MATHEMATICS

STUDY MATERIAL

PROPERTIES AND SOLUTIONS OF TRIANGLES & HEIGHTS AND DISTANCES

AIEEE

श्रमेव जयते

THE NARAYANA GROUP

NARAYANA INSTITUTE OF CORRESPONDENCE COURSES

FNS HOUSE, 63 KALU SARAI MARKET
SARVAPRIYA VIHAR, NEW DELHI-110016
PH.: (011) 32001131/32/50 • FAX : (011) 41828320
Website : www.narayanaicc.com
E-mail : info@narayanaicc.com

PREFACE

Dear Student,

Heartiest congratulations on making up your mind and deciding to be an engineer to serve the society.

As you are planning to take various Engineering Entrance Examinations, we are sure that this **STUDY PACKAGE** is going to be of immense help to you.

At NARAYANA we have taken special care to design this package according to the **Latest Pattern of AIEEE,** which will not only help but also guide you to compete for AIEEE & other State Level Engineering Entrance Examinations.

The salient features of this package include :

- Power packed division of units and chapters in a scientific way, with a correlation being there.
- Sufficient number of solved examples in Physics, Chemistry & Mathematics in all the chapters to motivate the students attempt all the questions.
- All the chapters are followed by various types of exercises (Level-I, Level-II, Level-III and Questions asked in AIEEE and other Engineering Exams).

These exercises are followed by answers in the last section of the chapter. *This package will help you to know* what to study, how to study, time management, your weaknesses and improve your performance.

We, at NARAYANA, strongly believe that quality of our package is such that the students who are not fortunate enough to attend to our Regular Classroom Programs, can still get the best of our quality through these packages.

We feel that there is always a scope for improvement. We would welcome your suggestions & feedback.

Wish you success in your future endeavours.

THE NARAYANA TEAM

ACKNOWLEDGEMENT

While preparing the study package, it has become a wonderful feeling for the NARAYANA TEAM to get the wholehearted support of our Staff Members including our Designers. They have made our job really easy through their untiring efforts and constant help at every stage.

We are thankful to all of them.

THE NARAYANA TEAM

CONTENTS

PROPERTIES AND SOLUTIONS OF TRIANGLES AND HEIGHTS & DISTANCES

PROPERTIES AND SOLUTIONS OF TRIANGLES & HEIGHTS AND DISTANCES

AIEEE Syllabus

Properties of triangles including centroid, incentre, circumcentre and orthocentre, solution of triangles, Heights and distances.

CONTENTS

INTRODUCTION

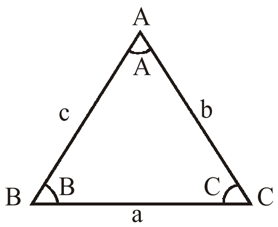
A triangle has three angles and three sides. There are many relations among these six quantities which help in studying the various properties of a triangle. For example, if any three quantities out of three angles and three sides (at least one of which is a side), are given then using some of these relations, the remaining three can be found, which is termed as the solution of the triangle.

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

PROPERTIES OF TRIANGLE

1. ELEMENTS OF A TRIANGLE

In a triangle ABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite to these angles are denoted by small letters a, b and c. Semi perimeter of the triangle is given by s = $\frac{a+b+c}{2}$ and its area is denoted by Δ.



Note : **(i)** A + B + C = π

(ii) a + b > c, b + c > a, c + a > b

2. SINE RULE

In a triangle ABC, the sides are proportional to the sines of the angles opposite to them

i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Illustration 1: In any triangle ABC, prove that a cosA + b cosB + c cosC = 2a sinB sinC

Solution : Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, Then, a = sinA, b = k sinB, c = k sinC

L.H.S. = a cosA + b cosB + c cosC

= k sinA cosA + k sinB cosB + k sinC cosC

= $\frac{k}{2}$ [sin2A + sin2B + sin2C] = $\frac{k}{2}$ (4sinA sinB sinC)

= 2k sinA sinB sinC = 2a sinB sinC = R.H.S. [$\because$ k sinA = a]

3. COSINE RULE

(1) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \, . \cos A$

(2) $\cos B = \frac{c^2+a^2-b^2}{2ca}$ or $b^2 = c^2 + a^2 - 2ac \, . \cos B$

(3) $\cos C = \frac{a^2+b^2-c^2}{2ab}$ or $c^2 = a^2 + b^2 - 2ab \, . \cos C$

Illustration 2: If the lengths of the sides of a triangle are $x^2 + x + 1$, $x^2 - 1$, and $2x + 1$ then find the greatest angle?

Solution : Given the sides $x^2 + x + 1$, $x^2 - 1$, $2x + 1$

putting x 1, we get the values of sides a, b, c as 3, 0, 3 respectively

Similarly putting x = 2, we get 7, 3, 5 respectively

$\therefore$ Greatest side opposite angle is greatest angle

$$\therefore \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{9+25-49}{2(3)(5)} = \frac{-15}{2(15)} = \frac{-1}{2} \quad \therefore \angle A = 120^o$$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

4. PROJECTION FORMULAE

In any ΔABC,

(1) $a = b \cos C + c \cos B$

(2) $b = c \cos A + a \cos C$

(3) $c = a \cos B + b \cos A$

Illustration 3 : ***If $A = 45^0$, $B = 75^0$, prove that $a + c\sqrt{2} = 2b$.***

Solution : As $A = 45^0$, $B = 75^0$ we have $C = 60^0$

$\therefore$ R.H.S. $= 2b = 2(a \cos C + c \cos A) = 2(a \cos 60^0 + c \cos 45^0)$

$= a + c\sqrt{2} =$ L.H.S.

5. NAPIER'S ANALOGY (TANGENT RULE)

In any ΔABC,

(1) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$

(2) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

(3) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

Illustration 4: In ΔABC, if $x = \tan\frac{(B-C)}{2}\tan\frac{A}{2}$, $y = \tan\frac{(C-A)}{2}\tan\frac{B}{2}$, $z = \tan\frac{(A-B)}{2}\tan\frac{C}{2}$

then find x + y + z (in terms of x, y, z) ?

Solution : $x = \frac{b-c}{b+c}$, $y = \frac{c-a}{c+a}$, $z = \frac{a-b}{a+b}$

then put a = 1, b = 2, c = 3

then $x = -\frac{1}{5}$, $y = \frac{1}{2}$, $z = -\frac{1}{3}$

$\therefore \quad x + y + z = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} = \frac{-6+15-10}{30} = -\frac{1}{30}$

$\therefore \quad xyz = -\frac{1}{30}$

6. HALF ANGLE FORMULAE

In any ΔABC,

(1) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

(2) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(3) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$, where $s = \frac{a+b+c}{2}$ and Δ = area of triangle.

Note : $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(4) (i) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

(ii) $\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ca}$

(iii) $\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}$,

Illustration 5: In a ΔABC if cotA/2 cotB/2 = c, cotB/2 cotC/2 = a and cotC/2 cotA/2 = b then $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} = ?$

Solution : cotA/2 cotB/2 = $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}\cdot\frac{s(s-b)}{(s-c)(s-a)}} = c$

$\Rightarrow \frac{s}{s-c} = c \Rightarrow \frac{1}{s-c} = \frac{c}{s}$

Similarly $\frac{1}{s-b} = \frac{b}{s}$, $\frac{1}{s-a} = \frac{a}{s}$

$\therefore \frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} = \frac{a+b+c}{s} = \frac{2s}{s} = 2$

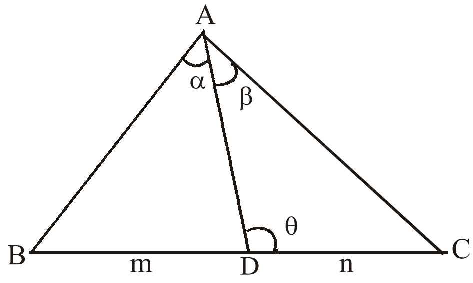
7. m-n THEOREM

Let D be a point on the side BC of a ΔABC such that BD : DC = m : n and $\angle$ADC = θ, $\angle$BAD = α and $\angle$DAC = β (as shown in figure). Then

(1) (m + n) cot θ = m cot α – n cot β

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

(2) $(m + n)\cot\theta = n\cot B - m\cot C$



8. AREA OF TRIANGLE

The area of a triangle ABC is given by

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$$ (Hero's formula)

Illustration 6: If in a triangle ABC, a = 6, b = 3 and $\cos(A-B) = \frac{4}{5}$, find the area of the triangle.

Solution : a = 6, b = 3

$$\cos(A-B) = \frac{4}{5} \Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

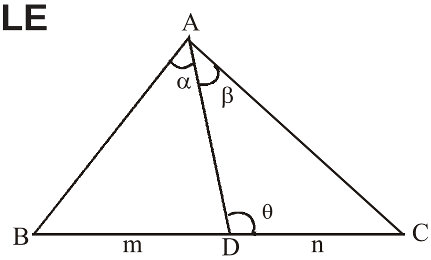
$$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{c}{2}$$

$$\Rightarrow \frac{1}{3} = \frac{6-3}{6+3}\cot\frac{c}{2} \quad \Rightarrow c = \frac{\pi}{2}$$

Area of $\Delta ABC = \frac{1}{2}\times 6\times 3 = 9$ sq.units

9. CENTROID AND MEDIANS OF A TRIANGLE

The line joining any vertex of a triangle to the mid point of the opposite side of the triangle is called the median of the triangle. The three medians of a triangle are concurrent and the point of concurrency of the medians of any triangle is called the centroid of the triangle. The centroid divides the median in the ratio 2 : 1.



