9. **Assertion :** In simple harmonic motion displacement and acceleration always have a constant ratio.

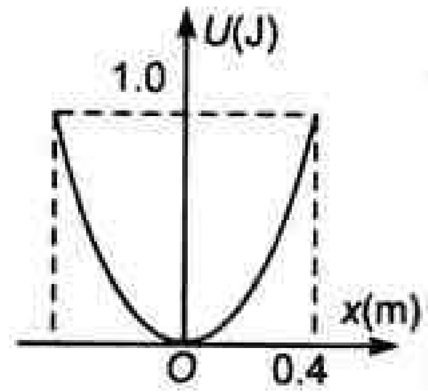
Reason : $T = 2\pi\sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$

10. **Assertion :** We can call circular motion also as simple harmonic motion.
Reason : Angular velocity in uniform circular motion and angular frequency in simple harmonic motion have the same meanings.

Objective Questions

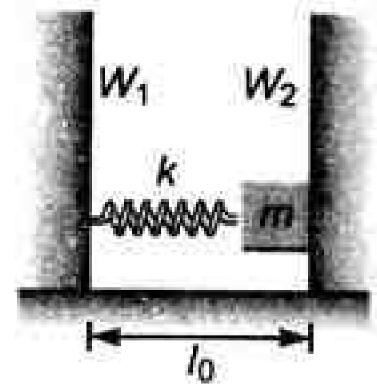
Single Correct Option

1. A particle of mass 2 kg moves in simple harmonic motion and its potential energy U varies with position x as shown. The period of oscillation of the particle is



(a) $\frac{2\pi}{5}$ s (b) $\frac{2\sqrt{2}\pi}{5}$ s
(c) $\frac{\sqrt{2}\pi}{5}$ s (d) $\frac{4\pi}{5}$ s

2. In the figure shown, a spring mass system is placed on a horizontal smooth surface in between two vertical rigid walls W_1 and W_2. One end of spring is fixed with wall W_1 and other end is attached with mass m which is free to move. Initially, spring is tension free and having natural length l_0. Mass m is compressed through a distance a and released. Taking the collision between wall W_2 and mass m as elastic and K as spring constant, the average force exerted by mass m on wall W_2 in one oscillation of block is



(a) $\frac{2aK}{\pi}$ (b) $\frac{2ma}{\pi}$ (c) $\frac{aK}{\pi}$ (d) $\frac{2aK}{m}$

3. Two simple harmonic motions are represented by the following equations $y = 40\sin\omega t$ and $y_2 = 10(\sin\omega t + c\cos\omega t)$. If their displacement amplitudes are equal, then the value of c (in appropriate units) is

(a) $\sqrt{13}$ (b) $\sqrt{15}$ (c) $\sqrt{17}$ (d) 4

4. A particle executes simple harmonic motion with frequency 2.5 Hz and amplitude 2 m. The speed of the particle 0.3 s after crossing, the equilibrium position is

(a) zero (b) 2π m/s (c) 4π m/s (d) π m/s

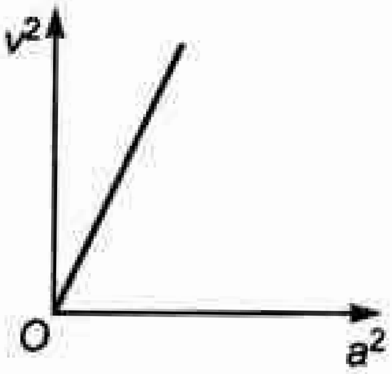
5. A particle oscillates simple harmonically with a period of 16 s. Two second after crossing the equilibrium position its velocity becomes 1 m/s. The amplitude is

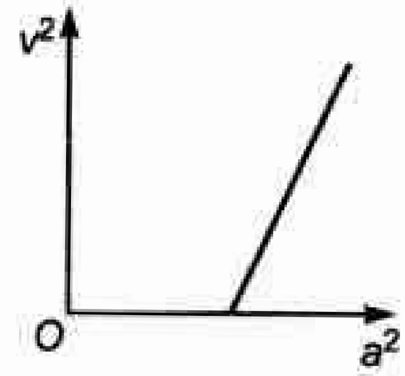
(a) $\frac{\pi}{4}$ m (b) $\frac{8\sqrt{2}}{\pi}$ m (c) $\frac{8}{\pi}$ m (d) $\frac{4\sqrt{2}}{\pi}$ m

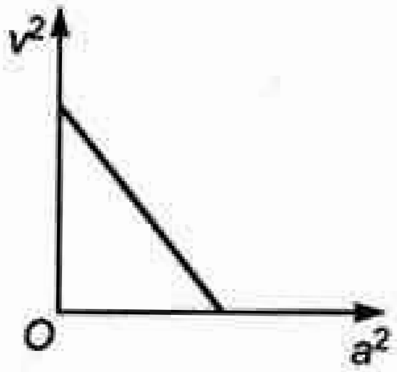
6. A seconds pendulum is suspended from the ceiling of a trolley moving horizontally with an acceleration of 4 m/s ^{2}. Its period of oscillation is

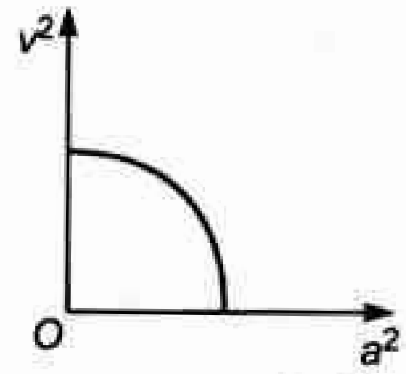
(a) 1.90 s (b) 1.70 s (c) 2.30 s (d) 1.40 s

7. A particle is performing a linear simple harmonic motion. If the instantaneous acceleration and velocity of the particle are a and v respectively, identify the graph which correctly represents the relation between a and v

(a) 

(b) 

(c) 

(d) 

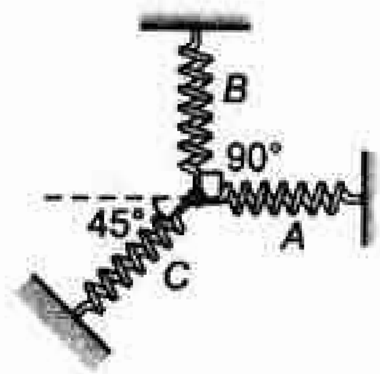
8. In a vertical U-tube a column of mercury oscillates simple harmonically. If the tube contains 1 kg of mercury and 1 cm of mercury column weighs 20 g, then the period of oscillation is

(a) 1 s (b) 2 s (c) $\sqrt{2}$ s (d) Insufficient data

9. A solid cube of side a and density ρ_0 floats on the surface of a liquid of density ρ. If the cube is slightly pushed downward, then it oscillates simple harmonically with a period of

(a) $2\pi\sqrt{\frac{\rho_0}{\rho}\frac{a}{g}}$ (b) $2\pi\sqrt{\frac{\rho}{\rho_0}\frac{a}{g}}$ (c) $2\pi\sqrt{\frac{a}{\left(1-\frac{\rho}{\rho_0}\right)g}}$ (d) $2\pi\sqrt{\frac{a}{\left(1+\frac{\rho}{\rho_0}\right)g}}$

10. A particle of mass m is attached with three springs A, B and C of equal force constants k as shown in figure. The particle is pushed slightly against the spring C and released, the time period of oscillation will be



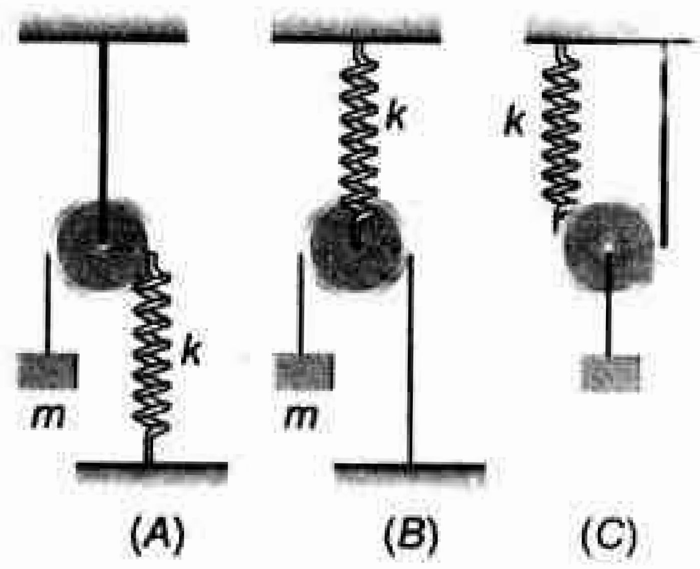
(a) $2\pi\sqrt{\frac{m}{k}}$ (b) $2\pi\sqrt{\frac{m}{2k}}$

(c) $2\pi\sqrt{\frac{m}{3k}}$ (d) $2\pi\sqrt{\frac{m}{5k}}$

11. A uniform stick of length l is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance d from the centre of mass. The time period of small oscillations has a minimum value when d/l is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{12}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

12. Three arrangements of spring-mass system are shown in figures (A), (B) and (C). If T_1, T_2 and T_3 represent the respective periods of oscillation, then correct relation is



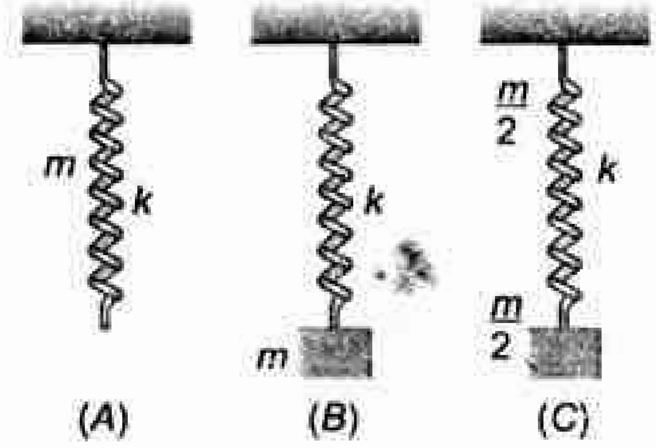
(a) $T_1 > T_2 > T_3$

(b) $T_3 > T_2 > T_1$

(c) $T_2 > T_1 > T_3$

(d) $T_2 > T_3 > T_1$

13. Three arrangements are shown in figure.



(A) A spring of mass m and stiffness k

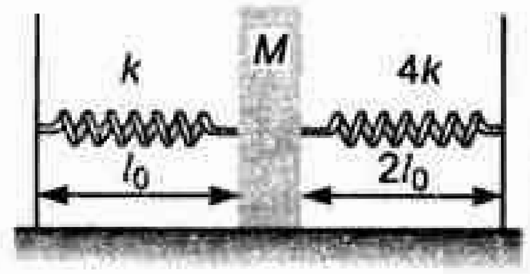
(B) A block of mass m attached to massless spring of stiffness k

(C) A block of mass $\frac{m}{2}$ attached to a spring of mass $\frac{m}{2}$ and stiffness k

If T_1, T_2 and T_3 represent the period of oscillation in the three cases respectively, then identify the correct relation

(a) $T_1 < T_2 < T_3$ (b) $T_1 < T_3 < T_2$ (c) $T_1 > T_3 > T_2$ (d) $T_3 < T_1 < T_2$

14. A block of mass M is kept on a smooth surface and touches the two springs as shown in the figure but not attached to the springs. Initially springs are in their natural length. Now, the block is shifted $(l_0/2)$ from the given position in such a way that it compresses a spring and released. The time-period of oscillation of mass will be



(a) $\frac{\pi}{2}\sqrt{\frac{M}{k}}$ (b) $2\pi\sqrt{\frac{M}{5k}}$

(c) $\frac{3\pi}{2}\sqrt{\frac{M}{k}}$ (d) $\pi\sqrt{\frac{M}{2k}}$

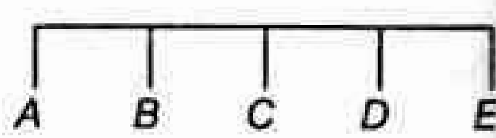
15. A particle moving on x-axis has potential energy $U = 2 - 20x + 5x^2$ Joule along x-axis. The particle is released at $x = -3$. The maximum value of x will be (x is in metre)

(a) 5 m (b) 3 m (c) 7 m (d) 8 m

16. A block of mass m, when attached to a uniform ideal spring with force constant k and free length L executes SHM. The spring is then cut in two pieces, one with free length nL and other with free length $(1-n)L$. The block is also divided in the same fraction. The smaller part of the block attached to longer part of the spring executes SHM with frequency f_1. The bigger part of the block attached to smaller part of the spring executes SHM with frequency f_2. The ratio f_1/f_2 is

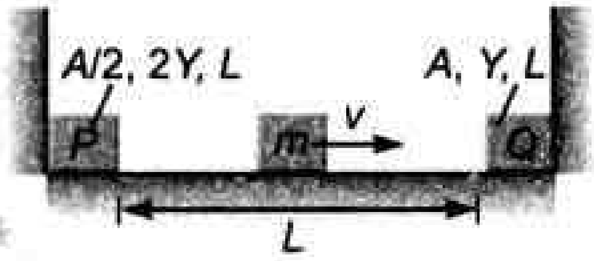
(a) 1 (b) $\frac{n}{1-n}$ (c) $\frac{1+n}{n}$ (d) $\frac{n}{1+n}$

17. A body performs simple harmonic oscillations along the straight line $ABCDE$ with C as the midpoint of AE. Its kinetic energies at B and D are each one fourth of its maximum value. If $AE = 2R$, the distance between B and D is



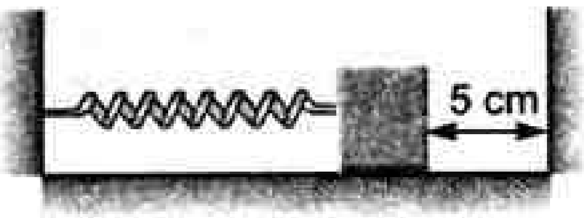
(a) $\frac{\sqrt{3}}{2}R$ (b) $\frac{R}{\sqrt{2}}$ (c) $\sqrt{3}R$ (d) $\sqrt{2}R$

18. In the given figure, two elastic rods P and Q are rigidly joined to end supports. A small mass m is moving with velocity v between the rods. All collisions are assumed to be elastic and the surface is given to be smooth. The time period of small mass m will be; (A = area of cross section, Y = Young's modulus, L = length of each rod)



(a) $\frac{2L}{v}+2\pi\sqrt{\frac{mL}{AY}}$ (b) $\frac{2L}{v}+2\pi\sqrt{\frac{2mL}{AY}}$ (c) $\frac{2L}{v}+\pi\sqrt{\frac{mL}{AY}}$ (d) $\frac{2L}{v}$

19. A block of mass 100 g attached to a spring of stiffness 100 N/m is lying on a frictionless floor as shown. The block is moved to compress the spring by 10 cm and released. If the collision with the wall is elastic the time period of motion is



(a) 0.2 s (b) 0.166 s (c) 0.155 s (d) 0.133 s

20. A particle executes SHM of period 1.2 s and amplitude 8 cm. Find the time it takes to travel 3 cm from the positive extremity of its oscillation. [$\cos^{-1}(5/8) = 0.9$ rad]

(a) 0.28 s (b) 0.32 s (c) 0.17 s (d) 0.42 s

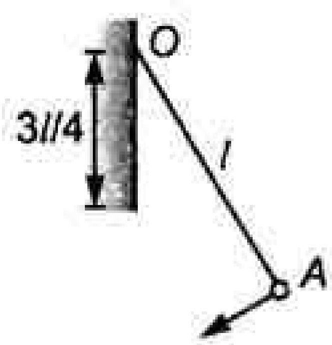
21. A wire frame in the shape of an equilateral triangle is hinged at one vertex so that it can swing freely in a vertical plane, with the plane of the triangle always remaining vertical. The side of the frame is $1/\sqrt{3}$ m. The time period in seconds of small oscillations of the frame will be ($g = 10\ m/s^2$)

(a) $\pi/\sqrt{2}$ (b) $\pi/\sqrt{3}$ (c) $\pi/\sqrt{6}$ (d) $\pi/\sqrt{5}$

22. A particle moves along the x-axis according to $x = A[1+\sin\omega t]$. What distance does is travel in time interval from $t = 0$ to $t = 2.5\,\pi/\omega$?

(a) 4 A (b) 6 A (c) 5 A (d) 3 A

23. A small bob attached to a light inextensible thread of length l has a periodic time T when allowed to vibrate as a simple pendulum. The thread is now suspended from a fixed end O of a vertical rigid rod of length $3l/4$. If now the pendulum performs periodic oscillations in this arrangement, the periodic time will be

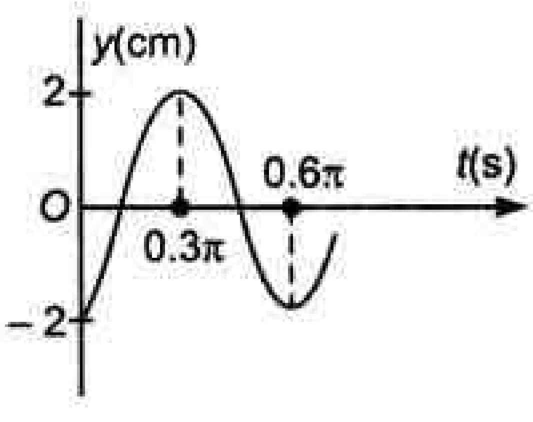


(a) $3T/4$ (b) $4T/5$
(c) $2T/3$ (d) $5T/6$

24. A stone is swinging in a horizontal circle of diameter 0.8 m at 30 rev/min. A distant light causes a shadow of the stone on a nearly wall. The amplitude and period of the SHM for the shadow of the stone are

(a) 0.4 m, 4s (b) 0.2 m, 2s (c) 0.4 m, 2s (d) 0.8 m, 2s

25. Part of SHM is graphed in the figure. Here y is displacement from mean position. The correct equation describing the SHM is



(a) $y=4\cos(0.6\,t)$ (b) $y=2\sin\left(\frac{10}{3}t-\frac{\pi}{2}\right)$

(c) $y=2\sin\left(\frac{\pi}{2}-\frac{10}{3}t\right)$ (d) $y=2\cos\left(0.6\,t+\frac{\pi}{2}\right)$

26. A particle performs SHM with a period T and amplitude a. The mean velocity of particle over the time interval during which it travels a distance $a/2$ from the extreme position is

(a) $6\,a/T$ (b) $2\,a/T$ (c) $3\,a/T$ (d) $a/2T$

27. A man of mass 60 kg is standing on a platform executing SHM in the vertical plane. The displacement from the mean position varies as $y=0.5\sin(2\pi ft)$. The value of f, for which the man will feel weightlessness at the highest point, is (y in metre)

(a) $g/4\pi$ (b) $4\pi g$ (c) $\frac{\sqrt{2g}}{2\pi}$ (d) $2\pi\sqrt{2g}$

28. A particle performs SHM on a straight line with time period T and amplitude A. The average speed of the particle between two successive instants, when potential energy and kinetic energy become same is

(a) $\frac{A}{T}$ (b) $\frac{4\sqrt{2}A}{T}$ (c) $\frac{2A}{T}$ (d) $\frac{2\sqrt{2}A}{T}$

29. The time taken by a particle performing SHM to pass from point A to B where its velocities are same is 2 s. After another 2 s it returns to B. The ratio of distance OB to its amplitude (where O is the mean position) is

(a) $1:\sqrt{2}$ (b) $(\sqrt{2}-1):1$ (c) $1:2$ (d) $1:2\sqrt{2}$

30. A particle is executing SHM according to the equation $x = A\cos\omega t$. Average speed of the particle during the interval $0 \le t \le \frac{\pi}{6\omega}$ is

(a) $\frac{\sqrt{3}A\omega}{2}$ (b) $\frac{\sqrt{3}A\omega}{4}$ (c) $\frac{3\,A\omega}{\pi}$ (d) $\frac{3\,A\omega}{\pi}(2-\sqrt{3})$

Passage : (Q 31 and Q 32)

A 2 kg block hangs without vibrating at the bottom end of a spring with a force constant of 400 N/m. The top end of the spring is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of 5 m/s^2 when the acceleration suddenly ceases at time $t=0$ and the car moves upward with constant speed ($g=10\ m/s^2$)

31. What is the angular frequency of oscillation of the block after the acceleration ceases?

(a) $10\sqrt{2}$ rad/s (b) 20 rad/s (c) $20\sqrt{2}$ rad/s (d) 32 rad/s

32. The amplitude of the oscillation is

(a) 7.5 cm (b) 5 cm (c) 2.5 cm (d) 1 cm

More than One Correct Options

1. A simple pendulum with a bob of mass m is suspended from the roof of a car moving with horizontal acceleration a

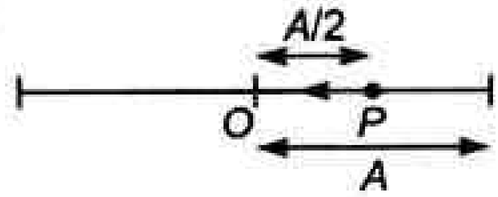
(a) The string makes an angle of $\tan^{-1}(a/g)$ with the vertical

(b) The string makes an angle of $\sin^{-1}\left(\frac{a}{g}\right)$ with the vertical

(c) The tension in the string is $m\sqrt{a^2+g^2}$

(d) The tension in the string is $m\sqrt{g^2-a^2}$

2. A particle starts from a point P at a distance of $A/2$ from the mean position O and travels towards left as shown in the figure. If the time period of SHM, executed about O is T and amplitude A then the equation of the motion of particle is



(a) $x = A\sin\left(\frac{2\pi}{T}t+\frac{\pi}{6}\right)$ (b) $x = A\sin\left(\frac{2\pi}{T}t+\frac{5\pi}{6}\right)$

(c) $x = A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{6}\right)$ (d) $x = A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{3}\right)$

3. A spring has natural length 40 cm and spring constant 500 N/m. A block of mass 1 kg is attached at one end of the spring and other end of the spring is attached to a ceiling. The block is released from the position, where the spring has length 45 cm

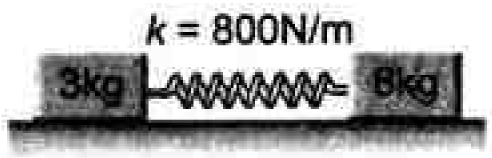
(a) the block will perform SHM of amplitude 5 cm

(b) the block will have maximum velocity $30\sqrt{5}$ cm/s

(c) the block will have maximum acceleration 15 m/s^2

(d) the minimum elastic potential energy of the spring will be zero

4. The system shown in the figure can move on a smooth surface. They are initially compressed by 6 cm and then released



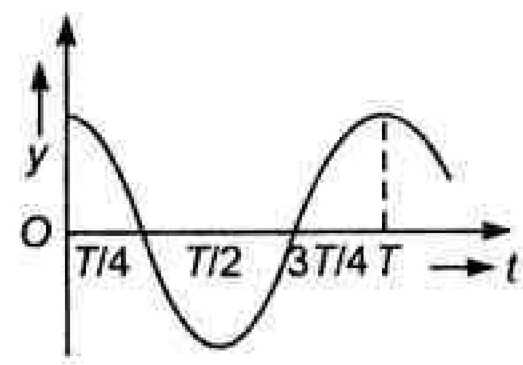
(a) The system performs, SHM with time period $\frac{\pi}{10}$ s

(b) The block of mass 3 kg perform SHM with amplitude 4 cm

(c) The block of mass 6 kg will have maximum momentum of 2.40 kg m/s

(d) The time periods of two blocks are in the ratio of $1:\sqrt{2}$

5. The displacement-time graph of a particle executing SHM is shown in figure. Which of the following statements is/are true?



(a) The velocity is maximum at $t = T/2$

(b) The acceleration is maximum at $t = T$

(c) The force is zero at $t = 3T/4$

(d) The kinetic energy equals the total oscillation energy at $t = T/2$

6. For a particle executing SHM, x = displacement from mean position, v = velocity and a = acceleration at any instant, then

(a) v-x graph is a circle (b) v-x graph is an ellipse

(c) a-x graph is a straight line (d) a-x graph is a circle

7. The acceleration of a particle is $a = -100x + 50$. It is released from $x = 2$. Here a and x are in SI units

(a) the particle will perform SHM of amplitude 2 m

(b) the particle will perform SHM of amplitude 1.5 m

(c) the particle will perform SHM of time period 0.63 s
(d) the particle will have a maximum velocity of 15 m/s

8. Two particles are performing SHM in same phase. It means that
(a) the two particles must have same distance from the mean position simultaneously
(b) two particles may have same distance from the mean position simultaneously
(c) the two particles must have maximum speed simultaneously
(d) the two particles may have maximum speed simultaneously

9. A particle moves along y-axis according to the equation

$$y\,(\text{in cm}) = 3\sin 100\pi t + 8\sin^2 50\pi t - 6$$

(a) the particle performs SHM
(b) the amplitude of the particle's oscillation is 5 cm
(c) the mean position of the particle is at $y = -2$ cm
(d) the particle does not perform SHM

Match the Columns

1. For the x-t equation of a particle in SHM along x-axis, match the following two columns.

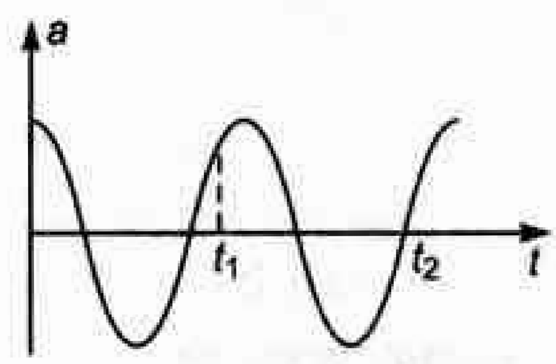
$$x = 2 + 2\cos\omega t$$

Column I	Column II
(a) Mean position	(p) $x = 0$
(b) Extreme position	(q) $x = 2$
(c) Maximum potential energy at	(r) $x = 4$
(d) Zero potential energy at	(s) Can't tell

2. Potential energy of a particle at mean position is 4 J and at extreme position is 20 J. Given that amplitude of oscillation is A. Match the following two columns.

Column I	Column II
(a) Potential energy at $x = \frac{A}{2}$	(p) 18 J
(b) Kinetic energy at $x = \frac{A}{4}$	(q) 16 J
(c) Kinetic energy at $x = 0$	(r) 8 J
(d) Kinetic energy at $x = \frac{A}{2}$	(s) None

3. Acceleration-time graph of a particle in SHM is as shown in figure. Match the following two columns.



Column I	Column II
(a) Displacement of particle at t_1	(p) zero
(b) Displacement of particle at t_2	(q) positive
(c) Velocity of particle at t_1	(r) negative
(d) Velocity of particle at t_2	(s) maximum

4. Mass of a particle is 2 kg. Its displacement-time equation in SHM is

$$x = 2\sin(4\pi t) \qquad \text{(SI Units)}$$

Match the following two columns for 1 second time interval.

Column I	Column II
(a) Speed becomes 30 m/s	(p) two times
(b) Velocity becomes + 10 m/s	(q) four times
(c) Kinetic energy becomes 400 J	(r) one time
(d) Acceleration becomes $-100\ m/s^2$	(s) None

5. x-t equation of a particle in SHM is,

$$x = 4 + 6\sin \pi t$$

Match the following tables corresponding to time taken in moving from

Column I	Column II
(a) $x = 10$ m to $x = 4$ m	(p) $\frac{1}{3}$ second
(b) $x = 10$ m to $x = 7$ m	(q) $\frac{1}{2}$ second
(c) $x = 7$ m to $x = 1$ m	(r) 1 second
(d) $x = 10$ m to $x = -2$ m	(s) None

Subjective Questions

1. A 1 kg block is executing simple harmonic motion of amplitude 0.1 m on a smooth horizontal surface under the restoring force of a spring of spring constant 100 N/m. A block of mass 3 kg is gently placed on it at the instant it passes through the mean position. Assuming that the two blocks move together. Find the frequency and the amplitude of the motion.

2. Two particles are in SHM along same line. Time period of each is T and amplitude is A. After how much time will they collide if at time $t = 0$.

(a) first particle is at $x_1 = +\frac{A}{2}$ and moving towards positive x-axis and second particle is at $x_2 = -\frac{A}{\sqrt{2}}$ and moving towards negative x-axis,

(b) rest information are same as mentioned in part (a) except that particle first is also moving towards negative x-axis.

3. A particle that hangs from a spring oscillates with an angular frequency of 2 rad/s. The spring particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.5 m/s. The car then stops suddenly.

(a) With what amplitude does the particle oscillate ?

(b) What is the equation of motion for the particle ?

(Choose upward as the positive direction)

4. A 2 kg mass is attached to a spring of force constant 600 N/m and rests on a smooth horizontal surface. A second mass of 1 kg slides along the surface toward the first at 6 m/s.

(a) Find the amplitude of oscillation if the masses make a perfectly inelastic collision and remain together on the spring. What is the period of oscillation ?

(b) Find the amplitude and period of oscillation if the collision is perfectly elastic.

(c) For each case, write down the position x as a function of time t for the mass attached to the spring, assuming that the collision occurs at time $t = 0$. What is the impulse given to the 2 kg mass in each case ?

5. A block of mass 4 kg hangs from a spring of force constant $k = 400$ N/m. The block is pulled down 15 cm below equilibrium and released. How long does it take the block to go from 12 cm below equilibrium (on the way up) to 9 cm above equilibrium?

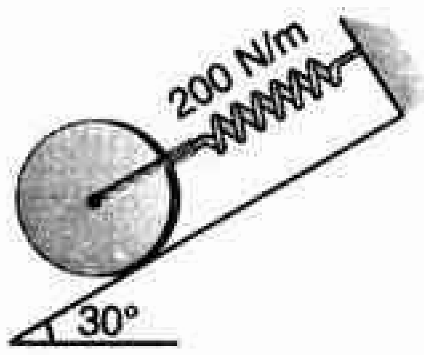
6. A plank with a body of mass m placed on it starts moving straight up according to the law $y = a(1 - \cos \omega t)$, where y is the displacement from the initial position, $\omega = 11$ rad/s. Find :

(a) The time dependence of the force that the body exerts on the plank.

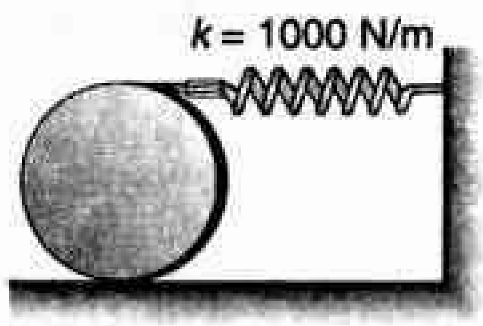
(b) The minimum amplitude of oscillation of the plank at which the body starts falling behind the plank.

7. A particle of mass m free to move in the x-y plane is subjected to a force whose components are $F_x = -kx$ and $F_y = -ky$, where k is a constant. The particle is released when $t = 0$ at the point (2, 3). Prove that the subsequent motion is simple harmonic along the straight line $2y - 3x = 0$.

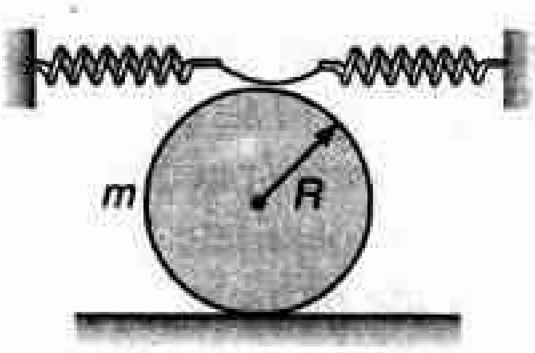
8. Determine the natural frequency of vibration of the 100 N disk. Assume the disk does not slip on the inclined surface.



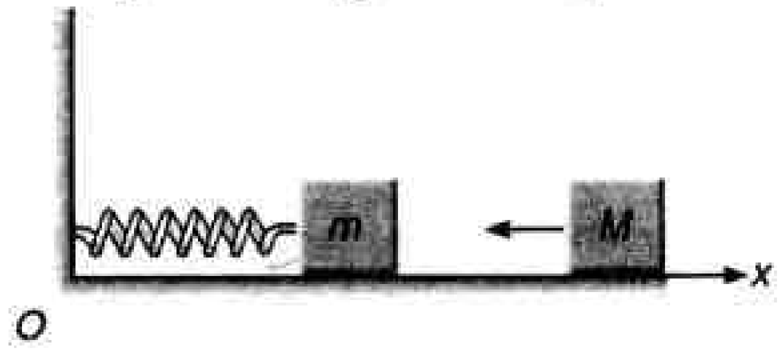
9. The disk has a weight of 100 N and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counterclockwise 0.4 rad, determine the equation which describes its oscillatory motion when it is released.



10. A solid uniform cylinder of mass m performs small oscillations due to the action of two springs of stiffness k each (figure). Find the period of these oscillations in the absence of sliding.

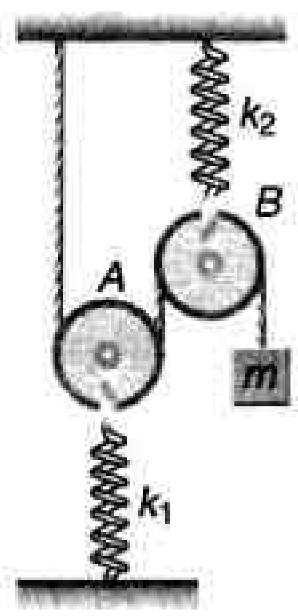


11. One end of an ideal spring is fixed to a wall at origin O and axis of spring is parallel to x-axis. A block of mass $m = 1$ kg is attached to free end of the spring and it is performing SHM. Equation of position of the block in co-ordinate system shown in figure is $x = 10 + 3\sin(10t)$. Here, t is in second and x in cm. Another block of mass $M = 3$ kg, moving towards the origin with velocity 30 cm/s collides with the block performing SHM at $t = 0$ and gets stuck to it. Calculate :

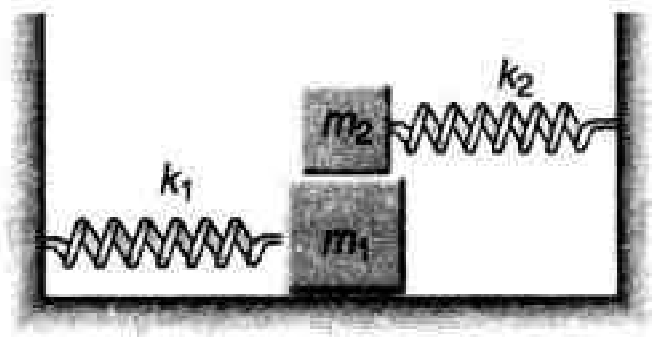


(a) new amplitude of oscillations,
(b) new equation for position of the combined body,
(c) loss of energy during collision. Neglect friction.

12. A block of mass m is attached to one end of a light inextensible string passing over a smooth light pulley B and under another smooth light pulley A as shown in the figure. The other end of the string is fixed to a ceiling. A and B are held by springs of spring constants k_1 and k_2. Find angular frequency of small oscillations of the system.

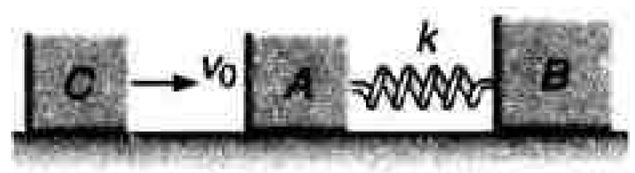


13. In the shown arrangement, both the springs are in their natural lengths. The coefficient of friction between m_2 and m_1 is μ. There is no friction between m_1 and the surface. If the blocks are displaced slightly, they together perform simple harmonic motion. Obtain



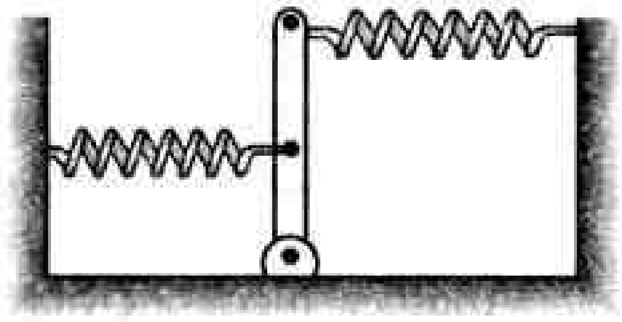
(a) Frequency of such oscillations.
(b) The condition if the frictional force on block m_2 is to act in the direction of its displacement from mean position.
(c) If the condition obtained in (b) is met, what can be maximum amplitude of their oscillations?

14. Two blocks A and B of masses $m_1 = 3$ kg and $m_2 = 6$ kg respectively are connected with each other by a spring of force constant $k = 200$ N/m as shown in figure. Blocks are pulled away from each other by $x_0 = 3$ cm and then released. When spring is in its natural length and blocks are moving towards each other, another block C of mass $m = 3$ kg moving with velocity $v_0 = 0.4$ m/s (towards right) collides with A and gets stuck to it. Neglecting friction, calculate

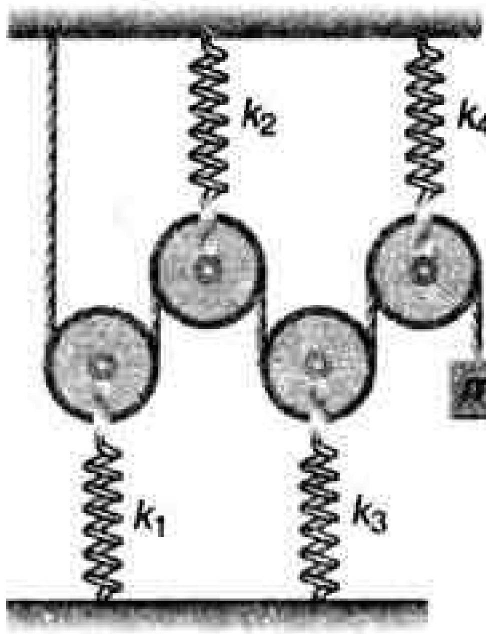


(a) velocities v_1 and v_2 of the blocks A and B respectively just before collision and their angular frequency.
(b) velocity of centre of mass of the system, after collision,
(c) amplitude of oscillations of combined body,
(d) loss of energy during collision.

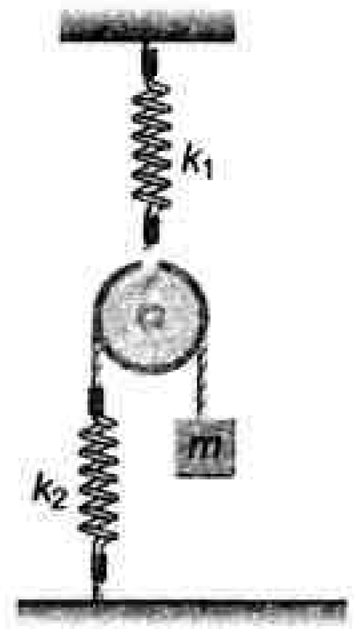
15. A rod of length l and mass m, pivoted at one end, is held by a spring at its mid-point and a spring at far end. The springs have spring constant k. Find the frequency of small oscillations about the equilibrium position.



