one-quarter of the string can follow the pendulum bob when it swings to the left of its resting position. The pendulum is released from rest at a certain point. How long will it take to return to that point ? In answering this question, you may assume that the angle between the moving string and the vertical stays small throughout the motion.



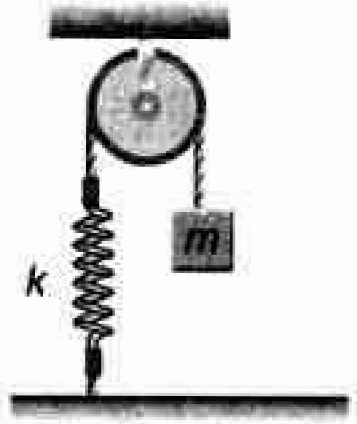
21. An object suspended from a spring exhibits oscillations of period T. Now, the spring is cut in half and the two halves are used to support the same object, as shown in figure. Show that the new period of oscillation is $T/2$.



22. Assume that a narrow tunnel is dug between two diametrically opposite points of the earth. Treat the earth as a solid sphere of uniform density. Show that if a particle is released in this tunnel, it will execute a simple harmonic motion. Calculate the time period of this motion.

23. A simple pendulum is taken at a place where its separation from the earth's surface is equal to the radius of the earth. Calculate the time period of small oscillations if the length of the string is 1.0 m. Take $g = \pi^2\ m/s^2$ at the surface of the earth.

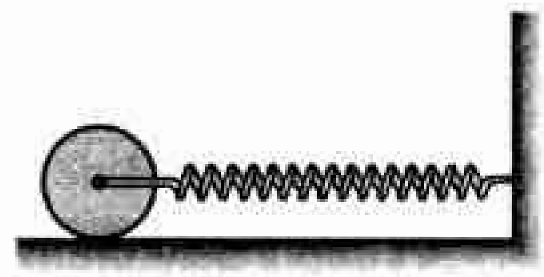
24. The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.



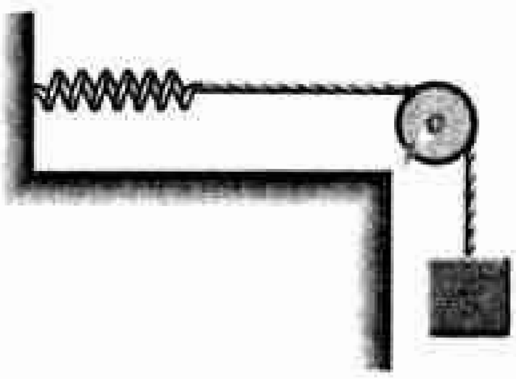
25. A solid cylinder of mass $M = 10\,kg$ and cross-sectional area $A = 20\,cm^2$ is suspended by a spring of force constant $k = 100$ N/m and hangs partially immersed in water. Calculate the period of small oscillations of the cylinder.

26. Pendulum A is a physical pendulum made from a thin, rigid and uniform rod whose length is d. One end of this rod is attached to the ceiling by a frictionless hinge, so that the rod is free to swing back and forth. Pendulum B is a simple pendulum whose length is also d. Obtain the ratio $\frac{T_A}{T_B}$ of their periods for small angle oscillations.

27. A solid cylinder of mass m is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. (See the accompanying figure.) Show that the centre of mass of the cylinder executes simple harmonic motion with a period $T = 2\pi\sqrt{\frac{3m}{2k}}$, if displaced from mean position.



28. A cord is attached between a 0.50 kg block and a spring with force constant $k = 20$ N/m. The other end of the spring is attached to the wall and the cord is placed over a pulley ($I = 0.60\ MR^2$) of mass 5.0 kg and radius 0.50 m. (See the accompanying figure). Assuming no slipping occurs, what is the frequency of the oscillations when the body is set into motion ?



Combination of two or more simple harmonic motions

29. Two linear SHM of equal amplitudes A and frequencies ω and 2ω are impressed on a particle along x and y-axes respectively. If the initial phase difference between them is $\pi/2$. Find the resultant path followed by the particle.

30. Three simple harmonic motions of equal amplitudes A and equal time periods in the same direction combine. The phase of the second motion is 60° ahead of the first and the phase of the third motion is 60° ahead of the second. Find the amplitude of the resultant motion.

