Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 1

BASIC MATHS & VECTORS - Theory

Index

BASIC MATHS,
VECTORS
&
UNIT DIMENSION

Serial No.	CONTENTS	Page No.
	BASIC MATHS, VECTORS & UNIT DIMENSION	
1.	Elementary Algebra	2
2.	Trigonometry	4
3.	Coordinate Geometry	10
4.	Calculus	12
5.	Function and Graph	21
6.	Vectors	28
7.	Unit & Dimensions	42
8.	**Exercise-1** Check your Grasp (Single Choice Objective Questions)	58
9.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	62
10.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	66
11.	**Exercise-4A** Conceptual Subjective Exercise	73
12.	**Exercise-4B** Brain Storming Exercise	77
13.	**Exercise-5A** Previous years questions	79
14.	**Exercise-5B** Previous years questions	80

ALLEN Materials Provided By - Material Point Available on Learnaf.com

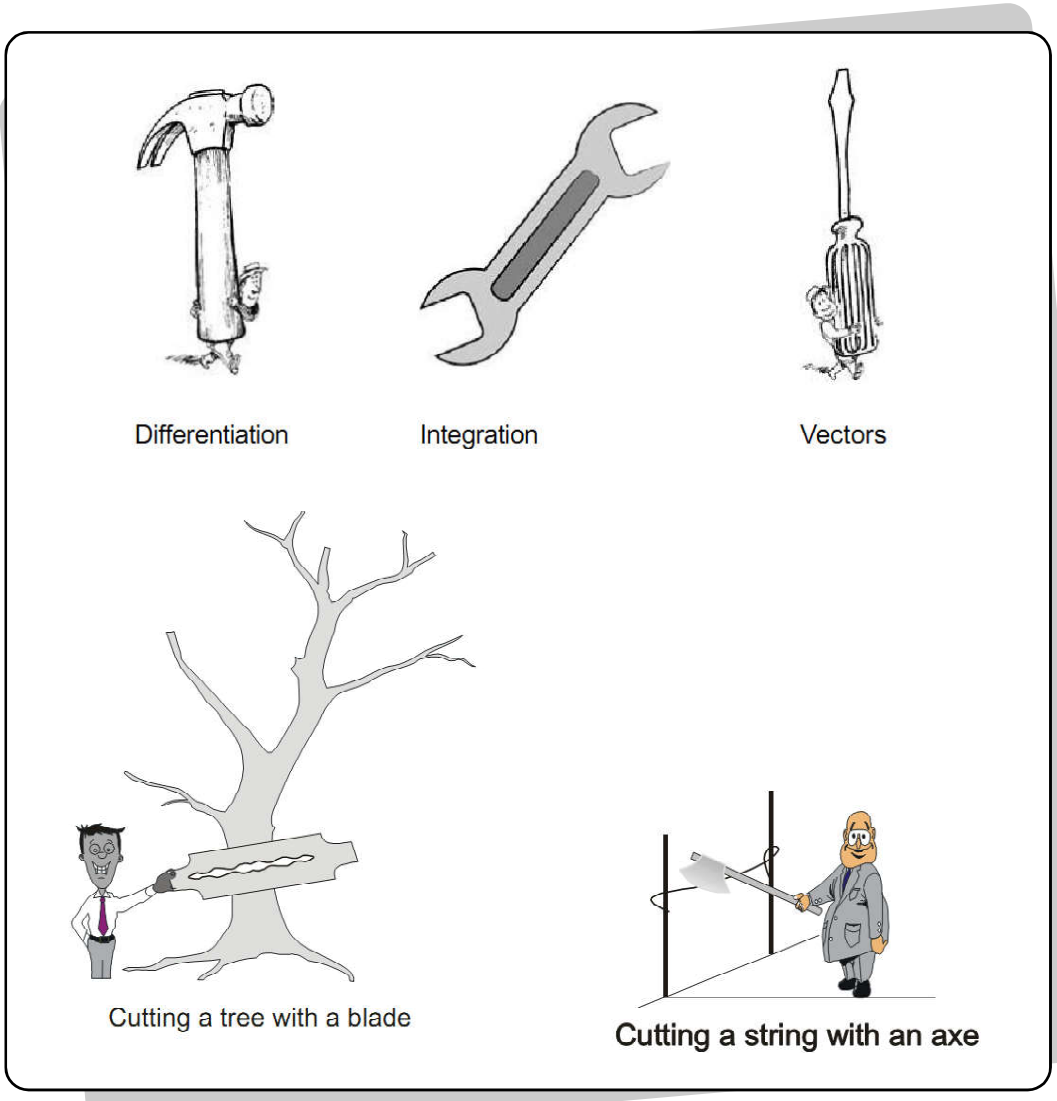
JEE Advanced : Units and dimensions, dimensional analysis;

JEE (Main) Syllabus :

S.I. units, Fundamental and derived units. Dimensions of Physical quantities, dimensional analysis and its applications.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

BASIC MATHS USED IN PHYSICS



Mathematics is the supporting tool of Physics. The elementary knowledge of basic mathematics is useful in problem solving in Physics. In this chapter we study *Elementary Algebra, Trigonometry, Coordinate Geometry* and *Calculus (differentiation and integration).*

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

SECTION [A] : ELEMENTARY ALGEBRA

QUADRATIC EQUATION

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. The equation $ax^2 + bx + c = 0$....(i) is the general form of quadratic equation where $a \neq 0$. The general solution of above equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If values of x be x_1 and x_2 then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Here x_1 and x_2 are called roots of equation (i). We can easily see that

$$\textit{sum of roots} = x_1 + x_2 = -\frac{b}{a} \quad \textit{and product of roots} = x_1 x_2 = \frac{c}{a}$$

Example :

Find roots of equation $2x^2 - x - 3 = 0$.

Solution :

Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$,

we have $a=2$, $b=-1$, $c=-3$.

Now from $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $x = \frac{-(1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} \Rightarrow x = \frac{6}{4} \Rightarrow x = \frac{-4}{4} \Rightarrow x = \frac{3}{2} \text{ or } x = -1$$

Example :

In quadratic equation $ax^2 + bx + c = 0$, if discriminant $D = b^2 - 4ac$, then roots of quadratic equation are:

(a) real and distinct, if $D > 0$

(b) real and equal (repeated roots), if $D = 0$

(c) non-real (imaginary), if $D < 0$

(d) None of the above

Ans. (ABC)

BINOMIAL EXPRESSION

An algebraic expression containing two terms is called a binomial expression.

For example $(a+b)$, $(a+b)^3$, $(2x--3y)^{-1}$, $\left(x + \frac{1}{y}\right)$ etc. are binomial expressions.

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2 \times 1}a^{n-2}b^2 + ... \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1}x^2 + ...$$

Binomial Approximation

If x is very small, then terms containing higher powers of x can be neglected so

$(1+x)^n \square\ 1 + nx$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example :

The mass m of a body moving with a velocity v is given by $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3 10^8 m/s. Find the value of m at v = 3 10^7 m/s.

Solution:

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2} = 10\left[1-\frac{1}{100}\right]^{-1/2}$$

$$\approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] = 10+\frac{10}{200} \approx 10.05 \text{ kg}$$

LOGARITHM

Common formulae :

- $\log mn = \log m + \log n$
- $\log \frac{m}{n} = \log m - \log n$
- $\log m^n = n \log m$
- $\log_e m = 2.303 \log_{10} m$

COMPONENDO AND DIVIDENDO RULE

If $\frac{p}{q} = \frac{a}{b}$ then $\frac{p+q}{p-q} = \frac{a+b}{a-b}$

ARITHMETIC PROGRESSION (AP)

General form : a, a + d, a +2d, ..., a + (n–1)d

Here a = first term, d = common difference

Sum of n terms $S_n = \frac{n}{2}[a+a+(n-1)d] = \frac{n}{2}[I^{st} \text{ term} + n^{th} \text{ term}]$

Example :

Find sum of first n natural numbers.

Solution:

Let sum be S_n then $S_n = 1 + 2 + 3 + \ldots + n$; $S_n = \frac{n}{2}[1+n] = \left[\frac{n(n+1)}{2}\right]$

GEOMETRICAL PROGRESSION (GP)

General form : a, ar, ar^2,..., ar^{n-1} Here a = first term, r = common ratio

Sum of n terms $S_n = \frac{a(1-r^n)}{1-r}$ Sum of ∞ term $S_\infty = \frac{a}{1-r}$

Example :

Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$upto ∞.

Solution :

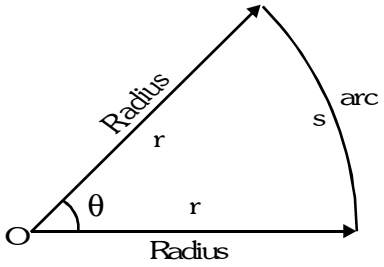
Here, a = 1, r = $\frac{1}{2}$ So, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

node06\E_NOD06\(E)\Data\2014\Kota\IIEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

SECTION [B] : TRIGONOMETRY

ANGLE :

it is measure of change in direction.



$$\text{Angle } (\theta) = \frac{\text{Arc}(s)}{\text{Radius}(r)}$$

Angels measured in anticlockwise and clockwise direction are usually taken positive and negative respectively.

System of measurement of an angle

[A] Sexagesimal system :

In this system, angle is measured in degrees.

In this system , 1 right angle = 90 , 1 = 60' (arc minutes), 1' = 60'' (arc seconds)

[B] Circular system :

In this system, angle is measured in radian.

if arc = radius then θ=1 rad

Relation between degrees and radian

2π rad = 360

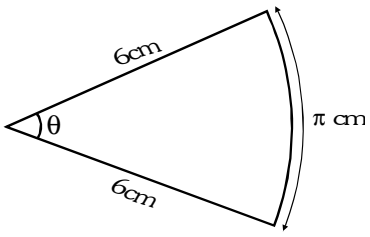
$$\pi \text{ rad} = 180 \Rightarrow 1 \text{ rad} = \frac{180^\circ}{\pi} = 57.3$$

To convert from degree to radian multiply by $\frac{\pi}{180^\circ}$

To convert from radian to degree multiply by $\frac{180^\circ}{\pi}$

Example :

A circular arc of length π cm. Find angle subtended by it at the centre in radian and degree.



Solution :

$$\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6}\text{rad} = 30$$

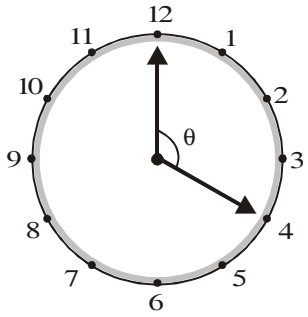
Example :

When a clock shows 4 o'clock, how much angle do its minute and hour needles make?

(a) 120 (b) $\frac{\pi}{3}$ rad (c) $\frac{2\pi}{3}$ rad (d) 160

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Solution : **Ans. (AC)**



From diagram angle $\theta = 4 \times 30° = 120° = \frac{2\pi}{3}$ rad

Example :

The moon's distance from the earth is 360000 km and its diameter subtends an angle of 42' at the eye of the observer. The diameter of the moon in kilometers is

(a) 4400 (b) 1000 (c) 3600 (d) 8800

Solution : **Ans. (A)**

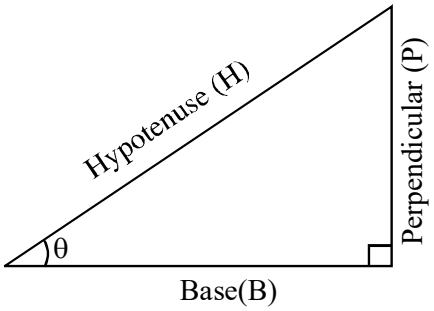
Here angle is very small so diameter $\approx$ arc

$$\theta = 42' = \left(42 \times \frac{1}{60}\right)^0 = 42 \times \frac{1}{60} \times \frac{\pi}{180} = \frac{7\pi}{1800} \text{ rad}$$

$$\text{Diameter} = R\theta = 360000 \times \frac{7}{1800} \times \frac{22}{7} = 4400 \text{ km}$$

Trigonometric Ratios (T-ratios)

Following ratios of the sides of a right angled triangle are known as trigonometrical ratios.



$\sin\theta = \frac{P}{H}$ $\cos\theta = \frac{B}{H}$ $\tan\theta = \frac{P}{B}$ $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{B}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$

Trigonometric Identities

In figure , $P^2 + B^2 = H^2$ Divide by H^2, $\left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1 \Rightarrow \sin^2\theta + \cos^2\theta = 1$

Divide by B^2, $\left(\frac{P}{B}\right)^2 + 1 = \left(\frac{H}{B}\right)^2 \Rightarrow 1 + \tan^2\theta = \sec^2\theta$

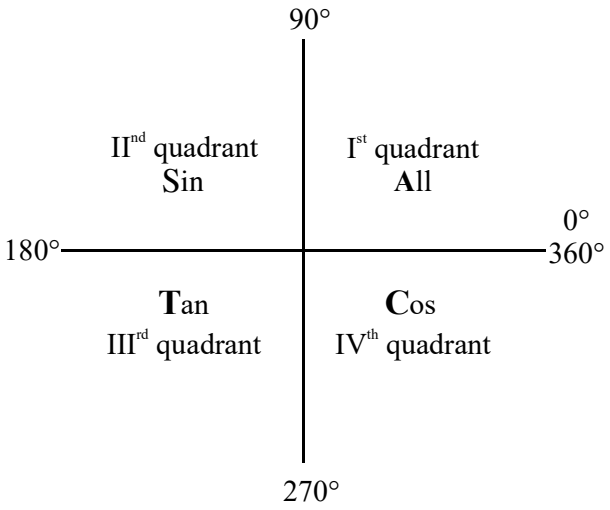
Divide by P^2, $1 + \left(\frac{B}{P}\right)^2 = \left(\frac{H}{P}\right)^2 \Rightarrow 1 + \cot^2\theta = \text{cosec}^2\theta$

node06\E_NODE65\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Commonly Used Values of Trigonometric Functions

Angle(θ)	**0°**	**30°**	**37°**	**45°**	**53°**	**60°**	**90°**
sin θ	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

Four Quadrants and ASTC Rule*



In first quadrant, all trigonometric ratios are positive.
In second quadrant, only sinθ and cosecθ are positive.
In third quadrant, only tanθ and cotθ are positive.
In fourth quadrant, only cosθ and secθ are positive

---

* Remember as **A**dd **S**ugar **T**o **C**offee or **A**fter **S**chool **T**o **C**ollege.

Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle 2nπ + θ where n=0, 1, 2, 3,.... will be remains same.

sin(2nπ +θ) = sinθ cos(2nπ+θ) = cosθ tan(2nπ+θ)=tanθ

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will remains same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

sin(π–θ) = + sinθ cos(π–θ) = –cosθ tan(π–θ)= –tanθ
sin(π+θ)= –sinθ cos(π+θ)= –cosθ tan(π+θ)= +tanθ
sin(2π–θ)= –sinθ cos(2π–θ)= +cosθ tan(2π–θ)= –tanθ

node06\E_NODE66 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2}+\theta\right)=+\cos\theta$ $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$ $\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos\theta$ $\cos\left(\frac{\pi}{2}-\theta\right)=+\sin\theta$ $\tan\left(\frac{\pi}{2}-\theta\right)=+\cot\theta$

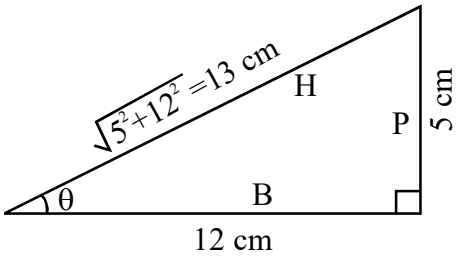
(iv) Trigonometric function of an angle $-\theta$ (negative angles)

$\sin(-\theta)=-\sin\theta$ $\cos(-\theta)=+\cos\theta$ $\tan(-\theta)=-\tan\theta$

Example :

The two shorter sides of right angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$.

Solution:



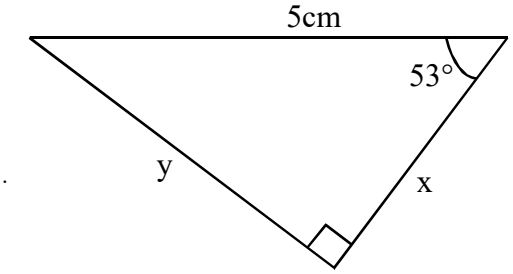
$$\sin\theta=\frac{P}{H}=\frac{5cm}{13cm}=\frac{5}{13}$$

$$\cos\theta=\frac{B}{H}=\frac{12cm}{13cm}=\frac{12}{13}$$

$$\tan\theta=\frac{P}{B}=\frac{5cm}{12cm}=\frac{5}{12}$$

Example :

Find x, y and perimeter of the triangle



Solution:

$\frac{y}{5}=\sin 53^\circ=\frac{4}{5}\Rightarrow y=4$ cm and $\frac{x}{5}=\cos 53^\circ=\frac{3}{5}\Rightarrow x=3$ cm

Perimeter of the triangle = x + y + 5 =3 + 4 + 5 = 12 cm

Example :

Find the value of :

(i) sin30 + cos60	(ii) sin 0 – cos 0	(iii) tan 45 –tan 37	(iv) sin 390
(v) cos 405	(iv) tan 420	(viii) sin 150	(viii) cos 120
(ix) tan 135	(x) sin (330)	(xi) cos 300	(xii) sin(–30)
(xiii) cos(–60)	(xiv) tan(–45)	(xvi) sin(–150)	

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Solution:

(i) $\sin 30 + \cos 60 = \frac{1}{2} + \frac{1}{2} = 1$

(ii) $\sin 0 - \cos 0 = 0 - 1 = -1$

(iii) $\tan 45 - \tan 37 = 1 - \frac{3}{4} = \frac{1}{4}$

(iv) $\sin 390 = \sin(360 + 30) = \sin 30 = \frac{1}{2}$

(v) $\cos 405 = \cos(360 + 45) = \cos 45 = \frac{1}{\sqrt{2}}$

(vi) $\tan 420 = \tan(360 + 60) = \tan 60 = \frac{1}{2}$

(vii) $\sin 150 = \sin(90 + 60) = \cos 60 = \frac{1}{2}$ or $\sin 150 = \sin(180 - 30) = \sin 30 = \frac{1}{2}$

(viii) $\cos 120 = \cos(180 - 60) = -\cos 60 = -\frac{1}{2}$

(ix) $\tan 135 = \tan(180 - 45) = -\tan 45 = -1$

(x) $\sin 330 = \sin(360 - 30) = -\sin 30 = -\frac{1}{2}$

(xi) $\cos 300 = \cos(360 - 60) = \cos 60 = \frac{1}{2}$

(xii) $\sin(-30) = -\sin 30 = -\frac{1}{2}$

(xiii) $\cos(-60) = +\cos 60 = \frac{1}{2}$

(xiv) $\tan(-45) = -\tan 45 = -1$

(xv) $\sin(-150) = -\sin(150) = -\sin(180 - 30) = -\sin 30 = -\frac{1}{2}$

Example :

The values of $\sin\theta_1$, $\cos^2\theta_2$ and $\tan\theta_3$ are given as ½ ,– ½ and 3 (not in order), for some angles θ_1, θ_2 and θ_3. Choose incorrect statement.

(A) The value of $\tan\theta_3$ could be –½

(B) The value of $\sin\theta_1$ can not be 3.

(C) The value of $\cos^2\theta_2$ can't be –½

(D) The value of $\cos^2\theta_2$ could be 3.

Solution: **Ans. (D)**

$-1 \le \sin\theta_1 \le 1$, $0 \le \cos^2\theta_2 \le 1$, $-\infty < \tan\theta_3 < \infty$

Addition/Subtraction Formulae for Trigonometrical Ratios

- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\sin(A-B) = \sin A \cos B - \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Example :

By using above basic addition/ subtraction formulae, prove that

(i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(iii) $\sin 2\theta = 2\sin\theta \cos\theta$

(iv) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

(v) $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Solution :

(i) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

$$= \frac{\cos A \cos B\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)}{\cos A \cos B\left(1 - \frac{\sin A \sin B}{\cos A \cos B}\right)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) $\tan (A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\cos A \cos B\left[\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}\right]}{\cos A \cos B\left[1 + \frac{\sin A \sin B}{\cos A \cos B}\right]} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(iii) $\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

(iv) $\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$= 1 - 2(1 - \cos^2\theta) = 2\cos^2\theta - 1$

(v) $\tan 2\theta = \tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$

Example :

Find the value of

(i) sin 74 (ii) cos 106 (iii) sin 15 (iv) cos 75

Solution:

(i) $\sin 74 = \sin(2 \cdot 37) = 2\sin 37 \cos 37 = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

(ii) $\cos 106 = \cos(2 \cdot 53) = \cos^2 53 - \sin^2 53 = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9-16}{25} = -\frac{7}{25}$

(iii) $\sin 15 = \sin(45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 75 = \cos(45 + 30) = \cos 45 \cos 30 - \sin 45 \sin 30 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Small Angle Approximation

If θ is small (say < 5) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$

Note : here θ must be in radian.

Example :

Find the approximate values of (i) sin 1 (ii) tan 2 (iii) cos 1 .

Solution:

(i) $\sin 1^\circ = \sin\left(1^\circ \times \frac{\pi}{180^\circ}\right) = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$

(ii) $\tan 2^\circ = \tan\left(2^\circ \times \frac{\pi}{180^\circ}\right) = \tan\frac{\pi}{90} \approx \frac{\pi}{90}$

(iii) $\cos 1^\circ = \cos\left(1^\circ \times \frac{\pi}{180^\circ}\right) = \cos\frac{\pi}{180} = 1$

node06\E_NOD06\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Maximum and Minimum Values of Some useful Trigonometric Functions

• $-1 \le \sin\theta \le 1$ • $-1 \le \cos\theta \le 1$ $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Example :

Find maximum and minimum values of y :

(i) $y = 2\sin x$ (ii) $y = 4-\cos x$ (iii) $y = 3\sin x + 4\cos x$

Solution:

(i) $y_{max} = 2(1) = 2$ and $y_{min} = 2(-1) = -2$ (ii) $y_{max} = 4 - (-1) = 4+1 = 5$ and $y_{min} = 4-(1) = 3$

(iii) $y_{max} = \sqrt{3^2+4^2} = 5$ and $y_{min} = -\sqrt{3^2+4^2} = -5$

Example :

A ball is projected with speed u at an angleθ to the horizontal. The range R of the projectile is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

for which value of θ will the range be maximum for a given speed of projection? (Here g = constant)

(a) $\frac{\pi}{2}$ rad (b) $\frac{\pi}{4}$ rad (c) $\frac{\pi}{3}$ rad (d) $\frac{\pi}{6}$ rad

Solution: **Ans.(B)**

As $\sin 2\theta \le 1$ so range will be maximum if $\sin 2\theta = 1$. Therefore $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ rad.

Example :

The position of a particle moving along x-axis varies with time t according to equation $x = \sqrt{3}\sin\omega t - \cos\omega t$ where ω is constants. Find the region in which the particle is confined.

Solution:

$\because x = \sqrt{3}\sin\omega t - \cos\omega t$

$\therefore x_{max} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $x_{min} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = -2$

Thus, the particle is confined in the region $-2 \le x \le 2$

SECTION [C] : COORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

node6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

ORIGIN

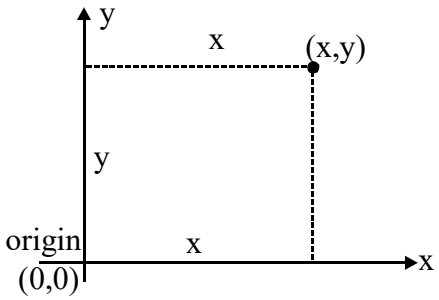
This is any fixed point which is convenient to you. All measurement are taken w.r.t. this fixed point.

Axis or Axes

Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure.



Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example :

For point (2, 14) find abscissa and ordinates. Also find distance from y and x-axis.

Solution:

Abscissa = x-coordinate = 2 = distance from y-axis.

Ordinate = y-coordinate = 14 = distance from x-axis.

Example :

Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Solution:

By using distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow 13 = \sqrt{[13 - (-9)]^2 + [3 - a]^2}$

$\Rightarrow 13^2 = 12^2 + (3-a)^2 \Rightarrow (3-a)^2 = 13^2 - 12^2 = 5^2 \Rightarrow (3-a) = \pm 5 \Rightarrow a = -2$ cm or 8 cm

Example :

A dog wants to catch a cat. The dog follows the path whose equation is y–x = 0 while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

(a) (2, –2) (b) (2, 2) (c) (–2, 2) (d) (–2, –2)

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Solution: **Ans. (BD)**

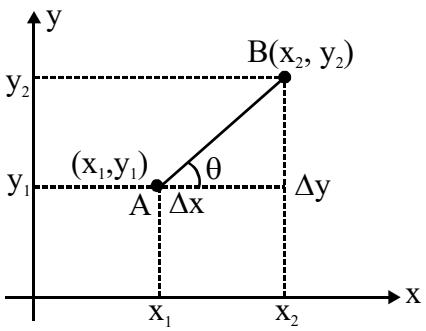
Let catching point be (x_1, y_1) then, $y_1 - x_1 = 0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; So possible points are (2, 2) and (–2, –2).

Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$ [If both axes have identical scales]



Here θ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.

Example :

Distance between two points (8, – 4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.

Solution: **Ans. 2**

From distance formula $(8-0)^2 + (-4-a)^2 = 100 \Rightarrow (4+a)^2 = 36 \Rightarrow a = 2$

SECTION [D] : CALCULUS

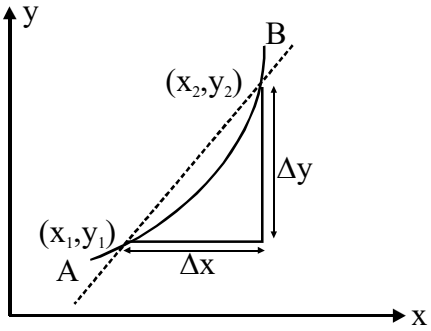
Calculus is the study of how things change. In this we study the relationship between continuously varying functions.

(A) DIFFERENTIAL CALCULUS

The purpose of differential calculus to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently.

Average rate of change :

Let a function y = f(x) be plotted as shown in figure. Average rate of change in y w.r.t. x in interval $[x_1, x_2]$ is

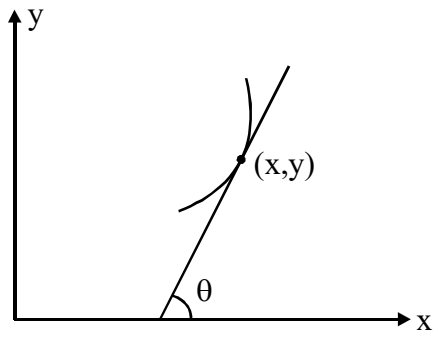


Average rate of change = $\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ = slope of chord AB.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Instantaneous rate of change :

It is defined as the rate of change in y with x at a particular value of x. It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x,y) and algebraically by the first derivative of function y =f(x).



Instantaneous rate of change = $\frac{dy}{dx}$ = slope of tangent = tanθ

First Derivatives of Commonly used Functions :

- y = constant $\Rightarrow \frac{dy}{dx}=0$
- $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$
- $y=e^x \Rightarrow \frac{dy}{dx} = e^x$
- $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$
- $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$
- $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$
- $y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$
- $y = \cot x \Rightarrow \frac{dy}{dx} = -\text{cosec}^2 x$

Method of Differentiation or Rules of Differentiation :

(i) Function multiplied by a constant i.e., y = kf(x) $\Rightarrow \frac{dy}{dx} = kf'(x)$

Example :

Find derivatives of the following functions :

(i) $y = 2x^3$ (ii) $y = \frac{4}{x}$ (iii) $y = 3e^x$ (iv) $y = 6 \ln x$ (v) $y = 5 \sin x$

Solution:

(i) $y = 2x^3 \Rightarrow \frac{dy}{dx} = 2\left[3x^{3-1}\right] = 6x^2$

(ii) $y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = 4\left[(-1)x^{-1-1}\right] = -\frac{4}{x^2}$

(iii) $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$

(iv) $y = 6\ln x \Rightarrow \frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{6}{x}$

(v) $y = 5\sin x \Rightarrow \frac{dy}{dx} = 5(\cos x) = 5\cos x$

node06\E_NOD06\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

(ii) Sum or Subtraction of Two functions i.e., $y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$

Example :

Find differentiation of y w.r.t x.

(i) $y = x^2 - 6x$ (ii) $y = x^5 + 2e^x$ (iii) $y = 4 \ln x + \cos x$

Solution:

(i) $\frac{dy}{dx} = 2x^{2-1} - 6(1) = 2x - 6$ (ii) $\frac{dy}{dx} = 5x^{5-1} + 2e^x = 5x^4 + 2e^x$

(iii) $\frac{dy}{dx} = 4\left(\frac{1}{x}\right) + (-\sin x) = \frac{4}{x} - \sin x$

(iii) Product of two functions : Product rule

$$y = f(x)\ g(x) \Rightarrow \frac{dy}{dx} = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Example :

Find first derivative of y w.r.t. x. (i) $y = x^2 \sin x$ (ii) $y = 4(e^x)\cos x$

Solution:

(i) $\frac{dy}{dx} = x^2(\cos x) + (2x)(\sin x) = x^2 \cos x + 2x \sin x$

(ii) $\frac{dy}{dx} = 4\left[(e^x)(\cos x) + (e^x)(-\sin x)\right] = 4e^x[\cos x - \sin x]$

(iv) Division of two functions : Quotient rule

$$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Example :

Find differentiation of y w.r.t. x. (i) $y = \frac{\sin x}{x}$ (ii) $y = \frac{4x^3}{e^x}$

Solution:

(i) Here $f(x) = \sin x$, $g(x) = x$ So $f'(x) = \cos x$,, $g'(x) = 1$

Therefore $\frac{dy}{dx} = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$

(ii) Here $f(x) = 4x^3$, $g(x) = e^x$ So $f'(x) = 12x^2, g'(x) = e^x$

Therefore, $\frac{dy}{dx} = \frac{12x^2(e^x) - 4x^3(e^x)}{(e^x)^2} = \frac{12x^2 - 4x^3}{e^x}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

(v) Function of Functions : Chain rule

Let f be a function of x, which in turn is a function of t. The first derivative of f w.r.t. t is equal to the product of $\frac{df}{dx}$ and $\frac{dx}{dt}$ Therefore $\frac{df}{dt}=\frac{df}{dx}\times\frac{dx}{dt}$

Example :

Find first derivative of y w.r.t. x.

(i) $y = e^{-x}$ (ii) $y = 4 \sin 3x$ (iii) $y=4e^{x^2-2x}$

Solution:

(i) $y = e^{-x} = e^z$ where $z = -x$ so $\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\left(e^z\right)(-1)=-e^z=-e^{-x}$

(ii) $y = 4 \sin 3x = 4 \sin z$ where $z = 3x$ so $\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=4(\cos z)(3)=12\cos 3x$

(iii) $y=4e^{x^2-2x}=4e^z$ where $z = x^2 - 2x$ so $\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=4(z)(2x-2)=(8x-8)e^{x^2-2x}$

Example :

The position of a particle moving along x-axis varies with time t as $x=4t -t^2 +1$. Find the time interval(s) during which the particle is moving along positive x-direction.

Solution:

If the particle moves along positive x-direction, its x-coordinate must increase with time t.

x-coordinate will increase with time t if $\frac{dx}{dt}>0$.

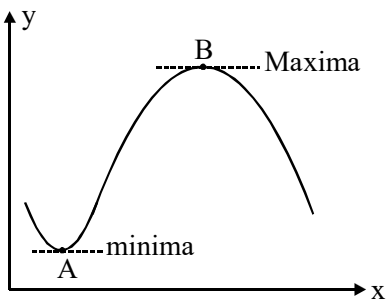
$$\frac{dx}{dt}=4-2t$$

$$\frac{dx}{dt}>0\Rightarrow 4-2t>0\Rightarrow t<2$$

Hence, the particle moves in positive x-direction during time-interval $0 < t < 2$.

MAXIMUM AND MINIMUM VALUE OF A FUNCTION

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative becomes zero.



node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

At point 'A' (minima) : As we see in figure, in the neighborhood of A, slope is increases so $\frac{d^2y}{dx^2} > 0$.

Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$

At point 'B' (maxima) : As we see in figure, in the neighborhood of B, slope is decreases so $\frac{d^2y}{dx^2} < 0$

Condition for maxima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

Example :

The minimum value of $y = 5x^2 - 2x + 1$ is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{4}{5}$ (d) $\frac{3}{5}$

Solution: **Ans. (C)**

For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$

Now at $x=\frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive so minima at $x = \frac{1}{5}$.

Therefore $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

Example :

The radius of a circular plate increase at the rate of 0.1 cm per second. At what rate does the area increase when the radius of plate is $\frac{5}{\pi}$ cm?

(A) 1 cm^2/s (B) 0.1 cm^2/s (C) 0.5 cm^2/s (D) 2 cm^2/s

Solution: **Ans. (A)**

Area of disk, $A = \pi r^2$ (where r = radius of disk)

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt} \text{ so } \frac{dA}{dt} = 2\pi \times \frac{5}{\pi} \times 0.1 = 1 \text{ cm}^2/\text{s}$$

Example :

A particle moves along the curve $12y = x^3$. Which coordinate changes at faster rate at x=10 ?

(A) x–coordinate (B) y–coordinate

(C) Both x and y–coordinate (D) Data insufficient

Solution: **Ans. (B)**

$12y = x^3 \Rightarrow 12dy = 3x^2dx \Rightarrow \frac{dy}{dt} = \left(\frac{x}{2}\right)^2\left(\frac{dx}{dt}\right)$

Therefore for $\left(\frac{x}{2}\right)^2 > 1$ or $x > 2$, y– coordinate changes at faster rate.

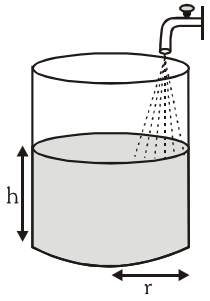
node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example :

Water pours out at the rate of Q from a tap, into a cylindrical vessel of radius r. The rate at which the height of water level rises when the height is h, is

(A) $\frac{Q}{\pi rh}$ (B) $\frac{Q}{\pi r^2}$ (C) $\frac{Q}{2\pi r^2}$ (D) $\frac{Q}{\pi r^2 h}$

Solution: **Ans. (B)**



$\because$ Volume : $V = \pi r^2 h$ $\therefore$ $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ But $\frac{dV}{dt} = Q$ so $\frac{dh}{dt} = \frac{Q}{\pi r^2}$

Example :

For conservative force field, $\vec{F} = -\frac{\partial U}{\partial x}\tilde{i} - \frac{\partial U}{\partial y}\tilde{j} - \frac{\partial U}{\partial z}\tilde{k}$

where F $\rightarrow$ Force, U $\rightarrow$ Potential energy and $\frac{\partial U}{\partial x}$ = Differentiation of U

w.r.t. x keeping y and z constant and so on.

	Column – I		Column – II
(A)	For $U = x^2 yz$, at (5,0,0)	(P)	$F_x = 0$
(B)	For $U = x^2 + yz$ at (5, 0,0)	(Q)	$F_y = 0$
(C)	For $U = x^2 (y+z)$ at (5, 0,0)	(R)	$F_z = 0$
(D)	For $U = x^2 y+z$ at (5,0,0)	(S)	$U = 0$

Solution: **Ans. (A – P,Q,R,S ; B – Q,R ; C – P,S ; D – P,S)**

For (A) : $\vec{F} = -2xyz\,\tilde{i} - x^2 z\tilde{j} - x^2 y\tilde{k} \Rightarrow F_x=0,\ F_y=0,\ F_z=0,\ U=0$

For (B) : $\vec{F} = -2x\tilde{i} - z\tilde{j} - y\tilde{k} \Rightarrow F_x \neq 0,\ F_y=0,\ F_z=0,\ U \neq 0$

For (C) : $\vec{F} = -2x(y+z)\tilde{i} - x^2\tilde{j} - x^2\tilde{k} \Rightarrow F_x=0,\ F_y \neq 0,\ F_z \neq 0,\ U=0$

For (D) : $\vec{F} = -2xy\,\tilde{i} - x^2\tilde{j} - \tilde{k} \Rightarrow F_x=0,\ F_y \neq 0,\ F_z \neq 0,\ U=0$

Example :

If surface area of a cube is changing at a rate of 5 m^2/s, find the rate of change of body diagonal at the moment when side length is 1 m.

(A) 5 m/s (B) $5\sqrt{3}$ m/s (C) $\frac{5}{2}\sqrt{3}$ m/s (D) $\frac{5}{4\sqrt{3}}$ m/s

Solution: **Ans. (D)**

Surface area of cube $S=6a^2$ (where a = side of cube)

Body diagonal $\ell = \sqrt{3}a$. Therefore $S=2\ell^2$

Differentiating it w.r.t. time $\frac{dS}{dt} = 2(2\ell)\frac{d\ell}{dt} \Rightarrow \frac{d\ell}{dt} = \frac{1}{4(\sqrt{3}a)}\frac{dS}{dt} = \frac{5}{4\sqrt{3}}$ m/s

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

(B) Integral Calculus

Integration is the reverse process of differentiation. By help of integration we can find a function whose derivative is known. Consider a function F(x) whose differentiation w.r.t. x is equal to f(x) then

$$\int f(x)dx = F(x) + c$$

here c is the constant of integration and this is called indefinite integration.

Few basic formulae of integration are :

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \int \frac{1}{x} dx = \ln x + c$$

$$\int \sin x dx = -\cos x + c \qquad \int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c \qquad \int e^x dx = e^x + c$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad \int \frac{dx}{ax+b} = \frac{\ln(ax+b)}{a} + c$$

$$\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a} + c \qquad \int \cos(ax+b)dx = \frac{\sin(ax+b)}{a} + c$$

$$\int \sec^2(ax+b)dx = \frac{\tan(ax+b)}{a} + c \qquad \int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$$

Example :

Integrate the following w.r.t. x.

(i) $4x^3$ (ii) $x - \frac{1}{x}$ (iii) $\frac{1}{2x+3}$ (iv) cos (4x+3) (v) $\cos^2 x$

Solution:

(i) $\int 4x^3 dx = 4\left(\frac{x^{3+1}}{3+1}\right) + c = \frac{4x^4}{4} + c = x^4 + c$

(ii) $\int \left(x - \frac{1}{x}\right)dx = \int x dx - \int \frac{1}{x} dx = \frac{x^2}{2} - \ln x + c$

(iii) $\int \frac{dx}{2x+3} = \frac{\ln(2x+3)}{2} + c$

(iv) $\int \cos(4x+3)dx = \frac{\sin(4x+3)}{4} + c$

(v) $\int \cos^2 x dx = \int \frac{2\cos^2 x}{2} dx = \int \frac{(1+\cos 2x)}{2} dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$

$$= \frac{x}{2} + \frac{1}{2}\frac{\sin 2x}{2} + c = \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

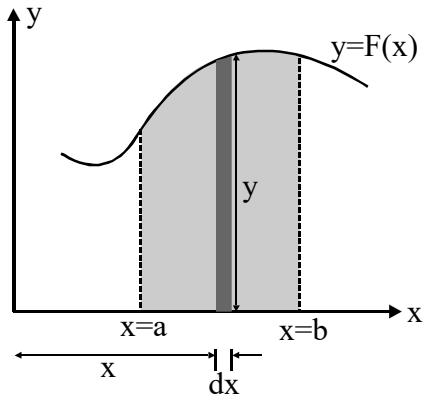
Definite Integration :

When a function is integrated between a lower limit and an upper limit, it is called a definite integral. Consider a function F(x) whose differentiation w.r.t. x is equal to f(x), in an interval $a \le x \le b$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Area under a curve and definite integration



Area of small shown element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then $\int_a^b f(x)dx$ = shaded area between curve and x-axis.

Example :

The integral $\int_1^5 x^2 dx$ is equal to

(A) $\frac{125}{3}$ (B) $\frac{124}{3}$ (C) $\frac{1}{3}$ (D) 45

Solution: **Ans. (B)**

$$\int_1^5 x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3} - \frac{1^3}{3}\right] = \frac{125}{3} - \frac{1}{3} = \frac{124}{3}$$

Example :

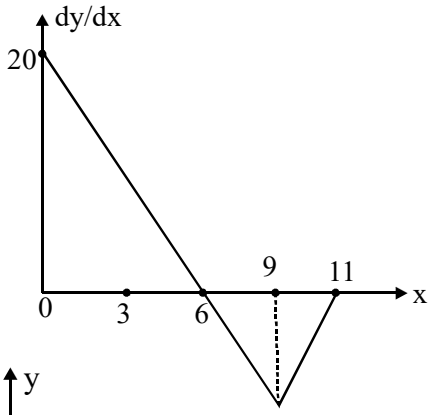
The following curve represent rate of change of a variable y w.r.t x. The change in the value of y when x changes from 0 to 11 is:

(a) 60

(b) 25

(c) 35

(d) 85



Solution:

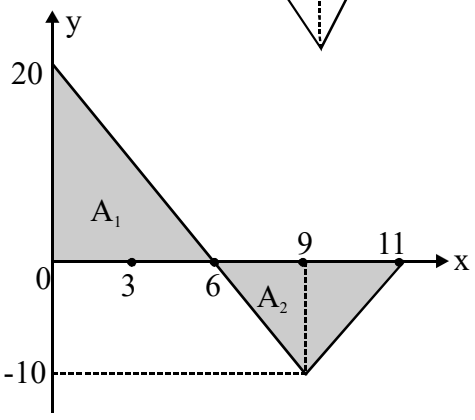
As $dy = \left(\frac{dy}{dx}\right)dx$ So $\Delta y = \int dy = \int_0^{11}\left(\frac{dy}{dx}\right)dx$

Area under the curve

$$A_1 = \frac{1}{2}\times 6 \times 20 = 60$$

$$A_2 = -\frac{1}{2}\times(11-6)(10) = -25$$

$$\Delta y = A_1 + A_2 = 60 - 25 = 35$$



node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Average value of a continuous function in an interval

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by

$$y_{av} = \frac{\int_a^b y\,dx}{\int_a^b dx} = \frac{\int_a^b y\,dx}{b-a}$$

Example :

Determine the average value of y = 2x +3 in the interval $0 \le x \le 1$.

(A) 1 (B) 5 (C) 3 (D) 4

Solution: **Ans. (D)**

$$y_{av} = \frac{\int_0^1 y\,dx}{1-0} = \int_0^1 (2x+3)\,dx = \left[2\left(\frac{x^2}{2}\right)+3x\right]_0^1 = 1^2+3(1)-0^2-3(0) = 1+3 = 4$$

Example :

The average value of alternating current $I=I_0 \sin\omega t$ in time interval $\left[0, \frac{\pi}{\omega}\right]$ is

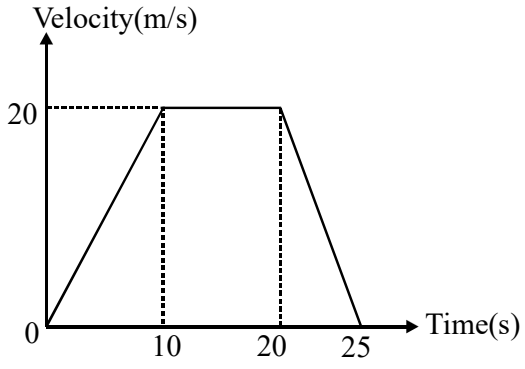
(A) $\frac{2I_0}{\pi}$ (B) $2I_0$ (C) $\frac{4I_0}{\pi}$ (D) $\frac{I_0}{\pi}$

Solution: **Ans. (A)**

$$I_{av} = \frac{\int_0^{\pi/\omega} I\,dt}{\frac{\pi}{\omega}-0} = \frac{\omega}{\pi}\int_0^{\pi/\omega} I_0 \sin\omega t\,dt = \frac{\omega}{\pi}\left[\frac{I_0(-\cos\omega t)}{\omega}\right]_0^{\pi/\omega} = -\frac{\omega}{\pi}\frac{I_0}{\omega}[\cos\pi - \cos 0] = -\frac{I_0}{\pi}[-1-1] = \frac{2I_0}{\pi}$$

Example :

The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car is first 25 seconds is



(A) 20 m/s (B) 14 m/s (C) 10 m/s (D) 17.5 m/s

Solution: **Ans. (B)**

$$\text{Average velocity} = \frac{\int_0^{25} v\,dt}{25-0} = \frac{\text{Area of v-t graph between t=0 to t = 25 s}}{25} = \frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right] = 14 \text{ m/s}$$

node06\E_NOD06\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

SECTION [E] : FUNCTION AND GRAPH

FUNCTION

Physics involves study of natural phenomena and describes them in terms of several physical quantities. A mathematical formulation of interdependence of these physical quantities is necessary for a concise and precise description of the phenomena. These mathematical formulae are expressed in form of equations and known as function.

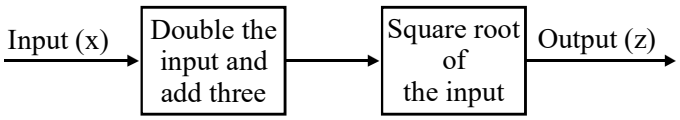
Thus, a function describing a physical process expresses an unknown physical quantity in terms of one or more known physical quantities. We call the unknown physical quantity as dependent variable and the known physical quantities as independent variables. For the sake of simplicity, we consider a function that involves a dependant variable y and only one independent variable x. It is denoted y=f(x) and is read as y equals to f of x. Here f(x) is the value of y for a given x. Following are some examples of functions.

$$y=2x+1, \qquad y=2x^2+3x+1, \qquad y=\sin x, \qquad y=\ell n\,(2x+1)$$

Knowledge of the dependant variable for different values of the independent variable, and how it changes when the independent variable varies in an interval is collectively known as behavior of the function.

Example :

In the given figure, each box represents a function machine. A function machine illustrates what it does with the input.



Which of the following statements are correct?

(A) $y=2x+3$ (B) $y=2(x+3)$ (C) $z=\sqrt{2x+3}$ (D) $z=\sqrt{2(x+3)}$

Ans. (C)

Graph of a Function.

Graph is diagrammatic representation of a function and allows us to visualize it. To plot a graph the dependant variable (here y) is usually taken on the ordinate and the independent variable (here x) on the abscissa. Graph being an alternative way to represent a function does not require elaborate calculations and explicitly shows behavior of the function in a concerned interval.

Example :

Consider a body moving with constant speed of 2 m/s in a straight line. When you start your stopwatch, you observe the body 1 m away from a fixed point on the line. Suggest suitable physical quantities, write a function and draw its graph describing motion of the body.

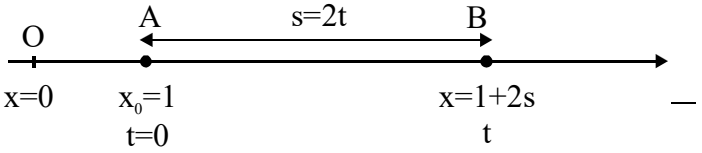
Solution :

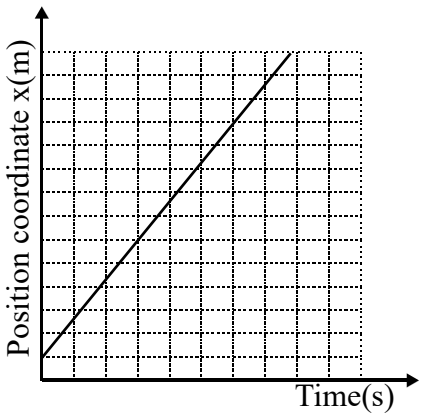
Distance x of the body from the given fixed point and time t measured by the stopwatch are the suitable variables. If we consider the fixed point as the origin, distance x is known as the position coordinate of the body.

In the following figure it is shown that the body is on point A at the instant t = 0 and after a time t it reaches another point B covering a distance, which equals to product of speed and time interval. Thus, distance s covered by the body in time t is given by the following equation.

$$s = 2t$$

node06\E_NOD06\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65





With the help of the above figure, position coordinate x of the body at any time t is given by the following equation, which is the required function describing motion of the body.

$$x = 2t + 1$$

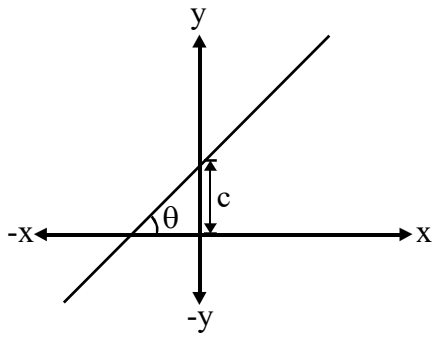
Graph of this equation is also shown in the adjoining figure.

Graphs of some commonly used Functions.

Linear, parabolic, trigonometric and exponential functions are the most common in use.

Straight line Equation and its Graph.

When the dependant variable y varies linearly with the independent variable x, the relationship between them is represented by a linear equation of the type given below. The equation is also shown in graph by an arbitrary line.

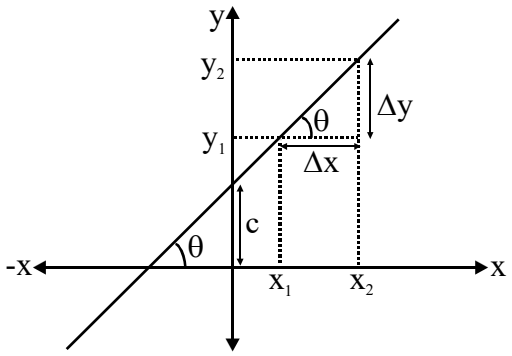


$y = mx + c$

Here m & c are known as slope of the line and intercept on the y-axis, respectively.

Slope

Slope of a line is a quantitative measure to express the inclination of the line. It is expressed by ratio of change in ordinate to change in obscissa.



Slope of a line $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$ = slope of tangent

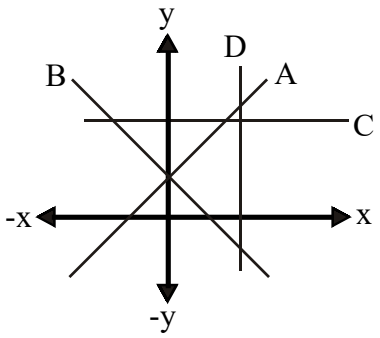
node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

When the x and the y axes are scaled identically, slope equals to tangent of the angle, which line makes with the positive x-axis.

$$m=\tan\theta$$

Sometimes the slope is also called gradient and expressed by the term "Δy in ℓ'" where ℓ is the length along the line $\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

It is positive if y increases with increase in x, negative if y decreases with increase in x, zero if y remains unchanged with change in x and infinite if y changes but x remains unchanged. For these cases the line is inclined up, inclined down, parallel to x-axis and parallel to y-axis respectively as shown in the adjoining figure by lines A, B, C and D respectively.

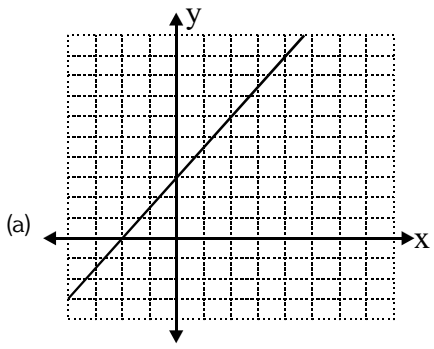


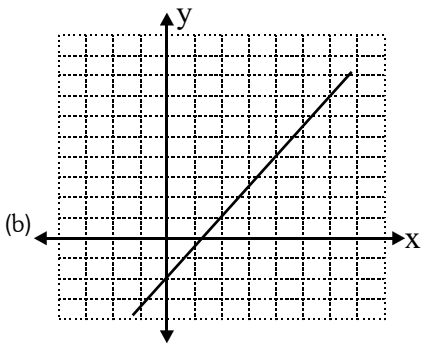
Intercept.

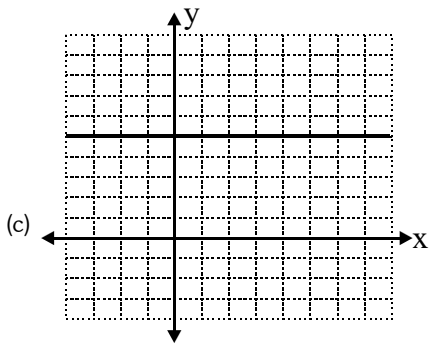
It equals to the value of ordinate y, where the line cuts the y-axis. It may be positive, negative or zero for lines crossing the positive y-axis, negative y-axis and passing through the origin respectively.

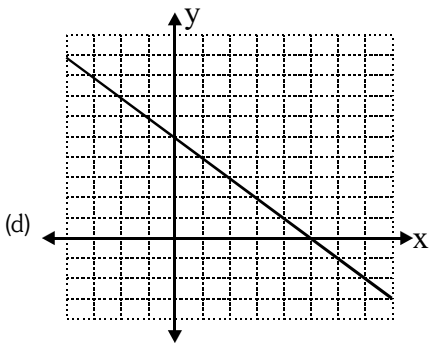
Example :

Write equations for the straight lines shown in the following graphs.

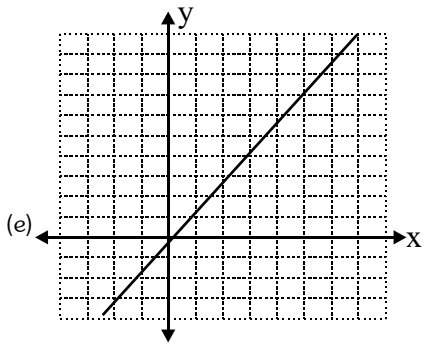


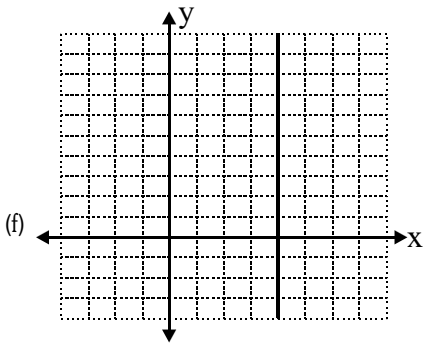






node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65



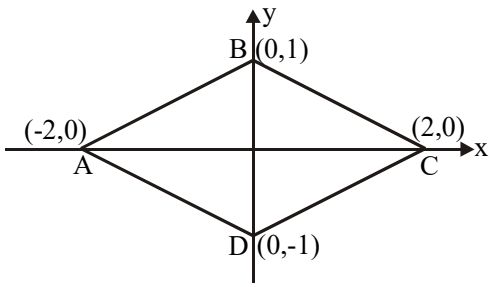


Solution : **Ans.**

(a) $y = \frac{3}{2}x + 3$; (b) $y = \frac{3}{2}x - 2$; (c) y=5; (d) y=–x+5 (e) $y = \frac{3}{2}x$; (f) x=4

Example :

A parallelogram ABCD is shown in figure.



	Column I		Column II
(A)	Equation of side AB	(P)	2y + x =2
(B)	Equation of side BC	(Q)	2y – x = 2
(C)	Equation of side CD	(R)	2y + x = –2
(D)	Equation of side DA	(S)	2y –x =–2
		(T)	y + 2x =2

Solution : **Ans. (A) Q (B) P (C) S (D) R**

For side AB : $m = \frac{1-0}{0-(-2)} = \frac{1}{2}, \quad c = 1 \Rightarrow y = \frac{1}{2}x + 1$

For side BC : $m = \frac{2-0}{0-1} = -2, \quad c = 1 \Rightarrow y = -2x + 1$

For side CD : $m = \frac{0-(-1)}{2-0} = \frac{1}{2}, \quad c = -1 \Rightarrow y = \frac{1}{2}x - 1$

For side DA : $m = \frac{-1-0}{0-(-2)} = -\frac{1}{2}, \quad c = -1 \Rightarrow y = -\frac{1}{2}x - 1$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

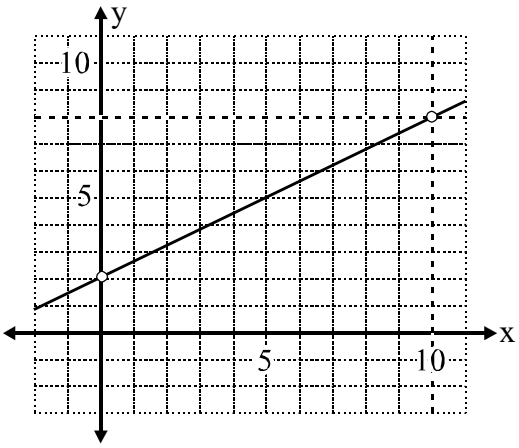
Example

A variable y increases from y_1 = 2 to y_2 = 8 linearly with another variable x in the interval x_1 = 0 to x_2 = 10. Express y as function of x and draw its graph.

Solution :

Linear variation is represented by a linear equation of the form y=mx+c. To represent the function on graph we have to join two points whose coordinates are (x_1, y_1) and (x_2, y_2) i.e. (0, 2) and (10, 8).

Slope of the line is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{10-0} = \frac{3}{5}$



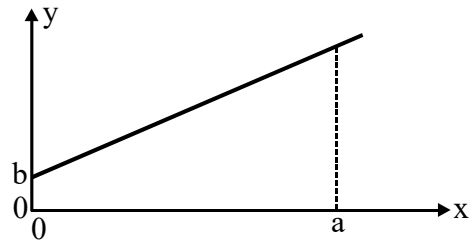
From the graph, intercept is c=2. Now the required equation is $y = \frac{3}{5}x + 2$

Example:

The graph shows a linear relation between variable y and x. Consider two quantities p and q defined by the equations .

$p = \frac{y}{x}$

$q = \frac{y-b}{x}$



As x changes from zero to a, which of the following statements are correct according to the graph?

(A) Quantity p increases and q decrease.
(B) Quantity p decrease and q increases.
(C) Quantity p decreases and q remain constant.
(D) Quantity p increases and q remain constant.

Solution :

Ans. (C)

q is slope of the given line, which is a constant for a straight line.

p is slope of the line joing origin and point on the line, which is decreaes as x increases.

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example:

Frequency f of a simple pendulum depends on its length ℓ and acceleration g due to gravity according to the following equation

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

Graph between which of the following quantities is a straight line?

(A) f on the ordinate and l on the abscissa

(B) f on the ordinate and $\sqrt{\lambda}$ on the abscissa

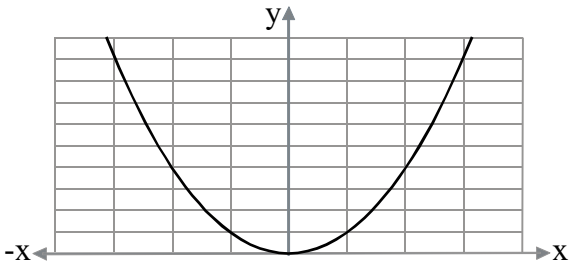
(C) f^2 on the ordinate and l on the abscissa

(D) f^2 on the ordinate and $1/\lambda$ on the abscissa

Ans. (D)

Parabola equation and its graph.

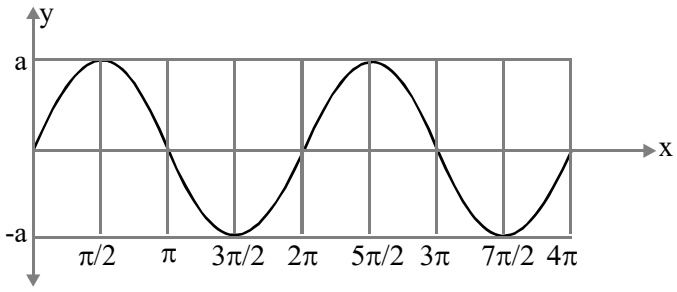
A function of the form $y=ax^2+bx+c$ is known as parabola. The simplest parabola has the form $y=ax^2$. Its graph is shown in the following figure.



Graph of some trigonometric functions.

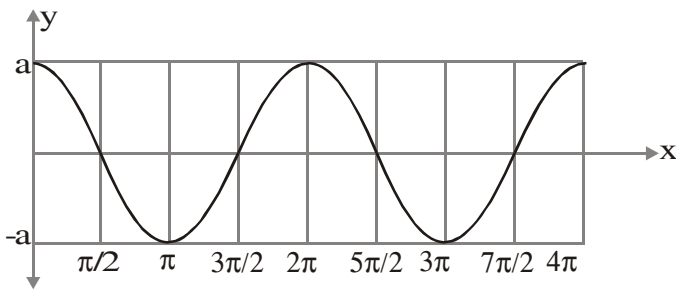
Among all the trigonometric functions, sinusoidal function, which includes sine and cosine both is most common in use.

Sine Function $y = a \sin x$



Here, a is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a sine function is shown, which has amplitude a units.

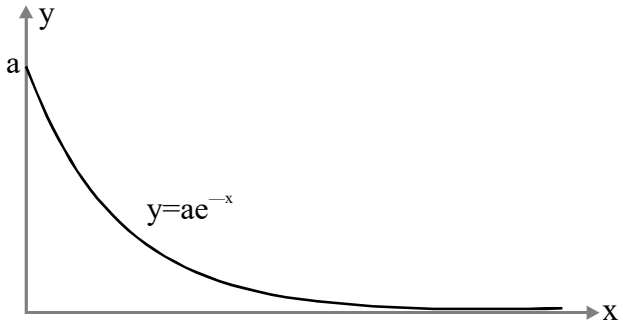
Cosine Function $y = a \cos x$



node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Here, a is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a cosine function is shown, which has amplitude a units.

Exponential function and its graph.

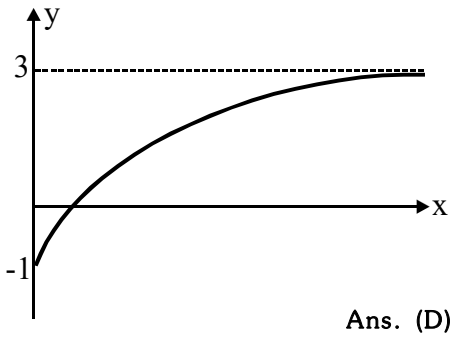


Behavior of several physical phenomena is described by exponential function to the base e. Here e is known as Euler's Number. e=2.718218

Most commonly used exponential function has the form $y=ae^{-x}$. In the adjoining figure graph of this function is shown.

Example :

In the given figure is shown a variable y varying exponentially on another variable x. Study the graph carefully. Which of the following equations best suits the shown graph?



(A) $y = 3 - e^{-x}$ (B) $y = 1 - 4e^{-x}$

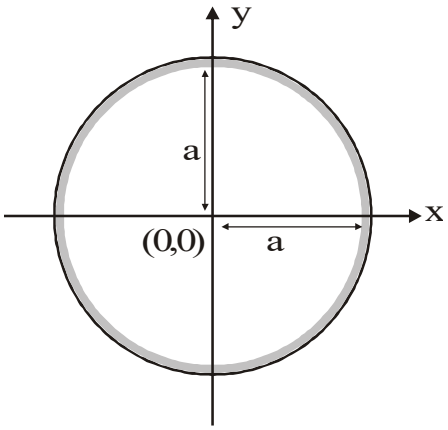
(C) $y = 1 - 3e^{-x}$ (D) $y = 3 - 4e^{-x}$

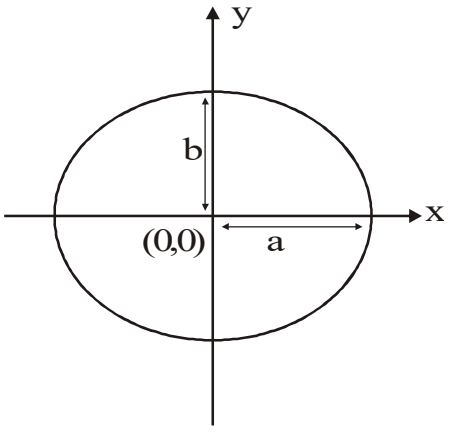
Solution: **Ans. (D)**

Shift the curve $(-4e^{-x})$ in positive y-direction by 3 units.

Circle and Ellipse

Circle : $x^2 + y^2 = a^2$ Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$





node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

SECTION [F] : VECTORS

Precise description of laws of physics and physical phenomena requires expressing them in form of mathematical equations. In doing so we encounter several physical quantities, some of them have only magnitude and other have direction in addition to magnitude. Quantities of the former kind are referred as scalars and the latter as vectors and mathematical operations with vectors are collectively known as vector analysis.

VECTORS

A vector has both magnitude and sense of direction, and follows triangle law of vector addition.

For example, displacement, velocity, and force are vectors.

Vector quantities are usually denoted by putting an arrow over the corresponding letter, as $\vec{A}$ or $\vec{a}$. Sometimes in print work (books) vector quantities are usually denoted by boldface letters as **A** or **a**.

Magnitude of a vector $\vec{A}$ is a positive scalar and written as $|\vec{A}|$ or A.

Unit Vector

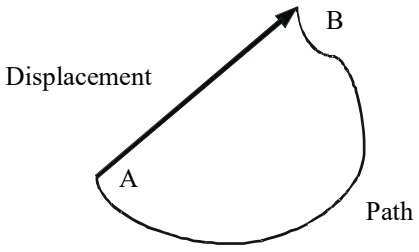
It is mathematical way to express direction of a vector and defined by the ratio of a vector to its magnitude. When a unit vector is multiplied with a scalar magnitude, we get a vector of corresponding magnitude in the direction of the unit vector. A unit vector is usually represented by putting a sign (^) known as cap, hat or caret over a letter assigned to the unit vector. This letter may be the same as used for the vector, or its lower case letter, or some other symbol. For example, if we assign lower case letter a to unit vector in the direction of vector $\vec{A}$, the unit vector denoted by $\hat{a}$ is expressed by the following equation.

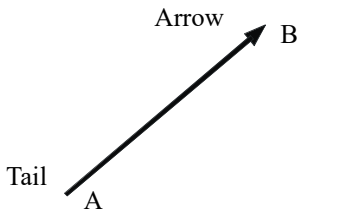
$$\vec{A} = A\hat{a}$$

Geometrical Representation of Vectors.

Geometrically a vector is represented by a directed straight-line segment drawn to a scale. Starting point of the directed line segment is known as tail and the end-point as arrow, head, or tip. The orientation of the line and the arrow collectively show the direction and the length of the line drawn to a scale shows the magnitude.

For example let a particle moves from point A to B following a curvilinear path shown in the figure. It displacement vector is straight line AB directed form A to B. If straight-line distance between A and B is 25 m, the directed line segment has to be drawn to suitable scale. If we assume the scale 1.0 cm = 10 m, the geometrical length of the displacement vector AB must be 2.5 cm.





Geometrical representation of Displacement Vector

Addition of Vectors: The Triangle Law

Use of geometry in solving problems involving vectors is of fundamental nature. The triangle law also uses principles of plane geometry. This law states:

The vectors to be added are drawn in such a manner that the tail of a vector coincides the tip of the preceding vector (in tip to tail fashion); their resultant is defined by the vector drawn from the tail of the first vector to the tip of the second vector. The two vectors to be added and their resultant are coplanar.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure-I. Using the triangle law, we obtain geometrical construction shown in the figure-II, where it is shown that two vectors and their sum $\vec{C} = \vec{A} + \vec{B}$ always make a closed triangle. If we change order of vector $\vec{A}$ and $\vec{B}$, it shown in figure-III that sum given by equation $\vec{C} = \vec{B} + \vec{A}$ remain unchanged. Therefore, vector addition is commutative.

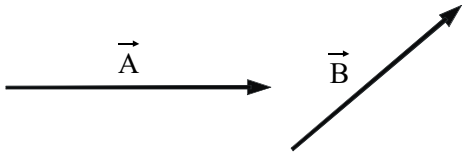


Fig. (i)

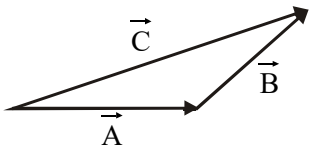


Fig. (ii)

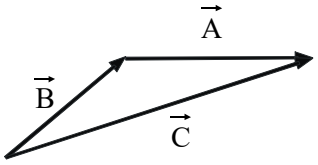
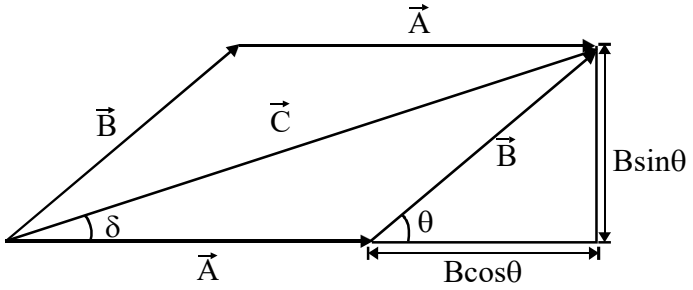


Fig. (iii)

Construction, which is combination of the figure-II and III, is in form of a parallelogram and is shown in figure-IV.



Geometry of the above figure suggests the following results.

$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}\ ;\quad \delta = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

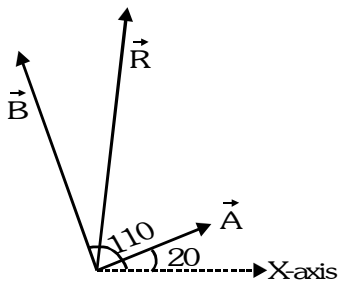
Example :

A vector $\vec{A}$ and $\vec{B}$ make angles of 20 and 110 respectively with the X–axis. The magnitudes of these vectors are 5m and 12m respectively. Find their resultant vector.

Solution :

Angle between the $\vec{A}$ and $\vec{B}$ = 110 – 20 = 90

$$R = \sqrt{A^2 + B^2 + 2AB\cos 90^\circ} = \sqrt{5^2 + 12^2} = 13\text{m}$$



Let angle of $\vec{R}$ from $\vec{A}$ is α

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} = \frac{12\sin 90^\circ}{5 + 12\cos 90^\circ} = \frac{12\times 1}{5 + 12\times 0} = \frac{12}{5}$$

or $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$ with vector $\vec{A}$ or (α + 20) with X–axis

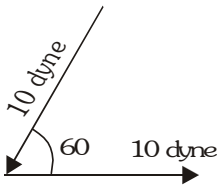
node06\E_NOD06 (E)\Data\2014\Kota\IIEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example :

Two forces each numerically equal to 10 dynes are acting as shown in the figure, then find resultant of these two vectors.

Solution :

The angle θ between the two vectors is 120 and not 60 .



$$\therefore R = \sqrt{(10)^2 + (10)^2 + 2(10)(10)(\cos 120^\circ)}$$

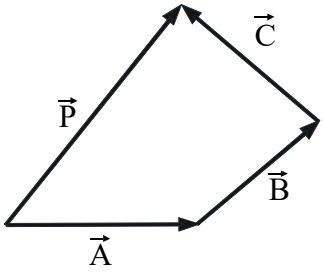
$$= \sqrt{100 + 100 - 100} = 10 \text{ dyne}$$

Addition of more than two Vectors

The triangle law can be extended to define addition of more than two vectors. Accordingly, if vectors to be added are drawn in tip to tail fashion, resultant is defined by a vector drawn from the tail of the first vector to the tip of the last vector. This is also known as the polygon rule for vector addition.

Operation of addition of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and their resultant $\vec{P}$ are shown in figure.

$$\vec{A} + \vec{B} + \vec{C} = \vec{P}$$

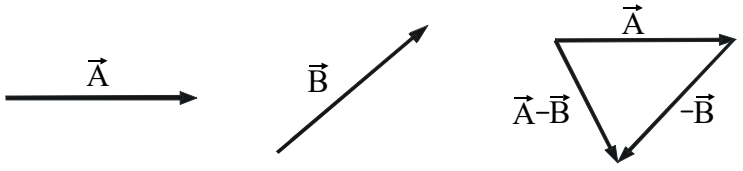


Here it is not necessary that three or more vectors and their resultant are coplanar. In fact, the vectors to be added and their resultant may be in different planes. However if all the vectors to be added are coplanar, their resultant must also be in the same plane containing the vectors.

Subtraction of Vectors

A vector opposite in direction but equal in magnitude to another vector $\vec{A}$ is known as negative vector of $\vec{A}$. It is written as $-\vec{A}$. Addition of a vector and its negative vector results a vector of zero magnitude, which is known as a null vector. A null vector is denoted by arrowed zero $(\vec{0})$.

The idea of negative vector explains operation of subtraction as addition of negative vector. Accordingly to subtract a vector from another consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure. To subtract $\vec{B}$ from $\vec{A}$, the negative vector $-\vec{B}$ is added to $\vec{A}$ according to the triangle law as shown in figure-II.



Multiplying by a number

Multiplication by a positive number changes magnitude of the vector but not the direction and multiplication by a negative number changes magnitude and reverses direction.

Thus multiplying a vector by a number n makes magnitude of the vector n times. $n\vec{A} = (nA)\hat{a}$

Here $\hat{a}$ denotes the unit vector in the direction of vector $\vec{A}$.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Resolution of a Vector into Components

Following laws of vector addition, a vector can be represented as a sum of two (in two-dimensional space) or three (in three-dimensional space) vectors each along predetermined directions. These directions are called axes and parts of the original vector along these axes are called components of the vector.

Cartesian components in two dimensions

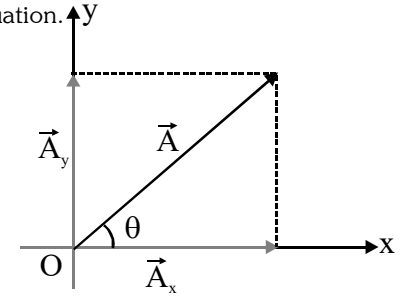
If a vector is resolved into its components along mutually perpendicular directions, the components are called Cartesian or rectangular components.

In figure is shown, a vector $\vec{A}$ resolved into its Cartesian components $\vec{A}_x$ and $\vec{A}_y$ along the x and y-axis. Magnitudes A_x and A_y of these components are given by the following equation.

$A_x = A\cos\theta$ and $A_y = A\sin\theta$

$\vec{A} = A_x\hat{i} + A_y\hat{j}$

$A = \sqrt{A_x^2 + A_y^2}$

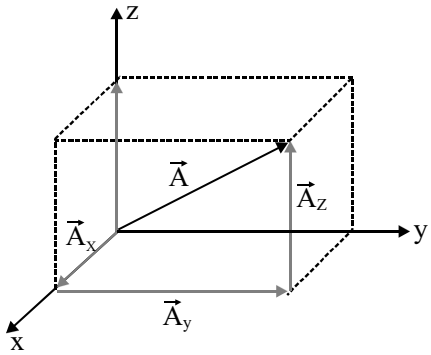


Here $\hat{i}$ and $\hat{j}$ are the unit vectors for x and y coordinates respectively.

Mathematical operations e.g. addition, subtraction, differentiation and integration can be performed independently on these components. This is why in most of the problems use of Cartesian components becomes desirable.

Cartesian components in three dimensions

A vector $\vec{A}$ resolved into its three Cartesian components one along each of the directions x, y, and z-axis is shown in the figure.



$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$; $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Equal Vectors

Two vectors of equal magnitudes and same directions are known as equal vectors. Their x, y and z components in the same coordinates system must be equal.

If two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ are equal vectors, we have

$$\vec{a} = \vec{b} \Rightarrow a_x = b_x,\ a_y = b_y \text{ and } a_z = b_z$$

Parallel Vectors

Two parallel vectors must have the same direction and may have unequal magnitudes. Their x, y and z components in the same coordinate system bear the same ratio.

Consider two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, if they are parallel, we have

$$\vec{a} \cdot \vec{b} \Rightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

node06\E_NODE65\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

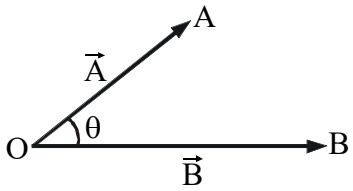
ALLEN Materials Provided By - Material Point Available on Learnaf.com

Product of Vectors

In all physical situation, whose description involve product of two vectors, only two categories are observed. One category where product is also a vector involves multiplication of magnitudes of two vectors and sine of the angle between them, while the other category where product is a scalar involves multiplication of magnitudes of two vectors and cosine of the angle between them. Accordingly, we define two kinds of product operation. The former category is known as vector or cross product and the latter category as scalar or dot product.

Scalar or dot product of two vectors

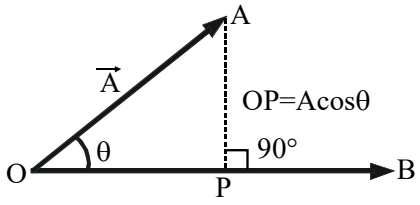
The scalar product of two vectors $\vec{A}$ and $\vec{B}$ equals to the product of their magnitudes and the cosine of the angle θ between them.

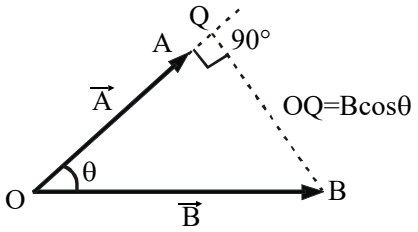


$$\vec{A}\cdot\vec{B} = AB\cos\theta = OA\cdot OB\cdot\cos\theta$$

The above equation can also be written in the following ways.

$$\vec{A}\cdot\vec{B} = (A\cos\theta)B = OP\cdot OB \qquad \vec{A}\cdot\vec{B} = A(B\cos\theta) = OA\cdot OQ$$

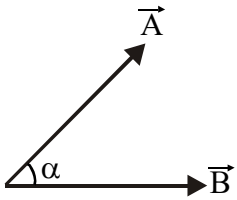


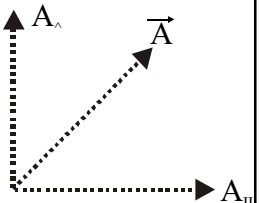


Above two equations and figures, suggest a scalar product as product of magnitude of the one vector and magnitude of the component of another vector in the direction of the former vector.

GOLDEN KEY POINTS

- Dot product of two vectors is commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- If two vectors are perpendicular, their dot product is zero. $\vec{A}\cdot\vec{B} = 0$, if $\vec{A}\perp\vec{B}$
- Dot product of a vector by itself is known as self-product. $\vec{A}\cdot\vec{A} = A^2 \Rightarrow A = \sqrt{\vec{A}\cdot\vec{A}}$
- The angle between the vectors $\theta = \cos^{-1}\left(\dfrac{\vec{A}\cdot\vec{B}}{AB}\right)$





- (a) Component of $\vec{A}$ in direction of $\vec{B}$

$$\vec{A}_{\parallel} = (|\vec{A}|\cos\theta)\tilde{B} = |\vec{A}|\left(\frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}\right)\tilde{B} = \left(\frac{\vec{A}.\vec{B}}{|\vec{B}|}\right)\tilde{B} = (\vec{A}.\tilde{B})\tilde{B}$$

(b) Component of $\vec{A}$ perpendicular to $\vec{B}$: $\vec{A}_{\perp} = \vec{A} - \vec{A}_{\parallel}$

- Dot product of Cartesian unit vectors: $\tilde{i}\cdot\tilde{i} = \tilde{j}\cdot\tilde{j} = \tilde{k}\cdot\tilde{k} = 1$

$$\tilde{i}\cdot\tilde{j} = \tilde{j}\cdot\tilde{k} = \tilde{k}\cdot\tilde{i} = 0$$

- If $\vec{A} = A_x\tilde{i} + A_y\tilde{j} + A_z\tilde{k}$ and $\vec{B} = B_x\tilde{i} + B_y\tilde{j} + B_z\tilde{k}$, their dot product is given by

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example :

If $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|$, then find the angle between $\vec{A}$ and $\vec{B}$.

Solution :

$\because |\vec{A}+\vec{B}|=|\vec{A}-\vec{B}| \quad \therefore \sqrt{A^2+B^2+2AB\cos\theta}=\sqrt{A^2+B^2-2AB\cos\theta}$

or $A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$ or $\cos\theta = 0$ $\therefore \theta = 90$

Example :

If $\vec{A}=4\tilde{i}+n\tilde{j}-2\tilde{k}$ and $\vec{B}=2\tilde{i}+3\tilde{j}+\tilde{k}$, then find the value of n so that $\vec{A}\perp\vec{B}$

Solution :

Dot product of two mutually perpendicular vectors is zero $\vec{A}\cdot\vec{B}=0$

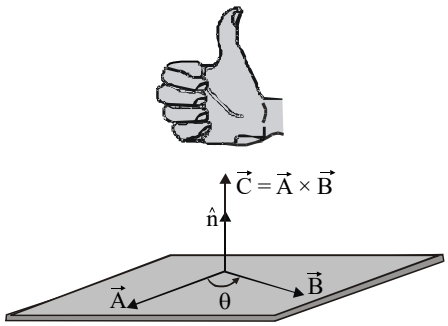
$\therefore (4\tilde{i}+n\tilde{j}-2\tilde{k}).(2\tilde{i}+3\tilde{j}+\tilde{k})=0 \Rightarrow (4\times2)+(n\times3)+(-2\times1)=0 \Rightarrow 3n = -6 \Rightarrow n = -2$

Vector or cross product of two vectors

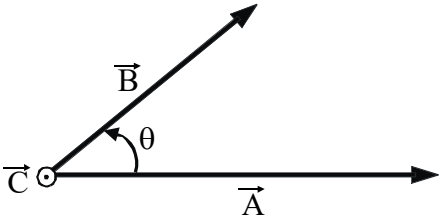
The vector product $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ is defined as

- Its magnitude is the product of magnitudes of $\vec{A}$ and $\vec{B}$ and of the sine of angle θ between vectors $\vec{A}$ and $\vec{B}$.
- Its direction is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and is decided by right hand rule by curling fingers in the direction from the first vector towards the second vector. In figure, where it is represented by $\tilde{n}$.

$\vec{C}=\vec{A}\times\vec{B}=(AB\sin\theta)\cdot\tilde{n}$



On paper vectors perpendicularly out and into the plane of paper are represented by encircled dot ⊙ and encircled cross ⊗ signs respectively. Following this convention, cross product $\vec{C}=\vec{A}\times\vec{B}$ is shown in the figure.

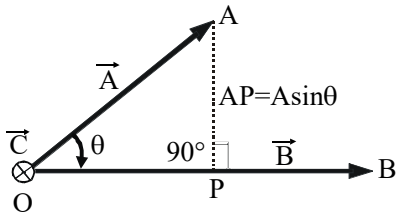


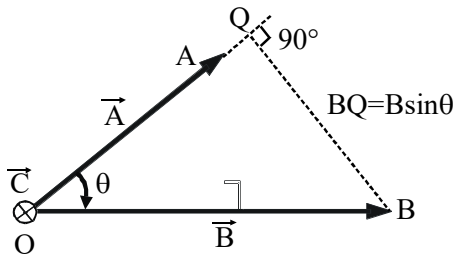
To have different symbols for scalar and vector products, symbols dot (·) and cross ( ) respectively are written between the vectors undergoing these operations.

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Cross product $\vec{C} = \vec{A} \times \vec{B}$, can also be written in the following ways.

$\vec{C} = \vec{A} \times \vec{B} = A(B\sin\theta)\hat{n}$ $\qquad$ $\vec{C} = \vec{A} \times \vec{B} = (A\sin\theta)B\hat{n}$





The above two equations and figures explain that the magnitude of vector or cross product is the product of magnitude of one vector and magnitude of the component of the other vector in the direction perpendicular to the first one.

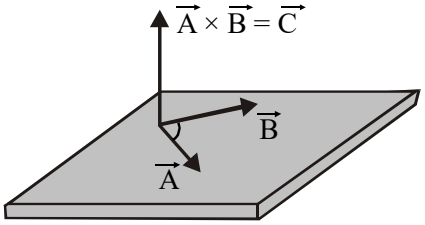
GOLDEN KEY POINTS

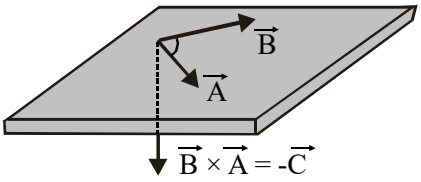
- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors, i.e., orthogonal (perpendicular) to both the vectors $\vec{A}$ and $\vec{B}$.

  Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is $\hat{n} = \pm\dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

- Vector product of two vectors is not commutative i.e. cross products $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ have equal magnitudes but opposite directions as shown in the figure.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$





- The vector product is distributive when the order of the vectors is strictly maintained,

  i.e. $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

- Angle θ between two vectors $\vec{A}$ and $\vec{B}$ is given by $\theta = \sin^{-1}\left[\dfrac{|\vec{A} \times \vec{B}|}{|\vec{A}|\,|\vec{B}|}\right]$

- The self cross product, i.e., product of a vector by itself is a zero vector or a null vector.

$$\vec{A} \times \vec{A} = (AA\sin 0°)\hat{n} = \vec{0} = \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$$

- In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$; according to right hand thumb rule

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j} \text{ and } \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

- If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, their cross-products is given by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

- If $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar, then $\vec{A}.(\vec{B} \times \vec{C}) = 0$.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example :

Find a unit vector perpendicular to both the vectors $(2\tilde{i}+3\tilde{j}+\tilde{k})$ and $(\tilde{i}-\tilde{j}+2\tilde{k})$.

Solution :

Let $\vec{A}=2\tilde{i}+3\tilde{j}+\tilde{k}$ and $\vec{B}=\tilde{i}-\tilde{j}+2\tilde{k}$

unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is $\tilde{n}=\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ 2 & 3 & 1\\ 1 & -1 & 2\end{vmatrix}=\tilde{i}(6+1)-\tilde{j}(4-1)+\tilde{k}(-2-3)=7\tilde{i}-3\tilde{j}-5\tilde{k}$$

$$\therefore |\vec{A}\times\vec{B}|=\sqrt{7^2+(-3)^2+(-5)^2}=\sqrt{83}\ \text{unit} \qquad \therefore \tilde{n}=\frac{1}{\sqrt{83}}(7\tilde{i}-3\tilde{j}-5k)$$

Rate of change of a vector with time

It is derivative of a vector function with respect to time. Cartesian components of a time dependent vector, if given as function of time as $\vec{r}(t)=x(t)\tilde{i}+y(t)\tilde{j}+z(t)\tilde{k}$, the time rate of change can be calculated according to equation

$$\frac{d\vec{r}(t)}{dt}=\frac{dx(t)\tilde{i}}{dt}+\frac{dy(t)\tilde{j}}{dt}+\frac{dz(t)\tilde{k}}{dt}$$

Methods of differentiation of vector functions

Methods of differentiation of scalar functions are also applicable to differentiation of vector functions.

1. $\dfrac{d}{dt}\left(\vec{F}\pm\vec{G}\right)=\dfrac{d\vec{F}}{dt}\pm\dfrac{d\vec{G}}{dt}$

2. $\dfrac{d}{dt}\left(\vec{F}\cdot\vec{G}\right)=\dfrac{d\vec{F}}{dt}\cdot\vec{G}+\vec{F}\cdot\dfrac{d\vec{G}}{dt}$

3. $\dfrac{d}{dt}\left(X\vec{F}\right)=\dfrac{dX}{dt}\vec{F}+X\dfrac{d\vec{F}}{dt}$ Here X is a scalar function of time.

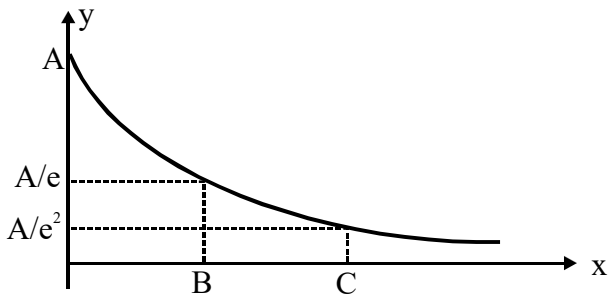
4. $\dfrac{d}{dt}\left(\vec{F}\times\vec{G}\right)=\dfrac{d\vec{F}}{dt}\times\vec{G}+\vec{F}\times\dfrac{d\vec{G}}{dt}$ Order of the vector functions $\vec{F}$ and $\vec{G}$ must be retained.

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

SOME WORKED OUT EXAMPLES

***Example* #1**

In the given figure, a function $y = 15e^{-x}$ is shown. What is the numerical value of expression A/(B+C)?



Solution: **Ans. 5**

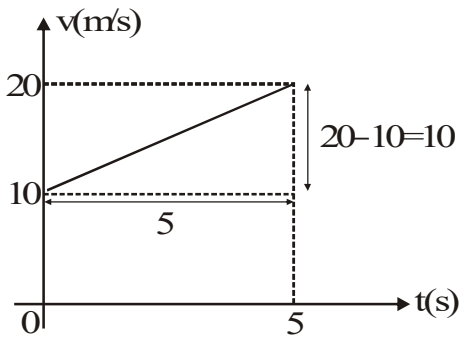
From graph A = 15 ; B =1 ; C =2 . Therefore [A/(B+C) = 15/3 = 5]

***Example* #2**

A car changes its velocity linearly from 10 m/s to 20 m/s in 5 seconds. Plot v-t graph and write velocity as a function of time.

Solution:

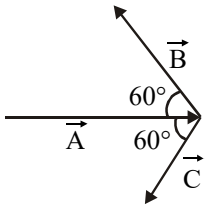
Slope = $\frac{20-10}{5-0} = 2 =$ m y-intercept = 10 = c $\Rightarrow$ v = 2t + 10



node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example #3

Three coplanar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitudes 4, 3 and 2 respectively. If the angle between any two vectors is 120° then which of the following vector may be equal to $\frac{3\vec{A}}{4}+\frac{\vec{B}}{3}+\frac{\vec{C}}{2}$



(A)  (B)  (C)  (D)

Solution : **Ans. (B)**

As $\left|\frac{\vec{B}}{3}\right|=\left|\frac{\vec{C}}{2}\right|$ so $\frac{\vec{B}}{3}+\frac{\vec{C}}{2}=-\frac{\vec{A}}{4}$ therefore $\frac{3\vec{A}}{4}+\frac{\vec{B}}{3}+\frac{\vec{C}}{2}=\frac{\vec{A}}{2}$

Example #4

The magnitude of pairs of displacement vectors are given. Which pairs of displacement vectors cannot be added to give a resultant vector of magnitude 13 cm?

(A) 4 cm, 16 cm (B) 20 cm, 7 cm (C) 1 cm, 15 cm (D) 6 cm, 8 cm

Solution : **Ans. (C)**

Resultant of two vectors $\vec{A}$ and $\vec{B}$ must satisfy $A \sim B \le R \le A + B$

Example #5

Three non zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ satisfy the relation $\vec{A}\cdot\vec{B}=0$ & $\vec{A}\cdot\vec{C}=0$. Then $\vec{A}$ can be parallel to :

(A) $\vec{B}$ (B) $\vec{C}$ (C) $\vec{B}\cdot\vec{C}$ (D)

Solution : **Ans. (D)**

$\vec{A}\cdot\vec{B}=0 \Rightarrow \vec{A}\perp\vec{B}$ & $\vec{A}\cdot\vec{C}=0 \Rightarrow \vec{A}\perp\vec{C}$

But $\vec{B}\times\vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ so $\vec{A}$ is parallel to $\vec{B}\times\vec{C}$.

Example #6

α and β are the angle made by a vector from positive x & positive y-axes respectively. Which set of α and β is not possible

(A) 45^0, 60^0 (B) 30^0, 60^0 (C) 60^0, 60^0 (D) 30^0, 45^0

Solution : **Ans. (D)**

α,β must satisfy $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Example #7

Let $\vec{A}$, $\vec{B}$ and $\vec{C}$, be unit vectors. Suppose that $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{C}=0$ and the angle between $\vec{B}$ and $\vec{C}$ is $\frac{\pi}{6}$ then

(A) $\vec{A}=(\vec{B}\times\vec{C})$ (B) $\vec{A}=2(\vec{B}\times\vec{C})$ (C) $\vec{A}=2(\vec{C}\times\vec{B})$ (D) $|\vec{B}\times\vec{C}|=\frac{\sqrt{3}}{2}$

node6\E_NOD6\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Solution : Ans. (BC)

As $\vec{A} \perp \vec{B}$ and $\vec{A} \perp \vec{C}$ so $\vec{A} = \pm\frac{(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|}$ But $|\vec{B}\times\vec{C}|$=BC sin30 = $\frac{1}{2}$

So $\vec{A} = \pm 2(\vec{B}\times\vec{C}) \Rightarrow \vec{A} = 2(\vec{B}\times\vec{C})$ and $\vec{A} = -2(\vec{B}\times\vec{C}) = 2(\vec{C}\times\vec{B})$

Example #8

Angle between $\vec{a}$ and $\vec{b}$ is 60 than

(A) The component of $\vec{a}-\vec{b}$ along $\vec{a}+\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+ab}}$

(B) $\vec{a}\times\vec{b}$ is perpendicular to resultant of $(\vec{a}+2\vec{b})$ and $(\vec{a}-\vec{b})$

(C) The component of $\vec{a}-\vec{b}$ along $\vec{a}+\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+2ab}}$

(D) The component of $\vec{a}+\vec{b}$ along $\vec{a}-\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+\sqrt{3}ab}}$

Solution : Ans. (A,B)

For (A) : Required component $= \frac{(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|} = \frac{a^2-b^2}{\sqrt{a^2+b^2+2ab\cos 60°}} = \frac{a^2-b^2}{\sqrt{a^2+b^2+ab}}$

For (B) : $\vec{a}+2\vec{b}+\vec{a}-\vec{b} = 2\vec{a}+\vec{b}$ which lies in the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow$ resultant is perpendicular to $\vec{a}\times\vec{b}$

Example #9

Which of the following sets of concurrent forces may be in equilibrium?

(A) F_1=3N, F_2=5N, F_3=1N (B) F_1=3N, F_2=5N, F_3=6N

(C) F_1=3N, F_2=5N, F_3=9N (D) F_1=3N, F_2=5N, F_3=16 N

Solution : Ans. (B)

For equilibrium, net resultant force must be zero. These forces form a closed traingle such that

$F_1 \sim F_2 \le F_3 \le F_1+F_2 \Rightarrow 2N \le F_3 \le 8N$

Example #10

Consider three vectors

$\vec{A} = 2\hat{i}+3\hat{j}-2\hat{k}$ $\quad$ $\vec{B} = 5\hat{i}+n\hat{j}+\hat{k}$ $\quad$ $\vec{C} = -\hat{i}+2\hat{j}+3\hat{k}$

If these three vectors are coplanar, then value of n will be

(A) 0 (B) 12 (C) 16 (D) 18

Solution: Ans. (D)

For coplanar vectors $\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 2 & 3 & -2 \\ 5 & n & 1 \\ -1 & 2 & 3 \end{vmatrix} = 2(3n-2)-3(15+1)-2(10+n) = 0 \Rightarrow 4n-72=0 \Rightarrow n=18$

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example #11 to 13

Vector product of three vectors is given by $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$

11. The value of $\tilde{i}\times(\tilde{j}\times\tilde{k})$ is

(A) 0 (B) $\vec{0}$ (C) 1 (D) 3

12. The plane of vector $\vec{A}\times(\vec{A}\times\vec{B})$ is lies in the plane of

(A) $\vec{A}$ (B) $\vec{B}$ (C) $\vec{A}\times\vec{B}$ (D) $\vec{A}$ and $\vec{B}$

13. The value of $\tilde{i}\times(\tilde{i}\times\tilde{j})+\tilde{j}\times(\tilde{j}\times\tilde{k})+\tilde{k}\times(\tilde{k}\times\tilde{i})$ is

(A) $\tilde{i}+\tilde{j}+\tilde{k}$ (B) $-\tilde{i}-\tilde{j}-\tilde{k}$ (C) $\vec{0}$ (D) $-3\tilde{i}-3\tilde{j}-3\tilde{k}$

Solution :

11. Ans. (B)

$\tilde{i}\times\left(\tilde{j}\times\tilde{k}\right)=\tilde{j}\left(\hat{i}\cdot\tilde{k}\right)-\tilde{k}\left(\hat{i}\cdot\hat{j}\right)=\vec{0}$

12. Ans. (D)

$\vec{A}\times\left(\vec{A}\times\vec{B}\right)=\vec{A}\left(\vec{A}\cdot\vec{B}\right)-\vec{B}\left(\vec{A}\cdot\vec{A}\right)$ $\Rightarrow$This vector lies in plane of $\vec{A}$ and $\vec{B}$

13. Ans. (B)

$\Sigma\hat{i}\times\left(\hat{i}\times\hat{j}\right)=\Sigma\hat{i}\left(\hat{i}\cdot\hat{j}\right)-\hat{j}\left(\hat{i}\cdot\hat{i}\right)=-\Sigma\hat{j}=-\left(\hat{i}+\hat{j}+\tilde{k}\right)$

Example #14

If $\vec{A}=\tilde{i}+2\tilde{j}+3\tilde{k}$, $\vec{B}=-\tilde{i}+\tilde{j}+4\tilde{k}$ and $\vec{C}=3\tilde{i}-3\tilde{j}-12\tilde{k}$, then find the angle between the vectors $\left(\vec{A}+\vec{B}+\vec{C}\right)$ and $\left(\vec{A}\times\vec{B}\right)$ in degrees.

Solution : **Ans. 90**

$\vec{P}=\vec{A}+\vec{B}+\vec{C}=3\tilde{i}-5\tilde{k}$ and $\vec{Q}=\vec{A}\times\vec{B}=\begin{vmatrix}\tilde{i}&\tilde{j}&\tilde{k}\\1&2&3\\-1&1&4\end{vmatrix}=5\tilde{i}-7\tilde{j}+3\tilde{k}$

Angle between $\vec{P}$ & $\vec{Q}$ is given by $\cos\theta=\frac{\vec{P}\cdot\vec{Q}}{PQ}=\frac{15-15}{PQ}=0\Rightarrow\theta=90^\circ$

Example #15

$\vec{a}$ and $\vec{b}$ are unit vectors and angle between them is $\frac{\pi}{k}$. If $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then find the integer value of k.

Solution : **Ans. 3**

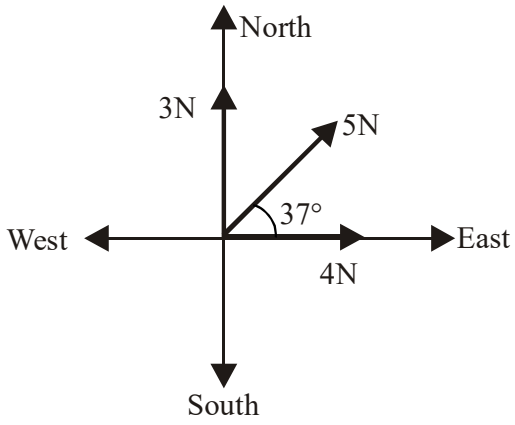
$(\vec{a}+2\vec{b}).(5\vec{a}-4\vec{b})=0\Rightarrow5a^2+10\vec{a}.\vec{b}-8\vec{b}^2-4\vec{a}.\vec{b}\Rightarrow-3+6\,\vec{a}.\vec{b}=0$

$$\Rightarrow ab\cos\theta=\frac{3}{6}\Rightarrow\cos\theta=\frac{1}{2}=\theta=\frac{\pi}{3}\Rightarrow k=3$$

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Example #16

For shown situation, what will be the magnitude of minimum force in newton that can be applied in any direction so that the resultant force is along east direction?



Solution :

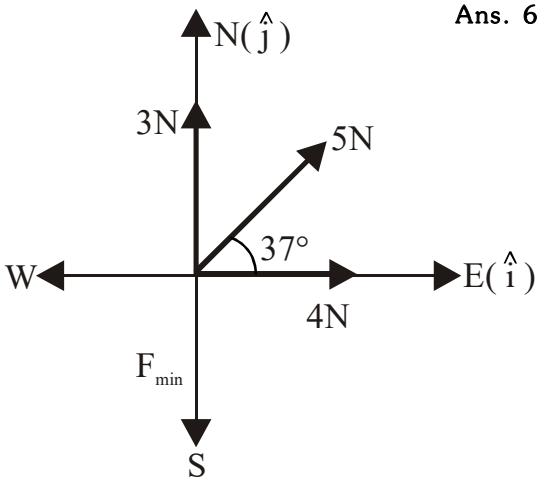
Ans. 6

Let force be F so resultant is in east direction

$$4\tilde{i}+3\tilde{j}+\left(5\cos 37^\circ\tilde{i}+5\sin 37^\circ\tilde{j}\right)+\vec{F}=k\tilde{i}$$

$$\Rightarrow 4\tilde{i}+3\tilde{j}+4\tilde{i}+3\tilde{j}+\vec{F}=k\tilde{i}\Rightarrow 8\tilde{i}+6\tilde{j}+\vec{F}=k\tilde{i}$$

$$\Rightarrow \vec{F}=(k-8)\tilde{i}-6\tilde{j}\Rightarrow F=\sqrt{(k-8)^2+(6)^2}\Rightarrow F_{min}=6N$$



Example #17

Column–I (Operation of nonzero vectors $\vec{P}$ and $\vec{Q}$)		Column II (Possible angle between $\vec{P}$ and $\vec{Q}$)	
(A)	$\lvert\vec{P}\times\vec{Q}\rvert=0$	(P)	90
(B)	$\lvert\vec{P}\times\vec{Q}\rvert=\sqrt{3}\vec{P}.\vec{Q}$	(Q)	180
(C)	$\vec{P}+\vec{Q}=\vec{R}$ and P+Q=R	(R)	60
(D)	$\lvert\vec{P}+\vec{Q}\rvert=\lvert\vec{P}-\vec{Q}\rvert$	(S)	0
		(T)	30

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Solution : **Ans. (A) → Q,S ; (B) → R ; (C) → S ; (D) → P**

For (A) $|\vec{P}\times\vec{Q}| = 0 \Rightarrow$ Angle between $\vec{P}$ and $\vec{Q}$ is 0^0 or 180^0

For (B) $|PQ\sin\theta| = \sqrt{3}\ PQ\cos\theta \Rightarrow |\sin\theta| = \sqrt{3}\ \cos\theta$

Here $\cos\theta$ must be positive so $\theta = 60$

For (C) Here $P^2 + Q^2 + 2PQ\cos\theta = P^2 + Q^2 + 2PQ \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

For (D) Here $P^2 + Q^2 + 2PQ\cos\theta = P^2 + Q^2 - 2PQ\cos\theta \Rightarrow \cos\theta = 0, \Rightarrow \theta = 90$

Example #18

The position of a particle moving in XY-plane varies with time t as x=t, y = 3t –5.

(i) What is the path traced by the particle?

(ii) When does the particle cross-x-axis?

Solution :

(i) x=t, y=3t–5 By eliminating t from above two equations y = 3x –5

This is the equation of a straight line.

(ii) The particle crosses x-axis when y=0. So $0 = 3t - 5 \Rightarrow t = \frac{5}{3}$

Example #19

Two particles A and B move along the straight lines x+2y +3 = 0 and 2x + y –3 = 0 respectively. Their position vector, at the time of meeting will be

(A) $3\tilde{i} + 3\tilde{j}$ (B) $3\tilde{i} - 3\tilde{j}$ (C) $\frac{\tilde{i}}{3} - \frac{\tilde{j}}{3}$ (D) Particles never meet

Solution : **Ans. (B)**

The particles meet at the point of intersection of lines.

By solving them x=3, y= –3 , So position vector of meeting point will be $3\tilde{i} - 3\tilde{j}$

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\1. Basic Maths Vectors.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 2

UNIT & DIMENSIONS - Theory

UNIT & DIMENSIONS

PHYSICAL QUANTITIES

All the quantities which are used to describe the laws of physics are known as *physical quantities.*

Classification : Physical quantities can be classified on the following bases :

I. Based on their directional properties

1. **Scalars:** The physical quantities which have only magnitude but no direction are called *scalar quantities*.

   Ex. mass, density, volume, time, etc.

2. **Vectors :** The physical quantities which have both magnitude and direction and obey laws of vector algebra are called *vector quantities*.

   Ex. displacement, force, velocity, etc.

II. Based on their dependency

1. **Fundamental or base quantities :** The quantities which do not depend upon other quantities for their complete definition are known as *fundamental or base quantities*.

   e.g. : length, mass, time, etc.

2. **Derived quantities :** The quantities which can be expressed in terms of the fundamental quantities are known as *derived quantities.*

   Ex. Speed (=distance/time), volume, acceleration, force, pressure, etc.

Physical quantities can also be classified as dimensional and dimensionless quantities or constants and variables.

Example :

Classify the quantities displacement, mass, force, time, speed, velocity, acceleration, moment of intertia, pressure and work under the following categories :

(a) base and scalar (b) base and vector (c) derived and scalar (d) derived and vector

Solution :

(a) mass, time (b) displacement (c) speed, pressure, work (d) force, velocity, acceleration

UNITS OF PHYSICAL QUANTITIES

The chosen reference standard of measurement in multiples of which, a physical quantity is expressed is called the *unit* of that quantity.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Classification of Units : The units of physical quantities can be classified as follows :

1. ***Fundamental or base units***

The units of fundamental quantities are called *base units*. In SI there are seven base units.

2. ***Derived units***

The units of derived quantities or the units that can be expressed in terms of the base units are called *derived* units

Some derived units are named in honour of great scientists.

• Unit of force – newton (N) • Unit of frequency – hertz (Hz) etc.

Example :

Find the SI unit of speed and acceleration.

Solution :

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{\text{meter (m)}}{\text{second (s)}} = \text{m/s} \quad \text{(called as meter per second)}$$

$$\text{acceleration} = \frac{\text{velocity}}{\text{time}} = \frac{\text{displacement /time}}{\text{time}}$$

$$= \frac{\text{displacement}}{(\text{time})^2} = \frac{\text{meter}}{\text{second}^2} = \text{m/s}^2 \text{ (called as meter per second square)}$$

SYSTEM OF UNITS

(a) FPS or British Engineering system :

In this system length, mass and time are taken as fundamental quantities and their base units are foot (ft), pound (lb) and second (s) respectively.

(b) CGS or Gaussian system :

In this system the fundamental quantities are length, mass and time and their respective units are centimetre (cm), gram (g) and second (s).

(c) MKS system :

In this system also the fundamental quantities are length, mass and time but their fundamental units are metre (m), kilogram (kg) and second (s) respectively.

Table 1: Units of some physical quantities in Different systems

Type of Physical Quantity	Physical Quantity	CGS (Originated in France)	MKS (Originated in France)	FPS (Originated in Britain)
Fundamental	Length	cm	m	ft
	Mass	g	kg	Lb
	Time	s	s	S
Derived	Force	dyne	newton(N)	Poundal
	Work or Energy	erg	joule(J)	ft-poundal
	Power	erg/s	watt(W)	ft-poundal/s

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

(d) International system (SI) of units :

This system is modification over the MKS system and so it is also known as *Rationalised MKS system. Besides* the three base units of MKS system four fundamental and two supplementary units are also included in this system.

The SI system is at present widely used throughout the world.

Table 2 : SI Base Quantities and Units

Base Quantity	SI Units		
	Name	Symbol	Definition
Length	meter	m	The meter is the length of the path traveled by light in vacuum during a time interval of 1/299, 792, 458 of a second (1983)
Mass	kilogram	kg	The kilogram is equal to the mass of the international prototype of the kilogram (a platinum-iridium alloy cylinder) kept at International Bureau of Weights and Measures, at Sevres, near Paris, France. (1889)
Time	second	s	The second is the duration of 9, 192, 631, 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom (1967)
Electric Current	ampere	A	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} Newton per metre of length. (1948)
Thermodynamic Temperature	kelvin	K	The kelvin, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water. (1967)
Amount of Substance	mole	mol	The mole is the amount of substance of a system, which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12. (1971)
Luminous Intensity	candela	Cd	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian (1979).

Example :

Find the SI unit of (i) area, (ii) density, and (iii) momentum.

Ans. (i) m^2 (ii)kg/m^3 (iii) kgm/s.

3. ***Supplementary units***

In SI, two supplementary units are also defined viz. (i) radian (rad) for plane angle and (ii) steradian (sr) for solid angle.

(i) radian : 1 radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

(ii) steradian : 1 steradian is the solid angle subtended at the centre of a sphere, by that surface of the sphere which is equal in area to the square of the radius of the sphere.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

4. *Practical units* :

Due to the fixed sizes of SI units, some *practical units* are also defined for both fundamental and derived quantities. e.g. light year (ly) is a practical unit of astronomical distance (a fundamental quantity) and horse power (hp) is a practical unit of power (a derived quantity). Practical units may or may not belong to a particular system of units but can be expressed in any system of units. e.g. 1 mile = 1.6 km = 1.6 10^3 m =1.6 10^5 cm.

5. *Improper units* :

These are the units which are not of the same nature as that of the physical quantities for which they are used. e.g. kg-wt is an improper unit of weight. Here kg is a unit of mass but it is used to measure the weight (force).

DIMENSIONS

Dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity.

1. **Dimensional formula :** The *dimensional formula* of any physical quantity is that expression which represents how and which of the base quantities are included in that quantity.

It is written by enclosing the symbols for base quantities with appropriate powers in square brackets i.e. [ ]

Example :

Write dimensional formula of mass and speed.

Solution :

Dimensional formula of mass is $[M^1L^0\ T^0]$ and that of speed (= distance/time) is $[M^0L^1T^{-1}]$

2. **Dimensional equation :** The equation obtained by equating a physical quantity with its dimensional formula is called a *dimensional equation*.

e.g. $[v] = [M^0L^1T^{-1}]$

For example $[F] = [MLT^{-2}]$ is a dimensional equation, $[MLT^{-2}]$ is the dimensional formula of the force and the dimensions of force are 1 in mass, 1 in length and –2 in time

3. **Applications of dimensional analysis :**

(i) To convert a physical quantity from one system of units to the other :

n_2 = numerical value in II system

n_1 = numerical value in I system

M_1 = unit of mass in I system

M_2 = unit of mass in II system

L_1 = unit of length in I system

L_2 = unit of length in II system

T_1 = unit of time in I system

T_2 = unit of time in II system

This is based on a fact that *magnitude of a physical quantity remains same whatever system is used for measurement* i.e. magnitude = numeric value (n) unit (u) = constant or $n_1u_1 = n_2u_2$

So if a quantity is represented by $[M^aL^bT^c]$

Then $\boxed{n_2 = n_1\left(\frac{u_1}{u_2}\right) = n_1\left(\frac{M_1}{M_2}\right)^a\left(\frac{L_1}{L_2}\right)^b\left(\frac{T_1}{T_2}\right)^c}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Example :

Convert 1 newton (SI unit of force) into dyne (CGS unit of force)

Solution :

The dimensional equation of force is $[F] = [M^1 L^1 T^{-2}]$

Therefore if n_1, u_1, and n_2, u_2 corresponds to SI & CGS units respectively, then

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^1\left[\frac{L_1}{L_2}\right]^1\left[\frac{T_1}{T_2}\right]^{-2} = 1\left[\frac{kg}{g}\right]\left[\frac{m}{cm}\right]\left[\frac{s}{s}\right]^{-2} = 1 \quad 1000 \quad 100 \quad 1 = 10^5$$

$\therefore$ 1 newton = 10^5 dyne.

Example :

The acceleration due to gravity is 9.8 m s^{-2}. Give its value in ft s^{-2}

Solution :

As 1m = 3.2 ft $\quad \therefore$ 9.8 m/s^2 = 9.8 3.28 ft/s^2 = 32.14 ft/s^2 $\approx$ 32 ft/s^2

To check the dimensional correctness of a given physical relation :

If in a given relation, the terms on both the sides have the same dimensions, then the relation is dimensionally correct. This is known as the *principle of homogeneity of dimensions*.

Example :

Check the accuracy of the relation $T = 2\pi\sqrt{\frac{L}{g}}$ for a simple pendulum using dimensional analysis.

Solution:

The dimensions of LHS = the dimension of $T = [M^0 L^0 T^1]$

The dimensions of RHS = $\left(\frac{\text{dimensions of length}}{\text{dimensions of acceleration}}\right)^{1/2}$ ($\because$ 2π is a dimensionless constant)

$$= \left[\frac{L}{LT^{-2}}\right]^{1/2} = [T^2]^{1/2} = [T] = [M^0 L^0 T^1]$$

Since the dimensions are same on both the sides, the relation is correct.

To derive relationship between different physical quantities :

Using the same principle of homogeneity of dimensions new relations among physical quantities can be derived if the dependent quantities are known.

Example :

It is known that the time of revolution T of a satellite around the earth depends on the universal gravitational constant G, the mass of the earth M, and the radius of the circular orbit R. Obtain an expression for T using dimensional analysis.

Solution :

We have $[T] = [G]^a [M]^b [R]^c$

$[M]^0 [L]^0 [T]^1 = [M]^{-a} [L]^{3a} [T]^{-2a} \quad [M]^b \quad [L]^c = [M]^{b-a} [L]^{c+3a} [T]^{-2a}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Comparing the exponents

For [T] : $1 = -2a \Rightarrow a = -\frac{1}{2}$ **For [M]** : $0 = b - a \Rightarrow b = a = -\frac{1}{2}$

For [L] : $0 = c + 3a \Rightarrow c = -3a = \frac{3}{2}$

Putting the values we get $T \propto G^{-1/2} M^{-1/2} R^{3/2} \propto \sqrt{\frac{R^3}{GM}}$

The actual expression is $T = 2\pi\sqrt{\frac{R^3}{GM}}$

DIMENSIONS OF SOME MATHEMATICAL FUNCTIONS

Dimensions of differential coefficients and integrals

In General $\left[\frac{d^n y}{dx^n}\right] = \left[\frac{y}{x^n}\right]$ and $[\int ydx] = [yx]$

Example :

Find dimensional formula :

(i) $\frac{dx}{dt}$ (ii) $m\frac{d^2x}{dt^2}$ (iii) $\int vdt$ (iv) $\int adt$

where x→ displacement, t→ time, v → velocity and a→ acceleration

Solution:

(i) $\left[\frac{dx}{dt}\right] = \left[\frac{x}{t}\right] = \left[\frac{L}{T}\right] = [M^0L^1T^{-1}]$

(ii) $\left[m\frac{d^2x}{dt^2}\right] = \left[m\frac{x}{t^2}\right] = \left[\frac{ML}{T^2}\right] = [M^1L^1T^{-2}]$

(iii) $\left[\int vdt\right] = [vt] = [LT^{-1}\ T] = [M^0L^1T^0]$

(iv) $\left[\int adt\right] = [at] = [LT^{-2}\ T] = [M^0L^1T^{-1}]$

Dimensions of trigonometric, exponential, logarithmic functions etc.

All trigonometric, exponential and logarithmic functions and their arguments are dimensionless.

Note : Trigonometric function sinθ and its argument θ are dimensionless.

Example :

If $\alpha = \frac{F}{v^2}\sin\beta t$, find dimensions of α and β. Here v = velocity, F = force and t = time.

Solution :

Here sin βt and βt must be dimensionless

So $[\beta t] = 1 \Rightarrow [\beta] = \left[\frac{1}{t}\right] = [T^{-1}]$; $[\alpha] = \left[\frac{F}{v^2}\sin\beta t\right] = \left[\frac{F}{v^2}\right] = \left[\frac{MLT^{-2}}{L^2T^{-2}}\right] = [ML^{-1}]$

node6\E_NODE6 (E)\Data\2014\Kota\VIEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

LIMITATIONS OF THIS METHOD

(i) In Mechanics the formula for a physical quantity depending on more than three physical quantities cannot be derived. It can only be checked.

(ii) This method can be used only if the dependency is of multiplication type. The formulae containing exponential, trigonometrical and logarithmic functions can't be derived using this method. Formulae containing more than one term which are added or subtracted like $s = ut + at^2/2$ also can't be derived.

(iii) The relation derived from this method gives no information about the dimensionless constants.

(iv) If dimensions are given, physical quantity may not be unique as many physical quantities have the same dimensions.

(v) It gives no information whether a physical quantity is a scalar or a vector.

SI PREFIXES

The magnitudes of physical quantities vary over a wide range. The CGPM recommended standard prefixes for magnitude too large or too small to be expressed more compactly for certain powers of 10.

Table 3 : Prefixes used for different powers of 10

Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10^{1}	deca	da	10^{-18}	atto	a

General Guidelines for using Symbols for SI Units, Some other Units, and SI prefixes

(i) Symbols for units of physical quantities are printed/written in Roman (upright type), and not in italics

For Example :

1 N is correct but 1 *N* is incorrect.

(ii) (i) Unit is never written with capital initial letter if it is named after a scientist.

For example :

SI unit of force is newton (correct) Newton (incorrect)

(ii) For a unit named after a scientist, the symbol is a capital letter.

But for other units, the symbol is NOT a capital letter.

For example :

force	→	newton (N)
energy	→	joule (J)
electric current	→	ampere (A)
temperature	→	kelvin (K)
frequency	→	hertz (Hz)

For example :

length	→	meter (m)
mass	→	kilogram (kg)
luminous intensity	→	candela (cd)
time	→	second (s)

Note : The single exception is L, for the unit litre.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

(iii) Symbols for units do not contain any final full stop at the end of recommended latter and remain unaltered in the plural, using only singular form of the unit.

For example :

Quantity	Correct	Incorrect
25 centimeters	25 cm	25 cm. 25 cms

(iv) Use of solidus (/) is recommended only for indicating a division of one letter unit symbol by another unit symbol. Not more than one solidus is used.

For example :

Correct	Incorrect
m / s^2	m / s / s
$N s / m^2$	N s / m / m
J / K mol	J / K / mol
kg / m s	kg / m / s

(v) Prefix symbols are printed in roman (upright) type without spacing between the prefix symbol and the unit symbol. Thus certain approved prefixes written very close to the unit symbol are used to indicate decimal fractions or multiples of a SI unit, when it is inconveniently small or large.

For example :

megawatt	1 MW = 10^6 W
centrimetre	1 cm = 10^{-2} m
kilometre	1 km = 10^3 m
millivolt	1 mV = 10^{-3} V
kilowatt-hour	1 kW h = 10^3 W h = 3.6 MJ = 3.6 10^6 J
microampere	1 μA = 10^{-6} A
angstrom	1 Å = 0.1 nm = = 10^{-10} m
nanosecond	1 ns = 10^{-9} s
plcofarad`	1 pF = 10^{-12} F
microsecond	1 μs = 10^{-6} s
gigahertz	1 GHz = 10^9 Hz
micron	1 μm = 10^{-6} m

The unit 'fermi', equal to a femtometre or 10^{-15} m has been used as the convenient length unit in nuclear studies.

(vi) When a prefix is placed before the symbol of a unit, the combination of prefix and symbol is considered as a new symbol, for the unit, which can be raised to a positive or negative power without using brackets. These can be combined with other unit symbols to form compound unit.

node6\E_NODE6 (E)\Data \2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

For example :

Quantity	Correct	Incorrect
cm^3	$(cm)^3 = (0.01\ m)^3 = (10^{-2}\ m)^3 = 10^{-6}\ m^3$	$0.01\ m^3$ or $10^{-2}\ m^3$ or $1\ cm^3$
mA^2	$(mA)^2 = (0.001\ A)^2 = (10^{-3}\ A)^2 = 10^{-6}\ A^2$	$0.001\ A^2$ or mA^2

(a) A prefix is never used alone. It is always attached to a unit symbol and written or fixed before the unit symbol.

For example : 10^3 / m^3 = 1000 /m^3 or 1000 m^{-3}, but not k/m^3 or k m^{-3} .

(vii) Prefix symbol is written very close to the unit symbol without spacing between them, while unit symbols are written separately with spacing when units are multiplied together.

For example :

Quantity	Correct	Incorrect
1 ms^{-1}	1 metre per second	1 milli per second
1 ms	1 millisecond	1 metre second.
1 Cm	1 coulomb metre	1 centimetre
1 cm	1 centimetre	1 coulomb metre

(viii) The use of double prefixes is avoided when single prefixes are available.

For example :

Quantity	Correct	Incorrect
10^{-9} m	1 nm (nanometre)	1 mμm (millimicrometre)
10^{-6} m	1 μm (micron)	1 mmm (millimillimetre)
10^{-12} F	1 pF (picofarad)	1 μμF (micromicrofarad)
10^9 W	1 GW (giga watt)	1 kMW (kilomegawatt)

(ix) The use of a combination of unit and the symbols for units is avoided when the physical quantity is expressed by combining two or more units.

Quantity	Correct	Incorrect
joule per mole Kelvin	J/mol K or J mol^{-1} K^{-1}	joule / mole K or J /mol Kelvin or J/mole K
newton metre second	N m s	newton m second or N m second or N metre s or newton metre s

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Table 4 : Units of important Physical Quantities

Physical quantity	Unit	Physical quantity	Unit
Angular acceleration	rad s^{-2}	Frequency	hertz
Moment of inertia	kg – m^2	Resistance	kg m^2 A^{-2} s^{-3}
Self inductance	henry	Surface tension	newton/m
Magnetic flux	weber	Universal gas constant	joule K^{-1} mol^{-1}
Pole strength	A–m	Dipole moment	coulomb–meter
Viscosity	poise	Stefan constant	watt m^{-2} K^{-4}
Reactance	ohm	Permittivity of free space	$coulomb^2/N–m^2$
Specific heat	J/kg C		
Strength of magnetic field	newton A^{-1} m^{-1}	Planck's constant	joule–sec
Astronomical distance	parsec		

Table 4 : Dimensions of Important Physical Quantities

Physical quantity	Dimensions	Physical quantity	Dimensions
Momentum	$M^1 L^1 T^{-1}$	Capacitance	$M^{-1} L^{-2} T^4 A^2$
Calorie	$M^1 L^2 T^{-2}$	Modulus of rigidity	$M^1 L^{-1} T^{-2}$
Latent heat	$M^0 L^2 T^{-2}$	Magnetic permeability	$M^1 L^1 T^{-2} A^{-2}$
Self inductance	$M^1 L^2 T^{-2} A^{-2}$	Pressure	$M^1 L^{-1} T^{-2}$
Coefficient of thermal conductivity	$M^1 L^1 T^{-3} K^{-1}$	Planck's constant	$M^1 L^2 T^{-1}$
Power	$M^1 L^2 T^{-3}$	Solar constant	$M^1 L^0 T^{-3}$
Impulse	$M^1 L^1 T^{-1}$	Magnetic flux	$M^1 L^2 T^{-2} A^{-1}$
Hole mobility in a semi conductor	$M^{-1} L^0 T^2 A^1$	Current density	$M^0 L^{-2} T^{-1} Q$
Bulk modulus of elasticity	$M^1 L^{-1} T^{-2}$	Young modulus	$M^1 L^{-1} T^{-2}$
Potential energy	$M^1 L^2 T^{-2}$	Magnetic field intensity	$M^0 L^{-1} T^0 A^1$
Gravitational constant	$M^{-1} L^3 T^{-2}$	Magnetic Induction	$M^1 T^{-2} A^{-1}$
Light year	$M^0 L^1 T^0$	Permittivity	$M^{-1} L^{-3} T^4 A^2$
Thermal resistance	$M^{-1} L^{-2} T^{-3} \theta$	Electric Field	$M^1 L^1 T^{-3} A^{-1}$
Coefficient of viscosity	$M^1 L^{-1} T^{-1}$	Resistance	$ML^2 T^{-3} A^{-2}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Table 6 : Sets of Quantities having same dimensions

S.N.	Quantitie	Dimensions
1.	Strain, refractive index, relative density, angle, solid angle, phase, distance gradient, relative permeability, relative permittivity, angle of contact, Reynolds number, coefficient of friction, mechanical equivalent of heat, electric susceptibility, etc.	$[M^0 L^0 T^0]$
2.	Mass and inertia	$[M^1 L^0 T^0]$
3.	Momentum and impulse.	$[M^1 L^1 T^{-1}]$
4.	Thrust, force, weight, tension, energy gradient.	$[M^1 L^1 T^{-2}]$
5.	Pressure, stress, Young's modulus, bulk modulus, shear modulus, modulus of rigidity, energy density.	$[M^1 L^{-1} T^{-2}]$
6.	Angular momentum and Planck's constant (h).	$[M^1 L^2 T^{-1}]$
7.	Accelaration, g and gravitational field intensity.	$[M^0 L^1 T^{-2}]$
8.	Surface tension, free surface energy (energy per unit area), force gradient, spring constant.	$[M^1 L^0 T^{-2}]$
9.	Latent heat and gravitational potential.	$[M^0 L^2 T^{-2}]$
10.	Thermal capacity, Boltzman constant, entropy.	$[ML^2 T^{-2}K^{-1}]$
11.	Work, torque, internal energy, potential energy, kinetic energy, moment of force, (q^2/C), (LI^2), (qV), (V^2C), (I^2Rt), $\frac{V^2}{R}t$, (VIt), (PV), (RT), (mL), (mc ΔT)	$[M^1 L^2 T^{-2}]$
12.	Frequency, angular frequency, angular velocity, velocity gradient, radioactivity $\frac{R}{L}, \frac{1}{RC}, \frac{1}{\sqrt{LC}}$	$[M^0 L^0 T^{-1}]$
13.	$\left(\frac{\ell}{g}\right)^{1/2}, \left(\frac{m}{k}\right)^{1/2}, \left(\frac{R}{g}\right)^{1/2}, \left(\frac{L}{R}\right), (RC), (\sqrt{LC})$, time	$[M^0 L^0 T^1]$
14.	(VI), (I^2R), (V^2/R), Power	$[ML^2 T^{-3}]$

Table 7 : Some Fundamental Constants

Gravitational constant (G)	6.67×10^{-11} N m^2 kg^{-2}
Speed of light in vacuum (c)	3×10^8 ms^{-1}
Permeability of vacuum (μ_0)	$4\pi \times 10^{-7}$ H m^{-1}
Permittivity of vacuum (ε_0)	8.85×10^{-12} F m^{-1}
Planck constant (h)	6.63×10^{-34} Js
Atomic mass unit (amu)	1.66×10^{-27} kg
Energy equivalent of 1 amu	931.5 MeV
Electron rest mass (m_e)	9.1×10^{-31} kg $\equiv$ 0.511 MeV
Avogadro constant (N_A)	6.02×10^{23} mol^{-1}
Faraday constant (F)	9.648×10^4 C mol^{-1}
Stefan–Boltzman constant (σ)	5.67×10^{-8} W m^{-2} K^{-4}
Wien constant (b)	2.89×10^{-3} mK
Rydberg constant (R_∞)	1.097×10^7 m^{-1}
Triple point for water	273.16 K (0.01°C)

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

SOME WORKED OUT EXAMPLES

Example #1

Which of the following sets cannot enter into the list of fundamental quantities in any newly proposed system of units?

(A) length, mass and velocity (B) pressure, density and velocity

(C) force, veloicty and time (D) force, momentum and time

Solution : ***Ans. (B,D)***

For (A): Length [L], mass [M] and velocity [LT^{-1}] are independent.

For (B): Pressure [$M^1L^{-1}T^{-2}$], density [$M^1L^{-3}T^0$] and velocity [M^0LT^{-1}] are dependent as

$$\begin{vmatrix} 1 & -1 & -2 \\ 1 & -3 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 1(3-0)+1(-1-0)-2(1-0)=0$$

Note : Like coplanar vectors, for dependent quantites, determinant of powers of M, L, T's must be zero.

For (C): Force [MLT^{-2}], velocity [LT^{-1}] and time [T] are independent as

$$\begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0)-1(0-0)-2(0-0)=1\neq 0$$

For (D): $\text{Force} = \dfrac{\text{momentum}}{\text{time}} \Rightarrow$ force, momentum and time are dependent.

Example #2

If A and B are two physical quantities having different dimensions then which of the following can denote a new physical quantity?

(A) $A+\dfrac{A^3}{B}$ (B) $\exp\left(-\dfrac{A}{B}\right)$ (C) AB^2 (D) $\dfrac{A}{B^4}$

Solution : **Ans. (A,C,D)**

For (A) : A and $\dfrac{A^3}{B}$ may have same dimension.

For (B) : As A and B have different dimension so $\exp\left(-\dfrac{A}{B}\right)$ is meaningless.

For (C) : AB^2 is meaningful.

For (D) : AB^{-4} is meaningful.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example #3

The dimensional formula of product and quotient of two physical quantities A and B are given by

$[AB] = [ML^2T^{-2}]$; $\left[\frac{A}{B}\right] = [MT^{-2}]$. The quantities A and B respectively are

(A) Force and velocity

(B) Force and displacement

(C) Momentum and displacement

(D) Work and velocity

Solution: **Ans. (B)**

$$[A]^2 = [AB]\cdot\left[\frac{A}{B}\right] = [M^2L^2T^{-4}] \Rightarrow [A] = [MLT^{-2}] \equiv \text{Force}$$

$$[B] = \frac{[AB]}{[A]} = [L] \Rightarrow [B] = [L] \equiv \text{Displacement}$$

Example #4

If the dimensions of a physical quantity are given by $M^xL^yT^z$, then physical quantity may be

(A) acceleration due to gravity, if x =0, y=1, z=–2

(B) atmospheric pressure, if x=1, y=1, z=–2

(C) linear momentum, if x=1, y=1, z=–1

(D) potential energy, if x=1, y=2, z=–2

Solution : **Ans. (A,C,D)**

For (A) : [Acceleration] = $[LT^{-2}]$

For (B) : [Pressure] = $[ML^{-1}T^{-2}]$

For (C) : [Momentum] = $[MLT^{-1}]$

For (D) : [Energy] = $[ML^2T^{-2}]$

Example #5

The Vanderwall's equation for n moles of a real gas is given by $\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$, where

P = pressure of gas

V = volume of gas

T = temperature of gas

R = molar gas constant

a & b = Vander wall's constants

Which of the following have the same dimensions as those of nRT.

(A) PV (B) $\frac{aV}{b^2}$ (C) $\frac{PV^2}{nb}$ (D) $\frac{na}{b}$

Solution : **Ans. (A,B,C,D)**

Here $[P] = \left[\frac{n^2a}{V^2}\right], [V] = [nb]$ so $[PV] = [nRT]$

$$\text{Also } \left[\frac{aV}{b^2}\right] = \left[\left(\frac{PV^2}{n^2}\right)\frac{V}{(V/n)^2}\right] = [PV] = [nRT] \Rightarrow \left[\frac{PV^2}{nb}\right] = n^2a$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Example #6

Force on a particle in one-dimensional motion is given by $F = Av + Bt + \frac{Cx}{At+D}$, where F = force, v= speed, t=time, x=position and A,B,C and D are constants. Dimension of C will be-

(A) $M^2L^{-2}T^0$ (B) $ML^{-1}T^0$ (C) $M^2L^0T^{-2}$ (D) None of these

Solution : **Ans. (C)**

$$\dim(A) = \dim\left(\frac{F}{v}\right) \Rightarrow \dim(C) = \dim\left[\frac{F(At+D)}{x}\right] = \dim\left[\frac{F}{x}\left(\frac{Ft}{v}\right)\right] = \dim\left[\frac{F^2t}{xv}\right] = \dim\left[\frac{F^2}{v^2}\right] = [M^2T^{-2}]$$

Example #7

Consider the equation $\frac{d}{dt}\left(\int \vec{F}\cdot d\vec{S}\right) = A\left(\vec{F}\cdot\vec{p}\right)$ where $\vec{F} \equiv$ force, $\vec{s} \equiv$ displacement, t≡time and $\vec{p} =$ momentum. The dimensional formula of A will be

(A) $M^0L^0T^0$ (B) ML^0T^0 (C) $M^{-1}L^0T^0$ (D) $M^0L^0T^{-1}$

Solution : **Ans. (C)**

$$\left[\frac{d}{dt}\left(\int \vec{F}\cdot d\vec{S}\right)\right] = \left[A\left(\vec{F}\cdot\vec{p}\right)\right] \Rightarrow \left[\frac{Fs}{t}\right] = [AFp] \Rightarrow [A] = \left[\frac{s}{pt}\right] = \frac{L}{MLT^{-1}\times T} = M^{-1}$$

Example # 8 to 10 :

A physical quantity X depends on another physical quantities as $X = YFe^{-\beta r^2} + ZW\sin(\alpha r)$ where r, F and W represents distance, force and work respectively & Y and Z are unknown physical quantities and α, β are positive constants.

8. If Y represent displacement then $\dim\left(\frac{\alpha YZ}{\beta F}\right)$ is equal to

(A) $M^{-1}LT^2$ (B) $M^{-1}L^2T^{-2}$ (C) $M^1L^1T^{-2}$ (D) None of these

9. If Y represent velocity then dim (X) is equal to

(A) ML^2T^{-3} (B) $M^{-1}L^2T^{-3}$ (C) ML^2T^{-2} (D) None of these

10. If Z represent frequency then choose the correct alternative

(A) The dimension of X is $[ML^1T^{-3}]$ (B) The dimension of Y is $[M^0LT^{-1}]$

(C) The dimension of β is $[M^0L^{-1}T^0]$ (D) The dimension of α is $[M^0L^1T^0]$

Solutions :

8. Ans. (A)

Given $X = YFe^{-\beta r^2} + ZW\sin(\alpha r)$ Here $\text{Dim}(\beta)=L^{-2}$, $\text{Dim}(\alpha)=L^{-1}$

Also $\text{Dim}(X) = \text{Dim}(YF)$ ...(i) & $\text{Dim}(X) = \text{Dim}(ZW)$ ...(ii)

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

Now If Dim(Y) = L

So Dim(X) = $[LMLT^{-2}] = [ML^2T^{-2}]$ from equation (i)

& Dim(Z) = $\text{Dim}\left(\frac{X}{W}\right)=\left[\frac{ML^2T^{-2}}{ML^2T^{-2}}\right]=[M^0L^0T^0]$ From equation (ii)

Thus $\text{Dim}\left(\frac{\alpha YZ}{\beta F}\right)=\left[\frac{L^{-1}L}{L^{-2}MLT^{-2}}\right]=[M^{-1}LT^2]$

9. Ans. (A)

If D(Y) = $[LT^{-1}]$. Than D(X) = $[LT^{-1}MLT^{-2}] = [ML^2T^{-3}]$ from equation (i)

10. Ans. (B)

If D(Z) = T^{-1}. Then D(Y) = ?

from equation (ii) : D(X) = $[T^{-1}ML^2T^{-2}] = [ML^2T^{-3}]$

So from equation (i) : D(Y) = $\frac{D(X)}{D(F)}=\left[\frac{ML^2T^{-3}}{MLT^{-2}}\right]=[LT^{-1}]$

Example #11

In a new system (say TK system) of measurement, the fundamental quantities length, mass and time are measured in Akshay, Shahrukh and Aamir respectively.

1 Akshay = 1 km

1 Shahrukh = 1 Quintal (100 kg)

1 Aamir = 1 minute

Column I		*Column II*	
(A)	One unit of acceleration in TK system.	(P)	$\frac{5}{18}\times10^{-4}$ SI unit
(B)	One unit of kinetic energy in TK system	(Q)	$\frac{5}{18}\times10^{-3}$ MKS unit
(C)	One unit of pressure in TK system	(R)	$\frac{5}{18}\times10^{0}$ MKS unit
(D)	One unit of work in TK system	(S)	$\frac{5}{18}\times10^{4}$ MKS unit
		(T)	$\frac{5}{18}\times10^{5}$ SI unit

Solution :

Ans. (A) →(R); (B) →(T); (C) →(P); (D) →(T)

For (A) : ∵[Acceleration] = $[LT^{-2}]$ ∴ unit of acceleration = (1km) $(1min)^{-2} = \frac{5}{18}$ m/s^2

For (B) : ∵[Kinetic energy] = $[ML^2T^{-2}]$

∴ Unit of kinetic energy = (1 Quintal) $(1km)^2(1min)^{-2} = \frac{5}{18}$ 10^5 kgm^2s^{-2}

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

y **For (C) :** $\because$ [Pressure] = $]ML^{-1}T^{-2}]$

$\therefore$ Unit of pressure = (1 Quintal) $(1km)^{-1}(1min)^{-2} = \frac{5}{18}\ 10^{-4}\ kgm^{-1}s^{-2}$

For (D) : $\because$ [Work] = $[ML^2T^{-2}]$

$\therefore$ Unit of work = (1 Quintal) $(1km)^2(1min)^{-2} = \frac{5}{18}\ 10^5\ kgm^2s^{-2}$

Example #12

The density of a material in CGS system is 2g /cm^3. In a system of units in which unit of length is 2 cm and unit of mass is 4 g, what is the numerical value of the density of the material?

Solution : **Ans. 4**

$$n_1u_1 = n_2u_2 \Rightarrow n_2 = n_1\left[\left[\frac{M_1}{M_2}\right]^1\left[\frac{L_1}{L_2}\right]^{-3}\right] = 2\left[\left(\frac{1g}{4g}\right)\left(\frac{1cm}{2cm}\right)^{-3}\right] = 2\times\frac{1}{4}\times 8 = 4$$

Example #13

A problem in I. E. Irodov's book is given as below:

"A particle of mass m is located in a region where its potential energy [U(x)] depends on the position x as Potential Energy [U(x)] = $\frac{a}{x^2}-\frac{b}{x}$ here a & b are positive constants ..."

(i) Write dimensional formula of a & b.

(ii) If the time period of oscillation which is calculated from above formula is stated by a student as $T = 4\pi a\sqrt{\frac{ma}{b^2}}$, check whether his answer is dimensionally correct.

Solution :

(i) $[a] = [Ux^2] = ML^2T^{-2}L^2 = \mathbf{[ML^4T^{-2}]}$; $[b] = [Ux] = ML^2T^{-2}L = \mathbf{[ML^3T^{-2}]}$

(ii) $T = 4\pi a\sqrt{\frac{ma}{b^2}} = 4\pi\sqrt{\frac{ma^3}{b^2}}$; Dimenison of RHS $= \left[4\pi\sqrt{\frac{ma^3}{b^2}}\right] = \sqrt{\frac{MM^3L^{12}T^{-6}}{M^2L^6T^{-4}}} \neq T$

So, his answer is dimensionally incorrect

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Basic Maths\English\2. Unit Dimensions.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 3

KINEMATICS - Theory

Index

KINEMATICS

Serial No.	CONTENTS	Page No.
	KINEMATICS	
1.	Kinematics	1
2.	Curvilinear, translation in Cartesian coordinate system	4
3.	Rectilinear Motion	5
4.	Projectile Motion	10
5.	Relative Motion	16
6.	Constraint Motion	17
7.	Kinematics of circular Motion	20
8.	**Exercise-1** Check your Grasp (Single Choice Objective Questions)	38
9.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	46
10.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	54
11.	**Exercise-4A** Conceptual Subjective Exercise	65
12.	**Exercise-4B** Brain Storming Exercise	69
13.	**Exercise-5A** Previous years questions	71
14.	**Exercise-5B** Previous years questions	72

ALLEN Materials Provided By - Material Point Available on Learnaf.com

JEE(Advanced) Syllabus :

Kinematics in one and two dimensions (Cartesian coordinates only), projectiles; Uniform Circular motion; Relative velocity.

JEE(Main) Syllabus :

Frame of reference. Motion in a straight line: position-time graph, speed and velocity. uniform and non-uniform motion, average speed and instantaneous velocity. Uniformly accelerated motion, velocity-time, position-time graphs for uniformly accelerated motion. Scalars and Vectors, Vector addition and Subtraction, Zero Vector, Scalar and Vector products, Unit Vector, Resolution of a Vector. Relative Velocity, Motion in a plane, Projectile Motion, Uniform Circular Motion.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Particle Kinematics

Kinematics

In kinematics we study how a body moves without knowing why it moves. All particles of a rigid body in translation motion move in identical fashion hence any of the particles of a rigid body in translation motion can be used to represent translation motion of the body. This is why, while analyzing its translation motion, a rigid body is considered a particle and kinematics of translation motion as particle kinematics.

Particle kinematics deals with nature of motion i.e. how fast and on what path an object moves and relates the position, velocity, acceleration, and time without any reference to mass, force and energy. In other words, it is study of geometry of motion.

Types of Translation Motion

A body in translation motion can move on either a straight–line path or curvilinear path.

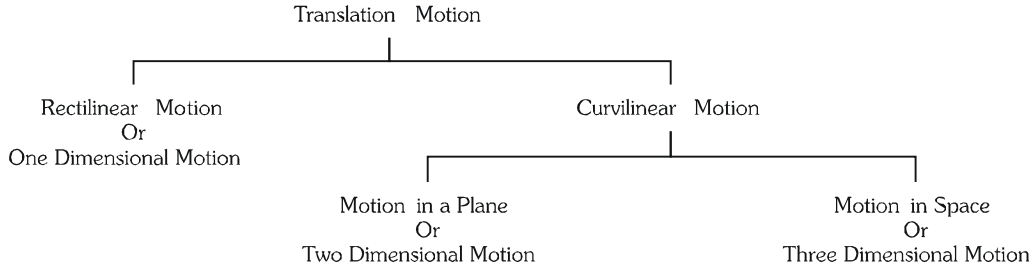
Rectilinear Motion

Translation motion on straight–line path is known as rectilinear translation. It is also known as one–dimensional motion. A car running on a straight road, train running on a straight track and a ball thrown vertically upwards or dropped from a height etc are very common examples of rectilinear translation.

Curvilinear Motion

Translation motion of a body on curvilinear path is known as curvilinear translation. If the trajectory is in a plane, the motion is known as two–dimensional motion. A ball thrown at some angle with the horizontal describes a curvilinear trajectory in a vertical plane; a stone tied to a string when whirled describes a circular path and an insect crawling on the floor or on a wall are examples of two–dimensional motion.

If path is not in a plane and requires a region of space or volume, the motion is known as three–dimensional motion or motion is space. An insect flying randomly in a room, motion of a football in soccer game over considerable duration of time etc are common examples of three–dimensional motion.



Reference Frame

Motion of a body can only be observed if it changes its position with respect to some other body. Therefore, for a motion to be observed there must be a body, which is changing its position with respect to other body and a person who is observing motion. The person observing motion is known as observer. The observer for the purpose of investigation must have its own clock to measure time and a point in the space attached with the other body as origin and a set of coordinate axes. These two things the time measured by the clock and the coordinate system are collectively known as reference frame.

In this way, motion of the moving body is expressed in terms of its position coordinates changing with time.

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

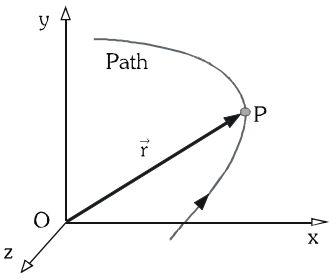
Position Vector, Velocity and Acceleration Vector

For analyzing translation motion, we assume the moving body as a particle and represent it as mathematical point. Consider a particle P moving on a curvilinear path.

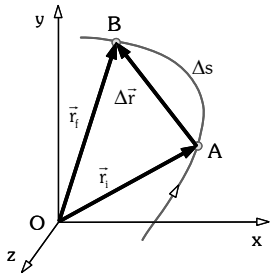
Position–Vector

It describes position of a particle relative to other particle and is a vector from the later towards the first. To study motion of a particle we have to assume a reference frame fixed with some other body. The vector drawn from the origin of the coordinate system representing the reference frame to the location of the particle P is known as position vector of the particle P.

Consider a particle P moving in space traces a path shown in the figure. Its position continuously changes with time and so does the position vector. At an instant of time, its position vector $\vec{r}$ is shown in the following figure.



Position Vector



Displacement Vector & Distance Traveled

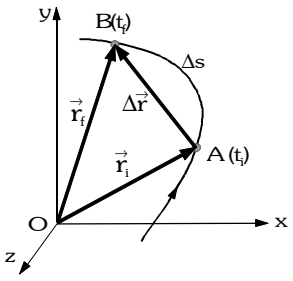
Displacement and distance traveled

Displacement is measure of change in place i.e. position of particle. It is defined by a vector from the initial position to the final position. Let the particle moves from point A to B on the curvilinear path. The vector $\overrightarrow{AB} = \Delta\vec{r}$ is displacement.

Distance traveled is length of the path traversed. We can say it "path length". Here in the figure length of the curve Δs from A to B is the distance traveled.

Distance traveled between two places is greater than the magnitude of displacement vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal.

Average Velocity and Average Speed



Average velocity of a particle in a time interval is that constant velocity with which particle would have covered the same displacement in the same time interval as it covers in its actual motion. It is defined as the ratio of displacement to the concerned time interval.

If the particle moves from point A to point B in time interval t_i to t_f, the average velocity $\vec{v}_{av}$ in this time interval is given by the following equation.

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Similar to average velocity, average speed in a time interval is that constant speed with which particle would travel the same distance on the same path in the same time interval as it travels in its actual motion. It is defined as the ratio of distance traveled to the concerned time interval.

If in moving from point A to B, the particle travels path length i.e. distance Δs in time interval t_i to t_f, its average speed c_{av} is given by the following equation.

$$c_{av} = \frac{\Delta s}{\Delta t} = \frac{\text{Path Length}}{t_f - t_i}$$

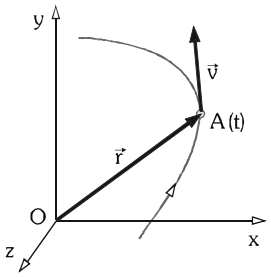
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Average speed in a time interval is greater than the magnitude of average velocity vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal.

Instantaneous Velocity and speed

If we assume the time interval Δt to be infinitesimally small i.e. $\Delta t \to 0$, the point B approaches A making the chord AB to coincide with the tangent at A. Now we can express the instantaneous velocity $\vec{v}$ by the following equations.



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity equals to the rate of change in its position vector $\vec{r}$ with time. Its direction is along the tangent to the path. Instantaneous speed is defined as the time rate of distance traveled.

$$c = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

You can easily conceive that when $\Delta t \to 0$, not only the chord AB but also the arc AB both approach to coincide with each other and with the tangent. Therefore $ds = |d\vec{r}|$. Now we can say that speed equals to magnitude of instantaneous velocity.

Instantaneous speed tells us how fast a particle moves at an instant and instantaneous velocity tells us in what direction and with what speed a particle moves at an instant of time.

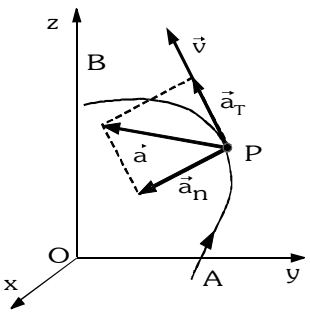
Acceleration

Instantaneous acceleration $\vec{a}$ is measure of how fast velocity of a body changes i.e. how fast direction of motion and speed change with time.

At an instant, it equals to the rate of change in velocity vector $\vec{v}$ with time.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

A vector quantity changes, when its magnitude or direction or both change. Accordingly, acceleration vector may have two components, one responsible to change only speed and the other responsible to change only direction of motion.



node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Component of acceleration responsible to change speed must be in the direction of motion. It is known as tangential component of acceleration $\vec{a}_T$. The component responsible to change direction of motion must be perpendicular to the direction of motion. It is known as normal component of acceleration $\vec{a}_n$. Acceleration vector $\vec{a}$ of a particle moving on a curvilinear path and its tangential and Normal components are shown in the figure.

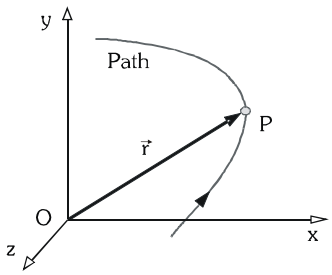
Curvilinear Translation in Cartesian coordinate system: Superposition of three rectilinear Motions

Consider a particle moving on a three dimensional curvilinear path AB. At an instant of time t it is at point P (x, y, z) moving with velocity $\vec{v}$ and acceleration $\vec{a}$. Its position vector is defined by equations

$$\vec{r} = x\tilde{i} + y\tilde{j} + z\tilde{k}$$

Differentiating it with respect to time, we get velocity vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\tilde{i} + \frac{dy}{dt}\tilde{j} + \frac{dz}{dt}\tilde{k} = v_x\tilde{i} + v_y\tilde{j} + v_z\tilde{k}$$



Here $v_x = dx/dt$, $v_y = dy/dt$ and $v_z = dz/dt$ are the components of velocity vectors in the x, y and z– directions respectively.

Now the acceleration can be obtained by differentiating velocity vector $\vec{v}$ with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\tilde{i} + \frac{dv_y}{dt}\tilde{j} + \frac{dv_z}{dt}\tilde{k} = a_x\tilde{i} + a_y\tilde{j} + a_z\tilde{k}$$

Acceleration vector can also be obtained by differentiating position vector twice with respect to time.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\tilde{i} + \frac{d^2x}{dt^2}\tilde{j} + \frac{d^2x}{dt^2}\tilde{k} = a_x\tilde{i} + a_y\tilde{j} + a_z\tilde{k}$$

In the above two equations, $a_x = d^2x/dt^2 = dv_x/dt$, $a_y = d^2y/dt^2 = dv_y/dt$ and $a_z = d^2z/dt^2 = dv_z/dt$ are the components of acceleration vectors in the x, y and z– directions respectively.

In the above equations, we can analyze each of the components x, y and z of motion as three individual rectilinear motions each along one of the axes x, y and z.

Along the x–axis $\quad v_x = \frac{dx}{dt} \quad$ and $\quad a_x = \frac{dv_x}{dt}$

Along the y–axis $\quad v_y = \frac{dy}{dt} \quad$ and $\quad a_y = \frac{dv_y}{dt}$

Along the z–axis $\quad v_z = \frac{dz}{dt} \quad$ and $\quad a_z = \frac{dv_z}{dt}$

A curvilinear motion can be analyzed as superposition of three simultaneous rectilinear motions each along one of the coordinate axes.