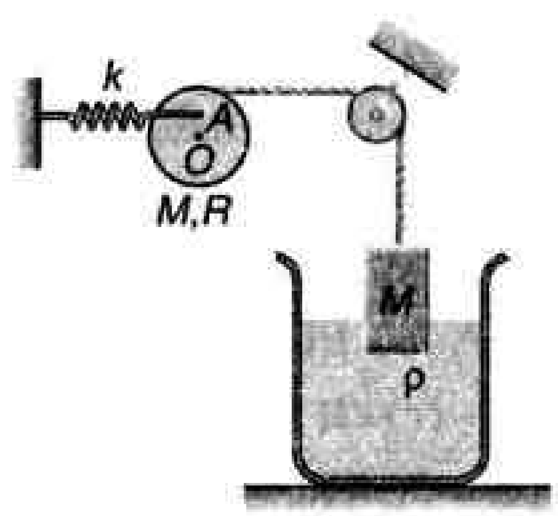
16. In the arrangement shown in figure, pulleys are light and springs are ideal. k_1, k_2, k_3 and k_4 are force constants of the springs. Calculate period of small vertical oscillations of block of mass m.



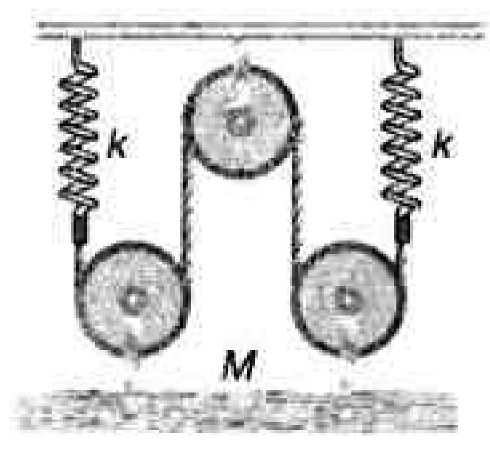
17. A light pulley is suspended at the lower end of a spring of constant k_1, as shown in figure. An inextensible string passes over the pulley. At one end of string a mass m is suspended, the other end of the string is attached to another spring of constant k_2. The other ends of both the springs are attached to rigid supports, as shown. Neglecting masses of springs and any friction, find the time period of small oscillations of mass m about equilibrium position.



18. Figure shows a solid uniform cylinder of radius R and mass M, which is free to rotate about a fixed horizontal axis O and passes through centre of the cylinder. One end of an ideal spring of force constant k is fixed and the other end is higned to the cylinder at A. Distance OA is equal to $\frac{R}{2}$. An inextensible thread is wrapped round the cylinder and passes over a smooth, small pulley. A block of equal mass M and having cross sectional area A is suspended from free end of the thread. The block is partially immersed in a non-viscous liquid of density ρ. If in equilibrium, spring is horizontal and line OA is vertical, calculate frequency of small oscillations of the system.



19. Find the natural frequency of the system shown in figure. The pulleys are smooth and massless.



ANSWERS

Introductory Exercise 11.1

1. 6.0 N/m **2.** $\frac{1}{4}, \frac{3}{4}$ **3.** 10.0 cm, $\frac{\pi}{6}$ rad

4. (a) 15.0 cm (b) 0.726 s (c) 1.38 Hz (d) 1.69 J (e) 1.30 m/s **5.** 1.4×10^{-3} s

6. (a) 1.39 J (b) 1.1 s **7.** (a) 0.28 s (b) 0.4%

Introductory Exercise 11.2

1. $2\pi \sqrt{\frac{l}{\left\{g^2 + \left(\frac{v^2}{r}\right)^2\right\}^{1/2}}}$ and inclined to the vertical at an angle $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$ away from the centre

2. (a) $2\pi \sqrt{\frac{l}{(g+a)}}$ (b) $2\pi \sqrt{\frac{l}{(g-a)}}$ (c) Infinite (d) $2\pi \sqrt{\frac{l}{(g^2+a^2)^{1/2}}}$

3. The clock will lose 10.37 s **4.** $\left(\sqrt{\frac{10}{9}}\right) T$

Introductory Exercise 11.3

1. $T = 2\pi \sqrt{\frac{m}{k}}$ **2.** 0.314 s **3.** $\frac{mv_0}{\sqrt{k(M+m)}}$ **4.** $\frac{1}{\sqrt{2}}$ times

Introductory Exercise 11.4

1. $T = 2\pi \sqrt{\frac{\frac{3R}{2} + \frac{r^2}{2R}}{g}}$ **2.** 1.4×10^5 g-cm^2

Introductory Exercise 11.5

1. (a) 7.0 cm (b) 6.1 cm (c) 5.0 cm (d) 1.0 cm

2. (a) $\frac{3\sqrt{3}}{2}$ unit (b) $100\pi\sqrt{37}$ units (c) $(100\pi)^2\sqrt{37}$ units.

For JEE Main

Subjective Questions

1. (a) 2.8 N/m (b) 0.84 s **2.** 0.78 s **3.** 0.101 m/s, 1.264 m/s^2, 0.632 N

4. 0.4 s, 0.102 m **5.** 0.58 m/s, – 0.45 m/s^{-2}, 0.60 m/s^2, zero **6.** $\frac{\pi}{6}$ **7.** $\theta = \left(\frac{\pi}{10}\text{ rad}\right)\cos[(40\pi s^{-1})t]$

8. (a) $\frac{\pi}{120}$ sec (b) $\frac{\pi}{30}$ sec (c) $\frac{\pi}{30}$ sec **9.** 7.2 m **10.** (a) equal (b) equal

11. (a) 0.08 m (b) 1.57 rad/s (c) 1.97 N/s (d) zero (e) 0.197 m/s^2

13. (a) $U = E_0/4, K = 3E_0/4$ (b) $x = \frac{x_0}{\sqrt{2}}$ **15.** $A = d\sqrt{\frac{m_1}{m_1+m_2}}$ **16.** (a) $\frac{F}{k}, 2\pi\sqrt{\frac{M}{k}}$ (b) $\frac{F^2}{2k}$ (c) $\frac{F^2}{2k}$

17. (a) 10 cm (b) 2.5 J (c) $\frac{\pi}{5}$ sec (d) 20 cm (e) 4.5 J (f) 0.5 J **18.** $y = (0.1\text{ m})\sin\left[(4s^{-1})t + \frac{\pi}{4}\right]$

19. (a) 1 rad/s (b) $U_{mean} = 10$ J, $K_{mean} = 16$ J, $U_{extreme} = 26$ J, $K_{extreme} = 0$ (c) 4 m (d) $x = 6$ m and $x = -2$ m

20. $3T/4$ **22.** $T = 2\pi\sqrt{\frac{R^3}{GM}}$ **23.** 4 s **24.** $T = 2\pi\sqrt{\frac{m}{k}}$ **25.** 1.8 s **26.** 0.816 **28.** 0.38 Hz

29. Parabola, $y = A\left(1 - \frac{2x^2}{A^2}\right)$ **30.** 2 A **31.** $\frac{2\pi}{3}$ **32.** (a) –2.41 cm (b) 0.27 cm

Objective Questions

1. (a)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (d)	**6.** (b)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (d)
11. (d)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (c)	**20.** (b)
21. (b)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (a)
31. (a)	**32.** (b)								

For JEE Advanced

Assertion and Reason

1. (d)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (d)	**6.** (d)	**7.** (d)	**8.** (a)	**9.** (a)	**10.** (d)

Objective Questions

1. (d)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (b)
11. (b)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (d)	**20.** (c)
21. (d)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (d)
31. (a)	**32.** (c)								

More than One Correct Options

1. (a,c)	**2.** (b,d)	**3.** (b,c,d)	**4.** (a,b,c)	**5.** (b,c)	**6.** (b,c)	**7.** (b,c,d)
8. (b,c)	**9.** (a,b,c)					

Match the Columns

1.	(a) → q	(b) → p,r	(c) → p,r	(d) → s
2.	(a) → r	(b) → s	(c) → q	(d) → s
3.	(a) → r	(b) → p	(c) → r	(d) → r,s
4.	(a) → s	(b) → q	(c) → s	(d) → q
5.	(a) → q	(b) → p	(c) → p	(d) → r

Subjective Questions

1. 0.8 Hz, 0.05 m **2.** (a) $\frac{19}{48}$ T (b) $\frac{11}{48}$ T **3.** (a) $A = 0.75$ m (b) $x = -0.75 \sin 2t$

4. (a) 14.1 cm, 0.44 s (b) 23 cm, 0.36 s (c) $x = \pm (14.1 \text{ cm}) \sin(10\sqrt{2}\,t)$, $x = \pm (23 \text{ cm}) \sin(10\sqrt{3}\,t)$, 4 N-s, 8 N-s

5. $\frac{\pi}{20}$ s = 0.157 s **6.** (a) $N = m(g + a\omega^2 \cos \omega t)$ (b) 8.1 cm **8.** 0.56 Hz **9.** $\theta = 0.4 \cos (16.16\,t)$

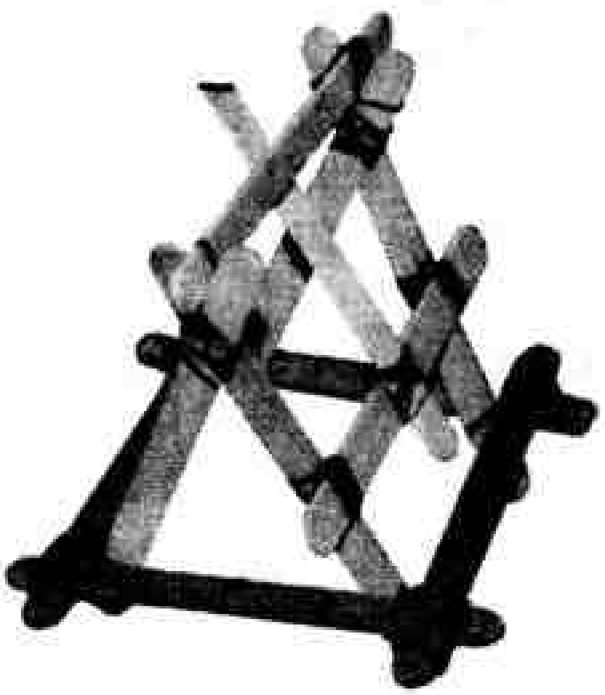
10. $T = \frac{\pi}{2}\sqrt{\frac{3m}{k}}$ **11.** (a) 3 cm (b) $x = 10 - 3 \sin 5t$ (c) 0.135 J **12.** $\sqrt{\frac{k_1 k_2}{4m(k_1 + k_2)}}$

13. (a) $\frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m_1 + m_2}}$ (b) $\frac{k_1}{k_2} < \frac{m_1}{m_2}$ (c) $\frac{\mu(m_1 + m_2)m_2 g}{m_1 k_2 - m_2 k_1}$

14. (a) 0.2 m/s, 0.1 m/s, 10 rad/s (b) 0.1 m/s (towards right) (c) 4.8 cm (d) 0.03 J

15. $\frac{1}{2\pi}\sqrt{\frac{15k}{4m}}$ **16.** $T = 4\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}\right)}$ **17.** $T = 2\pi\sqrt{\frac{m(4k_2 + k_1)}{k_1 k_2}}$ **18.** $f = \frac{1}{2\pi}\sqrt{\frac{k + 4A\rho g}{6M}}$

19. $\frac{1}{\pi}\sqrt{\frac{2k}{M}}$



12

ELASTICITY

Chapter Contents

12.1 Introduction

In day to day work an engineer comes across certain materials *e.g.*, steel girders, angle irons, circular bars, cement etc., which are used in his projects. While selecting a suitable material for his project, an engineer is always interested to know its strength. The strength of a material may be defined as an ability to resist its failure and behaviour under the action of external forces. It has been observed that, under the action of these forces, the material is first deformed and then its failure takes place. As a matter of fact the properties of material under the action of external forces are very essential, for an engineer, to enable him, in designing him all types of structures and machines.

Whenever a load is attached to a thin hanging wire it elongates and the load moves downwards (sometimes through a negligible distance). The amount by which the wire elongates depends upon the amount of load and the nature of wire material. Cohesive force, between the molecules of the hanging wire offer resistance against the deformation, and the force of resistance increases with the deformation. The process of deformation stops when the force of resistance is equal to the external force (*i.e.*, the load attached). Sometimes the force of resistance offered by the molecules is less than the external force. In such a case, the deformation continues until failure takes place.

Thus, we may conclude that if some external force is applied to a body it has two effects on it, namely :

(i) deformation of the body,
(ii) internal resistance (restoring) forces are developed.

12.2 Elasticity

As we have already discussed that whenever a single force (or a system of forces) acts on a body it undergoes some deformation and the molecules offer some resistance to the deformation. When the external force is removed, the force of resistance also vanishes and the body returns back to its original shape. But it is only possible if the deformation is within a certain limit. Such a limit is called elastic limit. This property of materials of returning back to their original position is called the elasticity.

A body is said to be perfectly elastic if it returns back completely to its original shape and size after removing the external force (s). If a body remains in the deformed state and does not even partially regain its original shape after the removal of the deforming forces, it is called a perfectly inelastic or plastic body. Quite often, when the external forces are removed, the body partially regains the original shape. Such bodies are partially elastic. If the force acting on the body is increased and the deformation exceeds the elastic limit, the body loses to some extent, its property of elasticity. In this case the body will not return to its original shape and size even after removal of the external force. Some deformation is left permanently.

12.3 Stress and Strain

Stress

When an external force is applied to a body then at each cross-section of the body an internal restoring force is developed which tends to restore the body to its original state. The internal restoring force per unit area of cross-section of the deformed body is called stress. It is usually denoted by σ (sigma).

Thus,
$$\text{Stress } (\sigma) = \frac{\text{Restoring force}}{\text{Area}}$$

Depending upon the way the deforming forces are applied to a body, there are three types of stress: longitudinal stress, shearing stress and volume stress.

Longitudinal and Shearing Stress

The body of figure is in static equilibrium under an arbitrary set of external forces. In Fig. 12.1(b), we see the same body with an imaginary sectional cut at CC'. Since each of the two individual parts of the body is also in static equilibrium, both internal forces and internal torques are developed at the cross section. Those on the right portion are due to the left portion and vice-versa. On the left portion, the normal and tangential components of the internal forces are $\vec{\mathbf{F}}_n$ and $\vec{\mathbf{F}}_t$ respectively, and the net internal torque is $\vec{\tau}$. From Newton's third law, the right portion is subjected at this same cross-section to force components $-\vec{\mathbf{F}}_n$ and $-\vec{\mathbf{F}}_t$ and the torque $-\vec{\tau}$. We define the **normal stress or longitudinal stress** over the area as,

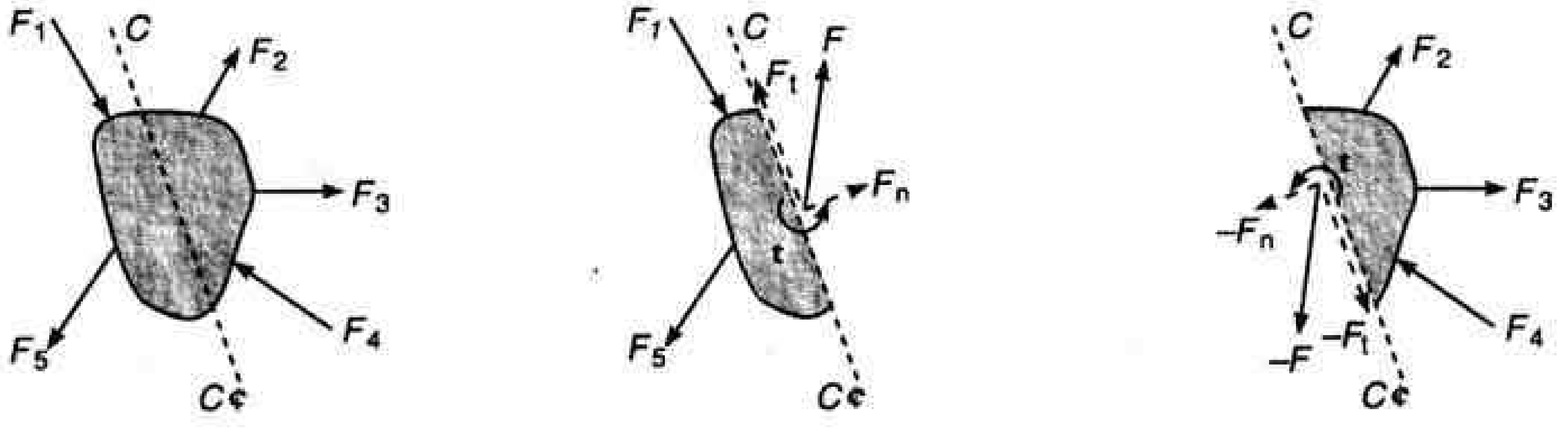


Fig. 12.1

$$\sigma_n = \frac{F_n}{A}$$

and the **tangential stress or shearing stress** over the area as,

$$\sigma_t = \frac{F_t}{A}$$

Here, A is the cross-section area of the body at CC'. The longitudinal stress can be of two types. The two parts of the body on two sides of a cross section may pull each other. The longitudinal stress is then called the tensile stress. This is the case when a rod or a wire is stretched by equal and opposite forces. In case of tensile stress in a wire or a rod, the force F_n is just the tension.

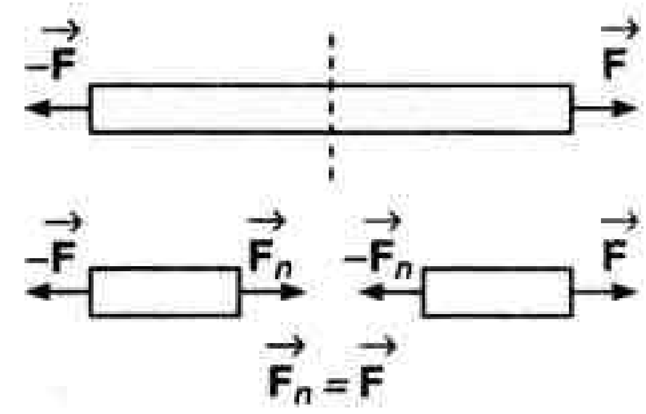


Fig. 12.2

If the rod is pushed at the two ends with equal and opposite forces, it will be under compression. Taking any cross-section of the rod the two parts on the two sides push each other. The longitudinal stress in this case is called the compressive stress.

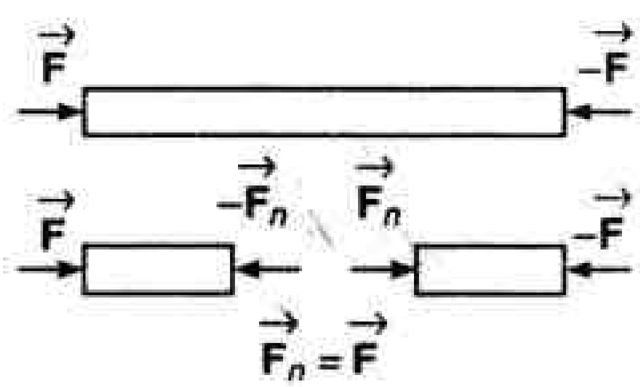


Fig. 12.3

Volume Stress

When a body is acted upon by forces in such a manner that,

(i) the force at any point is normal to the surface.

(ii) the magnitude of the force on any small area is proportional to the area.

The force per unit area is then called the volume stress, *i.e.*,

$$\sigma_v = \frac{F}{A}$$

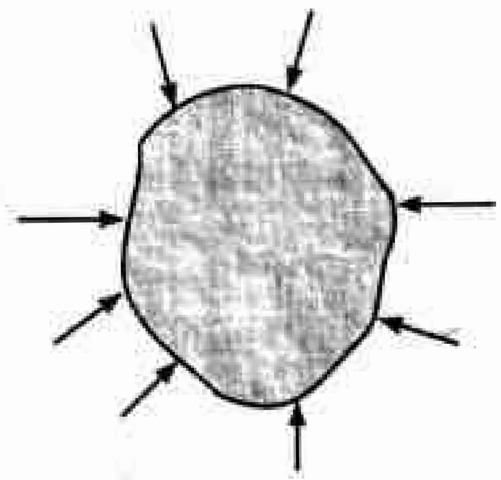


Fig. 12.4

which is same as the pressure. This is the case when a body is immersed in a liquid.

Strain

When the size or shape of a body is changed under an external force, the body is said to be strained. The change occurred in the unit size of the body is called strain. Usually it is denoted by ε. Thus,

$$\varepsilon = \frac{\Delta x}{x}$$

Here, Δx is the change (may be in length, volume etc.) and x the original value of the quantity in which change has occurred. For example, when the length of a suspended wire increases under an applied load, the value of strain is,

$$\varepsilon = \frac{\Delta l}{l}$$

and it is called **longitudinal strain.**

Similarly, if the change has occurred in the volume of a body, it is called **volumetric strain** and is given by,

$$\varepsilon = \frac{\Delta V}{V}$$

Shearing Strain

This type of strain is produced when a shearing stress is present.

Consider a body of square cross section $ABCD$. Four forces of equal magnitude F are applied as shown in figure. Net resultant force and net torque is zero. Hence, the body is in translational as well as rotational equilibrium. Because of the forces the shape of the cross section changes from a square to a parallelogram.

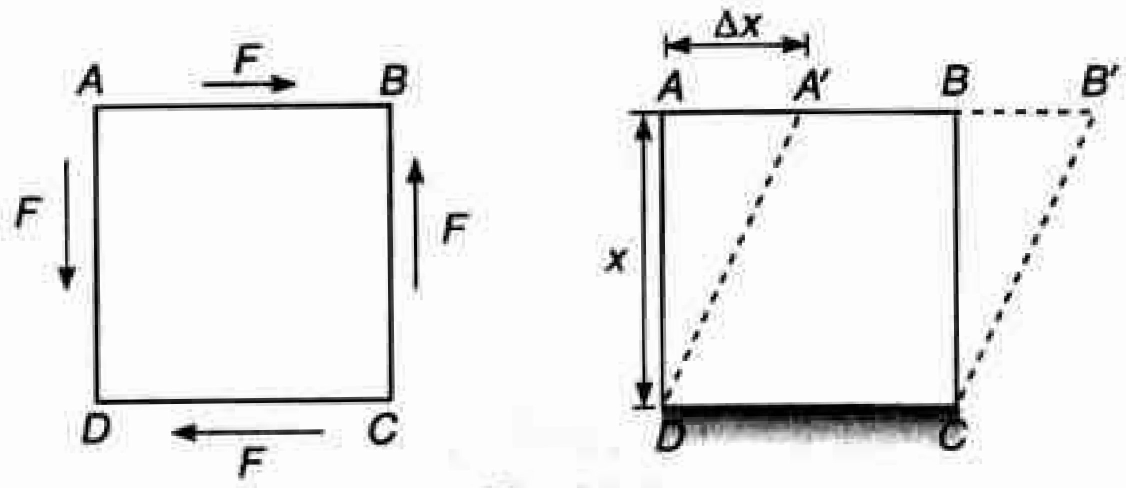


Fig. 12.5

We define the shearing strain as the displacement of a layer divided by its distance from the fixed layer. Thus, shearing strain,

$$\varepsilon = \frac{\Delta x}{x}$$

12.4 Hooke's Law and the Modulus of Elasticity

According to Hooke's law,

"For small deformation, the stress in a body is proportional to the corresponding strain." *i.e.*,

$$\text{stress} \propto \text{strain}$$

or

$$\text{stress} = (E)\,(\text{strain})$$

Here, $E = \dfrac{\text{stress}}{\text{strain}}$ is a constant called the modulus of elasticity. Depending upon the nature of force applied on the body, the modulus of elasticity is classified in following three types:

Young's Modulus of Elasticity (Y)

When a wire is acted upon by two equal and opposite forces in the direction of its length, the length of the body is changed. The change in length per unit length $\left(\dfrac{\Delta l}{l}\right)$ is called the longitudinal strain and the restoring force (which is equal to the applied force in equilibrium) per unit area of cross section of the wire is called the longitudinal stress.

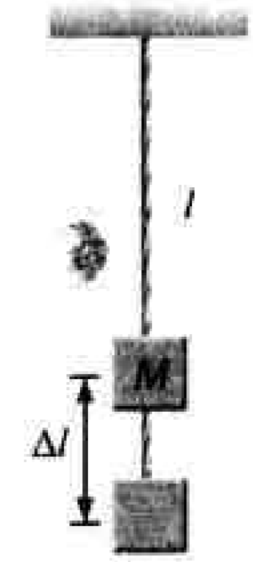


Fig. 12.6

For small change in the length of the wire, the ratio of the longitudinal stress to the corresponding strain is called the Young's modulus of elasticity (Y) of the wire. Thus,

$$Y = \frac{\dfrac{F}{A}}{\dfrac{\Delta l}{l}}$$

or

$$Y = \frac{Fl}{A\Delta l}$$

Let there be a wire of length l and radius r. Its one end is clamped to a rigid support and a mass M is attached at the other end. Then,

$$F = Mg \qquad \text{and} \qquad A = \pi r^2$$

Substituting in above equation, we have

$$Y = \frac{Mgl}{(\pi r^2)\Delta l}$$

Bulk Modulus of Elasticity (B)

When a uniform pressure (normal force) is applied all over the surface of a body, the volume of the body changes. The change in volume per unit volume of the body is called the 'volume strain' and the normal force acting per unit area of the surface (pressure) is called the normal stress or volume stress. For small strains, the ratio of the volume stress to the volume strain is called the 'bulk modulus' of the material of the body. It is denoted by B. Then,

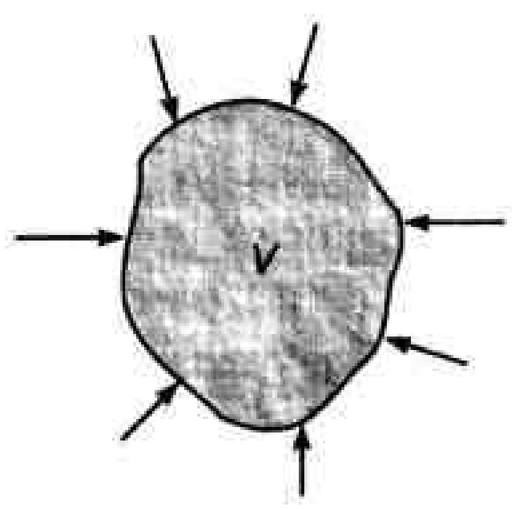


Fig. 12.7

$$B = \frac{-P}{\Delta V / V}$$

Here, negative sign implies that when the pressure increases volume decreases and *vice-versa*.

Compressibility

The reciprocal of the bulk modulus of the material of a body is called the 'compressibility' of that material. Thus,

$$\text{Compressibility} = \frac{1}{B}$$

Modulus of Rigidity (η)

When a body is acted upon by an external force tangential to a surface of the body, the opposite surface being kept fixed, it suffers a change in shape, its volume remaining unchanged. Then the body is said to be sheared.

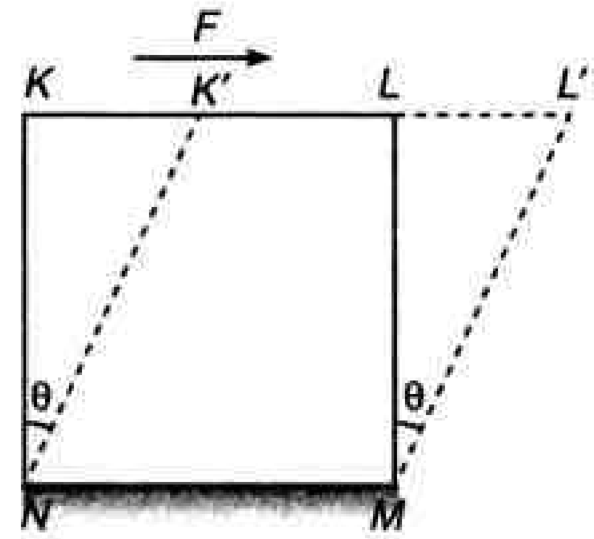


Fig. 12.8

The ratio of the displacement of a layer in the direction of the tangential force and the distance of the layer from the fixed surface is called the shearing strain and the tangential force acting per unit area of the surface is called the "shearing stress".

For small strain the ratio of the shearing stress to the shearing strain is called the "modulus of rigidity" of the material of the body. It is denoted by η.

Thus, $$\eta = \frac{F/A}{KK'/KN}$$

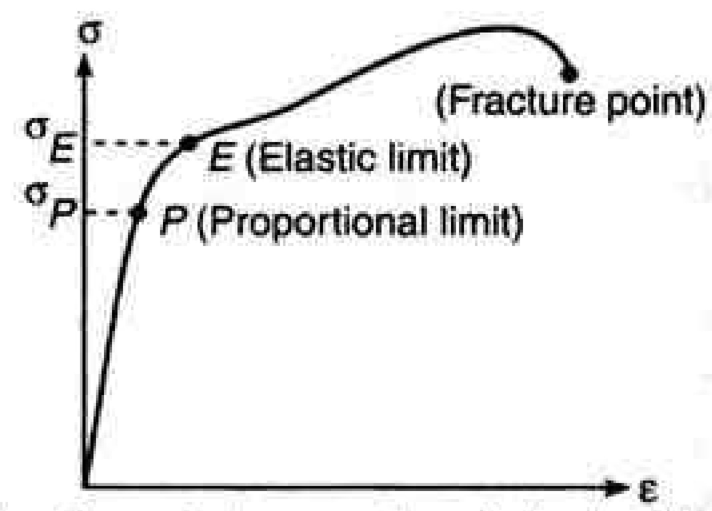
Here, $$\frac{KK'}{KN} = \tan\theta \approx \theta$$

∴ $$\eta = \frac{F/A}{\theta}$$

or $$\eta = \frac{F}{A\theta}$$

12.5 The Stress-Strain Curve

A plot of normal stress (either tensile or compressive) versus normal strain for a typical solid is shown in figure. The strain is directly proportional to the applied stress for values of stress upto σ_P. In this linear region, the material returns to its original size when the stress is removed. Point P is known as the proportional limit of the solid. For stresses between σ_P and σ_E, where point E is called the elastic limit, the material also returns to its original size.



The stress-strain curve for a typical solid

Fig. 12.9

However, notice that stress and strain are not proportional in this region. For deformations beyond the elastic limit, the material does not return to its original size when the stress is removed, it is permanently distorted. Finally, further stretching beyond the elastic limit leads to the eventual fracture of the solid. The proportionality constant for linear region or the slope of stress-strain curve in this curve is called the Young's modulus of elasticity Y.

Sample Example 12.1 *Determine the elongation of the steel bar 1 m long and 1.5 cm^2 cross-sectional area when subjected to a pull of 1.5×10^4 N. (Take $Y = 2.0\times10^{11}\ N/m^2$)*

Solution
$$Y = \frac{F/A}{\Delta l/l}$$

$\therefore$
$$\Delta l = \frac{Fl}{AY}$$

Substituting the values,
$$\Delta l = \frac{(1.5\times10^4)(1.0)}{(1.5\times10^{-4})(2.0\times10^{11})} = 0.5\times10^{-3}\ \text{m}$$

or
$$\Delta l = 0.5\ \text{mm}$$
Ans.

Sample Example 12.2 *A bar of mass m and length l is hanging from point A as shown in figure. Find the increase in its length due to its own weight. The Young's modulus of elasticity of the wire is Y and area of cross-section of the wire is A.*

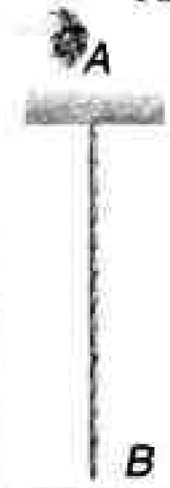


Fig. 12.10

Solution Consider a small section dx of the bar at a distance x from B. The weight of the bar for a length x is,

$$W = \left(\frac{mg}{l}\right)x$$

Elongation in section dx will be

$$dl = \left(\frac{W}{AY}\right)dx = \left(\frac{mg}{lAY}\right)x\,dx$$

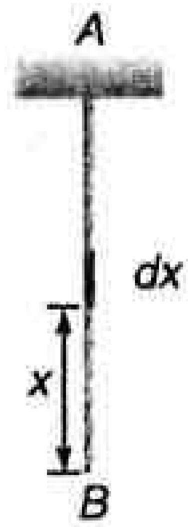


Fig. 12.11

Total elongation in the bar can be obtained by integrating this expression for $x = 0$ to $x = l$.

$\therefore$
$$\Delta l = \int_{x=0}^{x=l} dl = \left(\frac{mg}{lAY}\right)\int_0^l x\,dx$$

or
$$\Delta l = \frac{mgl}{2AY}$$
Ans.

Sample Example 12.3 *A brass bar, having cross sectional area 10 cm^2 is subjected to axial forces as shown in figure. Find the total elongation of the bar. (Take $Y = 8\times10^2\ t/cm^2$).*

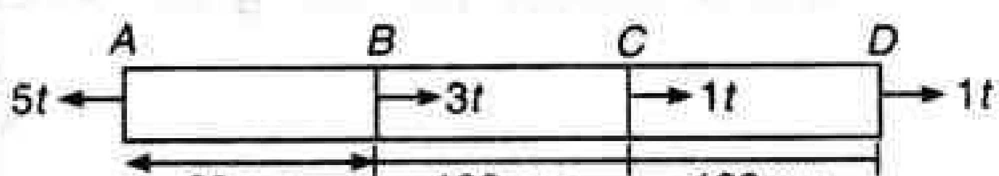


Fig. 12.12

Solution Given, $A = 10\,\text{cm}^2$, $Y = 8 \times 10^2\ t/\text{cm}^2$

Let, Δl = total elongation of the bar. For the sake of simplicity the force of $3t$ acting at B may be split into two forces of $5t$ and $2t$ as shown in figure. Similarly, the force of $1t$ acting at C may be split into two forces of $2t$ and $1t$.

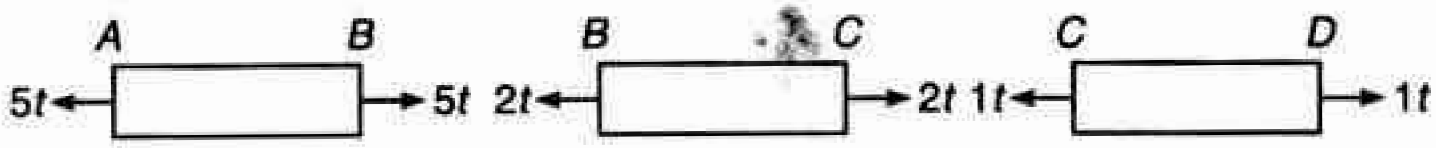


Fig. 12.13

Using the equation, $\Delta l = \frac{1}{AY}(F_1 l_1 + F_2 l_2 + F_3 l_3)$ with usual notations

$$\Delta l = \frac{1}{10 \times 8 \times 10^2}[5 \times 60 + 2 \times 100 + 1 \times 120]$$

$$= 0.0775\ \text{cm}$$ **Ans.**

Introductory Exercise 12.1

1. Two wires A and B of same dimensions are stretched by same amount of force. Young's modulus of A is twice that of B. Which wire will get more elongation?
2. A rod 100 cm long and of 2 cm × 2 cm cross-section is subjected to a pull of 1000 kg force. If the modulus of elasticity of the material is 2.0×10^6 kg/cm^2, determine the elongation of the rod.
3. A cast iron column has internal diameter of 200 mm. What should be the minimum external diameter so that it may carry a load of 1.6 MN without the stress exceeding 90 N/mm^2?

12.6 Potential Energy in a Stretched Wire

When a wire is stretched, work is done against the inter atomic forces. This work is stored in the wire in the form of elastic potential energy. Suppose on applying a force F on a wire of length l, the increase in length is Δl. The area of cross-section of the wire is A. The potential energy stored in the wire should be,

$$U = \frac{1}{2} k (\Delta l)^2$$

Here, $$k = \frac{YA}{l}$$

$\therefore$ $$U = \frac{1}{2} \frac{YA}{l} (\Delta l)^2$$

Elastic potential energy per unit volume of the wire is,

$$u = \frac{U}{\text{volume}}$$

or $$u = \frac{\frac{1}{2}\frac{YA}{l}(\Delta l)^2}{Al} \quad \text{or} \quad u = \frac{1}{2}\left(\frac{\Delta l}{l}\right)\left(Y \cdot \frac{\Delta l}{l}\right)$$

or $$u = \frac{1}{2}(\text{strain})(Y \times \text{strain})$$

or $$u = \frac{1}{2}(\text{strain}) \times (\text{stress})$$

12.7 Thermal Stresses and Strains

Whenever there is some increase or decrease in the temperature of the body, it causes the body to expand or contract. If the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

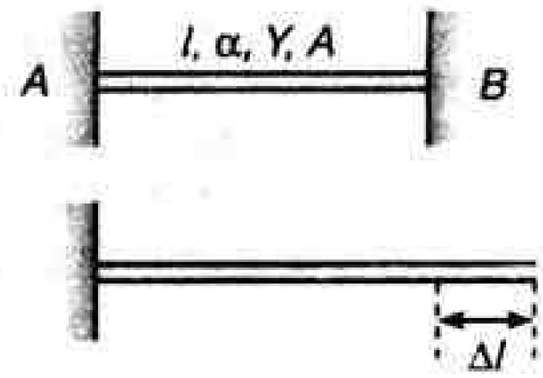


Fig. 12.14

Consider a rod AB fixed at two supports as shown in figure.

Let l = length of rod

A = area of cross-section of the rod

Y = Young's modulus of elasticity of the rod

and α = thermal coefficient of linear expansion of the rod

Let the temperature of the rod is increased by an amount t. The length of the rod would had increased by an amount Δl, if it were not fixed at two supports. Here

$$\Delta l = l\alpha t$$

But since the rod is fixed at the supports a compressive strain will be produced in the rod. Because at the increased temperature, the natural length of the rod is $l + \Delta l$, while being fixed at two supports its actual length is l. Hence, thermal strain

$$\varepsilon = \frac{\Delta l}{l} = \frac{l\alpha t}{l} = \alpha t$$

or $$\varepsilon = \alpha t$$

Therefore, thermal stress $$\sigma = Y\varepsilon \qquad (\text{stress} = Y \times \text{strain})$$

or $$\sigma = Y\alpha t$$

or force on the supports,

$$F = \sigma A = YA\alpha t$$

This force F is in the direction shown below :

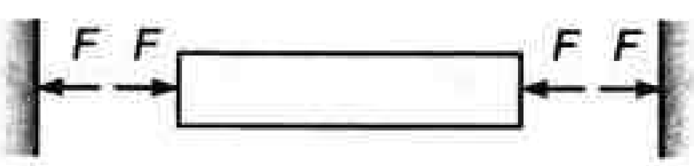


Fig. 12.15

Extra Points

- Modulus of elasticity E (whether it is Y, B or η) is given by

$$E = \frac{\text{stress}}{\text{strain}}$$

Following conclusions can be made from the above expression :

(i) $E \propto$ stress (for same strain), *i.e.*, if we want the equal amount of strain in two different materials, the one which needs more stress is having more E.

(ii) $E \propto \frac{1}{\text{strain}}$ (for same stress), *i.e.*, if the same amount of stress is applied on two different materials, the one having the less strain is having more E. Rather we can say that, the one which offers more resistance to the external forces is having greater value of E. So, we can see that modulus of elasticity of steel is more than that of rubber or

$$E_{\text{steel}} > E_{\text{rubber}}$$

(iii) $E =$ stress for unit strain $\left(\frac{\Delta x}{x} = 1 \text{ or } \Delta x = x\right)$, *i.e.*, suppose the length of a wire is 2 m, then the Young's modulus of elasticity (Y) is the stress applied on the wire to stretch the wire by the same amount of 2 m.