31. A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions. Find the phase difference between the individual motions.

32. A particle is subjected to two simple harmonic motions given by

$$x_1 = 2.0 \sin(100\pi t)$$

and $$x_2 = 2.0 \sin(120\pi t + \pi/3)$$

where, x is in cm and t in second. Find the displacement of the particle at

(a) $t = 0.0125$, (b) $t = 0.025$.

Objective Questions

Single Correct Option

1. A simple harmonic oscillation has an amplitude A and time period T. The time required to travel from $x = A$ to $x = \frac{A}{2}$ is

(a) $\frac{T}{6}$ (b) $\frac{T}{4}$ (c) $\frac{T}{3}$ (d) $\frac{T}{12}$

2. The potential energy of a particle executing SHM varies sinusoidally with frequency f. The frequency of oscillation of the particle will be

(a) $\frac{f}{2}$ (b) $\frac{f}{\sqrt{2}}$ (c) f (d) $2f$

3. For a particle undergoing simple harmonic motion, the velocity is plotted against displacement. The curve will be

(a) a straight line (b) a parabola (c) a circle (d) an ellipse

4. A simple pendulum is made of bob which is a hollow sphere full of sand suspended by means of a wire. If all the sand is drained out, the period of the pendulum will

(a) increase (b) decrease (c) remain same (d) become erratic

5. Two simple harmonic motions are given by $y_1 = a\sin\left[\left(\frac{\pi}{2}\right)t + \phi\right]$ and $y_2 = b\sin\left[\left(\frac{2\pi}{3}\right)t + \phi\right]$. The phase difference between these after 1 s is

(a) zero (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

6. A particle starts performing simple harmonic motion. Its amplitude is A. At one time its speed is half that of the maximum speed. At this moment the displacement is

(a) $\frac{\sqrt{2}A}{3}$ (b) $\frac{\sqrt{3}A}{2}$ (c) $\frac{2A}{\sqrt{3}}$ (d) $\frac{3A}{\sqrt{2}}$

7. The length of a simple pendulum is decreased by 21%. The percentage change in its time period is

(a) 11% decrease (b) 11% increase (c) 21% decrease (d) 21% increase

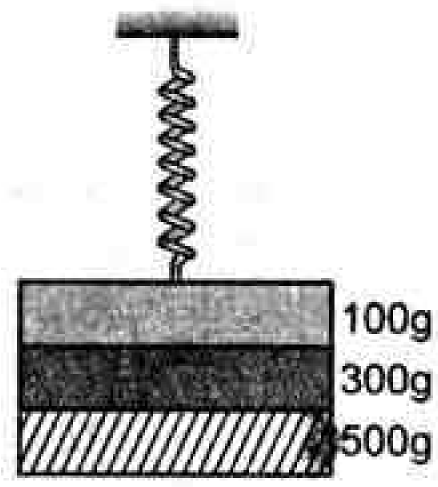
8. Which of the following is not simple harmonic function?

(a) $y = a\sin 2\omega t + b\cos^2 \omega t$
(b) $y = a\sin \omega t + b\cos 2\omega t$
(c) $y = 1 - 2\sin^2 \omega t$
(d) $y = (\sqrt{a^2 + b^2})\sin \omega t \cos \omega t$

9. A mass M, attached to a spring, oscillates with a period of 2 s. If the mass is increased by 4 kg, the time period increases by one second. Assuming that Hooke's law is obeyed, the initial mass M was

(a) 3.2 kg (b) 1 kg (c) 2 kg (d) 8 kg

10. Three masses of 500 g, 300 g and 100 g are suspended at the end of a spring as shown and are in equilibrium. When the 500 g mass is removed suddenly, the system oscillates with a period of 2 seconds. When the 300 g mass is also removed, it will oscillate with period