Example

Position vector $\vec{r}$ of a particle varies with time t according to the law $\vec{r} = \left(\frac{1}{2}t^2\right)\tilde{i} - \left(\frac{4}{3}t^{1.5}\right)\tilde{j} + (2t)\tilde{k}$, where r is in meters and t is in seconds.

(a) Find suitable expression for its velocity and acceleration as function of time.

(b) Find magnitude of its displacement and distance traveled in the time interval t = 0 to 4 s.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

Solution

(a) Velocity $\vec{v}$ is defined as the first derivative of position vector with respect to time.

$$\vec{v} = \frac{d\vec{r}}{dt} = t\tilde{i} - 2\sqrt{t}\tilde{j} + 2\tilde{k} \text{ m/s}$$

Acceleration $\vec{a}$ is defined as the first derivative of velocity vector with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \tilde{i} - \frac{1}{\sqrt{t}}\tilde{j} \text{ m/s}^2$$

(b) Displacement $\Delta\vec{r}$ is defined as the change in place of position vector.

$$\Delta\vec{r} = 8\tilde{i} - \tfrac{32}{3}\tilde{j} + 8\tilde{k} \text{ m}$$

Magnitude of displacement $\Delta r = \sqrt{8^2 + \left(\frac{32}{3}\right)^2 + 8^2} = 15.55$ m

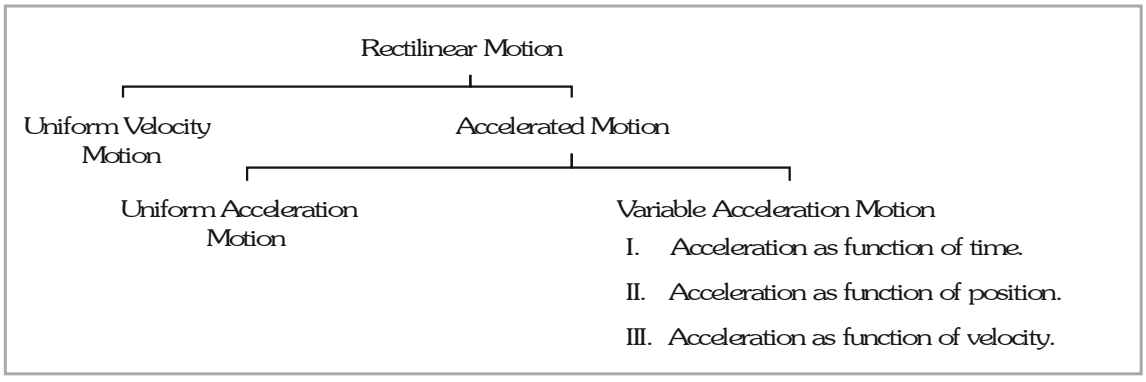
Distance Δs is defined as the path length and can be calculated by integrating speed over the concerned time interval.

$$\Delta s = \int_0^4 v\,dt = \int_0^4 \sqrt{t^2 + 4t + 4}\,dt = \int_0^4 (t+2)\,dt = 16 \text{ m}$$

Rectilinear Motion

Curvilinear motion can be conceived as superposition of three rectilinear motions each along one of the Cartesian axes. Therefore, we first study rectilinear motion in detail.

We can classify rectilinear motion problems in following categories according to given information.

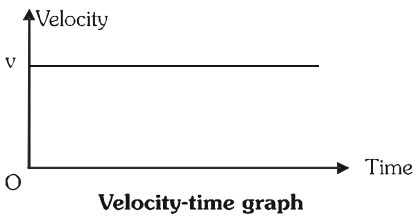


Uniform Velocity Motion

In uniform velocity motion, a body moves with constant speed on a straight–line path without change in direction.

If a body starting from position $x = x_o$ at the instant $t = 0$, moves with uniform velocity v in the positive x–direction, its equation of motion at any time t is $x = x_o + vt$

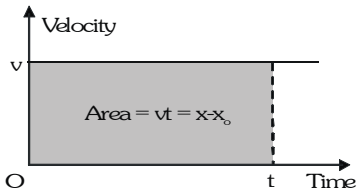
Velocity–time (v–t) graph for this motion is shown in the following figure.

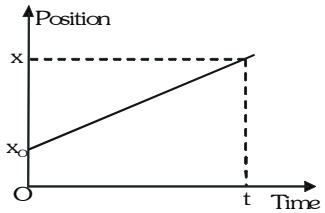


Velocity-time graph

node6\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

As we know that, the area between v–t graph and the time axes equals to change in position i.e displacement, the position–time relationship or position at any instant can be obtained.

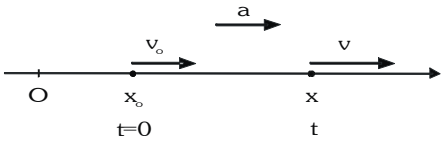




Position-time graph

Uniform Acceleration Motion

Motion in which acceleration remains constant in magnitude as well as direction is called uniform acceleration motion. In the motion diagram, is shown a particle moving in positive x–direction with uniform acceleration a. It passes the position x_o, moving with velocity v_o at the instant t = 0 and acquires velocity v at a latter instant t.



$$dv = adt \Rightarrow \int_{v_o}^{v} dv = a\int_{0}^{t} dt \Rightarrow v - v_o = at$$

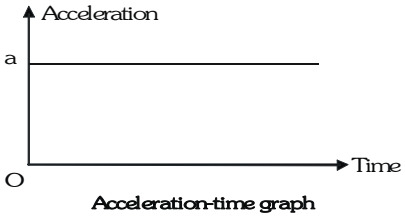
$$v = v_o + at \qquad \ldots(i)$$

Now from the above equation, we have $dx = vdt \Rightarrow \int_{x_o}^{x} dx = \int_{0}^{t} (v_o + at)dt \Rightarrow x - x_o = v_o t + \frac{1}{2}at^2$

$$x = x_o + v_o t + \frac{1}{2}at^2 \qquad \ldots(ii)$$

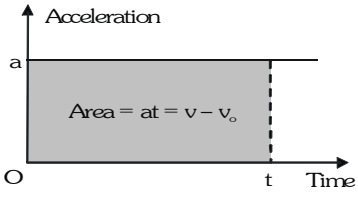
Eliminating time t, from the above two equations, we have $v^2 = v_o^2 + 2a(x - x_o)$ ...(iii)

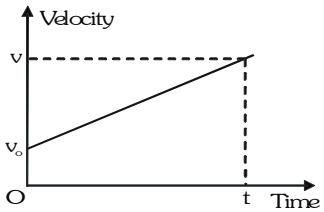
Equations (i), (ii) and (iii) are known as the first, second and third equations of motion for uniformly accelerated bodies. Acceleration–time (a–t) graph for this motion is shown in the following figure.



Acceleration-time graph

As we know that, the area between a–t graph and the time axes equals to change in velocity, velocity–time relation or velocity at any instant can be obtained.

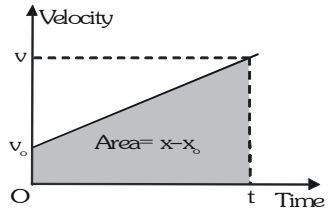




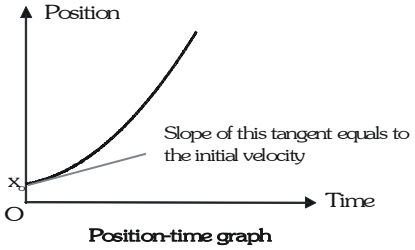
Velocity-time graph

The area between v–t graph and the time axes equals to change in position. Therefore, position–time relation or position at any instant can be obtained.

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65



$$\text{Area} = x - x_o = \left(\frac{v_o + v}{2}\right)t$$



Position-time graph

Example

A particle moving with uniform acceleration passes the point x = 2 m with velocity 20 m/s at the instant t = 0. Some time latter it is observed at the point x = 32 m moving with velocity 10 m/s.

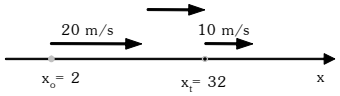
(a) What is its acceleration?

(b) Find its position and velocity at the instant t = 8 s.

(c) What is the distance traveled during the interval t = 0 to 8 s?

Solution

In the adjoining figure the given and required information shown are not to a scale. As motion diagram is a schematic representation only.



(a) Using the third equation of uniform acceleration motion, we have

$$v_t^2 = v_o^2 + 2a(x_t - x_o) \rightarrow a = \frac{v_t^2 - v_o^2}{2(x_t - x_o)} = \frac{10^2 - 20^2}{2(32-2)} = -5 \text{ m/s}^2$$

(b) Using second equation of uniform acceleration motion, we have

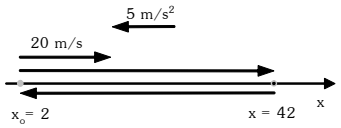
$$x_t = x_o + v_o t + \tfrac{1}{2}at^2 \rightarrow x_8 = 2 + 20\times 8 + \tfrac{1}{2}(-5)8^2 = 2 \text{ m}$$

Using the first equation of uniform acceleration motion, we have

$$v_t = v_o + at \rightarrow v_8 = 20 + (-5)\times 8 = -20 \text{ m/s}$$

(c) Where the particle returns, its velocity must be zero. Using the third equation of uniform acceleration motion, we have

$$v^2 = v_o^2 + 2a(x - x_o) \rightarrow x = x_o + \frac{v^2 - v_o^2}{2a} = 2 + \frac{0 - 20^2}{2(-5)} = 42 \text{ m}$$



This location is shown in the adjoining modified motion diagram.

The distance-traveled Δs is $\Delta s = |x - x_o| + |x_o - x| = 80$ m

Example

A ball is dropped from the top of a building. The ball takes 0.50 s to fall past the 3 m length of a window, which is some distance below the top of the building.

(a) How fast was the ball going as it passed the top of the window?

(b) How far is the top of the window from the point at which the ball was dropped?

Assume acceleration g in free fall due to gravity be 10 m/s² downwards.

Solution

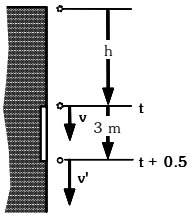
The ball is dropped, so it start falling from the top of the building with zero initial velocity (v_o= 0). The motion diagram is shown with the given information in the adjoining figure.

Using the first equation of the constant acceleration motion, we have

node06\E_NOD65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

$v_t = v_o + at \rightarrow \quad v = 0 + 10t = 10t$ ...(i)

$v' = 0 + 10(t + 0.5) = 10t + 5$ ...(ii)



Using values of v and v' in following equation, we have

$x - x_o = \left(\frac{v_o + v}{2}\right)t \rightarrow$ window height $\left(\frac{v + v'}{2}\right) \times 0.5 \Rightarrow t = 0.35s$

(o) From equation (i), we have $v = 10t = 3.5$ m/s

(p) From following equation, we have

$$x - x_o = \left(\frac{v_o + v}{2}\right)t \rightarrow h = \left(\frac{0 + v}{2}\right)t = 61.25 \text{ cm}$$

Variable Acceleration Motion

More often, problems in rectilinear motion involve acceleration that is not constant. In these cases acceleration is expressed as a function of one or more of the variables t, x and v. Let us consider three common cases.

Acceleration given as function of time

If acceleration is a given function of time say a = f(t), from equation a = dv/dt we have

$$dv = f(t)dt \Rightarrow \int dv = \int f(t)dt$$

The above equation expresses v as function of time, say v = g(t). Now substituting g(t) for v in equation v = dx/dt, we have

$$dx = g(t)dt \Rightarrow \int dx = \int g(t)dt$$

The above equation yield position as function of time.

Example

The acceleration of a particle moving along the x-direction is given by equation a = (3–2t) m/s^2. At the instants t = 0 and t = 6 s, it occupies the same position.

(a) Find the initial velocity v_o.

(b) What will be the velocity at t = 2 s?

Solution

By substituting the given equation in equation $a = dv/dt$, we have

$$dv = (3 - 2t)dt \Rightarrow \int_{v_o}^{v} dv = \int_0^t (3 - 2t)dt \Rightarrow v = v_o + 3t - t^2 \quad \text{...(i)}$$

By substituting eq. (i) in equation v = dx/dt , we have

$$dx = (v_o + 3t - t^2)dt \Rightarrow \int_{x_o}^{x} dx = \int_0^t (v_o + 3t - t^2)dt \Rightarrow x = x_o + v_o t + \tfrac{3}{2}t^2 - \tfrac{1}{3}t^3 \quad \text{...(ii)}$$

(a) Applying the given condition that the particle occupies the same x coordinate at the instants t = 0 and t = 6 s in eq. (ii), we have

$x_o = x_6 \Rightarrow x_o = x_o + 6v_o + 54 - 72 \Rightarrow v_o = 3$ m/s

(b) Using v_o in eq. (i), we have $v = 3 + 3t - t^2 \Rightarrow v_2 = 5$ m/s

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Acceleration as function of position

If acceleration is a given function of position say a = f(x), we have to use equation a = vdv/dx. Rearranging term in this equation we have vdv = adx. Now substituting f(x) for a, we have

$$vdv = f(x)dx \Rightarrow \int vdv = \int f(x)dx$$

The above equation provides us with velocity as function of position. Let relation obtained in this way is v = g(x). Now substituting g(x) for v in equation v = dx/dt, we have

$$dt = \frac{dx}{g(x)} \Rightarrow \int dt = \int \frac{dx}{g(x)}$$

The above equation yields the desired relation between x and t.

Example

Acceleration of a particle moving along the x-axis is defined by the law $a = -4x$, where a is in m/s^2 and x is in meters. At the instant t = 0, the particle passes the origin with a velocity of 2 m/s moving in the positive x-direction.

(a) Find its velocity v as function of its position coordinates.

(b) Find its position x as function of time t.

(c) Find the maximum distance it can go away from the origin.

Solution

(a) By substituting given expression in the equation a = v dv/dx and rearranging, we have

$$vdv = -4xdx \Rightarrow \int_2^v vdv = -4\int_0^x xdx \Rightarrow v = \pm 2\sqrt{1-x^2} \rightarrow v = 2\sqrt{1-x^2}$$

Since the particle passes the origin with positive velocity of 2 m/s, so the minus sign in the eq. (i) has been dropped.

(b) By substituting above obtained expression of velocity in the equation v = dx/dt and rearranging, we have

$$\frac{dx}{\sqrt{1-x^2}} = 2dt \Rightarrow \int_0^x \frac{dx}{\sqrt{1-x^2}} = 2\int_0^t dt \Rightarrow \sin^{-1}(x) = 2t \rightarrow x = \sin 2t$$

(c) The maximum distance it can go away from the origin is 1m because maximum magnitude of sine function is unity.

Acceleration as function of velocity

If acceleration is given as function of velocity say a=f(v), by using equation a = dv/dt we can obtain velocity as function of time.

$$dt = \frac{dv}{f(v)} \Rightarrow \int dt = \int \frac{dv}{f(v)}$$

Now using equation v = dx/dt we can obtain position as function of time

In another way if we use equation a = vdv/dx, we obtain velocity as function of position.

$$dx = \frac{vdv}{f(v)} \Rightarrow \int dx = \int \frac{vdv}{f(v)}$$

Now using equation v = dx/dt we can obtain position as function of time

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example

Acceleration of particle moving along the x-axis varies according to the law $a = -2v$, where a is in m/s^2 and v is in m/s. At the instant t = 0, the particle passes the origin with a velocity of 2 m/s moving in the positive x-direction.

(a) Find its velocity v as function of time t.

(b) Find its position x as function of time t.

(c) Find its velocity v as function of its position coordinates.

(d) Find the maximum distance it can go away from the origin.

(e) Will it reach the above-mentioned maximum distance?

Solution

(a) By substituting the given relation in equation $a = dv/dt$, we have

$$\frac{dv}{v} = -2dt \Rightarrow \int_2^v \frac{dv}{v} = -2\int_0^t dt \rightarrow v = 2e^{-2t} \qquad \ldots(i)$$

(b) By substituting the above equation in v = dx/dt, we have

$$dx = 2e^{-2t}dt \Rightarrow \int_0^x dx = 2\int_0^t e^{-2t}dt \rightarrow x = 1 - 2e^{-2t} \qquad \ldots(ii)$$

(c) Substituting given expression a in the equation $a = v\,dv/dx$ and rearranging, we have

$$dv = -2dx \Rightarrow \int_2^v dv = -2\int_0^x dx \rightarrow v = 2(1 - x) \qquad \ldots(iii)$$

(d) Eq. (iii) suggests that it will stop at x = 1 m. Therefore, the maximum distance away from the origin it can go is 1 m.

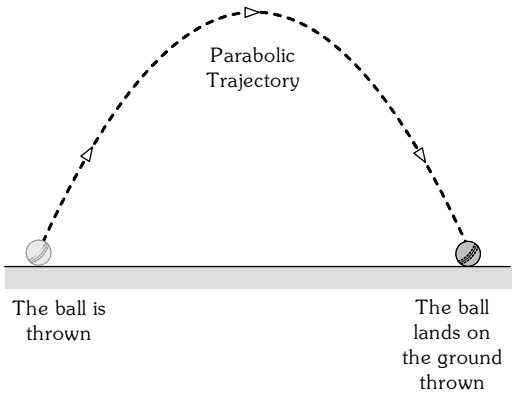
(e) Eq. (ii) suggests that to cover 1 m it will take time whose value tends to infinity. Therefore, it can never cover this distance.

Projectile Motion

An object projected by an external force when continues to move by its own inertia is known as projectile and its motion as projectile motion.

A football kicked by a player, an arrow shot by an archer, water sprinkling out a water–fountain, an athlete in long jump or high jump, a bullet or an artillery shell fired from a gun are some examples of projectile motion.

In simplest case when a projectile does reach great heights above the ground as well as does not cover a very large distance on the ground, acceleration due to gravity can be assumed uniform throughout its motion. Moreover, such a projectile does not spend much time in air not permitting the wind and air resistance to gather appreciable effects. Therefore, while analyzing them, we can assume gravity to be uniform and neglect effects of wind as well as air resistance. Under these circumstances when an object is thrown in a direction other than the vertical, its trajectory assumes shape of a parabola. In the figure, a ball thrown to follow a parabolic trajectory is shown as an example of projectile motion.



node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

At present, we study projectiles moving on parabolic trajectories and by the term projectile motion; we usually refer to this kind of motion.

For a projectile to move on parabolic trajectory, the following conditions must be fulfilled.

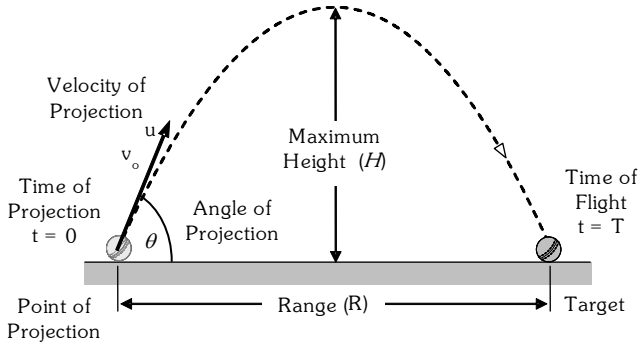
- Acceleration vector must be uniform.
- Velocity vector never coincides with line of acceleration vector.

Analyzing Projectile motion

Since parabola is a plane curve, projectile motion on parabolic trajectory becomes an example of a two–dimensional motion. It can be conceived as superposition of two simultaneous rectilinear motions in two mutually perpendicular directions, which can be analyzed separately as two Cartesian components of the projectile motion.

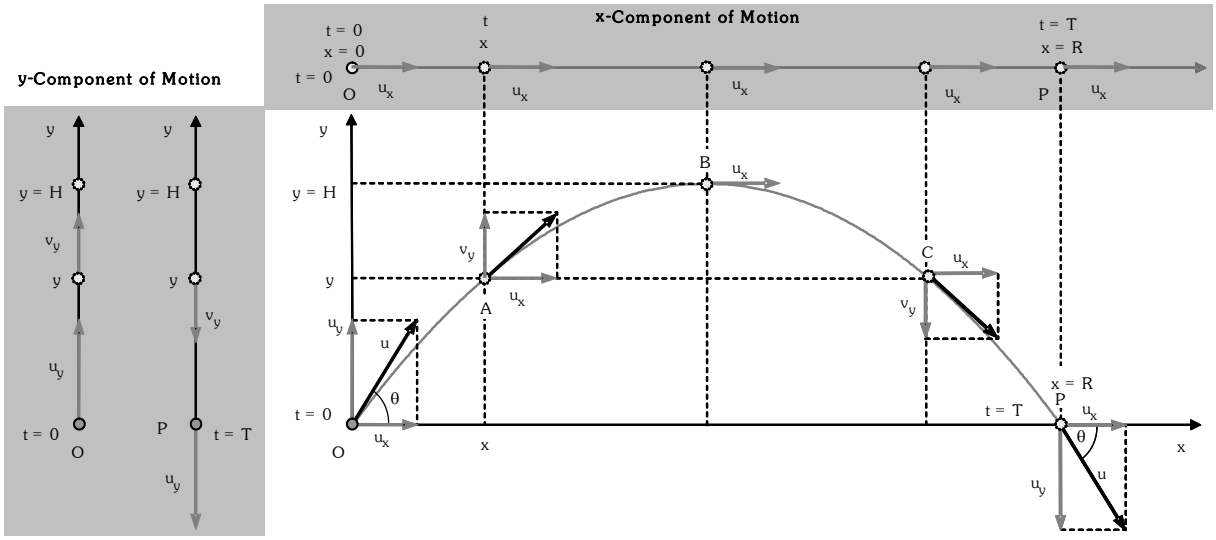
Projectile Motion near the Horizontal or Flat Ground using Cartesian components

Consider motion of a ball thrown from ground as shown in the figure. The point from where it is projected is known as point of projection, the point where it falls on the ground is known as point of landing or target. The distance between these two points is known as horizontal range or range, the height from the ground of the highest point it reaches during flight is known as maximum height and the duration for which it remain in the air is known as air time or time of flight. The velocity with which it is thrown is known as velocity of projection and angle which velocity of projection makes with the horizontal is known as angle of projection.



A careful observation of this motion reveals that when a ball is thrown its vertical component of velocity decreases in its upward motion, vanishes at the highest point and thereafter increases in its downward motion due to gravity similar to motion of a ball thrown vertically upwards. At the same time, the ball continues to move uniformly in horizontal direction due to inertia. The actual projectile motion on its parabolic trajectory is superposition of these two simultaneous rectilinear motions.

In the following figure, the above ideas are shown representing the vertical by y-axis and the horizontal by x-axis.



Projectile motion resolved into its two Cartesian components.
Projectile motion as superposition of two rectilinear motions one in vertical and other in horizontal direction.

node6\E_NOD65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Vertical or y-component of motion.

Component of initial velocity in the vertical direction is u_y. Since forces other than gravitational pull of the earth are negligible, vertical component of acceleration a_y of the ball is g vertically downwards. This component of motion is described by the following three equations. Here v_y denotes y-component of velocity, y denotes position coordinate y at any instant t.

$$v_y = u_y - gt \quad ...(i) \qquad y = u_y t - \tfrac{1}{2}gt^2 \quad ...(ii) \qquad v_y^2 = u_y^2 - 2gy \quad ...(iii)$$

Horizontal or x-component of motion.

Since effects of wind and air resistance are assumed negligible as compared to effect of gravity, the horizontal component of acceleration of the ball becomes zero and the ball moves with uniform horizontal component of velocity u_x. This component of motion is described by the following equation.

$$x = u_x t \quad ...(iv)$$

Equation of trajectory

Equation of the trajectory is relation between the x and the y coordinates of the ball without involvement of time t. To eliminate t, we substitute its expression from equation (iv) into equation (ii).

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \quad ...(v)$$

Every projectile motion can be analyzed using the above five equations. In a special case of interest, if the projectile lands the ground again, its time of flight, the maximum height reached and horizontal range are obtained using the above equations.

Time of Flight

At the highest point of trajectory when $t = \frac{1}{2}T$, the vertical component of velocity becomes zero. At the instant t =T, the ball strikes the ground with vertical component of velocity $v_y = -u_y$. By substituting either of these conditions in equation (i), we obtain the time of flight.

$$T = \frac{2u_y}{a_y} = \frac{2u_y}{g}$$

Maximum Height

At the highest point of trajectory where y = H, the vertical component of velocity becomes zero. By substituting this information in equation (iii), we obtain the maximum height.

$$H = \frac{u_y^2}{2g}$$

Horizontal Range

The horizontal range or simply the range of the projectile motion of the ball is distance traveled on the ground in its whole time of flight.

$$R = u_x T = \frac{2u_x u_y}{g} = \frac{u^2\sin 2\theta}{g}$$

Maximum Range

It is the maximum distance traveled by a projectile in the horizontal direction for a certain velocity of projection.

The above expression of range makes obvious that to obtain maximum range the ball must be projected at angle $\theta = 45°$.

Substituting this condition in the expression of range, we obtain the maximum range R_m.

$$R_m = \frac{u^2}{g}$$

Trajectory Equation alternate form

If range is known in advance, the equation of trajectory can be written in an alternative form involving horizontal range.

$$y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

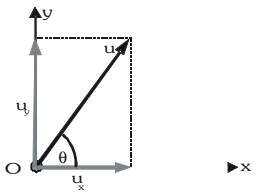
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example

A ball is thrown with 25 m/s at an angle 53 above the horizontal. Find its time of flight, maximum height and range.

Solution

In the adjoining figure velocity of projection u = 25 m/s, angle of projection θ= 53 , the horizontal and vertical components u_x and u_y of velocity of projection are shown. From these information we have



$u_x = u\cos 53^\circ = 15$ m/s and $u_y = u\sin 53^\circ = 20$ m/s

Using equations for time of flight T, maximum height H and range R, we have

$$T = \frac{2u_y}{g} = \frac{2\times 20}{10} = 4 \text{ s}$$

$$H = \frac{u_y^2}{2g} = \frac{20^2}{2\times 10} = 20 \text{ m}$$

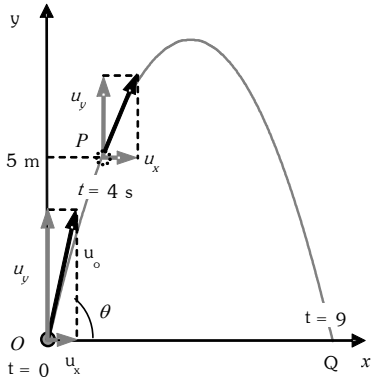
$$R = \frac{2u_x u_y}{g} = \frac{2\times 15\times 20}{10} = 60 \text{ m}$$

Example

A ball 4 s after the instant it was thrown from the ground passes through a point P, and strikes the ground after 5 s from the instant it passes through the point P. Assuming acceleration due to gravity to be 9.8 m/s^2 find height of the point P above the ground.

Solution

The ball projected with velocity $\vec{u} = u_x\hat{i} + u_y\hat{j}$ form O reaches the point P with velocity $\vec{v} = u_x\hat{i} + v_y\hat{j}$ and hits the ground at point Q at the instant T = 4 + 5 = 9 s as shown in the adjoining motion diagram.



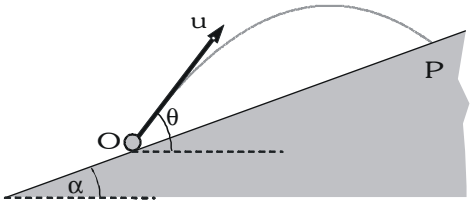
From equation of time of flight, we have its initial y-component of velocity u_y $T = \frac{2u_y}{g} \rightarrow u_y = \frac{1}{2}gT$

Substituting above in eq. (ii) and rearranging terms, we have the height y of the point P.

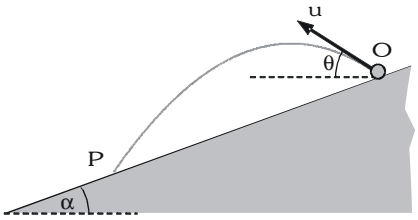
$$y = u_y t - \tfrac{1}{2}gt^2 \rightarrow y = \tfrac{1}{2}gt(T-t) = \tfrac{1}{2}\times 9.8\times 4(9-4) = 98\text{m}$$

Projectile on inclined plane

Artillery application often finds target either up a hill or down a hill. These situations can approximately be modeled as projectile motion up or down an inclined plane.



Projectile up an inclined plane



Projectile down an inclined plane

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

In the above left figure is shown a shell projected from a point O with velocity u at an angle θ to hit a target at point P uphill. This projectile motion is called projectile up a hill or inclined plane. Similarly in the above right figure is shown a projectile down a hill or inclined plane.

Analyzing projectile motion up an inclined plane using Cartesian components

Consider the projectile motion up an inclined plane described earlier. Assume a Cartesian coordinate system whose x-axis coincides with the line of fire OP and the origin with the point of projection as shown. The line OP is along the line of the greatest slope.

Velocity of projection makes angle $\theta - \alpha$ with the positive x-axis, therefore its x and y-components u_x and u_y are

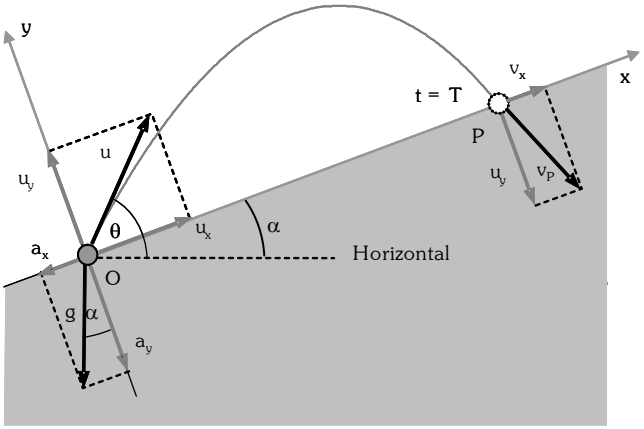
$$u_x = u\cos(\theta - \alpha)$$

$$u_y = u\sin(\theta - \alpha)$$

Acceleration due to gravity g being vertical makes the angle α with the negative y-axis, therefore x and y-components of acceleration vector are

$$a_x = g\sin\alpha$$

$$a_y = g\cos\alpha$$



Projectile motion up an inclined plane resolved into its two Cartesian components.

Motion component along the y-axis

The projectile starts with initial y-component of velocity u_y in the positive y-direction and has uniform y-component of acceleration $a_y = g\cos\alpha$ in the negative y-direction. This component of motion is described by the following three equations. Here v_y denotes y-component of velocity, y denotes position coordinate y at any instant t.

$$v_y = u_y - a_y t \qquad \ldots(i)$$

$$y = u_y t - \tfrac{1}{2} a_y t^2 \qquad \ldots(ii)$$

$$v_y^2 = u_y^2 - 2a_y y \qquad \ldots(iii)$$

Motion component along the x-axis

The x-component of motion is also uniformly accelerated motion. The projectile starts with initial x-component of velocity u_x in the positive x-direction and has uniform x-component of acceleration a_x in the negative x-direction. This component of motion is described by the following three equations. Here v_x denotes x-component of velocity, y denotes position coordinate x at any instant t.

$$v_x = u_x - a_x t \qquad \ldots(iv)$$

$$x = u_x t - \tfrac{1}{2} a_x t^2 \qquad \ldots(v)$$

$$v_x^2 = u_x^2 - 2a_x x \qquad \ldots(vi)$$

Every projectile motion up an incline can be analyzed using the above six equations. Quantities of interest in artillery applications and hence in projectile on incline plane are time of flight, range on the incline plane and the angle at which the shell hits the target.

Time of flight.

Moving in air for time interval T the projectile when hits the target P, its y-component of velocity u_y becomes in the negative y-direction. Using this information in equation (i), we obtain the time of flight.

$$T = \frac{2u_y}{a_y} = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$$

When the projectile hits the target P, its y component of displacement also becomes zero. This information with equation (ii) also yield the time of flight.

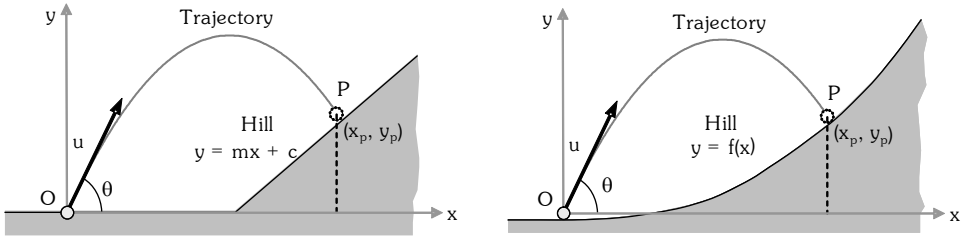
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Range on the plane. The range of a projectile on an incline plane is the distance between the point of projection and the target. It equals to displacement in the x-direction during whole flight. By substituting time of flight in equation (v), we obtain expression for the range R.

$$R = \frac{2u\sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}$$

Analysis of projectile on an incline plane using Equation of trajectory

Sometimes the hill may be away from the point of projection or the hill may not have uniform slope as shown in the following two figures.



In these cases, the shape of the hill can be expressed by a suitable equation of the form y = mx + c for uniform slope hill or y = f(x) for nonuniform slope hill. The target P where the projectile hits the hill is the intersection of trajectory of the projectile and the hill. Therefore, coordinates (x_p, y_p) of the target can be obtained by simultaneously solving equation of the hill and equation of trajectory of the projectile.

Time of flight

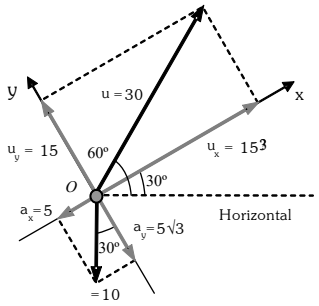
Since a projectile move with uniform horizontal component of the velocity (u_x), its time of flight T can be calculated from the following equation.

$$T = \frac{x_p}{u_x} = \frac{x_p}{u\cos\theta}$$

Example

A particle is projected with a velocity of 30 m/s at an angle 60 above the horizontal on a slope of inclination 30 . Find its range, time of flight and angle of hit.

Sol. The coordinate system, projection velocity and its component, and acceleration due to gravity and its component are shown in the adjoining figure.



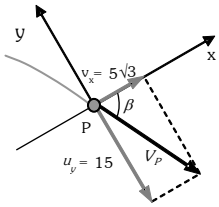
Substituting corresponding values in following equation, we get the time of flight.

$$T = \frac{2u_y}{a_y} \rightarrow T = \frac{2\times 15}{5\sqrt{3}} = 2\sqrt{3}\ \text{s}$$

Substituting value of time of flight in following equation, we get the range R.

$$R = u_x T - \tfrac{1}{2}a_x T^2 \Rightarrow R = 15\sqrt{3}\times 2\sqrt{3} - \tfrac{1}{2}\times 5\times(2\sqrt{3})^2 = 60\text{m}$$

In the adjoining figure, components of velocity $\vec{v}_P$ when the projectile hits the slope at point P are shown. The angle β which velocity vector makes with the x-axis is known as angle of hit. The projectile hits the slope with such a velocity $\vec{v}_P$, whose y-component is equal in magnitude to that of velocity of projection. The x-component of velocity v_x is calculated by substituting value of time of flight in following equation.



$$v_x = u_x - a_x t \rightarrow v_x = 15\sqrt{3} - 5\times 2\sqrt{3} = 5\sqrt{3}$$

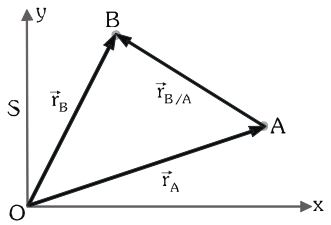
$$\beta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \rightarrow \beta = 60°$$

node06\E_NOD65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Relative Motion

Motion of a body can only be observed, when it changes its position with respect to some other body. In this sense, motion is a relative concept. To analyze motion of a body say A, therefore we have to fix our reference frame to some other body say B. The result obtained is motion of body A relative to body B.

Relative position, Relative Velocity and Relative Acceleration



Let two bodies represented by particles A and B at positions defined by position vectors $\vec{r}_A$ and $\vec{r}_B$, moving with velocities $\vec{v}_A$ and $\vec{v}_B$ and accelerations $\vec{a}_A$ and $\vec{a}_B$ with respect to a reference frame S. For analyzing motion of terrestrial bodies the reference frame S is fixed with the ground.

The vectors $\vec{r}_{B/A}$ denotes position vector of B relative to A.

Following triangle law of vector addition, we have

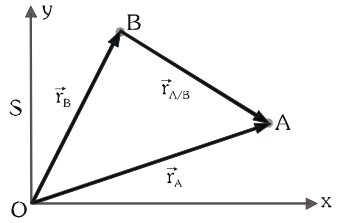
$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \qquad \text{...(i)}$$

First derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to velocity of particle A and velocity of particle B relative to frame S and first derivative of $\vec{r}_{B/A}$ with respect to time defines velocity of B relative to A.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \text{...(ii)}$$

Second derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to acceleration of particle A and acceleration of particle B relative to frame S and second derivative of $\vec{r}_{B/A}$ with respect to time defines acceleration of B relative to A.

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad \text{...(iii)}$$



In similar fashion motion of particle A relative to particle B can be analyzed with the help of adjoining figure. You can observe in the figure that position vector of A relative to B is directed from B to A and therefore

$\vec{r}_{B/A} = -\vec{r}_{A/B}$, $\vec{v}_{B/A} = -\vec{v}_{A/B}$ and $\vec{a}_{B/A} = -\vec{a}_{A/B}$.

The above equations elucidate that how a body A appears moving to another body B is opposite to how body B appears moving to body A.

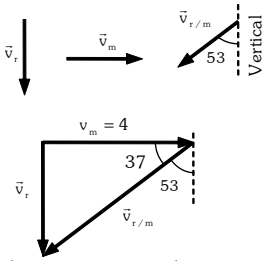
Example

A man when standstill observes the rain falling vertically and when he walks at 4 km/h he has to hold his umbrella at an angle of 53 from the vertical. Find velocity of the raindrops.

Solution

Assigning usual symbols $\vec{v}_m$, $\vec{v}_r$ and $\vec{v}_{r/m}$ to velocity of man, velocity of rain and velocity of rain relative to man, we can express their relationship by the following eq.

$$\vec{v}_r = \vec{v}_m + \vec{v}_{r/m}$$



The above equation suggests that a standstill man observes velocity $\vec{v}_r$ of rain relative to the ground and while he is moving with velocity $\vec{v}_m$, he observes velocity of rain relative to himself $\vec{v}_{r/m}$. It is a common intuitive fact that umbrella must be held against $\vec{v}_{r/m}$ for optimum protection from rain. According to these facts, directions of the velocity vectors are shown in the adjoining figure.

The addition of velocity vectors is represented according to the above equation is also represented. From the figure we have

$$v_r = v_m \tan 37^\circ = 3 \text{ km/h}$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example

A boat can be rowed at 5 m/s on still water. It is used to cross a 200 m wide river from south bank to the north bank. The river current has uniform velocity of 3 m/s due east.

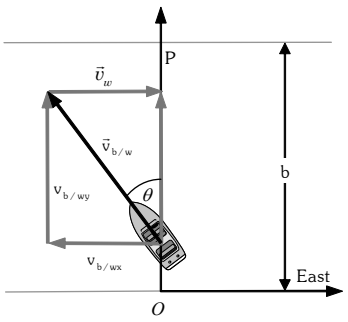
(a) In which direction must it be steered to cross the river perpendicular to current?

(b) How long will it take to cross the river in a direction perpendicular to the river flow?

(c) In which direction must the boat be steered to cross the river in minimum time? How far will it drift?

Solution

(a) Velocity of a boat on still water is its capacity to move on water surface and equals to its velocity relative to water.



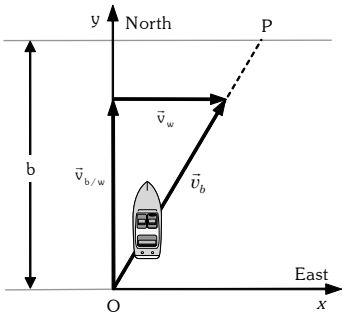
$\vec{v}_{b/w}$ = Velocity of boat relative to water = Velocity of boat on still water

On flowing water, the water carries the boat along with it. Thus velocity $\vec{v}_b$ of the boat relative to the ground equals to vector sum of $\vec{v}_{b/w}$ and $\vec{v}_w$. The boat crosses the river with the velocity $\vec{v}_b$.

$$\vec{v}_b = \vec{v}_{b/w} + \vec{v}_w$$

(b) To cross the river perpendicular to current the boat must be steered in a direction so that one of the components of its velocity ($\vec{v}_{b/w}$) relative to water becomes equal and opposite to water flow velocity $\vec{v}_w$ to neutralize its effect. It is possible only when velocity of boat relative to water is grater than water flow velocity. In the adjoining figure it is shown that the boat starts from the point O and moves along the line OP (y-axis) due north relative to ground with velocity $\vec{v}_b$. To achieve this it is steered at an angle θ with the y-axis.

$$v_{b/w}\sin\theta = v_w \rightarrow 5\sin\theta = 3 \Rightarrow \theta = 37^\circ$$



(c) The boat will cover river width b with velocity

$v_b = v_{b/wy} = v_{b/w}\sin 37^\circ = 4$ m/s in time t, which is given by

$$t = b / v_b \rightarrow t = 50\text{s}$$

(d) To cross the river in minimum time, the component perpendicular to current of its velocity relative to ground must be kept to maximum value. It is achieved by steering the boat always perpendicular to current as shown in the adjoining figure. The boat starts from O at the south bank and reaches point P on the north bank. Time t taken by the boat is given by

$$t = b / v_{b/w} \rightarrow t = 40\text{s}$$

Drift is the displacement along the river current measured from the starting point. Thus, it is given by the following equation. We denote it by x_d.

$$x_d = v_{bx}t$$

Substituting $v_{bx} = v_w = 3$ m/s, from the figure, we have

$$x_d = 120 \text{ m}$$

Dependant Motion or Constraint Motion

Effect of motion of one body on another, when they are interconnected through some sort of physical link of a definite property is what we study in dependant motion.

The definite property of the connecting link is a constraint that decides how motion of one body depends on that of the other. Therefore, dependant motion is also known as constraint motion.

node6\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

In various physical situations, we often encounter interconnected bodies affecting motion of each other. The variety of connecting link may be a string, a rod or a direct contact. A string has a definite length and can only pull a body, it cannot push; a rod also has definite length and can pull or push a body, bodies in direct smooth contact can only push each other. These problems are analyzed by the following methods.

Method of constraint equation

In this method, a property of connecting link is expressed in terms of position coordinates of the bodies. This equation is known as constraint equation. Differentiating the constraint equation once with respect to time we get relationship between their velocities and again differentiating the velocity relation with respect to time we get relationship between their accelerations.

Method of Virtual Work

In this method, we use concepts of force and work. Work is defined as scalar product of force and displacement of the point of application of force. If two bodies are connected by inextensible links or links of constant length, the sum of scalar products of forces applied by connecting links and displacement of contact points at the ends of the connecting links equals to zero in every infinitesimally small time interval.

Let the forces applied by the connecting links on connected bodies are $\vec{F}_1$, $\vec{F}_2$,..... $\vec{F}_i$ .... $\vec{F}_n$ and displacements of corresponding contact points in an infinitesimally small time interval dt are $d\vec{r}_1$, $d\vec{r}_2$, ... $d\vec{r}_i$,.... $d\vec{r}_n$.

The principle suggest that $$\sum_{i=1}^{n} \vec{F}_i \cdot d\vec{r}_i = 0$$

The relation between speeds of the contact point can directly be obtained by dividing the equation with time interval dt.

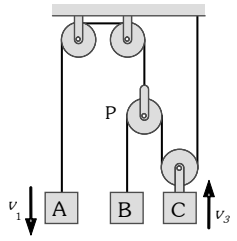
$$\sum_{i=1}^{n} \vec{F}_i \cdot \vec{v}_i = 0$$

When angle between force vectors and velocity vectors do not vary with time, we can differentiate the above equation to obtain relationship between accelerations. However, care must be taken in deciding acceleration relation, when angle between force vectors and velocity vectors vary with time. In these circumstances, we may get an additional term involving the derivative of the angle between the force and the velocity. Therefore, at present we restrict ourselves to use this method when angle between force and velocity vectors remain constant. In these situations, we have

$$\sum_{i=1}^{n} \vec{F}_i \cdot \vec{a}_i = 0$$

Example

In the system shown, the block A is moving down with velocity v_1 and C is moving up with velocity v_3. Express velocity of the block B in terms of velocities of the blocks A and C.

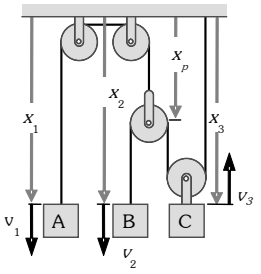


Solution

Method of Constrained Equations.

The method requires assigning position coordinate to each of the moving bodies and making constraint equation for each string.

In the given system, there are four separately moving bodies and two strings. The moving bodies are the three blocks and one pulley P. We assign position coordinates x_1, x_2, x_3 and x_p all measured from the fixed reference ceiling as shown in the figure. The required constraint equation for string connecting block A and pulley P is $x_1 + x_p = \ell_1$ ...(i)



And the required constraint equation for the other string is $x_2 + 2x_3 - 2x_p = \ell_2$ ...(i)

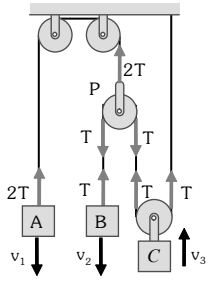
node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

Let the block B is moving down with velocity v_2. The velocities are defined as $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$, and $v_3 = -\dot{x}_3$

Differentiating terms of eq. (i) and (ii), eliminating $\dot{x}_p$ and substituting above values of velocities, we have $v_2 = 2(v_3 - v_1)$

Method of Virtual Work.

The tension forces applied by the strings on each contact point and displacements of the blocks are shown in the adjacent figure.



Let the tension in the string connected to block B is T. The tension in the string connecting the block A and the pulley P must be 2T in order to justify Newton's Second Law for massless pulley P.

$$\sum \vec{F}_i \cdot \vec{v}_i = 0 \rightarrow -2Tv_1 - Tv_2 + 2Tv_3 = 0 \Rightarrow v_2 = 2(v_3 - v_1)$$

Visual Inspection with Supperposition.

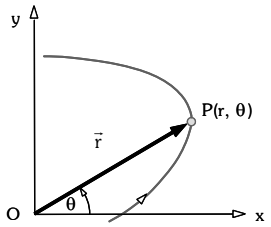
Motion of a body in an interconnected system equals to sum of individual effects of all other bodies. So velocity of block B equals to addition of individual effects of motion of A and C.

The individual effect of motion of A is velocity of B due to motion of A only and can easily be predicted by visual inspection of the system. Let this individual effect be denoted by v_{BA}. $v_{BA} = 2v_1$ ...(i)

Individual effect of motion of C on motion of B is $v_{BC} = 2v_3$ ...(ii)

According to the principle of superposition, velocity v_2 of block B equals to $v_2 = 2(v_3 - v_1)$

Describing Translation Motion by Angular Variables

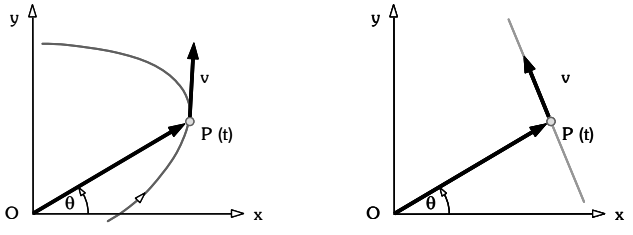


Position of particle can completely be specified by its position vector $\vec{r}$, if magnitude r of the position vector and its orientation relative to some fixed reference direction is known. In the given figure is shown a particle P at location shown by position vector $\vec{r} = \overrightarrow{OP}$. Magnitude of the position vector is distance r = OP of the particle from the origin O and orientation of the position vector is the angle θ made by line OP with the positive x-axis. We now specify position of a particle by these to variables r and θ, known as polar coordinates.

When the particle moves, either or both of these coordinates change with time. If a particle moves radially away from the origin, magnitude r of its position vector $\vec{r}$ increases without any change in angle θ. Similarly, if a particle moves radially towards the origin, r decreases without any change in angle θ. If a particle moves on a circular path with center at the origin, only the angle θ changes with time. If the particle moves on any path other than a radial straight line or circle centered at the origin, both of the coordinates r and θ change with time.

Angular Motion :

Change in direction of position vector $\vec{r}$ is known as angular motion. It happens when a particle moves on a curvilinear path or straight-line path not containing the origin as shown in the following figures.



Angular Motion

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Angular position : The coordinate angle θ at an instant is known as angular position of the particle.

Angular Displacement : A change in angular position θ in a time interval is known as angular displacement.

Angular Velocity : The instantaneous rate of change in angular position θ with respect to time is known as angular velocity.

We denote angular velocity by symbol ω. $\omega = \frac{d\theta}{dt}$

Angular Acceleration : The instantaneous rate of change in angular velocity ω with respect to time is known as angular acceleration.

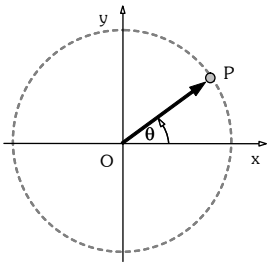
We denote angular acceleration by symbol α. $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$

If a particle moves in a plane, the position vector turns either in clockwise or anticlockwise sense. Assuming one of these direction positive and other negative, problems of angular motion involving angular position θ, angular velocity ω and angular acceleration α can be solved in fashion similar to problems of rectilinear motion involving position x, velocity v, and acceleration a.

Kinematics of Circular Motion

A body in circular motion moves on a circular path. At present, we discuss only translation motion of a body on circular path and disregard any rotation; therefore, we represent the body as a particle.

In the given figure, a particle P is shown moving on a circular path of radius r. Here only for simplicity center of the circular path is assumed at the origin of a coordinate system. In general, it is not necessary to assume center at the origin. Position vector of the particle is shown by a directed radius $\overrightarrow{OP} = \vec{r}$. Therefore, it is also known as radius vector. The radius vector is always normal to the path and has constant magnitude and as the particle moves, it is the angular position θ, which varies with time.



Angular Variables in Circular Motion

Angular position θ, angular velocity ω and angular acceleration α known as angular variables vary in different manner depending on how the particle moves.

Motion with uniform angular velocity

If a particle moves with constant angular velocity, its angular acceleration is zero and position vector turns at constant rate. It is analogous to uniform velocity motion on straight line. The angular position θ at any instant of time t is expressed by the following equation.

$$\theta = \theta_0 + \omega t$$

Motion with uniform angular acceleration

If a particle moves with constant angular acceleration, its angular velocity changes with time at a constant rate. The angular position θ, angular velocity ω and the angular acceleration α bear relations described by the following equations, which have forms similar to corresponding equations that describe uniform acceleration motion.

$$\omega = \omega_o + \alpha t$$

$$\theta = \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2$$

$$\theta = \theta_o + \tfrac{1}{2}(\omega_o + \omega)t$$

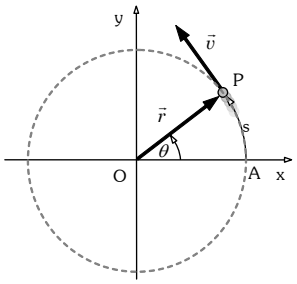
$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

Motion with variable angular acceleration

Variable angular acceleration of a particle is generally specified as function of time, angular position or angular velocity. Problems involving variable angular acceleration can also be solved in a way analogous to corresponding rectilinear motion problems in which acceleration is specified as function of time, position or velocity.

Linear Velocity and Acceleration in circular Motion



The instantaneous velocity $\vec{v}$ and the instantaneous acceleration $\vec{a}$ are also known as linear velocity and linear acceleration.

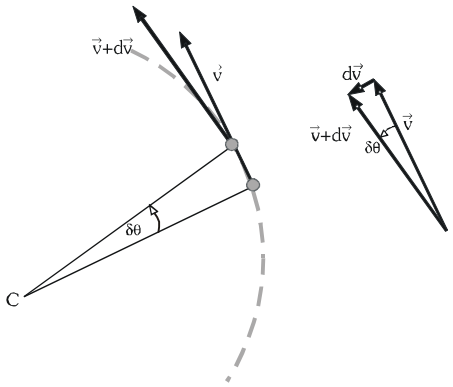
In the figure is shown a particle moving on a circular path. As it moves it covers a distance s (arc length).

$$s = \theta r$$

Linear velocity $\vec{v}$ is always along the path. Its magnitude known as linear speed is obtained by differentiating s with respect to time t.

$$v = \frac{d\theta}{dt} r = \omega r$$

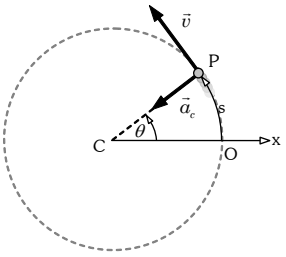
If speed of the particle is uniform, the circular motion is known as uniform circular motion. In this kind of motion as the particle precedes further, only direction of velocity changes. Therefore, instantaneous acceleration or linear acceleration accounts for only change in direction of motion.



Consider a particle in uniform circular motion. It is shown at two infinitely close instants t and t + dt, where its velocity vectors are $\vec{v}$ and $\vec{v} + d\vec{v}$. These two velocity vectors are equal in magnitude and shown in adjacent figure. From this figure, it is obvious that the change $d\vec{v}$ in velocity vector is perpendicular to velocity vector $\vec{v}$ i.e towards the center. It can be approximated as arc of radius equal to magnitude of $\vec{v}$. Therefore we can write $|d\vec{v}| = d\theta \cdot v$. Hence acceleration of this particle is towards the center. It is known as normal component of acceleration or more commonly centripetal acceleration.

Dividing $|d\vec{v}|$ by time interval dt we get magnitude of centripetal acceleration a_c.

$$a_c = \frac{d\theta}{dt} v = \omega v = \omega^2 r = \frac{v^2}{r}$$



Acceleration and velocity of a particle in uniform circular motion are shown in the following figure.

To keep the particle in uniform circular motion net force acting on it must be towards the center, therefore it is known as centripetal force.

If particle moves with varying speed, the net force on it must have a component along the direction of velocity vector in addition to the centripetal force. This component force is along the tangent to the path and produces a component of acceleration in the tangential direction. This component known as tangential component of acceleration a_T, accounts for change in speed.

$$a_T = \frac{dv}{dt} = \frac{d\omega}{dt} r = \alpha r$$

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example

Angular position θ of a particle moving on a curvilinear path varies according to the equation $\theta = t^3 - 3t^2 + 4t - 2$, where θ is in radians and time t is in seconds. What is its average angular acceleration in the time interval t = 2s to t = 4s?

Solution

Like average linear acceleration, the average angular acceleration α_{av} equals to ratio of change in angular velocity Δω to the concerned time interval Δt.

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_{final} - \omega_{initial}}{t_{final} - t_{initial}} \quad ...(i)$$

The angular velocity ω being rate of change in angular position can be obtained by equation

$$\omega = \frac{d\theta}{dt}$$

Substituting the given expression of the angular position θ, we have

$$\omega = 3t^2 - 6t + 4 \quad ...(ii)$$

From the above eq. (ii), angular velocities ω_2 and ω_4 at the given instants $t = 2$ s and 4s are

ω_4 = 4 rad/s and ω_4=28 rad/s.

Substituting the above values in eq. (1), we have α_{av}=12 rad/s^2

Example

A particle starts form rest and moves on a curve with constant angular acceleration of 3.0 rad/s^2. An observer starts his stopwatch at a certain instant and record that the particle covers an angular span of 120 rad at the end of 4th second. How long the particle had moved when the observer started his stopwatch?

Solution

Let the instants when the particle starts moving and the observer starts his stopwatch, are t_0=0 to t=t_1. Denoting angular positions and angular velocity at the instant $t = t_1$ by θ_1 and ω_1 and the angular position at the instant $t_2 = t_1 + 4$ s by θ_2, we can express the angular span covered during the interval from eq.

$$\theta - \theta_o = \omega_o t + \tfrac{1}{2}\alpha t^2 \rightarrow \theta_2 - \theta_1 = \omega_1 (t_2 - t_1) + \tfrac{1}{2}\alpha (t_2 - t_1)^2$$

Substituting values θ_1, θ_2, $t_1$and t_2, we have $\omega_1 = 24$ rad/s

From eq. $\omega=\omega_0+at$, we have $\omega_1=\omega_0+at_1$

Now substituting ω_0=0, ω_1= 24 and α=3 rad/s^2, we have t_1 = 8.0 s

Example

A particle moves on a circular path of radius 8 m. Distance traveled by the particle in time t is given by the equation $s = \frac{2}{3}t^3$. Find its speed when tangential and normal accelerations have equal magnitude.

Solution

The speed v, tangential acceleration a_τ and the normal acceleration a_n are expressed by the following equations.

$$v = \frac{ds}{dt}$$

Substituting the given expression for s, we have $v = 2t^2$

$$a_\tau = \frac{d^2s}{dt^2} \quad ...(i)$$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

Substituting the given expression for s, we have $a_\tau = 4t$

$$a_n = \frac{v^2}{r} \qquad \text{...(ii)}$$

Substituting v from eq. (i) and r =8m, we have $a_n = \frac{1}{2}t^4$ ...(iii)

The instant when the tangential and the normal accelerations have equal magnitude, can be obtained by equating their expressions given in eq. (ii) and (iii).

$a_n = a_T \rightarrow$ t = 2s

Substituting the above value of t in eq. (i), we obtain v = 8 m/s

Example

A particle is moving on a circular path of radius 1.5 m at a constant angular acceleration of 2 rad/s^2. At the instant t = 0, angular speed is $60/\pi$ rpm. What are its angular speed, angular displacement, linear velocity, tangential acceleration and normal acceleration at the instant t = 2 s.

Solution

Initial angular speed is given in rpm (revolution per minute). It is expressed in rad/s as

$$1 \text{ rpm} = \frac{2\pi \text{ rad}}{60 \text{ s}}$$

$$\omega_o = \left(\frac{60}{\pi}\right) \times \frac{2\pi \text{ rad}}{60 \text{ s}} = 2 \text{ rad/s}$$

At the instant t = 2 s, angular speed ω_2 and angular displacement θ_2 are calculated by using eq.

$$\omega_2 = \omega_o + \alpha t$$

Substituting values $\omega_o = 2$ rad/s, $\alpha = 2$ rad/s^2, $t = 2$ s, we have

$$\omega_2 = 6 \text{ rad/s}$$

$$\theta_2 = \theta_o + \tfrac{1}{2}(\omega_o + \omega_2)t$$

Substituting values $\theta_o = 0$ rad, $\omega_o = 2$ rad/s, $\omega_2 = 6$ rad/s and t =2 s, we have

θ_2 = 8 rad

Linear velocity at t = 2 s, can be calculated by using eq.

$v_2 = r\omega_2$

Substituting $r = 1.5$ m and $\omega_2 = 6$ rad/s, we have

v_2 = 9 m/s

Tangential acceleration a_τ and normal acceleration a_n can be calculated by using eq. and respectively.

$a_\tau = r\alpha$

Substituting $r = 1.5$ m and $\alpha = 2$ rad/s^2, we have

a_τ = 3 m/s^2

$a_n = \omega^2 r$

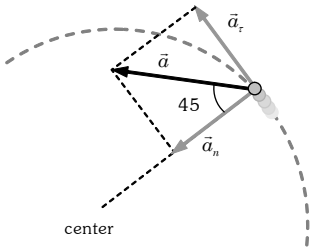
Substituting ω_2 = 6 rad/s and r = 1.5 m, we have

a_n = 54 m/s^2

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example

A particle is moving in a circular orbit with a constant tangential acceleration. After 2 s from the beginning of motion, angle between the total acceleration vector and the radius R becomes 45 . What is the angular acceleration of the particle?



Solution

In the adjoining figure are shown the total acceleration vector $\vec{a}$ and its components the tangential accelerations $\vec{a}_\tau$ and normal accelerations $\vec{a}_n$ are shown. These two components are always mutually perpendicular to each other and act along the tangent to the circle and radius respectively. Therefore, if the total acceleration vector makes an angle of 45 with the radius, both the tangential and the normal components must be equal in magnitude. Now from eq. and , we have

$$a_\tau = a_n \rightarrow \alpha R = \omega^2 R \Rightarrow \alpha = \omega^2 \quad \text{...(i)}$$

Since angular acceleration is uniform, form eq., we have $\omega = \omega_o + \alpha t$

Substituting ω_0=0 and t = 2 s, we have $\omega = 2\alpha$ ...(ii)

From eq. (i) and (ii), we have α = 0.25 rad/s^2

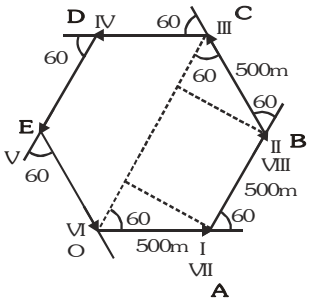
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

SOME WORKED OUT EXAMPLES

Example#1

On an open ground a motorist follows a track that turns to his left by an angle of 60 after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of displacement with the total path length covered by the motorist in each case.

Solution



At III turn

Displacement $= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC} = 500 \cos 60 + 500 + 500 \cos 60$

$= 500 \quad \frac{1}{2} + 500 + 500 \quad \frac{1}{2} = 1000$ m from O to C

Distance $= 500 + 500 + 500 = 1500$ m. So $\frac{\text{Displacement}}{\text{Distance}} = \frac{1000}{1500} = \frac{2}{3}$

At VI turn : $\because$ initial and final positions are same so displacement

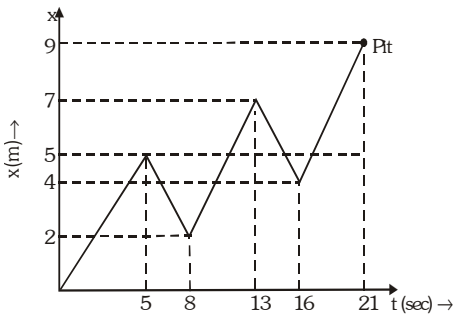
$= 0$ and distance $= 500 \quad 6 = 3000$ m $\therefore \frac{\text{Displacement}}{\text{Distance}} = \frac{0}{3000} = 0$

At VIII turn : Displacement $= 2(500)\cos\left(\frac{60°}{2}\right) = 1000 \quad \cos 30 = 1000 \times \frac{\sqrt{3}}{2} = 500\sqrt{3}$ m

Distance $= 500 \quad 8 = 4000$ m $\quad \therefore \frac{\text{Displacement}}{\text{Distance}} = \frac{500\sqrt{3}}{4000} = \frac{\sqrt{3}}{8}$

Example#2

A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1m long and requires 1s. Plot the x–t graph of his motion. Determine graphically or otherwise how long the drunkard takes to fall in a pit 9m away from the start.



node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Solution

from x–t graph time taken = 21 s

OR

(5m – 3m) + (5m – 3m) + 5m = 9m $\Rightarrow$ total steps = 21 $\Rightarrow$ time = 21 s

Example#3

A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km/h. On reaching the market he instantly turns and walks back with a speed of 7.5 km/h. What is the

(a) magnitude of average velocity and

(b) average speed of the man, over the interval of time (i) 0 to 30 min. (ii) 0 to 50 min (iii) 0 to 40 min.

Solution

Time taken by man to go from his home to market, $t_1 = \frac{\text{distance}}{\text{speed}} = \frac{2.5}{5} = \frac{1}{2}$ h

Time taken by man to go from market to his home, $t_2 = \frac{2.5}{7.5} = \frac{1}{3}$ h

$\therefore$ Total time taken = $t_1 + t_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ h = 50 min.

(i) 0 to 30 min

Average velocity $= \frac{\text{displacement}}{\text{time interval}} = \frac{2.5}{\frac{30}{60}} = 5$ km/h towards market

Average speed $= \frac{\text{distance}}{\text{time interval}} = \frac{2.5}{\frac{30}{60}} = 5$ km/h

(ii) 0 to 50 min

Total displacement = zero so average velocity = 0

So, average speed $= \frac{5}{50/60} = 6$ km/h

Total distance travelled = 2.5 + 2.5 = 5 km.

(iii) 0 to 40 min

Distance covered in 30 min (from home to market) = 2.5 km.

Distance covered in 10 min (from market to home) with speed 7.5 km/h = $7.5 \frac{10}{60} = 1.25$ km

So, displacement = 2.5 – 1.25 = 1.25 km (towards market)

Distance travelled = 2.5 + 1.25 = 3.75 km

Average velocity $= \frac{1.25}{\frac{40}{60}} = 1.875$ km/h. (towards market)

Average speed $= \frac{3.75}{\frac{40}{60}} = 5.625$ km/h.

Note : Moving body with uniform speed may have variable velocity. e.g. in uniform circular motion speed is constant but velocity is non–uniform.

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example#4

A driver takes 0.20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving a car at a speed of 54 km/h and the brakes cause a deceleration of 6.0m/s^2, find the distance travelled by the car after he sees the need to put the brakes on ?

Solution

Distance covered by the car during the application of brakes by driver

$$s_1 = ut = \left(54 \times \frac{5}{18}\right)(0.2) = 15 \text{ x } 0.2 = 3.0 \text{ m}$$

After applying the brakes; v = 0, u = 15 m/s, a = 6 m/s^2 s_2 = ?

Using $v^2 = u^2 - 2as \Rightarrow 0 = (15)^2 - 2 \quad 6 \quad s_2 \Rightarrow 12\ s_2 = 225 \Rightarrow s_2 = \frac{225}{12} = 18.75\,m$

Distance travelled by the car after driver sees the need for it $s = s_1 + s_2 = 3 + 18.75 = 21.75$ m.

Example#5

A passenger is standing d distance away from a bus. The bus begins to move with constant acceleration a. To catch the bus, the passenger runs at a constant speed u towards the bus. What must be the minimum speed of the passenger so that he may catch the bus?

Sol. Let the passenger catch the bus after time t.

The distance travelled by the bus, $s_1 = 0 + \frac{1}{2}\ at^2$ ....(i)

and the distance travelled by the passenger $s_2 = ut + 0$ ....(ii)

Now the passenger will catch the bus if $d + s_1 = s_2$ ....(iii)

$$\Rightarrow d + \frac{1}{2}\ at^2 = ut \Rightarrow \frac{1}{2}at^2 - ut + d = 0 \Rightarrow t = \frac{[u \pm \sqrt{u^2 - 2ad}]}{a}$$

So the passenger will catch the bus if t is real, i.e., $u^2 \geq 2$ ad $\Rightarrow u \geq \sqrt{2\ ad}$

So the minimum speed of passenger for catching the bus is $\sqrt{2ad}$.

Example#6

If a body travels half its total path in the last second of its fall from rest, find : (a) The time and (b) height of its fall. Explain the physically unacceptable solution of the quadratic time equation. (g = 9.8 m/s^2)

Solution

If the body falls a height h in time t, then

$$h = \frac{1}{2}gt^2 \text{ [u = 0 as the body starts from rest]} \quad \text{... (i)}$$

Now, as the distance covered in (t – 1) second is $h' = \frac{1}{2}g(t-1)^2$ ... (ii)

So from Equations (i) and (ii) distance travelled in the last second.

$$h - h' = \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 \text{ i.e., } h - h' = \frac{1}{2}g(2t-1)$$

But according to given problem as $(h - h') = \frac{h}{2}$

$$\Rightarrow \left(\frac{1}{2}\right)h = \left(\frac{1}{2}\right)g(2t-1) \text{ or } \left(\frac{1}{2}\right)gt^2 = g(2t-1) \text{ [as from equation (i) } h = \left(\frac{1}{2}\right)gt^2\text{]}$$

$\Rightarrow t^2 - 4t + 2 = 0$ or $t = [4 \pm \sqrt{(4^2 - 4 \times 2)}/2 \Rightarrow t = 2 \pm \sqrt{2} \Rightarrow t = 0.59$ s or 3.41 s

0.59 s is physically unacceptable as it gives the total time t taken by the body to reach ground lesser than one sec while according to the given problem time of motion must be greater than 1s.

so t=3.41s and h= ½ (9.8) (3.41)^2=57 m

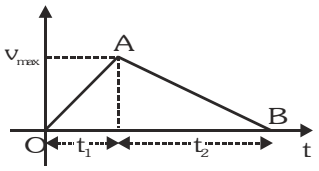
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example#7

A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t evaluate (a) the maximum velocity attained and (b) the total distance travelled.

Solution

(a) Let the car accelerates for time t_1 and decelerates for time t_2 then $t = t_1 + t_2$....(i)
and corresponding velocity–time graph will be as shown in. fig.



From the graph α = slope of line OA $= \frac{v_{max}}{t_1} \Rightarrow t_1 = \frac{v_{max}}{\alpha}$

and β = – slope of line AB = $\frac{v_{max}}{t_2} \Rightarrow t_2 = \frac{v_{max}}{\beta}$

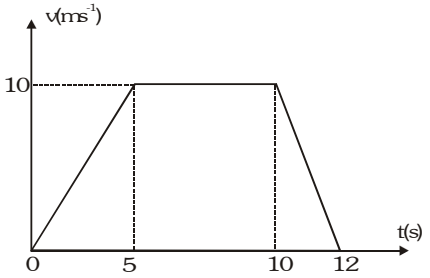
$$\Rightarrow \frac{v_{max}}{\alpha} + \frac{v_{max}}{\beta} = t \Rightarrow v_{max}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = t \Rightarrow v_{max} = \frac{\alpha\beta t}{\alpha+\beta}$$

(b) Total distance = area under v–t graph $= \frac{1}{2}\ t\ v_{max} = \frac{1}{2}\ t\ \frac{\alpha\beta t}{\alpha+\beta} = \frac{1}{2}\left(\frac{\alpha\beta t^2}{\alpha+\beta}\right)$

Note: This problem can also be solved by using equations of motion (v = u + at, etc.).

Example#8

Draw displacement time and acceleration – time graph for the given velocity–time graph



Solution

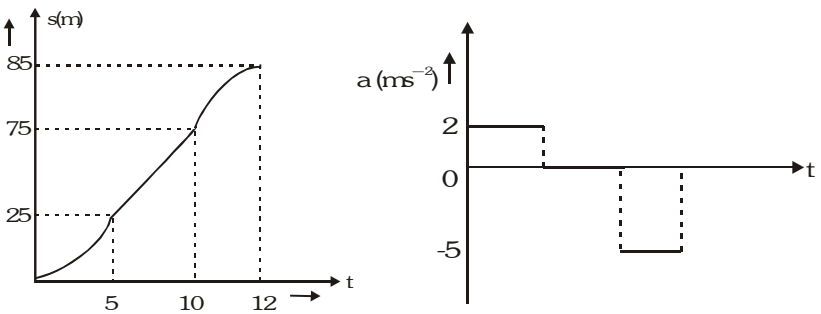
For $0 \le t \le 5$ $v \propto t \Rightarrow s \propto t^2$ and a_1=constant $\frac{10}{5} = 2\ ms^{-2}$

for whole interval s_1 = Area under the curve = $\frac{1}{2}\ 5\ 10 = 25$ m

For $5 \le t \le 10$, $v = 10 ms^{-1} \Rightarrow a = 0$
for whole interval s_2 = area under the curve = 5 10 = 50 m

For $10 \le t \le 12$ v linearly decreases with time $\Rightarrow a_3 = -\frac{10}{2} = -5\ ms^{-2}$

for whole interval s_3= Area under the curve = $\frac{1}{2}\ 2\ 10 = 10$ m



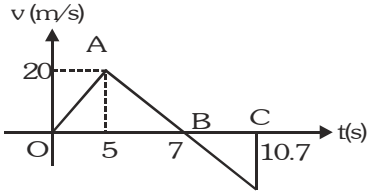
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

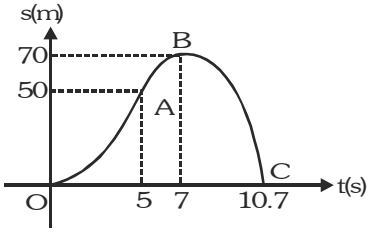
ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example#9

A rocket is fired upwards vertically with a net acceleration of 4 m/s^2 and initial velocity zero. After 5 seconds its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then it accelerates downwards with acceleration g and return back to ground. Plot velocity–time and displacement–time graphs for the complete journey. Take g = 10 m/s^2.

Solution





In the graphs, $v_A = at_{OA} = (4)(5) = 20$ m/s $\quad v_B = 0 = v_A - gt_{AB}$

$\therefore t_{AB} = \frac{v_A}{g} = \frac{20}{10} = 2s$ $\quad \therefore t_{OAB} = (5+2)s = 7s$

Now, s_{OAB} = area under v–t graph between 0 to 7 s = $\frac{1}{2}(7)(20) = 70$ m

Now, $s_{OAB} = s_{BC} = \frac{1}{2}gt_{BC}^2$ $\quad \therefore 70 = \frac{1}{2}(10)\,t_{BC}^2$

$\therefore t_{BC} = \sqrt{14} = 3.7$ s $\quad \therefore t_{OABC} = 7+3.7 = 10.7$s

Also s_{OA} = area under v–t graph between OA = $\frac{1}{2}(5)(20) = 50$ m

Example#10

At the height of 500m, a particle A is thrown up with v = 75 ms^{-1} and particle B is released from rest. Draw, acceleration –time, velocity–time, speed–time and displacement–time graph of each particle.

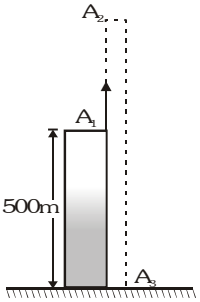
For particle A :

Time of flight

$-500 = +75\,t - \frac{1}{2}\,10t^2$

$\Rightarrow t^2 - 15t - 100 = 0$

$\Rightarrow t = 20$ s



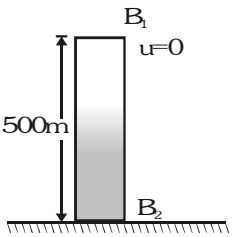
Time taken for A_1A_2

$v = 0 = 75 - 10t \Rightarrow t = 7.5$ s

Velocity at A_3, $v = 75 - 10 \quad 20 = -125$ ms^{-1}

Height $A_2A_1 = 75 \quad 7.5 - \frac{1}{2}(10)(7.5)^2 = 281.25$ m

For Particle B



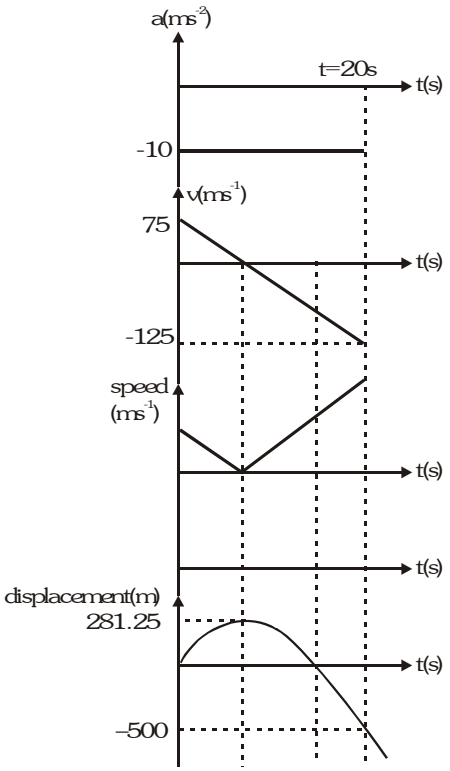
Time of flight

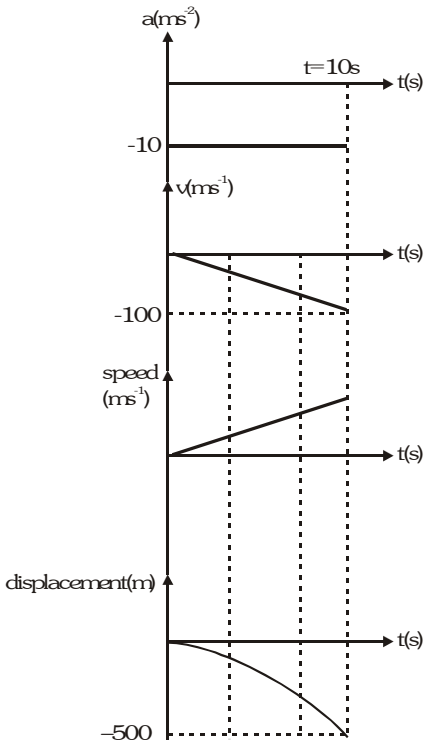
$500 = \frac{1}{2}(10)t^2 \Rightarrow t = 10$ s

Velocity at B_2

$v = 0 - (10)(10) = -100$ ms^{-1}

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65





Example#11

Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 km/h and ship B is streaming north at 20 km/h. What is their distance of closest approach and how long do they take to reach it?

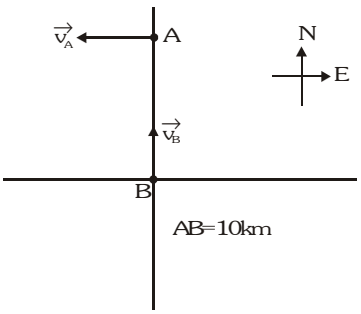
Solution

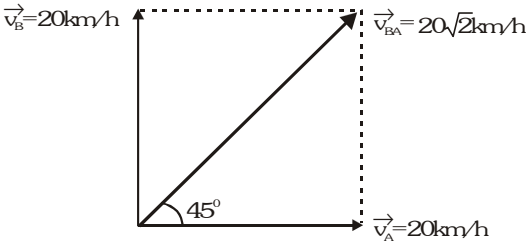
Ships A and B are moving with same speed 20 km/h in the directions shown in figure. It is a two dimensional, two body problem with zero acceleration. Let us find $\vec{v}_{BA}$

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

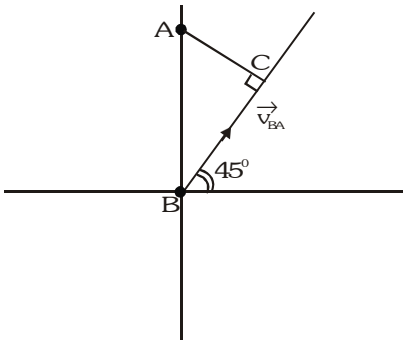
Here, $|\vec{v}_{BA}| = \sqrt{(20)^2 + (20)^2} = 20\sqrt{2}$ km/h

i.e., $\vec{v}_{BA}$ is $20\sqrt{2}$ km/h at an angle of 45 from east towards north. Thus, the given problem can be simplified as :





A is at rest and B is moving with $\vec{v}_{BA}$ in the direction shown in figure. Therefore, the minimum distance between the two is



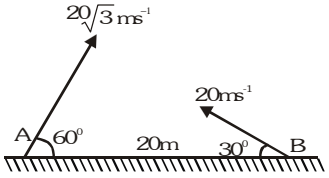
s_{min} = AC = AB sin 45^0 = 10 $\left(\frac{1}{\sqrt{2}}\right)$ km = 5 $\sqrt{2}$ km

and the desired time is $t = \frac{BC}{|\vec{v}_{BA}|} = \frac{5\sqrt{2}}{20\sqrt{2}} = \frac{1}{4}$ h = 15 min

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

Example#12

In the figure shown, the two projectile are fired simultaneously. Find the minimum distance between them during their flight.



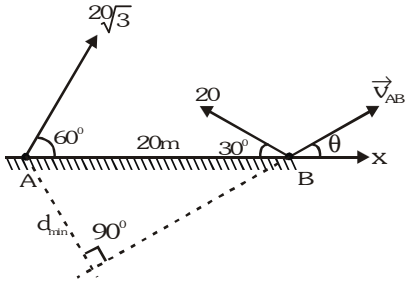
Solution

Taking origin at A and x axis along AB

Velocity of A w.r.t. B

$$= 20\sqrt{3}\left(\cos 60\tilde{i} + \sin 60\tilde{j}\right) - 20\left(\cos 150\tilde{i} + \sin 150\tilde{j}\right)$$

$$= 20\sqrt{3}\left(\frac{1}{2}\tilde{i} + \frac{\sqrt{3}}{2}\tilde{j}\right) - 20\left(-\frac{\sqrt{3}}{2}\tilde{i} + \frac{1}{2}\tilde{j}\right) = 20\sqrt{3}\tilde{i} + 20\tilde{j}$$



$$\tan\theta = \frac{20}{20\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^0 \text{ so } \frac{d_{min}}{20} = \sin\theta = \sin 30° = \frac{1}{2} \Rightarrow d_{min} = 10\text{m}$$

Example #13

A particle is dropped from the top of a high building of height 360 m. The distance travelled by the particle in ninth second is (g = 10 m/s^2)

(A) 85 m (B) 60 m (C) 40 m (D) can't be determined

Solution **Ans. (C)**

Total time taken by particle to reach the ground T = $\sqrt{\frac{2H}{g}} = \sqrt{\frac{2\times 360}{10}} = 6\sqrt{2} = 8.484$ s

Distance travelled in 8 seconds = $\frac{1}{2}$ gt^2 = $\frac{1}{2}$ (10) (8)^2 = 320 m

Therefore distance travelled in ninth second = 360 – 320 = 40 m

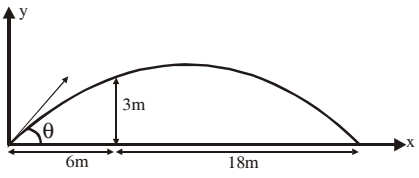
Example #14

A ball is thrown from the ground to clear a wall 3 m high at a distance of 6 m and falls 18 m away from the wall, the angle of projection of ball is

(A) $\tan^{-1}\left(\frac{3}{2}\right)$ (B) $\tan^{-1}\left(\frac{2}{3}\right)$ (C) $\tan^{-1}\left(\frac{1}{2}\right)$ (D) $\tan^{-1}\left(\frac{3}{4}\right)$

Solution **Ans. (B)**

From equation of trajectory y = xtanθ $\left[1-\frac{x}{R}\right] \Rightarrow 3 = 6\tan\theta\left[1-\frac{1}{4}\right] \Rightarrow \tan\theta = \frac{2}{3}$



node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example #15

A particle moves in XY plane such that its position, velocity and acceleration are given by

$$\vec{r} = x\tilde{i} + y\tilde{j}\,; \quad \vec{v} = v_x\tilde{i} + v_y\tilde{j}\,; \quad \vec{a} = a_x\tilde{i} + a_y\tilde{j}$$

which of the following condition is correct if the particle is speeding down?

(A) $xv_x + yv_y < 0$ (B) $xv_x + yv_y > 0$ (C) $a_xv_x + a_yv_y < 0$ (D) $a_xv_x + a_yv_y > 0$

Solution **Ans. (C)**

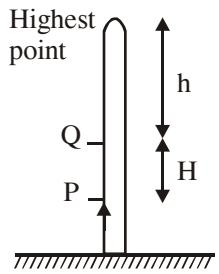
For speeding down $\vec{a}.\vec{v} < 0 \Rightarrow a_xv_x + a_yv_y < 0$

Example #16

A particle is thrown vertically upwards from the surface of the earth. Let T_P be the time taken by the particle to travel from a point P above the earth to its highest point and back to the point P. Similarly, let T_Q be the time taken by the particle to travel from another point Q above the earth to its highest point and back to the same point Q. If the distance between the points P and Q is H, the expression for acceleration due to gravity in terms of T_P, T_Q and H, is :-

(A) $\frac{6H}{T_P^2 + T_Q^2}$ (B) $\frac{8H}{T_P^2 - T_Q^2}$ (C) $\frac{2H}{T_P^2 + T_Q^2}$ (D) $\frac{H}{T_P^2 - T_Q^2}$

Solution **Ans. (B)**



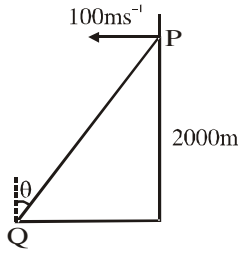
Time taken from point P to point P $T_P = 2\sqrt{\frac{2(h+H)}{g}}$

Time taken from point Q to point Q $T_Q = 2\sqrt{\frac{2h}{g}}$

$\Rightarrow T_P^2 = \frac{8(h+H)}{g}$ and $T_Q^2 = \frac{8h}{g} \Rightarrow T_P^2 = T_Q^2 + \frac{8H}{g} \Rightarrow g = \frac{8H}{T_P^2 - T_Q^2}$

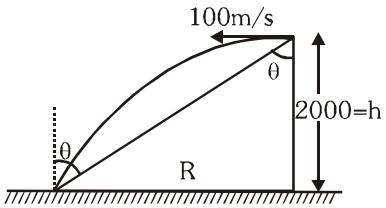
Example #17

An aeroplane is travelling horizontally at a height of 2000 m from the ground. The aeroplane, when at a point P, drops a bomb to hit a stationary target Q on the ground. In order that the bomb hits the target, what angle θ must the line PQ make with the vertical ? [g = 10ms^{-2}]



(A) 15 (B) 30 (C) 90 (D) 45

Solution : **Ans. (D)**



Let t be the time taken by bomb to hit the target.

$h = 2000 = \frac{1}{2}gt^2 \Rightarrow t = 20$ sec

$R = ut = (100)(20) = 2000$ m

$\because \tan\theta = \frac{R}{h} = \frac{2000}{2000} = 1 \Rightarrow \theta = 45°$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example #18

Some informations are given for a body moving in a straight line. The body starts its motion at t=0.

Information I : The velocity of a body at the end of 4s is 16 m/s

Information II : The velocity of a body at the end of 12s is 48 m/s

Information III : The velocity of a body at the end of 22s is 88 m/s

The body is certainly moving with

(A) Uniform velocity (B) Uniform speed

(C) Uniform acceleration (D) Data insufficient for generalization

Solution **Ans. (D)**

Here average acceleration = $\frac{16-0}{4-0}=\frac{48-16}{12-4}=\frac{88-48}{22-12}=4$

But we can't say certainly that body have uniform acceleration.

Example #19

A large number of particles are moving each with speed v having directions of motion randomly distributed. What is the average relative velocity between any two particles averaged over all the pairs?

(A) v (B) $(\pi/4)v$ (C) $(4/\pi)v$ (D) Zero

Solution : **Ans. (C)**

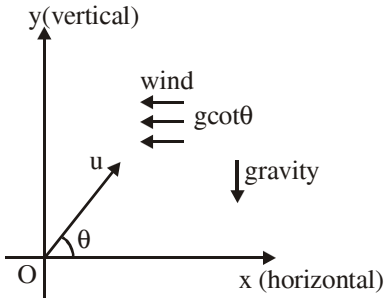
Relative velocity , $v_r=|\vec{v}_1-\vec{v}_2|$ where $v_1 = v_2 = v$

If angle between them be θ, then $v_r=\sqrt{v^2+v^2-2v^2\cos\theta}=\sqrt{2v^2(1-\cos\theta)}=2v\sin\left(\frac{\theta}{2}\right)$

Hence, average relative velocity $\vec{v}_r=\frac{\int_0^{2\pi}2v\sin\frac{\theta}{2}d\theta}{\int_0^{2\pi}d\theta}=\frac{4v}{\pi}$

Example #20

A ball is projected as shown in figure. The ball will return to point :



(A) O (B) left to point O (C) right to point O (D) none of these

Solution : **Ans. (A)**

Here $\frac{a_x}{a_y}=\frac{g\cot\theta}{g}=\frac{1}{\tan\theta}=\frac{u_x}{u_y}$ $\Rightarrow$ Initial velocity & acceleration are opposite to each other.

$\Rightarrow$ Ball will return to point O.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example #21

Throughout a time interval, while the speed of a particle increases as it moves along the x-axis, its velocity and acceleration might be

(A) positive and positive respectively. (B) positive and negative respectively.

(C) negative and negative respectively. (D) negative and positive respectively.

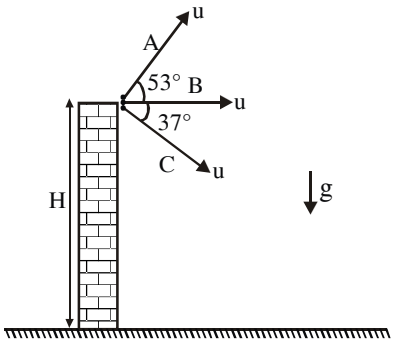
Solution : **Ans. (A, C)**

Speed increases if both velocity & acceleration have same signs.

Example #22

Three point particles A, B and C are projected from same point with same speed at t=0 as shown in figure.

For this situation select correct statement(s).



(A) All of them reach the ground at same time.

(B) All of them reach the ground at different time.

(C) All of them reach the ground with same speed.

(D) All of them have same horizontal displacement when they reach the ground.

Solution : **Ans. (B, C)**

Vertical component of initial velocities are different $\Rightarrow$reach the ground at different time.

Example #23

A projectile is thrown with speed u into air from a point on the horizontal ground at an angle θ with horizontal. If the air exerts a constant horizontal resistive force on the projectile then select correct alternative(s).

(A) At the farthest point, the velocity is horiozntal. (B) The time for ascent equals the time for descent.

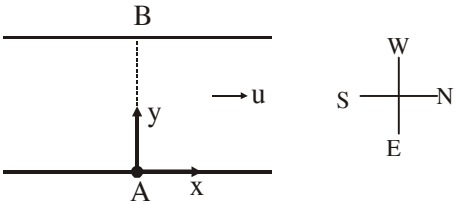
(C) The path of the projectile may be parabolic. (D) The path of the projectile may be a straight line.

Solution : **Ans. (C,D)**

Here total acceleration a = $\sqrt{g^2 + a_x^2}$ = constant, so path may be parabolic or straight line.

Example #24 to 26

A river of width 'd' with straight parallel banks flows due North with speed u. A boat, whose speed is v relative to water, starts from A and crosses the river. If the boat is steered due West and u varies with y as u = $\frac{y(d-y)v}{d^2}$ then answer the following questions.



24. The time taken by boat to cross the river is

(A) $\frac{d}{\sqrt{2}v}$ (B) $\frac{d}{v}$ (C) $\frac{d}{2v}$ (D) $\frac{2d}{v}$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

25. Absolute velocity of boat when it reaches the opposite bank is

(A) $\frac{4}{3}v$, towards East (B) v, towards West (C) $\frac{4}{3}v$, towards West (D) v, towards East

26. Equation of trajectory of the boat is

(A) $y = \frac{x^2}{2d}$ (B) $x = \frac{y^2}{2d}$ (C) $y = \frac{x^2}{2d} - \frac{x^3}{3d^2}$ (D) $x = \frac{y^2}{2d} - \frac{y^3}{3d^2}$

Solution :

24. Ans. (B)

Time taken = $\frac{d}{v_y} = \frac{d}{v}$

25. Ans. (B)

At y=d, u =0 so absolute velocity of boat = v towards West.

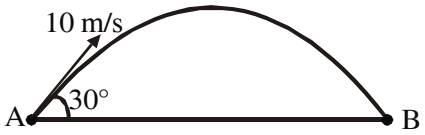
26. Ans. (D)

For boat (w.r.t. ground) $v_y = v$, $v_x = u = \frac{y(d-y)}{d^2}v \Rightarrow \frac{dy}{dt} = v$ and $\frac{dx}{dt} = \frac{y(d-y)}{d^2}v$

$$\Rightarrow \frac{dx}{dy} = \frac{y(d-y)}{d^2} \Rightarrow \int_0^x dx = \int_0^y \frac{(yd-y^2)}{d^2}dy \Rightarrow x = \frac{y^2}{2d} - \frac{y^3}{3d^2}$$

Example #27

As shown in the figure there is a particle of mass $\sqrt{3}$ kg, is projected with speed 10 m/s at an angle 30 with horizontal (take g = 10 m/s^2) then match the following

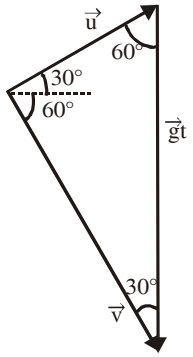


Column I	Column II
(A) Average velocity (in m/s) during half of the time of flight, is	(P) $\frac{1}{2}$
(B) The time (in sec) after which the angle between velocity vector and initial velocity vector becomes $\pi/2$, is	(Q) $\frac{5}{2}\sqrt{13}$
(C) Horizontal range (in m), is	(R) $5\sqrt{3}$
(D) Change in linear momentum (in N-s) when particle is at highest point, is	(S) At an angle of $\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$ from horizontal
	(T) 2

Solution : **Ans. (A) → (Q,S); (B) → (T); (C) → (R); (D) → (R)**

For (A) : $v_{av} = \sqrt{(v_{avx})^2 + (v_{avy})^2} = \sqrt{(10\cos 30°)^2 + \left(\frac{10\sin 30° + 0}{2}\right)^2} = \sqrt{75 + \frac{25}{4}} = \frac{5}{2}\sqrt{13}$ m/s

Angle with horizontal $\theta = \tan^{-1}\left(\frac{v_{avy}}{v_{avx}}\right) = \tan^{-1}\left(\frac{5/2}{5\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$



For (B) : By using $\vec{v} = \vec{u} + \vec{a}t$ We have $\frac{u}{gt} = \sin 30° \Rightarrow t = \frac{10}{(10)(1/2)} = 2$

For (C) : Horizontal range(R) = $\frac{u^2 \sin 2\theta}{g} = \frac{100 \times \sqrt{3}/2}{10} = 5\sqrt{3}$ m

For (D) : Change in linear momentum = $mu_y = \sqrt{3}$ 10sin30 = $5\sqrt{3}$ N-s

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Example #28

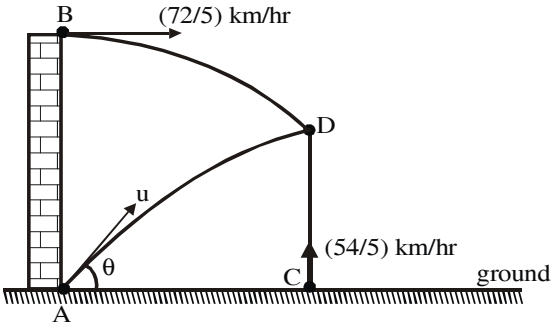
A particle is moving along a straight line along x-axis with an initial velocity of 2 m/s towards positive x-axis. A constant acceleration of 0.5 m/s^2 towards negative x-axis starts acting on particle at t=0. Find velocity (in m/s) of particle at t = 2s.

Solution : **Ans. 1**

$v = u + at \Rightarrow v = 2 + (-0.5)(2) = 1$ m/s

Example #29

In the given figure points A and C are on the horizontal ground & A and B are in same vertical plane. Simultaneously bullets are fired from A, B and C and they collide at D. The bullet at B is fired horizontally with speed of $\frac{72}{5}$ km/hr and the bullet at C is projected vertically upward at velocity of $\frac{54}{5}$ km/hr. Find velocity of the bullet projected from A in m/s.

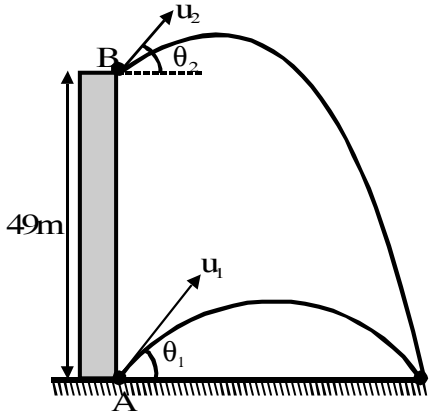


Solution : **Ans. 5**

For collision $u = \sqrt{u_B^2 + u_C^2} = \sqrt{\left(\frac{72}{5}\times\frac{5}{18}\right)^2 + \left(\frac{54}{5}\times\frac{5}{18}\right)^2} = \sqrt{4^2+3^2} = 5$ m/s

Example #30

Two stones A and B are projected simultaneously as shown in figure. It has been observed that both the stones reach the ground at the same place after 7 sec of their projection. Determine difference in their vertical components of initial velocities in m/s. (g = 9.8 m/s^2)

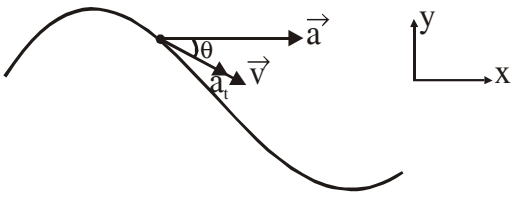


Solution : **Ans. 7**

In time of flight i.e. 7 s, the vertical displacement of A is zero and that of B is 49 m so for relative motion of B w.r.t. A $(u_2\sin\theta_2 - u_1\sin\theta_1)\ 7 = 49 \Rightarrow u_2\sin\theta_2 - u_1\sin\theta_1 = 7$ m/s

Example #31

A particle moves with a tangential acceleration $a_t = \vec{a}.\tilde{v}$ where $\vec{a} = (5\tilde{i})$ m/s^2 . If the speed of the particle is zero at x=0, then find v (in m/s) at x = 4.9 m.



node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Solution : **Ans. 7**

As $vdv = \vec{a}.\vec{dr} = adx = 5dx \Rightarrow \int_0^v vdv = 5\int_0^{4.9} dx \Rightarrow \frac{v^2}{2} = 5(4.9) \Rightarrow v^2 = 49 \Rightarrow v = 7$ m/s

Example #32

A body is thrown up with a speed 49 m/s. It travels 5 m in the last second of its upward journey. If the same body is thrown up with a velocity 98 m/s, how much distance (in m) will it travel in the last second. ($g = 10$ m/s^2)

Solution : **Ans. 5**

In last second of upward journey, all bodies travel same distance (= g/2 = 5m)

Example #33

A particle is moving in a circle of radius R in such a way that at any instant the normal and the tangential component of its acceleration are equal. If its speed at t=0 is v_0 then the time it takes to complete the first revolution is $\frac{R}{\alpha v_0}(1 - e^{-\beta\pi})$. Find the value of $(\alpha + \beta)$.

Solution : **Ans. 3**

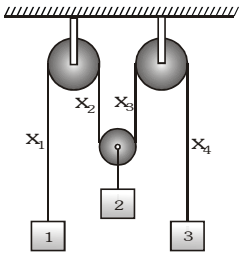
$$\frac{dv}{dt} = \frac{v^2}{R} \Rightarrow \int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \Rightarrow \left(-\frac{1}{v}\right)_{v_0}^{v} = \frac{t}{R} \Rightarrow v = \frac{v_0}{1 - \frac{v_0}{R}t} \Rightarrow \frac{ds}{dt} = \frac{v_0}{1 - \frac{v_0}{R}t} \Rightarrow \int_0^{2\pi R} ds = \int_0^t \frac{v_0}{\left(1 - \frac{v_0}{R}t\right)} dt$$

$$\Rightarrow 2\pi R = -R\left[\ell n\left(1 - \frac{v_0}{R}t\right)\right]_0^t \Rightarrow 2\pi = -\ell n\left(1 - \frac{v_0}{R}t\right) \Rightarrow 1 - \frac{v_0}{R}t = e^{-2\pi} \Rightarrow t = \frac{R}{v_0}(1 - e^{-2\pi})$$

$$\Rightarrow \alpha = 1, \beta = 2 \Rightarrow (\alpha + \beta) = (1 + 2) = 3$$

Example #34

Find the relation between acceleration of blocks a_1, a_2 and a_3.



Solution

$x_1 + x_2 + x_3 + x_4 = \ell$

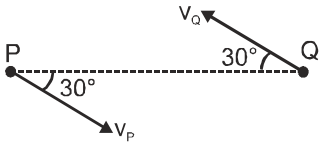
$\Rightarrow \ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 + \ddot{x}_4 = 0$

$\Rightarrow a_1 + a_2 + a_2 + a_3 = 0$

$\Rightarrow a_1 + 2a_2 + a_3 = 0$

Example#35

Two moving particles P and Q are 10 m apart at any instant. Velocity of P is 8 m/s at 30°, from line joining the P and Q and velocity of Q is 6m/s at 30°.Calculate the angular velocity of P w.r.t. Q



Solution

$$\omega_{PQ} = \frac{8\sin 30^\circ - (-6\sin 30^\circ)}{10} = 0.7 \text{ rad/s.}$$

node6\E_NOD65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Kinematics\English\theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 4

NEWTON'S LAWS OF MOTION & FRICTION - Theory

Index

NEWTON'S LAWS OF MOTION & FRICTION

Serial No.	CONTENTS	Page No.
	NLM & FRICTION	
1.	Motion	1
2.	Momentum	1
3.	Force	2
4.	Newton's Ist law of motion	2
5.	Newton's IInd law of motion	2
6.	Newton's IIIrd law of motion	3
7.	Concept of free body diagram	3
8.	Various Contact Force	4
9.	Translational Equilibrium	7
10.	Dynamic of particles	12
11.	System of Interconnected bodies	18
12.	Friction	20
13.	Inertial and Non-inertial reference frame	32
14.	Dynamics of Circular Motion	37
15.	**Exercise-I** : Check your Grasp (Single Choic Objective questions)	51
16.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	58
17.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	63
18.	**Exercise-4A** Conceptual Subjective Exercise	70
19.	**Exercise-4B** Brain Storming Exercise	73
20.	**Exercise-5-A** Previous years questions	76
21.	**Exercise-5-B** Previous years questions	79

ALLEN Materials Provided By - Material Point Available on Learnaf.com

JEE (Advanced) Syllabus :

Newton's laws of motion; Inertial and uniformly accelerated frames of reference; Static and dynamic friction.

JEE (Main) Syllabus :

Force and Inertia, Newton's First Law of motion; Momentum, Newton's Second Law of motion; Newton's Third Law of motion, Equilibrium of concurrent forces. Static and Kinetic friction, laws of friction. Dynamics of uniform circular motion: Centripetal force and its applications

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Newton's Laws of Motion

Motion

Motion of a body is its movement and is identified by change in either its location or orientation or both, relative to other objects.

Location

Location of a rigid body tells us where it is placed and can be measured by position coordinates of any particle of the body or its mass center. It is also known as position.

Orientation

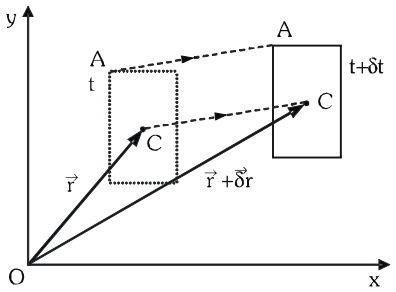
Orientation of a body tells us how it is placed with respect to the coordinate axes. Angles made with the coordinate axes by any linear dimension of the body or a straight line drawn on it, provide suitable measure of orientation.

Translation and Rotation Motion

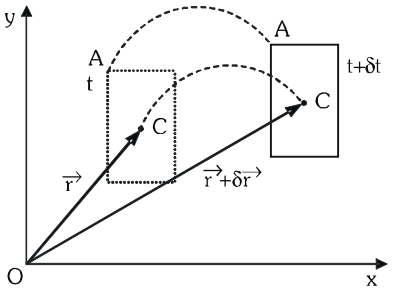
If a body changes its location without change in orientation, it is in pure translation motion and if it changes orientation without change in location, it is in pure rotation motion.

Translation Motion

Let us consider the motion of a plate, which involves only change in position without change in orientation. It is in pure translation motion. The plate is shown at two different instants t and t+δt. The coordinate axes shown are in the plane of the plate and represent the reference frame. A careful observation makes the following points obvious.



Rectilinear Translation



Curvilinear Translation

- None of the linear dimension or any line drawn on the body changes its angles with the coordinate. Therefore, there is no rotation motion.
- All the particles of the body including its mass center move on identical parallel trajectories. Here trajectories of corner A and center C are shown by dashed lines.
- All the particles and mass center of the body cover identical segments of their trajectories in a given time interval. Therefore, at any instant of time all of them have identical velocities and accelerations.

Pure translation motion of a body can be represented by motion of any of its particle. This is why, we usually consider a body in pure translation motion as a particle.

Momentum: Amount of Motion

Amount of motion in a body depends on its velocity and mass.

Linear momentum of a body is defined as product of its mass and velocity. It provides measure of amount of motion.

Linear momentum $\vec{p}$ of a body of mass m, moving with velocity by $\vec{v}$ is expressed by the following equation.

$$\vec{p} = m\vec{v}$$

SI unit of momentum is kg-m/s.

Dimensions of momentum are MLT^{-1}

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Force

The concept of force is used to explain mutual interaction between two material bodies as the action of one body on another in form of push or pull, which brings out or tries to bring out a change in the state of motion of the two bodies. A mutual interaction between two bodies, which creates force on one body, also creates force on the other body. Force on body under study is known as *action* and the force applied by this body on the other is known as *reaction*.

Contact and Field Forces:

When a body applies force on other by direct contact, the force is known as contact force. When two bodies apply force on each other without any contact between them, the force is known as field force.

When you lift something, you first hold it to establish contact between your hand and that thing, and then you apply the necessary force to lift. When you pull bucket of water out of a well, the necessary force you apply on the rope by direct contact between your hand and the rope and the rope exerts the necessary force on the bucket through a direct contact. When you deform a spring, you have to hold the spring and establish contact between your hand and the spring and then you apply the necessary force. In this way, you can find countless examples of contact forces.

Things left free, fall on the ground, planets orbit around the sun, satellites orbit around a planet due to *gravitational force*, which can act without any contact between the concerned bodies. A plastic comb when rubbed with dry hair, becomes electrically charged. A charged plastic comb attracts small paper pieces without any physical contact due to *electrostatic force*. A bar magnet attracts iron nails without any physical contact between them. This force is known as *magnetic force*. The gravitation, electrostatic and magnetic forces are examples of field forces.

Basic Characteristics of a Force:

Force is a vector quantity therefore has *magnitude* as well as *direction*. To predict how a force affects motion of a body we must know its magnitude, direction and point on the body where the force is applied. This point is known a *point of application* of the force. The direction and the point of application of a force both decide *line of action* of the force. Magnitude and direction decide effect on translation motion and magnitude and line of action decides effects on rotation motion.

Newton's Laws of Motion

Newton has published three laws, which describe how forces affect motion of a body on which they act. These laws are fundamental in nature in the sense that the first law gives concept of force, inertia and the inertial frames; the second law defines force and the third law action and reaction as two aspects of mutual interaction between two bodies.

The First Law

Every material body has tendency to preserve its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by external forces impressed on it.

- **Inertia**

  The tendency of a material body to preserve its present state of uniform motion or of rest is known as inertia of the body. It was first conceived by Galileo.

  Inertia is a physical quantity and mass of a material body is measure of its inertia.

- **Inertial Frame of Reference**

  The first law requires a frame of reference in which only the forces acting on a body can be responsible for any acceleration produced in the body and not the acceleration of the frame of reference. These frames of reference are known as inertial frames.

The Second Law

The rate of change in momentum of a body is equal to, and occurs in the direction of the net applied force.

A body of mass m in translational motion with velocity $\vec{v}$, if acted upon with a net external force $\vec{F}$, the second law suggests:

Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

$$\vec{F} = \frac{d}{dt}(m\vec{v})$$

If mass of the body is constant, the above equation relates the acceleration $\vec{a}$ of the body with the net force $\vec{F}$ acting on it.

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\vec{a}$$

The first law provides concept of force and the second law provides the quantitative definition of force, therefore the second law is also valid only in inertial frames.

SI unit of force is newton. It is abbreviated as N. One newton equals to one kilogram-meter per second square.

$$1 \text{ N} = 1 \text{ kg-m/s}^2$$

Dimensions of force are MLT^{-2}

The Third Law

Force is always a two-body interaction. The first law describes qualitatively and the second law describes quantitatively what happens to a body if a force acts on it, but do not reveal anything about what happens to the other body participating in the interaction responsible for the force.

The third law accounts for this aspect of the force and states that every action on a body has equal and opposite reaction on the other body participating in the interaction.

Concept of Free Body Diagram (FBD)

A force on a body can only exists when there is another body to create it, therefore in every physical situation of concern there must be two or more bodies applying forces on each other. On the other hand the three laws of Newton, describe motion of a single body under action of several forces, therefore, to analyze a given problem, we have to consider each of the bodies separately one by one. This idea provides us with the concept of free body diagram.

A free body diagram is a pictorial representation in which the body under study is assumed free from rest of the system i.e. assumed separated from rest of the interacting bodies and is drawn in its actual shape and orientation and all the forces acting on the body are shown.

How to draw a Free Body Diagram (FBD)

- Separate the body under consideration from the rest of the system and draw it separately in actual shape and orientation.
- Show all the forces whether known or unknown acting on the body at their respective points of application.

  For the purpose count every contact where we separate the body under study from other bodies. At every such point, there may be a contact force. After showing, all the contact forces show all the field forces.

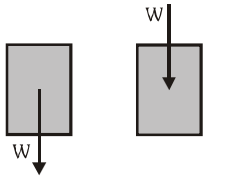
Various Field Forces

Field forces include the gravitational force (weight) electrostatic forces and magnetic forces, which can easily be identified. At present, we consider only gravitational pull from the earth i.e. weight of the body.

Weight: The net gravitational pull of the Earth

The gravitational pull from the earth acts on every particle of the body hence it is a distributed force. The net gravitational pull of the Earth on a body may be considered as weight of the body. It is assumed to act on the center of gravity of the body. For terrestrial bodies or celestial bodies of small size, this force can be assumed uniform throughout its volume. Under such circumstances, center of gravity and center of mass coincide and the weight is assumed to act on them. Furthermore, center of mass of uniform bodies lies at their geometrical center. At present, we discuss only uniform bodies and assume their weight to act on their geometrical center. In the figure weight of a uniform block is shown acting on its geometrical centre that coincides with the center of mass and the centre of gravity of the body.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65



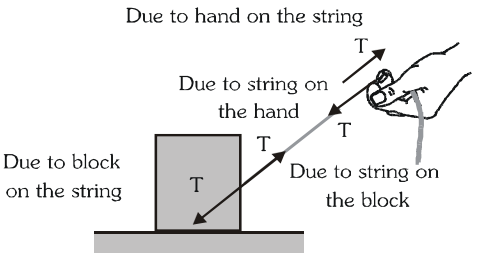
Weight can be shown in either of the above two ways

Various Contact Forces

At every point where a body under consideration is supposed to be separated from other bodies to draw its free-body diagram, there may be a contact force. Most common contact forces, which we usually encounter, are tension force of a string, normal reaction on a surface in contact, friction, spring force etc.

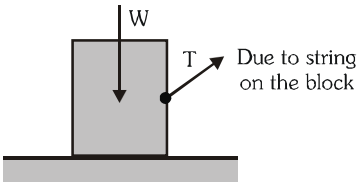
Tension Force of Strings

A string or similar flexible connecting links as a thread or a chain etc. we use to transmit a force. Due to flexibility, a string can be used only to pull a body connected to it by applying a force always along the string. According to the third law, the connected body must also apply an equal and opposite force on the string, which makes the string taut. Therefore, this force is known as tension force T of the string. In the given figure is shown a block pulled by a string, which is being pulled by a person.

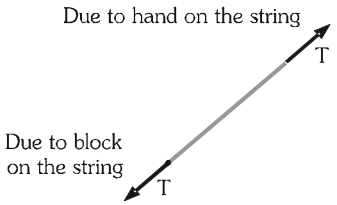


The tension force applied by string on the block and the force applied by the block on the string shown in the figure constitute a third law action-reaction pair. Similarly, tension force applied by the string on hand and force applied by the hand on string is another third law action-reaction pair.

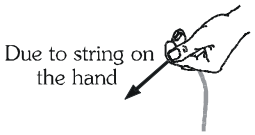
While studying motion of the block, the force applied by the string on it, weight of the block and a reaction from the floor has to be considered. In the figure only weight and tension of string are shown.



To study motion of the string, the force applied by the block on the string and the force applied by the hand on the string must be considered. These forces are shown in the FBD of string.

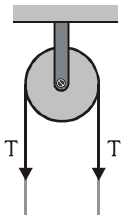


To study conditions of motion of the person, the force applied by the string on the hand has to be considered as shown in the figure.



String passing over a pulley

A pulley is a device consisting of a wheel, which can rotate freely on its axel. A single pulley changes direction of tension force. At present for simplicity, we discuss only ideal pulley, which is massless i.e. has negligible mass and rotates on its axel without any friction. An ideal pulley offers no resistance to its rotation, therefore tension force in the string on both sides of it are equal in magnitude. Such a pulley is known as ideal pulley.



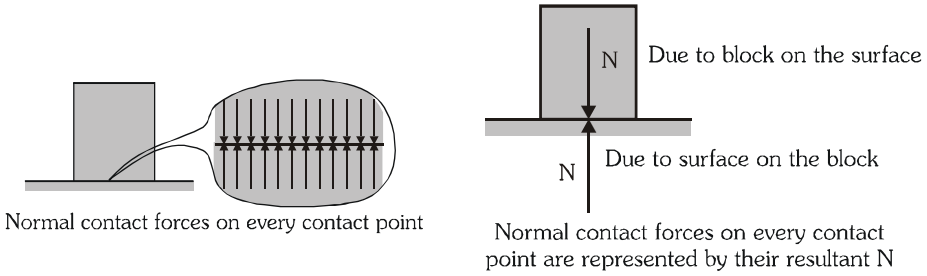
Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

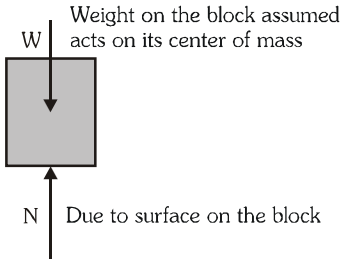
Normal Reaction

Two bodies in contact, when press each other, must apply equal and opposite forces on each other. These forces constitute a third law action-reaction pair. If surfaces of the bodies in contact are frictionless, this force acts along normal to the surface at the point of contact. Therefore, it is known as normal reaction.

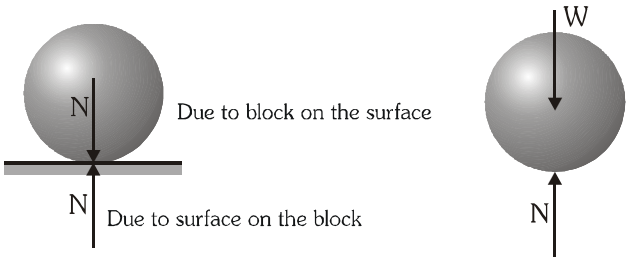
Consider a block of weight W placed on a frictionless floor. Because of its weight it presses the floor at every point in contact and the floor also applies equal and opposite reaction forces on every point of contact. We show all of them by a single resultant N obtained by their vector addition.



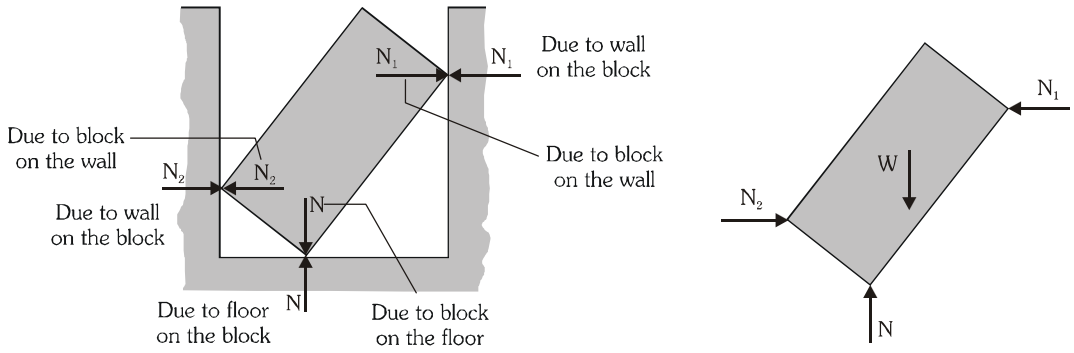
To apply Newton's laws of motion (NLM) on the block, its weight W and normal reaction N applied by the floor on the block must be considered as shown in the following figure. It is the FBD of the block.



Consider a spherical ball of weight W placed on a floor. The normal reaction from the floor on the ball and from the ball on the floor makes third law action-reaction pair. These forces are shown in the left figure.



To apply Newton's laws of motion (NLM) on the ball, its weight W and normal reaction N applied by the floor on the ball must be considered as shown in the above right figure. It is the FBD of the ball.
When two surfaces make contact, the normal reaction acts along the common normal and when a surface and a sharp corner make a contact the normal reaction acts along the normal to the surface. Consider a block placed in a rectangular trough as shown in the figure.

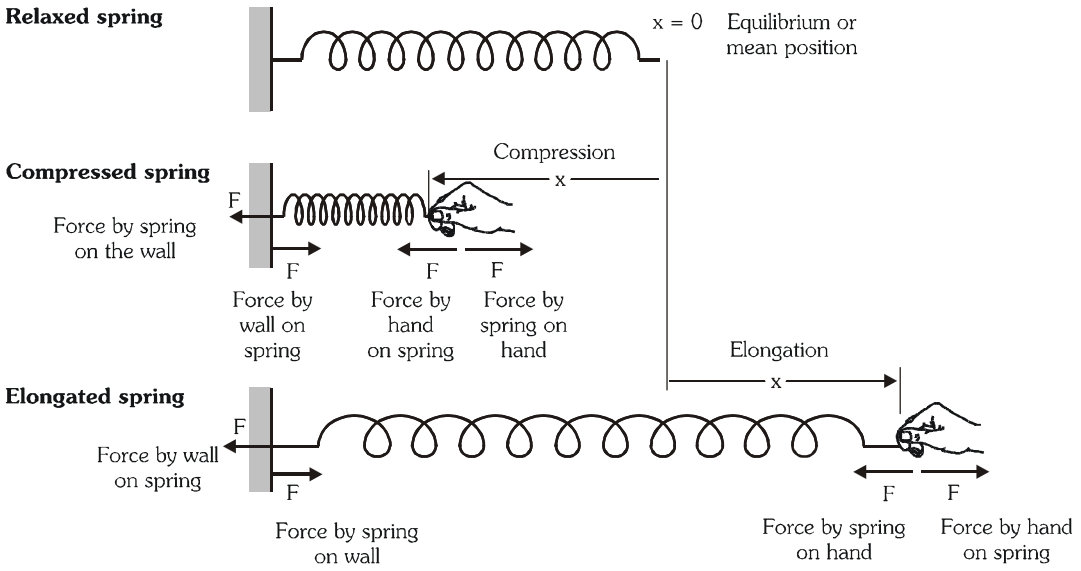


Node6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

To apply Newton's laws of motion (NLM) on the block, its free body diagram (FBD) is shown in the above right figure.

Spring Force

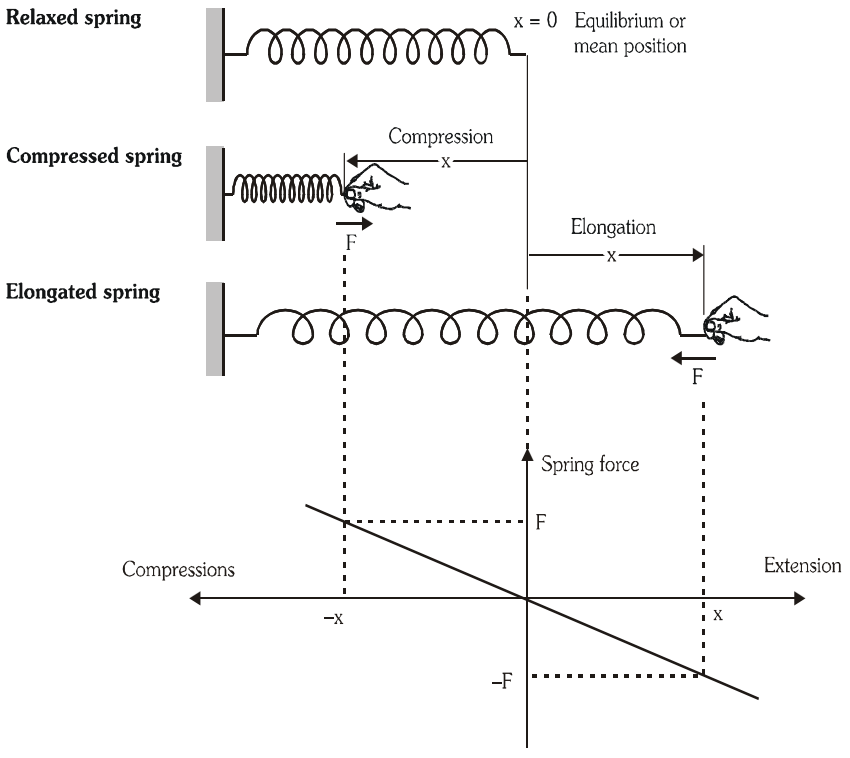
When no force acts on a spring, it is in relaxed condition i.e. neither compressed nor elongated. Consider a spring attached to a fixed support at one of its end and the other end is free. If we neglect gravity, it remains in relaxed state. When it is pushed by a force F, it is compressed and displacement x of its free end is called compression. When the spring is pulled by a force F, it is elongated and displacement x of its free end is called elongation. Various forces developed in these situations are shown in the following figure.



The force applied by the spring on the wall and the force applied by the wall on the spring make a third law action-reaction pair. Similarly, force by hand on the spring and the force by spring on the hand make another third law action-reaction pair.

Hooke's Law:

How spring force varies with deformation in length x of the spring is also shown in the following figure.



Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NLM\English\Theory.p65

The force F varies linearly with x and acts in a direction opposite to x. Therefore, it is expressed by the following equation

$$F = - kx$$

Here, the minus (–) sign represents the fact that force F is always opposite to x.

The constant of proportionality k is known as *force constant of the spring* or simply as *spring constant.* The slope modulus of the graph equals to the spring constant.

SI unit of spring constant is newton per meter or (N/m).

Dimensions of spring constant are MT^{-2}.

Translational Equilibrium

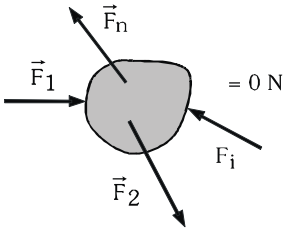
A body in state of rest or moving with constant velocity is said to be in translational equilibrium. Thus if a body is in translational equilibrium in a particular inertial frame of reference, it must have no linear acceleration. When it is at rest, it is in *static equilibrium*, whereas if it is moving at constant velocity it is in *dynamic equilibrium.*

Conditions for translational equilibrium

For a body to be in translational equilibrium, no net force must act on it i.e. vector sum of all the forces acting on it must be zero.

If several external forces $\vec{F}_1$, $\vec{F}_2$ ..... $\vec{F}_i$ ..... and $\vec{F}_n$ act simultaneously on a body and the body is in translational equilibrium, the resultant of these forces must be zero.

$$\sum \vec{F}_i = \vec{0}$$



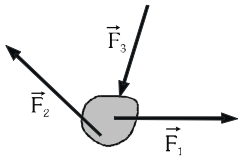
If the forces $\vec{F}_1$, $\vec{F}_2$ ..... $\vec{F}_i$ .....and $\vec{F}_n$ are expressed in Cartesian components, we have :

$$\sum F_{ix} = 0 \qquad \sum F_{iy} = 0 \qquad \sum F_{iz} = 0$$

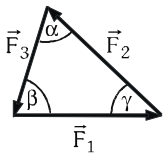
If a body is acted upon by a single external force, it cannot be in equilibrium.

If a body is in equilibrium under the action of only two external forces, the forces must be equal and opposite.

If a body is in equilibrium under action of three forces, their resultant must be zero; therefore, according to the triangle law of vector addition they must be coplanar and make a closed triangle.



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0} \quad \Rightarrow$$

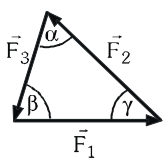


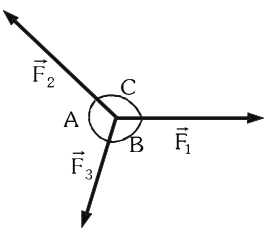
The situation can be analyzed by either graphical method or analytical method.

- Graphical method makes use of sine rule or Lami's theorem.

Sine rule : $\dfrac{F_1}{\sin\alpha} = \dfrac{F_2}{\sin\beta} = \dfrac{F_3}{\sin\gamma}$

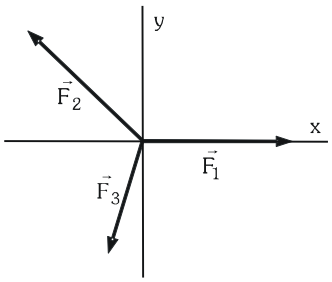
Lami's theorem : $\dfrac{F_1}{\sin A} = \dfrac{F_2}{\sin B} = \dfrac{F_3}{\sin C}$





Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

- Analytical method makes use of Cartesian components. Since the forces involved make a closed triangle, they lie in a plane and a two-dimensional Cartesian frame can be used to resolve the forces. As far as possible orientation of the x-y frame is selected in such a manner that angles made by forces with axes should have convenient values.



$$\sum F_x = 0 \Rightarrow F_{1x} + F_{2x} + F_{3x} = 0$$

$$\sum F_y = 0 \Rightarrow F_{1y} + F_{2y} + F_{3y} = 0$$

Problems involving more than three forces should be analyzed by analytical method. However, in some situations, there may be some parallel or anti-parallel forces and they should be combined first to minimize the number of forces. This may sometimes lead a problem involving more than three forces to a three-force system.

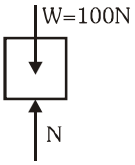
Example

Consider a box of mass 10 kg resting on a horizontal table and acceleration due to gravity to be 10 m/s^2.

(a) Draw the free body diagram of the box.
(b) Find value of the force exerted by the table on the box.
(c) Find value of the force exerted by the box on the table.
(d) Are force exerted by table on the box and weight of the box third law action-reaction pair?

Solution

(a) N : Force exerted by table on the box.



(b) The block is in equilibrium. $\sum \vec{F} = \vec{0} \Rightarrow W - N = 0 \Rightarrow N = 100\ N$

(c) N = 100 N : Because force by table on the box and force by box on table make Newton's third law pair.

(d) No

Example

Consider a spring attached at one of its ends to a fixed support and at other end to a box, which rests on a smooth floor as shown in the figure. Denote mass of the box by m, force constant of the spring by k and acceleration due to gravity by g.

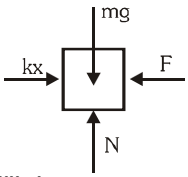


The box is pushed horizontally displacing it by distance x towards the fixed support and held at rest.

(a) Draw free body diagram of the box.
(b) Find force exerted by hand on the box.
(c) Write all the third law action-reaction pairs.

Solution

(a) F is push by hand.



(b) Since the block is in equilibrium $\Sigma F_x = 0 \Rightarrow F = kx$

(c) (i) Force by hand on box and force by box on hand.
(ii) Force by spring on box and force by box on spring.
(iii) Normal reaction by box on floor and normal reaction by floor on box.
(iv) Weight of the box and the gravitational force by which box pulls the earth.
(v) Force by spring on support and force by support on spring.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

(a) A box of weight $10\sqrt{3}$ N is held in equilibrium with the help of two strings OA and OB as shown in figure-I. The string OA is horizontal. Find the tensions in both the strings.

Fig. I

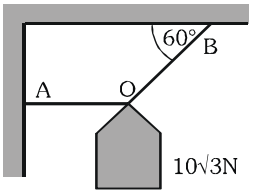
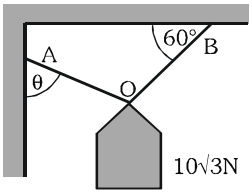


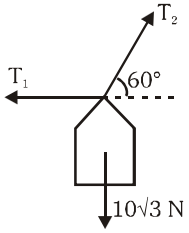
Fig. II



(b) If you can change location of the point A on the wall and hence the orientation of the string OA without altering the orientation of the string OB as shown in figure-II. What angle should the string OA make with the wall so that a minimum tension is developed in it?

Solution

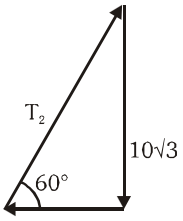
(a) Free body diagram of the box



Graphical Method : Use triangle law

$T_2\sin60 = 10\sqrt{3} \Rightarrow T_2=20N$

$T_1\tan60 = 10\sqrt{3} \Rightarrow T_1=10N$



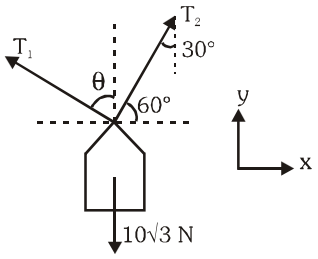
Analytical Method: Use Cartesian components

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos60° = T \qquad \text{...(i)}$$

$$\sum F_y = 0 \quad \Rightarrow \quad T_2\sin60° = 10\sqrt{3} \qquad \text{...(ii)}$$

From equation (i) & (ii) we have T_1=10N and T_2=20N

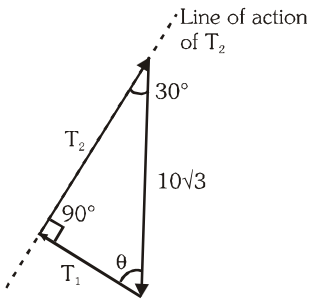
(b) Free body diagram of the box



Graphical Method : Use triangle law

For T_1 to be minimum, it must be perpendicular to T_2.

From figure θ = 60



Analytical Method : Use Cartesian components

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos60° = T_1\sin\theta \qquad \text{...(i)}$$

$$\sum F_y = 0 \quad \Rightarrow \quad T_1\cos\theta + T_2\sin60° = 10\sqrt{3} \qquad \text{...(ii)}$$

From equation (i) and (ii), we have $T_1 = \dfrac{10\sqrt{3}}{\sqrt{3}\sin\theta + \cos\theta}$

If T_1 is minimum, $\sqrt{3}\sin\theta + \cos\theta$ must be maximum. Maximum value of $\sqrt{3}\sin\theta + \cos\theta$ is 2.

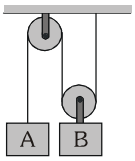
$$\sqrt{3}\sin\theta + \cos\theta = 2$$

Solving the above equation we get θ = 60

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

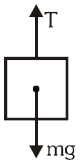
Example

Two boxes A and B of masses m and M are suspended by a system of pulleys are in equilibrium as shown. Express M in terms of m.

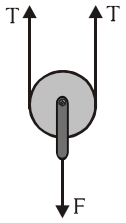


Solution

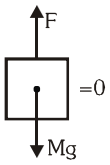
Since tension on both sides of a pulley are equal and string is massless therefore tension everywhere on the string must have same magnitude.



FBD of block A



FBD of pulley



FBD of Block B

For equilibrium of block A $\sum\vec{F}=\vec{0} \Rightarrow T = mg$ ...(i)

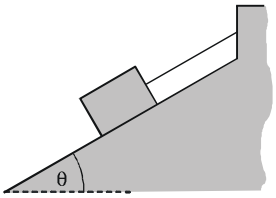
For equilibrium of pulley attached to block B $\sum\vec{F}=\vec{0} \Rightarrow F = 2T$ ...(ii)

For equilibrium of block B $\sum\vec{F}=\vec{0} \Rightarrow F = Mg$ ...(iii)

From equation (i), (ii) and (iii), we have M = 2m

Example

A box of mass m rests on a smooth slope with help of a thread as shown in the figure. The thread is parallel to the incline plane.



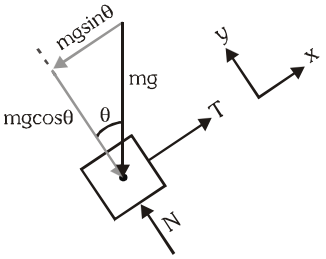
(a) Draw free body diagram of the box.

(b) Find tension in the thread.

(c) If the thread is replaced by a spring of force constant k, find extension in the spring.

Solution

(a) Free body diagram of the block



(b) The block is in equilibrium, therefore $\sum F_x = 0 \Rightarrow T = mg\sin\theta$ ...(i)

(c) If the thread is replaced by a spring, spring force must be equal to T, therefore

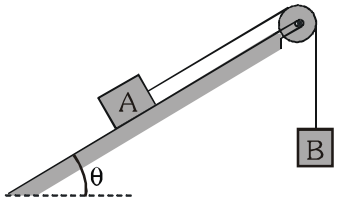
$T = kx$ ...(ii)

From equation (i) and (ii), we have $x = \frac{mg\sin\theta}{k}$

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

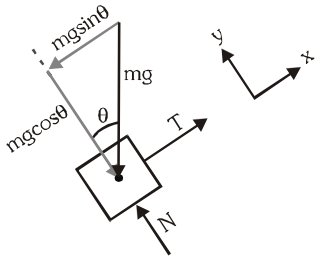
Block A of mass m placed on a smooth slope is connected by a string with another block B of mass M as shown in the figure. If the system is in equilibrium, express M in terms of m.



Solution

For equilibrium of the block A, net force on it must be zero.

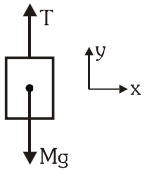
N: Normal reaction from slope



$\sum F_x = 0 \quad \Rightarrow \quad T = mg\sin\theta$ ...(i)

$\sum F_y = 0 \quad \Rightarrow \quad N = mg\cos\theta$ ...(ii)

For equilibrium of block B, the net force on it must be zero.

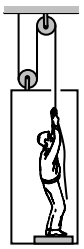


$\sum F_y = 0 \quad \Rightarrow \quad T = mg$ ... (ii)

From equations (i) and (ii), we have $M = m\sin\theta$

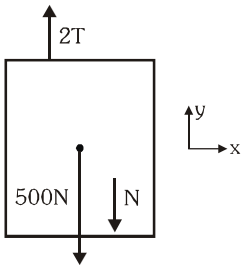
Example

A 70 kg man standing on a weighing machine in a 50 kg lift pulls on the rope, which supports the lift as shown in the figure. Find the force with which the man should pull on the rope to keep the lift stationary and the weight of the man as shown by the weighing machine.



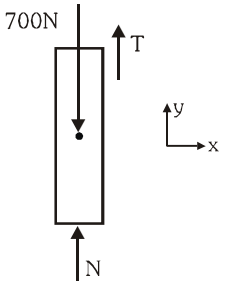
Solution

Tension magnitude everywhere in the string is same. For equilibrium of the lift.



$\sum F_y = 0 \quad \Rightarrow \quad 500 + N = 2T$ ...(i)

To analyse equilibrium of the man let us assume him as a block



$\sum F_y = 0 \quad \Rightarrow \quad N + T = 700$ ...(ii)

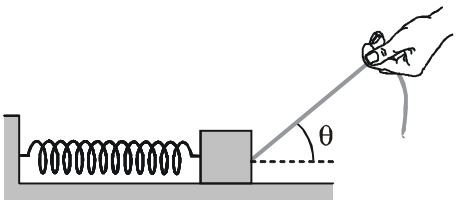
From equations (i) & (ii), we have T = 400 N and N = 300 N

Here, T is the pull of mass and N is reading of the weighing machine.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

A block of mass m placed on a smooth floor is connected to a fixed support with the help of a spring of force constant k. It is pulled by a rope as shown in the figure. Tension force T of the rope is increased gradually without changing its direction, until the block losses contact from the floor. The increase in rope tension T is so gradual that acceleration in the block can be neglected.



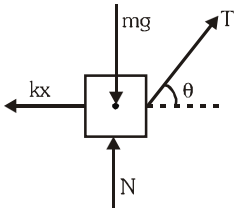
(a) Well before the block losses contact from the floor, draw its free body diagram.

(b) What is the necessary tension in the rope so that the block looses contact from the floor?

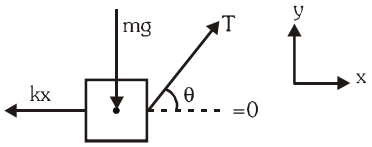
(c) What is the extension in the spring, when the block looses contact with the floor?

Solution

(a) Free body diagram of the block, well before it looses contact with the floor.



(b) When the block is about to leave the floor, it is not pressing the floor.
Therefore N = 0 and the block is in equilibrium.



$$\sum F_x = 0 \quad \Rightarrow \quad T\cos\theta = kx \qquad \text{...(i)}$$

$$\sum F_y = 0 \quad \Rightarrow \quad T\sin\theta = mg \qquad \text{...(ii)}$$

From equations (ii), we have $T = mg\,\text{cosec}\,\theta$

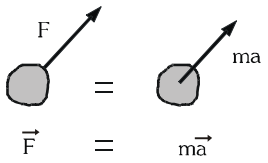
(c) From equation (i) and (ii), we have $x = \dfrac{mg\cot\theta}{k}$

Dynamics of Particles: Translation motion of accelerated bodies

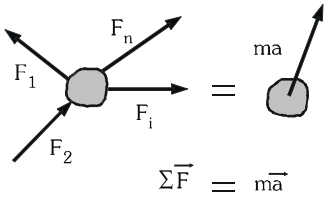
Newton's laws are valid in inertial frames, which are un-accelerated frames. At present, we are interested in motion of terrestrial bodies and for this purpose; ground can be assumed a satisfactory inertial frame.

In particle dynamics, according to Newton's second law, forces acting on the body are considered as cause and rate of change in momentum as effect. For a rigid body of constant mass, the rate of change in momentum equals to product of mass and acceleration vector. Therefore, forces acting on it are the cause and product of mass and acceleration vector is the effect.

To write the equation of motion it is recommended to draw the free body diagram, put a sign of equality and in front of it draw the body attached with a vector equal to mass times acceleration produced. In the figure is shown a body of mass m on which a single force $\vec{F}$ acts and an observer in an inertial frame of reference observes the body moving with acceleration $\vec{a}$.



Acceleration imparted to a body by a force is independent of other forces, therefore when several forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_n$ act simultaneously on a body, the acceleration imparted to the body is the same as a single force equal to the vector sum of these forces could produce. The vector sum of these forces is known as the net resultant of these forces.



Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots\ldots\ldots\ldots + \vec{F}_n = m\vec{a}$$

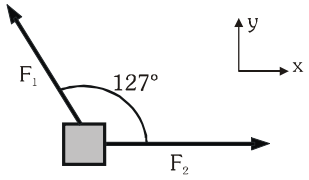
$$\sum \vec{F} = m\vec{a}$$

In Cartesian coordinate system the vector quantities in the above equation is resolved into their components along x, y, and z axes as follows:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

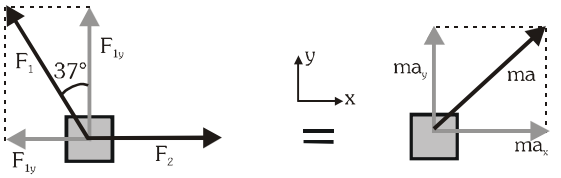
Example

Two forces F_1 and F_2 of magnitudes 50 N and 60 N act on a free body of mass m = 5 kg in directions shown in the figure. What is acceleration of object with respect to the free space?



Solution

In an inertial frame of reference with its x-axis along the force F_2, the forces are expressed in Cartesian components.



$\vec{F}_1 = \left(-30\hat{i} + 40\hat{j}\right)$ N and $\vec{F}_2 = 60\hat{i}$ N

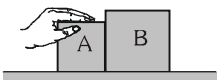
$\sum F_x = ma_x \Rightarrow a_x = 6$ m/s^2

$\sum F_y = ma_y \Rightarrow a_y = 8$ m/s^2

$\vec{a} = \left(6\hat{i} + 8\hat{j}\right)$ m/s^2

Example

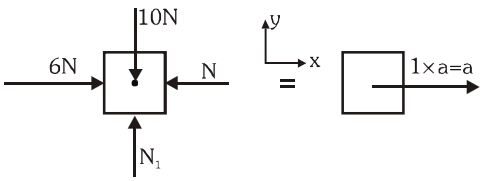
Boxes A and B of mass m_A = 1 kg and m_B = 2 kg are placed on a smooth horizontal plane. A man pushes horizontally the 1 kg box with a force F = 6 N. Find the acceleration and the reaction force between the boxes.



Solution

Since both the blocks move in contact it is obvious that both of them have same acceleration. Say it is 'a'.

Applying NLM to block A



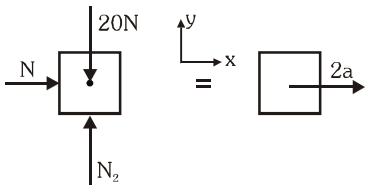
N: Normal reaction from B

N_1: Normal reaction from floor

$\sum F_x = ma_x \Rightarrow 6 - N = a$ ...(i)

$\sum F_y = 0 \Rightarrow N_1 = 10$ N ...(ii)

Applying NLM to block B



N : Normal reaction from A

N_2: Normal reaction from ground

$\sum F_x = ma_x \Rightarrow N = 2a$ ...(iii)

$\sum F_y = 0 \Rightarrow N_2 = 20$ N ...(iv)

From equations (i) & (iii), we have a = 2 m/s^2 and N = 4 N

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

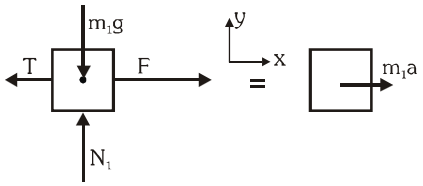
Two blocks A and B of masses m_1 and m_2 connected by light strings are placed on a smooth floor as shown in the figure. If the block A is pulled by a constant force F, find accelerations of both the blocks and tension in the string connecting them.



Solution

String connecting the blocks remain taut keeping separation between them constant. Therefore it is obvious that both of them move with the same acceleration. Say it is 'a'.

Applying NLM to block A



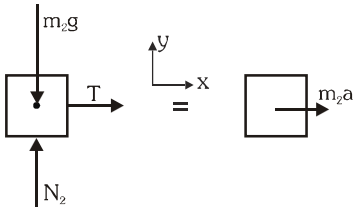
T: Tension of string

N_1 : Normal reaction from ground

$\sum F_x = ma \quad \Rightarrow \quad F - T = m_1 a$ ...(i)

$\sum F_y = 0 \quad \Rightarrow \quad N_1 = m_1 g$ ...(ii)

Applying NLM to block B.



T : Tension of string

N_2 : Normal reaction from ground.

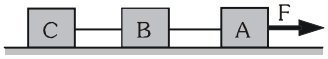
$\sum F_x = ma_x \quad \Rightarrow \quad T = m_2 a$ . ..(iii)

$\sum F_y = 0 \quad \Rightarrow \quad N_2 = m_2 g$ ...(iv)

From equation (i) and (iii), we have $a = \frac{F}{m_1 + m_2}$ and $T = \frac{m_2 F}{m_1 + m_2}$

Example

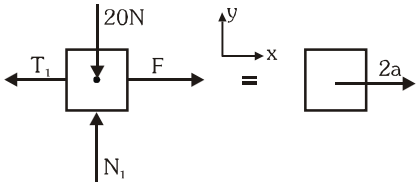
Three identical blocks A, B and C, each of mass 2.0 kg are connected by light strings as shown in the figure. If the block A is pulled by an unknown force F, the tension in the string connecting blocks A and B is measured to be 8.0 N. Calculate magnitude of the force F, tension in the string connecting blocks B and C, and accelerations of the blocks.



Solution

It is obvious that all the three blocks move with the same acceleration. Say it is 'a'.

Applying NLM to the block A.



T_1: Tension of string connecting blocks A and B.

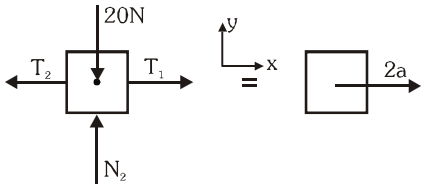
N_1 : Normal reaction from floor.

$\sum F_x = ma_x \quad \Rightarrow \quad F - T_1 = 2a$ ...(i)

$\sum F_y = 0 \quad \Rightarrow \quad N_1 = 20N$ ...(ii)

Applying NLM to the block B.

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65



T_1: Tension of string connecting blocks A and B.

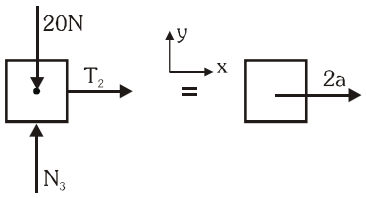
T_2 : Tension of string connecting B & C.

N_1 : Normal reaction from floor.

$\sum F_x = ma_x \quad \Rightarrow \quad T_1 - T_2 = 2a$ ...(iii)

$\sum F_y = 0 \quad \Rightarrow \quad N_2 = 20N$ ...(iv)

Applying NLM to the block C.



T_3 : Tension of string connecting B & C.

N_3 : Normal reaction from floor.

$\sum F_x = ma_x \quad \Rightarrow \quad T_2 = 2a$ ...(v)

$\sum F_y = 0 \quad \Rightarrow \quad N_3 = 20N$ ...(vi)

From equations (i), (iii) and (v), we have $F = 6a$ ...(vii)

Now using the fact that $T_1 = 8N$ with equation (i), We have $a = 2\ m/s^2$

Now from equation (i) we have F =12 N From equation (iii), we have $T_2 = 4$ N

Example

Two blocks A and B of masses m_1 and m_2 connected by uniform string of mass m and length ℓ are placed on smooth floor as shown in the figure. The string also lies on the floor. The block A is pulled by a constant force F.



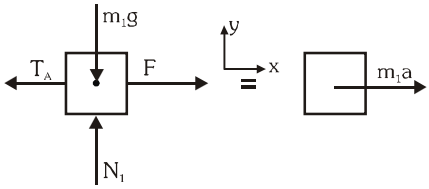
(a) Find accelerations a of both the blocks and tensions T_A and T_B at the ends of the string.

(b) Find an expression for tension T in the string at a distance x from the rear block in terms of T_A, T_B, m, ℓ and x.

Solution

It is obvious that both the blocks and the whole string move with the same acceleration say it is 'a'. Since string has mass it may have different tensions at different points.

(a) Applying NLM to block A.



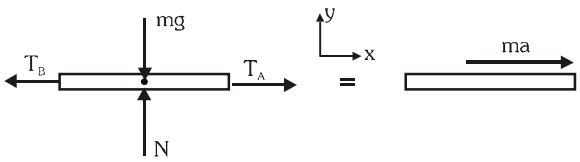
T_A : Tension of the string at end connected to block A.

N_1 : Normal reaction of floor

$\sum F_x = ma_x \quad \Rightarrow \quad F - T_A = m_1 a$ ...(i)

$\sum F_y = 0 \quad \Rightarrow \quad N_1 = m_1 g$ ...(ii)

Applying NLM to the rope



T_B : Tension of string at end connected to block B.

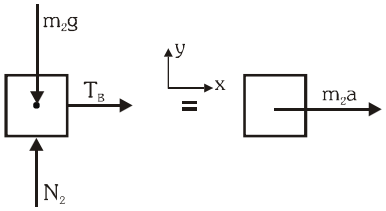
N : Normal reaction of floor

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

$$\sum F_x = ma_x \quad \Rightarrow \quad T_A - T_B = ma \qquad \text{...(iii)}$$

$$\sum F_y = 0 \quad \Rightarrow \quad N = mg \qquad \text{...(iv)}$$

Applying NLM to the block B



T_B : Tension of string

N_2 : Normal reaction from floor

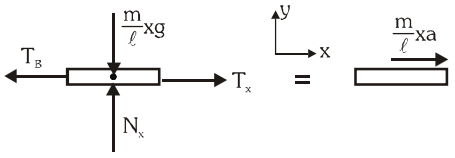
$$\sum F_x = ma_x \quad \Rightarrow \quad T_B = m_2 a \qquad \text{...(v)}$$

$$\sum F_y = 0 \quad \Rightarrow \quad N_2 = m_2 g \qquad \text{...(vi)}$$

From equations (i), (iii) and (v), we have

$$a = \frac{F}{m + m_1 + m_2} \text{ ...(vii)} \qquad T_A = \frac{(m + m_2)F}{m + m_1 + m_2} \text{ ...(viii)} \qquad T_B = \frac{m_2 F}{m + m_1 + m_2} \qquad \text{...(ix)}$$

(b) To find tension at a point x distance away from block B, we can consider string of length x or ℓ–x. Let as consider length of string x and apply NLM.



$\frac{mx}{\ell}$: mass of length x.

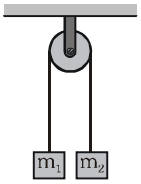
T_x = Tension at distance x

N_x = Normal reaction of floor

$$\sum F_x = ma_x \quad \Rightarrow \quad T_x - T_B = \frac{mx}{\ell} a \qquad \text{...(x)}$$

From equation (vii), (viii), (ix) and (x), we have $T_x = \left(\frac{m}{\ell} x + m_2\right) \frac{F}{m + m_1 + m_2}$

Example



The system shown in the figure is released from rest. Assuming mass m_2 more than the mass m_1, find the accelerations of the blocks and the tension in the string.

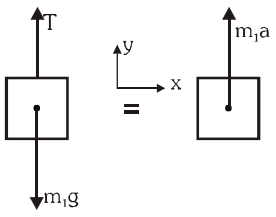
Solution

It obvious that both blocks move with same acceleration magnitudes. Say it is 'a'.

Since m_2 is heavier, it moves downwards and m_1 moves upwards.

Tension at both the ends of the string has same magnitude. Say it is 'T'.