- The material which has smaller value of Y is more ductile, *i.e.*, it offers less resistance in framing it into a wire. Similarly, the material having the smaller value of B is more malleable. Thus, for making wire we choose a material having less value of Y.
- A solid will have all the three modulii of elasticity Y, B and η. But in case of a liquid or a gas only B can be defined as a liquid or a gas can not be framed into a wire or no shear force can be applied on them.
- For a liquid or a gas,

$$B = \left(\frac{-dP}{dV/V}\right)$$

So, instead of P we are more interested in change in pressure dP.

- In case of a gas,

$$B = XP$$

in the process $PV^X =$ constant

For example, for $x = 1$, or $PV =$ constant (isothermal process) $B = P$.

i.e., isothermal bulk modulus of a gas (denoted by B_T) is equal to the pressure of the gas at that instant of time or

$$B_T = P$$

Similarly, for $x = \gamma = \frac{C_P}{C_V}$ or $PV^{\gamma} =$ constant (adiabatic process) $B = \gamma P$.

i.e., adiabatic bulk modulus of a gas (denoted by B_s) is equal to γ times the pressure of the gas at that instant of time or

$$B_s = \gamma P$$

For a gas $$B \propto P$$

whether it is an isothermal process or an adiabatic process. Physically this can be understood as under:

Suppose we have two containers A and B. Some gas is filled in both the containers. But the pressure in A is more than the pressure in B, *i.e.*,

$$P_1 > P_2$$

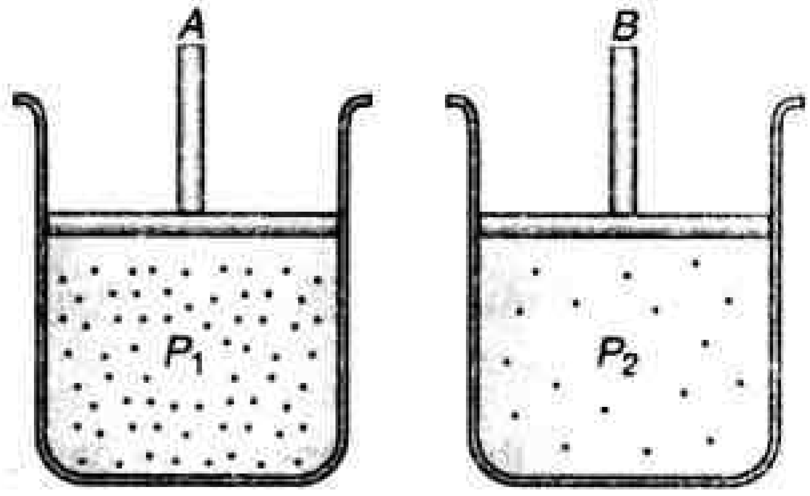


Fig. 12.16

So, bulk modulus of A should be more than the bluk modulus of B, or

$$B_1 > B_2$$

and this is quiet obvious, because it is more difficult to compress the gas in chamber A, *i.e.*, it provides more resistance to the external forces. And as we have said in point number 1(ii) the modulus of elasticity is greater for a substance which offers more resistance to external forces.

If a spring is stretched or compressed by an amount Δl, the restoring force produced in it is,

$$F_s = k\,\Delta l \qquad \text{...(i)}$$

Here, k = force constant of spring

Similarly, if a wire is stretched by an amount Δl, the restoring force produced in it is,

$$F = \left(\frac{YA}{l}\right)\Delta l \qquad \text{...(ii)}$$

Comparing Eqs. (i) and (ii), we can see that force constant of a wire is,

$$k = \frac{YA}{l} \qquad \text{...(iii)}$$

i.e., a wire is just like a spring of force constant $\frac{YA}{l}$.

So, all formulae which we use in case of a spring can be applied to a wire also.

From Eq. (iii), we may also conclude that force constant of a spring is inversely proportional to the length of the spring l or,

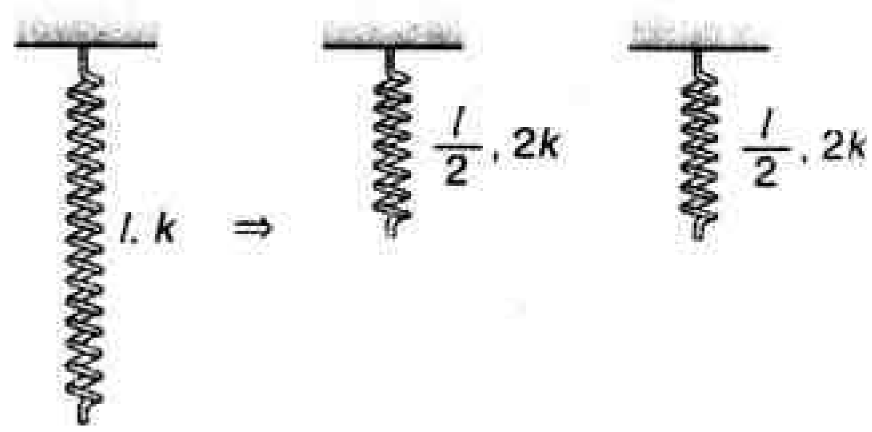


Fig. 12.17

$$k \propto \frac{1}{l}$$

i.e., if a spring is cut into two equal pieces its force constant is doubled.

When a pressure (dP) is applied on a substance its density is changed. The change in density can be calculated as under :

$$\rho = \frac{\text{mass}}{\text{volume}} \qquad (\rho = \text{density})$$

or $$\rho \propto \frac{1}{V} \qquad (\text{mass} = \text{constant})$$

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV}$$

or

$$\rho' = \rho\left(\frac{V}{V + dV}\right)$$

$$= \rho\left(\frac{V}{V - (dP/B)V}\right) \qquad \text{as } B = -\frac{dP}{dV/V}$$

$$\rho' = \frac{\rho}{1 - \frac{dP}{B}}$$

From this expression we can see that ρ' increases as pressure is increased (dP is positive) and *vice-versa*.

- **Poisson's ratio :** When a longitudinal force is applied on a wire, its length increases but its radius decreases. Thus two strains are produced by a single force :

(i) Longitudinal strain $= \frac{\Delta l}{l}$ and

(ii) Lateral strain $= \frac{\Delta R}{R}$

The ratio of these two strains is called the Poisson's ratio.

Thus, the Poisson's ratio

$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = -\frac{\Delta R/R}{\Delta l/l}$$

Following points are worthnothing in case of Poisson's ratio :

(i) Negative sign in σ indicates that radius of the wire decreases as the length increases.

(ii) Theoretical value of σ lies between -1 and $+\frac{1}{2}$.

(iii) Practical value of σ lies between 0 and $+\frac{1}{2}$.

- **Relation between Y, B, η and σ :** Following are some relations between the four

(a) $B = \frac{Y}{3(1 - 2\sigma)}$ (b) $\eta = \frac{Y}{2(1 + \sigma)}$ (c) $\sigma = \frac{3B - 2\eta}{2\eta + 6B}$ (d) $\frac{9}{Y} = \frac{1}{B} + \frac{3}{\eta}$

Solved Examples

For JEE Main

Example 1 *A steel wire of length 4 m and diameter 5 mm is stretched by 5 kg-wt. Find the increase in its length, if the Young's modulus of steel is 2.4×10^{12} dyne/cm^2.*

Solution Here, $l = 4\text{ m} = 400\text{ cm}$, $2r = 5\text{ mm}$

or $$r = 2.5\text{ mm} = 0.25\text{ cm}$$

$$f = 5\text{ kg-wt} = 5000\text{ g-wt} = 5000 \times 980\text{ dyne}$$

$$\Delta l = ?,\quad Y = 2.4 \times 10^{12}\text{ dyne/cm}^2$$

As $$Y = \frac{F}{\pi r^2} \times \frac{l}{\Delta l}$$

or $$\Delta l = \frac{Fl}{\pi r^2 Y} = \frac{(5000 \times 980) \times 400}{(22/7) \times (0.25)^2 \times 2.4 \times 10^{12}}$$

$$= 0.0041\text{ cm}$$ **Ans.**

Example 2 *The bulk modulus of water is 2.3×10^9 N/m^2.*

(a) Find its compressibility.

(b) How much pressure in atmospheres is needed to compress a sample of water by 0.1% ?

Solution Here, $B = 2.3 \times 10^9\text{ N/m}^2$

$$= \frac{2.3 \times 10^9}{1.01 \times 10^5} = 2.27 \times 10^4\text{ atm}$$

(a) Compressibility $= \dfrac{1}{B} = \dfrac{1}{2.27 \times 10^4} = 4.4 \times 10^{-5}\text{ atm}^{-1}$ **Ans.**

(b) Here, $\dfrac{\Delta V}{V} = -0.1\% = -0.001$

Required increase in pressure,

$$\Delta P = B \times \left(-\frac{\Delta V}{V}\right) = 2.27 \times 10^4 \times 0.001 = 22.7\text{ atm}$$ **Ans.**

Example 3 *A steel wire 4.0 m in length is stretched through 2.0 mm. The cross-sectional area of the wire is 2.0 mm^2. If Young's modulus of steel is 2.0×10^{11} N/m^2. Find :*

(a) the energy density of wire,

(b) the elastic potential energy stored in the wire.

Solution Here, $l = 4.0$ m, $\Delta l = 2 \times 10^{-3}$ m, $A = 2.0 \times 10^{-6}$ m^2, $Y = 2.0 \times 10^{11}$ N/m^2

(a) The energy density of stretched wire

$$U = \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{1}{2} \times Y \times (\text{strain})^2$$

$$= \frac{1}{2} \times 2.0 \times 10^{11} \times \left(\frac{(2 \times 10^{-3})}{4} \right)^2$$

$$= 0.25 \times 10^5 = 2.5 \times 10^4 \text{ J/m}^3$$ **Ans.**

(b) Elastic potential energy = energy density × volume

$$= 2.5 \times 10^4 \times (2.0 \times 10^{-6}) \times 4.0 \text{ J}$$

$$= 20 \times 10^{-2} = 0.20 \text{ J}$$ **Ans.**

Example 4 *Find the greatest length of steel wire that can hang vertically without breaking. Breaking stress of steel* $= 8.0 \times 10^8$ N/m^2. *Density of steel* $= 8.0 \times 10^3$ kg/m^3. *Take* $g = 10$ m/s^2.

Solution Let l be the length of the wire that can hang vertically without breaking. Then the stretching force on it is equal to its own weight. If therefore, A is the area of cross-section and ρ the density, then

$$\text{Maximum stress } (\sigma_m) = \frac{\text{weight}}{A} \qquad \left(\text{Stress} = \frac{\text{force}}{\text{area}} \right)$$

or $$\sigma_m = \frac{(Al\rho)g}{A}$$

$\therefore$ $$l = \frac{\sigma_m}{\rho g}$$

Substituting the values $$l = \frac{8.0 \times 10^8}{(8.0 \times 10^3)(10)} = 10^4 \text{ m}$$ **Ans.**

Example 5 *(a) A wire* 4 *m long and* 0.3 *mm in diameter is stretched by a force of* 100 *N. If extension in the wire is* 0.3 *mm, calculate the potential energy stored in the wire.*

(b) Find the work done in stretching a wire of cross-section 1 mm^2 *and length* 2 *m through* 0.1 *mm. Young's modulus for the material of wire is* 2.0×10^{11} N/m^2.

Solution (a) Energy stored $$U = \frac{1}{2} (\text{stress})(\text{strain})(\text{volume})$$

or $$U = \frac{1}{2} \left(\frac{F}{A} \right) \left(\frac{\Delta l}{l} \right) (Al)$$

$$= \frac{1}{2} F \cdot \Delta l$$

$$= \frac{1}{2} (100)(0.3 \times 10^{-3})$$

$$= 0.015 \text{ J}$$ **Ans.**

(b) Work done = Potential energy stored

$$= \frac{1}{2} k (\Delta l)^2$$

$$= \frac{1}{2}\left(\frac{YA}{l}\right)(\Delta l)^2 \qquad \left(\text{as } k = \frac{YA}{l}\right)$$

Substituting the values, we have

$$W = \frac{1}{2} \frac{(2.0 \times 10^{11})(10^{-6})}{(2)} (0.1 \times 10^{-3})^2$$

$$= 5.0 \times 10^{-4} \text{ J}$$ **Ans.**

Example 6 *A rubber cord has a cross-sectional area* 1 mm^2 *and total unstretched length* 10.0 *cm. It is stretched to* 12.0 *cm and then released to project a missile of mass* 5.0 *g. Taking Young's modulus Y for rubber as* 5.0×10^8 N/m^2. *Calculate the velocity of projection.*

Solution Equivalent force constant of rubber cord.

$$k = \frac{YA}{l} = \frac{(5.0 \times 10^8)(1.0 \times 10^{-6})}{(0.1)}$$

$$= 5.0 \times 10^3 \text{ N/m}$$

Now, from conservation of mechanical energy, elastic potential energy of cord

= kinetic energy of missile

$$\therefore \quad \frac{1}{2} k (\Delta l)^2 = \frac{1}{2} mv^2$$

$$\therefore \quad v = \left(\sqrt{\frac{k}{m}}\right) \Delta l$$

$$= \left(\sqrt{\frac{5.0 \times 10^3}{5.0 \times 10^{-3}}}\right)(12.0 - 10.0) \times 10^{-2}$$

$$= 20 \text{ m/s}$$ **Ans.**

Note *Following assumptions have been made in this Problem :*

(*i*) *k has been assumed constant, even though it depends on the length* (*l*).

(*ii*) *The whole of the elastic potential energy is converting into kinetic energy of missile.*

Example 7 *What is the density of lead under a pressure of* 2.0×10^8 N/m^2, *if the bulk modulus of lead is* 8.0×10^9 N/m^2 *and initially the density of lead is* 11.4 g/cm^3?

Solution The changed density,

$$\rho' = \frac{\rho}{1 - \frac{dP}{B}}$$

Substituting the value, we have

$$\rho' = \frac{11.4}{1 - \dfrac{2.0 \times 10^8}{8.0 \times 10^9}}$$

or $$\rho' = 11.69 \text{ g/cm}^3$$ **Ans.**

For JEE Advanced

Example 1 *A light rod of length* 2.00 *m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section* 10^{-3} m^2 *and the other is of brass of cross-section* 2×10^{-3} m^2*. Find out the position along the rod at which a weight may be hung to produce,*

(a) equal stresses in both wires

(b) equal strains on both wires.

Young's modulus for steel is 2×10^{11} N/m^2 *and for brass is* 10^{11} N/m^2.

Solution (a) Given,

Stress in steel = stress in brass

$$\therefore \quad \frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{A_S}{A_B}$$

$$= \frac{10^{-3}}{2 \times 10^{-3}} = \frac{1}{2} \qquad \ldots\text{(i)}$$

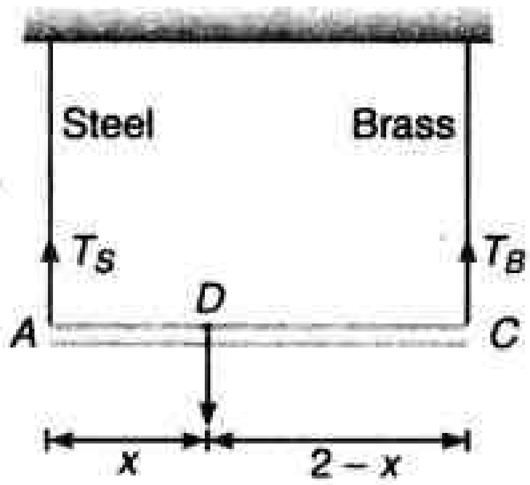


Fig. 12.18

As the system is in equilibrium, taking moments about *D*, we have

$$T_S \cdot x = T_B (2 - x)$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{2 - x}{x} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get $x = 1.33$ m **Ans.**

(b) $$\text{Strain} = \frac{\text{stress}}{Y}$$

Given, strain in steel = strain in brass

$$\therefore \quad \frac{T_S / A_S}{Y_S} = \frac{T_B / A_B}{Y_B}$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{A_S Y_S}{A_B Y_B} = \frac{(1 \times 10^{-3})(2 \times 10^{11})}{(2 \times 10^{-3})(10^{11})} = 1 \qquad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we have

$$x = 1.0 \text{ m}$$ **Ans.**

Example 2 *A steel rod of length 6.0 m and diameter 20 mm is fixed between two rigid supports. Determine the stress in the rod, when the temperature increases by 80° C if*

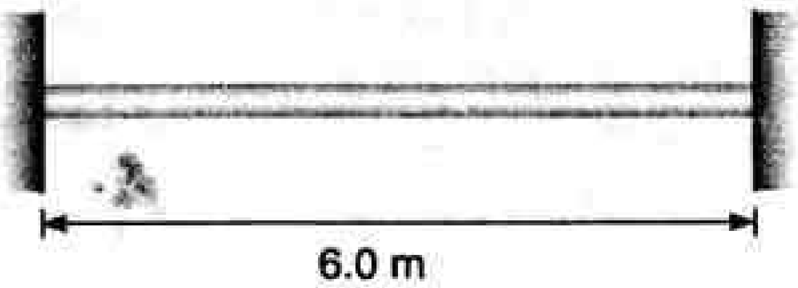


Fig. 12.19

(a) the ends do not yield
(b) the ends yield by 1 mm.
Take $Y = 2.0 \times 10^6$ *kg/cm²* *and* $\alpha = 12 \times 10^{-6}$ *per°C.*

Solution Given, length of the rod $l = 6 \text{ m} = 600 \text{ cm}$

Diameter of the rod $d = 20 \text{ mm} = 2 \text{ cm}$

Increase in temperature $t = 80°\text{C}$

Young's modulus $Y = 2.0 \times 10^6 \text{ kg/cm}^2$

and thermal coefficient of linear expansion

$$\alpha = 12 \times 10^{-6} \text{ per°C}$$

When the ends do not yield

Let, σ_1 = stress in the rod

Using the relation $\sigma = \alpha t Y$

$$\therefore \quad \sigma_1 = (12 \times 10^{-6})(80)(2 \times 10^6)$$

$$= 1920 \text{ kg/cm}^2 \qquad \textbf{Ans.}$$

When the ends yield by 1 mm

Increase in length due to increase in temperature

$$\Delta l = l\alpha t$$

of this 1 mm or 0.1 cm is allowed to expand. Therefore, net compression in the rod

$$\Delta l_{net} = (l\alpha t - 0.1)$$

or compressive strain in the rod,

$$\varepsilon = \frac{\Delta l_{net}}{l} = \left(\alpha t - \frac{0.1}{l}\right)$$

$$\therefore \quad \text{stress } \sigma_2 = Y\varepsilon = Y\left(\alpha t - \frac{0.1}{l}\right)$$

Substituting the values,

$$\sigma_2 = 2 \times 10^6 \left(12 \times 10^{-6} \times 80 - \frac{0.1}{600}\right)$$

$$= 1587 \text{ kg/cm}^2 \qquad \textbf{Ans.}$$

Example 3 *A steel rod of cross-sectional area* $16\ cm^2$ *and two brass rods each of cross-sectional area* $10\ cm^2$ *together support a load of 5000 kg as shown in figure. Find the stress in the rods. Take Y for steel* $=2.0\times10^6\ kg/cm^2$ *and for brass* $=1.0\times10^6\ kg/cm^2$.

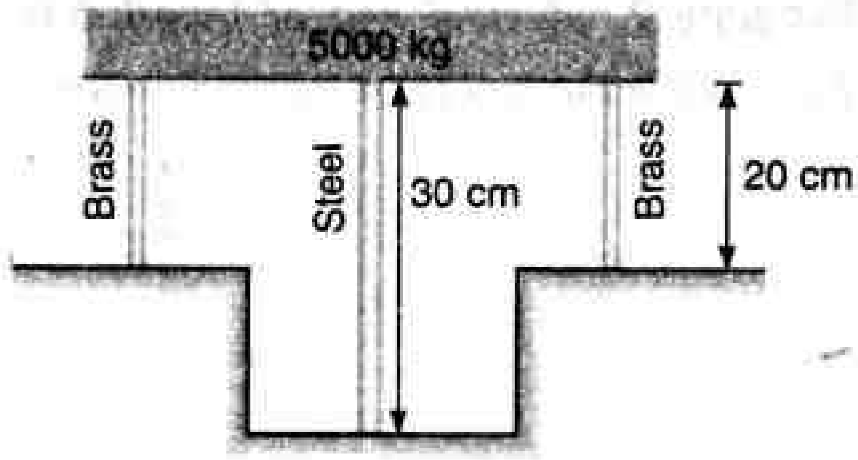


Fig. 12.20

Solution Given area of steel rod

$$A_S = 16\,\text{cm}^2$$

Area of two brass rods $\quad A_B = 2\times10 = 20\,\text{cm}^2$

Load, $\quad F = 5000\ \text{kg}$

Y for steel $\quad Y_S = 2.0\times10^6\ \text{kg/cm}^2$

Y for brass $\quad Y_B = 1.0\times10^6\ \text{kg/cm}^2$

Length of steel rod $\quad l_S = 30\,\text{cm}$

Length of brass rod $\quad l_B = 20\,\text{cm}$

Let $\quad \sigma_S$ = stress in steel

and $\quad \sigma_B$ = stress in brass

Decrease in length of steel rod = decrease in length of brass rod

or
$$\frac{\sigma_S}{Y_S}\times l_S = \frac{\sigma_B}{Y_B}\times l_B$$

or
$$\sigma_S = \frac{Y_S}{Y_B}\times\frac{l_B}{l_S}\times\sigma_B$$

$$= \frac{2.0\times10^6}{1.0\times10^6}\times\frac{20}{30}\times\sigma_B$$

$\therefore$
$$\sigma_S = \frac{4}{3}\sigma_B \qquad \text{...(i)}$$

Now, using the relation,

$$F = \sigma_S A_S + \sigma_B A_B$$

or
$$5000 = \sigma_S\times16 + \sigma_B\times20 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\sigma_B = 120.9\ \text{kg/cm}^2$$

and
$$\sigma_S = 161.2\ \text{kg/cm}^2$$

Ans.

Example 4 *A sphere of radius* 0.1 *m and mass* 8π *kg is attached to the lower end of a steel wire of length* 5.0 *m and diameter* 10^{-3} *m. The wire is suspended from* 5.22 *m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is* 1.994×10^{11} N/m^2.

Solution Let Δl be the extension of wire when the sphere is at mean position. Then, we have

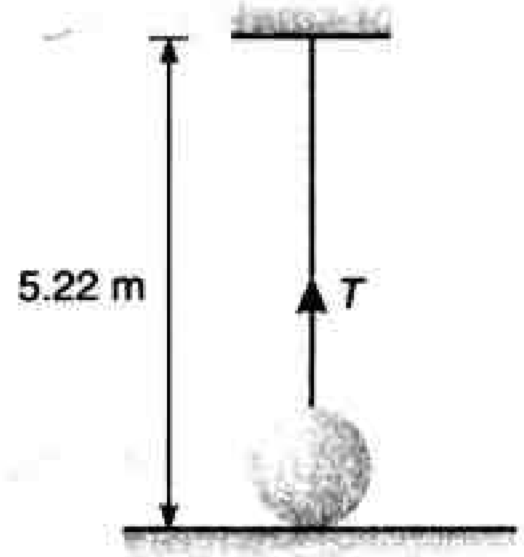


Fig. 12.21

$$l + \Delta l + 2r = 5.22$$

or
$$\Delta l = 5.22 - l - 2r$$
$$= 5.22 - 5 - 2 \times 0.1$$
$$= 0.02 \text{ m}$$

Let T be the tension in the wire at mean position during oscillations, then

$$Y = \frac{T/A}{\Delta l/l}$$

∴
$$T = \frac{YA\Delta l}{l} = \frac{Y\pi r^2 \Delta l}{l}$$

Substituting the values, we have

$$T = \frac{(1.994 \times 10^{11}) \times \pi \times (0.5 \times 10^{-3})^2 \times 0.02}{5}$$
$$= 626.43 \text{ N}$$

The equation of motion at mean position is,

$$T - mg = \frac{mv^2}{R} \quad \text{...(i)}$$

Here,
$$R = 5.22 - r = 5.22 - 0.1 = 5.12 \text{ m}$$

and
$$m = 8\pi \text{ kg} = 25.13 \text{ kg}$$

Substituting the proper values in Eq. (i), we have

$$(626.43) - (25.13 \times 9.8) = \frac{(25.13)v^2}{5.12}$$

Solving this equation, we get $v = 8.8$ m/s **Ans.**

Example 5 *A thin ring of radius R is made of a material of density* ρ *and Young's modulus Y. If the ring is rotated about its centre in its own plane with angular velocity* ω, *find the small increase in its radius.*

Solution Consider an element PQ of length dl. Let T be the tension and A the area of cross-section of the wire.

$$\text{Mass of element } dm = \text{volume} \times \text{density}$$
$$= A(dl)\rho$$

The component of T, towards the centre provides the necessary centripetal force

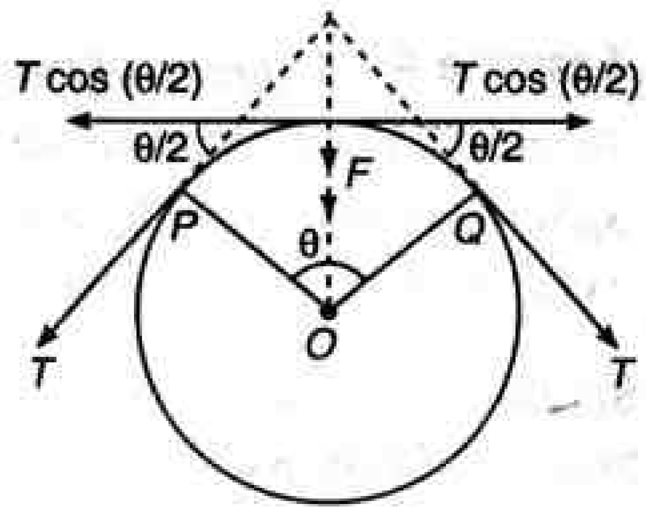


Fig. 12.22

$$\therefore \qquad 2T\sin\left(\frac{\theta}{2}\right) = (dm)R\omega^2 \qquad \text{...(i)}$$

For small angles $\sin\frac{\theta}{2} \approx \frac{\theta}{2} = \frac{(dl/R)}{2}$

Substituting in Eq. (i), we have

$$T\cdot\frac{dl}{R} = A(dl)\rho R\omega^2$$

or

$$T = A\rho\omega^2 R^2$$

Let ΔR be the increase in radius,

$$\text{Longitudinal strain} = \frac{\Delta l}{l} = \frac{\Delta(2\pi R)}{2\pi R} = \frac{\Delta R}{R}$$

Now,

$$Y = \frac{T/A}{\Delta R/R}$$

$$\therefore \qquad \Delta R = \frac{TR}{AY} = \frac{(A\rho\omega^2R^2)R}{AY}$$

or

$$\Delta R = \frac{\rho\omega^2R^3}{Y}$$

Ans.

Example 6 *A member ABCD is subjected to point loads F_1, F_2, F_3 and F_4 as shown in figure. Calculate the force F_2 for equilibrium if $F_1 = 4500$ kg, $F_3 = 45000$ kg and $F_4 = 13000$ kg. Determine the total elongation of the member, assuming modulus of elasticity to be 2.1×10^6 kg/cm^2.*

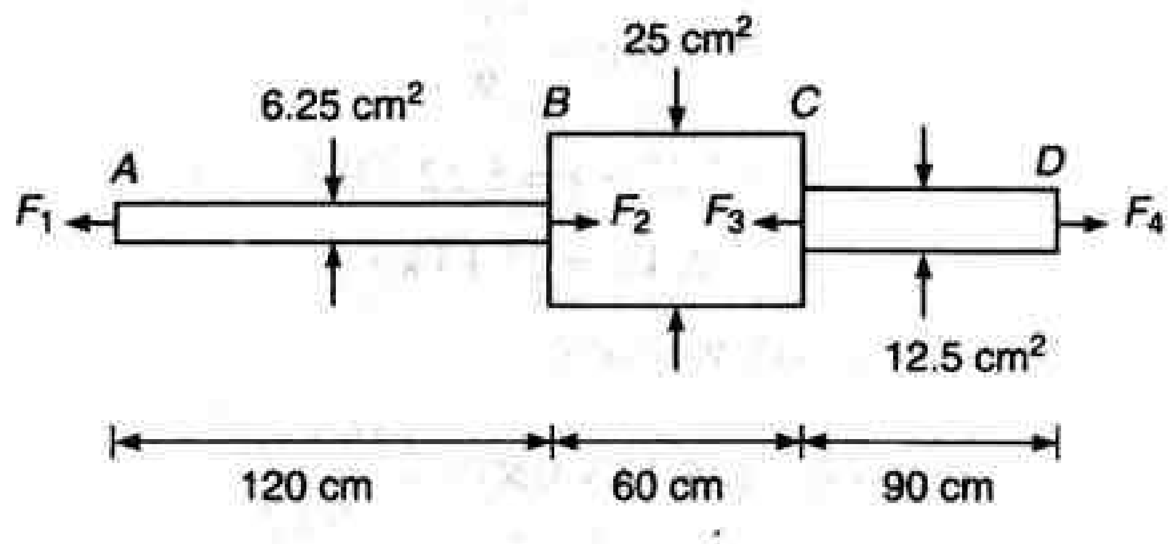


Fig. 12.23

Solution Given

Area of part AB, $A_1 = 6.25\ \text{cm}^2$

Area of part BC, $A_2 = 25\ \text{cm}^2$

Area of part CD, $A_3 = 12.5\ \text{cm}^2$

Length of part AB, $l_1 = 120$ cm

Length of part BC, $l_2 = 60$ cm

Length of part *CD*, $l_3 = 90\,\text{cm}$

Young's modulus of elasticity $Y = 2.1 \times 10^6\ \text{kg/cm}^2$

Magnitude of the force F_2 for Equilibrium

The magnitude of force F_2 may be found by equating the forces acting towards right to those acting towards left,

$$F_2 + F_4 = F_1 + F_3$$

$$F_2 + 13000 = 4500 + 45000$$

$$\therefore \quad F_2 = 36500\ \text{kg} \qquad \textbf{Ans.}$$

Total Elongation of the Member

For the sake of simplicity, the force of 36500 kg (acting at *B*) may be split up into two forces of 4500 kg and 32000 kg. The force of 45000 kg acting at *C* may be split into two forces of 32000 kg and 13000 kg. Now, it will be seen that the part *AB* of the member is subjected to a tension of 4500 kg, part *BC* is subjected to a compression of 32000 kg and part *CD* is subjected to a tension of 13,000 kg. Using the relation,

$$\Delta l = \frac{1}{Y}\left(\frac{F_1 l_1}{A_1} - \frac{F_2 l_2}{A_2} + \frac{F_3 l_3}{A_3}\right) \quad \text{with usual notations}$$

$$\Delta l = \frac{1}{2.1 \times 10^6}\left(\frac{4500 \times 120}{6.25} - \frac{32000 \times 60}{25} + \frac{13000 \times 90}{12.5}\right)\text{cm}$$

$$= 0.049\,\text{cm}$$

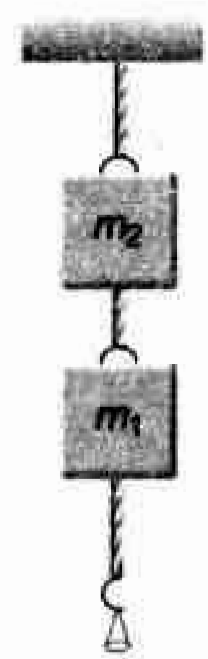
or $$\Delta l = 0.49\,\text{mm} \qquad \textbf{Ans.}$$

EXERCISES

For **JEE Main**

Subjective Questions

1. A cylindrical steel wire of 3 m length is to stretch no more than 0.2 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire ?
   $Y_{steel} = 2.1 \times 10^{11}\ N/m^2$
2. The elastic limit of a steel cable is $3.0 \times 10^8\ N/m^2$ and the cross-section area is $4\ cm^2$. Find the maximum upward acceleration that can be given to a 900 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.
3. If the elastic limit of copper is $1.5 \times 10^8\ N/m^2$, determine the minimum diameter a copper wire can have under a load of 10.0 kg, if its elastic limit is not to be exceeded.
4. Find the increment in the length of a steel wire of length 5 m and radius 6 mm under its own weight. Density of steel $= 8000\ kg/m^3$ and Young's modulus of steel $= 2 \times 10^{11}\ N/m^2$. What is the energy stored in the wire ? (Take $g = 9.8\ m/s^2$)
5. Two wires shown in figure are made of the same material which has a breaking stress of $8 \times 10^8\ N/m^2$. The area of cross-section of the upper wire is $0.006\ cm^2$ and that of the lower wire is $0.003\ cm^2$. The mass $m_1 = 10\ kg$, $m_2 = 20$ kg and the hanger is light. Find the maximum load that can be put on the hanger without breaking a wire. Which wire will break first if the load is increased ? (Take $g = 10\ m/s^2$)



6. A steel wire and a copper wire of equal length and equal cross-sectional area are joined end to end and the combination is subjected to a tension. Find the ratio of :

(a) the stresses developed in the two wires,

(b) the strains developed. (Y of steel $= 2\times10^{11}\ \text{N/m}^2$ and Y of copper $= 1.3\times10^{11}\ \text{N/m}^2$)

7. Calculate the approximate change in density of water in a lake at a depth of 400 m below the surface. The density of water at the surface is 1030 kg/m^3 and bulk modulus of water is $2\times10^9\ \text{N/m}^2$.

8. A wire of length 3 m, diameter 0.4 mm and Young's modulus $8\times10^{10}\ \text{N/m}^2$ is suspended from a point and supports a heavy cylinder of volume $10^{-3}\ \text{m}^3$ at its lower end. Find the decrease in length when the metal cylinder is immersed in a liquid of density 800 kg/m^3.

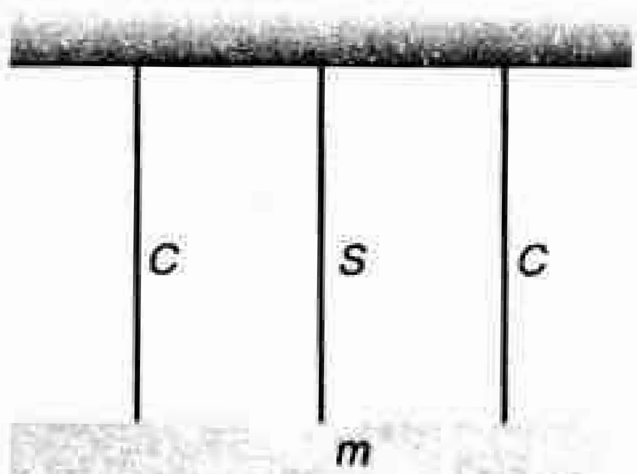
9. In taking a solid ball of rubber from the surface to the bottom of a lake of 180 m depth, reduction in the volume of the ball is 0.1%. The density of water of the lake is $1\times10^3\ \text{kg/m}^3$. Determine the value of the bulk modulus of elasticity of rubber. ($g = 9.8\ \text{m/s}^2$)

10. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire of length 5 m and diameter 4 mm which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor. Calculate the velocity of the ball at its lowest position. ($Y_{\text{steel}} = 2\times10^{11}\ \text{N/m}^2$)

11. A uniform ring of radius R and made up of a wire of cross-sectional radius r is rotated about its axis with a frequency ν. If density of the wire is ρ and Young's modulus is Y. Find the fractional change in radius of the ring.

12. A 6 kg weight is fastened to the end of a steel wire of unstretched length 60 cm. It is whirled in a vertical circle and has an angular velocity of 2 rev/s at the bottom of the circle. The area of cross-section of the wire is $0.05\ \text{cm}^2$. Calculate the elongation of the wire when the weight is at the lowest point of the path. Young's modulus of steel $= 2\times10^{11}\ \text{N/m}^2$.

13. A homogeneous block with a mass m hangs on three vertical wires of equal length arranged symmetrically. Find the tension of the wires if the middle wire is of steel and the other two are of copper. All the wires have the same cross-section. Consider the modulus of elasticity of steel to be double than that of copper.



14. A uniform copper bar of density ρ, length L, cross-sectional area S and Young's modulus Y is moving horizontally on a frictionless surface with constant acceleration a_0. Find :

(a) the stress at the centre of the wire,

(b) total elongation of the wire.

15. A 5 m long cylindrical steel wire with radius 2×10^{-3} m is suspended vertically from a rigid support and carries a bob of mass 100 kg at the other end. If the bob gets snapped, calculate the change in temperature of the wire ignoring radiation losses. (Take $g = 10\,\text{m/s}^2$)
(For the steel wire : Young's modulus $= 2.1 \times 10^{11}\,\text{N/m}^2$; Density $= 7860\,\text{kg/m}^3$; Specific heat $= 420\,\text{J/kg-°C}$).

Objective Questions

Single Correct Option

1. The bulk modulus for an incompressible liquid is
(a) zero (b) unity (c) infinity (d) between 0 and 1

2. The Young's modulus of a wire of length (L) and radius (r) is Y. If the length is reduced to $\frac{L}{2}$ and radius $\frac{r}{2}$, then its Young's modulus will be
(a) $\frac{Y}{2}$ (b) Y (c) $2Y$ (d) $4Y$

3. The maximum load that a wire can sustain is W. If the wire is cut to half its value, the maximum load it can sustain is
(a) W (b) $\frac{W}{2}$ (c) $\frac{W}{4}$ (d) $2W$

4. Identify the case when an elastic metal rod does not undergo elongation
(a) it is pulled with a constant acceleration on a smooth horizontal surface
(b) it is pulled with constant velocity on a rough horizontal surface
(c) it is allowed to fall freely
(d) All of the above

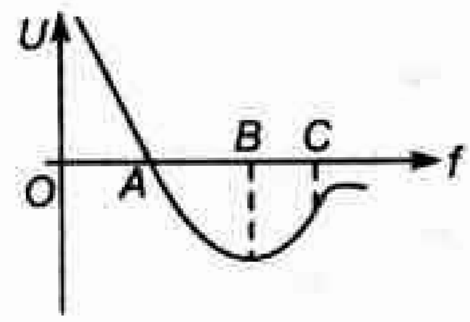
5. Vessel of $1 \times 10^{-3}\,\text{m}^3$ volume contains an oil. If a pressure of $1.2 \times 10^5\,\text{N/m}^2$ is applied on it, then volume decreases by $0.3 \times 10^{-3}\,\text{m}^3$. The bulk modulus of oil is
(a) $6 \times 10^{10}\,\text{N/m}^2$ (b) $4 \times 10^5\,\text{N/m}^2$ (c) $2 \times 10^7\,\text{N/m}^2$ (d) $1 \times 10^6\,\text{N/m}^2$

6. A bob of mass 10 kg is attached to a wire 0.3 m long. Its breaking stress is $4.8 \times 10^7\,\text{N/m}^2$. The area of cross-section of the wire is $10^{-6}\,\text{m}^2$. What is the maximum angular velocity with which it can be rotated in a horizontal circle?
(a) 8 rad/s (b) 4 rad/s (c) 2 rad/s (d) 1 rad/s

7. A uniform steel rod of cross-sectional area A and length L is suspended so that it hangs vertically. The stress at the middle point of the rod is
(a) $\frac{1}{2}\rho g L$ (b) $\frac{1}{4}\rho g L$ (c) $\rho g L$ (d) None of these

8. The bulk modulus of water is $2.0 \times 10^9\,\text{N/m}^2$. The pressure required to increase the density of water by 0.1% is
(a) $2.0 \times 10^3\,\text{N/m}^2$ (b) $2.0 \times 10^6\,\text{N/m}^2$
(c) $2.0 \times 10^5\,\text{N/m}^2$ (d) $2.0 \times 10^7\,\text{N/m}^2$

9. The potential energy U of diatomic molecules as a function of separation r is shown in figure. Identify the correct statement.