The lengths of the medians through A, B and C respectively are given by

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \text{ and } m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

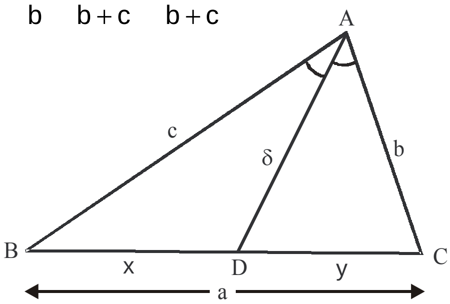
Note : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

10. BISECTORS OF THE ANGLES

If AD bisects the angle A and divide the base into portions x and y, we have, by Geometry,

$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b} \qquad \therefore \quad \frac{x}{c} = \frac{y}{b} = \frac{x+y}{b+c} = \frac{a}{b+c}$$



$$\Rightarrow \quad x = \frac{ac}{b+c} \text{ and } y = \frac{ab}{b+c}$$

Also let δ be the length of AD

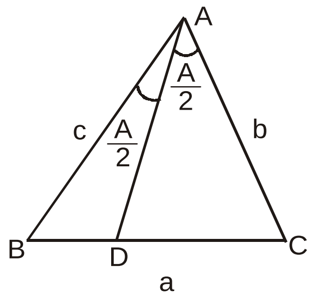
we have Δ ABD + Δ ACD = Δ ABC

$$\therefore \quad \frac{1}{2}c\delta\sin\frac{A}{2} + \frac{1}{2}b\delta\sin\frac{A}{2} = \frac{1}{2}bc\sin A,$$

i.e., $AD = \delta = \dfrac{bc}{b+c}\dfrac{\sin A}{\sin\frac{A}{2}} = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

Illustration 7: If the bisector of angle A of the triangle ABC makes an angle θ with BC, then $\sin\theta = ?$

Solution : We have $BD = \dfrac{C}{b+c}\cdot a$, $DC = \dfrac{b}{b+c}\cdot a$



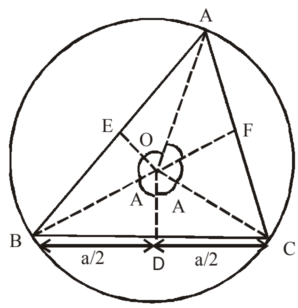
From triangle ADC, $\dfrac{\sin\theta}{b} = \dfrac{\sin A/2}{\frac{ba}{b+c}}$

$$\Rightarrow \quad \sin\theta = \frac{b+c}{a}\sin A/2 = \frac{\sin B + \sin C}{\sin A}\cdot \sin A/2$$

$$= \frac{2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}{2\sin A/2\cos A/2}\cdot \sin A/2 = \cos\frac{(B-C)}{2}$$

11. CIRCUM CIRCLE

The circle which passes through the angular points of a Δ ABC, is called its circumcircle. The centre of this circle i.e., the point of concurrency of the perpendicular bisectors of the sides of the Δ ABC, is called the circumcentre.



Radius (R) of the circumcircle is given by the following formulae

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

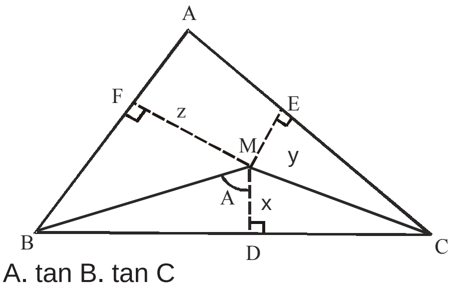
Illustration 8 : If the distances of the sides of a Δ ABC from its circumcentre be x, y and z respectively, then prove that $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = \dfrac{abc}{4xyz}$.

Solution : Let M be the circumcentre. MD $\perp$ BC. So BD = DC = $\dfrac{a}{2}$ and $\angle$BMD = A.

In Δ BDM, $\dfrac{BD}{MD}$ = tan A or $\dfrac{\frac{a}{2}}{x}$ = tan A, i.e., $\dfrac{a}{2x}$ = tan A,

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Similarly, $\frac{b}{2y} = \tan B$, $\frac{c}{2z} = \tan C$



$\therefore \quad \tan A + \tan B + \tan C = \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z}$

and $\quad \tan A.\ \tan B.\ \tan C = \frac{a}{2x}.\frac{b}{2y}.\frac{c}{2z}$

But in a triangle ABC, $\quad \tan A + \tan B + \tan C = \tan A.\ \tan B.\ \tan C$

$\therefore \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.

12. INCIRCLE

The circle which can be inscribed within the triangle so as to touch each of the sides of the triangle is called its incircle. The centre of this circle i.e., the point of concurrency of angle bisectors of the triangle is called the incentre of the Δ ABC.

Radius of the Incircle is given by the following formulae

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

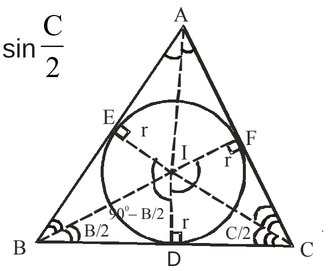


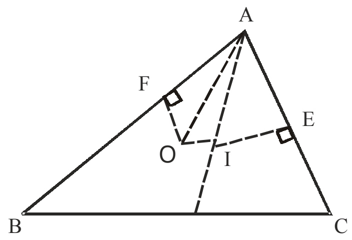
Illustration 9 : Find the distance between the circumcentre and the incentre.

Solution : Let O be the circumcentre and I be the incentre of Δ ABC.
Let OF be perpendicular to AB and IE be perpendicular to AC.

$\angle OAF = 90^0 - C.$

$\therefore \quad \angle OAI = \angle IAF - \angle OAF$

$= \frac{A}{2} - \left(90^0 - C\right) = \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}$



Also, $\quad AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore \quad OI^2 = OA^2 + AI^2 - 2OA.\ AI\cos OAI$

$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$

$\therefore \quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$

$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right) = 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$ ...(i)

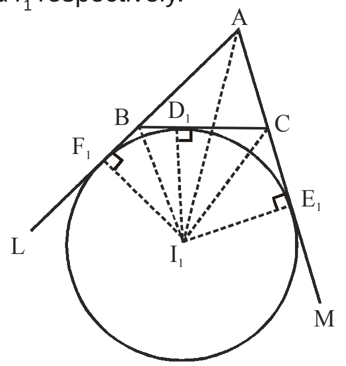
$\therefore \quad OI = R\sqrt{1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}}$.

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

13. ESCRIBED CIRCLES

The circle which touches the side BC and the two sides AB and AC produced is called the escribed circle opposite the angle A. Its centre and radius will be denoted by I_1 and r_1 respectively.

Radii of the excircles are given by the following formuale



(1) $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{a\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}}$

(2) $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{b\cos\dfrac{A}{2}\cos\dfrac{C}{2}}{\cos\dfrac{B}{2}}$

(3) $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{c\cos\dfrac{A}{2}\cos\dfrac{B}{2}}{\cos\dfrac{C}{2}}$

Illustration 10: If d_1, d_2, d_3 are the diameters of the three escribed circles of a triangle ABC, then $d_1d_2 + d_2d_3 + d_3d_1 = ?$

Solution : $d_1d_2 + d_2d_3 + d_3d_1 = 4(r_1r_2 + r_2r_3 + r_3r_1) = 4s^2 = (2s)^2 = (a + b + c)^2$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

SOLUTION OF TRIANGLES

When any three of the six elements (except all the three angles) of a triangle are given, the triangle is known completely. This process is called the solution of triangles.

(i) If the sides a, b and c are given, then $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. B and C can be obtained in the similar way.

(ii) If two sides b and c and the included angle A are given, then using

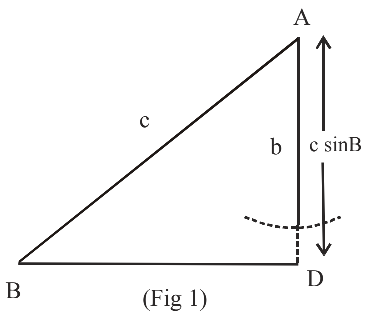
$\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2}$, we get $\left(\dfrac{B-C}{2}\right)$.

Also $\left(\dfrac{B+C}{2}\right) = 90^0 - \dfrac{A}{2}$, so that B and C can be evaluated.

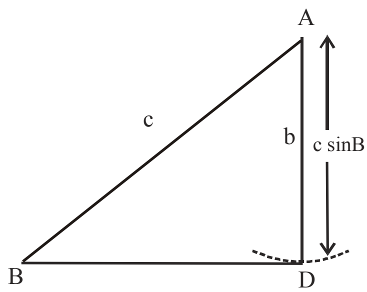
The third side is given by $a = \dfrac{b\sin A}{\sin B}$.

(iii) If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \dfrac{c}{b}\sin B$,

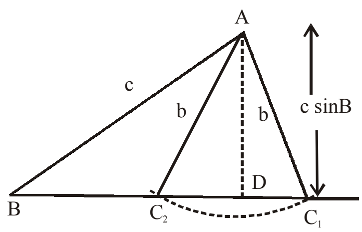
$A = 180^0 - (B + C)$ and $a = \dfrac{b\sin A}{\sin B}$ give the remaining elements. If $b < c \sin B$, there is no triangle possible (fig 1). If $b = c \sin B$ and B is an acute angle, then there is only one triangle possible (fig 2). If $c \sin B < b < c$ and B is an acute angle, then there are two values of angle C (fig 3). If $c < b$ and B is an acute angle, then there is only one triangle (fig 4).



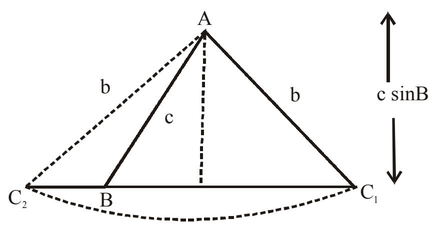
(Fig 1)



(Fig 2)



(Fig 3)



(Fig 4)

This case is, sometimes, called an ambiguous case.

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Illustration 11 : ***In any triangle ABC, the sides are 6 cm, 10 cm and 14 cm. Show that the triangle is obtuse-angled with the obtuse angle equal to 120⁰.***

Solution : Let a = 14, b = 10, c = 6

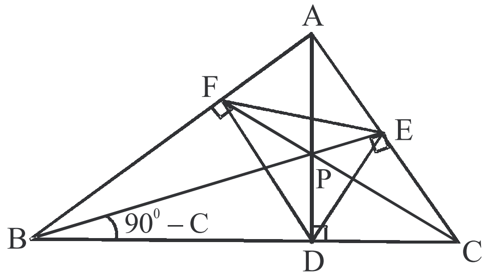
$\Rightarrow$ The largest angle is opposite the largest side.

$\therefore$ $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{100 + 36 - 196}{120} = -\dfrac{1}{2} \Rightarrow A = 120^o$

14. ORTHOCENTRE AND PEDAL TRIANGLE OF A TRIANGLE

In a triangle the altitudes drawn from the three vertices to the opposite sides are concurrent and the point of cuncurrency of the altitudes of the triangle is called the orthocentre of the triangle. The triangle formed by joining the feet of these perpendiculars is called the pedal triangle i.e.

Δ DEF is the pedal triangle of Δ ABC.



The triangle DEF which is formed by joining the feet of the altitudes is called the pedal triangle.