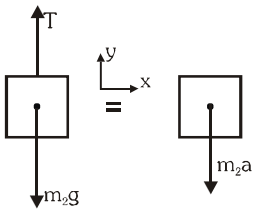
Apply NLM to block A of mass m_1



$$\sum F_y = ma_y \quad \Rightarrow \quad T - m_1 g = m_1 a \qquad \text{...(i)}$$

Apply NLM to block of mass m_2

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NLM\English\Theory.p65

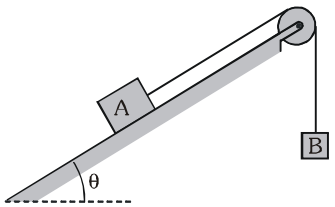


$$\sum F_y = ma_y \quad \Rightarrow \quad m_2 g - T = m_2 a \qquad ...(ii)$$

From equations (i) & (ii), we have $a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$, $T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right)g$

Example

Block A of mass m placed on a smooth slope is connected by a string with another block B of mass M (> msinθ) as shown in the figure. Initially the block A is held at rest and then let free. Find acceleration of the blocks and tension in the string.



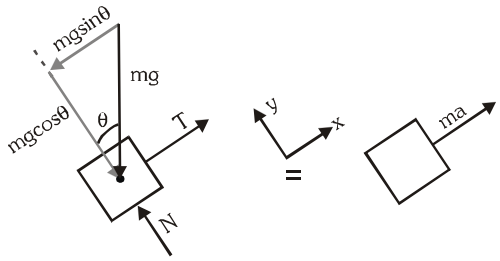
Solution

Both the blocks must move with the same magnitude of acceleration.

Since M > m sinθ, block B move downward pulling block A up the plane. Let acceleration magnitude is 'a'.

Tension at both the ends of the string is same. Say it is 'T'.

Apply NLM to block A



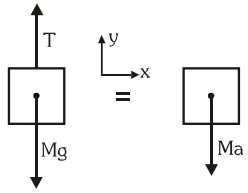
N: Normal reaction from slope

T : Tension of string

$$\sum F_x = ma_x \quad \Rightarrow \quad T - mg\sin\theta = ma \qquad ...(i)$$

$$\sum F_y = 0 \quad \Rightarrow \quad N = mg\cos\theta \qquad ...(ii)$$

Apply NLM to block B



T: Tension of string

$$\sum F_y = ma_y \quad \Rightarrow \quad Mg - T = Ma \qquad ...(iii)$$

From equation (i) & (iii), we have $a = \frac{(M - m\sin\theta)g}{M + m}$, $T = \frac{(1 + \sin\theta)mMg}{m + M}$

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

System of Interconnected bodies

In system of interconnected bodies, several bodies are interconnected in various manners through some sort of physical links. Sometimes these physical links includes ropes and pulleys and sometimes the bodies under investigation are pushing each other through direct contact. In systems consisting of bodies interconnected through ropes and pulleys, relation between their accelerations depends on the arrangement of the ropes and pulleys. In addition, in system where bodies push each other, they affect relation between their accelerations due to their shapes.

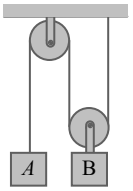
In kinematics while dealing with dependant motion or constrained motion, we have already learnt how to find relations between velocities and accelerations of interconnected bodies.

Analysis of physical situations involving interconnected bodies often demands relation between accelerations of these bodies in addition to the equations obtained by application of Newton's laws of motion. Therefore, while analyzing problems of interconnected bodies, it is recommended to explore first the relations between accelerations and then apply Newton's laws of motion.

In following few examples, we learn how to deal with problems of interconnected bodies.

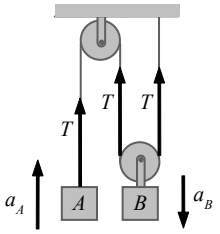
Example

Two boxes A and B of masses m and M interconnected by an ideal rope and ideal pulleys, are held at rest as shown. When it is released, box B accelerates downwards. Find accelerations of the blocks.

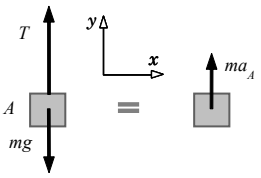


Solution.

We first show tension forces applied by the string on the box A and the pulley connected to box B. Since the string, as well as the pulleys, both are ideal; the string applies tension force of equal magnitude everywhere. Denoting the tension force by T, we show it in the adjacent figure.



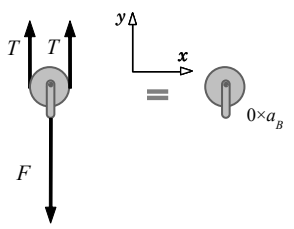
We first explore relation between accelerations a_A and a_B of the boxes A and B, which can be written either by using constrained relation or method of virtual work or by inspection.



$$a_A = 2a_B \quad \text{...(i)}$$

Applying Newton's Laws of motion to box A

$$\sum F_y = ma_y \rightarrow T - mg = ma_A \quad \text{...(i)}$$

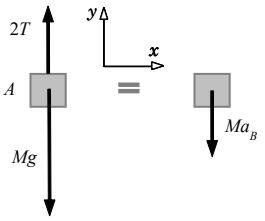


Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Applying Newton's Laws of motion to the pulley

$$\sum F_y = ma_y \rightarrow 2T - F = 0 \times a_B$$

$$F = 2T \quad \text{...(iii)}$$



Applying Newton's Laws of motion to box B

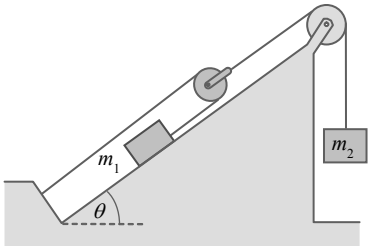
$$\sum F_y = ma_y \rightarrow Mg - 2T = Ma_B \quad \text{...(iv)}$$

From equations (i), (ii), (iii) and (iv), we have

$$a_A = 2\left(\frac{M-2m}{M+4m}\right)g \text{ and } a_B = \left(\frac{M-2m}{M+4m}\right)g$$

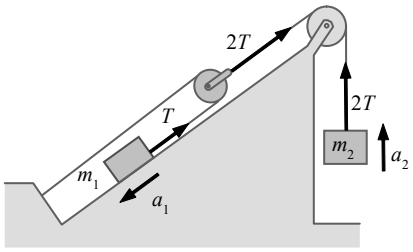
Example

In the system shown in figure, block m_1 slides down a friction less inclined plane. The pulleys and strings are ideal. Find the accelerations of the blocks.



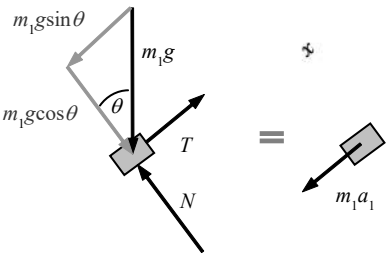
Solution

Tension forces applied by the strings are shown in the adjacent figure.



Let the block m_1 is moving down the plane with an acceleration a_1 and m_2 is moving upwards with accelerations a_2. Relation between accelerations a_1 and a_2 of the blocks can be obtained easily by method of virtual work.

$$a_1 = 2a_2 \quad \text{...(i)}$$

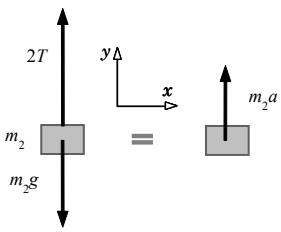


Applying Newton's laws to analyze motion of block m_1

$$\sum F_x = ma_x \rightarrow m_1 g\sin\theta - T = m_1 a_1 \quad \text{...(ii)}$$

Applying Newton's laws to analyze motion of block m_2

$$\sum F_y = ma_y \rightarrow 2T - m_2 g = m_2 a_2 \quad \text{...(iii)}$$



From equation (i), (ii) and (iii), we have

$$a_1 = \frac{2(2m_1\sin\theta - m_2)}{4m_1 + m_2}g \quad a_2 = \frac{2m_1 g\sin\theta - m_2 g}{4m_1 + m_2}$$

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

FRICTION

Whenever surfaces in contact are pressing each other slide or tend to slide over each other, opposing forces are generated tangentially to the surfaces in contact. These tangential forces, which oppose sliding or tendency of sliding between two surfaces are called frictional forces. Frictional forces on both bodies constitute third law action-reaction pair.

Types of Friction

Before we proceed further into detailed account of frictional phenomena, it is advisable to become familiar with different types of frictional forces. All types of frictional phenomenon can be categorized into dry friction, fluid friction, and internal friction.

Dry Friction

It exists when two solid un-lubricated surfaces are in contact under the condition of sliding or tendency of sliding. It is also known as Coulomb friction.

Fluid Friction

Fluid friction is developed when adjacent layers of a fluid move at different velocities and gives birth to phenomena, which we call viscosity of the fluid. Resistance offered to motion of a solid body in a fluid also comes in this category and commonly known as viscous drag. We will study this kind of friction in fluid mechanics.

Internal Friction

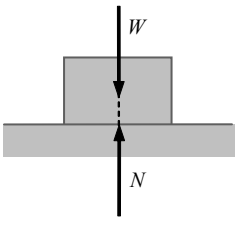
When solid materials are subjected to deformation, internal resistive forces developed because of relative movement of different parts of the solid. These internal resistive forces constitute a system of force, which is defined as internal friction. They always cause loss of energy.

Frictional forces exist everywhere in nature and result in loss of energy that is primarily dissipated in form of heat. Wear and tear of moving bodies is another unwanted result of friction. Therefore, sometimes, we try to reduce their effects – such as in bearings of all types, between piston and the inner walls of the cylinder of an IC engine, flow of fluid in pipes, and aircraft and missile propulsion through air. Though these examples create a negative picture of frictional forces, yet there are other situations where frictional forces become essential and we try to maximize the effects. It is the friction between our feet and the earth surface, which enables us to walk and run. Both the traction and braking of wheeled vehicles depend on friction.

Types of Dry Friction

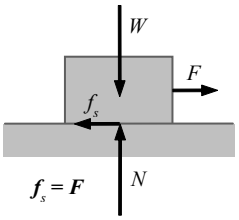
In mechanics of non-deformable bodies, we are always concerned with the dry friction. Therefore, we often drop the word "dry" and simply call it friction.

To understand nature of friction let us consider a box of weight W placed on a horizontal rough surface. The forces acting on the box are its weight and reaction from the horizontal surface. They are shown in the figure. The weight does not have any horizontal component, so the reaction of the horizontal surface on the box is normal to the surface. It is represented by N in the figure. The box is in equilibrium therefore both W and N are equal in magnitude, opposite in direction, and collinear.

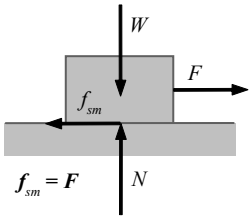


Now suppose the box is being pulled by a gradually increasing horizontal force F to slide the box. Initially when the force F is small enough, the box does not slide. This can only be explained if we assume a frictional force, which is equal in magnitude and opposite in direction to the applied force F acts on the box. The force F produces in the box a tendency of sliding and the friction force is opposing this tendency of sliding. The frictional force developed before sliding initiates is defined as static friction. It opposes tendency of sliding.

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\UNIT\English\Theory.p65



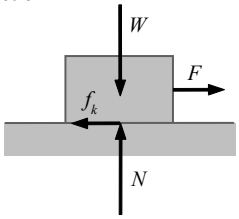
Static Friction



Limiting friction: The maximum Static Friction

As we increase F, the box remains stationary until a value of F is reached when the box starts sliding. Before the box starts sliding, the static friction increases with F and counterbalances F until the static friction reaches its maximum value known as limiting friction or maximum static friction f_{sm}.

When the box starts sliding, to maintain it sliding still a force F is needed to over come frictional force. This frictional force is known as kinetic friction (f_k). It always opposes sliding.



Kinetic friction

Dry Friction

Static Friction	Kinetic Friction
Opposes tendency of sliding	Opposes sliding and has a constant characteristic value.

Laws of Friction

When a normal force N exists between two surfaces, and we try to slide them over each other, the force of static friction (f_s) acts before sliding initiates. It can have a value maximum up to the limiting friction (f_{sm}).

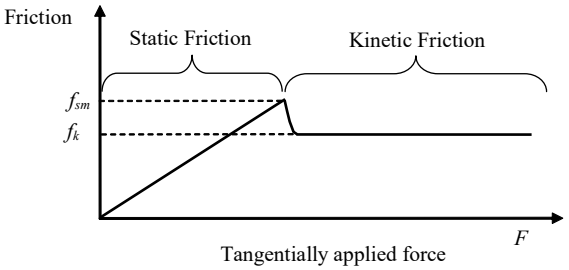
$$f_s \le f_{sm}$$

The limiting friction is experimentally observed proportional to the normal reaction between surfaces in contact.

$$f_{sm} = \mu_s N$$

Here μ_S is the constant of proportionality. It is known as the coefficient of static friction for the two surfaces involved.

When sliding starts between the surfaces, the frictional force rapidly drops to a characteristic value, which always opposes the sliding. This characteristic frictional force is known as kinetic friction (f_k). Kinetic friction is experimentally found proportional to the normal reaction between surfaces in contact.



$$f_k = \mu_k N$$

Here μ_k is the constant of proportionality. It is known as the coefficient of kinetic friction for the two surfaces involved.

The frictional forces between any pair of surfaces are decided by the respective coefficients of friction. The coefficients of friction are dimensionless constants and have no units. The coefficient of static fiction (μ_s) is generally larger than the coefficient of kinetic friction (μ_k) but never become smaller; at the most both of them may be equal. Therefore, the magnitude of kinetic friction is usually smaller than the limiting static friction (f_{sm}) and sometimes kinetic friction becomes equal to the limiting static friction but it can never exceed the limiting friction.

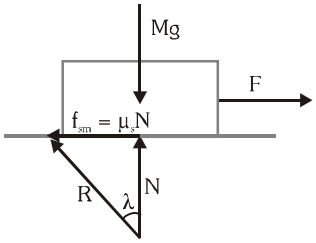
The limiting static friction and the kinetic friction between any pair of solid surfaces follow these two empirical laws.

- Frictional forces are independent of measured area of contact.
- Both the limiting static friction and kinetic friction are proportional to the normal force pressing the surfaces in contact.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Angle of Friction

The angle of friction is the angle between resultant contact force of and normal reaction N, when sliding is initiating. It is denoted by λ

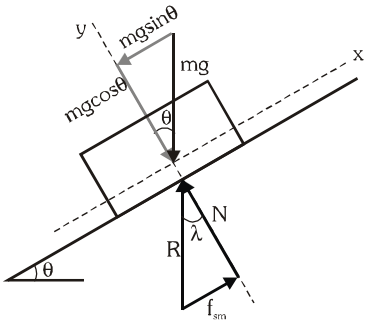


$$\tan\lambda = \frac{f_{sm}}{N} = \frac{\mu_s N}{N} = \mu_s$$

- For smooth surface $\lambda = 0$

Angle of Repose (θ)

A body is placed on an inclined plane and the angle of inclination is gradually increased. At some angle of inclination θ the body starts sliding down the plane due to gravity. This angle of inclination is called angle of repose (θ).



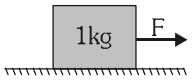
Angle of repose is that minimum angle of inclination at which a body placed on the inclined starts sliding down due to its own weight.

Thus, angle of repose = angle of friction.

Example

A block of mass 1 kg is at rest on a rough horizontal surface, where coefficients of static and kinetic friction are 0.2 and 0.15. Find the frictional forces if a horizontal force

(a) F = 1N (b) F = 1.96 N (c) F = 2.5 N is applied on a block



Solution

Maximum force of friction is the limiting friction f_{sm} = 0.2 1 9.8 N = 1.96 N

(a) For F = 1 N, $F < f_{sm}$

So, body is in rest means static friction is present and hence f_s = F= 1 N

(b) For F = 1.96 N, F = f_{sm} = 1.96 N. The block is about to slide, therefore f = 1.96 N

(c) For F = 2.5 N, So $F > f_{sm}$

Now body is sliding and kinetic friction acts.

Therefore $f = f_k = \mu_k N = \mu_k$ mg = 0.15 1 9.8 = 1.47 N

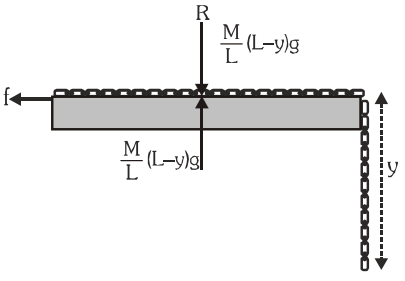
Example

Length of a uniform chain is L and coefficient of static friction is μ between the chain and the table top. Calculate the maximum length of the chain which can hang from the table without sliding.

Solution

Let y be the maximum length of the chain that can hang without causing the portion of chain on table to slide.

Length of chain on the table = (L – y)



Weight of part of the chain on table $= \frac{M}{L}(L-y)g$

Weight of hanging part of the chain $= \frac{M}{L}yg$

For equilibrium with maximum portion hanging, limiting friction = weight of hanging part of the chain

$$\mu\frac{M}{L}(L-y)g = \frac{M}{L}yg \Rightarrow y = \frac{\mu L}{1+\mu}$$

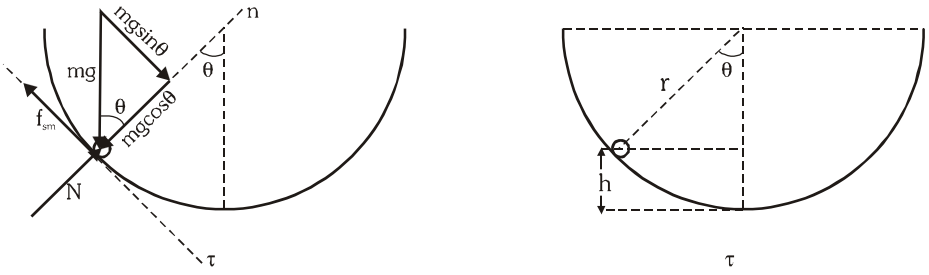
Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

An insect crawls on the inner surface of hemispherical bowl of radius r. If the coefficient of friction between an insect and bowl is μ and the radius of the bowl is r, find the maximum height to which the insect can crawl up.

Solution

The insect can crawl up, the bowl till the component of its weight tangent to the bowl is balanced by limiting frictional force.



$\sum F_n = 0 \rightarrow$ $N = mg\cos\theta$ ...(i)

$\sum F_\tau = 0 \rightarrow$ $f_{sm} = mg\sin\theta$ ...(ii)

Force of limiting friction $f_{sm} = \mu N$ ...(iii)

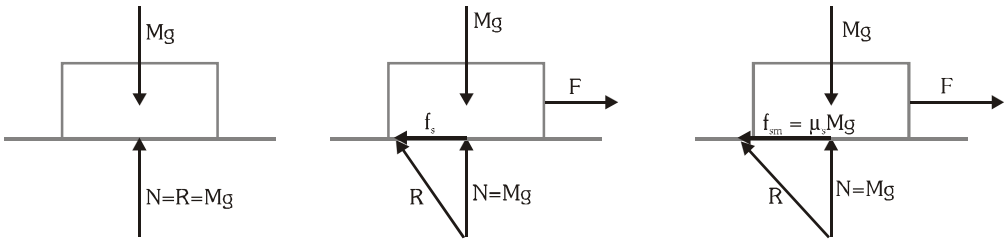
From equation (i), (ii) and (iii), $\tan\theta = \mu$ ...(iv)

$$h = r - r\cos\theta = r\left[1 - \frac{1}{\sqrt{1+\mu^2}}\right]$$

Example

A body of mass M is kept on a rough horizontal ground (static friction coefficient = μ_s). A person is trying to pull the body by applying a horizontal force F, but the body is not moving. What is the contact force between the ground and the block.

Solution



When F = 0

R = Mg

When $F < f_{sm}$

$R = \sqrt{(Mg)^2 + f_s^2}$

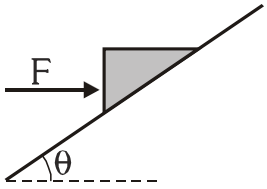
When $F = f_{sm}$

$R = Mg\sqrt{1+\mu_s^2}$

Therefore $Mg \le R \le Mg\sqrt{1+\mu_s^2}$

Example

A block rest on a rough inclined plane as shown in fig. A horizontal force F is applied to it (a) Find the force of normal reaction, (b) Can the force of friction be zero, if yes when? and (c) Assuming that friction is not zero find its magnitude and direction of its limiting value.



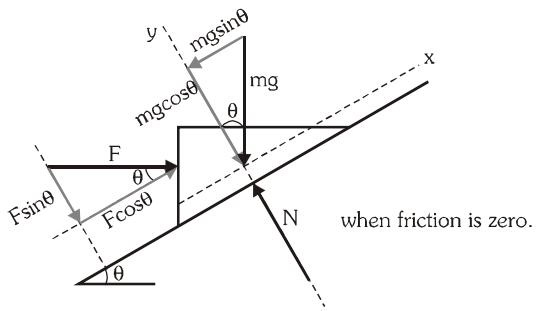
Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution

(a) $\sum F_y = 0 \rightarrow \quad N = mg\cos\theta + F\sin\theta$

(b) $\sum F_x = 0 \rightarrow \quad F\cos\theta = mg\sin\theta \Rightarrow F = mg\tan\theta$

(c) Limiting friction $f_{sm} = \mu N = \mu\,(mg\cos\theta + F\sin\theta)$;

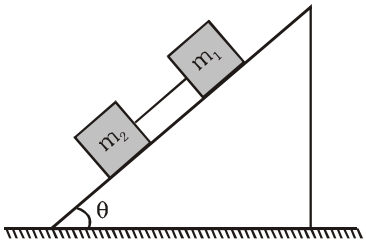


when friction is zero.

It acts down the plane if body has tendency to slide up and acts up the plane if body has tendency to slide down.

Example

Two blocks with masses m_1=1 kg and m_2 = 2 kg are connected by a string and slide down a plane inclined at an angle θ=45 with the horizontal. The coefficient of sliding friction between m_1 and plane is μ_1=0.4 and that between m_2 and plane is μ_2=0.2. Calculate the common acceleration of the two blocks and the tension in the string.



Solution

As $\mu_2 < \mu_1$, block m_2 has greater acceleration than m_1 if we separately consider the motion of blocks. But they are connected so they move together as a system with common acceleration.

So acceleration of the blocks :

$$a = \frac{(m_1+m_2)g\sin\theta - \mu_1 m_1 g\cos\theta - \mu_2 m_2 g\cos\theta}{m_1+m_2}$$

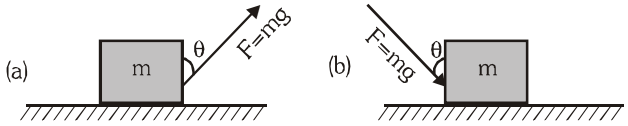
$$= \frac{(1+2)(10)\left(\frac{1}{\sqrt{2}}\right) - 0.4\times1\times10\times\frac{1}{\sqrt{2}} - 0.2\times2\times10\times\frac{1}{\sqrt{2}}}{1+2} = \frac{22}{3\sqrt{2}}\ ms^{-2}$$

For block m_2 : $m_2 g\sin\theta - \mu_2 m_2 g\cos\theta - T = m_2 a \Rightarrow \quad T = m_2 g\sin\theta - \mu_2 m_2 g\cos\theta - m_2 a$

$$= 2\ 10\ \frac{1}{\sqrt{2}} - 0.2\ \ 2\ \ 10\ \ \frac{1}{\sqrt{2}} - 2\ \ \frac{22}{3\sqrt{2}} = \frac{2}{3\sqrt{2}}N$$

Example

A block of mass m rests on a rough horizontal surface as shown in figure (a) and (b). Coefficient of friction between block and surface is μ. A force F = mg acting at an angle θ with the vertical side of the block. Find the condition for which block will move along the surface.

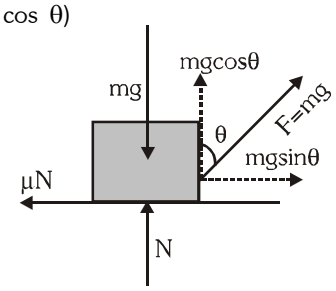


Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution

For (a) : normal reaction N = mg – mg cos θ frictional force = μN = μ(mg – mg cos θ)

Now block can be pulled when : Horizontal component of force ≥ frictional force i.e. mg sin θ ≥ μ(mg – mg cos θ)

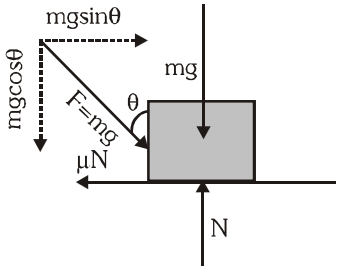


or $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq \mu(1-\cos\theta)$

or $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq 2\mu\sin^2\frac{\theta}{2}$ or $\cot\frac{\theta}{2} \geq \mu$

For (b) : Normal reaction N = mg + mg cos θ = mg (1 + cos θ)

Hence, block can be pushed along the horizontal surface when horizontal component of force ≥ frictional force



i.e. mg sin θ ≥ μ mg(1 + cos θ)

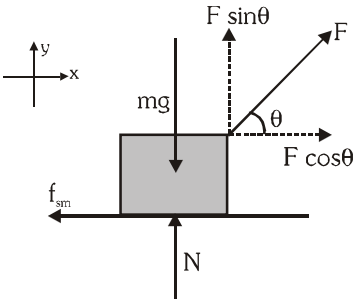
or $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq \mu \times 2\cos^2\frac{\theta}{2} \Rightarrow \tan\frac{\theta}{2} \geq \mu$

Example

A body of mass m rests on a horizontal floor with which it has a coefficient of static friction μ. It is desired to make the body move by applying the minimum possible force F. Find the magnitude of F and the direction in which it has to be applied.

Solution

Let the force F be applied at an angle θ with the horizontal as shown in figure.



$\sum F_y = 0 \rightarrow$ N = mg – F sin θ ...(i) N = mg – F sin θ ...(i)

$\sum F_x = 0 \rightarrow$ F cos θ ≥ $f_{sm} \Rightarrow$ F cos θ ≥ μN [as f_{sm} = μN] ...(ii)

Substituting value of N from equation (i) in (ii),

$$F \geq \frac{\mu mg}{(\cos\theta + \mu\sin\theta)} \qquad \text{...(iii)}$$

For the force F to be minimum (cos θ + μ sin θ) must be maximum,

maximum value of cos θ + μ sin θ is $\sqrt{1+\mu^2}$ so that $F_{min} = \frac{\mu mg}{\sqrt{1+\mu^2}}$ with θ = $\tan^{-1}$ (μ)

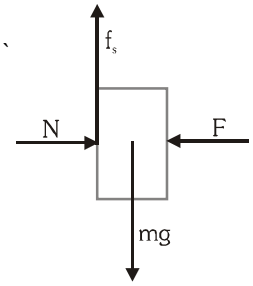
Example

A book of 1 kg is held against a wall by applying a force F perpendicular to the wall. If μ_S = 0.2, what is the minimum value of F ?

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution

The situation is shown in fig. The forces acting on the book are–



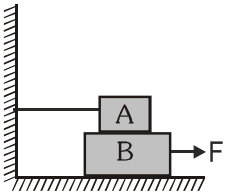
For book to be at rest it is essential that $Mg = f_s$

But $f_{s\ max} = \mu_S N$ and $N = F$

$$\therefore\ Mg = \mu_S F \ \Rightarrow\ F = \frac{Mg}{\mu_s} = \frac{1\times 9.8}{0.2} = 49\ N$$

Example

A is a 100 kg block and B is a 200 kg block. As shown in fig., the block A is attached to a string tied to a wall. The coefficient of friction between A and B is 0.2 and the coefficient of friction between B and floor is 0.3. Then calculate the minimum force required to move the block B.($g = 10\ m/s^2$).



Solution

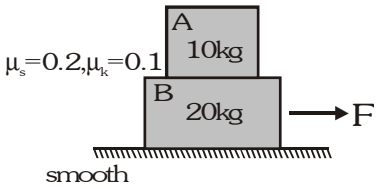
When B is tied to move, by applying a force F, then the frictional forces acting on the block B are f_1 and f_2 with limiting values, $f_1 = (\mu_S)_A m_A g$ and $f_2 = (\mu_S)_B\ (m_A + m_B)g$

Then minimum value of F should be (for just tending to move),

$$F = f_1 + f_2 = 0.2\quad 100\ g + 0.3\quad 300\ g = 110\ g = 1100\ N$$

Example

In the given figure block A is placed on block B and both are placed on a smooth horizontal plane. Assume lower block to be sufficiently long.



The force F pulling the block B horizontally is increased according to law F = 10t N

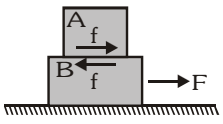
(a) When does block A start slipping on block B? What will be force F and acceleration just before slipping starts?

(b) When F is increased beyond the value obtained in part (a), what will be acceleration of A ?

(c) Draw acceleration-time graph.

Solution

Direction of friction forces



Block A moves forward always, due to friction, therefore friction on it must be in forward direction.

Friction between two adjacent surfaces are equal and opposite because they make Newton's is third law action reaction pair.

Range of Value of friction

Before slipping starts, friction is static $f_s \le 20\ N$

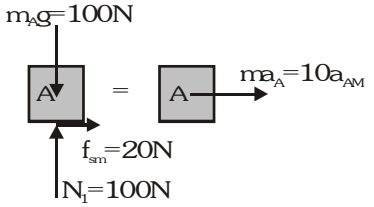
After slipping starts, friction is kinetic $f_k = 10\ N$

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Maximum possible acceleration

A can accelerate only due to friction, its maximum possible acceleration is a_{AM} (when $f_s = f_{sm} = 20$ N)

Block A



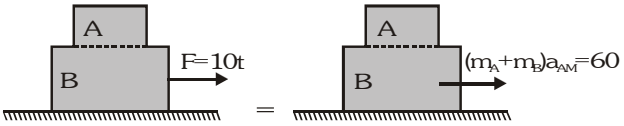
So $20 = 10a_{AM} \Rightarrow a_{AM} = 2\ m/s^2$

Sequence of slipping :

Since ground is smooth, block B first starts slipping on the ground and carries A together with it. When acceleration of A & B becomes equal to a_{AM}, Block A starts slipping on B.

(a) Just before the moment A starts slipping, both were moving together with acceleration a_{AM}.

Considering them as a one body.

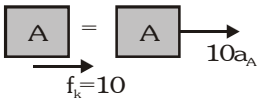


(On a smooth stationary surface we will not show the normal forces i.e. FBD of combined block showing horizontal forces only).

Value of F $\quad F = 60$ N

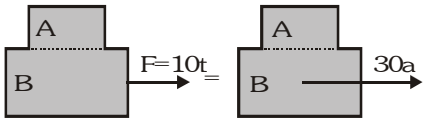
and Time $\quad 10t = 60 \Rightarrow t = 6s$

(b) If F is increased beyond 60 N, A slides and kinetic friction acts on it. Now acceleration of A



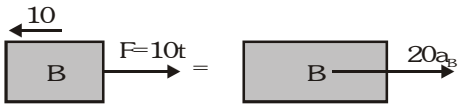
$\Rightarrow 10 = 10a_A \Rightarrow a_A = 1\ m/s^2$

(c) When $F \leq 60$ N, both are moving with same acceleration a. We treat them as one body.



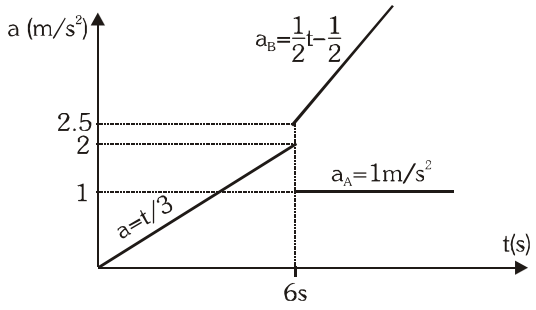
So $10t = 30a \Rightarrow a = \frac{1}{3}t$

This acceleration increases to $a_{AM} = 2\ m/s^2$, when F = 60 N at t = 6 s. Thereafter A starts slipping and its acceleration provided by kinetic friction, drops to a constant value $a_A = 1\ m/s^2$. However acceleration of B keeps on increasing according to equation



$10t - 10 = 20a_B \Rightarrow a_B = \frac{1}{2}t - \frac{1}{2}$

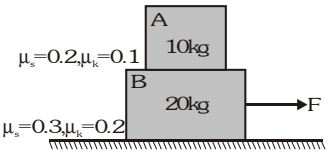
Graph between acceleration and time



Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

Block A is placed on another block B, which rests on a rough horizontal ground. Horizontal force F pulling the block B is increased gradually.



(a) Find the maximum value of F so that no motion occurs.

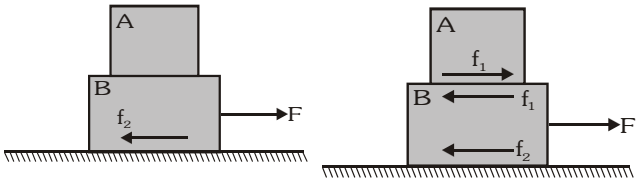
(b) Find maximum F so that A does not slide on B.

(c) If F is increased beyond the value obtained in part (b) what are acceleration of both the blocks ? Explain your answer in terms of F.

(d) If F is increased according to law F =10t N draw a–t graph

Solution

Directions of friction forces



Before B starts slipping After B starts slipping

Range of values of frictional forces

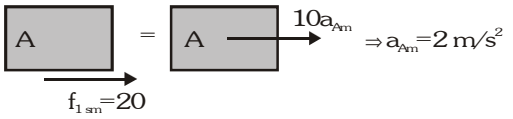
f_{1k} = 10 N (A is slipping)

f_{1s} = 20 N (A is not slipping)

f_{2k} = 60 N (B is slipping)

$f_{2s} \leq$ 90 N (B is at rest)

Maximum possible acceleration : A can move only due to friction. Its maximum possible acceleration is

Block A 

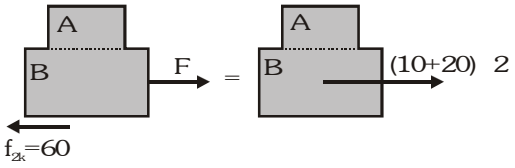
$\Rightarrow a_{Am}=2\ m/s^2$

Sequence of slipping : When $F \geq f_{2s}$, block B starts slipping on ground and carries block A together with it till its acceleration reaches value a_{AM}. Thereafter A also starts slipping on B.

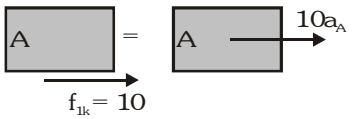
(a) F = 90 N

(b) When A does not slide on B, both move with the same acceleration (a_{Am}) and can be treated as one body, which can have maximum acceleration a_{AM} = 2 m/s^2.

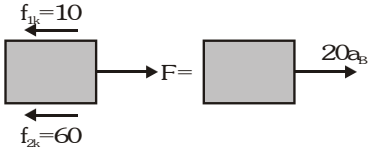
F – 60 = 60 $\Rightarrow$ F = 120 N



(c) When F is increased beyond F = 120 N, block A starts sliding and friction between A & B drops to f_{1K} = 10 N. Both the blocks now move with different acceleration so we treat them separate bodies. Now acceleration A also drops to a constant value a_A.

Acceleration of A : 

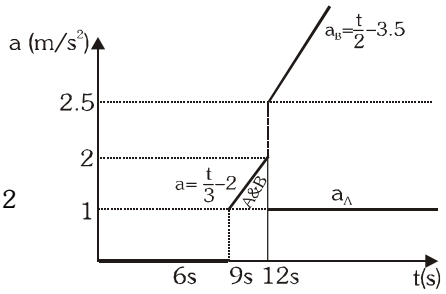
$\Rightarrow 10 = 10\ a_A \Rightarrow a_A = 1\ m/s^2$

Acceleration of B : 

So F–70 = $20a_B \Rightarrow a_B = \dfrac{F-70}{20}$

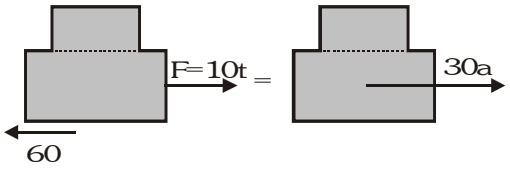
Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

(d) If F = 10t, values of acceleration of both the blocks in different time intervals are as under:



- $F \leq 90\ N \Rightarrow$ $t \leq 9\ s$ $a_A = a_B = 0$
- $90\ N < F \leq 120\ N$ $9\ s < t \leq 12\ s$ $a_A = a_B = \frac{t}{3} - 2$

  In the interval both the blocks move as one body

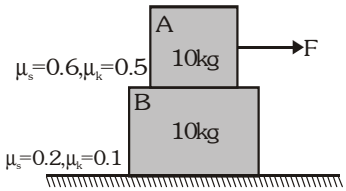


$$a = \frac{10t - 60}{30} = \frac{t}{3} - 2$$

- $F > 120\ N$ $\Rightarrow t > 12s$ $a_A = 1;\ a_B = \frac{t}{2} - 3.5$

Example

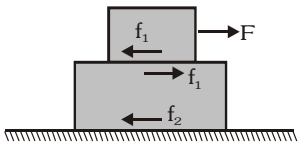
Block A is placed on another block B, which rests on a rough horizontal ground. Horizontal force pulling A is increased gradually



(a) Find maximum F so that none of the blocks move. Which block starts sliding first ?
(b) Express acceleration of each block as function of F for all positive values of F.
(c) If F=10t draw a-t graph

Solution

Directions of friction forces



Range of values of friction forces

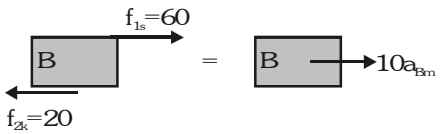
$f_{1s} \leq 60\ N$

$f_{1k} = 50\ N$

$f_{2s} \leq 40\ N$

$f_{2k} = 20\ N$

Maximum possible acceleration of B : Block B acceleration due to friction only. Its maximum acceleration is

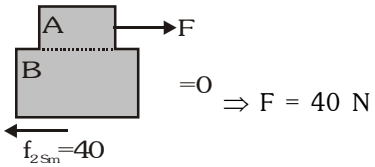


$60-20 = 10a_{Bm} \Rightarrow a_{Bm} = 4\ m/s^2$

Sequence of slipping : Smaller, limiting friction is between B and ground so it will start sliding first. Then both will move together till acceleration B reaches its maximum possible values 4 m/s^2. Thereafter A starts sliding on B

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

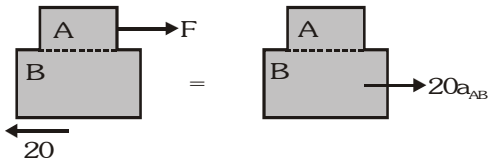
(a) Till the F reaches the limiting friction between block B and the ground none of the blocks move.



$\Rightarrow$ F = 40 N

(b) If $F \leq 40 \Rightarrow a_A = a_B = 0$ ...(i)

If F > 40 N, block B starts sliding and carries A together with it with the same acceleration till acceleration reach to 4 m/s^2. At this moment A starts slipping. Before this moment we may treat both of them as single body.



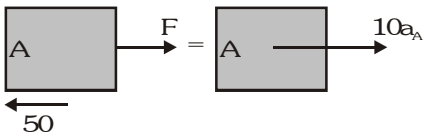
$\Rightarrow F - 20 = 20a_{AB} \Rightarrow a_{AB} = a_A = a_B = \frac{F-20}{20}$

When A starts sliding on B, $a_A = a_B = 4$, from the above equation, we have F = 100 N.

When F ≥ 100 N block A also starts slipping on B and friction between A & B drops to value 50 N.

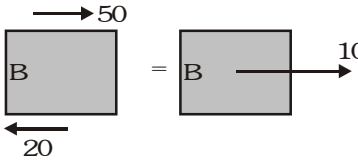
Now since they move with different acceleration we treat them separately.

Block A



$\Rightarrow a_A = \frac{F-50}{10}$

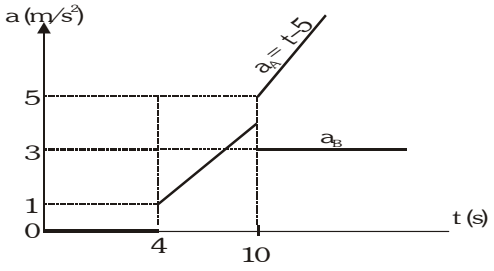
Block B



$\Rightarrow a_B = \frac{50-20}{10} = 3\ m/s^2$

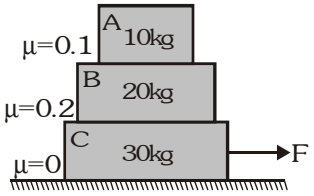
(c)

$F \leq 40$ N	$t \leq 4$ s	$a_A = a_B = 0$
$40 \leq F \leq 100$	$4 < t \leq 10$	$a_A = a_B = \frac{t}{2} - 1$
$100 < F$	$\Rightarrow t > 10$	$a_A = t - 5$, $a_B = 3\ m/s^2$



Example

Block A is placed on B and B is placed on block C, which rests on smooth horizontal ground as shown in the figure. Block A is pulled horizontally by a force F which increases gradually.



(a) Decide sequence of slipping.

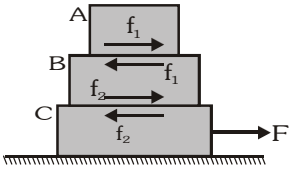
(b) If F is increased gradually find acceleration of each block for all values of F.

(c) If F = 15t N, draw a–t graph.

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution

Direction of friction forces :



Range of values of friction forces

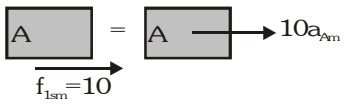
$f_1 \leq 10$ N (A does not slides on B)

$f_1 = 10$ N (A slides on B)

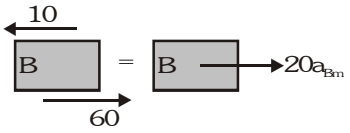
$f_2 \leq 60$ N (B does not slides on C)

$f_2 = 60$ N (B slides on C)

Maximum possible acceleration : Blocks A and B move due to friction forces only, we find their maximum possible acceleration.

Block A 

$a_{Am} = 1$ m/s^2

Block B 

$a_{Bm} = \dfrac{60-10}{20} = 2.5$ m/s^2

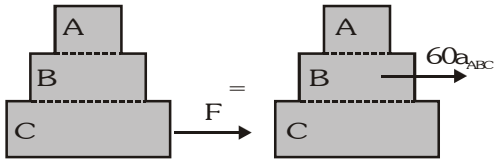
(a) Sequence of slipping

Since ground is smooth the block C starts sliding first

A starts slipping on B secondly till that moment all the three blocks move with same acceleration, which can achieve maximum value of $a_{AM} = 1$ m/s^2.

Thirdly B starts sliding on C, till that moment B & C move with the same acceleration $a_{Bm} = 2.5$ m/s^2

(b) Before A starts slipping, all the three were moving with the same acceleration $a_{AM} = 1$ m/s^2. We therefore treat then as a single body.

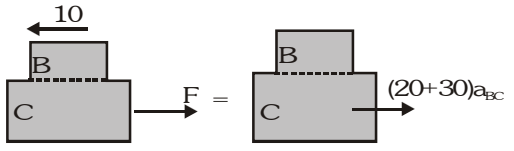


$a_{ABC} = \dfrac{F}{60}$

When A starts sliding $a_{ABC} \leq a_{AM} \Rightarrow \dfrac{F}{60} \leq 1 \Rightarrow F \leq 60$ N

When $F \geq 60$ N, block A starts slipping on B and its acceleration decided by friction f_1, achieves a constant value $a_A = 1$ m/s^2.

Now, F is increased beyond 60 N and B and C will continue to move together till their acceleration a_{BC} becomes $a_{Bm} = 2.5$ m/s^2, when slipping between B and C starts. Till this moment, we treat B and C as one body.

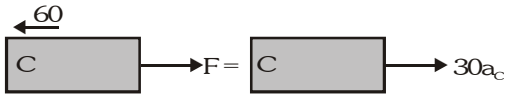


$\Rightarrow a_{BC} = \dfrac{F-10}{50}$

When slipping between B & C starts : $a_{BC} = a_{Bm} \Rightarrow \dfrac{F-10}{50} = 2.5 \Rightarrow F \leq 135$ N

When $F > 135$ N, block B also starts slipping on C. Now acceleration of A & B achieves the maximum value $a_{Bm} = 2.5$ m/s^2 and acceleration of block C is decided by F.

Node6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65



$\Rightarrow F - 60 = 30\ a_C \Rightarrow a_C = \dfrac{F-60}{30}$

Acceleration of blocks for different values of force.

- $F \le 60$ N $\quad a_A = a_B = a_C = a_{ABC} = \dfrac{F}{60}$
- $60 < F \le 135$ N $\quad a_A = a_{Am} = 1\ m/s^2,\ a_B = a_C = a_{BC} = \dfrac{F-10}{50}$
- $135 < F$ $\quad a_A = a_{Am} = 1,\ a_B = a_{Bm} = 2.5,$ and $a_C = \dfrac{F-60}{30}$

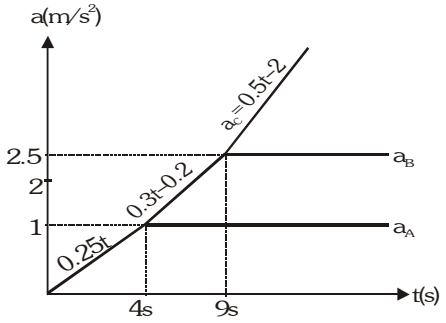
(c) If $F = 15t$

- $F \le 60$ $\quad t \le 4$ s $\quad a_A = a_B = a_C = \dfrac{F}{60} = \dfrac{t}{4} = 0.25t$
- $60 < F \le 135$ $\quad 4 < t \le 9$ $\quad a_A = 1\ m/s^2$

  $a_B = a_C = \dfrac{F-10}{50} = 0.3t - 0.2$
- $135 < F$ $\quad 9 < t$ $\quad a_A = 1\ m/s^2$

  $a_B = 2.5\ m/s^2$

  $a_C = \dfrac{F-60}{30} = 0.5t - 2$



Inertial and Non-inertial Reference Frames

A body is observed in motion, when it changes its position or orientation relative to another body or another set of bodies. A frame of reference consists of a set of coordinate axes fixed with the body relative to which the motion is observed and a clock. The coordinate axes are required to measure position of the moving body and the clock is required to measure time.

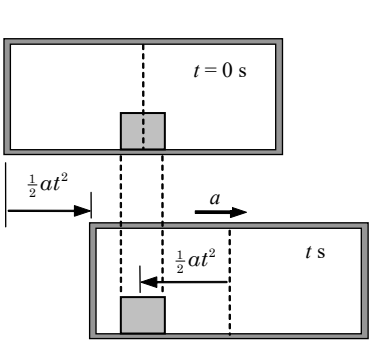
All the kinematical variables position, velocity, and acceleration are measured relative to a reference frame; therefore depend on the state of motion of the reference frame and we say that motion is essentially a relative concept. When the reference frame and a body both are stationary or move identically, the body is observed stationary relative to the reference frame. It is only when the reference frame and the body move in different manner, the body is observed moving relative the reference frame.

Now think about the whole universe where the planets, stars, galaxies and other celestial bodies all are in motion relative to each other. If any one of them can be assumed in state of rest, we can attach a reference frame to it and define motion of all other bodies relative to it. Such a body, which we assume in state of rest with respect to all other bodies in the universe, is known in absolute rest and the reference frame attached to it as most preferred reference frame. Unfortunately, the very notion of the reference frame and the idea motion as a relative concept, make it impossible to find a body anywhere in the universe at absolute rest. Therefore, the idea of absolute rest and a preferred reference frame become essentially meaningless. Now we can have only two categories of reference frames. In one category, we can have reference frames that move with uniform velocities and in the other category; we can have reference frames that are in accelerated motion.

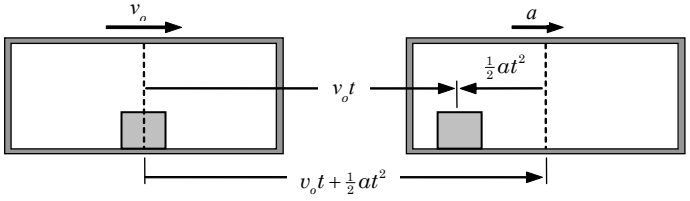
To understand the above ideas let us think an experiment. Consider a closed container on a goods train either at rest or moving with constant velocity v_o on a level track. The floor of the container is smooth and a block is placed in the center of the container. We observe the situation relative to two reference frames, one fixed with

Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

the ground and other fixed with the container. Relative to the ground frame both the container and the block are at rest or move together with the same velocity and relative to the container frame the block is at rest as shown in the figure.



Container accelerates from rest



Uniformly moving container accelerates

Now let the driver of the train accelerates the container at uniform rate a. If the train were initially at rest, relative to the ground, the block remains at rest and the container moves forward. Relative to the container the block moves backwards with the same magnitude of acceleration as with the container moves forward. If the train were initially moving uniformly, relative to the ground the block continues to move with the same original velocity and train accelerates and becomes ahead in space. Relative to the container the block appears moving backward with acceleration that is equal in magnitude to the acceleration of the container.

Now consider a man sitting on a fixed chair in the container. He is always stationary relative to the container. If he does not look outs side, in no way he can know whether the container is at rest or moving uniformly. However, he can definitely say whether the container accelerates or not, by observing motion of the block relative to the container. Because net forces acting on the block are still zero, therefore observed acceleration of the block can only be due to acceleration of the container as per Newton's laws of motion.

Now we can conclude that there can be only two kinds of reference frames either non-accelerated or accelerated. The reference frames that are non-accelerated i.e. at rest or moving with uniform velocities are known as inertial reference frames and those in accelerated motion as non-inertial reference frames.

Inertial Reference Frames and Newton's laws of motion

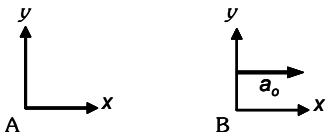
In Newton's laws of motion, force is conceived as two-body interaction that can be the only agent producing acceleration in a body. As far as we observe motion of a body from an inertial frame, any acceleration observed in a body can only be due to some forces acting on the body. All the three laws are in perfect agreement with the observed facts and we say that all the laws holds true in inertial reference frames.

Non-Inertial Reference Frames and Newton's laws of motion

A body if at rest or in uniform velocity motion relative to some inertial frame net forces acting on it must be zero. Now if motion of the same body is observed relative to a non-inertial frame, it will be observed moving with acceleration that is equal in magnitude and opposite in direction to the acceleration of the non-inertial frame. This observed acceleration of the body is purely a kinematical effect. But to explain this observed acceleration relative to the non-inertial frame according to Newton's laws of motion, we have assume a force must be acting on the body. This force has to be taken equal to product of mass of the body and opposite of acceleration vector of the non-inertial frame. Since this force is purely an assumption and not a result of interaction of the body with any other body, it is a fictitious force. This fictitious force is known as pseudo force or inertial force.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Until now, we have learnt the idea that how we can apply Newton's laws of motion in non-inertial frame to a body in equilibrium in inertial frame. Now it is turn to discuss how we can apply Newton's laws of motion to analyze motion in non-inertial frame of a body, which is in accelerated motion relative to an inertial frame.



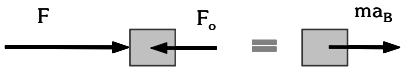
Reference frames *A* is an inertial frame and *B* is a non inertial frame.

Consider a net physical force $\vec{F}$ in positive x-direction applied on the box. Here by the term physical force, we refer forces produced by two body interactions. Relative to inertial frame A, the box is observed to have an acceleration $\vec{a} = \vec{F}/m$ defined by the second law of motion and a force equal in magnitude and opposite in direction to $\vec{F}$ can be assigned to the body exerting $\vec{F}$ on the box constituting Newton's third law pair. All the three laws of motion hold equally well in inertial frame A.



A net physical force imparts acceleration to the box in inertial frame *A*.

Relative to non-inertial frame B, the box is observed moving in x-direction with acceleration $\vec{a}_B = \vec{a} - \vec{a}_o$, which can satisfy Newton's second law, only if the fictitious force $\vec{F}_o = -m\vec{a}_o$ is assumed acting together with the net physical force $\vec{F}$ as shown in the figure. Now we can write modified equation of Newton's second law in non-inertial frame.



A net physical force and pseudo force imparts acceleration to the box in non-inertial frame B.

$$\vec{F} + \vec{F}_o = m\vec{a}_B \rightarrow F - F_o = ma_B$$

From the above discussion, we can conclude that in non-inertial frames Newton's second law is made applicable by introducing pseudo force in addition to physical forces. The pseudo force equals to the product of mass of the concerned body and the acceleration of the frame of reference in a direction opposite to the acceleration of the frame of reference.

Practical inertial frame of reference

The definition of an inertial frame of reference is based on the concept of a free body in uniform velocity motion or absolute rest. It is impossible to locate a body anywhere in the universe, where forces from all other bodies exactly balance themselves and lead to a situation of uniform velocity motion according to the first law. Thus, we cannot find anywhere in the universe a body, to which we attach a frame of reference and say that it is a perfect inertial frame of reference. It is the degree of accuracy, required in analyzing a particular physical situation that decides which body in the universe is to be selected a preferred inertial frame of reference.

The earth and other planets of the sun are rotating about their own axis and revolving around the sun and the sun is moving too. In fact, all celestial bodies in the universe the sun, its planets, other stars, our galaxy the Milky Way, and other galaxies are in accelerated motion whose nature is not known exactly. Therefore, none of them can be used as a perfect inertial frame of reference. However, when the acceleration of any one of the above-mentioned celestial bodies becomes negligible as compared to the accelerations involved in a physical situation, a frame of reference attached with the corresponding celestial body may be approximated as an inertial frame of reference and the physical situation under consideration may be analyzed with acceptable degree of accuracy.

The centripetal acceleration due to rotation of the earth at any point on its surface varies from zero at the poles to a maximum value of approximately 0.034 m/s^2 at the equator. The physical phenomena, which we usually observe describe motion of a body on or near the earth surface such as motion of transport vehicles, short-

Nodes\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

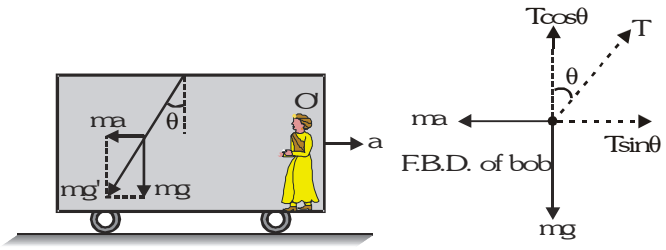
range missiles, an oscillating pendulum etc. In these phenomena, the acceleration due to rotation of the earth may be neglected and a frame of reference attached at any point on the earth surface may be considered as a preferred inertial frame of reference. If the moving body is at considerable distance from the earth as in the case of satellites, long-range missiles etc., the effect of rotation of the earth become significant. For these situations or like ones we can attach the frame of reference at the earth's center and consider it as an inertial frame of reference. In astronomical field and space exploration programs, we require very high accuracy. Therefore, a frame of reference attached to distant stars is used as an inertial frame of reference. These stars are situated at such a vast distance from the earth that they appear as a motionless point source of light thus closely approach to the condition of absolute rest.

Example

A pendulum of mass m is suspended from the ceiling of a train moving with an acceleration 'a' as shown in figure. Find the angle θ in equilibrium position.

Solution

Non–inertial frame of reference (Train)



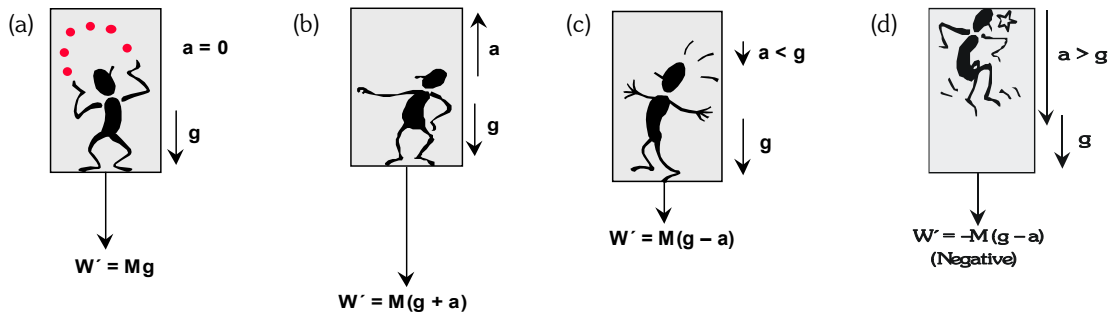
F.B.D. of bob w.r.t. train. (physical forces + pseudo force) : with respect to train, bob is in equilibrium

$\therefore \Sigma F_y = 0 \Rightarrow T\cos\theta = mg$ and $\Sigma F_x = 0 \Rightarrow T\sin\theta = ma \Rightarrow \tan\theta = \frac{a}{g} \Rightarrow \theta = \tan^{-1}\left(\frac{a}{g}\right)$

Example

The weight of a body is simply the force exerted by earth on the body. If body is on an accelerated platform, the body experiences fictitious force, so the weight of the body appears changed and this new weight is called apparent weight. Let a man of weight W = Mg be standing in a lift.

We consider the following cases :



Case

(a) If the lift moving with constant velocity v upwards or downwards. In this case there is no accelerated motion hence no pseudo force experienced by observer inside the lift. So apparent weight W′ = Mg Actual weight.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

(b) If the lift is accelerated upward with constant acceleration a. Then net forces acting on the man are (i) weight W = Mg downward (ii) fictitious force F_0=Ma downward.

So apparent weight W′ = W + F_0 = Mg + Ma = M(g + a)

(c) If the lift is accelerated downward with acceleration a < g . Then fictitious force F_0 = Ma acts upward while weight of man W = Mg always acts downward. So apparent weight W′=W+F_0 = Mg – Ma = M(g–a)

Special Case :

If a = g then W′ = 0 (condition of weightlessness).Thus, in a freely falling lift the man will experience weightlessness.

(d) If lift accelerates downward with acceleration a > g . Then as in Case c . Apparent weight W′ =M(g–a) is negative, i.e., the man will be accelerated upward and will stay at the ceiling of the lift.

Example

A spring weighing machine inside a stationary lift reads 50 kg when a man stands on it. What would happen to the scale reading if the lift is moving upward with (i) constant velocity, and (ii) constant acceleration?

Solution

(i) In the case of constant velocity of lift, there is no fictitious force; therefore the apparent weight = actual weight. Hence the reading of machine is 50 kgwt.

(ii) In this case the acceleration is upward, the fictitious force ma acts downward, therefore apparent weight is more than actual weight i.e. W′= m (g + a).

Hence scale shows a reading = m (g + a) = $\frac{mg\left(1+\frac{a}{g}\right)}{g}=\left(50+\frac{50a}{g}\right)$ kgwt.

Example

Two objects of equal mass rest on the opposite pans of an arm balance. Does the scale remain balanced when it is accelerated up or down in a lift?

Solution

Yes, since both masses experience equal fictitious forces in magnitude as well as direction.

Example

A passenger on a large ship sailing in a quiet sea hangs a ball from the ceiling of her cabin by means of a long thread. Whenever the ship accelerates, she notes that the pendulum ball lags behind the point of suspension and so the pendulum no longer hangs vertically. How large is the ship's acceleration when the pendulum stands at an angle of 5 to the vertical?

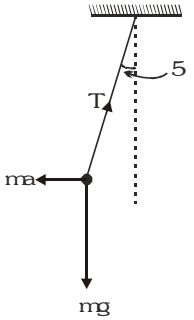
Solution

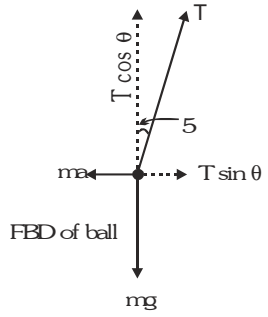
The ball is accelerated by the force T sin5 .

Therefore T sin5 = ma

Vertical component ΣF = 0, so T cos5 = mg

By solving a = g tan 5 = 0.0875 g = 0.86 m/s

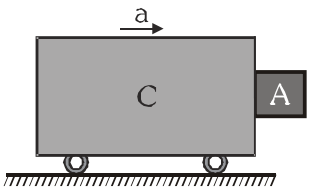




Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

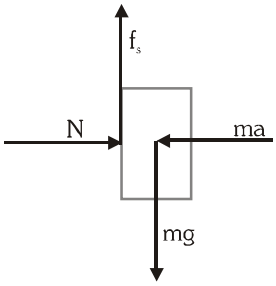
Example

Consider the figure shown here of a moving cart C. If the coefficient of friction between the block A and the cart is μ, then calculate the minimum acceleration a of the cart C so that the block A does not fall.



Solution

The forces acting on the block A (in block A's frame (i.e. non inertial frame) are :

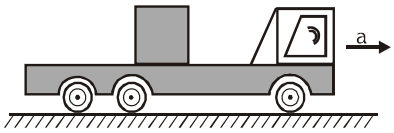


For A to be at rest in block A's frame i.e. no fall,

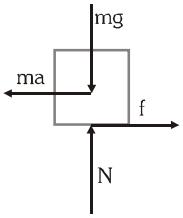
We require $W = f_s \Rightarrow mg = \mu(ma)$ Thus $a = \frac{g}{\mu}$

Example

A block of mass 1kg lies on a horizontal surface in a truck, the coefficient of static friction between the block and the surface is 0.6, What is the force of friction on the block. If the acceleration of the truck is 5 m/s^2.



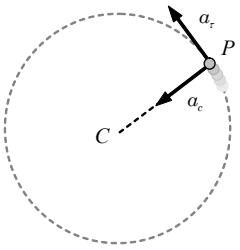
Solution



Fictitious force on the block F = ma = 1 5 = 5N
While the limiting friction force
F = $\mu_s N = \mu_S mg$ = 0.6 1 9.8 = 5.88 N
As applied force F lesser than limiting friction force. The block will remain at rest in the truck and the force of friction will be equal to 5N and in the direction of acceleration of the truck.

Dynamics of Circular Motion

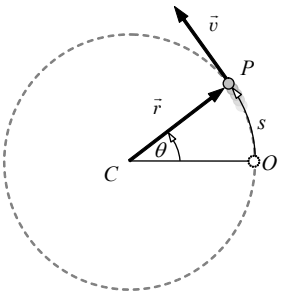
Velocity vector points always tangent to the path and continuously change its direction, as a particle moves on a circular path even with constant speed and give rise to normal component of acceleration, which always points toward the center of the circular path. This component of acceleration is known as centripetal (center seeking) acceleration and denoted by a_c. Moreover, if speed also changes the particle will have an additional acceleration component along the tangent to the path. This component of acceleration is known as tangential acceleration and denoted by a_{τ}.



The centripetal and tangential accelerations

The centripetal acceleration accounts only for continuous change in the direction of motion whereas the tangential acceleration accounts only for change in speed.

Consider a particle moving on circular path of radius r. It passes point O with velocity v_o at the instant t = 0 and point P with velocity v at the instant t traveling distance s along the path and angular displacement θ as shown in the figure. Kinematics of this circular motion is described in terms of linear variables as well as angular variables.



Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

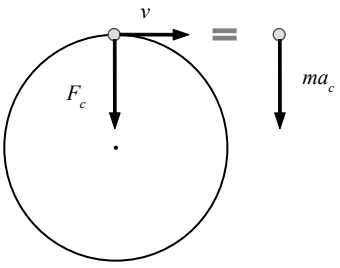
Kinematics of Circular Motion

Angular Variables	Linear Variables
Angular displacement θ	Distance traveled s
Angular velocity $\omega = \frac{d\theta}{dt}$	Speed $v = \frac{ds}{dt}$
Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$	Tangential acceleration $a_\tau = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$
	Centripetal acceleration $a_C = \frac{v^2}{r}$

Relations between angular and linear variable in circular motion

Distance traveled	$s = \theta r$
Speed	$v = \omega r$
Tangential acceleration	$a_\tau = ar$
Centripetal acceleration	$a_C = \frac{v^2}{r} = \omega^2 r = \omega v$

Application of Newton's law in Circular Motion

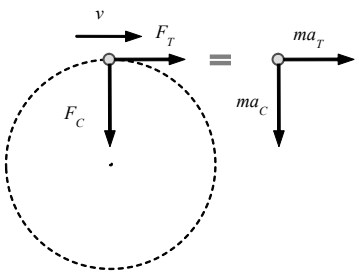


Consider a particle of mass m moving with uniform speed v in a circle of radius r as shown in figure. It necessarily posses a centripetal acceleration and hence there must be a net force $\left(\sum\vec{F} = \vec{F}_C\right)$ acting always towards the center according to the second law. This net force F_C acting towards the center is known as centripetal force.

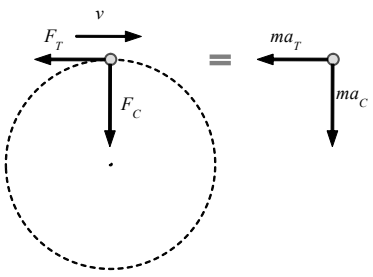
$$\sum\vec{F} = m\vec{a} \rightarrow F_C = ma_C$$

When a particle is whirled with the help of a string in a horizontal circle, the required centripetal force is the tension in the string. The gravitational attraction between a satellite and the earth, between moon and the earth, between the sun and its planets, and the electrostatic attraction between the nucleus and electrons are the centripetal forces and provide the necessary centripetal acceleration.

Now consider a particle moving on circular path with varying speed. The net acceleration has two components, the tangential acceleration and the centripetal acceleration. Therefore, the net force must also have two components, one component in tangential direction to provide the tangential acceleration and the other component towards the center to provide the centripetal acceleration. The former one is known as tangential force and the latter one as centripetal force.



Particle moving with increasing speed.



Particle moving with decreasing speed.

When the particle moves with increasing speed the tangential force acts in the direction of motion and when the particle moves with decreasing speed the tangential force acts in direction opposite to direction of motion.

To write equations according to the second law, we consider the tangential and the radial directions as two

Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

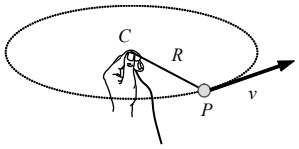
mutually perpendicular axes. The components along the tangential and the radial directions are designated by subscripts T and C.

$$F_C = ma_C$$

$$F_T = ma_T$$

Example

In free space, a man whirls a small stone P of mass m with the help of a light string in a circle of radius R as shown in the figure. Establish the relation between the speed of the stone and the tension developed in the string. Also, find the force applied on the string by the man.



Solution.

The system is in free space therefore no force other than the tension acts on the stone to provide necessary centripetal force. The tension does not have any component in tangential direction therefore tangential component of acceleration is zero. In the adjoining figure it is shown that how tension (T) in the string produces necessary centripetal force.

Applying Newton's second law of motion to the stone, we have $F_C = ma_C \rightarrow T = ma_C = \frac{mv^2}{R}$

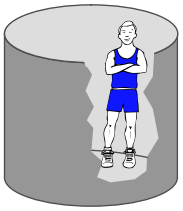


The end where man holds the string is stationary and tension applied by the string to this end is T towards the stone, therefore the man must apply a force equal to the tension in magnitude but in a direction away from the stone.

$$T = \frac{mv^2}{R}$$

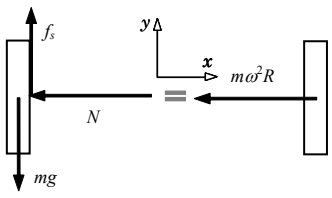
Example

A boy stands on a horizontal platform inside a cylindrical container of radius R resting his back on the inner surface of the container. The container can be rotated about the vertical axis of symmetry. The coefficient of static friction between his back and the inner surface of the container is μ_s. The angular speed of the container is gradually increased. Find the minimum angular speed at which if the platform below his feet removed, the boy should not fall.



Solution.

As the container rotates at angular speed ω the boy moves in a circular path of radius R with a speed $v = \omega R$. Since the angular speed is increased gradually the angular acceleration can be ignored and hence the tangential acceleration of the boy too. Thus, the boy has a centripetal acceleration of $\omega^2 R$, provided by the normal reaction N applied by the wall of the container. The weight of the boy is balanced by the force of static friction. All these force are shown in the adjoining figure where the boy is shown schematically by a rectangular box of mass m.



$$\sum F_x = ma_x \rightarrow N = m\omega^2 R \qquad \text{...(i)}$$

$$\sum F_y = ma_y \rightarrow f_s = mg \qquad \text{...(ii)}$$

Since force of static friction cannot be greater than the limiting friction $\mu_S N$,

we have $f_s \le \mu_s N$ ...(iii)

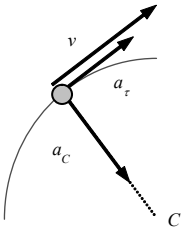
From the above equations, the minimum angular speed is $\omega_{min} = \sqrt{\frac{g}{\mu_s R}}$

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

A motorcyclist wishes to travel in circle of radius R on horizontal ground and increases speed at constant rate a. The coefficient of static and kinetic frictions between the wheels and the ground are μ_s and μ_k. What maximum speed can he achieve without slipping?

Solution.



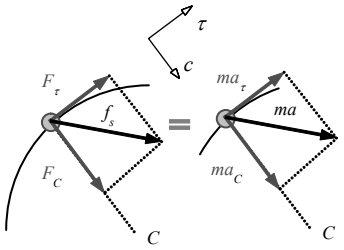
The motorcyclist and the motorcycle always move together hence they can be assumed to behave as a single rigid body of mass equal to that of the motorcyclist and the motorcycle. Let the mass of this body is m. The external forces acting on it are its weight (mg), the normal reaction N on wheels from ground, and the force of static friction f_s. The body has no acceleration in vertical direction therefore; the normal reaction N balances the weight (mg).

$$N = mg \qquad \ldots(i)$$

The frictional force cannot exceed the limiting friction.

$$f_{sm} \le \mu_s N \qquad \ldots(ii)$$

During its motion on circular path, the only external force in horizontal plane is the force of static friction, which is responsible to provide the body necessary centripetal and tangential acceleration. These conditions are shown in the adjoining figure where forces in vertical direction are not shown.



$$F_\tau = ma_\tau \qquad \ldots(iii)$$

$$F_C = ma_C = \frac{mv^2}{r} \qquad \ldots(iv)$$

The above two forces are components of the frictional force in tangential and normal directions. Therefore, we have

$$f_s = \sqrt{F_\tau^2 + F_C^2} = m\sqrt{a_\tau^2 + a_C^2} \qquad \ldots(v)$$

The centripetal acceleration increases with increase in speed and the tangential acceleration remains constant. Therefore, their resultant increases with speed. At maximum speed the frictional force achieves its maximum value (limiting friction f_{sm}), therefore from eq. (i), (ii), (iii), (iv), and (v), we have

$$v = \left[r^2\left\{(\mu_s g)^2 - a^2\right\}\right]^{\frac{1}{4}}$$

Circular Turning of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways:

• By friction only. • By banking of roads only. • By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both.

• **By Friction only**

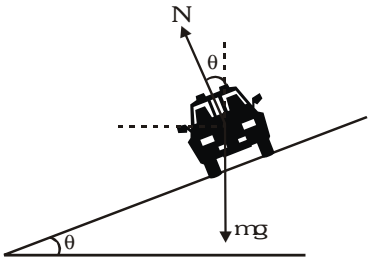
Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards centre.

Thus, $f = \frac{mv^2}{r}$ $\because$ $f_{max} = \mu N = \mu mg$

Therefore, for a safe turn without sliding $\frac{mv^2}{r} \le f_{max} \Rightarrow \frac{mv^2}{r} \le \mu mg \Rightarrow v \le \sqrt{\mu rg}$

Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

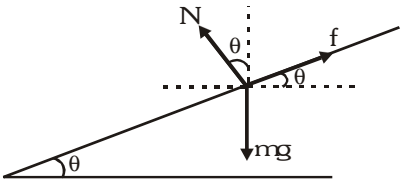
- **By Banking of Roads only**



Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

$N\sin\theta = \frac{mv^2}{r}$ and $N\cos\theta = mg \Rightarrow \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$

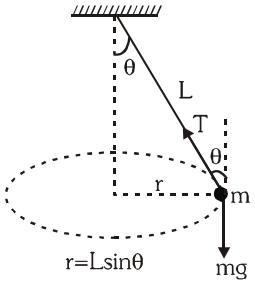
- **Friction and Banking of Road both**



If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction. The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force, i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_{max} = \mu N$). So, the magnitude of normal reaction N and direction plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre.

Conical Pendulum

If a small particle of mass m tied to a string is whirled in a horizontal circle, as shown in figure. The arrangement is called the 'conical pendulum'. In case of conical pendulum the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force. Thus,,



$T\sin\theta = \frac{mv^2}{r}$ and $T\cos\theta = mg \Rightarrow v = \sqrt{rg\tan\theta}$

$\therefore$ Angular speed $\omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}}$

So, the time period of pendulum is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

Example

Find the maximum speed at which a car can turn round a curve of 30 m radius on a level road if the coefficient of friction between the tyres and the road is 0.4 [acceleration due to gravity = 10 m/s^2]

Solution

Here centripetal force is provided by friction so

$\frac{mv^2}{r} \le \mu mg \Rightarrow v_{max} = \sqrt{\mu rg} = \sqrt{120} \approx 11\ ms^{-1}$

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example

For traffic moving at 60 km/hr, if the radius of the curve is 0.1 km, what is the correct angle of banking of the road ? ($g = 10\ m/s^2$)

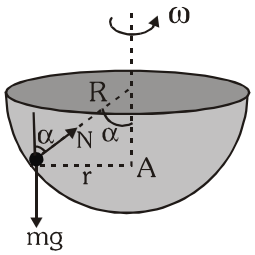
Solution

In case of banking $\tan\theta = \frac{v^2}{rg}$ Here v= 60 km/hr = 60 $\frac{5}{18}$ ms^{-1} = $\frac{50}{3} ms^{-1}$ r = 0.1 km = 100 m

So $\tan\theta = \frac{50/3 \times 50/3}{100 \times 10} = \frac{5}{18} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{18}\right)$

Example

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.



Solution

$N\cos\alpha = mg$ and $N\sin\alpha = mr\omega^2$ but $r = R\sin\alpha$

$\Rightarrow N\sin\alpha = mR\sin\alpha\omega^2 \Rightarrow N = mR\omega^2$

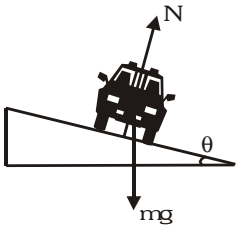
$\Rightarrow (mR\omega^2)\cos\alpha = mg \qquad \Rightarrow \omega = \sqrt{\frac{g}{R\cos\alpha}}$

Example

A car is moving along a banked road laid out as a circle of radius r.

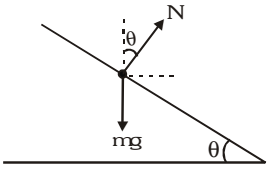
(a) What should be the banking angle θ so that the car travelling at speed v needs no frictional force from the tyres to negotiate the turn?

(b) The coefficients of friction between tyres and road are μ_s = 0.9 and μ_k = 0.8. At what maximum speed can a car enter the curve without sliding towards the top edge of the banked turn?



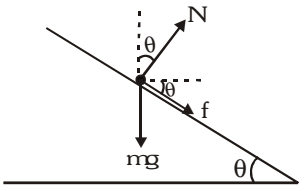
Solution

(a) $N\sin\theta = \frac{mv^2}{r}$ and $N\cos\theta = mg \Rightarrow \tan\theta = \frac{v^2}{rg}$



Note : In above case friction does not play any role in negotiating the turn.

(b) If the driver moves faster than the speed mentioned above, a friction force must act parallel to the road, inward towards centre of the turn.



$\Rightarrow F\cos\theta + N\sin\theta = \frac{mv^2}{r}$ and $N\cos\theta = mg + f\sin\theta$

For maximum speed of $f = \mu N$

$\Rightarrow N(\mu\cos\theta + \sin\theta) = \frac{mv^2}{r}$ and $N(\cos\theta - \mu\sin\theta) = mg$

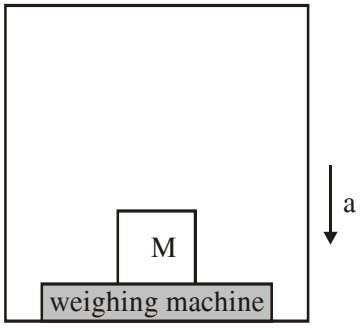
$\Rightarrow \frac{v^2}{rg} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} \Rightarrow v = \sqrt{\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)rg}$

Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

SOME WORKED OUT EXAMPLES

Example #1

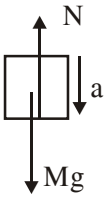
With what acceleration 'a' shown the elevator descends so that the block of mass M exerts a force of $\frac{Mg}{10}$ on the weighing machine? [g= acceleration due to gravity]



(A) 0.3 g (B) 0.1 g (C) 0.9 g (D) 0.6 g

Solution **Ans. (C)**

FBD of block :



Mg – N = Ma; Now according to question N = $\frac{Mg}{10}$ so a = $\frac{Mg-\frac{Mg}{10}}{M} = 0.9\ g$

Example #2

An astronaut accidentally gets separated out of his small spaceship accelerating in inter-stellar space at a constant acceleration of 10 m/s^2. What is the acceleration of the astronaut at the instant he is outside the spaceship?

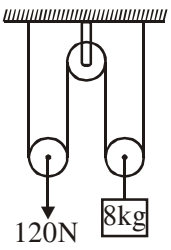
(A) 10 m/s^2 (B) 9.8 m/s^2 (C) $\approx$ 0 m/s^2 (D) could be anything

Solution **Ans. (C)**

When the astronaut is outside the spaceship, the net external force (except negligible gravitational force due to spaceship) is zero as he is isolated from all interactions.

Example #3

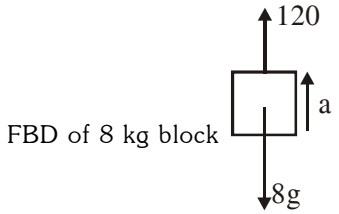
If the string is pulled down with a force of 120 N as shown in the figure, then the acceleration of 8 kg block would be



(A) 10 m/s^2 (B) 5 m/s^2 (C) 0 m/s^2 (D) 4 m/s^2

Node6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

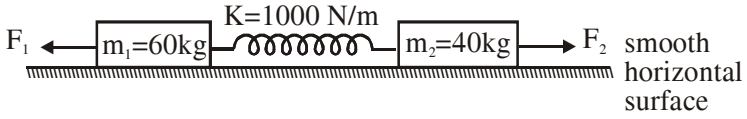
Solution **Ans. (B)**



FBD of 8 kg block $a = \frac{120-80}{8} = 5\ m/s^2$

Example #4

In the shown situation, which of the following is/are possible?



(A) Spring force = 52 N, if F_1 = 40 N and F_2 = 60 N

(B) Spring force = 52 N, if F_1 = 60 N and F_2 = 40 N

(C) Spring force = 0, if $F_1 = F_2$ = 100 N

(D) Spring force ≠ 0, if F_1 = 0.2 N and F_2 = 0.3 N

Solution **Ans. (A,B,D)**

If $F_1 \neq F_2$, then system will move with acceleration so spring force ≠ 0

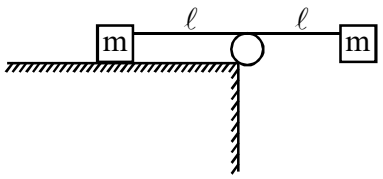
If F_1 = 40 N & F_2 = 60 N then $a = \frac{F_2 - F_1}{m_1 + m_2} = \frac{20}{100} = \frac{1}{5}\ m/s^2$

and spring force = $F_1 + m_1 a = 40 + \frac{1}{5}(60) = 52$ N

If F_1 = 60 N & F_2 = 40 N then spring force = 52 N

Example #5

As shown in figure, the left block rests on a table at distance ℓ from the edge while the right block is kept at the same level so that thread is unstretched and does not sag and then released. What will happen first ?



(A) Left block reach the edge of the table

(B) Right block hit the table

(C) Both (A) & (B) happens at the same time

(D) Can't say anything

Solution **Ans. (A)**

Net force in horizontal direction is more for left block so it will reach the edge of the table first.

Example #6

The force exerted by the floor of an elevator on the foot of a person standing there is less than the weight of the person if the elevator is

(A) going up and slowing down

(B) going up and speeding up

(C) going down and slowing down

(D) going down and speeding up

Solution **Ans. (A,D)**

If N < mg then N = m (g–a) $\Rightarrow$ elevator is going down with acceleration or elevator is going up with retardation.

Node6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example #7

If a body is placed on a rough inclined plane, the nature of forces acting on the body is(are)

(A) gravitational (B) electromagnetic (C) nuclear (D) weak

Solution **Ans. (A,B)**

When a body is placed on a rough inclined plane, it is acted upon by a reactional force due to plane [electromagnetic in nature], a frictional force due to roughness of plane [electromagnetic in nature] and a gravitational force (due to its weight).

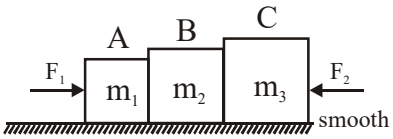
Example #8

For shown situation let

N_1 = Normal reaction between A & B

N_2 = Normal reaction between B & C

Which of the following statement (s) is/are correct ?



(A) If $F_1 > F_2$ then $N_1 \neq N_2$ and $F_2 < \sqrt{N_1N_2} < F_1$ (B) If $F_1 < F_2$ then $N_2 > N_1$

(C) If $F_1 = F_2$ then $N_1 = N_2$ (D) If $F_1 = F_2$ then $N_1 \neq N_2$

Solution **Ans. (A, B, C)**

If $F_1 > F_2$, the system moves towards right so $N_1 < F_1$, $N_2 < N_1$ & $F_2 < N_2$

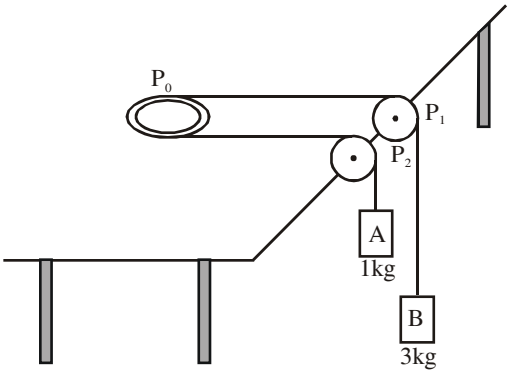
$F_2 < N_1$ or $N_2 < F_1$

If $F_1 < F_2$, the system moves towards left so $N_1 < N_2$

If $F_1 = F_2$, the system does not move.

Example #9 to 11

A smooth pulley P_0 of mass 2 kg is lying on a smooth table. A light string passes round the pulley and has masses 1 kg and 3kg attached to its ends. The two portions of the string being perpendicular to the edge of the table so that the masses hang vertically. Pulleys P_1 and P_2 are of negligible mass. [$g=10m/s^2$]



9. Tension in string is

(A) 12 N (B) 6 N (C) 24 N (D) 18 N

10. Acceleration of pulley P_0 is

(A) 2 m/s^2 (B) 4 m/s^2 (C) 3 m/s^2 (D) 6 m/s^2

11. Acceleration of block A is

(A) 6 m/s^2 (B) 4 m/s^2 (C) 3 m/s^2 (D) 8 m/s^2

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution

9. Ans. (B)

Let acceleration of $P_0 = a_0$; acceleration of A = a_1, acceleration of B = a_2

By constraint relations $a_0 = \frac{a_1 + a_2}{2}$ ...(i)

Now for pulley P_0 : $2T = 2a_0 \Rightarrow T = a_0$...(ii)

For block A : $1\,g - T = 1(a_1) \Rightarrow 10 - T = a_1$...(iii)

For block B : $3g - T = 3\,(a_2) \Rightarrow 30 - T = 3a_2$...(iv)

By putting the values of a_0, a_1 & a_2 in equation (i)

$$T = \frac{(10 - T) + \left(10 - \frac{T}{3}\right)}{2} \Rightarrow T = 6N$$

10. Ans. (D)

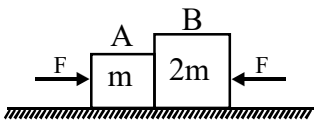
Acceleration of pulley; $a_0 = T = 6\ m/s^2$

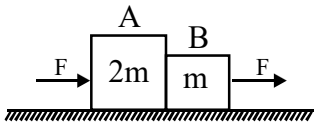
11. Ans. (B)

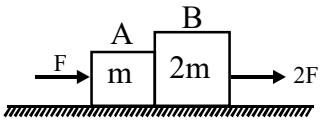
Acceleration of block A; $a_1 = 10 - T = 10 - 6 = 4\ m/s^2$

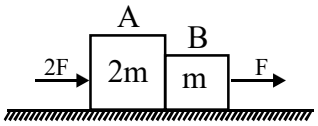
Example 12

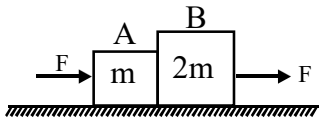
Five situations are given in the figure (All surfaces are smooth)

I → 

II → 

III → 

IV → 

V → 

Column–I		Column–II	
(A)	Accelerations of A & B are same	(P)	I
(B)	Accelerations of A & B are different	(Q)	II
(C)	Normal reaction between A & B is zero	(R)	III
(D)	Normal reaction between A & B is non zero	(S)	IV
		(T)	V

Node-6\E:\Data\2014 Vkas\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Solution **Ans. (A) → (P,R,S,T) ; (B) → (Q) ; (C) → (Q,R,S) ; (D) → (P, T)**

I : $a_A = a_B = 0$ & $N \neq 0$

II : $a_A = \frac{F}{2m}$, $a_B = \frac{F}{m}$ & $N = 0$

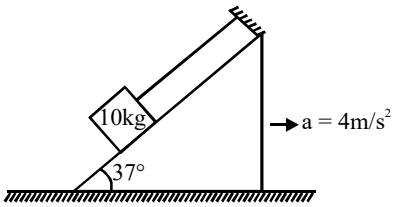
III : $a_A = \frac{F}{m}$, $a_B = \frac{2F}{2m} = \frac{F}{m}$ & $N = 0$

IV : $a_A = \frac{2F}{2m} = \frac{F}{m}$, $a_B = \frac{F}{m}$ & $N = 0$

V : $a_A = a_B = \frac{2F}{3m}$ & $N \neq 0$

Example #13 to 15

A body of mass 10 kg is placed on a smooth inclined plane as shown in figure. The inclined plane is moved with a horizontal acceleration a.



13. The normal reaction between block and inclined plane is :-

(A) 92 N (B) 44 N (C) 56 N (D) Can't be determined

14. The tension in thread is :-

(A) 92 N (B) 44 N (C) 56 N (D) Can't be determined

15. At what acceleration 'a' will the body lose contact with the inclined plane ?

(A) 10 m/s² (B) 13.33 m/s² (C) 3.33 m/s² (D) 6.66 m/s²

Solution

13. Ans. (C)

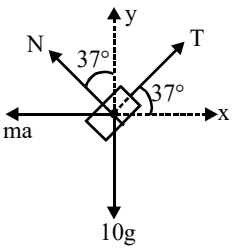
FBD of block w.r.t. inclined plane

$\sum F_x = 0$ $\Rightarrow T\cos 37 - N\sin 37 - ma = 0$

$\Rightarrow 4T - 3N = 200$ .....(i)

$\sum F_y = 0$ $\Rightarrow N\cos 37 + T\sin 37 - 10g = 0$

$\Rightarrow 4N + 3T = 500$ .....(ii)



By solving equation (i) & (ii) , N = 56 newton

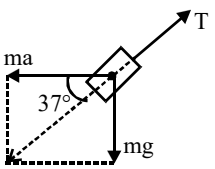
14. Ans. (A)

From above equation, T = 92 newton

15. Ans. (B)

To lose contact, N = 0

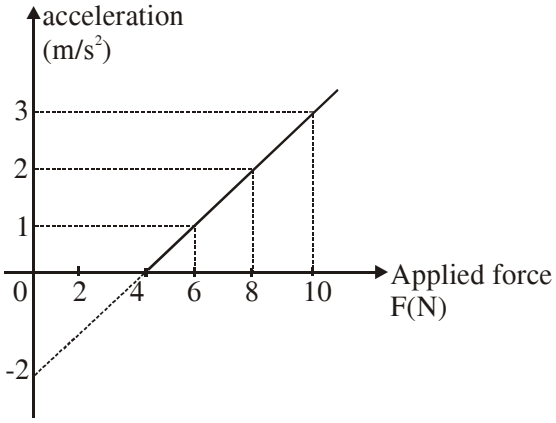
$\tan 37 = \frac{mg}{ma}$ $\Rightarrow \frac{3}{4} = \frac{10}{9}$ $\Rightarrow a = \frac{40}{3} = 13.33 m/s^2$



Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example #16

A block of unknown mass is at rest on a rough horizontal surface. A force F is applied to the block. The graph in the figure shows the acceleration of the block w.r.t. the applied force.



The mass of the block and coefficient of friction are ($g = 10\ m/s^2$)

(A) 2 kg, 0.1 (B) 2 kg, 0.2 (C) 1 kg, 0.1 (D) can't be determined

Solution **Ans. (B)**

Acceleration of block, $a = \frac{F - \mu mg}{m} \Rightarrow a = \left(\frac{1}{m}\right)F - \mu g$

From graph ; slope = $\frac{1}{m} = \frac{1}{2} \Rightarrow m = 2kg$ and y-intercept; $-\mu g = -2 \Rightarrow \mu = 0.2$

Example #17

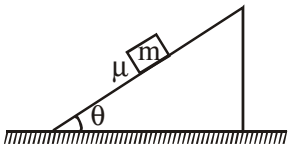
A body is placed on an inclined plane. The coefficient of friction between the body and the plane is μ. The plane is gradually tilted up. If θ is the inclination of the plane, then frictional force on the body is

(A) constant upto $\theta = \tan^1(\mu)$ and decreases after that

(B) increases upto $\theta = \tan^1(\mu)$ and decreases after that

(C) decreases upto $\theta = \tan^1(\mu)$ and constant after that

(D) constant throughout



Solution **Ans. (B)**

Friction force $F = mg\sin\theta$ if $\theta \le \tan^{-1}(\mu)$ and $F = \mu mg\cos\theta$ if $\theta \ge \tan^{-1}(\mu)$

which increases upto $\theta = \tan^{-1}(\mu)$ and then decreases.

Example #18

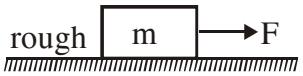
A block is placed on a rough horizontal surface and a horizontal force F is applied to it as shown in figure. The force F is increased from zero in small steps. The graph between applied force and frictional force f is plotted by taking equal scales on axes. The graph is

(A) a straight line of slope 45

(B) a straight line parallel to F-axis

(C) a straight line parallel to f-axis

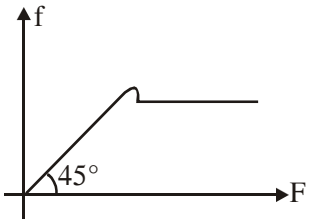
(D) a straight line of slope 45 for small F and a straight line parallel to F-axis for large F.



Node-6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

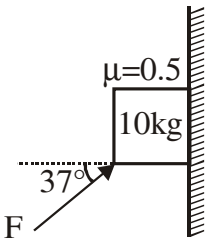
ALLEN Materials Provided By - Material Point Available on Learnaf.com

Solution Ans. (D)



Example #19

A force F pushes a block weighing 10 kg against a vertical wall as shown in the figure. The coefficient of friction between the block and wall is 0.5. The minimum value of F to start the upward motion of block is [g = 10 m/s^2]

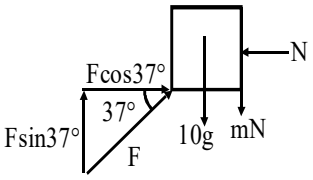


(A) 100 N (B) 500 N (C) $\frac{500}{3}$ N (D) can't be determined

Solution Ans. (B)

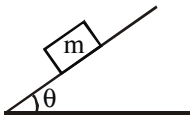
N = Fcos37 = $\frac{4}{5}$F and to start upward motion Fsin37 = 10 g + μN

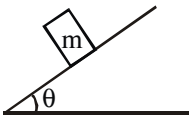
$\frac{3}{5}F = 100 + (0.5)\left(\frac{4}{5}F\right) \Rightarrow F = 500$ N



Example #20

A block is first placed on its long side and then on its short side on the same inclined plane (see figure). The block slides down in situation II but remains at rest in situation I. A possible explanation is

Situation–I 

Situation–II 

(A) The normal contact force is less in situation-II.

(B) The frictional force is less in situation-II because the contact area is less.

(C) The shorter side is smoother.

(D) In situation-I, frictional force is more.

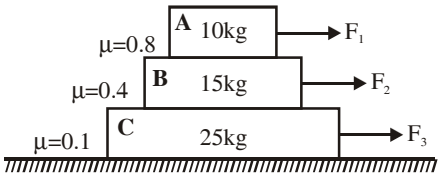
Solution Ans. (C, D)

This is due to less frictional force or low friction coefficient.

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Example #21

For the situation shown in the figure below, match the entries of column I with the entries of column II. [g = 10 m/s^2]



Column I	Column II
(A) If F_1 = 160 N, F_2 = 0 & F_3 = 0, then	(P) There is no relative motion between A & B
(B) If F_1 = 140 N, F_2=0 & F_3 = 0 N, then	(Q) There is no relative motion between B & C
(C) If F_1 = F_2 = 0 & F_3 = 45 N, then	(R) There is a relative motion between A & B
(D) If F_1 = 0, F_2 = 160 N and F_3 = 0, then	(S) There is a relative motion between B & C
	(T) There is a relative motion between C & ground

Solution **Ans. (A) → (Q,R,T) ; (B) → (Q,R,T) ; (C) → (P,Q) ; (D) → (S,T)**

Maximum frictional force between A and B = (0.8) (10) (10) = 80 N

Maximum frictional force between B and C = (0.4) (25) (10) = 100 N

Maximum frictional force between C and ground = (0.1) (50) (10) = 50 N

Therefore there is no relative motion between B and C if F_2 = 0 & F_3 =0

To start motion $(F_1 + F_2 + F_3)_{min}$ = 50 N

To start slipping between B and C (if F_3=0), $(F_1 + F_2)_{min}$ = 150 N

If F_2 = F_3 =0 then to start slipping beween A & B $F_{1\,min}$ = 87.5 N

Node-6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\NUM\English\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 5
WORK, ENERGY & POWER With CIRCULAR MOTION - Theory

Index

WORK, ENERGY & POWER AND CIRCULAR MOTION

Serial No.	CONTENTS	Page No.
1.	Work	1
2.	Work and Energy	6
3.	Work energy theorem	7
4.	Conservative & Non-conservative forces	11
5.	Potential Energy	12
6.	Conservation of Mechanical Energy	15
7.	Power	18
8.	Circular Motion in Vertical Plane	19
9.	**Exercise-1** Single Choice Objective Questions	31
10.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	34
11.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	40
12.	**Exercise-4A** Conceptual Subjective Exercise	49
13.	**Exercise-4B** Brain Storming Exercise	52
14.	**Exercise-5-A** Previous years questions	55
15.	**Exercise-5-B** Previous years questions	57

ALLEN Materials Provided By - Material Point Available on Learnaf.com

JEE(Advanced) Syllabus :

Kinetic and potential energy; Work and power; Conservation of mechanical energy.

JEE(Main) Syllabus :

Work done by a constant force and a variable force; kinetic and potential energies, work- energy theorem, power.

Potential energy of a spring, conservation of mechanical energy, conservative and non-conservative forces.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

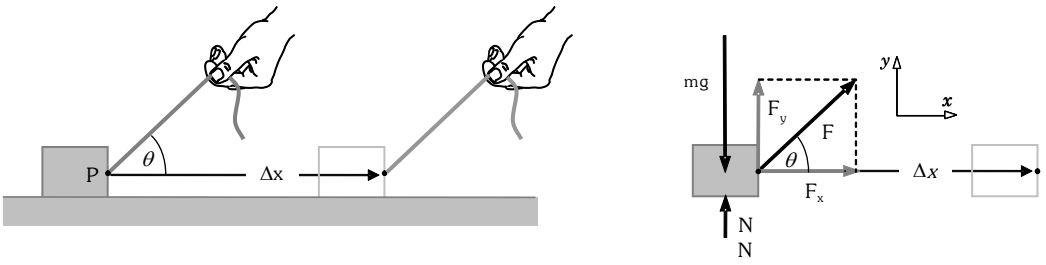
Methods of Work and Energy

Work of a Force

In everyday life by the word 'work', we refer to a vast category of jobs. This meaning is not precise enough to be used as a physical quantity. It was the practical need of scientists and engineers of the late 18^{th} century at the start of Industrial Revolution that made necessary to define work quantitatively as a physical quantity. Physical concept of work involves a force and displacement produced.

Work of a constant Force on a body in rectilinear motion

To understand concept of work, consider a block being pulled with the help of a string on frictionless horizontal ground. Let pull $\vec{F}$ of the string on the box is constant in magnitude as well as direction. The vertical component F_y of $\vec{F}$, the weight (mg) and the normal reaction N all act on the box in vertical direction but none of them can move it unless F_y becomes greater than the weight (mg). Consider that F_y is smaller than the weight of the box. Under this condition, the box moves along the plane only due to the horizontal component F_x of the force $\vec{F}$. The weight mg, the normal reaction N from the ground and vertical component F_y all are perpendicular to the displacement therefore have no contribution in its displacement. Therefore, work is done on the box only by the horizontal component F_x of the force $\vec{F}$.



Here we must take care of one more point that is the box, which is a rigid body and undergoes translation motion therefore, displacement of every particle of the body including that on which the force is applied are equal. The particle of a body on which force acts is known as point of application of the force.

Now we observer that block is displaced & its speed is increased. And work W of the force $\vec{F}$ on the block is proportional to the product of its component in the direction of the displacement and the magnitude of the displacement Δx.

$$W \propto F_x \cdot \Delta x = F\cos\theta \cdot \Delta x$$

If we chose one unit of work as newton-meter, the constant of proportionality becomes unity and we have

$$W = F\cos\theta \cdot \Delta x = \vec{F} \cdot \Delta\vec{x}$$

The work W done by the force $\vec{F}$ is defined as scalar product of the force $\vec{F}$ and displacement $\Delta\vec{x}$ of point of application of the force.

Unit and Dimensions of Work of a Force

SI unit of work is "joule", named after famous scientist James Prescott Joule. It is abbreviated by letter J.

1 joule = 1 newton 1 meter

CGS unit of work is "erg". Its name is derived from the Greek ergon, meaning work.

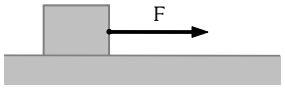
1 erg = 1 dyne 1 centimeter

Dimensions of work are ML^2T^{-2}

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory p6.5

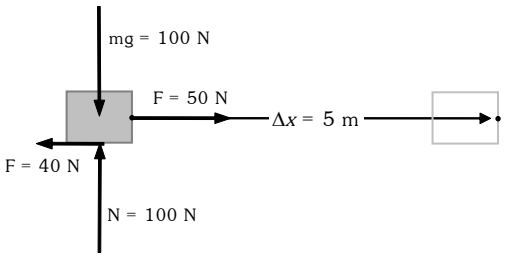
Example

A 10 kg block placed on a rough horizontal floor is being pulled by a constant force 50 N. Coefficient of kinetic friction between the block and the floor is 0.4. Find work done by each individual force acting on the block over displacement of 5 m.



Solution.

Forces acting on the block are its weight (mg = 100 N), normal reaction (N = 100 N) from the ground, force of kinetic friction (f = 40 N) and the applied force (F = 50 N) and displacement of the block are shown in the given figure.



All these force are constant force, therefore we use equation $W_{i\to f} = \vec{F}\cdot\Delta\vec{r}$.

Work done W_g by the gravity i.e. weight of the block $W_g = 0$ J $\left(\because m\vec{g} \perp \Delta\vec{x}\right)$

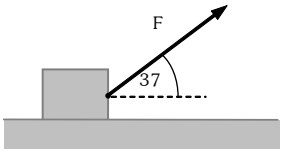
Work done W_N by the normal reaction $W_N = 0$ J $\left(\because \vec{N} \perp \Delta\vec{x}\right)$

Work done W_F by the applied force $W_F = 250$ J $\left(\because \vec{F} \square \Delta\vec{x}\right)$

Work done W_f by the force of kinetic friction $W_f = -200$ J $\left(\because \vec{f} \uparrow\downarrow \Delta\vec{x}\right)$

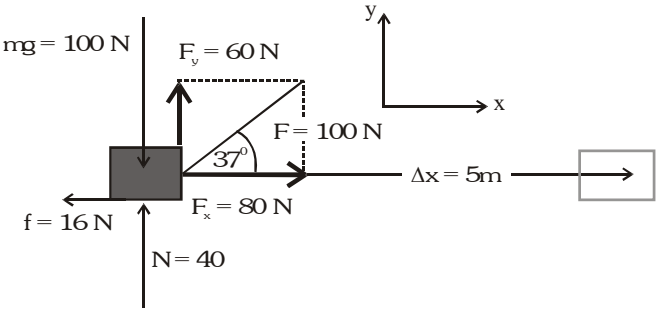
Example

A 10 kg block placed on a rough horizontal floor is being pulled by a constant force 100 N acting at angle 37 . Coefficient of kinetic friction between the block and the floor is 0.4. Find work done by each individual force acting on the block over displacement of 5 m.



Solution.

Forces acting on the block are its weight (mg = 100 N), normal reaction (N = 40 N) from the ground, force of kinetic friction (f = 16 N) and the applied force (F = 100 N) and displacement of the block are shown in the given figure.



All these force are constant force, therefore we use equation $W_{i\to f} = \vec{F}\cdot\Delta\vec{r}$.

Work done W_g by the gravity i.e. weight of the block $W_g = 0$ J $\left(\because m\vec{g} \perp \Delta\vec{x}\right)$

Work done W_N by the normal reaction $W_N = 0$ J $\left(\because \vec{N} \perp \Delta\vec{x}\right)$

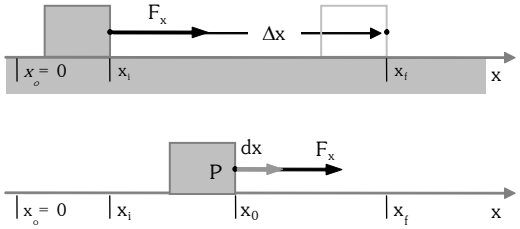
Work done W_F by the applied force $W_F = \vec{F}\cdot\Delta\vec{x} = F_x\Delta x = 400$ J

Work done W_f by the force of kinetic friction $W_f = -80$ J $\left(\because \vec{f} \uparrow\downarrow \Delta\vec{x}\right)$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Work of a variable Force on a body in rectilinear motion

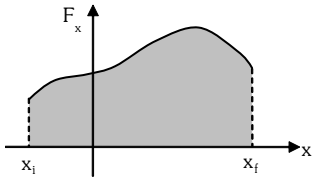
Usually a variable force does not vary appreciably during an infinitely small displacement of its point of application and therefore can be assumed constant in that infinitely small displacement.



Consider a box being pulled by a variable horizontal force F_x which is known as function of position x. We now calculate work done by this force in moving the box from position x_i to x_f. Over any infinitely small displacement dx the force does not vary appreciably and can be assumed constant. Therefore to calculate work done dW by the force F_x during infinitely small displacement dx is given by $dW = \vec{F}_x.d\vec{x} = F_x dx$. Integrating dW from x_i to x_f we obtain work done by the force in moving the box from position x_i to x_f.

$$W_{i\to f} = \int_{x_i}^{x_f} F_x.dx$$

The above equation also suggests that in rectilinear motion work done by a force equals to area under the force-position graph and the position axis.



In the given figure is shown how a force F_x varies with position coordinate x. Work done by this force in moving its point of application from position x_i to x_f equals to area of the shaded portion.

Example

A force which varies with position coordinate x according to equation F_x = (4x+2) N. Here x is in meters. Calculate work done by this force in carrying a particle from position x_i = 1 m to x_f = 2 m.

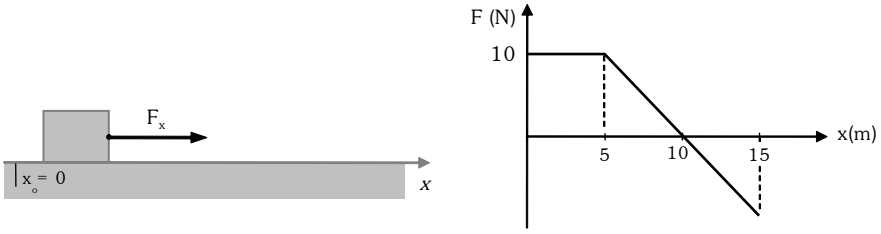
Solution.

Using the equation $W_{i\to f} = \int_{x_i}^{x_f} F_x.dx$, we have $W_{i\to f} = \int_1^2 (4x+2).dx = 8\ J$

The above problem can also be solved by using graph

Example

A horizontal force F is used to pull a box placed on floor. Variation in the force with position coordinate x measured along the floor is shown in the graph.



node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

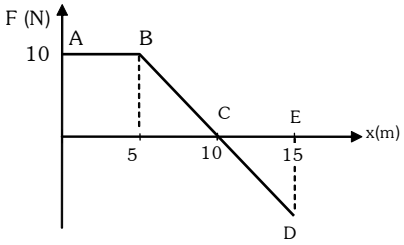
(a) Calculate work done by the force in moving the box from x = 0 m to x = 10 m.

(b) Calculate work done by the force in moving the box from x = 10 m to x = 15 m.

(c) Calculate work done by the force in moving the box from x = 0 m to x = 15 m.

Solution.

In rectilinear motion work done by a force equals to area under the force-position graph and the position axis



(a) $W_{0\to10}$ = Area of trapazium OABC = 75 J

(b) $W_{10\to15}$ = –Area of triangle CDE = –25 J

(c) $W_{0\to15}$ = Area of trapazium OABC – Area of triangle CDE = 50 J

Example

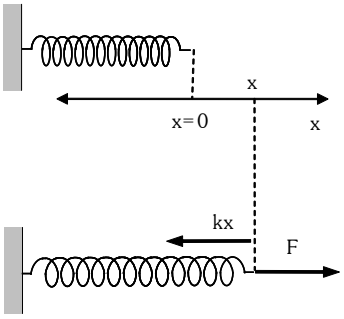
A coiled spring with one end fixed has a realaxed length l_0 and a spring constant k. What amount of work must be done to stretch the spring by an amount s?

Solution.

In order to stretch the free end of the spring to a point x, some agency must exert a force F, which must everywhere be equal to spring force.

$$F = kx$$

The applied force and the spring force are shown in the adjoining figure.

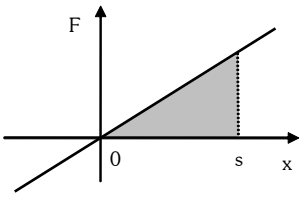


The work done W_F by the applied force in moving the free end of the spring from x = 0 to x = s be

$$W_F = \int_0^s \vec{F}\cdot d\vec{x} = \tfrac{1}{2}ks^2$$ **Ans.**

Use of Graph.

The variation in F with extension x in the spring is linear therefore area under the force extension graph can easily be calculated. This area equals to the work done by the applied force. The graph showing variation in F with x is shown in the adjoining figure.

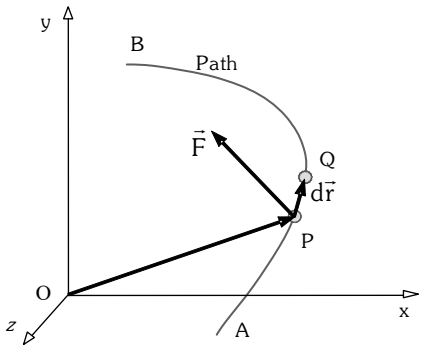


Variation in F with extension x.

$$W_F = \int_0^s \vec{F}\cdot d\vec{x} = \text{ Area of the shaded portion } = \tfrac{1}{2}ks^2$$

Work of a variable Force on a body in curvilinear translation motion

Till now we have learnt how to calculate work of a force in rectilinear motion. We can extend this idea to calculate work of a variable force on any curvilinear path. To understand this let us consider a particle moving from point A to B. There may be several forces acting on it but here we show only that force whose work we want to calculate. This force may be constant or variable. Let this force is denoted by $\vec{F}$. Consider an infinitely small path length PQ. Over this infinitely small path length, the force can be assumed constant. Work of this force $\vec{F}$ over this path length PQ is given by



$$dW = \vec{F}\cdot d\vec{r}$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

The whole path from A to B can be divided in several such infinitely small elements and work done by the force over the whole path from A to B is sum of work done over every such infinitely small element. This we can calculate by integration. Therefore, work done $W_{A\to B}$ by the force $\vec{F}$ is given by the following equation.

$$W_{A\to B} = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot d\vec{r}$$

Work of a Variable Force

For a generalization, let point A be the initial point and point B be the final point. Now we can express work $W_{i\to f}$ of a force $\vec{F}$ when its point of application moves from position vector $\vec{r}_i$ to $\vec{r}_f$ over a path by the following equation.

$$W_{i\to f} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r}$$

The integration involved in the above equation must be carried over the path followed. Such kind of integration is known as path integrals.

Work of a constant Force

In simple situations where force $\vec{F}$ is constant, the above equation reduces to a simple form.

$$W_{i\to f} = \vec{F}\cdot\left(\int_{\vec{r}_i}^{\vec{r}_f} d\vec{r}\right) = \vec{F}\cdot\left(\vec{r}_f - \vec{r}_i\right) = \vec{F}\cdot\Delta\vec{r}$$

Example

Calculate work done by the force $\vec{F} = \left(3\tilde{i} - 2\tilde{j} + 4\tilde{k}\right)N$ in carrying a particle from point (–2 m, 1 m, 3 m) to (3 m, 6 m, –2 m).

Solution.

The force $\vec{F}$ is a constant force, therefore we can use equation $W_{i\to f} = \vec{F}\cdot\Delta\vec{r}$.

$W = \vec{F}.\Delta\vec{r} = (3\hat{i} - 2\hat{j} + 4\hat{k}).(5\hat{i} + 5\hat{j} - 5\hat{k}) = -15$ J

Example

A particle is shifted form point (0, 0, 1 m) to (1 m, 1 m, 2 m), under simultaneous action of several forces. Two of the forces are $\vec{F}_1 = \left(2i + 3\tilde{j} - \tilde{k}\right)N$ and $\vec{F}_2 = \left(\tilde{i} - 2\tilde{j} + 2\tilde{k}\right)N$. Find work done by these two forces.

Solution.

Work done by a constant force equals to dot product of the force and displacement vectors.

$$W = \vec{F}\cdot\Delta\vec{r} \rightarrow W = \left(\vec{F}_1 + \vec{F}_2\right)\cdot\Delta\vec{r}$$

Substituting given values, we have

$$W = \left(3\tilde{i} + \tilde{j} + \tilde{k}\right)\cdot\left(\tilde{i} + \tilde{j} + \tilde{k}\right) = 3 + 1 + 1 = 5\ J$$

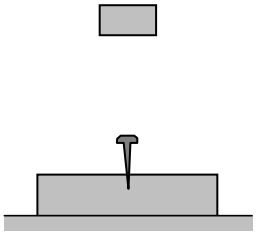
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Work of a force depends on frame of reference

A force does not depend on frame of reference and assumes same value in all frame of references, but displacement depends on frame of reference and may assume different values relative to different reference frames. Therefore, work of a force depends on choice of reference frame. For example, consider a man holding a suitcase stands in a lift that is moving up. In the reference frame attached with the lift, the man applies a force equal to weight of the bag but the displacement of the bag is zero, therefore work of this force on the bag is zero. However, in a reference frame attached with the ground the bag has displacement equal to that of the lift and the force applied by the man does a nonzero work.

Work and Energy

Suppose you have to push a heavy box on a rough horizontal floor. You apply a force on the box it moves and you do work. If you continue pushing, after some time you get tired and become unable to maintain your speed and eventually become unable to push the box further. You take rest and next day you can repeat the experiment and same thing happens. Why you get tired and eventually become unable to pull the box further? The explanation lies in fact that you have a capacity to do work, and when it is used up, you become unable to do work further. Next day you recollect this capacity and repeat the experiment. This capacity of doing work is known as energy. Here it comes from chemical reactions occurring with food in our body and is called chemical energy.



Consider another experiment in which we drop a block on a nail as shown in the figure. When set free, weight of the block accelerates it through the distance it falls and when it strikes the nail, its motion vanishes and what appear are the work that drives the nail, heat that increases temperature of the surrounding, and sound that causes air molecules to oscillate. If the block were placed on the nail and pressed hard, it would not have been so effective. Actually, the weight and the distance through which the hammer falls on the nail decide its effectiveness. We can explain these events by assuming that the block possesses energy due to its position at height against gravity.

This energy is known as gravitational potential energy. When the block falls, this potential energy is converted into another form that is energy due to motion. This energy is known as kinetic energy. Moreover, when the block strikes the nail this kinetic energy is converted into work driving the nail, increasing temperature and producing sound.

Potential, Kinetic and Mechanical Energy

If a material-body is moved against a force like gravitational, electrostatic, or spring, a work must be done. In addition, if the force continues to acts even after the displacement, the work done can be recovered in form of energy, if the body is set loose. This recoverable stored energy by virtue of position in a force field is defined as potential energy, a name given by William Rankine.

All material bodies have energy due to their motion. This energy is known as kinetic energy, a name given by Lord Kelvin.

These two forms of energies - the kinetic energy and the potential energy are directly connected with motion of the body and force acting on the body respectively. They are collectively known as mechanical energy.

Other forms of Energy

Thermal energy, sound energy, chemical energy, electrical energy and nuclear energy are examples of some other forms of energy. Actually, in very fundamental way every form of energy is either kinetic or potential in nature. Thermal energy which is contribution of kinetic energy of chaotic motion of molecules in a body and potential energy due to intermolecular forces within the body. Sound energy is contribution kinetic energy of oscillating molecules and potential energy due intermolecular forces within the medium in which sound propagates. Chemical energy is contribution of potential energy due inter-atomic forces. Electric energy is

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

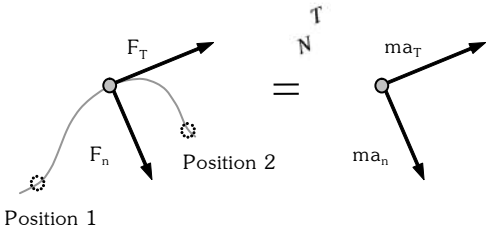
kinetic energy of moving charge carries in conductors. In addition, nuclear energy is contribution of electrostatic potential energy of nucleons.

In fact, every physical phenomenon involves in some way conversion of one form of energy into other. Whenever mechanical energy is converted into other forms or vice versa it always occurs through forces and displacements of their point of applications i.e. work. Therefore, we can say that work is measure of transfer of mechanical energy from one body to other. That is why the unit of energy is usually chosen equal to the unit of work.

Work-Kinetic Energy Theorem

Consider the situation described in the figure. The body shown is in translation motion on a curvilinear path with increasing speed. The net force acting on the body must have two components – the tangential component necessary to increase the speed and the normal component necessary to change the direction of motion. Applying Newton's laws of motion in an inertial frame, we have

$$\sum F_T = ma_T \text{ and } \sum F_N = ma_N$$



Let the body starts at position 1 with speed v_1 and reaches position 2 with speed v_2. If an infinitely small path increment is represented by vector $d\vec{s}$, the work done by the net force during the process is

$$W_{1\to2} = \int_1^2 \vec{F}\cdot d\vec{s} = \int_1^2 \left(\vec{F}_T + \vec{F}_N\right)\cdot d\vec{s} = \int_1^2 F_T ds$$

$$W_{1\to2} = \int_1^2 ma_T ds = \int_{v_1}^{v_2} mvdv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

The terms $\frac{1}{2}mv_1^2$ and $\frac{1}{2}mv_2^2$ represent the kinetic energies K_1 and K_2 of the particle at position-1 and 2 respectively. With this information the above equation reduces to $W_{1\to2} = K_2 - K_1$

The above equation expresses that the work done by all external forces acting on a body in carrying it from one position to another equals to the change in the kinetic energy of the body between these positions. This statement is known as the work kinetic energy theorem.

How to apply work kinetic energy theorem

The work kinetic energy theorem is deduced here for a single body moving relative to an inertial frame, therefore it is recommended at present to use it for a single body in inertial frame. To use work kinetic energy theorem the following steps should be followed.

- Identify the initial and final positions as position 1 and 2 and write expressions for kinetic energies, whether known or unknown.
- Draw the free body diagram of the body at any intermediate stage between positions 1 and 2. The forces shown will help in deciding their work. Calculate work by each force and add them to obtain work done $W_{1\to2}$ by all the forces.
- Use the work obtained in step 2 and kinetic energies in step 1 into $W_{1\to2} = K_2 - K_1$.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory pt.5

Example

A 5kg ball when falls through a height of 20 m acquires a speed of 10 m/s. Find the work done by air resistance.

Solution.

The ball starts falling from position 1, where its speed is zero; hence, kinetic energy is also zero.

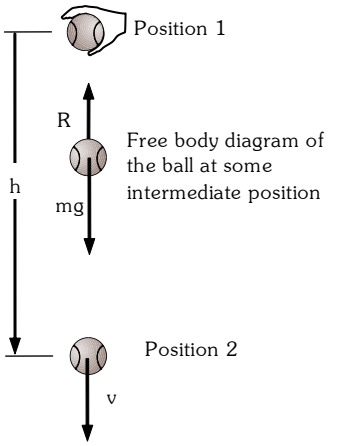
$$K_1 = 0 \text{ J} \qquad \text{...(i)}$$

During downwards motion of the ball constant gravitational force mg acts downwards and air resistance R of unknown magnitude acts upwards as shown in the free body diagram. The ball reaches position 20 m below the position-1 with a speed v = 10 m/s, so the kinetic energy of the ball at position 2 is

$$K_2 = \tfrac{1}{2}mv^2 = 250 \text{ J} \qquad \text{...(ii)}$$

The work done by gravity

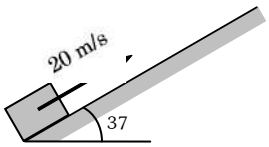
$$W_{g,1\to2} = mgh = 1000 \text{ J} \qquad \text{...(iii)}$$



Denoting the work done by the air resistance $W_{R,1\to2}$ and making use of eq. 1, 2, and 3 in work kinetic energy theorem, we have $W_{1\to2} = K_2 - K_1 \to W_{g,1\to2} + W_{R,1\to2} = K_2 - K_1 \Rightarrow W_{R,1\to2} = -750\text{J}$

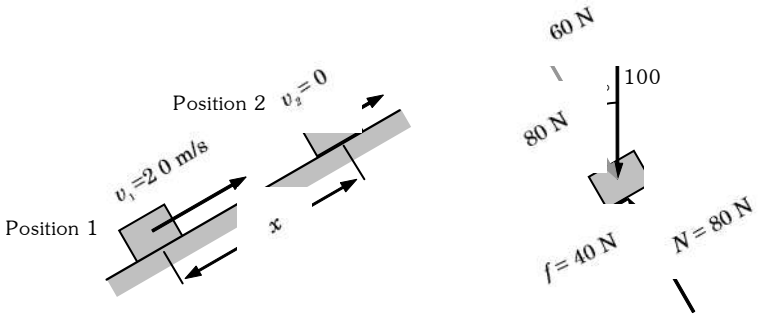
Example

A box of mass m = 10 kg is projected up an inclined plane from its foot with a speed of 20 m/s as shown in the figure. The coefficient of friction μ between the box and the plane is 0.5. Find the distance traveled by the box on the plane before it stops first time.



Solution.

The box starts from position 1 with speed v_1 = 20 m/s and stops at position 2.



Kinetic energy at position 1: $K_1 = \tfrac{1}{2}mv_1^2 = 2000 \text{ J}$

Kinetic energy at position 2: K_2 = 0 J

Work done by external forces as the box moves from position 1 to position 2:

$$W_{1\to2} = W_{g,1\to2} + W_{f,1\to2} = -60x - 40x = -100x \text{ J}$$

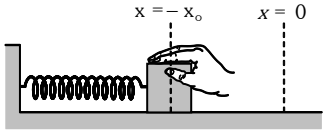
Applying work energy theorem for the motion of the box from position 1 to position 2, we have

$$W_{1\to2} = k_2 - k_1 \Rightarrow -100\,x = 0 - 2000 \Rightarrow x = 20 \text{ m}$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

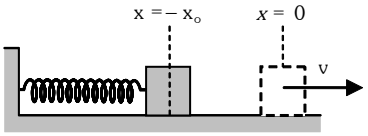
Example

A box of mass m is attached to one end of a coiled spring of force constant k. The other end of the spring is fixed and the box can slide on a rough horizontal surface, where the coefficient of friction is μ. The box is held against the spring force compressing the spring by a distance x_o. The spring force in this position is more than force of limiting friction. Find the speed of the box when it passes the equilibrium position, when released.

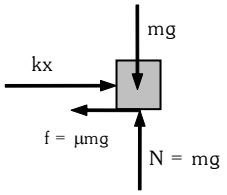


Solution.

Before the equilibrium position, when the box passes the position coordinate –x, forces acting on it are its weight mg, normal reaction N from the horizontal surface, the force of kinetic friction f, and spring force F = kx as shown in the free body diagram. Let the box passes the equilibrium position with a speed v_o.



Applying work energy theorem on the box when it moves from position 1 ($-x_o$) to position 2 (x = 0), we have

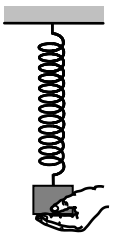


$$W_{1\to2} = K_2 - K_1 \Rightarrow W_{F,1\to2} + W_{f,1\to2} = K_2 - K_1$$

$$\tfrac{1}{2}kx_o^2 - \mu mgx_o = \tfrac{1}{2}mv_o^2 - 0 \Rightarrow v_o = \sqrt{\left(kx_o^2 - 2\mu mgx_o\right)}$$

Example

A block of mass m is suspended from a spring of force constant k. it is held to keep the spring in its relaxed length as shown in the figure.

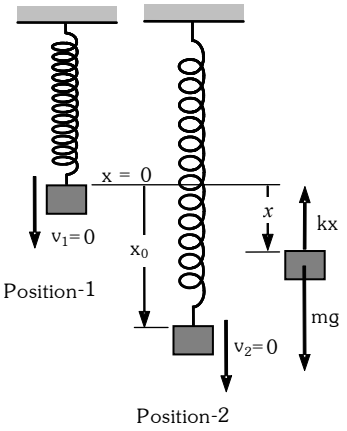


(a) The applied force is decreased gradually so that the block moves downward at negligible speed. How far below the initial position will the block stop?

(b) The applied force is removed suddenly. How far below the initial position, will the block come to an instantaneous rest?

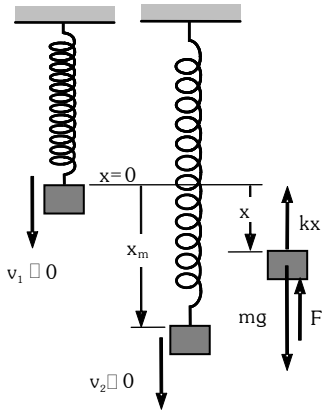
Solution.

(a) As the applied force (F) is decreased gradually, everywhere in its downward motion the block remains in the state of translational equilibrium and moves with negligible speed. Its weight (mg) is balanced by the upward spring force (kx) and the applied force. When the applied force becomes zero the spring force becomes equal to the weight and the block stops below a distance x_o from the initial position. The initial and final positions and free body diagram of the block at any intermediate position are shown in the adjoining figure. Applying the conditions of equilibrium, we have



$$x_o = \frac{mg}{k}$$

(b) In the previous situation the applied force was decreased gradually keeping the block everywhere in equilibrium. If the applied force is removed suddenly, the block will accelerate downwards. As the block moves, the increase in spring extension increases the upward force, due to which acceleration decreases until extension becomes x_o. At this extension, the block will acquire its maximum speed and it will move further downward. When extension becomes more than x_o spring force becomes more than the weight (mg) and the block decelerates and ultimately stops at a distance x_m below the initial position. The initial position-1, the final position-2, and the free body diagram of the block at some intermediate position when spring extension is x are shown in the adjoining figure.



node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Kinetic energy in position-1 is $K_1 = 0$

Kinetic energy in position-2 is $K_2 = 0$

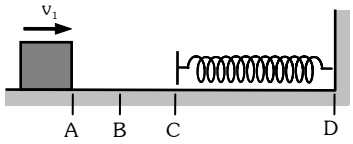
Work done $W_{1\to2}$ by gravity and the spring force is $W_{1\to2} = W_{g,1\to2} + W_{spring,1\to2} = mgx_m - \frac{1}{2}kx_m^2$

Using above values in the work energy theorem, we have $W_{1\to2} = K_2 - K_1 \to mgx_m - \frac{1}{2}kx_m^2 = 0$

$x_m = 2mg/k$

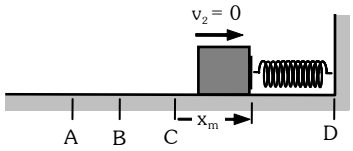
Example

A block of mass m = 0.5 kg slides from the point A on a horizontal track with an initial sped of v_1 = 3 m/s towards a weightless horizontal spring of length 1 m and force constant k = 2 N/m. The part AB of the track is frictionless and the part BD has the coefficients of static and kinetic friction as 0.22 and 0.2 respectively. If the distance AB and BC are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. g = 10 m/s^2.



Solution

Since portion AB of the track is smooth, the block reaches B with velocity v_1. Afterward force of kinetic friction starts opposing its motion. As the block passes the point C the spring force also starts opposing its motion in addition to the force of kinetic friction. The work done by these forces decrease the kinetic energy of the block and stop the block momentarily at a distance x_m after the point C.



Kinetic energy of the block at position-1 is $K_1 = \frac{1}{2}mv_1^2 = 2.25$ J.

Kinetic energy of the block at position-2 is $K_2 = \frac{1}{2}mv_2^2 = 0$ J.

Work $W_{f,1\to2}$ done by the frictional force before the block stops is $W_{f,1\to2} = \mu mg(BC + x_m) = 2.14 + x_m$

Work $W_{s,1\to2}$ done by the spring force before the block stops is $W_{s,1\to2} = \int_{x=0}^{x_m} kxdx = \frac{1}{2}kx_m^2 = x_m^2$

Using above information and the work energy principle, we have

$$W_{1\to2} = K_2 - K_1 \Rightarrow 2.14 + x_m + x_m^2 = 2.25 \Rightarrow x_m = 0.1 \text{ m.}$$

The motion of block after it stops momentarily at position-2 depends upon the condition whether the spring force is more than or less than the force of limiting friction. If the spring force in position-2 is more than the force of limiting friction the block will move back and if the spring force in position-2 is less than the force of static friction the block will not move back and stop permanently.

Spring force F_s at position-2 is $F_s = kx_m = 0.2$ N.

The force of limiting friction f_m is $f_m = \mu_s mg = 1.1$ N.

The force of limiting friction is more than the spring force therefore the block will stop at position-2 permanently.

The total distance traveled by the block = AB + BC + x_m =4.24 m.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Conservative and Non-conservative Forces

Gravitational, electrostatic, and restoring force of a spring are some of the natural forces with a property in common that work done by them in moving a particle from one point to another depends solely on the locations of the initial and final points and not on the path followed irrespective of pair of points selected. On the other hand, there are forces such as friction, whose work depends on path followed. Accordingly, forces are divided into two categories – one whose work is path independent and other whose work is path dependant. The forces of the former category are known as conservative forces and of the later one as non-conservative forces.

A force, whose finite non-zero work $W_{1\to2}$ expended in moving a particle from a position 1 to another position 2 is independent of the path followed, is defined as a conservative force.

Consider a particle moving from position 1 to position 2 along different paths A, B, and C under the action of a conservative force F as shown in figure. If work done by the force along path A, B, and C are $W_{1\to2,A}$, $W_{1\to2,B}$ and $W_{1\to2,C}$ respectively, we have

$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C}$$

If these works are positive, the work done $W_{2\to1,D}$ by the same force in moving the particle from position 2 to 1 by any other path say D will have the same magnitude but negative sign. Hence, we have

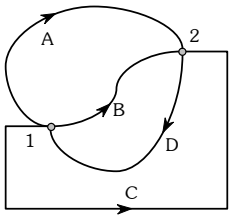
$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C} = -W_{2\to1,D}$$

$$W_{1\to2,A} + W_{2\to1,D} = W_{1\to2,B} + W_{2\to1,D} = W_{1\to2,C} + W_{2\to1,D} = 0$$

The above equations are true for any path between any pair of points-1 and 2.

Representing the existing conservative force by $\vec{F}$, an infinitely path increment by $d\vec{r}$ and integration over a closed path by $\oint(\ )\cdot d\vec{r}$, the above equation can be represented in an alternative form as $\oint\vec{F}\cdot d\vec{r} = 0$

The equation () shows that the total work done by a conservative force in moving a particle from position 1 to another position 2 and moving it back to position 1 i.e. around a closed path is zero. It is used as a fundamental property of a conservative force.

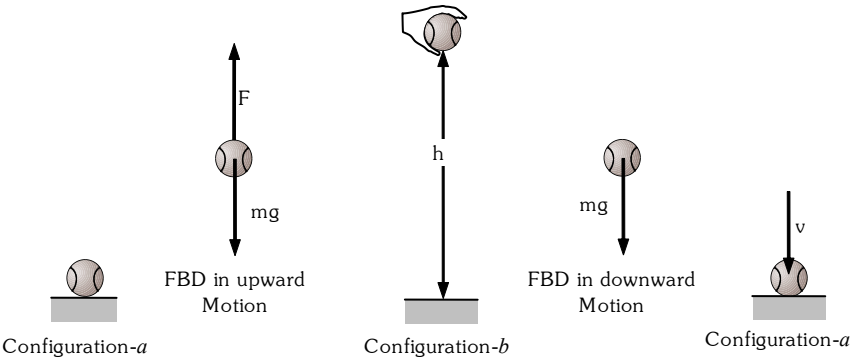


- A conservative force must be function only of position not of velocity or time.
- All uniform and constant forces are conservative forces. Here the term uniform means same magnitude and direction everywhere in the space and the term constant means same magnitude and direction at all instants of time.
- All central forces are conservative forces. A central force at any point acts always towards or away from a fixed point and its magnitude depends on the distance from the fixed point.
- All forces, whose magnitude or direction depends on the velocity, are non-conservative. Sliding friction, which acts in opposite direction to that of velocity, and viscous drag of fluid depends in magnitude of velocity and acts in opposite direction to that of velocity are non-conservative.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Potential Energy

Consider a ball of mass m placed on the ground and someone moves it at negligible speed through a height h above the ground as shown in figure. The ball remains in the state of equilibrium therefore the upward force F applied on it everywhere equals to the weight (mg) of the ball. The work $W_{a\to b}$ done by the applied force on the ball is $W_{a\to b} = \vec{F}\cdot\vec{h} = mgh$.



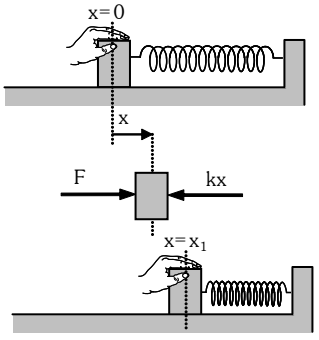
Now if the ball is dropped from the height h it starts moving downwards due to its weight and strikes the ground with speed v. The work $W_{b\to a}$ done by its weight during its downward motion imparts it a kinetic energy K_c, which is obtained by using work energy principle and the above equation as

$$W_{b\to a} = K_a - K_b \Rightarrow mgh = \tfrac{1}{2}mv^2 - 0 \Rightarrow K_a = \tfrac{1}{2}mv^2 = mgh$$

Instead of raising the ball to height h, if it were thrown upwards with a speed v it would have reached the height h and returned to the ground with the same speed. Now if we assume a new form of energy that depends on the separation between the ball and the ground, the above phenomena can be explained. This new form of energy is known as potential energy of the earth-ball system. When ball moves up, irrespective of the path or method how the ball has been moved, potential energy of the earth-ball system increases. This increase equals to work done by applied force F in moving the ball to height h or negative of work done by gravity. When the ball descends, potential energy of the earth ball system decreases; and is recovered as the kinetic energy of the ball when separation vanishes. During descend of the ball gravity does positive work, which equals to decrease in potential energy.

Potential energy of the earth ball system is due to gravitation force and therefore is call gravitation potential energy. Change in gravitational potential energy equals to negative of work done by gravitational force. It is denoted by ΔU.

In fact, when the ball is released both the ball and the earth move towards each other and acquire momenta of equal magnitude but the mass of the earth is infinitely large as compared to that of the ball, the earth acquires negligible kinetic energy. It is the ball, that acquires almost all the kinetic energy and therefore sometimes the potential energy is erroneously assigned with the ball and called the potential energy of the ball. Nevertheless, it must be kept in mind that the potential energy belongs to the entire system.



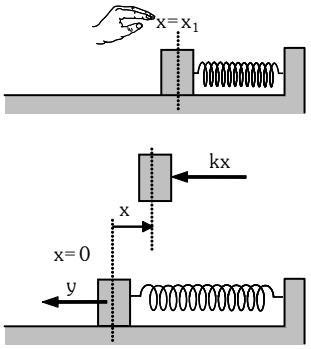
node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

As another instance, consider a block of mass m placed on a smooth horizontal plane and connected to one end of a spring of force constant k, whose other end is connected to a fixed support. Initially, when the spring is relaxed, no net force acts on the block and it is in equilibrium at position x = 0. If the block is pushed gradually against the spring force and moves at negligible speed without acceleration, at every position x, the applied force F balances the spring force kx. The work done $W_{0\to1}$ by this force in moving the block from position x =0 to x = x_1 is

$$W_{0\to1} = \int_{x=0}^{x=x_1} \vec{F}\cdot d\vec{x} = \tfrac{1}{2}kx_1^2$$

If the applied force is removed, the block moves back and reaches its initial position with a kinetic energy $K_{0,}$ which is obtained by applying work energy theorem together with the above equation.

$$W_{1\to0} = K_0 - K_1 \Rightarrow K_0 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kx_1^2$$



The above equation shows that the work done on the block by the applied force in moving it from x = 0 to x = x_1 is stored in the spring block system as increase in potential energy and when the block returns to its initial position x = 0 this stored potential energy decreases and is recovered as the kinetic energy of the block. The same result would have been obtained if the block were pulled elongating the spring and then released. The change in potential energy of the spring block system when the spring length is increased or decreased by x equals to negative of work done by the spring force.

In both the above cases forces involved were conservative. In fact, work done against all conservative forces is recoverable. With every conservative force, we can associate a potential energy, whose change equals to negative of work done by the conservative force. For an infinitely small change in configuration, change in potential energy dU equals to the negative of work done dW_C by conservative forces.

$$dW = dU = - dW_C$$

Since a force is the interaction between two bodies, on very fundamental level potential energy is defined for every pair of bodies interacting with conservative forces. The potential energy of a system consisting of a large number of bodies thus will be sum of potential energies of all possible pairs of bodies constituting the system. Because only change in potential energy has significance, we can chose potential energy of any configuration as reference value.

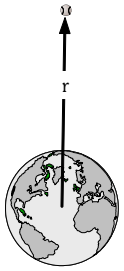
Gravitational potential energy for uniform gravitational force

Near the earth surface for heights small compared to the radius of the earth, the variation in the gravitational force between a body of mass m and the ground can be neglected. For such a system, change in gravitational potential energy in any vertically upward displacement h of mass m is given by $\Delta U=mgh$ and in vertical downward displacement h is given by $\Delta U = - mgh$.

node06\E_NODE65 (E)\Data\2014\Kota\IEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Gravitational potential energy for non-uniform gravitational force

When motion of a body of mass m involves distances from the earth surface large enough, the variation in the gravitational force between the body and the earth cannot be neglected. For such physical situations the configuration, when the body is at infinitely large distance from the earth center is taken as the reference configuration and potential energy of this configuration is arbitrarily assumed zero (U_∞=0).



The ball at a distance *r* from the center of the earth.

If the body is brought at negligible speed to a distance r from infinitely large distance from the earth center, the work done W_g by the gravitational force is given by the following equation.

$$W_g = \int_{\infty}^{r} \vec{F}_g \cdot d\vec{r} = \left[\frac{GMm}{r}\right]_{\infty}^{r}$$

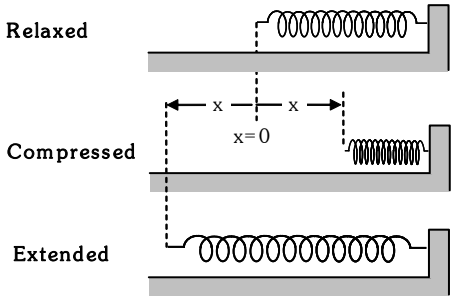
Negative of this work done equals to change in potential energy of the system. Denoting potential energies in configuration of separation r and ∞ by U_r and U_∞, we have

$$U_r - U_\infty = -W_g \rightarrow U_r = -\frac{GMm}{r}$$

Potential energy associated with spring force

The potential energy associated with a spring force of an ideal spring when compressed or elongated by a distance x from its natural length is defined by the following equation

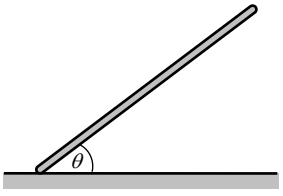
$$U = \tfrac{1}{2}kx^2$$

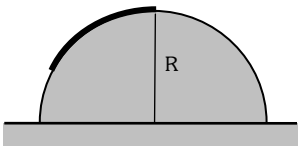


Example

Find the gravitational potential energies in the following physical situations. Assume the ground as the reference potential energy level.

(a) A thin rod of mass m and length L kept at angle θ with one of its end touching the ground.





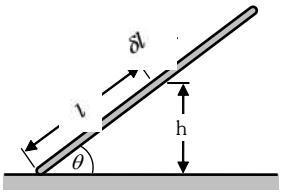
(b) A flexible rope of mass m and length L placed on a smooth hemisphere of radius R and one of the ends of the rope is fixed at the top of the hemisphere.

Solution

In both the above situations, mass is distributed over a range of position coordinates. In such situations calculate potential energy of an infinitely small portion of the body and integrate the expression obtained over the entire range of position coordinates covered by the body.

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

(a) Assume a small portion of length $\delta\ell$ of the rod at distance ℓ from the bottom end and height of the midpoint of this portion from the ground is h. Mass of this portion is δm. When $\delta\ell$ approaches to zero, the gravitational potential energy dU of the assumed portion becomes

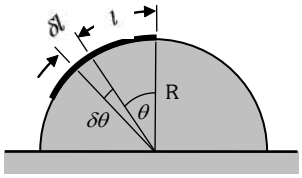


$$dU = \frac{m}{L} ghd\ell = \frac{m}{L} g\ell \sin\theta d\ell$$

The gravitational potential energy U of the rod is obtained by carrying integration of the above equation over the entire length of the rod.

$$U = \int_{\ell=0}^{L} \frac{m}{L} g\ell \sin\theta d\ell = \tfrac{1}{2} mgL \sin\theta$$

(b) The gravitational potential energy dU of a small portion of length $\delta\ell$ shown in the adjoining figure, when $\delta\ell$ approaches to zero is



$$dU = \frac{m}{L} gR^2 \sin\theta d\theta$$

The gravitational potential energy U of the rope is obtained by carrying integration of the above equation over the entire length of the rope.

$$U = \int_{\theta=0}^{\theta=\frac{L}{R}} \frac{m}{L} gR^2 \sin\theta d\theta = \frac{m}{L} gR^2 \left\{1 - \cos\left(\frac{L}{R}\right)\right\}$$

Conservation of mechanical energy

The total potential energy of the system and the total kinetic energy of all the constituent bodies together are known as the mechanical energy of the system. If E, K, and U respectively denote the total mechanical energy, total kinetic energy, and the total potential energy of a system in any configuration, we have

$$E = K + U$$

Consider a system on which no external force acts and all the internal forces are conservative. If we apply work-kinetic energy $\left(W_{1\to2} = K_2 - K_1\right)$ theorem, the work $W_{1\to2}$ will be the work done by internal conservative forces, negative of which equals change in potential energy. Rearranging the kinetic energy and potential energy terms, we have

$$E = K_1 + U_1 = K_2 + U_2$$

The above equation takes the following forms

$$E = K + U = \text{constant}$$

$$\Delta E = 0 \Rightarrow \Delta K + \Delta U = 0$$

Above equations, express the principle of conservation of mechanical energy.

If there is no net work done by any external force or any internal non-conservative force, the total mechanical energy of a system is conserved.

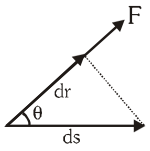
The principle of conservation of mechanical energy is developed from the work energy principle for systems where change in configuration takes place under internal conservative forces only. Therefore, in physical situations, where external forces or non-conservative internal forces are involved, the use of work energy principle should be preferred.

In systems, where external forces or internal nonconservative forces do work, the net work done by these forces becomes equal to change in the mechanical energy of the system.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory p6.5

Potential energy and the associated conservative force

We know how to find potential energy associated with a conservative force. Now we learn how to obtain the conservative force if potential energy function is known. Consider work done dW by a conservative force in moving a particle through an infinitely small path length $d\vec{s}$ as shown in the figure.



$$dU = -dW = -\vec{F}\cdot d\vec{s} = -Fds\cos\theta$$

From the above equation, the magnitude F of the conservative force can be expressed.

$$F = -\frac{dU}{ds\cos\theta} = -\frac{dU}{dr}$$

If we assume an infinitely small displacement $d\vec{r}$ in the direction of the force, magnitude of the force is given by the following equation.

$$F = -\frac{dU}{dr}$$

Here minus sign suggest that the force acts in the direction of decreasing potential energy. Therefore if we assume unit vector $\tilde{e}_r$ in the direction of $d\vec{r}$, force vector $\vec{F}$ is given by the following equation.

$$\vec{F} = -\frac{dU}{dr}\tilde{e}_r$$

Example

Force between the atoms of a diatomic molecule has its origin in the interactions between the electrons and the nuclei present in each atom. This force is conservative and associated potential energy U(r) is, to a good approximation, represented by the Lennard – Jones potential.

$$U(r) = U_o\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

Here r is the distance between the two atoms and U_o and a are positive constants. Develop expression for the associated force and find the equilibrium separation between the atoms.

Solution.

Using equation $F = -\frac{dU}{dr}$, we obtain the expression for the force

$$F = \frac{6U_0}{a}\left\{2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right\}$$

At equilibrium the force must be zero. Therefore the equilibrium separation r_o is

$$r_o = 2^{\frac{1}{6}}a$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

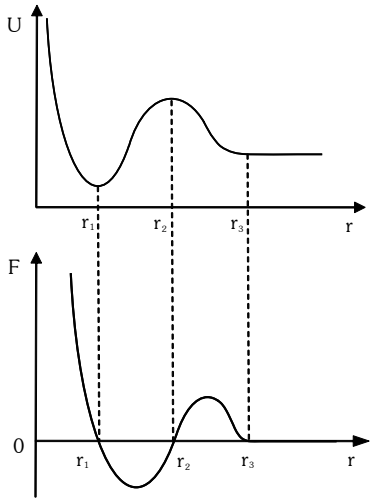
Potential energy and nature of equilibrium

The above equation suggests that on every location where the potential energy function assumes a minimum or a maximum value or in every region where the potential energy function assumes a constant value, the associated conservative force becomes zero and a body under the action of only this conservative force must be in the state of equilibrium. Different status of potential energy function in the state of equilibrium suggests us to define three different types of equilibriums – the stable, unstable and neutral equilibrium.

The state of stable and unstable equilibrium is associated with a point location, where the potential energy function assumes a minimum and maximum value respectively, and the neutral equilibrium is associated with region of space, where the potential energy function assumes a constant value.

For the sake of simplicity, consider a one dimensional potential energy function U of a central force F. Here r is the radial coordinate of a particle. The central force F experienced by the particle equals to the negative of the slope of the potential energy function. Variation in the force with r is also shown in the figure.

At locations $r = r_1$, $r = r_2$, and in the region $r \geq r_3$, where potential energy function assumes a minimum, a maximum, and a constant value respectively, the force becomes zero and the particle is in the state of equilibrium.



Force is negative of the slope of the potential energy function.

Stable Equilibrium.

At $r = r_1$ the potential energy function is a minima and the force on either side acts towards the point $r = r_1$. If the particle is displaced on either side and released, the force tries to restore it at $r = r_1$. At this location the particle is in the state of stable equilibrium. The dip in the potential energy curve at the location of stable equilibrium is known as potential well. A particle when disturbed from the state of stable within the potential well starts oscillations about the location of stable equilibrium. At the locations of stable equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{ ; and } \frac{\partial F}{\partial r} < 0 \text{ ; and } \frac{\partial^2 U}{\partial r^2} > 0$$

Unstable Equilibrium.

At $r = r_2$ the potential energy function is a maxima, the force acts away from the point $r = r_2$. If the particle is displaces slightly on either side, it will not return to the location $r = r_2$. At this location, the particle is in the state of unstable equilibrium. At the locations of unstable equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{ therefore } \frac{\partial F}{\partial r} > 0 \text{ ; and } \frac{\partial^2 U}{\partial r^2} < 0$$

Neutral Equilibrium.

In the region $r \geq r_3$, the potential energy function is constant and the force is zero everywhere. In this region, the particle is in the state of neutral equilibrium. At the locations of neutral equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{ therefore } \frac{\partial F}{\partial r} = 0 \text{ and } \frac{\partial^2 U}{\partial r^2} = 0$$

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

POWER

When we purchase a car or jeep we are interested in the horsepower of its engine. We know that usually an engine with large horsepower is most effective in accelerating the automobile.

In many cases it is useful to know not just the total amount of work being done, but how fast the work is done. We define power as the rate at which work is being done.

$$\text{Average Power} = \frac{\text{Work done}}{\text{Time taken to do work}} = \frac{\text{Total change in kinetic energy}}{\text{Total change in time}}$$

If ΔW is the amount of work done in the time interval Δt. Then $P = \frac{\Delta W}{\Delta t} = \frac{W_2 - W_1}{t_2 - t_1}$

When work is measured in joules and t is in seconds, the unit for power is the joule per second, which is called watt. For motors and engines, power is usually measured in horsepower, where horsepower is 1 hp = 746 W. The definition of power is applicable to all types of work like mechanical, electrical, thermal.

$$\text{Instanteneous power } P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{r}}{dt} = \vec{F}.\vec{v}$$

Where v is the instantaneous velocity of the particle and dot product is used as only that component of force will contribute to power which is acting in the direction of instantaneous velocity.

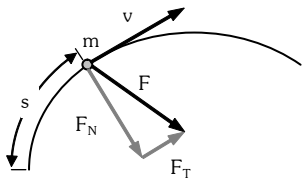
- Power is a scalar quantity with dimension $M^1L^2T^{-3}$
- SI unit of power is J/s or watt
- 1 horsepower = 746 watt

Example

A vehicle of mass m starts moving such that its speed v varies with distance traveled s according to the law $v = k\sqrt{s}$, where k is a positive constant. Deduce a relation to express the instantaneous power delivered by its engine.

Solution

Let the particle is moving on a curvilinear path. When it has traveled a distance s, the force F acting on it and its speed v are shown in the adjoining figure.



Instantaneous power delivered by the engine: $P = \vec{F}.\vec{v} = (\vec{F}_T + \vec{F}_N)\cdot\vec{v} = F_T v = ma_T v$

Tangential acceleration of the vehicle: $a_T = v\frac{dv}{ds} = \frac{k^2}{2}$

From above equations, we have $P = \frac{mk^3}{2}\sqrt{s}$

node06\E_NODE06 [E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

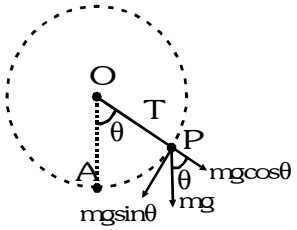
ALLEN Materials Provided By - Material Point Available on Learnaf.com

CIRCULAR MOTION IN VERTICAL PLANE

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point P of the circle as shown in figure. Here, $h = R(1 - \cos\theta)$ ...(i)

From conservation of mechanical energy

$\frac{1}{2}m(u^2 - v^2) = mgh \Rightarrow v^2 = u^2 - 2gh$ ....(ii)



The necessary centripetal force is provided by the resultant of tension T and $mg\cos\theta$

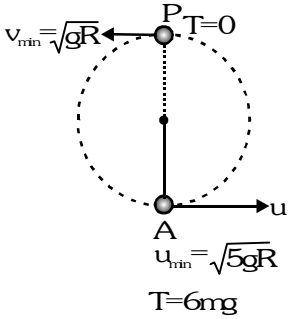
$T - mg\cos\theta = \frac{mv^2}{R}$ ...(iii)

Since speed of the particle decreases with height, hence tension is maximum at the bottom, where $\cos\theta = 1$ (as $\theta = 0^0$)

$\Rightarrow T_{max} = \frac{mv^2}{R} + mg$; $T_{min} = \frac{mv'^2}{R} - mg$ at the top. Here, v' = speed of the particle at the top.