(a) The atoms are in equilibrium if $r = OA$
(b) The force is repulsive only if r lies between A and B
(c) The force is attractive if r lies between A and B
(d) The atoms are in equilibrium if $r = OB$

10. The length of a steel wire is l_1 when the stretching force is T_1 and l_2 when the stretching force is T_2. The natural length of the wire is

(a) $\dfrac{l_1T_1 + l_2T_2}{T_1 + T_2}$ (b) $\dfrac{l_2T_1 + l_1T_2}{T_1 + T_2}$ (c) $\dfrac{l_2T_1 - l_1T_2}{T_1 - T_2}$ (d) $\dfrac{l_1T_1 - l_2T_2}{T_1 - T_2}$

11. A mass m is suspended from a wire. Change in length of the wire is Δl. Now the same wire is stretched to double its length and the same mass is suspended from the wire. The change in length in this case will become (it is assumed that elongation in the wire is within the proportional limit)

(a) Δl (b) $2\Delta l$ (c) $4\Delta l$ (d) $8\Delta l$

12. A uniform metal rod fixed at its ends of $2\ \text{mm}^2$ cross-section is cooled from 40°C to 20°C. The coefficient of the linear expansion of the rod is 12×10^{-6} per degree celsius and its Young's modulus of elasticity is $10^{11}\ \text{N/m}^2$. The energy stored per unit volume of the rod is

(a) $2880\ \text{J/m}^3$
(b) $1500\ \text{J/m}^3$
(c) $5760\ \text{J/m}^3$
(d) $1440\ \text{J/m}^3$

13. A rod of length 1000 mm and coefficient of linear expansion $\alpha = 10^{-4}$ per degree celsius is placed in horizontal smooth surface symmetrically between fixed walls separated by 1001 mm. The Young's modulus of rod is $10^{11}\ \text{N/m}^2$. If the temperature is increased by 20°C, then the stress developed in the rod is (in N/m^2)

(a) 10^5
(b) 10^8
(c) 10^7
(d) 10^6

14. A uniform elastic plank moves due to a constant force F_0 distributed uniformly over the end face whose area is S. The Young's modulus of the plank is Y. The strain produced in the direction of force is

(a) $\dfrac{F_0}{2SY}$
(b) $\dfrac{F_0}{SY}$
(c) $\dfrac{2F_0}{SY}$
(d) $\dfrac{\sqrt{2}F_0}{SY}$

For JEE Advanced

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Steel is more elastic than rubber.
Reason : For same strain, steel requires more stress to be produced in it.

2. **Assertion :** If pressure is increased, bulk modulus of gases will increase.
Reason : With increase in pressure, temperature of gas also increases.

3. **Assertion :** From the relation $Y = \frac{Fl}{A\Delta l}$, we can say that, if length of a wire is doubled, its Young's modulus of elasticity will also becomes two times.
Reason : Modulus of elasticity is a material property.

4. **Assertion :** Bulk modulus of elasticity can be defined for all three states of matter, solid liquid and gas.
Reason : Young's modulus is not defined for liquids and gases.

5. **Assertion :** Every wire is like a spring, whose spring constant, $K \propto \frac{1}{l}$ where l is length of wire.
Reason : It follows from the relation $K = \frac{YA}{l}$

6. **Assertion :** Ratio of stress and strain is always constant for a substance.
Reason : This ratio is called modulus of elasticity.

7. **Assertion :** Ratio of isothermal bulk modulus and adiabatic bulk modulus for a monoatomic gas at a given pressure is $\frac{3}{5}$.
Reason : This ratio is equal to $\gamma = \frac{C_p}{C_V}$.

More than One Correct Options

1. A metal wire of length L, area of cross-section A and Young's modulus Y is stretched by a variable force F such that F is always slightly greater than the elastic forces of resistance in the wire. When the elongation of the wire is l

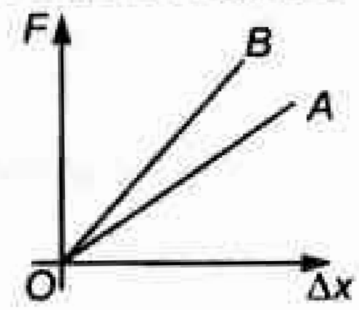
(a) the work done by F is $\frac{YAl^2}{2L}$

(b) the work done by F is $\frac{YAl^2}{L}$

(c) the elastic potential energy stored in the wire is $\frac{YAl^2}{2L}$

(d) the elastic potential energy stored in the wire is $\frac{YAl^2}{4L}$

2. Two wires A and B of same length are made of same material. The figure represents the load F *versus* extension Δx graph for the two wires. Then :



(a) The cross sectional area of A is greater than that of B
(b) The elasticity of B is greater than that of A
(c) The cross-sectional area of B is greater than that of A
(d) The elasticity of A is greater than that of B

3. A body of mass M is attached to the lower end of a metal wire, whose upper end is fixed. The elongation of the wire is l.
(a) Loss in gravitational potential energy of M is Mgl
(b) The elastic potential energy stored in the wire is Mgl
(c) The elastic potential energy stored in the wire is $\frac{1}{2}Mgl$
(d) Heat produced is $\frac{1}{2}Mgl$

Match the Columns

1. Match the following two columns. (dimension wise)

Column I	Column II
(a) Stress	(p) coefficient of friction
(b) Strain	(q) relative density
(c) Modulus of elasticity	(r) energy density
(d) Force constant of a wire	(s) None

2. A wire of length l, area of cross section A and Young's modulus of elasticity Y is stretched by a longitudinal force F. The change in length is Δl. Match the following two columns.

Note *In column I, corresponding to every option, other factors remain constant.*

Column I	Column II
(a) F is increased	(p) Δl will increase
(b) l is increased	(q) stress will increase
(c) A is increased	(r) Δl will decrease
(d) Y is increased	(s) stress will decrease

ANSWERS

Introductory Exercise 12.1

1. Wire B 2. 0.0125 cm 3. 250.2 mm

For JEE Main

Subjective Questions

1. 1.91 mm 2. 34.64 m/s^2 3. Diameter, $d = 0.912$ mm 4. $\Delta l = 4.9 \times 10^{-6}$ m, $U = 5.43 \times 10^{-5}$ J
5. Load = 14 kg, lower string 6. (a) 1 (b) Ratio = $\frac{13}{20}$ 7. 2.0 kg/m^3 8. $\Delta l = 2.34 \times 10^{-3}$m
9. $B = 1.76 \times 10^9$N/m^2 10. $v = 31.23$ m/s 11. $\frac{4\pi^2 v^2 \rho R^3}{Y}$ 12. $\Delta l = 3.8 \times 10^{-4}$ m
13. $T_C = \frac{mg}{4}$, $T_S = 2T_C$ 14. (a) $\frac{1}{2} L\rho a_0$ (b) $\frac{1}{2} \frac{\rho a_0 L^2}{Y}$ 15. 4.568×10^{-3} °C

Objective Questions

1. (c) 2. (b) 3. (a) 4. (c) 5. (b) 6. (b) 7. (a) 8. (b) 9. (d) 10. (c)
11. (c) 12. (a) 13. (b) 14. (a)

For JEE Advanced

Assertion and Reason

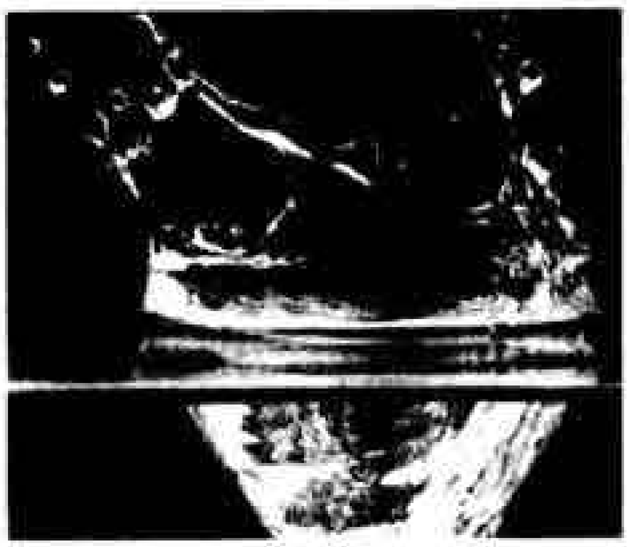
1. (a) 2. (c) 3. (d) 4. (b) 5. (a) 6. (d) 7. (c)

More than One Correct Options

1. (a,c) 2. (c) 3. (a,c,d)

Match the Columns

1. (a) → r (b) → p,q (c) → r (d) → s
2. (a) → pq (b) → p (c) → rs (d) → r



13

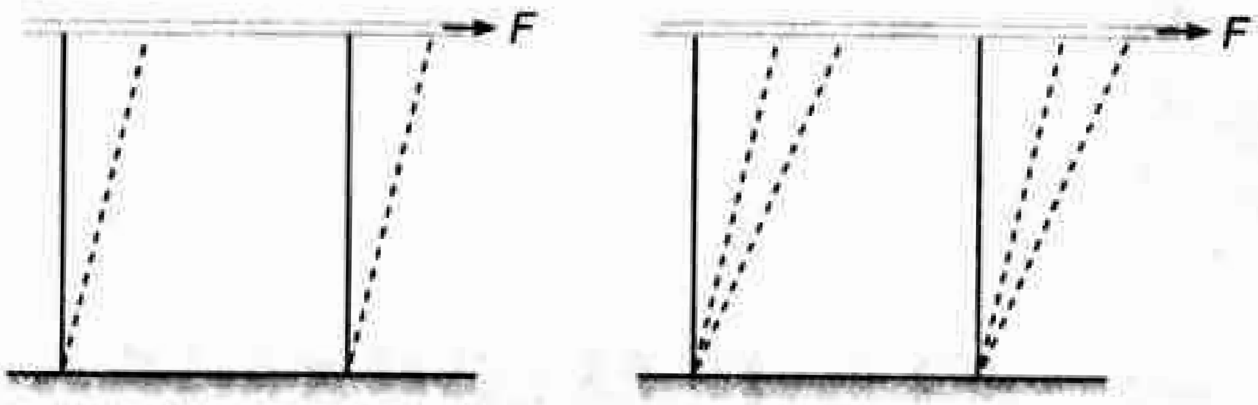
FLUID MECHANICS

Chapter Contents

13.1 Definition of a Fluid

Fluid mechanics deals with the behaviour of fluids at rest and in motion. A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be.

Thus, fluids comprise the liquid and gas (or vapor) phases of the physical forms in which matter exists. The distinction between a fluid and the solid state of matter is clear if you compare fluid and solid behaviour. A solid deforms when a shear stress is applied but it does not continue to increase with time. However if a shear stress is applied to a fluid, the deformation continues to increase as long as the stress is applied. We may alternatively define a fluid as a substance that **cannot sustain a shear stress when at rest.**



(a) Solid (b) Fluid

Fig. 13.1 Behaviour of a solid and a fluid, under the action of a constant shear force

In the present chapter we shall deal with liquids. An ideal liquid is incompressible and nonviscous in nature. An incompressible liquid means the density of the liquid is constant, it is independent of the variations in pressure. A nonviscous liquid means that, parts of the liquid in contact do not exert any tangential force on each other. Thus, there is no friction between the adjacent layers of a liquid. The force by one part of the liquid on the other part is **perpendicular to the surface of contact.**

13.2 Density of a Liquid

Density (ρ) of any sustance is defined as the mass per unit volume or

$$\rho = \frac{\text{mass}}{\text{volume}}$$

or

$$\rho = \frac{m}{V}$$

Relative Density (RD)

In case of a liquid, sometimes an another term **relative density** (RD) is defined. It is the ratio of density of the substance to the density of water at 4°C. Hence,

$$\text{RD} = \frac{\text{Density of substance}}{\text{Density of water at } 4°\text{C}}$$

RD is a pure ratio. So, it has no units. It is also sometimes referred as specific gravity. Density of water at 4°C in CGS is 1 g/cm^3. Therefore, numerically the RD and density of substance (in CGS) are equal. In SI units the density of water at 4°C is 1000 kg/m^3.

Sample Example 13.1 *Relative density of an oil is 0.8. Find the absolute density of oil in CGS and SI units.*

Solution Density of oil (in CGS) = (RD) g/cm^3

$$= 0.8 \text{ g/cm}^3$$
$$= 800 \text{ kg/m}^3$$ **Ans.**

Density of a mixture of two or more liquids

Here, we have two cases.

Case 1 : Suppose two liquids of densities ρ_1 and ρ_2 having masses m_1 and m_2 are mixed together. Then the density of the mixture will be

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{(m_1 + m_2)}{(V_1 + V_2)}$$

$$= \frac{(m_1 + m_2)}{\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right)}$$

If $m_1 = m_2$, then
$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

Case 2 : If two liquids of densities ρ_1 and ρ_2 having volumes V_1 and V_2 are mixed, then the density of the mixture is,

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{m_1 + m_2}{V_1 + V_2}$$

$$= \frac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2}$$

If, $V_1 = V_2$, then
$$\rho = \frac{\rho_1 + \rho_2}{2}$$

Effect of Temperature on Density

As the temperature of a liquid is increased, the mass remains the same while the volume is increased and hence, the density of the liquid decreases $\left(\text{as } \rho \propto \frac{1}{V}\right)$. Thus,

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV} = \frac{V}{V + V\gamma\Delta\theta}$$

or
$$\frac{\rho'}{\rho} = \frac{1}{1 + \gamma\Delta\theta}$$

Here, γ = thermal coefficient of volume expansion

and $\Delta\theta$ = rise in temperature

$$\therefore \quad \rho' = \frac{\rho}{1 + \gamma\Delta\theta}$$

Effect of pressure on Density

As pressure is increased, volume decreases and hence density will increase. Thus,

$$\rho \propto \frac{1}{V}$$

$$\therefore \quad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV} = \frac{V}{V - \left(\frac{dP}{B}\right)V}$$

or

$$\frac{\rho'}{\rho} = \frac{1}{1 - \frac{dP}{B}}$$

Here, dP = change in pressure

and B = bulk modulus of elasticity of the liquid

Therefore,

$$\rho' = \frac{\rho}{1 - \frac{dP}{B}}$$

13.3 Pressure in a Fluid

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid.

While the fluid as a whole is at rest, the molecules that makes up the fluid are in motion, the force exerted by the fluid is due to molecules colliding with their surroundings.

If we think of an imaginary surface within the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface, otherwise the surface would accelerate and the fluid would not remain at rest.

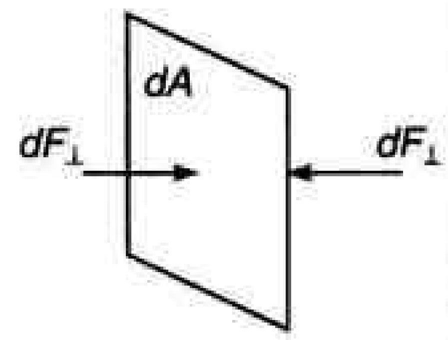


Fig. 13.2

Consider a small surface of area dA centered on a point on the fluid, the normal force exerted by the fluid on each side is $dF_{\perp}$. The pressure P is defined at that point as the normal force per unit area, *i.e.*,

$$P = \frac{dF_{\perp}}{dA}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$P = \frac{F_{\perp}}{A}$$

where $F_{\perp}$ is the normal force on one side of the surface. The SI unit of pressure is pascal, where

$$1 \text{ pascal} = 1 \text{ Pa} = 1.0 \text{ N/m}^2$$

One unit used principally in meterology is the Bar which is equal to 10^5 Pa.

$$1 \text{ Bar} = 10^5 \text{ Pa}$$

Atmospheric Pressure (P_o)

It is pressure of the earth's atmosphere. This changes with weather and elevation. Normal atmospheric pressure at sea level (an average value) is 1.013×10^5 Pa. Thus,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

Note *Fluid pressure acts perpendicular to any surface in the fluid no matter how that surface is oriented. Hence, pressure has no intrinsic direction of its own, its a* ***scalar.*** *By contrast, force is a vector with a definite direction.*

Absolute Pressure and Gauge Pressure

The excess pressure above atmospheric pressure is usually called gauge pressure and the total pressure is called absolute pressure. Thus,

Gauge pressure = absolute pressure – atmospheric pressure

Absolute pressure is always greater than or equal to zero. While gauge pressure can be negative also.

Variation in Pressure with depth

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. But often the fluid's weight is not negligible and under such condition pressure increases with increasing depth below the surface.

Let us now derive a general relation between the pressure P at any point in a fluid at rest and the elevation y of that point. We will assume that the density ρ and the acceleration due to gravity g are the same throughout the fluid. If the fluid is in equilibrium, every volume element is in equilibrium.

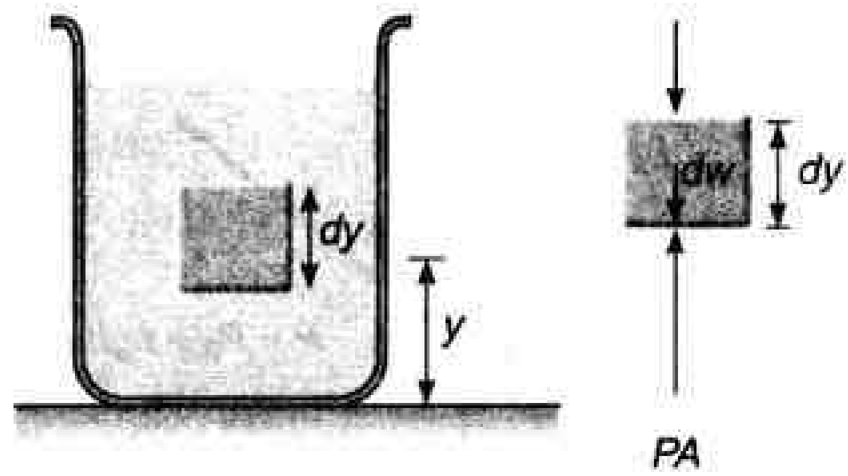


Fig. 13.3

Consider a thin element of fluid with height dy. The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The weight of the fluid element is

$$dW = (\text{volume})\ (\text{density})\ (g) = (A\ dy)(\rho)(g)$$

or

$$dW = \rho g A dy$$

What are the other forces in y-direction on this fluid element? Call the pressure at the bottom surface P, the total y component of upward force is PA. The pressure at the top surface is $P + dP$ and the total y-component of downward force on the top surface is $(P + dP)A$. The fluid element is in equilibrium, so the total y-component of force including the weight and the forces at the bottom and top surfaces must be zero.

$$\Sigma F_y = 0$$

∴

$$PA - (P + dP)A - \rho g A\ dy = 0$$

or

$$\frac{dP}{dy} = -\rho g \qquad \text{...(i)}$$

This equation shows that when y increases, P decreases, *i.e.*, as we move upward in the fluid, pressure decreases.

If P_1 and P_2 be the pressures at elevations y_1 and y_2 and if ρ and g are constant, then integrating Eq. (i), we get

$$\int_{P_1}^{P_2} dP = -\rho g \int_{y_1}^{y_2} dy$$

or $$P_2 - P_1 = -\rho g(y_2 - y_1) \qquad \text{...(ii)}$$

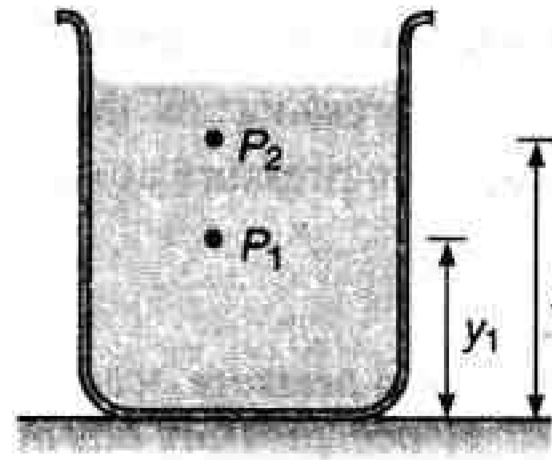


Fig. 13.4

It's often convenient to express Eq. (ii) in terms of the depth below the surface of a fluid. Take point 1 at depth h below the surface of fluid and let P represents pressure at this point. Take point 2 at the surface of the fluid, where the pressure is P_0 (subscript zero for zero depth). The depth of point 1 below the surface is,

$$h = y_2 - y_1$$

and Eq. (ii) becomes

$$P_0 - P = -\rho g(y_2 - y_1) = -\rho gh$$

$\therefore$ $$P = P_0 + \rho gh \qquad \text{...(iii)}$$

Thus, pressure increases linearly with depth, if ρ and g are uniform. A graph between P and h is shown below.

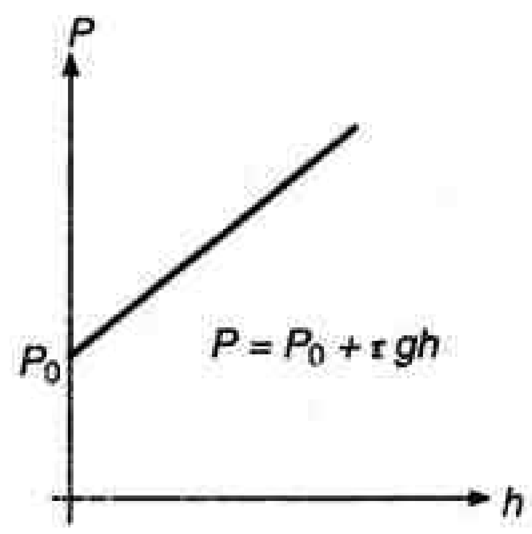


Fig. 13.5

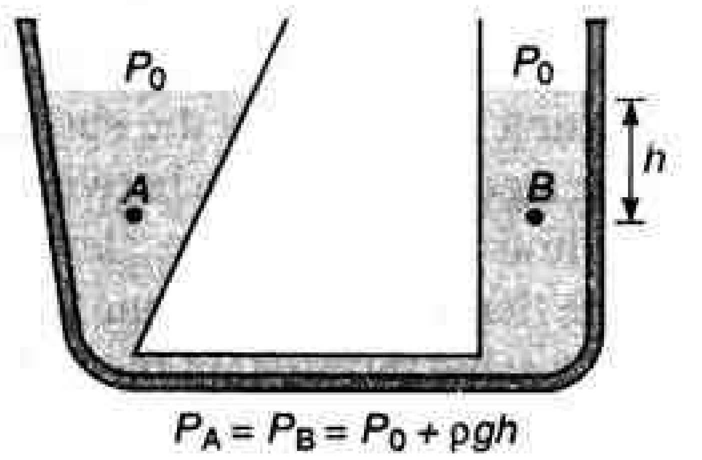


Fig. 13.6

Further, the pressure is the same at any two points at the same level in the fluid. The shape of the container does not matter.

Pascal's Law

Pascal's law or the **principle of transmission of fluid-pressure** is a principle in fluid mechanics that states that pressure exerted anywhere in a confined incompressible fluid is transmitted equally in all directions throughout the fluid.

The simplest instance of this is stepping on a balloon; the balloon bulges out on all sides under the foot and not just on one side. This is precisely what Pascal's Law is all about – the air which is the fluid in this case, was confined by the balloon, and you applied pressure with your foot causing it to get displaced uniformly.

A well known application of Pascal's law is the hydraulic lift used to support or lift heavy objects. It is schematically illustrated in figure.

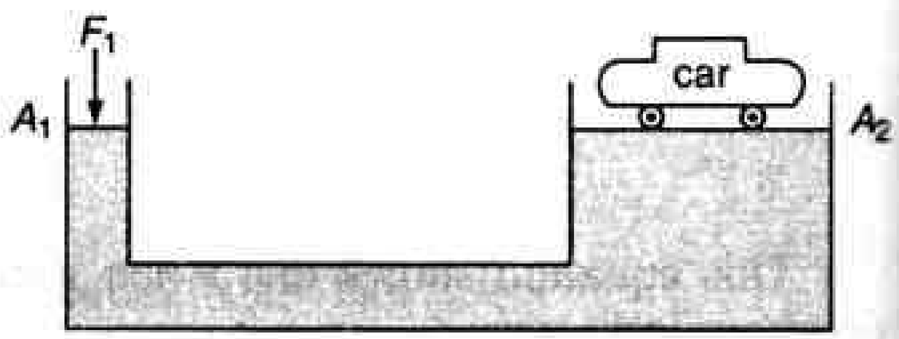


Fig. 13.7

A piston with small cross section area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $P = \frac{F_1}{A_1}$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \text{or} \quad F_2 = \frac{A_2}{A_1} \cdot F_1$$

Now, since $A_2 > A_1$, therefore, $F_2 > F_1$. Thus, hydraulic lift is a force multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators and hydraulic brakes all use this principle.

Sample Example 13.2 *Figure shows a hydraulic press with the larger piston of diameter 35 cm at a height of 1.5 m relative to the smaller piston of diameter 10 cm. The mass on the smaller piston is 20 kg. What is the force exerted on the load by the larger piston? The density of oil in the press is 750 kg/m^3. (Take g = 9.8 m/s^2)*

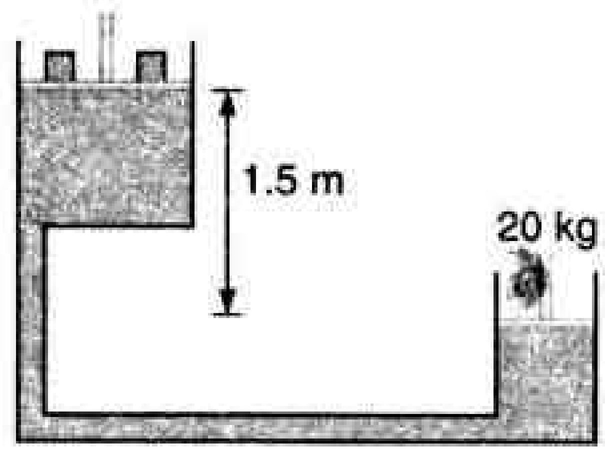


Fig. 13.8

Solution Pressure on the smaller piston $= \frac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2}$ N/m^2

Pressure on the larger piston $= \frac{F}{\pi \times (17.5 \times 10^{-2})^2}$ N/m^2

The difference between the two pressures = $h\rho g$

where $h = 1.5$ m and $\rho = 750$ kg/m^3

Thus, $$\frac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2} - \frac{F}{\pi \times (17.5 \times 10^{-2})^2} = 1.5 \times 750 \times 9.8$$

which gives, $F = 1.3 \times 10^3$ N **Ans.**

Note *Atmospheric pressure is common to both pistons and has been ignored.*

• *Important points* in Pressure

1. At same point on a fluid pressure is same in all directions. In the figure,

$$P_1 = P_2 = P_3 = P_4$$

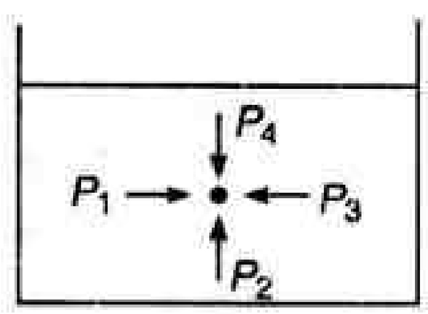


Fig. 13.9

2. Forces acting on a fluid in equilibrium have to be perpendicular to its surface. Because it cannot sustain the shear stress.
3. In the **same liquid** pressure will be same at all points at the same level. For example, in the figure:

$$P_1 \neq P_2$$

Further, $$P_3 = P_4$$

$\therefore$ $$P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$$

or $$\rho_1 h_1 = \rho_2 h_2 \quad \text{or} \quad h \propto \frac{1}{\rho}$$

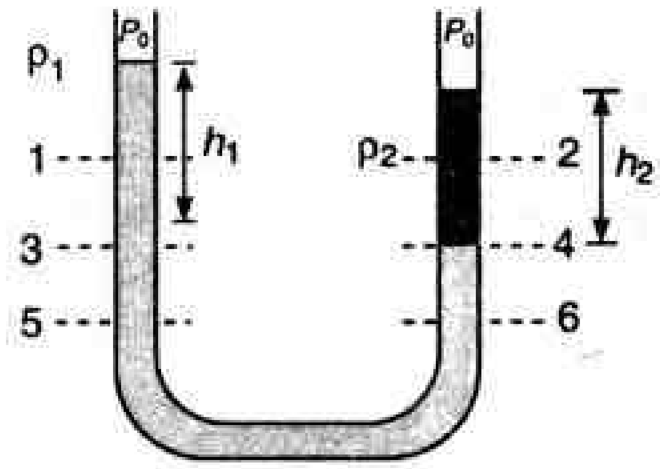


Fig. 13.10

4. **Barometer :** It is a device used to measure atmospheric pressure.

In principle, any liquid can be used to fill the barometer, but mercury is the substance of choice because its great density makes possible an instrument of reasonable size.

$$P_1 = P_2$$

Here, $P_1 =$ atmospheric pressure (P_0)

and $P_2 = 0 + \rho g h = \rho g h$

Here, $\rho =$ density of mercury

$\therefore$ $$P_0 = \rho g h$$

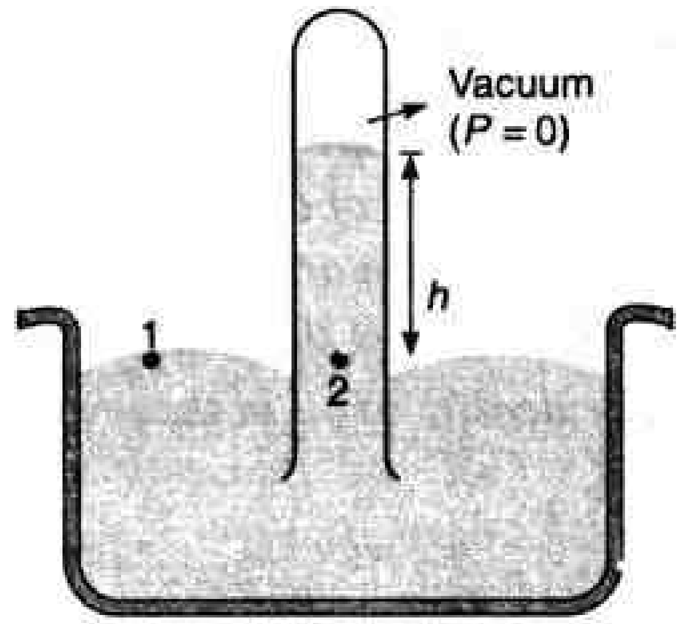


Fig. 13.11

Thus, the mercury barometer reads the atmospheric pressure (P_0) directly from the height of the mercury column.

For example if the height of mercury in a barometer is 760 mm, then atmospheric pressure will be,

$$P_0 = \rho g h = (13.6 \times 10^3)(9.8)(0.760) = 1.01 \times 10^5 \text{ N/m}^2$$

5. **Manometer :** It is a device used to measure the pressure of a gas inside a container.

The U-shaped tube often contains mercury.

$$P_1 = P_2$$

Here, $P_1 =$ pressure of the gas in the container (P)

and $P_2 =$ atmospheric pressure $(P_0) + \rho g h$

$\therefore$ $$P = P_0 + h\rho g$$

This can also be written as

$$P - P_0 = \text{gauge pressure} = h\rho g$$

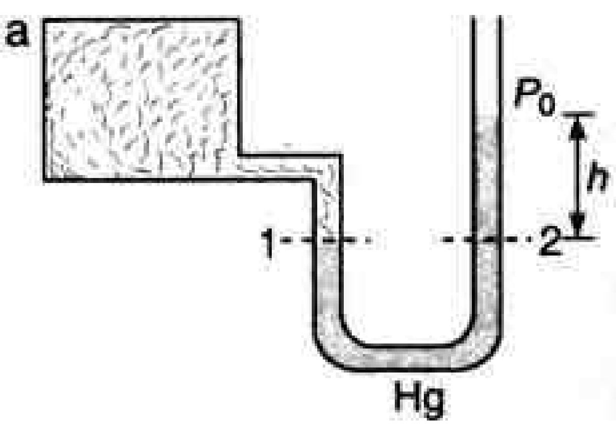


Fig. 13.12

Here, ρ is the density of the liquid used in U-tube.

Thus by measuring h we can find absolute (or gauge) pressure in the vessel.

6. **Free body diagram of a liquid :** The free body diagram of the liquid (showing the vertical forces only) is shown in Fig. (b). For the equilibrium of liquid.

Net downward force = net upward force

$\therefore$ $$P_0 A + W = (P_0 + \rho g h)A$$

or $$W = \rho g h A$$

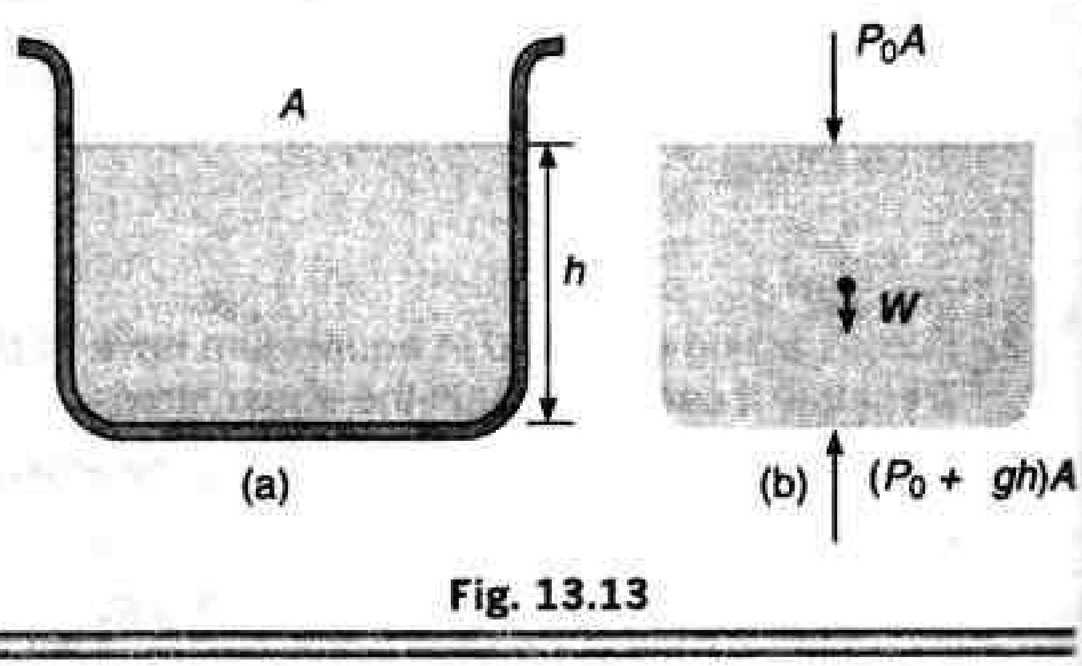


Fig. 13.13

Sample Example 13.3 *A glass full of water upto a height of* 10 *cm has a bottom of area* 10 cm^2, *top of area* 30 cm^2 *and volume* 1 *litre.*

(a) *Find the force exerted by the water on the bottom.*

(b) *Find the resultant force exerted by the sides of the glass on the water.*

(c) *If the glass is covered by a jar and the air inside the jar is completely pumped out, what will be the answers to parts (a) and (b).*

(d) *If a glass of different shape is used provided the height, the bottom area and the volume are unchanged, will the answers to parts (a) and (b) change.*

Take $g = 10\ m/s^2$, *density of water* $= 10^3\ kg/m^3$ *and atmospheric pressure* $= 1.01 \times 10^5\ N/m^2$.

Solution (a) Force exerted by the water on the bottom

$$F_1 = (P_0 + \rho g h) A_1 \qquad \text{...(i)}$$

Here, P_0 = atmospheric pressure = $1.01 \times 10^5\ \text{N/m}^2$

ρ = density of water = $10^3\ \text{kg/m}^3$

$g = 10\ \text{m/s}^2$, $\quad h = 10\ \text{cm} = 0.1\ \text{m}$

and A_1 = area of base = $10\ \text{cm}^2 = 10^{-3}\ \text{m}^2$

Substituting in Eq. (i), we get

$$F_1 = (1.01 \times 10^5 + 10^3 \times 10 \times 0.1) \times 10^{-3}$$

or

$$F_1 = 102\ \text{N (downwards)} \qquad \textbf{Ans.}$$

(b) Force exerted by atmosphere on water:

$$F_2 = (P_0) A_2$$

Here, A_2 = area of top = $30\ \text{cm}^2 = 3 \times 10^{-3}\ \text{m}^2$

$\therefore$

$$F_2 = (1.01 \times 10^5)(3 \times 10^{-3})$$

$$= 303\ \text{N (downwards)}$$

Force exerted by bottom on the water

$$F_3 = -F_1$$

or

$$F_3 = 102\ \text{N (upwards)}$$

weight of water $W = (\text{volume})(\text{density})(g) = (10^{-3})(10^3)(10)$

$$= 10\ \text{N (downwards)}$$

Let F be the force exerted by side walls on the water (upwards). Then, from equilibrium of water

Net upward force = net downward force

or

$$F + F_3 = F_2 + W$$

$\therefore$

$$F = F_2 + W - F_3 = 303 + 10 - 102$$

or

$$F = 211\ \text{N (upwards)} \qquad \textbf{Ans.}$$

(c) If the air inside the jar is completely pumped out,

$$F_1 = (\rho g h) A_1 \qquad (\text{as } P_0 = 0)$$

$$= (10^3)(10)(0.1)(10^{-3})$$

$$= 1\text{ N (downwards)}$$ **Ans.**

In this case, $F_2 = 0$

and $F_3 = 1\text{ N (upwards)}$

$\therefore$ $F = F_2 + W - F_3$

$$= 0 + 10 - 1$$

$$= 9\text{ N (upwards)}$$ **Ans.**

(d) No, the answer will remain the same. Because the answers depend upon P_0, ρ, g, h, A_1 and A_2.

Sample Example 13.4 *Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill up to a particular common height. Is the force exerted by water on the base of the vessel the same in the two cases? If so, why do the vessels filled with water to that same height give different readings on a weighing scale?*

Solution Pressure (and therefore force) on the two equal base areas are identical. But force is exerted by water on the sides of the vessels also, which has a non-zero vertical component when the sides of the vessel are not perfectly normal to the base. This net vertical component of force by water on the sides of the vessel is greater for the first vessel than the second. Hence, the vessels weigh different when force on the base is the same in the two cases.

Introductory Exercise 13.1

1. Two vessels A and B have same base area. Equal volumes of a liquid are poured in the two vessels to different heights h_A and $h_B (> h_A)$. In which vessel, the force on the base of vessel will be more.
2. A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross section of the piston carrying the load is 425 cm^2. What maximum pressure would the smaller piston have to bear?
3. A U tube contains water and methylated spirit seprated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the relative density of sprit?
4. In the above question if 15.0 cm of water and spirit each are further, poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms? (Relative density of mercury = 13.6)
5. A manometer reads the pressure of a gas in an enclosure as shown in figure (a). When some of the gas is removed by a pump, the manometer reads as in (b). The liquid used in the manometers is mercury and the atmospheric pressure is 76 cm of mercury.
   (i) Give the absolute and gauge pressure of the gas in the enclosure for cases (a) and (b) in units of cm of mercury.
   (ii) How would the levels change in case (b) if 13.6 cm of water are poured into the right limb of the manometer?