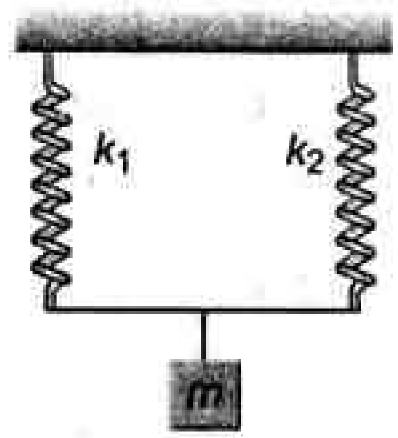


(a) 2 s
(b) 1.25 s
(c) 1.5 s
(d) 1 s

11. The displacement of a particle varies according to the relation $y = 4(\cos \pi t + \sin \pi t)$. The amplitude of the particle is

(a) 8 units (b) 2 units (c) 4 units (d) $4\sqrt{2}$ units

12. A mass is suspended separately by two springs of spring constant k_1 and k_2 in successive order. The time periods of oscillations in the two cases are T_1 and T_2 respectively. If the same mass be suspended by connecting the two springs in parallel (as shown in figure) then the time period of oscillations is T.



The correct relation is

(a) $T^2 = T_1^2 + T_2^2$
(b) $T^{-2} = T_1^{-2} + T_2^{-2}$
(c) $T^{-1} = T_1^{-1} + T_2^{-1}$
(d) $T = T_1 + T_2$

13. A particle executes simple harmonic motion. Its instantaneous acceleration is given by $a = -px$, where p is a positive constant and x is the displacement from the mean position. Angular frequency of the particle is given by

(a) $\frac{1}{p}$ (b) p (c) $\sqrt{p}$ (d) $\frac{1}{\sqrt{p}}$

14. Two pendulums X and Y of time periods 4 s and 4.2 s are made to vibrate simultaneously. They are initially in same phase. After how many vibrations of X, they will be in the same phase again?

(a) 30 (b) 25 (c) 21 (d) 26

15. A mass M is suspended from a massless spring. An additional mass m stretches the spring further by a distance x. The combined mass will oscillate with a period

(a) $2\pi\sqrt{\left\{\frac{(M+m)x}{mg}\right\}}$
(b) $2\pi\sqrt{\left\{\frac{mg}{(M+m)x}\right\}}$
(c) $2\pi\sqrt{\left\{\frac{(M+m)}{mgx}\right\}}$
(d) $\frac{\pi}{2}\sqrt{\left\{\frac{mg}{(M+m)x}\right\}}$

16. Two bodies P and Q of equal masses are suspended from two separate massless springs of force constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal. The ratio of the amplitude of P to that of Q is

(a) $\sqrt{\frac{k_1}{k_2}}$ (b) $\frac{k_1}{k_2}$ (c) $\sqrt{\frac{k_2}{k_1}}$ (d) $\frac{k_2}{k_1}$

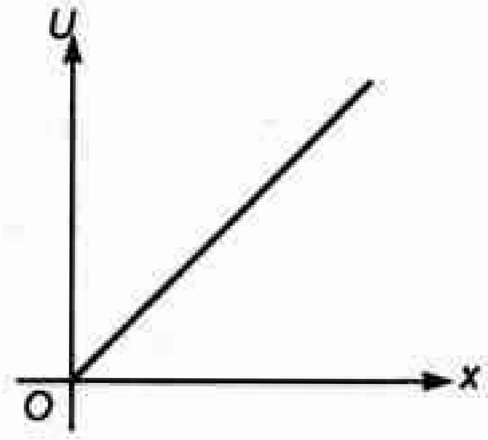
17. A simple harmonic oscillator has an acceleration of 1.25 m/s^2 at 5 cm from the equilibrium. Its period of oscillation is

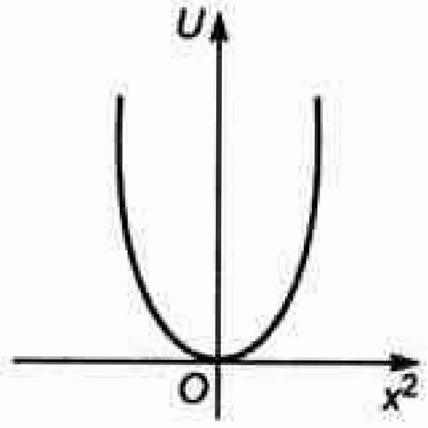
(a) $\frac{4\pi}{5}$ s (b) $\frac{5\pi}{2}$ s (c) $\frac{2\pi}{5}$ s (d) $\frac{2\pi}{25}$ s