(1) The distances of the orthocentre from the angular points of the ΔABC are
2 R cos A , 2 R cos B and 2 R cos C

(2) The distances of P from sides are 2 R cos B cos C, 2 R cos C cos A and
2 R cos A cos B

(3) The sides of the pedal triangle are a cos A (= R sin 2 A), b cos B (= R sin 2 B) and c cos C (= R sin 2 C) and its angles are π – 2A, π – 2B and π – 2C.

(4) Circumradii of the triangles PBC, PCA, PAB and ABC are all equal.

Illustration 12 : Find the distance of the orthocentre from the sides and angular points of a triangle ABC.

Solution : PD = DB tan $\angle$PBD = DB tan (90^0 – C)

$= AB \cos B \cot C = \dfrac{c}{\sin C} \cos B \cos C = 2R \cos B \cos C$

Similarly,

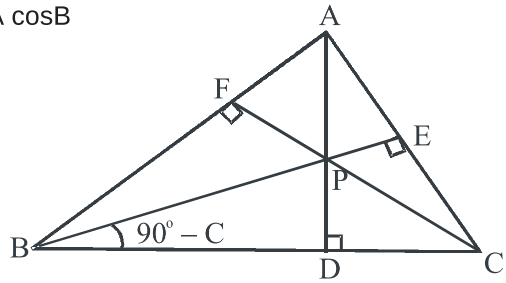
PE = 2R cosA cosC and PF = 2R cosA cosB

Again,

AP = AE sec DAC = c cos A cosec C

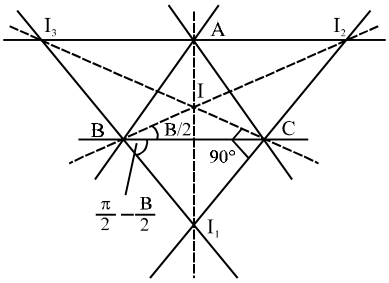
$= \dfrac{c}{\sin C} \cos A = 2 R \cos A$

So, BP = 2R cos B and CP = 2R cos C



FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

15. EXCENTRAL TRIANGLE



The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

Note that :

(1) Incentre I of ΔABC is the orthocenter of the excentral $\Delta I_1I_2I_3$.

(2) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(3) the sides of the excentral triangle are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$ and its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(4) $II_1 = 4R\sin\frac{A}{2}$: $II_2 = 4R\sin\frac{B}{2}$: $II_3 = 4R\sin\frac{C}{2}$

16. THE DISTANCES BETWEEN THE SPECIAL POINTS

(1) The distance between circumcenter and orthocenter is = R. $\sqrt{1-8\cos A\cos B\cos C}$

(2) The distance between circumcenter and incentre is = $\sqrt{R^2-2Rr}$.

(3) The distance between incentre and orthocenter is $\sqrt{2r^2-4R^2\cos A\cos B\cos C}$.

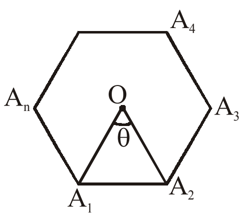
17. RESULTS RELATED WITH POLYGONS

Some geometrical results related with polygons are as follows:

(1) In a regular polygon, all sides are equal.

(2) Number of vertices of a polygon = number of sides of the polygon. i.e. if a polygon is of n sides, number of its vertices will be n.

(3) Let $A_1A_2A_3...A_n$ be a regular polygon of n sides and let O be the centre of this polygon.
Clearly, this O will also be the centre of the circumscribing circle to this polygon.



Now, $\angle A_1OA_2 = \angle A_2OA_3 = \angle A_3OA_4 = ... = \angle A_nOA_1 = \theta$ (suppose).

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Then, $n \cdot \theta = 2\pi \Rightarrow \theta = \frac{2\pi}{n}$.

i.e., angle subtended by each side of a regular polygon of n sides at its centre $= \frac{2\pi}{n}$.

(4) Sum of interior angles of the polygon $+2\pi = n\pi$

$\Rightarrow$ sum of Interior angles of the polygon of n sides

$$= (n-2)\pi = (2n-4)\frac{\pi}{2}$$

(5) Perimeter (P) and area (A) of a regular polygon of n sides **inscribed** in a circle of radius r are given by

$$P = 2nr\sin\frac{\pi}{n}$$

$$A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$$

Perimeter and area of a regular polygen of n sides **circumscribed** about a given circle of radius r is given by

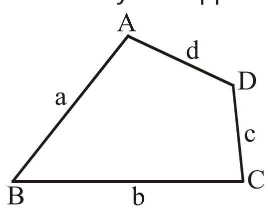
$$P = 2nr\tan\frac{\pi}{n}$$

$$A = nr^2\tan\frac{\pi}{n}$$

(6) If a, b, c, d be the sides and s the semiperimeter of a cyclic quadrilateral, then its area

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Note : If we have any quadrilateral, not necessarily inscribable in a circle, we can express its area in terms of its sides and the sum of any two opposite angles.



For let the sum of the two angles B and D be denoted by 2α, then the area of the quadrilateral

$$= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$$

18. TIPS AND TRICKS

(1) In an equilateral triangle

a = b = c = 1 and $\angle A = \angle B = \angle C = 60^o$

(i) $S = \frac{a+b+c}{2} = \frac{3}{2}$ (ii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{\sqrt{3}}{4}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

(iii) $r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}}{\frac{3}{2}} = \frac{1}{2\sqrt{3}}$ (iv) $R = \frac{abc}{4\Delta} = \frac{1.1.1}{4.\frac{\sqrt{3}}{4}} = \frac{1}{\sqrt{3}}$

(v) $r_1 = \frac{\Delta}{s-a} = \frac{\frac{\sqrt{3}}{4}}{1/2} = \frac{\sqrt{3}}{2}$ $\therefore$ $r_1 = r_2 = r_3 = \frac{\sqrt{3}}{2}$

$\therefore$ $h = p_1 = p_2 = p_3 = \frac{\sqrt{3}}{2}$ ($\because$ p_1, p_2, p_3 are the lengths of the attitudes of a triangle)

(vi) $r : R : r_1 = 1 : 2 : 3$

(2) In a right angled triangle, right angled at A, and a = 5, b = 4, c = 3 then

(i) $s = \frac{5+4+3}{2} = 6$ (ii) $\Delta = \frac{1}{2}(3)(4) = 6$

(iii) $r = \frac{\Delta}{s} = 1$ (iv) $R = \frac{abc}{4\Delta} = \frac{(5)(4)(3)}{4(6)} = \frac{5}{2}$

(v) $r_1 = \frac{\Delta}{s-a} = \frac{6}{1} = 6$ (vi) $r_2 = \frac{\Delta}{s-b} = \frac{6}{2} = 3$

(vii) $r_3 = \frac{\Delta}{s-c} = \frac{6}{3} = 2$ (viii) $\tan A/2 = \frac{\Delta}{s(s-a)} = \frac{6}{6(1)} = 1$

(ix) $\tan B/2 = \frac{\Delta}{s(s-b)} = \frac{6}{6(2)} = 1/2$ (x) $\tan C/2 = \frac{\Delta}{s(s-c)} = \frac{6}{6(3)} = 1/3$

(3) (i) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ (ii) $\Delta^2 = r r_1 r_2 r_3$

(4) If $\frac{a}{b+c} + \frac{b}{c+a} = 1$ then C = 60° when a = b = c = 1

19. OTHER IMPORTANT RESULTS :

(1) If $\frac{1}{a+b} + \frac{1}{a+c} = \frac{3}{a+b+c}$ then $\angle A = 60$

(2) If $\frac{b}{a^2-c^2} + \frac{c}{a^2-b^2} = 0$ then $A = 60^o$

(3) If (a + b + c) (b + c – a) = 3bc then $\angle A = 60$

(In the right hand side missing letter is 'a' and the corresponding side is 60°)

(4) (i) If $rr_1 = r_2r_3$ then $\angle A = 90$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

(ii) If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$ then $\angle A = 90$

(iii) If $(r_2 - r_1)(r_3 - r_1) = 2\, r_2\, r_3$ then $\angle A = 90$

(iv) $r_2 + r_3 + r - r_1 = 0$ then $\angle A = 90$

(v) If $R + r = r_1$ then $A = 90^o$

(vi) If $r : R : r_1 = 2 : 5 : 12$ then $\angle A = 90$

(a) The above all missing OR repeated letter that corresponding side is 90^o

(b) If cotA/2 : cotB/2 : cotC/2 = x : y : z then a : b : c = y + z : z + x : x + y

(c) If $x\, r_1 = y\, r_2 = z\, r_3$ then a : b : c = y + z : z + x : x + y

(d) $(r_1 - r)(r_2 + r_3) = a^2$

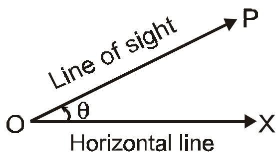
(e) $r_1 + r_2 + r_3 - r = 4R$

(f) $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

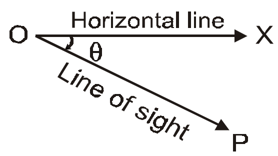
HEIGHTS AND DISTANCES

(1) Let '*O*' be the observer's eye and *OX* be the horizontal line through *O*.

(2) If the object *P* is at a higher level than *O*, then angle *POX* (= θ) is called the ***angle of elevation*** .



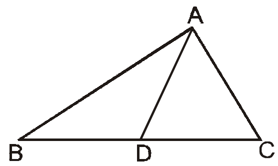
(3) If the object *P* is at a lower level than *O*, then angle *POX* is called the ***angle of depression***.



20. SOME USEFUL RESULTS :

In a triangle ABC,

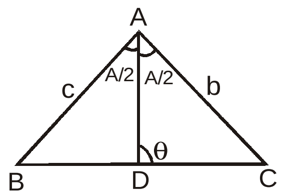
(1) If AD is median, then $AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

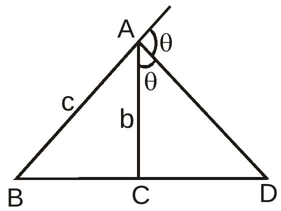


FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

(2) If AD is the angle bisector of $\angle BAC$, or if AD is the external angle bisector of $\angle$A then

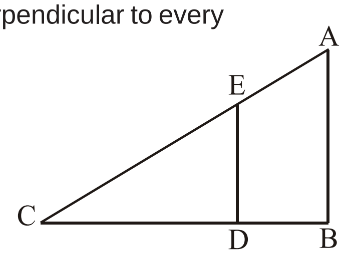
$$\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$$



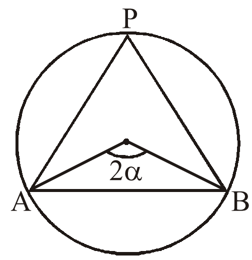


(3) If a line is perpendicular to a plane, then it is perpendicular to every line lying in that plane .