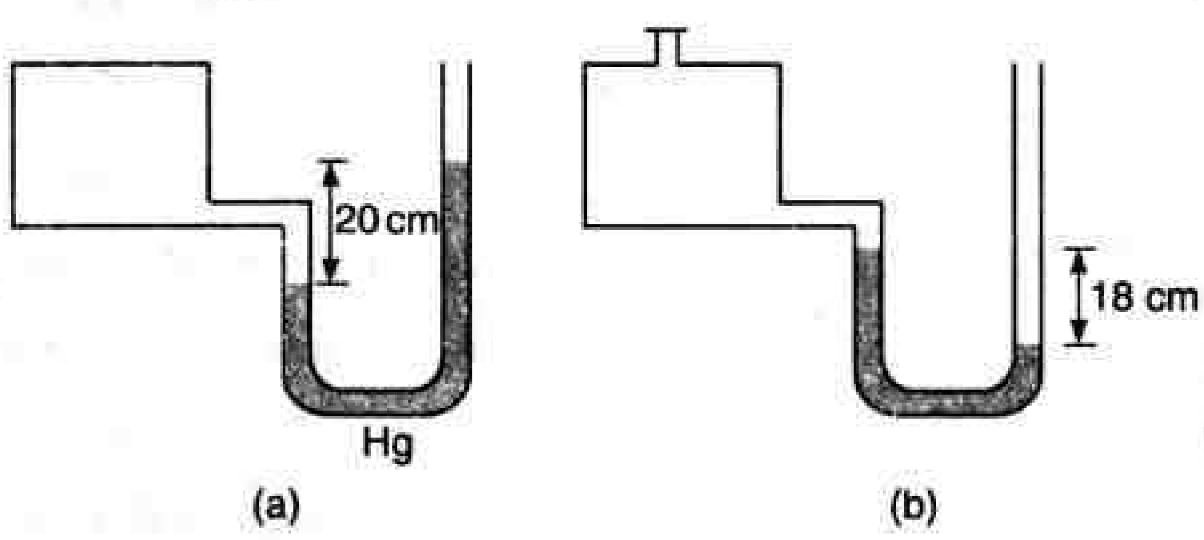


Fig. 13.14

13.4 Pressure Difference in Accelerating Fluids

Consider a liquid kept at rest in a beaker as shown in figure (a). In this case we know that pressure do not change in horizontal direction (x-direction), it decreases upwards along y-direction. So, we can write the equations,

$$\frac{dP}{dx} = 0 \quad \text{and} \quad \frac{dP}{dy} = -\rho g \qquad \ldots\text{(i)}$$

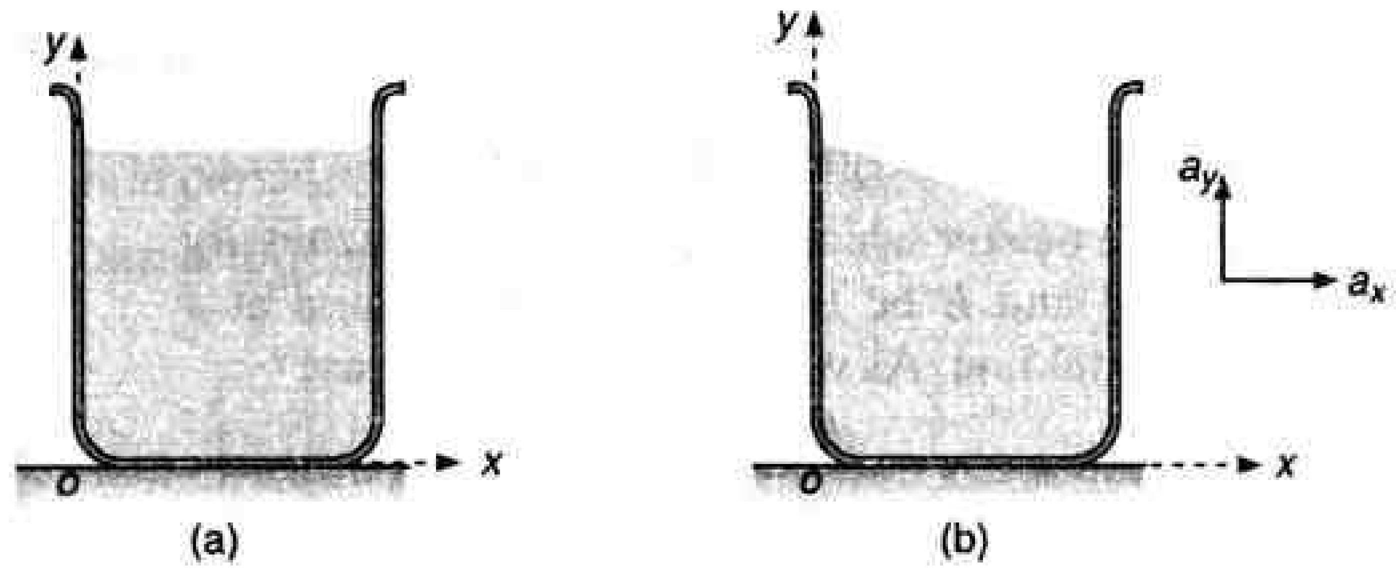


Fig. 13.15

But, suppose the beaker is accelerated and it has components of acceleration a_x and a_y in x and y directions respectively, then the pressure decreases along both x and y directions. The above equation in that case reduces to,

$$\frac{dP}{dx} = -\rho a_x$$

and

$$\frac{dP}{dy} = -\rho(g + a_y) \qquad \ldots\text{(ii)}$$

These equations can be derived as under :

Consider a beaker filled with some liquid of density ρ accelerating upwards with an acceleration a_y along positive y-direction. Let us draw the free body diagram of a small element of fluid of area A and length dy as shown in figure.

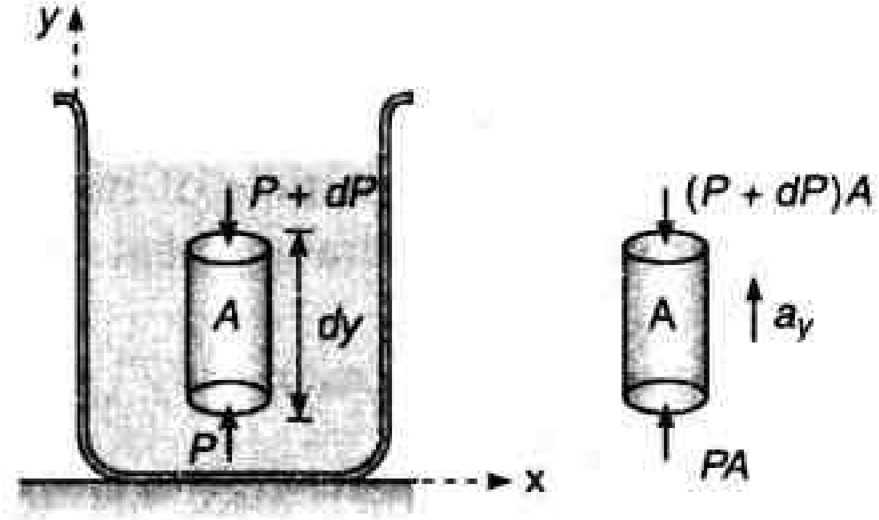


Fig. 13.16

Equation of motion for this fluid element is,

$$PA - W - (P + dP)A = (\text{mass})(a_y)$$

or $$-W - (dP)A = (A\rho\, dy)(a_y)$$

or $$-(A\rho g\, dy) - (dP)A = (A\rho\, dy)(a_y)$$

or $$\frac{dP}{dy} = -\rho(g + a_y)$$

Similarly, if the beaker moves along positive x-direction with acceleration a_x, the equation of motion for the fluid element shown in figure is,

$$PA-(P+dP)A=(\text{mass})(a_x)$$

or $$-(dP)A=(A\rho\, dx)a_x$$

or $$\frac{dP}{dx}=-\rho a_x$$

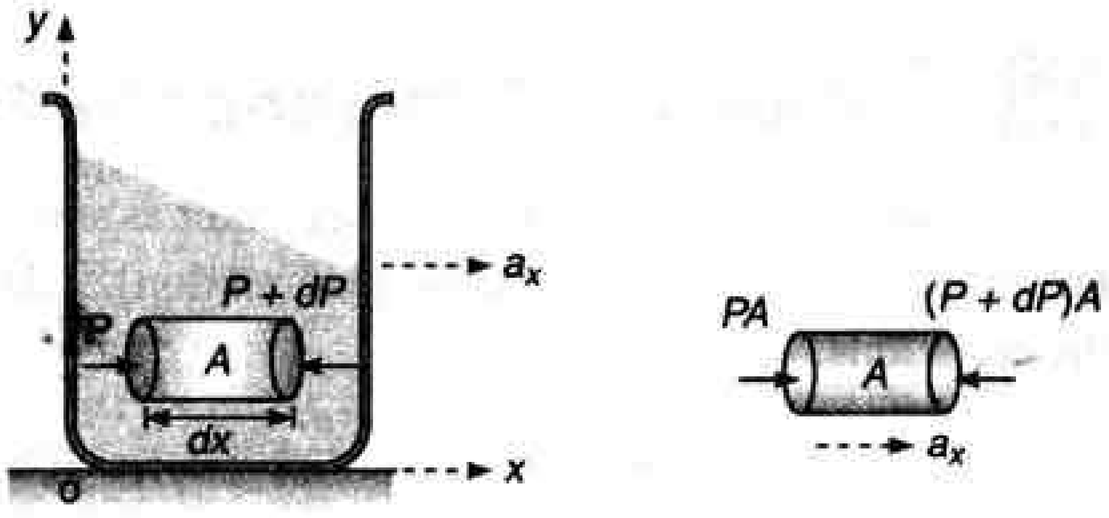


Fig. 13.17

Free Surface of a Liquid Accelerated in Horizontal Direction

Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration 'a'. Let A and B be two points in the liquid at a separation x in the same horizontal line. As we have seen in this case

$$\frac{dP}{dx}=-\rho a$$

or $$dP=-\rho a\, dx$$

Integrating this with proper limits, we get

$$P_A-P_B=\rho ax \qquad \ldots\text{(iii)}$$

Further, $$P_A=P_0+\rho g h_1$$

and $$P_B=P_0+\rho g h_2$$

Substituting, in Eq. (iii), we get

$$\rho g(h_1-h_2)=\rho ax$$

$$\therefore \qquad \frac{h_1-h_2}{x}=\frac{a}{g}=\tan\theta$$

$$\therefore \qquad \tan\theta=\frac{a}{g}$$

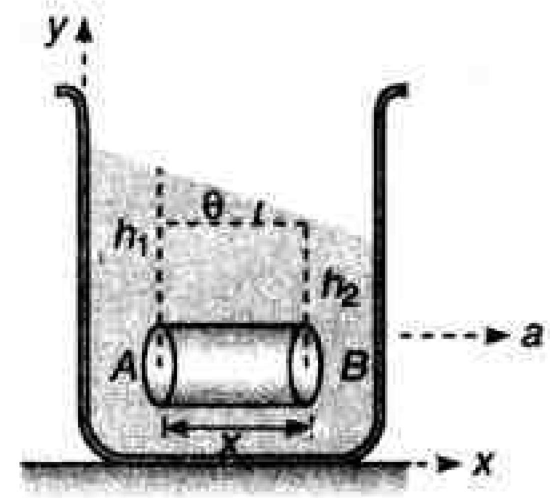


Fig. 13.18

Alternate Method

Consider a fluid particle of mass m at point P on the surface of liquid. From the accelerating frame of reference, two forces are acting on it,

(i) pseudo force (ma)

(ii) weight (mg)

As we said earlier also, net force in equilibrium should be perpendicular to the surface.

$$\therefore \qquad \tan\theta=\frac{ma}{mg}$$

or $$\tan\theta=\frac{a}{g}$$

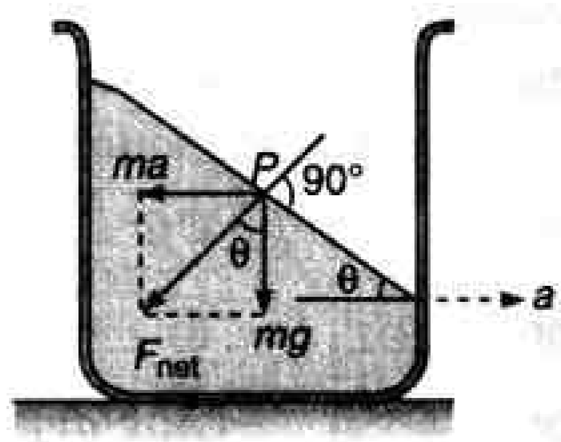


Fig. 13.19

Sample Example 13.5 *A liquid of density ρ is in a bucket that spins with angular velocity ω as shown in figure. Show that the pressure at a radial distance r from the axis is*

$$P = P_0 + \frac{\rho\omega^2 r^2}{2}$$

where P_0 is the atmospheric pressure.

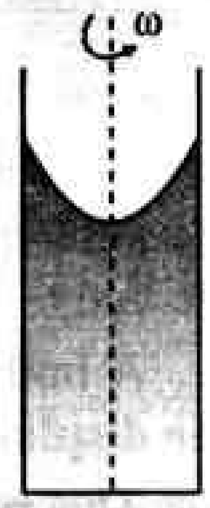


Fig. 13.20

Solution Consider a fluid particle P of mass m at coordinates (x, y). From a non-inertial rotating frame of reference two forces are acting on it,

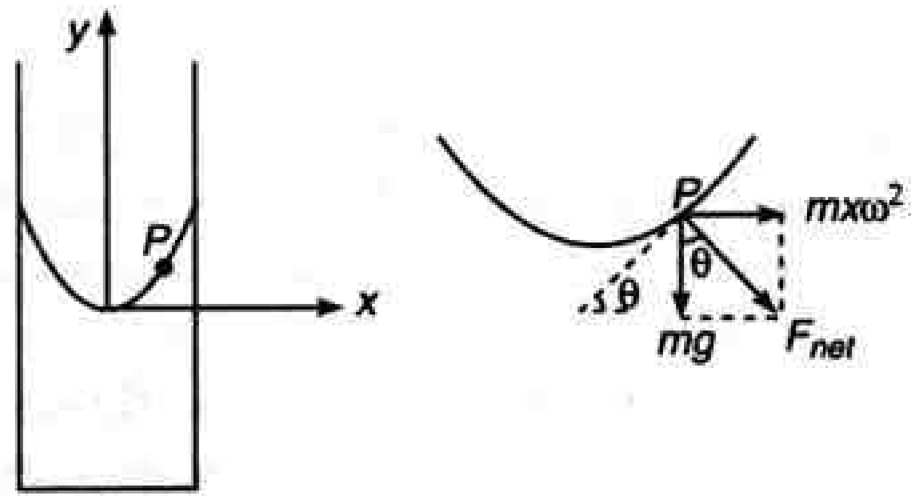


Fig. 13.21

(i) pseudo force $(mx\omega^2)$

(ii) weight (mg)

in the directions shown in figure.

Net force on it should be perpendicular to the free surface (in equilibrium). Hence,

$$\tan\theta = \frac{mx\omega^2}{mg} = \frac{x\omega^2}{g} \quad \text{or} \quad \frac{dy}{dx} = \frac{x\omega^2}{g}$$

$$\therefore \qquad \int_0^y dy = \int_0^x \frac{x\omega^2}{g}\,.\,dx$$

$$\therefore \qquad y = \frac{x^2\omega^2}{2g}$$

This is the equation of the free surface of the liquid, which is a parabola.

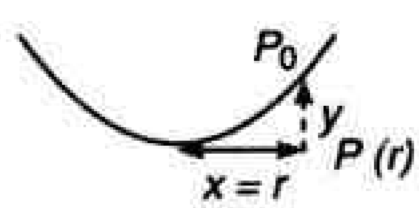


Fig. 13.22

At $x = r$,
$$y = \frac{r^2\omega^2}{2g}$$

$$\therefore \qquad P(r) = P_0 + \rho g y$$

or
$$P(r) = P_0 + \frac{\rho\omega^2 r^2}{2}$$

Hence proved.

13.5 Archimedes' Principle

If a heavy object is immersed in water, it seems to weigh less than when it is in air. This is because the water exerts an upward force called **buoyant force.** It is equal to the weight of the fluid displaced by the body.

A body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.

This result is known as **Archimedes' principle.**

Thus, the magnitude of buoyant force (F) is given by,

$$F = V_i \rho_L g$$

Here, V_i = immersed volume of solid $\quad \rho_L$ = density of liquid

and $\quad g$ = acceleration due to gravity

Proof

Consider an arbitrarily shaped body of volume V placed in a container filled with a fluid of density ρ_L. The body is shown completely immersed, but complete immersion is not essential to the proof. To begin with, imagine the situation before the body was immersed. The region now occupied by the body was filled with fluid, whose weight was $V\rho_L g$. Because the fluid as a whole was in hydrostatic equilibrium, the net upwards force (due to difference in pressure at different depths) on the fluid in that region was equal to the weight of the fluid occuping that region.

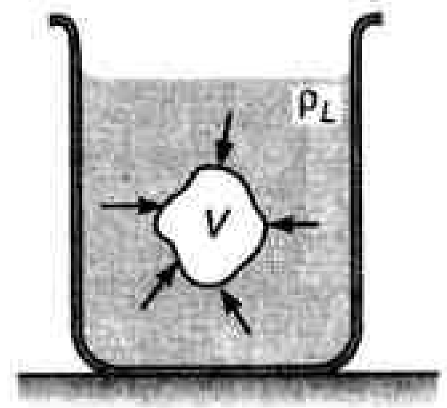


Fig. 13.23

Now, consider what happens when the body has displaced the fluid. The pressure at every point on the surface of the body is unchanged from the value at the same location when the body was not present. This is because the pressure at any point depends only on the depth of that point below the fluid surface. Hence, the net force exerted by the surrounding fluid on the body is exactly the same as that exerted on the region before the body was present. But we know the latter to be $V\rho_L g$, the weight of the displaced fluid. Hence, this must also be the buoyant force exerted on the body. Archimedes' principle is thus, proved.

Law of Floatation

Consider an object of volume V and density ρ_S floating in a liquid of density ρ_L. Let V_i be the volume of object immersed in the liquid.

For equilibrium of object,

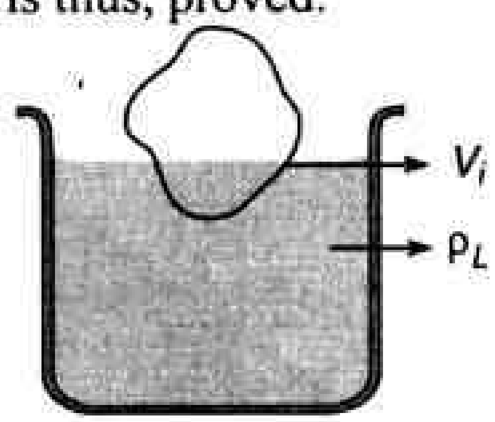


Fig. 13.24

$$\text{Weight} = \text{Upthrust}$$

$$\therefore \quad V\rho_S g = V_i \rho_L g$$

$$\therefore \quad \frac{V_i}{V} = \frac{\rho_S}{\rho_L} \qquad \ldots\text{(i)}$$

This is the fraction of volume immersed in liquid.

Percentage of volume immersed in liquid $= \dfrac{V_i}{V} \times 100 = \dfrac{\rho_S}{\rho_L} \times 100$

Three possibilities may now arise.

(i) If $\rho_S < \rho_L$, only fraction of body will be immersed in the liquid. This fraction will be given by the above equation.

(ii) If $\rho_S = \rho_L$, the whole of the rigid body will be immersed in the liquid. Hence, the body remains floating in the liquid wherever it is left.

(iii) If $\rho_S > \rho_L$, the body will sink.

Apparent Weight of a Body inside a Liquid

If a body is completely immersed in a liquid its effective weight gets decreased. The decrease in its weight is equal to the upthrust on the body. Hence,

$$W_{\text{app}} = W_{\text{actual}} - \text{Upthrust}$$

or

$$W_{\text{app}} = V\rho_S g - V\rho_L g$$

Here, V = total volume of the body

ρ_S = density of body

and ρ_L = density of liquid

Thus,

$$W_{\text{app}} = Vg(\rho_S - \rho_L)$$

If the liquid in which body is immersed, is water, then

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \text{Relative density of body (R. D.)}$$

This can be shown as under :

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \frac{\text{Weight in air}}{\text{Upthrust}} = \frac{V\rho_S g}{V\rho_w g}$$

$$= \frac{\rho_S}{\rho_w}$$

$$= \text{RD}$$

Hence proved.

Buoyant Force in Accelerating Fluids

Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with an acceleration $\vec{\mathbf{a}}$. The buoyant force F in this case becomes,

$$F = V\rho_L g_{\text{eff}}$$

Here, $g_{\text{eff}} = |\vec{\mathbf{g}} - \vec{\mathbf{a}}|$

For example, if the lift is moving upwards with an acceleration a, the value of g_{eff} is $g + a$ and if it is moving downwards with acceleration a, the g_{eff} is $g - a$. In a freely falling lift g_{eff} is zero (as $a = g$) and hence, net buoyant force is zero. This is why, in a freely falling vessel filled with some liquid, the air bubbles do not rise up (which otherwise move up due to buoyant force). The above result can be derived as follows.

Suppose a body is dipped inside a liquid of density ρ_L in an elevator moving up with an acceleration a. As was done earlier also, replace the body into the liquid by the same liquid of equal volume. The replaced liquid is at rest with respect to the elevator. Thus, this replaced liquid is also moving up with an acceleration a together with the rest of the liquid.

The forces acting on the replaced liquid are,

(i) the buoyant force F and

(ii) the weight mg of the substituted liquid.

From Newton's second law,

$$F - mg = ma \quad \text{or} \quad F = m(g + a)$$

Here, $m = V\rho_L$

$$\therefore \quad F = V\rho_L (g + a) = V\rho_L g_{\text{eff}}$$

where $$g_{\text{eff}} = g + a$$

Sample Example 13.6 *Density of ice is 900 kg/m³. A piece of ice is floating in water of density 1000 kg/m³. Find the fraction of volume of the piece of ice outside the water.*

Solution Let V be the total volume and V_i the volume of ice piece immersed in water. For equilibrium of ice piece,

$$\text{weight} = \text{upthrust}$$

$$\therefore \quad V\rho_i g = V_i \rho_w g$$

Here, ρ_i = density of ice = 900 kg/m³

and ρ_w = density of water = 1000 kg/m³

Substituting in above equation, we get

$$\frac{V_i}{V} = \frac{900}{1000} = 0.9$$

i.e., the fraction of volume outside the water,

$$f = 1 - 0.9 = 0.1$$ **Ans.**

Sample Example 13.7 *A piece of ice is floating in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?*

Solution Let m be the mass of ice piece floating in water.

In equilibrium, weight of ice piece = upthrust

or $$mg = V_i \rho_w g$$

or $$V_i = \frac{m}{\rho_w} \quad \text{...(i)}$$

Here, V_i is the volume of ice piece immersed in water

When the ice melts, let V be the volume of water formed by m mass of ice. Then,

$$V = \frac{m}{\rho_w} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we see that

$$V_i = V$$

Hence, the level will not change. **Ans.**

Sample Example 13.8 *A piece of ice having a stone frozen in it floats in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?*

Solution Let, m_1 = mass of ice, m_2 = mass of stone

ρ_S = density of stone and ρ_w = density of water

In equilibrium, when the piece of ice floats in water,

weight of (ice + stone) = upthrust

or $$(m_1 + m_2)g = V_i \rho_w g$$

$\therefore$ $$V_i = \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w} \quad \text{...(i)}$$

Here, V_i = Volume of ice immersed

When the ice melts, m_1 mass of ice converts into water and stone of mass m_2 is completely submerged.

Volume of water formed by m_1 mass of ice,

$$V_1 = \frac{m_1}{\rho_w}$$

Volume of stone (which is also equal to the volume of water displaced)

$$V_2 = \frac{m_2}{\rho_S}$$

Since, $$\rho_S > \rho_w$$

Therefore, $$V_1 + V_2 < V_i$$

or, the level of water will decrease. **Ans.**

Sample Example 13.9 *An ornament weighing* 50 g *in air weighs only* 46 g *is water. Assuming that some copper is mixed with gold to perpare the ornament. Find the amount of copper in it. Specific gravity of gold is* 20 *and that of copper is* 10.

Solution Let m be the mass of the copper in ornament. Then mass of gold in it is $(50 - m)$.

Volume of copper $$V_1 = \frac{m}{10} \quad \left(\text{volume} = \frac{\text{mass}}{\text{density}}\right)$$

and volume of gold $$V_2 = \frac{50 - m}{20}$$

When immersed in water ($\rho_w = 1\ \text{g/cm}^3$)

$$\text{Decrease in weight} = \text{upthrust}$$

$\therefore$ $$(50-46)g = (V_1 + V_2)\rho_w g$$

or $$4 = \frac{m}{10} + \frac{50-m}{20}$$

or $$80 = 2m + 50 - m$$

$\therefore$ $$m = 30\text{ g}$$ **Ans.**

Sample Example 13.10 *The tension in a string holding a solid block below the surface of a liquid (of density greater than that of solid) as shown in figure is T_0 when the system is at rest. What will be the tension in the string if the system has an upward acceleration a?*

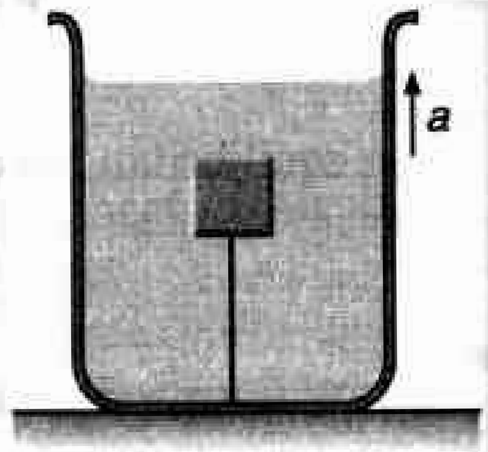


Fig. 13.25

Solution Let m be the mass of block.

Initially for the equilibrium of block,

$$F = T_0 + mg \quad \text{...(i)}$$

Here, F is the upthrust on the block.

When the lift is accelerated upwards, g_{eff} becomes $g + a$ instead of g. Hence,

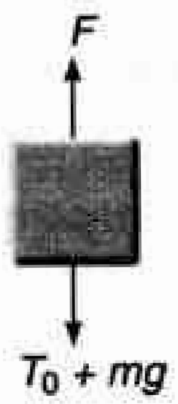


Fig. 13.26

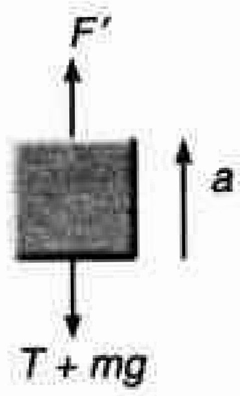


Fig. 13.27

$$F' = F\left(\frac{g+a}{g}\right) \quad \text{...(ii)}$$

From Newton's second law,

$$F' - T - mg = ma \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$T = T_0\left(1 + \frac{a}{g}\right)$$ **Ans.**

Introductory Exercise 13.2

1. A metallic sphere floats in an immissible mixure of water ($\rho_w = 10^3$ kg/m^3) and a liquid ($\rho_L = 13.5\times 10^3$ kg/m^3) such that its $\frac{4}{5}$ th volume is in water and $\frac{1}{5}$ th volume in the liquid. Find the density of metal.
2. A metallic sphere weighs 210 g in air, 180 g in water and 120 g in an unknown liquid. Find the density of metal and of liquid.
3. A block of wood floats in a bucket of water placed in a lift. Will the block sink more or less if the lift starts accelerating up?
4. A weightless balloon is filled with water. What will be its apparent weight when weighed in water?
5. A body of weight W_1 displaces an amount of water W_2. Then $W_1 < W_2$, is this statement true or false?
6. A raft of wood (density = 600 kg/m^3) of mass 120 kg floats in water. How much weight can be put on the raft to make it just sink?
7. A man is sitting in a boat which is floating in a pond. If the man drinks some water from the pond, how will the level of water in the pond change?
8. A cubical block of ice floating in water has to support a metal piece weighing 0.5 kg. What can be the minimum edge of the block so that it does not sink in water? Specific gravity of ice = 0.9.
9. When a cube of wood floats in water, 60% of its volume is submerged. When the same cube floats in an unknown fluid 85% of its volume is submerged. Find the densities of wood and the unknown fluid.
10. A glass tube of radius 0.8 cm floats vertical in water, as shown in figure. What mass of lead pellets would cause the tube to sink a further 3 cm?

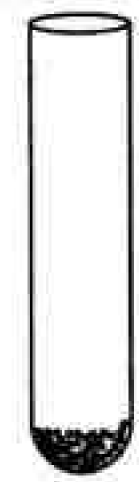


Fig. 13.28

13.6 Flow of Fluids

Steady Flow

If the velocity of fluid particles at any point does not vary with time, the flow is said to be steady. Steady flow is also called streamlined or laminar flow. The velocity at different points may be different. Hence, in the figure,

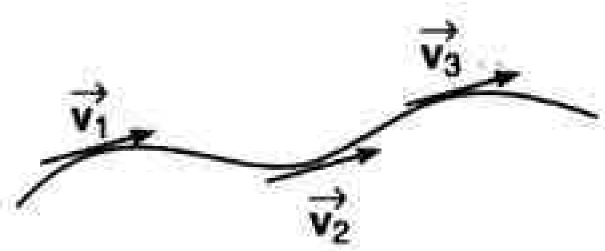


Fig. 13.29

$\vec{v}_1$ = constant, $\vec{v}_2$ = constant, $\vec{v}_3$ = constant

but $\vec{v}_1 \neq \vec{v}_2 \neq \vec{v}_3$

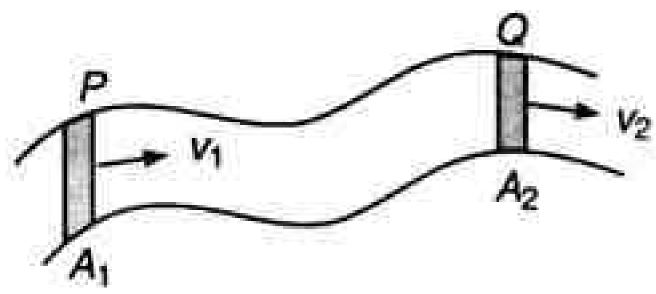


Fig. 13.30

Principle of Continuity

It states that, when an incompressible and non-viscous liquid flows in a stream lined motion through a tube of non-uniform cross section, then the product of the area of cross section and the velocity of flow is same at every point in the tube.

Thus, $$A_1 v_1 = A_2 v_2$$

or $$Av = \text{constant}$$

or $$v \propto \frac{1}{A}$$

This is basically the law of conservation of mass in fluid dynamics.

Proof

Let us consider two cross sections P and Q of area A_1 and A_2 of a tube through which a fluid is flowing. Let v_1 and v_2 be the speeds at these two cross sections. Then being an incompressible fluid, mass of fluid going through P in a time interval Δt = mass of fluid passing through Q in the same interval of time Δt.

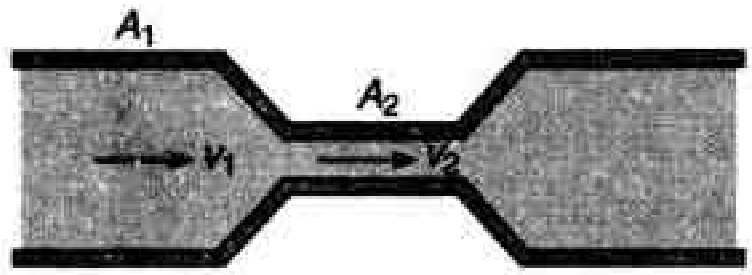


Fig. 13.31

$$\therefore \quad A_1 v_1 \rho \Delta t = A_2 v_2 \rho \, \Delta t \quad \text{or} \quad A_1 v_1 = A_2 v_2 \quad \textbf{Proved.}$$

Therefore, the velocity of the liquid is smaller in the wider parts of the tube and larger in the narrower parts.

or $$v_2 > v_1 \quad \text{as} \quad A_2 < A_1$$

Note *The product Av is the volume flow rate* $\frac{dV}{dt}$, *the rate at which volume crosses a section of the tube. Hence*

$$\frac{dV}{dt} = \text{volume flow rate} = Av$$

The mass flow rate is the mass flow per unit time through a cross section. This is equal to density (ρ) times the volume flow rate $\frac{dV}{dt}$.

We can generalize the continuity equation for the case in which the fluid is not incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2 then,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

So, this is the continuity equation for a compressible fluid.

Sample Example 13.11 *Water is flowing through a horizontal tube of non-uniform crross section. At a place the radius of the tube is 1.0 cm and the velocity of water is 2 m/s. What will be the velocity of water where the radius of the pipe is 2.0 cm?*

Solution Using equation of continuity, $A_1 v_1 = A_2 v_2$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1 \quad \text{or} \quad v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right) v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$$

Substituting the values, we get

$$v_2 = \left(\frac{1.0 \times 10^{-2}}{2.0 \times 10^{-2}}\right)^2 (2)$$

or $$v_2 = 0.5 \text{ m/s}$$ **Ans.**

13.7 Bernoulli's Equation

Bernoulli's equation relates the pressure, flow speed and height for flow of an ideal (incompressible and non-viscous) fluid. The pressure of a fluid depends on height as in the static situation, and it also depends on the speed of flow.

The dependence of pressure on speed can be understood from the continuity equation. When an incompressible fluid flows along a tube with varying cross section, its speed must change, and so, an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure must be different in regions of different cross section. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving towards a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes additional pressure difference.

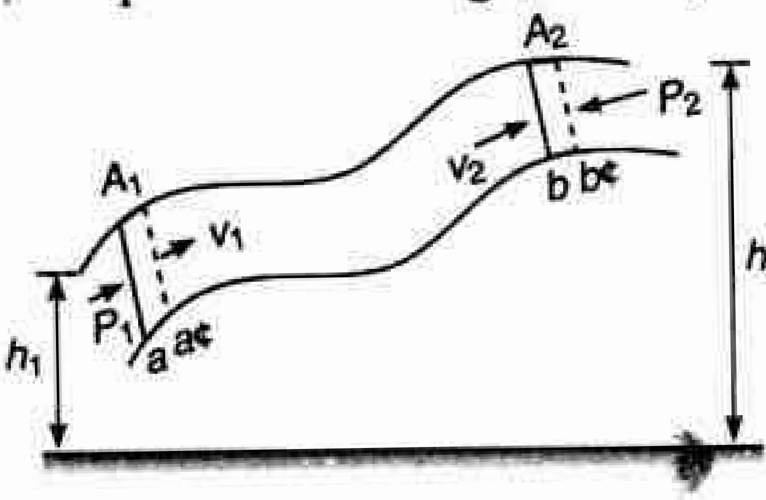


Fig. 13.32

To derive Bernoulli's equation, we apply the work-energy theorem to the fluid in a section of the fluid element. Consider the element of fluid that at some initial time lies between two cross sections a and b. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval the fluid that is initially at a moves to a' a distance $aa' = ds_1 = v_1\ dt$ and the fluid that is initially at b moves to b' a distance $bb' = ds_2 = v_2\ dt$. The cross-section areas at the two ends are A_1 and A_2 as shown. The fluid is incompressible, hence, by the continuity equation, the volume of fluid dV passing through any cross-section during time dt is the same.

That is,

$$dV = A_1 ds_1 = A_2 ds_2$$

Work done on the Fluid Element

Let us calculate the work done on this fluid element during time interval dt. The pressure at the two ends are P_1 and P_2, the force on the cross section at a is $P_1 A_1$ and the force at b is $P_2 A_2$. The net work done dW on the element by the surrounding fluid during this displacement is,

$$dW = P_1 A_1 ds_1 - P_2 A_2 ds_2 = (P_1 - P_2)dV \qquad \text{...(i)}$$

The second term is negative, because the force at b opposes the displacement of the fluid.

This work dW is due to forces other than the conservative force of gravity, so it equals the change in total mechanical energy (kinetic plus potential). The mechanical energy for the fluid between sections a and b does not change.

Change in Potential Energy

At the beginning of dt the potential energy for the mass between a and a' is $dmgh_1 = \rho dVgh_1$. At the end of dt the potential energy for the mass between b and b' is $dmgh_2 = \rho dVgh_2$. The net change in potential energy dU during dt is,

$$dU = \rho(dV)g(h_2 - h_1) \qquad \text{...(ii)}$$

Change in Kinetic Energy

At the beginning of dt the fluid between a and a' has volume $A_1 ds_1$, mass $\rho A_1 ds_1$ and kinetic energy $\frac{1}{2}\rho(A_1 ds_1)v_1^2$. At the end of dt the fluid between b and b' has kinetic energy $\frac{1}{2}\rho(A_2 ds_2)v_2^2$. The net change in kinetic energy dK during time dt is,

$$dK = \frac{1}{2}\rho(dV)(v_2^2 - v_1^2) \qquad \text{...(iii)}$$

Combining Eqs. (i), (ii) and (iii) in the energy equation,

$$dW = dK + dU$$

We obtain,

$$(P_1 - P_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho(dV)g(h_2 - h_1)$$

or

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \qquad \text{...(iv)}$$

This is Bernoulli's equation. It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We can also express Eq. (iv) in a more convenient form as,

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2 \qquad \text{Bernoulli's equation}$$

The subscripts 1 and 2 refer to any two points along the flow tube, so we can also write

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{constant}$$

Note *When the fluid is not moving* ($v_1 = 0 = v_2$). *Bernoulli's equation reduces to,*

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

$$\therefore \quad P_1 - P_2 = \rho g(h_2 - h_1)$$

This is the pressure relation we derived for a fluid at rest.

• *Points to be Remembered* in Bernoulli's Theorem

(1) Energy of a flowing fluid

There are following three types of energies in a flowing fluid.

(i) Pressure energy

If P is the pressure on the area A of a fluid, and the liquid moves through a distance l due to this pressure, then

$$\text{Pressure energy of liquid} = \text{work done} = \text{force} \times \text{displacement} = PAl$$

The volume of the liquid is Al.

$$\therefore \text{ Pressure energy per unit volume of liquid} = \frac{PAl}{Al} = P$$

(ii) Kinetic energy

If a liquid of mass m and volume V is flowing with velocity v, then the kinetic energy is $\frac{1}{2}mv^2$.

$\therefore$ Kinetic energy per unit volume of liquid

$$=\frac{1}{2}\left(\frac{m}{V}\right)v^2=\frac{1}{2}\rho v^2$$

Here, ρ is the density of liquid.

(iii) Potential energy

If a liquid of mass m is at a height h from the reference line ($h = 0$), then its potential energy is mgh.

$\therefore$ Potential energy per unit volume of the liquid $=\left(\frac{m}{V}\right)gh=\rho gh$

Thus, the Bernoulli's equation

$$P+\frac{1}{2}\rho v^2+\rho gh=\text{constant (J/m}^3\text{)}$$

can also be written as:

Sum of total energy per unit volume (pressure + kinetic + potential) is constant for an ideal fluid.

(2) Pressure head, velocity head and gravitational head of a flowing Liquid

Dividing the Bernoulli's equation by ρg, we have

$$\frac{P}{\rho g}+\frac{v^2}{2g}+h=\text{constant (m)}$$

In this expression $\frac{P}{\rho g}$ is called the 'pressure head', $\frac{v^2}{2g}$ the velocity head and h the gravitational head. The SI unit of each of these three is metre. Therefore, Bernoulli's equation may also be stated as,
Sum of pressure head, velocity head and gravitational head is constant for an ideal fluid.