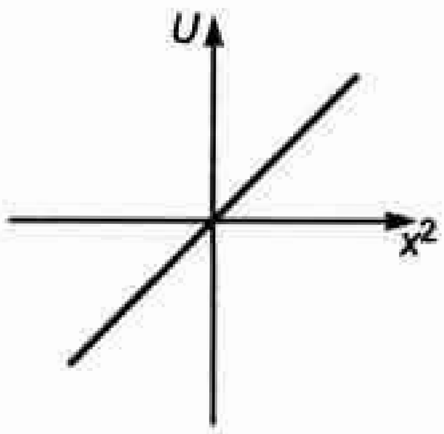
18. A disc of radius R is pivoted at its rim. The period for small oscillations about an axis perpendicular to the plane of disc is

(a) $2\pi\sqrt{\frac{R}{g}}$ (b) $2\pi\sqrt{\frac{2R}{g}}$ (c) $2\pi\sqrt{\frac{2R}{3g}}$ (d) $2\pi\sqrt{\frac{3R}{2g}}$

19. Identify the correct variation of potential energy U as a function of displacement x from mean positioin of a harmonic oscillator (U at mean position $= 0$)

(a) 

(b) 

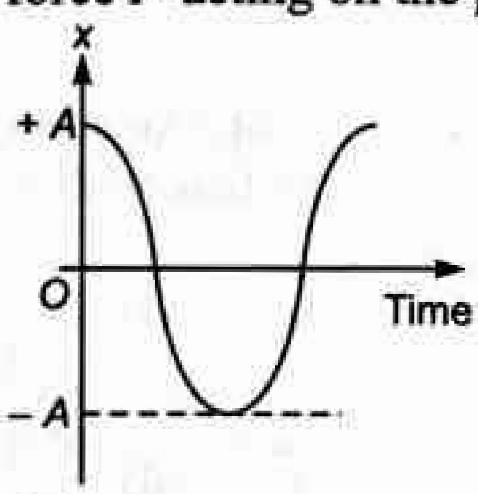
(c) 

(d) None of these

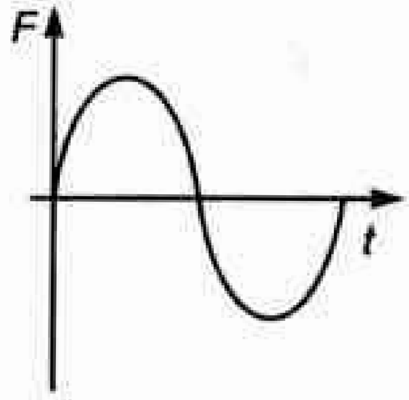
20. If the length of a simple pendulum is equal to the radius of the earth, its time period will be

(a) $2\pi\sqrt{R/g}$ (b) $2\pi\sqrt{R/2g}$ (c) $2\pi\sqrt{2R/g}$ (d) infinite

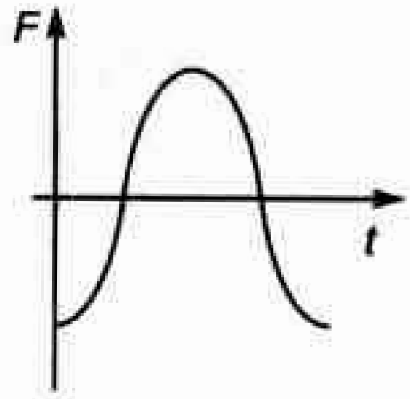
21. The displacement-time (x-t) graph of a particle executing simple harmonic motion is shown in figure. The correct variation of net force F acting on the particle as a function of time is



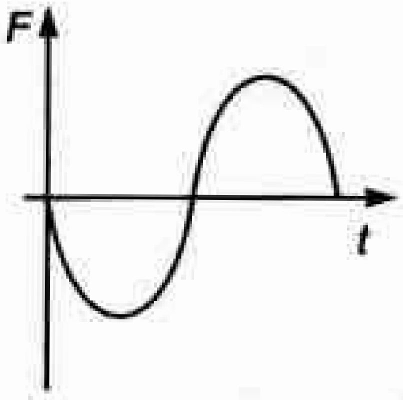
(a)