(4) If DE || AB, then $\frac{AB}{DE}=\frac{BC}{DC}=\frac{AC}{EC}$

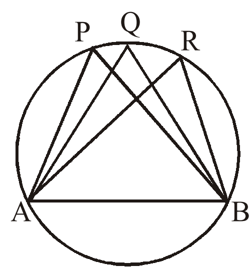


(5) The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.

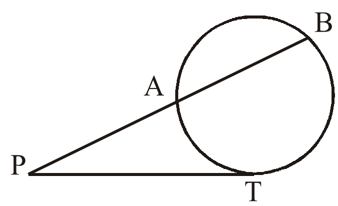


(6) Angles in the same segment of a circle are equal

i.e. $\angle$APB = $\angle$AQB = $\angle$ARB



(7) PAB is a secant of a circle and PT is tangent then PA.PB = PT^2



SOLVED EXAMPLES

Example - 1 In a ΔABC, if cotA/2 : cotB/2 : cotC/2 = 3 : 5 : 7 then a : b : c =

(1) 4 : 5 : 6 (2) 5 : 6 : 4
(3) 6 : 5 : 4 (4) 6 : 4 : 5

Solution : Ans. (3)

$$\frac{s(s-a)}{\Delta}:\frac{s(s-b)}{\Delta}:\frac{s(s-c)}{\Delta}=3:5:7$$

$\Rightarrow$ s – a : s – b : s – c = 3 : 5 : 7

$\Rightarrow$ s – a = 3k, s – b = 5k, s – c = 7k

$\Rightarrow$ s – a + s – b = 8k $\Rightarrow$ c = 8k

$\Rightarrow$ s – b + s – c = 12k $\Rightarrow$ a = 12k

$\Rightarrow$ s – c + s – a = 10k $\Rightarrow$ b = 10k

$\therefore$ a : b : c = 12k : 10k : 8k = 6 : 5 : 4

(OR)

If cotA/2 : cotB/2 : cotC/2 = x : y : z then a : b : c = y + z : z + x : x + y

= 12 : 10 : 8 = 6 : 5 : 4

Example - 2 In a ΔABC, if $\angle C = 60^o$, then $\frac{a}{b+c}+\frac{b}{c+a}=$

(1) 2 (2) 1
(3) 4 (4) 3

Solution : Ans. (2)

In an equilateral triangle a = b = c = 1 and $\angle A = \angle B = \angle C = 60$

$$\therefore \quad \frac{a}{b+c}+\frac{b}{c+a}=\frac{1}{2}+\frac{1}{2}=1$$

(OR)

$$\frac{a}{b+c}+\frac{b}{c+a}=\frac{ac+a^2+b^2+bc}{bc+ab+c^2+ac}=\frac{ac+bc+c^2+ab}{bc+ab+c^2+ac}=1$$

If $\angle C = 60$ then $c^2 = a^2 + b^2 - 2ab\cos c = a^2 + b^2 - 2ab\cos 60^o$

$c^2 = a^2 + b^2 - ab$

$\therefore$ $c^2 + ab = a^2 + b^2$

Example - 3 If n, n + 1, n + 2, where n is any natural number, represents the sides of a triangle ABC in which the largest angle is twice the smallest, then n =

(1) 1 (2) 2
(3) 3 (4) 4

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Solution : **Ans. (4)**

On verification if n = 4, the sides are 4, 5, 6

$\Rightarrow$ C = 2A

$\therefore$ cosC = cos2A = 2 cos²A – 1 = $2\left[\frac{25+36-16}{2.5.6}\right]^2 - 1 = 2\left(\frac{9}{16}\right) - 1 = \frac{1}{8}$

$\therefore$ $\cos C = \frac{16+25-36}{2(4)(5)} = \frac{1}{8}$

$\therefore$ $\boxed{C = 2A}$

Example - 4 If in a triangle ABC, 5 cosC + 6 cosB = 4 and 6 cosA + 4 cosC = 5 then tanA/2 tanB/2 =

(1) 2/3 (2) 3/2

(3) 1/5 (4) 5

Solution : **Ans. (3)**

Adding the given relations we get

9 = 9 cosC + 6 (cosA + cosB)

$\Rightarrow$ $9(1 - \cos C) = 6\ .\ 2\cos\frac{(A+B)}{2}\cos\frac{(A-B)}{2}$

$\Rightarrow$ $9.2\sin^2 C/2 = 12\sin C/2\cos\frac{(A-B)}{2}$ $\Rightarrow$ $\frac{\cos\frac{(A+B)}{2}}{\cos\frac{(A-B)}{2}} = \frac{2}{3}$

$\Rightarrow$ $\frac{\cos A/2\cos B/2 - \sin A/2\sin B/2}{\cos A/2\cos B/2 + \sin A/2\sin B/2} = \frac{2}{3}$ $\Rightarrow$ $\frac{1 - \tan A/2\tan B/2}{1 + \tan A/2\tan B/2} = \frac{2}{3}$

$\Rightarrow$ 3 – 3 tanA/2 tanB/2 = 2 + 2 tanA/2 tanB/2

$\Rightarrow$ 1 = 5 tanA/2 tanB/2 $\Rightarrow$ tanA/2 tanB/2 = 1/5

Example - 5 The perimeter of a triangle right angled at 'C' is 70 and the in radius '6' then |a – b| =

(1) 1 (2) 2

(3) 8 (4) 9

Solution : **Ans. (1)**

We know that $\Delta = sr = \frac{70}{2}\times 6 = 210$

$\Rightarrow$ $\frac{1}{2}ab = 210$ ($\because$ sinC = sin90° = 1, C = 90°)

$\Rightarrow$ ab = 420

Now a + b + c = 70

$\Rightarrow$ a + b = 70 – c $\Rightarrow$ $(a + b)^2 = (70 - c)^2$

$\Rightarrow$ $a^2 + b^2 + 2ab = 4900 - 140c + c^2$ ($\because$ $a^2 + b^2 = c^2$)

$\Rightarrow$ 140c = 4900 – 2(420) = 4060

c = 29

$\Rightarrow$ $(a - b)^2 = a^2 + b^2 - 2ab = c^2 - 2ab = 841 - 840 = 1$

$\therefore$ |a – b| = 1

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Example - 6 If the area of ΔABC is $a^2 - (b-c)^2$ then its circumradius R =

(1) $\frac{a}{6}\sin^2 A/2$ (2) $\frac{a}{16}\operatorname{cosec}^2 A/2$

(3) $\frac{b}{16}\sin^2 B/2$ (4) $\frac{c}{16}\sin^2 C/2$

Solution : **Ans. (2)**

$\Delta = a^2 - (b-c)^2 = a^2 - b^2 - c^2 + 2bc$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-\Delta}{2bc}$ $\Rightarrow \cos A = 1 - \frac{\Delta}{2bc} = 1 - \frac{abc}{8Rbc} = 1 - \frac{a}{8R}$

$\Rightarrow 8R = \frac{a}{1-\cos A} = \frac{a}{2\sin^2 A/2}$

$\therefore R = \frac{a}{16}\operatorname{cosec}^2 A/2$

Example - 7 If in a triangle ABC, $r_1 = 2r_2 = 3r_3$; D is the midpoint of BC then $\cos\angle ADC =$

(1) 7/25 (2) –7/25

(3) 24/25 (4) –24/25

Solution : **Ans. (2)**

$r_1 = 2r_2 = 3r_3 \Rightarrow a : b : c = 5 : 4 : 3$

$\therefore$ a = 5k, b = 4k, c = 3k

$\Rightarrow a^2 = b^2 + c^2 = 25k^2$

$\Rightarrow$ ABC is a right angled triangle with $\angle A = 90$ since D is the midpoint of BC, the hypotenuse, AD = DC

$\Rightarrow \angle DAC = \angle C$ and $\cos\angle ADC = \cos(180 - 2C) = -\cos 2C = 1 - 2\cos^2 C$

$= 1 - 2\frac{b^2}{a^2} = 1 - 2\left(\frac{16}{25}\right) = -7/25$

Example - 8 In a triangle ABC, the sides a, b, c are respectively 13, 14, 15. If r_1 is the radius of the escribed circle touching BC and the sides AB and AC produced then r_1 is equal to

(1) 21/2 (2) 12

(3) 14 (4) 4

Solution : **Ans. (1)**

$s = \frac{1}{2}(13+14+15) = 21$

$\Delta^2 = s(s-a)(s-b)(s-c)$

$= 21 \times 8 \times 7 \times 6 = 21^2 \times 4^2 = 21 \times 4$

$\Delta = 84$

$\therefore r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = \frac{21}{2}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

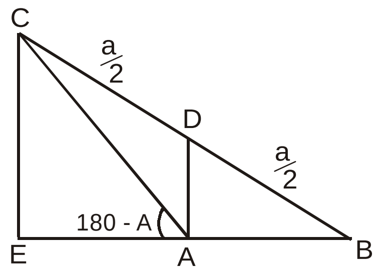
Example - 9 If the median of the triangle ABC through A is perpendicular to AB, then tanA + 2tanB =

(1) tanc (2) sinc

(3) cosc (4) 0

Solution : **Ans. (4)**

Let CE be perpendicular to BA produced, then from similar triangles BAD and BEC, we find EC = 2AD and EA = AB = C



$\tan B = \frac{AD}{C}$ (from triangle ABD)

$\Rightarrow \quad \tan(\pi - A) = \frac{CE}{EA} = \frac{2AD}{C} \Rightarrow \tan A = -\frac{2AD}{C}$ so that tanA + 2tanB = 0

Example - 10 If r = 1, R = 4, $\Delta = 8$ then ab + bc + ca =

(1) 73 (2) 81

(3) 84 (4) 78

Solution : **Ans. (2)**

We know that $r_1 + r_2 + r_3 - r = 4R$

$\Rightarrow \quad r_1 + r_2 + r_3 = 4R + r = 17$

We know that $r = \frac{\Delta}{s} \Rightarrow s = \frac{\Delta}{r} = \frac{8}{1} = 8$

We know that $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

$\Rightarrow \quad 17 = ab + bc + ca - 64 \quad \Rightarrow \quad ab + bc + ca = 81$