Sample Example 13.12 *Calculate the rate of flow of glycerine of density* 1.25×10^3 kg/m^3 *through the conical section of a pipe, if the radii of its ends are* 0.1 *m and* 0.04 *m and the pressure drop across its length is* 10 N/m^2.

Solution From continuity equation,

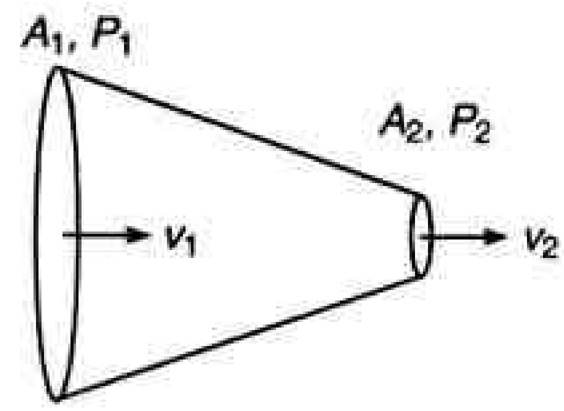


Fig. 13.33

$$A_1v_1=A_2v_2$$

or
$$\frac{v_1}{v_2}=\frac{A_2}{A_1}=\frac{\pi r_2^2}{\pi r_1^2}=\left(\frac{r_2}{r_1}\right)^2=\left(\frac{0.04}{0.1}\right)^2=\frac{4}{25} \quad \text{...(i)}$$

From Bernoulli's equation,

$$P_1+\frac{1}{2}\rho v_1^2=P_2+\frac{1}{2}\rho v_2^2$$

or
$$v_2^2-v_1^2=\frac{2(P_1-P_2)}{\rho}$$

or
$$v_2^2-v_1^2=\frac{2\times10}{1.25\times10^3}=1.6\times10^{-2}\ \text{m}^2/\text{s}^2 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 \approx 0.128 \text{ m/s}$$

$\therefore$ Rate of volume flow through the tube

$$Q = A_2 v_2 = (\pi r_2^2) v_2$$
$$= \pi (0.04)^2 (0.128)$$
$$= 6.43 \times 10^{-4} \text{ m}^3\text{/s}$$ **Ans.**

13.8 Applications Based on Bernoulli's Equation

(a) Venturimeter

Figure shows a venturimeter used to measure flow speed in a pipe of non-uniform cross-section. We apply Bernoulli's equation to the wide (point 1) and narrow (point 2) parts of the pipe, with $h_1 = h_2$

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

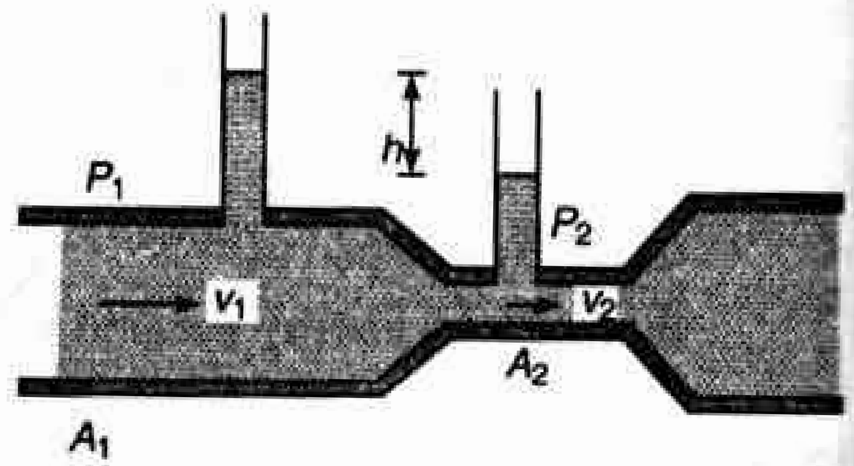


Fig. 13.34

From the continuity equation $v_2 = \frac{A_1 v_1}{A_2}$

Substituting and rearranging, we get

$$P_1 - P_2 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right) \quad \text{...(i)}$$

Because A_1 is greater than A_2, v_2 is greater than v_1 and hence the pressure P_2 is less than P_1. A net force to the right accelerates the fluid as it enters the narrow part of the tube (called throat) and a net force to the left slows as it leaves. The pressure difference is also equal to ρgh, where h is the difference in liquid level in the two tubes. Substituting in Eq. (i), we get

$$v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

Note *(i) The discharge or volume flow rate can be obtained as,*

$$\frac{dV}{dt} = A_1 v_1 = A_1 \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

(ii) The venturi effect can be used to give a qualitative understanding of the lift of an air plane wing and the path of a pitcher's curve ball. An airplane wing is designed so that air moves faster over the top of the wing that it does under the wing, thus, making the air pressure less on top than underneath. This difference in pressure results in a net force upward on the wing.

(b) Speed of Efflux

Suppose, the surface of a liquid in a tank is at a height h from the orifice O on its sides, through which the liquid issues out with velocity v. The speed of the liquid coming out is called the speed of efflux. If the dimensions of the tank be sufficiently large, the velocity of the liquid at its surface may be taken to be zero and since the pressure there as well as at the orifice O is the same viz atmospheric it plays no part in the flow of the liquid, which thus occurs purely in consequence of the hydrostatic pressure of the liquid itself. So that, considering a tube of flow, starting at the liquid surface and ending at the orifice, as shown in figure. Applying Bernoulli's equation, we have

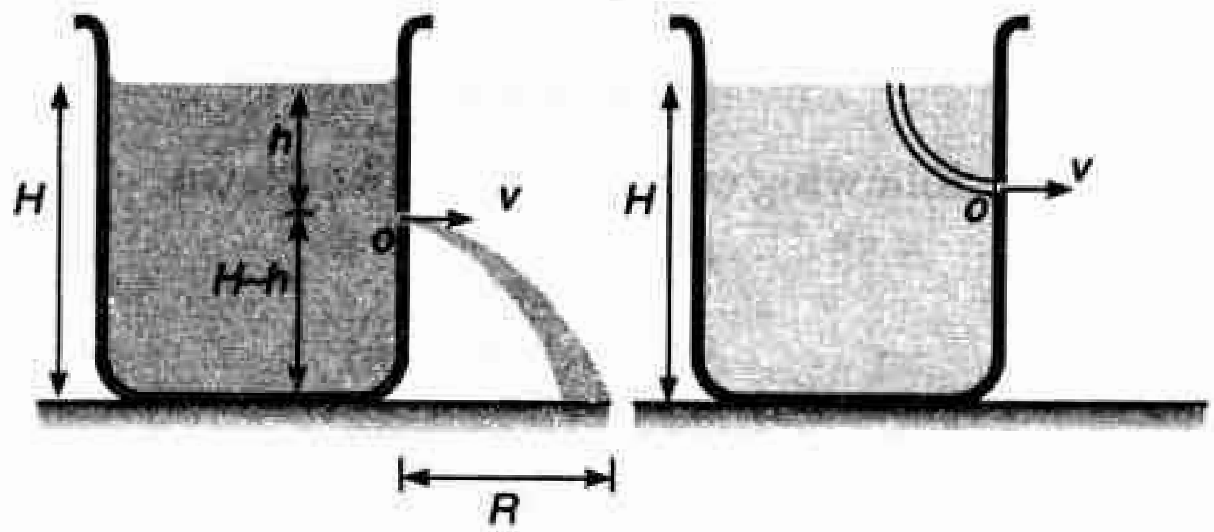


Fig. 13.35

Total energy per unit volume of the liquid at the surface

$$= \text{KE} + \text{PE} + \text{pressure energy} = 0 + \rho gh + P_0 \quad \text{...(i)}$$

and total energy per unit volume at the orifice

$$= \text{KE} + \text{PE} + \text{pressure energy} = \frac{1}{2}\rho v^2 + 0 + P_0 \quad \text{...(ii)}$$

Since total energy of the liquid must remain constant in steady flow, in accordance with Bernoulli's equation, we have

$$\rho gh + P_0 = \frac{1}{2}\rho v^2 + P_0$$

or

$$v = \sqrt{2gh}$$

Evangelista Torricelli showed that this velocity is the same as the liquid will attain in falling freely through the vertical height (h) from the surface to the orifice. This is known as **Torricelli's theorem** and may be stated as, "The velocity of efflux of a liquid issuing out of an orifice is the same as it would attain if allowed to fall freely through the vertical height between the liquid surface and orifice."

Range (R)

Let us find the range R on the ground.

Considering the vertical motion of the liquid,

$$(H - h) = \frac{1}{2}gt^2$$

or $$t=\sqrt{\frac{2(H-h)}{g}}$$

Now, considering the horizontal motion,

$$R = vt$$

or $$R=(\sqrt{2gh})\left(\sqrt{\frac{2(H-h)}{g}}\right)$$

or $$R=2\sqrt{h(H-h)}$$

From the expression of R, following conclusions can be drawn,

(i) $$R_h = R_{H-h}$$

as $$R_h = 2\sqrt{h(H-h)}$$

and $$R_{H-h} = 2\sqrt{(H-h)h}$$

This can be shown as in Fig. 13.36.

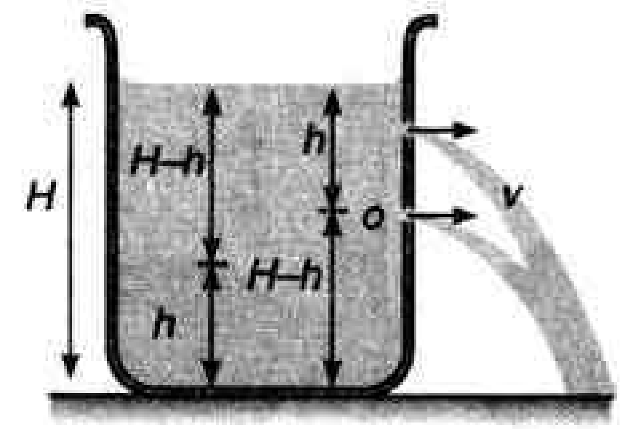


Fig. 13.36

(ii) R is maximum at $h=\frac{H}{2}$ and $R_{max} = H$.

Proof : $$R^2 = 4(Hh - h^2)$$

For R to be maximum, $$\frac{dR^2}{dh}=0$$

or $$H-2h=0 \quad \text{or} \quad h=\frac{H}{2}$$

That is, R is maximum at $$h=\frac{H}{2}$$

and $$R_{max} = 2\sqrt{\frac{H}{2}\left(H-\frac{H}{2}\right)} = H$$ **Proved.**

Time taken to empty a tank

We are here interested in finding the time required to empty a tank if a hole is made at the bottom of the tank.

Consider a tank filled with a liquid of density ρ upto a height H. A small hole of area of cross section a is made at the bottom of the tank. The area of cross-section of the tank is A.

Let at some instant of time the level of liquid in the tank is y. Velocity of efflux at this instant of time would be,

$$v=\sqrt{2gy}$$

Now, at this instant volume of liquid coming out of the hole per second is $\left(\frac{dV_1}{dt}\right)$.

Volume of liquid coming down in the tank per second is $\left(\frac{dV_2}{dt}\right)$.

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$\therefore$
$$av = A\left(-\frac{dy}{dt}\right)$$

$\therefore$
$$a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right)$$

or
$$\int_0^t dt = -\frac{A}{a\sqrt{2g}}\int_H^0 y^{-1/2}\,dy \qquad \text{...(i)}$$

$\therefore$
$$t = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_0^H$$

$\therefore$
$$t = \frac{A}{a}\sqrt{\frac{2H}{g}}$$

Sample Example 13.13 *A tank is filled with a liquid upto a height H. A small hole is made at the bottom of this tank. Let t_1 be the time taken to empty first half of the tank and t_2 the time taken to empty rest half of the tank. Then find $\frac{t_1}{t_2}$.*

Solution Substituting the proper limits in Eq. (i), derived in the theory, we have

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}}\int_H^{H/2} y^{-1/2}\,dy$$

or
$$t_1 = \frac{2A}{a\sqrt{2g}}[\sqrt{y}]_{H/2}^H$$

or
$$t_1 = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$

or
$$t_1 = \frac{A}{a}\sqrt{\frac{H}{g}}(\sqrt{2}-1) \qquad \text{...(ii)}$$

Similarly,
$$\int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}}\int_{H/2}^0 y^{-1/2}\,dy$$

or
$$t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\frac{t_1}{t_2} = \sqrt{2} - 1$$

or
$$\frac{t_1}{t_2} = 0.414 \qquad \textbf{Ans.}$$

Note *From here we see that $t_1 < t_2$. This is because initially the pressure is high and the liquid comes out with greater speed.*

13.9 Viscosity

Viscosity is internal friction in a fluid. Viscous forces opposes the motion of one portion of a fluid relative to the other.

The simplest example of viscous flow is motion of a fluid between two parallel plates.

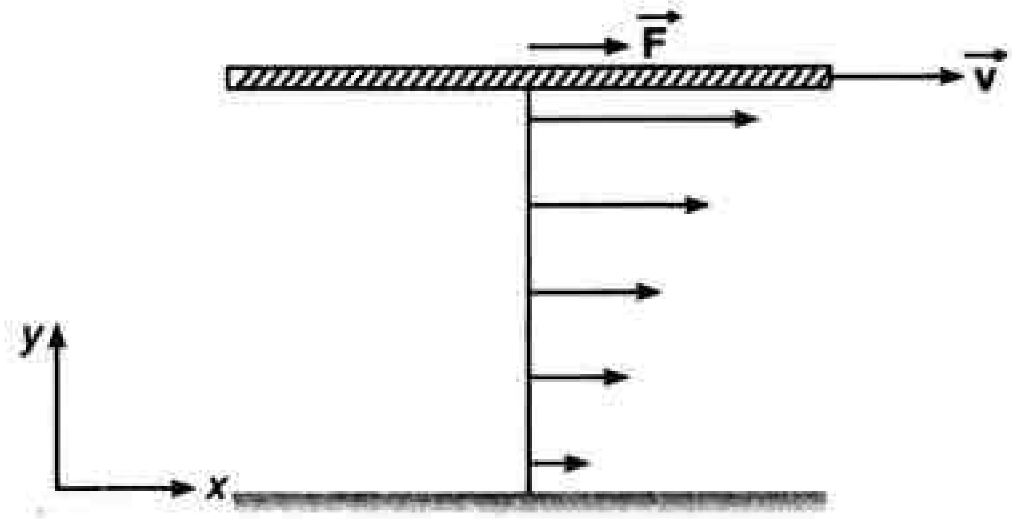


Fig. 13.37

The bottom plate is stationary and the top plate moves with constant velocity $\vec{v}$. The fluid in contact with each surface has same velocity at that surface. The flow speeds of intermediate layers of fluid increase uniformly from bottom to top, as shown by arrows. So the fluid layers slide smoothly over one another.

According to Newton, the frictional force F (or viscous force) between two layers depends upon the following factors,

(i) Force F is directly proportional to the area (A) of the layers in contact, *i.e.*,

$$F \propto A$$

(ii) Force F is directly proportional to the velocity gradient $\left(\frac{dv}{dy}\right)$ between the layers. Combining these two, we have

$$F \propto A\frac{dv}{dy}$$

or
$$F = -\eta A\frac{dv}{dy}$$

Here, η is constant of proportionality and is called **coefficient of viscosity.** Its value depends on the nature of the fluid. The negative sign in the above equation shows that the direction of viscous force F is opposite to the direction of relative velocity of the layer.

The SI unit of η is N-s/m^2. It is also called decapoise or pascal second. Thus,

$$1 \text{ decapoise} = 1 \text{ N-s/m}^2 = 1 \text{ Pa-s} = 10 \text{ poise}$$

Dimensions of η are $[ML^{-1}T^{-1}]$.

Coefficient of viscosity of water at 10°C is $\eta = 1.3 \times 10^{-3}$ N-s/m^2. Experiments show that coefficient of viscosity of a liquid decreases as its temperature rises.

Sample Example 13.14 *A plate of area 2 m^2 is made to move horizontally with a speed of 2 m/s by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.*

Solution Velocity gradient $= \frac{\Delta v}{\Delta y}$

$$= \frac{2-0}{1-0} = 2\,\frac{\text{m/s}}{\text{m}}$$

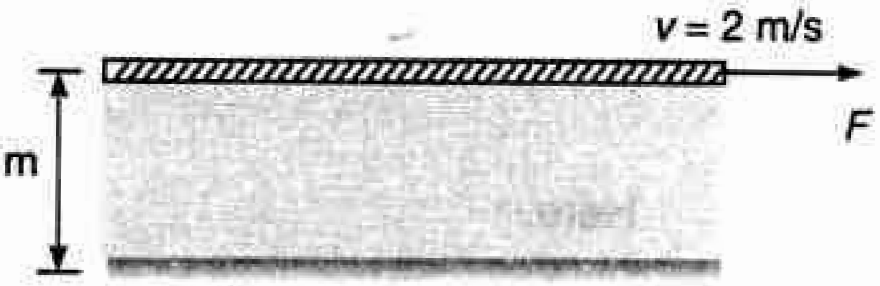


Fig. 13.38

From, Newton's law of viscous force,

$$|F| = \eta A \frac{\Delta v}{\Delta y}$$

$$= (0.01 \times 10^{-1})(2)(2)$$

$$= 4 \times 10^{-3}\ \text{N}.$$

So, to keep the plate moving, a force of 4×10^{-3} N must be applied **Ans.**

Flow of liquid through a Cylindrical pipe

Figure shows the flow speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. The speed is greatest along the axis and zero at the pipe walls. The flow speed v at a distance r from the axis of a pipe of radius R is,

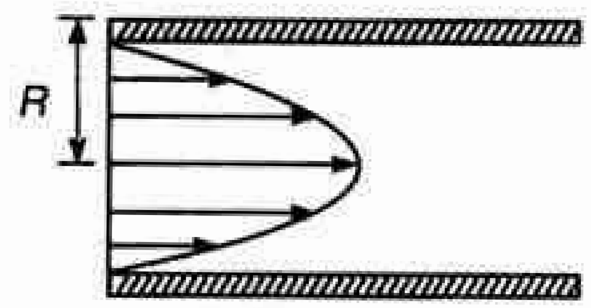


Fig. 13.39

$$v = \frac{P_1 - P_2}{4\eta L}(R^2 - r^2)$$

where P_1 and P_2 are the pressure at the two ends of a pipe with length L. The flow is always in the direction of decreasing pressure.

From the above equation we can see that v-r graph is a parabola.

$$v = 0 \quad \text{at} \quad r = R \quad \text{(along the walls)}$$

and

$$v = \frac{(P_1 - P_2)R^2}{4\eta L} = v_{max} \quad \text{at} \quad r = 0 \quad \text{(along the axis)}$$

Volume flow rate $\left(Q \text{ or } \frac{dV}{dt}\right)$

To find the total volume flow rate through the pipe, we consider a ring with inner radius r, outer radius $r + dr$ and cross sectional area $dA = 2\pi r\,dr$. The volume flow rate through this element is $v\,dA$. The total volume flow rate is found by integrating from $r = 0$ to $r = R$. The result is,

$$Q = \frac{dV}{dt} = \frac{\pi}{8}\left(\frac{R^4}{\eta}\right)\left(\frac{P_1 - P_2}{L}\right)$$

The relation was first derived by Poiseuille and is called **Poiseuille's equation.**

• *Important point in* Poiseuille's Equation

1. Poiseuille's equation can also be written as,

$$Q = \frac{P_1 - P_2}{\left(\frac{8\eta L}{\pi R^4}\right)} = \frac{\Delta P}{X}$$

Here, $$X = \frac{8\eta L}{\pi R^4}$$

This equation can be compared with the current equation through a resistance, *i.e.*,

$$i = \frac{\Delta V}{R}$$

Here, ΔV = potential difference and R = electrical resistance

For current flow through a resistance, potential difference is a requirement similarly for flow of liquid through a pipe pressure difference is must.

Problems of series and parallel combination of pipes can be solved in the similar manner as is done in case of an electrical circuit. The only difference is,

(i) Potential difference (ΔV) is replaced by the pressure difference (ΔP)

(ii) The electrical resistance $R\left(= \rho \frac{L}{A}\right)$ is replaced by $X\left(= \frac{8\eta L}{\pi R^4}\right)$ and

(iii) The electrical current i is replaced by volume flow rate Q or $\frac{dV}{dt}$. The following example will illustrate the above theory.

Sample Example 13.15 *A liquid is flowing through horizontal pipes as shown in figure.*

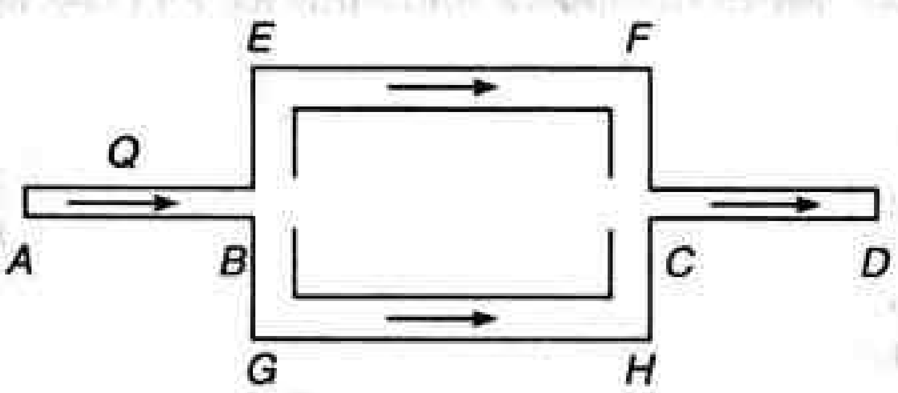


Fig. 13.40

Length of different pipes has the following ratio

$$L_{AB} = L_{CD} = \frac{L_{EF}}{2} = \frac{L_{GH}}{2}$$

Similarly, radii of different pipes has the ratio,

$$R_{AB} = R_{EF} = R_{CD} = \frac{R_{GH}}{2}$$

Pressure at A is $2P_0$ and pressure at D is P_0. The volume flow rate through the pipe AB is Q. Find,

(a) volume flow rates through EF and GH.

(b) pressure at E and F.

Solution The equivalent electrical circuit can be drawn as under,

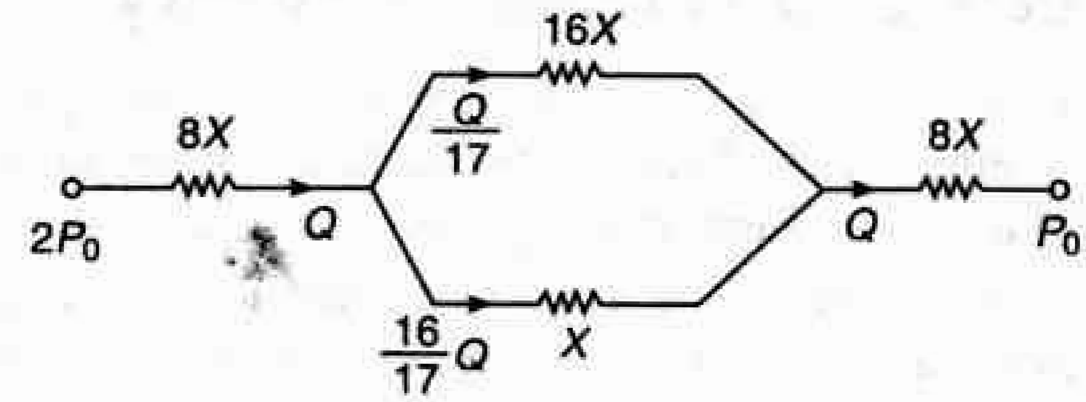


Fig. 13.41

$$X \propto \frac{L}{R^4} \qquad \left(\text{as } X = \frac{8\eta L}{\pi R^4}\right)$$

$$\therefore \quad X_{AB} : X_{CD} : X_{EF} : X_{GH} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^4} : \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^4} : \frac{(1)}{\left(\frac{1}{2}\right)^4} : \frac{(1)}{(1)^4}$$

$$= 8 : 8 : 16 : 1$$

(a) As the current is distributed in the inverse ratio of the resistance (in parallel). The Q will be distributed in the inverse ratio of X.

Thus, volume flow rate through EF will be $\frac{Q}{17}$ and that from GH will be $\frac{16}{17}Q$. **Ans.**

(b)
$$X_{\text{net}} = 8X + \left[\frac{(16X)(X)}{(16X)+(X)}\right] + 8X = \frac{288}{17}X$$

$$\therefore \quad Q = \frac{\Delta P}{X_{\text{net}}} \qquad \left(\text{as } i = \frac{\Delta V}{R}\right)$$

$$= \frac{(2P_0 - P_0)}{\frac{288}{17}X} = \frac{17P_0}{288X}$$

Now, let P_1 be the pressure at E, then

$$2P_0 - P_1 = 8QX = \frac{8 \times 17P_0}{288}$$

$$\therefore \quad P_1 = \left(2 - \frac{17 \times 8}{288}\right)P_0 = 1.53P_0$$ **Ans.**

Similarly, if P_2 be the pressure at F, then

$$P_2 - P_0 = 8QX$$

$$\therefore \quad P_2 = P_0 + \frac{8 \times 17}{288}P_0$$

or
$$P_2 = 1.47P_0$$ **Ans.**

13.10 Stoke's Law and Terminal Velocity

When an object moves through a fluid, it experiences a viscous force which acts in opposite direction of its velocity. The mathematics of the viscous force for an irregular object is difficult, we will consider here only the case of a small sphere moving through a fluid.

The formula for the viscous force on a sphere was first derived by the English physicist G. Stokes in 1843. According to him, a spherical object of radius r moving at velocity v experiences a viscous force given by

$$F = 6\pi\eta r v \qquad (\eta = \text{coefficient of viscosity})$$

This law is called **Stoke's Law.**

Terminal velocity (v_T)

Consider a small sphere falling from rest through a large column of viscous fluid. The forces acting on the sphere are,

(i) Weight W of the sphere acting vertically downwards
(ii) Upthrust F_t acting vertically upwards
(iii) Viscous force F_v acting vertically upwards, *i.e.*, in a direction opposite to velocity of the sphere.

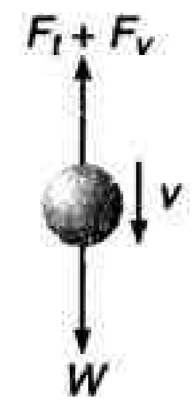


Fig. 13.42

Initially, $F_v = 0$ (as $v = 0$)

and $W > F_t$

and the sphere accelerates downwards. As the velocity of the sphere increases, F_v increases. Eventually a stage in reached when

$$W = F_t + F_v \qquad \ldots\text{(i)}$$

After this net force on the sphere is zero and it moves downwards with a constant velocity called **terminal velocity (v_T).**

Substituting proper values in Eq. (i) we have,

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_T \qquad \ldots\text{(ii)}$$

Here, ρ = density of sphere, σ = density of fluid

and η = coefficient of viscosity of fluid

From Eq. (ii), we get $$v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$$

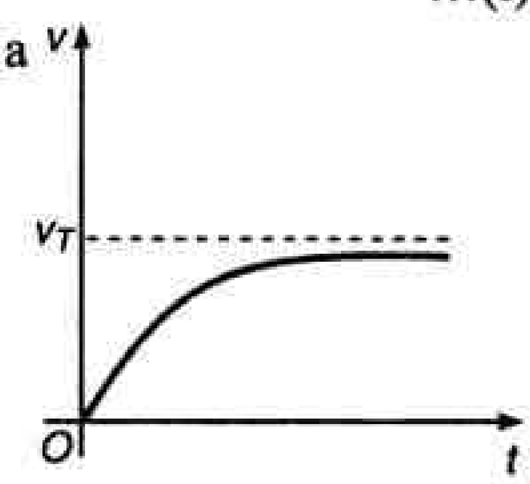


Fig. 13.43

Figure shows the variation of the velocity v of the sphere with time.

Note *From the above expression we can see that terminal velocity of a spherical body is directly proportional to the difference in the densities of the body and the fluid ($\rho - \sigma$). If the density of fluid is greater than that of body (i.e., $\sigma > \rho$), the terminal velocity is negative. This means that the body instead of falling, moves upward. This is why air bubbles rise up in water.*

Sample Example 13.16 *Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 1 m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?*

Solution $$v_T \propto r^2 \qquad \text{...(i)}$$

Let r be the radius of small rain drops and R the radius of large drop.
Equating the volumes, we have

$$\frac{4}{3}\pi R^3 = 2\left(\frac{4}{3}\pi r^3\right)$$

$$\therefore \quad R = (2)^{1/3}.r \quad \text{or} \quad \frac{R}{r} = (2)^{1/3}$$

$$\therefore \quad \frac{v_T'}{v_T} = \left(\frac{R}{r}\right)^2 = (2)^{2/3}$$

$$\therefore \quad v_T' = (2)^{2/3} v_T = (2)^{2/3}(1.0) \text{ m/s} = 1.587 \text{ m/s}$$ **Ans.**

13.11 Surface Tension

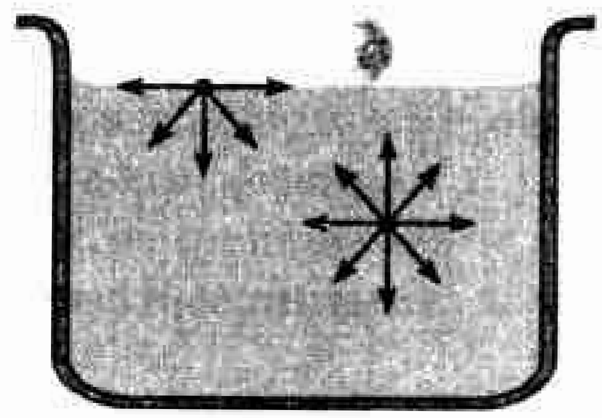


Fig. 13.44

A needle can made to float on a water surface if it is placed there carefully. The forces that support the needle are not buoyant forces but are due to surface tension. The surface of a liquid behaves like a membrane under tension. The molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid.But a surface molecule is drawn into the volume. Thus, the liquid tends to minimize its surface area, just as a stretched membrane does.

Freely falling raindrops are spherical because a sphere has a smaller surface area for a given volume than any other shape. Hence, the surface tension can be defined as the **property of a liquid at rest by virtue of which its free surface behaves like a stretched membrane under tension and tries to occupy as small area as possible.**

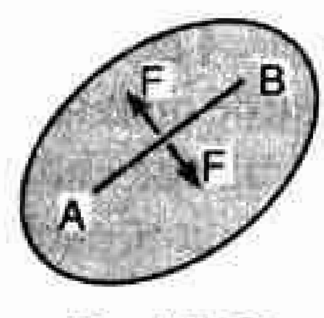


Fig. 13.45

Let an imaginary line AB be drawn in any direction in a liquid surface. The surface on either side of this line exerts a pulling force on the surface on the other side. This force is at right angles to the line AB. The magnitude of this force per unit length of AB is taken as a measure of the surface tension of the liquid. Thus, if F be the total force acting on either side of the line AB of length L, then the surface tension is given by,

$$T = \frac{F}{L}$$

Hence, the surface tension of a liquid is defined as the force per unit length in the plane of the liquid surface, acting at right angles on either side of an imaginary line drawn on that surface.

Few Examples of Surface Tension

Example 1 *Take a ring of wire and dip it in a soap solution. When the ring is taken out, a soap film is formed. Place a loop of thread gently on the soap film. Now, prick a hole inside the loop.*

The thread is radially pulled by the film surface outside and it takes a circular shape.

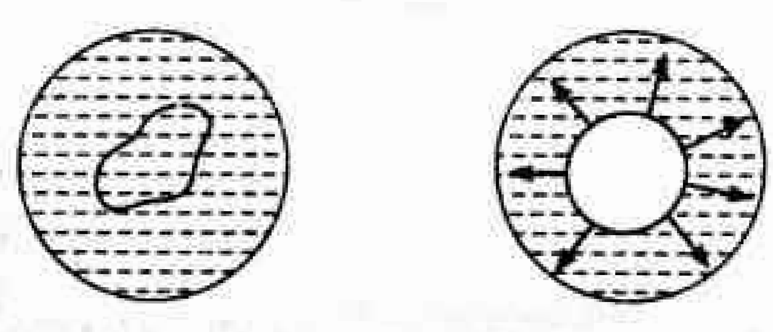


Fig. 13.46

Reason : Before the pricking, there were surfaces both inside and outside the thread loop. Surfaces on both sides pull it equally and the net force is zero. Once the surface inside was punctured, the outside surface pulled the thread to take the circular shape so that area outside the loop becomes minimum (because for given perimeter area of circle is maximum).

Example 2 *A piece of wire is bent into a U-shape, and a second piece of wire slides on the arms of the U. When the apparatus is dipped into a soap solution and removed, a liquid film is formed. The film exerts a surface tension force on the slider and if the frame is kept in a horizontal position, the slider quickly slides towards the closing arm of the frame. If the frame is kept vertical, one can have some weight to keep it in equilibrium. This shows that the soap surface in contact with the slider pulls it parallel to the surface.*

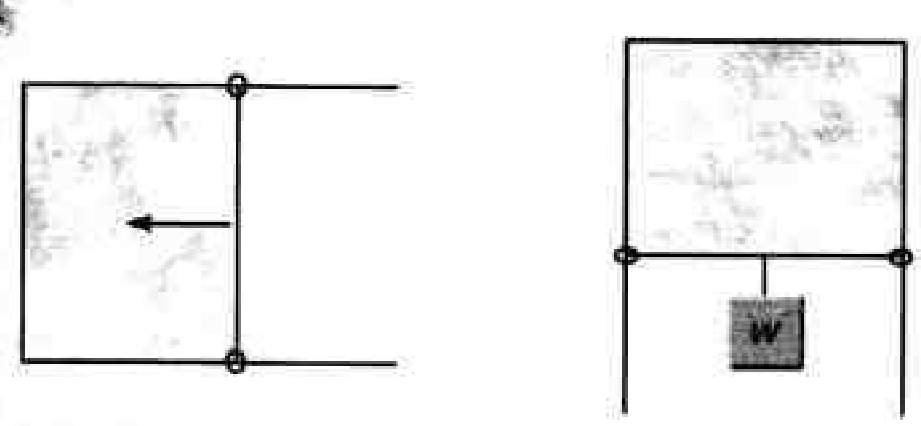


Fig. 13.47

Note *The surface tension of a particular liquid usually decreases as temperature increases. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibres. This requires increasing the surface area of the water ,which is difficult to do because of surface tension. Hence, hot water and soapy water is better for washing.*

Surface Energy

When the surface area of a liquid is increased, the molecules from the interior rise to the surface. This requires work against force of attraction of the molecules just below the surface. This work is stored in the form of potential energy. Thus, the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called 'surface energy'. The surface energy is related to the surface tension as discussed below:

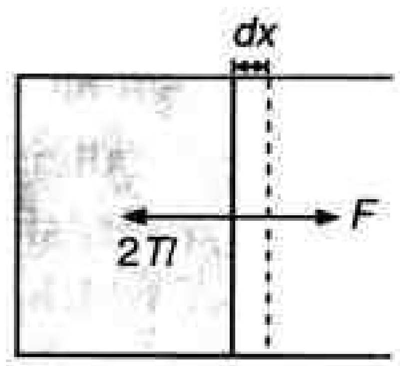


Fig. 13.48

Let a liquid film be formed on a wire frame and a straight wire of length l can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solution, each surface will pull the wire parallel to itself with a force Tl. Thus, net force on the wire due to both the surfaces is $2Tl$. One has to apply an external force F equal and opposite to it to keep the wire in equilibrium. Thus,

$$F = 2Tl$$

Now, suppose the wire is moved through a small distance dx, the work done by the force is,

$$dW = F\ dx = (2Tl)\ dx$$

But $(2l)(dx)$ is the total increase in area of both the surfaces of the film. Let it be dA. Then,

$$dW = T\ dA$$

or

$$T = \frac{dW}{dA}$$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by unity.

Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface.

$$\therefore \qquad T = \frac{dU}{dA} \qquad \text{(as } dW = dU\text{)}$$

Thus, the surface tension of a liquid is equal to the surface energy per unit surface area.

Sample Example 13.17 *How much work will be done in increasing the diameter of a soap bubble from 2 cm to 5 cm? Surface tension of soap solution is* 3.0×10^{-2} *N/m.*

Solution Soap bubble has two surfaces. Hence,

$$W = T\,\Delta A$$

Here,

$$\Delta A = 2[4\pi\{(2.5\times10^{-2})^2 - (1.0\times10^{-2})^2\}]$$

$$= 1.32\times10^{-2}\ \text{m}^2$$

$$\therefore \quad W = (3.0\times10^{-2})(1.32\times10^{-2})\ \text{J}$$

$$= 3.96\times10^{-4}\ \text{J}$$

Ans.

Sample Example 13.18 *Calculate the energy released when* 1000 *small water drops each of same radius* 10^{-7} *m coalesce to form one large drop. The surface tension of water is* 7.0×10^{-2} *N/m.*

Solution Let r be the radius of smaller drops and R of bigger one. Equating the initial and final volumes, we have

$$\frac{4}{3}\pi R^3 = (1000)\left(\frac{4}{3}\pi r^3\right)$$

or

$$R = 10r = (10)(10^{-7})\ \text{m}$$

or

$$R = 10^{-6}\ \text{m}$$

Further, the water drops have only one free surface. Therefore,

$$\Delta A = 4\pi R^2 - (1000)(4\pi r^2)$$

$$= 4\pi\ [(10^{-6})^2 - (10^3)(10^{-7})^2]$$

$$= -36\pi(10^{-12})\ \text{m}^2$$

Here, negative sign implies that surface area is decreasing. Hence, energy released in the process.

$$U = T|\Delta A| = (7\times10^{-2})(36\pi\times10^{-12})\ \text{J}$$

$$= 7.9\times10^{-12}\ \text{J}$$

Ans.

Excess Pressure inside a bubble or liquid drop

Surface tension causes a pressure difference between the inside and outside of a soap bubble or a liquid drop.

(i) Excess pressure inside soap bubble

A soap bubble consists of two spherical surface films with a thin layer of liquid between them. Because of surface tension, the film tend to contract in an attempt to minimize their surface area. But as the bubble contracts, it

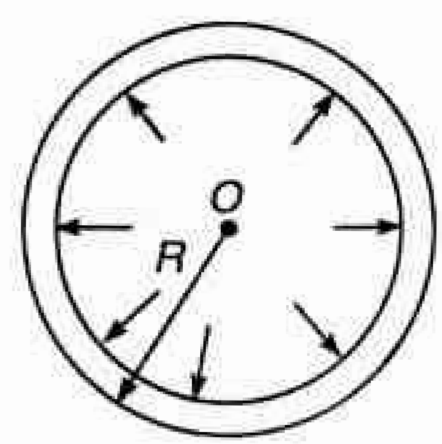


Fig. 13.49

compresses the inside air, eventually increasing the interior pressure to a level that prevents further contraction.

We can derive an expression for the excess pressure inside a bubble in terms of its radius R and the surface tension T of the liquid.