Condition of Looping the Loop $\left(u \geq \sqrt{5gR}\right)$

The particle will complete the circle if the string does not slack even at the highest point $(\theta = \pi)$. Thus, tension in the string should be greater than or equal to zero ($T \geq 0$) at $\theta = \pi$. In critical case substituting T=0 and $\theta = \pi$ in Eq. (iii), we get $mg = \frac{mv_{min}^2}{R} \Rightarrow v_{min} = \sqrt{gR}$ (at highest point)



Substituting $\theta = \pi$ in Eq. (i), Therefore, from Eq. (ii)

$u_{min}^2 = v_{min}^2 + 2gh = gR + 2g(2R) = 5gR \Rightarrow u_{min} = \sqrt{5gR}$

Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At $u = \sqrt{5gR}$, velocity at highest point is $v = \sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0^0$ and $v = \sqrt{5gR}$ in Eq. (iii), we get T =6 mg or in the critical condition tension in the string at lowest position is 6 mg. This is shown in figure. If $u < \sqrt{5gR}$, following two cases are possible.

Condition of Leaving the Circle $\left(\sqrt{2gR} < u < \sqrt{5gR}\right)$

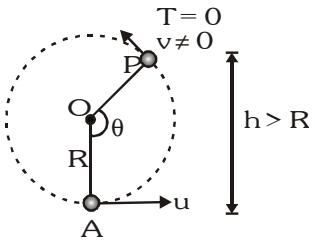
If $u < \sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From Eq. (iii), tension in the string becomes zero (T=0) where, $\cos\theta = \frac{-v^2}{Rg} \Rightarrow \cos\theta = \frac{2gh - u^2}{Rg}$

Substituting, this value of $\cos\theta$ in Eq. (i), we get $\frac{2gh - u^2}{Rg} = 1 - \frac{h}{R} \Rightarrow h = \frac{u^2 + Rg}{3g} = h_1$ (say) .....(iv)

node6\E_NOD06\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle becomes zero when $0 = u^2 - 2gh \Rightarrow h = \frac{u^2}{2g} = h_2$(say)...(v) i.e., at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero. or T = 0 but $v \neq 0$. This is possible only when $h_1 < h_2$



$\Rightarrow \frac{u^2 + Rg}{3g} < \frac{u^2}{2g} \Rightarrow 2u^2 + 2Rg < 3u^2 \Rightarrow u^2 > 2Rg \Rightarrow u > \sqrt{2Rg}$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

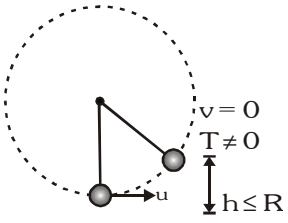
From Eq. (iv), we can see that h > R if $u^2 > 2gR$. Thus, the particle, will leave the circle when h > R or $90^0 < \theta < 180^0$. This situation is shown in the figure

$\sqrt{2gR} < u < \sqrt{5gR}$ or $90^0 < \theta < 180^0$

Note : After leaving the circle, the particle will follow a parabolic path.

Condition of Oscillation $\left(0 < u \leq \sqrt{2gR}\right)$

The particle will oscillate if velocity of the particle becomes zero but tension in the string is not zero or v = 0, but $T \neq 0$. This is possible when $h_2 < h_1$



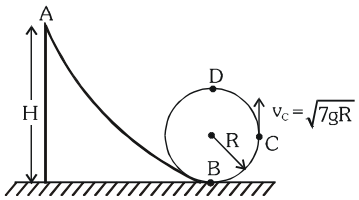
$\Rightarrow \frac{u^2}{2g} < \frac{u^2 + Rg}{3g} \Rightarrow 3u^2 < 2u^2 + 2Rg \Rightarrow u^2 < 2Rg \Rightarrow u < \sqrt{2Rg}$

Moreover, if $h_1 = h_2$, $u = \sqrt{2Rg}$ and tension and velocity both becomes zero simultaneously. Further, from Eq. (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$.

Thus, for $0 < u \leq \sqrt{2gR}$, particle oscillates in lower half of the circle ($0^0 < \theta \leq 90$)

This situation is shown in the figure. $0 < u \leq \sqrt{2gR}$ or $0 < \theta \leq 90$

Example



Calculate following for shown situation :–

(a) Speed at D (b) Normal reaction at D (c) Height H

Solution

(a) $v_D^2 = v_C^2 - 2gR = 5gR \Rightarrow v_D = \sqrt{5gR}$

(b) $mg + N_D = \frac{mv_D^2}{R} \Rightarrow N_D = \frac{m(5gR)}{R} - mg = 4mg$

(c) by energy conservation between point A & C

$mgH = \frac{1}{2}mv_C^2 + mgR = \frac{1}{2}mv_D^2 + mg2R = \frac{1}{2}m(5gR) + mg2R = \frac{9}{2}mgR \Rightarrow H = \frac{9}{2}R$

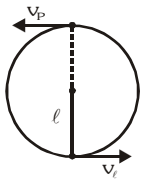
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Example

A stone of mass 1 kg tied to a light string of length $\ell = \frac{10}{3}$ m is whirling in a circular path in vertical plane. If the ratio of the maximum to minimum tension in the string is 4, find the speed of the stone at the lowest and highest points

Solution

$\because \frac{T_{max}}{T_{min}} = 4 \therefore \frac{\frac{mv_\ell^2}{\ell} + mg}{\frac{mv_p^2}{\ell} - mg} = 4 \quad \Rightarrow \frac{v_\ell^2 + g\ell}{v_p^2 - g\ell} = 4$



We know $v_\ell^2 = v_p^2 + 4g\ell \Rightarrow \frac{v_P^2 + 5g\ell}{v_P^2 - g\ell} = 4 \Rightarrow 3v_P^2 = 9g\ell$

$\Rightarrow v_P = \sqrt{3g\ell} = \sqrt{3 \times 10 \times \frac{10}{3}} = 10 \text{ ms}^{-1} \Rightarrow v_\ell = \sqrt{7g\ell} = \sqrt{7 \times 10 \times \frac{10}{3}} = 15.2 \text{ ms}^{-1}$

$\Rightarrow T = (mg + ma) + 2m(g+a)(1-\cos\theta) = m(g+a)(3 - 2\cos\theta)$

Example

A heavy particle hanging from a fixed point by a light inextensible string of length ℓ, is projected horizontally with speed $\sqrt{(g\ell)}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.

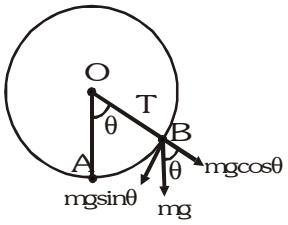
Solution

Let tension in the string becomes equal to the weight of the particle when particle reaches the point B and deflection of the string from vertical is θ. Resolving mg along the string and perpendicular to the string, we get net radial force on the particle at B i.e.

$F_R = T - mg\cos\theta$ ....(i)

If v_B be the speed of the particle at B, then

$F_R = \frac{mv_B^2}{\ell}$ ....(ii)



From (i) and (ii), we get, $T - mg\cos\theta = \frac{mv_B^2}{\ell}$ ....(iii)

Since at B, T = mg $\Rightarrow$ $mg(1-\cos\theta) = \frac{mv_B^2}{\ell} \Rightarrow v_B^2 = g\ell(1-\cos\theta)$ ...(iv)

Applying conservation of mechanical energy of the particle at point A and B, we have

$\frac{1}{2}mv_A^2 = mg\ell(1-\cos\theta) + \frac{1}{2}mv_B^2$; where $v_A = \sqrt{g\ell}$ and $v_B = \sqrt{g\ell(1-\cos\theta)}$

$\Rightarrow g\ell = 2g\ell(1-\cos\theta) + g\ell(1-\cos\theta) \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$

Putting the value of $\cos\theta$ in equation (iv), we get : $v_B = \sqrt{\frac{g\ell}{3}}$

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

SOME WORKED OUT EXAMPLES

Example #1

When a conservative force does positive work on a body, then

(A) its potential energy must increase. (B) its potential energy must decrease.

(C) its kinetic energy must increase. (D) its total energy must decrease.

Solution **Ans. (B)**

Work done by conservative force = $-\Delta U$ = positive $\Rightarrow \Delta U\downarrow$

Example #2

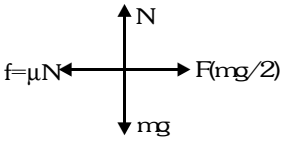
A box of mass m is initially at rest on a horizontal surface. A constant horizontal force of mg/2 is applied to the box directed to the right. The coefficient of friction of the surface changes with the distance pushed as $\mu = \mu_0 x$ where x is the distance from the initial location. For what distance is the box pushed until it comes to rest again?

(A) $\frac{2}{\mu_0}$ (B) $\frac{1}{\mu_0}$ (C) $\frac{1}{2\mu_0}$ (D) $\frac{1}{4\mu_0}$

Solution **Ans. (B)**

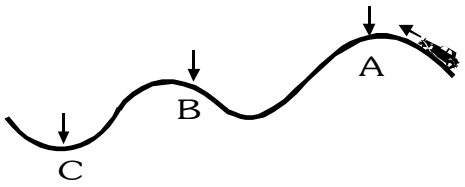
Net change in kinetic energy = 0 $\Rightarrow$ net work W = 0

$$W = \int dW = \int F dx - \int \mu N dx = \frac{mg}{2}x - mg\mu_0 \int_0^x x dx = 0 \Rightarrow x = \frac{1}{\mu_0}$$



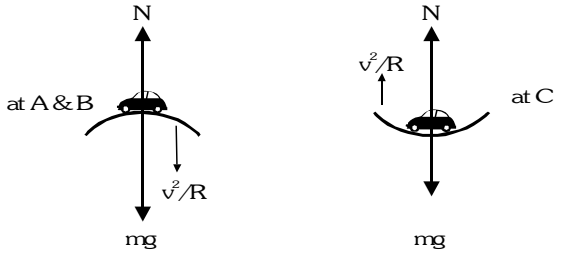
Example #3

A car is moving along a hilly road as shown (side view). The coefficient of static friction between the tyres and pavement is constant and the car maintains a steady speed. If, at one of the points shown the driver applies the brakes as hard as possible without making the tyres slip, the magnitude of the frictional force immediately after the brakes are applied will be maximum if the car was at



(A) point A (B) point B (C) point C (D) friction force same for positions A, B and C

Solution **Ans. (C)**

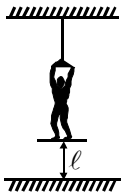


At A & B, $N = mg - mv^2/R$ & at C, $N = mg + mv^2/R$ $\therefore f_{max} = \mu_s N \rightarrow$ maximum for C

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Example #4

One end of a light rope is tied directly to the ceiling. A man of mass M initially at rest on the ground starts climbing the rope hand over hand upto a height ℓ. From the time he starts at rest on the ground to the time he is hanging at rest at a height ℓ, how much work was done on the man by the rope?



(A) 0 (B) $Mg\ell$

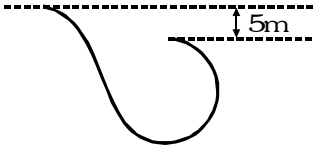
(C) $-Mg\ell$ (D) It depends on how fast the man goes up.

Solution **Ans. (B)**

Total work done on man = 0 $\Rightarrow$ Work done by string = – work done by gravity = $-(-Mg\ell) = Mg\ell$

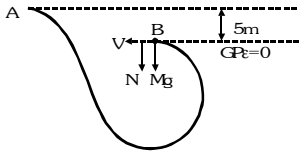
Example #5

Consider a roller coaster with a circular loop. A roller coaster car starts from rest from the top of a hill which is 5 m higher than the top of the loop. It rolls down the hill and through the loop. What must the radius of the loop be so that the passengers of the car will feel at highest point, as if they have their normal weight?



(A) 5 m (B) 10 m (C) 15 m (D) 20 m

Solution **Ans. (A)**



According to mechanical energy conservation between A and B

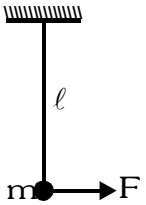
$$mg(5) = O + \frac{1}{2}mv^2 \Rightarrow v^2 = 10g \quad ...(i)$$

According to centripetal force equation

$$N + mg = \frac{mv^2}{r} \text{ for } N = mg;\ 2mg = \frac{mv^2}{r} \Rightarrow r = \frac{v^2}{2g} = \frac{10g}{2g} = 5m$$

Example #6

A pendulum bob of mass m is suspended at rest. A cosntant horiozntal force F = mg/2 starts acting on it. The maximum angular deflection of the string is

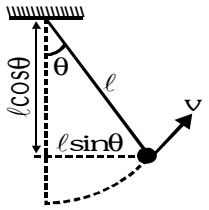


(A) 90 (B) 53 (C) 37 (D) 60

node6\E_NOD65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Solution **Ans. (C)**

Let at angular deflection θ its velocity be v then by work energy theorem W= ΔKE



$$\frac{1}{2}mv^2 = -mg(\ell - \ell\cos\theta) + F\ell\sin\theta$$

At maximum angular deflection, v = 0

$$0 = -mg\ell(1-\cos\theta) + \frac{mg}{2}\sin\theta \Rightarrow 2-2\cos\theta = \sin\theta \Rightarrow 4 + 4\cos^2\theta - 8\cos\theta = \sin^2\theta = 1-\cos^2\theta$$

$$\Rightarrow 5\cos^2\theta - 8\cos\theta + 3 = 0 \Rightarrow 5\cos^2\theta - 5\cos\theta - 3\cos\theta + 3 = 0$$

$$\Rightarrow 5\cos\theta(\cos\theta-1) - 3(\cos\theta-1) = 0 \Rightarrow (5\cos\theta-3)(\cos\theta-1) = 0$$

$$\Rightarrow \cos\theta = \frac{3}{5} \text{ or } \cos\theta = 1 \Rightarrow \theta = 37 \text{ or } \theta = 0$$

Example #7

The potential energy for the force $\vec{F} = yz\tilde{i} + xz\tilde{j} + xy\tilde{k}$, if the zero of the potential energy is to be chosen at the point (2, 2, 2), is

(A) 8 + xyz (B) 8 – xyz (C) 4–xyz (D) 4 + xyz

Solution **Ans. (B)**

$$\because \vec{F} = -\frac{\partial U}{\partial x}\tilde{i} - \frac{\partial U}{\partial y}\tilde{j} - \frac{\partial U}{\partial z}\tilde{k} \qquad \therefore \frac{\partial U}{\partial x} = -yz, \frac{\partial U}{\partial y} = -xz, \frac{\partial U}{\partial z} = -xy$$

Therefore U = – xyz + C where C = constant As at (2, 2, 2), U =0 so C =8

OR

Objective question approach : Check that U =0 at (2, 2, 2)

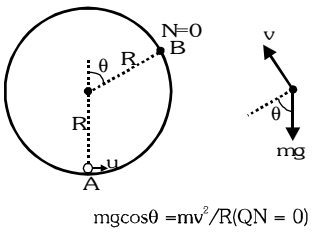
Example #8

A particle is projected along the inner surface of a smooth vertical circle of radius R, its velocity at the lowest point being $\frac{1}{5}\sqrt{95Rg}$. It will leave the circle at an angular distance.... from the highest point

(A) 37 (B) 53 (C) 60 (D) 30

Solution **Ans. (B)**

By conservation of mechanical energy [between point A and B]



$$\frac{1}{2}mu^2 = mgR(1+\cos\theta) + \frac{1}{2}mv^2$$

$$\frac{1}{2}m\left(\frac{1}{5}\sqrt{95Rg}\right)^2 = mgR(1+\cos\theta) + \frac{1}{2}mgR\cos\theta$$

$$\Rightarrow \frac{95}{25} = 2 + 2\cos\theta + \cos\theta \Rightarrow 3\cos\theta = \frac{45}{25} \Rightarrow \cos\theta = \frac{15}{25} = \frac{3}{5} \Rightarrow \theta = 53°$$

$mg\cos\theta = mv^2/R$(QN = 0)

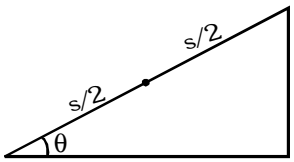
node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Example #9

The upper half of an inclined plane with inclination θ is perfectly smooth while the lower half is rough. A body starting from rest at the top will again come to rest at the bottom if the coefficient of friction for the lower half is given by

(A) tanθ (B) 2tanθ (C) 2cosθ (D) 2sinθ

Solution **Ans. (B)**



Refer to figure. In the journey over the upper half of incline, $v^2 - u^2 = 2$ as

$v^2 - 0 = 2\,(g\sin\theta)\,\frac{s}{2} = g\sin\theta.s$

In the journey over the lower half of incline $v^2 - u^2 = 2$ as

$0 - g\sin\theta.s = 2\,g(\sin\theta - \mu\cos\theta)\frac{s}{2} \Rightarrow -\sin\theta = \sin\theta - \mu\cos\theta \Rightarrow \mu = \frac{2\sin\theta}{\cos\theta} = 2\tan\theta$

Example #10

Simple pendulums P_1 and P_2 have lengths $\ell_1 = 80$ cm and $\ell_2 = 100$ cm respectively. The bobs are of masses m_1 and m_2. Initially both are at rest in equilibrium position. If each of the bobs is given a displacement of 2 cm, the work done is W_1 and W_2 respectively. Then,

(A) $W_1 > W_2$ if $m_1 = m_2$ (B) $W_1 < W_2$ if $m_1 = m_2$ (C) $W_1 = W_2$ if $\frac{m_1}{m_2} = \frac{5}{4}$ (D) $W_1 = W_2$ if $\frac{m_1}{m_2} = \frac{4}{5}$

Solution **Ans. (A,D)**

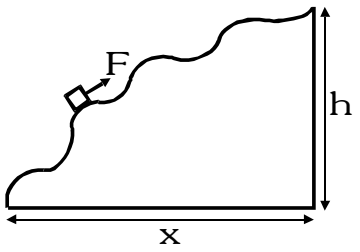
With usual notation, the height through which the bob falls is $h = \ell(1-\cos\theta) = \ell\left(2\sin^2\frac{\theta}{2}\right) = 2\ell\left(\frac{\theta^2}{4}\right)$ since θ is

small. Therefore, we can write $h = \frac{\ell\theta^2}{2} = \frac{\ell}{2}\left(\frac{a}{\ell}\right)^2 = \frac{a^2}{2\ell}$. where a = amplitude

Thus, the work done W= P.E. = mgh = $\frac{mga^2}{2\ell} \Rightarrow W \propto \frac{1}{\ell}$

Example #11

A body of mass m is slowly halved up the rough hill by a force F at which each point is directed along a tangent to the hill.



Work done by the force

(A) independent of shape of trajectory. (B) depends upon x.

(C) depends upon h. (D) depends upon coefficient of friction (μ)

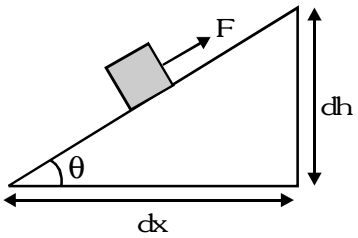
node6\E_NOD06\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory p6.5

Solution **Ans. (ABCD)**

Work done by the force = Work done against gravity (W_g) + work done against friction (W_f)

$= W_g = \int (mg\sin\theta)ds = mg\int ds\sin\theta = mg\int dh = mgh$

and $W_f = \int (\mu mg\cos\theta)ds = \mu mg\int ds\cos\theta = \mu mg\int dx = \mu mgx$



Example #12

The kinetic energy of a particle continuously increase with time. Then

(A) the magnitude of its linear momentum is increasing continuously.

(B) its height above the ground must continuously decrease.

(C) the work done by all forces acting on the particle must be positive.

(D) the resultant force on the particle must be parallel to the velocity at all times.

Solution **Ans. (A, C)**

For (A) : $p = \sqrt{2mK}$ if K ↑ then p ↑

For (B) : Its height may ↑ or ↓

For (C) : W = ΔK if ΔK = positive then W = positive

For (D) : The resultant force on the particle must be at an angle less than 90 all times

Example #13

A particle moves in one dimensional field with total mechanical energy E. If potential energy of particle is U(x), then

(A) Particle has zero speed where U(x) = E

(B) Particle has zero acceleration where U(x) = E

(C) Particle has zero velocity where $\frac{dU(x)}{dx} = 0$

(D) Particle has zero acceleration where $\frac{dU(x)}{dx} = 0$

Solution **Ans. (A,D)**

Mechanical energy = kinetic energy + potential energy $E = K + U(x)$ where $K = \frac{1}{2}mv^2$

If K = 0 then E = U(x)

If F = 0 then $F = -\frac{dU(x)}{dx} = 0 \Rightarrow \frac{dU(x)}{dx} = 0$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Example #14

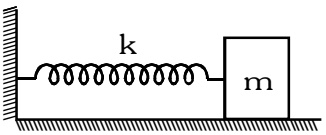
A spring block system is placed on a rough horizontal surface having coefficient of friction μ. The spring is given initial elongation $\frac{3\mu mg}{k}$ and the block is released from rest. For the subsequent motion

(A) Initial acceleration of block is 2μg.

(B) Maximum compression in spring is $\frac{\mu mg}{k}$.

(C) Minimum compression in spring is zero.

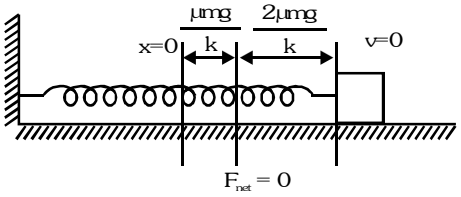
(D) Maximum speed of the block is $2\mu g\sqrt{\frac{m}{k}}$



Solution **Ans. (A,B,C,D)**

For (A) : Initial acceleration = $\frac{k\left(\frac{3\mu mg}{k}\right)-\mu mg}{m}=2\mu g$

For (B,C) :



Therefore maximum compresion = $\frac{2\mu mg}{k}-\frac{\mu mg}{k}=\frac{\mu mg}{k}$ and minimum compression = 0

For (D) : At maximum speed F_{net} = 0 so by using work energy therorem

$$\frac{1}{2}mv^2=\frac{1}{2}k\left(\frac{3\mu mg}{k}\right)^2-\frac{1}{2}k\left(\frac{\mu mg}{k}\right)^2-\mu mg\left(\frac{2\mu mg}{k}\right) \Rightarrow v = 2\mu g\sqrt{m/k}$$

Example #15 to 17

A particle of mass m = 1 kg is moving along y-axis and a single conservative force F(y) acts on it. The potential energy of particle is given by U(y) = (y^2–6y+14) J where y is in meters. At y = 3 m the particle has kinetic energy of 15 J.

15. The total mechanical energy of the particle is

(A) 15 J (B) 5 J (C) 20 J (D) can't be determined

16. The maximum speed of the particle is

(A) 5 m/s (B) $\sqrt{30}$ m/s (C) $\sqrt{40}$ m/s (D) $\sqrt{10}$ m/s

17. The largest value of y (position of particle) is

(A) $3+\sqrt{5}$ (B) $3-\sqrt{5}$ (C) $3+\sqrt{15}$ (D) $6+\sqrt{15}$

node6\E_NOD06\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Solution

15. Ans. (C)

Total mechanical energy = kinetic energy + potential energy = 15+ [3^2–6(3)+14] = 15 +5 = 20 J

16. Ans. (B)

At maximum speed (i.e. maximum kinetic energy), potential energy is minimum U = y^2 – 6y + 14 = 5 + $(y-3)^2$ which is minimum at y=3 m so U_{min} = 5J

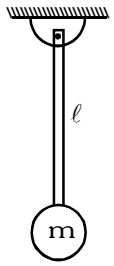
Therefore K_{max} = 20 – 5 = 15 J $\Rightarrow \frac{1}{2}mv_{max}^2 = 15 \Rightarrow v_{max} = \sqrt{30}$ m/s

17. Ans. (C)

For particle $K \geq 0 \Rightarrow E - U \geq 0 \Rightarrow 20 - (5+(y-3)^2] \geq 0 \Rightarrow (y-3)^2 \leq 15 \Rightarrow y-3 \leq \sqrt{15} \Rightarrow y \leq 3 + \sqrt{15}$

Example #18 to 20

A rigid rod of length ℓ and negligible mass has a ball with mass m attached to one end and its other end fixed, to form a pendulum as shown in figure. The pendulum is inverted, with the rod straight up, and then released.



18. At the lowest point of trajectory, what is the ball's speed?

(A) $\sqrt{2g\ell}$ (B) $\sqrt{4g\ell}$ (C) $2\sqrt{2g\ell}$ (D) $\sqrt{8g\ell}$

19. What is the tension in the rod at the lowest point of trajectory of ball?

(A) 6 mg (B) 3 mg (C) 4 mg (D) 5 mg

20. Now, if the pendulum is released from rest from a horizontal position. At what angle from the vertical does the tension in the rod equal to the weight of the ball?

(A) $\cos^{-1}\left(\frac{2}{3}\right)$ (B) $\cos^{-1}\left(\frac{1}{3}\right)$ (C) $\cos^{-1}\left(\frac{1}{2}\right)$ (D) $\cos^{-1}\left(\frac{1}{4}\right)$

Solution

18. Ans. (B)

From COME : $2mg\ell = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{4g\ell} = 2\sqrt{g\ell}$

19. Ans. (D)

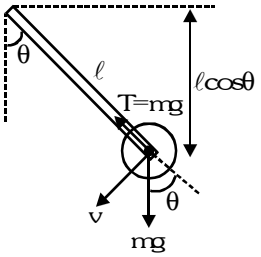
At the lowest point $T - mg = \frac{mv^2}{\ell} \Rightarrow T = mg + \frac{m}{\ell}(4g\ell) = 5mg$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

20. Ans. (B)

Force equation $T - mg\cos\theta = \frac{mv^2}{\ell}$

Energy equation $mg\ell\cos\theta = \frac{1}{2}mv^2$



Therefore $mg - mg\cos\theta = 2mg\cos\theta \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{3}$

Example #21

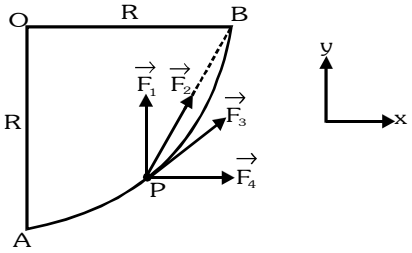
AB is a quarter of a smooth horizontal circular track of radius R. A particle P of mass m moves along the track from A to B under the action of following forces :

$\vec{F}_1$ = F (always towards y-axis)

$\vec{F}_2$ = F (always towards point B)

$\vec{F}_3$ = F (always along the tangent to path AB)

$\vec{F}_4$ =F (always towards x-axis)

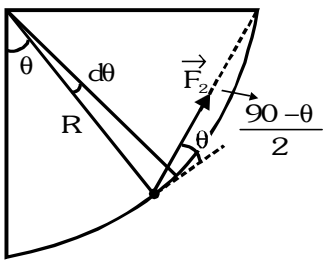


	Column I		Column II
(A)	Work done by $\vec{F}_1$	(P)	$\sqrt{2}$ FR
(B)	Work done by $\vec{F}_2$	(Q)	$\frac{1}{\sqrt{2}}$FR
(C)	Work done by $\vec{F}_3$	(R)	FR
(D)	Work done by $\vec{F}_4$	(S)	$\frac{\pi FR}{2}$
		(T)	$\frac{2FR}{\pi}$

Solution **Ans. (A) →(R); (B) →(P); (C) →(S); (D) →(R)**

For (A) : Work done by $\vec{F}_1 = FR$

For (B) : $dW = \vec{F}.d\vec{s} = (FRd\theta)\cos\left(45 - \frac{\theta}{2}\right) = FR\left(45 - \frac{\theta}{2}\right)d\theta$



$$W = \int_0^{\pi/4} FR\cos\left(45 - \frac{\theta}{2}\right) = -2FR\left(\sin 45° - \frac{\theta}{2}\right)_0^{\pi/2} = \sqrt{2}FR$$

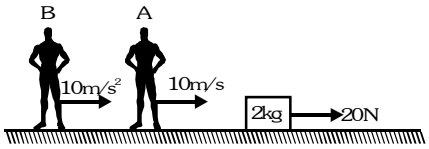
For (C) : $W = \int \vec{F}.d\vec{s} = F\left(\frac{\pi R}{2}\right) = \frac{\pi FR}{2}$

For (D) : $W = \int \vec{F}.d\vec{s} = (F)(R) = FR$

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Example #22

A block of mass 2 kg is dragged by a force of 20 N on a smooth horizontal surface. It is observed from three reference frames ground, observer A and observer B. Observer A is moving with constant velocity of 10 m/s and B is moving with constant acceleration of 10 m/s^2. The observer B and block starts simultaneously at t =0.



Column I	Column II
(A) Work energy theorem is applicable in	(P) 100 J
(B) Work done on block in 1s as observed by ground is	(Q) – 100 J
(C) Work done on block is 1 s as observed by observer A is	(R) zero
(D) Work done on block in 1 s as observed by observer B is	(S) only ground & A
	(T) all frames ground, A & B

Solution **Ans. (A) →(T); (B) →(P); (C) →(Q); (D) →(R)**

For (A) : Work energy theorem is applicable in all reference frames.

For (B) : w.r.t. ground : At t =0, u =0 and t = 1 s, $v = at = \left(\frac{20}{2}\right)(1) = 10$ m/s

Work done = change in kinetic energy = $\frac{1}{2}(2)(10)^2 - \frac{1}{2}(2)(0)^2 = 100$ J

For (C) : w.r.t. observer A : Initial velocity = 0 – 10 = – 10 m/s, Final velocity = 10 – 10 = 0

Work done = $\frac{1}{2}(2)(0)^2 - \frac{1}{2}(2)(-10)^2 = -100$ J

For (D) : w.r.t. observer B : Initial velocity = 0 – 0 = 0

Final velocity = 10 – 10 = 0; Work done = 0

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Work Power Energy\Eng\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 6
CENTER OF MASS & COLLISION - Theory

Index

CENTRE OF MASS AND COLLISIONS

Serial No.	CONTENTS	Page No.
1.	Impulse of a force	1
2.	Impulse Momentum Principle	2
3.	Impulsive Motion	5
4.	Momentum and Kinetic energy	6
5.	Internal and external forces and system of particles	6
6.	Principle of conservation of linear momentum	7
7.	Impact between two bodies	11
8.	System of Particles	20
9.	System of particles and mass centre	20
10.	Center of Mass Frame of Reference or Centroidal Frame	30
11.	Application of Newton's laws of motion to a system of particles.	31
12.	Application of methods of impulse and momentum to a system of particles.	34
13.	Application of methods of work and energy to a system of particles	35
14.	**Exercise-I** Check your Grasp (Objective)	51
15.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	55
16.	**Exercise-3** Miscellaneous Type Questions (T/F, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	62
17.	**Exercise-4A** Conceptual Subjective Exercise	70
18.	**Exercise-4B** Brain Storming Exercise	74
19.	**Exercise-5A** Previous years questions	76
20.	**Exercise-5B** Previous years questions	78

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Main : Centre of mass of a two-particle system, Centre of mass of a rigid body; Momentum; Impulse; Law of conservation of linear momentum and its applications, Elastic and inelastic collisions in one and two dimensions

JEE Advanced : Conservation of linear momentum, Systems of particles; Centre of mass and its motion; Impulse; Elastic and inelastic collisions.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

METHOD OF IMPULSE AND MOMENTUM

Until now, we have studied kinematics of particles, Newton's laws of motion and methods of work and energy. Newton's laws of motion describe relation between forces acting on a body at an instant and acceleration of the body at that instant. Therefore, it only helps us do analyze what is happening at an instant. The work kinetic energy theorem is obtained by integrating equation of motion ($\vec{F} = m\vec{a}$) over a path. Therefore, methods of work and energy help us to in exploring change in speed over a position interval. Now, we direct our attention on another principle – principle of impulse and momentum. It is obtained when equation of motion ($\vec{F} = m\vec{a}$) is integrated with respect to time. Therefore, this principle facilitates us with method to explore what is happening over a time interval.

Impulse of a Force

Net force applied on a rigid body changes momentum i.e. amount of motion of that body. A net force for a longer duration cause more change in momentum than the same force acting for shorter duration. Therefore duration in which a force acts on a body together with magnitude and direction of the force decide effect of the force on the change in momentum of the body.

Linear impulse or simply impulse of a force is defined as integral of the force with respect to time.

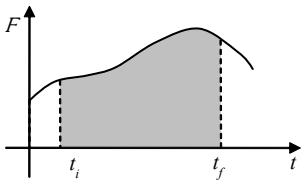
If a force $\vec{F}$ acts on a body, its impulse in a time interval from t_i to t_f is given by the following equation.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F} dt$$

If the force is constant, its impulse equals to product of the force vector $\vec{F}$ and time interval Δt.

$$\vec{I}_{mp} = \vec{F}(\Delta t)$$

For one-dimensional force, impulse equals to area between force-time graph and the time axis. In the given figure is shown how a force F along x-axis varies with time t. Impulse of this force in time interval t_i to t_f equals to area of the shaded portion.



If several forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ ..... $\vec{F}_n$ act on a body in a time interval, the total impulse $\vec{I}_{mp}$ of all these forces equals to impulse of the net force.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F}_1 dt + \int_{t_i}^{t_f} \vec{F}_2 dt + \ldots\ldots\ldots\ldots + \int_{t_i}^{t_f} \vec{F}_n dt = \int_{t_i}^{t_f} \left(\vec{F}_1 + \vec{F}_2 + \ldots\ldots\ldots\ldots + \vec{F}_n\right) dt$$

Impulse is measured in newton-second.

Dimensions of impulse are MLT^{-1}

Example

Calculate impulse of force $\vec{F} = \left(3t^2\tilde{i} - (2t-1)\tilde{j} + 2\tilde{k}\right)$ N over the time interval from t = 1 s to t = 3 s.

Solution.

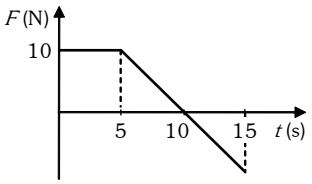
$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F} dt \rightarrow \quad \vec{I}_{mp} = \int_1^3 \left(3t^2\tilde{i} - (2t-1)\tilde{j} + 2\tilde{k}\right) dt = \left[t^3\tilde{i} - (t^2 - t)\tilde{j} + 2t\tilde{k}\right]_1^3 = \tilde{i}[t^3]_1^3 - \tilde{j}\left[(t^2 - t)\right]_1^3 + \tilde{k}[2t]_1^3$$

$$= \left(26\tilde{i} - 6\tilde{j} + 4\tilde{k}\right) \text{ N-s}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Example

A one-dimensional force F varies with time according to the given graph. Calculate impulse of the force in following time intervals.



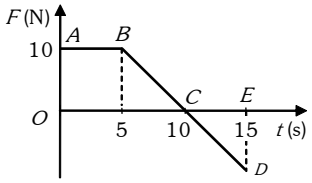
(a) From t = 0 s to t = 10 s.

(b) From t = 10 s to t = 15 s.

(c) From t = 0 s to t = 15 s.

Solution.

For one-dimensional force, impulse equals to area between force-time graph and the time axis.



(a) $I_{0\to10}$ = Area of trapazium OABC = 75 N-s

(b) $I_{10\to15}$ = – Area of triangle CDE = – 25 N-s

(c) $I_{0\to15}$ = Area of trapazium OABC – Area of triangle CDE = 50 N-s

Impulse Momentum Principle

Consider body of mass m in translational motion. When it is moving with velocity $\vec{v}$, net external force acting on it is $\vec{F}$. Equation of motion as suggested by Newton's second law can be written in the form

$$\vec{F}dt = d(m\vec{v})$$

If the force acts during time interval from t_i to t_f and velocity of the body changes from $\vec{v}_i$ to $\vec{v}_f$, integrating the above equation with time over the interval from t_i to t_f, we have

$$\int_{t_i}^{t_f} \vec{F}dt = m\vec{v}_f - m\vec{v}_i$$

Here left hand side of the above equation is impulse $\vec{I}_{mp}$ of the net force $\vec{F}$ in time interval from t_i to t_f, and quantities $m\vec{v}_i$ and $m\vec{v}_f$ on the right hand side are linear momenta of the particle at instants t_i and t_f. If we denote them by symbols $\vec{p}_i$ and $\vec{p}_f$, the above equation can be written as

$$\vec{I}_{mp} = \vec{p}_f - \vec{p}_i$$

The idea expressed by the above equation is known as *impulse momentum principle*. It states that change in the momentum of a body in a time interval equals to the impulse of the net force acting on the body during the concerned time interval.

For the ease of application to physical situations the above equation is rearranged as

$$\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$$

This equation states that impulse of a force during a time interval when added to momentum of a body at the beginning of an interval of time we get momentum of the body at the end of the interval concerned.

Since impulse and momentum both are vector quantities, the impulse momentum theorem can be expressed by there scalar equations making use of Cartesian components.

$$p_{1x} + \sum I_{mp,x} = p_{2x}$$

$$p_{1y} + \sum I_{mp,y} = p_{2y}$$

$$p_{1z} + \sum \mathrm{I}_{mp,z} = p_{2z}$$

The impulse momentum principle is deduced here for a single body moving relative to an inertial frame, therefore impulses of only physical forces are considered. If we are using a non-inertial reference frame, impulse of corresponding pseudo force must also be considered in addition to impulse of the physical forces.

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

How to apply Impulse Momentum Principle

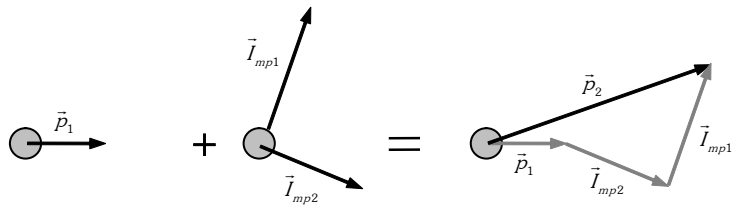
The impulse momentum principle is deduced here for a single body, therefore it is recommended at present to use it for a single body. To use this principle the following steps should be followed.

(i) Identify the initial and final positions as position 1 and 2 and show momenta $\vec{p}_1$ and $\vec{p}_2$ of the body at these instants.

(ii) Show impulse of each force acting on the body at an instant between positions 1 and 2.

(iii) Use the impulse obtained in step (ii) and momenta obtained in step (i) into equation $\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$.

Consider a particle moving with momentum $\vec{p}_1$ in beginning. It is acted upon by two forces, whose impulses in a time interval are $\vec{I}_{mp1}$ and $\vec{I}_{mp2}$. As a result, at the end of the time interval, momentum of the particle becomes $\vec{p}_2$. This physical situation is shown in the following diagram. Such a diagram is known as *impulse momentum diagram*.



Example

A particle of mass 2 kg is moving with velocity $\vec{v}_o = (2\tilde{i} - 3\tilde{j})$ m/s in free space. Find its velocity 3 s after a constant force $\vec{F} = (3\tilde{i} + 4\tilde{j})$ N starts acting on it.

Solution.

$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \quad m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

Substituting given values, we have

$2\vec{v}_f = 2(2\tilde{i} - 3\tilde{j}) + (3\tilde{i} + 4\tilde{j}) \times 3 = 13\tilde{i} + 6\tilde{j}$

$\vec{v}_f = (6.5\tilde{i} + 3\tilde{j})$ m/s

Example

A particle of mass 2 kg is moving in free space with velocity $\vec{v}_o = (2\tilde{i} - 3\tilde{j} + \tilde{k})$ m/s is acted upon by force $\vec{F} = (2\tilde{i} + \tilde{j} - 2\tilde{k})$ N. Find velocity vector of the particle 3 s after the force starts acting.

Solution.

$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \quad m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

Substituting given values, we have

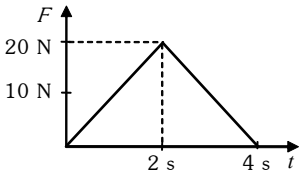
$2\vec{v}_f = 2(2\tilde{i} - 3\tilde{j} + \tilde{k}) + (2\tilde{i} + \tilde{j} - 2\tilde{k}) \times 3 = 10\tilde{i} - 3\tilde{j} - 4\tilde{k}$

$\vec{v}_f = (5\tilde{i} - 1.5\tilde{j} - 2\tilde{k})$ m/s

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Example

A box of mass m = 2 kg resting on a frictionless horizontal ground is acted upon by a horizontal force F, which varies as shown. Find speed of the particle when the force ceases to act.



Solution.

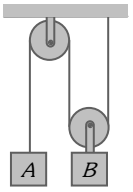
$$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \quad m\vec{v}_f = m\vec{v}_i + \int_{t_i}^{t_f} \vec{F}dt$$

$$2v = 2 \times 0 + \tfrac{1}{2} \times 20 \times 4$$

$$v = 20 \text{ m/s}$$

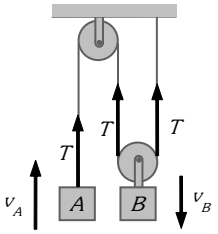
Example

Two boxes A and B of masses m and M interconnected by an ideal rope and ideal pulleys are held at rest as shown. When it is released, box B accelerates downwards. Find velocities of box A and B as function of time t after system has been released.



Solution.

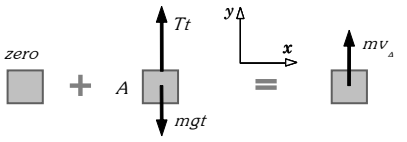
We first explore relation between accelerations a_A and a_B of the boxes A and B, which can be written either by using constrained relation or method of virtual work or by inspection.



$$v_A = 2v_B \qquad \text{...(i)}$$

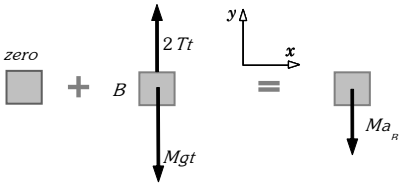
Applying impulse momentum principle to box A

$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow \qquad Mv_A = 0 + Tt - mgt \quad \text{...(ii)}$$



Applying impulse momentum principle to box B

$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow \qquad mv_B = 0 + Mgt - 2Tt \quad \text{...(iii)}$$



From equations (i), (ii) and (iii), we have

$$v_A = 2\left(\frac{M-2m}{M+4m}\right)gt \text{ and } v_B = \left(\frac{M-2m}{M+4m}\right)gt$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

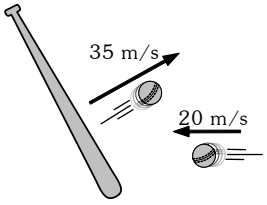
Impulsive Motion



Sometimes a very large force acts for a very short time interval on a particle and produces finite change in momentum. Such a force is known as impulsive force and the resulting motion as impulsive motion. When a batsman hits a ball by bat, the contact between the ball and the bat lasts for a very small duration Δt, but the average value of the force F exerted by the bat on the ball is very large, and the resulting impulse $F\Delta t$ is large enough to change momentum of the ball.

During an impulsive motion, some other forces of magnitudes very small in comparison to that of an impulsive force may also act. Due to negligible time interval of the impulsive motion, impulse of these forces becomes negligible. These forces are known as non-impulsive forces. Effect of non-impulsive forces during an impulsive motion is so small that they are neglected in analyzing impulsive motion of infinitely small duration. Non-impulsive forces are of finite magnitude and include weight of a body, spring force or any other force of finite magnitude. When duration of the impulsive motion is specified, care has to be taken in neglecting any of the non-impulsive force. In analyzing motion of the ball for very small contact duration (usually in mili-seconds), impulse of the weight of the ball has to be neglected. Unknown reaction forces may be impulsive or non-impulsive; their impulse must therefore be included.

Example



A 100 gm ball moving horizontally with 20 m/s is struck by a bat, as a result it starts moving with a speed of 35 m/s at an angle of 37 above the horizontal in the same vertical plane as shown in the figure.

(a) Find the average force exerted by the bat if duration of impact is 0.30 s.

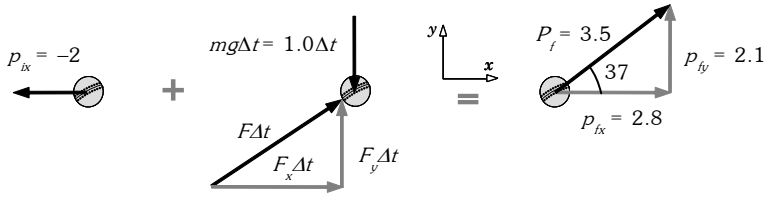
(b) Find the average force exerted by the bat if duration of impact is 0.03 s.

(c) Find the average force exerted by the bat if duration of impact is 0.003 s.

(d) What do you conclude for impulse of weight of the ball as duration of contact decreases?

Solution.

The impulse momentum diagram of the ball is shown in the figure below. Here F, mg, and Δt represent the average value of the force exerted by the bat, weight of the ball and the time interval.



Applying principle of impulse and momentum in x- direction, we have

$$p_{fx} = p_{ix} + \sum I_{mp,x} \rightarrow \qquad -2 + F_x\Delta t = 2.8$$

$$F_x = \frac{4.8}{\Delta t}\ \text{N} \qquad \text{...(i)}$$

Applying principle of impulse and momentum in y- direction, we have

$$p_{fy} = p_{iy} + \sum I_{mp,y} \rightarrow \qquad 0.0 + F_y\Delta t - 1.0\Delta t = 2.1$$

$$F_y = \left(\frac{2.1}{\Delta t} + 1.0\right)\ \text{N} \qquad \text{...(ii)}$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory\p65

(a) Substituting Δt = 0.30 s, in equations (i) and (ii), we find $\vec{F} = 16\tilde{i} + 8\tilde{j}$ N

(b) Substituting Δt = 0.03 s, in equations (i) and (ii), we find $\vec{F} = 160\tilde{i} + 71\tilde{j}$ N

(c) Substituting Δt = 0.003 s, in equations 1 and 2, we find $\vec{F} = 1600\tilde{i} + 701\tilde{j}$ N

(d) It is clear from the above results that as the duration of contact between the ball and the bat decreases, effect of the weight of the ball also decreases as compared with that of the force of the bat and for sufficiently short time interval, it can be neglected.

Momentum and Kinetic Energy

A moving particle possesses momentum as well as kinetic energy. If a particle of mass m is moving with velocity v, magnitude of its momentum p and its kinetic energy K bear the following relation.

$$K = \frac{p^2}{2m} = \frac{1}{2}pv$$

Example

An object is moving so that its kinetic energy is 150 J and the magnitude of its momentum is 30.0 kg-m/s. With what velocity is it traveling?

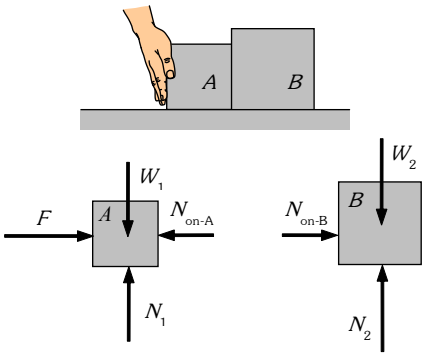
Solution.

$$K = \frac{p^2}{2m} = \frac{1}{2}pv \rightarrow v = \frac{2\times 150}{30.0} = 10.0 \text{ m/s}$$

Internal and external Forces and System of interacting Particles

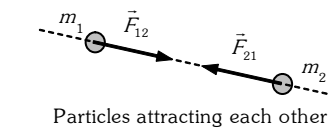
Bodies applying forces on each other are known as interacting bodies. If we consider them as a system, the forces, which they apply on each other, are known as internal forces and all other forces applied on them by bodies not included in the system are known as external forces.

Consider two blocks A and B placed on a frictionless horizontal floor. Their weights W_1 and W_2 are counterbalanced by normal reactions N_1 and N_2 on each of them from the floor. Push F by the hand is applied on A. The forces of normal reaction $N_{on\text{-}A}$ and $N_{on\text{-}B}$ constitute Newton's third law action-reaction pair, therefore are equal in magnitude and opposite in direction. Among these forces weights W_1 and W_2 applied by the earth, normal reactions N_1 and N_2 applied by the ground and the push F applied by the hand are external forces and normal reactions $N_{on\text{-}A}$ and $N_{on\text{-}B}$ are internal forces.

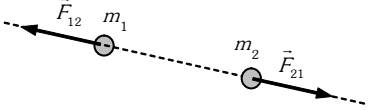


If the blocks are connected by a spring and the block A is either pushed or pulled, the forces W_1, W_2, N_1 and N_2 still remain external forces for the two block system and the forces, which the spring applies on each other are the internal forces. Here force of gravitational interaction between them being negligible has been neglected.

We can conceive a general model of two interacting particles. In the figure is shown a system of two particles of masses m_1 and m_2. Particle m_1 attracts m_2 with a force $\vec{F}_{12}$ and m_2 attracts (or pulls) m_1 with a force $\vec{F}_{21}$. These forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are the internal forces of this two-particle system and are equal in magnitude and opposite in directions. Instead of attraction may repeal each other. Such a system of two particles repealing each other is also shown.



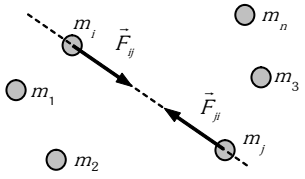
Particles attracting each other



Particles repealing each other

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

In similar way we may conceive a model of a system of n interacting particles having masses $m_1, m_2, \ldots m_i, \ldots m_j, \ldots$ and m_n respectively. The forces of interaction $\vec{F}_{ij}$ and $\vec{F}_{ji}$ between m_i and m_j are shown in the figure. Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles are internal forces of the system. Any of the two interacting particles always apply equal and opposite forces on each other. Here fore simplicity only the forces $\vec{F}_{ij}$ and $\vec{F}_{ji}$ are shown.



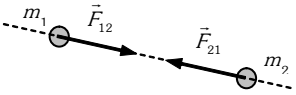
System of n interacting particles.

Principle of Conservation of linear momentum

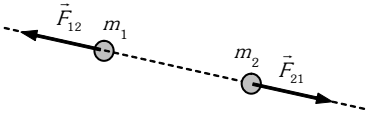
The principle of conservation of linear momentum or simply conservation of momentum for two or more interacting bodies is one of guiding principles of the classical as well as the modern physics.

To understand this principle, we first discuss a system of two interacting bodies, and then extend the ideas developed to a system consisting of many interacting bodies.

Consider a system of two particles of mass m_1 and m_2. Particle m_1 attracts m_2 with a force $\vec{F}_{12}$ and m_2 attracts m_1 with a force $\vec{F}_{21}$. These forces have equal magnitudes and opposite directions as shown in the figure. If the bodies are let free i.e. without any external force acting on any of them, each of them move and gain momentum equal to the impulse of the force of interaction. Since equal and opposite interaction forces act on both of them for the same time interval, the momenta gained by them are equal in magnitude and opposite in direction resulting no change in total momentum of the system.



Particles attracting each other

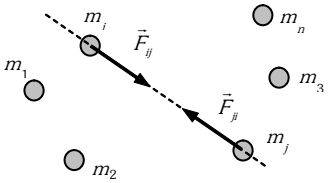


Particles repealing each other

However, if an external force acts on any one of them or different forces with a nonzero resultant act on both of them, the total momentum of the bodies will certainly change. If the system undergoes an impulsive motion, total momentum will change only under the action of external impulsive force or forces. Internal impulsive forces also exist in pairs of equal and opposite forces and cannot change the total momentum of the system. Non-impulsive forces if act cannot change momentum of the system by appreciable amount. For example, gravity is a non-impulsive force, therefore in the process of collision between two bodies near the earth the total momentum remains conserved.

The total momentum of a system of two interacting bodies remains unchanged under the action of the forces of interaction between them. It can change only if a net impulse of external force is applied.

In similar way we may conceive a model of a system of n interacting particles having masses $m_1, m_2, \ldots m_i, \ldots m_j, \ldots$ and m_n respectively. The forces of interaction $\vec{F}_{ij}$ and $\vec{F}_{ji}$ between m_i and m_j are shown in the figure. Since internal forces exist in pairs of equal and opposite forces, in any time interval of concern each of them have a finite impulse but their total impulse is zero. Thus if the system is let free, in any time interval momentum of every individual particle changes but the total momentum of the system remains constant.



System of n interacting particles.

It can change only if external forces are applied to some or all the particles. Under the action of external forces, the change in total momentum of the system will be equal to the net impulse of all the external forces.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Thus, total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final}$$

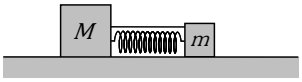
The above statement is known as the principle of conservation of momentum. It is applicable only when the net impulse of all the external forces acting on a system of particles becomes zero in a finite time interval. It happens in the following conditions.

- When no external force acts on any of the particles or bodies.
- When resultant of all the external forces acting on all the particles or bodies is zero.
- In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act, total component of momentum in that direction remains conserved.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

Example

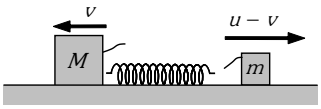
Two blocks of masses m and M are held against a compressed spring on a frictionless horizontal floor with the help of a light thread. When the thread is cut, the smaller block leaves the spring with a velocity u relative to the larger block. Find the recoil velocity of the larger block.



Solution.

When the thread is cut, the spring pushes both the block, and impart them momentum. The forces applied by the spring on both the block are internal forces of the two-block system. External forces acting on the system are weights and normal reactions on the blocks from the floor. These external forces have zero net resultant of the system. In addition to this fact no external force acts on the system in horizontal direction, therefore, horizontal component of the total momentum of the system remains conserved.

Velocities of both the objects relative to the ground (inertial frame) are shown in the adjoining figure.



Since before the thread is cut system was at rest, its total momentum was zero. Principle of conservation of momentum for the horizontal direction yields

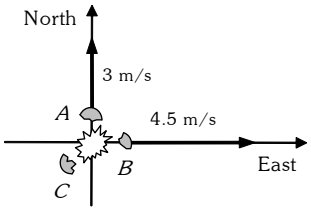
$$\sum_{i=1}^{n} p_{horizontal} = 0 \rightarrow \qquad -Mv + m(u - v) = 0$$

$$v = \frac{mu}{M + m}$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example

A shell fired vertically up, when reaches its highest point, explodes into three fragments A, B and C of masses m_A = 4 kg, m_B = 2 kg and m_C = 3kg. Immediately after the explosion, A is observed moving with velocity v_A = 3 m/s towards north and B with a velocity v_B = 4.5 m/s towards east as shown in the figure. Find the velocity v_C of the piece C.



Solution.

Explosion takes negligible duration; therefore, impulse of gravity, which is a finite external force, can be neglected. The pieces fly off acquiring above-mentioned velocities due to internal forces developed due to expanding gases produced during the explosion. The forces applied by the expanding gases are internal forces; hence, momentum of the system of the three pieces remains conserved during the explosion and total momentum before and after the explosion are equal.

Assuming the east as positive *x*-direction and the north as positive y-direction, the momentum vectors $\vec{p}_A$ and $\vec{p}_B$ of pieces A and B become

$$\vec{p}_A = m_A v_A \tilde{j} = 12 \text{ kg-m/s} \quad \text{and} \quad \vec{p}_B = m_B v_B \tilde{i} = 9 \text{ kg-m/s}$$

Before the explosion, momentum of the shell was zero, therefore from the principle of conservation of momentum, the total momentum of the fragments also remains zero.

$$\vec{p}_A + \vec{p}_B + \vec{p}_C = \vec{0} \rightarrow \qquad \vec{p}_C = -(9\tilde{i} + 12\tilde{j})$$

From the above equation, velocity of the piece C is

$$\vec{v}_C = \frac{\vec{p}_C}{m_C} = -(3\tilde{i} + 4\tilde{j}) = 5 \text{ m/s, } 53^\circ \text{ south of west.}$$

Example

In free space, three identical particles moving with velocities $v_o\tilde{i}$, $-3v_o\tilde{j}$ and $5v_o\tilde{k}$ collide successively with each other to form a single particle. Find velocity vector of the particle formed.

Solution.

Let m be the mass of a single particle before any of the collisions. The mass of particle formed after collisions must be 3m. In free space, no external forces act on any of the particles, their total momentum remains conserved.

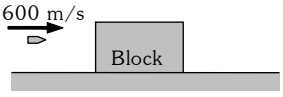
Applying principle of conservation of momentum, we have

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \rightarrow \qquad mv_o\tilde{i} - 3mv_o\tilde{j} + 5mv_o\tilde{k} = 3m\vec{v}$$

$$\vec{v} = \tfrac{1}{3}v_o\left(\tilde{i} - 3\tilde{j} + 5\tilde{k}\right) \text{ m/s}$$

Example

A bullet of mass 50 g moving with velocity 600 m/s hits a block of mass 1.0 kg placed on a rough horizontal ground and comes out of the block with a velocity of 400 m/s. The coefficient of friction between the block and the ground is 0.25. Neglect loss of mass of the block as the bullet pierces through it.



(a) In spite of the fact that friction acts as an external force, can you apply principle of conservation of momentum during interaction of the bullet with the block?

(b) Find velocity of the block immediately after the bullet pierces through it.

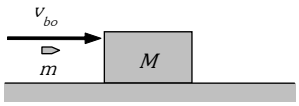
(c) Find the distance the block will travel before it stops.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

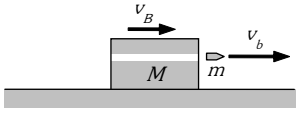
Solution.

(a) There is no net external force in the vertical direction and in the horizontal direction, only external force friction is non-impulsive, therefore momentum of the bullet-block system during their interaction remains conserved.

(b) Let us denote velocities of the bullet before it hits the block and immediately after it pierces through the block by v_{bo} and v_b, velocity of the block immediately after the bullet pierces through it is v_B and masses of the bullet and the block by m and M respectively. These are shown in the adjacent figure.



Immediately before the bullet hits the block



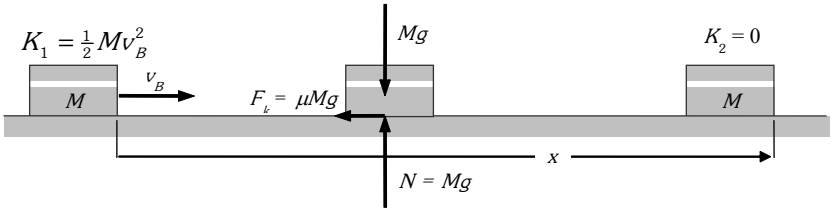
Immediately after the bullet pierces the block

Applying principle of conservation of momentum for horizontal component, we have

$$mv_{bo} = mv_b + Mv_B \rightarrow \qquad v_B = \frac{m(v_{bo} - v_b)}{M}$$

Substituting the given values, we have v_B = 10 m/s

(c) To calculate distance traveled by the block before it stops, work kinetic energy theorem has to be applied.



During sliding of the box on the ground only the force of kinetic friction does work.

$$W_{1\rightarrow 2} = K_2 - K_1 \rightarrow \qquad -\mu Mgx = 0 - \tfrac{1}{2}Mv_B^2$$

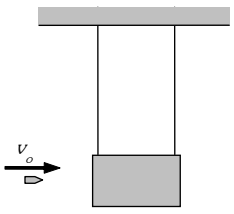
$$x = \frac{v_B^2}{2\mu g}$$

Substituting given values, we have x = 20 m

Example

Ballistic Pendulum : A ballistic pendulum is used to measure speed of bullets. It consists of a wooden block suspended from fixed support.

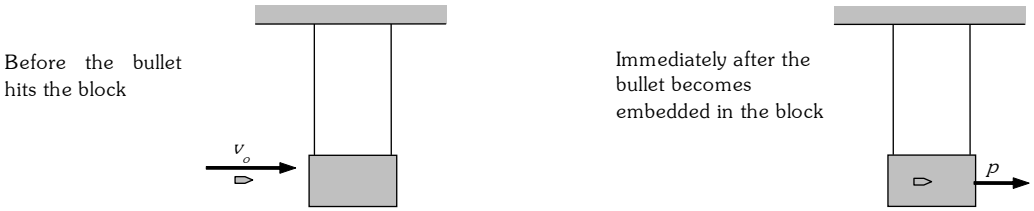
A wooden block of mass M is suspended with the help of two threads to prevent rotation while swinging. A bullet of mass m moving horizontally with velocity v_o hits the block and becomes embedded in the block. Receiving momentum from the bullet, the bullet-block system swings to a height h. Find expression for speed of the bullet in terms of given quantities.



Solution.

When the bullet hits the block, in a negligible time interval, it becomes embedded in the block and the bullet-block system starts moving with horizontally. During this process, net force acting on the bullet-block system in vertical direction is zero and no force acts in the horizontal direction. Therefore, momentum of the bullet-block system remains conserved.

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65



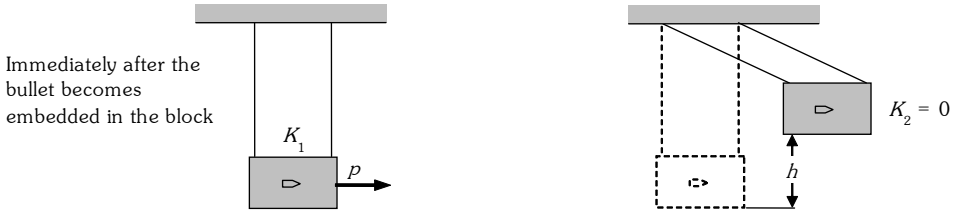
Let us denote momentum of the bullet-block system immediately after the bullet becomes embedded in the block by p and apply principle of conservation of momentum to the system for horizontal component of momentum.

$$p = mv_o$$

Using equation $K = p^2 / 2m$, we can find kinetic energy K_1 of the bullet-block system immediately after the bullet becomes embedded in the block.

$$K_1 = \frac{(mv_o)^2}{2(M+m)}$$

During swing, only gravity does work on the bullet-block system. Applying work-kinetic energy theorem during swing of the bullet-block system, we have



$$W_{1\to 2} = K_2 - K_1 \rightarrow \qquad -(M+m)gh = 0 - \frac{(mv_o)^2}{2(M+m)}$$

Rearranging terms, we have

$$v_o = \frac{(M+m)}{m}\sqrt{2gh}$$

Impact between two bodies

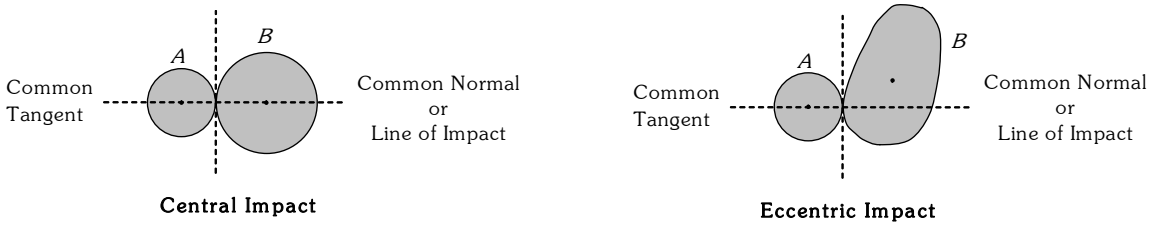
Impact or collision is interaction of very small duration between two bodies in which the bodies apply relatively large forces on each other.

Interaction forces during an impact are created due to either direct contact or strong repulsive force fields or some connecting links. These forces are so large as compared to other external forces acting on either of the bodies that the effects of later can be neglected. The duration of the interaction is short enough as compared to the time scale of interest as to permit us only to consider the states of motion just before and after the event and not during the impact. Duration of an impact ranges from 10^{-23} s for impacts between elementary particles to millions of years for impacts between galaxies. The impacts we observe in our everyday life like that between two balls last from 10^{-3} s to few seconds.

Central and Eccentric Impact

The common normal at the point of contact between the bodies is known as line of impact. If mass centers of the both the colliding bodies are located on the line of impact, the impact is called central impact and if mass centers of both or any one of the colliding bodies are not on the line of impact, the impact is called eccentric impact.

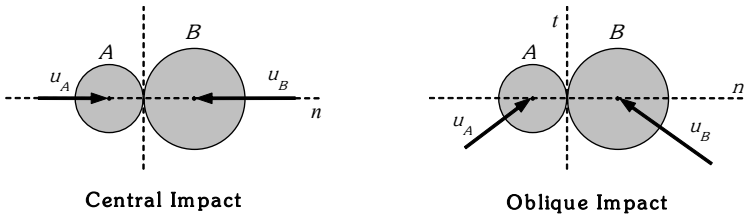
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65



Central impact does not produce any rotation in either of the bodies whereas eccentric impact causes the body whose mass center is not on the line of impact to rotate. Therefore, at present we will discuss only central impact and postpone analysis of eccentric impact to cover after studying rotation motion.

Head–on (Direct) and Oblique Central Impact

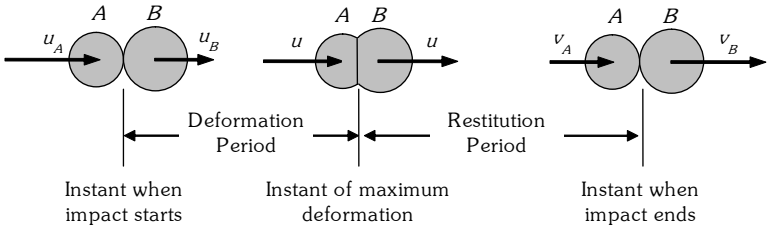
If velocities vectors of the colliding bodies are directed along the line of impact, the impact is called a direct or head-on impact; and if velocity vectors of both or of any one of the bodies are not along the line of impact, the impact is called an oblique impact.



In this chapter, we discuss only central impact, therefore the term central we usually not use and to these impacts, we call simply head-on and oblique impacts. Furthermore, use of the line of impact and the common tangent is so frequent in analysis of these impacts that we call them simply *t*-axis and *n*-axis.

Head–on (Direct) Central Impact

To understand what happens in a head-on impact let us consider two balls A and B of masses m_A and m_B moving with velocities u_A and u_B in the same direction as shown. Velocity u_A is larger than u_B so the ball A hits the ball B. During impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they tries to regain their original shape due to elastic behaviors of the materials forming the balls.

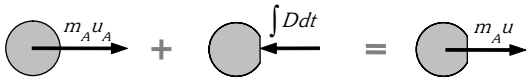


The time interval when deformation takes place is called the deformation period and the time interval in which the ball try to regain their original shapes is called the restitution period. Due to push applied by the balls on each other during period of deformation speed of the ball A decreases and that of the ball B increases and at the end of the deformation period, when the deformation is maximum both the ball move with the same velocity say it is u. Thereafter, the balls will either move together with this velocity or follow the period of restitution. During the period of restitution due to push applied by the balls on each other, speed of the ball A decrease further and that of ball B increase further till they separate from each other. Let us denote velocities of the balls A and B after the impact by v_A and v_B respectively.

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

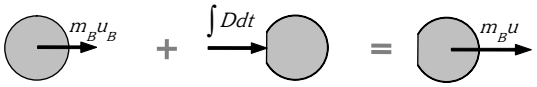
Equation of Impulse and Momentum during impact

Impulse momentum principle describes motion of ball A during deformation period.



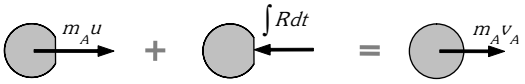
$$m_A u_A - \int Ddt = m_A u \qquad \text{...(i)}$$

Impulse momentum principle describes motion of ball B during deformation period.



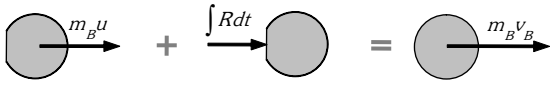
$$m_B u_B + \int Ddt = m_B u \qquad \text{...(ii)}$$

Impulse momentum principle describes motion of ball A during restitution period.



$$m_A u - \int Rdt = m_A v_A \qquad \text{...(iii)}$$

Impulse momentum principle describes motion of ball B during restitution period.



$$m_B u + \int Rdt = m_B v_B \qquad \text{...(iv)}$$

Conservation of Momentum during impact

From equations, (i) and (ii) we have $m_A u_A + m_B u_B = (m_A + m_B)u$ ...(v)

From equations, (iii) and (iv) we have $(m_A + m_B)u = m_A v_A + m_B v_B$ ...(vi)

From equations, (v) and (vi) we obtain the following equation.

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad \text{...(vii)}$$

The above equation elucidates the principle of conservation of momentum.

Coefficient of Restitution

Usually the force D applied by the bodies A and B on each other during period differs from the force R applied by the bodies on each other during period of restitution. Therefore, it is not necessary that magnitude of impulse $\int Ddt$ of deformation equals to the magnitude of impulse $\int Rdt$ restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

$$e = \frac{\int Rdt}{\int Ddt} \qquad \text{...(viii)}$$

Now from equations (i), (ii), (iii) and (iv), we have

$$e = \frac{v_B - v_A}{u_A - u_B} \qquad \text{...(ix)}$$

Coefficient of restitution depend on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collision where kinetic energy is generated its value may exceed one.

Depending on values of coefficient of restitution, two particular cases are of special interest.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Perfectly Plastic or Inelastic Impact For these impacts $e = 0$, and bodies undergoing impact stick to each other after the impact.

Perfectly Elastic Impact For these impacts $e = 1$.

Strategy to solve problems of head-on impact

Write momentum conservation equation

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \text{...(A)}$$

Write rearranging terms of equation of coefficient of restitution

$$v_B - v_A = e(u_A - u_B) \quad \text{...(B)}$$

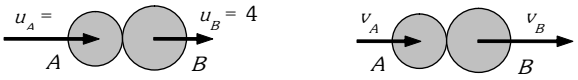
Use the above equations A and B.

Example

A ball of mass 2 kg moving with speed 5 m/s collides directly with another of mass 3 kg moving in the same direction with speed 4 m/s. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution.

Denoting the first ball by A and the second ball by B velocities immediately before and after the impact are shown in the figure.



Immediately before impact starts

Immediately after impact ends

Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow \; 3v_B + 2v_A = 2 \times 5 + 3 \times 4$$

$$3v_B + 2v_A = 22 \quad \text{...(i)}$$

Applying equation of coefficient of restitution, we have

$$v_B - v_A = e(u_A - u_B) \rightarrow \qquad v_B - v_A = \tfrac{2}{3}(5 - 4)$$

$$3v_B - 3v_A = 2 \quad \text{...(ii)}$$

From equation (i) and (ii), we have v_A = 4 m/s and v_B = 4.67 m/s **Ans.**

Example

A block of mass 5 kg moves from left to right with a velocity of 2 m/s and collides with another block of mass 3 kg moving along the same line in the opposite direction with velocity 4 m/s.

(a) If the collision is perfectly elastic, determine velocities of both the blocks after their collision.

(b) If coefficient of restitution is 0.6, determine velocities of both the blocks after their collision.

Solution.

Denoting the first block by A and the second block by B velocities immediately before and after the impact are shown in the figure.



Immediately before impact starts

Immediately after impact ends

Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow \; 3v_B + 5v_A = 5 \times 2 + 3 \times (-4)$$

$$3v_B + 5v_A = -2 \quad \text{...(i)}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Applying equation of coefficient of restitution, we have

$v_B - v_A = e(u_A - u_B) \rightarrow \quad v_B - v_A = e\{2-(-4)\}$

$v_B - v_A = 6e \quad \ldots(ii)$

(a) For perfectly elastic impact $e = 1$. Using this value in equation (ii), we have

$v_B - v_A = 6 \quad \ldots(iia)$

Now from equation (i) and (iia), we obtain

$v_A = -2.5$ m/s and $v_B = 3.5$ m/s

(b) For value $e = 0.6$, equation 2 is modified as

$v_B - v_A = 3.6 \quad$ (iib)

Now from equation (i) and (iib), we obtain

$v_A = -1.6$ m/s and $v_B = 2.0$ m/s

Block A reverse back with speed 1.6 m/s and B also move in opposite direction to its original direction with speed 2.0 m/s.

Example

Two identical balls A and B moving with velocities u_A and u_B in the same direction collide. Coefficient of restitution is e.

(a) Deduce expression for velocities of the balls after the collision.

(b) If collision is perfectly elastic, what do you observe?

Solution.

Equation expressing momentum conservation is

$v_A + v_B = u_A + u_B \quad \ldots(A)$

Equation of coefficient of restitution is

$v_B - v_A = eu_A - eu_B \quad \ldots(B)$

(a) From the above two equations, velocities v_A and v_B are

$$v_A = \left(\frac{1-e}{2}\right)u_A + \left(\frac{1+e}{2}\right)u_B \quad \ldots(i)$$

$$v_B = \left(\frac{1+e}{2}\right)u_A + \left(\frac{1-e}{2}\right)u_B \quad \ldots(ii)$$

(b) For perfectly elastic impact $e = 1$, velocities v_A and v_B are

$v_A = u_B \quad \ldots(iii)$

$v_B = u_A \quad \ldots(iv)$

Identical bodies exchange their velocities after perfectly elastic impact.

Conservation of kinetic energy in perfectly elastic impact

For perfectly elastic impact equation for conservation of momentum and coefficient of restitution are

$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \ldots(A)$

$v_B - v_A = u_A - u_B \quad \ldots(B)$

Rearranging the terms of the above equations, we have

$m_A(v_A - u_A) = m_B(u_B - v_B)$

$u_A + v_A = v_B + u_B$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Multiplying LHS of both the equations and RHS of both the equations, we have

$$m_A\left(v_A^2 - u_A^2\right) = m_B\left(u_B^2 - v_B^2\right)$$

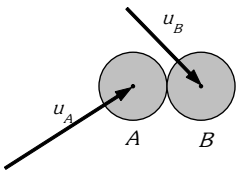
Multiplying by ½ and rearranging term of the above equation, we have

$$\tfrac{1}{2}m_A u_A^2 + \tfrac{1}{2}m_B u_B^2 = \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2$$

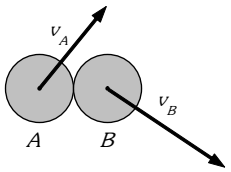
In perfectly elastic impact total kinetic energy of the colliding body before and after the impact are equal. In inelastic impacts, there is always loss of kinetic energy.

Oblique Central Impact

In oblique central impact, velocity vectors of both or of any one of the bodies are not along the line of impact and mass center of bodies are on the line of impact. Due to impact speeds and direction of motion of both the balls change. In the given figure is shown two balls A and B of masses m_A and m_B moving with velocities u_A and u_B collide obliquely. After the collision let they move with velocities v_A and v_B as shown in the nest figure.

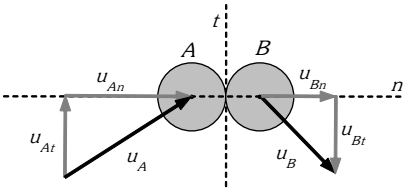


Immediately before Impact

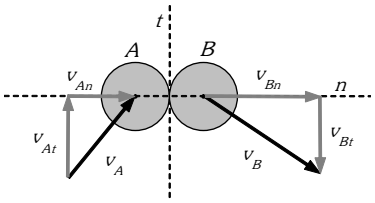


Immediately after Impact

To analyze the impact, we show components of velocities before and after the impact along the common tangent and the line of impact. These components are shown in the following figure.



Immediately before Impact



Immediately before Impact

Component along the t-axis

If surfaces of the bodies undergoing impact are smooth, they cannot apply any force on each other along the *t*-axis and component of momentum along the *t*-axis of each bodies, considered separately, is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_{At} = u_{At} \text{ and } v_{Bt} = u_{Bt} \qquad \text{...(A)}$$

Component along the n-axis

For components of velocities along the n-axis, the impact can be treated same as head-on central impact.

The component along the n-axis of the total momentum of the two bodies is conserved

$$m_B v_{Bn} + m_A v_{An} = m_B u_{Bn} + m_A u_{An} \qquad \text{...(B)}$$

Concept of coefficient of restitution e is applicable only for the *n*-component velocities.

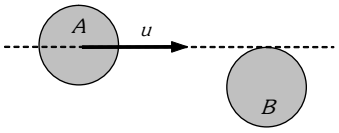
$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \qquad \text{...(C)}$$

The above four independent equation can be used to analyze oblique central impact of two freely moving bodies.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

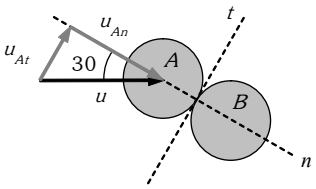
Example

A disk sliding with velocity u on a smooth horizontal plane strikes another identical disk kept at rest as shown in the figure. If the impact between the disks is perfectly elastic impact, find velocities of the disks after the impact.

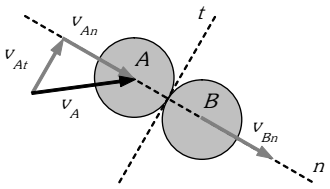


Solution.

(a) We first show velocity components along the t and the n-axis immediately before and after the impact. angle that the line of impact makes with velocity u is 30 .



Immediately before Impact



Immediately after Impact

Component along t-axis

Components of momentum along the *t*-axis of each disk, considered separately, is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_{At} = u_{At} = \frac{u}{2} \text{ and } v_{Bt} = u_{Bt} = 0 \qquad \text{...(i)}$$

Component along n-axis

The component along the *n*-axis of the total momentum of the two bodies is conserved

$$m_B v_{Bn} + m_A v_{An} = m_B u_{Bn} + m_A u_{An} \rightarrow m v_{Bn} + m v_{An} = m \times 0 + m\frac{u\sqrt{3}}{2}$$

$$v_{Bn} + v_{An} = \frac{u\sqrt{3}}{2} \qquad \text{...(ii)}$$

Concept of coefficient of restitution *e* is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \quad v_{Bn} - v_{An} = \frac{u\sqrt{3}}{2} \qquad \text{...(iii)}$$

From equations (ii) and (iii), we have $v_{An} = 0$ and $v_{Bn} = \frac{u\sqrt{3}}{2}$ ...(iv)

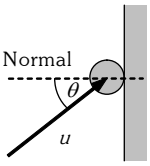
From equations (i) and (iv) we can write velocities of both the disks.

Example

A ball collides with a frictionless wall with velocity u as shown in the figure. Coefficient of restitution for the impact is *e*.

(a) Find expression for the velocity of the ball immediately after the impact.

(b) If impact is perfectly elastic what do you observe?

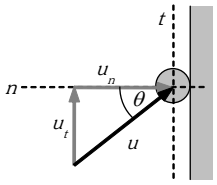


Solution.

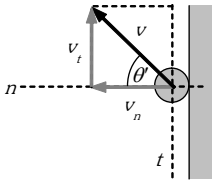
(a) Let us consider the ball as the body *A* and the wall as the body *B*. Since the wall has infinitely large inertia (mass) as compared to the ball, the state of motion of the wall, remains unaltered during the impact i.e. the wall remain stationary.

Now we show velocities of the ball and its t and n-components immediately before and after the impact. For the purpose we have assumed velocity of the ball after the impact *v*.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65



Immediately before Impact



Immediately after Impact

Component along t-axis

Components of momentum along the *t*-axis of the ball is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_t = u_t = u\sin\theta \quad ...(i)$$

Component along n-axis

Concept of coefficient of restitution *e* is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \qquad -v_n = eu_n$$

$$v_n = -eu\cos\theta \quad ...(ii)$$

From equations (i) and (ii), the *t* and *n*-components of velocity of the ball after the impact are

$$v_t = u\sin\theta \quad \text{and} \quad v_n = eu\sin\theta$$

(b) If the impact is perfectly elastic, we have $v_t = u\sin\theta$, $v_n = u\sin\theta$ and $\theta' = \theta$

The ball will rebound with the same speed making the same angle with the vertical at which it has collided. In other words, a perfectly elastic collision of a ball with a wall follows the same laws as light follows in reflection at a plane mirror.

Oblique Central Impact when one or both the colliding bodies are constrained in motion

In oblique collision, we have discussed how to analyze impact of bodies that were free to move before as well as after the impact. Now we will see what happens if one or both the bodies undergoing oblique impact are constrained in motion.

Component along the t-axis

If surfaces of the bodies undergoing impact are smooth, the *t*-component of the momentum of the body that is free to move before and after the impact remain conserved.

If both the bodies are constrained, the *t*-component of neither one remains conserved.

Momentum Conservation

We may find a direction in which no external force acts on both the bodies. The component of total momentum of both the bodies along this direction remains conserved.

Coefficient of restitution

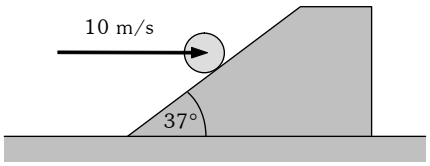
Concept of coefficient of restitution *e* is applicable only for the n-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right)$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

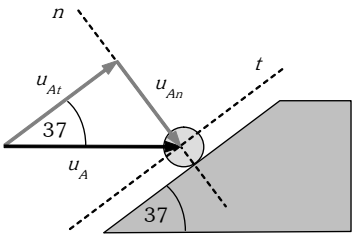
Example

A 250 g ball moving horizontally with velocity 10.0 m/s strikes inclined surface of a 720 g smooth wedge as shown in the figure. The wedge is placed at rest on a frictionless horizontal ground. If the coefficient of restitution is 0.8, calculate the velocity of the wedge after the impact.

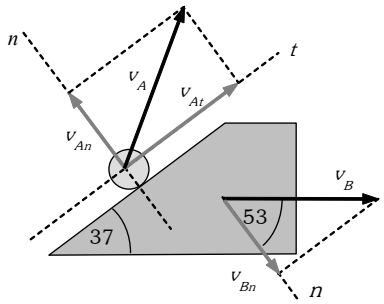


Solution.

Let us consider the ball as the body *A* and the wedge as the body *B*. After the impact, the ball bounces with velocity v_A and the wedge advances in horizontal direction with velocity v_B. These velocities and their *t* and *n*-components are immediately before and after the impact are shown in the following figures.



Immediately before Impact



Immediately after Impact

Component along t-axis

The ball is free to move before and after the impact, therefore its *t*-component of momentum conserved. Hence, *t*-component of velocities of the ball remains unchanged.

$$v_{At} = u_{At} = 10\cos 37° = 8 \text{ m/s} \qquad \text{...(i)}$$

Momentum Conservation

In the horizontal direction, there is no external force on both the bodies. Therefore horizontal component of total momentum of both the bodies remain conserved.

$$m_A u_A = m_A\left(v_{At}\cos 37° - v_{An}\cos 53°\right) + m_B v_B$$

$$0.25\times 10 = 0.25\left(8\times\frac{4}{5} - \frac{3v_{An}}{5}\right) + 0.72 v_B \qquad \text{...(ii)}$$

Coefficient of restitution

Concept of coefficient of restitution *e* is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \quad \frac{3v_B}{5} - v_{An} = 0.8(6-0) \qquad \text{...(iii)}$$

From equations (i), (ii) and (iii), we obtain v_B = 2.0 m/s

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

SYSTEM OF PARTICLES

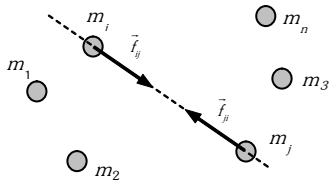
Study of kinematics enables us to explore nature of translation motion without any consideration to forces and energy responsible for the motion. Study of kinetics enables us to explore effects of forces and energy on motion. It includes Newton's laws of motion, methods of work and energy and methods of impulse and momentum. The methods of work and energy and methods of impulse and momentum are developed using equation $\vec{F} = m\vec{a}$ together with the *methods of kinematics*. The advantage of these methods lie in the fact that they make determination of acceleration unnecessary. Methods of work and energy directly relate force, mass, velocity and displacement and enable us to explore motion between two points of space i.e. in a space interval whereas methods of impulse and momentum enable us to explore motion in a time interval. Moreover methods of impulse and momentum provides only way to analyze impulsive motion.

The work energy theorem and impulse momentum principle are developed from Newton's second law, and we have seen how to apply them to analyze motion of single particle i.e. translation motion of rigid body. Now we will further inquire into possibilities of applying these principles to a system of large number of particles or rigid bodies in translation motion.

System of Particles

By the term system of particles, we mean a well defined collection of several or large number of particles, which may or may not interact or be connected to each other.

As a schematic representation, consider a system of n particles of masses $m_1, m_2, \ldots m_i \ldots m_j \ldots$ and m_n respectively. They may be actual particles of rigid bodies in translation motion. Some of them may interact with each other and some of them may not. The particles, which interact with each other, apply forces on each other. The forces of interaction $\vec{f}_{ij}$ and $\vec{f}_{ji}$ between a pair of i^{th} and j^{th} particles are shown in the figure. Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles of the system are *internal forces* of the system.



System of n interacting particles.

These internal forces always exist in pairs of forces of equal magnitudes and opposite directions. It is not necessary that all the particles interact with each other; some of them, which do not interact with each other, do not apply mutual forces on each other. Other than internal forces, external forces may also act on all or some of the particles. Here by the term *external force* we mean a force that is applied on any one of the particle included in the system by some other body out-side the system.

In practice we usually deal with extended bodies, which may be deformable or rigid. An extended body is also a system of infinitely large number of particles having infinitely small separations between them. When a body undergoes deformation, separations between its particles and their relative locations change. A rigid body is an extended body in which separations and relative locations of all of its particles remain unchanged under all circumstances.

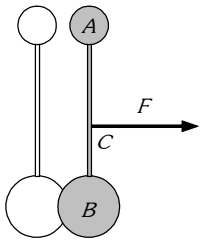
System of Particles and Mass Center

Until now we have deal with translation motion of rigid bodies, where a rigid body can be treated as a particle. When a rigid body undergoes rotation, all of its particles do not move in identical fashion, still we must treat it a system of particles in which all the particles are rigidly connected to each other. On the other hand we may have particles or bodies not connected rigidly to each other but may be interacting with each other through internal forces. Despite the complex motion of which a system of particles is capable, there is a single point, known as **center of mass** or **mass center** (CM), whose translation motion is characteristic of the system.

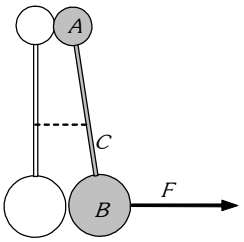
The existence of this special point can be demonstrated in the following examples dealing with a rigid body. Consider two disks A and B of unequal masses connected by a very light rigid rod. Place it on a very smooth

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

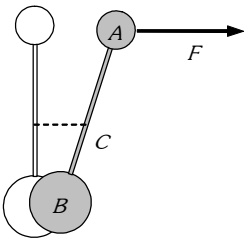
table. Now pull it horizontally applying a force at different points. You will find a point nearer to the heavier disk, on which if the force is applied the whole assembly undergoes translation motion. Furthermore you cannot find any other point having this property. This point is the mass center of this system. We can assume that all the mass were concentrated at this point. In every rigid body we can find such a point. If you apply the force on any other point, the system moves forward and rotates but the mass center always translates in the direction of the force.



Force applied on the mass center

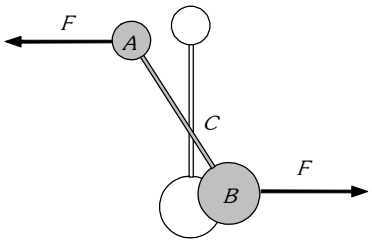


Force not applied on the mass center



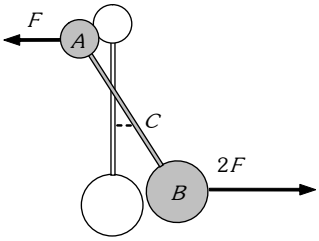
Force not applied on the mass center

In another experiment, if two forces of equal magnitudes are applied on the disks in opposite directions, the system will rotate, but the mass center *C* remains stationary as shown in the following figure.



Body rotates but the mass center remains stationary under action of equal and opposite forces.

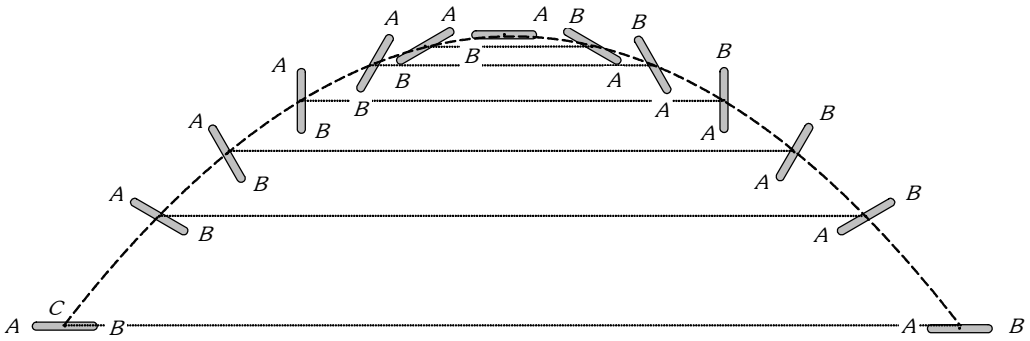
If the above experiment is repeated with both disks *A* and *B* of identical masses, the mass center will be the mid point. And if the experiment is repeated with a uniform rod, the mass center again is the mid point.



Body rotates and the mass center translates under action of unbalanced forces applied at different points.

As another example let us throw a uniform rod in air holding it from one of its ends so that it rotates also. Snapshots taken after regular intervals of time are shown in the figure. The rod rotates through 360 . As the rod moves all of its particles move in a complex manner except the mass center *C*, which follows a parabolic trajectory as if it were a particle of mass equal to that of the rod and force of gravity were acting on it.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65



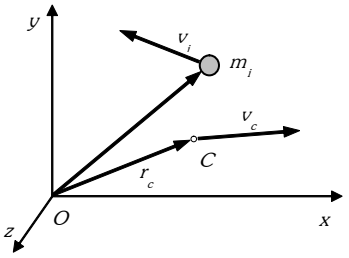
Thus mass center of a rigid body or system of particles is a point, whose translation motion under action of unbalanced forces is same as that of a particle of mass equal to that of the body or system under action the same unbalanced forces. And if different forces having a net resultant are applied at different particles, the system rotates but the mass center translates as if it were a particle of the mass same as that of the system and the net resultant were applied on it.

Concept of mass center provides us a way to look into motion of the system as a whole as superposition of translation of the mass center and motion of all the particles relative to the mass center. In case of rigid bodies all of its particles relative to the mass center can move only on circular paths because they cannot changes their separations.

The concept of mass center is used to represent gross translation of the system. Therefore total linear momentum of the whole system must be equal to the linear momentum of the system due to translation of its mass center.

Center of Mass of System of Discrete Particles

A system of several particles or several bodies having finite separations between them is known as system of discrete particles. Let at an instant particles of such a system $m_1, m_2, \ldots m_i, \ldots\ldots$ and m_n are moving with velocities $\vec{v}_1$, $\vec{v}_1, \vec{v}_2, \ldots\ldots \vec{v}_i, \ldots\ldots$ and $\vec{v}_n$ at locations $\vec{r}_1, \vec{r}_2, \ldots\ldots \vec{r}_i, \ldots\ldots$ and $\vec{r}_n$ respectively. For the sake of simplicity only i^{th} particle and the mass center C are shown in the figure. The mass center C located at $\vec{r}_c$ is moving with velocity $\vec{v}_c$ at this instant.



As the mass center represents gross translation motion of the whole system, the total linear i.e. sum of linear momenta of all the particles must be equal to linear momentum of the whole mass due to translation of the mass center.

$$m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots\ldots\ldots + m_i\vec{v}_i + \ldots\ldots\ldots\ldots\ldots + m_n\vec{v}_n = M\vec{v}_c$$

We can write the following equation in terms of masses and position vectors as an analogue to the above equation. This equation on differentiating with respect to time yields the above equation therefore can be thought as solution of the above equation.

$$m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots\ldots\ldots + m_i\vec{r}_i + \ldots\ldots\ldots\ldots\ldots + m_n\vec{r}_n = M\vec{r}_c$$

If $M = \Sigma m_i$ denotes total mass of the system, the above two equations can be written in short as

$$\Sigma m_i\vec{v}_i = M\vec{v}_c \quad (1)$$

$$\Sigma m_i\vec{r}_i = M\vec{r}_c \quad (2)$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

The above equation suggests location of mass center of a system of discreet particles.

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots\ldots\ldots + m_i\vec{r}_i + \ldots\ldots\ldots\ldots\ldots + m_n\vec{r}_n}{M} = \frac{\Sigma m_i\vec{r}_i}{M} \quad (3)$$

Cartesian coordinate (x_c, y_c, z_c) of the mass center are components of the position vector $\vec{r}_c$ of the mass center.

$$x_c = \frac{\Sigma m_i x_i}{M};\ y_c = \frac{\Sigma m_i y_i}{M};\ z_c = \frac{\Sigma m_i z_i}{M} \quad (4)$$

Example

Center of Mass of Two Particle System

(a) Find expression of position vector of mass center of a system of two particles of masses m_1 and m_2 located at position vectors $\vec{r}_1$ and $\vec{r}_2$.

(b) Express Cartesian coordinates of mass center, if particle m_1 at point (x_1, y_1) and particle m_2 at point (x_2, y_2).

(c) If you assume origin of your coordinate system at the mass center, what you conclude regarding location of the mass center relative to particles.

(d) Now find location of mass center of a system of two particles masses m_1 and m_2 separated by distance *r*.

Solution.

(a) Consider two particles of masses m_1 and m_2 located at position vectors $\vec{r}_1$ and $\vec{r}_2$. Let their mass center *C* at position vector $\vec{r}_c$.

From eq. , we have

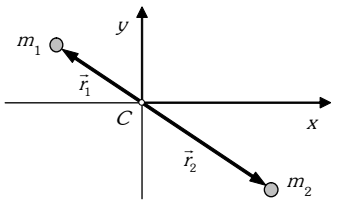
$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{M} \rightarrow \qquad \vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

(b) From result obtained in part *(a)*, we have

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \text{ and } y_c = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

(c) If we assume origin at the mass center vector $\vec{r}_c$ vanishes and we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$



Since either of the masses m_1 and m_2 cannot be negative, to satisfy the above equation, vectors $\vec{r}_1$ and $\vec{r}_2$ must have opposite signs. It is geometrically possible only when mass center *C* lies between the two particles on the line joining them as shown in the figure.

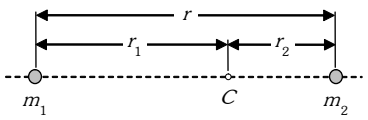
If we substitute magnitudes r_1 and r_2 of vectors $\vec{r}_1$ and $\vec{r}_2$ in the above equation, we have $m_1r_1 = m_2r_2$, which suggest

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

Now we conclude that mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

(d) Consider two particles masses m_1 and m_2 at distance r from each other. There mass center C must lie in between them on the line joining them. Let distances of these particles from the mass center are r_1 and r_2.



Since mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles, we can write

$$r_1 = \frac{m_2 r}{m_1 + m_2} \text{ and } r_2 = \frac{m_1 r}{m_1 + m_2}$$

Example

Mass centre of several particles

Find position vectors of mass center of a system of three particle of masses 1 kg, 2 kg and 3 kg located at position vectors $\vec{r}_1 = \left(4\tilde{i} + 2\tilde{j} - 3\tilde{k}\right)$ m, $\vec{r}_2 = \left(\tilde{i} - 4\tilde{j} + 2\tilde{k}\right)$ m and $\vec{r}_3 = \left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)$ m respectively.

Solution.

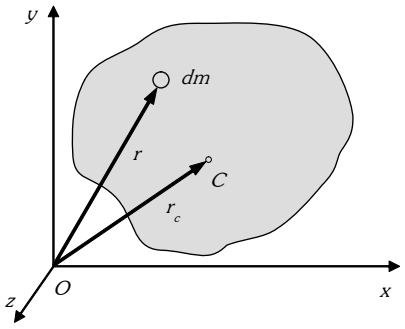
From eq. , we have

$$\vec{r}_c = \frac{\Sigma m_i \vec{r}_i}{M} \rightarrow \qquad \vec{r}_c = \frac{1\left(4\tilde{i} + 2\tilde{j} - 3\tilde{k}\right) + 2\left(\tilde{i} - 4\tilde{j} + 2\tilde{k}\right) + 3\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)}{1 + 2 + 3} = 2\tilde{i} - 2\tilde{j} + \frac{2}{3}\tilde{k}$$

Center of Mass of an Extended Body or Continuous Distribution of Mass

An extended body is collection of infinitely large number of particles so closely located that we neglect separation between them and assume the body as a continuous distribution of mass. A rigid body is an extended body in which relative locations of all the particles remain unchanged under all circumstances. Therefore a rigid body does not get deformed under any circumstances.

Let an extended body is shown as a continuous distribution of mass by the shaded object in the figure. Consider an infinitely small portion of mass dm of this body. It is called a mass element and is shown at position given by position vector $\vec{r}$. Total mass M of the body is $M = \int dm$. The mass center C is assumed at position given by position vector $\vec{r}_c$. Position vector of centre of mass of such a body is given by the following equation.



$$\vec{r}_c = \frac{\int \vec{r} dm}{M} \qquad (5)$$

Cartesian coordinate (x_c, y_c, z_c) of the mass center are components of the position vector $\vec{r}_c$ of the mass center.

$$x_c = \frac{\int x dm}{M}\ ;\ y_c = \frac{\int y dm}{M}\ ;\ z_c = \frac{\int z dm}{M} \qquad (6)$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example

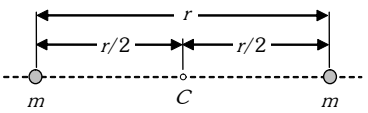
Mass centre of uniform symmetrical bodies.

Show that mass center of uniform and symmetric mass distributions lies on axis of symmetry.

Solution.

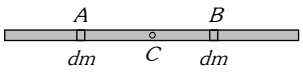
For simplicity first consider a system of two identical particles and then extend the idea obtained to a straight uniform rod, uniform symmetric plates and uniform symmetric solid objects.

Mass Center of a system of two identical particles



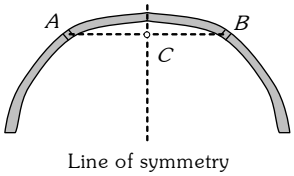
Mass center of a system of two identical particles lies at the midpoint between them on the lie joining them.

Mass Center of a system of a straight uniform rod



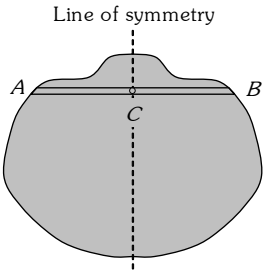
Consider two identical particles A and B at equal distances from the center C of the rod. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the mid point C of the rod. Therefore mass center of the whole rod must be at its mid point.

Mass Center of a system of a uniform symmetric curved rod



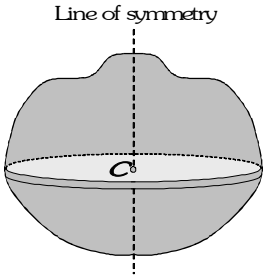
Consider two identical particles A and B located at equal distances from the line of symmetry. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the mid point C of the joining them. Therefore mass center of the whole rod must be on the axis of symmetry.

Mass Center of a uniform plate (lamina)



Consider a symmetric uniform plate. It can be assumed composed of several thin uniform parallel rods like rod AB shown in the figure. All of these rods have mass center on the line of symmetry, therefore the whole lamina has its mass center on the line of symmetry.

Mass Center of a uniform symmetric solid object



A uniform symmetric solid object occupies a volume that is made by rotating a symmetric area about its line of symmetry though 180º. Consider a uniform symmetric solid object shown in the figure. It can be assumed composed of several thin uniform parallel disks shown in the figure. All of these disks have mass center on the line of symmetry, therefore the whole solid object has its mass center on the line of symmetry.

Mass Center of uniform bodies

Following the similar reasoning, it can be shown that mass center of uniform bodies lies on their geometric centers.

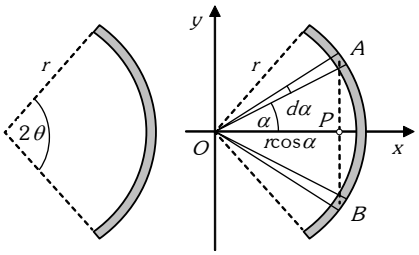
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Example

Mass Center of a system of a segment of a uniform circular rod (arc)

Find location of mass center of a thin uniform rod bent into shape of an arc.

Solution.



Consider a thin rod of uniform line mass density λ (mass per unit length) and radius r subtending angle 2θ on its center O.

The angle bisector OP is the line of symmetry, and mass center lies on it. Therefore if we assume the angle bisector as one of the coordinate axes say x-axis, y-coordinate of mass center becomes zero.

Let two very small segments A and B located symmetric to the line of symmetry (x-axis). Mass center of these two segments is on P at a distance $x = r\cos\alpha$ from center O. Total mass of these two elements is $dm = 2\lambda r d\alpha$. Now using eq. , we have

$$x_c = \frac{\int x dm}{M} \rightarrow \qquad x_c = \frac{\int_{-\theta}^{\theta} (r\cos\alpha)(2\lambda r d\alpha)}{\lambda r \theta} = \frac{r\sin\theta}{\theta}$$

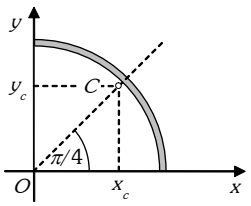
Mass center of a thin uniform arc shaped rod of radius r subtending angle 2θ at the center lies on its angle bisector at distance OC from the center.

$$OC = \frac{r\sin\theta}{\theta}$$

Example

Find coordinates of mass center of a quarter ring of radius r placed in the first quadrant of a Cartesian coordinate system, with centre at origin.

Solution.



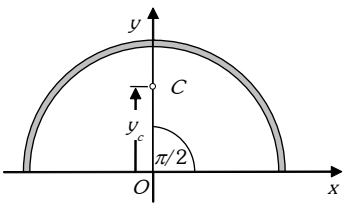
Making use of the result obtained in the previous example, distance OC of the mass center form the center is $OC = \frac{r\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}r}{\pi}$

Coordinates of the mass center (x_c, y_c) are $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$

Example

Find coordinates of mass center of a semicircular ring of radius r placed symmetric to the y-axis of a Cartesian coordinate system.

Solution.



The y-axis is the line of symmetry, therefore mass center of the ring lies on it making x-coordinate zero.

Distance OC of mass center from center is given by the result obtained in example 4. Making use of this result, we have

$$OC = \frac{r\sin\theta}{\theta} \rightarrow \qquad y_c = \frac{r\sin(\pi/2)}{\pi/2} = \frac{2r}{\pi}$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

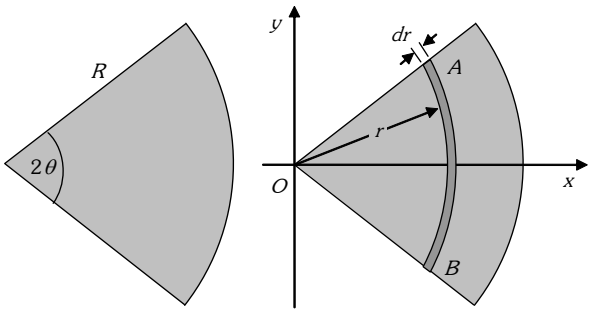
Example

Mass Center of a sector of a uniform circular plate

Find location of mass center of a sector of a thin uniform plate.

Solution.

Consider a sector of a thin uniform plate of surface mass density mass density σ (mass per unit area) and radius r subtending angle 2θ on its center.



Let a thin arc of radius r and width dr be an infinitely small part of the sector. Mass dm of the arc AB equals to product of mass per unit area and area of the arc.

$$dm = \sigma(2r\theta dr) = 2\sigma r\theta dr$$

Due to symmetry mass center of this arc must be on the angle bisector i.e. on x-axis at distance $x = \frac{r\sin\theta}{\theta}$.

Using above two information in eq. , we obtain the mass center of the sector.

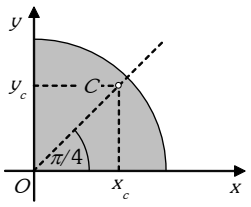
$$x_c = \frac{\int x dm}{M} \rightarrow \qquad x_c = \frac{\int_0^R \left(\frac{r\sin\theta}{\theta}\right)(\sigma r\theta dr)}{\sigma(\text{Area of thre sector})} = \frac{\int_0^R \left(\frac{r\sin\theta}{\theta}\right)(2\sigma r\theta dr)}{\sigma r\theta^2} = \frac{2r\sin\theta}{3\theta}$$

Example

Find coordinates of mass center of a quarter sector of a uniform disk of radius r placed in the first quadrant of a Cartesian coordinate system with centre at origin.

Solution.

Making use of the result obtained in the previous example, distance OC of the mass center form the center is



$$OC = \frac{2r\sin(\pi/4)}{3\pi/4} = \frac{4\sqrt{2}r}{3\pi}$$

Coordinates of the mass center (x_c, y_c) are $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$

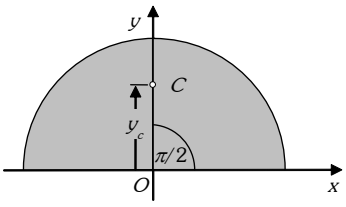
Example

Find coordinates of mass center of a uniform semicircular plate of radius r placed symmetric to the y-axis of a Cartesian coordinate system, with centre at origin.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Solution.

The y-axis is the line of symmetry, therefore mass center of the plate lies on it making x-coordinate zero.



Distance OC of mass center from center is given by the result obtained in example 7. Making use of this result, we have

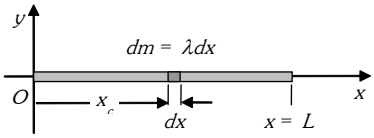
$$OC = \frac{2r\sin\theta}{3\theta} \rightarrow \quad y_c = \frac{2r\sin(\pi/2)}{3\pi/2} = \frac{4r}{3\pi}$$

Example

Find coordinates of mass center of a non-uniform rod of length L whose linear mass density λ varies as λ=a+bx, where x is the distance from the lighter end.

Solution.

Assume the rod lies along the x-axis with its lighter end on the origin to make mass distribution equation consistent with coordinate system.



Making use of eq. , we have $x_c = \frac{\int x dm}{M} \rightarrow x_c = \frac{\int_0^L x\lambda dx}{\int_0^L \lambda dx} = \frac{\int_0^L x(ax+b)dx}{\int_0^L (ax+b)dx} = \frac{(2aL+3b)L}{3(aL+2b)}$

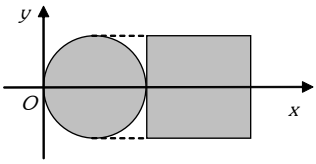
Example

Mass Center of composite bodies

A composite body is made of joining two or more bodies. Find mass center of the following composite body made by joining a uniform disk of radius r and a uniform square plate of the same mass per unit area.

Solution.

To find mass center the component bodies are assumed particle of masses equal to corresponding bodies located on their respective mass centers. Then we use equation to find coordinates of the mass center of the composite body.



To find mass center of the composite body, we first have to calculate masses of the bodies, because their mass distribution is given.

If we denote surface mass density (mass per unit area) by σ, masses of the bodies are

Mass of the disk $\quad m_d = \text{Mass per unit area} \times \text{Area} = \sigma(\pi r^2) = \sigma\pi r^2$

Mass of the square plate $\quad m_p = \text{Mass per unit area} \times \text{Area} = \sigma(r^2) = \sigma r^2$

Location of mass center of the disk $\quad x_d = \text{Center of the disk} = r \quad \text{and} \quad y_d = 0$

Location of mass center of the square plate $\quad x_p = \text{Center of the surface plate} = 3r \text{ and } y_d = 0$

Using eq. , we obtain coordinates (x_c, y_c) of the composite body.

$$x_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = \frac{r(\pi+3)}{(\pi+1)} \quad \text{and} \quad y_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = 0$$

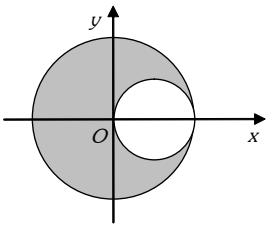
Coordinates of the mass center are $\left(\frac{r(\pi+3)}{(\pi+1)},\ 0\right)$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example

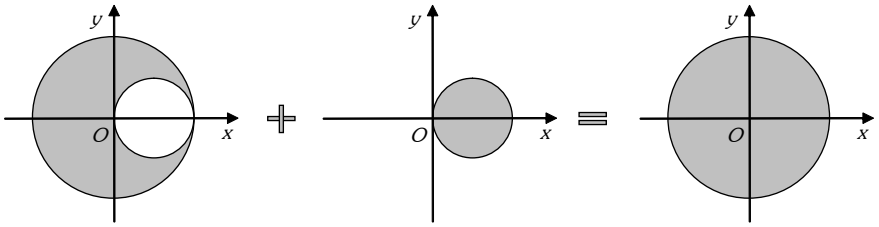
Mass Center of truncated bodies

A truncated body is made by removing a portion of a body. Find mass center of the following truncated disk made by removing disk of radius equal to half of the original disk as shown in the figure. Radius of the original uniform disk is *r*.



Solution.

To find mass center of truncated bodies we can make use of superposition principle that is, if we add the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

Denoting masses of the truncated body, removed portion and original body by m_{tb}, m_{rp} and m_{ob} and location of their mass centers by x_{tb}, x_{rp} and x_{ob}, we can write $m_{tb}x_{tb} + m_{rb}x_{rp} = m_{ob}x_{ob}$

From the above equation we obtain position co-ordinate x_{tb} of the mass center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} \qquad (1)$$

Denoting mass per unit area by σ , we can express the masses m_{tb}, m_{rp} and m_{ob}.

Mass of truncated body $m_{tb} = \sigma\left\{\pi\left(r^2 - \frac{r^2}{4}\right)\right\} = \frac{3\sigma\pi r^2}{4}$

Mass of the removed portion $m_{rp} = \frac{\sigma\pi r^2}{4}$

Mass of the original body $m_{ob} = \sigma\pi r^2$

Mass center of the truncated body x_{tb}

Mass center of the removed portion $x_{rp} = \frac{r}{2}$

Mass center of the original body $x_{ob} = 0$

Substituting the above values in equation (1), we obtain the mass the center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} = \frac{(\sigma\pi r^2)\times 0 - \left(\frac{\sigma\pi r^2}{4}\right)\left(\frac{r}{2}\right)}{\frac{3\sigma\pi r^2}{4}} = -\frac{r}{6}$$

Mass center of the truncated body is at point $\left(-\frac{r}{6},\ 0\right)$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Center of Mass Frame of Reference or Centroidal Frame

Center of mass frame of reference or centroidal frame is reference frame assume attached with the mass center of the system at its origin. It moves together with the mass center.

It is a special frame and presents simple interpretations and solutions to several phenomena. Let us first discuss some of its fundamental properties.

In centroidal frame center of mass is assumed at the origin, therefore position vector, velocity and acceleration of the mass center in centroidal frame all become zero.

- Sum of mass moments in centroidal frame vanishes.

  Mass moment of a particle is product of mass of the particle and its position vector.

$$\Sigma m_i \vec{r}_{i/c} = \vec{0} \quad \text{or} \quad m_1\vec{r}_{1/c} + m_2\vec{r}_{2/c} + \ldots\ldots\ldots\ldots + m_i\vec{r}_{i/c} + \ldots\ldots\ldots\ldots + m_n\vec{r}_{n/c} = \vec{0} \tag{7}$$

- Total linear momentum of the system in centroidal frame vanishes..

$$\Sigma m_i \vec{v}_{i/c} = \vec{0} \quad \text{or} \quad m_1\vec{v}_{1/c} + m_2\vec{v}_{2/c} + \ldots\ldots\ldots\ldots + m_i\vec{v}_{i/c} + \ldots\ldots\ldots\ldots + m_n\vec{v}_{n/c} = \vec{0} \tag{8}$$

Example

Motion of Mass Center in One Dimension

A jeep of mass 2400 kg is moving along a straight stretch of road at 80 km/h. It is followed by a car of mass 1600 kg moving at 60 km/h.

(a) How fast is the center of mass of the two cars moving?

(b) Find velocities of both the vehicles in centroidal frame.

Solution.

(a) Velocity of the mass center $\vec{v}_c = \dfrac{m_{jeep}\vec{v}_{jeep} + m_{car}\vec{v}_{car}}{m_{jeep} + m_{car}}$

Assuming direction of motion in the positive *x*-direction, we have

$$\vec{v}_c = \frac{m_{jeep}\vec{v}_{jeep} + m_{car}\vec{v}_{car}}{m_{jeep} + m_{car}} \rightarrow \qquad \vec{v}_c = \frac{2400\times 80 + 1600\times 60}{2400 + 1600} = 72 \text{ km/h}$$

(b) Velocity of the jeep in centroidal frame $v_{jeep/c} = 80 - 72 = 8$ km/h in positive *x*-direction.

Velocity of the car in centroidal frame $v_{car/c} = 60 - 72 = -12$ km/h

12 km/h negative *x*-direction direction.

Example

Motion of Mass Center in Vector Form

A 2.0 kg particle has a velocity of $\vec{v}_1 = \left(2.0\tilde{i} - 3.0\tilde{j}\right)$ m/s, and a 3.0 kg particle has a velocity

$\vec{v}_2 = \left(1.0\tilde{i} + 6.0\tilde{j}\right)$ m/s.

(a) How fast is the center of mass of the particle system moving?

(b) Find velocities of both the particles in centroidal frame.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Solution.

(a) Velocity of the mass center $\vec{v}_c = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

$\vec{v}_c = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \rightarrow$ $\vec{v}_c = \dfrac{2\left(2.0\tilde{i} - 3.0\tilde{j}\right) + 3\left(1.0\tilde{i} + 6.0\tilde{j}\right)}{2+3} = \left(1.4\tilde{i} + 2.4\tilde{j}\right)\text{m/s}$

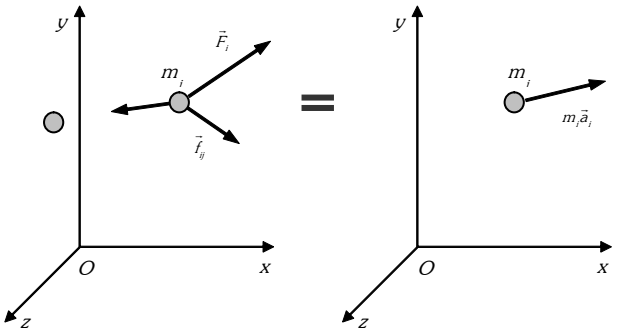
(b) Velocity of the first particle in centroidal frame

$\vec{v}_{1/c} = \vec{v}_1 - \vec{v}_c \rightarrow$ $\vec{v}_{1/c} = \left(2.0\tilde{i} - 3.0\tilde{j}\right) - \left(1.4\tilde{i} + 2.4\tilde{j}\right) = 0.6\left(\tilde{i} + \tilde{j}\right)\text{m/s}$

Velocity of the second particle in centroidal frame

$\vec{v}_{2/c} = \vec{v}_1 - \vec{v}_c \rightarrow$ $\vec{v}_{2/c} = \left(1.0\tilde{i} + 6.0\tilde{j}\right) - \left(1.4\tilde{i} + 2.4\tilde{j}\right) = -\left(0.4\tilde{i} + 3.6\tilde{j}\right)\text{m/s}$

Application of Newton's Laws of Motion to a System of Particles



In order to write equation of motion for a system of particles, we begin by applying Newton's second law to an individual particle.

Consider i^{th} particle of mass m_i. Internal force applied on it by j^{th} particle is shown by $\vec{f}_{ij}$. Other particles of the system may also apply internal forces on it. One of them is shown in the figure by an unlabeled vector. In addition to these internal forces, external forces may also be applied on it by bodies out side the system. Resultant of all these external forces is shown by vector $\vec{F}_i$. If under the action of these forces this particles has acceleration $\vec{a}_i$ relative to an inertial frame *Oxyz*, its free body diagram and kinetic diagram can be represented by the following figure and Newton's second law can be written by the following equation.

$$\vec{F}_i + \sum \vec{f}_{ij} = m_i\vec{a}_i$$

In similar fashion, we can write Newton's second law for all the particles of the system. These equations are

For 1st particle	$\vec{F}_1 + \sum \vec{f}_{1j} = m_1\vec{a}_1$
For 2nd particle	$\vec{F}_2 + \sum \vec{f}_{2j} = m_2\vec{a}_2$
.....	.....
For i^{th} particle	$\vec{F}_i + \sum \vec{f}_{ij} = m_i\vec{a}_i$
.....	.....
For n^{th} particle	$\vec{F}_n + \sum \vec{f}_{nj} = m_n\vec{a}_n$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Every internal force $\vec{f}_{ij}$ on particle m_i due to particle m_j and $\vec{f}_{ji}$ on the particle m_j due to particle m_i constituting Newton's third law pair must be equal in magnitude and opposite in direction, therefore the sum all these internal forces for all the particles must be zero. Keeping this fact in mind and denoting the mass of the whole system by M and acceleration of the mass center C by $\vec{a}_C$ relative to the inertial frame, Newton's second law representing translation motion of the system of particles particle can be represented by the following equation.

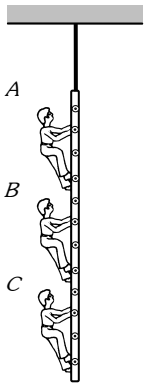
$$\sum \vec{F}_i = \sum \left(m_i \vec{a}_i\right) = M\vec{a}_C \tag{9}$$

$$\sum \vec{F}_i = M\vec{a}_C = \frac{d\vec{p}_c}{dt} \tag{10}$$

Example

Newton' Laws of Motion and System of Particles

A ladder of mass 20 kg is hanging from ceiling as shown in figure. Three men A, B and C of masses 40 kg, 60 kg, and 50 kg are climbing the ladder. Man A is climbing with upward retardation 2 m/s^2, B is climbing up with a constant speed of 0.5 m/s and C is climbing with upward acceleration of 1 m/s^2. Find the tension in the string supporting the ladder.



Solution.

External forces acting on the system are weights of the men, weight of the ladder and tension supporting the ladder. Denoting masses of men A, B, C and ladder by m_A, m_B, m_C and m_L, acceleration due to gravity by g , tension in the string by T and accelerations of the men A, B, C and ladder by a_A, a_B, a_C and a_L respectively, we can write the following equation according to equation .

$$\Sigma\vec{F}_i = \Sigma\left(m_i\vec{a}_i\right) \rightarrow \qquad T - m_A g - m_B g - m_C g - m_L g = m_A a_A + m_B a_A + m_C a_A + m_L a_A$$

Substituting given values of masses $m_A = 40$ kg, $m_B = 60$ kg, $m_C = 50$ kg, $m_L = 20$ kg,

given values of accelerations $g = 10$ m/s^2, $a_A = -2$ m/s^2, $a_B = 0$ m/s^2, $a_C = 1$ m/s^2, and $a_L = 0$ m/s^2,

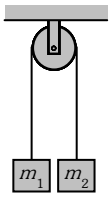
we obtain $\qquad T - 400 - 600 - 500 - 200 = -80 + 0 + 50 + 0$

$$T = 1670 \text{ N}$$

Example

Simple Atwood Machine as System of Particles

The system shown in the figure is known as simple Atwood machine. Initially the masses are held at rest and then let free. Assuming mass m_2 more than the mass m_1, find acceleration of mass center and tension in the string supporting the pulley.



Solution.

We know that accelerations a_1 and a_2 are given by the following equations.

$$a_2 = \frac{m_2 - m_1}{m_2 + m_1} g \downarrow \quad \text{and} \quad a_1 = \frac{m_2 - m_1}{m_2 + m_1} g \uparrow$$

Making use of eq. , we can find acceleration a_C of the mass center. We denote upward direction positive and downward direction negative signs respectively.

$$M\vec{a}_C = \Sigma\left(m_i\vec{a}_i\right) \rightarrow \qquad \left(m_1 + m_2\right)a_c = m_1 a_1 - m_2 a_2$$

Substituting values of accelerations a_1 and a_2, we obtain

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

$$a_C = \frac{2m_1m_2 - \left(m_1^2 + m_2^2\right)}{\left(m_1 + m_2\right)^2} g$$

To find tension T in the string supporting the pulley, we again use eq. (9)

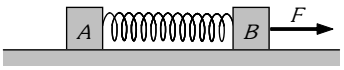
$\Sigma\vec{F}_i = M\vec{a}_C \rightarrow$ $\qquad T - m_1g - m_2g = \left(m_1 + m_2\right)a_c$

Substituting expression obtained for a_C, we have

$$T = \frac{4m_1m_2}{m_1 + m_2} g$$

Example

Two blocks each of mass m, connected by an un-stretched spring are kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure.



(a) Find acceleration of the mass center.

(b) Find the displacement of the centre of mass as function of time t.

(c) If the extension of the spring is x_o at an instant t, find the displacements of the two blocks relative to the ground at this instant.

Solution.

(a) Forces in vertical direction on the system are weights of the blocks and normal reaction from the ground. They balance themselves and have no net resultant. The only external force on the system is the applied force F in the horizontal direction towards the right.

$\Sigma\vec{F}_i = M\vec{a}_C \rightarrow$ $\qquad F = (m + m)a_c$

$$a_c = \frac{F}{2m} \text{ towards right}$$

(b) The mass center moves with constant acceleration, therefore it displacement in time t is given by equation of constant acceleration motion.

$x = ut + \frac{1}{2}at^2 \rightarrow$ $\qquad x_c = \frac{Ft^2}{4m}$

(c) Positions x_A and x_B of particles A and B forming a system and position x_C mass center are obtained by following eq.

$$M\vec{r}_c = \Sigma m_i\vec{r}_i$$

Substituting values wee obtain $2mx_c = mx_A + mx_B$ $\qquad x_c = \frac{x_A + x_B}{2}$

Now using result obtained in part *(b)*, we have $\qquad x_A + x_B = \frac{Ft^2}{2m}$

Extension in the spring at this instant is $x_o = x_B - x_A$

From the above two equations, we have $\qquad x_A = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right)$ and $x_B = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right)$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Application of Methods of Impulse and Momentum to a System of Particles

In a phenomenon, when a system changes its configuration, some or all of its particles change their respective locations and momenta. Sum of linear momenta of all the particles equals to the linear momentum due to translation of mass center. Principle of impulse and moment suggests net impulse of all the external forces equals to change in momentum of mass center.

$$\sum \int \vec{F}_i dt = \vec{p}_{cf} - \vec{p}_{ci} \qquad (11)$$

Conservation of Linear momentum

The above event suggests that total linear momentum of a system of particle remains conserved in a time interval in which impulse of external forces is zero.

Total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \text{ or } \vec{p}_{ci} = \vec{p}_{cf} \qquad (12)$$

The above statement is known as the *principle of conservation of momentum*.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

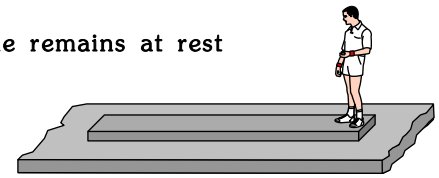
During an event the net impulse of external forces in a direction is zero in the following cases.

- When no external force acts in a particular direction on any of the particles or bodies.
- When resultant of all the external forces acting in a particular direction on all the particles or bodies is zero.
- In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act.

Example

No external force: Stationary mass relative to an inertial frame remains at rest

A man of mass m is standing at on end of a plank of mass M. The length of the plank is L and it rests on a frictionless horizontal ground. The man walks to the other end of the plank. Find displacement of the plank and man relative to the ground.



Solution.

Denoting x-coordinates of the man, mass center of plank and mass center of the man-plank system by x_m, x_p and x_c, we can write the following equation.

$$(\Sigma m_i)\vec{r}_c = \Sigma m_i \vec{r}_i \rightarrow \qquad (m+M)\vec{x}_c = m\vec{x}_m + M\vec{x}_p$$

Net force on the system relative to the ground is zero. Therefore mass center of the system which is at rest before the man starts walking, remains at rest $(\Delta\vec{x}_c = 0)$ after while the man walks on the plank.

$$(\Delta\vec{x}_c = 0) \rightarrow \qquad m\Delta\vec{x}_m + M\Delta\vec{x}_p = 0 \qquad (1)$$

The man walks displacement $\left(\Delta\vec{x}_{m/p} = -L\tilde{i}\right)$ relative to the plank. Denoting displacements of the man and the plank relative to the ground by $\Delta\vec{x}_m$ and $\Delta\vec{x}_p$, we can write

$$\Delta\vec{x}_{m/p} = \Delta\vec{x}_m - \Delta\vec{x}_p \rightarrow \quad \Delta\vec{x}_m - \Delta\vec{x}_p = -L\tilde{i} \qquad (2)$$

From the above equations (1) and (2), we have

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

$$\Delta\vec{x}_m = -\frac{ML\tilde{i}}{m+M}$$

The man moves a distance $\frac{ML}{m+M}$ towards left relative to the ground.

$$\Delta\vec{x}_p = \frac{mL\tilde{i}}{m+M}$$

The plank moves a distance $\frac{mL}{m+M}$ towards right relative to the ground.

Example

No external force: Mass center moving relative to an inertial frame moves with constant velocity

Two particles of masses 2 kg and 3 kg are moving under their mutual interaction in free space. At an instant they were observed at points (–2 m, 1 m, 4 m) and (2 m, –3 m, 6 m) with velocities $\left(3\tilde{\mathrm{i}}-2\tilde{\mathrm{j}}+\tilde{\mathrm{k}}\right)$ m/s and $\left(-\tilde{\mathrm{i}}+\tilde{\mathrm{j}}-2\tilde{\mathrm{k}}\right)$ m/s respectively. If after 10 sec, the first particle passes the point (6 m, 8 m, –6 m), find coordinate of the point where the second particle passes at this instant?

Solution.

System of these two particles is in free, therefore no external forces act on them. There total linear momentum remains conserved and their mass center moves with constant velocity relative to an inertial frame.

Velocity of the mass center

$$v_c = \frac{\Sigma m_i\vec{v}_i}{\Sigma m_i} = \frac{2\left(3\tilde{i}-2\tilde{j}+\tilde{k}\right)+3\left(-\tilde{i}+\tilde{j}-2\tilde{k}\right)}{2+3} = \frac{3\tilde{i}-\tilde{j}-4\tilde{k}}{5}\ \text{m/s}$$

Location $\vec{r}_{co}$ of the mass center at the instant t = 0 s

$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i} \rightarrow \qquad \vec{r}_{co} = \frac{2\left(-2\tilde{i}+\tilde{j}+4\tilde{k}\right)+3\left(2\tilde{i}-3\tilde{j}+6\tilde{k}\right)}{2+3} = \frac{2\tilde{i}-7\tilde{j}+26\tilde{k}}{5}$$

New location $\vec{r}_c$ of the mass center at the instant t = 10 s

$$\vec{r}_c = \vec{r}_{co} + \vec{v}_c t \rightarrow \qquad \vec{r}_c = \frac{2\tilde{i}-7\tilde{j}+26\tilde{k}}{5} + \frac{3\tilde{i}-\tilde{j}-4\tilde{k}}{5}\times 10 = \frac{32\tilde{i}-17\tilde{j}-14\tilde{k}}{5}$$

New location (x, y, z) of the second particle.

$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i} \rightarrow \qquad \frac{32\tilde{i}-17\tilde{j}-14\tilde{k}}{5} = \frac{2\left(6\tilde{i}+8\tilde{j}-6\tilde{k}\right)+3\left(x\tilde{i}+y\tilde{j}+z\tilde{k}\right)}{2+3}$$

Solving the above equation, we obtain the coordinates of the second particle (20/3, –11, –2/3)

Application of Methods of Work and Energy to a System of Particles

In a system of particles, all the particles occupy different locations at every instant of time and may change their locations with time. At an instant of time set of locations of all the particles of a system is known as *configuration* of the system. We say something has happened with the system only when some or all of its particles change their locations. It means that in every event or phenomena the system changes its configuration.

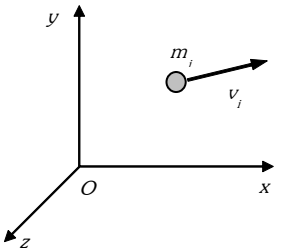
Methods of work and energy equips us to analyze what happens when a particle moves form one point of space to other. Now we will apply these methods to analyze a phenomenon in which a system of particle changes its configuration.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Kinetic Energy of a System of Particle

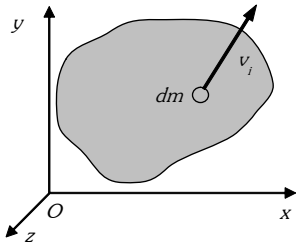
Kinetic energy of a system of particles is defined as sum of kinetic energies of all the particles of the system.

If at an instant particles of masses $m_1, m_2, \ldots m_i \ldots m_j \ldots\ldots$ and m_n are observed moving with velocities $\vec{v}_1$, $\vec{v}_2$, ... $\vec{v}_i$, ... $\vec{v}_n$ respectively relative to a reference frame, the kinetic energy of the whole system relative to the reference frame is given by the following equation.



$$K = \frac{1}{2}\sum m_i v_i^2 \qquad (13)$$

If the system consists of continuous distribution of mass, instead of discrete particles, expression of kinetic energy becomes

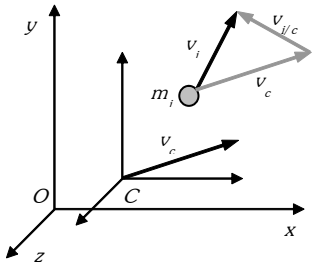


$$K = \frac{1}{2}\int v^2 dm \qquad (14)$$

Kinetic Energy of a System of Particle using Centroidal Frame

Centroidal frame of reference or center of mass frame is reference frame attached with the mass center of the system.

Let velocity of i^{th} particle of mass m_i is moving with velocity $\vec{v}_i$ relative to frame *Oxyz.* Mass center *C* and hence the centroidal frame *Cxyz* is moving with velocity $\vec{v}_c$. Therefore velocity of i^{th} particle relative to the centroidal frame is $\vec{v}_{i/c}$.



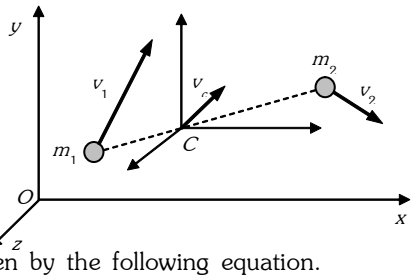
Kinetic energy of the whole system is given by the following equation.

$$K = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i v_c^2 + \frac{1}{2}\sum m_i v_{i/c}^2 \qquad (15)$$

Here the first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.

Kinetic Energy of a Two Particle System using Centroidal Frame

A two particle system consists of only two particles. Let a two particle system consists of particles of masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ relative to a frame *Oxyz*. Their mass center *C* lies on the line joining them and divides separation between them in reciprocal ratio of masses m_1 and m_2. The mass center and hence the centroidal frame is moving with velocity $\vec{v}_c$.



Kinetic energy of this two particle system relative to a frame *Oxyz* is given by the following equation.

$$K = \frac{1}{2}(m_1 + m_2)v_c^2 + \frac{1}{2}\mu v_{rel}^2 \qquad (16)$$

The first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.

Here symbol μ is known as reduced mass of the two particle system and symbol v_{rel} is magnitude of velocity of either of the particles relative to the other.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad v_{rel} = |\vec{v}_1 - \vec{v}_2| = |\vec{v}_2 - \vec{v}_1|$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Work Energy Theorem for a System of Particles

The work energy theorem can be applied to each particle of the system. For i^{th} particle of the system, we can write

$$K_{i,i} + W_{i,i\to f} = K_{i,f}$$

Here $W_{i,i'!f}$ is total work done by all the internal forces $\vec{f}_{ij}$ and resultant external force $\vec{F}_i$ on the i^{th} particle, when the system goes from one configuration to other.

Adding kinetic energies of all particles, we can write kinetic energies K_i and K_f of the whole system in the initial as well as the final configuration. Adding work done $W_{i,i\to f}$ by internal as well as external forces on every particle we find total work done $W_{i\to f}$ by all the internal as well as external forces on the system. Now we can write work energy theorem.

$$K_i + W_{i\to f} = K_f \qquad (17)$$

While applying the above equation to a system, care must be taken in calculating $W_{i\to f}$. In spite of the fact that the internal forces $\vec{f}_{ij}$ and $\vec{f}_{ji}$ being equal in magnitude and opposite in direction, the work done by them on the i^{th} and the j^{th} particles will not, in general, cancel out, since i^{th} and the j^{th} particles may undergo different amount of displacements.

The above description at first presents calculating of $W_{i\to f}$ as a cumbersome task. However for systems, which we usually encounter are not as complex as a general system of large number of particles may be. Systems which we usually face to analyze have limited number of particles or bodies interacting. For these systems we can simplify the task by calculating work of conservative internal forces as decrease in potential energy of the system. Total work of internal forces other than internal conservative forces vanishes, if these forces are due connecting inextensible links or links of constant length. These forces include string tension and normal reaction at direct contacts between the bodies included I the system. Work of internal forces of the kind other than these and work of external forces, can be calculated by definition of work.

Conservation of Mechanical Energy

If total work of internal forces other than conservative is zero and no external forces act on a system, total mechanical energy remains conserved.

$$K_i + U_i = K_f + U_f \qquad (18)$$

Since external forces are capable of changing mechanical energy of the system, under their presence total mechanical energy changes by amount equal to work $W_{ext,\ i'!f}$ done by all the external forces.

$$W_{ext,\ i\to f} = E_f - E_i = (K_f + U_f) - (K_i + U_i) \qquad (19)$$

Example

Total work of pseudo forces in centroidal frame.

Show that total work done in centroidal frame on all the particles of a system by pseudo forces due to acceleration of mass center is zero.

Solution.

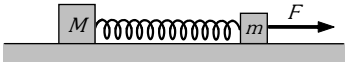
Let acceleration of mass centre relative to an inertial frame is $\vec{a}_c$. Pseudo force on i^{th} particle of mass m_i in centroidal frame is $\left(-m_i\vec{a}_c\right)$. Let displacement of i^{th} particle in a time interval is $\Delta\vec{r}_i$ relative to the centroidal frame.

Total work of pseudo forces on all the particles in centroidal frame can now be expressed by the expression

$$\Sigma\left(-m_i\vec{a}_c\right)\cdot\Delta\vec{r}_i = -\vec{a}_c\cdot\Sigma\left(m_i\Delta\vec{r}_i\right) = -\vec{a}_c\cdot\vec{0} = 0$$

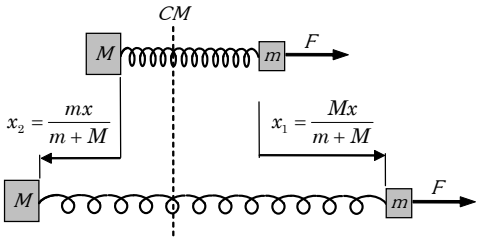
Example

Two blocks of masses m and M connected by a spring are placed on frictionless horizontal ground. When the spring is relaxed, a constant force F is applied as shown. Find maximum extension of the spring during subsequent motion.



Solution.

If we use ground as inertial frame as we usually do, solution of the problem becomes quite involved. Therefore, we prefer to use the centroidal frame, in which mass center remains at rest.



In the adjacent figure is shown horizontal position of mass center (CM) by dashed line. It remains unchanged in centroidal frame.

Mass center of two particle system divides separation between them in reciprocal ratio of the masses; therefore displacements x_1 and x_2 of the blocks must also be in reciprocal ratio of their masses. The extension x is sum of displacements x_1 and x_2 of the blocks as shown in the figure.

When extension of the spring achieves its maximum value, both the block must stop receding away from the mass center, therefore, velocities of both the blocks in centroidal frame must be zero.

During the process when spring is being extended, total work done by pseudo forces in centroidal frame become zero, negative work done by spring forces becomes equal to increase in potential energy and work done by the applied force evidently becomes Fx_1.

Using above fact in applying work energy theorem on the system relative to the centroidal frame, we obtain

$$K_i + W_{i\to f} = K_f \rightarrow \qquad 0 + W_{i\to f,\,springforce} + W_{i\to f,F} = 0$$

$$-\left(0 - \tfrac{1}{2}kx^2\right) + \frac{FMx}{m+m} = 0$$

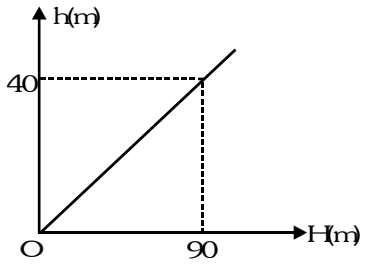
$$x = \frac{2FM}{k(m+m)}$$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

SOME WORKED OUT EXAMPLES

Example#1

A ball of mass 2 kg dropped from a height H above a horizontal surface rebounds to a height h after one bounce. The graph that relates H to h is shown in figure. If the ball was dropped from an initial height of 81 m and made ten bounces, the kinetic energy of the ball immediately after the second impact with the surface was



(A) 320 J (B) 480 J (C) 640 J (D) Can't be determined

Solution **Ans. (A)**

From graph $e = \sqrt{\frac{h}{H}} = \sqrt{\frac{40}{90}} = \frac{2}{3}$

Kinetic energy of the ball just after second bounce

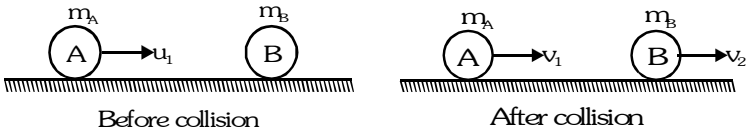
$$= \frac{1}{2}m(e^2u)^2 = \frac{1}{2}me^4u^2 = (e^4)(mgH) = \left(\frac{2}{3}\right)^4 (2)(10)(81) = 320J$$

Example#2

Consider an one dimensional elastic collision between a given incoming body A and body B, initially at rest. The mass of B in comparison to the mass of A in order that B should recoil with greatest kinetic energy is

(A) $m_B >> m_A$ (B) $m_B << m_A$ (C) $m_B = m_A$ (D) can't say anything

Solution **Ans. (C)**



Velocity of block B after collision $v_2 = \frac{2m_A u_1}{m_A + m_B}$

KE of block B = $\frac{1}{2}m_B v_2^2 = \frac{1}{2}m_B\left[\frac{4m_A^2u_1^2}{(m_A+m_B)^2}\right] = \frac{2m_A^2m_B}{(m_A+m_B)^2}u_1^2$

which is maximum if $m_A = m_B$

Example#3

An object is moving through air at a speed v. If the area of the object normal to the direction of velocity is A and assuming elastic collision with the air molecules, then the resistive force on the object is proportional to– (assume that molecules striking the object were initially at rest)

(A) 2Av (B) $2Av^2$ (C) $2Av^{1/2}$ (D) Can't be determined

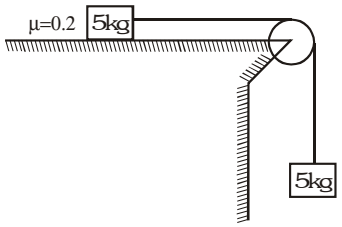
NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Solution **Ans. (B)**

Velocity of air molecule after collision = 2v . The number of air– molecules accelerated to a velocity 2v in time Δt is proportional to $Av\Delta t$. Therefore $F = \frac{\Delta p}{\Delta t} \propto (Av\Delta t)\left(\frac{2v}{\Delta t}\right) \Rightarrow F \propto 2Av^2$

Example#4

The magnitude of acceleration of centre of mass of the system is



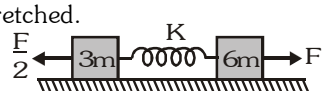
(A) 4 m/s^2 (B) 10 m/s^2 (C) 5 m/s^2 (D) 2 $\sqrt{2}$ m/s^2

Solution **Ans. (D)**

$$a = \frac{\text{Net force on system}}{\text{total mass of system}} = \frac{5g - \mu(5g)}{5+5} = \frac{50(1-0.2)}{10} = 4 \text{ m/s}^2 \; ; \; a_{cm} = \left|\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}\right| = \frac{a}{\sqrt{2}} = 2\sqrt{2} \text{ m/s}^2$$

Example#5

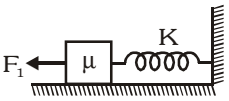
For shown situation find the maximum elongation in the spring. Neglect friction everywhere. Initially, the blocks are at rest and spring is unstretched.



(A) $\frac{4F}{3K}$ (B) $\frac{3F}{4K}$ (C) $\frac{4F}{K}$ (D) $\frac{2F}{K}$

Solution **Ans. (A)**

By using reduced mass concept this system can be reduced to



Where $\mu = \frac{(3m)(6m)}{3m+6m} = 2$ m and F_1 = Force on either block w.r.t. centre of mass of the system

$$= \frac{F}{2} + (3m)a_{cm} = \frac{F}{2} + (3m)\left(\frac{F - F/2}{9m}\right) = \frac{F}{2} + \frac{F}{6} = \frac{2}{3}F$$

Now from work energy theorem , $\frac{2F}{3}x_m = \frac{1}{2}Kx_m^2 \Rightarrow x_m = \frac{4F}{3K}$

Example#6

A small sphere of mass 1kg is moving with a velocity $(6\hat{i} + \hat{j})$ m/s. It hits a fixed smooth wall and rebounds with velocity $(4\hat{i} + \hat{j})$ m/s . The coefficient of restitution between the sphere and the wall is-

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{9}{16}$ (D) $\frac{4}{9}$

Solution **Ans. (B)**

Impulse = Change in momentum $= 1(4\hat{i} + \hat{j}) - 1(6\hat{i} + \hat{j}) = -2\hat{i}$

Which is perpendicular to the wall.

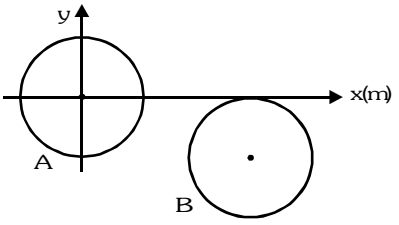
Component of initial velocity along $\hat{i} = 6\hat{i}$ $\Rightarrow$ Speed of approach = 6 m/s

Similarly speed of separation = $4ms^{-1}$ $\Rightarrow e = \frac{4}{6} = \frac{2}{3}$

NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

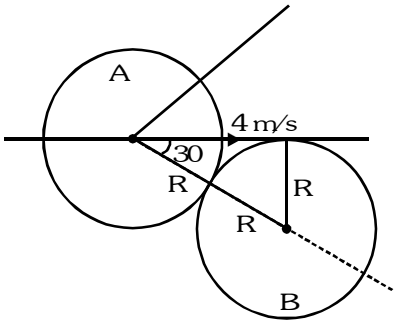
Example#7

Two smooth balls A and B, each of mass m and radius R, have their centre at (0, 0, R) and (5R, –R, R) respectively, in a coordinate system as shown. Ball A, moving along positive x-axis, collides with ball B. Just before the collision, speed of ball A is 4 m/s and ball B is stationary. The collision between the balls is elastic. Velocity of the ball A just after the collision is

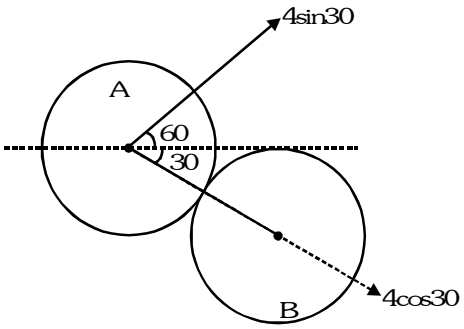


(a) $\left(\tilde{i}+\sqrt{3}\tilde{j}\right)$ m/s (b) $\left(\tilde{i}-\sqrt{3}\tilde{j}\right)$ m/s (c) $\left(2\tilde{i}+\sqrt{3}\tilde{j}\right)$ m/s (d) $\left(2\tilde{i}+2\tilde{j}\right)$ m/s

Solution **Ans. (A)**



Before collision

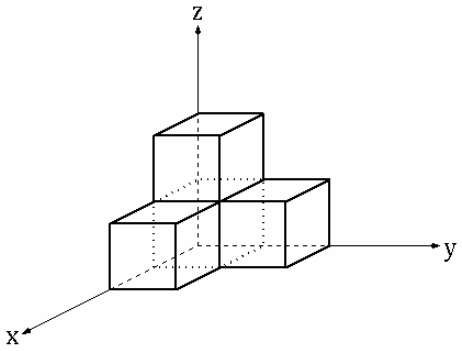


After collision

$$v_A = 4\sin 30°\left[\cos 60\tilde{i}+\sin 60\tilde{j}\right]=\tilde{i}+\sqrt{3}\tilde{j}$$

Example#8

Find the center of mass (x,y,z) of the following structure of four identical cubes if the length of each side of a cube is 1 unit.



(A) (1/2,1/2,1/2) (B)(1/3,1/3,1/3) (C) (3/4,3/4,3/4) (D)(1/2,3/4,1/2)

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Solution **Ans. (C)**

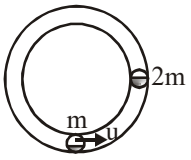
First we find the center of mass of each cube. It is located by symmetry: (0.5,0.5,0.5), (1.5,0.5,0.5), (0.5,1.5,0.5), (0.5,0.5,1.5) . Now we find the center of mass by treating the COM of each cube as a point particle:

$$x_{COM} = \frac{0.5+1.5+0.5+0.5}{4} = 0.75 \; ; \; y_{COM} = \frac{0.5+0.5+1.5+0.5}{4} = 0.75$$

$$z_{COM} = \frac{0.5+0.5+0.5+1.5}{4} = 0.75$$

Example#9

Two masses m and 2m are placed in fixed horizontal circular smooth hollow tube of radius r as shown. The mass m is moving with speed u and the mass 2m is stationary. After their first collision, the time elapsed for next collision. (coefficient of restitution $e=1/2$)



(A) $\frac{2\pi r}{u}$ (B) $\frac{4\pi r}{u}$ (C) $\frac{3\pi r}{u}$ (D) $\frac{12\pi r}{u}$

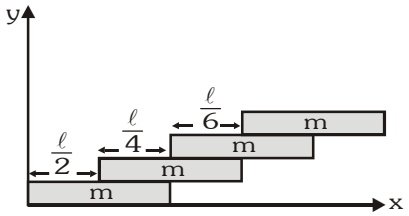
Solution **Ans. (B)**

Let the speeds of balls of mass m and 2m after collision be v_1 and v_2 as shown in figure. Applying conservation of momentum $mv_1 + 2mv_2 = mu$ & $-v_1 + v_2 = \frac{u}{2}$. Solving we get $v_1 = 0$ and $v_2 = \frac{u}{2}$

Hence the ball of mass m comes to rest and ball of mass 2m moves with speed $\frac{u}{2}$. $t = \frac{2\pi r}{u/2} = \frac{4\pi r}{u}$

Example#10

Find the x coordinate of the centre of mass of the bricks shown in figure :



(A) $\frac{24}{25}\ell$ (B) $\frac{25}{24}\ell$ (C) $\frac{15}{16}\ell$ (D) $\frac{16}{15}\ell$

Solution **Ans. (B)**

$$X_{cm} = \frac{m\left(\frac{\ell}{2}\right)+m\left(\frac{\ell}{2}+\frac{\ell}{2}\right)+m\left(\frac{\ell}{2}+\frac{\ell}{4}+\frac{\ell}{2}\right)+m\left(\frac{\ell}{2}+\frac{\ell}{4}+\frac{\ell}{6}+\frac{\ell}{2}\right)}{m+m+m+m} = \frac{25}{24}\ell$$

Example#11

Object A strikes the stationary object B with a certain given speed u head–on in an elastic collision. The mass of A is fixed, you may only choose the mass of B appropriately for following cases. Then after the collision :

(A) For B to have the greatest speed, choose $m_B = m_A$

(B) For B to have the greatest momentum, choose $m_B << m_A$

(C) For B to have the greatest speed, choose $m_B << m_A$

(D) For the maximum fraction of kinetic energy transfer, choose $m_B = m_A$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Solution **Ans. (B,C,D)**

$m_A u = m_A v_A + m_B v_B$ and $e = 1 = \frac{v_B - v_A}{u} \Rightarrow v_B = \frac{2m_A u}{m_A + m_B}$

For $m_A >> m_B$, $v_B = 2u$

For $m_A = m_B$, $v_B = u$

For $m_A << m_B$, $v_B = 0$

kinetic energy $K_B = \frac{1}{2} m_B v_B^2 = \frac{2m_B u^2}{\left(1 + \frac{m_B}{m_A}\right)^2}$

Example#12

A man is sitting in a boat floating in water of a pond. There are heavy stones placed in the boat.

(A) When the man throws the stones in water from the pond, the level of boat goes down.

(B) When the man throws the stones in water from the pond, the level of boat rises up.

(C) When the man drinks some water from the pond, the level of boat goes down

(D) When the man drinks some water from the pond, the level of boat remains unchanged.

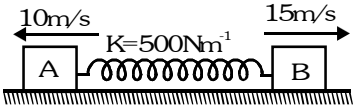
Solution **Ans. (B,D)**

For (A/B) : Force of buoyancy increases. Therefore level of boat rises up.

For (C/D): When man drinks some water, the level of boat remains unchanged.

Example#13

Two blocks A and B are joined together with a compressed spring. When the system is released, the two blocks appear to be moving with unequal speeds in the opposite directions as shown in figure.
Select incorrect statement(s) :



(A) The centre of mass of the system will remain stationary.

(B) Mass of block A is equal to mass of block B.

(C) The centre of mass of the system will move towards right.

(D) It is an impossible physical situation.

Solution **Ans. (BCD)**

As net force on system = 0 (after released)

So centre of mass of the system remains stationary.

Example#14

In which of the following cases, the centre of mass of a rod may be at its centre?

(A) The linear mass density continuously decreases from left to right.

(B) The linear mass density continuously increases from left to right.

(C) The linear mass density decreases from left to right upto centre and then increases.

(D) The linear mass density increases from left to right upto centre and then decreases.