Example - 11 If 4r = 3R then cosA + cosB + cosC =

(1) $\frac{1}{4}$ (2) $\frac{3}{4}$

(3) $\frac{5}{4}$ (4) $\frac{7}{4}$

Solution : **Ans. (4)**

$\cos A + \cos B + \cos C = 2\cos\frac{(A+B)}{2}\cos\frac{(A-B)}{2} + 1 - 2\sin^2\frac{C}{2}$

$= 2\sin C/2\cos\frac{(A-B)}{2} + 1 - 2\sin^2 C/2$

$= 1 + 2\sin C/2\left[\cos\frac{(A-B)}{2} - \cos\frac{(A+B)}{2}\right]$ = 1 + 4 sinA/2 sinB/2 sinC/2

$= 1 + \frac{4R\sin A/2\sin B/2\sin C/2}{R} = 1 + \frac{r}{R} = 1 + \frac{3}{4} = \frac{7}{4}$ $\left(\because \frac{r}{R} = \frac{3}{4}\right)$

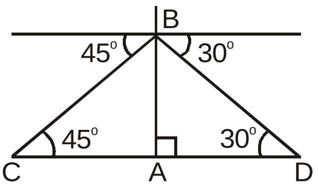
FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Example - 12 An observer on the top of a cliff 200 m above the sea level observes the angle of depression of two ships on opposite sides of the cliff to be 45° and 30° respectively. The line joining the ships points to the base of the cliff. The distance between the ships is

(1) 200 m (2) 546.4 m
(3) 346.4 m (4) 946.4 m

Solution : **Ans. (2)**

Let AB be the cliff and C, D be the positions of the ships, then AB = 200 m



$\angle ACB = 45^o$ and $\angle ADB = 30^o$

$\therefore \quad \frac{AC}{AB} = \cot 45^o \quad \Rightarrow \quad AC = 200$ m

Also $\frac{AD}{AB} = \cot 30^o \quad \Rightarrow \quad AD = 200\sqrt{3} = 346.4$ m

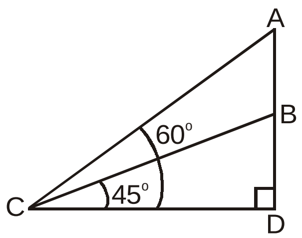
Distance between the ships = CD = AC + AD = 546.4 m

Example - 13 An aeroplane when 600 m high passes vertically above another aeroplane at an instant when their angles of elevation at the same observing point are 60° and 45° respectively. The difference of the heights of the two planes is

(1) 346.4 m (2) 600 m
(3) 100 m (4) 253.6 m

Solution : **Ans. (4)**

Let A be the position of the aeroplane moving 600 m high from the horizontal line CD and let B be the position of another plane at that instant.



Let C be the observing point

Then $\angle DCB = 45^o$ and $\angle DCA = 60^o$

$\frac{CD}{AD} = \cot 60^o = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad CD = \frac{600}{\sqrt{3}}$

Also $\frac{CD}{BD} = \cot 45^o \quad \Rightarrow \quad CD = BD$

$\therefore \quad BD = \frac{600}{\sqrt{3}}$

$\therefore \quad AB = AD - BD = 600 - 346.4 = 253.6$ m

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

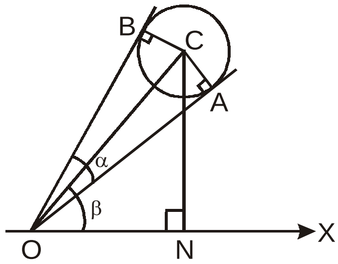
Example - 14 A spherical balloon whose radius is 4 cm, subtends an angle α at the observer's eye when the angular elevation of the centre is β. The height of the centre of the balloon is

(1) $r \sin\beta$ (2) $r \operatorname{cosec}\frac{\alpha}{2}$

(3) r (4) $r \operatorname{cosec}\frac{\alpha}{2}\sin\beta$

Solution : **Ans. (4)**

Let C be the centre of the balloon and O be the position of the observer at the horizontal line OX. Let OA and OB be the tangents to the balloon



Then, $\angle AOB = \alpha$ and $\angle XOC = \beta$ and CA = CB = r

$$\angle AOC = \angle BOC = \frac{\alpha}{2}$$

$$CN \perp OX$$

$$\frac{OC}{CA} = \operatorname{cosec}\frac{\alpha}{2}$$

$\therefore$ $OC = r \operatorname{cosec} \alpha/2$

$$\frac{CN}{OC} = \sin\beta \Rightarrow CN = OC\ \sin\beta = r\operatorname{cosec}\frac{\alpha}{2}\sin\beta$$

Example - 15 Sides of a triangle ABC are in A.P. If a < min.{b, c} then cosA is equal to

(1) $\frac{3c-4b}{2b}$ (2) $\frac{3c-4b}{2c}$

(3) $\frac{4c-3b}{2b}$ (4) $\frac{4c-3b}{2c}$

Solution : **Ans. (4)**

Since sides of a triangle ABC are in A.P. and a < min.{b, c}middle term of the A.P. is either b or c

Case(i) when 2c = a + b

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-(2c-b)^2}{2bc} = \frac{4bc-3c^2}{2bc} = \frac{4b-3c}{2b}$$

Case (ii) 2b = a + c

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-(2b-c)^2}{2bc} = \frac{4bc-3b^2}{2bc} = \frac{4c-3b}{2c}$$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

Example - 16 If a, b, c be the side lengths of triangle ABC then the value of the expression

$$\left(\frac{b+c}{a}-1\right)^a\left(\frac{c+a}{b}-1\right)^b\left(\frac{a+b}{c}-1\right)^c \text{ is always}$$

(1) ≤ 1 (2) ≥ 1

(3) ≤ 2 (4) ≥ 2

Solution : **Ans. (1)**

$$\left(\frac{b+c}{a}-1\right)^a\left(\frac{c+a}{b}-1\right)^b\left(\frac{a+b}{c}-1\right)^c$$

$$= \left(\frac{b+c-a}{a}\right)^a\left(\frac{c+a-b}{b}\right)^b\left(\frac{a+b-c}{c}\right)^c$$

Using weighted A.M. and G.M. inequality we get.

$$\frac{a.\left(\frac{b+c-a}{a}\right)+b.\left(\frac{c+a-b}{b}\right)+c.\left(\frac{a+b-c}{c}\right)}{a+b+c}$$

$$\geq \left[\left(\frac{b+c-a}{a}\right)^a\left(\frac{c+a-b}{b}\right)^b\left(\frac{a+b-c}{c}\right)^c\right]^{1/a+b+c}$$

$$\Rightarrow \quad \left(\frac{b+c}{a}-1\right)^a\left(\frac{c+a}{b}-1\right)^b\left(\frac{a+b}{c}-1\right)^a \leq 1$$

Example - 17 If the length of medians AA_1, BB_1 and CC_1 of triangle ABC are m_a, m_b,m_c respectively, then

(1) $\sum m_a > \frac{3}{2}s$ (2) $\sum m_a > 3s$

(3) $\sum m_a > \frac{5}{2}s$ (4) $\sum m_a > 2s$

where s is the semi-perimeter of triangle ABC.

Solution : **Ans. (1)**

If 'G' is the centroid then we have the following

$$BG + CG > a \Rightarrow \frac{2}{3}m_b + \frac{2}{3}m_c > a$$

Similarly, $\frac{2}{3}m_a + \frac{2}{3}m_b > c$, $\frac{2}{3}m_a + \frac{2}{3}m_c > b$

$$\Rightarrow \quad \frac{4}{3}(m_a + m_b + m_c) > a + b + c$$

$$\Rightarrow \quad m_a + m_b + m_c > \frac{3}{4}(a + b + c) \quad \Rightarrow m_a + m_b + m_c > \frac{3}{2}s$$

Example - 18 In a ΔABC, $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) =$

(1) 3 abc (2) 3(a + b + c)

(3) 0 (4) abc(a + b + c)

Solution: **Ans. (1)**

$a^3\cos(B - C) = 8R^3 \sin^3 A \cos(B - C)$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

$= 8R^3 \sin^2 A \sin(B + C) \cos(B - C) = 4R^3 \sin^2 A(\sin 2B + \sin 2C)$

$= 4R^3 \sin^2 A\,(2\sin B \cos B + 2 \cos C \sin C)$

$\therefore$ the given expression = $\sum 4R^3 \sin^2 A(2 \sin B \cos B + 2 \sin C \cos C)$

$= \sum 8R^3 \sin A \sin B (\sin A \cos B + \cos A \sin B) = \sum 8R^3 \sin A \sin B \sin(A + B)$

$= \sum 8R^3 \sin A \sin B \sin C = \sum abc = 3\,abc.$

Example - 19 If an angle α is divided into two parts A and B such that $A - B = \theta$, and $\tan A : \tan B = m : n$, then $\sin\theta =$

(1) $\frac{m+n}{m-n}\sin\alpha$ (2) $\frac{m}{m+n}\sin\alpha$

(3) $\frac{m-n}{m+n}\sin\alpha$ (4) $\frac{n}{m+n}\sin\alpha$

Solution : **Ans. (3)**

$A + B = \alpha$ and $A - B = \theta \Rightarrow A = \frac{\alpha+\theta}{2}$, $B = \frac{\alpha-\theta}{2}$

$$\frac{m}{n} = \frac{\tan A}{\tan B} = \frac{\tan\frac{\alpha+\theta}{2}}{\tan\frac{\alpha-\theta}{2}} = \frac{\sin\frac{\alpha+\theta}{2}\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}\sin\frac{\alpha-\theta}{2}} = \frac{\sin\alpha+\sin\theta}{\sin\alpha-\sin\theta}$$

$$\therefore \frac{m-n}{m+n} = \frac{\sin\theta}{\sin\alpha} \Rightarrow \sin\theta = \left(\frac{m-n}{m+n}\right)\sin\alpha.$$

Example - 20 In triangle ABC where A, B, C are acute, the distances of the orthocentre from the sides are in the proportion

(1) cos A : cos B : cos C (2) sin A : sin B : sin C

(3) sec A : sec B : sec C (4) tan A : tan B : tan C

Solution : **Ans. (3)**

$$HD = BD \tan\angle EBC = c \cos B \tan\left(\frac{\pi}{2} - c\right) = \frac{2R\sin C\cos B\cos C}{\sin C}$$

$$= 2R \cos B \cos C = \frac{2R\cos A\cos B\cos C}{\cos A}$$

Similarly, $HE = \frac{2R\cos A\cos B\cos C}{\cos B}$ and

$$HF = \frac{2R\cos A\cos B\cos C}{\cos C} \Rightarrow HD : HE : HF = \frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}.$$

Example 21 If A, B, C are the angles of a triangle, then $\sin A + \sin B - \cos C \le$

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$

(3) 1 (4) $\frac{5}{2}$

Solution: **Ans. (2)**

Let $\sin A + \sin B - \cos C = k$

or, $2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos(A+B) = k$ $[\because A + B = \pi - C]$

or, $2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\left(\frac{A+B}{2}\right) = k$

or, $2\sin^2\left(\frac{A+B}{2}\right) - 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right) + (k-1) = 0$

Since this is quadratic equation in $\sin\left(\frac{A+B}{2}\right)$ which is real

Applying cosine rule in triangle AOC, $\therefore$ discriminant ≥ 0

$\therefore$ $4\cos^2\left(\frac{A-B}{2}\right) - 8(k-1) \geq 0$ or $\cos^2\left(\frac{A-B}{2}\right) \geq 2(k-1)$

or, $1 \geq 2(k-1)$ $\left[\because \cos^2\left(\frac{A-B}{2}\right) \leq 1\right]$ or $k \leq \frac{3}{2} \Rightarrow \sin A + \sin B - \cos C \leq \frac{3}{2}$

Example - 22 Prove that the distance between the circum-centre and the orthocentre of a triangle ABC is R $\sqrt{1 - 8\cos A \cos B \cos C}$.