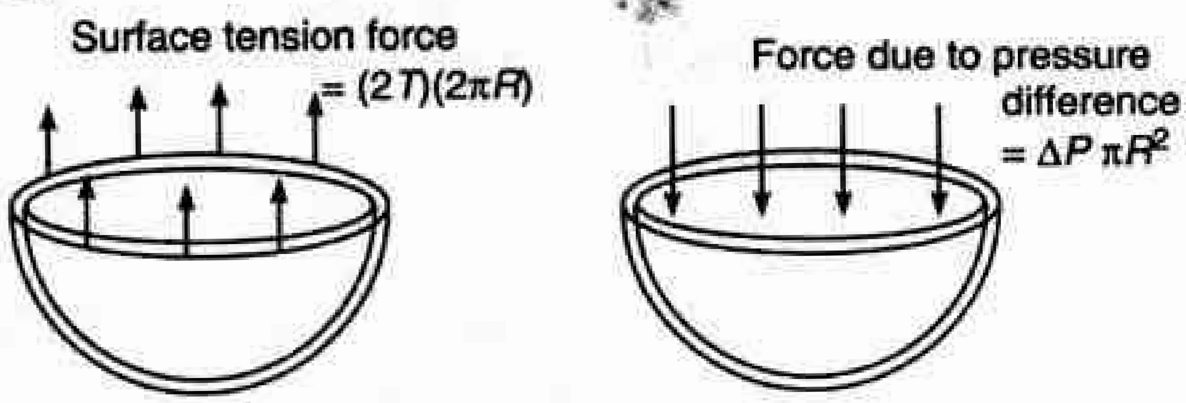


Fig. 13.50

Each half of the soap bubble is in equilibrium. The lower half is shown in figure. The forces at the flat cirular surface where this half joins the upper half are :

(i) The upward force of surface tension. The total surface tension force for each surface (inner and outer) is $T(2\pi R)$, for a total of $(2T)(2\pi R)$

(ii) downward force due to pressure difference.

The magnitude of this force is $(\Delta P)(\pi R^2)$. In equilibrium these two forces have equal magnitude.

$\therefore$ $$(2T)(2\pi R) = (\Delta P)(\pi R^2)$$

or $$\Delta P = \frac{4T}{R}$$

Note *Suppose, the pressure inside the air bubble is P, then*

$$P - P_0 = \frac{4T}{R}$$

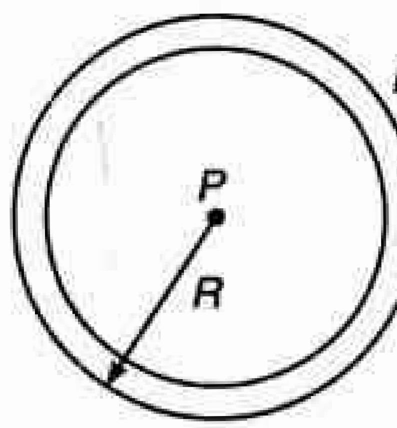


Fig. 13.51

(ii) Excess pressure inside a liquid drop

A liquid drop has only one surface film. Hence, the surface tension force is $T(2\pi R)$, half that for a soap bubble. Thus, in equilibrium,

$$T(2\pi R) = \Delta P(\pi R^2)$$

or $$\Delta P = \frac{2T}{R}$$

Note *(i) If we have an air bubble inside a liquid, a single surface is formed. There is air on the concave side and liquid on the convex side. The pressure in the concave side (that is in the air) is greater than the pressure in the convex side (that is in the liquid) by an amount* $\frac{2T}{R}$.

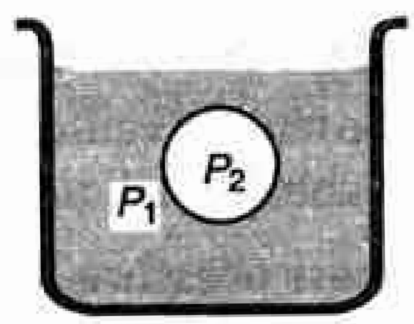


Fig. 13.52

$\therefore$ $$P_2 - P_1 = \frac{2T}{R}$$

The above expression has been written by assuming P_1 to be constant from all sides of the bubble. For small size bubbles this can be assumed.

(ii) From the above discussion, we can make a general statement. The pressure on the concave side of a spherical liquid surface is greater than the convex side by $\frac{2T}{R}$.

Sample Example 13.19 *What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface. Surface tension of water = 7.2×10^{-2} N/m and atmospheric pressure = 1.013×10^5 N/m^2.*

Solution Surface tension of water $T = 7.2 \times 10^{-2}$ N/m

Radius of air bubble $R = 0.1 \text{ mm} = 10^{-4}$ m

The excess pressure inside the air bubble is given by,

$$P_2 - P_1 = \frac{2T}{R}$$

$\therefore$ Pressure inside the air bubble,

$$P_2 = P_1 + \frac{2T}{R}$$

Substituting the values, we have

$$P_2 = (1.013 \times 10^5) + \frac{(2 \times 7.2 \times 10^{-2})}{10^{-4}}$$

$$= 1.027 \times 10^5 \text{ N/m}^2$$

Ans.

Shape of liquid Surface

The surface of a liquid when meets a solid, such as the wall of a container, it usually curves up or down near the solid surface. The angle θ at which it meets the surface is called the **contact angle.** The curved surface of the liquid is called meniscus. The shape of the meniscus (convex or concave) is determined by the relative strengths of what are called the cohesive and adhesive forces. The force between the molecules of the same material is known as **cohesive force** and the force between the molecules of different kinds of material is called **adhesive force.**

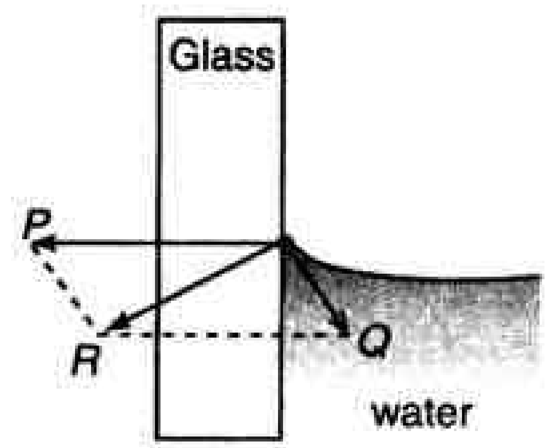


Fig. 13.53

When the adhesive force (P) between solid and liquid molecules is more than the cohesive force (Q) between liquid-liquid molecules (as with water and glass), shape of the meniscus is concave and the angle of contact θ is less than 90°. In this case the liquid wets or adheres to the solid surface. The resultant (R) of P and Q passes through the solid.

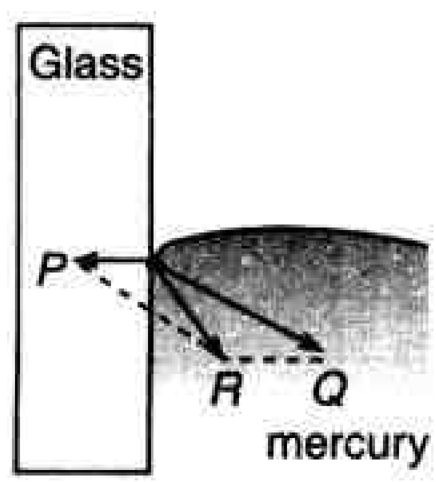


Fig. 13.54

On the other hand when $P < Q$ (as with glass and mercury), shape of the meniscus is convex and the angle of contact $\theta > 90°$. The resultant (R) of P and Q in this case passes through the liquid.

Let us now see why the liquid surface bends near the contact with a solid. A liquid in equilibrium can not sustain tangential stress. The resultant force on any small part of the surface layer must be perpendicular to the surface at that point. Basically three forces are acting on a small part of the liquid surface near its contact with solid. These forces are,

(i) P, attraction due to the molecule of the solid surface near it

(ii) Q, attraction due to liquid molecules near this part and

(iii) The weight W of the part considered.

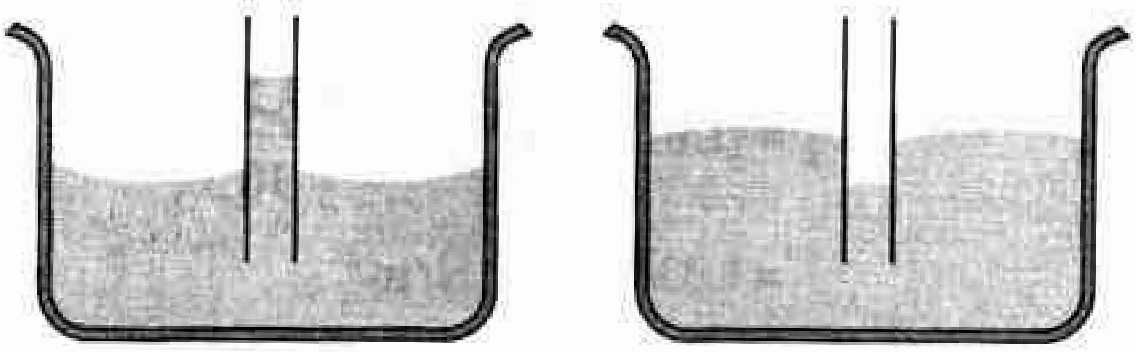
We have considered very small part, so weight of that part can be ignored for better understanding. As we have seen in the last figures, to make the resultant (R) of P and Q perpendicular to the liquid surface the surface becomes curved (convex or concave).

Note *The angle of contact between water and clean glass is zero and that between mercury and clean glass is 137°.*

Capillarity

Surface tension causes elevation or depression of the liquid in a narrow tube. This effect is called capillarity.

When a glass capillary tube (A tube of very small diameter is called a capillary tube) open at both ends is dipped vertically in water,the water in the tube will rise above the level of water in the vessel as shown in figure (a). In case of mercury, the liquid is depressed in the tube below the level of mercury in the vessel as shown in figure (b).



(a) (b)

Fig. 13.55

When the contact angle is less than 90° the liquid rises in the tube. For a nonwetting liquid angle of contact is greater than 90° and the surface is depressed, pulled down by the surface tension forces.

Explanation

When a capillary tube is dipped in water, the water meniscus inside the tube is concave. The pressure just below the meniscus is less than the pressure just above it by $\frac{2T}{R}$, where T is the surface tension of water and R is the radius of curvature of the meniscus. The pressure on the surface of water is P_0, the atmospheric pressure. The pressure just below the plane surface of water outside the tube is also P_0, but that just below the meniscus inside the tube is $P_0 - \frac{2T}{R}$. We know that pressure at all points in the same level of water must be the same. Therefore, to make up the deficiency of pressure $\frac{2T}{R}$ below the meniscus water begins to flow from outside into the tube. The rising of water in the capillary stops at a certain height h. In this position the pressure of water column of height h becomes equal to $\frac{2T}{R}$, *i.e.*,

$$h\rho g = \frac{2T}{R}$$

or $$h = \frac{2T}{R\rho g}$$

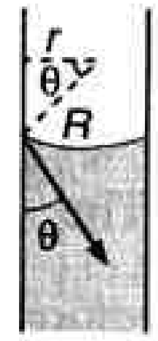


Fig. 13.56

If r is the radius of the capillary tube and θ the angle of contact, then

$$R = \frac{r}{\cos\theta}$$

∴ $$h = \frac{2T\cos\theta}{r\rho g}$$

Alternative proof for the formula of Capillary rise

As we have already seen, when the contact angle is less than 90°, the total surface tension force just balances the extra weight of the liquid in the tube.

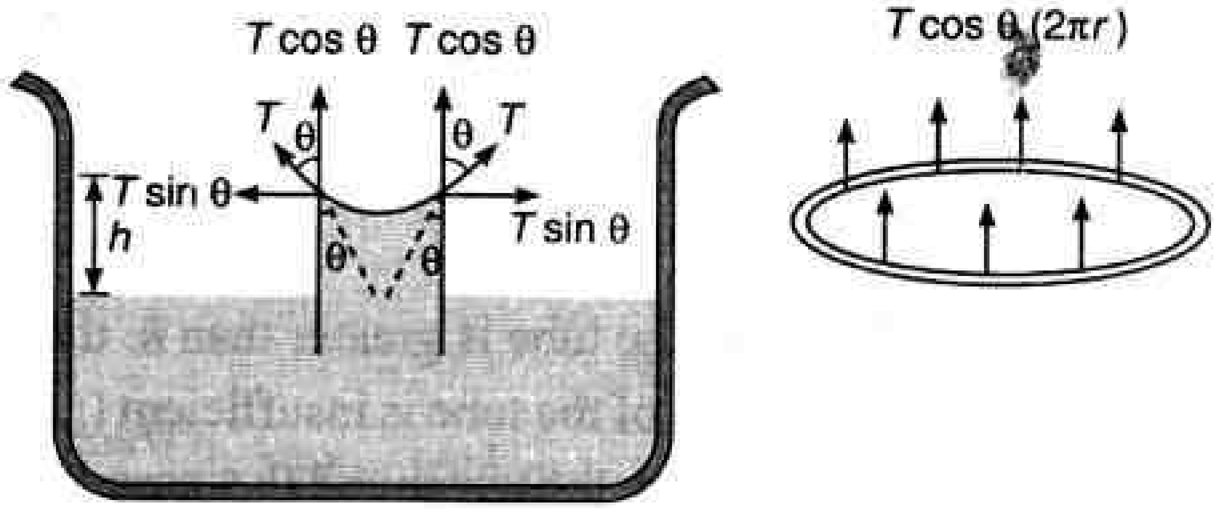


Fig. 13.57

The water meniscus in the tube is along a circle of circumference $2\pi r$ which is in contact with the glass. Due to the surface tension of water, a force equal to T per unit length acts at all points of the circle. If the angle of contact is θ, then this force is directed inward at an angle θ from the wall of the tube.

In accordance with Newton's third law, the tube exerts an equal and opposite force T per unit length on the circumference of the water meniscus. This force which is directed outward, can be resolved into two components. $T\cos\theta$ per unit length acting vertically upward and $T\sin\theta$ per unit length acting horizontally outward. Considering the entire circumference $2\pi r$, for each horizontal component $T\sin\theta$ there is an equal and opposite component and the two neutralise each other. The vertical components being in the same direction are added up to give a total upward force $(2\pi r)(T\cos\theta)$. It is this force which supports the weight of the water column so raised. Thus,

$$(T\cos\theta)(2\pi r) = \text{Weight of the liquid column.}$$

$$= (\pi r^2 \rho g h) \qquad \ldots\text{(i)}$$

∴ $$h = \frac{2T\cos\theta}{r\rho g}$$

The result has following notable features,

(i) If the contact angle θ is greater than 90°, the term $\cos\theta$ is negative and hence, h is negative. The expression then gives the depression of the liquid in the tube.

(ii) The **correction** due to weight of the liquid contained in the meniscus can be made for contact angle $\theta = 0°$. The meniscus is then hemispherical. The volume of the shaded part is

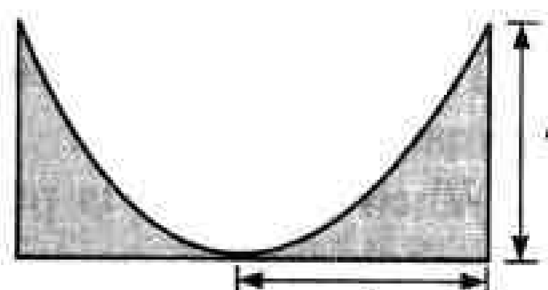


Fig. 13.58

$$V = (\pi r^2)(r) - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^3$$

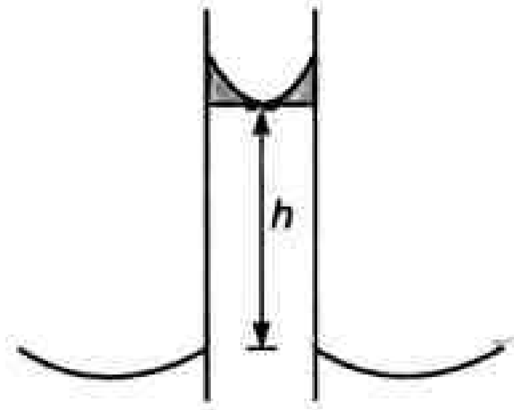


Fig. 13.59

The weight of the liquid contained in the meniscus is $\frac{1}{3}\pi r^3 \rho g$.

Therefore, we can write Eq. (i) as,

$$(T\cos 0^\circ)(2\pi r) = \pi r^2 \rho g h + \frac{1}{3}\pi r^3 \rho g$$

or $$h = \frac{2T}{r\rho g} - \frac{r}{3}$$

(iii) Suppose a capillary tube is held vertically in a liquid which has a concave meniscus, then capillary rise is given by,

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2T}{R\rho g} \qquad \left(\text{as } R = \frac{r}{\cos\theta}\right)$$

or $$hR = \frac{2T}{\rho g} \qquad \text{...(ii)}$$

When the length of the tube is greater than h, the liquid rises in the tube, so as to satisfy the above relation. But if the length of the tube is insufficient (*i.e.*, less than h) say h', the liquid does not emerge in the form of a fountain from the upper end (because it will violate the law of conservation of energy) but the angle made by the liquid surface and hence, the R changes in such a way that the force $2\pi rT\cos\theta$ equals the weight of the liquid raised. Thus,

$$2\pi rT\cos\theta' = \pi r^2 \rho g h'$$

$$h' = \frac{2T\cos\theta'}{r\rho g} \quad \text{or} \quad h' = \frac{2T}{R'\rho g}$$

or $$h'R' = \frac{2T}{\rho g} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii) $$hR = h'R' = \frac{2T}{\rho g}$$

Effect of Detergent on Surface Tension

As we have already learned that the surface of any liquid behaves as though it is covered by a stretched membrane. Small insects can walk on water without getting wet only due to this property of liquid called surface tension.

Detergents

The properties of detergents arise from their complicated molecular structure. This is illustrated schematically in figure.

When detergent is put into water the detergent molecules on the surface are aligned with their hydrophobic ends pointing up as shown in figure (c). Other detergent molecules are dispersed throughout the water. Along the surface there are water molecules and hydrophobic ends. the surface

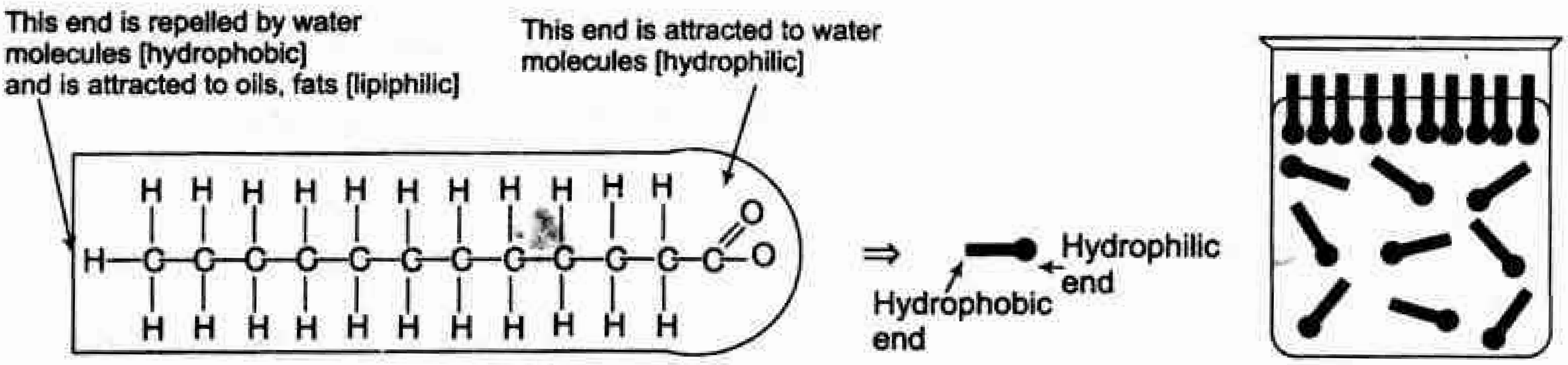


Fig. 13.60 (a) Detergent molecule (b) (c) Detergent molecules in water

membrane is broken by the detergent molecules. It is easier to pull this surface apart than it is to pull a surface of pure water apart. Due to this surface tension is decreased.

Sample Example 13.20 *A capillary tube whose inside radius is 0.5 mm is dipped in water having surface tension 7.0×10^{-2} N/m. To what height is the water raised above the normal water level? Angle of contact of water with glass is 0°. Density of water is 10^3 kg/m³ and $g = 9.8$ m/s².*

Solution

$$h=\frac{2T\cos\theta}{r\rho g}$$

Substituting the proper values, we have

$$h=\frac{(2)(7.0\times10^{-2})\cos 0^\circ}{(0.5\times10^{-3})(10^3)(9.8)}=2.86\times10^{-2}\text{ m}$$

$$= 2.86\text{ cm}$$ **Ans.**

Sample Example 13.21 *A glass tube of radius 0.4 mm is dipped vertically in water. Find upto what height the water will rise in the capillary? If the tube in inclined at an angle of 60° with the vertical, how much length of the capillary is occupied by water? Surface tension of water = 7.0×10^{-2} N/m, density of water = 10^3 kg/m³.*

Solution For glass-water, angle of contact $\theta = 0^\circ$.

Now,

$$h=\frac{2T\cos\theta}{r\rho g}=\frac{(2)(7.0\times10^{-2})\cos 0^\circ}{(0.4\times10^{-3})(10^3)(9.8)}$$

$$=3.57\times10^{-2}\text{ m}$$

$$=3.57\text{ cm}$$ **Ans.**

$$l=\frac{h}{\cos 60^\circ}=\frac{3.57}{\frac{1}{2}}=7.14\text{ cm}$$ **Ans.**

Introductory Exercise 13.3

1. A water film is made between two straight parallel wires of length 10 cm each, and at a distance of 0.5 cm from each other. If the distance between the wires is increased by 1 mm, how much work will be done? Surface tension of water = 7.2×10^{-2} N/m.
2. A soap bubble of radius R has been formed at normal temperature and pressure under isothermal conditions. Compute the work done. The surface tension of soap solution is T.
3. If a drop of radius r breaks up into 27 small drops then how much will be the change in the surface energy. The surface tension of liquid is T.
4. When wax is rubbed on cloth, the cloth becomes water proof why?
5. Water at 20°C is flowing in a pipe of radius 20.0 cm. The viscosity of water at 20°C is 1.005 centipoise. If the water's speed in the centre of the pipe is 3.00 m/s, what is water's speed:
   (a) 10.0 cm from the centre of the pipe (half way between the centre and the walls)
   (b) at the walls of the pipe?
6. Water at 20°C is flowing in a horizontal pipe that is 20.0 m long. The flow is laminar and the water completely fills the pipe. A pump maintains a gauge pressure of 1400 Pa, at a large tank at one end of the pipe. The other end of the pipe is open to the air, The viscosity of water at 20°C is 1.005 poise.
   (a) If the pipe has diameter 8.0 cm, what is the volume flow rate?
   (b) What gauge pressure must the pump provide to achieve the same volume flow rate for a pipe with a diameter of 4.0 cm?
   (c) For pipe in part (a) and the same gauge pressure maintained by the pump, what does the volume flow rate become if the water is at a temperature of 60°C (the viscosity of water at 60°C is 0.469 poise)

13.12 Laminar And Turbulent Flow, Reynolds Number

When a liquid flowing in a pipe is observed carefully, it will be seen that the pattern of flow becomes more disturbed as the velocity of flow increases. Perhaps this phenomenon is more commonly seen in a river or stream. When the flow is slow the pattern is smooth, but when the flow is fast, eddies develop and swirl in all directions.

At the low velocities, flow is calm. This is called **"laminar flow"**.

In a series of experiments, **Reynolds** showed this by injecting a thin stream of dye into the fluid and finding that it ran in a smooth stream in the direction of the flow at low speeds. As the velocity of flow increased, he found that the smooth line of dye was broken up, at high velocities, the dye was rapidly mixed into the disturbed flow of the surrounding fluid. This is called **"turbulent flow"**.

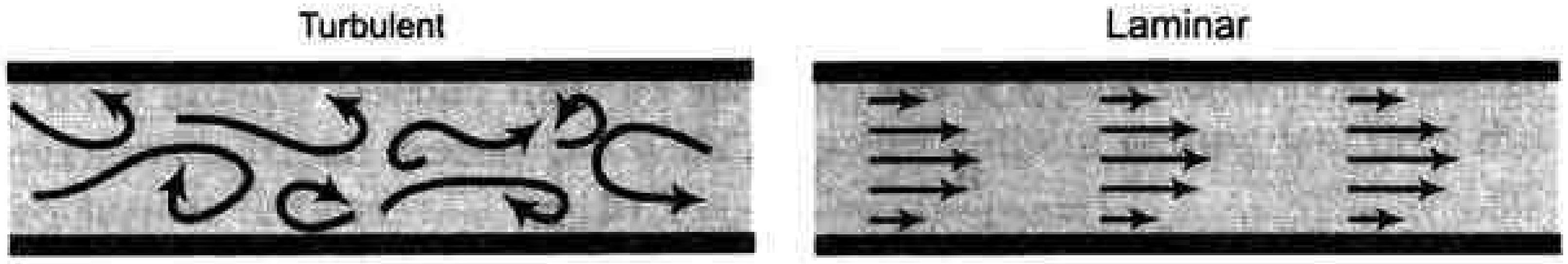


Fig. 13.61

After many experiments **Reynolds** saw that the expression:

$$\frac{\rho u d}{\eta}$$

where, ρ = density, u = mean velocity, d = diameter and η = viscosity would help in predicting the change in flow type. If the value is less than about 2000 then flow is laminar, if greater than 4000 then turbulent and in between these two in the transition zone.

This value is known as the Reynolds number, Re.

$$Re = \frac{\rho u d}{\eta}$$

Laminar flow: Re < 2000
Transitional flow: 2000 < Re < 4000
Turbulent flow: Re > 4000

SI Units Of Reynolds Number :

$$\rho = kg/m^3, \; u = m/s^3, \quad d = m$$
$$\eta = Ns/m^2 = kg/ms$$
$$Re = \frac{\rho u d}{\eta} = \frac{(kg/m^3)(m/s)(m)}{(kg/ms)} = 1$$

i.e. it has no units. A quantity that has no units is known as a non-dimensional (or dimensionless) quantity. Thus the Reynolds number, Re, is a non-dimensional number.

In summary:

Laminar flow

- Re< 2000
- 'low' velocity
- Dye does not mix with water
- Fluid particles move in straight lines
- Simple mathematical analysis possible
- Rare in practice in water systems.

Transitional flow

- 2000> Re< 4000
- 'medium' velocity
- Dye stream wavers in water - mixes slightly.

Turbulent flow

- Re> 4000
- 'high' velocity
- Dye mixes rapidly and completely
- Particle paths completely irregular
- Average motion is in the direction of the flow
- Mathematical analysis very difficult - so experimental measures are used
- Most common type of flow.

Extra Points

- **Torque due to hydrostatic forces** : Consider a dam in which water of density ρ is filled upto a height H. We are interested in finding the torque of hydrostatic forces on wall AB about point B. For this, consider a small length RS equal to dh of the wall AB at a depth h below the free surface of the dam. Further, let us assume a unit width perpendicular to paper inwards. Pressure on RS from left side is $P_0 + \rho g h$ and from right side is P_0.

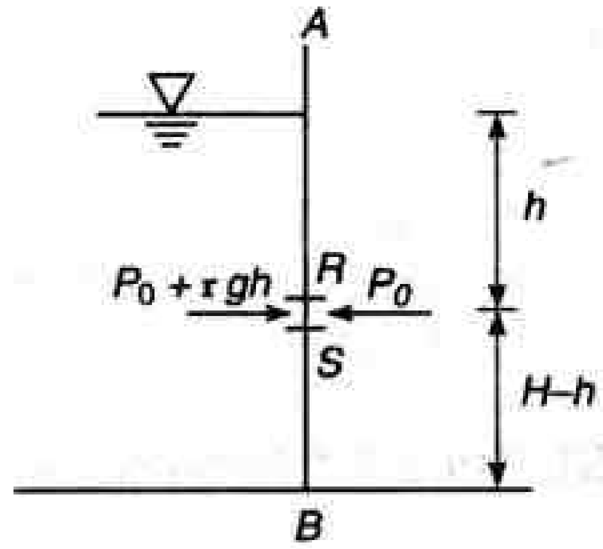


Fig. 13.62

Area of RS, $dA = (1)dh = dh$

Excess pressure $P = \rho g h$

∴ Net force $F = PdA = \rho g h \, dh$

Perpendicular distance of this force from point B is, $r_\perp = H - h$

∴ Torque of this force about B, $d\tau = Fr_\perp$

or $d\tau = \rho g h (H - h) dh$

Therefore, net torque $\tau = \int_0^H d\tau$

$$= \int_0^H \rho g h \,(H - h)\, dh$$

∴ $$\tau = \frac{\rho g H^3}{6}$$

This is the torque of hydrostatic forces per unit width of the wall.

Note

In the figure shown, torque of hydrostatic force about point O, the centre of a semicylindrical (or hemispherical) gate is zero as the hydrostatic force at all points passes through point O.

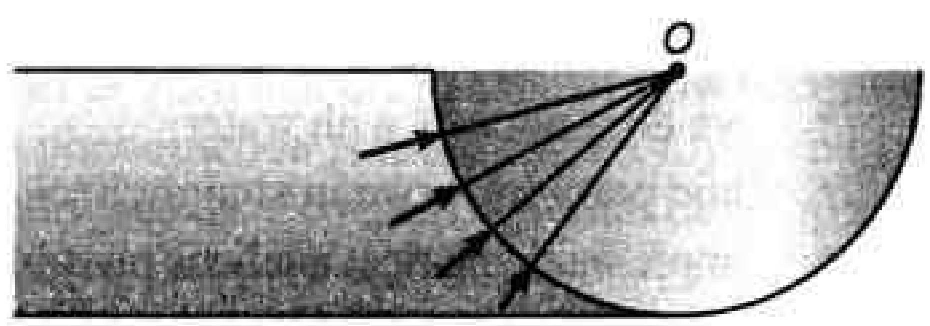


Fig. 13.63

- **Principle of Syphon**

In Fig. 13.62 $v_1 = v_2 = 0$

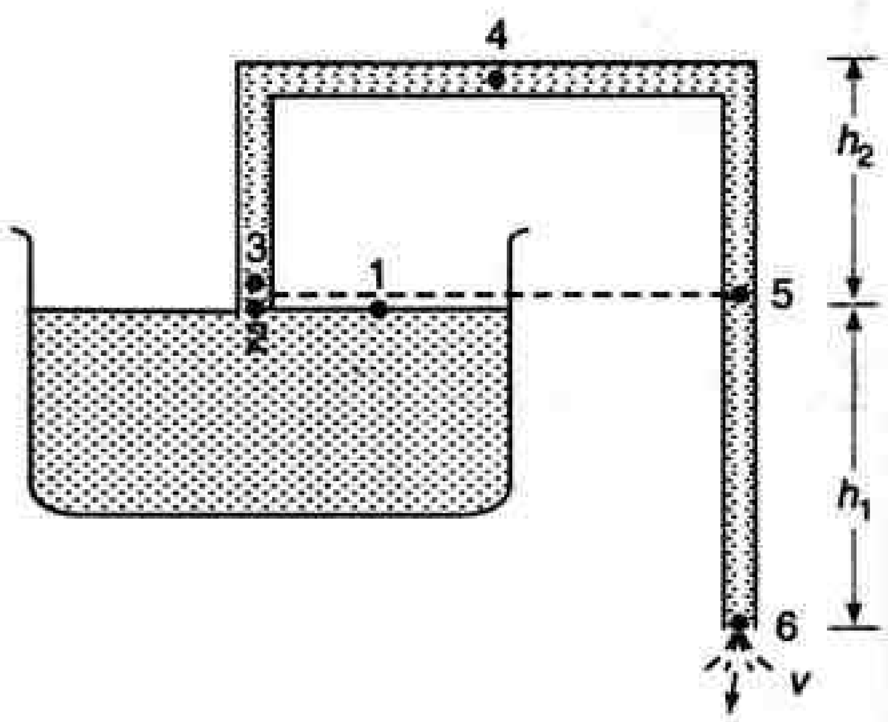
$P_1 = P_2 = P_0$

$v_3 = v_4 = v_5 = v_6 = v$ (say)

Applying Bernoulli's equation at 1 (or 2), 3, 4, 5 and 6, we have

$$P_0 + 0 + 0 = P_3 + \frac{1}{2}\rho v^2 + 0 = P_4 + \frac{1}{2}\rho v^2 + \rho g h_2$$

$$= P_5 + \frac{1}{2}\rho v^2 + 0 = P_0 + \frac{1}{2}\rho v^2 - \rho g h_1$$



Note Point 3 is just above point 2. $v_2 = 0$ but $v_3 = v$

Fig. 13.64

From this equation following conclusions can be made.

(i) $P_1 = P_2 = P_0$

(ii) $P_3 = P_5 < P_0$

(iii) $v_1 = v_2 = 0$

(vi) $v_3 = v_4 = v_5 = v_6 = v$

(v) $v = \sqrt{2gh_1}$ so, h_1 should be greater than zero

(vi) $P_4 = P_0 - \rho g (h_1 + h_2)$

From the last equation we can see that P_4 decreases as $(h_1 + h_2)$ increases. Minimum value of P_4 can be zero and this will occur at,

$$0 = P_0 - \rho g (h_1 + h_2)_{max}$$

Thus, $(h_1 + h_2)_{max} = \dfrac{P_0}{\rho g}$ and simultaneously $h_1 > 0$

Thus, syphon will work when $h_1 > 0$ and $(h_1 + h_2) < \dfrac{P_0}{\rho g}$.

- In projectile motion the path of the projectile is a parabola if the effects of air resistance are neglected. In more realistic model that includes the drag force due to the air, the path of the ball is non-parabolic and the ball lands short of range predicted with the simple model (without considering the air drag).

  Furthermore, the launch angle for maximum range is less than 45° and depends upon the initial speed of the ball.

Solved Examples

For JEE Main

Example 1 *For the arrangement shown in the figure, what is the density of oil ?*

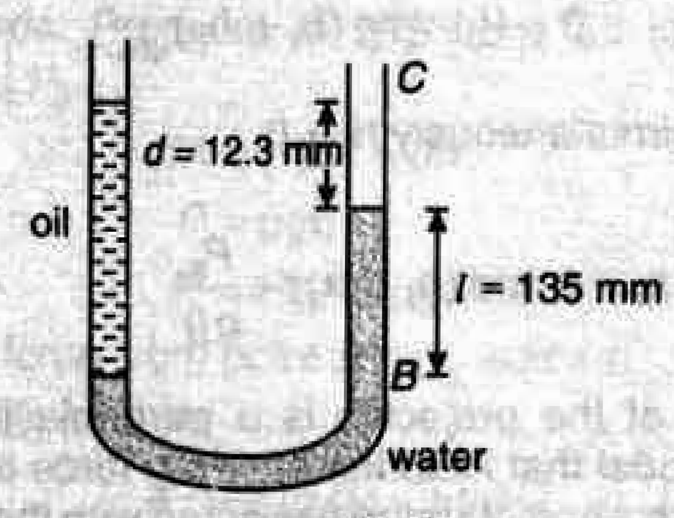


Fig. 13.65

Solution $P_0 + \rho_w gl = P_0 + \rho_{oil}\ (l + d)\, g$

$$\Rightarrow \qquad \rho_{oil} = \frac{\rho_w l}{l + d} = \frac{1000.(135)}{(135 + 12.3)} = 916\ \text{kg/m}^3 \qquad \textbf{Ans.}$$

Example 2 *A solid floats in a liquid of different material. Carry out an analysis to see whether the level of liquid in the container will rise or fall when the solid melts.*

Solution Let M = Mass of the floating solid

ρ_1 = density of liquid formed by the melting of the solid

ρ_2 = density of the liquid in which the solid is floating

The mass of liquid displaced by the solid is M. Hence, the volume of liquid displaced is $\frac{M}{\rho_2}$.

When the solid melts, the volume occupied by it is $\frac{M}{\rho_1}$. Hence, the level of liquid in container will rise or fall according as

$$\frac{M}{\rho_1} > \text{ or } < \frac{M}{\rho_2} \qquad i.e., \qquad \rho_1 < \text{ or } > \rho_2$$

There will be no change in the level if $\rho_1 = \rho_2$. In case of ice floating in water $\rho_1 = \rho_2$ and hence, the level of water remains unchanged when ice melts.

Example 3 *An iron casting containing a number of cavities weighs* 6000 *N in air and* 4000 *N in water. What is the volume of the cavities in the casting? Density of iron is* $7.87\ g/cm^3$. *Take* $g = 9.8\ m/s^2$ *and density of water* $= 10^3\ kg/m^3$.

Solution Let v be the volume of cavities and V the volume of solid iron. Then,

$$V = \frac{\text{mass}}{\text{density}} = \left(\frac{6000/9.8}{7.87 \times 10^3}\right) = 0.078\ \text{m}^3$$

Further, decrease in weight = upthrust

$$\therefore \quad (6000 - 4000) = (V + v)\rho_w g$$

or $$2000 = (0.078 + v) \times 10^3 \times 9.8$$

or $$0.078 + v \approx 0.2$$

$$\therefore \quad v = 0.12 \text{ m}^3$$ **Ans.**

Example 4 *A boat floating in a water tank is carrying a number of stones. If the stones were unloaded into water, what will happen to the water level?*

Solution Let weight of boat $= W$ and weight of stone $= w$.