Solution **Ans. (CD)**

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example#15

A man of mass 80 kg stands on a plank of mass 40 kg. The plank is lying on a smooth horizontal floor. Initially both are at rest. The man starts walking on the plank towards north and stops after moving a distance of 6 m on the plank. Then

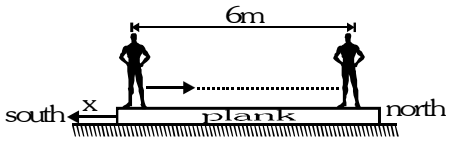
(A) The centre of mass of plank-man system remains stationary.

(B) The plank will slide to the north by a distance 4 m

(C) The plank will slide to the south by a distance 4 m

(D) The plank will slide to the south by a distance 12 m

Solution **Ans. (AC)**



Let x be the displacement of the plank.

Since CM of the system remains stationary

so 80 (6–x) = 40 x $\Rightarrow$ 12 – 2x = x $\Rightarrow$x = 4m

Example#16

A body moving towards a body of finite mass at rest, collides with it. It is impossible that

(A) both bodies come to rest

(B) both bodies move after collision

(C) the moving body stops and body at rest starts moving

(D) the stationary body remains stationary and the moving body rebounds

Solution **Ans. (AD)**

For (A) : Momentum can't destroyed by internal forces.

For (D) : If mass of stationary body is infinite then the moving body rebounds.

Example#17

Three interacting particles of masses 100 g, 200 g and 400 g each have a velocity of 20 m/s magnitude along the positive direction of x-axis, y-axis and z-axis. Due to force of interaction the third particle stops moving. The velocity of the second particle is $\left(10\tilde{j}+5\tilde{k}\right)$. What is the velocity of the first particle?

(A) $20\tilde{i}+20\tilde{j}+70\tilde{k}$ (B) $10\tilde{i}+20\tilde{j}+8\tilde{k}$ (C) $30\tilde{i}+10\tilde{j}+7\tilde{k}$ (D) $15\tilde{i}+5\tilde{j}+60\tilde{k}$

Solution **Ans. (A)**

Initial momentum = $m_1\vec{v}_1+m_2\vec{v}_2+m_3\vec{v}_3=2\tilde{i}+4\tilde{j}+8\tilde{k}$

When the third particle stops the final momentum = $m_1\vec{v}_1+m_2\vec{v}_2+m_3\vec{v}_3=0.1\vec{v}_1+0.2\left(10\tilde{j}+5\tilde{k}\right)+\vec{0}$

By principle of conservation of momentum $0.1\ \vec{v}_1+2\tilde{j}+\tilde{k}=2\tilde{i}+4\tilde{j}+8\tilde{k}$; $\vec{v}_1=20\tilde{i}+20\tilde{j}+70\tilde{k}$

Example#18 to 20

A bullet of mass m is fired with a velocity 10 m/s at angle θ with the horizontal. At the highest point of its trajectory, it collides head-on with a bob of mass 3m suspended by a massless string of length 2/5 m and gets embedded in the bob. After the collision the string moves through an angle of 60 .

18. The angle θ is

(A) 53 (B) 37 (C) 45 (D) 30

19. The vertical coordinate of the initial position of the bob w.r.t. the point of firing of the bullet is

(A) $\frac{9}{4}$m (B) $\frac{9}{5}$m (C) $\frac{24}{5}$m (D) None of these

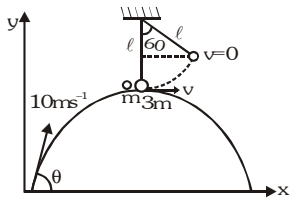
20. The horizontal coordinate of the initial position of the bob w.r.t. the point of firing of the bullet isn

(A) $\frac{9}{5}$m (B) $\frac{24}{5}$m (C) $\frac{9}{4}$m (D) None of these

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Solution

18. Ans. (B)



Velocity of combined mass just after collision

$m\ (10\ \cos\ \theta) = 4mv \Rightarrow v = \frac{5}{2}\cos\ \theta$

But from energy conservation $\frac{1}{2}(4m)v^2 = 4mg\ell(1-\cos 60)$

$\Rightarrow v = \sqrt{g\ell} = \frac{5}{2}\cos\theta \Rightarrow \cos\theta = \frac{2}{5}\sqrt{g\ell} = \frac{2}{5}\sqrt{10\times\frac{2}{5}} = \frac{4}{5} \Rightarrow \theta = 37$

19. Ans. (B)

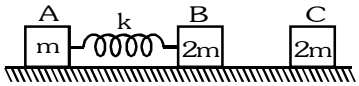
$H_{max} = \frac{u^2\sin^2\theta}{2g} = \frac{(100)(9/25)}{20} = \frac{9}{5}m$

20. Ans. (B)

$\frac{R}{2} = \frac{2u^2\sin\theta\cos\theta}{2g} = \frac{(100)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)}{10} = \frac{24}{5}m$

Example#21 to 23

Two blocks A and B of masses m and 2m respectively are connected by a spring of spring constant k. The masses are moving to the right with a uniform velocity v_0 each, the heavier mass leading the lighter one. The spring is of natural length during this motion. Block B collides head on with a third block C of mass 2m. at rest, the collision being completely inelastic.



21. The velocity of block B just after collision is-

(A) v_0 (B) $\frac{v_0}{2}$ (C) $\frac{3v_0}{5}$ (D) $\frac{2v_0}{5}$

22. The velocity of centre of mass of system of block A, B & C is-

(A) v_0 (B) $\frac{3v_0}{5}$ (C) $\frac{2v_0}{5}$ (D) $\frac{v_0}{2}$

23. The maximum compression of the spring after collision is -

(A) $\sqrt{\frac{mv_0^2}{12k}}$ (B) $\sqrt{\frac{mv_0^2}{5k}}$ (C) $\sqrt{\frac{mv_0^2}{10k}}$ (D) None of these

Solution

21. Ans. (B)

By applying conservation of linear momentum $2mv_0 = (2m + 2m)v \Rightarrow v = \frac{v_0}{2}$

22. Ans. (B)

$v_{cm} = \frac{mv_0 + 2mv_0}{m + 2m + 2m} = \frac{3v_0}{5}$

23. Ans. (B)

At maximum compression, velocity of all blocks are same & equal to velocity of centre of mass.

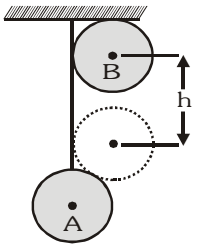
$\frac{1}{2}kx_m^2 = \left[\frac{1}{2}mv_0^2 + \frac{1}{2}(4m)\left(\frac{v_0}{2}\right)^2\right] - \frac{1}{2}(5m)\left(\frac{3v_0}{5}\right)^2 \Rightarrow \frac{1}{2}kx_m^2 = \frac{1}{10}mv_0^2 \Rightarrow x_m = \sqrt{\frac{mv_0^2}{5k}}$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example#24

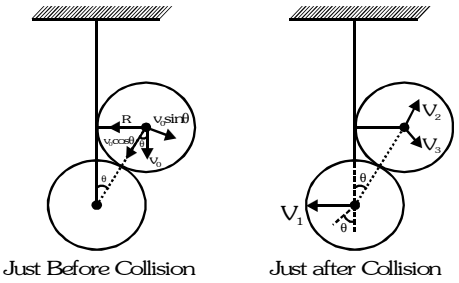
A smooth ball A of mass m is attached to one end of a light inextensible string, and is suspended from fixed point O. Another identical ball B, is dropped from a height h, so that the string just touches the surface of the sphere.



	Column I		Column II
(A)	If collision between balls is completely elastic then speed of ball A just after collision is	(P)	$\frac{3m}{5}\sqrt{2gh}$
(B)	If collision between balls is completely elastic then impulsive tension provided by string is	(Q)	$\frac{\sqrt{6gh}}{5}$
(C)	If collision between balls is completely inelastic then speed of ball A just after collision is	(R)	$\frac{6m}{5}\sqrt{2gh}$
(D)	If collision between balls is completely inelastic then impulsive tension provided by string is	(S)	$\frac{2\sqrt{6gh}}{5}$
		(T)	None of these

Solution **Ans. (A) → (S), (B) → (R), (C) → (Q), (D) → (P)**

For(A) $v_0 = \sqrt{2gh}$, $\sin\theta = \frac{R}{2R} = \frac{1}{2}$. By definition of e, $e = 1 = \frac{v_1 \sin\theta + v_2}{v_0 \cos\theta}$



Let impulse given by ball B be N. then by impulse momentum theorem
$N=m(v_2 + v_0\cos\theta)$ & $N\sin\theta = mv_1$

$$\Rightarrow v_1 = \frac{2v_0 \sin\theta\cos\theta}{1+\sin^2\theta} = \frac{\left(2\sqrt{2gh}\right)\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{1+\left(\frac{1}{2}\right)^2} = \frac{2\sqrt{6gh}}{5}$$

For(B) Impulsive tension = $N\cos\theta = \left(\frac{mv_1}{\sin\theta}\right)\cos\theta = mv_1\cot\theta = \frac{6m}{5}\sqrt{2gh}$

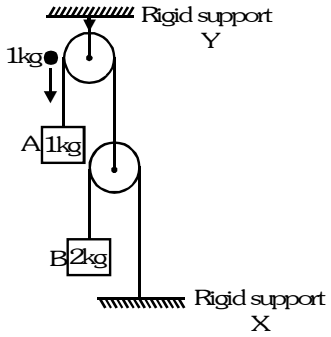
For (C) For completely inelastic collision $e=0$, so $v_1\sin\theta + v_2 = 0 \Rightarrow v_1 = \frac{v_0 \sin\theta\cos\theta}{1+\sin^2\theta} = \frac{\sqrt{6gh}}{5}$

For (D) Impulsive tension = $N\cos\theta = \left(\frac{mv_1}{\sin\theta}\right)\cos\theta = mv_1\cot\theta = \frac{3m}{5}\sqrt{2gh}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example#25

Collision between ball and block A is perfectly inelastic as shown. If impulse on ball (at the time of collision) is J then



	Column- I		Column-II
(A)	Net impulse on block A is	(P)	J
(B)	Net impulse on block B is	(Q)	4J/9
(C)	Impulse due to rigid support Y is	(R)	16J/9
(D)	Impulse due to rigid support X is	(S)	2J/9
		(T)	J/9

Solution **(A) T (B) Q (C) R (D) Q**

By using impulse momentum theorem :

on A : J–2T = 1(v)

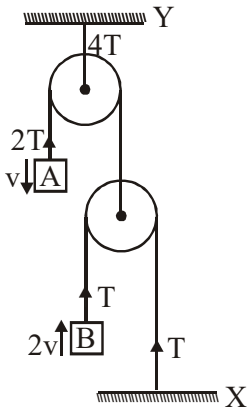
on B : T = 2(2v) Therefore J = 9 v

Net impulse on A = 1(v) = $\frac{J}{9}$

Net impulse on B = $4v = \frac{4J}{9}$

Impulse due to rigid support Y = 4T = $\frac{16J}{9}$

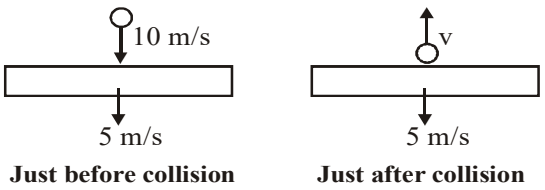
Impulse due to rigid support X = T = $\frac{4J}{9}$



Example#26

A ball moving vertically downward with a speed of 10 m/s collides with a platform. The platform moves with a velocity of 5 m/s in downward direction. If e = 0.8, find the speed (in m/s) of the ball just after collision.

Solution **Ans. 1**

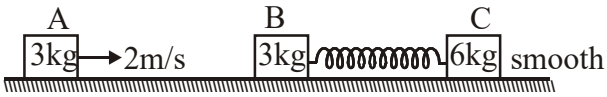


By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2}$; we have $0.8 = \frac{v+5}{10-5} \Rightarrow v = 1$ m/s

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example#27

For shown situation, if collision between block A and B is perfectly elastic, then find the maximum energy stored in spring in joules.



Solution **Ans. 4**

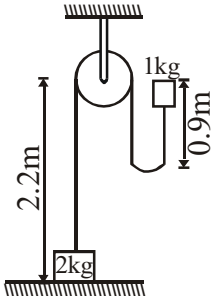
At maximum compression of spring, velocities of block B and C are same (say v_0)

then by conservation of linear momentum $3(2) = (3+6)v_0 \Rightarrow v_0 = \frac{2}{3}$ m/s

At this instant energy stored in spring = $\frac{1}{2}(3)(2)^2 - \frac{1}{2}(3+6)\left(\frac{2}{3}\right)^2$ = 6 – 2 = 4 J

Example#28

In the shown figure, the heavy block of mass 2 kg rests on the horizontal surface and the lighter block of mass 1 kg is dropped from a height of 0.9 m. At the instant the string gets taut, find the upward speed (in m/s) of the heavy block.



Solution **Ans. 2**

Velocity of lighter block at the instant the string just gets taut v = $\sqrt{2gh} = \sqrt{2\times10\times1.8}$ = 6 m/s

Now by impulse - momentum theorem, let common speed be v_1 then (2+1) v_1= (1) v $\Rightarrow v_1 = \frac{v}{3} = \frac{6}{3}$ = 2 m/s

Example#29

Two balls of equal mass have a head-on collision with speed 6 m/s. If the coefficient of restitution is $\frac{1}{3}$, find the speed of each ball after impact in m/s.

Solution **Ans. 2**

Just before collision 

Just after collision 

By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow \frac{1}{3} = \frac{v+v}{6+6} \Rightarrow v = 2$ m/s

Example#30

A thin rod of length 6 m is lying along the x-axis with its ends at x=0 and x = 6 m. Its linear density (mass/length) varies with x as kx^4. Find the position of centre of mass of rod in meters.

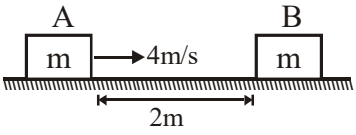
Solution **Ans. 5**

$$x_{cm} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^6 x(kx^4\,dx)}{\int_0^6 (kx^4\,dx)} = \frac{\int_0^6 x^5\,dx}{\int_0^6 x^4\,dx} = \frac{\left(\frac{x^6}{6}\right)_0^6}{\left(\frac{x^5}{5}\right)_0^6} = 5m$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Example#31

The friction coefficient between the horizontal surface and blocks A and B are $\frac{1}{15}$ and $\frac{2}{15}$ respectively. The collision between the blocks is perfectly elastic. Find the separation (in meters) between the two blocks when they come to rest.



Solution **Ans. 5**

Velocity of block A just before collision $v_A = \sqrt{u_A^2 - 2\mu g x} = \sqrt{16 - 2\left(\frac{1}{15}\right)(10)(2)} = \sqrt{\frac{40}{3}}$

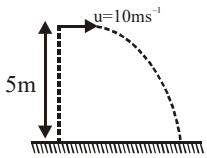
Velocity of Block B just after collision $v_B = v_A = \sqrt{\frac{40}{3}}$

Velocity of Block A just after collision = 0

Total distance travelled by block B = $\frac{v_B^2}{2\mu g} = \frac{40/3}{2\left(\frac{2}{15}\right)(10)} = 5m$

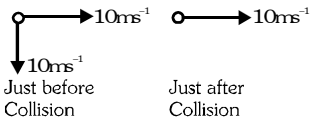
Example#32

A ball of mass 1 kg is projected horizontally as shown in figure. Assume that collision between the ball and ground is totally inelastic. The kinetic energy of ball (in joules) just after collision is found to be 10α. Find the value of α.



Solution **Ans. 5**

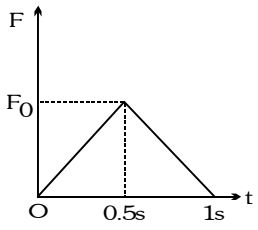
Vertical velocity just before collision $v_y = \sqrt{2gh} = \sqrt{2 \times 10 \times 5} = 10$ m/s



$\Rightarrow$ Kinetic energy of ball just after collision = $\frac{1}{2}$ 1 10^2 = 50 J

Example#33

A body of mass 1 kg moving with velocity 1 m/s makes an elastic one dimensional collision with an identical stationary body. They are in contact for brief time 1 sec. Their force of interaction increases from zero to F_0 linearly in time 0.5 s and decreases linearly to zero in further time 0.5 sec as shown in figure. Find the magnitude of force F_0 in newton.



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory p65

Solution **Ans. 2**

In the one dimensional elastic collision with one body at rest, the body moving initially comes to rest & the one which was at rest earlier starts moving with the velocity that first body had before collision.

so, if m & V_0 be the mass & velocity of body,

the change in momentum = $mV_0 \Rightarrow \int Fdt = mV_0 \Rightarrow \int Fdt = mV_0 \Rightarrow F = \frac{2mV_0}{\Delta t} = 2N$

Example#34

An object A of mass 1 kg is projected vertically upward with a speed of 20 m/s. At the same moment another object B of mass 3 kg, which is initially above the object A, is dropped from a height h = 20 m. The two point like objects (A and B) collide and stick to each other. The kinetic energy is K (in J) of the combined mass just after collision, find the value of K/25.

Solution **Ans. 2**

Using relative motion, the time of collision is $t = \frac{h}{20+0} = 1s$

By conservation of momentum for collision 3(10) + 1 (–10) = 4 (V) $\Rightarrow$V = 5 m/s

$KE = \frac{1}{2}(4)(5)^2 = 50J$

Example#35

An 80 kg man is riding on a 40 kg cart travelling at a speed of 2.5 m/s on a frictionless horizontal plane. He jumps off the cart, such that, his velocity just after jump is zero with respect to ground. The work done by him on the system during his jump is given as $\frac{A}{4}$ KJ (A$\in$ integer). Find the value of A.

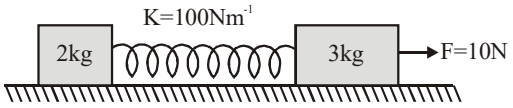
Solution **Ans. 3**

By conservation of linear momentum (80 + 40) (2.5) = 80 (0) + 40 (v) $\Rightarrow$ v = 7.5 m/s

work done = $\Delta KE = \frac{1}{2}40(7.5)^2 - \frac{1}{2}(80+40)(2.5)^2 = 750$ J

Example#36

At t=0, a constant force is applied on 3 kg block. Find out maximum elongation in spring in cm.

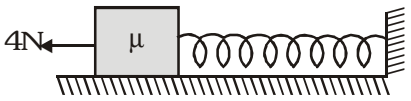


Solution **Ans. 8**

Given system can be reduced by using reduced mass concept $\mu = \frac{2\times3}{2+3} = \frac{6}{5}kg$

and $F_{reduced}$ = force on any block w.r.t. centre of mass = $\frac{m_1F_2 + m_2F_1}{m_1+m_2} = \frac{2\times10+3\times0}{2+3} = 4$ N

$\frac{1}{2}kx^2 = 4x \Rightarrow x = \frac{8}{k} = \frac{8}{100} = 8\times10^{-2}m = 8$ cm



NODE6 E : \Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Centre of mass & collision\Eng\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 7

ROTATIONAL MOTION - Theory

Index

ROTATION MOTION

Serial No.	CONTENTS	Page No.
1.	Rigid Body	1
2.	Types of motions involving rotation	2
3.	Kinematics of rotation about fixed axis	7
4.	Kinetics of rigid body	19
5.	Concept of rotational inertia	23
6.	Force and torque equations in general plane motion	28
7.	Dynamics of rigid body as a system of particles	29
8.	Energy methods	35
9.	Power	39
10.	Methods of impulse and momentum	41
11.	Eccentric impact	50
12.	**Exercise-I** : Check your Grasp (MCQ's with one correct answer)	67
13.	**Exercise-2** : Brain Teasers (MCQ's with one or more than one correct ans.)	73
14.	**Exercise-3** : Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, ComprehensionBased Question)	78
15.	**Exercise-4A** : Conceptual Sub. Exercise	89
16.	**Exercise-4B** :Brain Storming Exercise	92
17.	**Exercise-5A** : Previous years questions	94
18.	**Exercise-5B** : Previous years questions	98

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Main : Moment of inertia, radius of gyration. Values of moments of inertia for simple geometrical objects, parallel and perpendicular axes theorems and their applications. Rigid body rotation, equations of rotational motion.

JEE Advanced : Rigid body, moment of inertia, parallel and perpendicular axes theorems, moment of inertia of uniform bodies with simple geometrical shapes; Angular momentum; Torque; Conservation of angular momentum; Dynamics of rigid bodies with fixed axis of rotation; Rolling without slipping of rings, cylinders and spheres; Equilibrium of rigid bodies; Collision of point masses with rigid bodies.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

ROTATIONAL MOTION

Till so far we have learnt kinematics and kinetics of translation motion in which all the particles of a body undergo identical motions i.e. at any instant of time all of them have equal velocities and equal accelerations and in any interval of time they all follow identical trajectories. Therefore kinematics of any particle of a body or of its mass center in translation motion is representative of kinematics of the whole body. But when a body is in rotation motion, all of its particles and the mass center do not undergo identical motions. Newton's laws of motion, which are the main guiding laws of mechanics, are applicable to a point particle and if applied to a rigid body or system of particles, they predict motion of the mass center. Therefore, it becomes necessary to investigate how mass center and different particles of a rigid body move when the body rotates. In kinematics of rotation motion we investigate relations existing between time, positions, velocities and accelerations of different particles and mass center of a rigid body in rotation motion.

Rigid Body

A rigid body is an assemblage of a large number of material particles, which do not change their mutual distances under any circumstance or in other words, they are not deformed under any circumstance.

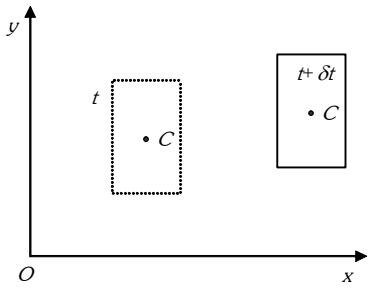
Actual material bodies are never perfectly rigid and are deformed under action of external forces. When these deformations are small enough to be considered during their course of motion, the body is assumed a rigid body. Hence, all solid objects such as stone, ball, vehicles etc are considered as rigid bodies while analyzing their translation as well as rotation motion.

To analyze rotation of a body relative motion between its particles cannot be neglected and size of the body becomes a considerable factor. This is why study of rotation motion is also known as *mechanics of rigid bodies*.

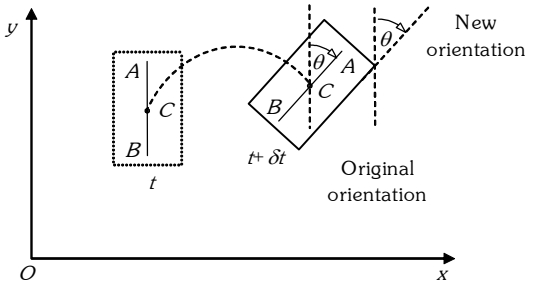
Rotation Motion of a Rigid Body

Any kind of motion of a body is identified by change in position or change in orientation or change in both. If a body changes its orientation during its motion it said to be in rotation motion.

In the following figures, a rectangular plate is shown moving in the *x-y* plane. The point *C* is its mass center. In the first case it does not changes orientation, therefore is in pure translation motion. In the second case it changes its orientation by during its motion. It is a combination of translation and rotation motion.



Pure Translation



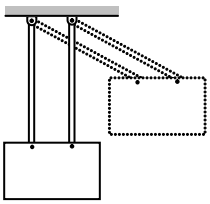
Combination of translation and rotation

Rotation i.e. change in orientation is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure this angle is shown by θ.

Example

Identify Translation and rotation motion

A rectangular plate is suspended from the ceiling by two parallel rods each pivoted at one end on the plate and at the other end on the ceiling. The plate is given a side-push to oscillate in the vertical plane containing the plate. Identify motion of the plate and the rods.



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution.

Neither of the linear dimensions of the plate turns during the motion. Therefore, the plate does not change its orientation. Here edges of the body easily fulfill our purpose to measure orientation; therefore, no line is drawn on it.

The plate is in *curvilinear translation motion* and the rods are in rotation motion.

Types of Motions involving Rotation

Motion of body involving rotation can be classified into following three categories.

I Rotation about a fixed axis.

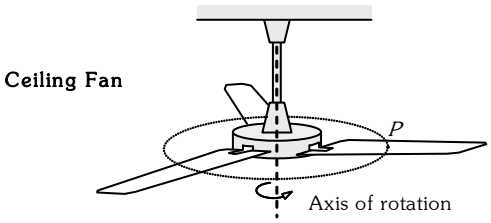
II Rotation about an axis in translation.

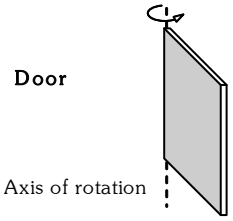
III Rotation about an axis in rotation

Rotation about a fixed axis

Rotation of ceiling fan, potter's wheel, opening and closing of doors and needles of a wall clock etc. come into this category.

When a ceiling fan it rotates, the vertical rod supporting it remains stationary and all the particles on the fan move on circular paths. Circular path of a particle P on one of its blades is shown by dotted circle. Centers of circular paths followed by every particle are on the central line through the rod. This central line is known as *axis of rotation* and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore the axis is stationary and the fan is in rotation about this fixed axis.



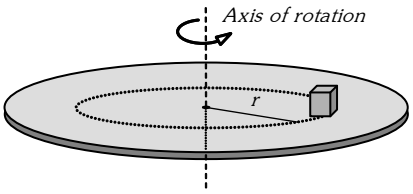


A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

Axis of rotation

An imaginary line perpendicular to plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as *axis of rotation*.

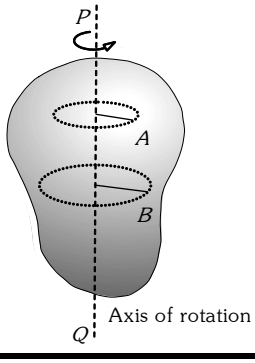
It is not necessary that the axis of rotation pass through the body. Consider system shown in the figure, where a block is fixed on a rotating disk. The axis of rotation passes through the center of the disk but not through the block.



Important observations

Let us consider a rigid body of arbitrary shape rotating about a fixed axis PQ passing through the body. Two of its particles A and B shown are moving on their circular paths.

⇨ All of its particles, not on the axis of rotation, move on circular paths with centers on the axis or rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.



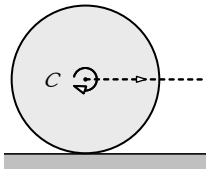
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

- ⇨ All the particles of the body cover same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.
- ⇨ Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.
- ⇨ All the particles on a line parallel to the axis of rotation move circular paths of the same radius therefore have same velocity and acceleration vectors.
- ⇨ Consider two particles in a plane perpendicular to the rotational axis. Every such particle on a rigid body in rotation motion moves on circular path relative to another one. Radius of the circular path equals to the distance between the particles. In addition, angular velocity and angular acceleration equals to that of rotation motion of the body.

Rotation about an axis in translation

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion. A rod lying on table when pushed from its one end in its perpendicular direction also executes this kind of motion. To understand more let us discuss few examples.

Consider rolling of wheels of a vehicle, moving on straight level road. Relative to a reference frame, moving with the vehicle wheel appears rotating about its stationary axel. The rotation of the wheel from this frame is rotation about fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axel, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions – rotation about the axel fixed with the vehicle and translation of the axel together with the vehicle.



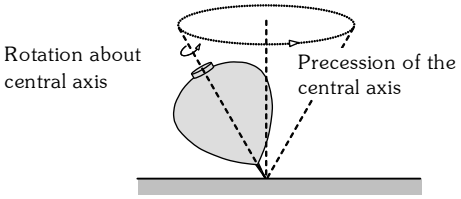
Important observations

- ⇨ Every particle of the body always remains in a plane perpendicular to the rotational axis. Therefore, this kind of motion is also known as *general plane motion*.
- ⇨ Relative to every particle another particle in a plane perpendicular to axis of rotation moves on circular path. Radius of the circular path equals to the distance between the particles and angular velocity and angular acceleration equals to that of rotation motion of the body.
- ⇨ Rotation about axis in translation is superposition of pure rotation about the axis and simultaneous translation motion of the axis.

Rotation about an axis in rotation.

In this kind of motion, the body rotates about an axis that also rotates about some other axis. Analysis of rotation about rotating axes is not in the scope of JEE, therefore we will discus it to have an elementary idea only.

As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore is in rotation motion. This type of rotation in which the axis of rotation also rotates and sweeps out a cone is known as *precession*.

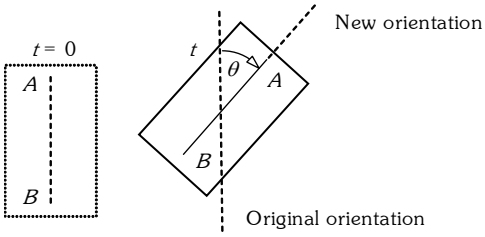


Another example of rotation about axis in rotation is a table-fan swinging while rotating. Table-fan rotates about its horizontal shaft along which axis of rotation passes. When the rotating table-fan swings, its shaft rotates about a vertical axis.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Angular displacement, angular velocity and angular acceleration

Rotation motion is the change in orientation of a rigid body with time.It is measured by turning of a linear dimension or a straight line drawn on the body.



In the figure is shown at two different instants $t = 0$ and t a rectangular plate moving in its own plane. Change in orientation during time t equals to the angle θ through which all the linear dimensions of the plate or a line AB turns.

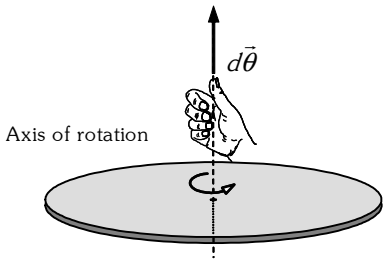
If the angle θ continuously changes with time t, instantaneous angular velocity ω and angular acceleration α for rotation of the body are defined by the following equations.

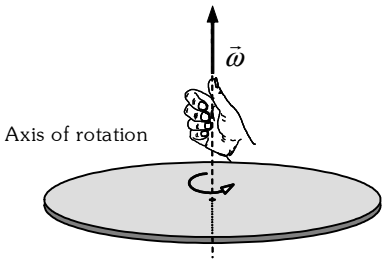
$$\omega = \frac{d\theta}{dt} \qquad [1]$$

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} \qquad [2]$$

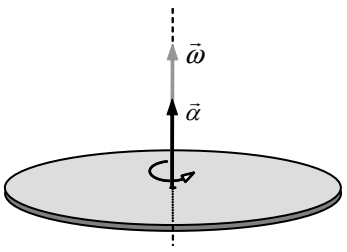
Direction of angular motion quantities

Angular displacement, angular velocity and angular acceleration are known as angular motion quantities. Infinitesimally small angular displacement, instantaneous angular velocity and angular acceleration are vector quantities. Direction of infinitesimally small angular displacement and instantaneous angular velocity is given by the right hand rule. For a disk rotating as shown in the figure, the angular velocity points upwards along the axis of rotation.

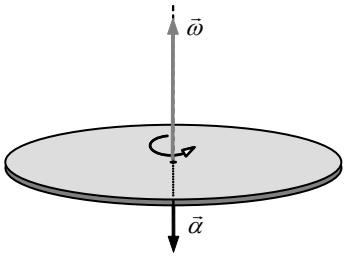




The direction of angular acceleration depends on whether angular velocity increases or decreases with time. For increasing angular velocity, the angular acceleration vector points in the direction of angular velocity vector and for decreasing angular velocity, the angular acceleration vector points opposite to the angular velocity vector.



Angular acceleration: Increasing angular speed



Angular acceleration: Decreasing angular speed

In rotation about fixed axis and rotation about axis in translation, the axis of rotation does not rotate and angular velocity and acceleration always point along the axis of rotation. Therefore, in dealing these kinds of motions, the angular motion quantities can used in scalar notations by assigning them positive sign for one direction and negative sign for the opposite direction.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

These quantities have similar mathematical relations as position coordinate, velocity, acceleration and time have in rectilinear motion.

⇨ A body rotating with constant angular velocity ω and hence zero angular acceleration is said to be uniform rotation. Angular position θ is given by equation

$$\theta = \theta_o + \omega t \quad [3]$$

⇨ Thus for a body rotating with uniform angular acceleration α, the angular position θ and angular velocity ω can be expressed by the following equation.

$$\omega = \omega_o + \alpha t \quad [4]$$

$$\theta = \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2 = \theta_o + \tfrac{1}{2}(\omega_o + \omega)t \quad [5]$$

$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o) \quad [6]$$

Angular motion quantities in rotation and assumption of axis of rotation

Rotation is identified by change in orientation, which is measured by turning of a linear dimension of the body or a line drawn on the body. It remains unchanged relative to all inertial frames.

Therefore, *if we assume axis of rotation anywhere but parallel to the original one, angular displacement, angular velocity and angular acceleration of rotation motion remain the same.*

Example

A wheel is rotating with angular velocity 2 rad/s. It is subjected to uniform angular acceleration 2.0 rad/s^2.

(a) How much angular velocity does the wheel acquire after 10 s?

(b) How many complete revolution it makes in this time interval?

Solution.

The wheel is in uniform angular acceleration, therefore from eq. [4]

$\omega = \omega_o + \alpha t \rightarrow$ Substituting values of ω_o, α and t, we have

$$\omega = 2 + 2 \times 10 = 22 \text{ rad/s}$$

From eq.[5], we have

$\theta = \theta_o + \tfrac{1}{2}(\omega_o + \omega)t \rightarrow$ Substituting $\theta_o = 0$ for initial position, and ω_o from above equation, we have

$$\theta = 0 + \tfrac{1}{2}(2 + 10)10 = 60 \text{ rad.}$$

In one revolution, the wheel rotates through 2π radians. Therefore, number of complete revolutions n is

$$n = \frac{\theta}{2\pi} = \frac{60}{2\pi} \approx$$

Example

A disk rotates about a fixed axis. Its angular velocity ω varies with time according to equation $\omega = at + b$. At the instant t = 0 its angular velocity is 1.0 rad/s and at angular position is 2 rad and at the instant t = 2 s, angular velocity is 5.0 rad/s. Determine angular position θ and angular acceleration α when t = 4 s.

Solution.

The given equation $\omega = at + b$ has form similar to eq.[4], therefore motion is rotation with uniform angular acceleration.

Initial angular velocity = $\omega_o = b = 1.0$ rad/s

Angular acceleration $\alpha = a$

Substituting these values in eq.[5], we get

$$\theta = \tfrac{1}{2}at^2 + \omega_0 t + c$$

Since at t = 0, ω = 1.0 rad/s, we obtain the constant c.

Initial angular position = $\theta_o = c = 2.0$ rad

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Since at t = 2.0 s angular velocity is 5.0 rad/s, from given expression of angular velocity, we have

$\omega = at + b \rightarrow$ Substituting b = 1.0 rad/s, t = 2.0 s and ω = 5.0 rad/s, we have

$a = 2.0$ rad/s^2

Now we can write expressions for angular position, angular velocity and angular acceleration.

$$\theta = t^2 + t + 2.0 \qquad (1)$$

$$\omega = 2.0t + 1.0 \qquad (2)$$

From the above equations, we can calculate angular position, angular velocity and angular acceleration at t =4.0 s

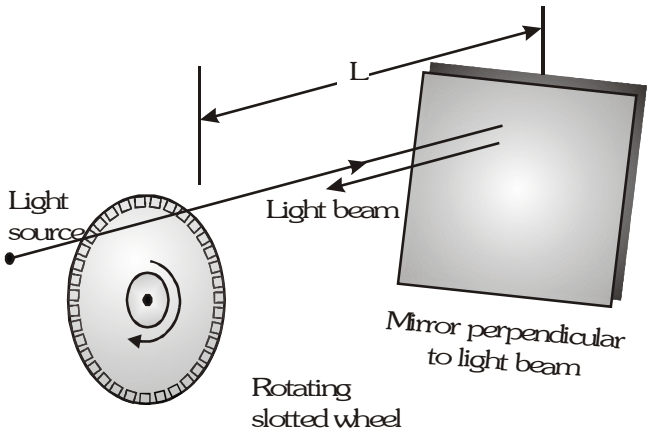
$$\theta_4 = 22 \text{ rad}, \quad \omega_4 = 9.0 \text{ rad/s}, \quad \alpha = 2.0 \text{ rad/s}^2$$

Example

An early method of measuring the speed of light makes use of a rotating slotted wheel. A beam of light passes through slot at the outside edge of the wheel, as shown in figure below, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such slotted wheel has a radius of 5.0 cm and 500 slots at its edge. Measurements taken when the mirror is L = 500 m from the wheel indicate a speed of light of 3.0 10^5 km/s.

(a) What is the (constant) angular speed of the wheel ?

(b) What is the linear speed of a point on the edge of the wheel?



Solution

(a) During the time light goes from the wheel to the mirror and comes back again, the wheel turns through an angle of $\theta = \frac{2\pi}{500}$ = 1.26 10^{-2} rad. That time is t = $\frac{2\ell}{c} = \frac{2(500\text{m})}{2.998\text{x}10^8\,\text{m/s}}$ = 3.34 10^{-4} s

So the angular speed of the wheel is $\omega = \frac{\theta}{t} = \frac{1.26\text{x}10^{-2}\text{rad}}{3.34\text{x}10^{-6}\text{s}}$ = 3.8 10^3 rad/s

(b) Linear speed of a point on the edge of a wheel v=ωr = 3.8 10^3 0.05 = 1.9 10^2 m/s

Example

A pulsar is rapidly rotating neutron star that emits a radio beam like a lighthouse emits a light beam. We receive a radio pulse for each rotation of the star. The period T of rotation is found by measuring the time between pulses. The pulsar in the Crab nebula has a period of rotation of T = 0.033 s that is increasing at the rate of 1.26 x 10^{-5} second/year.

(a) What is the pulsar's angular acceleration?

(b) If its angular acceleration is constant, how many years from now will the pulsar stop rotating?

(c) The pulsar originated in a supernova explosion seen in the year 1054. What was the initial T for the pulsar? (Assume constant angular acceleration since the pulsar originated.).

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution

(a) The angular velocity in rad/s $\omega = 2\pi / T$.

The angular acceleration = $\alpha = \frac{d\omega}{dt} = -\frac{2\pi}{T^2}\frac{dT}{dt}$

For the pulsar described $\frac{dT}{dt} = \frac{1.26\times10^{-5}\,s/y}{3.16\times10^{7}\,s/y} = 4.00 \quad 10^{-13}$

So $\alpha = -\left[\frac{2\pi}{(0.033s)^2}\right]\left(4.00\times10^{-13}\right) = -2.3 \quad 10^{-3}$ rad/s^2

The negative sign indicates that the angular acceleration is opposite the angular velocity and the pulsar is slowing down.

(b) $\omega = \omega_0 + \alpha t$ for the time t when ω=0.

$t = -\frac{\omega}{\alpha} = -\frac{2\pi}{aT} = -\frac{2\pi}{(-2.3\times10^{-9}\,red/s^2)(0.033s)} = 8.3 \quad 10^{10}$ s . This is about 2600 years.

(c) The pulsar was born 1992– 1054 = 938 years ago.
This is equivalent to (938 y) (3.16 10^7 s/y) = 2.96 x 10^{10} s. Its angular velocity was then

$\omega = \omega_0 + \alpha t = \frac{2\pi}{T} + \alpha t = \frac{2\pi}{0.033s} +$ (–2.3 x 10^{-9} rad/s^2) (–2.96 x 10^{10}s) = 258 rad/s.

Its period was $T = \frac{2\pi}{\omega} = 2.4 \quad 10^{-2}$ s.

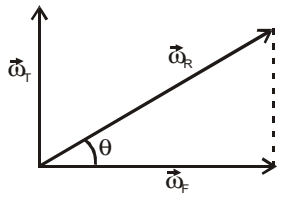
Example

A turn table is rotating in a horizontal plane about the vertical axis passing through its centre with an angular velocity 20 rad/s. It carries upon it a flywheel rotating with an angular velocity 40 rad/s about a horizontal axle mounted in bearings. Find the angular velocity of the wheel as seen by an observer in the room.

Solution

As the axis of the turn table is vertical its angular velocity ω_r is directed vertical. The axis of flywheel is horizontal therefore its angular velocity ω_F is directed horizontal, hence the resultant angular velocity is $\vec{\omega}_R = \vec{\omega}_F + \vec{\omega}_T$

$|\vec{\omega}_R| = \sqrt{\omega_F^2 + \omega_T^2} = \sqrt{40^2 + 20^2} = 20\sqrt{5}$ rad/s

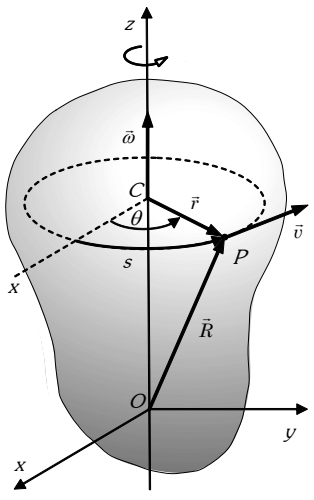


$\vec{\omega}_R$ lies in a plane which makes an angle θ with the horizontal plane, given by $\theta = \tan^{-1}\left(\frac{\omega_T}{\omega_F}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Kinematics of rotation about fixed axis

In figure is shown a rigid body of arbitrary shape rotating about the *z*-axis. In the selected frame (here the coordinate system) all the three axes are at rest, therefore the *z*-axis that is the axis of rotation is at rest and the body is in fixed axis rotation. All of its particles other than those on the *z*-axis move on circular paths with their centers on the *z*-axis. All these circular paths are parallel to the *x*-*y* plane. In the figure, one of its particles *P* is shown moving with velocity $\vec{v}$ on a circular path of radius *r* and center *C*. Its position vector is $\vec{R}$. It were at the line *Cx* at $t = 0$ and at the position shown at the instant *t*. During time interval *t*, it covers the circular arc of length *s* and its radius vector turns through angle θ.

In an infinitesimally small time interval *dt* let, the particle covers



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

infinitesimally small distance *ds* along its circular path.

$$d\vec{s} = d\vec{\theta} \times \vec{r} = d\vec{\theta} \times \vec{R} \qquad [7]$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}\,\frac{d\vec{\theta}}{dt} \times \vec{R} \qquad [8]$$

From eq. [7] and [8] we have

$$\vec{v} = \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{R} \qquad [9]$$

The above equation tells us the relation between the linear and angular velocity. Now we explore relation between the linear and angular accelerations. For the purpose, differentiate the above equation with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \qquad [10]$$

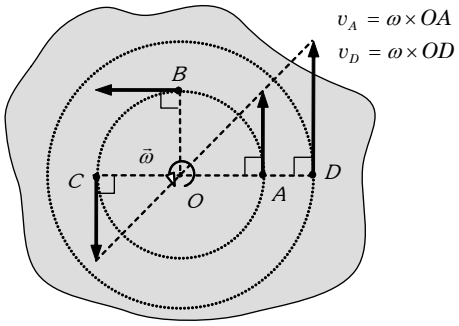
The first term on the RHS points along the tangent in the direction of the velocity vector and it is known as tangential acceleration $\vec{a}_T$ same as we have in circular motion. In addition, the second term point towards the center *C*. It is known as centripetal acceleration or normal component $\vec{a}_n$ of acceleration same as in circular motion. Now we have

Tangential acceleration $\vec{a}_T = \vec{\alpha} \times \vec{r}$ [11]

Normal acceleration $\vec{a}_n = \vec{\omega} \times \vec{v} = -\omega^2 \vec{r}$ [12]

How to Locate Axis of Rotation

Every particle in a plane perpendicular to the axis of rotation move with different velocities and accelerations, moreover, they all have the same angular velocity and angular acceleration. Such a section of a body in rotation is shown here. The particles *A*, *B* and *C* at equal distance from the axis of rotation move with equal speeds v_A and the particle *D* moves with speed v_D on concentric circular paths. The location of rotational axis can be determined by any of the two graphical techniques.

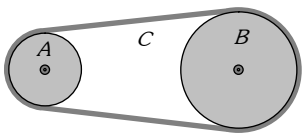


⇨ Lines perpendicular to velocity vectors and passing through the particles, whose velocity vectors are neither parallel nor antiparallel intersect at the axis of rotation. See pairs of particles *A* and *B*, *B* and *C* and *B* and *D*.

⇨ Lines perpendicular to velocity vectors and passing through the particles, whose velocity vectors are either parallel or antiparallel, coincide and intersect the line joining tips of their velocity vectors at the axis of rotation. Refer pairs of particles *A* and *C*, *A* and *D* and *C* and *D*.

Example

A belt moves over two pulleys *A* and *B* as shown in the figure. The pulleys are mounted on two fixed horizontal axels. Radii of the pulleys *A* and *B* are 50 cm and 80 cm respectively. Pulley *A* is driven at constant angular acceleration 0.8 rad/s^2 until the pulley *B* acquires an angular velocity of 10 rad/s. The belt does not slide on either of the pulleys.



(a) Find acceleration of a point *C* on the belt and angular acceleration of the pulley *B*.

(b) How long after the pulley *B* achieve angular velocity of 10 rad/s.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution.

Since the belt does not slide on the pulleys, magnitude of velocity and acceleration of any point on the belt are same as velocity and tangential acceleration of any point on periphery of either of the pulleys.

Using the above fact with eq.[11], we have

$\vec{a}_T = \vec{\alpha} \times \vec{r} \rightarrow$ $\quad a_C = \alpha_A r_A = \alpha_B r_B$

Substituting r_A = 0.5 m, $r_B = 0.8$ m and α = 0.8 rad/s^2, we have

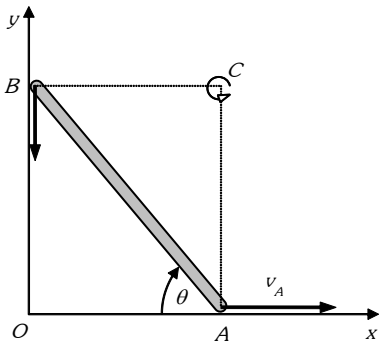
a_C = 0.4 m/s^2 and $\alpha_B = \dfrac{a_C}{r_B} = \dfrac{\alpha_A r_A}{r_B} = 0.5$ rad/s^2

From eq. [4], we have

$\omega = \omega_o + \alpha t \rightarrow$ $\quad t = \dfrac{\omega_B - \omega_{Bo}}{\alpha_B}$

Substituting $\omega_{Bo} = 0$, $\omega_B = 10$ rad/s and $\alpha_B = 0.5$ rad/s^2, we have t = 20 s

Kinematics of rotation about axis in translation



In this kind of motion, the body rotates about an axis and the axis moves without rotation. Rolling is a very common example of this kind of motion.

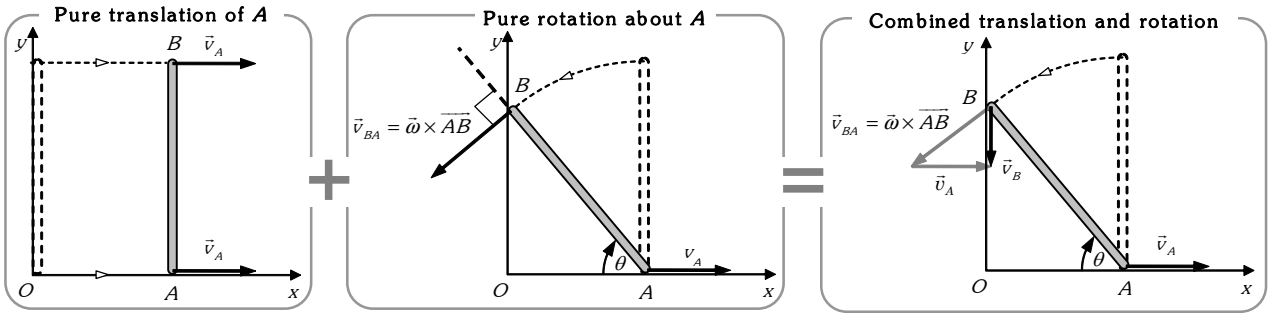
As an example consider a rod whose ends A and B are sliding on the x and y-axis as shown in the figure. Change in its orientation measured by change in angle θ indicates that the rod is in rotation. Perpendiculars drawn to velocity vector of its end points intersect at the axis of rotation, which is continuously changing it position.

Instantaneous Axis of Rotation (IAR)

It is a mathematical line about that a body in combined translation and rotation can be conceived in pure rotation at an instant. It continuously changes its location.

Now we explore how the combined translation and rotational motion of the rod is supperposition of translation motion of any of its particle and pure rotation about an axis through that particle.

Consider motion of the rod from beginning when it was parallel to the y-axis. In the following figure translation motion of point A is superimposed with pure rotation about A.



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

The motion of the rod can be conceived as superposition of translation of point A and simultaneous rotation about an axis through A.

The same experiment can be repeated to dnnnemonstrate that motion of the rod can be conceived as superposition of translation of any of its particle and simultaneous rotation about an axis through that particle.

Considering translation of A and rotation about A this fact can be expressed by the following equation.

Combined Motion = Translation of point A + Pure rotation about point A

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad [13]$$

Since point B moves relative A moving on circular path its velocity relative to A is given by the equation

$$\vec{v}_{BA} = \vec{\omega} \times \overrightarrow{AB} .$$

Now we have $$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overrightarrow{AB} \qquad [14]$$

The above fact is true for any rigid body in combined translation and rotation motion.

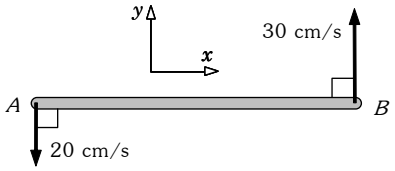
Rotation about an axis in translation of a rigid body can be conceived as well as analyzed as superposition of translation motion of any of its particle and simultaneous rotation about an axis passing through that particle provided that the axis is parallel to the actual one.

Similar to eq.[13], we can write equation for acceleration.

$$\left.\begin{aligned} \vec{a}_A &= \vec{a}_B + \vec{a}_{BA} \\ \vec{a}_A &= \vec{a}_B + \vec{a}_{BA_T} + \vec{a}_{BAn} \\ \vec{a}_A &= \vec{a}_B + \vec{\alpha} \times \overrightarrow{AB} + -\omega^2 \overrightarrow{AB} \end{aligned}\right\} \qquad [15]$$

Example

A 100 cm rod is moving on a horizontal surface. At an instant, when it is parallel to the x-axis its ends A and B have velocities 30 cm/s and 20 cm/s as shown in the figure.



(a) Find its angular velocity and velocity of its center.

(b) Locate its instantaneous axis of rotation.

Solution.

Let the rod is rotating anticlockwise, therefore its angular velocity is given by $\vec{\omega} = \omega\hat{k}$. Velocity vectors of all the points on the rod and its angular velocity must satisfy the relative motion eq.[14].

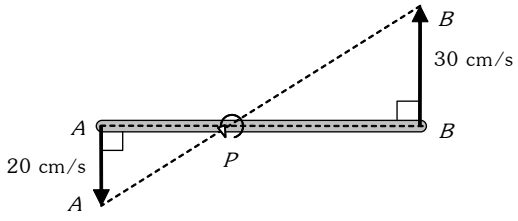
(a) Substituting velocities $\vec{v}_A = -20\tilde{j}$ cm/s and $\vec{v}_B = 30\tilde{j}$ cm/s and angular velocity $\vec{\omega}$ in eq.[14] , we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overrightarrow{AB} \rightarrow \qquad \omega = 0.5 \text{ rad/s}$$

Velocity vector of the center C of the rod also satisfy the following equation.

$$\vec{v}_C = \vec{v}_A + \vec{\omega} \times \overrightarrow{AC} \rightarrow \qquad \vec{v}_C = -20\tilde{j} + 0.5\tilde{k} \times 50\tilde{i} = 5.0\tilde{j} \text{ cm/s}$$

(b) Here velocity vectors of the particles A and B are antiparallel, therefore the instantaneous axis of rotation passes through intersection of the common perpendicular to their velocity vectors and a line joining tips of the velocity vectors. The required geometrical construction is shown in the following figure.



Since triangles $AA'P$ and $BB'P$ are similar and $AB = 100$ cm, we have $AP = 40$ cm.

The instantaneous axis of rotation passes through the point P, which is 40 cm from A.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Analytical Approach.

The instantaneous center of rotation is at instantaneous rest. Using this fact in eq.[14], we have

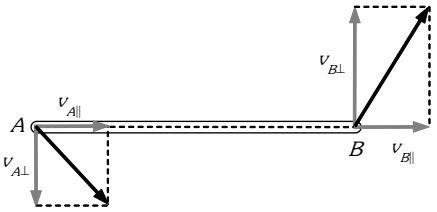
$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \overrightarrow{AP} \rightarrow \qquad \vec{0} = -20\tilde{j} + 0.5\tilde{k} \times (AP)\tilde{j} \Rightarrow AP = 40 \text{ cm}$

Example

Can you suggest a quick way to find angular velocity of a rod, if velocities of two of its points are known?

Solution.

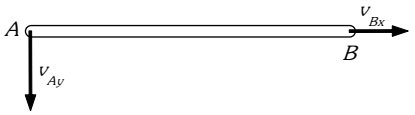
The eq.[14] suggest a quick way to determine angular velocity, when distance between two points and their velocity components perpendicular to the lining joining them are known.



Angular velocity of the rod $\omega = \dfrac{|\vec{v}_{BA}|}{AB} = \dfrac{|\vec{v}_{B\perp} - \vec{v}_{A\perp}|}{AB}$

Example

A 50 cm long rod AB is in combined translation and rotation motion on a table. At an instant velocity component of point A perpendicular the rod is 10 cm/s, velocity component of point B parallel to the rod is 6.0 cm/s and angular velocity of the rod is 0.4 rad/s in anticlockwise sense as shown in the given figure.

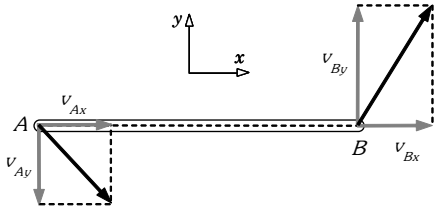


(a) Find velocity vectors of point A and B.

(b) Locate the instantaneous axis of rotation.

Solution.

Let x-y plane of a coordinate system coincides with the tabletop and the rod is parallel to the x-axis at the instant considered. The rod is shown in this coordinate frame in the following figure.



(a) Since distance between any two points remains unchanged, the velocity components of any two points parallel to the line joining them must be equal. Therefore, we have

$$v_{Ax} = v_{Bx} = 6.0 \text{ cm/s} \qquad (1)$$

Velocities of points A and B must satisfy eq.[14]. Substituting angular velocity $\vec{\omega} = 0.4\hat{k}$ rad/s, in this equation we have

$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overrightarrow{AB} \rightarrow \qquad 6.0\tilde{i} + v_{By}\tilde{j} = 6.0\tilde{i} - 10\tilde{j} + 0.4\tilde{k} \times 50\tilde{i}$

Equating y-components of both the sides, we have

$$v_{By} = 10 \text{ cm/s} \qquad (2)$$

From eq. (1), (2) and the given information, we can express the velocity vectors of the points A and B.

$\vec{v}_A = 6.0\tilde{i} - 10\tilde{j}$ cm/s, and $\vec{v}_B = 6.0\tilde{i} + 10\tilde{j}$ cm/s

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

(b) The instantaneous axis of rotation is at instantaneous rest. Let the end A of the rod is at the origin and coordinates of the point P in the x-y plane through which the IAR passes is (x, y). Now from eq.[14], we have

$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \overrightarrow{AP} \rightarrow \quad \vec{0} = 6.0\tilde{i} - 10\tilde{j} + 0.4\tilde{k} \times \left(x\tilde{i} + y\tilde{j}\right)$

Equating coefficients of x and y-components of both the sides, we have

$x = 25$ cm and $y = 15$ cm

Therefore, coordinates of the point P through which IAR passes the x-y plane are (25, 15).

Rolling as rotation about an axis in translation

Wheels of a moving vehicle roll on road. A ball rolls on ground when pushed. In fact, a body of round section can roll smoothly under favorable conditions. On the other hand, objects with corners, such as dice, roll by successive rotations about the edge or corner that is in contact with the ground. This type of motion is usually known as toppling.

If the point of contact of the of the rolling body does not slide it is known as *rolling without slipping* or *pure rolling* or simply *rolling* and if the point of contact slides it is known as *rolling with slipping*.

All kind of rolling motion is examples of rotation abut an axis in translation.

Rolling without slipping on stationary surface.

We first discuss velocity relations and thereafter accelerations relations of two points of a body of round section rolling on a stationary surface. For the purpose, we can use any of the following methods.

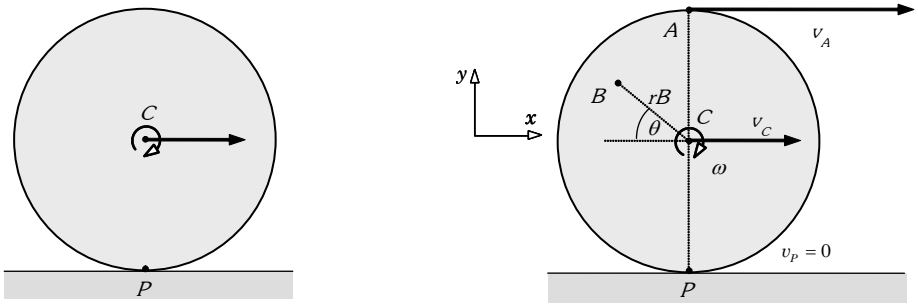
I Analytical Method: By using relative motion equations.

II Superposition Method: By superimposing translation of a point and pure rotation about that point.

III Use of ICR.

Velocity relations by Analytical Method

Its point of contact P does not slide on the surface, therefore velocity of the point of contact relative to the surface is zero. In the next figure, velocity vectors of its center C and top point A are shown.



Velocity of the center C can be obtained with the help of relative motion equation.

$\vec{v}_C = \vec{v}_P + \vec{\omega} \times \overrightarrow{PC} \rightarrow \quad \vec{v}_C = \vec{0} + \left(-\omega\tilde{k}\right) \times R\tilde{j}$

$\vec{v}_C = \omega R\tilde{i}$ [16]

The above equation is used as condition of rolling without slipping on stationary surface.

Velocity of the top point A can be obtained by relative motion equation.

$\vec{v}_A = \vec{v}_P + \vec{\omega} \times \overrightarrow{PA} \rightarrow \quad \vec{v}_C = \vec{0} + \left(-\omega\tilde{k}\right) \times \left(R\tilde{j}\right)$

$\vec{v}_A = 2\omega R\tilde{i} = 2\vec{v}_C$ [17]

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Once velocity of the center is obtained, we can use relative motion between A and C as well.

$$\vec{v}_A = \vec{v}_C + \vec{\omega}\times\overrightarrow{CA} \rightarrow \quad \vec{v}_C = \omega R\tilde{i} + (-\omega\tilde{k})\times(2R\tilde{j})$$

$$\vec{v}_A = 2\omega R\tilde{i} = 2\vec{v}_C \qquad [18]$$

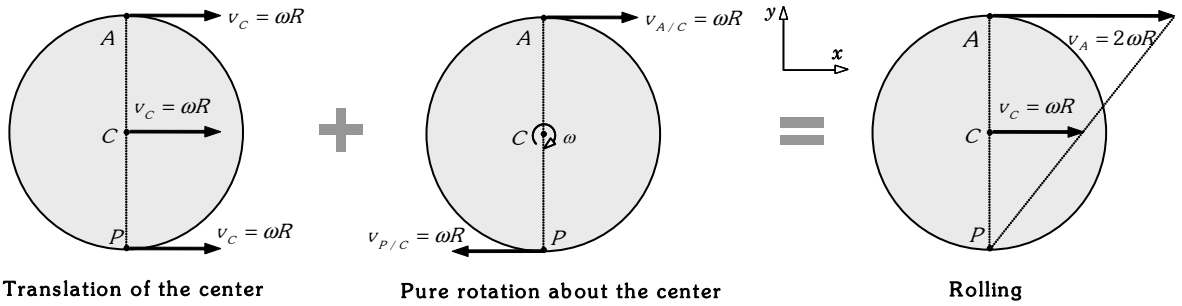
In similar fashion, velocity vector of an arbitrarily chosen point B.

$$\vec{v}_B = \vec{v}_C + \vec{\omega}\times\overrightarrow{CB} \rightarrow \quad \vec{v}_B = v_C\tilde{i} + (-\omega\tilde{k})\times(-r\cos\theta\tilde{i} + r\sin\theta\tilde{j})$$

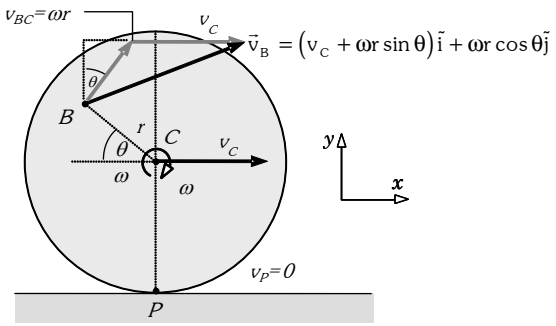
$$\vec{v}_B = (v_C + \omega r\sin\theta)\tilde{i} + \omega r\cos\theta\tilde{j} \qquad [19]$$

Velocity relations by Superposition Method

Now we will see that the above velocity relation can also be obtained by assuming rolling of the wheel as superposition of translation of its center and simultaneous rotation about the center.

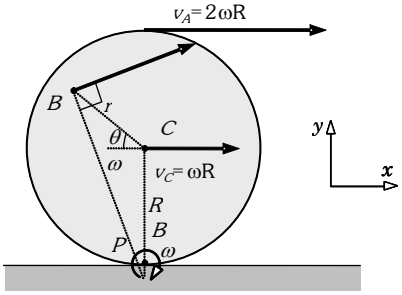


Velocity of an arbitrary point B as superposition of translation of the center and rotation about the center.



Velocity relations by Use of ICR

In rolling without slipping on stationary surface the point of contact is at instantaneous rest, therefore the ICR and the IAR passes through it. We will see how velocity of the center C, the top point A and an arbitrarily chosen point B can be calculated by assuming the body in state of pure rotation about the ICR.



Velocity of center C $\quad \vec{v}_C = \vec{\omega}\times\overrightarrow{PC} = \omega R\tilde{i}$

Velocity of the top point A $\quad \vec{v}_A = \vec{\omega}\times\overrightarrow{PA} = 2\omega R\tilde{i}$

Velocity of the point B $\quad \vec{v}_B = \vec{\omega}\times\overrightarrow{PB} = (v_C + \omega r\sin\theta)\tilde{i} + \omega r\cos\theta\tilde{j}$

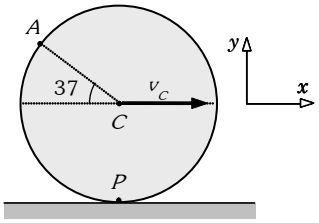
NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example

A cylinder of radius 5 m rolls on a horizontal surface. Velocity of its center is 25 m/s. Find its angular velocity and velocity of the point *A*.

Solution.

In rolling the angular velocity $\vec{\omega}$ and velocity of the center of a round section body satisfy condition described in the relative motion eq.[14]. So we have

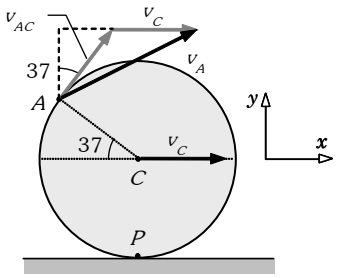


$$\vec{v}_C = \vec{\omega} \times \vec{r}_{C/P} \rightarrow \qquad 25\tilde{i} = \omega\tilde{k} \times 5\tilde{j} \Rightarrow \vec{\omega} = -5\tilde{k} \text{ rad/s}$$

Angular velocity vector points in the negative *z*-axis so the cylinder rotates in clockwise sense.

Velocity of the point *A* can be calculated by either analytical method, superposition method or by using method of ICR.

Analytical Method



$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \overrightarrow{CA} \rightarrow \qquad \vec{v}_A = 25\tilde{i} + (-5\tilde{k}) \times (-5\cos 37°\tilde{i} + 5\sin 37°\tilde{j})$$

$$\vec{v}_A = (40\tilde{i} + 20\tilde{j}) \text{ m/s}$$

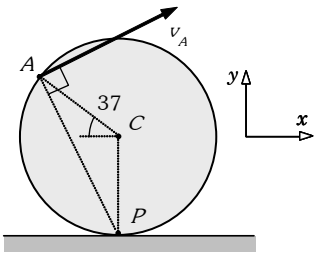
Superposition Method

In rolling $v_C = v_{AC} = \omega R = 25 m/s$. The superposition i.e. vector addition of the terms of equation $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$ are shown in the following figure. Resolving $v_{AC} = \omega R = 25 m/s$ into its Cartesian components and adding to $\vec{v}_C$, we obtain

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C} \rightarrow \qquad \vec{v}_A = 25\tilde{i} + 15\tilde{i} + 20\tilde{j} = (40\tilde{i} + 20\tilde{j}) \text{ m/s}$$

Use of ICR

The contact point *P* is the ICR in rolling. The cylinder is in pure rotation about the ICR at the instant under consideration, so from the relative motion equation, we have

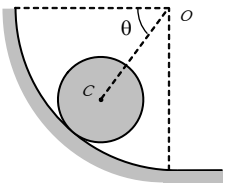


$$\vec{v}_A = \vec{\omega} \times \vec{r}_{AP} \rightarrow \qquad \vec{v}_A = \vec{\omega} \times (\overrightarrow{PC} + \overrightarrow{CA}) = (-5\tilde{k}) \times \{(5\tilde{j}) + (-4\tilde{i} + 3\tilde{j})\}$$

$$\vec{v}_A = (40\tilde{i} + 20\tilde{j}) \text{ m/s}$$

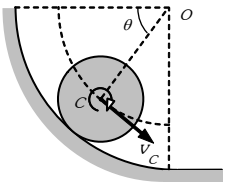
Example

A disk of radius *r* is rolling down a circular track of radius *R*. There is no slipping between the disk and the track. When line *OC* is at angle θ down the horizontal, center of the disk has velocity v_C. Assume center *O* of the track as origin of reference frame, find angular velocity of translation motion of the center of the disk and angular velocity of rotation motion.



Solution.

The angular velocity of translation motion of the center of the disk equals to the rate of change is θ. Let us denote it by ω_o.



$$\omega_o = \frac{d\theta}{dt}$$

The center of the disk moves on circular path of radius $R - r$. Relation between velocity v_C of the center of the disk, radius *R* of the circular track and radius of the disk is

$$v_C = \omega_o (R - r)$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Therefore, angular velocity of translation motion of center of the disk is $\omega_o = \frac{v_C}{R-r}$

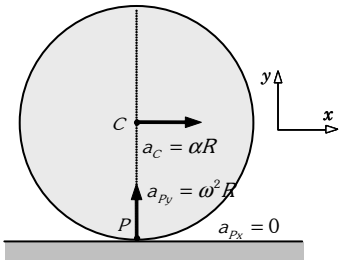
Since the disk is rolling without slipping on the circular track, its angular velocity of rotation ω is given by the following equation.

$$v_C = \omega r$$

Therefore, angular velocity of rotation of the disk is $\omega = \frac{v_C}{r}$

Acceleration relations by Analytical Method

The point of contact P does not slide on the surface, therefore component of its acceleration parallel to the surface must be zero. However, it has an acceleration component towards the center. The center always moves parallel to the horizontal surface and does not changes direction of its velocity; therefore, its acceleration can only be parallel to the surface.



Relation between acceleration of acceleration vector of the center C and point of contact P can be obtained with the help of relative motion [15] equation together with the above fact.

$\vec{a}_C = \vec{a}_P + \vec{\alpha} \times \overrightarrow{PC} - \omega^2 \overrightarrow{PC} \rightarrow$ $\quad a_C\tilde{i} = a_{Py}\tilde{j} + \left(-\alpha\tilde{k}\right) \times R\tilde{j} - \omega^2 R\tilde{j} = a_{Py}\tilde{j} + \alpha R\tilde{i} - \omega^2 R\tilde{j}$

Equating coefficients of x and y-components on both the sides of the above equation, we have

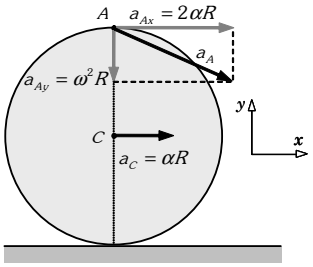
$$\vec{a}_C = \alpha R\tilde{i} \qquad [20]$$

$$\vec{a}_P = \omega^2 R\tilde{j} \qquad [21]$$

The eq. [20] *is used as condition for rolling without slipping together with eq.* [16]

In the given figure, acceleration vectors the point of contact; center and the top point are shown. Now we will see how these accelerations can be calculated by using relative motion equation.

Once velocity of the center is obtained, we can use relative motion between A and C as well. Now we calculate acceleration of the top point A.



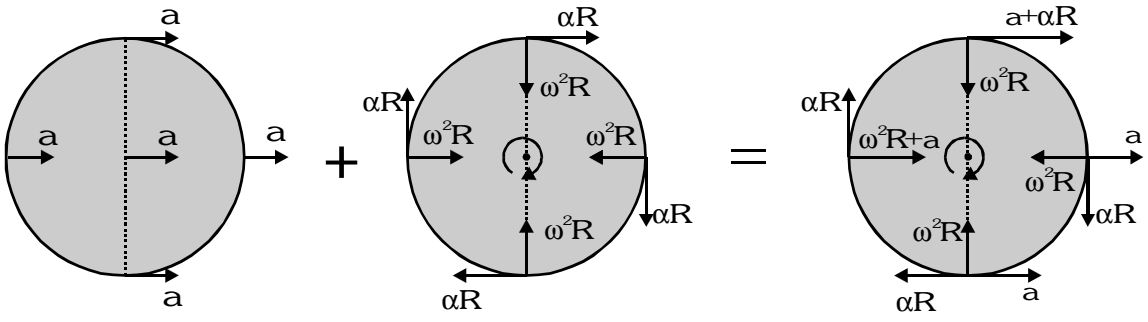
$\vec{a}_A = \vec{a}_C + \vec{\alpha} \times \overrightarrow{CA} - \omega^2 \overrightarrow{CA} \rightarrow$ $\quad \vec{a}_A = \alpha R\tilde{i} + \left(-\alpha\tilde{k}\right) \times R\tilde{j} - \omega^2 R\tilde{j}$

$$\vec{a}_A = 2\alpha R\tilde{i} - \omega^2 R\tilde{j} \qquad [22]$$

Acceleration vector of point A and its components are shown in the given figure.

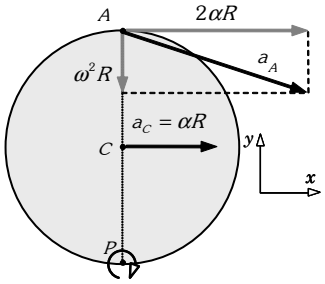
Acceleration relations by Superposition Method

Now we see how acceleration relations are expressed for a rolling wheel by assuming its rolling as superposition of its translation with the acceleration of center and simultaneous rotation about the centre.



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

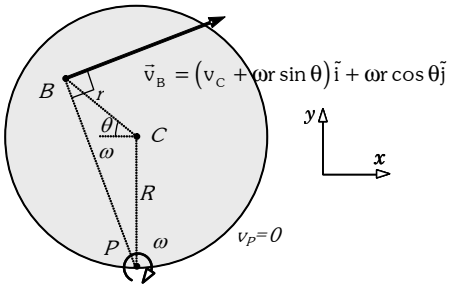
Acceleration relations by use of ICR



Acceleration relations can also be obtained by assuming the body in pure rotation about the ICR. Here we will use relative motion equation[15]. Always keep in mind that the acceleration of the ICR is not zero, it has value $\omega^2 R$ and points towards the center of the body. Now we will see how acceleration of the center C, the top point A and an arbitrarily chosen point B can be calculated by assuming the body in state of pure rotation about the ICR.

Acceleration of center C $\quad \vec{a}_C = \vec{a}_P + \vec{\alpha}\times\overrightarrow{PC} - \omega^2\overrightarrow{PC} = \alpha R\tilde{i}$

$$\vec{a}_C = \alpha R\tilde{i}$$



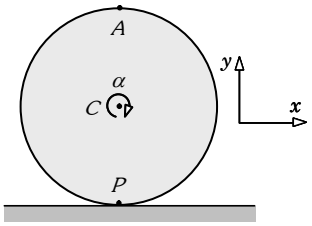
Acceleration of the top point A $\quad \vec{a}_A = \vec{a}_p + \vec{\alpha}\times\overrightarrow{PA} - \omega^2\overrightarrow{PA}\tilde{j}$

$$\vec{a}_A = 2\alpha R\tilde{i} - \omega^2 R\tilde{j}$$

Acceleration of the point B $\quad \vec{a}_B = \vec{a}_P + \vec{\alpha}\times\overrightarrow{PB} - \omega^2\overrightarrow{PB}$

$$\vec{a}_B = \left\{\alpha\left(R + r\sin\theta\right) + \omega^2\right\}\tilde{i} + \left\{\alpha r\cos\theta - \omega^2 r\sin\theta\right\}\tilde{i}$$

Example



A body of round section of radius 10 cm starts rolling on a horizontal stationary surface with uniform angular acceleration 2 rad/s^2.

(a) Find initial acceleration of the center C and top point A.

(b) Find expression for acceleration of the top point A as function of time.

Solution.

Initially when the body starts, it has no angular velocity; therefore, the last term in relative motion equation [15] for acceleration vanishes and for a pair of two points A and B the equation reduces to

$$\vec{a}_A = \vec{a}_B + \vec{\alpha}\times\overrightarrow{BA}$$

The angular acceleration vector is $\vec{\alpha} = -2\tilde{k}$ rad/s^2.

(a) Acceleration of the center C is obtained by using condition for rolling without slipping.

$\vec{a}_C = \vec{\alpha}\times\overrightarrow{PC} \rightarrow \qquad \vec{a}_C = -2\tilde{k}\times 10\tilde{j} = 20\tilde{i}$ cm/s^2

Acceleration of the point A can be obtained either by analytical method, superposition method or by use of ICR. These methods for calculation of acceleration of the top point are already described; therefore, we use the result.

$\vec{a}_A = 2\alpha R\tilde{i} \rightarrow \qquad \vec{a}_A = 40\tilde{i}$ cm/s^2

(b) Initially at the instant t = 0, when the body starts, its angular velocity is zero. At latter time it acquires angular velocity $\vec{\omega}$, therefore acceleration of any point on the body, other than its center, has an additional component of acceleration. This additional component is accounted for by the last term in the relative motion equation [15].

Angular velocity acquired by the body at time t is obtained by eq.[4] used for a body rotating with constant angular acceleration.

$\vec{\omega} = \vec{\omega}_o + \vec{\alpha}t \rightarrow$ Substituting $\omega_o = 0$, we have

$$\vec{\omega} = -2t\tilde{k}$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Analytical Method

Using the relative motion equation for the pair of points C and A, we have

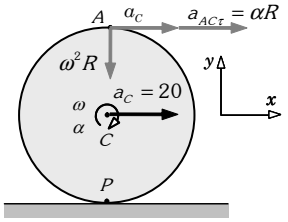
$\vec{a}_A = \vec{a}_C + \vec{\alpha}\times\overrightarrow{CA} - \omega^2\overrightarrow{CA} \rightarrow \quad \vec{a}_A = \alpha R\tilde{i} + \left(-\alpha\tilde{k}\right)\times R\tilde{j} - \omega^2 R\tilde{j} = 2\alpha R\tilde{i} - \omega^2 R\tilde{j}$

Substituting the known values

$\vec{\alpha} = -2\tilde{k}$ rad/s^2, $\vec{\omega} = -2t\tilde{k}$ rad/s and R = 10 cm

we have $\vec{a}_A = 40\tilde{i} - 40t^2\tilde{j}$ cm/s

Superposition Method



We superimpose translation motion of the center and rotation motion about the center. In fact it is vector addition of terms of above equation used in analytical method.

From the above figure, we have

$\vec{a}_A = \left(a_C + \alpha R\right)\tilde{i} - \omega^2 R\tilde{j}$

Substituting known values $\vec{\alpha} = -2\tilde{k}$ rad/s^2, $\vec{\omega} = -2t\tilde{k}$ rad/s and R=10 cm,

we have $\vec{a}_A = 40\tilde{i} - 40t^2\tilde{j}$ cm/s^2

Use of ICR

The point of contact P is the ICR, because the body is rolling without slipping. We use relative motion equation for pairs of points P and A.

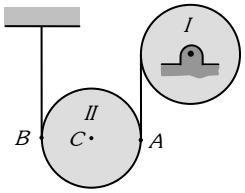
$\vec{a}_A = \vec{a}_P + \vec{\alpha}\times\overrightarrow{PA} - \omega^2\overrightarrow{PA} \rightarrow \quad \vec{a}_A = \omega^2 R\tilde{j} + 2\alpha R\tilde{i} - 2\omega^2 R\tilde{j} = 2\alpha R\tilde{i} - \omega^2 R\tilde{j}$

Substituting known values

$\alpha = 2$ rad/s^2, $\omega = 4t$ rad/s and $R = 10$ cm,

we have $\vec{a}_A = 40\tilde{i} - 40\tilde{j}$ cm/s^2

Example



Two identical disks, each of radius r, are connected by a cord as shown in the figure. The disk I rotates with constant angular acceleration α in anticlockwise direction. Find acceleration of the center of disk II and its angular acceleration.

Solution.

As the disk I rotates, the thread unwrap it. Acceleration of a point on the portion of the thread between the two disks equals to tangential acceleration of any point on the periphery of the disk I. The extreme left point A of the disk II must also descend with the same acceleration.

Downward acceleration a_B of point B = Tangential acceleration of a point on the periphery of disk I.

$a_B = \alpha_I r \rightarrow \qquad a_B = \alpha r$

The point B on the thread is at rest relative to the ground; therefore, it can be assumed that the second disk is in a motion similar to rolling without slipping on a vertical surface.

Now applying conditions of rolling without slipping we have

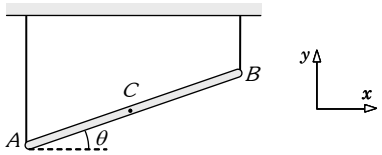
Acceleration of the center of the disk II $\qquad a_C = \frac{1}{2}a_B = \frac{1}{2}\alpha r$

Angular acceleration of the disk II $\qquad \alpha_{II} = \frac{a_C}{r} = \frac{1}{2}\alpha$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example

A uniform rod AB of length l is supported with the help of two light inextensible threads as shown in the figure. The thread supporting the end B is cut. If magnitude of acceleration of the center C of rod is a_C immediately after the thread is cut, find angular acceleration of the rod and acceleration of its end A.



Solution.

Immediately after the thread is cut, all the forces acting on the rod are in vertical direction; therefore, acceleration of its mass center is vertically downwards. The mass center of a uniform rod is at its center; therefore, acceleration of the center C immediately after the thread is cut is in vertically downward direction.

The end A can move on circular path of radius equal to length of thread supporting the end A. Therefore, acceleration a_A of the end A is in horizontal direction immediately after the thread supporting the end B is cut.

Analytical Method

If we assume angular acceleration α of the rod in clockwise direction, we can write relative motion equation for the pair of points A and C.

$\vec{a}_C = \vec{a}_A + \vec{\alpha} \times \overrightarrow{AC} - \omega^2 \overrightarrow{AC} \rightarrow$

The rod cannot acquire any angular velocity immediately after the thread is cut due to inertia therefore the last term in the relative motion equation vanishes.

Substituting $\vec{a}_C = -a_C\tilde{j}$, $\vec{\alpha} = -\alpha\tilde{k}$, $\vec{a}_A = a_A\tilde{i}$ and

$$\overrightarrow{AC} = \tfrac{1}{2}l\left(\cos\theta\tilde{i} + \sin\theta\tilde{j}\right),$$

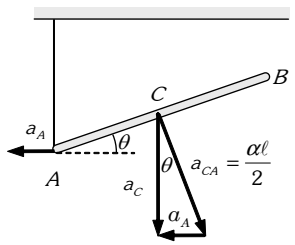
we have $-a_C\tilde{j} = a_A\tilde{i} - \tfrac{1}{2}\alpha l\cos\theta\tilde{j} + \tfrac{1}{2}\alpha l\sin\theta\tilde{i}$

Equating coefficients of x and y-components, we have

$$\alpha = \frac{2a_C}{l\cos\theta}\circlearrowright \text{ and } a_A = a_C\tan\theta$$

Superposition Method

We superimpose rod's translation with velocity of A and rotation about A. The rod cannot acquire any angular velocity immediately after the string is cut due to its inertia; therefore, point C cannot have any term involving radial acceleration. The acceleration vectors to be added are shown in the following figure.



From the figure, we have $a_C = \dfrac{\alpha\ell}{2}\cos\theta \Rightarrow \alpha = \dfrac{2a_C}{l\cos\theta}\circlearrowright$

$a_A = a_C\tan\theta$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

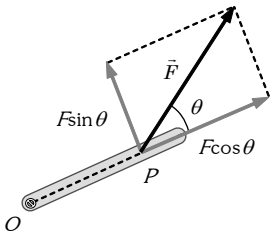
KINETICS OF RIGID BODY

In this section we deal with equilibrium of rigid bodies and kinetics of rigid bodies. Equilibrium of rigid bodies includes both the translational equilibrium and rotational equilibrium. In kinetics we account for causes affacting rotational motion.

Torque: Moment of a force

Torque is rotational analogue of force and expresses tendency of a force applied to an object to cause the object to rotate about a given point.

To investigate further let us discuss an experiment. Consider a rod pivoted at the point *O*. A force $\vec{F}$ is applied on it at the point *P*. The component $F\cos\theta$ of the force along the rod is counterbalanced by the reaction force of the pivot and cannot contribute in rotating the rod. It is the component $F\sin\theta$ of the force perpendicular to the rod, which is responsible for rotation of the rod. Moreover, farther is the point *P* from *O*, where the force is applied easier is to rotate the rod. This is why handle on a door is attached as far away as possible from the hinges.

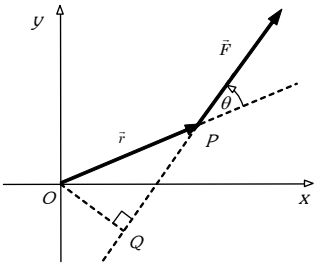


Magnitude of torque of a force is proportional to the product of distance of point of application of the force from the pivot and magnitude of the perpendicular component $F\sin\theta$ of the force. Denoting torque by symbol τ, the distance of point of application of force from the pivot by *r*, we can write

$$\tau_o \propto rF\sin\theta$$

Since rotation has sense of direction, torque should also be a vector. Its direction is given by right hand rule. Now we can express torque by the cross product of $\vec{r}$ and $\vec{F}$.

$$\vec{\tau}_o = \vec{r}\times\vec{F} \qquad [1]$$



Here constant of proportionality has been assumed a dimensionless number unity because a unit of torque has been chosen as product of unit of force and unit of length.

The geometrical construction shown in figure suggests a simple way to calculate torque. The line $OQ\ (= r\sin\theta)$ known as *moment arm*, is the length of perpendicular drawn from *O* on the line of action of the force. The magnitude of the torque equals to the product of *OQ* and magnitude of the force $\vec{F}$..

Torque about a Point and Torque about an Axes

We have defined *torque of a force about a point* as the moment of the force about that point. In dealing with rotation about a fixed axis we need to know torque about the axis rotation.

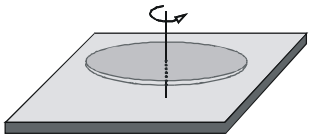
When a body is in plane motion the net torque of all the forces including the forces necessary to restrain rotation of the axis is along the axis of rotation. It is known as torque about the axis. Torque of a force about an axis of rotation equals to the moment of force about the point where plane of motion of the point of application of the force intersects the axis.

In analyzing plane motion we always consider torque about an axis under consideration and in rest of the book by the term torque of force we mean torque about an axis.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

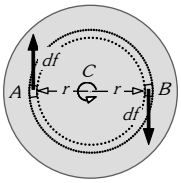
Example

A uniform disk of mass M and radius R rotating about a vertical axis passing through its center and perpendicular to its plane is placed gently on a rough horizontal ground, were coefficient of friction is μ. Calculate torque of the frictional forces.



Solution.

When the disk rotates on the ground, kinetic friction acts at every contact point.Since the gravity acts uniformly everywhere and the disk is also uniform, the normal reaction form the ground is uniformly distributed over the entire contact area. Consider two diametrically opposite identical portions A and B of the disk each of mass dm at distance r from the center as shown in the adjacent figure. The normal reaction form the ground on each of these portions equals to their weights and hence frictional forces are $df = \mu dmg$



Friction forces on these two and on all other diametrically opposite portions of the disk are equal and opposite, therefore net resultant friction force on the disk is zero. But torque of friction force on every portion is in same direction and all these torques add to contribute a net counterclockwise torque about the axis.

Consider a ring of radius r and width dr shown by dashed lines. Net torque $d\tau_C$ of friction force on this ring can easily be expressed by the following equation.

$$d\tau_C = r\mu(\text{mass of the ring})g = r\mu\left(\frac{\text{Mass of the disk}}{\text{Area of the disk}} \times \text{Area of the ring}\right)g = r\mu\left(\frac{2Mrdr}{R^2}\right)g$$

Integrating both sides of the above equation, we have

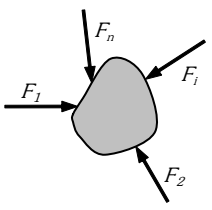
$$\tau_C = \frac{2\mu Mg}{R^2}\int_{r=0}^{R} r^2\,dr = \tfrac{2}{3}\mu MgR$$

Rotational equilibrium

A rigid body is said to be in state of rotational equilibrium if its angular acceleration is zero. Therefore a body in rotational equilibrium must either be in rest or rotation with constant angular velocity.

Since scope of JEE syllabus is confined only to rotation about a fixed axis or rotation about an axis in translation motion, the discussion regarding rotational equilibrium is limited here to situations involving only coplanar forces. Under these circumstances the necessary and sufficient condition for rotational equilibrium is

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.

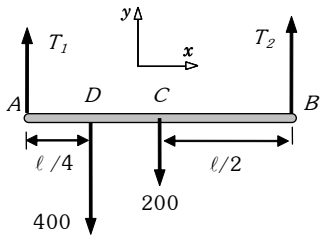


In the figure a body is shown under the action of several external coplanar forces F_1, F_2, ... F_i, and F_n.

$$\sum \vec{\tau}_P = 0$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Here P is a point in the plane of the forces about which we calculate torque of all the external forces acting on the body. The flexibility available in selection of the point P provides us with advantages that we can select such a point about which torques of several unknown forces will become zero or we can make as many number of equations as desired by selecting several different points. The first situation yields to a simpler equation to be solved and second situation though does not give independent equation, which can be used to determine additional unknowns yet may be used to check the solution.



The above condition reveals that a body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the mass center of the body.

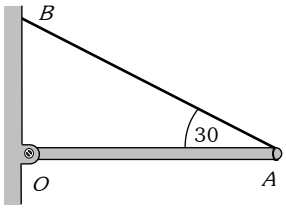
A case of particular interest arises where only three coplanar forces are involved and the body is in rotational equilibrium. It can be shown that *if a body is in rotational equilibrium under the action of three forces, the lines of action of the three forces must be either concurrent or parallel.* This condition provides us with a graphical technique to analyze rotational equilibrium.

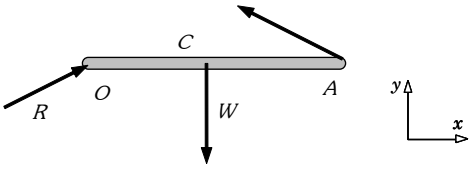
Equilibrium of Rigid Bodies

A rigid body is said to be in equilibrium, if it is in translational as well as rotational equilibrium both. To analyze such problems conditions for both the equilibriums must be applied.

Example

A 10 kg uniform rod OA is pivoted at O on a vertical wall with the help of a cable AB. Find the tension in the cable and reaction force applied by the pivot.





Free-body diagram of the rod

Solution.

The rod is in translational and rotational equilibrium under the action of three forces that are weight (W) of the rod, the tension (T) in the cable, and the reaction (R) of the pivot. These forces are shown in the free-body diagram of the rod.

Translational equilibrium

$\Sigma F_x = 0 \rightarrow \qquad R_x = T_x = T\cos 30° \qquad (1)$

$\Sigma F_y = 0 \rightarrow \qquad R_y + T_y = W \Rightarrow R_y + T\sin 30° = W \qquad (2)$

Rotational equilibrium: Let us apply the condition about O, because torque of the reaction R will become zero.

$\Sigma \vec{\tau}_O = \vec{0} \rightarrow \qquad Wl/2 = (T\sin 30°)l$

$T = W = 100$ N

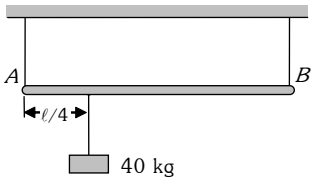
Now from equations (1) and (2) we have R = 100 N

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example

A uniform rod of 20 kg is hanging in horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tension developed in each thread.

Solution.



Free body diagram of the rod is shown in the figure.

Translational equilibrium

$\Sigma F_y = 0 \rightarrow \quad T_1 + T_2 = 400 + 200 = 600\ N \qquad (1)$

Rotational equilibrium: Applying the condition about A, we get T_2

$\Sigma \vec{\tau}_A = \vec{0} \rightarrow \quad 400(l/4) + 200(l/2) - T_2 l = 0$

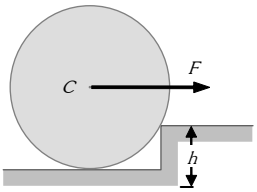
$T_2 = 200\ \text{N}$

Similarly writing torque equation about B, we have

$\Sigma \vec{\tau}_B = \vec{0} \rightarrow \quad T_1 = 400\ \text{N}.$

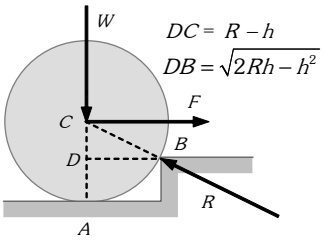
Example

A cylinder of radius R and weight W is to be raised against a step of height h by applying a horizontal force at its center as shown in the figure. Find the required minimum magnitude of this force. Assume sufficient friction between the cylinder and the corner of the step to prevent slipping.



Solution.

The forces acting on the sphere are its weight W_s the horizontal pull F, reaction R from the corner and the normal reaction from the ground. The reaction from the corner includes the normal reaction and friction. We need not to worry about this force because its torque about the corner vanishes. The moment it starts rising the normal reaction from the ground also vanishes. The requirement that the force F should be of minimum magnitude will cause the cylinder to rotate about B at very small angular vacuity and with negligible angular acceleration. Therefore the cylinder can be assumed in the state of rotational equilibrium as well as translational equilibrium.



The weight W, the pull F and the reaction R from the corner are shown in the free body diagram of the cylinder.

Rotational equilibrium: The cylinder is in rotational equilibrium under the action of three coplanar forces therefore these forces must be concurrent.

Torques equation of all the forces about the corner B to zero, we have

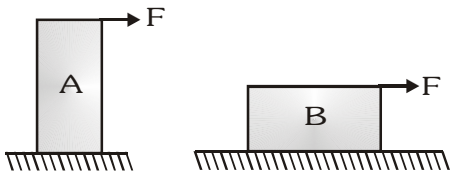
$\sum \vec{\tau}_B = 0 \rightarrow \quad F(CD) = W(DB)$

By solving above equation we have $F = W\dfrac{\sqrt{2Rh - h^2}}{R - h}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

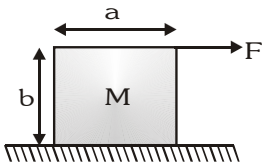
Toppling :

For shown situation (A) & (B), more chances of toppling in (A). In case of toppling, normal reaction must passes through end points.



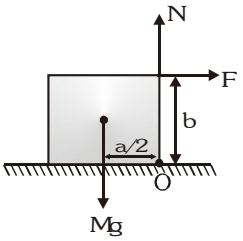
Example

Find the minimum value of F to topple about an edge.



Solution

In case of toppling



Taking torque about O

$F\,(b) = Mg\left(\frac{a}{2}\right) \Rightarrow F_{min} = \frac{Mga}{2b}$

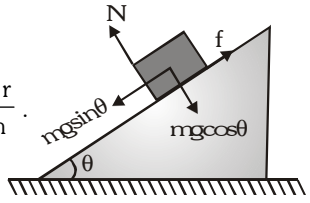
Example

A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will (a) slide before toppling (b) topple before sliding.

Solution

(a) The cylinder will slide if $mg\sin\theta > \mu mg\cos\theta \Rightarrow \tan\theta > \mu$ ...(i)

The cylinder will topple if $(mg\sin\theta)\ \frac{h}{2} > (mg\cos\theta)r \Rightarrow \tan\theta > \frac{2r}{h}$ ... (ii)



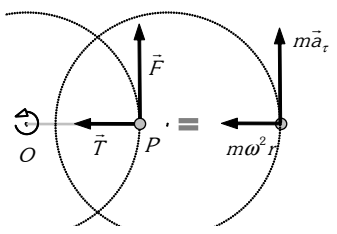
Thus, the condition of sliding is $\tan\theta > \mu$ & condition of toppling is $\tan\theta > \frac{2r}{h}$.

Hence, the cylinder will slide before toppling if $\mu < \frac{2r}{h}$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$

Concept of Rotational Inertia

Every particle of a rigid body in rotation moves on circular paths about the axis of rotation; therefore a rigid body in rotation can be thought as a group of large number of particle moving on circular paths.



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Let a particle of mass m constrained by a string to move on a circular path of radius r about a fixed point O in free space. To provide the particle tangential acceleration α there is a force $\vec{F}$ and to provide the particle necessary centripetal acceleration there is string tension $\vec{T}$ as shown in the figure.

$$\vec{F} = m\vec{a}_\tau \text{ and } \vec{T} = -m\omega^2\vec{r}_{P/O}$$

Taking moments about the center O of all the forces acting on the particle and then summing up them, the above equations yield

$$\vec{r}_{P/O} \times \vec{F} + \vec{r}_{P/O} \times \vec{T} = \vec{r}_{P/O} \times m\vec{a}_\tau + \vec{r}_{P/O} \times \left(-m\omega^2\vec{r}_{P/O}\right)$$

The left hand side equals to resultant torque about the center O of all the external force acting on the particle. The vector product $\vec{r}_{P/O} \times \vec{a}_\tau$ in the first term of the right hand side become $r^2\vec{\alpha}$. The second term on both the side vanishes.

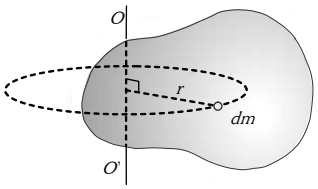
$$\Sigma\vec{\tau}_o = mr^2\vec{\alpha} \qquad [2]$$

The above equation is similar to Newton's second law of motion. It suggest that torque, angular acceleration and the term mr^2 play similar roles in rotation motion as the net force, the acceleration and mass plays in translation motion. Total mass of a body in translation motion is the measure of its inertia to translation motion, therefore the sum of all terms mr^2 arising due to all particle of the rigid body provides suitable measure of its inertia to rotation motion. *The inertial to rotation motion is known as rotational inertia or more commonly moment of inertia.*

Moment of inertia of a rigid body

A rigid body is continuous distribution of mass and can be assumed consisting of infinitely large number of point particles. If one of the point particle of infinitely small mass dm is at a distance r from the axis of rotation OO', the moment of inertia of this point particle is given by

$$dI_o = r^2 dm$$



The moment of inertia of the whole body about the axis OO' can now be obtained by integrating term of the above equation over the limits to cover whole of the body.

$$I_o = \int dI_o = \int r^2 dm \qquad [3]$$

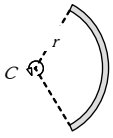
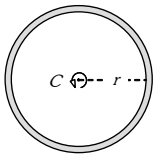
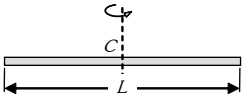
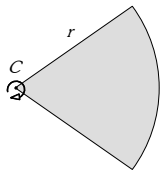
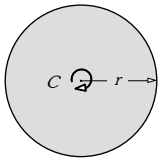
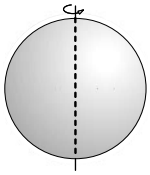
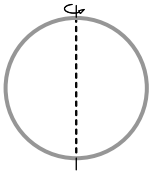
Expression for moment of inertia contains product of two terms. One of them is the mass of the body and the other is a characteristic dimension, which depends on the manner how mass of the body is distributed relative to the axis of rotation. Therefore moment of inertia of a rigid body depends on the mass of the body and distribution of the mass relative to the axis of rotation. Obviously for uniform bodies expression of moment of inertia depends on their shape and location and orientation of the axis of rotation. Based on these facts we can conclude

1. If mass distribution is similar for two bodies about an axis, expressions of their moment of inertia must be of the same form about that axis.

2. If the whole body or any of its portions is shifted parallel to the axis of rotation, moment of inertia remains unchanged.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Moment of Inertia for some commonly used bodies

Body	Axis	Moment of Inertia
Uniform thin rod bent into shape of an arc of mass m 	Passing through center and perpendicular to the plane containing the arc	$I_C = mr^2$
Uniform ring of mass m 	Passing through center and perpendicular to the plane containing the arc or the centroidal axis.	$I_C = mr^2$
Straight uniform rod 	Passing through center and perpendicular to the rod or the centroidal axis.	$I_C = \frac{mL^2}{12}$
Sector of a uniform disk of mass m 	Passing through center and perpendicular to the plane containing the sector.	$I_C = \frac{mr^2}{2}$
Uniform disk of mass m 	Passing through center and perpendicular to the plane containing the disk or the centroidal axis.	$I_C = \frac{mr^2}{2}$
Homogeneous cylinder of mass m 	Axis of the cylinder or the centroidal axis.	$I_C = \frac{mr^2}{2}$
Homogeneous sphere of mass m 	Diameter or the centroidal axis	$I_C = \frac{2}{5}mR^2$
Spherical shell of mass m 	Diameter or the centroidal axis	$I = \frac{2}{3}mR^2$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Theorems on Moment of Inertia

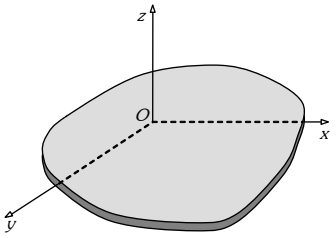
Moment of inertias of a rigid body about different axes may be different. There are two theorems known as *theorem of perpendicular axes* and *theorem of parallel axes*, which greatly simplify calculation of moment of inertia about an axis if moment of inertia of a body about another suitable axis is known.

Theorem of Perpendicular Axes

This theorem is applicable for a rigid body that lies entirely within a plane i.e. a laminar body or a rod bent into shape of a plane curve. The moment of inertia I_x, I_y and I_z of the body about the x, y and z-axis can be expressed by the following equations.

$$I_z = I_y + I_x$$

For a planar body, the moment of inertia about an axis perpendicular to the plane of the body is the sum of the moment of inertias about two perpendicular axes in the plane of the object provided that all the three axes are concurrent.

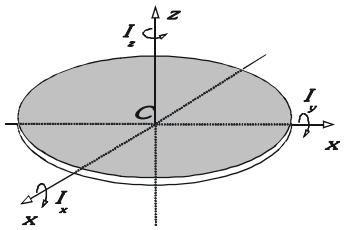


Example

Find moment of inertia of a uniform disk of mass m and radius r about one of its diameter.

Solution.

In the adjoining figure a disk is shown with two of its diameter perpendicular to each other. These diameters are along the x and the y-axis of a coordinate system. The x-axis is perpendicular to the plane of the disk and passes through its center is also shown.



Since the disk is symmetric about both the diameters, moment of inertias about both the diameters must be equal. Thus substituting this in the theorem of perpendicular axes, we have

$$I_z = I_y + I_x \rightarrow \qquad I_z = 2I_x = 2I_y$$

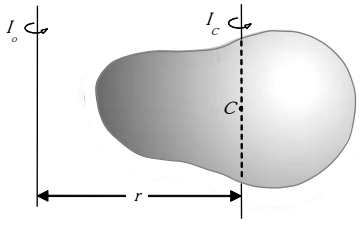
Moment of inertia of the disk about the z-axis is $I_z = \frac{1}{2}mr^2$. Substituting it in the above equation, we have

$$I_x = I_y = \tfrac{1}{2}I_z = \tfrac{1}{4}mr^2 \qquad \textbf{Ans.}$$

Theorem of Parallel Axes

This theorem also known as Steiner's theorem can be used to determine the moment of inertia of a rigid body about any axis, if the moment of inertia of the body about a parallel axis passing through mass center of the body and perpendicular distance between both the axes is known.

Consider a body of arbitrary shape and mass m shown in the figure. Its moment of inerta I_o and I_C are defined about two parallel axes. The axis about which moment of ienertia I_C is defined passes through the mass center C. Seperation between the axes is r. These two moment of inertias are related by the following equation.



$$I_O = I_C + Mx_C^2$$

The above equation is known as the theorem of parallel axes or Steiner's theorem.

⇨ *The moment of inertia about any axis parallel to an axis through the mass center is given by sum of moment of inertia about the axis through the mass center and product term of mass of the body and square of the distance between the axes.*

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

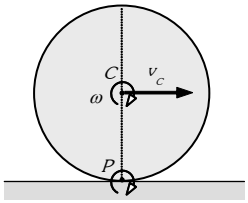
⇨ *Among all the parallel axes the moment of inertia of a rigid body about the axis through the mass center is the minimum moment of inertia.*

The second term added to the moment of inertia I_C about the centroidal axis in the above equation can be recognized as the moment of inertia of a particle of mass equal to that of the body and located at its mass center. It again reveals that *the plane motion of a rigid body is superposition of pure rotation about the mass center or centroidal rotation and translation of its mass center*.

Example

Find moment of inertia of a uniform ring, uniform disk, uniform cylinder and uniform sphere each of mass m and radius r about their instantaneous axis of rotation in rolling.

Solution.



In rolling the instantaneous axis of rotation passes through the point of contact P with the surface on which the body rolls. Each of these bodies has round section of radius r and can be represented in the adjoining figure. Denoting the moment of inertia about the instantaneous axis of rotation by I_P and through parallel centroidal axis by I_C, we have from the theorem of parallel axes.

Ring $\quad I_P = I_C + Mr^2 = Mr^2 + Mr^2 = 2Mr^2$

Disk $\quad I_P = I_C + Mr^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$

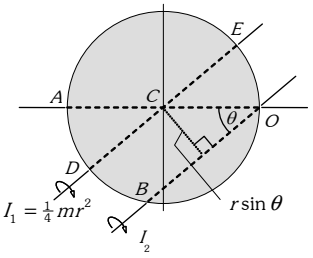
Cylinder $\quad I_P = I_C + Mr^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$

Sphere $\quad I_P = I_C + Mr^2 = \frac{2}{5}Mr^2 + Mr^2 = \frac{7}{5}Mr^2$

Example

Find expression for moment of inertia of a uniform disk of mass m, radius r about one of its secant making an angle θ with one of its diameter.

Solution.

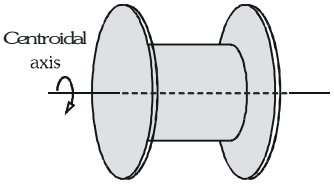


A disk, the secant OB and diameter OA are shown in the adjoining figure. The secant OB is parallel to another diameter DE. Moment of inertia of the disk about one of its diameter is $\frac{1}{4}mr^2$ and hence moment of inertia I_1 about the diameter DE. Distance between the secant OB and the parallel diameter DE is $r\sin\theta$.

Substituting above information in the theorem of parallel axes, we have

$$I_2 = I_1 + m(r\sin\theta)^2 \rightarrow \quad I_2 = mr^2\left(\tfrac{1}{4} + \sin^2\theta\right)$$

Example



Find moment of inertia about centroidal axis of a bobbin, which is constructed by joining coaxially two identical disks each of mass m and radius $2r$ to a cylinder of mass m and radius r as shown in the figure.

Solution.

The bobbin is a composite body made by joining two identical disks coaxially to cylinder. The moment of inertia I of the bobbin equals to the sum of moment of inertias of the two disks and moment of inertia of the cylinder about their centroidal axes. Using expressions for the moment of inertia for disk and cylinder, we have

$$I = 2I_{disk} + I_{cylinder} \rightarrow \qquad I = 2\left\{\tfrac{1}{2}m(2r)^2\right\} + \tfrac{1}{2}mr^2 = \tfrac{9}{2}mr^2$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example

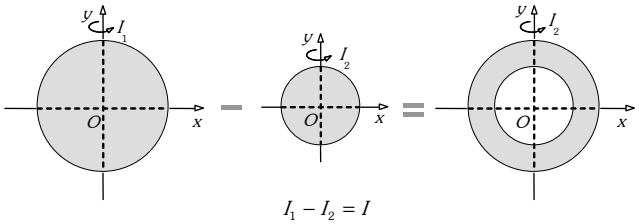
Find moment of inertia about one of diameter of a hollow sphere of mass *m*, inner radius *r* and outer radius *R*.

Solution.

The hollow sphere is assumed as if a concentric smaller sphere of radius *r* is removed from a larger sphere of radius *R*. Thus the moment inertia of the hollow sphere about any axes can be obtained by subtracting moment of inertia of the smaller sphere from that of the larger sphere. As shown in the following figure.

Let the mass of the hollow sphere is *m*.
Density of the material used is

$$\rho = \frac{3m}{4\pi\left(R^3 - r^3\right)}$$



$I_1 - I_2 = I$

Masses m_1 and m_2 of the smaller spheres are

$$m_1 = \rho\left(\tfrac{4}{3}\pi R^3\right) = \frac{mR^3}{R^3 - r^3} \text{ and } m_2 = \rho\left(\tfrac{4}{3}\pi r^3\right) = \frac{mr^3}{R^3 - r^3}$$

Subtracting I_2 from I_1 we have *I*.

$$I = I_1 - I_2 \rightarrow I = \frac{2m_1R^2}{5} - \frac{2m_2r^2}{5} = \frac{2m\left(R^5 - r^5\right)}{5\left(R^3 - r^3\right)}$$

Radius of Gyration

It is the radial distance from a rotation axis at which the mass of an object could be concentrated without altering the moment of inertia of the body about that axis.

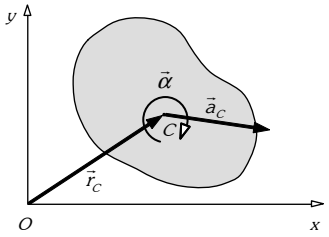
If the mass *m* of the body were actually concentrated at a distance *k* from the axis, the moment of inertia about that axis would be mk^2.

$$k = \sqrt{\frac{I}{m}}$$

The radius of gyration has dimensions of length and is measured in appropriate units of length such as meters.

Force and Torque equations in General Plane Motion

A rigid body is a system of particles in which separation between the particles remains unchanged under all circumstances. For a system of particles sum of all the external forces equals to product of mass of the whole system and acceleration of mass center. This fact we can express by the following equation and call it as force equation.



$$\Sigma\vec{F}_i = M\vec{a}_C$$

To make use of the above idea we conceive general plane motion as superposition of translation of the mass center and simultaneous centroidal rotation. In the figure is shown a body in general plane motion with acceleration of the mass center and angular acceleration about the centroidal axis. Therefore we can write torque of all the external forces about an axis parallel to the original one and passing through the origin of an inertial frame as sum of moment of effective force $\left(M\vec{a}_C\right)$ on mass center and effective torque $\left(I_C\vec{\alpha}\right)$ of centroidal rotation. Here I_C is the moment of inertia of the body about the centroidal axis parallel to the original one.

$$\Sigma\vec{\tau}_o = \vec{r}_C \times M\vec{a}_C + I_C\vec{\alpha}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

If we write torque of all the external forces about the centroidal axis the first term on the right hand side vanishes and we obtain torque equation describing the centroidal rotation.

$$\Sigma\vec{\tau}_C = I_C\vec{\alpha}$$

If instantaneous axis of rotation is known we can write the torque equation about it. For the purpose, we make use of parallel axis theorem of moment of inertia.

$$\Sigma\vec{\tau}_{IAR} = I_{IAR}\vec{\alpha}$$

Dynamics of Rigid Body as a system of particles

Motion of a rigid body either pure rotation or rotation about axis in translation can be thought and analyzed as superposition of translation of any of its particle and simultaneous rotation about an axis passing through that particle provided that the axis remain parallel to the original one. As far as kinematics in concerned this particle may or may not be the mass center. Whereas in dealing with kinetics, general plane motion is conceived as superposition of translation motion of the mass center and simultaneous centroidal rotation.

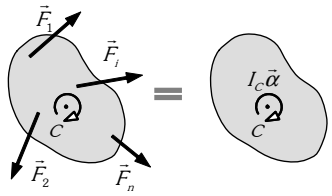
To make use the above idea and equations developed in the previous section we classify pure rotation i.e. rotation about fixed axis into two categories and deal with general plane motion as the third category.

Pure centroidal rotation: Rotation about fixed axis through mass centre

In this kind of rotation motion the axis of rotation passes through the mass center and remain fixed in space. Rotation of ceiling fan is a common example of this category. It is a subcategory of pure rotation. The axis of rotation passes through the mass center and remains fixed. In this kind of rotation the mass center of the body does not move.

In the figure, free body diagram and kinetic diagram of a body rotating about a fixed axis passing through its mass center C is shown. The mass center of the body does not accelerate; therefore we only need to write the torque equation.

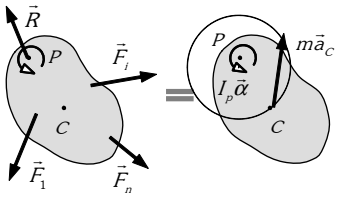
$$\Sigma\vec{\tau}_C = I_C\vec{\alpha}$$



Rotation about fixed axis not passing through mass center

In this kind of rotation the axis of rotation remains fixed and does not passes through the mass center. Rotation of door is a common example of this category. Doors are hinged about their edges; therefore their axis of rotation does not pass through the mass center. In this kind of rotation motion the mass center executes circular motion about the axis of rotation.

In the figure, free body diagram and kinetic diagram of a body rotating about a fixed axis through point P is shown. It is easy to conceive that as the body rotates its mass center moves on a circular path of radius $\vec{r}_{P/C}$. The mass center of the body is in translation motion with acceleration $\vec{a}_C$ on circular path of radius $r_{P/C}$. To deal with this kind of motion, we have to make use of both the force and the torque equations.



Translation of mass center $\quad \Sigma\vec{F}_i = M\vec{a}_C = M\vec{\alpha}\times\vec{r}_{C/P} - M\omega^2\vec{r}_{C/P}$

Centroidal Rotation $\quad \Sigma\vec{\tau}_C = I_C\vec{\alpha}$

Making use of parallel axis theorem $\quad \left(I_P = Mr_{P/C}^2 + I_C\right)$ and $\vec{a}_{C/P} = \vec{\alpha}\times\vec{r}_{C/P} - \omega^2\vec{r}_{C/P}$

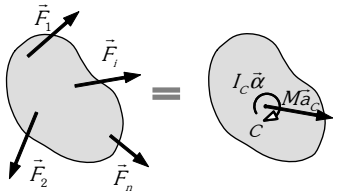
we can write the following equation also.

Pure Rotation about P $\quad \Sigma\vec{\tau}_P = I_P\vec{\alpha}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

General Plane Motion: Rotation about axis in translation motion

Rotation of bodies about an axis in translation motion can be dealt with either as superposition of translation of mass center and centroidal rotation or assuming pure rotation about the instantaneous axis of rotation. In the figure is shown the free body diagram and kinetic diagram of a body in general plane motion.



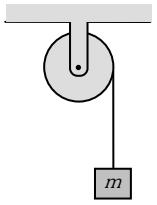
Translation of mass center $\sum_{i=0}^{n}\vec{F}_i = M\vec{a}_C$

Centroidal Rotation $\sum_{i-1}^{n}\vec{\tau}_C = I_C\vec{\alpha}$

This kind of situation can also be dealt with considering it rotation about IAR. It gives sometimes quick solutions, especially when IAR is known and forces if acting at the IAR are not required to be found.

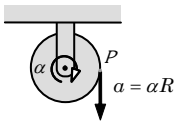
Example

A block of mass m is suspended with the help of a light cord wrapped over a cylindrical pulley of mass M and radius R as shown in the figure. The system is released from rest. Find the angular acceleration of the pulley and the acceleration of the block.



Solution.

After the system is released, the block is in translation motion and the pulley in rotation about an axis passing through its mass center i.e. in pure rotation.



Let the block moves vertically down with acceleration a pulling the cord down and causing the pulley to rotate clockwise. Since the cord is inextensible every point on its vertical portion and point of contact P of the pulley move down with acceleration a as shown in the adjacent figure. It is the tangential acceleration of point P so the angular acceleration α of the pulley rotating in clockwise sense is given by

$$a = \alpha R \qquad (1)$$

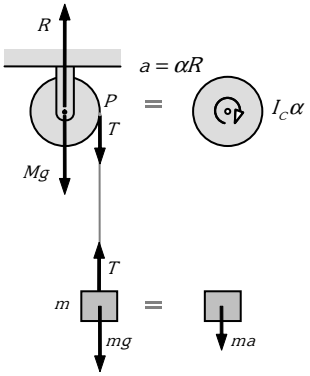
The forces acting on the pulley and on the block are shown in their free-body diagrams along with the effective torque $I_C\alpha$ of the pulley and effective force ma of the block. Here T is the tension in the string, R is the reaction by the axel of the pulley, Mg is weight of the pulley and mg is weight of the block.

The pulley is in rotation about fixed axis through its mass center so we use eq. .

$\sum\vec{\tau}_C = I_C\vec{\alpha}\rightarrow$ $\qquad TR = I_C\alpha$

After substituting $I_C = \frac{1}{2}MR^2$ and α from eq. (1), we have

$$T = \tfrac{1}{2}Ma \qquad (2)$$



The block is in translation motion, so we use Newton's second law

$\sum\vec{F} = m\vec{a}\rightarrow$ $\qquad mg - T = ma \qquad (3)$

From equation (2) and (3), we have

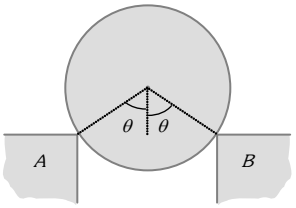
Acceleration of the block $\qquad a = \dfrac{2mg}{M+2m}$

From eq. (1) and the above, we have $\qquad \alpha = \dfrac{2mg}{R(M+2m)}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example

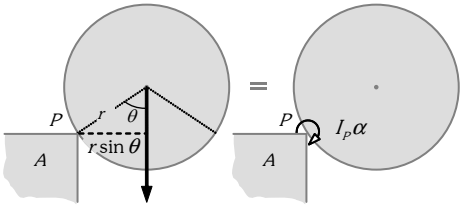
A cylinder of radius r and mass m rests on two horizontal parallel corners of two platforms. Both the platforms are of the same height. Platform B is suddenly removed. Assume friction between the corner of the platform A and cylinder to be sufficient enough to prevent sliding. Determine angular acceleration of the cylinder immediately after the removal of the platform B.



Solution.

Since the cylinder does not slide at the point of contact with the corner of platform A, it rotates about fixed axis through the point of contact in subsequent motion. Torque equation should be used.

Forces acting on the cylinder and the effective torque are shown in the adjacent figure. Since forces acting at the point of contact does not contribute any torque about it, the normal reaction form the corner and the friction force are not shown in the free body diagram.



Applying the torque equation about the fixed axis through P, we have

$$\sum \vec{\tau}_P = I_P\vec{\alpha} \rightarrow \qquad mgr\sin\theta = I_P\alpha \qquad (1)$$

Applying the theorem of parallel axes and expression for moment of inertia about centroidal axes, we obtain moment of inertia I_P about an axis through the point P.

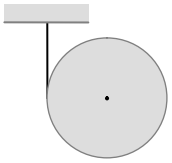
$$I_P = I_C + mr^2 \rightarrow \qquad I_P = \tfrac{1}{2}mr^2 + mr^2 = \tfrac{3}{2}mr^2 \qquad (2)$$

Substituting I_P from eq. (2) in (1), we have

Angular acceleration of the cylinder $\alpha = \dfrac{2g\sin\theta}{3r}$

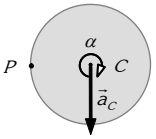
Example

A thread is wrapped around a uniform disk of radius r and mass m. One end of the thread is attached to a fixed support on the ceiling and the disk is held stationary in vertical plane below the fixed support as shown in the figure. When the disk is set free, it rolls down due to gravity. Find the acceleration of the center of the disk and tension in the thread.



Solution.

The point P, where the thread leaves the disk is always at instantaneous rest; therefore the disk can be assumed rolling without slipping with ICR at point P. Acceleration of the mass center $\vec{a}_C$ and angular acceleration of the disk are shown in the adjacent figure. Applying condition for rolling on stationary surface, we have



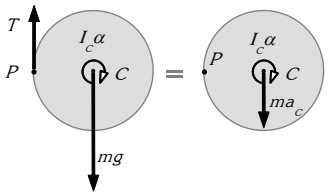
$$\vec{a}_C = \vec{\alpha}\times\vec{r}_{C/P} \rightarrow \qquad a_C = \alpha r \qquad (1)$$

The disk rolls down on the vertical stationary thread. Its motion can either be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation or a pure rotation about ICR. Since tension, which acts at the ICR is asked; we prefer superposition of translation of the mass center and simultaneous centroidal rotation.

Forces acting on the disk are tension T applied by the thread at point P and weight of the disk. These forces and the effective force ma_C and effective torque $I_C\alpha$ are shown in the adjacent figure.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Applying Newton's second law for translation of mass center, we have



$$\sum \vec{F}_i = M\vec{a}_C \rightarrow \qquad mg - T = ma_C \qquad (2)$$

Applying torque equation for centroidal rotation, we have

$$\sum \vec{\tau}_C = I_C\vec{\alpha} \rightarrow \qquad Tr = I_C\alpha$$

Substituting $\frac{1}{2}mr^2$ for I_C and α form eq. (1), we have

$$T = \tfrac{1}{2}ma_C \qquad (3)$$

From eq. (2) and (3), we have

Acceleration of the mass center $a_C = \frac{2}{3}g$

Tension in the string $T = \frac{1}{3}mg$

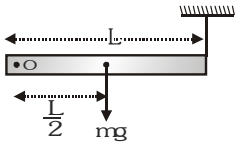
Example

A rod is pivoted at its one edge about point O. Other edge of rod is suspended from the ceiling through rope as shown. If the rope is suddenly cut then find the angular acceleration of rod.



Solution

When the rope is cut, weight of rod due to force of gravity will produce torque about point O. $\tau = I\alpha$, consider force mg acting on shown in figure at CM of rod (i.e. middle point of the rod)



$$mg\left[\frac{L}{2}\right] = \frac{mL}{3}\alpha \quad \Rightarrow \quad \frac{g}{2} = \frac{L\alpha}{3} \quad \Rightarrow \alpha = \frac{3}{2}\frac{g}{L}$$

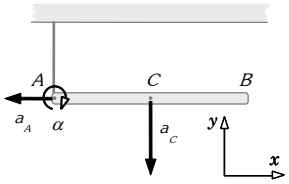
Example

A uniform rod AB of mass m and length ℓ is suspended in horizontal position with the help of two strings as shown in the figure. The string supporting the end B is cut. Find acceleration of the mass center and end A immediately after the string is cut.



Solution.

After the string is cut forces acting on the rod are tension in the string A and weight of the rod. Both of these forces are in vertical direction so acceleration of the mass center C must be vertically downwards. The string is inextensible so the point A can have acceleration only in horizontal direction. Let acceleration of the mass center C is denoted by a_C downwards and acceleration a_A of the point A towards the left as shown in the adjacent figure.



Applying the relative motion equation, we have

$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} \rightarrow \qquad \vec{a}_C = \vec{a}_A + \vec{\alpha}\times\overrightarrow{AC} - \omega^2\overrightarrow{AC}$$

Immediately after the string is cut the rod cannot acquire any angular velocity. So the last term in the above equation vanishes. Now we have

$$-a_C\tilde{j} = -a_A\tilde{i} + \left(-\alpha\tilde{k}\right)\times\left(\tfrac{1}{2}\ell\tilde{i}\right)$$

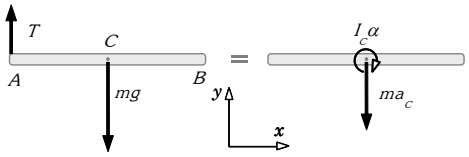
Comparing x and y-components on both the sides we have

$$a_C = \tfrac{1}{2}\alpha\ell \qquad (1)$$

$$a_A = 0$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

The rod is in plane motion, which can be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation. The forces acting on the rod immediately after the string at B is cut, the effective force ma_C and the effective torque $I_C\alpha$ are shown in the adjacent figure.



Applying Newton's second law for translation of mass center, we have

$\sum \vec{F}_i = M\vec{a}_C \rightarrow \qquad mg - T = ma_C \qquad (2)$

Applying torque equation for centroidal rotation, we have

$\sum \vec{\tau}_C = I_C\vec{\alpha} \rightarrow \qquad T\left(\frac{1}{2}\ell\right) = I_C\alpha$

Substituting $\frac{1}{12}m\ell^2$ for I_C and α form eq. (1), we have

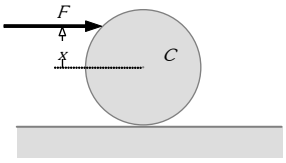
$T = \frac{1}{3}ma_C \qquad (3)$

From eq. (2) and (3), we have

Acceleration of the mass center $\quad a_C = \frac{3}{4}g$

Example

A uniform rigid body of mass m and round section of radius r rests on rough horizontal surface. The radius of gyration of the body about its centroidal axis is k. It is pushed by a constant horizontal force F. The height x of the point where the force is applied can be adjusted.



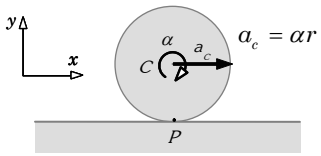
(a) Deduce suitable expression for magnitude and direction of the friction force necessary to ensure rolling.

(b) How direction of the friction force depends on x.

Solution

(a) The problem requires solution of the force and the torque equations consistent with the condition of rolling, so it is not necessary to decide the direction of friction as priory. To start with let the static friction f_s acts in the forward direction.

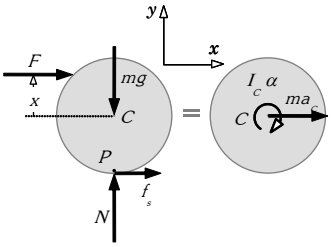
Let the mass center of the body moves towards right with acceleration a_C and angular acceleration α of the body is in clockwise sense as shown in the adjacent figure.



Necessary condition for rolling in terms of acceleration a_C of mass center and angular acceleration α of the body is

$\vec{a}_C = \vec{\alpha} \times \vec{r}_{C/P} \rightarrow \qquad \vec{a}_C = \left(-\alpha\tilde{k}\right) \times r\tilde{j} \Rightarrow a_C = \alpha r \qquad (1)$

The forces acting on the body are its weight mg, the normal reaction N from the ground and the force of static friction f_s. These forces are shown in the adjacent figure together with the effective force ma_C and the effective torque $I_C\alpha$ about the mass center. We analyze the problem as superposition of translation of the mass center and simultaneous centroidal rotation.



Applying Newton's second law for translation of mass center, we have

$\sum F_x = Ma_{Cx} \rightarrow \qquad F + f_s = ma_C \qquad (2)$

Applying torque equation for centroidal rotation, we have

$\sum \vec{\tau}_C = I_C\vec{\alpha} \rightarrow \qquad Fx - f_s r = I_C\alpha$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Substituting mk^2 for I_C, and value of α form eq. (1), we have

$$F\frac{x}{r} - f_s = \frac{mk^2 a_C}{r^2} \qquad (3)$$

From eq. (2) and (3), we have

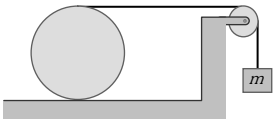
Force of static friction $\quad f_s = F\left(\frac{x - k^2/r}{r + k^2/r}\right)$

(b) The above expression shows that to ensure rolling

For $x > k^2/r$, the friction is in direction of the applied force F.
For $x = k^2/r$, no friction is required to ensure rolling.
For $x < k^2/r$, the friction must be in opposite to the applied force F.

Example

A block of mass m is attached at one end of a thin light cord, which passes over an ideal pulley. At the other end, it is wrapped around a cylinder of mass M, which can roll without slipping over a horizontal plane.



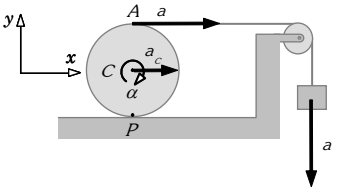
(a) What is the acceleration of the block?

(b) What is the friction force on the cylinder?

Solution.

The problem requires solution of the force and the torque equations consistent with the condition of rolling, so it is not necessary to decide the direction of friction as priory. To start with let the static friction f_s acts in the forward direction.

(a) Let the block descend with acceleration a. Since the cord is inextensible the top point A of the cylinder also moves with the same acceleration. Applying relative motion equation with the condition required for rolling that the particle of the cylinder at the point of contact has no acceleration parallel to the horizontal plane.

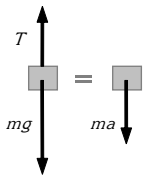


$$\vec{a}_A = \vec{\alpha} \times \vec{r}_{A/P} \rightarrow \qquad a_A\tilde{i} = \left(-\alpha\tilde{k}\right) \times \left(2r\tilde{j}\right) \Rightarrow \alpha = \frac{a}{2r} \qquad (1)$$

From eq. (1) and relative motion equation for P and the center C, we have

$$\vec{a}_C = \vec{\alpha} \times \vec{r}_{C/P} \rightarrow \qquad a_C\tilde{i} = \left(-\alpha\tilde{k}\right) \times \left(r\tilde{j}\right) \Rightarrow a_C = \tfrac{1}{2}a \qquad (2)$$

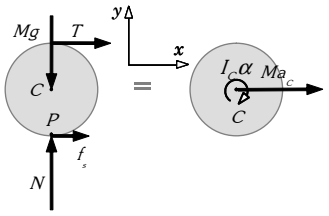
The block is in translation motion under the action of its weight mg and tension T in the string. These forces and the effective force ma are shown in the adjacent figure.



Applying Newton's second law for translation of mass center, we have

$$\sum \vec{F}_i = M\vec{a}_C \rightarrow \qquad mg - T = ma \qquad (3)$$

The cylinder is in rolling under the action of its weight Mg, normal reaction N form the ground; tension T in the cord and force of static friction f_s. These forces, the effective force Ma_C and the effective torque $I_C\alpha$ are shown in the adjacent figure.



Applying Newton's second law for translation of mass center, we have

$$\sum F_x = Ma_{Cx} \rightarrow \qquad T + f_s = Ma_C$$

Substituting a_C from eq. (2), we have

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

$$T + f_s = \tfrac{1}{2}Ma \qquad (4)$$

Applying torque equation for centroidal rotation, we have

$$\sum \vec{\tau}_C = I_C\vec{\alpha} \rightarrow \qquad Tr - f_s r = I_C\alpha$$

Substituting $\frac{1}{2}mr^2$ for I_C, and value of α form eq. (1), we have

$$T - f_s = \tfrac{1}{4}Ma \qquad (5)$$

From eq. (3), (4) and (5), we have

Acceleration of the block $a = \dfrac{8mg}{3M + 8m}$

(b) From eq. (4), (5) and above value of acceleration *a,* we have

Force of static friction $f_s = \dfrac{Mmg}{3M + 8m}$

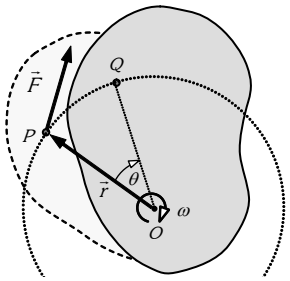
Energy Methods

Newton's laws of motion tell us what is happening at an instant, while method of work and energy equips us to analyze what happens when a body moves from one place to other or a system changes its configuration. In this section, we introduce how to use methods of work and energy to analyze motion of rigid bodies.

Concept of Work in rotation motion

Work of a force is defined as the scalar product of the force vector and displacement vector of the point of application of the force. If during the action of a force $\vec{F}$ its point of application moves from position $\vec{r}_1$ to $\vec{r}_2$, the work $W_{1\rightarrow 2}$ done by the force is expressed by the following equation.

$$W_{1\rightarrow 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$$



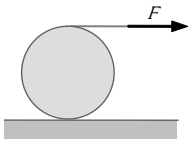
Either we can use of this idea to calculate work of a force or its modified version in terms of torque and angular displacement.

The work done by a torque during a finite rotation of the rigid body from initial value θ_i of the angle θ to final value θ_f, can be obtained by integrating both the sides of the equation given

$$W_{i\rightarrow f} = \int_{\theta_i}^{\theta_f} \vec{\tau}_o \cdot d\vec{\theta}$$

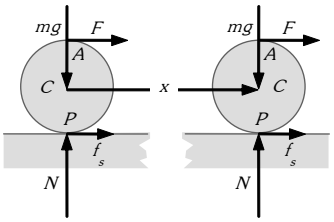
Example

A thin light cord is wound around a uniform cylinder placed on a rough horizontal ground. When free end of the cord is pulled by a constant force F the cylinder rolls. Denote radius of the cylinder by r and obtain expression for work done by each of the forces acting on the cylinder when center of the cylinder shifts by distance x.



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution.



Forces acting on the cylinder are its weight W, the normal reaction from the ground N, the tension T in the cord and the force of static friction f_s. The tension in the cord equals to the applied force F. These forces are shown in the adjacent figure.

In rolling point of contact P is at instantaneous rest, the center C moves with velocity $v_C = \omega r$ and the top point moves with velocity $v_A = 2v_C = 2\omega r$ both parallel to the surface on which body rolls. Since the cord is inextensible displacement of the top point equals to the displacement of the free end of the cord. These fact suggests that during displacement x of the center the free end of the cord shift through a distance $2x$.

Work done by the weight of the cylinder.

$W_g = 0$ The weigh is assumed to act on the center of gravity which coincides with the mass center in uniform gravitation field near the ground. The displacement x of the mass center and weight both are perpendicular to each other so the work done by gravity is zero.

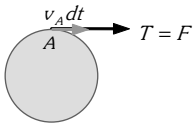
Work done by the normal reaction on the cylinder

$W_N = 0$ The normal reaction acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore normal reaction does no work.

Work done by the force of static friction.

$W_{fs} = 0$ The force of static friction f_s acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore force of static friction f_s does no work.

Work done by the tension in the cord.



$W_T = W_F$ The particle of the wheel on which the tension in the cord acts is at the top point. Though this particle is also continuously changing but it is not in instantaneous rest and has velocity v_A. So in every infinitesimally small time interval displacement of this particle is $v_A dt = 2v_C dt$, thus work done dW_T by the tension during a time interval dt

$$dW_T = T(2v_C dt) = 2F(v_C dt)$$

When the center shifts by a distance x the work done by the tension becomes

$$W_T = W_F = 2Fx$$

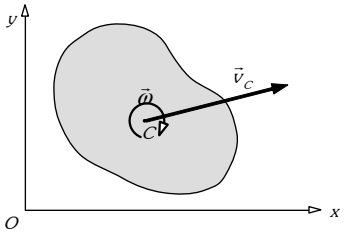
Potential Energy of a rigid body

Since potential energy of a system is function of its configuration and does not depend on the manner in which the system is brought into a particular configuration, hence it does not depends on motion involved whether it is translation, rotational or their combination.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Kinetic Energy of a rigid body in rotation motion

A rigid body can be represented as a system of large number of particles, which keep their mutual distances unchanged in all circumstances. Kinetic energy of the whole body must be sum of kinetic energies of all of its particles. In this section we develop expressions for kinetic energy of a rigid body.



Kinetic Energy of a rigid body in plane motion

In the figure is shown a body in plane motion. Its mass center at an instant is moving with velocity $\vec{v}_C$ and rotating with angular velocity $\vec{\omega}$. Both these motions are shown superimposed in the given figure.

Kinetic energy too can be written as sum of kinetic energy $\left(\frac{1}{2}Mv_C^2\right)$ due to translation motion of the mass center and kinetic energy $\left(\frac{1}{2}I_C\omega^2\right)$ due to centroidal rotation.

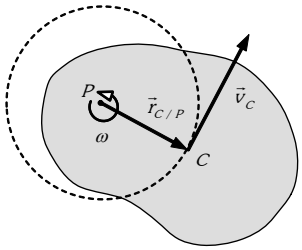
$$K = \tfrac{1}{2}Mv_C^2 + \tfrac{1}{2}I_C\omega^2$$

If location of the instantaneous axis of rotation (IAR) is known, making use of the parallel axis theorem we can write kinetic energy by the following equation also.

$$K = \tfrac{1}{2}I_{IAR}\omega^2$$

Kinetic Energy of a rigid body in rotation about fixed axis not passing through the mass centre

In this kind of motion the mass center is in circular motion about the axis of rotation. In the figure is shown a body rotation with angular velocity ω about a fixed axis through pint P and perpendicular to plane of the paper. Mass center moves with speed $v_C = \omega r$. Kinetic energy of the body can now be expressed by the following equation.



$$K = \tfrac{1}{2}Mv_C^2 + \tfrac{1}{2}I_C\omega^2$$

Making use of the parallel axis theorem $\left(I_P = Mr_{P/C}^2 + I_C\right)$ we can write kinetic energy by the following equation also.

$$K = \tfrac{1}{2}I_P\omega^2$$

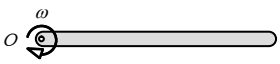
Kinetic Energy of a rigid body in pure centroidal rotation

In pure centroidal rotation the mass center remain at rest; therefore kinetic energy due to translation of mass center vanishes.

$$K = \tfrac{1}{2}I_C\omega^2$$

Example

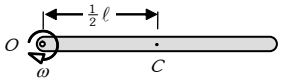
A rod of mass m and length ℓ is pivoted to a fixed support at one of its ends O. It is rotating with constant angular velocity ω. Write expression for its kinetic energy.



Solution.

If the point C is the mass center of the rod, from theorem of parallel axes, the moment of inertia I_O of the rod about the fixed axis is

$$I_O = I_C + m(OC)^2 \rightarrow \quad I_O = I_C + \tfrac{1}{4}m\ell^2$$



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Substituting $\frac{1}{12}m\ell^2$ for I_C, we have

$I_O = \frac{1}{3}m\ell^2$

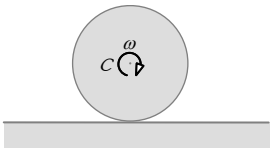
Kinetic energy of the rod equals to kinetic energy due to rotation about the fixed axis.

$K = \frac{1}{2}I_o\omega^2 \rightarrow$ Using above expression for I_O, we have

$K = \frac{1}{6}m\ell^2\omega^2$

Example

A uniform rigid body of mass m and round section of radius r is rolling on horizontal ground with angular velocity ω. Its radius of gyration about the centroidal axis is k.



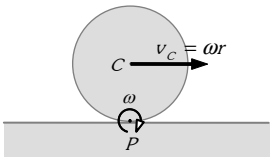
(a) Write expression of its kinetic energy.

(b) Also express the kinetic energy as sum of kinetic energy due to translation of mass center and kinetic energy due to simultaneous centroidal rotation.

Solution.

(a) The point of contact with ground of a body rolling on the ground is its ICR. Let the point P is the ICR as shown in the adjacent figure. The geometrical center C of a uniform body and the mass center coincide. Therefore moment of inertia I_P of the body about the ICR can be written by using the theorem of parallel axes.

$I_P = I_C + m(PC)^2 \rightarrow$ $I_P = I_C + mr^2$



Substituting $I_C = mk^2$, we have

$I_P = m\left(k^2 + r^2\right)$ (1)

Kinetic energy of a rigid body equals to kinetic energy due to rotation about the ICR.

$K = \frac{1}{2}I_P\omega^2 \rightarrow$ Substituting I_P from eq. (1), we have

$K = \frac{1}{2}m\left(k^2 + r^2\right)\omega^2$

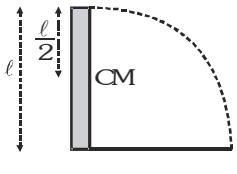
(b) Kinetic energy of the body also equals to sum of kinetic energy due to translation of its mass center and kinetic energy due to simultaneous centroidal rotation.

$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2 \rightarrow$ Substituting condition for rolling $v_C = \omega r$ and $I_C = mk^2$, we have

$$K = \frac{1}{2}m\left(\omega r\right)^2 + \frac{1}{2}mk^2\omega^2 = \frac{1}{2}m\left(r^2 + k^2\right)\omega^2$$

Example

A thin meter scale is kept vertical by placing its one end on floor, keeping the end in contact stationary, it is allowed to fall. Calculate the velocity of its upper end when it hit the floor .



Solution

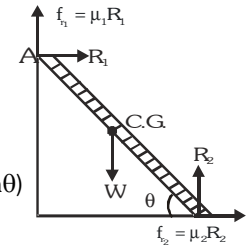
Loss in PE = gain in rotational KE $\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{m\ell^2}{3} \times \frac{v^2}{\ell^2} \Rightarrow v = \sqrt{3g\ell}$

Example

A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by $\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right)$ where μ_1 is the coefficient of friction between rod and wall, μ_2 is the coefficient of friction between rod and ground.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution



For equilibrium of rod $\Sigma F_x = 0 \Rightarrow R_1 = \mu_2 R_2$

$\Sigma F_y = 0 \Rightarrow \mu_1 R_1 + R_2 = W \Rightarrow (\mu_1 \mu_2 + 1) R_2 = W$

Taking torque about point A : $W\left(\frac{\ell}{2}\cos\theta\right) + \mu_2 R_2 (\ell \sin\theta) = R_2 (\ell \cos\theta) \Rightarrow W = 2R_2 (1 - \mu_2 \tan\theta)$

$$\therefore \mu_1 \mu_2 + 1 = 2 - 2\mu_2 \tan\theta \Rightarrow \tan\theta = \left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$$

Power

Power defined as the time rate of work done, takes into account the duration in which work is done. To calculate power we make use of the following equation.

$$P = \frac{dW}{dt}$$

Instantaneous power of a force can be expressed by the following equation. Here velocity $\vec{v}$ is the velocity of the point of application of the force $\vec{F}$.

$$P = \vec{F} \cdot \vec{v}$$

Work and Energy Theorem

Work energy theorem can be applied in similar fashion as it was applied to analyze translation motion of a single body or a system of several bodies.

The work energy theorem relates kinetic energy K_1 and K_2 of a body in its initial and final position with work $W_{1\to2}$ done by all the external forces acting on the body to carry it form the initial position to the final position according to the following equation.

$$W_{1\to2} = K_2 - K_1$$

This equation is applicable in all inertial as well as noninertial frames. To write equation of work energy theorem kinetic energy must be written relative to the frame under consideration. To calculate work consider only all the physical forces in inertial frame and all the physical forces as well as pseudo force in non-inertial frame and displacement of point of applications of these forces relative to the frame under consideration.

For a system of several bodies the corresponding equation of work energy theorem can be obtained by applying the theorem for each individual body and then adding all of them. In this way we obtain an equation of the following form.

$$\Sigma W_{1\to2} = K_{2,s} - K_{1,s}$$

Here the term $\Sigma W_{1\to2}$ equals to the work of all the forces acting on various bodies irrespective of whether the force are internal or external from point of view of the system under consideration. In systems of several bodies interconnected by links of constant length e.g. inextensible cords, rods etc or body in direct contact the total work of internal forces vanishes. The work done by internal conservative forces can be accounted for by decrease in corresponding potential energies. The terms $K_{2,s}$ and $K_{1,s}$ are total kinetic energies of all the bodies in initial and final configurations of the system.

Conservation of Mechanical Energy

The work of conservative forces equals to decrease in potential energy. When a single rigid body moves or a system of rigid body changes its configuration under the action of conservative forces and nonconservative forces are either not present or if present do no work, the work energy principle can be expressed as

$$U_{1,s} + K_{1,s} = U_{2,s} + K_{2,s}$$

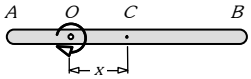
The above equation expresses the law of conservation of mechanical energy and states that *if a rigid body*

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

moves or a system consisting of several rigid bodies changes its configuration under action of conservative forces the mechanical energy i.e. sum of kinetic and potential energy remain constant; provided that nonconservative, if present, do no work,

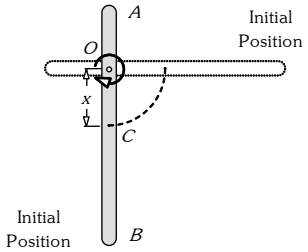
Though the work energy principle and the law of conservation of mechanical energy are equivalent, we prefer to use the former to account for nonconservative forces easily

Example



A uniform rod AB of mass m and length ℓ is pivoted at a point (O) to rotate in the vertical plane. The rod is held in horizontal position and released. Find the distance x of the pivot from the mass center (C) of the rod, so that angular speed ω of the rod as it passes through the vertical position is maximum.

Solution.



The problem involves change in angular velocity with change in position, therefore demands application of principle of work and energy.

The rod when released rotates about a fixed horizontal axis passing through the point O. Its initial and final positions are shown in the adjacent figure.

Moment of inertia of the rod about the pivot O can be calculated by theorem of parallel axes.

$I_o = I_C + mx^2 \rightarrow$ Substituting $\frac{1}{12}m\ell^2$ for I_C, we have

$$I_o = \tfrac{1}{12}m\left(\ell^2 + 12x^2\right) \qquad (1)$$

Kinetic energy in the initial position.

$K = \frac{1}{2}I_o\omega^2 \rightarrow \quad K_1 = 0 \qquad (2)$

Kinetic energy in the final position.

$K = \frac{1}{2}I_o\omega^2 \rightarrow$ Substituting for I_o form eq. (1), we have

$$K_2 = \tfrac{1}{24}m\left(\ell^2 + 12x^2\right)\omega^2 \qquad (3)$$

Only gravity does work when the rod moves from the initial to final position.

$W = \int \vec{F}\cdot d\vec{r} \rightarrow \quad W_{1\rightarrow 2} = mgx \qquad (4)$

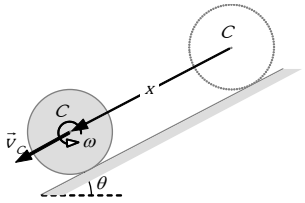
Substituting values form eq. (2), (3) and (4) in equation of work energy principle, we have

$W_{1\rightarrow 2} = K_2 - K_1 \rightarrow \quad \omega = \sqrt{24gx/\left(\ell^2 + 12x^2\right)} \qquad (5)$

The above equation expresses angular velocity of the rod when it passes the vertical position. For it to be maximum

$\frac{d\omega}{dx} = 0 \rightarrow \quad x = \frac{\ell}{\sqrt{12}}$

Example



A uniform rigid body of mass m and round section of radius r rolls down a slope inclined at an angle θ to the horizontal. The radius of gyration of the body about it central axis of symmetry is k.

(a) Derive suitable expressions for angular velocity and velocity of its mass center after it covers a distance x.

(b) Obtain expression for its angular acceleration and acceleration of the mass center.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution.

The problem involves change in angular velocity with change in position, therefore demands application of principle of work and energy

The geometrical center and mass center for uniform bodies coincide; therefore center C is the mass center.

In rolling the point of contact P must always be at instantaneous rest and angular velocity ω, velocity of center C, angular acceleration α and acceleration of the center must bear the following relations.

$$v_C = \omega r \quad \text{and} \quad a_C = \alpha r \qquad (1)$$

(a) The rolling motion can be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation.

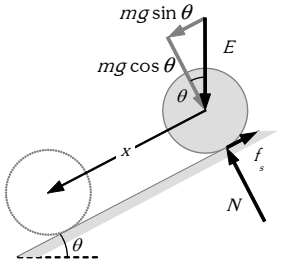
Kinetic energy in the initial position.

$$K_1 = 0 \qquad (2)$$

Kinetic energy in the final position.

$K = \frac{1}{2} m v_C^2 + \frac{1}{2} I_C \omega^2 \rightarrow$ Substituting for $I_C = mk^2$ and v_C form eq. (1), we have

$$K_2 = \tfrac{1}{2} m\left(r^2 + k^2\right)\omega^2 \qquad (3)$$



The forces acting on the body are its weight mg, the normal reaction N from the slope and the force of static friction f_s. These forces and displacement are shown in the adjacent figure. The normal reaction and the force of static friction do no work in rolling, it is the weight, which does work.

$$W = \int \vec{F} \cdot d\vec{r} \rightarrow \qquad W_{1\rightarrow 2} = mgx\sin\theta \qquad (4)$$

Substituting values form eq. (2), (3) and (4) in equation of work energy principle, we have

$$W_{1\rightarrow 2} = K_2 - K_1 \rightarrow \qquad \omega = \sqrt{\frac{2gx\sin\theta}{k^2 + r^2}}$$

Substituting v_C from eq. (1), we have

$$v_C = \sqrt{\frac{2gx\sin\theta}{\left(k^2/r^2\right)+1}}$$

Acceleration a, velocity v and position coordinate x bear the relation $a = v\left(dv/dx\right)$. Therefore acceleration of mass center of the body.

$$a_C = v_C \frac{dv_C}{dx} \rightarrow \qquad a_C = \frac{g\sin\theta}{\left(k^2/r^2\right)+1}$$

Substituting a_C in eq. (1), we have the angular acceleration of the body.

$$a_C = \alpha r \rightarrow \qquad \alpha = \frac{g\sin\theta}{\left(k^2/r\right)+r}$$

Methods of Impulse and Momentum

Methods of impulse and momentum describe what happens over a time interval. When motion of a body involves rotation we have to consider angular impulse as well as angular momentum. In this section we discuss concept of angular impulse, angular momentum of rigid body, angular impulse momentum principle and conservation of angular momentum.

Angular Impulse

Like impulse of a forcem angular impulse of a constant torque equals to product of the torque and concerned time interval and if the torque is not constant it must be integrated with time over the concerned time interval.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

If torque $\vec{\tau}_o$ about an axis passing through O is constant, its angular impulse during a time interval from t_1 to t_2 denoted by $\vec{J}_{o,1\to2}$ is given by the following equation.

$$\vec{J}_{o,1\to2} = \vec{\tau}_o\left(t_2 - t_1\right)$$

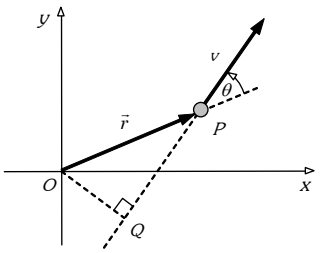
If torque $\vec{\tau}_o$ about an axis passing through O is time varying, its angular impulse during a time interval from t_1 to t_2 denoted by $\vec{J}_{o,1\to2}$ is given by the following equation.

$$\vec{J}_{o,1\to2} = \int_{t_1}^{t_2} \vec{\tau}_o\, dt$$

Angular momentum of a particle

Angular momentum $\vec{L}_o$ about the origin O of a particle of mass m moving with velocity $\vec{v}$ is defined as the moment of its linear momentum $\vec{p} = m\vec{v}$ about the point O.

$$\vec{L}_o = \vec{r} \times \left(m\vec{v}\right)$$



Angular Momentum of a Rigid Body

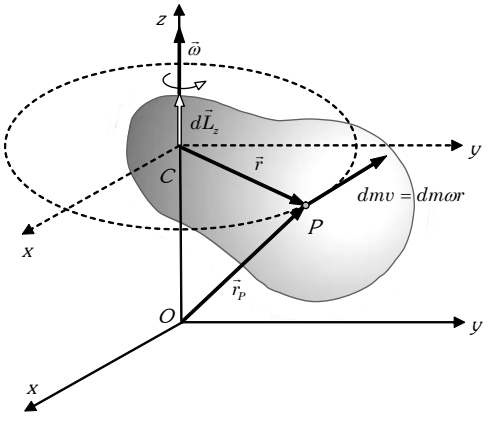
Angular momentum is quantity of rotation motion in a body. The angular momentum of a system of particles is the sum of angular momentum all the particles within the system. A rigid body is an assemblage of large number of particles maintaining their mutual distances intact under all circumstances, therefore angular momentum of a rigid body must be sum of angular momenta of all of its particles.

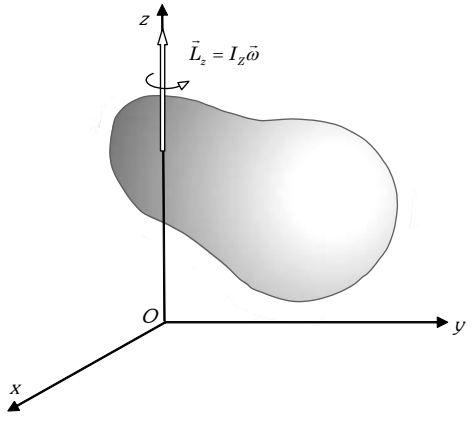
Angular Momentum about a point and about an axis

Angular momentum of a particle is not defined about an axis instead it is defined about a point. Therefore above idea of summing up angular momenta of all the particles about a point gives angular momentum of the rigid body about a point. But while dealing with fixed axis rotation or rotation about axis in translation we need angular momentum about an axis.