(1) $R\sqrt{1+8\cos A\cos B\cos C}$
(2) $R\sqrt{1-8\cos A\cos B\cos C}$
(3) $R\sqrt{1+8\sin A\sin B\sin C}$
(4) $R\sqrt{1-8\sin A\sin B\sin C}$

Solution : **Ans. (2)**

Let O and P be the circumcentre and the orthocentre respectively.
If OF is perpendicular to AB, we have

$$\angle OAF = 90^o - \angle AOF = 90^o - C$$

Also $= \angle OAP = A - \angle OAF - \angle PAL = A - 2(90^o - C) = A + 2C - 180^o$
$= A + 2C - (A + B + C) = C - B$

Also $OA = R$

and $PA = 2R\cos A$

Now in S ΔAOP, $OP^2 = OA^2 + PA^2 - 2OA.PA.\cos\angle OAP$
$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B) = R^2 + 4R^2\cos A\{\cos A - \cos(C-B)\}$
$= R^2 - 4R^2\cos A\{\cos(B+C) + \cos(C-B)\} = R^2 - 8R^2\cos A\cos B\cos C$

Hence, $OP = R\sqrt{1 - 8\cos A\cos B\cos C}$.

Example - 23 The two adjacent sides of a cyclic quadritateral are 2 and 5 and the angle between them is 60^o if the third side is 3. The remaining fourth side is

(1) 2
(2) 3
(3) 4
(4) 5

Solution : **Ans. (1)**

Let ABCD be a cyclic quadrilateral

AB = 2, BC= 5, CD = 3, $\angle A = 60^o$ $\therefore$ $\angle D = 120^o$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

In a ΔABC, and ACD we have

$AC^2 = AB^2 + BC^2 - 2AB\ BC \cos 60^o$

or $AC^2 = 29 - 10 = 19$

and $AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cos 120^o$

or $AC^2 = d^2 + 9 + 3d \quad \therefore\ d^2 + 3d - 10 = 0$

$\Rightarrow \quad d^2 + 3d - 10 = 0 \quad \Rightarrow (d + 5)(d - 2) = 0 \quad \Rightarrow \quad d = 2$ or -5

Example - 24 The area of the circle and the area of the regular polygon of n-sides and of perimeter equal to that of the Δ are in the ratio of

(1) $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ (2) $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

(3) $\sin\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ (4) $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

Solution : **Ans. (1)**

Let r be the ratio of the Δ and A_1 be its area, then $A_1 = \pi r^2$

Since the perimeter of the Δ is same is the perimeter of a regular polygon of n sides.

$\therefore\ 2\pi r = na$, where 'a' is the length of one side if the regular polygon $\Rightarrow\ a = \frac{2\pi r}{n}$

Let A_2 be the area of the polygon.

Then $A_2 = \frac{1}{4}\pi a^2 \cot\left(\frac{\pi}{a}\right) = \frac{\pi r^2}{n}\cot\left(\frac{\pi}{n}\right)$

$\therefore\ A_1 : A_2 = \pi r^2 : \frac{\pi^2 r^2}{n}\cot\left(\frac{\pi}{n}\right) \Rightarrow \tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

Example - 25 In a ΔABC, $\angle B = \pi/3$ and $\angle C = \pi/4$ and AD-divides BC internally in the ratio 1 : 3 then $\frac{\sin\angle BAD}{\sin\angle CAD} =$

(1) $\frac{\sqrt{2}}{3}$ (2) $\frac{1}{\sqrt{3}}$

(3) $\frac{1}{\sqrt{6}}$ (4) $\frac{1}{3}$

Solution : **Ans. (3)**

In ΔABD, applying sine law we get

$\frac{AD}{\sin\frac{\pi}{3}} = \frac{x}{\sin\alpha} \Rightarrow AD = \frac{\sqrt{3}x}{2\sin\alpha}$ ....(i)

In ΔACD, applying sine law, we get

$\frac{AD}{\sin\frac{\pi}{4}} = \frac{3x}{\sin\beta} \Rightarrow AD = \frac{3x}{\sqrt{2}\sin\beta}$ ...(ii)

From (i) and (ii), $\Rightarrow \frac{\sqrt{3}x}{2\sin\alpha} = \frac{3x}{\sqrt{2}\sin\beta} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

EXERCISES

LEVEL – I

1. If in ΔABC, $r_1 = 2r_2 = 3r_3$ then a : b : c =

(1) 3 : 4 : 5 (2) 5 : 4 : 3

(3) 5 : 3 : 4 (4) 3 : 5 : 4

2. In an equilateral triangle r : R : r_1 =

(1) 1 : 1 : 1 (2) 1 : 2 : 3

(3) 1 : 3 : 2 (4) 2 : 3 : 4

3. If in a triangle ABC, $\angle B = 60^o$, then

(1) $(a - b)^2 = c^2 - ab$ (2) $(b - c)^2 = a^2 - bc$

(3) $(c - a)^2 = b^2 + ac$ (4) $a^2 + b^2 + c^2 = 2b^2 + ac$

4. If the angles of a triangle ABC are in A.P. then B =

(1) 75^o (2) 90^o

(3) 60^o (4) 45^o

5. In a triangle ABC, if a = 3, b = 4 and sinA = 3/4, then B =

(1) 60^o (2) 90^o

(3) 45^o (4) 30^o

6. If a : b : c = 7 : 8 : 9 then cosA : cosB : cosC =

(1) 2 : 11 : 2 (2) 3 : 7 : 11

(3) 14 : 11 : 6 (4) 16 : 11 : 14

7. In a ΔABC, of a = 7, b = 8, c = 9 then the length of the line joining B to the midpoint of AC is

(1) 6 (2) 7

(3) 5 (4) 6

8. In a ΔABC, if r_1 = 36, r_2 = 18, r_3 = 12 then the area of the triangle is

(1) 216 (2) 316

(3) 326 (4) 256

9. The sines of two angles of a triangle are equal to 5/13 and 99/101. The cosine of the third angle is

(1) $\frac{255}{1315}$ (2) $\frac{251}{1313}$

(3) $\frac{255}{1313}$ (4) $\frac{199}{513}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

10. Given b = 2, c = $\sqrt{3}$, $\angle$A = 30°, then inradius of Δ ABC is

(1) $\frac{\sqrt{3}-1}{2}$ (2) $\frac{\sqrt{3}+1}{2}$

(3) $\frac{\sqrt{3}-1}{4}$ (4) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

11. In a right angled triangle ABC, $\sin^2A + \sin^2B + \sin^2C$ =

(1) 0 (2) 1

(3) –1 (4) 2

12. If the angles A,B and C of a triangle ABC are in AP and the sides a,b and c opposite to these angles are in G.P. ; then a^2, b^2 and c^2 are in

(1) G.P. (2) A.P.

(3) H.P. (4) all the above

13. If $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then the sides of ΔABC are in

(1) A.P. (2) G.P.

(3) H.P. (4) none of these

14. If the sum of the squares of the sides of a triangle is equal to twice the square of its circum diameter then $\sin^2A + \sin^2B + \sin^2C$ =

(1) 4 (2) 3

(3) 1 (4) 2

15. The base of a triangle is 80 cm and one of the base angle is 60°. If the sum of the lengths of the other two sides is 90 cm then the shortest side is

(1) 15 cm (2) 19 cm

(3) 21 cm (4) 17 cm

16. In a ΔABC, if $r_1 = 36$, $r_2 = 18$ and $r_3 = 12$, then the perimeter of the triangle is

(1) 36 (2) 18

(3) 72 (4) 54

17. If a cosA = b cosB, then ΔABC is

(1) isosceles (2) right angled

(3) equilateral (4) right angled isosceles

18. If $A = 75^o, B = 45^o$, then $b + c\sqrt{2} =$

(1) a (2) a + b + c

(3) 2a (4) $\frac{1}{2}(a+b+c)$

19. If Δ stands for the area of triangle ABC, then $a^2 \sin 2B + b^2 \sin 2A$ =

(1) 3Δ (2) 2Δ

(3) 4Δ (4) –4Δ

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

20. If $\cot\frac{A}{2} = \frac{b+c}{a}$ then the ΔABC is

(1) isosceles
(2) equilateral
(3) right angled
(4) none of these

21. In a Δ ABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Then the ratio of the sides opposite to the angles is

(1) 1 : 2
(2) 3 : 4
(3) 1 : 3
(4) 4 : 5

22. In a Δ ABC, $\sin^3A + \sin^3B + \sin^3C = 3 \sin A \sin B \sin C$ then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} =$

(1) 0
(2) $(a+b+c)^3$
(3) (a+b+c) (ab+bc+ca)
(4) $(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

23. In a triangle ABC O is a point inside the triangle such that $\angle OBC = \angle OCA = \angle OAB = 15^o$, then value of cot A + cot B + cot C is

(1) $2-\sqrt{3}$
(2) $\sqrt{2}-1$
(3) $\sqrt{2}+1$
(4) $2+\sqrt{3}$

24. On the level ground the angle of elevation of the top of a tower is 30°. On moving 20 mt. nearer the tower, the angle of elevation is found to be 60°. The height of the tower is