Assuming density of water = 1 g/cc

Volume of water displaced initially $= (w + W)$

Later, volume displaced $= \left(W + \dfrac{w}{\rho}\right)$ (ρ = density of stones)

$\Rightarrow$ *Water level comes down.*

Example 5 *Water rises in a capillary tube to a height of 2.0 cm. In another capillary tube whose radius is one third of it, how much the water will rise?*

Solution $$h = \frac{2T \cos\theta}{r\rho g}$$

$$\therefore \quad hr = \frac{2T\cos\theta}{\rho g} = \text{constant}$$

$$\therefore \quad h_1 r_1 = h_2 r_2 \quad \text{or} \quad h_2 = \frac{h_1 r_1}{r_2}$$

Substituting the values $$h_2 = (2.0)(3) \qquad \left(\frac{r_2}{r_1} = \frac{1}{3}\right)$$

$$= 6.0 \text{ cm}$$ **Ans.**

Example 6 *Mercury has an angle of contact of 120° with glass. A narrow tube of radius 1.0 mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside. Surface tension of mercury at the temperature of the experiment is 0.5 N/m and density of mercury is 13.6×10^3 kg/m^3. (Take $g = 9.8$ m/s^2).*

Solution $$h = \frac{2T\cos\theta}{r\rho g}$$

Substituting the values, we get $$h = \frac{2 \times 0.5 \times \cos 120°}{10^{-3} \times 13.6 \times 10^3 \times 9.8} = -3.75 \times 10^{-3} \text{ m}$$

or $$h = -3.75 \text{ mm}$$ **Ans.**

Note *Here, negative sign implies that mercury suffers capillary depression.*

Example 7 *Two narrow bores of radius 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water is 7.3×10^{-2} N/m. Take the angle of contact to be zero and density of water to be 10^3 kg/m^3. ($g=9.8$ m/s^2)*

Solution
$$h\rho g=\Delta P=\frac{2T\cos\theta}{r_1}-\frac{2T\cos\theta}{r_2}$$

or
$$h=\frac{2T\cos\theta}{\rho g}\left(\frac{r_2-r_1}{r_1r_2}\right)$$

Substituting the values, we have
$$h=\frac{2\times7.3\times10^{-2}\times\cos0^\circ}{10^3\times9.8}\left(\frac{6.0-3.0}{6.0\times3.0}\right)\times\frac{1}{10^{-3}}$$
$$=2.48\times10^{-3}\text{ m}$$
$$=2.48\text{ mm}$$ **Ans.**

Example 8 *With what terminal velocity will an air bubble 0.8 mm in diameter rise in a liquid of viscosity 0.15 N-s/m^2 and specific gravity 0.9? Density of air is 1.293 kg/m^3.*

Solution The terminal velocity of the bubble is given by,
$$v_T=\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

Here, $r=0.4\times10^{-3}$ m, $\sigma=0.9\times10^3$ kg/m^3, $\rho=1.293$ kg/m^3, $\eta=0.15$, N-s/m^2

and $g=9.8$ m/s^2

Substituting the values, we have
$$v_T=\frac{2}{9}\times\frac{(0.4\times10^{-3})^2(1.293-0.9\times10^3)\times9.8}{0.15}$$
$$=-0.0021\text{ m/s}$$

or
$$v_T=-0.21\text{ cm/s}$$ **Ans.**

Note *Here negative sign implies that the bubble will rise up.*

Example 9 *A spherical ball of radius 3.0×10^{-4} m and density 10^4 kg/m^3 falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, find h. Viscosity of water is 9.8×10^{-6} N-s/m^2.*

Solution Before entering the water the velocity of ball is $\sqrt{2gh}$. If after entering the water this velocity does not change then this value should be equal to the terminal velocity. Therefore,
$$\sqrt{2gh}=\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

$$\therefore \qquad h=\frac{\left\{\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}\right\}^2}{2g}$$

$$=\frac{2}{81}\times\frac{r^4(\rho-\sigma)^2g}{\eta^2}$$

$$=\frac{2}{81}\times\frac{(3\times10^{-4})^4(10^4-10^3)^2\times9.8}{(9.8\times10^{-6})^2}$$

$$=1.65\times10^3\ \text{m}$$ **Ans.**

For JEE Advanced

Example 1 *A solid ball of density half that of water falls freely under gravity from a height of 19.6 m and then enters water. Upto what depth will the ball go. How much time will it take to come again to the water surface? Neglect air resistance and viscosity effects in water. (Take $g=9.8\ m/s^2$)*

Solution $v=\sqrt{2gh}=\sqrt{2\times9.8\times19.6}=19.6$ m/s

Let ρ be the density of ball and 2ρ the density of water. Net retardation inside the water,

$$a=\frac{\text{upthrust}-\text{weight}}{\text{mass}}$$

$$=\frac{V(2\rho)g-V(\rho)(g)}{V(\rho)} \quad (V=\text{volume of ball})$$

$$=g=9.8\ \text{m/s}^2$$

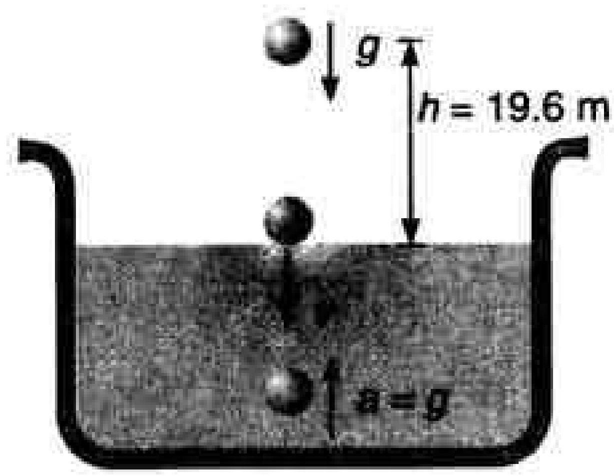


Fig. 13.66

Hence, the ball will go upto the same depth 19.6 m below the water surface. **Ans.**

Further, time taken by the ball to come back to water surface is,

$$t=2\left(\frac{v}{a}\right)=2\left(\frac{19.6}{9.8}\right)=4\ \text{s}$$ **Ans.**

Example 2 *A block of mass 1 kg and density 0.8 g/cm^3 is held stationary with the help of a string as shown in figure. The tank is accelerating vertically upwards with an acceleration $a=1.0\ m/s^2$. Find :*

(a) the tension in the string,

(b) if the string is now cut find the acceleration of block.

(Take $g=10\ m/s^2$ and density of water $=10^3\ kg/m^3$).

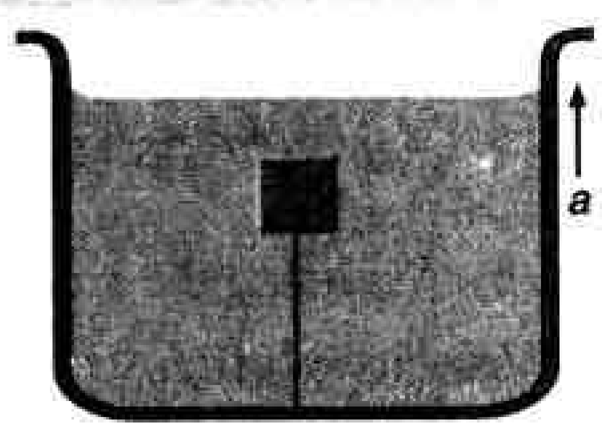


Fig. 13.67

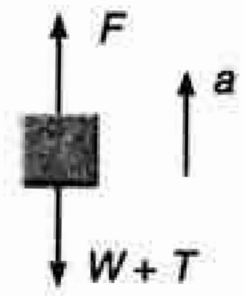


Fig. 13.68

Solution (a) Free body diagram of the block is shown in Fig. 13.66

In the figure,

$$F = \text{upthrust force}$$
$$= V\rho_\omega (g + a)$$
$$= \left(\frac{\text{mass of block}}{\text{density of block}}\right)\rho_\omega \ (g + a)$$
$$= \left(\frac{1}{800}\right)(1000)(10 + 1) = 13.75 \text{ N}$$
$$W = mg = 10 \text{ N}$$

Equation of motion of the block is,

$$F - T - W = ma$$
$$\therefore \quad 13.75 - T - 10 = 1 \times 1$$
$$\therefore \quad T = 2.75 \text{ N}$$ **Ans.**

(b) When the string is cut $T = 0$

$$\therefore \quad a = \frac{F - W}{m}$$
$$= \frac{13.75 - 10}{1}$$
$$= 3.75 \text{ m/s}^2$$ **Ans.**

Example 3 *A fresh water on a reservoir is 10 m deep. A horizontal pipe 4.0 cm in diameter passes through the reservoir 6.0 m below the water surface as shown in figure. A plug secures the pipe opening.*

(a) Find the friction force between the plug and pipe wall.

(b) The plug is removed. What volume of water flows out of the pipe in 1 h? Assume area of reservoir to be too large.

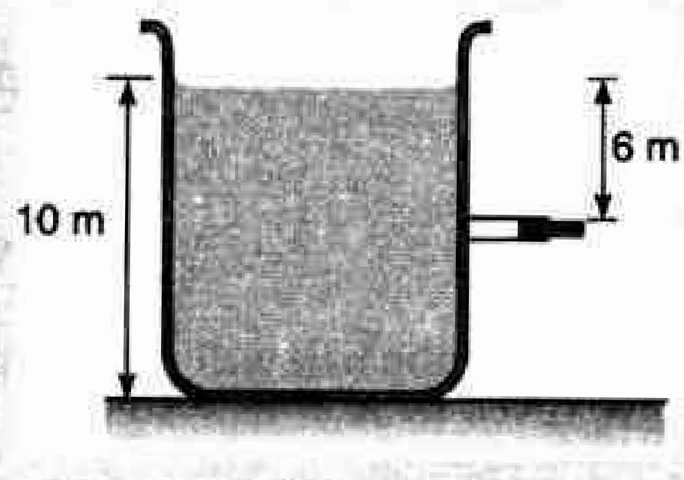


Fig. 13.69

Solution (a) Force of friction

$$= \text{pressure difference on the sides of the plug} \times \text{area of cross section of the plug}$$
$$= (\rho g h)A = (10^3)(9.8)(6.0)(\pi)(2 \times 10^{-2})^2$$
$$= 73.9 \text{ N}$$ **Ans.**

(b) Assuming the area of the reservoir to be too large,

Velocity of efflux $\quad v = \sqrt{2gh} = \text{constant}$

$$\therefore \quad v = \sqrt{2 \times 9.8 \times 6} = 10.84 \text{ m/s}$$

Volume of water coming out per sec,

$$\frac{dV}{dt} = Av$$
$$= \pi(2\times10^{-2})^2(10.84)$$
$$= 1.36\times10^{-2}\ \text{m}^3/\text{s}$$

∴ The volume of water flowing through the pipe in 1 h

$$V = \left(\frac{dV}{dt}\right)t$$
$$= (1.36\times10^{-2})(3600)$$
$$= 48.96\ \text{m}^3$$

Ans.

Example 4 *The U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface. The tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine the speed of the free jet and the minimum absolute pressure of the water in the bend. Given atmospheric pressure* $= 1.01\times10^5\ N/m^2$, $g = 9.8\ m/s^2$ *and density of water* $= 10^3\ kg/m^3$.

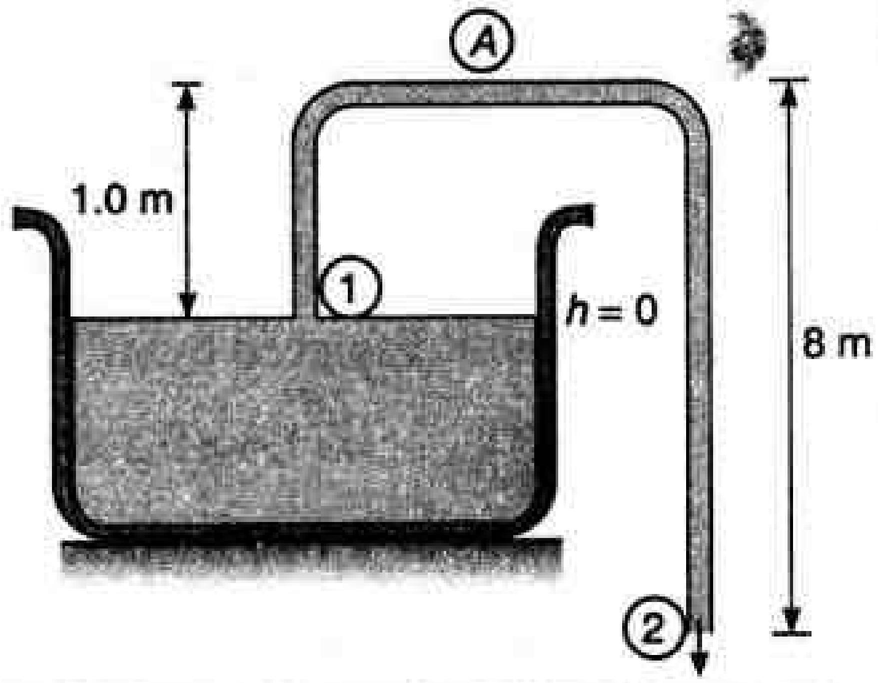


Fig. 13.70

Solution (a) Applying Bernoulli's equation between points (1) and (2)

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since, area of reservoir >> area of pipe

$$v_1 \approx 0, \quad \text{also} \quad P_1 = P_2 = \text{atmospheric pressure}$$

So, $$v_2 = \sqrt{2g(h_1 - h_2)} = \sqrt{2\times9.8\times7} = 11.7\ \text{m/s}$$ **Ans.**

(b) The minimum pressure in the bend will be at A. Therefore, applying Bernoulli's equation between (1) and (A)

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_A + \frac{1}{2}\rho v_A^2 + \rho g h_A$$

Again, $v_1 \approx 0$ and from conservation of mass $v_A = v_2$

or $$P_A = P_1 + \rho g(h_1 - h_A) - \frac{1}{2}\rho v_2^2$$

Therefore, substituting the values, we have

$$P_A = (1.01\times10^5) + (1000)(9.8)(-1) - \frac{1}{2}\times(1000)(11.7)^2 2$$

$$= 2.27\times10^4 \text{ N/m}^2$$ **Ans.**

Example 5 *A wooden rod weighing 25 N is mounted on a hinge below the free surface of water as shown. The rod is 3 m long and uniform in cross section and the support is 1.6 m below the free surface. At what angle α will it come to rest when allowed to drop from a vertical position. The cross-section of the rod is 9.5×10^{-4} m² in area. Density of water is 1000 kg/m³. Assume buoyancy to act at centre of immersion. g = 9.8 m/s². Also find the reaction on the hinge in this position.*

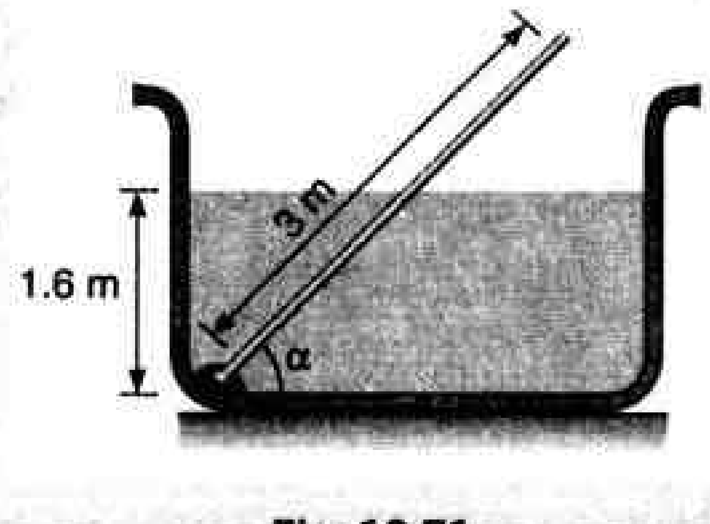


Fig. 13.71

Solution Let G be the mid-point of AB and E the mid point of AC (*i.e.*, the centre of buoyancy)

$$AC = 1.6 \operatorname{cosec}\alpha$$

$$\text{Volume of } AC = (1.6\times9.5\times10^{-4})\operatorname{cosec}\alpha$$

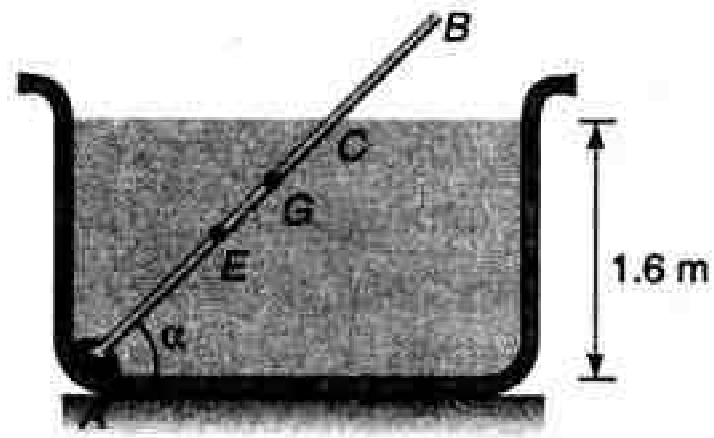


Fig. 13.72

Weight of water displaced by AC

$$= (1.6\times9.5\times10^{-4}\times10^3\times9.8)\operatorname{cosec}\alpha$$

$$= 14.896 \operatorname{cosec}\alpha$$

Hence, the buoyant force is $14.896 \operatorname{cosec}\alpha$ acting vertically upwards at E. While the weight of the rod is 25 N acting vertically downwards at G. Taking moments about A,

$$(14.896 \operatorname{cosec}\alpha)(AE\cos\alpha) = (25)(AG\cos\alpha)$$

or $$(14.896 \operatorname{cosec}\alpha)\left(\frac{1.6\operatorname{cosec}\alpha}{2}\right) = 25\times\frac{3}{2}$$

or $$\sin^2\alpha = 0.32$$

∴ $$\sin\alpha = 0.56$$

or $$\alpha = 34.3°$$ **Ans.**

Further, let F be the reaction at hinge in vertically downward direction. Then, considering the translatory equilibrium of rod in vertical direction we have,

$$F + \text{weight of the rod} = \text{upthrust}$$

∴ $$F = \text{upthrust} - \text{weight of the rod}$$

$$= 14.896 \operatorname{cosec}(34.3°) - 25$$

$$= 26.6 - 25$$

∴ $$F = 1.6 \text{ N} \quad \text{(downwards)}$$ **Ans.**

Example 6 *Two separate air bubbles (radii 0.004 m and 0.002 m) formed of the same liquid (surface tension 0.07 N/m) come together to form a double bubble. Find the radius and the sense of curvature of the internal film surface common to both the bubbles.*

Solution

$$P_1 = P_0 + \frac{4T}{r_1}$$

$$P_2 = P_0 + \frac{4T}{r_2}$$

$$r_2 < r_1$$

$$\therefore \quad P_2 > P_1$$

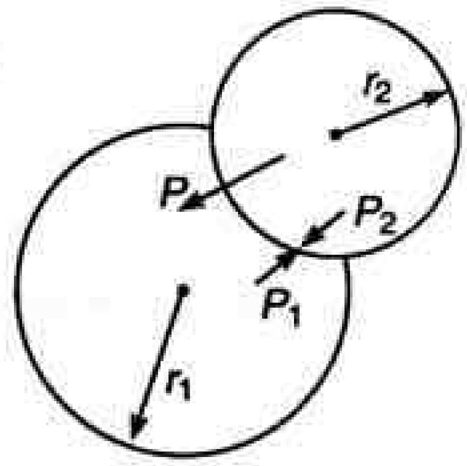


Fig. 13.73

i.e., pressure inside the smaller bubble will be more. The excess pressure

$$P = P_2 - P_1 = 4T\left(\frac{r_1 - r_2}{r_1 r_2}\right) \quad \text{...(i)}$$

This excess pressure acts from concave to convex side, the interface will be concave towards smaller bubble and convex towards larger bubble. Let R be the radius of interface then,

$$P = \frac{4T}{R} \quad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$R = \frac{r_1 r_2}{r_1 - r_2} = \frac{(0.004)(0.002)}{(0.004 - 0.002)}$$

$$= 0.004 \text{ m}$$ **Ans.**

Example 7 *Under isothermal condition two soap bubbles of radii r_1 and r_2 coalesce to form a single bubble of radius r. The external pressure is P_0. Find the surface tension of the soap in terms of the given parameters.*

Solution As mass of the air is conserved,

$$\therefore \quad n_1 + n_2 = n \qquad (\text{as } PV = nRT)$$

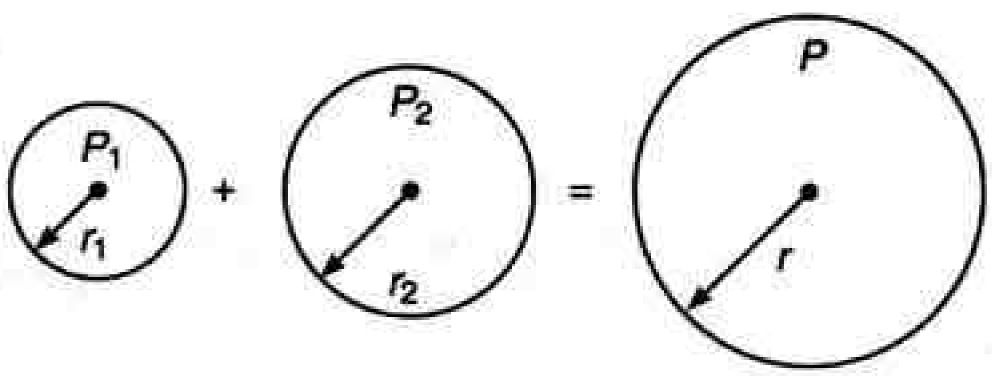


Fig. 13.74

$$\therefore \quad \frac{P_1 V_1}{RT_1} + \frac{P_2 V_2}{RT_2} = \frac{PV}{RT}$$

As temperature is constant, $T_1 = T_2 = T$

$\therefore$ $$P_1V_1 + P_2V_2 = PV$$

$\therefore$ $$\left(P_0 + \frac{4S}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(P_0 + \frac{4S}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right)$$
$$= \left(P_0 + \frac{4S}{r}\right)\left(\frac{4}{3}\pi r^3\right)$$

Solving, this we get

$$S = \frac{P_0(r^3 - r_1^3 - r_2^3)}{4(r_1^2 + r_2^2 - r^2)}$$ **Ans.**

Note *To avoid confusion with the temperature surface tension here is represented by S.*

Example 8 *A cylindrical tank of base area A has a small hole of area 'a' at the bottom. At time $t = 0$, a tap starts to supply water into the tank at a constant rate α m^3/s.*

(a) what is the maximum level of water h_{max} in the tank?

(b) find the time when level of water becomes h ($< h_{max}$).

Solution (a) Level will be maximum when

Rate of inflow of water = rate of outflow of water

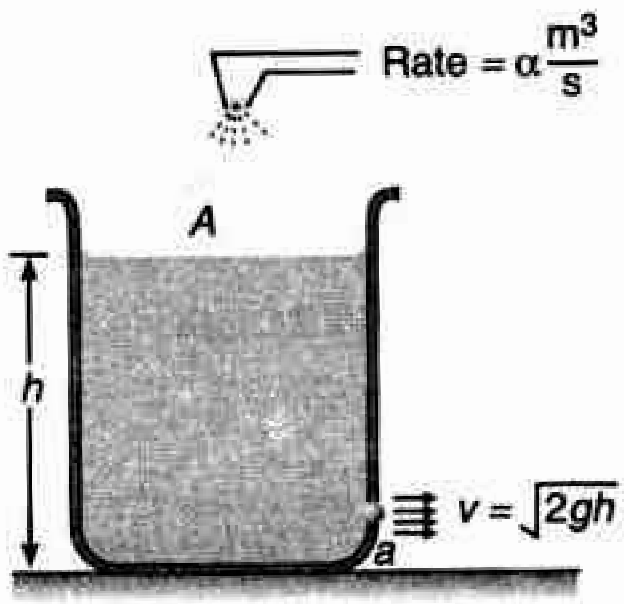


Fig. 13.75

i.e., $$\alpha = av$$

or $$\alpha = a\sqrt{2gh_{max}}$$

$\Rightarrow$ $$h_{max} = \frac{\alpha^2}{2ga^2}$$ **Ans.**

(b) Let at time t, the level of water be h. Then,

$$A\left(\frac{dh}{dt}\right) = \alpha - a\sqrt{2gh}$$

or $$\int_0^h \frac{dh}{\alpha - a\sqrt{2gh}} = \int_0^t \frac{dt}{A}$$

Solving this, we get

$$t = \frac{A}{ag}\left[\frac{\alpha}{a} \ln\left\{\frac{\alpha - a\sqrt{2gh}}{\alpha}\right\} - \sqrt{2gh}\right]$$ **Ans.**

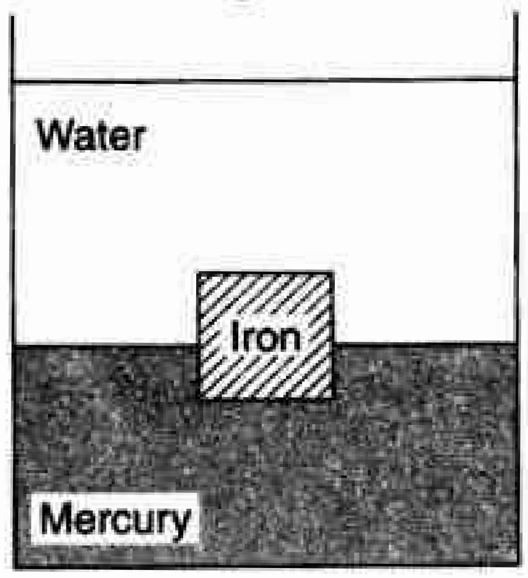
EXERCISES

For JEE Main

Subjective Questions

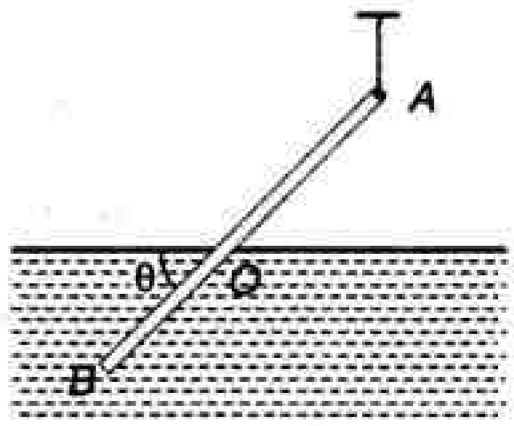
Upthrust

1. A block of material has a density ρ_1 and floats three-fourth submerged in a liquid of unknown density. Show that the density ρ_2 of the unknown liquid is given by $\rho_2 = \frac{4}{3}\rho_1$.
2. A metal ball weighs 0.096 N. When suspended in water it has an apparent weight of 0.071 N. Find the density of the metal.
3. A block of wood has a mass of 25 g. When a 5 g metal piece with a volume of 2 cm^3 is attached to the bottom of the block, the wood barely floats in water. What is the volume V of the wood?
4. A block of wood weighing 71.2 N and of specific gravity 0.75 is tied by a string to the bottom of a tank of water in order to have the block totally immersed. What is the tension in the string?
5. What is the minimum volume of a block of wood (density $= 850\,kg/m^3$) if it is to hold a 50 kg woman entirely above the water when she stands on it ?
6. An irregular piece of metal weighs 10.00 g in air and 8.00 g when submerged in water.
   (a) Find the volume of the metal and its density.
   (b) If the same piece of metal weighs 8.50 g when immersed in a particular oil, what is the density of the oil ?
7. A beaker when partly filled with water has total mass 20.00 g. If a piece of metal with density 3.00 g/cm^3 and volume 1.00 cm^3 is suspended by a thin string, so that it is submerged in the water but does not rest on the bottom of the beaker, how much does the beaker then appear to weigh if it is resting on a scale?
8. A tank contains water on top of mercury. A cube of iron, 60 mm along each edge, is sitting upright in equilibrium in the liquids. Find how much of it is in each liquid. The densities of iron and mercury are $7.7\times10^3\ kg/m^3$ and $13.6\times10^3\ kg/m^3$ respectively.

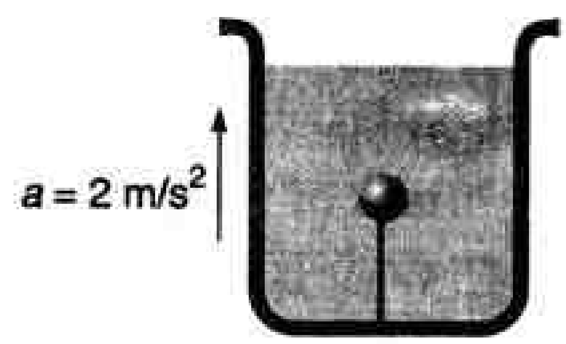


9. A small block of wood, of density 0.4×10^3 kg/m^3, is submerged in water at a depth of 2.9 m. Find :
   (a) the acceleration of the block toward the surface when the block is released and
   (b) the time for the block to reach the surface. Ignore viscosity.

10. A uniform rod AB, 4 m long and weighing 12 kg, is supported at end A, with a 6 kg lead weight at B. The rod floats as shown in figure with one-half of its length submerged. The buoyant force on the lead mass is negligible as it is of negligible volume. Find the tension in the cord and the total volume of the rod.



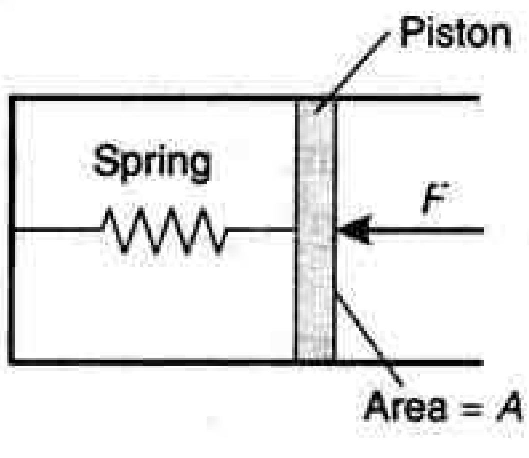
11. A solid sphere of mass $m = 2$ kg and density $\rho = 500\,\text{kg/m}^3$ is held stationary relative to a tank filled with water. The tank is accelerating upward with acceleration 2 m/s^2. Calculate :



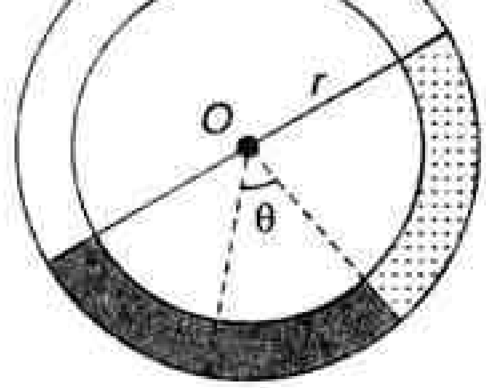
   (a) Tension in the thread connected between the sphere and the bottom of the tank.
   (b) If the thread snaps, calculate the acceleration of sphere with respect to the tank.
   [Density of water $= 1000\,\text{kg/m}^3$, $g = 10\,\text{m/s}^2$]

Pressure and Pascal's Law

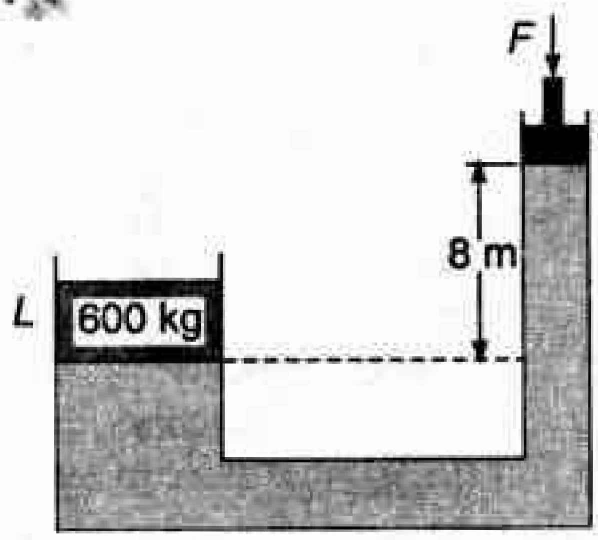
12. The pressure gauge shown in figure has a spring for which $k = 60$ N/m and the area of the piston is $0.50\,\text{cm}^2$. Its right end is connected to a closed container of gas at a gauge pressure of 30 kPa. How far will the spring be compressed if the region containing the spring is (a) in vacuum and (b) open to the atmosphere ? Atmospheric pressure is 101 kPa.



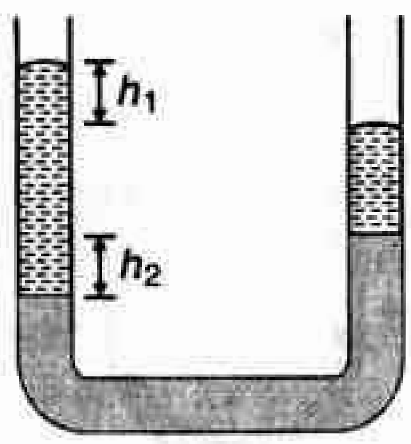
13. A small uniform tube is bent into a circle of radius r whose plane is vertical. Equal volumes of two fluids whose densities are ρ and $\sigma (\rho > \sigma)$ fill half the circle. Find the angle that the radius passing through the interface makes with the vertical.



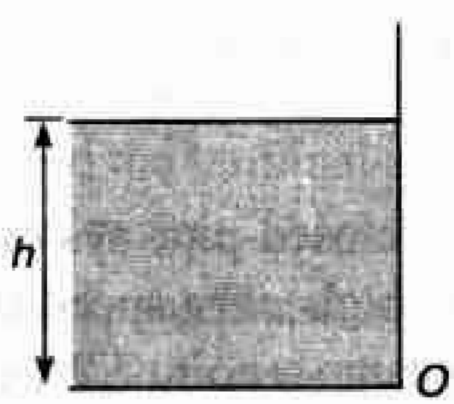
14. For the system shown in figure, the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm^2. The piston on the right, at S, has cross-sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil ($\rho = 0.78$ g/cm^3), what is the force F required to hold the system in equilibrium ?



15. A U-tube of uniform cross-sectional area and open to the atmosphere is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in figure with $h_2 = 1.0$ cm, determine the value of h_1.



16. Water stands at a depth h behind the vertical face of a dam. It exerts a resultant horizontal force on the dam tending to slide it along its foundation and a torque tending to overturn the dam about the point O. Find :



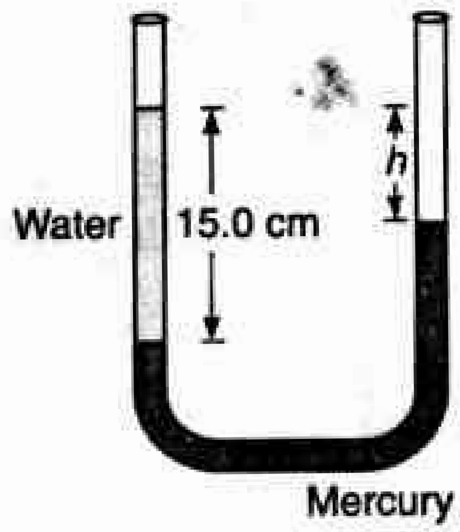
(a) horizontal force,
(b) torque about O,
(c) the height at which the resultant force would have to act to produce the same torque,

l = cross-sectional length and ρ = density of water.

17. A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm.

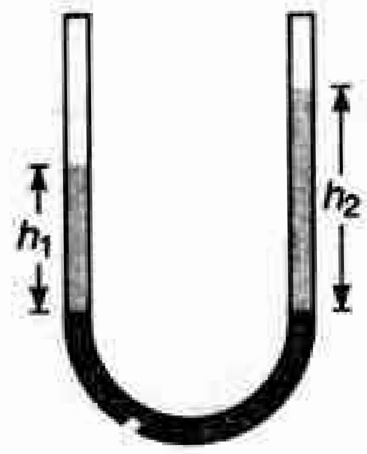
(a) What is the gauge pressure at the water mercury interface ?

(b) Calculate the vertical distance h from the top of the mercury in the right hand arm of the tube to the top of the water in the left-hand arm.



18. Water and oil are poured into the two limbs of a U-tube containing mercury. The interfaces of the mercury and the liquids are at the same height in both limbs.

Determine the height of the water column h_1 if that of the oil $h_2 = 20$ cm. The density of the oil is 0.9.



19. Mercury is poured into a U-tube in which the cross-sectional area of the left-hand limb is three times smaller than that of the right one. The level of the mercury in the narrow limb is a distance $l = 30$ cm from the upper end of the tube. How much will the mercury level rise in the right-hand limb if the left one is filled to the top with water ?

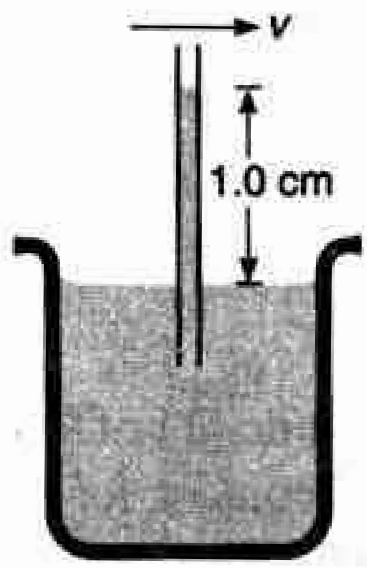
Fluids in Motion

20. Water is flowing smoothly through a closed-pipe system. At one point the speed of the water is 3.0 m/s, while at another point 1.0 m higher the speed is 4.0 m/s. If the pressure is 20 kPa at the lower point, what is the pressure at the upper point ? What would the pressure at the upper point be if the water were to stop flowing and the pressure at the lower point were 18 kPa ?

21. A water barrel stands on a table of height h. If a small hole is punched in the side of the barrel at its base, it is found that the resultant stream of water strikes the ground at a horizontal distance R from the barrel. What is the depth of water in the barrel ?

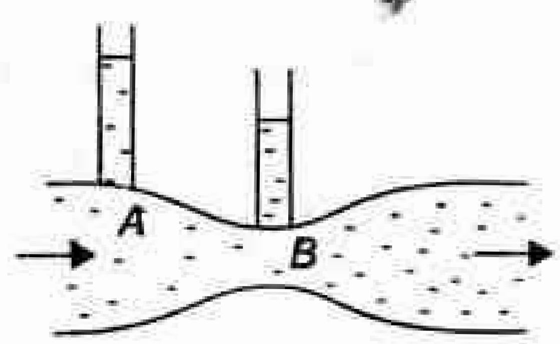
22. A pump is designed as a horizontal cylinder with a piston of area A and an outlet orifice of area a arranged near the cylinder axis. Find the velocity of out flow of the liquid from the pump if the piston moves with a constant velocity under the action of a constant force F. The density of the liquid is ρ.

23. When air of density $1.3\ \text{kg/m}^3$ flows across the top of the tube shown in the accompanying figure, water rises in the tube to a height of 1.0 cm. What is the speed of the air ?

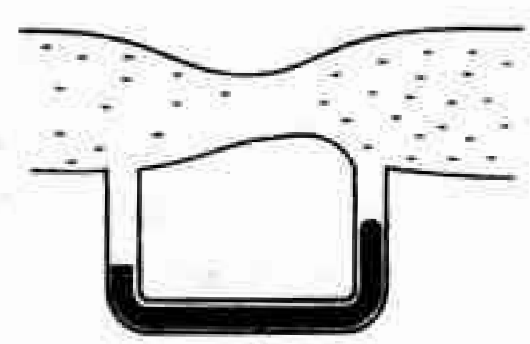


24. The area of cross-section of a large tank is $0.5\ \text{m}^2$. It has an opening near the bottom having area of cross-section $1\ \text{cm}^2$. A load of 20 kg is applied on the water at the top. Find the velocity of the water coming out of the opening at the time when the height of water level is 50 cm above the bottom. (Take $g = 10\ \text{m/s}^2$)

25. Water flows through a horizontal tube as shown in figure . If the difference of heights of water column in the vertical tubes is 2 cm and the areas of cross-section at A and B are $4\ \text{cm}^2$ and $2\ \text{cm}^2$ respectively. Find the rate of flow of water across any section.



26. Water flows through the tube as shown in figure. The areas of cross-section of the wide and the narrow portions of the tube are $5\ \text{cm}^2$ and $2\ \text{cm}^2$ respectively. The rate of flow of water through the tube is $500\ \text{cm}^3/\text{s}$. Find the difference of mercury levels in the U-tube.



Viscosity and Surface Tension

Note *1 Poise* $= 0.1\ N\text{-s/m}^2$ *and 1 Pl* $= 1.0\ \text{N-s/m}^2$.

27. A typical riverborne silt particle has a radius of 20 μm and a density of $2 \times 10^3\ \text{kg/m}^3$. The viscosity of water is 1.0 mPl. Find the terminal speed with which such a particle will settle to the bottom of a motionless volume of water.

28. What is the pressure drop (in mm Hg) in the blood as it passes through a capillary 1 mm long and 2μ m in radius if the speed of the blood through the centre of the capillary is 0.66 mm/s ? (The viscosity of whole blood is 4×10^{-3} Pl).

29. Two equal drops of water are falling through air with a steady velocity v. If the drops coalesced, what will be the new velocity ?

30. A long cylinder of radius R is displaced along its axis with a constant velocity u_0 inside a stationary co-axial cylinder of radius R_2. The space between the cylinders is filled with viscous liquid. Find the velocity of the liquid as a function of the distance r from the axis of the cylinders. The flow is laminar.