Angular momentum about an axis is calculated similar to torque abut an axis. To calculate angular momentum of a particle of rigid body about an axis we take moment of momentum of the particle about the point where plane of motion of the point of application of the force intersects the axis.

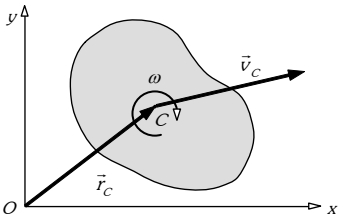
In the following figure is shown angular momentum $d\vec{L}_z = \vec{r} \times (dm\vec{v}) = r^2\, dm\omega$ of a particle P of a rigid body rotating about the z-axis. It is along the z-axis i.e. axis of rotation. In the next figure total angular momentum $\vec{L}_z = \int d\vec{L}_z = I_z\vec{\omega}$ about the axis of rotation is shown. It is also along the axis of rotation.





NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Angular Momentum in general plane motion



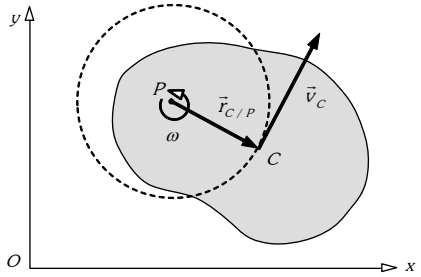
Angular momentum of a body in plane motion can also be written similar to torque equation or kinetic energy as sum of angular momentum about the axis due to translation of mass center and angular momentum of centroidal rotation about centroidal axis parallel to the original axis.

Consider a rigid body of mass M in plane motion. At the instant shown its mass center has velocity $\vec{v}$ and it is rotating with angular velocity $\vec{\omega}$ about an axis perpendicular to the plane of the figure. It angular momentum $\vec{L}_o$ about an axis passing though the origin and parallel to the original is expressed by the following equation.

$$\vec{L}_o = \vec{r}_C \times \left(M\vec{v}_C\right) + I_C\vec{\omega}$$

The first term of the above equation represent angular momentum due to translation of the mass center and the second term represents angular momentum in centroidal rotation.

Angular momentum in rotation about fixed axis

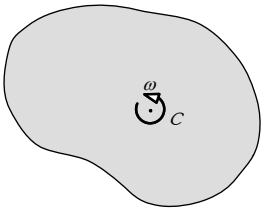


Consider a body of mass M rotating with angular velocity ω about a fixed axis perpendicular to plane of the figure passing through point P. Making use of the parallel axis theorem $I_P = Mr_{C/P}^2 + I_C$ and equation $\vec{v}_C = \vec{\omega} \times \vec{r}_{C/P}$ we can express the angular momentum $\vec{L}_P$ of the body about the fixed rotational axis.

$$\vec{L}_P = I_P\vec{\omega}$$

The above equation reveals that the angular momentum of a rigid body in plane motion can also be expressed in a single term due to rotation about the instantaneous axis of rotation.

Angular momentum in pure centroidal rotation



In pure centroidal rotation, mass center remains at rest, therefore angular momentum due to translation of the mass center vanishes.

$$\vec{L}_C = I_C\vec{\omega}$$

Rotational Equivalent of the Newton's Laws of Motion

Differentiating terms on both the sides of equation $\vec{L}_o = \vec{r}_C \times \left(M\vec{v}_C\right) + I_C\vec{\omega}$ with respect to time, and making substitution of $\vec{v}_C = d\vec{r}_C/dt$, $\vec{a}_C = d\vec{v}_C/dt$ and $\vec{\alpha} = d\vec{\omega}/dt$ we have

$$\frac{d\vec{L}_o}{dt} = \vec{v}_C \times \left(M\vec{v}_C\right) + \vec{r}_C \times M\vec{a}_C + I_C\vec{\alpha}$$

The first term on the right hand side vanishes, so we can write

$$\frac{d\vec{L}_o}{dt} = \vec{r}_C \times M\vec{a}_C + I_C\vec{\alpha}$$

Now comparing the above equation with torque equation $\Sigma\vec{\tau}_o = \vec{r}_C \times M\vec{a}_C + I_C\vec{\alpha}$, we have

$$\sum\vec{\tau}_o = \frac{d\vec{L}_o}{dt}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

The above equation though developed for plane motion only yet is valid for rotation about an axis in rotation also. It states that the net torque about the origin of an inertial frame equals to the time rate of change in angular momentum about the origin and can be treated as a parallel to Newton's second law which states that net external force on a body equals to time rate of change in its linear momentum.

Angular Impulse Momentum Principle

Rearranging the terms and integrating both the sides obtained form previous equation, we can write

$$\Sigma\int_{t_1}^{t_2}\vec{\tau}_o\,dt = \vec{L}_{o2} - \vec{L}_{o1}$$

The left hand side of the above equation is the angular impulse of torque of all the external forces in the time interval in the time interval t_1 to t_2.

$$\Sigma\vec{J}_{o,1\to2} = \vec{L}_{o2} - \vec{L}_{o1}$$

The idea expressed by the above equation is known as angular impulse momentum principle and states that increment in the angular momentum of a body about a point in a time interval equals to the net angular impulse of all the external forces acting on it during the concerned time interval.

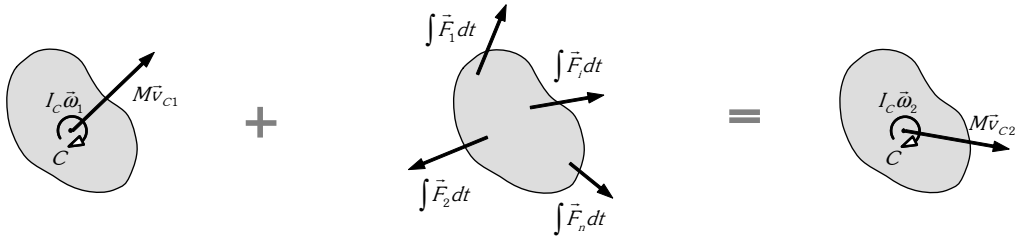
For the ease of application the above equation is rearranged as

$$\vec{L}_{o1} + \sum\vec{J}_{o,1\to2} = \vec{L}_{o2}$$

Like linear impulse momentum principle, the angular impulse momentum principle provides us solution of problems concerned with change in angular velocity in a time interval or change in angular velocity during very short interval interactions.

Method of Impulse Momentum Principle for Plane motion of a Rigid Body

Linear momentum and angular momentum serve as measures of amount of translation and rotation motion respectively. The external forces acting on a rigid body can change its state of translation as well as rotation motion which is reflected by change in linear as well as angular momentum according to the principles of linear impulse and momentum and angular impulse and momentum.



Linear and angular momenta at the instant t_1

Impulse of all the forces during time interval t_1 to t_2

Linear and angular momenta at the instant t_2

In the above figure is shown strategy to apply method of impulse and momentum. Consider a rigid body of mass M in plane motion. Its moment of inertial about the centroidal axis perpendicular to plane of motion is I_C. Let $\vec{v}_{C1}$ and $\vec{\omega}_1$ represent velocity of its mass center and its angular velocity at the beginning of a time interval t_1 to t_2. Under the action of several forces $\vec{F}_1$, $\vec{F}_2$ ..... $\vec{F}_i$ ..... $\vec{F}_n$ during the time interval its mass center velocity and angular velocity become $\vec{v}_{C2}$ and $\vec{\omega}_2$ respectively. The adjacent figure shows strategy representing how to write equations for linear and angular impulse momentum principles.

While applying the principle it becomes simpler to consider translation of the mass center and centroidal rotation separately. Thus in an alternative way we apply linear impulse momentum principle for translation of the mass center and angular impulse momentum principle for centroidal rotation.

Translation of mass center: Linear impulse momentum principle.

$$\vec{p}_1 + \sum\vec{I}_{mp1\to2} = \vec{p}_2$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Here $\vec{p}_1 = M\vec{v}_{C1}$ and $\vec{p}_2 = M\vec{v}_{C2}$ represent linear momentums at the beginning and end of the time interval and $\sum \vec{I}_{mp1\to2}$ stands for impulse of all the external forces during the time interval.

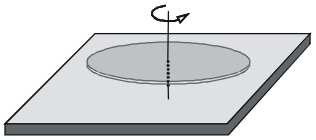
Centroidal rotation: Angular impulse momentum principle.

$$\vec{L}_{C1} + \sum \vec{J}_{C,1\to2} = \vec{L}_{C2}$$

Here $\vec{L}_{C1} = I_C\vec{\omega}_1$ and $\vec{L}_{C2} = I_C\vec{\omega}_2$ represent angular momentums about the centroidal axis at the beginning and end of the time interval and $\sum \vec{J}_{C,1\to2}$ stands for angular impulse of all the external forces about the centroidal axis during the time interval.

Example

A uniform disk of mass M and radius R rotating with angular velocity ω_o about a vertical axis passing through its center and perpendicular to its plane is placed gently on a rough horizontal ground, where coefficient of friction is μ. How long it will take to stop.



Solution.

Refer the worked out example 8.12. The torque of friction forces is

$$\tau_C = \tfrac{2}{3}\mu MgR \qquad (1)$$

The angular impulse of the torque of friction is responsible to stop the disk. Applying angular impulse momentum principle, we have

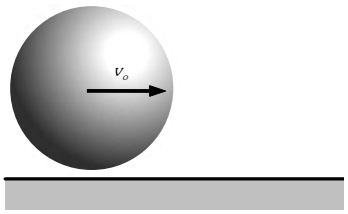
$$\vec{L}_{C1} + \sum \vec{J}_{C,1\to2} = \vec{L}_{C2} \to \quad I_C\omega_o - \tau_C t = 0$$

Substituting $I_C = \tfrac{1}{2}MR^2$ and τ_C from eq. (1), we have

$$t = \frac{3R\omega_o}{4\mu g}$$

Example

A uniform sphere of mass m and radius r is projected along a rough horizontal floor with linear velocity v_o and no angular velocity. The coefficients of kinetic and static frictions are represented by μ_s and μ_k respectively.



(a) How long the sphere will slide on the floor before it starts rolling.

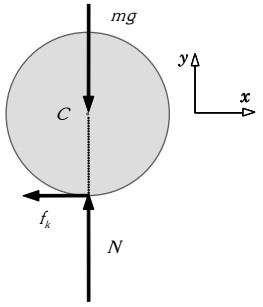
(b) How far the sphere will slide on the floor before it starts rolling.

(c) Find the linear and angular velocities of the sphere when it starts rolling.

(d) Find the work done by frictional forces during the process and thereafter.

Solution.

When the sphere touches the floor it is on translation motion. All the points including the bottom one are moving with the same velocity v_o. Thus the bottom point which makes the contact with the floor slide on it causing kinetic friction to act in backward direction. In the adjacent figure the forces acting on the sphere are shown. Here mg represent weight, N the normal reaction from the ground and f_k.



Since the sphere has no vertical component of acceleration, by applying Newton's law we have

$$\sum F_y = 0 \to \qquad N = mg$$

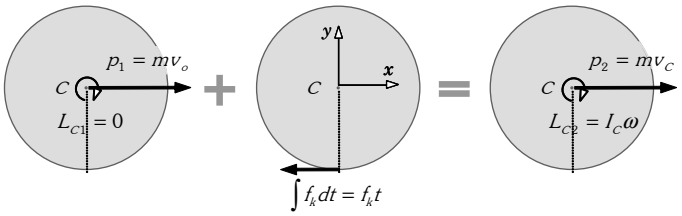
The kinetic friction $\qquad f_k = \mu N = \mu mg \qquad (1)$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

The only force which applied torque about the centroidal axis is the kinetic friction. Angular impulse of torque of kinetic friction increases the angular velocity ω and impulse of kinetic friction decreases the mass center velocity v_C till both bear following condition required for rolling. Thereafter the sphere will continue to roll with the unfirom velocity.

$$v_C = \omega r \qquad (2)$$

In the adjacent figure of impulse momentum diagram the impulse of kinetic friction is shown



Translation of mass center:Applying linear impulse momentum principle in x direction, we have

$\vec{p}_1 + \sum \vec{I}_{mp1\to 2} = \vec{p}_2 \to$ $\qquad p_1 - f_k t = p_2$

Substituting p_1, p_2 and f_k from eq. (1), we have

$$v_C = v_o - \mu g t \qquad (3)$$

Centroidal rotation: Angular impulse momentum principle about the centroidal axis.

$\vec{L}_{C1} + \sum \vec{J}_{C,1\to 2} = \vec{L}_{C2} \to$ $\qquad 0 + f_k r t = I_C \omega$

Substituting $\frac{2}{5}mr^2$ for I_C and f_k from eq. (1), we have

$$\omega = \frac{5\mu g t}{2r} \qquad (4)$$

(a) Substituting values of v_C and ω form eq. (3) and (4) into eq. (2), we have

Time when rolling starts $\quad t = \dfrac{2v_o}{7\mu g} \qquad (5)$

(b) Eq. (3) reveals that the mass center is in uniformly retarded motion. So its displacement in time t, when it starts rolling is given by the following equation.

$x = \frac{1}{2}(v_o + v_C)t \to$ Substituting values for v_C and t from eq. (3) and (5) respectively we have

$$x = \frac{12v_o^2}{49\mu g} \qquad (6)$$

(c) Linear and angular velocities of the sphere when it starts rolling can be obtained by substituting t from eq. (5) into (3) and (4) respectively.

Linear velocity when rolling starts $\quad v_C = \frac{5}{7}v_o \qquad (7)$

Angular velocity when rolling starts $\omega = \dfrac{5v_o}{7r} \qquad (8)$

(d) Work done by a force depends on displacement of point of application or displacement of the particle on which force is applied. The particle of the body in contact with the ground on which force of kinetic friction acts continuously changes; therefore it is recommended to calculate work done with the help of work energy theorem instead of using definition of work.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Kinetic energy in the initial position at the instant t_1,

$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2 \rightarrow \quad K_1 = \frac{1}{2}mv_o^2$ (9)

Since in the beginning angular velocity is zero.

Kinetic energy in the final position at the instant t_2,

$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2 \rightarrow$ Substituting values of v_C and ω form eq. (7) and (8) and $\frac{2}{5}mr^2$ for I_C, we can write

$$K_2 = \frac{35mv_o^2}{98} \quad (10)$$

Only force of kinetic friction does work during sliding. Denoting it by $W_{f1\rightarrow 2}$ in the equation of work energy theorem, we have

$K_1 + W_{1\rightarrow 2} = K_2 \rightarrow \quad W_{f1\rightarrow 2} = K_2 - K_1$

Substituting values of K_1 and K_2 form eq. (9) and (10), we have

$$W_{f1\rightarrow 2} = -\frac{1}{7}mv_o^2$$

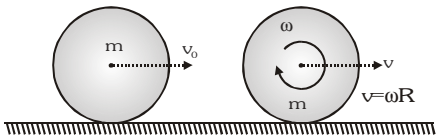
Example

A body of radius R and mass m is placed on horizontal rough surface with linear velocity v_0 , after some time it comes in the condition of pure rolling then determine :

(i) Time t at which body starts pure rolling.

(ii) Linear velocity of body at time t.

(iii) Work done by frictional force in this time t.

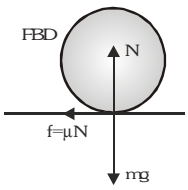


Solution

For translatory motion v = u + at

Initial velocity u = v_0

Let after time t pure rolling starts and at this time t final velocity = v and acceleration = a



From FBD :

Normal Reaction N = mg

Friction force f = μ N = μ mg $\Rightarrow$ ma = μ mg [$\because$ f = ma]

Retardation a = μ g

v = v_0 – at (–ve sign for retardation)

v = v_0 – μ gt ...(i)

For rotatory motion $\omega = \omega_0 + \alpha t$ (Initial angular velocity ω_0 = 0)

$\Rightarrow \quad \omega = \alpha t$ ...(ii)

$\because \tau = I\alpha \Rightarrow \alpha = \frac{\tau}{I} = \frac{fR}{mK^2} = \frac{\mu mgR}{mK^2} = \frac{\mu gR}{K^2}$ ...(iii)

From eq$^{n.}$ (ii) and eq$^{n.}$ (iii) $\omega = \frac{\mu gR}{K^2}t$ ...(iv)

$\because$ For pure rolling v = ωR $\Rightarrow \omega = \frac{v}{R}$ ...(v)

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

From eq. (iv) and (v) $\frac{v}{R}=\frac{\mu gR}{K^2}t$ or $v=\frac{\mu gR^2t}{K^2}$ ...(vi)

substitute v from eq$^{n.}$ (vi) into eq$^{n.}$(i) $\frac{\mu gR^2t}{K^2}=v_0-\mu gt \Rightarrow t=\frac{v_0}{\mu g\left[1+\frac{R^2}{K^2}\right]}$

Putting the value of t in equation (i) $v=v_0-\mu g\frac{v_0}{\mu g\left[1+\frac{R^2}{K^2}\right]}=v_0-\frac{v_0}{1+\frac{R^2}{K^2}}=\frac{v_0}{1+\frac{K^2}{R^2}}$

Work done in sliding by frictional force = Initial kinetic energy – Final kinetic energy

Work done by friction $W_f=\frac{1}{2}Mv_0^2-\frac{1}{2}Mv^2\left(1+\frac{K^2}{R^2}\right)=\frac{1}{2}Mv_0^2-\frac{1}{2}\frac{Mv_0^2}{\left(1+\frac{K^2}{R^2}\right)}=\frac{Mv_0^2}{2\left(1+\frac{R^2}{K^2}\right)}$

Conservation of Angular Momentum

If angular impulse of all the external forces about an axis in time interval vanishes, the angular momentum of the system about the same axis in that time interval remain unchanged.

If $\Sigma\int_{t_1}^{t_2}\vec{\tau}_o dt=0$, we have $\vec{L}_{o1}=\vec{L}_{o2}$

The condition of zero net angular impulse required for conservation of angular momentum can be fulfilled in the following cases.

- If no external force acts, the angular impulse about all axes will be zero and hence angular momentum remains conserved about all axes.
- If net torque of all the external forces or torques of each individual force about an axis vanishes the angular momentum about that axes will be conserved.
- If all the external forces are finite in magnitude and the concerned time interval is infinitely small, the angular momentum remain conserved.
- If a system of rigid bodies changes its moment of inertia by changing its configuration due to internal forces only its angular momentum about any axes remains conserved. If we denote the moment of inertias in two configurations by I_1 and I_2 and angular velocities by ω_1 and ω_2, we can write

$$I_1\vec{\omega}_1=I_2\vec{\omega}_2$$

The principle of conservation of angular momentum governs a wide range of physical processes from subatomic to celestial world. The following examples explicate some of these applications.

Spinning Ice Skater.

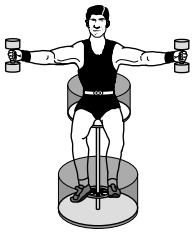


A spinning ice skater and ballet dancers can control her moment of inertia by spreading or bringing closer her hands and make use of conservation of angular momentum to perform their spins. In doing so no external forces is needed and if we ignore effects of friction from the ground and the air, the angular momentum can be assumed conserved. When she spreads her hand or leg away, her moment of inertia decreases therefore her angular velocity decreases and when she brings her hands or leg closer her moment of inertia increases therefore her angular velocity increases.

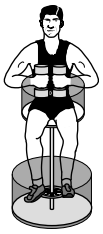
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Student on rotating turntable

The student, the turntable and dumbbells make an isolated system on which no external torque acts, if we ignore friction in the bearing of the turntable and air friction. Initially the student has his arm stretched on rotating turntable. When he pulls dumbbells close to his body, angular velocity increases due to conservation of angular momentum.



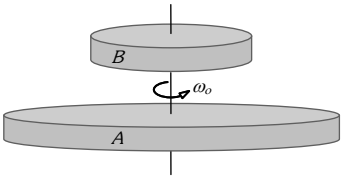
Larger moment of inertia and smaller angular velocity



Smaller moment of inertia and larger angular velocity

Example

Consider the disk A of moment of inertia I_1 rotating freely in horizontal plane about its axis of symmetry with angular velocity ω_o. Another disk B of moment of inertia I_2 held at rest above the disk A. The axis of symmetry of the disk B coincides with that of the disk A as shown in the figure. The disk B is released to land on the disk A. When sliding stops, what will be the angular velocity of both the disks?



Solution.

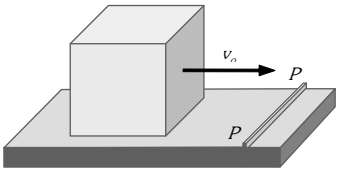
Both the disks are symmetric about the axis of rotation therefore does not require any external torque to keep the axis stationary. When the disk B lands on A slipping starts. The force of friction provides an internal torque to system of both the disk. It slows down rotation rate of A and increases that of B till both acquire same angular velocity ω.

Since there is no external torques on the system of both the disks about the axis of rotation, the total angular momentum of the system remains conserved. The total angular momentum of the system is the sum of angular momentum of both disks. Denoting the angular momentum of the disk A before B lands on it and long after slipping between them stops by symbols $\vec{L}_{A1}$, $\vec{L}_{B1}$, $\vec{L}_{A2}$ and $\vec{L}_{B2}$ respectively, we can express conservation of angular momentum by the following equation.

$$\vec{L}_{A1}+\vec{L}_{B1}=\vec{L}_{A2}+\vec{L}_{B2}\rightarrow \quad I_1\omega_o+0=I_1\omega+I_2\omega\Rightarrow\omega=\frac{I_1\omega_o}{I_1+I_2}$$

Example

A cube of mass m and edge length ℓ can slide freely on a smooth horizontal floor. Moving on the floor with velocity v_o, it strikes a long obstruction PP of small height. The obstruction is parallel to the leading bottom edge of the cube. The leading bottom edge gets pivoted with the obstruction and the cube starts rotating. Determine angular velocity of the cube immediately after the impact.



Solution.

Before the impact, there is no external force in the horizontal direction and the cube slides with uniform velocity v_o, and during the impact reaction forces of the obstruction stops its leading bottom edge and cause it to rotate about its leading bottom.

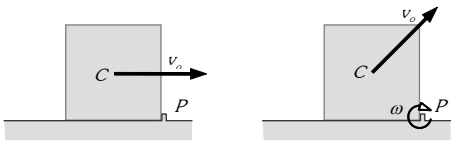
During the impact external forces acting on the cube are its weight, the normal reaction from the ground and reaction from the obstruction. The weight and the normal reaction from the ground both are finite in magnitude and the impact ends in infinitesimally small time interval so their impulses and angular impulses about any axes are negligible. It is the reaction from the obstruction which has finite impulse during the impact. Its horizontal

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

component changes the momentum of the cube during the impact, but its angular impulse about the obstruction is zero, therefore the angular momentum of the cube about an axis coincident with the leading bottom edge remain conserved.

Let the velocity of the mass center and angular velocity of the cube immediately after the impact are v_{Co} and ω_o. These velocities are shown in the adjacent figure.

We denote the angular momentum of the cube about axis coincident with the obstruction edge before and after the impact by L_{P1} and L_{P2}.



Immediately before the impact. Immediately after the impact.

Applying principle of conservation of angular momentum about an axis coincident with the obstruction, we have

$\vec{L}_{P1} = \vec{L}_{P2} \rightarrow \qquad \vec{r}_{C/P} \times m\vec{v}_o = I_P\vec{\omega}$

Using theorem of parallel axes for moment of inertia I_P about the leading bottom edge, we get $I_P = I_C + mr_{C/P}^2 = \frac{2}{3}m\ell^2$. Substituting this in the above equation, we have

Angular velocity immediately after the impact $\omega_o = \dfrac{3v_o}{4\ell}$

Angular momentum of a body in combined translational and rotational motion

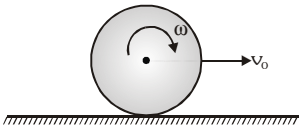
Suppose a body is rotating about an axis passing through its centre of mass with an angular velocity ω_{cm} and moving translationally with a linear velocity v. Then, the angular momentum of the body about a point P outside the body in the lab frame is given by, $\vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm}$ where r is the position vector of the centre of mass with respect to point P. Hence, $\vec{L}_P = I_{cm} + \vec{r} \times m\vec{v}_{cm}$

Example

A solid sphere rolls without slipping on a rough surface and the centre of mass has constant speed v_0. If mass of the sphere is m and its radius is R, then find the angular momentum of the sphere about the point of contact.

Solution

$\because \quad \vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R} \times m\vec{v}_{cm}$



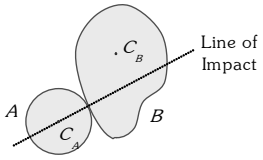
Since sphere is in pure rolling motion hence

$\omega = v_0 R \Rightarrow \quad \vec{L}_p = \left(\frac{2}{5}MR^2\frac{v_0}{R}\right)\left(-\tilde{k}\right) + Mv_0R = \frac{7}{5}Mv_0R\left(-\tilde{k}\right)$

Eccentric Impact

In eccentric impact the line of impact which is the common normal drawn at the point of impact does not passes through mass center of at least one of the colliding bodies. It involves change in state of rotation motion of either or both the bodies.

Consider impact of two A and B such that the mass center C_B of B does not lie on the line of impact as shown in figure. If we assume bodies to be frictionless their mutual forces must act along the line of impact. The reaction force of A on B does not passes through the mass center of B as a result state of rotation motion of B changes during the impact.



Problems of Eccentric Impact

Problems of eccentric impact can be divided into two categories. In one category both the bodies under going eccentric impact are free to move. No external force act on either of them. There mutual forces are responsible for change in their momentum and angular momentum. In another category either or both of the bodies are hinged.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Eccentric Impact of bodies free to move

Since no external force acts on the two body system, we can use principle of conservation of linear momentum, principle of conservation of angular momentum about any point and concept of coefficient of restitution.

The coefficient of restitution is defined for components of velocities of points of contacts of the bodies along the line of impact.

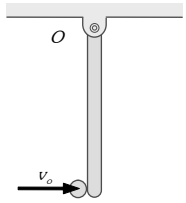
While applying principle of conservation of angular momentum care must be taken in selecting the point about which we write the equation. The point about which we write angular momentum must be at rest relative to the selected inertial reference frame and as far as possible its location should be selected on line of velocity of the mass center in order to make zero the first term involving moment of momentum of mass center.

Eccentric Impact of hinged bodies

When either or both of the bodies are hinged the reaction of the hinge during the impact act as external force on the two body system, therefore linear momentum no longer remain conserved and we cannot apply principle of conservation of linear momentum. When both the bodies are hinged we cannot also apply conservation of angular momentum, and we have to use impulse momentum principle on both the bodies separately in addition to making use of coefficient of restitution. But when one of the bodies is hinged and other one is free to move, we can apply conservation of angular momentum about the hinge.

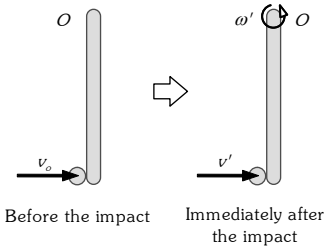
Example

A uniform rod of mass M and length ℓ is suspended from a fixed support and can rotate freely in the vertical plane. A small ball of mass m moving horizontally with velocity v_o strikes elastically the lower end of the rod as shown in the figure. Find the angular velocity of the rod and velocity of the ball immediately after the impact.



Solution.

The rod is hinged and the ball is free to move. External forces acting on the rod ball system are their weights and reaction from the hinge. Weight of the ball as well as the rod are finite and contribute negligible impulse during the impact, but impulse of reaction of the hinge during impact is considerable and cannot be neglected. Obviously linear momentum of the system is not conserved. The angular impulse of the reaction of hinge about the hinge is zero. Therefore angular momentum of the system about the hinge is conserved. Let velocity of the ball after the impact becomes v'_B and angular velocity of the rod becomes ω'.



Before the impact Immediately after the impact

We denote angular momentum of the ball and the rod about the hinge before the impact by L_{B1} and L_{R1} and after the impact by L_{B2} and L_{R2}.

Applying conservation of angular momentum about the hinge, we have

$$\vec{L}_{B1}+\vec{L}_{R1}=\vec{L}_{B2}+\vec{L}_{R2}\rightarrow \quad mv_o\ell+0=mv'_B\ell+I_o\omega'$$

Substituting $\frac{1}{3}M\ell^2$ for I_o, we have

$$3mv'_B+M\ell\omega'=3mv_o \qquad (1)$$

The velocity of the lower end of the rod before the impact was zero and immediately after the impact it becomes $\ell\omega'$ towards right. Employing these facts we can express the coefficient of restitution according to eq.

$$e=\frac{v'_{Qn}-v'_{Pn}}{v_{pn}-v_{Qn}}\rightarrow \qquad \ell\omega'-v'_B=ev_o \qquad (2)$$

From eq. (1) and (2), we have

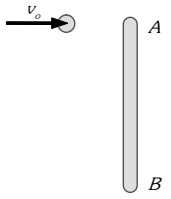
Velocity of the ball immediately after the impact $\quad v'_B=\dfrac{(3m-eM)v_o}{3m+M}$

Angular velocity of the rod immediately after the impact $\quad \omega'=\dfrac{3(1+e)mv_o}{(3m+M)\ell}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

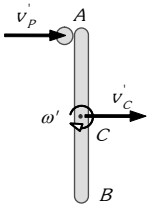
ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example



A uniform rod AB of mass M and length ℓ is kept at rest on a smooth horizontal plane. A particle P of mass m_o moving perpendicular to the rod with velocity v_o strikes the rod at one of its ends as shown in the figure. Derive suitable expressions for the coefficient of restitution, velocity of mass center of the rod and angular velocity of the rod immediately after the impact. Assume it is the coefficient of restitution.

Solution.



Both the bodies can move freely in the horizontal plane, therefore no horizontal external force acts on the particle-rod system. The linear momentum as well as angular momentum about any axis normal to the plane is conserved.
Let the velocity of the particle, angular velocity of the rod and velocity of the mass center of the rod immediately after the impact are v'_P towards right, ω' in clockwise sense and v'_C towards right as shown in the adjacent figure. Using relative motion equation, we can express the velocity of the end A of the rod.

$$v'_A = v'_C + \tfrac{1}{2}\omega'\ell \qquad (1)$$

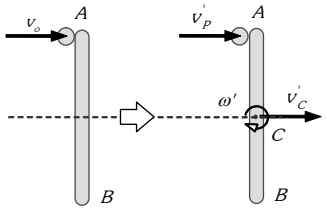
We denote linear momentum of the particle and rod before the impact by $\vec{p}_{P1}$ and $\vec{p}_{C1}$ and immediately after the impact by $\vec{p}_{P2}$ and $\vec{p}_{C2}$ respectively.

Applying conservation of linear momentum, we have

$$\vec{p}_{P2} + \vec{p}_{C2} = \vec{p}_{P1} + \vec{p}_{C1} \rightarrow \quad mv'_P + Mv'_C = mv_o \qquad (2)$$

The above equation shows that the mass center of the rod will move toward the right. If we write angular momentum of the rod about a stationary point O, which is in line with the velocity v'_C, the first term involving moment of momentum of rod vanishes and only angular momentum due to its centroidal rotation remains in the expression.

We denote angular momentum of the particle and the rod about the point O before the impact by L_{P1} and L_{R1} and after the impact by L_{P2} and L_{R2}.
Applying conservation of angular momentum about the hinge, we have



Before the impact
Immediately after the impact

$$\vec{L}_{P2} + \vec{L}_{R2} = \vec{L}_{P1} + \vec{L}_{R1} \rightarrow \tfrac{1}{2}mv'_P\ell + I_C\omega' = \tfrac{1}{2}mv_o\ell$$

Substituting $\frac{1}{12}M\ell^2$ for I_C, we have

$$6mv'_P + M\ell\omega' = 6mv_o \qquad (3)$$

The velocity of the end A of the rod before the impact was zero and immediately after the impact it becomes v'_A towards right. Employing these facts we can express the coefficient of restitution as

$$e = \frac{v'_{Qn} - v'_{Pn}}{v_{pn} - v_{Qn}} \rightarrow \qquad v'_A - v'_P = ev_o$$

Substituting v'_A form eq. (1), we have

$$2v'_C + \omega'\ell - 2v'_P = 2ev_o \qquad (4)$$

Eq. (2), (3) and (4) involves three unknowns v'_C, ω' and v'_P, which can be obtained by solving these equation.

Velocity of the ball immediately after the impact $\quad v'_P = \left(\dfrac{4m - eM}{4m + M}\right)v_o$

Velocity of mass center of rod immediately after the impact $\quad v'_C = \left\{\dfrac{m(1+e)}{4m+M}\right\}v_o$

Angular velocity of the rod immediately after the impact $\quad \omega' = \left\{\dfrac{6m(1+e)}{4m+M}\right\}\dfrac{v_o}{\ell}$

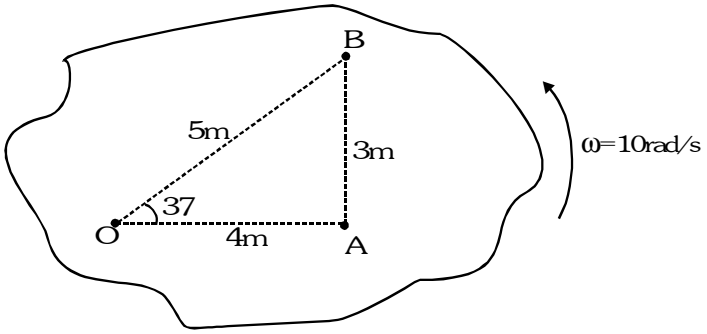
NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SOME WORKED OUT EXAMPLES

Example#1

A rigid lamina is rotating about an axis passing perpendiuclar to its plane through point O as shown in figure.



The angular velocity of point B w.r.t. A is

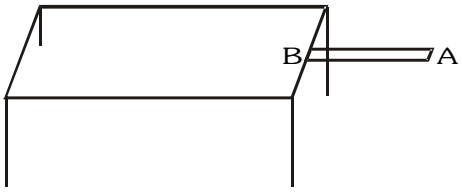
(A) 10 rad/s (B) 8 rad/s (C) 6 rad/s (D) 0

Solution **Ans. (A)**

In a rigid body, angular veloicty of any point on the rigid body w.r.t. any other point on the rigid body is constant and equal to angular velocity of rigid body.

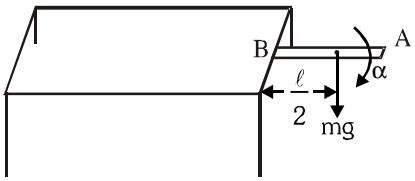
Example#2

A uniform thin stick of length ℓ and mass m is held horizontally with its end B hinged at a point B on the edge of a table. Point A is suddenly released. The acceleration of the centre of mass of the stick at the time of release, is :-



(A) $\frac{3}{4}g$ (B) $\frac{3}{7}g$ (C) $\frac{2}{7}r\ g$ (D) $\frac{1}{7}\ g$

Solution **Ans. (A)**



For angular motion of the stick $\tau = mg\left(\frac{\ell}{2}\right) = I\,\alpha$

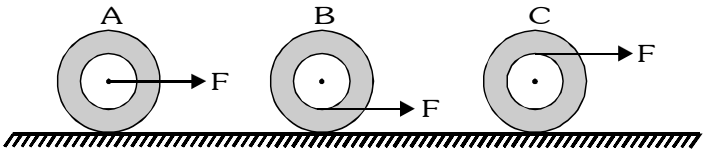
moment of intertia of stick about B is $I = \frac{m\ell^2}{3} \Rightarrow mg\left(\frac{\ell}{2}\right) = \left(\frac{m\ell^2}{3}\right)\alpha \Rightarrow \alpha = \frac{3g}{2\ell}$

Acceleration of centre of mass $= \alpha\left(\frac{\ell}{2}\right) = \left(\frac{3g}{2\ell}\right)\left(\frac{\ell}{2}\right) = \frac{3}{4}g$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example#3

Three spools A, B and C are placed on rough ground and acted by equal force F. Then which of the following statement is incorrect?



(A) Frictional force on spool A is in backward direction
(B) Frictional force on spool B is in backward direction
(C) Frictional force on spool C is in backward direction
(D) Frictional force on spool C is in forward direction

Solution **Ans. (D)**

For spool A, sliding tendency of point of contact is forward $\Rightarrow$ frictional force is in backward direction
For spool B, sliding tendency of point of contact is forward $\Rightarrow$ frictional force is in backward direction

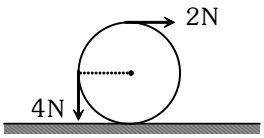
For spool C, slidin7g tendency of point of contact is forward as without friction

$$a = \frac{F}{m}, \alpha = \frac{\tau}{I} = \frac{Fr}{\frac{m}{2}(R^2 + r^2)} \Rightarrow \alpha R = \frac{2FRr}{m(R^2 + r^2)}$$

Here $a > \alpha R$ so acceleration of point of contact will be in forward direction.
$\Rightarrow$ frictional force is in backward direction.

Example#4

A uniform solid disc of mass 1 kg and radius 1m is kept on a rough horizontal surface. Two forces of magnitude 2 N and 4 N have been applied on the disc as shown in the figure. Linear acceleration of the centre of mass of the disc is if there is no slipping.



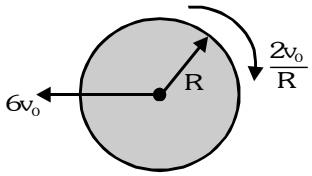
(A) 4 m/s^2 (B) 2 m/s^2 (C) 1 m/s^2 (D) zero

Solution **Ans. (D)**

Taking torque about contact point, $\tau = 4 \quad R - 2 \quad 2R = 0$, $F_{net} = 0$

Example#5

A disc of radius R = 2m moves as shown in the figure, with a velocity of translation of $6v_0$ of its centre of mass and an angular velocity of $\frac{2v_0}{R}$. The distance (in m) of instantaneous axis of rotation from its centre of mass is



(A) 3 (B) 4 (C) 5 (D) 6

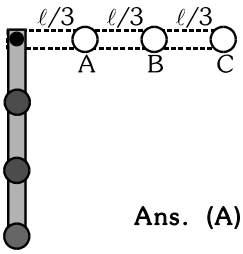
Solution **Ans. (D)**

Instantaneous axis of rotation lies above the centre of mass where $v - \omega r = 0 \Rightarrow r = \frac{v}{\omega} = \frac{6v_0}{\frac{2v_0}{R}} = 3R$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example#6

A light rod carries three equal masses A, B and C as shown in figure. What will be velocity of B in vertical position of rod, if it is released from horizontal position as shown in figure ?



(A) $\sqrt{\frac{8g\ell}{7}}$ (B) $\sqrt{\frac{4g\ell}{7}}$ (C) $\sqrt{\frac{2g\ell}{7}}$ (D) $\sqrt{\frac{10g\ell}{7}}$

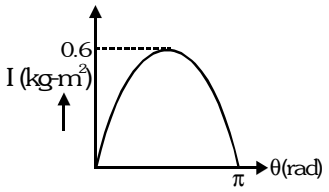
Solution **Ans. (A)**

Loss in P.E. = Gain in K.E. $mg\frac{\ell}{3} + mg\left(\frac{2\ell}{3}\right) + mg\ell = \frac{1}{2}\left(m\left(\frac{\ell}{3}\right)^2 + m\left(\frac{2\ell}{3}\right)^2 + m\ell^2\right)\omega^2$

$$\Rightarrow \omega = \sqrt{\frac{36g}{14\ell}} \Rightarrow v_B = \omega\ell_B = \frac{2\ell}{3}\sqrt{\frac{36g}{14\ell}} = \sqrt{\frac{8g\ell}{7}}$$

Example#7

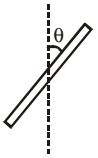
Figure shows the variation of the moment of inertia of a uniform rod, about an axis passing through its centre and inclined at an angle θ to the length. The moment of inertia of the rod about an axis passing through one of its ends and making an angle $\theta = \frac{\pi}{3}$ will be



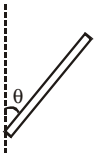
(A) 0.45 kg–m^2 (B) 1.8 kg–m^2 (C) 2.4 kg–m^2 (D) 1.5 kg–m^2

Solution **Ans. (B)**

$$I = \frac{ML^2}{12}\sin^2\theta \Rightarrow 0.6 = \frac{ML^2}{12}\sin\left(\frac{\pi}{2}\right) \Rightarrow ML^2 = 7.2$$

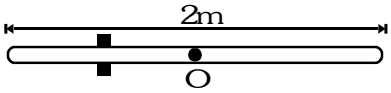


$$I = \frac{ML^2}{3}\sin^2\theta, \text{ at } \theta = \frac{\pi}{3}, I = \frac{ML^2}{3}\left(\frac{3}{4}\right) = \frac{ML^2}{4} = \frac{7.2}{4} = 1.8 \text{ kg-m}^2$$



Example#8

A 2m long rod of negligible mass is free to rotate about its centre. An object of mass 5 kg is threaded into the rod at a distance of 50 cm from its end in such a way that the object can move without friction. The rod is then released from its horizontal position. The speed of the rod's end in the rod's vertical position is (in m/s)

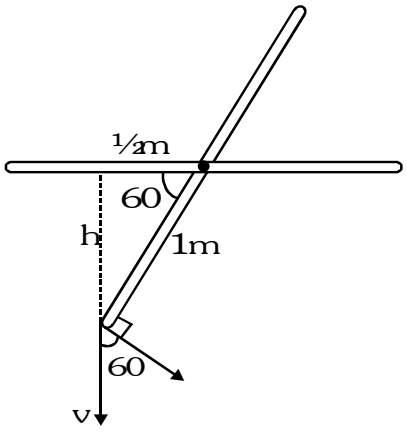


(A) $\sqrt{\frac{5\sqrt{3}}{2}}$ (B) $\sqrt{\frac{4\sqrt{3}}{2}}$ (C) $\sqrt{\frac{3\sqrt{5}}{2}}$ (D) $\sqrt{\frac{3\sqrt{3}}{2}}$

Solution **Ans. (A)**

Since friction and the rod's mass is negligible, the only force acting on the object is gravitational force, therefore the object undergoes free-fall.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65



The object-moves a distance of h = $\sqrt{1^2-\left(\frac{1}{2}\right)^2}=\frac{\sqrt{3}}{2}$m until it drops off from the rod.

Its velocity at this moment v = $\sqrt{2gh}$

The object's velocity perpendicular to the rod equals to the velocity of the rod's end at the moment when the object leaves the rod. After this moment the rod's end maintains its speed, so in vertical position its

$$\text{speed} = v\cos 60 = \left(\sqrt{2gh}\right)\left(\frac{1}{2}\right)=\sqrt{\frac{gh}{2}}=\sqrt{\frac{10\times\sqrt{3}/2}{2}}=\sqrt{\frac{5\sqrt{3}}{2}}\ \text{m/s}$$

Example#9

A child's top is spun with angular acceleration $\alpha = 4t^3 - 3t^2 + 2t$ where t is in seconds and α is in radian per second-squared. At t =0, the top has angular velocity ω_0 = 2 rad/s and a reference line on it is at angular position θ_0 = 1 rad.

Statement I : Expression for angular velocity $\omega = (2 + t^2 - t^3 + t^4)$ rad/s

Statement II : Expression for angular position $\theta = (1 + 2t - 3t^2 + 4t^3)$ rad

(A) Only statement-I is true
(B) Only statement-II is true
(C) Both of them are true
(D) None of them are true

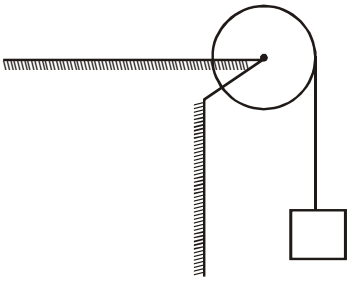
Solution **Ans. (A)**

$$\int_2^{\omega} d\omega = \int_0^t \alpha dt \Rightarrow \omega - 2 = \left(t^4 - t^3 + t^2\right)_0^t \Rightarrow \omega = 2 + t^2 - t^3 + t^4$$

$$\int_1^{\theta} d\theta = \int_0^t \omega dt \Rightarrow \theta - 1 = \left(2t + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}\right)_0^t \Rightarrow \theta = 1 + 2t + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}$$

Example#10

Figure shows a uniform disk, with mass M = 2.4 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. The tension in cord is



(A) 12 N
(B) 20 N
(C) 24 N
(D) None of these

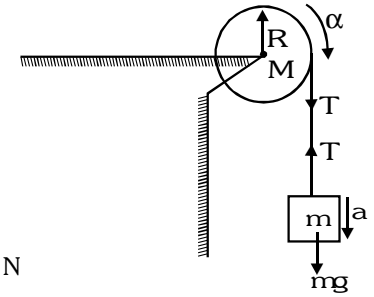
NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution. **Ans. (d)**

For block : mg – T = ma .....(i)

For disk (pulley) $TR = I\alpha = \frac{MR^2}{2}\alpha$

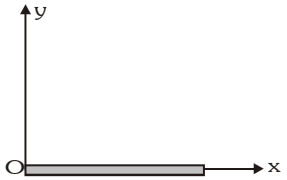
But $\alpha = \frac{a}{R}$ so $T = \frac{Ma}{2}$ ...(ii)



Therefore $\frac{mg-T}{T} = \frac{2m}{M} \Rightarrow \frac{mg}{T} = \frac{2m}{M}+1 \Rightarrow t = \frac{mg}{\left(\frac{2m}{M}+1\right)} = \frac{(1.2\times10)}{\left(\frac{211.2}{2.4}+1\right)} = 6N$

Example#11

The figure shows a uniform rod lying along the x-axis. The locus of all the points lying on the xy-plane, about which the moment of inertia of the rod is same as that about O is :



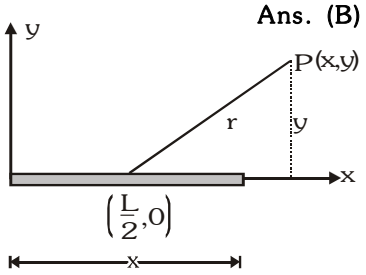
(A) an ellipse (B) a circle

(C) a parabola (D) a straight line

Solution **Ans. (B)**

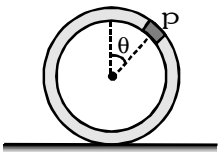
$I_P = I_{CM} + Mr^2 = \frac{ML^2}{12} + M\left[\left(x-\frac{L}{2}\right)^2 + y^2\right]$

$I_0 = \frac{ML^2}{3} \Rightarrow \left(x-\frac{L}{2}\right)^2 + y^2 = \left(\frac{L}{2}\right)^2 \Rightarrow$ Locus is a circle



Example#12

A small block of mass 'm' is rigidly attached at 'P' to a ring of mass '3m' and radius 'r'. The system is released from rest at θ = 90 and rolls without sliding. The angular acceleration of hoop just after release is–



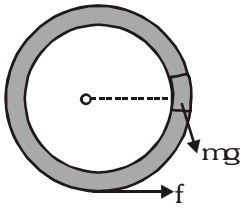
(A) $\frac{g}{4r}$ (B) $\frac{g}{8r}$ (C) $\frac{g}{3r}$ (D) $\frac{g}{2r}$

Solution **Ans. (B)**

f = 4 ma ...(i) $(mg - f)\, r = (3mr^2 + mr^2)\,\alpha$

mg – f = 4 ma ....(ii)

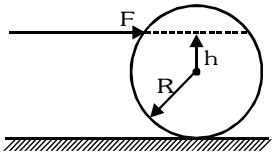
from (i) and (ii) $\Rightarrow$ 8 ma = mg $\Rightarrow a = \frac{g}{8} \Rightarrow \alpha = \frac{g}{8r}$



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example#13

An impulsive force F acts horizontally on a solid sphere of radius R placed on a horizontal surface. The line of action of the impulsive force is at a height h above the centre of the sphere. If the rotational and translational kinetic energies of the sphere just after the impulse are equal, then the value of h will be-



(A) $\frac{2}{5}R$ (B) $\sqrt{\frac{2}{3}}R$ (C) $\sqrt{\frac{2}{5}}R$ (D) None of these

Solution **Ans. (C)**

$F\Delta t = Mv$; $Fh\Delta t = I\omega \Rightarrow Mvh = I\,\omega = \frac{2}{5}MR^2\omega$ Also. $\frac{1}{2}Mv^2 = \frac{1}{2}I\omega^2 = \frac{1}{2}\ \frac{2}{5}\ MR^2\omega^2 \Rightarrow h = \sqrt{\frac{2}{5}}R$

Example#14

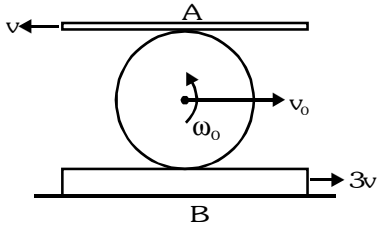
The disc of radius r is confined to roll without slipping at A and B. If the plates have the velocities shown, then

(A) linear velocity $v_0 = v$

(B) angular velocity of disc is $\frac{3v}{2r}$

(C) angular velocity of disc is $\frac{2v}{r}$

(D) None of these



Solution **Ans. (A,C)**

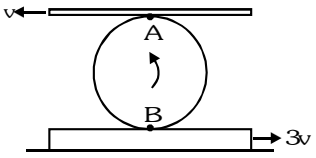
$v_A = \omega_0 r - v_0 = v$

$\omega_0 r - v_0 = v$ ....(i)

$v_B = \omega_0\, r + v_0 = 3v$ ...(ii)

from equation (i) & (ii) $2\omega_0 r = 4V \Rightarrow \omega_0 r = 2v$

$\omega_0 = \frac{2v}{r}$ from equation (i) $v_0 = v$



Example #15

A thin uniform rod of mass m and length ℓ is free to rotate about its upper end. When it is at rest, it receives an impulse J at its lowest point, normal to its length. Immediately after impact

(A) the angular momentum of the rod is $J\ell$. (B) the angular velocity of the rod is $\frac{3J}{m\ell}$

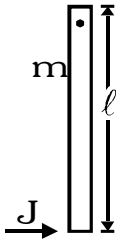
(C) the kinetic energy of the rod is $\frac{3J^2}{2m}$ (D) the linear velocity of the midpoint of the rod is $\frac{3J}{2m}$

Solution **Ans. (A,B,C,D)**

By impulse momentum theorem $J\ell = \frac{m\ell^2}{3}\omega \Rightarrow \omega = \frac{3J}{m\ell}$

KE of rod = $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\left(\frac{3J}{m\ell}\right)^2 = \frac{3J^2}{2m}$

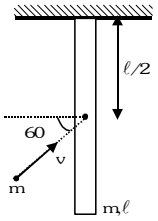
Linear velocity of midpoints = $\omega\left(\frac{\ell}{2}\right) = \frac{3J}{2m}$



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example#16

A thin rod of mass m and length ℓ is hinged to a ceiling and it is free to rotate in a vertical plane. A particle of mass m, moving with speed v strikes it as shown in the figure and gets stick with the rod. The value of v , for which the rod becomes horizontal after collision is



(A) The value of v, for which rod becomes horizontal after collision is $\sqrt{\frac{168}{9}g\ell}$

(B) The value of v, for which rod becomes horizontal after collision is $\sqrt{\frac{53}{3}g\ell}$

(C) Angular momentum of (rod + particle) system will remain constant about hinge just before and after collision

(D) Angular momentum of (rod + particle) system will remain same about centre of mass just before and after collision

Solution **Ans. (A,C)**

$$m.\frac{v}{2}.\frac{\ell}{2}=\left(m\frac{\ell^2}{3}+m\frac{\ell^2}{4}\right)\omega\ ; \qquad \frac{v}{4}=\frac{7}{12}\ell\omega\Rightarrow\omega=\frac{3}{7}\frac{v}{\ell}\ ...(i)$$

$$\frac{1}{2}\left(\frac{7}{12}m\ell^2\right)\omega^2=2mg\frac{\ell}{2}\Rightarrow\omega=\sqrt{\frac{24}{7}\frac{g}{\ell}}\ ...(ii)\quad \text{from equation (i) and (ii)}\ \ v=\sqrt{\frac{168}{9}g\ell}$$

Example#17

A bicycle is in motion. The force of friction exerted by the ground on its wheel is such that it acts:

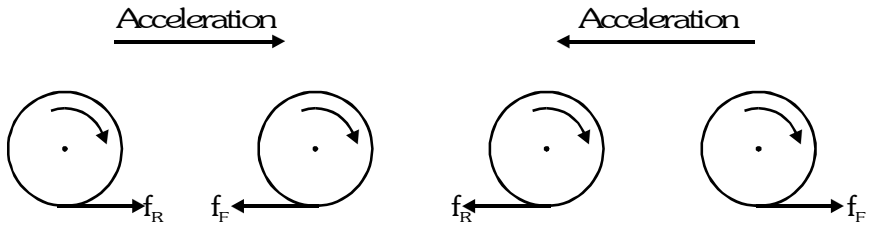
(A) in backward direction on front wheel and in forward direction on rear wheel when it is accelerating

(B) in forward direction on front wheel and in backward direction on rear wheel when brakes are applied on rear wheel only

(C) in backward direction on front wheel and in forward direction on rear wheel when brakes are applied on rear wheel only

(D) in backward direction on both the wheels when brakes are applied on front wheel

Solution **Ans. (A,B)**



Example#18

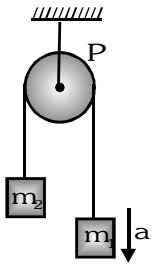
In the figure, the blocks have unequal masses m_1 and m_2 ($m_1 > m_2$). m_1 has a downward acceleration a. The pulley P has a radius r, and some mass. The string does not slip on the pulley–

(A) The two sections of the string have unequal tensions.

(B) The two blocks have accelerations of equal magnitude.

(C) The angular acceleration of P is $\frac{a}{r}$

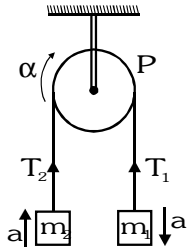
(D) $a<\left(\frac{m_1-m_2}{m_1+m_2}\right)g$



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

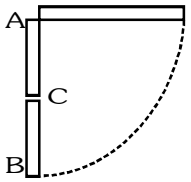
Solution **Ans. (A,B,C,D)**

In this situation $T_1 \neq T_2 \Rightarrow \alpha = \frac{a}{r} \Rightarrow a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{r^2}}$



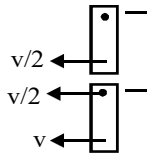
Example#19

A uniform rod AB of length ℓ is free to rotate about a horizontal axis passing through A. The rod is released from rest from horizontal position. If the rod gets broken at midpoint C when it becomes vertical, then just after breaking of the rod :



(A) Angular velocity of upper part starts to decrease while that of lower part remains constant.
(B) Angular velocity of upper part starts to decrease while that of lower part starts to increase
(C) Angular velocity of both the parts is identical
(D) Angular velocity of lower part becomes equal to zero

Solution . **Ans. (A,C)**



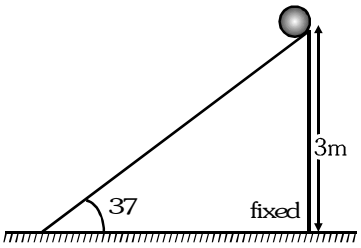
On upper part torque of mg about A will decrease the angular velocity.

Lower part of rod will not experience any couple hence its angular velocity can't change.

Initially both parts are having same angular velocities.

Example# 20 to 22

A uniform hollow sphere is released from the top of a fixed inclined plane of inclination 37 and height 3m. It rolls without sliding.



20. The acceleration of the centre of mass of the hollow sphere is

(A) $\frac{30}{7}$ m/s^2 (B) $\frac{18}{5}$ m/s^2 (C) $\frac{9}{5}$ m/s^2 (D) $\frac{15}{7}$ m/s^2

21. The speed of the point of contact of the sphere with the inclined plane when the sphere reaches half–way of the incline is

(A) $\sqrt{42}$ m/s (B) $\sqrt{21}$ m/s (C) $\sqrt{84}$ m/s (D) zero

22. The time taken by the sphere to reach the bottom is

(A) $\frac{3}{5}$s (B) $\frac{5}{3}$s (C) $\frac{5}{4}$s (D) None of these

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution

20. Ans. (B)

$$a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = \frac{(10)(3/5)}{1+\frac{2}{3}} = \frac{18}{5} ms^{-2}$$

21. Ans. (D)

Speed of point of contact in pure rolling is always zero

22. Ans. (B)

$$s = ut + \frac{1}{2}at^2 \Rightarrow \frac{3}{\sin 37°} = \frac{1}{2}\left(\frac{18}{5}\right)(t^2) \Rightarrow t = \frac{5}{3}s$$

Example#23 to 25

A mouse, searching for food, jumped onto the rim of a stationary circular disk mounted on a vertical axle. The disk is free to rotate without friction. The velocity of the mouse was tangent to the edge of the disk before it landed. When the mouse landed, it gripped the surface, remained fixed on the outer edge of the disk at a distance R from the center, and set it into rotation. The sketch indicates the situation.



Before After

The mass of the mouse is m = 0.10 kg, the radius of the disk is R = 0.20 m, and the rotational inertia of the disk is I = 0.0080 kg m . The speed of the mouse, just before it landed on the disk is v_o = 1.5 m/s.

23. Magnitude of the angular velocity of the disk plus mouse, after it landed becomes

(A) 0.25 rad/s (B) 2.5 rad/s (C) 0.375 rad/s (D) 3.75 rad/s

24. Find the magnitude of the impulse received by the mouse as it landed on the disk.

(A) 0.01 kg.m/s opposite to direction of motion (B) 0.01 kg.m/s in the direction of motion

(C) 0.10 kg.m/s opposite to direction of motion (D) 0.10 kg.m/s in the direction of motion

25. The mouse, still searching for food, crept to the center of the disk (where r = 0). Find angular velocity of the disk plus mouse, when the mouse was at the center of the disk.

(A) 0.25 rad/s (B) 2.5 rad/s (C) 0.375 rad/s (D) 3.75 rad/s

Solution

23. Ans. (B)

By conservation of angular momentum $mv_0R = (I + mR^2)\omega \Rightarrow \omega = \frac{mv_0R}{I+mR^2} = \frac{(0.1)(1.5)(0.2)}{0.008+0.004} = 2.5$ rad/s

24. Ans. (C)

Impulse received by mouse = change in momentum = 0.1 (2.5 0.2 – 1.5) = – 0.1 kg m/s

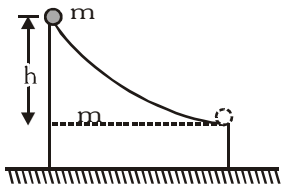
25. Ans. (D)

By conservation of angular momentum : $mv_0R = I\omega \Rightarrow \omega = \frac{(0.1)(1.5)(0.2)}{0.008} = 3.75$ rad/s

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Example#26 to 27

A hollow sphere is released from the top of a wedge, friction is sufficient for pure rolling of sphere on the wedge. There is no friction between the wedge and the ground. Radius of sphere is R. At the instant it leaves the wedge horizontally.



26. Velocity of centre of mass of sphere w.r.t. ground is-

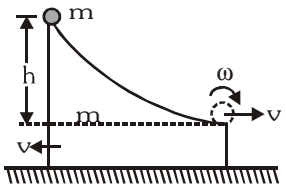
(A) $\sqrt{\frac{5}{7}gh}$ (B) $\sqrt{2gh}$ (C) $\sqrt{\frac{3}{7}gh}$ (D) $\sqrt{\frac{11}{7}gh}$

27. Angular velocity of sphere ω is-

(A) $\sqrt{\frac{12gh}{7R^2}}$ (B) $\sqrt{\frac{27}{7}\frac{gh}{R^2}}$ (C) $\sqrt{\frac{20gh}{7R^2}}$ (D) $\sqrt{\frac{44gh}{7R^2}}$

Solution

26. Ans. (C)



$$R\omega - v = v \Rightarrow \omega = \frac{2v}{R}$$

Enegy conservation

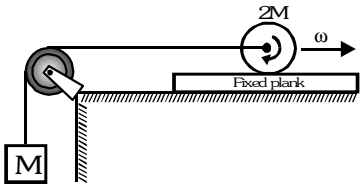
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{3}mR^2\right)\left(\frac{4v^2}{R^2}\right) \Rightarrow v = \sqrt{\frac{3}{7}gh}$$

27. Ans. (A)

$$\omega = \frac{2v}{R} = \sqrt{\frac{4}{R^2} \times \frac{3}{7}gh} = \sqrt{\frac{12gh}{7R^2}}$$

Example#28 to 30

A disc of mass 2 M and radius R is placed on a fixed plank (rough) of length L. The coefficient of friction between the plank and disc is μ = 0.5. String (light) is connected to centre of disc and passing over a smooth light pulley and connected to a block of mass M as shown in the figure. Now the disc is given an angular velocity ω_0 in clockwise direction and is gently placed on the plank. Consider this instant as t=0. Based on above information, answer the following questions :



28. Mark the correct statement w.r.t. motion of block and disc.

(A) The block remains at rest for some time, t > 0.

(B) The block starts accelerating just after placing of disc on plank.

(C) The disc is performing pure rotational motion for some time t > 0

(D) Both (A) and (C) are correct.

29. Time t_0 upto which the block remains stationary is

(A) $\frac{\omega_0 R}{g}$ (B) $\frac{4\omega_0 R}{g}$ (C) Zero (D) Question is irrelevant

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

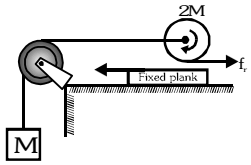
30. Time time t_{01} at which the disc will cross the other end of the plank is-

(A) $\sqrt{\frac{8L}{g}}$ (B) $\frac{\omega_0 R}{g}+\sqrt{\frac{8L}{g}}$ (C) $\sqrt{\frac{8L}{g}}+\frac{4\omega_0 R}{g}$ (D) $\sqrt{\frac{\omega_0 R}{g}}+\sqrt{\frac{4L}{g}}$

Solution

28. Ans.(D)

29. Ans. (A)



Frictional force = μMg

Friction will act up to the instant when the velocity of contact point becomes zero

$\omega_f = \omega_i + \alpha t \Rightarrow t = \frac{\omega_i}{\alpha}$ ....(i) $\alpha = \frac{f_r R}{I}$ ....(ii) By solving (i) and (ii) time $= \frac{\omega_0 R}{g}$

30. Ans. (B)

$Mg - T = Ma$ ...(i) $T - f_r = 2Ma$ ...(ii)

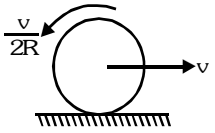
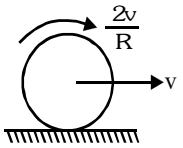
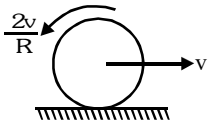
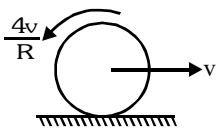
By adding (i) and (ii) $Mg - f_r = 3\,Ma$ ...(iii) $f_r R = I\alpha = \frac{2MR^2}{2}\frac{a}{R} \Rightarrow f_r = Ma$ ...(iv)

From equation (iii) and (iv) $a = g/4 \Rightarrow L = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{8L}{g}}$

Total time = $\frac{\omega_0 R}{g}+\sqrt{\frac{8L}{g}}$

Example#31

31. Four different situations of a moving disc are shown in column I and predictions about its final motion and forces acting on it are given in column - II.

Column I		Column II
(A)		(P) finally disc will roll along the initial direction of velocity (v)
(B)		(Q) finally, disc will roll in direction opposite to the initial direction of velocity (v)
(C)		(R) finally, disc stops
(D)		(S) Initially friction force acts in the direction opposite to that of initial velocity
		(T) None of these

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution **Ans. (A)→(S,P);(B)→(P);(C)→(S,R);(D)→(S,Q)**

Final direction of pure rolling will be in direction of initial angular momentum about point of contact $L = mv_{cm}r + I_{cm}\omega$

For (A) : $mvr - \frac{MR^2}{2}\left(\frac{v}{2R}\right) = \frac{3mvR}{4}$: hence in clockwise direction.

For (B) : $mvR + \frac{mR^2}{2}\left(\frac{2v}{R}\right) = 2mvR$: hence in clockwise direction.

For (C) : $mvR - \frac{mR^2}{2}\left(\frac{2v}{R}\right) = 0$

For (D) : $mvR - \frac{mR^2}{2}\left(\frac{4v}{R}\right) = -mRv$: hence in anticlockwise direction.

Direction of friction force :

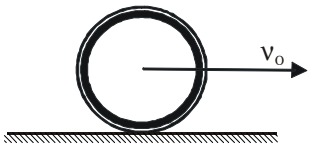
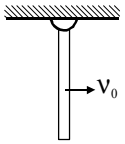
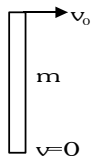
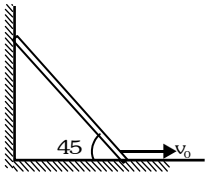
For (A) : Velocity of point of contact $\xrightarrow[v/2]{}$ v ⇒ Friction will be opposite to velocity

For (B) : Velocity of point of contact 2v ← • → v ⇒ Friction will be in the direction of velocity

For (C) : Velocity of point of contact $\xrightarrow{v}$ 2v ⇒ Friction will be opposite to the velocity

For (D) : Velocity of point of contact $\xrightarrow{v}$ 4v ⇒ Friction will be opposite to the velocity

Example#32

	Column I	Column II
(A)	A ring of mass m is projected on rough horizontal plane with velocity v_0. The magnitude of work done by frictional force to start rolling 	(P) $\frac{1}{3}mv_0^2$
(B)	Kinetic energy of pivoted rod of mass m, velocity of centre of mass is v_0. 	(Q) $\frac{1}{8}mv_0^2$
(C)	Kinetic energy of translation of a smooth rod of mass m, where velocity of one end is v_0. 	(R) $\frac{1}{4}mv_0^2$
(D)	Kinetic energy of a rod of mass m, as shown in figure. 	(S) $\frac{2}{3}mv_0^2$ (T) $\frac{1}{9}mv_0^2$

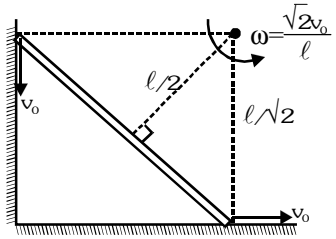
NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Solution **Ans. (A)→(R), (B)→(S), (C)→(Q), (D)→(P)**

For (A) : Final velocity $v = \frac{v_0}{2}$ $\left[\because mv_0R = mR^2\left(\frac{v}{R}\right) + mvR\right]$

So work done by friction = $\frac{1}{2}mv_0^2 - \left[\frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}m\left(\frac{v_0}{2}\right)^2\right] = \frac{1}{4}mv_0^2$

For (B) : $\omega = \frac{v_0}{\ell/2} = \frac{2v_0}{\ell}$ so KE = $\frac{1}{2}\left(\frac{m\ell^2}{3}\right)\left(\frac{2v_0}{\ell}\right)^2 = \frac{2}{3}mv_0^2$

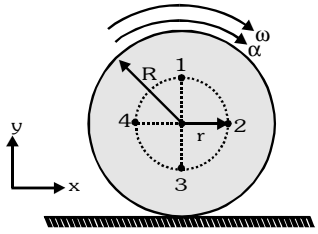


For (C) : $v_{cm} = \frac{v_0}{2}$ so KE = $\frac{1}{2}mv_{cm}^2 = \frac{1}{8}mv_0^2$

For (D) : KE of rod = $\frac{1}{2}I_C\omega^2 = \frac{1}{2}\left(\frac{m\ell^2}{12} + \frac{m\ell^2}{4}\right)\left(\frac{\sqrt{2}v_0}{\ell}\right) = \frac{1}{3}mv_0^2$

Example#33

A disc of radius R is rolling without slipping with an angular acceleration α, on a horizontal plane. Four points are marked at the end of horizontal and vertical diameter of a circle of radius r (<R) on the disc. If horizontal and vertical direction are chosen as x and y axis as shown in the figure, then acceleration of points 1, 2, 3 and 4 are $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ respectively, at the moment when angular velocity of the disc is ω. Match the following



	Column-I		Column-II
(A)	$\vec{a}_1$	(P)	$(R\alpha - r\alpha)\tilde{i} + (r\omega^2)\tilde{j}$
(B)	$\vec{a}_2$	(Q)	$(R\alpha + r\alpha)\tilde{i} - (r\omega^2)\tilde{j}$
(C)	$\vec{a}_3$	(R)	$(R\alpha - r\omega^2)\tilde{i} - (r\alpha)\tilde{j}$
(D)	$\vec{a}_4$	(S)	$(R\alpha + r\omega^2)\tilde{i} + (r\alpha)\tilde{j}$
		(T)	None of these

Solution **Ans. (A) Q; (B) R; (C) P; (D) S**

For (A): Acceleration of 1 w.r.t. centre of mass = $r\alpha\tilde{i} - \omega^2 r\tilde{j} \Rightarrow \vec{a}_1 = r\alpha\tilde{i} - \omega^2 r\tilde{j} + R\alpha\tilde{i} = (R + r)\alpha\tilde{i} - \omega^2 r\tilde{j}$

For (B) : $\vec{a}_2 = -r\alpha\tilde{j} - \omega^2 r\tilde{i} + R\alpha\tilde{i} = (R\alpha - \omega^2 r)\tilde{i} - r\alpha\tilde{j}$

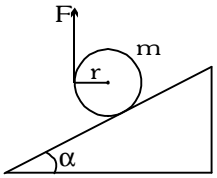
For (C) : $\vec{a}_3 = -r\alpha\tilde{i} + \omega^2 r\tilde{j} + R\alpha\tilde{i} = (R\alpha - r\alpha)\tilde{i} + \omega^2 r\tilde{j}$

For (D) : $\vec{a}_4 = r\alpha\tilde{j} + \omega^2 r\tilde{i} + R\alpha\tilde{i} = (R\alpha + \omega^2 r)\tilde{i} + r\alpha\tilde{j}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

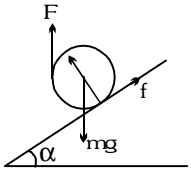
Example#34

A solid uniform cylinder of mass m = 6 kg and radius r = 0.1 m is kept in balance on a slope of inclination α = 37 with the help of a thread fastened to its jacket. The cylinder does not slip on the slope. The minimum required coefficient of friction to keep the cylinder in balance when the thread is held vertically is given as μ. Find the value of 4μ.



Solution **Ans. 3**

$Fr - fr = 0$ and $mg\sin\alpha - F\sin\alpha - f = 0$



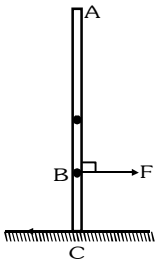
$$F = f = \frac{mg\sin\alpha}{1+\sin\alpha}$$

$F\cos\alpha + N - mg\cos\alpha = 0$

$$N = \left(mg - \frac{mg\sin\alpha}{1+\sin\alpha}\right)\cos\alpha = \frac{mg\cos\alpha}{1+\sin\alpha};\ f_{max} = \mu N;\ \frac{mg\sin\alpha}{1+\sin\alpha} = \frac{\mu mg\cos\alpha}{1+\sin\alpha} \Rightarrow \mu = \tan\alpha = 0.75$$

Example#35

A uniform rod ABC of mass M and length ℓ is placed vertically on a rough horizontal surface. The coefficient of friction between the rod and surface is μ.A force F = 1.2 μmg is applied on the rod at point B at a distance ℓ/3 below centre of rod horizontally as shown in figure. If the initially acceleration of point A is kμ then find value of k. (Friction is sufficient to prevent slipping)



Solution **Ans. 6**

Taking torque about C $F\frac{\ell}{6} = \left(\frac{m\ell^2}{3}\right)\alpha \Rightarrow \alpha = \frac{3}{5}\frac{\mu g}{\ell}$ So, $a_A = \alpha\ell = \frac{3}{5}\mu g = 6\mu$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-3\Rotation\Eng\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 8

GRAVITATION - Theory

Index

GRAVITATION

Serial No.	CONTENTS	Page No.
1.	Newton's Law of Gravitation	1
2.	Gravitational Field	5
3.	Acceleration due to gravity	6
4.	Gravitational Potential	8
5.	Gravitational Potential Energy	9
6.	Escape Velocity	9
7.	Kepler's Laws	11
8.	Satellite Motion	13
9.	Weightlessness	16
10.	**Exercise-I** Check your Grasp (Objective)	25
11.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	28
12.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	32
13.	**Exercise-4(A)** Conceptual Subjective Exercise	37
14.	**Exercise-4(B)** Brain Storming Exercise	39
15.	**Exercise-5(A)** Previous years questions	42
16.	**Exercise-5(B)** Previous years questions	44

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS

JEE Main : The universal law of gravitation. Acceleration due to gravity and its variation with altitude and depth. Kepler's laws of planetary motion. Gravitational potential energy; gravitational potential. Escape velocity. Orbital velocity of a satellite. Geo-stationary satellites.

JEE Advanced : Law of gravitation; Gravitational potential and field; Acceleration due to gravity; Motion of planets and satellites in circular orbits; Escape velocity.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

GRAVITATION

THE DISCOVERY OF THE LAW OF GRAVITATION

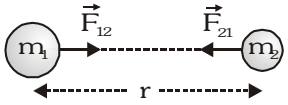
The way the law of universal gravitation was discovered is often considered the paradigm of modern scientific technique. The major steps involved were

- The hypothesis about planetary motion given by **Nicolaus Copernicus (1473–1543)**.
- The careful experimental measurements of the positions of the planets and the Sun by **Tycho Brahe (1546–1601)**.
- Analysis of the data and the formulation of empirical laws by **Johannes Kepler (1571–1630)**.
- The development of a general theory by **Isaac Newton (1642–1727)**.

NEWTON'S LAW OF GRAVITATION

It states that every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

$F \propto m_1 m_2$ and $F \propto \frac{1}{r^2}$ so $F \propto \frac{m_1 m_2}{r^2}$



$\therefore F = \frac{Gm_1m_2}{r^2}$ [G = Universal gravitational constant]

Note : This formula is only applicable for spherical symmetric masses or point masses.

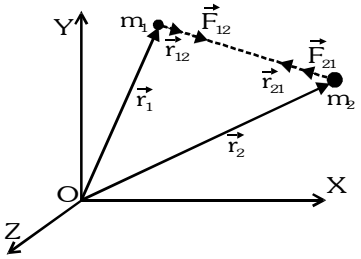
Vector form of Newton's law of Gravitation :

Let $\vec{r}_{12}$ = Displacement vector from m_1 to m_2

$\vec{r}_{21}$ = Displacement vector from m_2 to m_1

$\vec{F}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{F}_{12}$ = Gravitational force exerted on m_1 by m_2



$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21} = -\frac{Gm_1m_2}{r_{21}^3}\vec{r}_{21}$$

Negative sign shows that :

(i) The direction of $\vec{F}_{12}$ is opposite to that $\hat{r}_{21}$

(ii) The gravitational force is attractive in nature

Similarly $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$ or $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12}$ $\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

The gravitational force between two bodies are equal in magnitude and opposite in direction.

GRAVITATIONAL CONSTANT "G"

- Gravitational constant is a scalar quantity.
- **UNIT : S I :** G = 6.67 × 10^{-11} N–m^2/kg^2 **CGS :** 6.67 × 10^{-8} dyne–cm^2/g^2 **Dimensions :** $[M^{-1}L^3T^{-2}]$
- Its value is same throughout the universe, G does not depend on the nature and size of the bodies, it also does not depend upon nature of the medium between the bodies.
- Its value was first find out by the scientist **"Henry Cavendish"** with the help of "Torsion Balance" experiment.
- Value of G is small therefore gravitational force is weaker than electrostatic and nuclear forces.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Example

Two particles of masses 1 kg and 2 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial acceleration of heavier particle.

Solution

Force exerted by one particle on another $F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.34\times10^{-10}\,N$

Acceleration of heavier particle = $\frac{F}{m_2} = \frac{5.3\times10^{-10}}{2} = 2.67\times10^{-10}\,ms^{-2}$

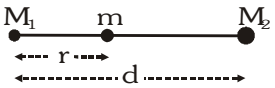
This example shows that gravitation is very weak but only this force keep bind our solar system and also this universe of all galaxies and other interstellar system.

Example

Two stationary particles of masses M_1 and M_2 are at a distance 'd' apart. A third particle lying on the line joining the particles, experiences no resultant gravitational forces. What is the distance of this particle from M_1.

Solution

The force on m towards M_1 is $F_1 = \frac{GM_1m}{r^2}$



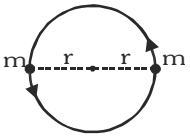
The force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

According to question net force on m is zero i.e. $F_1 = F_2$

$$\Rightarrow \frac{GM_1m}{r^2} = \frac{GM_2m}{(d-r)^2} \Rightarrow \left(\frac{d-r}{r}\right)^2 = \frac{M_2}{M_1} \Rightarrow \frac{d}{r} - 1 = \frac{\sqrt{M_2}}{\sqrt{M_1}} \Rightarrow r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right]$$

Example

Two particles of equal mass (m) each move in a circle of radius (r) under the action of their mutual gravitational attraction. Find the speed of each particle.



Solution

For motion of particle $\frac{mv^2}{r} = \frac{Gmm}{(2r)^2} \Rightarrow v^2 = \frac{Gm}{4r} \Rightarrow v = \frac{1}{2}\sqrt{\frac{Gm}{r}}$

Example

Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side 'a'. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining their original separation 'a'. Determine the initial velocity that should be given to each particle and time period of circular motion.

Solution

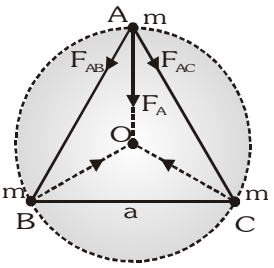
The resultant force on particle at A due to other two particles is

$$F_A = \sqrt{F_{AB}^2 + F_{AC}^2 + 2F_{AB}F_{AC}\cos60^\circ} = \sqrt{3}\frac{Gm^2}{a^2} \ldots(i) \quad \left[\because F_{AB} = F_{AC} = \frac{Gm^2}{a^2}\right]$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p5

Radius of the circle $r = \frac{2}{3} \times a \sin 60° = \frac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v, so that F acts as the centripetal force,



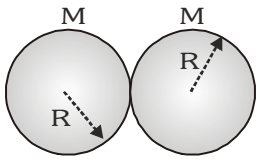
Now $\frac{mv^2}{r} = \sqrt{3}\frac{mv^2}{a}$ ...(ii)

From (i) and (ii) $\sqrt{3}\frac{mv^2}{a} = \frac{Gm^2\sqrt{3}}{a^2} \Rightarrow v = \sqrt{\frac{Gm}{a}}$

Time period $T = \frac{2\pi r}{v} = \frac{2\pi a}{\sqrt{3}}\sqrt{\frac{a}{Gm}} = 2\pi\sqrt{\frac{a^3}{3Gm}}$

Example

Two solid sphere of same size of a metal are placed in contact by touching each other. Prove that the gravitational force acting between them is directly proportional to the fourth power of their radius.



Solution

The weights of the spheres may be assumed to be concentrated at their centres.

So $F = \frac{G\left[\frac{4}{3}\pi R^3\rho\right] \times \left[\frac{4}{3}\pi R^3\rho\right]}{(2R)^2} = \frac{4}{9}(G\pi^2\rho^2)R^4 \quad \therefore F \propto R^4$

Example

A mass (M) is split into two parts (m) and (M–m). Which are then separated by a certain distance. What ratio $\frac{m}{M}$ will maximise the gravitational force between the parts ?

Solution

If r is the distance between m and (M – m), the gravitational force will be $F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(mM - m^2)$

For F to be maximum $\frac{dF}{dm} = 0$ as M and r are constant, i.e. $\frac{d}{dm}\left[\frac{G}{r^2}(mM - m^2)\right] = 0$ i.e. M – 2m = 0 $\left[\because \frac{G}{r^2} \neq 0\right]$

or $\frac{m}{M} = \frac{1}{2}$, i.e., the force will be maximum when the two parts are equal.

Example

A thin rod of mass M and length L is bent in a semicircle as shown in figure. (a) What is its gravitational force (both magnitude and direction) on a particle with mass m at O, the centre of curvature? (b) What would be the force on m if the rod is, in the form of a complete circle?

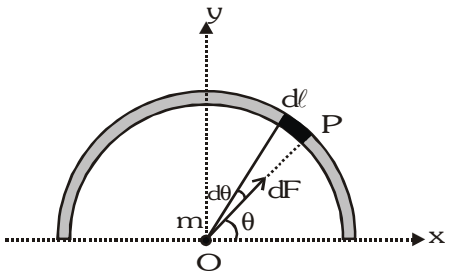
Solution

(a) Considering an element of rod of length $d\ell$ as shown in figure and treating it as a point of mass (M/L) $d\ell$ situated at a distance R from P, the gravitational force due to this element on the particle will be

$dF = \frac{Gm(M/L)(Rd\theta)}{R^2}$ along OP [as $d\ell = Rd\theta$]

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

So the component of this force along x and y–axis will be



$$dF_x = dF\cos\theta = \frac{GmM\cos\theta\, d\theta}{LR};\ dF_y = dF\sin\theta = \frac{GmM\sin\theta\, d\theta}{LR}$$

So that $F_x = \frac{GmM}{LR}\int_0^{\pi}\cos\theta\, d\theta = \frac{GmM}{LR}\left[\sin\theta\right]_0^{\pi} = 0$

and $F_y = \frac{GmM}{LR}\int_0^{\pi}\sin\theta\, d\theta = \frac{GmM}{LR}\left[-\cos\theta\right]_0^{\pi} = \frac{2\pi GmM}{L^2}\left[\text{as } R = \frac{L}{\pi}\right]$

So $F = \sqrt{F_x^2 + F_y^2} = F_y = \frac{2\pi GmM}{L^2}$ [as F_x is zero]

i.e., the resultant force is along the y–axis and has magnitude ($2\pi GmM/L^2$)

(b) If the rod was bent into a complete circle,

$$F_x = \frac{GmM}{LR}\int_0^{2\pi}\cos\theta\, d\theta = 0 \text{ and also } F_y = \frac{GmM}{LR}\int_0^{2\pi}\sin\theta\, d\theta = 0$$

i.e, the resultant force on m at O due to the ring is zero.

GOLDEN KEY POINT

- Gravitational force is always attractive.
- Gravitational forces are developed in form of action and reaction pair. Hence obey Newton's third law of motion.
- It is independent of nature of medium in between two masses and presence or absence of other bodies.
- Gravitational forces are central forces as they act along the line joining the centres of two bodies.
- The gravitational forces are conservative forces so work done by gravitational force does not depends upon path.
- If any particle moves along a closed path under the action of gravitational force then the work done by this force is always zero.
- Gravitational force is weakest force of nature.
- Force developed between any two masses is called gravitational force and force between Earth and any body is called gravity force.
- The total gravitational force on one particle due to number of particles is the resultant of forces of attraction exerted on the given particle due to individual particles i.e. $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots\ldots\ldots$ It means the principle of superposition is valid.
- Gravitational force holds good over a wide range of distances. It is found the true from interplanetary distances to interatomic distances.
- It is a two body interaction i.e. gravitational force between two particles is independent of the presence or absence of other particles.
- A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its centre.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

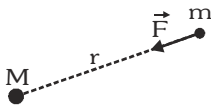
GRAVITATIONAL FIELD



The gravitational field is the space around a mass or an assembly of masses over which it can exert gravitational forces on other masses.

Theoretically speaking, the gravitational field extends up to infinity. However, in actual practice, the gravitational field may become too weak to be measured beyond a particular distance.

Gravitational Field Intensity [I_g or E_g]

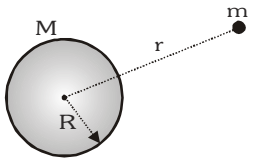


Gravitational force acting per unit mass at any point in the gravitational field is called Gravitational field intensity.

$$I_g = \left(\frac{GMm}{r^2}\Big/ m\right) = \frac{GM}{r^2} \qquad \text{Vector form} : \vec{I}_g = \frac{\vec{F}}{m} \quad \text{or} \quad \vec{I}_g = -\frac{GM}{r^2}\hat{r}$$

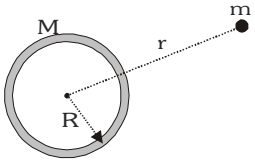
GOLDEN KEY POINT

- Gravitational field intensity is a vector quantity having dimension [LT^{-2}] and unit N/kg.
- As by definition $\vec{I}_g = \frac{\vec{F}}{m}$ i.e. $\vec{F} = m\vec{I}_g$ so force on a point mass (m) is multiplication of intensity of field and mass of point mass.
- **Solid Sphere**



- r > R $\quad I_{out} = \frac{GM}{r^2}$
- r = R $\quad I_{sur} = \frac{GM}{R^2}$
- r < R $\quad I_{inside} = \frac{GMr}{R^3} = \frac{4}{3}\pi G r \rho$

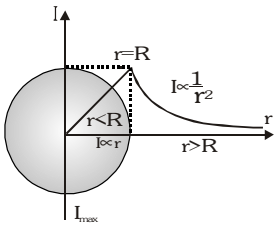
Spherical shell



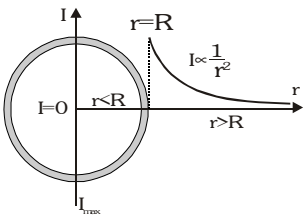
- r > R $\quad I_{out} = \frac{GM}{r^2}$
- r = R $\quad I_{sur} = \frac{GM}{R^2}$
- r < R $\quad I_{inside} = 0$

- **Graph between (I_g) and (r)**

For Solid Sphere



For Spherical shell



NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p6.5

GRAVITY

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is Earth then the gravitation is called 'gravity'. Hence, gravity is the force by which Earth attracts a body towards its centre. It is a special case of gravitation.

Gravitation near Earth's surface

Let us assume that Earth is a uniform sphere of mass M and radius R. The magnitude of the gravitational force from Earth on a particle of mass m, located outside Earth at a distance r from Earth is centre, is $F = \frac{GMm}{r^2}$

Now according to Newton's second laws $F = ma_g$

Therefore $a_g = \frac{GM}{r^2}$ ...(i)

At the surface of Earth, acceleration due to gravity $g = \frac{GM}{R^2} = 9.8\ m/s^2$

However any a_g value measured at a given location will differ from the a_g value calculated with equation (i) due to three reasons

(i) Earth's mass is not distributed uniformly.

(ii) Earth is not a perfect sphere and

(iii) Earth rotates

GOLDEN KEY POINT

- In form of density $g = \frac{GM_e}{R_e^2} = \frac{G}{R_e^2} \times \frac{4}{3}\pi R_e^3 \times \rho \quad \therefore g = \frac{4}{3}\pi G R_e \rho$

  If ρ is constant then $g \propto R_e$
- If M is constant $\therefore \ g \propto \frac{1}{R^2}$ % variation in 'g' (upto 5%) $\frac{\Delta g}{g} = -2\left(\frac{\Delta R_e}{R_e}\right)$
- If mass (M) and radius (R) of a planet, if small change is occurs in (M) and (R) then

  by $g = \frac{GM}{R^2}; \frac{\Delta g}{g} = \frac{\Delta M}{M} - 2\frac{\Delta R_e}{R_e}$

  If R is constant $\frac{\Delta g}{g} = \frac{\Delta M}{M}$ and if M is constant $\frac{\Delta g}{g} = -2\left(\frac{\Delta R_e}{R_e}\right)$

VARIATION IN ACCELERATION DUE GRAVITY

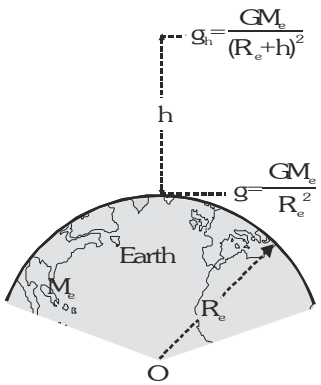
Due to Altitude (height)

From diagram

$$\frac{g_h}{g} = \frac{R_e^2}{(R_e+h)^2} = \frac{R_e^2}{R_e^2\left[1+\frac{h}{R_e}\right]^2} = \left(1+\frac{h}{R_e}\right)^{-2}$$

By binomial expansion $\left(1+\frac{h}{R_e}\right)^{-2} \approx \left(1-\frac{2h}{R_e}\right)$

[If $h << R_e$, then higher power terms are negligible] $\therefore \ g_h = g\left[1-\frac{2h}{R_e}\right]$



ALLEN Materials Provided By - Material Point Available on Learnaf.com

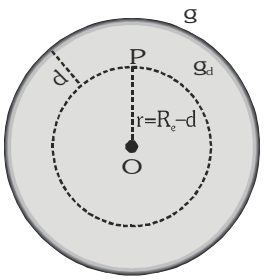
Example

Two equal masses m and m are hung from a balance whose scale pans differ in vertical height by 'h'. Determine the error in weighing in terms of density of the Earth ρ.

Solution

$$g_h = g\left[1-\frac{2h}{R_e}\right] \quad , \quad W_2-W_1 = mg_2-mg_1 = 2mg\left[\frac{h_1}{R_e}-\frac{h_2}{R_e}\right] = 2m\frac{GM}{R_e^2}\times\frac{h}{R_e}\left[\because g=\frac{GM}{R_e^2} \ \& \ h_1-h_2=h\right]$$

Error in weighing = $W_2 - W_1 = 2mG\frac{4}{3}\pi R_e^3\rho\frac{h}{R_e^3} = \frac{8\pi}{3}Gm\rho h$



Due to depth : Assuming density of Earth remains same throughout.

At Earth surface : $g = \frac{4}{3}\pi GR_e\rho$ ...(i) At depth d inside the Earth :

For point P only mass of the inner sphere is effective $g_d = \frac{GM'}{r^2}$

$$g_d = \frac{G}{r^2}\times\frac{Mr^3}{R_e^3} = \frac{GM}{R_e^2}\times\frac{r}{R_e} = \frac{GM}{R_e^2}\times\frac{R_e-d}{R_e}$$

$$g_d = g\left[1-\frac{d}{R_e}\right] \quad \text{valid for any depth}$$

Mass of sphere of radius r = M'

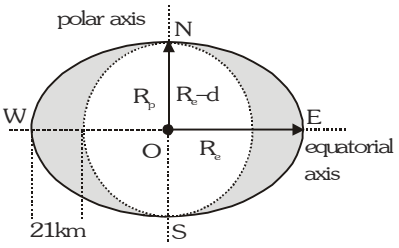
$$M' = \frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi r^3\times\frac{M}{4/3\pi R^3} = M' = \frac{M}{R^3}r^3$$

Example

At which depth from Earth surface, acceleration due to gravity is decreased by 1%

Solution

$$\frac{\Delta g_d}{g} = \frac{d}{R_e} \quad \Rightarrow \quad \frac{1}{100} = \frac{d}{6400} \quad \therefore \quad d = 64 \text{ km}$$



Due to shape of the Earth

From diagram

$R_p < R_e$ ($R_e = R_p + 21$ km) $g_p = \frac{GM_e}{R_p^2}$ & $g_e = \frac{GM_e}{(R_p+21)^2}$ $\therefore$ $g_e < g_p$

$\Rightarrow$ by putting the values $g_p - g_e = 0.02$ m/s^2

Due to Rotation of the Earth

Net force on particle at P $mg' = mg - mr\omega^2\cos\lambda$

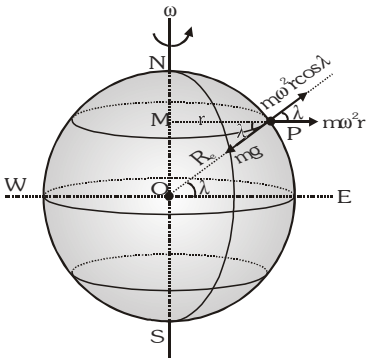
$g' = g - r\omega^2\cos\lambda$ from ΔOMP $r = R_e\cos\lambda$

Substituting value of r $g' = g - R_e\omega^2\cos^2\lambda$

If latitude angle $\lambda = 0$. It means at equator. $g'_{min} = g_e = g - R_e\omega^2$

If latitude angle $\lambda = 90$. it means at poles. $g'_{max.} = g_p = g \Rightarrow g_p > g_e$

Change in "g" only due to rotation $\Delta g_{rot.} = g_p - g_e = 0.03$ m/s^2



$$\Delta g_{total} = g_p - g_e = (0.05 \text{ m/s}^2) \rightarrow \begin{cases} 0.02 \text{ m/s}^2 & \text{(due to shape)} \\ 0.03 \text{ m/s}^2 & \text{(due to rotation)} \end{cases}$$

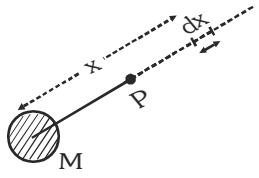
If rotation of Earth suddenly stops then acceleration due to gravity is increases at all places on Earth except the poles.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p6.5

GRAVITATIONAL POTENTIAL

Gravitational field around a material body can be described not only by gravitational intensity $\vec{I}_g$, but also by a scalar function, the gravitational potential V. Gravitational potential is the amount of work done in bringing a body of unit mass from infinity to that point without changing its kinetic energy. $V = \frac{W}{m}$

Gravitational force on unit mass at (A) will be = $\frac{GM(1)}{x^2} = \frac{GM}{x^2}$



Work done by this force , through the distance (AB) $dW = Fdx = \frac{GM}{x^2} . dx$

Total work done in bringing the body of unit mass from infinity to point (P) $W = \int_{\infty}^{r} \frac{GM}{x^2} dx = - \left(\frac{GM}{x}\right)_{\infty}^{r} = - \frac{GM}{r}$

This work done is the measure of gravitational potential at point (P) $\therefore V_P = - \frac{GM}{r}$

- If $r = \infty$ then $v_\infty = 0$. Hence gravitational potential is maximum at infinite (as it is negative quantity at point P)
- If $r = R_e$ at the surface of Earth $V_S = - \frac{GM_e}{R_e}$

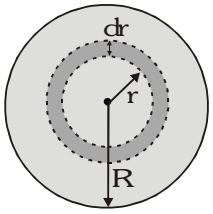
GOLDEN KEY POINTS

- Gravitational potential is a scalar quantity and its unit and its dimensions are J/kg and $[L^2T^{-2}]$
- $V = - \frac{W}{m} = - \int \frac{\vec{F}.\vec{dr}}{m}$ $\left[As \frac{\vec{F}}{m} = \vec{I}\right]$, $V = - \int \vec{I}.\vec{dr} \Rightarrow dV = - \vec{I} . dr \therefore I = - \frac{dV}{dr}$ = –ve potential gradient

	Solid Sphere	Hollow Spherical shell
Case I	r > R (outside the sphere)	r > R (outside the sphere)
	$V_{out} = - \frac{GM}{r}$	$V_{out} = - \frac{GM}{r}$
Case II	r = R (on the surface)	r = R (on the surface)
	$V_{surface} = - \frac{GM}{R}$	$V_{surface} = - \frac{GM}{R}$
Case III	r < R (inside the sphere)	r < R (Inside the sphere)
	$V_{in} = - \frac{GM}{2R^3} [3R^2 - r^2]$	Potential every where is same and equal
	It is clear that the potential |V| will be maximum at the centre (r = 0)	to its value at the surface $V_{in} = - \frac{GM}{R}$
	$\|V_{centre}\| = \frac{3}{2}\frac{GM}{R}$, $V_{centre} = \frac{3}{2} V_{surface}$	

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p6.5

Gravitational Self Energy of a Uniform Sphere



Consider a sphere of radius R and mass M uniformly distributed. Consider a stage of formation at which the radius of the spherical core is r. Its mass will be $\frac{4}{3}\pi r^3\rho$, where, ρ is the density of the sphere. Let, through an additional mass, the radius of the core be increased by dr in the form of spherical layer over the core.

The mass of this layer will be $4\pi r^2 dr.\rho$. The mutual gravitational potential energy of the above mass and the spherical core of radius r

$$dU = \frac{-G\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2 dr.\rho\right)}{r} = -\frac{16}{3}\pi^2\rho^2 G r^4 dr$$

Hence, total energy involved in the formation of the spherical body of radius R i.e. self energy

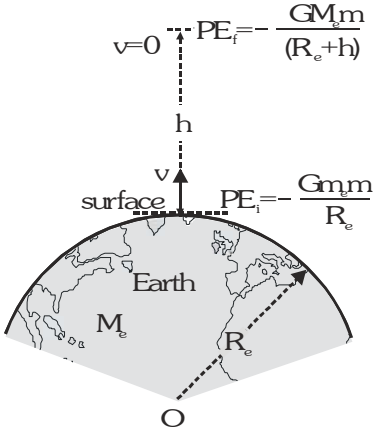
$$U = \int_0^R \frac{16}{3}\pi^2\rho^2 G r^4 dr = \frac{16}{15}\pi^2 G R^5 = \frac{3}{5}\frac{GM^2}{R}$$

GRAVITATIONAL POTENTIAL ENERGY

Work done by Gravitational force in shifting a mass from one place to another place. $W = U = -\frac{GMm}{r}$

(Here negative sign shows the boundness of the two bodies)

(Velocity (v) required to project a body till height "h")



by conservation energy $KE_i + PE_i = KE_f + PE_f$

$$\frac{1}{2}mv^2 + \left[-\frac{GM_em}{R_e}\right] = 0 + \left[-\frac{GM_em}{R_e+h}\right]$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{GM_em}{R_e} - \frac{GM_em}{R_e+h} = GM_em\left[\frac{1}{R_e} - \frac{1}{R_e+h}\right]$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{GM_emh}{R_e(R_e+h)} \Rightarrow v^2 = \frac{2GM_eh}{R_e^2\left(1+\frac{h}{R_e}\right)} \Rightarrow \quad v^2 = \frac{2gh}{1+\frac{h}{R_e}}$$

Maximum height reached by the body projected by velocity "v" from the Earth surface $h = \frac{v^2R_e}{2gR - v^2}$

If reference point is taken at Earth surface then U = mgh

ESCAPE VELOCITY (v_e)

It is the minimum velocity required for an object at Earth's surface so that it just escapes the Earth's gravitational field.

Consider a projectile of mass m, leaving the surface of a planet (or some other astronomical body or system), of radius R and mass M with esape speed v_e.

Projectile reaches inifinitely, it has no kinetic energy and no potential energy.

From conservation of mechanical energy $\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) = 0 + 0 \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Escape speed depends on :

(i) Mass (M) and size (R) of the planet

(ii) Position from where the particle is projected.

Escape speed does not depend on :

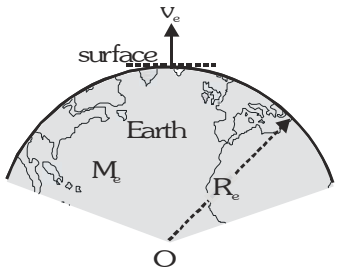
(i) Mass of the body which is projected (m)

(ii) Angle of projection.

If a body is thrown from Earth's surface with escape speed, it goes out of earth's gravitational field and never returns to the earth's surface.

Escape energy

Minimum energy given to a particle in form of kinetic energy so that it can just escape from Earth's gravitational field.



Magnitude of escape energy = $\frac{GMm}{R}$ (–ve of PE of Earth's surface)

Escape energy = Kinetic Energy $\Rightarrow \frac{GMm}{R} = \frac{1}{2}mv_e^2$

GOLDEN KEY POINT

- $v_e = \sqrt{\frac{2GM}{R}}$ (In form of mass) If M = constant $v_e \propto \frac{1}{\sqrt{R}}$
- $v_e = \sqrt{2gR}$ (In form of g) If g = constant $v_e \propto \sqrt{R}$
- $v_e = R\sqrt{\frac{8\pi G.\rho}{3}}$ (In form of density) If ρ = constant $v_e \propto R$
- Escape velocity does not depend on mass of body, angle of projection or direction of projection.
  $v_e \propto m^0$ and $v_e \propto \theta$
- Escape velocity at : Earth surface v_e = 11.2 km/s Moon surface v_e = 2.31 km/s
- Atmosphere on Moon is absent because root mean square velocity of gas particle is greater then escape velocity. $v_{rms} > v_e$

Example

A space–ship is launched into a circular orbit close to the Earth's surface. What additional speed should now be imparted to the spaceship so that orbit to overcome the gravitational pull of the Earth.

Solution

Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitation pull then

$$K = \frac{GMm}{2R}$$

Total kinetic energy = $\frac{GMm}{2R} + \Delta K = \frac{GMm}{2R} + \frac{GMm}{2R} = \frac{GMm}{R}$ then $\frac{1}{2}mv_2^2 = \frac{GMm}{R} \Rightarrow v_2 = \sqrt{\frac{2GM}{R}}$

But $v_1 = \sqrt{\frac{GM}{R}}$. So Additional velocity = $v_2 - v_1 = \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}} = \left(\sqrt{2}-1\right)\sqrt{\frac{GM}{R}}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Example

If velocity given to an object from the surface of the Earth is n times the escape velocity then what will be the residual velocity at infinity.

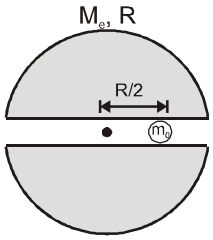
Solution

Let residual velocity be v then from energy conservation $\frac{1}{2}\ m\ (nv_e)^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 + 0$

$\Rightarrow v^2 = n^2v_e^{\ 2} - \frac{2GM}{R} = n^2v_e^{\ 2} - v_e^{\ 2} = (n^2 - 1)\ v^2_e \Rightarrow v = \left(\sqrt{n^2-1}\right)v_e$

Example

A very small groove is made in the earth, and a particle of mass m_0 is placed at $\frac{R}{2}$ distance from the centre. Find the escape speed of the particle from that place.



Solution

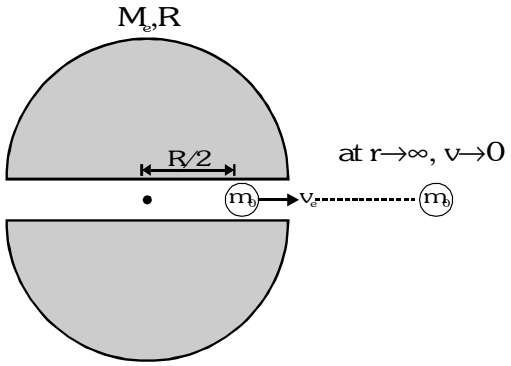
Suppose we project the particle with speed v_e, so that it just reaches at $(r \to \infty)$.

Applying energy conservation

$K_i + U_i = K_f + U_f$

$$\frac{1}{2}m_0v_e^2 + m_0\left[-\frac{GM_e}{2R^3}\left\{3R^2 - \left(\frac{R}{2}\right)^2\right\}\right] = 0$$

$$\Rightarrow v_e = \sqrt{\frac{11GM_e}{4R}}$$



KEPLER'S LAWS

Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

(a) First Law (Law of Orbits) :

All planets move around the Sun in elliptical orbits, having the Sun at one focus of the orbit.

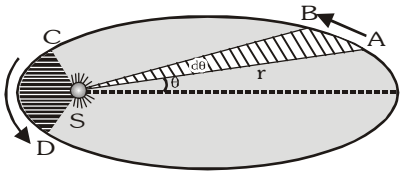
(b) Second Law (Law of Areas) :

A line joining any planet to the Sun sweeps out equal areas in equal times, that is, the areal speed of the planet remains constant.

According to the second law, when the planet is nearest to the Sun, then its speed is maximum and when it is farthest from the Sun, then its speed is minimum. In figure if a planet moves from A to B in a given time–interval, and from C to D in the same time–interval, then the areas ASB and CSD will be equal.

dA = area of the curved triangle SAB $= \frac{1}{2}(AB \times SA) = \frac{1}{2}(r\,d\theta \times r) = \frac{1}{2}r^2\,d\theta$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p6.5



Thus, the instantaneous areal speed of the planet is $\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}r^2\omega$ ...(i)

where ω is the angular speed of the planet.

Let J be the angular momentum of the planet about the Sun S and m the mass of the planet.

Then $J = I\omega = mr^2\omega$, ...(ii)

where I ($=mr^2$) is the instantaneous moment of inertia of the planet about the Sun S.

From eq. (i) and (ii), $\frac{dA}{dt}=\frac{J}{2m}$ ...(iii)

Now, the areal speed dA/dt of the planet is constant, according to Kepler's second law. Therefore, according to eq. (iii), the angular momentum J of the planet is also constant, that is, the angular momentum of the planet is conserved. Thus, Kepler's second law is equivalent to conservation of angular momentum.

(c) **Third Law : (Law of Periods) :** The square of the period of revolution (time of one complete revolution) of any planet around the Sun is directly proportional to the cube of the semi–major axis of its elliptical orbit.

Proof : If a and b are the semimajor and the semi–minor axes of the ellipse, then the area of the ellipse will be π ab. Hence if T be the period of revolution of the planet, then

$$T=\frac{\text{area of the ellipse}}{\text{areal speed}}=\frac{\pi ab}{J/2m} \quad \text{or} \quad T^2=\frac{4\pi^2m^2a^2b^2}{J^2}$$

Let ℓ be the semi–latus rectum of the elliptical orbit. Then $\ell=\frac{b^2}{a}$ $\therefore$ $T^2=\frac{4\pi^2m^2a^3\ell}{J^2}$ or $T^2 \propto a^3$

As all the other quantities are constant. So it is clear through this rule that the farthest planet from the Sun has largest period of revolution. The period of revolution of the closest planet Mercury is 88 days, while that of the farthest dwarf planet Pluto is 248 years.

NEWTON'S CONCLUSIONS FROM KEPLER'S LAWS

Newton found that the orbits of most of the planets (except Mercury and Pluto) are nearly circular. According to Kepler's second law, the areal speed of a planet remains constant. This means that in a circular orbit the linear speed of the planet (v) will be constant. Since the planet is moving on a circular path; it is being acted upon by a centripetal force directed towards the centre (Sun). This force is given by $F = mv^2/r$ where m is the mass of the planet, v is its linear speed and r is the radius of its circular orbit. If T is the period of revolution of the planet, then $v=\frac{\text{Linear distance travellel in one revolution}}{\text{Period of revolution}}=\frac{2\pi r}{T}$

$\therefore$ $F=\frac{m}{r}\left(\frac{2\pi r}{T}\right)^2=\frac{4\pi^2mr}{T^2}$ But, for circular orbit, according to Kepler's third law,

$T^2 = Kr^3$, where K is some constant. $\therefore$ $F=\frac{4\pi^2mr}{Kr^3}=\frac{4\pi^2}{K}\left(\frac{m}{r^2}\right)$ or $F \propto m/r^2$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p5.5

Example

Two satellites S_1 and S_2 are revolving round a planet in coplanar and concentric circular orbit of radii R_1 and R_2 in the same direction respectively. Their respective periods of revolution are 1 hr and 8 hr. The radius of the orbit of satellite S_1 is equal to 10^4 km. Find the relative speed in kmph when they are closest.

Solution

By Kepler's 3rd law, $\frac{T^2}{R^3}$ = constant $\quad \therefore \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3}$ or $\frac{1}{(10^4)^3} = \frac{64}{R_2^3}$ or $R_2 = 4 \quad 10^4$ km

Distance travelled in one revolution, $S_1 = 2\pi R_1 = 2\pi \quad 10^4$ and $S_2 = 2\pi R_2 = 2\pi \quad 4 \quad 10^4$

$$v_1 = \frac{S_1}{t_1} = \frac{2\pi \times 10^4}{1} = 2\pi \times 10^4 \text{ kmph} \quad \text{and} \quad v_2 = \frac{S_2}{t_2} = \frac{2\pi \times 4 \times 10^4}{8} = \pi \times 10^4 \text{ kmph}$$

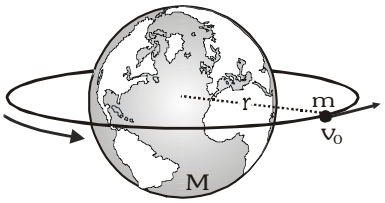
$\therefore$ Relative velocity = $v_1 - v_2 = 2\pi \quad 10^4 - \pi \quad 10^4 = \pi \quad 10^4$ kmph

SATELLITE MOTION

A light body revolving round a heavier body due to gravitational attraction, is called satellite. Earth is a satellite of the Sun while Moon is satellite of Earth.

Orbital velocity (v_0) : A satellite of mass m moving in an orbit of radius r with speed v_0 then required centripetal force is provided by gravitation.

$$F_{cp} = F_g \Rightarrow \frac{mv_0^2}{r} = \frac{GMm}{r^2} \Rightarrow v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R_e + h)}} \quad (r = R_e + h)$$



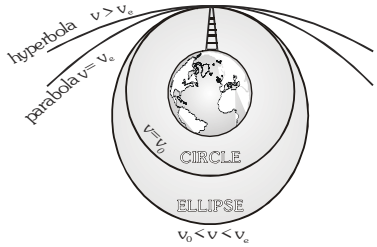
For a satellite very close to the Earth surface $h << R_e \therefore r = R_e$

$$v_0 = \sqrt{\frac{GM}{R_e}} = \sqrt{gR_e} = 8 \text{ km/s}$$

- If a body is taken at some height from Earth and given horizontal velocity of magnitude 8 km/sec then the body becomes satellite of Earth.
- v_o depends upon : Mass of planet, Radius of circular orbit of satellite, g (at planet), Density of planet
- If orbital velocity of a near by satellite becomes $\sqrt{2}\, v_o$ (or increased by 41.4%, or K.E. is doubled) then the satellite escapes from gravitational field of Earth.

Bound and Unbound Trajectories

Imagine a very tall tower on the surface of Earth where from a projectile is fired with a velocity v parallel to the surface of Earth. The trajectory of the projectile depends on its velocity.



Velocity	Trajectory
$v < \frac{v_0}{\sqrt{2}}$	Projectile does not orbit the Earth. It falls back on the Earth's surface.
$v = \frac{v_0}{\sqrt{2}}$	Projectile orbits the Earth in a circular path.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

$\frac{v_0}{\sqrt{2}} < v < v_e$ Projectile orbits in an elliptical path.

$v = v_e$ Projectile does not orbit. It escapes the gravitational field of Earth in a parabolic path.

$v > v_e$ Projectile does not orbit. It escape the gravitational field of Earth in a hyperbolic path.

Time Period of a Satellite $T = \frac{2\pi r}{v_0} = \frac{2\pi r^{3/2}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{R\sqrt{g}} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow T^2 \propto r^3 (r = R + h)$

For Geostationary Satellite T = 24 hr, h = 36,000 km $\sqcup$ 6 R_e ($r \sqcup 7\ R_e$), v_0 = 3.1 km/s

For Near by satellite $v_0 = \sqrt{\frac{GM_e}{R_e}} \sqcup 8$ km/s

$$T_{Ns} = 2\pi\sqrt{\frac{R_e}{g}} = 84 \text{ minute} = 1 \text{ hour } 24 \text{ minute} = 1.4 \text{ hr} = 5063 \text{ s}$$

In terms of density $T_{Ns} = \frac{2\pi(R_e)^{1/2}}{(G \times 4/3\,\pi R_e \times \rho)^{1/2}} = \sqrt{\frac{3\pi}{G\rho}}$

Time period of near by satellite only depends upon density of planet.

For Moon h_m = 380,000 km and T_m = 27 days

$$v_{om} = \frac{2\pi(R_e + h)}{T_m} = \frac{2\pi(386400 \times 10^3)}{27 \times 24 \times 60 \times 60} \sqcup 1.04 \text{ km/sec.}$$

Energies of a Satellite

Kinetic energy $K.E. = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2}$

Potential energy $P.E. = -\frac{GMm}{r} = -mv_0^2 = -\frac{L^2}{mr^2}$

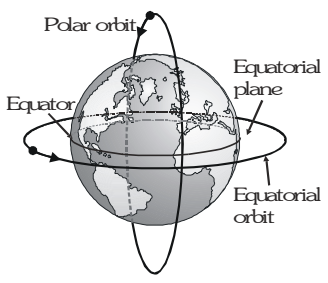
Total mechanical energy $T.E. = P.E. + K.E. = -\frac{mv_0^2}{2} = -\frac{GMm}{2r} = -\frac{L^2}{2mr^2}$

Essential Condition's for Satellite Motion

- Centre of satellite's orbit should coincide with centre of the Earth.
- Plane of orbit of satellite should pass through centre of the Earth.

Special Points about Geo–Stationary Satellite

- It rotates in equatorial plane.
- Its height from Earth surface is 36000 km. ($\tilde{}6R_e$)
- Its angular velocity and time period should be same as that of Earth.
- Its rotating direction should be same as that of Earth (West to East).
- Its orbit is called parking orbit and its orbital velocity is 3.1 km./sec.

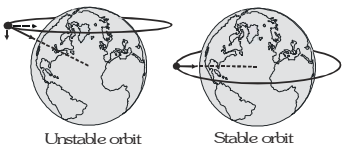


Polar Satellite (Sun – synchronous satellite)

It is that satellite which revolves in polar orbit around Earth. A polar orbit is that orbit whose angle of inclination with equatorial plane of Earth is 90 and a satellite in polar orbit will pass over both the north and south geographic poles once per orbit. Polar satellites are Sun–synchronous satellites.

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p5

The polar satellites are used for getting the cloud images, atmospheric data, ozone layer in the atmosphere and to detect the ozone hole over Antarctica.



BINDING ENERGY

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is bound or different parts of the system are bound to each other.

Binding energy of satellite (system)

$$\text{B.E.} = -\text{T.E.} \qquad \text{B.E.} = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2} \quad \text{Hence B.E.} = \text{K.E.} = -\text{T.E.} = \frac{-\text{P.E.}}{2}$$

Work done in Changing the Orbit of Satellite

$$W = \text{Change in mechanical energy of system but } E = \frac{-GMm}{2r} \quad \text{so} \quad W = E_2 - E_1 = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Example

A satellite moves eastwards very near the surface of the Earth in equatorial plane with speed (v_0). Another satellite moves at the same height with the same speed in the equatorial plane but westwards. If R = radius of the Earth and ω be its angular speed of the Earth about its own axis. Then find the approximate difference in the two time period as observed on the Earth.

Solution

$$T_{west} = \frac{2\pi R}{v_0 + R\omega} \text{ and } T_{east} = \frac{2\pi R}{v_0 - R\omega} \Rightarrow \Delta T = T_{east} - T_{west} = 2\pi R \left[\frac{2R\omega}{v_0^2 - R^2\omega^2}\right] = \frac{4\pi\omega R^2}{v_0^2 - R^2\omega^2}$$

Example

A planet of mass m moves along an ellipse around the Sun of mass M so that its maximum and minimum distances from Sun are a and b respectively. Prove that the angular momentum L of this planet relative to the centre of the Sun is $L = m\sqrt{\frac{2GMab}{(a+b)}}$.

Solution

Angular momentum at maximum distance from Sun = Angular momentum at minimum distance from Sun

$mv_1a = mv_2b \Rightarrow v_1 = \frac{v_2 b}{a}$ by applying conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$

From above equations $v_1 = \sqrt{\frac{2GMb}{a(a+b)}}$ Angular momentum of planet $L = mv_1a = ma\sqrt{\frac{2GMb}{(a+b)a}}$

Example

A body of mass m is placed on the surface of earth. Find work required to lift this body by a height

(i) $h = \frac{R_e}{1000}$ (ii) $h = R_e$

Solution :

(i) $h = \frac{R_e}{1000}$, as $h << R_e$, so we can apply $W_{ext} = mgh$; $W_{ext} = (m)\left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{1000}\right) = \frac{GM_em}{1000R_e}$

(ii) $h = R_e$, in this case h is not very less than R_e, so we cannot apply $\Delta U = mgh$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

$$W_{ext} = U_f - U_i = m(V_f - V_i)\ ;\ W_{ext} = m\left[\left(-\frac{GM_e}{R_e + R_e}\right) - \left(-\frac{GM_e}{R_e}\right)\right]\ ;\ W_{ext} = -\frac{GM_e m}{2R_e}$$

Example

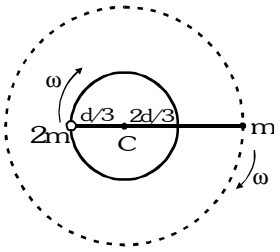
In a double star, two stars (one of mass m and the other of 2m) distant d apart rotate about their common centre of mass. Deduce an expression of the period of revolution. Show that the ratio of their angular momenta about the centre of mass is the same as the ratio of their kinetic energies.

Solution :

The centre of mass C will be at distances d/3 and 2d/3 from the masses 2m and m respectively. Both the stars rotate round C in their respective orbits with the same angular velocity ω. The gravitational force acting on each star due to the other supplies the necessary centripetal force.

For rotation of the smaller star, the centripetal force $\left[m\left(\frac{2d}{3}\right)\omega^2\right]$ is provided by gravitational force.

$$\therefore \quad \frac{G(2m)m}{d^2} = m\left(\frac{2d}{3}\right)\omega^2 \quad \text{or } \omega = \sqrt{\left(\frac{3Gm}{d^3}\right)}$$



Therefore, the period of revolution is given by $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{d^3}{3Gm}\right)}$

The ratio of the angular momenta is $\frac{(I\omega)_{big}}{(I\omega)_{small}} = \frac{I_{big}}{I_{small}} = \frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2} = \frac{1}{2}$ since ω is same for both.

The ratio of their kinetic energies is $\frac{\left(\frac{1}{2}I\omega^2\right)_{big}}{\left(\frac{1}{2}I\omega^2\right)_{small}} = \frac{I_{big}}{I_{small}} = \frac{1}{2}$,

which is the same as the ratio of their angular momenta.

WEIGHTLESSNESS

When the weight of a body (either true or apparent) becomes zero, the body is said to be in the state of weightlessness. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface then

True weight = $mg_h = \frac{mGM}{(R+h)^2} = \frac{mg}{\left(1+\frac{h}{R}\right)^2}$ and Apparent weight = $m(g_h - a)$

But $a = \frac{v_0^2}{r} = \frac{GM}{r^2} = \frac{GM}{(R+h)^2} = g_h \Rightarrow$ Apparent weight = $m(g_h - g_h) = 0$

Note : The condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

Condition of weightlessness on Earth surface

If apparent weight of body is zero then angular speed of Earth can be calculated as $mg' = mg - mR_e\omega^2\cos^2\lambda$

$$0 = mg - mR_e\omega^2\cos^2\lambda \Rightarrow \omega = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R_e}}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p65

SOME WORKED OUT EXAMPLES

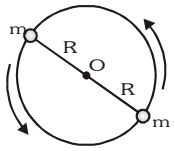
Example#1

Two particles of equal mass m go round a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(A) $\frac{1}{2R}\sqrt{\frac{1}{Gm}}$ (B) $\sqrt{\frac{Gm}{2R}}$ (C) $\frac{1}{2}\sqrt{\frac{Gm}{R}}$ (D) $\sqrt{\frac{4Gm}{R}}$

Solution **Ans. (C)**

Centripetal force provided by the gravitational force of attraction between two particles i.e. $\frac{mv^2}{R} = \frac{Gm \times m}{(2R)^2} \Rightarrow v = \frac{1}{2}\sqrt{\frac{Gm}{R}}$



Example#2

The escape velocity for a planet is v_e. A particle starts from rest at a large distance from the planet, reaches the planet only under gravitational attraction, and passes through a smooth tunnel through its centre. Its speed at the centre of the planet will be-

(A) $\sqrt{1.5}v_e$ (B) $\frac{v_e}{\sqrt{2}}$ (C) v_e (D) zero

Solution **Ans. (A)**

From mechanical energy conservation $0 + 0 = \frac{1}{2}mv^2 - \frac{3GMm}{2R} \Rightarrow v = \sqrt{\frac{3GM}{R}} = \sqrt{1.5}\ v_e$

Example#3

A particle is projected vertically upwards the surface of the earth (radius R_e) with a speed equal to one fourth of escape velocity. What is the maximum height attained by it from the surface of the earth ?

(A) $\frac{16}{15}R_e$ (B) $\frac{R_e}{15}$ (C) $\frac{4}{15}R_e$ (D) None of these

Solution **Ans. (B)**

From conservation of mechanical energy $\frac{1}{2}mv^2 = \frac{GMm}{R_e} - \frac{GMm}{R}$

Where R = maximum distance from centre of the earth Also $v = \frac{1}{4}v_e = \frac{1}{4}\sqrt{\frac{2GM}{R_e}}$

$$\Rightarrow \frac{1}{2}m \times \frac{1}{16} \times \frac{2GM}{R_e} = \frac{GMm}{R_e} - \frac{GMm}{R} \Rightarrow R = \frac{16}{15}R_e \Rightarrow h = R - R_e = \frac{R_e}{15}$$

Example#4

A mass 6 10^{24} kg (= mass of earth) is to be compressed in a sphere in such a way that the escape velocity from its surface is 3 10^8 m/s (equal to that of light). What should be the radius of the sphere?

(A) 9 mm (B) 8 mm (C) 7 mm (D) 6 mm

Solution **Ans. (A)**

As, $v_e = \sqrt{\left(\frac{2GM}{R}\right)}$, $R = \left(\frac{2GM}{v_e^2}\right)$, $\therefore R = \frac{2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{(3 \times 10^8)^2} = 9 \times 10^{-3}\,m = 9\ mm$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Example#5

Calculate the mass of the sun if the mean radius of the earth's orbit is 1.5 10^8 km and G = 6.67 10^{-11} N m^2/kg^2.

(A) $M \approx 2\times10^{30}$ kg (B) $M \approx 3\times10^{30}$ kg (C) $M \approx 2\times10^{15}$ kg (D) $M \approx 3\times10^{15}$ kg

Solution **Ans. (A)**

In case of orbital motion as $v=\sqrt{\left(\frac{GM}{r}\right)}$ so $T=\frac{2\pi r}{v}=2\pi r\sqrt{\frac{r}{GM}}$, i.e., $M=\frac{4\pi^2r^3}{GT^2}$

$M=\frac{4\times\pi^2\times(1.5\times10^{11})^3}{6.67\times10^{-11}\times(3.15\times10^7)^2}$ [as T = 1 year = 3.15 10^7s] i.e., $M \approx 2\times10^{30}$ kg

Example#6

Gravitational potential difference between a point on surface of planet and another point 10 m above is 4 J/kg. Considering gravitational field to be uniform, how much work is done in moving a mass of 2 kg from the surface to a point 5m above the surface?

(A) 4 J (B) 5 J (C) 6 J (D) 7 J

Solution **Ans. (A)**

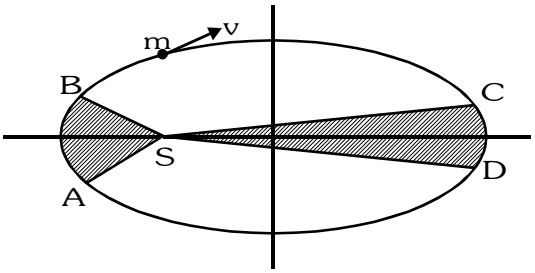
Gravitational field $g=-\frac{\Delta V}{\Delta x}=-\left(\frac{-4}{10}\right)=\frac{4}{10}$ J / kg m

Work done in moving a mass of 2 kg from the surface to a point 5 m above the surface,

W = mgh = (2kg) $\left(\frac{4}{10}\frac{J}{kgm}\right)$(5m) = 4J

Example#7

The figure shows elliptical orbit of a planet m about the sun S. The shaded area SCD is twice the shaded area SAB. If t_1 be the time for the planet to move from C to D and t_2 is the time to move from A to B, then :



(A) $t_1 = t_2$ (B) $t_1 = 8t_2$ (C) $t_1 = 4t_2$ (D) $t_1 = 2t_2$

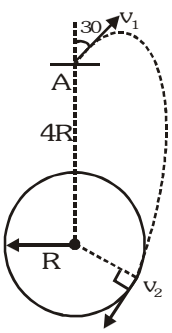
Solution **Ans. (D)**

From Kepler's law : Areal velocity = constant so $\frac{\text{Area SCD}}{t_1}=\frac{\text{Area SAB}}{t_2}\Rightarrow t_1=2t_2$

Example#8

A particle is projected from point A, that is at a distance 4R from the centre of the Earth, with speed v_1 in a direction making 30 with the line joining the centre of the Earth and point A, as shown. Find the speed v_1 of particle (in m/s) if particle passes grazing the surface of the earth. Consider gravitational interaction only between these two.

(use $\frac{GM}{R}$ = 6.4 10^7 m^2/s^2)



(A) $\frac{8000}{\sqrt{2}}$ (B) 800 (C) $800\sqrt{2}$ (D) None of these

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p6.5

Solution **Ans. (A)**

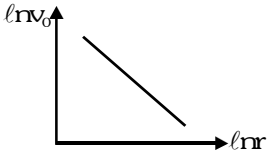
Conserving angular momentum : $m(v_1 \cos 60^\circ)4R = mv_2R \Rightarrow \frac{v_2}{v_1}=2$.

Conserving energy of the system : $-\frac{GMm}{4R}+\frac{1}{2}mv_1^2=-\frac{GMm}{R}+\frac{1}{2}mv_2^2$

$$\Rightarrow \frac{1}{2}v_2^2-\frac{1}{2}v_1^2=\frac{3}{4}\frac{GM}{R}\Rightarrow v_1^2=\frac{1}{2}\frac{GM}{R}\Rightarrow v_1=\frac{1}{\sqrt{2}}\sqrt{64\times10^6}=\frac{8000}{\sqrt{2}}m/s$$

Example#9

If the law of gravitation be such that the force of attraction between two particles vary inversely as the $5/2^{th}$ power of their separation, then the graph of orbital velocity v_0 plotted against the distance r of a satellite from the earth's centre on a log-log scale is shown alongside. The slope of line will be-



(A) $-\frac{5}{4}$ (B) $-\frac{5}{2}$ (C) $-\frac{3}{4}$ (D) -1

Solution **Ans. (C)**

$$\frac{mv_0^2}{r}=\frac{GMm}{r^{5/2}}\Rightarrow v_0=\frac{\sqrt{GM}}{r^{3/4}}\Rightarrow \ell n v_0=\ell n\sqrt{GM}-\frac{3}{4}\ell nr$$

Example#10

Two point objects of masses m and 4m are at rest at an infinite separation. They move towards each other under mutual gravitational attraction. If G is the universal gravitational constant, then at a separation r

(A) the total mechanical energy of the two objects is zero (B) their relative velocity is $\sqrt{\frac{10Gm}{r}}$

(C) the total kinetic energy of the objects is $\frac{4Gm^2}{r}$ (D) their relative velocity is zero

Solution **Ans. (A,B,C)**

By applying law of conservation of momentum (m) $\rightarrow v_1$ $v_2 \leftarrow$ (4m)

$mv_1 - 4mv_2 = 0 \Rightarrow v_1 = 4v_2$

By applying conservation of energy $\frac{1}{2}mv_1^2+\frac{1}{2}4mv_2^2=\frac{Gm4m}{r}\Rightarrow 10mv_2^2=\frac{G4m^2}{r}\Rightarrow v_2=2\sqrt{\frac{Gm}{10r}}$

$\therefore$ Total kinetic energy $=\frac{4Gm^2}{r}$; Relative velocity for the particle $\Rightarrow v_{rel} = |\vec{v}_1-\vec{v}_2| = 5v_2 = \sqrt{\frac{10Gm}{r}}$

Or

Mechanical energy of sysem = 0 = constant. By using reduced mass concept

$\frac{1}{2}\mu v_{rel}^2 = \frac{Gm(4m)}{r}$ where $\mu=\frac{(m)(4m)}{m+4m}=\frac{4}{5}m\Rightarrow v_{rel}=\sqrt{\frac{10Gm}{r}}$

Also total KE of system $=\frac{G(m)(4m)}{r}=\frac{4Gm^2}{r}$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Example#11

Which of the following statements are true about acceleration due to gravity?

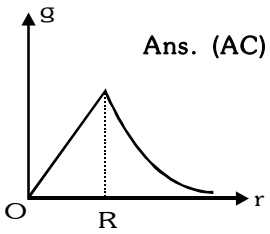
(A) 'g' decreases in moving away from the centre if r > R

(B) 'g' decreases in moving away from the centre if r < R

(C) 'g' is zero at the centre of earth

(D) 'g' decreases if earth stops rotating on its axis

Solution **Ans. (AC)**



Variation of g with distance

variation of g with ω : $g' = g - \omega^2 R\cos^2\lambda$

If ω=0 then g will not change at poles where $\cos\lambda = 0$.

Example#12

An astronaut, inside an earth satellite experiences weightlessness because:

(A) he is falling freely

(B) no external force is acting on him

(C) no reaction is exerted by floor of the satellite

(D) he is far away from the earth surface

Solution **Ans. (AC)**

As astronaut's acceleration = g so he is falling freely. Also no reaction is exerted by the floor of the satellite.

Example#13

If a satellite orbits as close to the earth's surface as possible

(A) its speed is maximum

(B) time period of its revolution is minimum

(C) the total energy of the 'earth plus satellite' system is minimum

(D) the total energy of the 'earth plus satellite' system is maximum

Solution **Ans. (ABC)**

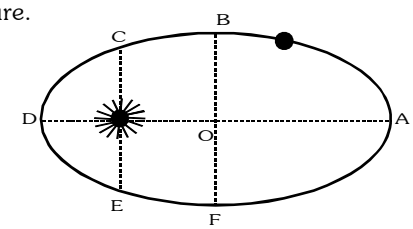
For (A) : orbital speed $v_0 = \sqrt{\frac{GM}{r}}$

For (B) : Time period of revolution $T^2 \propto r^3$

For (C/D): Total energy = $-\frac{GMm}{2r}$

Example#14

A planet is revolving around the sun is an elliptical orbit as shown in figure. Select correct alternative(s)



(A) Its total energy is negative at D.

(B) Its angular momentum is constant

(C) Net torque on planet about sun is zero

(D) Linear momentum of the planet is conserved

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Solution **Ans. (ABC)**

For (A) : For bounded system, the total energy is always negative.

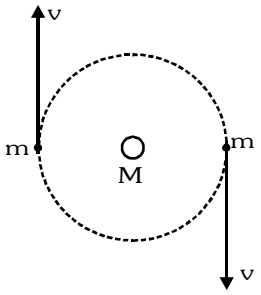
For (B) : For central force field, angular momentum is always conserved

For (C) :For central force field, torque = 0.

For (D) : In presence of external force, linear momentum is not conserved.

Example#15 to 17

A triple star system consists of two stars, each of mass m, in the same circular orbit about central star with mass M = 2 10^{30} kg. The two outer stars always lie at opposite ends of a diameter of their common circular orbit. The radius of the circular orbit is r = 10^{11} m and the orbital period of each star is 1.6 10^{7} s. [Take π^2 = 10 and G = $\frac{20}{3}$ 10^{-11} Nm^2kg^{-2}]



15. The mass m of the outer stars is

(A) $\frac{16}{15}\times10^{30}$ kg (B) $\frac{11}{8}\times10^{30}$ kg (C) $\frac{15}{16}\times10^{30}$ kg (D) $\frac{8}{11}\times10^{30}$ kg

16. The orbital velocity of each star is

(A) $\frac{5}{4}\sqrt{10}\times10^{3}$ m/s (B) $\frac{5}{4}\sqrt{10}\times10^{5}$ m/s (C) $\frac{5}{4}\sqrt{10}\times10^{2}$ m/s (D) $\frac{5}{4}\sqrt{10}\times10^{4}$ m/s

17. The total mechanical energy of the system is

(A) $-\frac{1375}{64}\times10^{35}$ J (B) $-\frac{1375}{64}\times10^{38}$ J (C) $-\frac{1375}{64}\times10^{34}$ J (D) $-\frac{1375}{64}\times10^{37}$ J

Solution

15. Ans. (B)

F_{mm} = Gravitational force between two outer stars = $\frac{Gm^2}{4r}$

F_{mM} = Gravitational force between central star and outer star = $\frac{GmM}{r^2}$

For circular motion of outer star, $\frac{mv^2}{r} = F_{mm} + F_{mM}$ $\therefore v^2 = \frac{G(m+4M)}{4r}$

T = period of orbital motion = $\frac{2\pi r}{v}$ $\therefore m = \frac{16\pi^2 r^3}{GT^2} - 4M = \left(\frac{150}{16} - 8\right)10^{30} = \frac{11}{8}\times10^{30}$ kg

16. Ans. (D)

$T = \frac{2\pi r}{v} \Rightarrow v = \frac{2\pi r}{T} = \frac{(2)(\sqrt{10})(10^{11})}{1.6\times10^{7}} = \frac{5}{4}\sqrt{10}\times10^{4}$ m/s

17. Ans. (B)

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p65

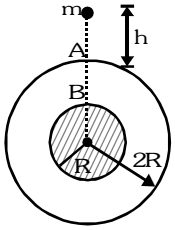
Total mechanical energy = K.E. + P.E.

$$= 2\left(\frac{1}{2}mv^2\right) - \frac{2GMm}{r} - \frac{Gm^2}{2r} = m\left[\frac{G(4M+m)}{4r} - \frac{2GM}{r} - \frac{Gm}{2r}\right] = -\frac{Gm}{r}\left[M + \frac{m}{4}\right]$$

$$= -\left(\frac{20}{3}\times10^{-11}\right)\left(\frac{11}{8}\times10^{30}\right)\times\frac{1}{10^{11}}\left(2\times10^{30} + \frac{11}{32}\times10^{30}\right) = -\frac{1375}{64}\times10^{38}\ J$$

Example#18 to 20

A solid sphere of mass M and radius R is surrounded by a spherical shell of same mass M and radius 2R as shown. A small particle of mass m is released from rest from a height h (<<R) above the shell. There is a hole in the shell.



18. In what time will it enter the hole at A :–

(A) $2\sqrt{\frac{hR^2}{GM}}$ (B) $\sqrt{\frac{2hR^2}{GM}}$ (C) $\sqrt{\frac{hR^2}{GM}}$ (D) none of these

19. What time will it take to move from A to B ?

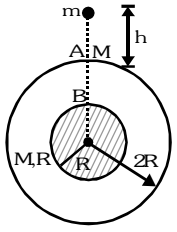
(A) $= \frac{4R^2}{\sqrt{GMR}}$ (B) $> \frac{4R^2}{\sqrt{GMR}}$ (C) $< \frac{4R^2}{\sqrt{GMR}}$ (D) none of these

20. With what approximate speed will it collide at B ?

(A) $\sqrt{\frac{2GM}{R}}$ (B) $\sqrt{\frac{GM}{2R}}$ (C) $\sqrt{\frac{3GM}{2R}}$ (D) $\sqrt{\frac{GM}{R}}$

Solution

18. Ans. (A)



$$a = \frac{GM}{(2R)^2} + \frac{GM}{(2R)^2} = \frac{GM}{2R^2} \quad \therefore t = \sqrt{\frac{2\times h}{a}} = \sqrt{\frac{2\times h\times 2R^2}{GM}} = 2\sqrt{\frac{hR^2}{GM}}$$

19. Ans. (C)

Given that (h<<R), so the velocity at A' is also zero.

We can see here that the acceleration always increases from 2R to R and its value must be greater than

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory p65

$a = \frac{GM}{4R^2}(\text{at A}) \quad \therefore t < \frac{v}{a} \Rightarrow t < \sqrt{\frac{GM}{R}} \times \frac{4R^2}{GM} \Rightarrow t < \frac{4R^2}{\sqrt{GMR}}$

20. Ans. (D)

Given that (h<<R), so the velocity at A' is also zero.

Loss in PE = gain in KE $\quad \therefore \frac{GMm}{2R} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{GM}{R}}$

Example#21

Imagine a light planet revolving around a very massive star in a circular orbit of radius R. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then match the following

	Column I		Column II
(A)	Time period of revolution is proportional to	(P)	R^0
(B)	Kinetic energy of planet is proportional to	(Q)	$R^{7/4}$
(C)	Orbital velocity of planet is proportional to	(R)	$R^{-1/2}$
(D)	Total mechanical energy of planet is proportional to	(S)	$R^{-3/2}$
		(T)	$R^{-3/4}$

Solution **Ans. (A) Q (B) S(C) T (D) S**

According to question $F = \frac{C}{R^{5/2}}$ where C is a constant so $\frac{mv^2}{R} = \frac{C}{R^{5/2}} \Rightarrow mv^2 \propto \frac{1}{R^{3/2}} \Rightarrow KE \propto R^{-3/2}$

Also $v \propto R^{-3/4}$ and $PE \propto R^{-3/2}$. Total mechanical energy = KE + PE $\propto R^{-3/2}$

Time period $T = \frac{2\pi R}{v} \propto R^{7/4}$

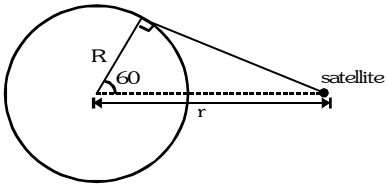
Example#22

A satellite is launched in the equatorial plane in such a way that it can transmit signals upto 60 latitude on the earth. The orbital velocity of the satellite is found to be $\sqrt{\frac{GM}{\alpha R}}$. Find the value of α.

Solution **Ans. 2**

$r\cos 60 = R \Rightarrow r = 2R$ orbital velocity $v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{2R}} \Rightarrow \alpha = 2$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65



Example#23

An artificial satellite (mass m) of a planet (mass M) revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming the force of resistance on satellite to depend on velocity as $F=av^2$ where 'a' is a constant, calculate how long the satellite will stay in the space before it falls onto the planet's surface.

Solution

Air resistance $F = -av^2$, where orbital velocity $v = \sqrt{\frac{GM}{r}}$

r = the distance of the satellite from planet's centre $\Rightarrow F = -\frac{GMa}{r}$

The work done by the resistance force $dW = Fdx = Fvdt = \frac{GMa}{r}\sqrt{\frac{GM}{r}}dt = \frac{(GM)^{3/2}a}{r^{3/2}}dt$ ....(i)

The loss of energy of the satellite = dE $\therefore \frac{dE}{dr} = \frac{d}{dr}\left[-\frac{GM\ m}{2r}\right] = \frac{GMm}{2r^2} \Rightarrow dE = \frac{GMm}{2r^2}dr$...(ii)

Since dE = – dW (work energy theorem) $-\frac{GMm}{2r^2}dr = \frac{(GM)^{3/2}}{r^{3/2}}dt$

$$\Rightarrow t = -\frac{m}{2a\sqrt{GM}}\int_{nR}^{R}\frac{dr}{\sqrt{r}} = \frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{GM}} = \left(\sqrt{n}-1\right)\frac{m}{a\sqrt{gR}}$$

NODE6 E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Gravitation\Eng\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 9
PROPERTIES OF MATTER – ELASTICITY - Theory

Index

PROPERTIES OF MATTER & FLUID MECHANICS

Serial No.	CONTENTS	Page No.
1.	Elasticity	1
2.	Surface tension	14
3.	Surface energy	17
4.	Capillary tube and Capillarity	22
5.	Fluid Statics	25
6.	Buoyancy and Archimede's Principle	36
7.	Fluid Dynamics	40
8.	Bernoulli's Theorem	41
9.	Viscosity	48
10.	**Exercise-I** Check your Grasp (Single Choice)	63
11.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	70
12.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	80
13.	**Exercise-4(A)** Conceptual Subjective Exercise	91
14.	**Exercise-4(B)** Conceptual Subjective Exercise	97
15.	**Exercise-5 (A)** Previous years questions	100
16.	**Exercise-5(B)** Previous years questions	103

ALLEN Materials Provided By - Material Point Available on Learnaf.com

JEE Main : Elastic behaviour, Stress-strain relationship, Hooke's Law, Young's modulus, bulk modulus, modulus of rigidity. Pressure due to a fluid column; Pascal's law and its applications.

Viscosity, Stokes' law, terminal velocity, streamline and turbulent flow, Reynolds number. Bernoulli's principle and its applications. Surface energy and surface tension, angle of contact, application of surface tension – drops, bubbles and capillary rise.

JEE Advanced : Pressure in a fluid; Pascal's law; Buoyancy; Surface energy and surface tension, capillary rise; Viscosity (Poiseuille's equation excluded), Stoke's law; Terminal velocity, Streamline flow, equation of continuity, Bernoulli's theorem and its applications.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

ELASTICITY

A body is said to be rigid if the relative positions of its constituent particles remains unchanged when external deforming forces are applied to it. The nearest approach to a rigid body is diamond or carborundum.

Actually no body is perfectly rigid and every body can be deformed more or less by the application of suitable forces. All these deformed bodies however regain their original shape or size, when the deforming forces are removed.

The property of matter by virtue of which a body tends to regain its original shape and size after the removal of deforming forces is called elasticity.

Some terms related to elasticity :

- **Deforming Force**

  External force which try to change in the length, volume or shape of the body is called deforming force.

- **Perfectly Elastic Body**

  The body which perfectly regains its original form on removing the external deforming force, is defined as a perfectly elastic body. Ex. : quartz – Very nearly a perfect elastic body.

- **Plastic Body**

  (a) The body which does not have the property of opposing the deforming force, is known as a plastic body.

  (b) The bodies which remain in deformed state even after removed of the deforming force are defined as plastic bodies.

- **Internal restoring force**

  When a external force acts at any substance then due to the intermolecular force there is a internal resistance produced into the substance called internal restoring force.

  At equilibrium the numerical value of internal restoring force is equal to the external force.

STRESS

The internal restoring force acting per unit area of cross–section of the deformed body is called stress.

$$\text{Stress} = \frac{\text{Internal restoring force}}{\text{Area of cross section}} = \frac{F_{\text{internal}}}{A} = \frac{F_{\text{external}}}{A}$$

Stress depends on direction of force as well as direction of area of application so it is tensor.

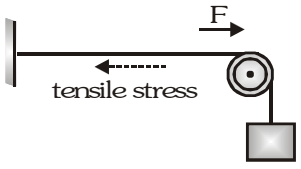
SI Unit : $N\text{–}m^{-2}$

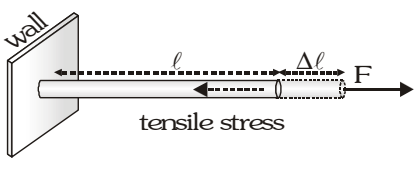
Dimensions : $M^1\ L^{-1}\ T^{-2}$

There are three types of stress :-

(a) Longitudinal Stress : When the stress is normal to the surface of body, then it is known as longitudinal stress. There are two types of longitudinal stress

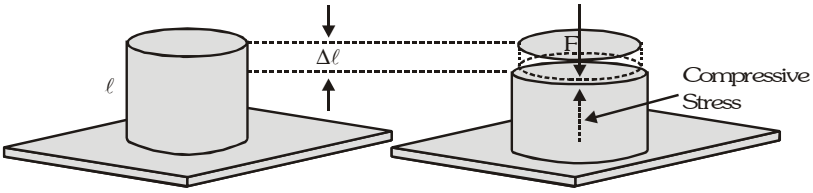
(i) Tensile Stress : The longitudinal stress, produced due to increase in length of a body, is defined as tensile stress.





node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

(ii) **Compressive stress :** The longitudinal stress, produced due to decrease in length of a body, is defined as compressive stress.

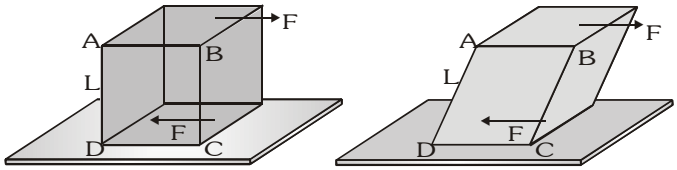


(b) Volume Stress

If equal normal forces are applied every one surface of a body, then it undergoes change in volume. The force opposing this change in volume per unit area is defined as volume stress.

(c) Tangential Stress or Shear Stress

When the stress is tangential or parallel to the surface of a body then it is known as shear stress. Due to this stress, the shape of the body changes or it gets twisted.



STRAIN

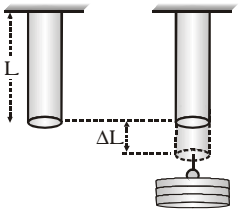
The ratio of change of any dimension to its original dimension is called strain.

$$\text{Strain} = \frac{\text{change in size of the body}}{\text{original size of the body}}$$

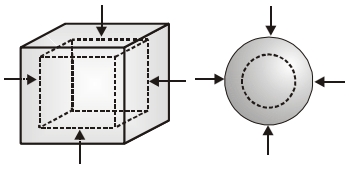
It is a unitless and dimensionless quantity.

There are three types of strain : Type of strain depends upon the directions of applied force.

(a) Longitudinal strain $= \dfrac{\text{change in length of the body}}{\text{initial length of the body}} = \dfrac{\Delta L}{L}$

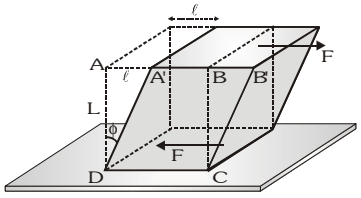


(b) Volume strain $= \dfrac{\text{change in volume of the body}}{\text{original volume of the body}} = \dfrac{\Delta V}{V}$



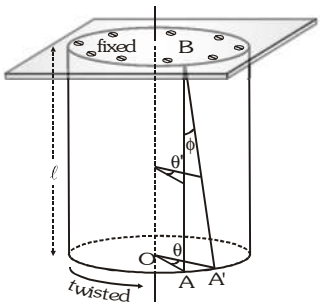
(c) Shear strain

When a deforming force is applied to a body parallel to its surface then its shape (not size) changes. The strain produced in this way is known as shear strain. The strain produced due to change of shape of the body is known as shear strain. $\tan\phi = \dfrac{\ell}{L}$ or $\phi = \dfrac{\ell}{L} = \dfrac{\text{displacement of upper face}}{\text{distance between two faces}}$



node06\E_NODE6 (E)\Data\ 2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Relation Between angle of twist and Angle of shear

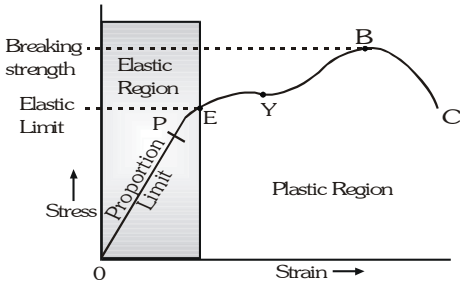


When a cylinder of length 'ℓ' and radius 'r' is fixed at one end and tangential force is applied at the other end, then the cylinder gets twisted. Figure shows the angle of shear ABA' and angle of twist AOA'. Arc AA' = r θ and Arc AA' = $\ell\phi$ so $r\theta = \ell\phi \Rightarrow \phi = \frac{r\theta}{\ell}$ where θ = angle of twist, ϕ = angle of shear

GOLDEN KEY POINTS

- When a material is under tensile stress restoring force are caused by intermolecular attraction while under compressive stress, the restoring force are due to intermolecular repulsion.
- If the deforming force is inclined to the surface at an angle θ such that $\theta \neq 0$ and $\theta \neq 90$ then both tangential and normal stress are developed.
- Linear strain in the direction of force is called **longitudinal** strain while in a direction perpendicular to force **lateral strain**.

Stress – Strain Graph



Proportion Limit :

The limit in which Hook's law is valid and stress is directly proportional to strain is called proportion limit. Stress $\propto$ Strain

Elastic limit :

That maximum stress which on removing the deforming force makes the body to recover completely its original state.

Yield Point :

The point beyond elastic limit, at which the length of wire starts increasing without increasing stress, is defined as the yield point.

Breaking Point :

The position when the strain becomes so large that the wire breaks down at last, is called breaking point. At this position the stress acting in that wire is called breaking stress and strain is called breaking strain.

Elastic after effect

We know that some material bodies take some time to regain their original configuration when the deforming force is removed. The delay in regaining the original configuration by the bodies on the removal of deforming force is called elastic after effect. The elastic after effect is negligibly small for quartz fibre and phosphor bronze. For this reason, the suspensions made from quartz and phosphor-bronze are used in galvanometers and electrometers. For glass fibre elastic after effect is very large. It takes hours for glass fibre to return to its original state on removal of deforming force.

node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit-No-4\Fluid Mechanics\English\Theory.p65

Elastic Fatigue :

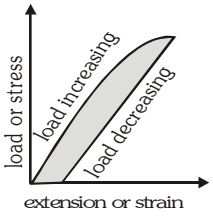
The loss of strength of the material due to repeated strains on the material is called elastic fatigue. That is why bridges are declared unsafe after a long time of their use.

Creep :

If a small force is applied for a long time then it causes breaking of metal. For example A fan is hung for 200 years then the shaft will break.

Elastic Hysteresis :

The strain persists even when the stress is removed. This lagging behind of strain is called elastic hysteresis. This is the reason why the values of strain for same stress are different while increasing the load and while decreasing the load.



Breaking Stress :

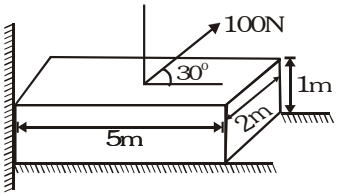
The stress required to cause actual facture of a material is called the breaking stress Breaking stress = F/A

GOLDEN KEY POINTS

- Breaking stress also measures the tensile strength.
- Metals with small plastic deformation are called brittle.
- Metals with large plastic deformation are called ductile.
- Elasticity restoring forces are strictly conservative only when the elastic hysteresis is zero. i.e. the loading and unloading stress – strain curves are identical.
- The material which have low elastic hysteresis have also low elastic relaxation time.