(1) 10 mt
(2) 20 mt
(3) $10\sqrt{3}$ mt
(4) none of these

25. A flag staff of 5 mt high stands on a building of 25 mt high. At an observer at a height of 30 mt, the flag staff and the building subtend equal angles. The distance of the observer from the top of the flag staff is

(1) $\frac{5\sqrt{3}}{2}$
(2) $5\sqrt{\frac{3}{2}}$
(3) $5\sqrt{\frac{2}{3}}$
(4) none of these

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

LEVEL - II

1. If the angles A, B, C of ΔABC are in A.P., then

(1) $c^2 = a^2 + b^2 - ab$ (2) $b^2 = a^2 + c^2 - ac$

(3) $c^2 = a^2 + b^2$ (4) none of these

2. In a triangle ABC, $\frac{c}{a^2 - b^2} + \frac{b}{a^2 - c^2} = 0$ then $\angle A =$

(1) 30^o (2) 45^o

(3) 60^o (4) 90^o

3. In a triangle ABC, if the median AD makes an angle θ with AC and AB = 2AD then $\sin\theta =$

(1) sinA (2) sinB

(3) sinC (4) sinAsinB

4. $r\cot\frac{B}{2}\cot\frac{C}{2} =$

(1) r_1 (2) r_2

(3) r_3 (4) Δ

5. In a Δ ABC, a = 5, b = 4 and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$. Then measure of C is

(1) 12 (2) 7

(3) 6 (4) 5

6. $\frac{b^2 - c^2}{2aR} =$

(1) cos (B – C) (2) sin (B – C)

(3) cos B – cos C (4) sin B – sin C

7. If $b^2 + c^2 = 3a^2$, then cotB + cotC – cotA =

(1) 1 (2) $\frac{ab}{4\Delta}$

(3) 0 (4) $\frac{ac}{4\Delta}$

8. In a ΔABC, a = 2b and $|A - B| = \frac{\pi}{3}$ the measure of $\angle C$ is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

9. Two sides of a Δ are given by the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$, the angle between the sides is $\pi/3$ the perimeter of the triangle is

(1) $6 + \sqrt{3}$ (2) $2\sqrt{3} + \sqrt{6}$

(3) $2\sqrt{3} + \sqrt{10}$ (4) $2\sqrt{3} + \sqrt{9}$

10. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in the circle is

(1) $a^2/4$ (2) $a^2/6$
(3) $a^2/9$ (4) $2a^2/3$

11. If the base angles of a triangle are $22\frac{1}{2}^o$ and $112\frac{1}{2}^o$ then the altitude of the triangle is equal to

(1) base (2) $\frac{1}{3}$rd of base
(3) $\frac{1}{2}$ of base (4) $\frac{1}{4}$th of base

12. If the sides a, b, c of ΔABC are in A.P., then $\cos A \cot\frac{1}{2}A, \cos B \cot\frac{1}{2}B, \cos C \cot\frac{1}{2}C$ are in

(1) A.P. (2) G.P.
(3) H.P. (4) none of these

13. In any triangle ABC, $\left[\frac{\sin^2 A + \sin A + 1}{\sin A}\right]$ is always greater than

(1) 9 (2) 3
(3) 27 (4) 10

14. If in a triangle ABC, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is

(1) right angled or isosceles (2) right angled and isosceles
(3) equilateral (4) none of these

15. If in a ΔABC, cosA + 2 cosB + cosC = 2, then a, b, c are in

(1) A.P. (2) H.P.
(3) G.P. (4) none of these

16. In a triangle, $a^2 + b^2 + c^2 = ca + ab\sqrt{3}$. Then the triangle is

(1) equilateral (2) right-angled and isosceles
(3) right-angled with $A = 90^o, B = 60^o, C = 30^o$ (4) none of these

17. In a ΔABC, A : B : C = 3 : 5 : 4, then $a + b + c\sqrt{2}$ is

(1) 2b (2) 2c
(3) 3b (4) 2a

18. In a ΔABC, A = 2π/3, b – c = $3\sqrt{3}$ cm and $ar(\Delta ABC) = \frac{9\sqrt{3}}{2}cm^2$ then a is

(1) $6\sqrt{3}$ cm (2) 9 cm
(3) 18 cm (4) 81 cm

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

19. If sin(A + B + C) = 1, tan(A – B) = $\frac{1}{\sqrt{3}}$ and sec(A + C) = 2 then

(1) A = 90°, B = 60°, C = 30°
(2) A = 120°, B = 60°, C = 0°
(3) A = 60°, B = 30°, C = 0°
(4) A = B = C = 60°

20. The sides of a triangle inscribed in a given circle subtend angle α, β and γ at the centre. The minimum value of the arithmetic mean of cos ($\alpha + \pi/2$), cos ($\beta + \pi/2$) and cos ($\gamma + \pi/2$) is equal to

(1) 0
(2) $1/\sqrt{2}$
(3) –1
(4) $-\sqrt{3}/2$

21. If the sides of a right angled triangle are in A.P. then the tangents of the acute angled triangled are

(1) $\sqrt{\sqrt{3}+\frac{1}{2}}, \sqrt{\sqrt{3}-\frac{1}{2}}$
(2) $\sqrt{\sqrt{5}+\frac{1}{2}}, \sqrt{\sqrt{5}-\frac{1}{2}}$
(3) $\sqrt{3}, \frac{1}{\sqrt{3}}$
(4) $\frac{3}{4}, \frac{4}{3}$

22. If a,b, A are given in a ΔABC, and C_1, C_2 are the possible values of the third side, then $C_1^2 + C_2^2 - 2C_1C_2 \cos 2A =$

(1) $4a^2 \sin^2 A$
(2) $4a^2 \cos^2 A$
(3) $4 \sin^2 A$
(4) $4 \cos^2 A$

23. The sides of a triangle are 3x+4y , 4x+3y and 5x+5y units, where x, y > 0. The triangle is

(1) right angled
(2) equilateral
(3) obtuse angled
(4) isosceles

24. An aeroplane flying at a height of 300 m above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. The height of the lower plane from the ground is

(1) $100\sqrt{3}$ m
(2) $100/\sqrt{3}$ m
(3) 50 m
(4) $150(\sqrt{3}+1)$ m

25. Two sides of a triangle are $2\sqrt{2}$cm and $2\sqrt{3}$cm and the angle opposite to the shorter side of the two is $\frac{\pi}{4}$ the largest possible length of the third side is

(1) $\sqrt{2}(\sqrt{3} \pm \sqrt{2})$cm
(2) $(6 \pm \sqrt{2})$cm
(3) $(\sqrt{6} \pm \sqrt{2})$cm
(4) $\sqrt{6} \pm \sqrt{3}$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

LEVEL - III

1. In a ΔABC, cosA = 3/5 and cosB = 5/13 then cosC =

(1) $\frac{7}{13}$ (2) $\frac{12}{13}$

(3) $\frac{33}{65}$ (4) $\frac{16}{5}$

2. The length of the sides of a triangle are x, y and $\sqrt{x^2+y^2+xy}$ the measure of the greatest angle is

(1) $\frac{2\pi}{3}$ (2) $\frac{5\pi}{6}$

(3) $\frac{3\pi}{4}$ (4) $\frac{5\pi}{3}$

3. In a ΔABC, the sides are in the ratio 4 : 5 : 6 the ratio of the circumradius and inradius is

(1) 8 : 7 (2) 3 : 2

(3) 7 : 3 (4) 16 : 7

4. If cosA + cosB + 2cosC = 2 then the sides of ΔABC are in

(1) A.P (2) G.P

(3) H.P (4) A.G.P

5. In a ΔABC, the sides a, b and c are such that they are the roots of $x^3 - 11x^2 + 38x - 40 = 0$ then $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=$

(1) $\frac{3}{4}$ (2) 1

(3) $\frac{9}{16}$ (4) $\frac{2}{9}$

6. The sides of a triangle are in the ratio $2:\sqrt{6}:\sqrt{3}+1$, then its angles are

(1) $45^o, 45^o, 90^o$ (2) $60^o, 30^o, 90^o$

(3) $45^o, 60^o, 75^o$ (4) none of these

7. If in a ΔABC, 3a = b + c then tanB/2 tanC/2 =

(1) tanA/2 (2) 1

(3) 2 (4) 1/2

8. In any ΔABC, $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=$

(1) $\frac{b^2+c^2}{a^2+c^2}$ (2) $\frac{a^2+b^2}{a^2+c^2}$

(3) $\frac{a^2+c^2}{a^2+b^2}$ (4) none of these

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

9. If $\angle A = 90^o$ in the triangle ABC, then $\tan^{-1}\left(\frac{c}{a+b}\right) + \tan^{-1}\left(\frac{b}{a+c}\right) =$

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

10. The area of ΔABC is $a^2 - (b-c)^2$, then tanA is equal to

(1) $\frac{1}{4}$ (2) $\frac{4}{3}$

(3) $\frac{3}{4}$ (4) $\frac{8}{15}$

11. If in a ΔABC $2\frac{\cos A}{a} + \frac{\cos B}{b} + 2\frac{\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$ then the value of the angle A is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

12. If the sides of a triangle ABC are in A.P. and 'a' is the smallest side then cos A in terms of b,c is

(1) $\frac{a+2b}{c}$ (2) $\frac{4c-3b}{2c}$

(3) $\frac{c-b}{c+b}$ (4) $\frac{c+b}{c-b}$

13. In a triangle ABC (a + b + c) (a + b – c) = 3ab the measure of angle $\angle C$ is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$

(3) $\frac{2\pi}{3}$ (4) $\frac{\pi}{2}$

14. The equation $ax^2 + bx + c = 0$, where a, b, c are the sides of a ΔABC and the equation $x^2 + \sqrt{2}x + 1 = 0$ have a common root, the measure of $\angle C$ is

(1) 90^o (2) 45^o

(3) 60^o (4) 30^o

15. If the radius of the circumcircle of an isosceles ΔPQR is equal to PQ(= PR) then $\angle P$ =

(1) $\pi/6$ (2) $\pi/3$

(3) $\pi/2$ (4) $2\pi/3$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

16. In a ΔABC, a = 1 and the perimeter is six times the A.M of the sines of the angles the measure of $\angle A$ is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$

(3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

17. If A, A_1, A_2, A_3 are areas of excircles and incircles of a triangle, then $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=$