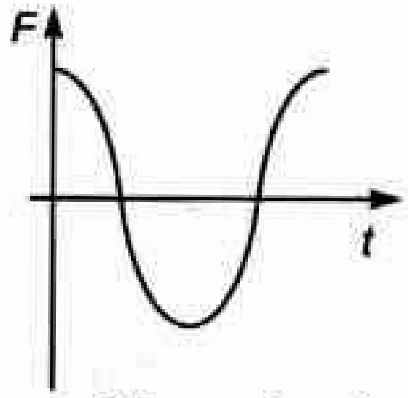
(b)



(c)



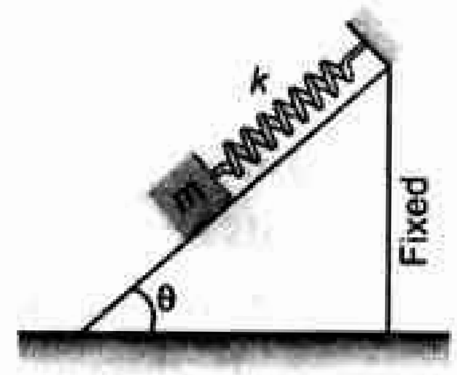
(d)



22. A mass M is suspended from a spring of negligible mass. The spring is pulled a little then released, so that the mass executes simple harmonic motion of time period T. If the mass is increased by m, the time period becomes $\frac{5T}{3}$. The ratio of m/M is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{16}{9}$ (d) $\frac{25}{9}$

23. In the figure shown the time period and the amplitude respectively, when m is left from rest when spring is relaxed are (the inclined plane is smooth)



(a) $2\pi\sqrt{\frac{m}{k}}, \frac{mg\sin\theta}{k}$

(b) $2\pi\sqrt{\frac{m\sin\theta}{k}}, \frac{2\,mg\sin\theta}{k}$

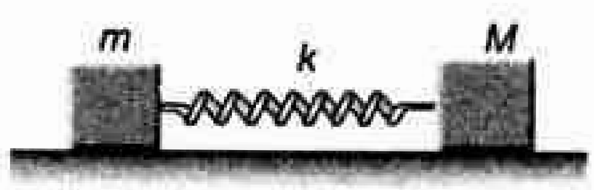
(c) $2\pi\sqrt{\frac{m}{k}}, \frac{mg\cos\theta}{k}$

(d) None of the above

24. The equation of motion of a particle of mass 1 g is $\frac{d^2x}{dt^2} + \pi^2 x = 0$, where x is displacement (in m) from mean position. The frequency of oscillation is (in Hz)

(a) 1/2 (b) 2 (c) $5\sqrt{10}$ (d) $1/5\sqrt{10}$

25. The spring as shown in figure is kept in a stretched position with extension x when the system is released. Assuming the horizontal surface to be frictionless, the frequency of oscillation is



(a) $\frac{1}{2\pi}\sqrt{\left[\frac{k(M+m)}{Mm}\right]}$ (b) $\frac{1}{2\pi}\sqrt{\left[\frac{mM}{k(M+m)}\right]}$

(c) $\frac{1}{2\pi}\sqrt{\left[\frac{kM}{m+M}\right]}$ (d) $\frac{1}{2\pi}\sqrt{\left[\frac{km}{M+m}\right]}$

26. A particle executes SHM on a straight line path. The amplitude of oscillation is 2 cm. When the displacement of the particle from the mean position is 1 cm, the numerical value of magnitude of acceleration is equal to the numerical value of magnitude of velocity. The frequency of SHM (in Hz) is

(a) $\frac{1}{\pi}$ (b) $\frac{\sqrt{2}}{\pi}$ (c) $\frac{\sqrt{3}}{2\pi}$ (d) $\frac{1}{2\pi\sqrt{3}}$

27. The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet. It is a second's pendulum on earth?

(a) $\sqrt{2}$ s (b) $2\sqrt{2}$ s (c) $\frac{1}{\sqrt{2}}$ s (d) $\frac{1}{2\sqrt{2}}$ s