Example

Find out longitudinal stress and tangential stress on a fixed block.



Solution

Longitudinal or normal stress $\sigma_1 = \frac{100\sin 30^0}{5\times 2} = 5\ N/m^2$

Tangential stress $\sigma_2 = \frac{100\cos 30^0}{5\times 2} = 5\sqrt{3} N/m^2$

Example

The breaking stress of aluminium is 7.5 10^8 dyne cm^{-2}. Find the greatest length of aluminium wire that can hang vertically without breaking. Density of aluminium is 2.7 g cm^{-3}. [Given : g = 980 cm s^{-2}]

Solution

Let ℓ be the greatest length of the wire that can hang vertically without breaking.

Mass of wire m = cross–sectional area (A) length (ℓ) density (ρ), Weight of wire = mg = A$\ell\rho$g

This is equal to the maximum force that the wire can withstand.

$$\therefore \text{ Breaking stress } = \frac{\ell A\rho g}{A} = \ell\rho g \Rightarrow 7.5 \quad 10^8 = \ell \quad 2.7 \quad 980$$

$$\Rightarrow \ell = \frac{7.5\times 10^8}{2.7\times 980} \text{ cm } = 2.834 \quad 10^5 \text{cm} = 2.834 \text{ km}$$

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

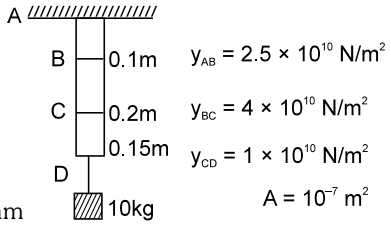
Hooke's Law

If the deformation is small, the stress in a body is proportional to the corresponding strain, this fact is known as Hooke's Law. Within elastic limit : stress $\propto$ strain $\Rightarrow \frac{\text{stress}}{\text{strain}}$ =constant

This constant is known as modulus of elasticity or coefficient of elasticity.
The modulus of elasticity depends only on the type of material used. It does not depend upon the value of stress and strain.

Example

Find out the shift in point B, C and D



Sol. $\Delta L_B = \Delta L_{AB} = \frac{FL}{AY} = \frac{MgL}{AY} = \frac{10 \times 10 \times 0.1}{10^{-7} \times 2.5 \times 10^{10}} = 4 \quad 10^{-3}$ m = 4mm

$\Delta L_C = \Delta L_B + \Delta L_{BC} = 4 \quad 10^{-3} + \frac{100 \times 0.2}{10^{-7} \times 4 \times 10^{10}} = 4 \quad 10^{-3} + 5 \quad 10^{-3} = 9$mm

$\Delta L_D = \Delta L_C + \Delta L_{CD} = 9 \quad 10^{-3} + \frac{100 \times 0.15}{10^{-7} \times 1 \times 10^{10}} = 9 \quad 10^{-3} + 15 \quad 10^{-3} = 24$ mm

Young's Modulus of Elasticity 'Y'

Within elastic limit the ratio of longitudinal stress and longitudinal strain is called Young's modulus of elasticity.

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{\ell/L} = \frac{FL}{\ell A}$$

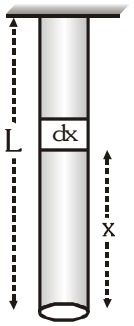
Within elastic limit the force acting upon a unit area of a wire by which the length of a wire becomes double, is equivalent to the Young's modulus of elasticity of material of a wire. If L is the length of wire, r is radius and ℓ is the increase in length of the wire by suspending a weight Mg at its one end then Young's modulus of elasticity of the material of wire $Y = \frac{(Mg/\pi r^2)}{(\ell/L)} = \frac{MgL}{\pi r^2 \ell}$

Unit of Y : N/m^2 Dimensions of Y : $M^1L^{-1}T^{-2}$

Increment of length due to own weight

Let a rope of mass M and length L is hanged vertically. As the tension of different point on the rope is different. Stress as well as strain will be different at different point.

(i) maximum stress at hanging point (ii) minimum stress at lower point



Consider a dx element of rope at x distance from lower end then tension $T = \left(\frac{M}{L}\right) x\, g$

So stress = $\frac{T}{A} = \left(\frac{M}{L}\right)\frac{xg}{A}$

Let increase in length of dx is dy then strain = $\frac{dy}{dx}$

So Young modulus of elasticity $Y = \frac{\text{stress}}{\text{strain}} = \frac{\frac{M}{L}\frac{xg}{A}}{dy/dx} \Rightarrow \left(\frac{M}{L}\right)\frac{xg}{A} dx = Y\, dy$

For full length of rope $\frac{Mg}{LA}\int_0^L x dx = Y\int_0^{\Delta\ell} dy \Rightarrow \frac{Mg}{LA}\frac{L^2}{2} = Y\Delta\ell \Rightarrow \Delta\ell = \frac{MgL}{2AY}$

[Since the stress is varying linearly we may apply average method to evaluate strain.]

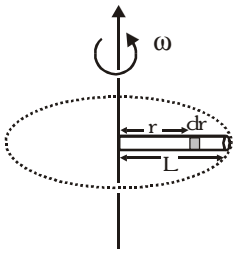
node06\E_NOD06\[E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics\English\Theory.p65

Example

A thin uniform metallic rod of length 0.5 m and radius 0.1 m rotates with an angular velocity 400 rad/s in a horizontal plane about a vertical axis passing through one of its ends. Calculate (a) tension in the rod and (b) the elongation of the rod. The density of material of the rod is 10^4 kg/m^3 and the Young's modulus is 2 10^{11} N/m^2.

Solution

(a) Consider an element of length dr at a distance r from the axis of rotation as shown in figure. The centripetal force acting on this element will be $dT = dm r\omega^2 = (\rho A dr) r\omega^2$. As this force is provided by tension in the rod (due to elasticity), so the tension in the rod at a distance r from the axis of rotation will be due to the centripetal force due to all elements between x = r to x = L i.e.,



$$T = \int_r^L \rho A\omega^2 r dr = \frac{1}{2}\rho A\omega^2\left[L^2 - r^2\right] \ldots(i)$$

So here $T = \frac{1}{2}$ 10^4 π 10^{-2} $(400)^2\left[\left(\frac{1}{2}\right)^2 - r^2\right] = 8\pi\times10^6\left[\frac{1}{4} - r^2\right]$N

(b) Now if dy is the elongation in the element of length dr at position r then strain

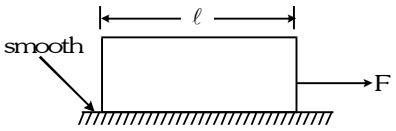
$$\frac{dy}{dr} = \frac{\text{stress}}{Y} = \frac{T}{AY} = \frac{1}{2}\frac{\rho\omega^2}{Y}[L^2 - r^2]$$

So the elongation of the whole rod

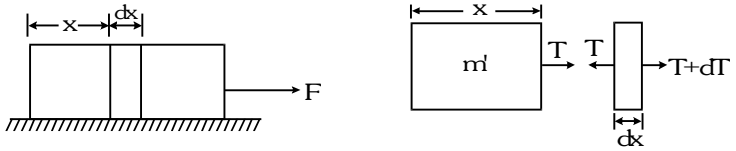
$$\Delta L = \frac{\rho\omega^2}{2Y}\int_0^L\left(L^2 - r^2\right)dr = \frac{1}{3}\frac{\rho\omega^2L^3}{Y} = \frac{1}{3}\frac{10^4\times(400)^2(0.5)^3}{2\times10^{11}} = \frac{1}{3}\times10^{-3}\text{m}$$

Example

Find out the elongation in block. If mass, area of cross-section and young modulus of block are m, A and y respectively.



Solution



Acceleration, $a = \frac{F}{m}$ then $T = m'a$ where $\Rightarrow m' = \frac{m}{\ell}x$; $T = \frac{m}{\ell}x\frac{F}{m} = \frac{Fx}{\ell}$

Elongation in element 'dx' = $\frac{Tdx}{Ay}$, total elongation, $\delta = \int_0^\ell\frac{Tdx}{Ay} = \int_0^\ell\frac{Fxdx}{A\ell y} = \frac{F\ell}{2Ay}$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Bulk modulus of elasticity 'K' or 'B'

Within elastic limit the ratio of the volume stress and the volume strain is called bulk modulus of elasticity.

$$K \text{ or } B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{F/A}{\frac{-\Delta V}{V}} = \frac{\Delta P}{\frac{-\Delta V}{V}}$$

The minus sign indicates a decrease in volume with an increase in stress.

Unit of K : N m^{-2} or pascal

Bulk modulus of an ideal gas is process dependence.

For isothermal process PV = constant $\Rightarrow$ PdV + VdP = 0 $\Rightarrow$ $P = \frac{-dP}{dV/V}$ So bulk modulus = P

For adiabatic process PV^{γ} = constant $\Rightarrow \gamma PV^{\gamma-1}dV + V^{\gamma}dP = 0 \Rightarrow \gamma PdV + VdP = 0 \Rightarrow \gamma P = \frac{-dP}{dV/V}$

So bulk modulus = γP

For any polytropic process PV^{n} = constant $\Rightarrow$ $nPV^{n-1}dV | V^{n}dP=0$ $\Rightarrow$ PdV+VdP=0 $\Rightarrow$ $nP = \frac{-dP}{dV/V}$

So bulk modulus = nP

Compressibility : The reciprocal of bulk modulus of elasticity is defined as compressibility. $C = \frac{1}{K}$

Modulus of Rigidity 'η'

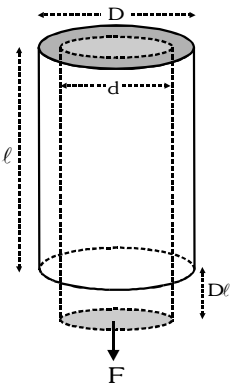
Within elastic limit the ratio of shearing stress and shearing strain is called modulus of rigidity of a material.

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \left(\frac{\frac{F_{\text{tangential}}}{A}}{\phi}\right) = \frac{F_{\text{tangential}}}{A\phi}$$

Note : Angle of shear 'ϕ' always take in radian

Poisson's Ratio (σ)

In elastic limit, the ratio of lateral strain and longitudinal strain is called Poisson's ratio.



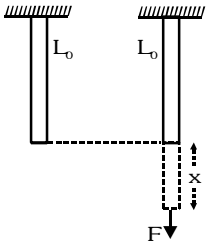
$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\beta}{\alpha}, \text{ where } \beta = \frac{-\Delta D}{D} = \frac{d-D}{D} \text{ & } \alpha = \frac{\Delta L}{L}$$

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics\English\Theory.p65

Work done in stretching a wire (Potential energy of a stretched wire)

When a wire is stretched, work is done against the interatomic forces, which is stored in the form of elastic potential energy.

For a wire of length L_o stretched by a distance x, the restoring elastic force is :



$$F = \text{stress} \quad \text{area} = Y\left[\frac{x}{L_o}\right]A$$

The work has to be done against the elastic restoring forces in stretching dx

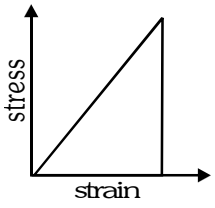
$$dW = F\,dx = \frac{YA}{L_0}x\,dx$$

The total work done in stretching the wire from x = 0 to x = $\Delta\ell$ is, then

$$W = \int_0^{\Delta\ell}\frac{YA}{L_o}x dx = \frac{YA}{L_o}\left[\frac{x^2}{2}\right]_0^{\Delta\ell} = \frac{YA(\Delta\ell)^2}{2L_o}$$

$$W = \frac{1}{2} \quad Y \quad (\text{strain})^2 \quad \text{original volume} = \frac{1}{2}(\text{stress})\ (\text{strain})\ (\text{volume})$$

GOLDEN KEY POINTS
• The value of K is maximum for solids and minimum for gases.
• For any ideal rigid body all three elastic modulus are infinite .
• η is the characteristic of solid material only as the fluids do not have fixed shape.
• Potential energy density = area under the stress–strain curve.
• Young's modulus = Slope of the stress–strain curve



Example

A steel wire of 4.0 m in length is stretched through 2.00 mm. The cross–sectional area of the wire is 2.0 mm^2. If Young's modulus of steel is 2.0 10^{11} N/m^2 find (i) the energy density of wire (ii) the elastic potential energy stored in the wire.

Solution

(i) The energy density of stretched wire

$$= \frac{1}{2} \quad \text{stress} \quad \text{strain} = \frac{1}{2} \quad Y \quad (\text{strain})^2 = \frac{1}{2} \quad 2.0 \quad 10^{11}\left[\frac{2\times10^{-3}}{4}\right]^2 = 2.5 \quad 10^4 \ J/m^3$$

(ii) Elastic potential energy = energy density volume = 2.5 10^4 $(2.0 \quad 10^{-6})$ 4.0 J = 20 10^{-2} = 0.20J

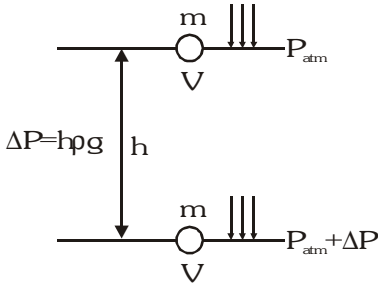
node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Example

Find the depth of lake at which density of water is 1% greater than at the surface. Given compressibility K = 50 10^{-6} /atm.

Solution

$$B = \frac{\Delta P}{-\Delta V / V} \Rightarrow \frac{\Delta V}{V} = -\frac{\Delta P}{B}$$



We know $P = P_{atm} + h\rho g$ and $m = \rho V$ = constant

$$d\rho V + dV\rho = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V} \text{ i.e. } \frac{\Delta\rho}{\rho} = \frac{\Delta P}{B} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{1}{100} = \frac{h\rho g}{B}$$

[assuming ρ = constant]; $h\rho g = \frac{B}{100} = \frac{1}{100K} \Rightarrow h\rho g = \frac{1\times1\times10^5}{100\times50\times10^{-6}}$

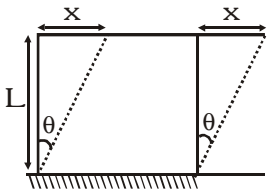
$$h = \frac{10^5}{5000\times10^{-6}\times1000\times10} = \frac{100\times10^3}{50}\text{ m} = 2\text{km}$$

Example

A rubber cube of side 5 cm has one side fixed while a tangential force equal to 1800 N is applied to opposite face. Find the shearing strain and the lateral displacement of the strained face. Modulus of rigidity for rubber is 2.4 10^6 N/m^2.

Solution

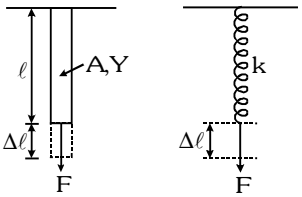
$$\text{Stress} = \frac{F}{A} = \eta\frac{x}{L}$$



$$\text{Strain} = \theta = \frac{F}{A\eta} = \frac{1800}{25\times10^{-4}\times2.4\times10^6} = \frac{180}{25\times24} = \frac{3}{10} = 0.3 \text{ rad}$$

as $\frac{x}{L} = 0.3 \Rightarrow x = 0.3 \quad 5 \quad 10^{-2} = 1.5 \quad 10^{-2}$ m = 1.5 mm

Analogy of Rod as a spring

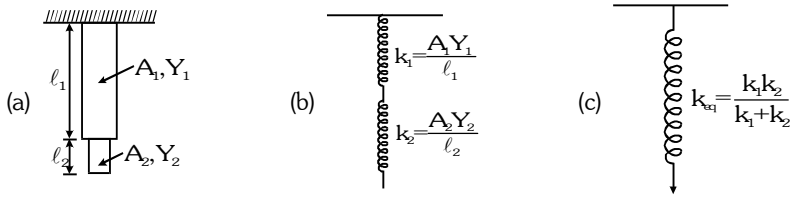


$$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow Y = \frac{F\ell}{A\Delta\ell} \text{ or } F = \frac{AY}{\ell}\Delta\ell$$

$\frac{AY}{\ell}$ = constant, depends on type of material and geometry of rod.

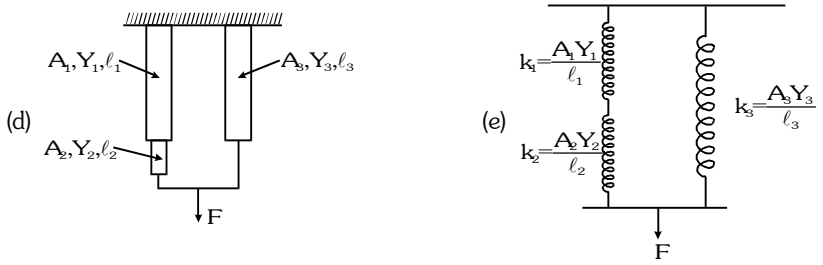
$F = k\Delta\ell$ where $k = \frac{AY}{\ell}$ = equivalent spring constant.

For the system of rods shown in figure (a), the replaced spring system is shown in figure (b) [two spring in series]. Figure (c) represents equivalent spring system.



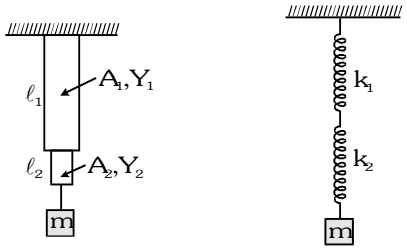
node06\E_NOD06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Figure (d) represents another combination of rods and their replaced spring system.



Example

A mass 'm' is attached with rods as shown in figure. This mass is slightly stretched and released whether the motion of mass is S.H.M., if yes then find out the time period.

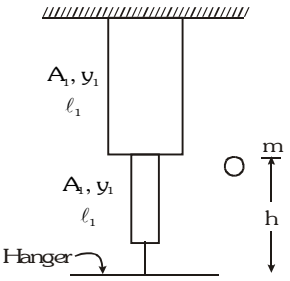


Solution

Here $k_{eq} = \frac{k_1k_2}{k_1+k_2}$ and $T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$ where $k_1 = \frac{A_1Y_1}{\ell_1}$ and $k_2 = \frac{A_2Y_2}{\ell_2}$

Example

Hanger is mass-less. A ball of mass 'm' drops from a height 'h', which sticks to hanger after striking. Neglect over turning, find out the maximum extension in rod. Assuming rod is massless. Let maximum extension be x_{max}.

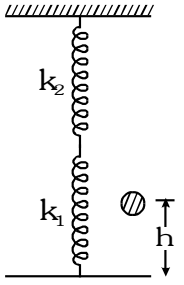


Sol. Applying energy conservation $mg(h + x_{max}) = \frac{1}{2}\frac{k_1k_2}{k_1+k_2}x^2$

where $k_1 = \frac{A_1y_1}{\ell_1}$; $k_2 = \frac{A_2y_2}{\ell_2}$ & $K_{eq} = \frac{A_1A_2y_1y_2}{A_1y_1\ell_2 + A_2y_1\ell_1}$

therefore $k_{eq}x^2_{max} - 2mgx_{max} - 2mgh = 0$

$$x_{max} = \frac{2mg \pm \sqrt{4m^2g^2 + 8mghk_{eq}}}{2k_{eq}} = \frac{mg}{k_{eq}} + \sqrt{\frac{m^2g^2}{k_{eq}^2} + \frac{2mgh}{k_{eq}}}$$

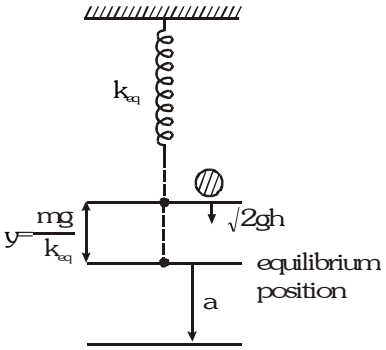


node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Otherway by S.H.M.

$\omega = \sqrt{\frac{k_{eq}}{m}}$ and $v = \omega \sqrt{a^2 - y^2}$

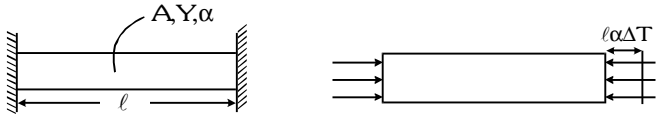
$\Rightarrow \sqrt{2gh} = \sqrt{\frac{keq}{m}}\sqrt{a^2 - y^2} \Rightarrow \sqrt{\frac{2mgh}{k_{eq}} + \frac{m^2g^2}{k_{eq}^2}} = a$



Maximum extension $= a + y = \frac{mg}{k_{eq}} + \sqrt{\frac{m^2g^2}{k_{eq}} + \frac{2mgh}{k_{eq}}}$

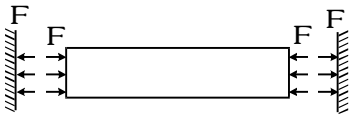
Thermal Stress :

If temperature of rod is increased by ΔT, then change in length $\Delta\ell = \ell\ \alpha\ \Delta T$; strain $= \frac{\Delta\ell}{\ell} = \alpha\Delta T$



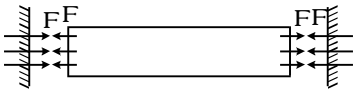
But due to rigid support, there is no strain. Supports provide force on stresses to keep the length of rod same

$Y = \frac{\text{stress}}{\text{strain}}$



If ΔT = (+) tive

Thermal stress = y strain = $Y\alpha\ \Delta T$

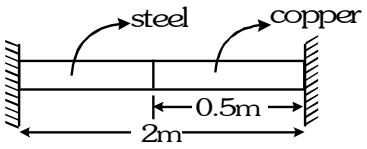


If ΔT = (–) tive

$\frac{F}{A} = Y\alpha\Delta T$ $\qquad$ $F = AY\alpha\ \Delta T$

Example

When composite rod is free, then composite length increases to 2.002 m for temperature 20 C to 120 C. When composite rod is fixed between the support, there is no change in component length find y and α of steel, if $y_c = 1.5 \quad 10^{13}\ N/m^2$ $\alpha_c = 1.6 \quad 10^{-5}/\ C$.

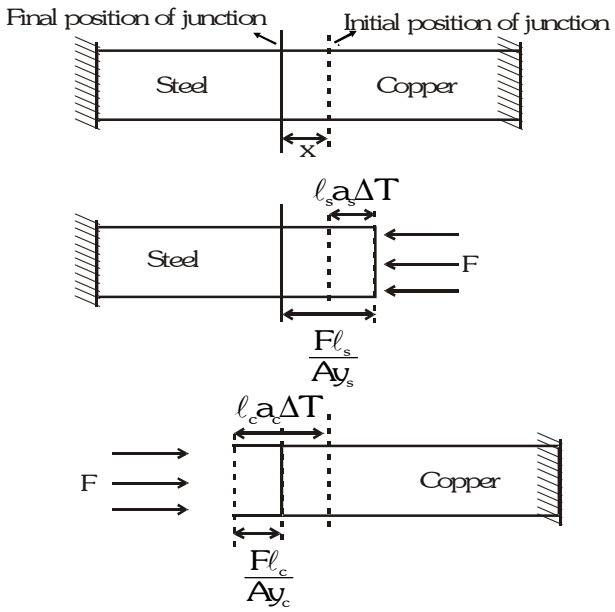


Solution

$\Delta\ell = \ell_s\ \alpha_s\ \Delta T + \ell_c\ \alpha_c\Delta T \Rightarrow 0.002 = [1.5\ \alpha_s + 0.5 \quad 1.6 \quad 10^{-5}] \quad 100 \qquad \Rightarrow \alpha_s = \frac{1.2\times10^{-5}}{1.5} = 8 \quad 10^{-6}/\ C$

there is no change in component length

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65



For steel $x = \ell_s \alpha_s \Delta T - \frac{F\ell_s}{AY_s} = 0 \Rightarrow \frac{F}{AY_s} = \alpha_s \Delta T$ ....(i)

For copper $x = \frac{F\ell_c}{AY_c} - \ell_c \alpha_c \Delta T = 0 \Rightarrow \frac{F}{Ay_c} = \alpha_c \Delta T$ ...(ii)

$$(ii)/(i) \Rightarrow Y_s = Y_c \frac{\alpha_c}{\alpha_s} = \frac{1.5 \times 10^{13} \times 16 \times 10^{-5}}{8 \times 10^{-6}}$$

$Y_s = 3 \quad 10^{13}$ N/m^2

Applications of Elasticity

Some of the important applications of the elasticity of the materials are discussed as follows :

The material used in bridges lose its elastic strength with time bridges are declared unsafe after long use.

To estimate the maximum height of a mountain :

The pressure at the base of the mountain = $h\rho g$ = stress.

The elastic limit of a typical rock is $3 \quad 10^8$ N m^{-2}

The stress must be less than the elastic limits, otherwise the rock begins to flow.

$h < \frac{3 \times 10^8}{\rho g} < \frac{3 \times 10^8}{3 \times 10^3 \times 10} < 10^4$ m ($\because \rho = 3 \quad 10^3$ kg m^{-3} ; g = 10 ms^{-2}) or h = 10 km

It may be noted that the height of Mount Everest is nearly 9 km.

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Torsion constant of a wire

$C = \frac{\pi\eta r^4}{2\ell}$ Where η is modulus of rigidity r and ℓ is radius and length of wire respectively.

(a) Toque required for twisting by angle θ, $\tau = C\theta$.

(b) Work done in twisting by angle θ, $W = \frac{1}{2} C\theta^2$.

Effect of Temperature on elasticity

When temperature is increased then due to weakness of inter molecular force the elastic properties in general decreases i.e. elastic constant decreases. Plasticity increases with temperature. For example, at ordinary room temperature, carbon is elastic but at high temperature, carbon becomes plastic. Lead is not much elastic at room temperature but when cooled in liquid nitrogen exhibit highly elastic behaviour.

For a special kind of steel, elastic constants do not vary appreciably temperature. This steel is called 'INVAR steel'.

Effect of Impurity on elasticity

Y is slightly increase by impurity. The inter molecular attraction force inside wire effectively increase by impurity due to this external force can be easily opposed.

node06\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics\English\Theory.p65

SURFACE TENSION

Surface tension is basically a property of liquid. The liquid surface behaves like a stretched elastic membrane which has a natural tendency to contract and tends to have a minimum possible area. This property of liquid is called *surface tension*.

INTERMOLECULAR FORCES

- The force which acts between the atoms or the molecules of different substances is called intermolecular force. This force is of two types.

  (a) Cohesive force – The force acting between the molecules of one type of molecules of same substance is called cohesive force.

  (b) Adhesive force – The force acting between different types of molecules or molecules of different substance is called adhesive force.

- Intermolecular forces are different from the gravitational forces not obey the inverse–square law

- The distance upto which these forces effective, is called molecular range. This distance is nearly 10^{-9} m. Within this limit this increases very rapidly as the distance decreases.

- Molecular range depends on the nature of the substance

Examples:

- **Water Glass :** Water wets glass surface but mercury does not. Because when water comes in contact with glass the adhesive force acts between water and glass molecules. As adhesive force is greater than the cohesive force of water molecules, the water molecules, cling with glass surface and surface become wet. In case of mercury adhesive force is less than that of cohesive force and Hg–molecules do not cling with glass surface and surface does not wet with Hg–molecules.

- **Oil–water :** Since Cohesive force of water >Adhesive force oil–water> Cohesive force of oil.

  (i) If water drop is poured on the surface of oil, it contracts in the form of globule.

  (ii) If oil drop is poured on the surface of water it spreads to a larger area in the form of thin film.

- **Ink–paper** : Since adhesive force between ink–paper > cohesive force on ink, the ink sticks to the paper.

EXPLANATION OF SURFACE TENSION (Molecular theory of surface Tension)

Laplace firstly explained the phenomenon of surface tension on the basis of intermolecular forces. According to him surface tension is a molecular phenomenon and its root causes are electromagnetic forces. He explained the cause of surface tension as described under. If the distance between two molecules is less than the molecular range C ($\approx 10^{-9}$ m) then they attract each other, but if the distance is more than this then the attraction becomes negligible. If a sphere of radius C with a molecule at centre is drawn, then only those molecules which are enclosed within this sphere can attract or be attracted by the molecule at the centre of the sphere. This sphere is called sphere of molecular activity or sphere of influence. In order to understand the tension acting at the free surface of liquid, let us consider four liquid molecules like A, B, C and D along with their spheres of molecular activity.

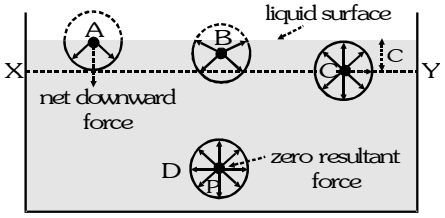
(a) According to figure D sphere is completely inside liquid. So molecule is attracted equally in all directions and hence resultant force is equal to zero.

(b) According to figure sphere of molecule C is just below the liquid surface. So resultant force is equal to zero.

(c) The molecule B which is a little below the liquid surface is more attracted downwards due to excess of molecules downwards. Hence the resultant force is acting downwards.

node06\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit\No-4\Fluid Mechanics \English\Theory.p65

(d) Molecule A is situated at surface so that its sphere of molecular activity is half outside the liquid and half inside. Only down portion has liquid molecules. Hence it experiences a maximum downward force. Thus all the molecules situated between the surface and a plane XY, distant C below the surface, experience a resultant downward cohesive force. When the surface area of liquid is increased molecules from the interior of the liquid rise to the surface.



As these molecules reach near the surface, work is done against the downward cohesive force. This work is stored in the molecules in the form of potential energy. Thus the potential energy of the molecules lying in the surface is greater than that of the molecules in the interior of the liquid. A system is in stable equilibrium when its potential energy is minimum. Hence in order to have minimum potential energy the liquid surface tends to have minimum number of molecules in it. In other words the surfaces tends to contract to a minimum possible area. This tendency is exhibited as surface tension.

GOLDEN KEY POINTS
• Surface tension is a scalar quantity.
• It acts tangential to liquid surface.
• Surface tension is always produced due to cohesive force.
• More is the cohesive force, more is the surface tension.
• When surface area of liquid is increased, molecules from the interior of the liquid rise to the surface. For this, work is done against the downward cohesive force. As a result, its potential energy increases and internal energy decreases. So on increase in surface area cooling occurs. If liquid temperature remains same, then extra energy may be given by external agency. So the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called Surface Energy, free surface energy or surface energy density.

DEPENDENCY OF SURFACE TENSION

● On Cohesive Force

Those factors which increase the cohesive force between molecules increase the surface tension and those which decrease the cohesive force between molecules decrease the surface tension.

● On Impurities

If the impurity is completely soluble then on mixing it in the liquid, its surface tension increases. e.g., on dissolving ionic salts in small quantities in a liquid, its surface tension increases. If the impurity is partially soluble in a liquid then its surface tension decreases because adhesive force between insoluble impurity molecules and liquid molecules decreases cohesive force effectively, e.g.

(a) On mixing detergent in water its surface tension decreases.

(b) Surface tension of water is more than (alcohol + water) mixture.

● On Temperature

On increasing temperature surface tension decreases. At critical temperature and boiling point it becomes zero.

Note : Surface tension of water is maximum at 4 C

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

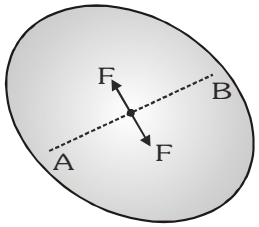
- **On Contamination**

The dust particles or lubricating materials on the liquid surface decreases its surface tension.

- **On Electrification**

The surface tension of a liquid decreases due to electrification because a force starts acting due to it in the outward direction normal to the free surface of liquid.

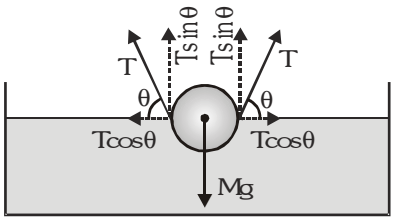
DEFINITION OF SURFACE TENSION



Surface tension can be defined in the form of an imaginary line on the surface or by relating it to the work done. The force acting per unit length of an imaginary line drawn on the free liquid surface at right angles to the line and in the plane of liquid surface, is defined as surface tension. Let an imaginary line AB be drawn in any direction on a liquid surface. The surface on either side of this line exerts a pulling force, which is perpendicular to line AB. If force is F and length of AB is L then $T = \frac{F}{L}$

SI UNITS : N/m and J/m^2 **CGS UNITS :** dyne/cm and erg/cm^2 **Dimensions :** $M^1L^0T^{-2}$

Illustrations:



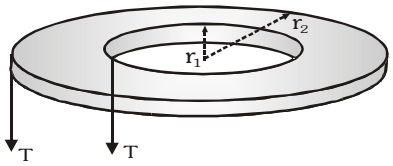
- When any needle floats on the liquid surface then $2T\ell \sin\theta = mg$

Ex. A mosquito sitting on a liquid surface.

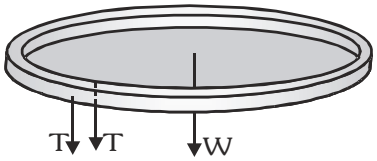


- If the needle is lifted from the liquid surface then required excess force will be $F_{excess} = 2T\ell$

Minimum force required $F_{min} = mg + 2T\ell$

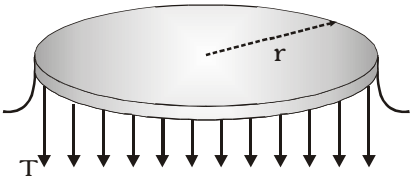


- Required excess force for a circular thick ring (or hollow disc) having internal and external radii r_1 and r_2 is dipped in and taken out from liquid $F_{excess} = F_1 + F_2 = T(2\pi r_1) + T(2\pi r_2) = 2\pi T(r_1 + r_2)$



- Required excess force for a circular ring ($r_1 = r_2 = r$)

$$F_{excess} = 2\pi T(r + r) = 4\pi r T$$

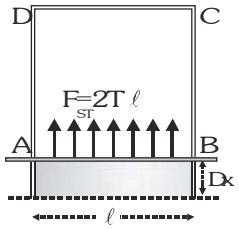


- Required excess force for a circular disc ($r_1 = 0$, $r_2 = r$)

$$F_{excess} = 2\pi rT$$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Surface Energy



According to molecular theory of surface tension the molecule in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called ' Surface energy". Let a liquid film be formed on a wire frame and a straight wire of length ℓ can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solution, each surface will pull the wire parallel to itself with a force $T\ell$.

Thus, net force on the wire due to both the surfaces is $2T\ell$. Apply an external force F equal and opposite to it to keep the wire is equilibrium. Thus, $F=2T\ell$ Now, suppose the wire is moved through a small distance dx, the work done by the force is, $dW = F\,dx = (2T\ell)\,dx$ But $(2\ell)\,(dx)$ is the total increase in area of both the surfaces of the film. Let it be dA. Then, $dW = T\,dA \Rightarrow T = dW/dA$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by unity. Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface. $T = \frac{dU}{dA}$ [as dW = dU]

Special Cases

- Work done (surface energy) in formation of a drop of radius r = Work done against surface tension

  W = Surface tension T change in area ΔA = T $4\pi r^2 = 4\pi r^2 T$

- Work done (surface energy) in formation of a soap bubble of radius r :

  W = T ΔA or W = T 2 $4\pi r^2 = 8\pi r^2 T$ [$\because$ soap bubble has two surfaces]

Example

The length of a needle floating on water is 2.5 cm. Calculate the added force required to pull the needle out of water. [T = 7.2 10^{-2} N/m]

Solution

The force of surface tension F = T 2ℓ ($\because$ Two free surfaces are there)

$\Rightarrow$ F = 7.2 10^{-2} 2 2.5 10^{-2} = 3.6 10^{-3} N

Example

A paper disc of radius R from which a hole of radius r is cut out, is floating in a liquid of surface tension, T. What will be force on the disc due to surface tension?

Solution

$$T = \frac{F}{L} = \frac{F}{2\pi(R+r)} \qquad \therefore F = 2\pi (R + r)T$$

Example

Calculate the work done against surface tension in blowing a soap bubble from a radius 10 cm to 20 cm if the surface tension of soap solution is 25 10^{-3} N/m. Then compare it with liquid drop of same radius.

Solution

(i) For soap bubble : Extension in area = 2 $4\pi r_2^2$ –2 $4\pi r_1^2$ $=8\pi [(0.2)^2 - (0.1)^2] = 0.24\ \pi\ m^2$

Work done W_1 = surface tension extension in area = 25 10^{-3} 0.24 π = 6π 10^{-3} J

(ii) For Liquid Drop : in case of liquid drop only one free surface, so extension in area will be half of soap bubble

$$\therefore W_2 = \frac{W_1}{2} = 3\pi \quad 10^{-3} \text{ J}$$

SPLITTING OF BIGGER DROP INTO SMALLER DROPLETS

If bigger drop is spitted into smaller droplets then in this process volume of liquid always remain conserved. Let bigger drop has radius R. It is splitted into n smaller drops of radius r then by conservation of volume

(i) $\frac{4}{3}\pi R^3 = n\left[\frac{4}{3}\pi r^3\right] \Rightarrow n = \left[\frac{R}{r}\right]^3 \Rightarrow r = \frac{R}{n^{1/3}}$

(ii) Initial surface area = $4\pi R^2$ and final surface area = $n(4\pi r^2)$

Therefore initial surface energy $E_i = 4\pi R^2 T$ and final surface energy $E_f = n(4\pi r^2 T)$

Change in area $\Delta A = n4\pi r^2 - 4\pi R^2 = 4\pi (nr^2 - R^2)$

Therefore the amount of surface energy absorbed i.e. $\Delta E = E_f - E_i = 4\pi T (nr^2 - R^2)$

$\therefore$ Magnitude of work done against surface tension i.e. $W = 4\pi (nr^2 - R^2)T$

$$W = 4\pi T (nr^2 - R^2) = 4\pi R^2 T (n^{1/3} - 1) = 4\pi R^2 T\left[\frac{R}{r} - 1\right] \Rightarrow W = 4\pi R^3 T \left[\frac{1}{r} - \frac{1}{R}\right]$$

In this process temperature of system decreases as energy gets absorbed in increasing surface area.

$$W = J\, m\, s\, \Delta\theta = 4\pi R^3 T\left[\frac{1}{r} - \frac{1}{R}\right] \Rightarrow \Delta\theta = \frac{4\pi R^3 T}{\frac{4}{3}\pi R^3 J\rho s}\left[\frac{1}{r} - \frac{1}{R}\right] = \frac{3T}{J\rho s}\left[\frac{1}{r} - \frac{1}{R}\right]$$

Where ρ = liquid density, s = liquid's specific heat

Thus in this process area increasing, surface energy increasing, internal energy decreasing, temperature decreasing, and energy is absorbed.

Example

A big drop is formed by coalescing 1000 small droplets of water .What will be the change in surface energy. What will be the ratio between the total surface energy of the droplets and the surface energy of the big drop ?

Solution

By conservation of volume $\frac{4}{3}\pi R^3 = 1000 \quad \frac{4}{3}\pi r^3 \Rightarrow r = \frac{R}{10}$

Surface energy of 1000 droplets = 1000 T $4\pi\left[\frac{R}{10}\right]^2$ = 10 (T $4\pi R^2$)

Surface energy of the big drop = T $4\pi R^2$

Surface energy will decrease in the process of formation of bigger drop, hence energy is released and temperature increases . $\therefore$ $\frac{\text{total surface energy of 1000 droplets}}{\text{surface energy of big drop}} = \frac{10(T \times 4\pi R^2)}{T \times 4\pi R^2} = \frac{10}{1}$

Example

A water drop of radius 1mm is broken into 10^6 identical drops. Surface tension of water is 72 dynes/cm. Find the energy spent in this process.

Solution

As volume of water remains constant, so $\frac{4}{3}\pi R^3 = n\frac{4}{3}\pi r^3 \Rightarrow r = \frac{R}{n^{1/3}}$

Increase in surface area $\Delta A = n(4\pi r^2) - 4\pi R^2 = 4\pi(n^{1/3} - 1)R^2 = 4\pi(100 - 1)10^{-6}$

$\therefore$ Energy spent = $T\Delta A = 4\pi \quad 99 \quad 10^{-6} \quad 72 \quad 10^{-3} = 89.5 \quad 10^{-6}$ J

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

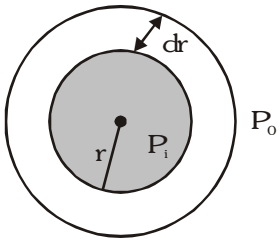
EXCESS PRESSURE INSIDE A CURVED LIQUID SURFACE

The pressure on the concave side of curved liquid surface is greater than that on the convex side. Therefore pressure difference exists across two sides of a curved surface. This pressure difference is called *excess pressure.*

Excess pressure inside the drop

Let a drop of radius r having internal and external pressure P_i and P_0 respectively, so that excess pressure

$P_{ex} = (P_i - P_0)$.



If the radius of the drop is changed from r to (r+dr) then

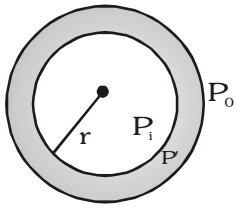
Work done = F.dr = (PA) dr = P $4\pi r^2 dr$

Change in surface area = $r\pi(r + dr)^2 - 4\pi r^2 = 8\pi r dr$

So by definition of surface energy $T = \frac{W}{\Delta A} = \frac{4\pi r^2 P dr}{8\pi r dr} \Rightarrow P_{ex} = (P_i - P_0) = \frac{2T}{r}$

Excess pressure inside soap bubble :

Since the soap bubble has two surfaces. The excess pressure will get doubled as compared to a drop



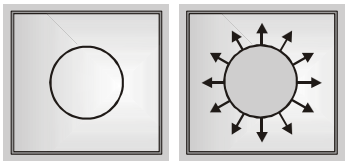
$P_i - P' = \frac{2T}{r}$, $P' - P_0 = \frac{2T}{r} \Rightarrow$ excess pressure = $P_i - P_0 = \frac{4T}{r}$

GOLDEN KEY POINTS

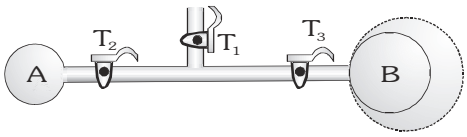
- For liquid surface, pressure on concave side is always higher than convex side



- If a bubble is formed inside a liquid, the pressure inside the bubble is more than the pressure outside the bubble.
- A soap film is formed in a wire frame. A loop of thread is lying on the film. If the film inside the loop is broken then the tension in the thread will be 2Tr.



- In the following arrangement, air will flow from bubble A to B if T_2 and T_3 are opened, because pressure in A is greater than in B.



- The force required to separate two plates of Area A is given by $F = \frac{2AT}{d}$

node06\E_NOD06\E]\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics\English\Theory.p65

Example

Prove that If two bubbles of radii r_1 and r_2 coalesce isothermally in vacuum then the radius of new bubble will be

$$r = \sqrt{r_1^2 + r_2^2}$$

Solution

When two bubbles coalesce then number of molecules of air will remain constant and temperature also constant

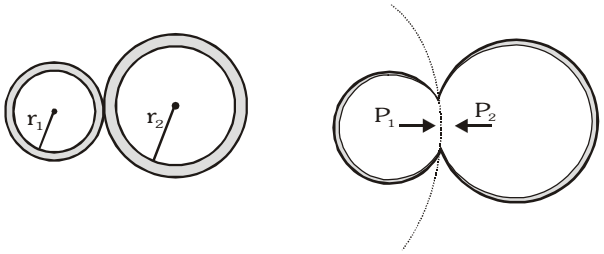
so $n_1 + n_2 = n \Rightarrow P_1V_1 + P_2V_2 = PV \Rightarrow \frac{4T}{r_1}\left(\frac{4}{3}\pi r_1^3\right) + \frac{4T}{r_2}\left(\frac{4}{3}\pi r_2^3\right) = \frac{4T}{r}\left(\frac{4}{3}\pi r^3\right) \Rightarrow r = \sqrt{r_1^2 + r_2^2}$

Example

Prove that If two bubbles of radii r_1 and r_2 ($r_1 < r_2$) come in contact with each other then the radius of curvature of the common surface $r = \frac{r_1 r_2}{r_2 - r_1}$.

Solution

$\because r_1 < r_2 \qquad \therefore P_1 > P_2$ Small part of bubbles is in equilibrium



$$\Rightarrow P_1(\Delta A) - P_2(\Delta A) = \frac{4T}{r}\Delta A \Rightarrow \frac{4T}{r_1} - \frac{4T}{r_2} = \frac{4T}{r} \Rightarrow r = \frac{r_1 r_2}{r_2 - r_1}$$

Example

Calculate the excess pressure within a bubble of air of radius 0.1 mm in water. If the bubble had been formed 10 cm below the water surface on a day when the atmospheric pressure was 1.013 10^5 Pa, then what would have been the total pressure inside the bubble? Surface tension of water = 73 10^{-3} N/m

Solution

Excess pressure $P_{excess} = \frac{2T}{r} = \frac{2 \times 73 \times 10^{-3}}{0.1 \times 10^{-3}} = 1460$ Pa

The pressure at a depth d, in liquid is P = hdg. Therefore, the total pressure inside the air bubble is

$$P_{in} = P_{atm} + hdg + \frac{2T}{r} = (1.013 \quad 10^5) + (10 \quad 10^{-2} \quad 10^3 \quad 9.8) + 1460$$

$$= 101300 + 980 + 1460 = 103740 = 1.037 \quad 10^5 \text{ Pa}$$

Example

The limbs of a manometer consist of uniform capillary tubes of radii are 10^{-3} & 7.2 10^{-4} m. Find out the correct pressure difference if the level of the liquid in narrower tube stands 0.2 m above that in the broader tube.(Density of liquid =10^3 kg/m^3, Surface tension = 72 10^{-3} N/m)

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Solution

If P_1 and P_2 are the pressures in the broader and narrower tubes of radii r_1 and r_2 respectively, the pressure just below the meniscus in the respective tubes will be $P_1 - \frac{2T}{r_1}$ and $P_2 - \frac{2T}{r_2}$

So that $\left[P_1 - \frac{2T}{r_1}\right] - \left[P_2 - \frac{2T}{r_2}\right] = h\rho g$ or $P_1 - P_2 = h\rho g - 2T\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$

$P_1 - P_2 = 0.2 \quad 10^3 \quad 9.8 \quad -2 \quad 72 \quad 10^{-3} \left[\frac{1}{7.2\times10^{-4}} - \frac{1}{14\times10^{-4}}\right] = 1960 - 97 = 1863$ Pa

ANGLE OF CONTACT (θ_c)

The angle enclosed between the tangent plane at the liquid surface and the tangent plane at the solid surface at the point of contact inside the liquid is defined as the *angle of contact*. The angle of contact depends the nature of the solid and liquid in contact.

- **Effect of Temperature on angle of contact**

On increasing temperature surface tension decreases, thus $\cos\theta_c$ increases $\left[\because \cos\theta_c \propto \frac{1}{T}\right]$ and θ_c decrease.

So on increasing temperature, θ_c decreases.

- **Effect of Impurities on angle of contact**

(a) Solute impurities increase surface tension, so $\cos\theta_c$ decreases and angle of contact θ_c increases.

(b) Partially solute impurities decrease surface tension, so angle of contact θ_c decreases.

- **Effect of Water Proofing Agent**

Angle of contact increases due to water proofing agent. It gets converted acute to obtuse angle.

Table of angle of contact of various solid–liquid pairs

Solid - Liquid Pair	θ C
Glass -Normal water	8
Glass -Distilled water	0
Glass - Alcohol	0
Glass - Mercury	135
Paraffin wax - Water	108
Silver - Water	90

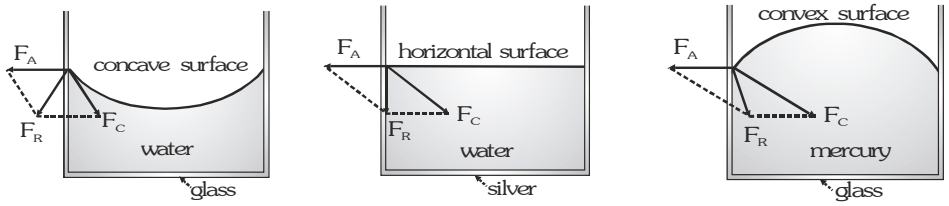
Shape of Liquid Surface

When a liquid is brought in contact with a solid surface, the surface of the liquid becomes curved near the place of contact. The shape of the surface (concave or convex) depends upon the relative magnitudes of the cohesive force between the liquid molecules and the adhesive force between the molecules of the liquid and the solid.

The free surface of a liquid which is near the walls of a vessel and which is curved because of surface tension is known as *meniscus*. The cohesive force acts at an angle 45 from liquid surface whereas the adhesive force acts at right angles to the solid surface. The relation between the shape of liquid surface, cohesive/adhesive forces, angle of contact, etc are summarised in the table below :

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

- Relation between cohesive and adhesive force



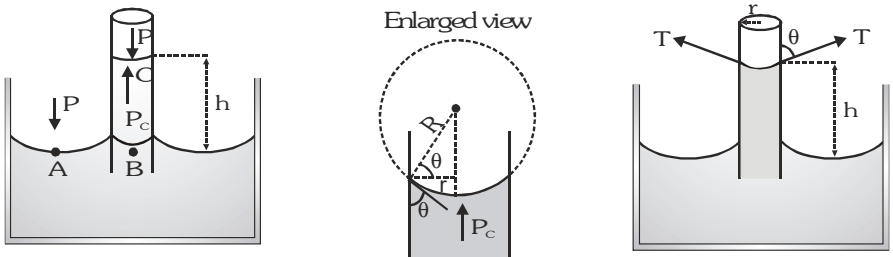
$F_A > \frac{F_C}{\sqrt{2}}$ $\quad$ $F_A = \frac{F_C}{\sqrt{2}}$ $\quad$ $F_A < \frac{F_C}{\sqrt{2}}$

• Shape of meniscus	Concave	Plane	Convex
• Angle of contact	$\theta_C < 90$ (Acute angle)	$\theta_C = 90$ (Right angle)	$\theta_C > 90$ (Obtuse angle)
• Shape of liquid drop			
• Level of liquid	Liquid rises up nor falls	Liquid neither rises	Liquid falls
• Wetting property	Liquid wets the solid surface	Liquid does not wet the solid surface	Liquid does not wet the solid surface
• Example	Glass – Water	Silver – Water	Glass – Mercury

CAPILLARY TUBE AND CAPILLARITY

A glass tube with fine bore and open at both ends is known as ***capillary tube***. The property by virtue of which a liquid rise or depress in a capillary tube is known as ***capillarity***. Rise or fall of liquid in tubes of narrow bore (capillary tube) is called ***capillary action***.

Calculation of Capillary Rise



When a capillary tube is first dipped in a liquid as shown in the figure, the liquid climbs up the walls curving the surface. The liquid continues to rise in the capillary tube until the weight of the liquid column becomes equal to force due to surface tension. Let the radius of the meniscus is R and the radius of the capillary tube is r. The angle of contact is θ, surface tension is T, density of liquid is ρ and the liquid rises to a height h.

Now let us consider two points A and B at the same horizontal level as shown. By Pascal's law

$P_A = P_B \Rightarrow \quad P_A = P_C + \rho gh \quad \Rightarrow \quad P_A - P_C = \rho gh \qquad (\because P_B = P_C + \rho gh)$

node06\E_NODE06 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Now, the point C is on the curved meniscus which has P_A and P_C as the two pressures on its concave and convex sides respectively.

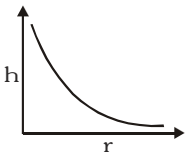
$$\therefore P_A - P_C = \frac{2T}{R} = \frac{2T}{r/\cos\theta} \Rightarrow \frac{2T}{r/\cos\theta} = \rho gh \Rightarrow 2T\cos\theta = r\rho gh \Rightarrow h = \frac{2T\cos\theta}{r\rho g}$$

Zurin's Law :

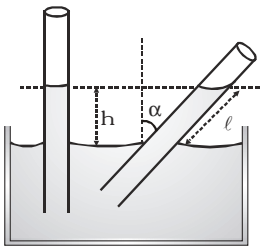
The height of rise of liquid in a capillary tube is inversely proportional to the radius of the capillary tube, if T, θ, ρ and g are constant $h \propto \frac{1}{r}$ or *rh* = constant. It implies that liquid will rise more in capillary tube of less radius and vice versa.

GOLDEN KEY POINTS

- For water and clean glass capillary $\theta_c \approx 0°$
- If angle of contact θ_c is acute then $\cos\theta_c$ is positive, so h is positive and liquid rises. If θ_c is obtuse then $\cos\theta_c$ is negative, so h is negative, therefore liquid depresses.
- Rise of liquid in a capillary tube does not obey the law of conservation of energy.
- For two capillaries of radii r_1 and r_2, the capillary rise h_1 and h_2 are such that $\frac{h_2}{h_1} = \frac{r_1}{r_2}$. The relation between h and r is graphically represented as



- Inside a satellite, water will rise upto top level but will not come outside. Radius of curvature(R') increases in such a way that final height h' is reduced and given by $h' = \frac{hR}{R'}$. (It is in accordance with Zurin's law).
- If a capillary tube is dipped into a liquid and tilted at an angle α from vertical then the vertical height of liquid column remains same whereas the length of liquid column in the capillary tube increases.



$$h = \ell\cos\alpha \Rightarrow \ell = \frac{h}{\cos\alpha}$$

- The height 'h' is measured from the bottom of the meniscus. However, there exist some liquid above this line also. If correction of this is applied then the formula will be $T = \frac{r\rho g\left[h + \frac{1}{3}r\right]}{2\cos\theta}$
- If a hollow sphere of radius r which has a fine hole, drowned in a vessel upto h depth, then liquid will not enter upto critical height h, given by $h\rho g = \frac{2T\cos\theta}{r}$ [normally $\theta \approx 0°$ therefore $\cos\theta \approx 1$]

node06\E_NODE65 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics\English\Theory.p65

Example

Calculate the height to which water will rise in a capillary tube of diameter 1 10^{-3} m. [Given : surface tension of water is 0.072 N m^{-1}, angle of contact is 0 , g = 9.8 m s^{-2} and density of water = 1000 kg m^{-3}]

Solution

Height of capillary rise $h = \frac{2T\cos\theta}{r\rho g} = \frac{2\times0.072\times\cos0^0}{5\times10^{-4}\times1000\times9.8}m = 2.94\times10^{-2}m$

Example

Water rises to a height of 20 mm in a capillary. If the radius of the capillary is made one third of its previous value then what is the new value of the capillary rise?

Solution

Since $h = \frac{2T\cos\theta}{rdg}$ and for the same liquid and capillaries of difference radii $h_1r_1 = h_2r_2$

$\therefore \frac{h_2}{h_1} = \frac{r_1}{r_2} = \frac{1}{(1/3)} = 3$ hence $h_2 = 3h_1 = 3$ 20 mm = 60 mm

node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-4\Fluid Mechanics \English\Theory.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 2

For – JEE (Advanced)

CHAPTER – 10

WAVE OPTICS – Theory & Exercise

Index

WAVE OPTICS

Serial No.	CONTENTS	Page No.
	Interference of Light	
1.	Properties of Light	1
2.	Wave Front	3
3.	Interference of Light	4
4.	Young's Double Slit Experiment	8
5.	Fresnel's Biprism	11
6.	Interference due to Thin Films	14
7.	**Exercise-1** (Single choice obj. questions)	23
8.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	27
9.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	30
10.	**Exercise-4A** (Conceptual Sub. Exercise)	38
11.	**Exercise-4B** (Brain Storming Exercise)	40
12.	**Exercise-5A** (Previous years questions)	42
13.	**Exercise-5B** (Previous years questions)	44
	Diffraction of Light	
14.	Diffraction of light	48
15.	Types of Diffrraction	49
16.	Resolving Power	51
	Polarization	
17.	Polarization of light	54
18.	Methods of obtaining plane polarised Light	55
19.	Optical Activity	58
20.	Exercise-I : Single Choice Objectives	60
21.	Exercise-II : Previous Years Questions	62

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Main

Wavefront and Huygens' principle, Laws of reflection and refraction using Huygen's principle. Interference, Young's double slit experiment and expression for fringe width, coherent sources and sustained interference of light. Diffraction due to a single slit, width of central maximum. Resolving power of microscopes and astronomical telescopes, Polarisation, plane polarized light; Brewster's law, uses of plane polarized light and Polaroids.

JEE Advanced

Wave nature of light: Huygen's principle, interference limited to Young's double-slit experiment.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

INTERFERENCE OF LIGHT

LIGHT

The physical cause, with the help of which our eyes experience the sensation of vision, is known as light or the form of energy, which excites our retina and produce the sensation of vision, is known as light.

PROPERTIES OF VISIBLE LIGHT

- No material medium is required for the propagation of light energy i.e. it travels even in vacuum.
- Its velocity is constant in all inertial frames i.e. it is an absolute constant. It is independent of the relative velocity between source and the observer.
- Its velocity in vacuum is maximum whose value is 3×10^8 m/s.
- It lies in the visible region of electromagnetic spectrum whose wavelength range is from 4000 Å to 8000 Å.
- Its energy is of the order of eV.
- It propagates in straight line.
- It exhibits the phenomena of reflection, refraction, interference, diffraction, polarisation and double refraction.
- It can emit electrons from metal surface i.e. it can produce photoelectric effect.
- It produces thermal effect and exerts pressure when incident upon a surface. It proves that light has momentum and energy.
- Its velocity is different in different media. In rarer medium it is more and in denser medium it is less.
- Light energy propagates via two processes.
  (a) The particles of the medium carry energy from one point of the medium to another.
  (b) The particles transmit energy to the neighbouring particles and in this way energy propagates in the form of a disturbance.

DIFFERENT THEORIES OF LIGHT

- Newton's corpuscular theory of light.
- Maxwell's electromagnetic theory of light.
- De-Broglie's dual theory of light.
- Hygen's wave theory of light.
- Plank's Quantum theory of light.

NEWTON'S CORPUSCULAR THEORY OF LIGHT

This theory was enuciated by Newton.

- **Characteristics of the theory**
  (i) Extremely minute, very light and elastic particles are being constantly emitted by all luminous bodies (light sources) in all directions
  (ii) These corpuscles travel with the speed of light..
  (iii) When these corpuscles strike the retina of our eye then they produce the sensation of vision.
  (iv) The velocity of these corpuscles in vacuum is 3×10^8 m/s.
  (v) The different colours of light are due to different size of these corpuscles.
  (vi) The rest mass of these corpuscles is zero.
  (vii) The velocity of these corpuscles in an isotropic medium is same in all directions but it changes with the change of medium.
  (viii) These corpuscles travel in straight lines.
  (ix) These corpuscles are invisible.

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

- **The phenomena explained by this theory**

  (i) Reflection and refraction of light. (ii) Rectilinear propagation of light.

  (iii) Existence of energy in light.

- **The phenomena not explained by this theory**

  (i) Interference, diffraction, polarisation, double refraction and total internal reflection.

  (ii) Velocity of light being greater in rarer medium than that in a denser medium.

  (iii) Photoelectric effect and Crompton effect.

WAVE THEORY OF LIGHT

This theory was enunciated by Hygen in a hypothetical medium known as luminiferrous ether.

Ether is that imaginary medium which prevails in all space, in isotropic, perfectly elastic and massless.

The different colours of light are due to different wave lengths of these waves.

The velocity of light in a medium is constant but changes with change of medium.

This theory is valid for all types of waves.

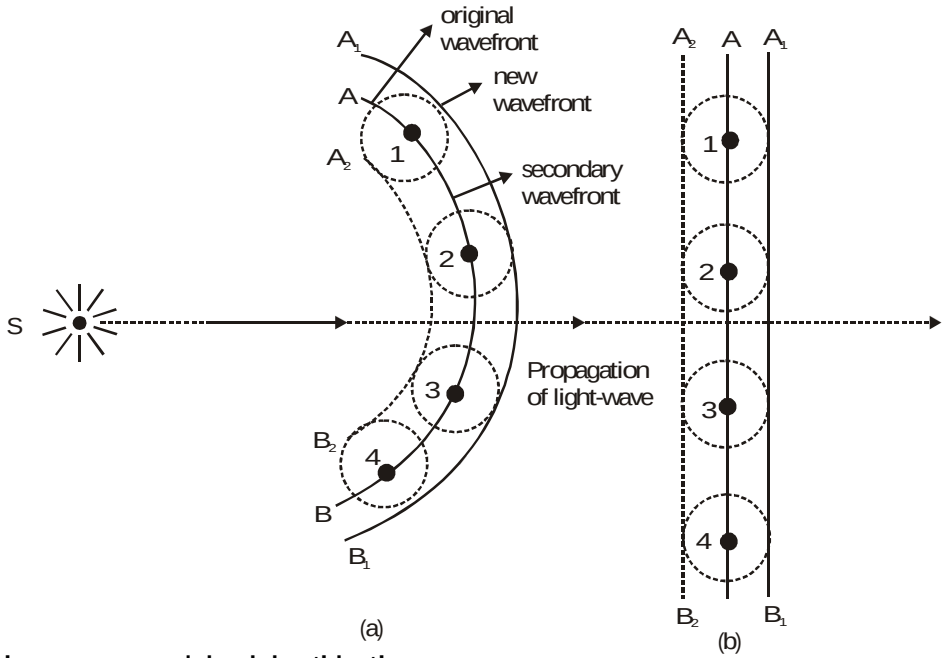
(i) The locus of all ether particles vibrating in same phase is known as wavefront.

(ii) Light travels in the medium in the form of wavefront.

(iii) When light travels in a medium then the particles of medium start vibrating and consequently a disturbance is created in the medium.

(iv) Every point on the wave front becomes the source of secondary wavelets. It emits secondary wavelets in all directions which travel with the speed of light (v),

The tangent plane to these secondary wavelets represents the new position of wave front.



The phenomena explained by this theory

(i) Reflection, refraction, interference, diffraction, polarisation and double refraction.

(ii) Rectilinear propagation of light.

(iii) Velocity of light in rarer medium being grater than that in denser medium.

Phenomena not explained by this theory

(i) Photoelectric effect, Compton effect and Raman effect.

(ii) Backward propagation of light.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

WAVE FRONT, VARIOUS TYPES OF WAVE FRONT AND RAYS

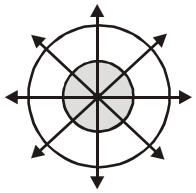
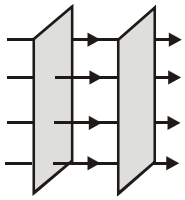
- **Wavefront**

  The locus of all the particles vibrating in the same phase is known as wavefront.

- **Types of wavefront**

  The shape of wavefront depends upon the shape of the light source originating that wavefront. On the basis of there are three types of wavefront.

Comparative study of three types of wavefront

S.No.	Wavefront	Shape of light source	Diagram of shape of wavefront	Variation of amplitude with distance	Variation of intensity with distance
1.	Spherical	Point source		$A \propto \frac{1}{d}$ or $A \propto \frac{1}{r}$	$I \propto \frac{1}{r^2}$
2.	cylindrical	Linear or slit		$A \propto \frac{1}{\sqrt{d}}$ or $A \propto \frac{1}{\sqrt{r}}$	$I \propto \frac{1}{r}$
3.	Plane	Extended large source situated at very large distance		A = constant	I = constant

CHARACTERISTIC OF WAVEFRONT

(a) The phase difference between various particles on the wavefront is zero.

(b) These wavefronts travel with the speed of light in all directions in an isotropic medium.

(c) A point source of light always gives rise to a spherical wavefront in an isotropic medium.

(d) In an anisotropic medium it travels with different velocities in different directions.

(e) Normal to the wavefront represents a ray of light.

(f) It always travels in the forward direction of the medium.

RAY OF LIGHT

The path of the light energy from one point to another is known as a ray of light.

(a) A line drawn at right angles to the wavefront is defined as a ray of light, which is shown by arrows in previous diagram of shape of wavefront.

(b) It represents the direction of propagation of light.

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

INTERFERENCE OF LIGHT

When two light waves of same frequency with zero initial phase difference or constant phase difference superimpose over each other, then the resultant amplitude (or intensity) in the region of superimposition is different from the amplitude (or intensity) of individual waves.

This modification in intensity in the region of superposition is called interference.

(a) Constructive interference

When resultant intensity is greater than the sum of two individual wave intensities [$I > (I_1 + I_2)$], then the interference is said to be constructive.

(b) Destructive interference

When the resultant intensity is less than the sum of two individual wave intensities [$I < (I_1 + I_2)$], then the interference is said to destructive.

There is no violation of the law of conservation of energy in interference. Here, the energy from the points of minimum energy is shifted to the points of maximum energy.

TYPES OF SOURCES

- **Coherent source**

Two sources are said to be coherent if they emit light waves of the same wavelength and start with same phase or have a constant phase difference.

Note : Laser is a source of monochromatic light waves of high degree of coherence.

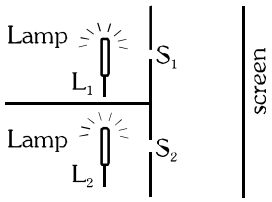
Main points :

1. They are obtained from the same single source.
2. Their state of polarization is the same

- **Incoherent source**

Two independent monochromatic sources, emit waves of same wavelength.

But the waves are not in phase. So they are incoherent. This is because, atoms cannot emit light waves in same phase and these sources are said to be incoherent sources. By using two independent laser beams it has been possible to record the interference pattern.



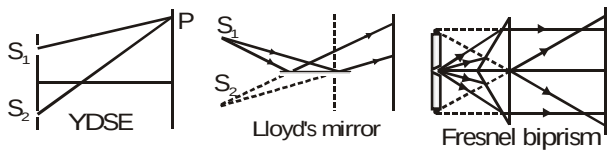
Independent Source
(non-coherent source)

METHOD FOR OBTAINING COHERENT SOURCE

- **Division of wave front**

In this method, the wavefront is divided into two or more parts by use of mirrors, lenses or prisms.

Example : Young's double slit experiment. Fresnel's Biprism and Lloyd's single mirror method.

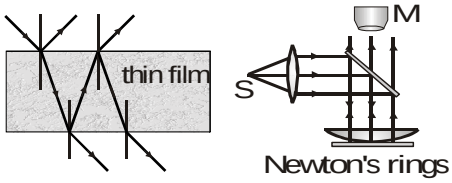


- **Division of amplitude**

The amplitude of incoming beam is divided into two or more parts by partial reflection or refraction. These divided parts travel different paths and are finally brought together to produce interference.

Example : The brilliant colour seen in a thin film of transparent material like soap film, oil film, Michelson's Interferro Meter, Newtons' ring etc.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65



Condition for sustained interference

To obtain the stationary interference pattern, the following conditions must be fulfilled :

(a) The two sources should be coherent, i.e., they should vibrate in the same phase or there should be a constant phase difference between them.

(b) The two sources must emit continuously waves of same wavelength and frequency.

(c) The separation between two coherent sources should be small.

(d) The distance of the screen from the two sources should be small.

(e) For good contrast between maxima and minima, the amplitude of two interfering waves should be as nearly equal as possible and the background should be dark.

(f) For a large number of fringes in the field of view, the sources should be narrow and monochromatic.

ANALYSIS OF INTERFERENCE OF LIGHT

When two light waves having same frequency and equal or nearly equal amplitude are moving in the same direction, They superimpose each other, at some point the intensity of light is maximum and at some point it is minimum this phenomenon is known as interference of light.

Let two waves having amplitude a_1 and a_2 and same frequency, same phase difference ϕ superpose. Let their displacement are : $y_1 = a_1 \sin \omega t$ and $y_2 = a_2 \sin (\omega t + \phi)$

By principle of superposition.

$$y = y_1 + y_2 = a_1 \sin \omega t + a_2 \sin (\omega t + \phi) = a_1 \sin \omega t + a_2 [\sin \omega t \cos \phi + \cos \omega t \sin \phi]$$

$$= \sin \omega t (a_1 + a_2 \cos \phi) + a_2 \cos \omega t \sin \phi$$

Let, $a_1 + a_2 \cos \phi = A \cos \theta$ and $a_2 \sin \phi = A \sin \theta$

Hence $y = A \sin \omega t \cos \theta + A \cos \omega t \sin \theta = A \sin (\omega t + \theta)$

Resultant amplitude $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos\phi}$ and Phase angle $\theta = \tan^{-1} \dfrac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$

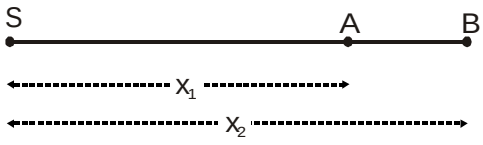
Intensity $\propto$ (Amplitude)^2 $\Rightarrow I \propto A^2 \Rightarrow$ $I = KA^2$ so $I_1 = Ka_1^2$ & $I_2 = Ka_2^2$ $\therefore$ $I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \cos \phi$

here, $2\sqrt{I_1}\sqrt{I_2} \cos \phi$ is known as interference factor.

- If the distance of a source from two points A and B is x_1 and x_2 then

Path difference $\delta = x_2 - x_1$

Phase difference $\phi = \dfrac{2\pi}{\lambda} (x_2 - x_1) \Rightarrow \phi = \dfrac{2\pi}{\lambda} \delta$



Time difference $\Delta t = \dfrac{\phi}{2\pi} t$

$$\frac{\text{Phase difference}}{2\pi} = \frac{\text{Path difference}}{\lambda} = \frac{\text{Time difference}}{T} \Rightarrow \frac{\phi}{2\pi} = \frac{\delta}{\lambda} = \frac{\Delta t}{T}$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

TYPES OF INTERFERENCE

Constructive Interference

When both waves are in same phase. So phase difference is an even multiple of $\pi \Rightarrow \phi = 2n\pi$; n = 0,1,2 ...

- When path difference is an even multiple of $\frac{\lambda}{2}$

$\because \frac{\phi}{2\pi} = \frac{\delta}{\lambda} \Rightarrow \frac{2n\pi}{2\pi} = \frac{\delta}{\lambda} \Rightarrow \delta = 2n\left(\frac{\lambda}{2}\right) \Rightarrow \delta = n\lambda$ (where n = 0,1,2...)

- When time difference is an even multiple of $\frac{T}{2}$ $\quad \therefore \Delta t = 2n\left(\frac{T}{2}\right)$

- In this condition the resultant amplitude and Intensity will be maximum.

$A_{max} = (a_1 + a_2) \Rightarrow I_{max} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} = (\sqrt{I_1} + \sqrt{I_2})^2$

Destructive Interference

When both the waves are in opposite phase. So phase difference is an odd multiple of π .

$\phi = (2n-1)\pi$; n = 1, 2 ...

- When path difference is an odd multiple of $\frac{\lambda}{2}$, $\delta = (2n-1)\frac{\lambda}{2}$, n = 1, 2 ...

- When time difference is an odd multiple of $\frac{T}{2}$, $\Delta t = (2n-1)\frac{T}{2}$, (n=1,2...)

In this condition the resultant amplitude and intensity of wave will be minimum.

$A_{min} = (a_1 - a_2) \Rightarrow I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$

GOLDEN KEY POINTS

- Interference follows law of conservation of energy.
- Average Intensity $I_{av} = \frac{I_{max} + I_{min}}{2} = I_1 + I_2 = a_1^2 + a_2^2$
- Intensity $\propto$ width of slit $\propto$ (amplitude)$^2 \Rightarrow I \propto w \propto a^2 \Rightarrow \frac{I_1}{I_2} = \frac{w_1}{w_2} = \frac{a_1^2}{a_2^2}$
- $\frac{I_{max}}{I_{min}} = \left[\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right]^2 = \left[\frac{a_1 + a_2}{a_1 - a_2}\right]^2 = \left[\frac{a_{max}}{a_{min}}\right]^2$
- Fringe visibility $V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \times 100\%$ when $I_{min} = 0$ then fringe visibility is maximum

i.e. when both slits are of equal width the fringe visibility is the best and equal to 100%.

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Example

If two waves represented by $y_1 = 4 \sin \omega t$ and $y_2 = 3 \sin \left(\omega t + \frac{\pi}{3}\right)$ interfere at a point. Find out the amplitude of the resulting wave.

Solution

Resultant amplitude $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\phi} = \sqrt{(4)^2 + (3)^2 + 2.(4)(3)\cos\frac{\pi}{3}} \Rightarrow A \simeq 6$

Example

Two beams of light having intensities I and 4I interferer to produce a fringe pattern on a screen. The phase difference between the beam is $\frac{\pi}{2}$ at point A and 2π at point B. Then find out the difference between the resultant intensities at A and B.

Solution

Resultant intensity $I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\phi$

Resultant intensity at point A is $I_A = I + 4I + 2\sqrt{I}\sqrt{4I}\cos\frac{\pi}{2} = 5I$

Resultant intensity at point B, $I_B = I + 4I + 2\sqrt{I}\sqrt{4I}\cos 2\pi = 9I$ $(\because \cos 2\pi = 1)$ $\therefore I_B - I_A = 9I - 5I \Rightarrow 4I$

Example

In interference pattern, if the slit widths are in the ratio 1:9. Then find out the ratio of minimum and maximum intensity.

Solution

Slit width ratio

$$\frac{w_1}{w_2} = \frac{1}{9} \because \frac{I_1}{I_2} = \frac{w_1}{w_2} = \frac{a_1^2}{a_2^2} = \frac{1}{9} \Rightarrow \frac{a_1}{a_2} = \frac{1}{3} \Rightarrow 3a_1 = a_2 \therefore \frac{I_{min}}{I_{max}} = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} = \frac{(a_1 - 3a_1)^2}{(a_1 + 3a_1)^2} = \frac{4}{16} = 1:4$$

Example

The intensity variation in the interference pattern obtained with the help of two coherent source is 5% of the average intensity. Find out the ratio of intensities of two sources.

Solution

$$\frac{I_{max}}{I_{min}} = \frac{105}{95} = \frac{21}{19} \Rightarrow \frac{(a_1 + a_2)^2}{(a_1 - a_2)^2} = \frac{21}{19} \Rightarrow \frac{a_1 + a_2}{a_1 - a_2} = \sqrt{\frac{21}{19}} = 1.05 \Rightarrow a_1 + a_2 = 1.05\, a_1 - 1.05\, a_2$$

$$0.05\, a_1 = 2.05\, a_2 \Rightarrow \frac{a_1}{a_2} = \frac{2.05}{0.05} = \frac{41}{1} \quad \therefore \frac{I_1}{I_2} = \frac{a_1^2}{a_2} = \frac{1680}{1}$$

Example

Waves emitted by two identical sources produces intensity of K unit at a point on screen where path difference between these waves is λ, calculate the intensity at that point on screen at which path difference is $\frac{\lambda}{4}$.

Solution

$$\phi_1 = \frac{2\pi\delta}{\lambda} \Rightarrow \frac{2\pi}{\lambda} \times \lambda = 2\pi \text{ and } \phi_2 = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2} \quad I_1 = I_0 + I_0 + 2\sqrt{I_0}\sqrt{I_0}\cos 2\pi = 4I_0$$

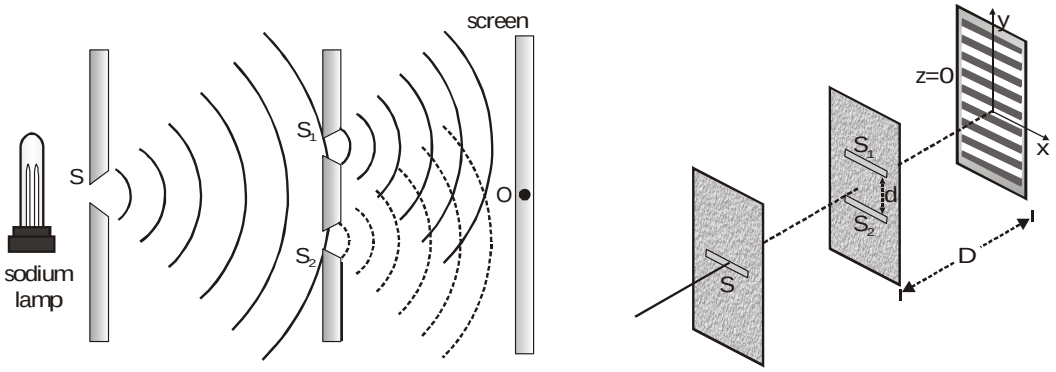
and $I_2 = I_0 + I_0 + 2\sqrt{I_0}\sqrt{I_0}\frac{\cos 2\pi}{2} = 2I_0 \quad \therefore \frac{I_1}{I_2} = \frac{4I_0}{2I_0} = 2 \quad \Rightarrow I_2 = \frac{I_1}{2} = \frac{K}{2}$ unit [$\because I_1 = K$ unit]

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

YOUNG'S DOUBLE SLIT EXPERIMENT (YDSE)

According to Huygen, light is a wave. It is proved experimentally by YDSE.

S is a narrow slit illuminated by a monochromatic source of light sends wave fronts in all directions. Slits S_1 and S_2 become the source of secondary wavelets which are in phase and of same frequency. These waves are superimposed on each other gave rise to interference. Alternate dark and bright bands are obtained on a screen (called interference fringes) placed certain distance from the plane of slit S_1 and S_2. Central fringe is always bright (due to path from S_1O and S_2O centre is equal) called central maxima.



Energy is conserved in interference. This indicated that energy is redistributed from destructive interference region to the constructive interference region .

- If one of the two slit is closed. The interference pattern disappears. It shows that two coherent sources are required to produce interference pattern.
- If white light is used as parent source, then the fringes will be coloured and of unequal width.

(i) Central fringe will be white.

(ii) As the wave length of violet colour is least, so fringe nearest to either side of the central white fringe is violet and the fringe farthest from the central white fringe is red.

CONDITION FOR BRIGHT AND DARK FRINGES

Bright Fringe

D = distance between slit and screen, d = distance between slit S_1 and S_2

Bright fringe occurs due to constructive interference.

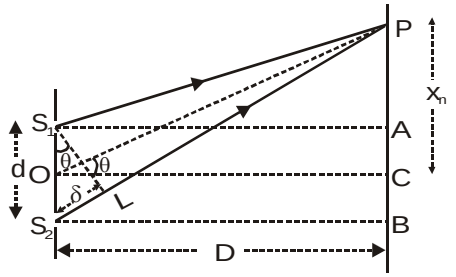
$\because$ For constructive interference path difference should be even multiple of $\frac{\lambda}{2}$

$\therefore$ Path difference $\delta = PS_2 - PS_1 = S_2L = (2n)\frac{\lambda}{2}$

In ΔPCO $\tan\theta = \frac{x_n}{D}$; In ΔS_1S_2L $\sin\theta = \frac{\delta}{d}$

$\delta = n\lambda$ for bright fringes

If θ is small then $\tan\theta \simeq \sin\theta \Rightarrow \frac{x_n}{D} = \frac{\delta}{d}$



The distance of n^{th} bright fringe from the central bright fringe $x_n = n\frac{D\lambda}{d}$

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Dark Fringe

Dark fringe occurs due to destructive interference.

$\because$ For destructive interference path difference should be odd multiple of $\frac{\lambda}{2}$.

$\therefore$ Path difference $\delta = (2m - 1)\frac{\lambda}{2}$

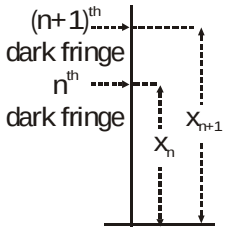
The distance of the m^{th} dark fringe from the central bright fringe $x_m = \frac{(2m-1)D\lambda}{2d}$

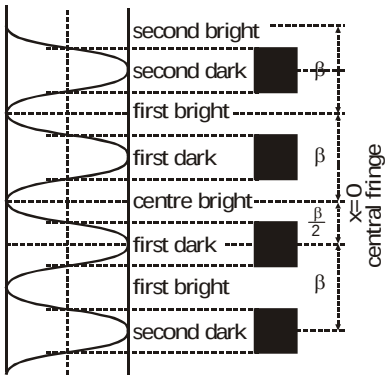
FRINGE WIDTH

The distance between two successive bright or dark fringe is known as fringe width.

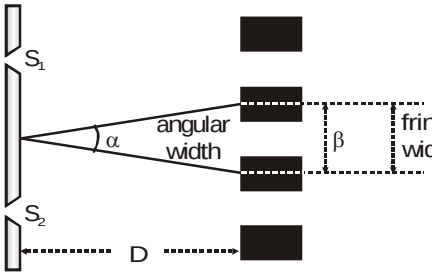
$$\beta = x_{n+1} - x_n = \frac{(n+1)D\lambda}{d} - \frac{nD\lambda}{d}$$

Fringe Width $\beta = \frac{D\lambda}{d}$





ANGULAR FRINGE WIDTH



Angular Fringe Width $\alpha = \frac{\beta}{D}$, $\alpha = \frac{\lambda}{d}$ $\left[\because \frac{\beta}{D} = \frac{\lambda}{d}\right]$

- The distance of n^{th} bright fringe from the central bright fringe $x_n = \frac{n\lambda D}{d} = n\beta$
- The distance between n_1 and n_2 bright fringe $x_{n_2} - x_{n_1} = n_2\frac{\lambda D}{d} - n_1\frac{\lambda D}{d} = (n_2 - n_1)\beta$
- The distance of m^{th} dark fringe from central fringe $x_m = \frac{(2m-1)D\lambda}{2d} = \frac{(2m-1)\beta}{2}$
- The distance of n^{th} bright fringe from m^{th} dark fringe $x_n - x_m = n\frac{D\lambda}{d} - \frac{(2m-1)D\lambda}{2d} = n\beta - \frac{(2m-1)\beta}{2}$

$$x_n - x_m = \left[n - \frac{(2m-1)}{2}\right]\beta$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

GOLDEN KEY POINTS

- If the whole apparatus is immersed in a liquid of refractive index μ, then wavelength of light $\lambda' = \frac{\lambda}{\mu}$ since $\mu > 1$ so $\lambda' < \lambda \Rightarrow$ wavelength will decrease. Hence fringe width ($\beta \propto \lambda$) will decrease
  $\Rightarrow$ fringe width in liquid $\beta' = \beta/\mu$ angular width will also decrease.
- With increase in distance between slit and screen D, angular width of maxima does not change, fringe width β increase linearly with D but the intensity of fringes decreases.
- If an additional phase difference of π is created in one of the wave then the central fringe become dark.
- When wavelength λ_1 is used to obtain a fringe n_1. At the same point wavelength λ_2 is required to obtain a fringe n_2 then $n_1\lambda_1 = n_2\lambda_2$
- When waves from two coherent sources S_1 and S_2 interfere in space the shape of the fringe is hyperbolic with foci at S_1 and S_2.

Example

Laser light of wavelength 630 nm incident on a pair of slits produces an interference pattern in which the bright fringes are separated by 8.1 mm. A second light produces an interference pattern in which the fringes are separated by 7.2 mm. Calculate the wavelength of the second light.

Solution

Fringe separation is given by $\beta = \frac{\lambda D}{d}$ i.e. $\frac{\beta_2}{\beta_1} = \frac{\lambda_2}{\lambda_1} \Rightarrow \lambda_2 = \frac{\beta_2}{\beta_1} \times \lambda_1 = \frac{7.2}{8.1} \times 630 = 560$ nm

Example

A double slit is illuminated by light of wave length 6000Å. The slit are 0.1 cm apart and the screen is placed one metre away. Calculate :

(i) The angular position of the 10th maximum in radian and

(ii) Separation of the two adjacent minima.

Solution

(i) $\lambda = 6000$ Å $= 6 \times 10^{-7}$ m, $d = 0.1$ cm $= 1 \times 10^{-3}$ m, $D = 1$m, $n = 10$

Angular position $\theta_n = \frac{n\lambda}{d} = \frac{10 \times 6 \times 10^{-7}}{10^{-3}} = 6 \times 10^{-3}$ rad.

(ii) Separation between two adjacent minima = fringe width β

$$\beta = \frac{\lambda D}{d} = \frac{6 \times 10^{-7} \times 1}{1 \times 10^{-3}} = 6 \times 10^{-4} \text{ m} = 0.6 \text{ mm}$$

Example

In Young's double slit experiment the fringes are formed at a distance of 1m from double slit of separation 0.12 mm. Calculate

(i) The distance of 3rd dark band from the centre of the screen.

(ii) The distance of 3rd bright band from the centre of the screen, given $\lambda = 6000$Å

Solution

(i) For mth dark fringe $x'_m = (2m-1)\frac{D\lambda}{2d}$ given, D = 1m = 100 cm, d = 0.12 mm = 0.012 cm

$$x'_3 = \frac{(2 \times 3 - 1) \times 100 \times 6 \times 10^{-7}}{2 \times 0.012} = 1.25 \text{ cm} \quad [\because m = 3 \text{ and } \lambda = 6 \times 10^{-7} \text{ m}]$$

(ii) For nth bright fringe $x_n = \frac{nD\lambda}{d} \Rightarrow x_3 = \frac{3 \times 100 \times 6 \times 10^{-7}}{0.012} = 1.5 \times 10^{-2}\text{ m} = 1.5 \text{ cm} \quad [\because n = 3]$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Example

In Young's double slit experiment the two slits are illuminated by light of wavelength 5890Å and the distance between the fringes obtained on the screen is 0.2°. The whole apparatus is immersed in water, then find out angular fringe width, (refractive index of water = $\frac{4}{3}$).

Solution

$$\alpha_{air} = \frac{\lambda}{d} \Rightarrow \alpha_{air} = 0.2^\circ \Rightarrow \alpha \propto \lambda \Rightarrow \frac{\alpha_w}{\alpha_{air}} = \frac{\lambda_w}{\lambda_{air}} \Rightarrow \lambda_w = \frac{\lambda_{air}}{\mu} \Rightarrow \alpha_w = \frac{\alpha_{air}\,\lambda}{\mu.\lambda} = \frac{0.2\times 3}{4} = 0.15$$

Example

The path difference between two interfering waves at a point on screen is 171.5 times the wavelength. If the path difference is 0.01029 cm. Find the wavelength.

Solution

Path difference = 171.5 $\lambda = \frac{343}{2}\lambda$ = odd multiple of half wavelength . It means dark finge is observed

According to question $0.01029 = \frac{343}{2}\lambda \Rightarrow \lambda = \frac{0.01029\times 2}{343} = 6\times 10^{-5}$ cm $\Rightarrow \lambda$ = 6000 Å

Example

In young's double slit interference experiment, the distance between two sources is $0.1/\pi$ mm. The distance of the screen from the source is 25 cm. Wavelength of light used is 5000Å. Then what is the angular position of the first dark fringe ?

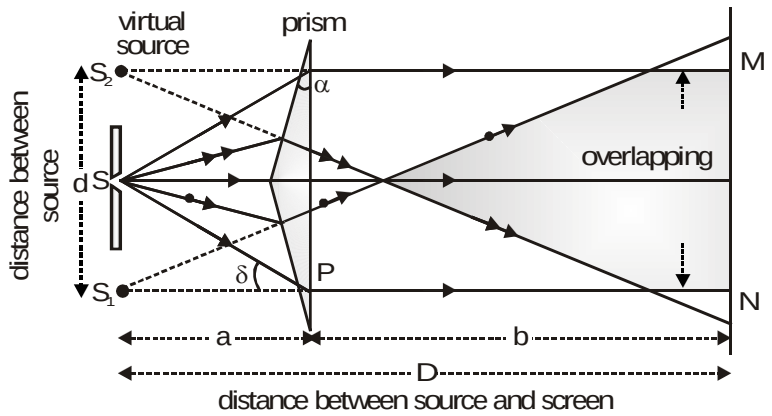
Solution

The angular position $\theta = \frac{\beta}{D} = \frac{\lambda}{d}$ $(\because \beta = \frac{\lambda D}{d})$ The first dark fringe will be at half the fringe width from the mid point of central maximum. Thus the angular position of first dark fringe will be-

$$\alpha = \frac{\theta}{2} = \frac{1}{2}\left[\frac{\lambda}{d}\right] = \frac{1}{2}\left[\frac{5000\times\pi}{.1\times 10^{-3}}\times 10^{-10}\right]\frac{180}{\pi} = 0.45^\circ.$$

FRESNEL'S BIPRISM

It is an optical device to obtain two coherent sources by refraction of light. It is prepared by rubbing an optically pure glass plate slightly on two sides so that each angle of prism is generally $\frac{1^\circ}{2}$ or 1°. The fringes of equal width are observed in the limited region MN due to superposition.



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

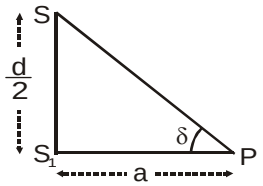
Distance between source and biprism = a

Distance between biprism and eye piece (screen) = b

The distance between source and screen D = a + b

Refracting angle = α , refractive index of the material of prism = μ

The distance between two coherent source = d



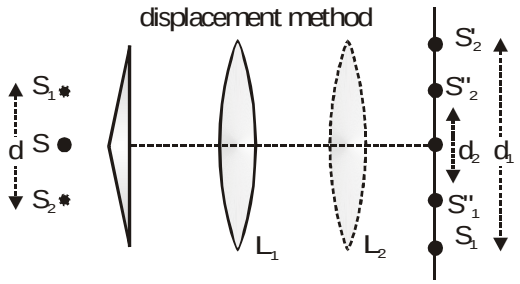
From ΔSS_1P $\tan\delta = \frac{d/2}{a}$ for very-very small δ hence tan $\delta = \delta$ so $\delta = \frac{d}{2a} \Rightarrow$ d = 2aδ

For prism $\delta = (\mu-1)\alpha$ $\therefore$ d = 2a$(\mu-1)\alpha$, Fringe width $\beta = \frac{\lambda D}{d}$ $\therefore$ $\beta = \frac{(a+b)\lambda}{2a(\mu-1)\alpha}$

- **To calculate the value of d by displacement method**

In this method a convex lens is placed between prism and screen. The lens is adjusted in two position L_1 and L_2 and image is obtained on screen. Let d_1 an d_2 be the real image in these two cases.

The distance d between the virtual source $d = \sqrt{d_1 d_2}$ Fringe width $\beta = \frac{(a+b)\lambda}{\sqrt{d_1 d_2}}$



GOLDEN KEY POINTS

- If the fresnel biprism experiment is performed in water instead of air then

(i) Fringe width in water increases $\left[\beta_w = \frac{\mu_g - 1}{\mu_g - \mu_w}\beta_{air}\right]$ $\beta_w = 3\,\beta_{air}$ $\left[\because \mu_g = \frac{3}{2}, \mu_w = \frac{4}{3}\right]$

(ii) Separation between the two virtual sources decreases.

(but in Young's double slit experiment it does not change.)

$\because$ d_{air} = 2 a $(\mu_g - 1)\,\alpha$ $\therefore$ d_w = 2 a $({}_w\mu_g - 1)\,\alpha$ $\Rightarrow$ $d_w = 2a\left[\frac{\mu_g}{\mu_w} - 1\right]\alpha$

$\therefore$ $\frac{d_w}{d_{air}} = \frac{\frac{\mu_g}{\mu_w} - 1}{\mu_g - 1} = \frac{3/2 - 1}{4/3 - 1} = \frac{1}{4} \Rightarrow$ $d_w = \frac{1}{4}\,d_{air}$

- If we use white light instead of monochromatic light then coloured fringes of different width are obtained. Central fringe is white.
- With the help of this experiment the wavelength of monochromatic light, thickness of thin films and their refractive index and distance between apparent coherent sources can be determined.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Example

Fringes are obtained with the help of a biprism in the focal plane of an eyepiece distant 1m from the slit. A convex lens produces images of the slit in two position between biprism and eyepiece. The distances between two images of the slit in two positions are 4.05×10^{-3} m and 2.9×10^{-3} m respectively. Calculate the distance between the slits.

Solution

$$d = \sqrt{d_1 d_2} = \sqrt{4.05 \times 10^{-3} \times 2.9 \times 10^{-3}} = 3.43 \times 10^{-3} \text{ m}$$

Example

In fresnel's biprism experiment a mica sheet of refractive index 1.5 and thickness 6×10^{-6} m is placed in the path of one of interfering beams as a result of which the central fringe gets shifted through five fringe widths. Then calculate the wavelength of light.

Solution

$$x = \frac{(\mu-1)t\beta}{\lambda} = \frac{(1.5-1)t\beta}{\lambda} \text{ but, } t = 5\beta \ \therefore \ 5\beta = \frac{0.5\ t\beta}{\lambda} \Rightarrow \lambda = \frac{t}{10} = \frac{6\times10^{-6}}{10} = 6000\text{Å}$$

Example

A whole biprism experiment is immersed in water. If the fringe width in air is β_a and refractive index of biprism material and water are 1.5 and 1.33 respectively. Find the value of the fringe width.

Solution

$$\beta_w = \frac{\frac{\mu_g}{1}}{\frac{\mu_g}{\mu_w}} \beta_a = \frac{\frac{3}{2}}{\frac{3}{2}\ \frac{1}{\frac{4}{2}}} \beta_a$$

Example

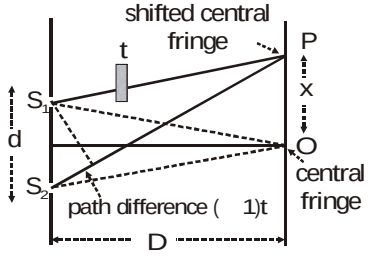
In fresnel's biprism experiment the distance between the source and the screen is 1m and that between the source and biprism is 10 cm. The wavelength of light used is 6000Å. The fringe width obtained is 0.03 cm and the refracting angle of biprism is 1°. Then calculate the refractive index of the material of biprism.

Solution

$$\beta = \frac{D\lambda}{2a(\mu-1)\alpha} \therefore (\mu-1) = \frac{D\lambda}{2a\beta\alpha} = \frac{1\times6\times10^{-7}\times180}{2\times0.1\times3\times10^{-4}\times3.14} \Rightarrow (\mu-1) = 0.573 \Rightarrow \mu = 1.573$$

THICKNESS OF THIN FILMS

When a glass plate of thickness t and refractive index μ is placed in front of the slit in YDSE then the central fringe shifts towards that side in which glass plate is placed because extra path difference is introduced by the glass plate. In the path S_1P distance travelled by wave in air = $S_1P - t$



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Distance travelled by wave in the sheet = t

Time taken by light to reach up to point P will be same from S_1 and S_2

$$\frac{S_2P}{c}=\frac{S_1P-t}{c}+\frac{t}{c/\mu}\Rightarrow\frac{S_2P}{c}=\frac{S_1P+(\mu-1)t}{c} \Rightarrow S_2P = S_1P + (\mu-1)t \Rightarrow S_2P - S_1P = (\mu-1)t$$

Path difference = $(\mu-1)t \Rightarrow$ Phase difference $\phi = \frac{2\pi}{\lambda}(\mu-1)t$

Distance of shifted fringe from central fringe $x = \frac{D(\mu-1)t}{d}$ $\left[\because \frac{xd}{D}=(\mu-1)t\right]$

$\therefore$ $x = \frac{\beta(\mu-1)t}{\lambda}$ and $\beta = \frac{D\lambda}{d}$ Number of fringes displaced = $\frac{(\mu-1)t}{\lambda}$

Example

When a mica sheet of thickness 7 microns and μ = 1.6 is placed in the path of one of interfering beams in the biprism experiment then the central fringe gets at the position of seventh bright fringe. What is the wavelength of light used ?

Solution

$$\lambda=\frac{(\mu-1)t}{n}=\frac{(1.6-1)\,7\times10^{-6}}{7}=6\times10^{-7}\ \text{meter}$$

GOLDEN KEY POINTS

- If a glass plate of refractive index μ_1 and μ_2 having same thickness t is placed in the path of ray coming from S_1 and S_2 then path difference $x = \frac{D}{d}(\mu_1-\mu_2)t$
- Distance of displaced fringe from central fringe $x = \frac{\beta(\mu_1-\mu_2)t}{\lambda}$ $\because \frac{\beta}{\lambda}=\frac{D}{d}$

COLOURS IN THIN FILMS

When white light is made incident on a thin film (like oil film on the surface of water or a soap bubble) Then interference takes place between the waves reflected from its two surfaces and waves refracted through it. The intensity becomes maximum and minimum as a result of interference and colours are seen.

(i) The source of light must be an extended source

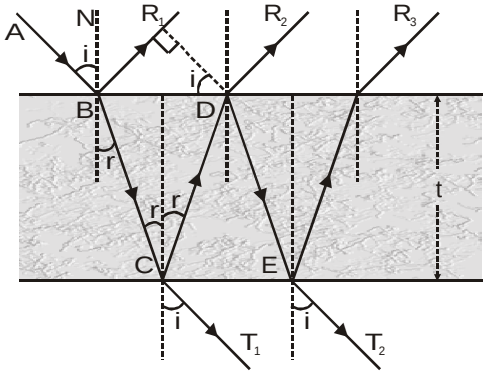
(ii) The colours obtained in reflected and transmitted light are mutually complementary.

(iii) The colours obtains in thin films are due to interference whereas those obtained in prism are due to dispersion.

INTERFERENCE DUE TO THIN FILMS

Consider a thin transparent film of thickness t and refractive index μ . Let a ray of light AB incident on the film at B. At B, a part of light is reflected along BR_1, and a part of light refracted along BC. At C a part of light is reflected along CD and a part of light transmitted along CT_1. At D, a part of light is refracted along DR_2 and a part of light is reflected along DE. Thus interference in this film takes place due to reflected light in between BR_1 and DR_2 also in transmitted light in between CT_1 and ET_2.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65



- **Reflected System**

The path difference between BR_1 and DR_2 is $x = 2\mu t \cos r$ due to reflection from the surface of denser medium involves an additional phase difference of π or path difference $\lambda/2$. Therefore the exact path difference between BR_1 and DR_2 is. $\Rightarrow x' = 2\ \mu t \cos r - \lambda/2$ maximum or constructive Interference occurs when path difference between the light waves is $n\lambda$.$2\ \mu t \cos r - \lambda/2 = n\lambda \Rightarrow 2\ \mu t \cos r = n\lambda + \lambda/2$

So the film will appear bright if $2\ \mu t \cos r = (2n + 1)\ \lambda/2$ (n = 0, 1, 2, 3 .....)

- **For minima or destructive interference :**

When path difference is odd multiple of $\frac{\lambda}{2} \Rightarrow 2\ \mu t \cos r - \frac{\lambda}{2} = (2n - 1)\frac{\lambda}{2}$

So the film will appear dark if $2\ \mu t \cos r = n\ \lambda$

- **For transmitted system**

Since No additional path difference between transmitted rays C T_1 and E T_2 .

So the net path difference between them is $x = 2\ \mu t \cos r$

For maxima $2\ \mu t \cos r = n\lambda$, n = 0, 1, 2.....

Minima $2\ \mu t \cos r = (2n + 1)\ \frac{\lambda}{2}$, n = 0, 1, 2.....s

USES OF INTERFERENCE EFFECT

Thin layer of oil on water and soap bubbles show different colours due to interference of waves reflected from two surfaces of their films. Similarly when a lens of large radius of curvature is placed on a plane glass plate, an air film exist between the plate and the lens. If sodium light is put on this film, concentric bright and dark interference rings are formed. These rings are called as Newton's rings.

Uses :

- Used to determine the wavelength of light precisely.
- Used to determine refractive index or thickness of transparent sheet.
- Used to test the flatness of plane surfaces. These surfaces are knows as optically plane surfaces.
- Used to calibrate meters in terms of wavelength of light.
- Used to design optical filter which allows a narrow band of wavelength to pass through it.
- Used in holography to produce 3-D images.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Example

Light of wavelength 6000Å is incident on a thin glass plate of refractive index 1.5 such that angle of refraction into the plate is 60°. Calculate the smallest thickness of plate which will make it appear dark by reflection.

Solution

$$2\mu t \cos r = n\lambda \Rightarrow t = \frac{n\lambda}{2\mu\cos r} = \frac{1\times 6\times 10^{-7}}{2\times 1.5\times \cos 60} = \frac{6\times 10^{-7}}{1.5} = 4 \times 10^{-7}\ m$$

Example

Light is incident on a glass plate (μ = 1.5) such that angle of refraction is 60°. Dark band is observed corresponding to the wavelength of 6000Å . If the thickness of glass plate is 1.2×10^{-3} mm. calculate the order of the interference band.

Solution

μ= 1.5, r = 60°, λ = 6000Å = 6×10^{-7} m $\Rightarrow$ t = $1.2 \times 10^{-3} = 1.2 \times 10^{-6}$ m

For dark band in the reflected light $2\ \mu t \cos r = n\lambda$

$$n = \frac{2\mu t\cos r}{\lambda} = \frac{2\times 1.5\times 1.2\times 10^{-6}\times \cos 60^\circ}{6\times 10^{-7}} = \frac{2\times 1.5\times 1.2\times 10^{-6}\times \frac{1}{2}}{6\times 10^{-7}} = 3$$

Thus third dark band is observed.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

SOME WORKED OUT EXAMPLES

Example#1

State two conditions to obtain sustained interference of light. In Young's double slit experiment, using light of wavelength 400 nm, interference fringes of width 'X' are obtained. The wavelength of light is increased to 600 nm and the separation between the slits is halved. If one wants the observed fringe width on the screen to be the same in the two cases, find the ratio of the distance between the screen and the plane of the slits in the two arrangements.

Sol. Conditions for sustained interference of light

(i) Sources should be coherent. (ii) There should be point sources

$\because$ fringe width $\beta = \frac{\lambda D}{d}$ Here, $\beta_1 = \frac{\lambda_1 D_1}{d_1}$ and $\beta_2 = \frac{\lambda_2 D_2}{d_2}$

As $\beta_1 = \beta_2 \Rightarrow \frac{\lambda_1 D_1}{d_1} = \frac{\lambda_2 D_2}{d_2} \Rightarrow \frac{D_1}{D_2} = \frac{\lambda_2 d_1}{\lambda_1 d_2} = \frac{600}{400} \times \frac{1}{1/2} = \frac{6}{2} = \frac{3}{1}$

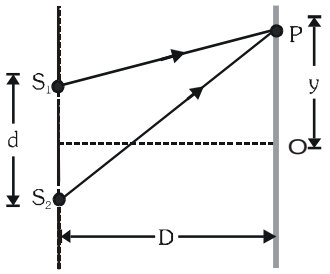
Example#2

Young's double slit experiment is carried out using microwaves of wavelength λ = 3 cm. Distance in between plane of slits and the screen is D = 100 cm. and distance in between the slits is 5 cm. Find

(a) the number of maximas and (b) their positions on the screen

Sol. (a) The maximum path difference that can be produced = distance between the sources or 5 cm. Thus, in this case we can have only three maximas, one central maxima and two on its either side for a path difference of λ or 3 cm.

(b) For maximum intensity at P, $S_2P - S_1P = \lambda \Rightarrow \sqrt{(y + d/2)^2 + D^2} - \sqrt{(y \quad d/2)^2 \quad D^2} = \lambda$



substiuting d = 5 cm, D = 100 cm and λ = 3cm we get y = $\pm$ 75 cm

Thus, the three maximas will be at y = 0 and y = $\pm$ 75 cm

Example#3

A beam of light consisting of two wavelengths 6500 Å and 5200 Å is used to obtain interference fringes in a young's double slit experiment. The distance between the slits is 2 mm and the distance between the plane of the slits and screen is 120 cm.

(a) Find the distance of the third bright fringe on the screen from the central maxima for the wavelength 6500 Å.

(b) What is the least distance from the central maxima where the bright fringes due to both the wave-lengths coincide ?

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Sol. (a) Distance of third bright fringe from centre of screen

$$x_3 = \frac{nD\lambda}{d} = \frac{3\times120\times10^{-2}\times6500\times10^{-10}}{2\times10^{-3}} = 1.17 \times 10^{-3}\ m = 1.17\ mm$$

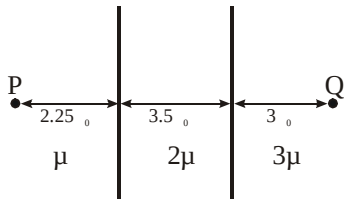
(b) When bright fringes coincide to each other then $n_1\lambda_1 = n_2\lambda_2 \Rightarrow \frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} = \frac{5200Å}{6500Å} = \frac{4}{5}$

for minimum value of n_1 & n_2 $n_1 = 4$, $n_2 = 5$

$$\text{So } x = \frac{n_1\lambda_1 D}{d} = \frac{4\times6500\times10^{-10}\times120\times10^{-2}}{2\times10^{-3}} = 0.156 \times 10^{-2}\ m = 0.156\ cm$$

Example#4

An electromagnetic wave of wavelength λ_0 (in vacuum) passes from P towards Q crossing three different media of refractive index μ, 2μ and 3μ respectively as shown in figure. ϕ_P and ϕ_Q be the phase of the wave at points P and Q. Find the phase difference $\phi_Q - \phi_P$. [Take : μ=1]



(A) 0 (B) $\frac{}{4}$ (C) $\frac{}{2}$ (D) π

Solution **Ans. (C)**

Optical path difference between (OPD) P & Q

(O.P.D.) = $2.25\ \lambda_0 \times 1 + (3.5\ \lambda_0) \times 2 + 3\lambda_0 \times 3 = 18.25\ \lambda_0$ and phase difference $\frac{2}{0}$ x $\frac{}{2}$

Example#5

Two slits separated by a distance of 1 mm are illuminated with red light of wavelength 6.5×10^{-7} m. The interference fringes are observed on a screen placed 1m from the slits. The distance between the third dark fringe and the fifth bright fringe is equal to

(A) 0.65 mm (B) 1.625 mm (C) 3.25 mm (D) 0.975 mm

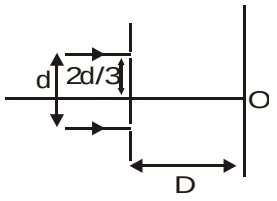
Solution **Ans. (B)**

Distance between third dark fringe and the fifth bright fringe

$$= 2.5 \quad 2.5\frac{D}{d} \quad 2.5\frac{6.5\ \ 10^{7}\ \ 1}{10^{3}} \quad 1.625\ mm$$

Example#6

In the figure shown if a parallel beam of white light is incident on the plane of the slits then the distance of the only white spot on the screen from O is :[assume d << D , λ << d]



(A) 0 (B) d/2 (C) d/3 (D) d/6

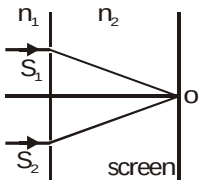
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Solution **Ans. (D)**

White spot will be at the symmetrical point w.r.t. slits . Its distance from O will be ,(2d/3) – (d/2) = d/6 .

Example#7

In a Young's double slit experiment the slits S_1 & S_2 are illuminated by a parallel beam of light of wavelength 4000 Å, from the medium of refractive index n_1 = 1.2. A thin film of thickness 1.2μm and refractive index n = 1.5 is placed infront of S_1 perpendicular to path of light. The refractive index of medium between plane of slits & screen is n_2 = 1.4. If the light coming from the film and S_1 & S_2 have equal intensities I then intensity at geometrical centre of the screen O is



(A) 0 (B) 2I (C) 4I (D) None of these

Solution **Ans. (B)**

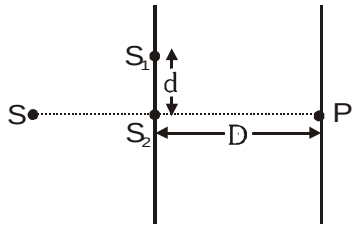
Path difference at O : $(\mu_{rel} - 1)\, t = \frac{n}{n_2} \quad 1 \quad t$

$\therefore$ Phase difference at O: $t \ \frac{n}{n_2} \quad 1 \ \frac{2}{\ \ _2}$ where $n_1\lambda_1 = n_2\ \lambda_2$

$\Rightarrow$ Phase difference= $\frac{}{2} \Rightarrow$ Resultant intensity = 2I

Example#8

In a YDSE experiment two slits S_1 and S_2 have separation of d = 2 mm. The distance of the screen is D = $\frac{8}{5}$ m. Source S starts moving from a very large distance towards S_2 perpendicular to S_1S_2 as shown in figure. The wavelength of monochromatic light is 500 nm. The number of maximas observed on the screen at point P as the source moves towards S_2 is



(A) 4001 (B) 3999 (C) 3998 (D) 4000

Solution **Ans. (D)**

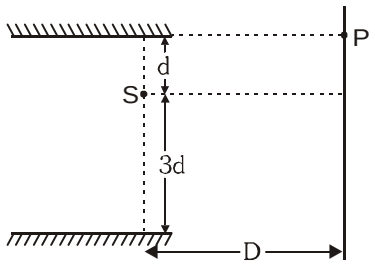
$S_1P - S_2P = \frac{d^2}{2D} = \frac{2 \quad 10^{\ 3} \quad 2 \quad 0^{\ 3}}{2 \quad \frac{8}{5}} = \frac{5}{2}$ 500nm

So when S is at ∞ there is I[st] minima and when S is at S_2 there is last minima because d/λ=4000
So the number of minima's will be 4001 and number of maxima's will be 4000.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

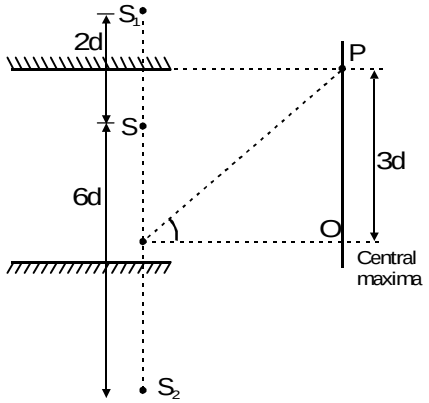
Example#9

Consider the optical system shown in figure. The point source of light S is having wavelength equals to λ.The light is reaching screen only after reflection. For point P to be 2nd maxima, the value of λ would be (D>>d and d>>λ)



(A) $\frac{12d^2}{D}$ (B) $\frac{6d^2}{D}$ (C) $\frac{3d^2}{D}$ (D) $\frac{24d^2}{D}$

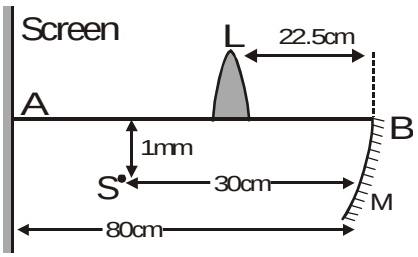
Solution **Ans. (A)**



At P, x $\frac{8d \quad 3d}{D}$; For 2nd maxima, $\Delta x = 2\lambda$ $\frac{24d^2}{D}$ 2 $\frac{12d^2}{D}$

Example#10 to12

In the figure shown, S is a point monochromatic light source of frequency 6 x 10^{14} Hz.M is a concave mirror of radius of curvature 20 cm and L is a thin converging lens of focal length 3.75 cm. AB is the principal axis of M and L.



Light reflected from the mirror and refracted from the lens in succession reaches the screen. An interference pattern is obtained on the screen by this arrangement.

10. Distance between two coherent sources which makes interference pattern on the screen is-

(A) 1 mm (B) 0.5 mm (C) 1.5 mm (D) 0.25 mm

11. Fringe width is-

(A) 1mm (B) 0.5 mm (C) 1.5 mm (D) 0.25 mm

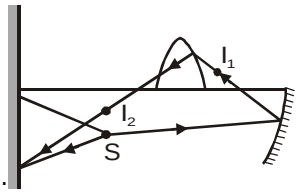
NODE6 (E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

12. If the lens is replaced by another converging lens of focal length $\frac{10}{3}$ cm and the lens is shifted towards right by 2.5 cm then-

(A) Fringe width remains same
(B) Intensity of pattern will remain same
(C) Fringe width will change
(D) No interference pattern will form.

Solution

10. Ans. (B)



Wave length of light $\frac{c}{f}$ 5 10^{7} m

Image formed by M : $\frac{1}{v}$ $\frac{1}{30}$ $\frac{1}{10}$ $\Rightarrow$ v = –15 cm also M = $-\frac{v}{u}$ $\frac{15}{30}$ $\frac{1}{2}$.

This will be located at 15 cm left of M and 0.5 mm above the line AB.

This will act as an object for the lens L.

Now for the lens u = –7.5cm and m = $\frac{v}{u}$ $\frac{7.5}{7.5}$ 1

So it will be at 7.5 cm to the left of L and 0.5 mm below line AB. See the ray diagram. Second image I_2 and source S will act as two slits (as in YDSE) to produce the interference pattern . Distance between them = 0.5 mm (= d)

11. Ans. (B)

$$\beta = \frac{5\ 10^{7}\ 50\ 10^{2}}{0.5\ 10^{3}} = 5 \times 10^{-4}\ m = 0.5\ mm$$

12. Ans. (D)

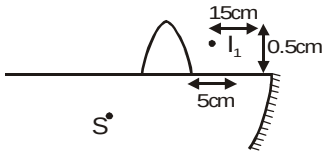


Image formed by the combination is I_2 at

$\frac{1}{v} - \frac{1}{5}$ $\frac{3}{10}$ $\Rightarrow$ v = 10 cm . It will coincide with S●,

so no interference pattern on the screen .

Example#13

Statement–1: In Young's double slit experiment the two slits are at distance d apart. Interference pattern is observed on a screen at distance D from the slits. At a point on the screen when it is directly opposite to one of the slits, a dark fringe is observed. Then, the wavelength of wave is proportional to square of distance of two slits.

and

Statement–2 : In Young's double slit experiment, for identical slits, the intensity of a dark fringe is zero.

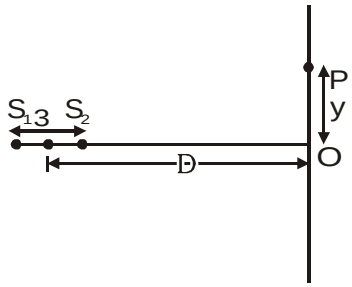
(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1
(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1
(C) Statement–1 is True, Statement–2 is False.
(D) Statement–1 is False, Statement–2 is True.

Solution **Ans. (B)**

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

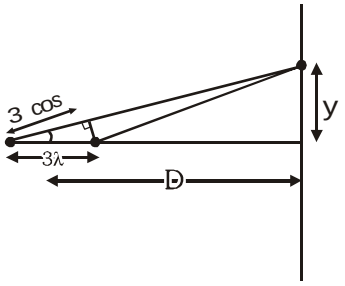
Example#14

Figure shows two coherent microwave source S_1 and S_2 emitting waves of wavelength λ and separated by a distance 3λ. For $\lambda << D$ and $y \neq 0$, the minimum value of y for point P to be an intensity maximum is $\frac{\sqrt{m}\,D}{n}$. Determine the value of m + n, if m and n are coprime numbers.



Solution **Ans. (7)**

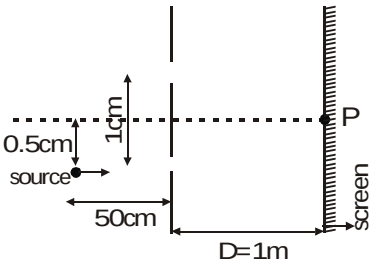
Path difference = $3\lambda \cos\theta = 2\lambda \Rightarrow \cos\theta = \frac{2}{3}$



$y = D\tan\theta = \frac{D\sqrt{5}}{2} \Rightarrow m + n = 5 + 2 = 7$

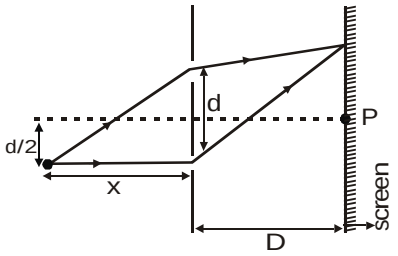
Example#15

In a typical Young's double slit experiment a point source of monochromatic light is kept as shown in the figure. If the source is given an instantaneous velocity v=1 mm per second towards the screen, then the instantaneous velocity of central maxima is given as $\alpha \times 10^{-\beta}$ cm/s upward in scientific notation. Find the value of $\alpha+\beta$.



Solution **Ans. 5**

The central maxima $\frac{dy}{D}$ $\sqrt{d^2 \quad x^2}$ x x 1 $\frac{d^2}{2x^2}$ x $\frac{d^2}{2x}$



y $\frac{Dd}{2x}$ $\frac{dy}{dt}$ $\frac{Dd}{2x^2}$ $\frac{dx}{dt}$ $\frac{1 \quad 0.01}{2 \quad 0.5 \quad 0.5}$ 0.001 0.02mm / s

$\Rightarrow y = 2 \times 10^{-3}$ cm/s $\Rightarrow \alpha + \beta = 5$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Select the correct alternative (only one correct answer)

1. Which of the following phenomenon can not be explained by the Huygen's theory-
(A) Refraction (B) Reflection (C) Diffraction (D) Formation of spectrum

2. Huygen's principle is applicable to-
(A) Only light waves (B) Only sound waves
(C) Only mechanical waves (D) For all the above waves

3. According to huygen's theory of secondary waves, following can be explained-
(A) Propagation of light in medium (B) Reflection of light
(C) Refraction of light (D) All of the above

4. Huygen's theory of secondary waves can be used to find-
(A) Velocity of light (B) The wavelength of light
(C) Wave front geometrically (D) Magnifying power of microscope

5. The main drawback of huygen's theory was-
(A) Failure in explanation of rectilinear propagation of lignt
(B) Failure of explain the spectrum of white light
(C) Failure to explain the formation of newton's rings
(D) A failure of experimental verification of ether medium

6. Light has a wave nature, because-
(A) the light travel in a straight line
(B) Light exhibts phenomenon of reflection and refraction
(C) Light exhibits phenomenon interference
(D) Light exhibits phenomenon of photo electric effect

7. The colour are characterized by which of following character of light–
(A) Frequency (B) Amplitude (C) Wavelength (D) Velocity

8. Two coherent sources of intensities I_1 and I_2 produce an interference pattern. The maximum intensity in the interference pattern will be :–
(A) $I_1 + I_2$ (B) $I_1^2 + I_2^2$ (C) $(I_1 + I_2)^2$ (D) $(\sqrt{I_1} \quad \sqrt{I_2})^2$

9. Two wave are represented by the equations $y_1 = a \sin \omega t$ and $y_2 = a \cos \omega t$. The first wave :–
(A) leads the second by π (B) lags the seconds by π (C) leads the second by $\frac{\pi}{2}$ (D) lags the seconds by $\frac{\pi}{2}$

10. The resultant amplitude of a vibrating particle by the superposition of the two waves
$y_1 = a\sin \quad t \quad \frac{}{3}$ and $y_2 = a \sin \omega t$ is :–
(A) a (B) $\sqrt{2}$ a (C) 2a (D) $\sqrt{3}$ a

11. The energy in the phenomenon of interference :–
(A) is conserved, gets redistributed (B) is equal at every point
(C) is destroyed in regions of dark fringes (D) is created at the place of bright fringes

12. The phase difference corresponding to path difference of x is :–
(A) $\frac{2\pi x}{\lambda}$ (B) $\frac{2\pi\lambda}{x}$ (C) $\frac{\pi x}{\lambda}$ (D) $\frac{\pi\lambda}{x}$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

13. The resultant amplitude in interference with two coherent sources depends upon :–
(A) only amplitude (B) only phase difference
(C) on both the previous option (D) none of the above

14. Phenomenon of interference is observed :–
(A) only for light waves (B) only for sound waves
(C) for both sound and light waves (D) none of above

15. Two coherent sources must have the same :–
(A) amplitude (B) phase difference (C) frequency (D) both (B) and (C)

16. For the sustained interference of light, the necessary condition is that the two sources should :–
(A) have constant phase difference (B) be narrow
(C) be close to each other (D) of same amplitude

17. If the ratio of the intensity of two coherent sources is 4 then the visibility $[(I_{max} - I_{min})/(I_{max} + I_{min})]$ of the fringes is
(A) 4 (B) 4/5 (C) 3/5 (D) 9

18. Two monochromatic and coherent point sources of light are placed at a certain distance from each other in the horizontal plane. The locus of all those points in the horizontal plane which have constructive interference will be–
(A) A hyperbola (B) Family of hyperbolas (C) Family of straight lines (D) Family of parabolas

19. If the distance between the first maxima and fifth minima of a double slit pattern is 7 mm and the slits are separated by 0.15 mm with the screen 50 cm from the slits, then wavelength of the light used is
(A) 600 nm (B) 525 nm (C) 467 nm (D) 420 nm

20. In Young's double slit experiment, the separation between the slits is halved and the distance between the slits and the screen is doubled. The fringe width is :–
(A) unchanged (B) halved (C) doubled (D) quadrupled

21. In Young's double slit experiment using sodium light (λ = 5898Å), 92 fringes are seen. If given colour (λ = 5461Å) is used, how many fringes will be seen
(A) 62 (B) 67 (C) 85 (D) 99

22. In Young's experiment, one slit is covered with a blue filter and the other (slit) with a yellow filter. Then the interference pattern :–
(A) will be blue (B) will be yellow (C) will be green (D) will not be formed

23. In Young's double slit experiment, a mica sheet of thickness t and refractive index μ is introduced in the path of ray from the first source S_1. By how much distance the fringe pattern will be displaced

(A) $\frac{d}{D}(\mu-1)t$ (B) $\frac{D}{d}(\mu-1)t$ (C) $\frac{d}{(\mu-1)D}$ (D) $\frac{D}{d}(\mu-1)$

24. In Young's double slit experiment, if monochromatic light is replaced by white light :–
(A) all bright fringes become white
(B) all bright fringes have coloures between violet and red
(C) only the central fringe is white, all other fringes are coloured
(D) no fringes are observed

25. In the young's double slit experiment the central maxima is observed to be I_0. If one of the slits is covered, then intensity at the central maxima will become :–

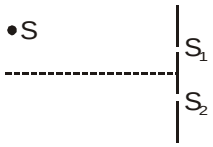
(A) $\frac{I_0}{2}$ (B) $\frac{I_0}{\sqrt{2}}$ (C) $\frac{I_0}{4}$ (D) I_0

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

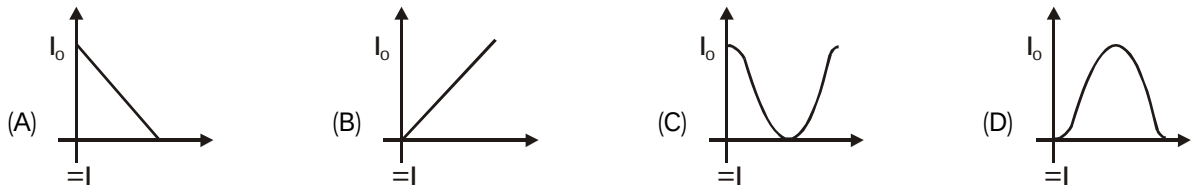
26. In Young's double slit experiment, one of the slits is so painted that intensity of light emitted from it is half of that of the light emitted from other slit. Then
(A) fringe system will disappear
(B) bright fringes will become brighter and dark fringes will be darker
(C) both bright and dark fringes will become darker
(D) dark fringes will become less dark and bright fringes will become less bright.

27. In YDSE how many maxima can be obtained on the screen if wavelength of light used is 200 nm and d = 700 nm :
(A) 12 (B) 7 (C) 18 (D) None of these

28. In YDSE, the source placed symmetrically with respect to the slit is now moved parallel to the plane of the slits it is closer to the upper slit, as shown. Then ,
(A) the fringe width will increase and fringe pattern will shift down.
(B) the fringe width will remain same but fringe pattern will shift up.
(C) the fringe width will decrease and fringe pattern will shift down.
(D) the fringe width will remain same but fringe pattern will shift down.

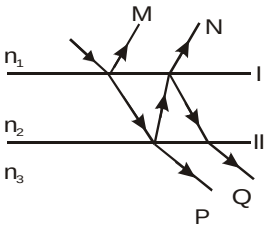


29. In a YDSE experiment if a slab whose refractive index can be varied is placed in front of one of the slits then the variation of resultant intensity at mid–point of screen with 'μ' will be best represented by ($\mu \geq 1$). [Assume slits of equal width and there is no absorption by slab]



30. In a double slit experiment, instead of taking slits of equal widths, one slit is made twice as wide as the other. Then in the interference pattern.
(A) the intensifies of both the maxima and minima increase.
(B) the intensity of the maxima increases and the minima has zero intensity.
(C) the intensity of the maxima decreases and that of minima increases
(D) the intensity of the maxima decreases and the minima has zero intensity.

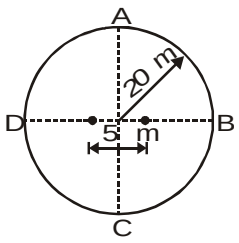
31. A ray of light is incident on a thin film. As shown in figure M, N are two reflected rays and P, Q are two transmitted rays, Rays N and Q undergo a phase change of π. Correct ordering of the refracting indices is :
(A) $n_2 > n_3 > n_1$ (B) $n_3 > n_2 > n_1$
(C) $n_3 > n_1 > n_2$ (D) none of these, the specified changes can not occur



32. Let S_1 and S_2 be the two slits in Young's double slit experiment. If central maxima is observed at P and angle $\angle S_1PS_2 = \theta$, then the fringe width for the light of wavelength λ will be. (Assume θ to be a small angle)
(A) λ/θ (B) $\lambda\theta$ (C) $2\lambda/\theta$ (D) $\lambda/2\theta$

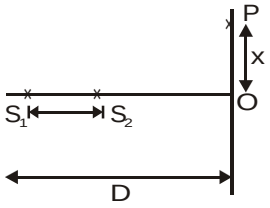
33. When light is refracted into a denser medium-
(A) Its wavelength and frequency both increase.
(B) Its wavelength increases but frequency remains unchanged.
(C) Its wavelength decreases but frequency remains unchanged.
(D) its wavelength and frequency both decrease.

34. Two point source separated by d = 5 μm emit light of wavelength λ=2 μm in phase. A circular wire of radius 20 μm is placed around the source as shown in figure.
(A) Points A and B are dark and points C and D are bright.
(B) Points A and B are bright and point C and D are dark.
(C) Points A and C are dark and points B and D are bright.
(D) Points A and C are bright and points B and D are dark.



N:\DD65\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

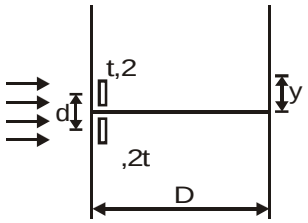
35. Two coherent narrow slits emitting light of wavelength λ in the same phase are placed parallel to each other at a small separation of 3λ. The light is collected on a screen S which is placed at a distance D (>>λ) from the slits. The smallest distance x such that the P is a maxima.



(A) $\sqrt{3D}$ (B) $\sqrt{8}D$ (C) $\sqrt{5}D$ (D) $\sqrt{5}\frac{D}{2}$

36. Minimum thickness of a mica sheet having $\mu = \frac{3}{2}$ which should be placed in front of one of the slits in YDSE is required to reduce the intensity at the centre of screen to half of maximum intensity is-
(A) $\lambda/4$ (B) $\lambda/8$ (C) $\lambda/2$ (D) $\lambda/3$

37. In the YDSE shown the two slits are covered with thin sheets having thickness t & 2t and refractive index 2μ and μ. Find the position (y) of central maxima



(A) zero (B) $\frac{tD}{d}$ (C) $\frac{tD}{d}$ (D) None of these

38 In a YDSE with two identical slits, when the upper slit is covered with a thin, perfectly transparent sheet of mica, the intensity at the centre of screen reduces to 75% of the initial value. Second minima is observed to be above this point and third maxima below it. Which of the following can not be a possible value of phase difference caused by the mica sheet

(A) $\frac{}{3}$ (B) $\frac{13}{3}$ (C) $\frac{17}{3}$ (D) $\frac{11}{3}$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

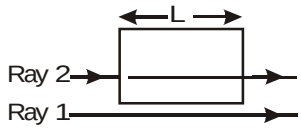
CHECK YOUR GRASP — ANSWER KEY — EXERCISE –1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	D	D	C	D	C	A	D	D	D	A	A	C	C	D	A	B	B	A	D
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
Ans.	D	D	B	C	C	D	B	D	C	A	B	A	C	D	B	C	B	A		

Select the correct alternatives (one or more than one correct answers)

1. As shown in arrangement waves with identical wavelengths and amplitudes and that are initially in phase travel through different media, Ray 1 travels through air and Ray 2 through a transparent medium for equal length L, in four different situations. In each situation the two rays reach a common point on the screen. The number of wavelengths in length L is N_2 for Ray 2 and N_1 for Ray 1. In the following table, values of N_1 and N_2 are given for all four situations, The order of the situations according to the intensity of the light at the common point in descending order is :

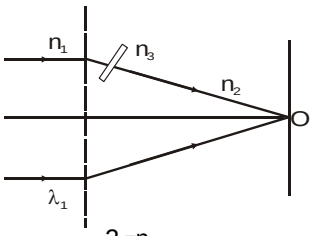
Situations	1	2	3	4
N_1	2.25	1.80	3.00	3.25
N_2	2.75	2.80	3.25	4.00



(A) $I_3 = I_4 > I_2 > I_1$ (B) $I_1 > I_3 = I_4 > I_2$ (C) $I_1 > I_2 > I_3 > I_4$ (D) $I_2 > I_3 = I_4 > I_1$

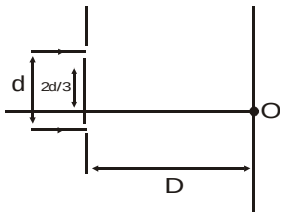
2. The path difference between two interfering waves at a point on the screen is $\lambda/6$. The ratio of intensity at this point and that at the central bright fringe will be : (Assume that intensity due to each slit is same)
(A) 0.853 (B) 8.53 (C) 0.75 (D) 7.5

3. In the figure shown in a YDSE, a parallel beam of light is incident on the slits from a medium of refractive index n_1. The wavelength of light in this medium is λ_1. A transparent slab of thickness 't' and refractive index n_3 is put in front of one slit. The medium between the screen and the plane of the slits is n_2. The phase difference between the light waves reaching point O (Symmetrical , relative to the slits) is :



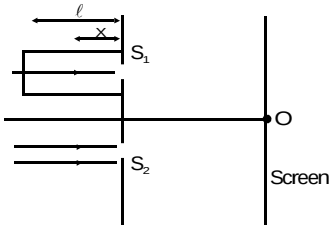
(A) $\frac{2\pi}{n_1\lambda_1}(n_3 - n_2)\,t$ (B) $\frac{2\pi}{\lambda_1}(n_3 - n_2)\,t$ (C) $\frac{2\pi n_1}{n_2\lambda_1}\left(\frac{n_3}{n_2}-1\right)t$ (D) $\frac{2\pi n_1}{\lambda_1}(n_3 - n_2)\,t$

4. In the figure shown if a parallel beam of white light is incident on the plane of the slits then the distance of the nearest white spot on the screen from O is : [assume d << D, λ << d]



(A) 0 (B) d/2 (C) d/3 (D) d/6

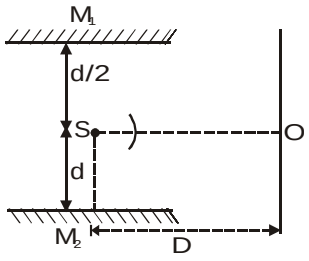
5. In the figure shown, a parallel beam of light is incident on the plane of the slits of Young's double slit experiment. Light incident on the slit, S_1 passes through a medium of variable refractive index $\mu = 1 + ax$ (where 'X' is the distance from the plane of slits as shown), upto a distance 'ℓ' before falling on S_1 . Rest of the space is filled with air. If at 'O' a minima is formed, then the minimum value of the positive constant a (in terms of ℓ and wavelength 'λ' in air) is:



(A) $\frac{\lambda}{\ell}$ (B) $\frac{\lambda}{\ell^2}$ (C) $\frac{\ell^2}{\lambda}$ (D) none of these

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

6. M_1 and M_2 are plane mirrors and kept parallel to each other. At point O there will be a maxima for wavelength λ. Light from monochromatic source S of wavelength λ is not reaching directly on the screen. Then λ is : [D >> d >> λ]



(A) $\frac{3d^2}{D}$ (B) $\frac{3d^2}{2D}$

(C) $\frac{d^2}{D}$ (D) $\frac{2d^2}{D}$

7. If the first minima in a Young's slit experiment occurs directly in front of one of the slits, (distance between slit & screen D = 12 cm and distance between slits d = 5 cm) then the wavelength of the radiation used can be :

(A) 2 cm (B) 4 cm (C) $\frac{2}{3}$ cm (D) $\frac{4}{3}$ cm

8. In young's double slit experiment, slits are arranged in such a way that besides central bright fringe, there is only one bright fringe on either side of it. Slit separation d for the given condition cannot be (if λ is wavelength of the light used) :

(A) λ (B) $\lambda/2$ (C) 2λ (D) $3\lambda/2$

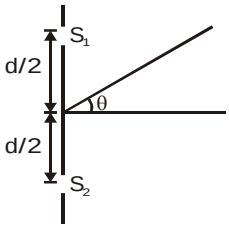
9. If one of the slits of a standard Young's double slit experiment is covered by a thin parallel sided glass slab so that it transmits only one half the light intensity of the other, then :

(A) The fringe pattern will get shifted towards the covered slit
(B) The fringe pattern will get shifted away from the covered slit
(C) The bright fringes will become less bright and the dark ones will becomes more bright
(D) The fringe width will remain unchanged

10. White light is used to illuminate the two slits in a Young's double slit experiment. The separation between the slits is b and the screen is at a distance d (>> b) from the slits. At a point on the screen directly in front of one of the slits, certain wavelengths are missing. Some of these missing wavelengths are :

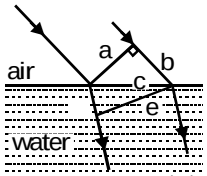
(A) $\lambda = b^2/d$ (B) $\lambda = 2b^2/d$ (C) $\lambda = b^2/3d$ (D) $\lambda = 2b^2/3d$

11. In an interference arrangement similar to Young's double–slit experiment, the slits S_1 and S_2 are illuminated with coherent microwave sources, each of frequency 10^6 Hz. The sources are synchronized to have zero phase difference. The slits are separated by a distance d = 150.0m. The intensity $I(\theta)$ is measured as a function of θ, where θ is defined as shown. If I_0 is the maximum intensity, then $I(\theta)$ for $0 \le \theta \le 90°$ is given by :



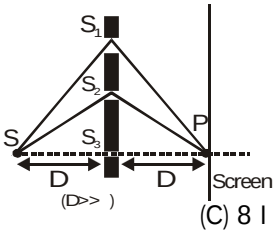
(A) $I(\theta) = I_0/2$ for $\theta = 30°$ (B) $I(\theta) = I_0/4$ for $\theta = 90°$
(C) $I(\theta) = I_0$ for $\theta = 0°$ (D) $I(\theta)$ is constant for all values of θ

12. Figure shows plane waves refracted from air to water using Huygen's principle a,b,c,d,e are lengths on the diagram. The refractive index of water w.r.t air is the ratio.



(A) a/e (B) b/e (C) b/d (D) d/b

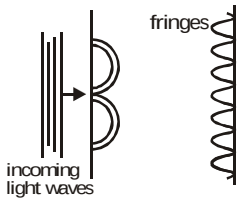
13. A monochromatic light source of wavelength λ is placed at S. Three slits S_1,S_2 and S_3 are equidistant from the source S and the point P on the screen. $S_1P - S_2P = \lambda/6$ and $S_1P - S_3P = 2\lambda/3$. If I be the intensity at P when only one slit is open, the intensity at P when all the three slits are open is–



(A) 3 I (B) 5 I (C) 8 I (D) zero

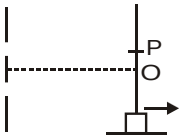
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

14. In a Young's double slit experiment, green light is incident on the two slits. The interference pattern is observed on a screen. Which of the following changes would cause the observed fringes to be more closely spaced ?
(A) Reducing the separation between the slits
(B) Using blue light instead of green light
(C) Used red light instead of green light
(D) Moving the light source further away from the slits.



15. Two monochromatic (wavelength = a/5) and coherent sources of electromagnetic waves are placed on the x-axis at the points (2a,0) and (–a,0). A detector moves in a circle of radius R(>>2a) whose centre is at the origin. The number of maximas detected during one circular revolution by the detector are-
(A) 60 (B) 15 (C) 64 (D) None

16. In a Young's Double slit experiment, first maxima is observed at a fixed point P on the screen. Now the screen is continuously moved away from the plane of slits. The ratio of intensity at point P to the intensity at point O (centre of the screen)-
(A) Remains constant
(B) Keeps on decreasing
(C) First decreases and then increases
(D) First decreases and then becomes constant



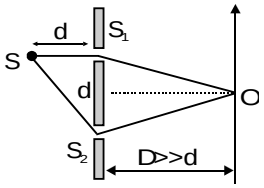
17. To make the central fringe at the centre O, a mica sheet of refractive index 1.5 is introduced. Choose the correct statements (s)-

(A) The thickness of sheet is $2(\sqrt{2}\ \ 1)\,d$ infront of S_1.

(B) The thickness of sheet is $(\sqrt{2}\ \ 1)\,d$ infront of S_2.

(C) The thickness of sheet is $2\sqrt{2}d$ infront of S_1.

(D) The thickness of sheet is $(2\sqrt{2}\ \ 1)\,d$ infront of S_1.



18. A thin slice is cut out of a glass cylinder along a plane parallel to its axis. The slice is placed on a flat glass plate with the curved surface downwards. Monochromatic light is incident normally from the top. The observed interference fringes from the combination do not follow on of the following statements.
(A) The fringes are straight and parallel to the length of the piece.
(B) The line of contact of the cylindrical glass piece and the glass plate appears dark.
(C) The fringe spacing increases as we go outwards.
(D) The fringes are formed due to the interference of light rays reflected from the curved surface of the cylindrical piece and the top surface of the glass plate.

19. A circular planar wire loop is dipped in a soap solution and after taking it out, held with its plane vertical in air. Assuming thickness of film at the very small, as sunlight falls on the soap film, & observer receive reflected light.
(A) The top portion appears dark while the first colour to be observed as one moves down is red.
(B) The top portion appears violet while the first colour to be observed as one moves down in indigo.
(C) The top potion appears dark while the first colour to be observed as one move down in violet.
(D) The top portion appears dark while the first colour to be observed as one move down depends on the refractive index of the soap solution.

20. Soap bubble appears coloured due to the phenomenon of :–
(A) interference (B) diffraction (C) dispersion (D) reflection

21. A parallel coherent beam of light falls on fresnel biprism of refractive index μ and angle α. The fringe width on a screen at a distance D from biprism will be (wavelength = λ)

(A) $\frac{}{2(\quad 1)}$ (B) $\frac{D}{2(\quad 1)}$ (C) $\frac{D}{2(\quad 1)}$ (D) None of these

BRAIN TREASURE **ANSWER KEY** **EXERCISE –2**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Ans.	D	C	A	D	B	B	A,C	A,B	A,C,D	A,C	A,C	C	A	B	A	C	A	C	C	A	A

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

EXERCISE–03 MISCELLANEOUS TYPE QUESTIONS

True / False

1. The intensity of light at a distance r from the axis of a long cylindrical source is inversely proportional to r.

2. Two slits in a Young's double slit experiment are illuminated by two different sodium lamps emitting light of the same wavelength. No interference pattern will be observed on the screen.

3. In a Young's double slit experiment performed with a source of white light, only black and white fringes are observed.

Fill in the blanks

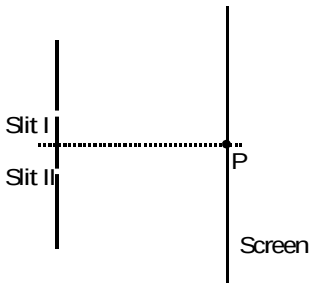
1. A light wave of frequency 5×10^{14} Hz enters a medium of refractive index 1.5. In the medium the velocity of the light wave is ..... and its wavelength is.....

2. A monochromatic beam of light of wavelength 6000Å in vacuum enters a medium of refractive index 1.5. In the medium its wavelength is....., its frequency is .....

3. In Young's double–slit experiment, the two slits act as coherent sources of equal amplitude A and of wavelength λ. In another experiment with the same set–up the two slits are sources of equal amplitude A and wavelength λ, but are incoherent. The ratio of the intensity of light at the mid–point of the screen in the first case to that in the second case is .....

Match the column

1. A double slit interference pattern is produced on a screen, as shown in the figure, using monochromatic light of wavelength 500nm. Point P is the location of the central bright fringe, that is produced when light waves arrive in phase without any path difference. A choice of three strips A, B and C of transparent materials with different thicknesses and refractive indices is available, as shown in the table. These are placed over one or both of the slits, singularly or in conjunction, causing the interference pattern to be shifted across the screen from the original pattern. In the column–I, how the strips have been placed, is mentioned whereas in the column–II, order of the fringe at point P on the screen that will be produced due to the placement of the strips(s), is shown. Correctly match both the column.



Film	A	B	C
Thickness (in μm)	5	1.5	0.25
Refractive index	1.5	2.5	2

Column I	Column II
(A) Only strip B is placed over slit–I	(p) First Bright
(B) Strip A is placed over slit–I and strip C is placed over slit–II	(q) Fourth Dark
(C) Strip A is placed over the slit–I and strip B and strip C are placed over the slit–II in conjunction	(r) Fifth Dark
(D) Strip A and strip C are placed over slit–I (in conjuction) and strip B is placed over Slit–II	(s) Central Bright
	(t) Fifth bright

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Assertion–Reason

1. **Statement–1 :** If white light is used in YDSE, then the central bright fringe will be white

and

Statement–2 : In case of white light used in YDSE, all the wavelengths produce their zero order maxima at the same position

(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1

(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

2. **Statement–1 :** In YDSE, if a thin film is introduced in front of the upper slit, then the fringe pattern shifts in the downward direction.

and

Statement–2 : In YDSE if the slit widths are unequal, the minima will be completely dark

(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1

(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

3. **Statement–1:** In YDSE, interference pattern disappears when one of the slits is closed.

and

Statement–2 : In YDSE, interference occurs due to super–imposition of light wave from two coherent sources.

(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1

(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

4. **Statement–1 :** In YDSE central maxima means the maxima formed with zero optical path difference. It may be formed anywhere on the screen.

and

Statement–2 : In an interference pattern, whatever energy disappears at the minimum, appears at the maximum**.**

(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1

(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

5. **Statement–1:** The phase difference between any two points on a wave front is zero.

and

Statement–2 : Light from the source reaches every point of the wave front at the same time.

(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1

(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1

(C) Statement–1 is True, Statement–2 is False.

(D) Statement–1 is False, Statement–2 is True.

N:\DD65\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

6. **Statement–1 :** As light travels from one medium to another, the frequency of light doesn't change.
and
Statement–2 : Frequency is the characteristic of source.
(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1
(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1
(C) Statement–1 is True, Statement–2 is False.
(D) Statement–1 is False, Statement–2 is True.

7. **Statement–1:**No interference pattern is detected when two coherent sources are infinitely close to each other.
and
Statement–2 : The fringe width is inversely proportional to the distance between the two slits.
(A) Statement–1 is True, Statement–2 is True ; Statement–2 is a correct explanation for Statement–1
(B) Statement–1 is True, Statement–2 is True ; Statement–2 is not a correct explanation for Statement–1
(C) Statement–1 is True, Statement–2 is False.
(D) Statement–1 is False, Statement–2 is True.

Comprehension based question

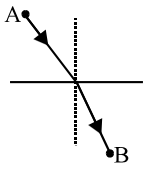
Comprehension#1

The lens governing the behavior of the rays namely rectilinear propagation, laws of reflection and refraction can be summarised in one fundamental law known as Fermat's principle. According to this principle a ray of light travels from one point to another such that the time taken is at a stationary value (maximum or minimum). If c is the velocity of light in a vacuum, the velocity in a medium of refractive index μ is $\frac{c}{\mu}$, hence time taken to travel a distance ℓ is $\frac{\ell}{c}$. If the light passes through a number of media, the total time taken is $\frac{1}{c}\sum \ell$ or $\frac{1}{c}\int d\ell$ if refractive index varies continuously. Now, ℓ is the total optical path, so that Fermat's principle states that the path of a ray is such that the optical path in at a stationary value. This principle is obviously in agreement with the fact that the ray are straight lines in a homogenous isotropic medium. It is found that it also agrees with the classical laws of reflection and refraction.

1. If refractive index of a slab varies as $\mu = 1 + x^2$ where x is measured from one end, then optical path length of a slab of thickness 1 m is :

(A) $\frac{4}{3}$m (B) $\frac{3}{4}$m (C) 1 m (D) None

2. The optical path length followed by ray from point A to B given that laws of refraction are obeyed as shown in figure.



(A) Maximum (B) Minimum (C) Constant (D) None

3. The optical path length followed by ray from point A to B given that laws of reflection are obeyed as shown in figure is



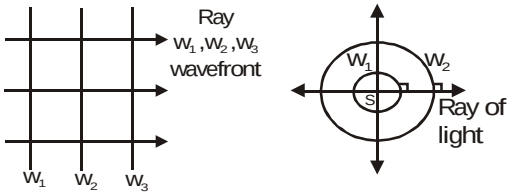
(A) Maximum (B) Minimum (C) Constant (D) None

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

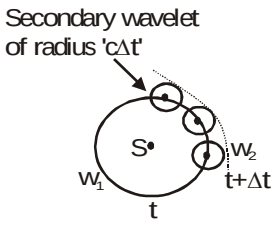
Comprehension#2

Huygen was the first scientist who proposed the idea of wave theory of light. He said that the light propagates in form of wavefronts. A wavefront is an imaginary surface of every point of which waves are in the same phase.

For example the wavefronts for a point source of light is collection of concentric spheres which have centre at the origin. w_1 is a wavefront. w_2 is another wavefront.



The radius of the wavefront at time 't' is 'ct' in this case where 'c' is the speed of light. The direction of propagation of light is perpendicular to the surface of the wavefront. The wavefronts are plane wavefronts in case of a parallel beam of light.



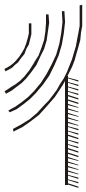
Huygen also said that every point of the wavefront acts as the source of secondary wavelets. The tangent drawn to all secondary wavelets at a time is the new wavefront at that time. The wavelets are to be considered only in the forward direction (i.e. the direction of propagation of light) and not in the reverse direction. If a wavefront w_1 at time t is given, then to draw the wavefront at time $t + \Delta t$ take some points on the wavefront w_1 and draw spheres of radius'$c\Delta t$'. They are called secondary wavelets.

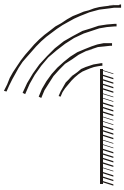
Draw a surface w_2 which is tangential to all these secondary wavelets. w_2 is the wavefront at time '$t + \Delta t$'. Huygen proved the laws of reflection and laws of refraction using concept of wavefronts.

1. A point source of light is placed at origin, in air. The equation of wavefront of the wave at time t, emitted by source at t = 0, is (Take refractive index of air as 1)

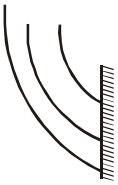
(A) $x + y + z = ct$ (B) $x^2 + y^2 + z^2 = t^2$ (C) $xy + yz + zx = c^2t^2$ (D) $x^2+ y^2 + z^2 = c^2 t^2$

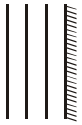
2. Spherical wavefronts shown in figure, strike a plane mirror. Reflected wavefront will be as shown in



(A) 

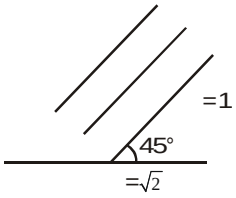
(B) 

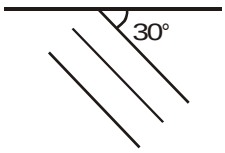
(C) 

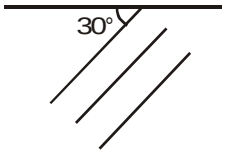
(D) 

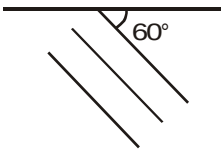
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

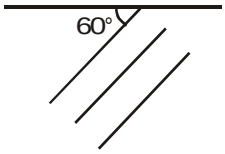
3. Wavefronts incident on an interface between the media are shown in the figure. The refracted wavefront will be as shown in



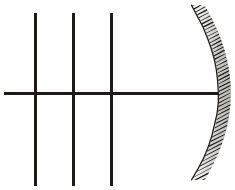
(A) 

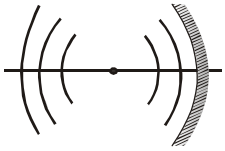
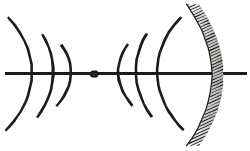
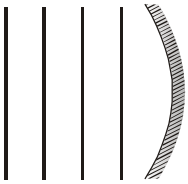
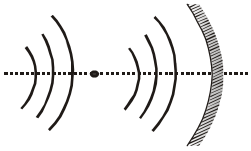
(B) 

(C) 

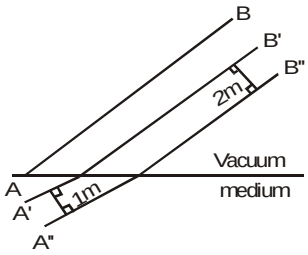
(D) 

4. Plane wavefronts are incident on a spherical mirror as shown in the figure. The reflected wavefronts will be



(A) 
(B) 
(C) 
(D) 

5. Certain plane wavefronts are shown in figure. The refractive index of medium is



(A) 2 (B) 4 (C) 1.5 (D) Cannot be determined

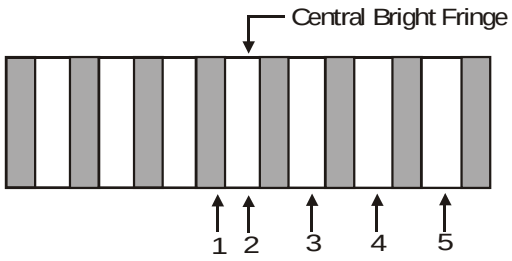
6. The wavefront of a light beam is given by the equation x + 2y + 3z = c, (where c is arbitrary constant) then the angle made by the direction of light with the y–axis is:

(A) $\cos^{1}\frac{1}{\sqrt{14}}$ (B) $\sin^{-1}\frac{2}{\sqrt{14}}$ (C) $\cos^{-1}\frac{2}{\sqrt{14}}$ (D) $\sin^{-1}\frac{3}{\sqrt{14}}$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Comprehension#3

The figure shows the interference pattern obtained in a double–slit experiment using light of wavelength 600nm.