(1) $\frac{2}{\sqrt{A}}$ (2) $\frac{3}{\sqrt{A}}$

(3) $\frac{1}{\sqrt{A}}$ (4) none

18. The area of the triangle whose sides are $\sqrt{b^2+c^2}, \sqrt{c^2+a^2}, \sqrt{a^2+b^2}$ where a,b,c > 0 is

(1) $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$ (2) $\frac{1}{2}\sqrt{a^4+b^4+c^4}$

(3) $\frac{\sqrt{3}}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$ (4) $\frac{\sqrt{3}}{2}(bc+ca+ab)$

19. If α, β, γ are the lengths of the altitudes of ΔABC, then $\alpha^{-2}+\beta^{-2}+\gamma^{-2}=$

(1) Δ (2) cotA + cotB + cotC

(3) $\frac{\cot A+\cot B+\cot C}{\Delta}$ (4) $\frac{\Delta}{\cot A+\cot B+\cot C}$

20. In a triangle ABC, a = 5, b = 7 and $\sin A=\frac{3}{4}$, how many such triangles are possible

(1) 1 (2) 0

(3) 2 (4) infinite

21. If in triangle ABC, (a + b + c) (a + b – c) = λ ab then exhaustive range of 'λ' is

(1) (2, 4) (2) (0, 4)

(3) (0, 2) (4) (1, 4)

22. In triangle ABC the value of the expression $r_1r_2 + r_2r_3 + r_3r_1$ is always equal to

(1) $\frac{1}{2}(a+b+c)^2$ (2) $(a+b+c)^2$

(3) $\frac{1}{16}(a+b+c)^2$ (4) $\frac{1}{4}(a+b+c)^2$

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

23. If Δ denotes the area of any triangle and S its semiperimeter, then

(1) $\Delta < \frac{s^2}{2}$ (2) $\Delta > \frac{s^2}{4}$

(3) $\Delta < \frac{s^2}{4}$ (4) none of these

24. A man in a boat rowing away from a cliff 150 metres high observes that it takes 2 minutes to change the anlge of elevation of the top of the cliff from 60° to 45°. The speed of the boat is

(1) $(1/2)\,(9-3\sqrt{3})$ km/h (2) $(1/2)\,(9+3\sqrt{3})$ km/h

(3) $(1/2)\,(9\sqrt{3})$ km/h (4) none of these

25. If p_1, p_2, p_3 are the lengths of the altitudes of a triangle from the vertices A, B, C, then $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=$

(1) $\frac{2ab\cos^2 C/2}{\Delta(a+b+c)}$ (2) $\frac{1}{R}$

(3) $\frac{\cot A+\cot B+\cot C}{\Delta}$ (4) 2R

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

QUESTIONS ASKED IN AIEEE & OTHER ENGINEERING EXAMS

1. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (= a) subtends an angle 60° at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30°. THe height of the tower is

(1) $a/\sqrt{3}$ (2) $a\sqrt{3}$
(3) $2a/\sqrt{3}$ (4) $2a\sqrt{3}$ **[AIEEE - 2007]**

2. In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle ABC, then 2(r + R) equals

(1) b + c (2) a + b
(3) a + b + c (4) c + a **[AIEEE - 2005]**

3. If in a ΔABC, the altitudes from the vertices A, B, C on opposite sides are in H.P., then sinA, sinB, sinC are in

(1) G.P. (2) A.P.
(3) A.G.P. (4) H.P. **[AIEEE - 2005]**

4. The sides of a triangle are sinα, cosα and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0 < \alpha < \pi/2$. Then the greatest angle of the triangle is

(1) 90° (2) 60°
(3) 120° (4) 150° **[AIEEE - 2004]**

5. A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 metres away from the tree, the angle of elevation becomes 30°. The breadth of the river is

(1) 40m (2) 30m
(3) 20m (4) 60 m **[AIEEE - 2004]**

6. The upper $\frac{3}{4}^{th}$ portion of a vertical pole subtends an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot and at a distance 40m from the foot. A possible height of the vertical pole is

(1) 20m (2) 40m
(3) 60m (4) 80m **[AIEEE - 2003]**

7. In a triangle ABC, medians AD are BE are drawn. If AD = 4, $\angle DAB = \pi/6$ and $\angle ABE = \pi/3$, then the area of the ΔABC is

(1) $16/3\sqrt{3}$ (2) $32/3\sqrt{3}$
(3) 32/3 (4) 64/3 **[AIEEE - 2003]**

8. If in a triangle ABC, $a\cos^2\left(\frac{C}{2}\right)+c\cos^2\left(\frac{A}{2}\right)=\frac{3b}{2}$ then the sides, a, b and c

(1) are in A.P. (2) are in G.P.
(4) are in H.P. (4) satisfy a + b = c **[AIEEE - 2003]**

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

9. The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side 'a' is

(1) $a\cot\left(\frac{\pi}{n}\right)$ (2) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$

(3) $a\cot\left(\frac{\pi}{2n}\right)$ (4) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$ **[AIEEE - 2003]**

10. In a triangle ABC, a = 4, b = 3, $\angle A = 60^o$, then c is the root of the equation

(1) $c^2 - 3c - 7 = 0$ (2) $c^2 + 3c + 7 = 0$

(3) $c^2 - 3c + 7 = 0$ (4) $c^2 + 3c - 7 = 0$ **[AIEEE - 2002]**

11. In a ΔABC, $\tan\frac{A}{2} = \frac{5}{6}$, $\tan\frac{C}{2} = \frac{2}{5}$, then

(1) a, c, b are in A.P. (2) a, b, c are in A.P.

(3) b, a, c are in A.P. (4) none of these **[AIEEE - 2002]**

12. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$ then n =

(1) 5 (2) 7

(3) 6 (4) 4 **[AIEEE - 2002]**

13. In a triangle ABC, 2ca $\sin\frac{A - B + C}{2}$ is equal to

(1) $a^2 + b^2 - c^2$ (2) $c^2 + a^2 - b^2$

(3) $b^2 - c^2 - a^2$ (4) $c^2 - a^2 - b^2$ **[AIEEE - 2002]**

14. The sides of a triangle are 4, 5 and 6 cm. The area of the triangle is equal to

(1) $\frac{15}{4}cm^2$ (2) $\frac{15}{4}\sqrt{7}cm^2$

(3) $\frac{4}{15}\sqrt{7}cm^2$ (4) none of these **[UPSEAT - 2004]**

15. In a triangle ABC if b = 2, B = 30° then the area of the circumcircle of triangle ABC in square units is

(1) π (2) 2π

(3) 4π (4) 6π **[CET (Karnataka) - 2004]**

16. If R is the radius of the circumcircle of the ΔABC, and Δ is its area then

(1) $R = \frac{a+b+c}{\Delta}$ (2) $R = \frac{a+b+c}{4\Delta}$

(3) $R = \frac{abc}{4\Delta}$ (4) $R = \frac{abc}{\Delta}$ **[CET (Karnataka) - 2000]**

17. Let the angles A, B, C of ΔABC be in A.P. and let b : c = $\sqrt{3}$: $\sqrt{2}$. Then angle A is

(1) 75° (2) 45°

(3) 60° (4) none of these **[CEET (Haryana) - 2000]**

18. In a ΔABC, if a cosA = b cosB then triangle is

(1) right angled (2) isosceles

(3) equilateral (4) none of these **[CEET (Haryana) - 2000]**

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

19. If in a triangle ABC, AD, BE and CF are the altitudes and R is the circum radius, then the radius of the circle DEF is

(1) $\frac{R}{2}$ (2) 2R

(3) R (4) none of these **[CEET (Haryana) - 2000]**

20. If D is the mid point of the side BC of a triangle ABC and AD is perpendicular to AC, then

(1) $b^2 = a^2 - c^2$ (2) $a^2 + b^2 = 5c^2$

(3) $3b^2 = a^2 - c^2$ (4) $3a^2 = b^2 - 3c^2$ **[CEET (Haryana) - 1999]**

21. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} =$

(1) $2 - \frac{r}{R}$ (2) $2 - \frac{r}{2R}$

(3) $2 + \frac{r}{2R}$ (4) none of these **[CEET (Haryana) - 1999]**

22. The perimeter of a ΔABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then the angle A is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{2}$ (4) π **[CEET (Haryana) - 1998]**

23. If r_1, r_2, r_3 in a triangle be in H.P., then the sides are in

(1) H.P. (2) A.P.

(3) G.P. (4) none of these **[CEET (Delhi) - 2000]**

24. If a = (b – c) secθ, then $\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2} =$

(1) $\cos\theta$ (2) $\cot\theta$

(3) $\tan\theta$ (4) $\sin\theta$ **[CEET (Delhi) - 2000]**

25. In a ΔABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and a = 2. Area of the triangle is

(1) 1 (2) 2

(3) $\frac{\sqrt{3}}{2}$ (4) $\sqrt{3}$ **[PET (MP) - 2000]**

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320

ANSWERS

EXERCISES

LEVEL – I

1. (2)	2. (2)	3. (4)	4. (3)	5. (2)
6. (3)	7. (2)	8. (1)	9. (3)	10. (1)
11. (4)	12. (2)	13. (1)	14. (1)	15. (4)
16. (3)	17. (4)	18. (3)	19. (3)	20. (3)
21. (3)	22. (1)	23. (4)	24. (3)	25. (2)

LEVEL – II

1. (2)	2. (3)	3. (1)	4. (1)	5. (3)
6. (2)	7. (3)	8. (2)	9. (2)	10. (2)
11. (3)	12. (1)	13. (3)	14. (2)	15. (1)
16. (4)	17. (3)	18. (2)	19. (3)	20. (4)
21. (4)	22. (2)	23. (3)	24. (4)	25. (3)

LEVEL - III

1. (3)	2. (1)	3. (4)	4. (1)	5. (3)
6. (3)	7. (3)	8. (2)	9. (2)	10. (4)
11. (3)	12. (2)	13. (1)	14. (2)	15. (4)
16. (2)	17. (3)	18. (1)	19. (3)	20. (2)
21. (2)	22. (4)	23. (3)	24. (1)	25. (3)

QUESTIONS ASKED IN AIEEE & OTHER ENGINEERING EXAMS

1. (1)	2. (2)	3. (2)	4. (3)	5. (3)
6. (2)	7. (2)	8. (1)	9. (2)	10. (1)
11. (2)	12. (2)	13. (2)	14. (2)	15. (3)
16. (3)	17. (1)	18. (1)	19. (1)	20. (3)
21. (4)	22. (1)	23. (2)	24. (3)	25. (4)

FNS House, 63, Kalu Sarai Market, Sarvapriya Vihar, New Delhi-110016 ● Ph.: (011) 32001131/32 Fax : (011) 41828320