28. The resultant amplitude due to superposition of three simple harmonic motions $x_1 = 3\sin\omega t$, $x_2 = 5\sin(\omega t + 37°)$ and $x_3 = -15\cos\omega t$ is

(a) 18 (b) 10 (c) 12 (d) None of these

29. Two SHMs $s_1 = a\sin\omega t$ and $s_2 = b\sin\omega t$ are superimposed on a particle. The s_1 and s_2 are along the directions which makes 37° to each other

(a) the particle will perform SHM (b) the path of particle is straight line
(c) Both (a) and (b) are correct (d) Both (a) and (b) are wrong

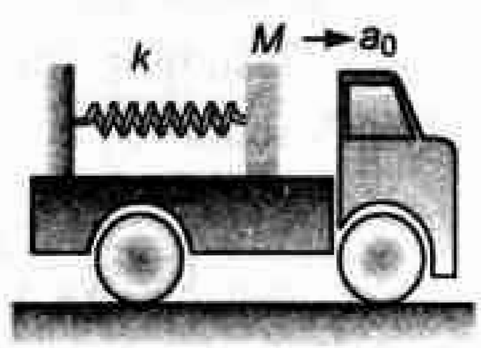
30. The amplitude of a particle executing SHM about O is 10 cm. Then

(a) when the KE is 0.64 times of its maximum KE, its displacement is 6 cm from O
(b) its speed is half the maximum speed when its displacement is half the maximum displacement
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

31. A particle is attached to a vertical spring and is pulled down a distance 4 cm below its equilibrium and is released from rest. The initial upward acceleration is 0.5 ms^{-2}. The angular frequency of oscillation is

(a) 3.53 rad/s (b) 0.28 rad/s (c) 1.25 rad/s (d) 0.08 rad/s

32. A block of mass 1 kg is kept on smooth floor of a truck. One end of a spring of force constant 100 N/m is attached to the block and other end is attached to the body of truck as shown in the figure. At $t = 0$, truck begins to move with constant acceleration 2 m/s^2. The amplitude of oscillation of block relative to the floor of truck is



(a) 0.06 m (b) 0.02 m (c) 0.04 m (d) 0.03 m

For **JEE Advanced**

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion and Reason** are true and the **Reason** is correct explanation of the **Assertion.***
(*b*) *If both **Assertion and Reason** are true but **Reason** is not the correct explanation of **Assertion.***
(*c*) *If **Assertion** is true, but the **Reason** is false.*
(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** In $x = A\cos\omega t$, x is the displacement measured from extreme position.
Reason : In the above equation $x = A$ at time $t = 0$.

2. **Assertion :** A particle is under SHM along the x-axis. Its mean position is $x = 2$, amplitude is $A = 2$ and angular frequency ω. At $t = 0$, particle is at origin, then x-co-ordinate versus time equation of the particle will be $x = -2\cos\omega t + 2$.
Reason : At $t = 0$, particle is at rest.

3. **Assertion :** A spring block system is kept over a smooth surface as shown in figure. If a constant horizontal force F is applied on the block it will start oscillating simple harmonically.

Reason : Time period of oscillation is less than $2\pi\sqrt{\frac{m}{k}}$.

4. **Assertion :** Time taken by a particle in SHM to move from $x = A$ to $x = \frac{\sqrt{3}A}{2}$ is same as the time taken by the particle to move from $x = \frac{\sqrt{3}A}{2}$ to $x = \frac{A}{2}$.
Reason : Corresponding angles rotated in the reference circle are same in the given time intervals.

5. **Assertion :** Path of a particle in SHM is always a straight line.
Reason : All straight line motions are not simple harmonic.

6. **Assertion :** In spring block system if length of spring and mass of block both are halved, then angular frequency of oscillations will remain unchanged.
Reason : Angular frequency is given by $\omega = \sqrt{\frac{k}{m}}$

7. **Assertion :** All small oscillations are simple harmonic in nature.
Reason : Oscillations of spring block system are always simple harmonic whether amplitude is small or large.

8. **Assertion :** In $x = A\cos\omega t$, the dot product of acceleration and velocity is positive for time interval $0 < t < \frac{\pi}{2\omega}$.
Reason : Angle between them is 0°.