1. The third order bright fringe is

(A) 2 (B) 3 (C) 4 (D) 5

2. Which fringe results from a phase difference of 4π between the light waves incidenting from two slits?

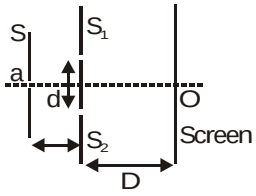
(A) 2 (B) 3 (C) 4 (D) 5

3. Let ΔX_A and ΔX_C represent path differences between waves interfering at 1 and 3 respectively then $(|\Delta X_C| - (|\Delta X_A|)$ is equal to

(A) 0 (B) 300 mn (C) 600 nm (D) 900 nm

Comprehension#4

The figure shows a schematic diagram showing the arrangement of Young's Double Slit experiment



1. Choose the correct statement (s) related to the wavelength of light used.

(A) Larger the wavelength of light larger the fringe width

(B) The position of central maxima depends on the wavelength of light used

(C) If white light is used in YDSE, then the violet colour forms its first maxima closest to the central maxima

(D) The central maxima of all the wavelength coincide

2. If the distance D is varied, then choose the correct statement (s)

(A) The angular fringe width does not change (B) The fringe width changes in direct proportion

(C) The change in fringe width is same for all wavelengths (D) The position of central maxima remains unchanged

3. If the distance d is varied, then identify the correct statement–

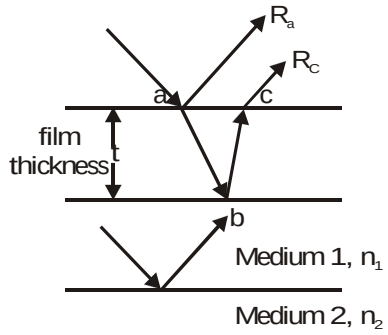
(A) The angular width does not change (B) The fringe width changes in inverse proportion

(C) The positions of all maxima change (D) The positions of all minima change

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Comprehension#5

Thin films, including soap bubbles and oil slicks, show patterns of alternating dark and bright regions resulting from interference among the reflected light waves. If two waves are in phase their crest and troughs will coincide. The interference will be constructive and the amplitude of the resultant wave will be greater than the amplitude of either constituent wave. If the two waves are out of phase, the crests of one wave will coincide with the troughs of the other wave. The interference will be destructive and the amplitude of the resultant wave will be less than that of either constituent wave. At the interface between two transparent media, some light is reflected and some light is refracted.



- When incident light, reaches the surface at point a, some of the light is reflected as ray R_a and some is refracted following the path ab to the back of the film.
- At point b some of the light is refracted out of the film and part is reflected back through the film along path bc. At point c some of the light is reflected back into the film and part is refracted out of the film as ray R_c.

  R_a and R_c are parallel. However, R_c has travelled the extra distance within the film of abc. If the angle of incidence is small, then abc is approximately twice the film's thickness. If R_a and R_c are in phase, they will undergo constructive interference and the region ac will be bright. If R_a and R_c are out of phase, they will undergo destructive interference.
- Refraction at an interface never changes the phase of the wave.
- For reflection at the interface between two media 1 and 2, if $n_1 < n_2$ the reflected wave will change phase by π. If $n_1 > n_2$ the reflected wave will not undergo a phase change. For reference $n_{air} = 1.00$.
- If the waves are in phase after reflection at all interfaces, then the effects of path length in the film are:

  Constructive interference occur when : (n = refractive index) $2t = m\lambda/n$ m = 0, 1, 2, 3.....

  Destructive interference occurs when : $2t = (m + 1/2)\lambda/n$ m = 0, 1, 2, 3.....

  If the waves are 180° out of phase after reflection at all interfaces then the effects of path length on the film are :

  Constructive interference occurs when : $2t = (m + 1/2)\lambda/n$ m = 0, 1, 2, 3.....

  Destructive interference occurs when : $2t = m\lambda/n$ m = 0, 1, 2, 3.....

1. A thin film with index of refraction 1.50 coats a glass lens with index of refraction 1.80. What is the minimum thickness of the thin film that will strongly reflect light with wavelength 600 nm ?

(A) 150 nm (B) 200 nm (C) 300 nm (D) 450 nm

2. A thin film with index of refraction 1.33 coats a glass lens with index of refraction 1.50. Which of the following choices is the smallest film thicknesses that will not reflect light with wavelength 640 nm ?

(A) 160 nm (B) 240 nm (C) 360 nm (D) 480 nm

3. A soap film of thickness t is surrounded by air and is illuminated at near normal incidence by monochromatic light with wavelength λ in the film. With respect to the wavelength of the monochromatic light in the film, what film thickness will produce maximum constructive interference in the reflected light

(A) $\frac{\lambda}{4}$ (B) $\frac{\lambda}{2}$ (C) $\mu\lambda$ (D) $2\mu\lambda$

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

4. The average human eye sees colors with wavelengths between 430 nm to 680 nm. For what visible wavelength will a 350 nm thick n = 1.35 soap film produce maximum destructive interference ?

(A) 560 nm (B) 473 nm (C) 610 nm (D) none of these

5. A 600 nm light is perpendicularly incident on a soap film suspended in air. The film is 1.00 μm thick with n = 1.35. Which statement most accurately describes the interference of the light reflected by the two surfaces of the film ?

(A) The waves are close to destructive interference (B) The waves are close to constructive interference

(C) The waves show complete destructive interference (D) The waves show complete constructive interference

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** **EXERCISE –3**

- ***True / False*** **1.** T **2.** T **3.** F
- ***Fill in the Blanks*** **1.** 2×10^8 m/s, 4×10^{-7} m **2.** 400Å, 5×10^{14} Hz **3.** 2
- ***Match the Column*** **1.** (A) t (B) r (C) s (D) p
- ***Assertion – Reason*** **1.** A **2.** E **3.** B **4.** B **5.** A **6.** A **7.** A
- ***Comprehension Based***

Comprehension #1: **1.** A **2.** B **3.** C

Comprehension #2 : **1.** D **2.** C **3.** B **4.** A **5.** A **6.** C

Comprehension #3 : **1.** D **2.** C **3.** B

Comprehension #4 : **1.** A,C,D **2.** A,B,D **3.** B,D

Comprehension #5 : **1.** B **2.** C **3.** A **4.** B **5.** D

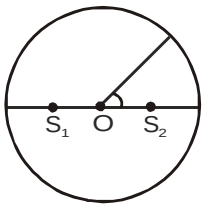
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

EXERCISE–04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Consider interference between two sources of intensity I and 4I. Find out resultant intensity where phase difference is (i) $\pi/4$ (ii) π (iii) 4π

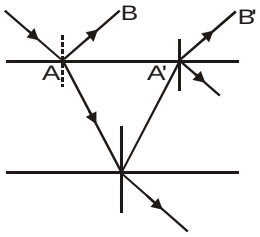
2. Two coherent sources S_1 and S_2 separated by distance 2λ emit light of wavelength λ in phase as shown in the figure. A circular wire of radius 100 λ is placed in such a way that S_1 S_2 lies in its plane and the mid–point of S_1S_2 is at the centre of wire.



(i) Find the angular positions θ on the wire for which constructive interference takes place. Hence or otherwise find the number of maxima.

(ii) Find the angular positions θ on the wire for which intensity reduces to half of its maximum value.

3. A ray of light of intensity I is incident on a parallel glass–slab at a point A as shown in figure. It undergoes partial reflection and refraction. At each reflection 20% of incident energy is refracted. The rays AB and A'B' undergo interference. Find the ratio $I_{max}/I_{min.}$



4. In Young's experiment for interference of light the slits 0.2 cm apart are illuminated by yellow light (λ = 5896 A°). What would be the fringe width on a screen placed 1m from the plane of slits ? What will be the fringe width if the system is immersed in water. (Refractive index = 4/3)

6. In a double–slit experiment, fringes are produced using light of wavelength 4800 A°. One slit is covered by a thin plate of glass of refractive index 1.4 and the other slit by another plate of glass of double thickness and of refractive index 1.7. On doing so, the central bright fringe shifts to a position originally occupied by the fifth bright fringe from the centre. Find the thickness of the glass plates.

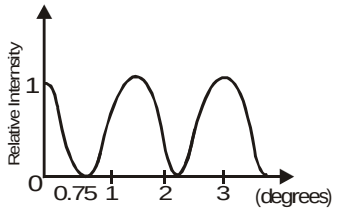
7. In a two–slit experiment with monochromatic light, fringes are obtained on a screen placed at some distance from the slits. If the screen is moved by 5×10^{-2} m towards the slits, the change in fringe width is 3×10^{-5}. If the distance between the slits is 10^{-3} m, calculate the wavelength of the light used.

8. Young's double slit experiment is carried out using microwaves of wavelength λ = 3 cm. Distance in between plane of slits and the screen is D = 100 cm and distance in between the slits is 5 cm. Find :(i) The number of maximas and (ii) Their positions on the screen

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

9. A thin glass plate of thickness t and refractive index μ is inserted between screen and one of the slits in a Young's experiment. If the intensity at the centre of the screen is I, what was the intensity at the same point prior to the introduction of the sheet.

10. Light of wavelength 520nm passing through a double slit, produces interference pattern of relative intensity versus deflection angle θ as shown in the figure. Find the separation d between the slits.



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

CONCEPTUAL SUBJECTIVE EXERCISE **ANSWER KEY** **EXERCISE–4(A)**

1. (i) 7.8 I (ii) I (iii) 9I

2. (i) 0°, 60°, 90°, 120°, 180°, 240°, 270°, 300°, 8 (ii) $\cos\theta = \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{5}{8}, \pm\frac{7}{8}$

3. 81 :1

4. 0.3 mm, 0.225 mm

6. 2.4 μm and 4.8μm

7. 6000Å

8. (i) 3 (ii) y=0 and y = ± 75 cm

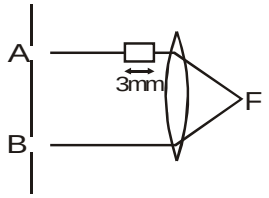
9. (i) $I_0 = I\cos^2\frac{1\ t}{\quad}$

10. 1.98×10^{-2} mm

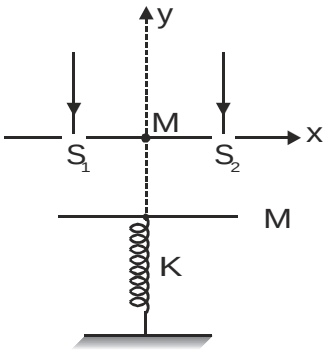
ALLEN Materials Provided By - Material Point Available on Learnaf.com

EXERCISE–04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. Two identical monochromatic light sources A & B intensity 10^{-15}W/m² produce wavelength of light $4000\sqrt{3}$Å . A glass of thickness 3mm is placed in the path of the ray as shown in figure. The glass has a variable refractive index n 1 $\sqrt{x}$ where x (in mm) is distance of plate from left to right. Calculate total intensity at focal point F of the lens.



2. Two slits S_1 and S_2 on the x–axis and symmetric with respect to y–axis are illuminated by a parallel monochromatic light beam of wavelength λ. The distance between the slits is d (>> λ). Point M is the mid point of the line S_1S_2 and this point is considered as the origin. The slits are in horizontal plane. The interference pattern is observed on a horizontal plate (acting as screen) of mass M, which is attached to one end of a vertical spring of spring constant K. The other end of the spring is fixed to ground. At t=0 the plate is at a distance D(>>d) below the plane of slits and the spring is in its natural length. The plate is left from rest from its initial position. Find the x and y co–ordinates of the n^{th} maxima on the plate as a function of time. Assume that spring is light and plate always remains horizontal.



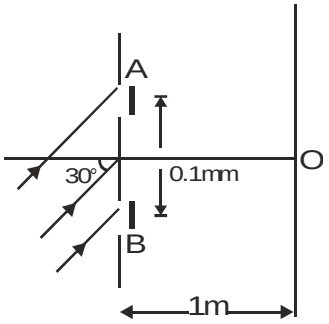
3. In a YDSE a parallel beam of light of wavelength 6000Å is incident on slits at angle of incidence 30°. A and B are two thin transparent films each of refractive index 1.5. Thickness of A is 20.4 μm. Light coming through A and B have intensities I and 4I respectively on the screen. Intensity at point O which is symmetric relative to the slits is 3I. The central maxima is above O.

(i) What is the maximum thickness of B to do so. Assuming thickness of B to be that found in part (i) answer the following parts.

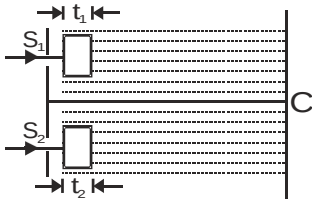
(ii) Find fringe width, maximum intensity and minimum intensity on screen.

(iii) Distance of nearest minima from O.

(iv) Intensity at 5 cm on either side of O.

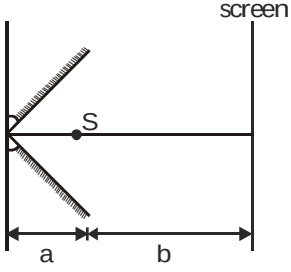


4. A screen is at a distance D=80 cm from a diaphragm having two narrow slits S_1 and S_2 which are d=2 mm apart. Slit S_1 is covered by a transparent sheet of thickness t_1=2.5 μm and S_2 by another sheet of thickness t_2 = 1.25 μm as shown in figure. Both sheets are made of same material having refractive index μ=1.40.



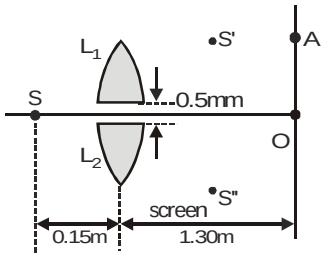
Water is filled in space between diaphragm and screen. A monochromatic light beam of wavelength λ=5000Å is incident normally on the diaphragm. Assuming intensity of beam to be uniform and slits of equal width, calculate ratio of intensity at C to maximum intensity of interference pattern obtained on the screen, where C is foot of perpendicular bisector of S_1S_2. (Refractive index of water, $\mu_w=\frac{4}{3}$)

5. Two plane mirrors, a source S of light, emitting monochromatic rays of wavelength λ and a screen are arranged as shown in figure. If angle θ is very small, calculate fringe width of the interference pattern formed by reflected rays.



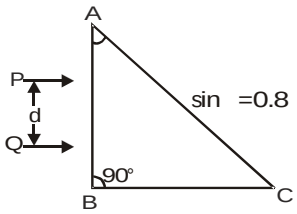
NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

6. In the figure shown S is a monochromatic point source emitting light of wavelength = 500 nm. A thin lens of circular shape and focal length 0.10 m is cut into two identical halves L_1 and L_2 by a plane passing through a diameter. The two halves are placed symmetrically about the central axis SO with a gap of 0.5 mm. The distance along the axis from S to L_1 and L_2 is 0.15 m, while that from L_1 and L_2 to O is 1.30 m.



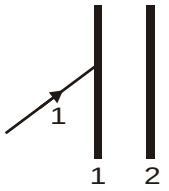
The screen at O is normal to SO. (i) If the third intensity maximum occurs at the point A on the screen, find the distance OA. (ii) If the gap between L_1 and L_2 is reduced from its original value of 0.5 mm, will the distance OA increase decrease or remain the same?

7. Two parallel beams of light P and Q (separation d) containing radiations of wavelengths 4000Å and 5000Å (which are mutually coherent in each wavelength separately) are incident normally on a prism as shown in figure The refractive index of the prism as a function of wavelength is given by the relation, $\mu(\ ) \quad 1.20 \quad \frac{b}{2}$, where λ is in Å and b is a positive constant.



The value of b is such that the condition for total reflection at the face AC is just satisfied for one wavelength and is not satisfied for the other. A convergent lens is used to bring these transmitted beams into focus. If the intensities of the upper and the lower beams immediately after transmission from the face AC, are 4I and I respectively, find the resultant intensity at the focus.

8. A narrow monochromatic beam of light of intensity I is incident on a glass plate as shown in figure. Another identical glass plate is kept close to the first one & parallel to it. Each glass plate reflects 25% of the light incident on it & transmits the remaining. Find the ratio of the minimum & the maximum intensities in the interference pattern formed by the two beams obtained after one reflection at each plate.



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** **EXERCISE-4(B)**

1. 4×10^{-15} W/m^2 **2.** $\frac{n\ D}{d}$, –D' where D' = D+ $\frac{Mg}{K}$ 1 cos t

3. (i) t_B = 120 μm (ii) β = 6mm, I_{max} =9I, I_{min} = I (iii) $\frac{}{6}$ = 1mm (iv) 9I, 3I

4. 3:4 **5.** $\frac{2\ a\ b}{4a}$ **6.** (i) 1mm (ii) increases **7.** 9I **8.** 1:49

EXERCISE–05(A) PREVIOUS YEARS QUESTIONS

1. To demonstrate the phenomenon of interference we require two sources which emit radiations of-
(1) nearly the same frequency (2) the same frequency **[AIEEE - 2003]**
(3) different wavelength (4)the same frequency and having a definite phase relationship

2. The maximum number of possible interference maxima for slit-separation equal to twice the wavelength in Young's double-slit experiment, is- **[AIEEE - 2004]**
(1) infinite (2) five (3) three (4) zero

3. A Young's double slit experiment uses a monochromatic source. The shape of the interference fringes formed on a screen is- **[AIEEE - 2005]**
(1) hyperbola (2) circle (3) straight line (4) parabola

4. In a Young's double slit experiment the intensity at a point where the path difference is $\frac{\lambda}{6}$ (λ being the wavelength of the light used) is I. If I_0 denotes the maximum intensity, I/I_0 is equal to- **[AIEEE - 2007]**
(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{3}{4}$

5. A mixture of light, consisting of wavelength590 nm and an unknown wavelength, illuminates Young's double slit and gives rise to two overlapping interference patterns on the screen. The central maximum of both lights coincide. Further, it is observed that the third bright fringe of known light coincides with the 4th bright fringe of the unknown light. From this data, the wavelength of the unknown light is :- **[AIEEE - 2009]**
(1) 442.5 nm (2) 776.8 nm (3) 393.4 nm (4) 885.0 nm

Direction : Questions are based on the following paragraph.

An initially parallel cylindrical beam travels in a medium of refractive index $\mu(I) = \mu_0 + \mu_2 I$, where μ_0 and μ_2 are positive constants and I is the intensity of the light beam. The intensity of the beam is decreasing with increasing radius. **[AIEEE - 2010]**

6. The initial shape of the wavefront of the beam is :-
(1) planar (2) convex
(3) concave (4) convex near the axis and concave near the periphery

7. The speed of the light in the medium is :-
(1) maximum on the axis of the beam (2) minimum on the axis of the beam
(3) the same everywhere in the beam (4) directly proportional to the intensity I

8. As the beam enters the medium, it will :
(1) travel as a cylindrical beam (2) diverge
(3) converge (4) diverge near the axis and converge near the periphery

9. At two points P and Q on screen in Young's double slit experiment, waves from slits S_1 and S_2 have a path difference of 0 and $\frac{}{4}$ respectively. the ratio of intensities at P and Q will be : **[AIEEE - 2011]**
(1) 3 : 2 (2) 2 : 1 (3) $\sqrt{2}$: 1 (4) 4 : 1

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

10. In a Young's double slit experiment, the two slits act as coherent sources of waves of equal amplitude A and wavelength λ. In another experiment with the same arrangement the two slits are made to act as incoherent sources of waves of same amplitude and wavelength. If the intensity at the middle point of the screen in the first case is I_1 and in the second case I_2, then the ratio $\frac{I_1}{I_2}$ is :- **[AIEEE - 2011]**

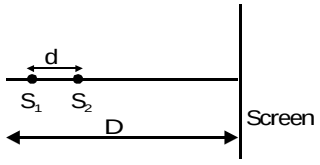
(1) 4 (2) 2 (3) 1 (4) 0.5

11. **Direction :** The question has a paragraph followed by two statement, Statement-1 and statement-2. Of the given four alternatives after the statements, choose the one that describes the statements.
A thin air film is formed by putting the convex surface of a plane-convex lens over a plane glass plate. With monochromatic light, this film gives an interference pattern due to light reflected from the top (convex) surface and the bottom (glass plate) surface of the film **[AIEEE - 2011]**
Statement-1: When light reflects from the air-glass plate interface, the reflected wave suffers a phase change of π.
Statement-2: The centre of the interference pattern is dark.
(1) Statement-1 is true, Statement-2 is true and Statement-2 is not the correct explanation of Statement-1.
(2) Statement-1 is false, Statement-2 is true
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is true, Statement-2 is true and Statement-2 is the correct explanation of statement-1.

12. In Young's double slit experiment, one of the slit is wider than other, so that the amplitude of the light from one slit is double of that from other slit. If I_m be the maximum intensity, the resultant intensity I when they interfere at phase difference ϕ is given by : **[AIEEE - 2012]**

(1) $\frac{I_m}{9}(1 + 8\cos^2\frac{\phi}{2})$ (2) $\frac{I_m}{9}(4 + 5\cos\phi)$ (3) $\frac{I_m}{3}(1 + 2\cos^2\frac{\phi}{2})$ (4) $\frac{I_m}{5}(1 + 4\cos^2\frac{\phi}{2})$

13. Two coherent point sources S_1 and S_2 are separated by a small distance 'd' as shown. The fringes obtained on the screen will be : **[JEE Mains - 2013]**



(1) points (2) straight lines (3) semi-circles (4) concentric circles

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE –5(A)

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	4	2	1	4	1	1	2	3	2	2	4	1	4

NODE6\E:\Data\2014 Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

EXERCISE–05(B) PREVIOUS YEARS QUESTIONS

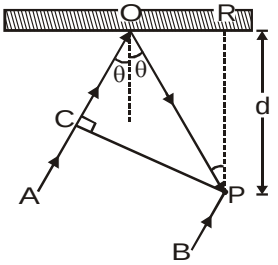
MCQ's (only one correct answers)

1. Two beams of light having intensifies I and 4I interfere to produce a fringe pattern on a screen. The phase difference between the beams on a screen. The phase difference between the beams is $\pi/2$ at point A and π at point B. Then the difference between resultant intensifies at A and B is **[IIT-JEE 2001]**
(A) 2I (B) 4I (C) 5I (D) 7I

2. In the ideal double–slit experiment, when a glass–plate (refractive index 1.5) of thickness t is introduced in the path of one of the interfering beams (wavelength λ), the intensity at the position where the central maximum occurred previously remains unchanged. The minimum thickness of the glass–plate is :– **[IIT-JEE 2002]**

(A) 2λ (B) $\frac{2\lambda}{3}$ (C) $\frac{\lambda}{3}$ (D) λ

3. In the adjacent diagram, CP represent a wavefront and AO and BP, the corresponding two rays. Find the condition on θ for constructive interference at P between the ray BP and reflected ray OP :– **[IIT-JEE 2003]**



(A) $\cos\theta = \frac{3\lambda}{2d}$ (B) $\cos\theta = \frac{\lambda}{4d}$ (C) $\sec\theta - \cos\theta = \frac{\lambda}{d}$ (D) $\sec\theta - \cos\theta = \frac{4\lambda}{d}$

4. In a YDSE bi–chromatic light of wavelengths 400nm and 560 nm are used. The distance between the slits is 0.1 mm and the distance between the plane of the slits and the screen is 1m. The minimum distance between two successive regions of complete darkness is :– **[IIT-JEE 2004]**
(A) 4 mm (B) 5.6 mm (C) 14 mm (D) 28 mm

5. In Young's double slit experiment intensity at a point is (1/4) of the maximum intensity. Angular position of this point is :– **[IIT-JEE 2005]**

(A) $\sin^{-1}\left(\frac{\lambda}{d}\right)$ (B) $\sin^{-1}\left(\frac{\lambda}{2d}\right)$ (C) $\sin^{-1}\left(\frac{\lambda}{3d}\right)$ (D) $\sin^{-1}\left(\frac{\lambda}{4d}\right)$

6. In the Young's double slit experiment using a monochromatic light of wavelength λ, the path difference (in terms of an integer n) corresponding to any point having half the peak intensity is :- **[IIT-JEE 2013]**

(A) $(2n+1)\frac{\lambda}{2}$ (B) $(2n+1)\frac{\lambda}{4}$ (C) $(2n+1)\frac{\lambda}{8}$ (D) $(2n+1)\frac{\lambda}{16}$

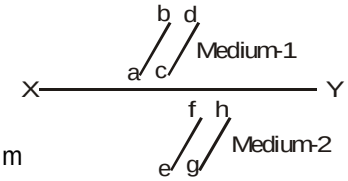
MCQ's (one or more than one correct answers)

1. In a Young's double slit experiment, the separation between the two slits is d and the wavelength of the light is λ. The intensity of light falling on slit 1 is four times the intensity of light falling on slit 2. Choose the correct choice (s) **[IIT-JEE 2008]**
(A) If $d = \lambda$, the screen will contain only one maximum
(B) If $\lambda < d < 2\lambda$, at least one more maximum (besides the central maximum) will be observed on the screen
(C) If the intensity of light falling on slit 1 is reduced so that it becomes equal to that of slit 2, the intensities of the observed dark and bright fringes will increase.
(D) If the intensity of light falling on slit 2 is increased so that it becomes equal to that of slit 1, the intensities of the observed dark and bright fringes will increase

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

Comprehension Based Question

The figure shows a surface XY separating two transparent media, medium–1 and medium–2. The lines ab and cd represent wavefronts of a light wave travelling in medium–1 and incident XY. The lines ef and gh represent wavefronts of the light wave in medium–2 after refraction. **[IIT-JEE 2007]**



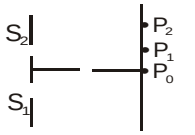
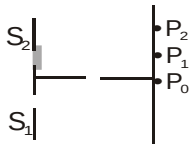
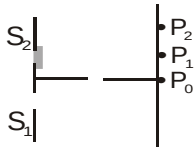
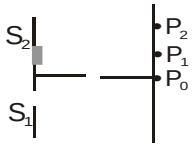
1. Light travels as a :–
(A) parallel beam in each medium
(B) convergent beam in each medium
(C) divergent beam in each medium
(D) divergent beam in one medium and convergent beam in the other medium

2. The phases of the light wave at c, d, e and f are ϕ_c, ϕ_d, ϕ_e and ϕ_f respectively. It is given that $\phi_c \neq \phi_f$:–
(A) ϕ_c cannot be equal to ϕ_d
(B) ϕ_d can be equal to ϕ_e
(C) $(\phi_d - \phi_f)$ is equal to $(\phi_c - \phi_e)$
(D) $(\phi_d - \phi_c)$ is not equal to $(\phi_f - \phi_e)$

3. Speed of light is :–
(A) the same in medium–1 and medium–2
(B) larger in medium–1 then in medium–2
(C) larger in medium–2 than in medium–1
(D) different at b and d

Match the Column

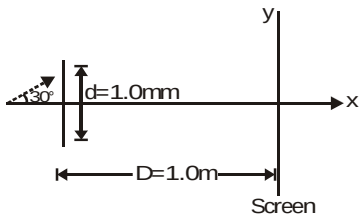
1. Column I shows four situations of standard Young's double slit arrangement with the screen placed far away from the slits S_1 and S_2. In each of these cases $S_1P_0 = S_2P_0$, $S_1P_1 - S_2P_1 = \lambda/4$ and $S_1P_2 - S_2P_2 = \lambda/3$, where λ is the wavelength of the light used. In the cases B, C and D, a transparent sheet of refractive index μ and thickness t is passed on slit S_2. The thicknesses of the sheets are different in different cases. The phase difference between the light waves reaching a point P on the screen from the two slits is denoted by $\delta(P)$ and the intensity by I(P). Match each situation given in **Column I** with the statement(s) in Column II valid for that situation. **[IIT-JEE 2009]**

Column-I			**Column-II**	
(A)			(p)	$\delta(P_0) = 0$
(B)		$(\mu - 1)t = \lambda/4$	(q)	$\delta(P_1) = 0$
(C)		$(\mu - 1)t = \lambda/2$	(r)	$I(P_1) = 0$
(D)		$(\mu - 1)t = 3\lambda/4$	(s)	$I(P_0) > I(P_1)$
			(t)	$I(P_2) > I(P_1)$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

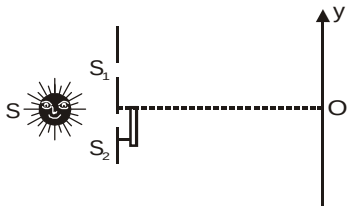
Subjective Questions

1. A coherent parallel beam of microwaves of wavelength λ = 0.5 mm falls on a Young's double slit apparatus. The separation between the slits is 1.0 mm. The intensity of microwaves is measured on a screen placed parallel to the plane of the slits at a distance of 1.0m from it as shown in the figure. **[IIT-JEE 1998]**



(i) If the incident beam falls normally on the double slit apparatus, find the y–coordinates of all the interference minima on the screen.
(ii) If the incident beam makes an angle of 30° with the x–axis (as in the dotted arrow shown in figure), find the y–coordinates of the first minima on either side of the central maximum.

2. The Young's double slit experiment is done in a medium of refractive index 4/3. A light of 600 nm wavelength is falling on the slits having 0.45 mm separation. The lower slit S_2 is covered by a thin glass sheet of thickness 10.4 µm and refractive index 1.5. The interference pattern is observed on a screen placed 1.5 m from the slits as shown in the figure. **[IIT-JEE 1999]**



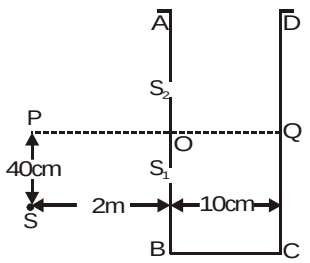
(i) Find the location of central max. (bright fringe with zero path difference) on the y–axis. (ii) Find the light intensity of point O relative to the maximum fringe intensity.
(iii) Now, if 600 nm light is replaced by white light of range 400 to 700 nm, find the wavelength of the light that form maxima exactly at point O. [All wavelengths in the problem are for the given medium of refractive index 4/3. Ignore dispersion]

3. A glass plate of refractive index 1.5 is coated with a thin layer of thickness t and refractive index 1.8. Light of wavelength λ travelling in air is incident normally on the layer. It is partly reflected at the upper and the lower surfaces of the layer and the two reflected rays interfere. Write the condition for their constructive interference. If λ = 648 nm, obtain the least value of t for which the rays interfere constructively.

[IIT-JEE 2000]

4. A vessel ABCD of 10cm width has two small slits S_1 and S_2 sealed with identical glass plates of equal thickness. The distance between the slits is 0.8 mm. POQ is the line perpendicular to the plane AB and passing through O, the middle point of S_1 and S_2. A monochromatic light source is kept at S, 40cm below P and 2m from the vessel, to illuminate the slits as shown in the figure alongside. Calculate the position of the central bright fringe on the other wall CD with respect to the line OQ. Now, a liquid is poured into the vessel and filled upto OQ. The central bright fringe is found to be at Q. Calculate the refractive index of the liquid.

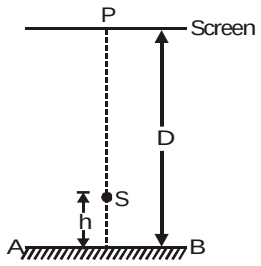
[IIT-JEE 2001]



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

5. A point source S emitting light of wavelength 600nm is placed at a very small height h above a flat reflecting surface AB (see figure). The intensity of the reflected light is 36% of the incident intensity. Interference fringes are observed on a screen placed parallel to the reflecting surface at a very large distance D from it.

[IIT-JEE 2002]



(i) What is the shape of the interference fringes on the screen ?
(ii) Calculate the ratio of the minimum to the maximum intensities in the interference fringes formed near the point P (shown in the figure).
(iii) If the intensity at point P corresponds to a maximum, calculate the minimum distance through which the reflecting surface AB should be shifted so that the intensity at P again becomes maximum.

6. In a Young's double slit experiment, two wavelengths of 500 nm and 700 nm were used. What is the minimum distance from the central maximum where their maximas coincide again ? Take $D/d = 10^3$. Symbols have their usual meanings. **[IIT-JEE 2004]**

PREVIOUS YEARS QUESTIONS **ANSWER KEY** **EXERCISE –5(B)**

- **MCQ's One correct answers** **1** B **2** A **3** B **4** D **5**C **6.** B
- **MCQ's One correct answers** **1** A,B
- **Comprehension** **1.** A **2.** C **3.** B
- **Match The Column** **1.** (A)- p, s (B) - q (C) - t (D) - r,s,t
- **Subjective**

1. (i) ± 0.26 m, ± 1.13 m (ii) 0.26m, 1.13 m

2. (i) 4.33 m (ii) $I = \frac{3I_{max}}{4}$ (iii) 650 nm; 433.33 nm

3. $2\mu t = n\frac{1}{2}$ with μ = 1.8 and n= 1,2,3,...90 , t_{min} = 90 nm

4. 2 cm above point Q on side CD, μ= 1.0016

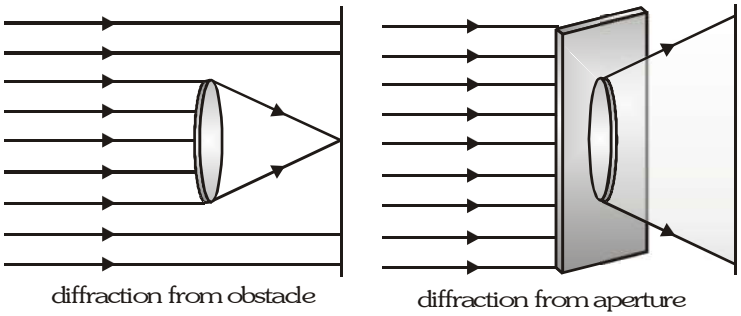
5. (i) Circular (ii) $\frac{1}{16}$ (iii) 300 nm

6. 3.5 mm

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\01_Interference of Light.p65

DIFFRACTION OF LIGHT

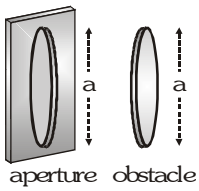
Bending of light rays from sharp edges of an opaque obstacle or aperture and its spreading in the eometricle shaddow region is defined as diffraction of light or deviation of light from its rectilinear propogation tendency is defined as diffraction of light.

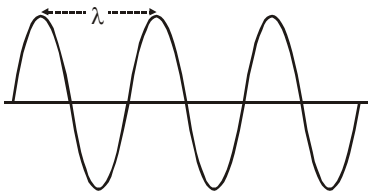


- Diffraction was discovered by Grimaldi
- Theoritically explained by Fresnel
- Diffraction is possible in all type of waves means in mechanical or electromagnetic waves shows diffraction.

Diffraction depends on two factors :

(i) Size of obstacles or aperture (ii) Wave length of the wave



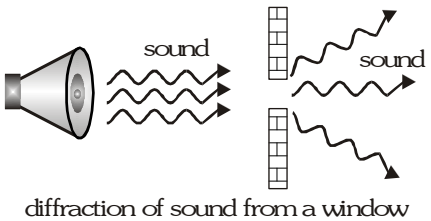


Condition of diffraction Size of obstacle or aperture should be nearly equal to the wave length of light

$\lambda \tilde{-} a$ $\frac{a}{\lambda} \simeq 1$

If size of obstacle is much greater than wave length of light, then rectilinear motion of light is observed.

- It is practically observed when size of aperture or obstacle is greater than 50 λ then obstacle or aperature does not shows diffraction.
- Wave length of light is in the order 10^{-7} m. In general obstacle of this wave length is not present so light rays does not show diffraction and it appears to travel in straight line Sound wave shows more diffraction as compare to light rays because wavelength of sound is high (16 mm to 16m). So it is generally diffracted by the objects in our daily life.
- Diffraction of ultrasonic wave is also not observed as easily as sound wave because their wavelength is of the order of about 1 cm. Diffraction of radio waves is very conveniently observed because of its very large wavelength (2.5 m to 250 m). X-ray can be diffracted easily by crystel. It was discovered by Lave.



diffraction of sound from a window

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

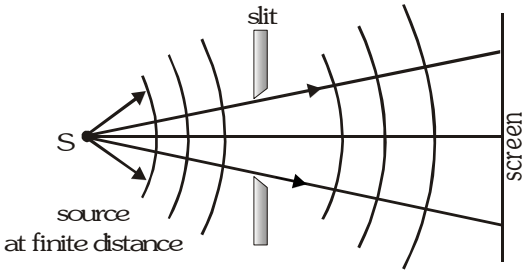
TYPES OF DIFFRACTION

(i) There are two type of diffraction of light : (a) Fresnel's diffraction. (b) Fraunhofer's diffraction.

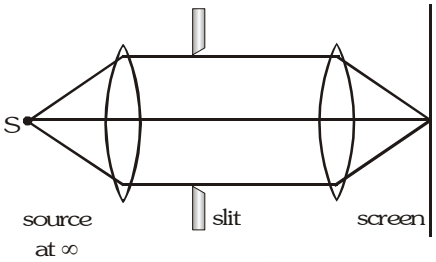
(a) Fresnel diffraction

If either source or screen or both are at finite distance from the diffracting device (obstacle or aperture), the diffraction is called fresnel diffraction and the pattern is the shadow of the diffracting device modified by diffraction effect.

Example :- Diffraction at a straight edge, small opaque disc, narrow wire are examples of fresnel diffraction.



Fresnel's diffraction



Fraunhofer's diffraction

(b) Fraunhofer diffraction

Fraunhofer diffraction is a particular limiting case of fresnel diffraction.In this case, both source and screen are effectively at infinite distance from the diffracting device and pattern is the image of source modified by diffraction effects.

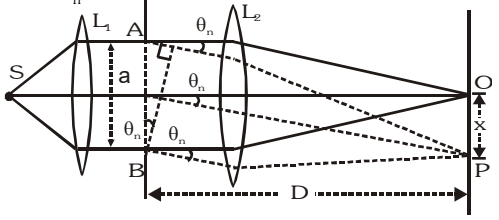
Example :- Diffraction at single slit, double slit and diffraction grating are the examples of fraunhofer diffraction.

Comparison between fresnel and fraunhofer diffraction

	Fresnel Diffraction	Fraunhofer Diffraction
(a)	Source and screen both are at finite distance from the diffractor.	Source and screen both are at infinite distance from the diffractor.
(b)	Incident and diffracted wave fronts are spherical or cylinderical.	Incident and diffracted wavefronts are plane due to infinite distance from source.
(c)	Mirror or lenses are not used for obtaining the diffraction pattern.	Lens are used in this diffraction pattern.
(d)	Centre of diffraction pattern is sometime bright and sometime dark depending on size of diffractor and distance of observation point.	Centre of diffraction is always bright.
(e)	Amplitude of wave coming from different half period zones are different due to difference of obliquity.	Amplitude of waves coming from different half period zones are same due to same obliquity.

FRAUNHOFER DIFFRACTION DUE TO SINGLE SLIT

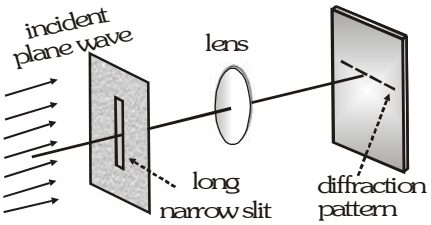
AB is single slit of width a, Plane wavefront is incident on a slit AB. Secondary wavelets coming from every part of AB reach the axial point P in same phase forming the central maxima. The intensity of central maxima is maximum in this diffrection. where θ_n represents direction of n^{th} minima Path difference BB' = a sin θ_n



for n^{th} minima a sin $\theta_n = n\lambda$ $\quad \therefore \quad \sin\theta_n \approx \theta_n = \frac{n\lambda}{a}$ (if θ_n is small)

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

- When path difference between the secondary wavelets coming from A and B is $n\lambda$ or $2n\left[\frac{\lambda}{2}\right]$ or even multiple of $\frac{\lambda}{2}$ then minima occurs



For minima $a\sin\theta_n = 2n\left[\frac{\lambda}{2}\right]$ where n = 1, 2, 3 ...

- When path difference between the secondary wavelets coming from A and B is $(2n+1)\frac{\lambda}{2}$ or odd multiple of $\frac{\lambda}{2}$ then maxima occurs

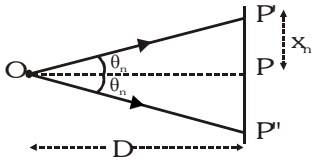
For maxima $a\sin\theta_n = (2n+1)\frac{\lambda}{2}$ where n = 1, 2, 3 ...

n = 1 $\rightarrow$ first maxima and n = 2 $\rightarrow$ second maxima

- In alternate order minima and maxima occurs on both sides of central maxima.

For n^{th} minima

If distance of n^{th} minima from central maxima = x_n
distance of slit from screen = D , width of slit = a



Path difference $\delta = a\sin\theta_n = \frac{2n\lambda}{2} \Rightarrow \sin\theta_n = \frac{n\lambda}{a}$

In $\Delta POP'$ $\tan\theta_n = \frac{x_n}{D}$ If θ_n is small $\Rightarrow \sin\theta_n \approx \tan\theta_n \approx \theta_n$

$x_n = \frac{n\lambda D}{a} \Rightarrow \theta_n = \frac{x_n}{D} = \frac{n\lambda}{a}$ First minima occurs both sides on central maxima.

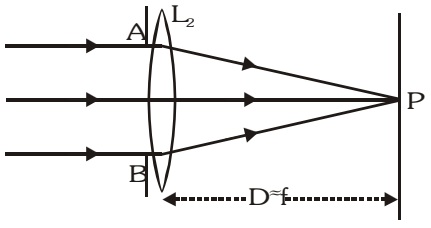
For first minima $x = \frac{D\lambda}{a}$ and $\theta = \frac{x}{D} = \frac{\lambda}{a}$

- Linear width of central maxima $w_x = 2x \Rightarrow w_x = \frac{2D\lambda}{a}$

- Angular width of central maxima $w_\theta = 2\theta = \frac{2\lambda}{a}$

SPECIAL CASE

Lens L_2 is shifted very near to slit AB. In this case distance between slit and screen will be nearly equal to the focal length of lense L_2 (i.e. D $\approx$ f) $\theta_n = \frac{x_n}{f} = \frac{n\lambda}{a} \Rightarrow x_n = \frac{n\lambda f}{a}$



$w_x = \frac{2\lambda f}{a}$ and angular width of central maxima $w_B = \frac{2x}{f} = \frac{2\lambda}{a}$

Fringe width : Distance between two consecutive maxima (bright fringe) or minima (dark fringe) is known as fringe width. Fringe width of central maxima is doubled then the width of other maximas i.e.,

$$\beta = x_{n+1} - x_n = (n+1)\frac{\lambda D}{a} - \frac{n\lambda D}{a} = \frac{\lambda D}{a}$$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

Intensity curve of Fraunhofer's diffraction

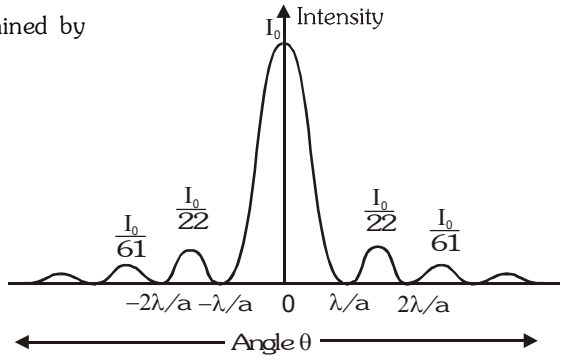
Intensity of maxima in Fraunhofer's diffrection is determined by

$$I = \left[\frac{2}{(2n+1)\pi}\right]^2 I_0$$

I_0 =intensity of central maxima
n = order of maxima

intensity of first maxima $I_1 = \frac{4}{9\pi^2} I_0 \approx \frac{I_0}{22}$

intensity of second maxima $I_2 = \frac{4}{25\pi^2} I_0 \approx \frac{I_0}{61}$



☯ *Diffraction occurs in slit is always fraunhofer diffraction as diffraction pattern obtained from the cracks between the fingers, when viewed a distant tubelight and in YDSE experiment are fraunhofer diffraction.*

GOLDEN KEY POINTS

- The width of central maxima $\propto \lambda$, that is, more for red colour and less for blue.

  i.e., $w_x \propto \lambda$ as $\lambda_{blue} < \lambda_{red} \Rightarrow w_{blue} < w_{red}$
- For obtaining the fraunhofer diffraction, focal length of second lens (L_2) is used.

  $w_x \propto \lambda \propto f \propto 1/a$ width will be more for narrow slit
- By decreasing linear width of slit, the width of central maxima increase.

RESOLVING POWER (R.P.)

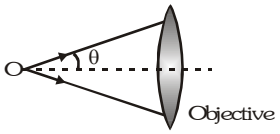
A large number of images are formed as a consequence of light diffraction from a source. If two sources are separated such that their central maxima do not overlap, their images can be distinguished and are said to be resolved R.P. of an optical instrument is its ability to distinguish two neighbouring points.

Linear R.P. = $d/\lambda D$ here D = Observed distance

Angular R.P. = d/λ d = Distance between two points

(1) **Microscope :** In reference to a microscope, the minimum distance between two lines at which they are just distinct is called Resolving limit (RL) and it's reciprocal is called Resolving power (RP)

$$R.L. = \frac{\lambda}{2\mu\sin\theta} \text{ and } R.P. = \frac{2\mu\sin\theta}{\lambda} \Rightarrow R.P. \propto \frac{1}{\lambda}$$



λ = Wavelength of light used to illuminate the object

μ = Refractive index of the medium between object and objective,

θ = Half angle of the cone of light from the point object, $\mu\sin\theta$ = Numerical aperture.

(2) **Telescope :** Smallest angular separations ($d\theta$) between two distant object, whose images are separated in the telescope is called resolving limit. So resolving limit $d\theta = \frac{1.22\lambda}{a}$ and resolving power

$$(RP) = \frac{1}{d\theta} = \frac{a}{1.22\lambda} \Rightarrow R.P. \propto \frac{1}{\lambda}$$ where a = aperture of objective.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

Example

Light of wavelength 6000Å is incident normally on a slit of width 24 10^{-5} cm. Find out the angular position of second minimum from central maximum ?

Sol. $a \sin\theta = 2\lambda$ given $\lambda = 6 \quad 10^{-7}$ m, $a = 24 \quad 10^{-5} \quad 10^{-2}$ m

$$\sin\theta = \frac{2\lambda}{a} = \frac{2\times6\times10^{-7}}{24\times10^{-7}} = \frac{1}{2} \qquad \therefore \qquad \theta = 30$$

Example

Light of wavelength 6328Å is incident normally on a slit of width 0.2 mm. Calculate the angular width of central maximum on a screen distance 9 m ?

Sol. given $\lambda = 6.328 \quad 10^{-7}$ m, $a = 0.2 \quad 10^{-3}$ m

$$w_\theta = \frac{2\lambda}{a} = \frac{2\times6.328\times10^{-7}}{2\times10^{-4}} \text{ radian} = \frac{6.328\times10^{-3}\times180}{3.14} = 0.36$$

Example

Light of wavelength 5000Å is incident on a slit of width 0.1 mm. Find out the width of the central bright line on a screen distance 2m from the slit ?

Sol. $w_x = \frac{2f\lambda}{a} = \frac{2\times2\times5\times10^{-7}}{10^{-4}} = 20$ mm

Example

The fraunhofer diffraction pattern of a single slit is formed at the focal plane of a lens of focal length 1m. The width of the slit is 0.3 mm. If the third minimum is formed at a distance of 5 mm from the central maximum then calculate the wavelength of light.

Sol. $x_n = \frac{nf\lambda}{a} \Rightarrow \qquad \lambda = \frac{ax_n}{fn} = \frac{3\times10^{-4}\times5\times10^{-3}}{3\times1} = 5000\text{Å} \qquad [\because \ n = 3]$

Example

Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width 12 10^{-5} cm when the slit is illuminated by monochromatic light of wavelength 6000 Å.

Sol. $\because \qquad \sin\theta = \frac{\lambda}{a}$ θ = half angular width of the central maximum.

$a = 12 \quad 10^{-5}$ cm, $\lambda = 6000$ Å $= 6 \quad 10^{-5}$ cm $\qquad \therefore \qquad \sin\theta = \frac{\lambda}{a} = \frac{6\times10^{-5}}{12\times10^{-5}} = 0.50 \qquad \Rightarrow \ \theta = 30$

Example

Light of wavelength 6000 Å is incident on a slit of width 0.30 mm. The screen is placed 2 m from the slit. Find (a) the position of the first dark fringe and (b) the width of the central bright fringe.

Sol. The first fringe is on either side of the central bright fringe.

here $n = \pm1$, $D = 2$ m, $\lambda = 6000$ Å $= 6 \quad 10^{-7}$ m

$\therefore \qquad \sin\theta = \frac{x}{D} \qquad \Rightarrow \ a = 0.30 \text{ mm} = 3 \quad 10^{-4} \text{ m} \ \Rightarrow \ a\sin\theta = n\lambda \ \Rightarrow \ \frac{ax}{D} = n\lambda$

(a) $\qquad x = \frac{n\lambda D}{a} \qquad \Rightarrow \qquad x = \pm\left[\frac{1\times6\times10^{-7}\times2}{3\times10^{-4}}\right] = \pm4\times10^{-3}\text{m}$

The positive and negative signs corresponds to the dark fringes on either side of the central bright fringe.

(b) The width of the central bright fringe $y = 2x = 2 \quad 4 \quad 10^{-3} = 8 \quad 10^{-3}$ m $= 8$ mm

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

DIFFERENCE BETWEEN INTERFERENCE AND DIFFRACTION :

	Interference	Diffraction
(1)	It is the phenomenon of superposition of two waves coming from two different coherent sources.	(1) It is the phenomenon of superposition of two waves coming from two different parts of the same wave front.
(2)	In interference pattern, all bright lines are equally bright and equally spaced.	(2) All bright lines are not equally bright and equally wide. Brightness and width goes on decreasing with the angle of diffraction.
(3)	All dark lines are totally dark	(3) Dark lines are not perfectly dark. Their contrast with bright lines and width goes on decreasing with angle of diffraction.
(4)	In interference bands are large in number	(4) In diffraction bands are a few in number

Example

A Slit of width a is illuminated by monochromatic light of wavelength 650nm at normal incidence. Calculate the value of a when -

(a) the first minimum falls at an angle of diffraction of 30

(b) the first maximum falls at an angle of diffraction of 30 .

Sol. (a) for first minimum $\sin\theta_1 = \frac{\lambda}{a}$ $\therefore a = \frac{\lambda}{\sin\theta_1} = \frac{650\times10^{-9}}{\sin 30^\circ} = \frac{650\times10^{-9}}{0.5} = 1.3 \quad 10^{-6}$m

(b) For first maximum $\sin\theta_1 = \frac{3\lambda}{2a}$ $\therefore$ $a = \frac{3\lambda}{2\sin\theta} = \frac{3\times650\times10^{-9}}{2\times0.5} = 1.95 \quad 10^{-6}$m

Example

Red light of wavelength 6500Å from a distant source falls on a slit 0.50 mm wide. What is the distance between the first two dark bands on each side of the central bright of the diffraction pattern observed on a screen placed 1.8 m. from the slit.

Sol. Given λ = 6500Å = 65 10^{-8} m, a = 0.5 mm = 0.5 10^{-3} m., D = 1.8 m.

Required distance between first two dark bands will be equal to width of central maxima.

$W_x = \frac{2\lambda D}{a} = \frac{2\times6500\times10^{-10}\times1.8}{0.5\times10^{-3}} = 468 \quad 10^{-5}$ m = 4.68 mm

Example

In a single slit diffraction experiment first minimum for λ_1 = 660 nm coincides with first maxima for wavelength λ_2. Calculate λ_2.

Sol. For minima in diffraction pattern $d\sin\theta = n\lambda$

For first minima $d\sin\theta_1 = (1)\lambda_1 \Rightarrow \sin\theta_1 = \frac{\lambda_1}{d}$

For first maxima $d\sin\theta_2 = \frac{3}{2}\lambda_2 \Rightarrow \sin\theta_2 = \frac{3\lambda_2}{2d}$

The two will coincide if, $\theta_1 = \theta_2$ or $\sin\theta_1 = \sin\theta_2$

$\therefore \quad \frac{\lambda_1}{d} = \frac{3\lambda_2}{2d} \Rightarrow \lambda_2 = \frac{2}{3}\lambda_1 = \frac{2}{3}$ 660 nm = 440 nm.

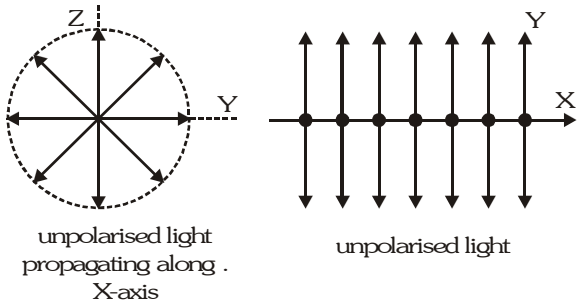
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

POLARISATION

Experiments on interference and diffraction have shown that light is a form of wave motion. These effects do not tell us about the type of wave motion i.e., whether the light waves are longitudinal or transverese.
The phenomenon of polarization has helped to establish beyond doubt that light waves are transverse waves.

UNPOLARISED LIGHT

An ordinary beam of light consists of a large number of waves emitted by the atoms of the light source. Each atom produces a wave with its own orientation of electric vector $\vec{E}$ so all direction of vibration of $\vec{E}$ are equally probable.



The resultant electromagnetic wave is a super position of waves produced by the individual atomic sources and it is called unpolarised light. In ordinary or unpolarised light, the vibrations of the electric vector occur symmetrically in all possible directions in a plane perpendicular to the direction of propagation of light.

POLARISATION

The phenomenon of restricting the vibration of light (electric vector) in a particular direction perpendicular to the direction of propagation of wave is called polarisation of light. In polarised light, the vibration of the electric vector occur in a plane perpendicular to the direction of propagation of light and are confined to a single direction in the plane (do not occur symmetrically in all possible directions). After polarisation the vibrations become asymmetrical about the direction of propagation of light.

POLARISER

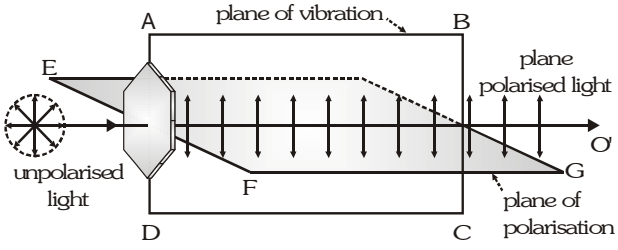
Tourmaline crystal : When light is passed through a tomaline crystal cut parallel to its optic axis, the vibrations of the light carrying out of the tourmaline crystal are confined only to one direction in a plane perpendicular to the direction of propagation of light. The emergent light from the crystal is said to be plane polarised light.

Nicol Prism : A nicol prism is an optical device which can be used for the production and detection of plane polarised light. It was invented by William Nicol in 1828.

Polaroid : A polaroid is a thin commercial sheet in the form of circular disc which makes use of the property of selective absorption to produce an intense beam of plane polarised light.

PLANE OF POLARISATION AND PLANE OF VIBRATION :

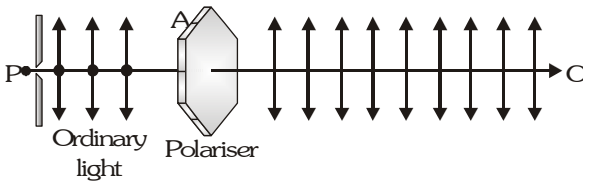
The plane in which vibrations of light vector and the direction of propogation lie is known as plane of vibration
A plane normal to the plane of vibration and in which no vibration takes place is known as plane of polarisation



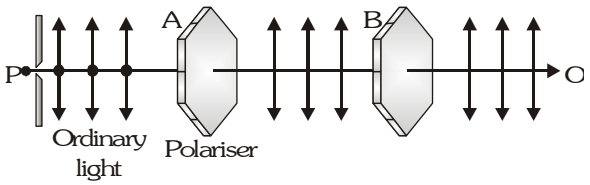
NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

EXPERIMENTAL DEMONSTRATION OF POLARISATION OF LIGHT

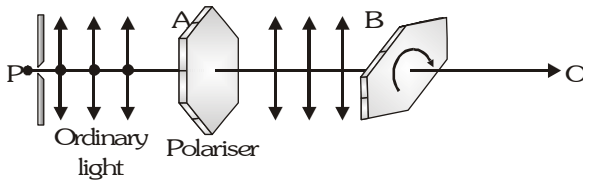
Take two tourmaline crystals cut parallel to their crystallographic axis (optic axis).



First hold the crystal A normally to the path of a beam of colour light. The emergent beam will be slightly coloured. Rotate the crystal A about PO. No change in the intensity or the colour of the emergent beam of light. Take another crystal B and hold it in the path of the emergent beam of so that its axis is parallel to the axis of the crystal A. The beam of light passes through both the crystals and outcoming light appears coloured.



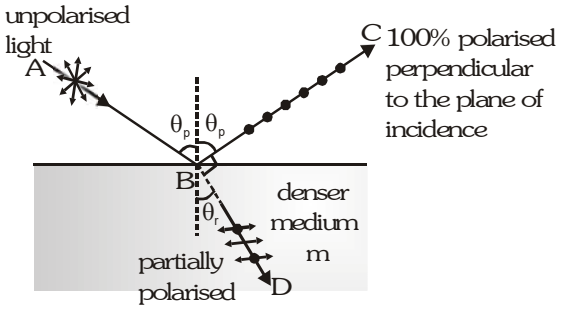
Now, rotate the crystal B about the axis PO. It will be seen that the intensity of the emergent beam decreases and when the axes of both the crystals are at right angles to each other no light comes out of the crystal B.



If the crystal B is further rotated light reappears and intensity becomes maximum again when their axes are parallel. This occurs after a further rotation of B through 90 . This experiment confirms that the light waves are transverse in nature. The vibrations in light waves are perpendicular to the direction of propogation of the the wave. First crystal A polarises the light so it is called polariser. Second crystal B, analyses the light whether it is polarised or not, so it is called analyser.

METHODS OF OBTAINING PLANE POLARISED LIGHT

- **Polarisation by reflection :** The simplest method to produce plane polarised light is by reflection. This method was discovered by Malus in 1808. When a beam of ordinary light is reflected from a surface, the reflected light is partially polarised. The degree of polarisation of the polarised light in the reflected beam is greatest when it is incident at an angle called polarising angle or Brewster's angle.



- **Polarising angle** : Polarising angle is that angle of incidence at which the reflected light is completely plane polarisation.
- **Brewster's Law** : When unpolarised light strikes at polarising angle θ_p on an interface separating a rare medium from a denser medium of refractive index μ , such that $\mu = \tan \theta_p$ then the reflected light (light in rare medium) is completely polarised. Also reflected and refracted rays are normal to each other. This relation is known as Brewster's law. The law state that the tangent of the polarising angle of incidence of a transparent medium is equal to its refractive index $\mu = \tan \theta_p$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

In case of polarisation by reflection :

(i) For $i = \theta_p$ refracted light is partially polarised.

(ii) For $i = \theta_p$ reflected and refracted rays are perpendicular to each other.

(iii) For $i < \theta_p$ or $i > \theta_p$ both reflected and refracted light become partially polarised.

According to snell's law $\mu = \dfrac{\sin\theta_p}{\sin\theta_r}$ .....(i)

But according to Brewster's law $\mu = \tan\theta_p = \dfrac{\sin\theta_p}{\cos\theta_p}$ .....(ii)

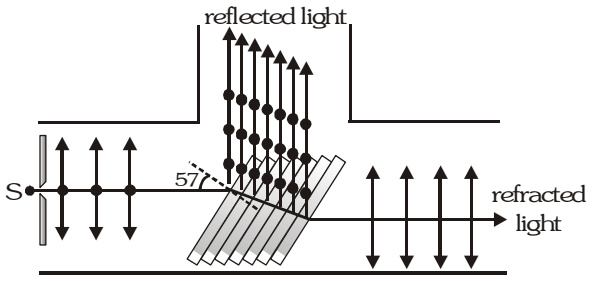
From equation (i) and (ii) $\dfrac{\sin\theta_p}{\sin\theta_r} = \dfrac{\sin\theta_p}{\cos\theta_p} \Rightarrow \sin\theta_r = \cos\theta_p$

$\therefore \sin\theta_r = \sin(90 - \theta_p) \Rightarrow \theta_r = 90 - \theta_p$ or $\theta_p + \theta_r = 90$

Thus reflected and refracted rays are mutually perpendicular

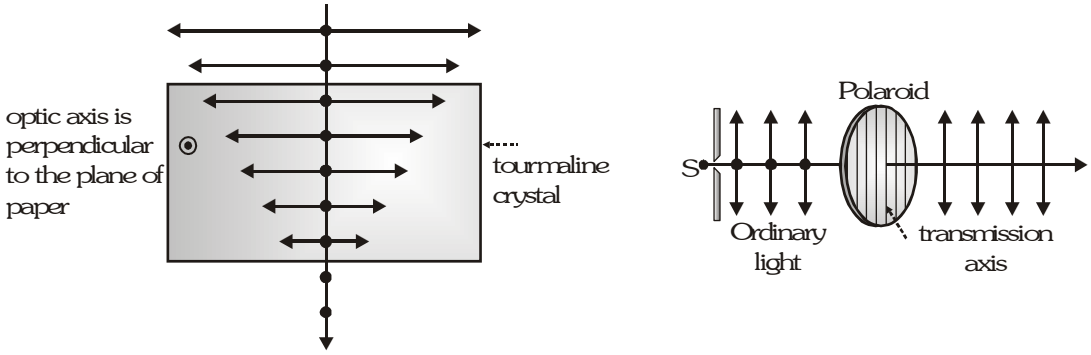
By Refraction

In this method, a pile of glass plates is formed by taking 20 to 30 microscope slides and light is made to be incident at polarising angle 57 . According Brewster law, the reflected light will be plane polarised with vibrations perpendicular to the plane of incidence and the transmitted light will be partially polarised. Since in one reflection about 15% of the light with vibration perpendicular to plane of paper is reflected therefore after passing through a number of plates emerging light will become plane polarised with vibrations in the plane of paper.



By Dichroism

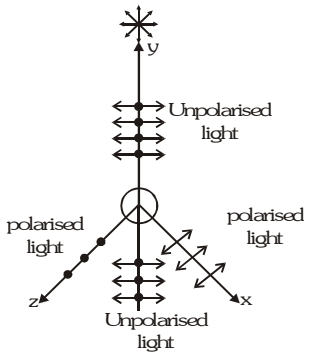
Some crystals such as tourmaline and sheets of iodosulphate of quinone have the property of strongly absorbing the light with vibrations perpendicular of a specific direction (called transmision axis) and transmitting the light with vibration parallel to it. This selective absorption of light is called dichroism. So if unpolarised light passes through proper thickness of these, the transmitted light will plane polarised with vibrations parallel to transmission axis. Polaroids work on this principle.



NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

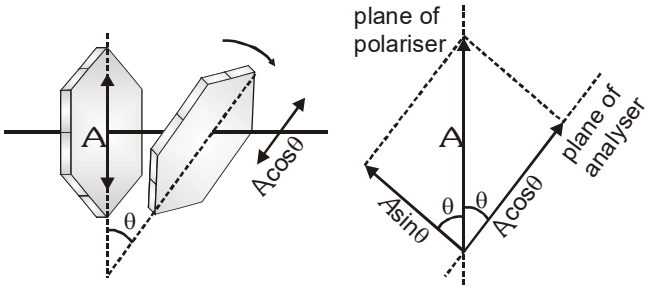
By scattering :

When light is incident on small particles of dust, air molecule etc. (having smaller size as compared to the wavelength of light), it is absorbed by the electrons and is re-radiated in all directions. The phenomenon is called as scattering. Light scattered in a direction at right angles to the incident light is always plane-polarised.



Law of Malus

When a completely plane polarised light beam light beamis incident analyser, then intensity intensity of emergent light varies as the square of cosine of the angle between the planes of transmission of the analyser and the polarizer. $I \propto \cos^2\theta \Rightarrow I = I_0 \cos^2\theta$



(i) If $\theta = 0$ then $I = I_0$ maximum value (Parallel arrangement)

(ii) If $\theta = 90$ then $I = 0$ minimum value (Crossed arrangement)

If plane polarised light of intensity $I_0 (= KA^2)$ is incident on a polaroid and its vibrations of amplitude A make angle θ with transmission axis, then the component of vibrations parallel to transmission axis will be A cos θ while perpendicular to it will be A sin θ.

Polaroid will pass only those vibrations which are parallel to transmission axis i.e. A cos θ, $\because I_0 \propto A^2$

So the intensity of emergent light $I = K(A\cos\theta)^2 = KA^2\cos^2\theta$

If an unpolarised light is converted into plane polarised light its intensity becomes half.

If light of intensity I_1, emerging from one polaroid called polariser is incident on a second polaroid (called analyser) the intensity of light emerging from the second polaroid is

$I_2 = I_1 \cos^2\theta$ θ = angle between the transmission axis of the two polaroids.

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

Optical Activity

When plane polarised light passes through certain substances, the plane of polarisation of the emergent light is rotated about the direction of propagation of light through a certain angle. This phenomenon is optical rotation.

The substance which rotate the plane of polarision rotates the plane of polarisation is known as optical active substance. Ex. Sugar solution, sugar crystal, soldium chlorate etc.

Optical activity of a substance is measured with the help of polarimeter in terms of specific rotation which is defined as the rotation produced by a solution of length 10 cm (1dm) and of unit concentration (1g/cc) for a given wave length of light at a given temp.

specific rotation $[\alpha]^{\lambda}_{t^\circ C} = \frac{\theta}{L \times C}$ θ = rotation in length L at concentration

Types of optically active substances

(a) Dextro rotatory substances

Those substances which rotate the plane of polarisation in clockwise direction are called dextro rotatory of right handed substances.

(b) Laveo rotatory substances

These substances which rotate the plane of polarisation in the anticlockwise direction are called laveo rotatory or left handed subsances.

The amount of optical rotation depends upon the thickness and density of the crystal or concentration in case of solutions, the temperature and the wavelength of light used.

Rotation varies inversely as the square of the wavelenth of light.

APPLICATIONS AND USES OF POLARISATION

- By determining the polarising angle and using Brewster's Law $\mu = \tan\theta_p$ refractive index of dark transparent substance can be determined.
- In calculators and watches, numbers and letters are formed by liquid crystals through polarisation of light called liquid crystal display (L.C.D.)
- In CD player polarised laser beam acts as needle for producing sound from compact disc.
- It has also been used in recording and reproducing three dimensional pictures.
- Polarised light is used in optical stress analysis known as photoelasticity.
- Polarisation is also used to study asymmetries in molecules and crystals through the phenomenon of optical activity.

Example

Two polaroids are crossed to each other. When one of them is rotated through 60 , then what percentage of the incident unpolarised light will be transmitted by the polaroids ?

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Sol. Initially the polaroids are crossed to each other, that is the angle between their polarising directions is 90 . When one is rotated through 60 , then the angle between their polarising directions will become 30 .

Let the intensity of the incident unpolarised light = I_0

Then the intensity of light emerging from the first polaroid is $I_1 = \frac{1}{2}I_0$

This light is plane polarised and passes through the second polaroid.

The intensity of light emerging from the second polaroid is $I_2 = I_1 \cos^2\theta$

θ = the angle between the polarising directions of the two polaroids.

$$I_1 = \frac{1}{2}I_0 \quad \text{and} \quad \theta = 30 \quad \text{so} \quad I_2 = I_1 \cos^2 30° = \frac{1}{2}I_0 \cos^2 30° \quad \Rightarrow \quad \frac{I_2}{I_0} = \frac{3}{8}$$

$$\therefore \quad \text{transmission percentage} = \frac{I_2}{I_0} \times 100 = \frac{3}{8} \times 100 = 37.5\%$$

Example

At what angle of incidence will the light reflected from water (μ = 1.3) be completely polarised ?

Sol. μ = 1.3,

From Brewster's law $\tan\theta_p = \mu = 1.3 \quad \Rightarrow \theta = \tan^{-1} 1.3 = 53$

Example

If light beam is incident at polarising angle (56.3) on air-glass interface, then what is the angle of refraction in glass ?

Sol. $\because \; i_p + r_p = 90 \quad \therefore \; r_p = 90 - i_p = 90 - 56.3 = 33.7$

Example

A polariser and an analyser are oriented so that maximum light is transmitted, what will be the intensity of outcoming light when analyer is rotated through 60 .

Sol. According to Malus Law $I = I_0 \cos^2\theta = I_0 \cos^2 60 = I_0\left[\frac{1}{2}\right]^2 = \frac{I_0}{4}$

Example

A 300 mm long tube containing 60 cm^3 of sugar solution produces an optical rotations of 10 when placed in a saccharimeter. If specific rotation of sugar is 60 , calculate the quantity of sugar contained in the tube solution.

Sol. ℓ = 300 mm = 30 cm = 3 decimetre, θ = 10 , $\left[\alpha\right]_\lambda^T = 60°$, volume of solution = 60 cm^3

$$\theta = \left[\alpha\right]_\lambda^T \ell C \;\Rightarrow\; C = \frac{\theta}{\left[\alpha\right]_\lambda^T \ell} = \frac{10°}{60° \times 3} = \frac{1}{18}\,g\ cm^{-3}$$

Quantity of sugar contained = $\frac{1}{18}$ 60 = 3.33 g

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

CHECK YOUR GRASP — EXERCISE-I

1. Diffraction and interference of light refers to :–
(A) quantum nature of light (B) wave nature of light
(C) transverse nature of light (D) electromagnetic nature of light

2. The phenomenon of diffraction of light was discovered by :–
(A) Huygens (B) Newton (C) Fresnel (D) Grimaldi

3. Sound waves shows more diffraction as compare to light rays :–
(A) wavelength of sound waves is more as compare to light rays
(B) wavelength of light rays is more as compare to sound waves
(C) wavelength of sound waves and light ray is same
(D) none of these

4. The conversation going on, in some room, can be heared by the person outside the room. The reason for it is :–
(A) interference of sound (B) reflection of sound (C) diffraction of sound (D) refraction of sound

5. Diffraction initiated from obstacle, depends upon the
(A) size of obstacle (B) wave length, size of obstacle
(C) wave length and distance of obstacle from screen (D) size of obstacle and its distance from screen

6. Phenomenon of diffraction occurs :–
(A) only in case of light and sound waves (B) for all kinds of waves
(C) for electro-magnetic waves and not for matter waves (D) for light waves only

7. Diffraction of light is observed only, when the obstacle size is :–
(A) very large (B) very small
(C) of the same order that of wavelength of light (D) any size

8. Which of the following ray gives more distinct diffraction :–
(A) X-ray (B) light ray (C) γ –ray (D) Radio wave

9. All fringes of diffraction are of :–
(A) the same intensity (B) unequal width (C) the same width (D) full darkness

10. A single slit of width d is placed in the path of beam of wavelength λ. The angular width of the principal maximum obtained is :–
(A) $\frac{d}{\lambda}$ (B) $\frac{\lambda}{d}$ (C) $\frac{2\lambda}{d}$ (D) $\frac{2d}{\lambda}$

11. Direction of the second maximum in the Fraunhofer diffraction pattern at a single slit is given by (a is the width of the slit) :–
(A) $a\sin\theta = \frac{\lambda}{2}$ (B) $a\cos\theta = \frac{3\lambda}{2}$ (C) $a\sin\theta = \lambda$ (D) $a\sin\theta = \frac{3\lambda}{2}$

12. Angular width (θ) of central maximum of a diffraction pattern of a single slit does not depend upon :–
(A) distance between slit and source (B) wavelength of light used
(C) width of the slit (D) frequency of light used

13. Red light is generally used to observe diffraction pattern from single slit. If green light is used instead of red light, then diffraction pattern :–
(A) will be more clear (B) will be contract (C) will be expanded (D) will not visualize

14. Calculate angular width of central maxima if λ = 6000 Å, a = 18 10^{-5} cm :–
(A) 20 (B) 40 (C) 30 (D) 260

15. In single slit Fraunhoffer diffraction which type of wavefront is required :–
(A) cylindrical (B) spherical (C) elliptical (D) plane

16. In the diffraction pattern of a single slit aperture, the width of the central fringe compared to widths of the other fringes, is :–
(A) equal (B) less (C) little more (D) double

17. Central fringe obtained in diffraction pattern due to a single slit :–
(A) is of minimum intensity (B) is of maximum intensity
(C) intensity does not depend upon slit width (D) none of the above

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

18. In a single slit diffraction pattern, if the light source is used of less wave length then previous one. Then width of the central fringe will be :–
(A) less (B) increase (C) unchanged (D) none of the above

19. In the laboratory, diffraction of light by a single slit is being observed. If slit is made slightly narrow, then diffraction pattern will :–
(A) be more spreaded than before (B) be less spreaded than before
(C) be spreaded as before (D) be disappeared

20. Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width 12 10^{-5} cm when the slit is illuminated by monochromatic light of wavelength 6000 Å.
(A) 40 (B) 45 (C) 30° (D) 60

21. In a Fraunhofer's diffraction by a slit, if slit width is a, wave length λ, focal length of lens is f, linear width of central maxima is :–

(A) $\frac{f\lambda}{a}$ (B) $\frac{fa}{\lambda}$ (C) $\frac{2f\lambda}{a}$ (D) $\frac{f\lambda}{2a}$

22. In a Fraunhofer's diffraction obtained by a single slit aperture, the value of path difference for n^{th} order of minima is :–

(A) nλ (B) 2nλ (C) $\frac{(2n-1)\lambda}{2}$ (D) (2n–1)λ

23. A polariser is used to :
(A) Reduce intensity of light (B) Produce polarised light
(C) Increase intensity of light (D) Produce unpolarised light

24. Light waves can be polarised as they are :
(A) Transverse (B) Of high frequency (C) Longitudinal (D) Reflected

25. Through which character we can distinguish the light waves from sound waves :
(A) Interference (B) Refraction (C) Polarisation (D) Reflection

26. The angle of polarisation for any medium is 60 , what will be critical angle for this :

(A) $\sin^{-1}\sqrt{3}$ (B) $\tan^{-1}\sqrt{3}$ (C) $\cos^{-1}\sqrt{3}$ (D) $\sin^{-1}\frac{1}{\sqrt{3}}$

27. The angle of incidence at which reflected light is totally polarized for reflection from air to glass (refractive index n)

(A) $\sin^{-1}(n)$ (B) $\sin^{-1}\left(\frac{1}{n}\right)$ (C) $\tan^{-1}\left(\frac{1}{n}\right)$ (D) $\tan^{-1}(n)$

28. A polaroid is placed at 45 to an incoming light of intensity I_0. Now the intensity of light passing through polaroid after polarisation would be :
(A) I_0 (B) $I_0/2$ (C) $I_0/4$ (D) Zero

29. Plane polarised light is passed through a polaroid. On viewing through the polariod we find that when the polariod is given one complete rotation about the direction of the light, one of the following is observed.
(A) The intensity of light gradually decreases to zero and remains at zero
(B) The intensity of light gradually increases to a maximum and remains at maximum
(C) There is no change in intensity
(D) The intensity of light is twice maximum and twice zero

30. A ray of light is incident on the surface of a glass plate at an angle of incidence equal to Brewster's angle ϕ. If μ represents the refractive index of glass with respect to air, then the angle between reflected and refracted rays is :
(A) 90 + ϕ (B)$\sin^{-1}(\mu\cos\phi)$ (C) 90 (D) 90 – $\sin^{-1}(\sin\phi/\mu)$

31. A beam of light strikes a glass plate at an angle of incident 60 and reflected light is completely polarised than the refractive index of the plate is:-

(A) 1.5 (B) $\sqrt{3}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-11\WaveOptics\Eng\02_Diffraction of Light & Polarisation.p65

32. Polarised glass is used in sun glasses because :
(A) It reduces the light intensity to half an account of polarisation
(B) It is fashionable
(C) It has good colour
(D) It is cheaper

33. When unpolarized light beam is incident from air onto glass (n=1.5) at the polarizing angle :
(A) Reflected beam is polarized 100 percent
(B) Reflected and refracted beams are partially polarized
(C) The reason for (A) is that almost all the light is reflected
(D) All of the above

34. When the angle of incidence on a material is 60 , the reflected light is completely polarized. The velocity of the refracted ray inside the material is (in ms^{-1}) :

(A) 3 10^8 (B) $\left(\frac{3}{\sqrt{2}}\right)$ 10^8 (C) $\sqrt{3}$ 10^8 (D) 0.5 10^8

CHECK YOUR GRASP — ANSWER-KEY — EXERCISE-I

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	D	A	C	B	B	C	D	B	C	D	A	B	B	D	D	B	A	A	C
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34						
Ans.	C	A	B	A	C	D	D	B	D	C	B	A	A	C						

PREVIOUS YEARS QUESTIONS — EXERCISE-II

1. Electromagnetic waves are transverse in nature is evident by- **[AIEEE - 2002]**
(1) polarization (2) interference (3) reflection (4) diffraction

2. When an unpolarized light of intensity I_0 is incident on a polarizing sheet, the intensity of the light which does not get transmitted is- **[AIEEE - 2005]**

(1) $\frac{1}{2}I_0$ (2) $\frac{1}{4}I_0$ (3) zero (4) I_0

3. If I_0 is the intensity of the principle maximum in the single slit diffraction pattern, then what will be its intensity when the slit width is doubled- **[AIEEE - 2005]**

(1) $2I_0$ (2) $4I_0$ (3) I_0 (4) $\frac{I_0}{2}$

4. **Statement-1:** On viewing the clear blue portion of the sky through a Calcite Crystal, the intensity of transmitted light varies as the crystal is rotated. **[AIEEE - 2011]**
Statement-1: The light coming from the sky is polarized due to scattering of sun light by particles in the atmosphere. The scattering is largest for blue light.
(1) Statement-1 is false, statement-2 is true
(2) Statement-1 is true, statement-2 is false
(3) Statement-1 is true, statement-2 true; statement-2 is the correct explanation of statement-1
(4) Statement-1 is true, statement-2 is true; statement -2 is not correct explanation of statement-1.

5. A beam of unpolarised light of intensity I_0 is passed through a polaroid A and then through another polaroid B which is oriented so that its principal plane makes an angle of 45 relative to that of A. The intensity of the emergent light is :
(1) I_0 (2) $I_0/2$ (3) $I_0/4$ (4) $I_0/8$

PREVIOUS YEARS QUESTIONS — ANSWER-KEY — EXERCISE-II

Que.	1	2	3	4	5
Ans.	1	1	3	3	3

NODE6\E:\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-11\Wave Optics\Eng\02_Diffraction of Light & Polarisation.p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution



PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 11

THERMAL PHYSICS - Theory

Index

THERMAL PHYSICS

- TEMPERATURE SCALES
- THERMAL EXPANSION
- HEAT TRANSFER
- CALORIMETRY
- KINETIC THEORY OF GASES
- THERMODYNAMICS

Serial No.	CONTENTS	Page No.
1.	Temperature Scales	1
2.	Thermal Expansion	2
3.	Mode of Heat Transfer	7
4.	Calorimetry	27
6.	Kinetic Theory of Gases	34
7.	Degree of Freedom	41
8.	Different Speed of gas molecules	46
9.	Thermodynamics	49
10.	First law of Thermodynamics	53
11.	Application of first law of thermodynamics	54
12.	Carnot Cycle	67
13.	Exercise-I Check your Grasp (Objective)	90
14.	Exercise-2 Brain Teasers (MCQ's with one or more than one correct answer)	97
15.	Exercise-3 Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	110
16.	Exercise-4A Conceptual Subjective Exercise	128
17.	Exercise-4B Brain Storming Exercise	137
18.	Exercise-5A Previous years questions	140
19.	Exercise-5B Previous years questions	147

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Mains : Thermal equilibrium, zeroth law of thermodynamics, concept of temperature. Heat, work and internal energy. First law of thermodynamics. Second law of thermodynamics: reversible and irreversible processes. Carnot engine and its efficiency.

Equation of state of a perfect gas, work doneon compressing a gas.Kinetic theory of gases - assumptions, concept of pressure. Kinetic energy and temperature: rms speed of gas molecules; Degrees of freedom, Law of equipartition of energy,applications to specific heat capacities of gases; Mean free path, Avogadro's number.

JEE Advanced : Thermal expansion of solids, liquids and gases; Calorimetry, latent heat; Heat conduction in one dimension; Elementary concepts of convection and radiation; Newton's law of cooling; Ideal gas laws; Specific heats (Cv and Cp for monoatomic and diatomic gases); Isothermal and adiabatic processes, bulk modulus of gases; Equivalence of heat and work; First law of thermodynamics and its applications (only for ideal gases); Blackbody radiation: absorptive and emissive powers; Kirchhoff's law; Wien's displacement law, Stefan's law.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

TEMPERATURE SCALES

TEMPERATURE

- Temperature is a macroscopic physical quantity related to our sense of hot and cold.
- The natural flow of heat is from higher temperature to lower temperature, i.e. temperature determines the thermal state of a body whether it can give or receive heat.

TEMPERATURE SCALES

- The Kelvin temperature scale is also known as thermodynamic scale. The SI unit of temperature is the kelvin and is defined as (1/273.16) of the temperature of the triple point of water. The triple point of water is that point on a P–T diagram where the three phase of water, the solid, the liquid and the gas, can coexist in equilibrium.
- In addition to Kelvin temperature scale, there are other temperature scales also like Celsius, Fahrenheit, Reaumur, Rankine, etc. Temperature on one scale can be converted into other scale by using the following identity $\frac{\text{Reading on any scale} - \text{lower fixed point (LFP)}}{\text{Upper fixed point (UFP)} - \text{lower fixed point (LFP)}}$ = constant for all scales

  Hence $\frac{C-0}{100-0} = \frac{F-32}{212-32} = \frac{K-273.15}{373.15-273.15}$
- Different temperature scales :

Name of the scale	Symbol for each degree	Lower fixed point (LFP)	Upper fixed point (UFP)	Number of divisions on the scale
Celsius	C	0 C	100 C	100
Fahrenheit	F	32 F	212 F	180
Kelvin	K	273.15 K	373.15 K	100

Example

Express a temperature of 60 F in degree celsius and in kelvin.

Solution

By using $\frac{C-0}{100-0} = \frac{F-32}{212-32} = \frac{K-273.15}{373.15-273.15}$

$\Rightarrow \frac{C-0}{100-0} = \frac{60°-32}{212-32} = \frac{K-273.15}{373.15-273.15} \Rightarrow$ C= 15.15 C and K = 288.7 K

Example

The temperature of an iron piece is heated from 30 to 90 C. What is the change in its temperature on the fahrenheit scale and on the kelvin scale?

Solution

$\Delta C = 90 - 30 = 60$ C

Temperature difference on Fahrenheit Scale $\Delta F = \frac{9}{5}\Delta C = \frac{9}{5}(60°C) = 108°F$

Temperature difference on Kelvin Scale $\Delta K = \Delta C = 60K$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

THERMAL EXPANSION

THERMAL EXPANSION

When matter is heated without any change in it's state, it usually expands. According to atomic theory of matter, asymmetry in potential energy curve is responsible for thermal expansion. As with rise in temperature the amplitude of vibration increases and hence energy of atoms increases, hence the average distance between the atom increases. So the matter as a whole expands.

- Thermal expansion is minimum in case of solids but maximum in case of gases because intermolecular force is maximum in solids but minimum in gases.
- Solids can expand in one dimension (Linear expansion), two dimension (Superficial expansion) and three dimension (Volume expansion) while liquids and gases usually suffers change in volume only.

Linear expansion :

$\ell = \ell_0 (1 + \alpha\Delta\theta) \Rightarrow \Delta\ell = \ell_0\alpha\Delta\theta$

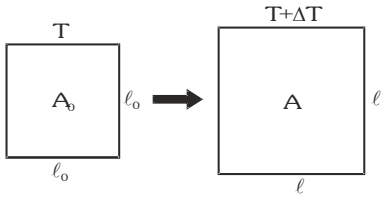


Superficial (areal) expansion :

$A = A_0 (1 + \beta\Delta\theta)$

Also $A_0 = \ell_0^2$ and $A = \ell^2$

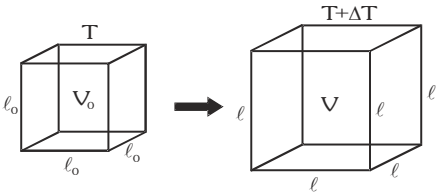
So $\ell^2 = \ell_0^2(1 + \beta\Delta\theta) = [\ell_0(1 + \alpha\Delta\theta)]^2 \Rightarrow \beta = 2\alpha$



Volume expansion :

$V = V_0 (1 + \gamma\Delta\theta)$ Also $V = \ell^3$ and $V_0=\ell_0^3$ so $\gamma = 3\alpha$

$\Rightarrow 6\alpha = 3\beta = 2\gamma$ or $\alpha : \beta : \gamma = 1 : 2 : 3$

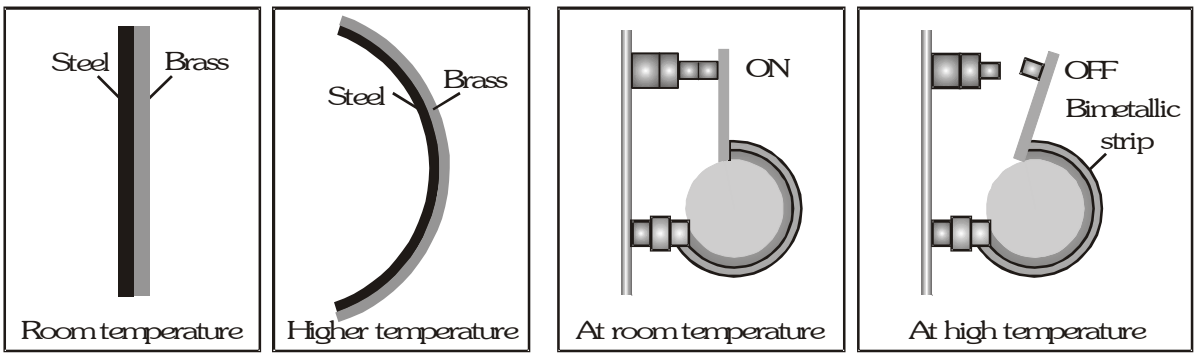


Contraction on heating :

Some rubber like substances contract on heating because transverse vibration of atoms of substance dominate over longitudinal vibration which is responsible for expansion.

Application of thermal Expansion in Solids

(a) Bi–metallic strip : Two strips of equal length but of different materials (different coefficient of linear expansion) when join together, it is called "Bi–metallic strip" and can be used in thermostat to break or make electrical contact. This strip has the characteristic property of bending on heating due to unequal linear expansion of the two metals. The strip will bend with metal of greater α on outer side.



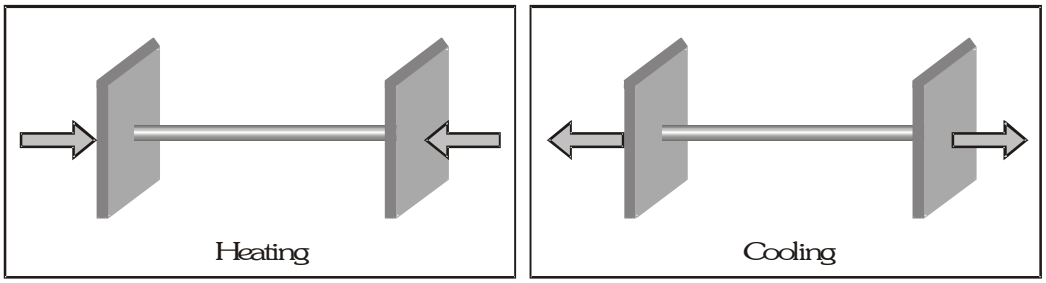
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

(b) **Effect of temperature on the time period of a simple pendulum :** A pendulum clock keeps proper time at temperature θ. If temperature is increased to θ' (> θ) then due to linear expansion, length of pendulum increases and hence its time period will increase.

Fractional change in time period $\frac{\Delta T}{T}=\frac{1}{2}\alpha\Delta\theta$ $\left(\because T\propto\sqrt{\ell}\ \ \therefore\ \frac{\Delta T}{T}=\frac{1}{2}\frac{\Delta\ell}{\ell}\right)$

- Due to increment in its time period, a pendulum clock becomes slow in summer and will lose time. Loss of time in a time period $\Delta T=\frac{1}{2}\alpha\,\Delta\theta T$
- The clock will lose time i.e. will become slow if θ' > θ (in summer) and will gain time i.e will become fast if θ' < θ (in winter).
- Since coefficient of linear expansion (α) is very small for invar, hence pendulums are made of invar to show the correct time in all seasons.

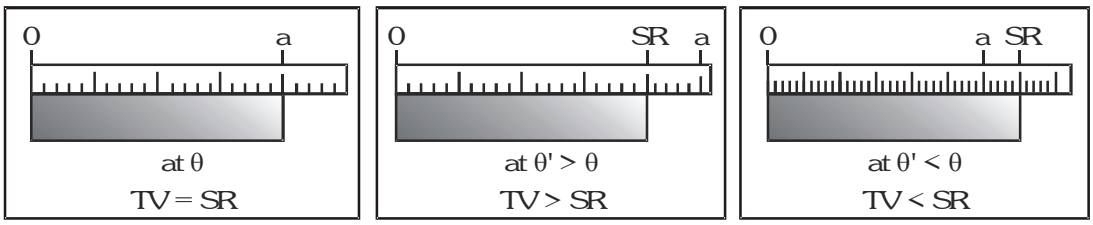
(c) When a rod whose ends are rigidly fixed such as to prevent expansion or contraction, undergoes a change in temperature due to thermal expansion or contraction, a compressive or tensile stress is developed in it. Due to this thermal stress the rod will exert a large force on the supports. If the change in temperature of a rod of length L is Δθ then :–



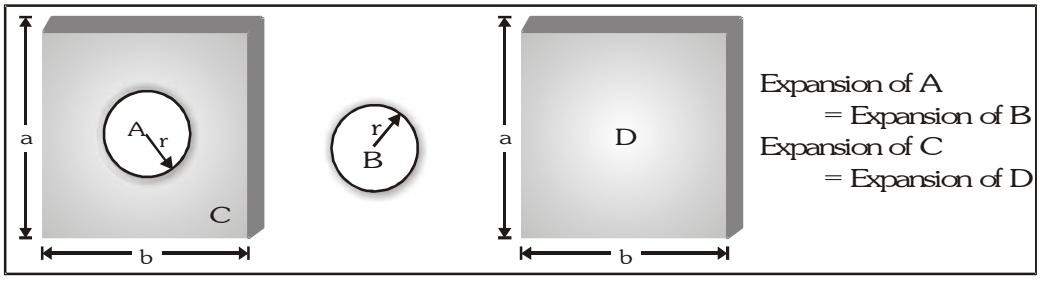
Thermal strain $=\frac{\Delta L}{L}=\alpha\Delta\theta$ $\because\ \alpha=\frac{\Delta L}{L}\times\frac{1}{\Delta\theta}$ So thermal stress = $Y\alpha\Delta\theta$ $\because\ Y=\frac{\text{stress}}{\text{strain}}$

So force on the supports $F=YA\alpha\Delta\theta$

(d) **Error in scale reading due to expansion or contraction:** If a scale gives correct reading at temperature θ. At temperature θ'(>θ) due to linear expansion of scale, the scale will expand and scale reading will be lesser than true value so that,



(e) **Expansion of cavity:** Thermal expansion of an isotropic object may be imagined as a photographic enlargement.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

(f) Some other application

- When rails are laid down on the ground, space is left between the ends of two rails
- The transmission cable are not tightly fixed to the poles
- Test tubes, beakers and cubicles are made of pyrex–glass or silica because they have very low value of coefficient of linear expansion
- The iron rim to be put on a cart wheel is always of slightly smaller diameter than that of wheel
- A glass stopper jammed in the neck of a glass bottle can be taken out by warming the neck of the bottle.

Example

A steel ruler exactly 20 cm long is graduated to give correct measurements at 20^0C. ($\alpha_{steel} = 1.2 \times 10^{-5}\ ^\circ C^{-1}$)

(a) Will it give readings that are too long or too short at lower temperatures?

(b) What will be the actual length of the ruler be when it is used in the desert at a temperature of 40^0C?

Solution

(a) If the temperature decreases, the length of the ruler also decreases through thermal contraction. Below 20^0C, each centimeter division is actually somewhat shorter than 1.0 cm, so the steel ruler gives readings that are too long.

(b) At 40^0C, the increases in length of the ruler is

$$\Delta\ell = \ell\alpha\Delta T = (20)\ (1.2 \quad 10^{-5})\ (40^0 - 20^0) = 0.48 \quad 10^{-2}\ \text{cm}$$

$\therefore$ The actual length of the ruler is, $\ell' = \ell + \Delta\ell = 20.0048$ cm

Example

A second's pendulum clock has a steel wire. The clock is calibrated at 20^0C. How much time does the clock lose or gain in one week when the temperature is increased to 30^0C? ($\alpha_{steel} = 1.2 \times 10^{-5}\ ^\circ C^{-1}$)

Solution

The time period of second's pendulum is 2 second. As the temperature increases length time period increases. Clock becomes slow and it loses the time. The change in time period is

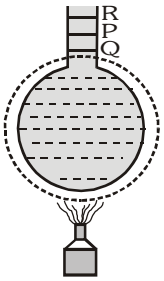
$$\Delta T = \frac{1}{2} T\alpha\Delta\theta = \left(\frac{1}{2}\right)(2)(1.2 \times 10^{-5})\ (30^0 - 20^0) = 1.2 \quad 10^{-4}\ \text{s}$$

$\therefore$ New Time period is $T' = T + \Delta T = (2 + 1.2 \times 10^{-4}) = 2.00012$ s

$\therefore$ Time lost in one week $\Delta t = \left(\frac{\Delta T}{T'}\right) t = \frac{(1.2 \times 10^{-4})}{(2.00012)}\ (7 \times 24 \times 3600) = 36.28$ s

Thermal Expansion in Liquids

- Liquids do not have linear and superficial expansion but these only have volume expansion.
- Since liquids are always to be heated along with a vessel which contains them so initially on heating the system (liquid + vessel), the level of liquid in vessel falls (as vessel expands more since it absorbs heat and liquid expands less) but later on, it starts rising due to faster expansion of the liquid.



PQ $\rightarrow$ represents expansion of vessel

QR $\rightarrow$ represents the real expansion of liquid.

- The actual increase in the volume of the liquid
  = The apparent increase in the volume of liquid + the increase in the volume of the vessel.
- Liquids have two coefficients of volume expansion.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

(i) Co–efficient of apparent expansion (γ_a)

- It is due to apparent (that appears to be, but in not) increase in the volume of liquid if expansion of vessel containing the liquid is not taken into account.

$$\gamma_a = \frac{\text{Apparent expansion in volume}}{\text{Initial volume} \times \Delta\theta} = \frac{(\Delta V)}{V \times \Delta\theta}$$

(ii) Co–efficient of real expansion (γ_r)

- It is due to the actual increase in volume of liquid due to heating.

$$\gamma_r = \frac{\text{Real increase in volume}}{\text{Initial volume} \times \Delta\theta} = \frac{(\Delta V)}{V \times \Delta\theta}$$

- Also coefficient of expansion of flask $\gamma_{Vessel} = \frac{(\Delta V)_{Vessel}}{V \times \Delta\theta}$
- $\gamma_{Real} = \gamma_{Apparent} + \gamma_{Vessel}$
- Change (apparent change) in volume in liquid relative to vessel is

  $\Delta V_{app} = V(\gamma_{Real} - \gamma_{Vessel})\ \Delta\theta = V(\gamma_r - 3\alpha)\Delta\theta$

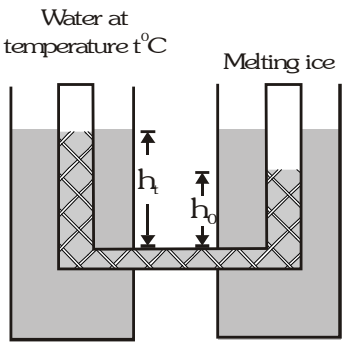
  α = Coefficient of linear expansion of the vessel.

- **Different level of liquid in vessel**

γ	ΔV	Level
$\gamma_{Real} > \gamma_{Vessel}(=3\alpha) \Rightarrow \gamma_{app} > 0$	ΔV_{app} is positive	Level of liquid in Vessel will rise on heating
$\gamma_{Real} < \gamma_{Vessel}(=3\alpha) \Rightarrow \gamma_{app} < 0$	ΔV_{app} is negative	Level of liquid in vessel will fall on heating
$\gamma_{Real} = \gamma_{Vessel}(=3\alpha) \Rightarrow \gamma_{app} = 0$	$\Delta V_{app} = 0$	Level of liquid in vessel will remain same

Example

In figure shown, left arm of a U–tube is immersed in a hot water bath at temperature t C, and right arm is immersed in a bath of melting ice; the height of manometric liquid in respective columns is h_t and h_0. Determine the coefficient of expansion of the liquid.



Solution

The liquid is in hydrostatic equilibrium $\Rightarrow \rho_t g h_t = \rho_0 g h_0$

Where, ρ_t is density of liquid in hot bath, ρ_0 is density of liquid in cold bath.

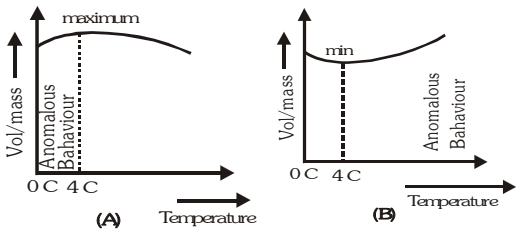
Volumes of a given mass M of liquid at temperatures t and 0^0C

are related by $V_t = V_0(1+\gamma t)$ Since $\rho_t V_t = \rho_0 V_0 \Rightarrow \rho_t = \frac{\rho_0 V_0}{V_t} = \frac{\rho_0}{(1+\gamma t)}$

Since $h_t = \frac{\rho_0 h_0}{\rho_t} = h_0(1+\gamma t)$ which on solving for γ, yields $\gamma = \frac{(h_t - h_0)}{h_0 t}$

Anomalous expansion of water

Generally matter expands on heating and contracts on cooling. In case of water, it expands on heating if its temperature is greater than 4 C. In the range 0 C to 4 C, water contracts on heating and expands on cooling, i.e. γ is negative. This behaviour of water in the range from 0 C to 4 C is called anomalous expansion.



This anomalous behaviour of water causes ice to form first at the surface of a lake in cold weather. As winter approaches, the water temperature increases initially at the surface. The water there sinks because of its increased density. Consequently, the surface reaches 0 C first and the lake becomes covered with ice. Aquatic

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

life is able to survive the cold winter as the lake bottom remains unfrozen at a temperature of about 4 C. At 4 C, density of water is maximum while its specific volume is minimum.

Example

The difference between lengths of a certain brass rod and of a steel rod is claimed to be constant at all temperatures. Is this possible ?

Solution

If L_B and L_S are the lengths of brass and steel rods respectively at a given temperature, then the lengths of the rods when temperature is changed by θ C.

$L'_B = L_B(1 + \alpha_B \Delta\theta)$ and $L'_S = L_S(1 + \alpha_B \Delta\theta)$ So that $L'_B = L'_S(L_B - L_S) + (L_B \alpha_B - L_S\alpha_S) \Delta\theta$

So $(L'_B - L'_S)$ will be equal to $(L_B - L_S)$ at all temperatures if , $L_B\alpha_B - L_S\alpha_S = 0$ [as $\Delta\theta \neq 0$] or $\frac{L_B}{L_S} = \frac{\alpha_S}{\alpha_B}$

i.e., the difference in the lengths of the two rods will be independent of temperature if the lengths are in the inverse ratio of their coefficients of linear expansion.

Example

There are two spheres of same radius and material at same temperature but one being solid while the other hollow. Which sphere will expand more if

(a) they are heated to the same temperature, (b) same heat is given to them ?

Solution

(a) As thermal expansion of isotropic solids is similar to true photographic enlargement, expansion of a cavity is same as if it had been a solid body of the same material



i.e. $\Delta V = V\gamma \Delta\theta$

As here V, γ and $\Delta\theta$ are same for both solid and hollow spheres treated (cavity) ; so the expansion of both will be equal.

(b) If same heat is given to the two spheres due to lesser mass, rise in temperature of hollow sphere will be more [as $\Delta\theta = \frac{Q}{mc}$] and hence its expansion will be more [as $\Delta V = V\gamma\Delta\theta$].

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

MODE OF HEAT TRANSFER

Heat is a form of energy which transfers from a body at higher temperature to a body at lower temperature.

The transfer of heat from one body to another may take place by any one of the following modes :.

- **Conduction**

  The process in which the material takes an active part by molecular action and energy is passed from one particle to another is called conduction. It is predominant in solids.

- **Convection**

  The transfer of energy by actual motion of particle of medium from one place to another is called convection. It is predominant is fluids (liquids and gases).

- **Radiation**

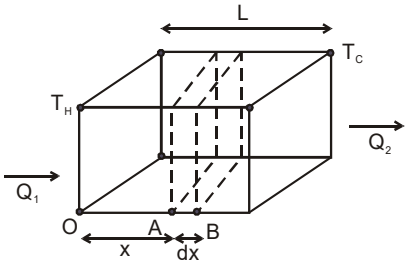
  Quickest way of transmission of heat is known as radiation. In this mode of energy transmission, heat is transferred from one place to another without effecting the inter–venning medium.

Conduction	Convection	Radiation
Heat Transfer due to Temperature difference	Heat transfer due to density difference	Heat transfer with out any medium
Due to free electron or vibration motion of molecules	Actual motion of particles	Electromagnetic radiation
Heat transfer in solid body (in mercury also)	Heat transfer in fluids (Liquid + gas)	All
Slow process	Slow process	Fast process (3 10^8 m/sec)
Irregular path	Irregular path	Straight line (like light)

THERMAL CONDUCTION

The process by which heat is transferred from hot part to cold part of a body through the transfer of energy from one particle to another particle of the body without the actual movement of the particles from their equilibrium positions is called conduction. The process of conduction only in solid body (except Hg) Heat transfer by conduction from one part of body to another continues till their temperatures become equal.

The process of transmission of heat energy in which heat is transferred from one particle of the medium to the other, but each particle of the medium stays at its own position is called conduction, for example if you hold an iron rod with one of its end on a fire for some time, the handle will get hot. The heat is transferred from the fire to the handle by conduction along the length of iron rod. The vibrational amplitude of atoms and electrons of the iron rod at the hot end takes on relatively higher values due to the higher temperature of their environment. These increased vibrational amplitude are transferred along the rod, from atom to atom during collision between adjacent atoms. In this way a region of rising temperature extends itself along the rod to your hand.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Consider a slab of face area A, Lateral thickness L, whose faces have temperatures T_H and $T_C(T_H > T_C)$.

Now consider two cross sections in the slab at positions A and B separated by a lateral distance of dx. Let temperature of face A be T and that of face B be T + ΔT. Then experiments show that Q, the amount of heat crossing the area A of the slab at position x in time t is given by

$$\boxed{\frac{Q}{t} = -KA\frac{dT}{dx}}$$

Here K is a constant depending on the material of the slab and is named thermal conductivity of the material, and the quantity $\left(\frac{dT}{dx}\right)$ is called temperature gradiant. The (–) sign in equation (2.1) shows heat flows from high to low temperature (ΔT is a –ve quantity)

STEADY STATE

If the temperature of a cross-section at any position x in the above slab remains constant with time (remember, it does vary with position x), the slab is said to be in steady state.

Remember steady-state is distinct from thermal equilibrium for which temperature at any position (x) in the slab must be same.

For a conductor in steady state there is no absorption or emission of heat at any cross-section. (as temperature at each point remains constant with time). The left and right face are maintained at constant temperatures T_H and T_C respectively, and all other faces must be covered with adiabatic walls so that no heat escapes through them and same amount of heat flows through each cross-section in a given Interval of time. Hence $Q_1 = Q = Q_2$. Consequently the temperature gradient is constant throughout the slab.

Hence, $$\frac{dT}{dx} = \frac{\Delta T}{L} = \frac{T_f - T_i}{L} = \frac{T_C - T_H}{L}$$

and $$\frac{Q}{t} = -KA\frac{\Delta T}{L} \Rightarrow \frac{Q}{t} = KA\left(\frac{T_H - T_C}{L}\right)$$

Here Q is the amount of heat flowing through a cross-section of slab at any position in a time interval of t.

Example

One face of an aluminium cube of edge 2 metre is maintained at 100 C and the other end is maintained at 0 C. All other surfaces are covered by adiabatic walls. Find the amount of heat flowing through the cube in 5 seconds. (thermal conductivity of aluminium is 209 W/m– C)

Solution

Heat will flow from the end at 100ºC to the end at 0 C.

Area of cross-section perpendicular to direction of heat flow, A = $4m^2$ then $\frac{Q}{t} = KA\frac{(T_H - T_C)}{L}$

$$Q = \frac{(209 W/m^{\circ}C)(4m^2)(100^{\circ}C - 0^{\circ}C)(5 sec)}{2m} = 209 kJ$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Thermal conductivity (K) :

- It's depends on nature of material.

  Order of thermal conductivity Ag > Cu > Au > Al $K\begin{cases}\text{For Ag maximum is (410 W/mK)}\\ \text{For Freon minimum is 12 (0.008 W/mK)}\end{cases}$

- **SI UNIT :** $J\ s^{-1}\ m^{-1}\ K^{-1}$ **Dimensions :** $M^1\ L^1\ T^{-3}\ \theta^{-1}$
- For an ideal or perfect conductor of heat the value of $K = \infty$
- For an ideal or perfect bad conductor or insulator the value of $K = 0$
- For cooking the food, low specific heat and high conductivity utensils are most suitable.

APPLICATION OF THERMAL CONDUCTION

- In winter, the iron chairs appear to be colder than the wooden chairs.
- Cooking utensils are made of aluminium and brass whereas their handles are made of wood.
- Ice is covered in gunny bags to prevent melting of ice.
- We feel warm in woollen clothes and fur coat.
- Two thin blankets are warmer than a single blanket of double the thickness.
- Birds often swell their feathers in winter.
- A new quilt is warmer than old one.

THERMAL RESISTANCE TO CONDUCTION

If you are interested in insulating your house from cold weather or for that matter keeping the meal hot in your tiffin-box, you are more interested in poor heat conductors, rather than good conductors. For this reason, the concept of thermal resistance R has been introduced.

For a slab of cross-section A, Lateral thickness L and thermal conductivity K, $R = \frac{L}{KA}$

In terms of R, the amount of heat flowing though a slab in steady-state (in time t) $\frac{Q}{t} = \frac{(T_H - T_L)}{R}$

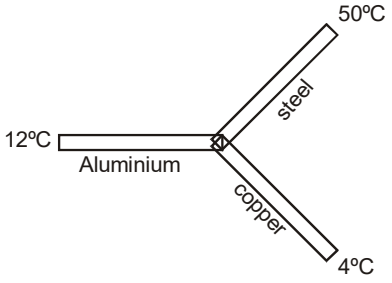
If we name $\frac{Q}{t}$ as thermal current i_T then, $i_T = \frac{T_H - T_L}{R}$

This is mathematically equivalent to OHM's law, with temperature donning the role of electric potential. Hence results derived from OHM's law are also valid for thermal conduction.

More over, for a slab in steady state we have seen earlier that the thermal current i_L remains same at each cross-section. This is analogous to kirchoff's current law in electricity, which can now be very conveniently applied to thermal conduction.

Example

Three identical rods of length 1m each, having cross-section area of $1cm^2$ each and made of Aluminium, copper and steel respectively are maintained at temperatures of 12ºC, 4ºC and 50ºC respectively at their separate ends. Find the temperature of their common junction. [K_{Cu}=400 W/m-K , K_{Al} = 200 W/m-K , K_{steel} = 50 W/m-K]



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

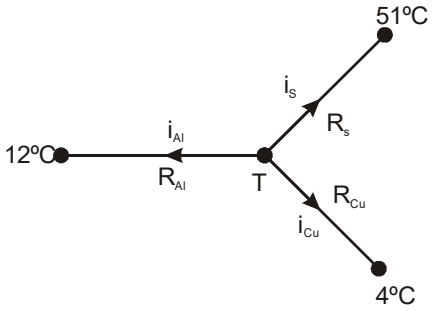
Solution

$$R_{Al} = \frac{L}{KA} = \frac{1}{10^{-4} \times 200} = \frac{10^4}{200}$$

Similarly $R_{steel} = \frac{10^4}{50}$ and $R_{copper} = \frac{10^4}{400}$

Let temperature of common junction = T

then from Kirchoff;s current laws, $i_{Al} + i_{steel} + i_{Cu} = 0$



$$\Rightarrow \quad \frac{T-12}{R_{Al}} + \frac{T-51}{R_{steel}} + \frac{T-4}{R_{Cu}} = 0$$

$$\Rightarrow \quad (T - 12)\ 200 + (T - 50)\ 50 + (T - 4)\ 400$$

$$\Rightarrow \quad 4(T - 12) + (T - 50) + 8\ (T - 4) = 0$$

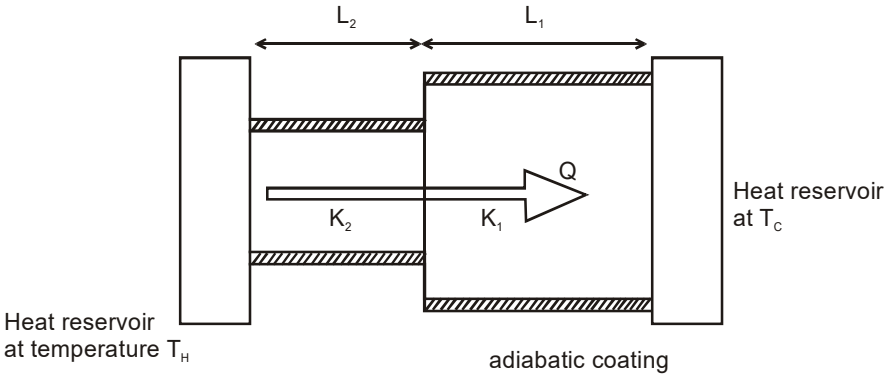
$$\Rightarrow \quad 13T = 48 + 50 + 32 = 130$$

$$\Rightarrow \quad T = 10\ C$$

SLABS IN PARALLEL AND SERIES

Slabs in series (in steady state)

Consider a composite slab consisting of two materials having different thickness L_1 and L_2 different cross-sectional areas A_1 and A_2 and different thermal conductivities K_1 and K_2. The temperature at the outer surface of the states are maintained at T_H and T_C, and all lateral surfaces are covered by an adiabatic coating.



Let temperature at the junction be T, since steady state has been achieved thermal current through each slab will be equal. Then thermal current through the first slab.

$$I = \frac{Q}{t} = \frac{T_H - T}{R_1} \Rightarrow T_H - T = IR_1$$

and that through the second slab, $i = I = \frac{Q}{t} = \frac{T - T_C}{R_2} \Rightarrow T - T_C = IR_2$

Adding equation $T_H - T_L = (R_1 + R_2)I \Rightarrow I = \frac{T_H - T_C}{R_1 + R_2}$

Thus these two slabs are equivalent to a single slab of thermal resistance $R_1 + R_2$.

If more than two slabs are joined in series and are allowed to attain steady state, then equivalent thermal resistance is given by

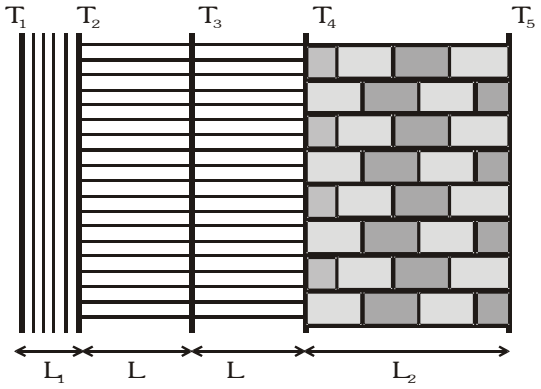
$$R = R_1 + R_2 + R_3 + ...$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory\P65

Example

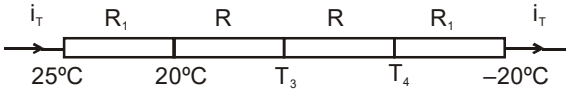
The figure shows the cross-section of the outer wall of a house built in a hill-resort to keep the house insulated from the freezing temperature of outside. The wall consists of teak wood of thickness L_1 and brick of thickness ($L_2 = 5L_1$), sandwitching two layers of an unknown material with identical thermal conductivities and thickness. The thermal conductivity of teak wood is K_1 and that of brick is ($K_2 = 5K$). Heat conduction through the wall has reached a steady state with the temperature of three surfaces being known.

(T_1 = 25 C, T_2 = 20 C and T_5 = –20 C). Find the interface temperature T_4 and T_3.



Sol. Let interface area be A. then thermal resistance of wood, $R_1 = \frac{L_1}{K_1 A}$ and that of brick wall $R_2 = \frac{L_2}{K_2 A} = \frac{5L_1}{5K_1 A} = R_1$

Let thermal resistance of the each sand witch layer = R. Then the above wall can be visualised as a circuit



Thermal current through each wall is same.

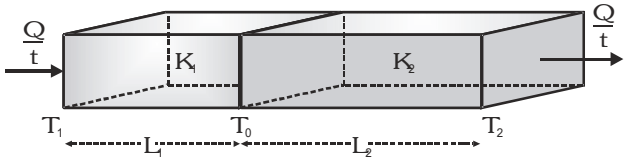
Hence $\frac{25-20}{R_1} = \frac{20-T_3}{R} = \frac{T_3-T_4}{R} = \frac{T_4+20}{R_1} \Rightarrow$ $25 - 20 = T_4 + 20 \Rightarrow T_4 = -15$ C

also, $20 - T_3 = T_3 - T_4 \Rightarrow$ $T_3 = \frac{20+T_4}{2} = 2.5$ C

- **Equivalent conductivity for Heat flow through slabs in series**

$R_{eq} = R_1 + R_2$

$\frac{L_1+L_2}{K_{eq}A} = \frac{L_1}{K_1 A} + \frac{L_2}{K_2 A}$



Equivalent thermal conductivity of the system is

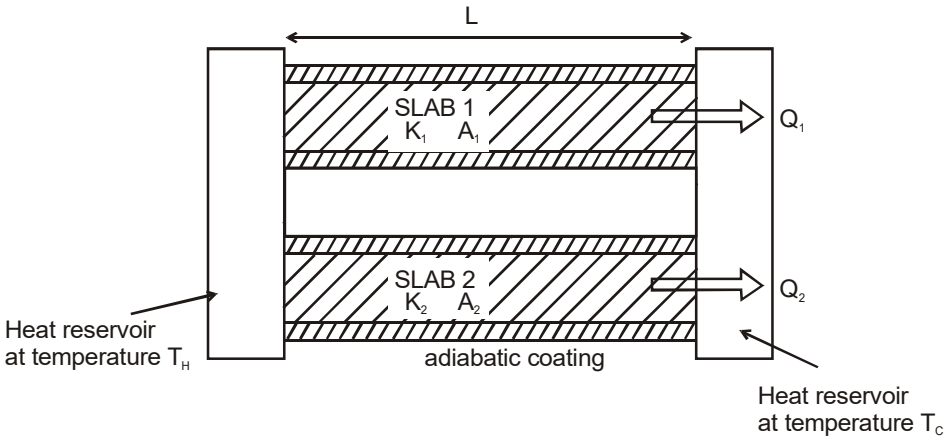
$K_{eq} = \frac{L_1+L_2}{\frac{L_1}{K_1}+\frac{L_2}{K_2}} = \frac{\Sigma L_i}{\Sigma \frac{L_i}{K_i}}$

equivalent to 

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

Slabs in parallel (in steady state):

Consider two slabs held between the same heat reservoirs, their thermal conductivities K_1 and K_2 and cross-sectional areas A_1 and A_2



then $R_1 = \frac{L}{K_1A_1}$, $R_2 = \frac{L}{K_2A_2}$

thermal current through slab 1 : $I_1 = \frac{T_H - T_C}{R_1}$ and that through slab 2 : $I_2 = \frac{T_H - T_C}{R_2}$

Net heat current from the hot to cold reservoir $I = I_1 + I_2 = (T_H - T_C)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

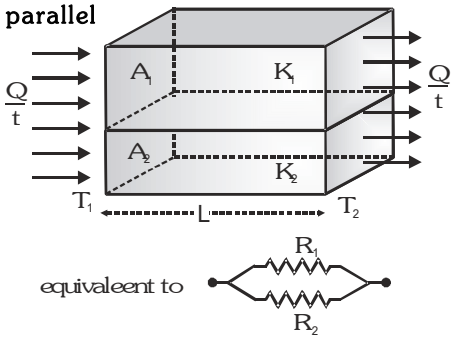
Comparing with $I = \frac{T_H - T_C}{R_{eq}}$, we get, $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$

If more than two rods are joined in parallel, the equivalent thermal resistance is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...$$

- **Equivalent thermal condctivity for Heat flow through slabs in parallel**

$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$, $R = \frac{L}{KA}$; $\frac{K_{eq}}{L}(A_1 + A_2) = \frac{K_1A_1}{L} + \frac{K_2A_2}{L}$



Equivalent thermal conductivity

$K_{eq} = \frac{K_1A_1 + K_2A_2}{A_1 + A_2} = \frac{\Sigma K_iA_i}{\Sigma A_i}$

Example

Three rods of material X and three rods of material Y are connected as shown in figure. All the rods are identical in length and cross–sectional area. If the end A is maintained at 60^0C and the junction E at at 10^0C, calculate the temp. of the junctions B,C,D. The thermal conductivity of X is 0.92 CGS units and that of Y is 0.46 CGS units.

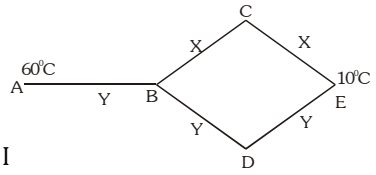
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Solution

$R_X \propto \frac{1}{K_X}$, $R_Y \propto \frac{1}{K_Y} \Rightarrow \frac{R_X}{R_Y} = \frac{K_Y}{K_X} = \frac{0.46}{0.92} = \frac{1}{2}$ Let $R_X = R$ $\therefore R_Y = 2R$

The total resistance $\Sigma R = R_Y$ + effective resistance in the bridge

$$\Sigma R = 2R + \frac{2R \times 4R}{2R + 4R} = 2R + \frac{4}{3}R = \frac{10}{3}R \quad \& \quad \because \Delta\theta = \ell \times R$$



Further $I_{BCE}(2R) = I_{BDE}(4R)$ and $I_{BCE} + I_{BDE} = I \Rightarrow I_{BCE} = \frac{2}{3}I$ and $I_{BDE} = \frac{1}{3}I$

For A and B $\theta_A - \theta_B = 60^0 - \theta_B \Rightarrow 60 - \theta_B = 2R \times I$ ...(i)

For B and C $\theta_B - \theta_C = \frac{2}{3}(I \times R)$ ....(ii) $\theta_C - \theta_E = \frac{2}{3} \times R \times I$

For A and E $\theta_A - \theta_E = 60 - 10 = 50 \Rightarrow \frac{10}{3}(R \times I) = 50$ ....(iii) $\therefore R \times I = 15$

$\therefore \theta_A - \theta_B - 2 \times 15 = 30$, $\theta_B = 60 - 30 = 30^0$C, $\theta_B - \theta_C = \left(\frac{2}{3}\right) \times 15 = 10$

$\therefore \theta_C = 30 - 10 = 20^0$C Obviously, $\theta_C = \theta_D = 20^0$C

GROWTH OF ICE ON LAKES

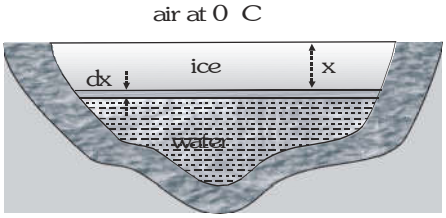
In winter atmospheric temperature falls below 0 C and water in the lake start freezing.

Let at time t thickness of ice on the surface of the lake = x and air temperature = $-\theta$ C

The temperature of water in contact with the lower surface of ice = 0 C

Let area of the lake = A

Heat escaping through ice in time dt is $dQ = KA\frac{[0-(-\theta)]}{x}dt$



Due to escape of this heat increasing extra thickness of ice = dx

Mass of this extra thickness of ice is $m = \rho V = \rho$ A.dx

$dQ = mL = (\rho$ A.dx) L

$\therefore \quad KA\frac{\theta}{x}dt = (\rho$ A.dx) L $\Rightarrow \quad dt = \frac{\rho L}{K\theta}x\,dx$

So time taken by ice to grow a thickness x is $t = \frac{\rho L}{K\theta}\int_0^x x\,dx = \frac{1}{2}\frac{\rho L}{K\theta}x^2$

So time taken by ice to grow from thickness x_1 to thickness x_2 is

$$t = t_2 - t_1 = \frac{1}{2}\frac{\rho L}{KT}(x_2^2 - x_1^2) \quad \text{and} \quad t \propto (x_2^2 - x_1^2)$$

Time taken to double and triple the thickness ratio $t_1 : t_2 : t_3 :: 1^2 : 2^2 : 3^2$ So $t_1 : t_2 : t_3 :: 1 : 4 : 9$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5.5

Example

One end of a brass rod 2m long and having 1 cm radius is maintained at 250 C. When a steady state is reached, the rate of heat flow across any cross–section is 0.5 cal s^{-1}. What is the temperature of the other end K = 0.26 cal s^{-1} cm^{-1} C^{-1}.

Solution

$\frac{Q}{t}$ = 0.5 cal s^{-1}; r = 1 cm $\quad\therefore$ Area A = πr^2 = 3.142 1 cm^2 = 3.142 cm^2

L = Length of rod = 2m = 200 cm, T_1 = 250 C, T_2 = ?

We know $\frac{Q}{t} = \frac{KA(T_1 - T_2)}{L}$ or $(T_1 - T_2) = \frac{Q}{t}\ \frac{\Delta x}{kA} = \frac{0.5 \times 200}{0.26\,C^{-1} \times 3.142}$ =122.4 C

$\therefore$ T_2 = 250 C – 122.4 C = 127.6 C

Example

Steam at 373 K is passed through a tube of radius 10 cm and length 2 m. The thickness of the tube is 5 mm and thermal conductivity of the material is 390 W m^{-1} K^{-1}, calculate the heat lost per second. The outside temp. is 0 C.

Solution

Using the relation Q = $\frac{KA(T_1 - T_2)t}{L}$

Here, heat is lost through the cylindrical surface of the tube.

A = 2πr (radius of the tube) (length of the tube) = 2π 0.1 2 = 0.4 πm^2

K = 390 W m^{-1} K^{-1}

T_1 = 373 K, T_2 = 0 C = 273 K,L = 5 mm = 0.005 m and t = 1 s

$\therefore$ Q = $\frac{390 \times 0.4\,\pi \times (373 - 273) \times 1}{0.005} = \frac{390 \times 0.4\pi \times 100}{0.005}$ = 98 10^5 J.

Example

The thermal conductivity of brick is 1.7 W m^{-1} K^{-1}, and that of cement is 2.9 W m^{-1} K^{-1}. What thickness of cement will have same insulation as the brick of thickness 20 cm.

Solution

Since Q = $\frac{KA(T_1 - T_2)t}{L}$. For same insulation by the brick and cement Q, A $(T_1 - T_2)$ and t do not change.

Hence, $\frac{K}{L}$ remain constant. If K_1 and K_2 be the thermal conductivities of brick and cement respectively and

L_1 and L_2 be the required thickness then $\frac{K_1}{L_1} = \frac{K_2}{L_2}$ or $\frac{1.7}{20} = \frac{2.9}{L_2}$ $\quad\therefore$ $L_2 = \frac{2.9}{1.7}$ 20 = 34.12 cm

Example

Two vessels of different material are identical in size and wall–thickness. They are filled with equal quantities of ice at 0 C. If the ice melts completely, in 10 and 25 minutes respectively then compare the coefficients of thermal conductivity of the materials of the vessels.

Solution

Let K_1 and K_2 be the coefficients of thermal conductivity of the materials, and t_1 and t_2 be the time in which ice melts in the two vessels. Since both the vessels are identical, so A and x in both the cases is same.

Now, Q = $\frac{K_1A(\theta_1 - \theta_2)t_1}{L} = \frac{K_2A(\theta_1 - \theta_2)t_2}{L} \Rightarrow \frac{K_1}{K_2} = \frac{t_2}{t_1} = \frac{25\text{ min}}{10\text{ min}} = \frac{5}{2}$

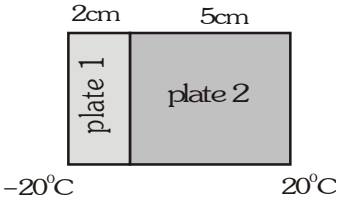
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

Two plates of equal areas are placed in contact with each other. Their thickness are 2.0 cm and 5.0 cm respectively. The temperature of the external surface of the first plate is –20 C and that of the external surface of the second plate is 20 C. What will be the temperature of the contact surface if the plate (i) are of the same material, (ii) have thermal conductivities in the ratio 2 : 5.

Solution

Rate of flow of heat in the plates is $\frac{Q}{t} = \frac{K_1A(\theta_1-\theta)}{L_1} = \frac{K_2A(\theta-\theta_2)}{L_2}$ ...(i)



(i) Here $\theta_1 = -20$ C, $\theta_2 = 20$ C,

$L_1 = 2$ cm $= 0.02$ m, $L_2 = 5$ cm $= 0.05$ m and $K_1 = K_2 = K$

$\therefore$ equation (i) becomes $\frac{KA(-20-\theta)}{0.02} = \frac{KA(\theta-20)}{0.05}$

$\therefore 5(-20-\theta) = 2(\theta - 20) \Rightarrow -100 - 5\theta = 2\theta - 40 \Rightarrow 7\theta = -60 \Rightarrow \theta = -8.6$ C

(ii) $\frac{K_1}{K_2} = \frac{2}{5}$ or $K_1 = \frac{2}{5}K_2$

$\therefore$ from equation (i) $\frac{2/5\,K_2A(-20-\theta)}{0.02} = \frac{K_2A(\theta-20)}{0.05}$ $-20 - \theta = \theta - 20$ or $-2\theta = 0$ $\quad\therefore\theta = 0$ C

Example

An ice box used for keeping eatables cold has a total wall area of 1 metre^2 and a wall thickness of 5.0 cm. The thermal conductivity of the ice box is K = 0.01 joule/metre– C. It is filled with ice at 0 C along with eatables on a day when the temperature is 30 C. The latent heat of fusion of ice is 334 10^3 joule/kg. Calculate the amount of ice melted in one day.

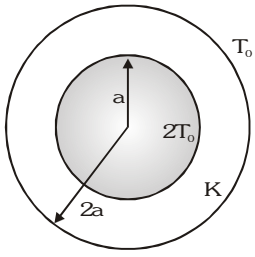
Solution

$\frac{dQ}{dt} = \frac{KA}{L}d\theta = \frac{0.01\times1}{0.05}\times30 = 6$ joule@ So $\frac{dQ}{dt}\times86400 = 6\times86400$

Q = mL (L – latent heat), $m = \frac{Q}{L} = \frac{6\times86400}{334\times10^3} = 1.552$ kg

Example

A hollow spherical ball of inner radius a and outer radius 2a is made of a uniform material of constant thermal conductivity K. The temperature within the ball is maintained at $2T_0$ and outside the ball it is T_0. Find, (a) the rate at which heat flows out of the ball in the steady state, (b) the temperature at r = 3a/2, where r is radial distance from the centre of shell. Assume steady state condition.



Solution

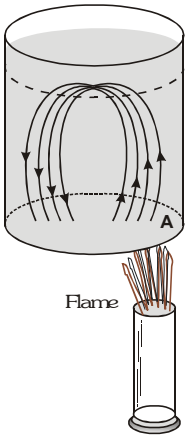
In the steady state, the net outward thermal current is constant, and does not depend on the radial position.

Thermal current, $C_1 = \left(\frac{dQ}{dt}\right) = -K.(4\pi r^2)\frac{dT}{dr} \Rightarrow \frac{dT}{dr} = -\frac{C_1}{4\pi K}\frac{1}{r^2}$ $\qquad T = \frac{C_1}{4\pi kr} + C_2$

At r=a, $T = 2T_0$ and r =2a, $T=T_0$ $\Rightarrow T = \frac{2a}{r}T_0$ (a) $\frac{dQ}{dt} = 8\pi aKT_0$ (b) $T(r = 3a/2) = 4T_0/3$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

CONVECTIONS



Convection requires a medium and is the process in which heat is transferred from one place to other by actual movement of heated substance (usually fluid).The type of convection which results from difference in densities is called natural convection (for example, a fluid in a container heated through its bottom). However, if a heated fluid is forced to move by a blower, fan or pump, the process is called forced convection. The rate of heat convection from an object is proportional to the temperature difference ($\Delta\theta$) between the object and convective fluid and the area of contact A, i.e., $\left[\frac{dQ}{dt}\right]_{convection} = hA\,\Delta\theta$ where, h represents a constant of proportionality called convection coefficient and depends on the properties of fluid such as density, viscosity, specific heat and thermal conductivity, etc.

PHENOMENA BASED ON CONVECTION :

(i) Land and sea breezes :

The heat from the Sun is absorbed more rapidly by land than by sea–water. Moreover, the specific heat of land is low as compared to that of sea–water. Consequently, the rise in temperature of land is higher as compared to that of sea–water. To sum–up, land is hotter than the sea during day time. As a result of this, the colder air over the sea blows towards the land. This is called sea–breeze.

At night, air blows from land towards sea. This is called land breeze.

(ii) Formation of trade winds :

The surface of Earth near the equator gets heated strongly. So, the air in contact with the surface of Earth at the expands and rises upwards. As a result of this, a low pressure is created at the equator.

At the poles, the air in the upper atmosphere gets cooled and comes down. So, a high pressure is created at the poles. Due to difference of pressures at the poles and equator, the air at the poles moves towards the equator, rises up, moves the poles and so on. In this way, a wind is formed in the atmosphere.

The rotation of the Earth also affects the motion of the wind. Due to anti–clockwise rotation of Earth the warm wind blowing from equator to north drifts towards east. The steady wind blowing from north–Earth to equator, near the surface of Earth, is called trade wind.

(iii) Monsoons :

In summer, the peninsular mass of central Asia becomes more strongly heated than the water of the Indian Ocean. This is due to the fact that the specific heat of water is much higher than that of the soil and rocks. Hot air from the heated land mass rises up and moves towards the Indian ocean. Air filled with moisture flows over the Indian ocean on the south towards heated land mass. When obstructed by mountains, the moist air rushes upwards to great height. In the process, it gets cooled. Consequently, the moisture condenses and falls as rain.

(iv) Ventilation :

Ventilator of exhaust fan in a room help of remove impure and warm air from a room. The fresh air from outside blows into the room. This is all due to the convection current set up in the room.

(v) To regulate temperature in the human body :

Heat transfer in the human body involves a combination of mechanisms. These together maintain a remarkably uniform temperature in the human body inspite of large changes in environmental conditions.

The chief internal mechanism is forced convection. The heart serves as the pump and the blood as the circulating fluid.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

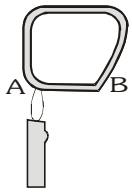
ALLEN Materials Provided By - Material Point Available on Learnaf.com

Some important points :

- ❒ Natural convection takes place from bottom to top while forced convection in any direction.
- ❒ In case of natural convection, convection currents move warm air upwards and cool air downwards. This is why heating is done from base, while cooling from the top.
- ❒ Natural convection is not possible in a gravity free region such as a freely falling lift or an orbiting satellite.
- ❒ Natural convection plays an important role in ventilation, in changing climate and weather and in forming land and sea breezes and trade winds.
- ❒ The forced convection of blood in our body by a pump (heart) helps in keeping the temperature of body constant.

Example

Water in a closed tube is heated with one arm vertically placed above the lamp. In what direction water will begin the circulate along the tube ?



Solution

On heating the liquid at A will become lighter and will rise up. This will push the liquid in the tube upwards and so the liquid in the tube will move clockwise i.e. form B to A.

GOLDEN KEY POINTS

- For heat propagation via convection, temperature gradient exists in vertical direction and not in horizontal direction.
- Most of heat transfer that is taking place on Earth is by convection, the contribution due to conduction and radiation is very small.

Thermal Radiation

The process of the transfer of heat from one place to another place without heating the intervening medium is called radiation. When a body is heated and placed in vacuum, it loses heat even when there is no medium surrounding it. The heat can not go out from the body by the process of conduction or convection since both of these process require the presence of a material medium between source and surrounding objects. The process by which heat is lost in this case is called radiation. This does not require the presence of any material medium.

It is by radiation that the heat from the Sun reaches the Earth. Radiation has the following properties:

(a) Radiant energy travels in straight lines and when some object is placed in the path, it's shadow is formed at the detector.

(b) It is reflected and refracted or can be made to interfere. The reflection or refraction are exactly as in case of light.

(c) It can travel through vacuum.

(d) Intensity of radiation follows the law of inverse square.

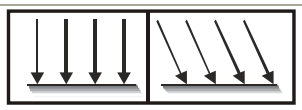
(e) Thermal radiation can be polarised in the same way as light by transmission through a nicol prism.

All these and many other properties establish that heat radiation has nearly all the properties possessed by light and these are also electromagnetic waves with the only difference of wavelength or frequency. The wavelength of heat radiation is larger than that of visible light.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Types of thermal Radiation :– Two types of thermal radiation.

Plane Radiation	Diffuse Radiation
	
Radiations which are incident on a surface at certain angle	Incident on the surface at all angles

- When radiation passes through any medium then radiations slightly absorbed by medium according to its absorptive power so temperature of medium slightly increases.
- In order to obtain a spectrum of radiation, a special prism used like KCℓ prism, Rock salt prism Flourspar prism. Normal glass prism or Quartz prism can not be used (because it absorbed some radiation).
- Radiation intensity measured with a specific device named as Bolometer.
- Heat radiation are always obtained in infra–red region of electromagnetic wave spectrum so they are called Infra red rays.
- Thermal radiations is incident on a surface, it exerts pressure on the surface, which is known as Radiation Pressure.

BASIC FUNDAMENTAL DEFINITIONS

- **Energy Density (u)**

  The radiation energy of whole wavelength (0 to ∞) present in unit volume at any point in space is defined as energy density. **S I UNIT :** J/m^3

- **Spectral energy density (u_λ) :** Energy density per unit spectral region. $u = \int_0^\infty u_\lambda d\lambda$ **SI UNIT :** J/m^3 Å

- **Absorptive power or absorptive coefficient 'a' :** The ratio of amount of radiation absorbed by a surface (Q_a) to the amount of radiation incident (Q) upon it is defined as the coefficient of absorption $a = \frac{Q_a}{Q}$ .It is unitless

- **Spectral absorptive power (a_λ)** $a_\lambda = \frac{Qa_\lambda}{Q_\lambda}$: Also called monochromatic absorptive coefficient

  At a given wavelength $a = \int_0^\infty a_\lambda d\lambda$. For ideal black body a_λ and $a = 1$, a and a_λ are unitless

- **Emissive power (e)** : The amount of heat radiation emitted by unit area of the surface in one second at a particular temperature. **SI UNIT :** J/m^2s

- **Spectral Emmisive power (e_λ)** : The amount of heat radiation emitted by unit area of the body in one second in unit spectral region at a given wavelength. Emissive power or total emissive power $e = \int_0^\infty e_\lambda \, d\lambda$

  SI UNIT : W/m^2 Å

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-5\Thermal physics\Eng\Theory P65

EMISSIVITY (e)

- **Absolute emissivity or emissivity :** Radiation energy given out by a unit surface area of a body in unit time corresponding to unit temperature difference w.r.t. the surroundings is called Emissivity.

  S I UNIT : W/m^2 K

- **Relative emissivity (e_r) :** $e_r = \frac{Q_{GB}}{Q_{IBB}} = \frac{e_{GB}}{E_{IBB}} = \frac{\text{emitted radiation by gray body}}{\text{emitted radiation by ideal black body}}$

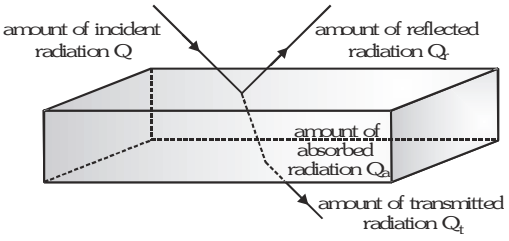
  GB = gray or general body, IBB = Ideal black body

  (i) No unit (ii) For ideal black body $e_r = 1$ (iii) range $0 < e_r < 1$

SPECTRAL, EMISSIVE, ABSORPTIVE AND TRANSMITTIVE POWER OF A GIVEN BODY SURFACE

Due to incident radiations on the surface of a body following phenomena occur by which the radiation is divided into three parts. (a) Reflection (b) Absorption (c) Transmission

- **From energy conservation**



$Q = Q_r + Q_a + Q_t \Rightarrow \frac{Q_r}{Q} + \frac{Q_a}{Q} + \frac{Q_t}{Q} = 1 \Rightarrow r + a + t = 1$

Reflective Coefficient $r = \frac{Q_r}{Q}$, Absorptive Coefficient $a = \frac{Q_a}{Q}$,

Transmittive Coefficient $t = \frac{Q_t}{Q}$

$r = 1$ and $a = 0$, $t = 0$ $\Rightarrow$ Perfect reflector

$a = 1$ and $r = 0$, $t = 0$ $\Rightarrow$ Ideal absorber (ideal black body)

$t = 1$ and $a = 0$, $r = 0$ $\Rightarrow$ Perfect transmitter (daithermanons)

Reflection power (r) = $\left[\frac{Q_r}{Q} \times 100\right]\%$, Absorption power (a) = $\left[\frac{Q_a}{Q} \times 100\right]\%$

Transmission power (t) = $\left[\frac{Q_t}{Q} \times 100\right]\%$

Example

Total radiations incident on body = 400 J, 20% radiation reflected and 120 J absorbs. Then find out % of transmittive power

Solution

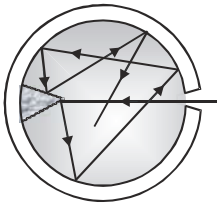
$Q = Q_t + Q_r + Q_a \Rightarrow 400 = 80 + 120 + Q_t \Rightarrow Q_t = 200$. So % of transmittive power is 50%

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

IDEAL BLACK BODY

- For a body surface which absorbs all incident thermal radiations at low temperature irrespective of their wave length and emitted out all these absorbed radiations at high temperature assumed to be an ideal black body surface.
- The identical parameters of an ideal black body is given by

  $a = a_\lambda = 1$ and $r = 0 = t$, $e_r = 1$
- The nature of emitted radiations from surface of ideal black body only depends on its temperature
- The radiations emitted from surface of ideal black body called as either full or white radiations.


Ferry's ideal black body

- At any temperature the spectral energy distribution curve for surface of an ideal black body is always continuous and according to this concept if the spectrum of a heat source obtained to be continuous then it must be placed in group of ideal black body like kerosene lamp; oil lamp Heating filament etc.
- There are two experimentally ideal black body

  (a) Ferry's ideal black body (b) Wien's ideal black body.
- At low temperature surface of ideal black body is a perfect absorber and at a high temperature it proves to be a good emitter.
- An ideal black body need not be black colour (eg. Sun)

PREVOST'S THEORY OF HEAT ENERGY EXCHANGE

According to Prevost at every possible temperature (Not absolute temperature) there is a continuous heat energy exchange between a body and its surrounding and this exchange carry on for infinite time.

The relation between temperature difference of body with its surrounding decides whether the body experience cooling effect or heating effect.

When a cold body is placed in the hot surrounding : The body radiates less energy and absorbs more energy from the surrounding, therefore the temperature of body increases.

When a hot body placed in cooler surrounding : The body radiates more energy and absorb less energy from the surroundings. Therefore temperature of body decreases.

When the temperature of a body is equal to the temperature of the surrounding

The energy radiated per unit time by the body is equal to the energy absorbed per unit time by the body, therefore its temperature remains constant.

GOLDEN KEY POINTS

- At absolute zero temperature (0 kelvin) all atoms of a given substance remains in ground state, so, at this temperature emission of radiation from any substance is impossible, so Prevost's heat energy exchange theory does not applied at this temperature, so it is called **limited temperature** of prevosts theory.
- With the help of Prevost's theory rate of cooling of any body w.r.t. its surroundings can be worked out (applied to Stefen Boltzman law, Newton's law of cooling.)

KIRCHHOFF'S LAW

At a given temperature for all bodies the ratio of their spectral emissive power (e_λ) to spectral absorptive power (a_λ) is constant and this constant is equal to spectral emissive power (E_λ) of the ideal black body at same temperature

$$\frac{e_\lambda}{a_\lambda} = E_\lambda = \text{constant} \qquad \left[\frac{e_\lambda}{a_\lambda}\right]_1 = \left[\frac{e_\lambda}{a_\lambda}\right]_2 = \text{constant} \quad e_\lambda \propto a_\lambda$$

Good absorbers are good emitters and bad absorbers are bad emitters

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory\P65

GOLDEN KEY POINTS

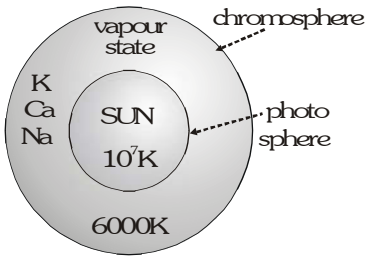
- For a constant temperature the spectral emmisive power of an ideal black body is a constant parameter
- The practical confirmation of Kirchhoff's law carried out by Rishi apparatus and the main base of this apparatus is a Lessilie container.
- The main conclusion predicted from Kirchhof's law can be expressed as

Good absorber	⇒	Good emitter
Bad absorber	⇒	Bad emitter
(at Low temperature)		(at high temperature)

APPLICATIONS OF KIRCHOFF LAW

- **Fraunhoffer's lines**

Fraunhoffer lines are dark lines in the spectrum of the Sun. When white light emitted from the central core of the Sun (Photosphere) passes. Through its atmosphere (Chromosphere) radiations of those wavelengths will be absorbed by the gases present, resulting in dark lines in the spectrum of Sun. At the time of total solar eclipse direct light rays emitted from photosphere cannot reach on the Earth and only rays from chromosphere are able to reach on the Earth surface. At that time we observe bright fraunhoffer lines.



- **In deserts days are hot and nights cold**

Sand is rough and black, so it is a good absorber and hence in deserts, days (When radiation from Sun is incident on sand) will be very hot. Now in accordance with Kirchhoff's Law, good absorber is a good emitter.

So nights (when send emits radiation) will be cold.

STEFAN'S LAW

The amount of radiation emitted per second per unit area by a black body is directly proportional to the fourth power of its absolute temperature.

Amount of radiation emitted $E \propto T^4$ where T = temperature of ideal black body (in K)

$E = \sigma T^4$ This law is true for only ideal black body

SI Unit : E = watt/m² σ = Stefen's constant = 5.67 x10^{-8} watt /m^2 K^4

Dimensions of σ : $M^1 L^0 T^{-3} \theta^{-4}$

Total radiation energy emitted out by surface of area A in time t :

Ideal black body $Q_{IBB} = \sigma A T^4 t$ and for any other body $Q_{GB} = e_r \sigma A T^4 t$

Rate of emission of radiation

When Temperature of surrounding T_0 (Let $T_0 < T$)

Rate of emission of radiation from ideal black body surface $E_1 = \sigma T^4$

Rate of emission of radiation from surrounding $E_2 = \sigma T_0^4$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Net rate of loss of radiation from ideal black body surface is $E = E_1 - E_2 = \sigma T^4 - \sigma T_0^4 = \sigma (T^4 - T_0^4)$

Net loss of radiation energy from entire surface area in time t is $Q_{IBB} = \sigma A (T^4 - T_0^4) t$

For any other body $Q_{GB} = e_r A\sigma (T^4 - T_0^4) t$

If in time dt the net heat energy loss for ideal black body is dQ and because of this its temperature falls by $d\theta$

Rate of loss of heat $$R_H = \frac{dQ}{dt} = \sigma A(T^4 - T_0^4)$$

It is also equal to emitted power or radiation emitted per second

Rate of fall in temperature (Rate of cooling) $$R_F = \frac{d\theta}{dt} = \frac{\sigma A}{msJ}(T^4 - T_0^4) \left[\because \frac{dQ}{dt} = msJ\frac{d\theta}{dt}\right]$$

Note :

(i) If all of T, T_0, m, s, V, ρ, are same for different shape body then R_F and R_H will be maximum in the flat surface.

(ii) If a solid and hollow sphere are taken with all the parameters same then hollow will cool down at fast rate.

(iii) Rate of temperature fall , $R_F \propto \frac{1}{s} \propto \frac{d\theta}{dt}$ so dt $\propto$ s. If condition in specific heat is $\Rightarrow s_1 > s_2 > s_3$

If all cooled same temperature i.e. temperature fall is also identical for all then required time

$t \propto s \therefore t_1 > t_2 > t_3$

- **When a body cools by radiation the cooling depends on :**

(i) Nature of radiating surface : greater the emissivity (e_r), faster will be the cooling.

(ii) Area of radiating surface : greater the area of radiating surface, faster will be the cooling.

(iii) Mass of radiating body : greater the mass of radiating body slower will be the cooling.

(iv) Specific heat of radiating body : greater the specific heat of radiating body slower will be the cooling.

(v) Temperature of radiating body : greater the temperature of radiating body faster will be the cooling.

Example

The operating temperature of a tungesten filament in an incandescent lamp is 2000 K and its emissivity is 0.3. Find the surface area of the filament of a 25 watt lamp. Stefan's constant $\sigma = 5.67 \quad 10^{-8}\ Wm^{-2}K^{-4}$

Solution

$\because$ Rate of emission = wattage of the lamp

$$\therefore W = Ae\sigma T^4 \Rightarrow A = \frac{W}{e\sigma T^4} = \frac{25}{0.3\times5.67\times10^{-8}\times(200)^4} = 0.918\ m^2$$

NEWTON'S LAW OF COOLING

Rate of loss of heat $\left(\frac{dQ}{dt}\right)$ is directly proportional to excess of temperature of the body over that of surrounding.

[(when $(\theta - \theta_0) \not> 35$ C] $\frac{dQ}{dt} \propto (\theta - \theta_0) \Rightarrow \frac{dQ}{dt} = ms\frac{d\theta}{dt}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

θ = temperature of body [in C], θ_o = temperature of surrounding, $\theta - \theta_0$ = excess of temperature ($\theta > \theta_0$)

If the temperature of body decrease $d\theta$ in time dt then rate of fall of temperature $-\frac{d\theta}{dt} \propto (\theta - \theta_0)$

Where negative sign indictates that the rate of cooling is decreasing with time.

Excess of temperature

If the temperature of body decreases from θ_1 to θ_2 and temperature of surroundings is θ_0 then average excess of temperature = $\left[\frac{\theta_1 + \theta_2}{2} - \theta_0\right] \Rightarrow \left[\frac{\theta_1 - \theta_2}{t}\right] = -K\left[\frac{\theta_1 + \theta_2}{2} - \theta_0\right]$

Example

If a liquid takes 30 seconds in cooling of 80 C to 70 C and 70 seconds in cooling 60 C to 50 C, then find the room temperature.

Solution

$$\frac{\theta_1 - \theta_2}{t} = K\left(\frac{\theta_1 + \theta_2}{2} - \theta_0\right)$$

In first case, $\frac{80 - 70}{30} = K\left(\frac{80 + 70}{2} - \theta_0\right)$ $\quad \frac{1}{3} = K(75 - \theta_0)$ $\quad$ ...(i)

In second case , $\frac{60 - 50}{70} = K\left(\frac{60 + 50}{2} - \theta_0\right)$ $\quad \frac{1}{7} = K(55 - \theta_0)$ ... (ii)

Equation (i) divide by equation (ii) $\frac{7}{3} = \frac{(75 - \theta_0)}{(55 - \theta_0)} \Rightarrow 385 - 7\theta_0 = 225 - 3\theta_0 \Rightarrow \theta_0 = \frac{160}{4} = 40$ C

Limitations of Newton's Law

- Temperature difference should not exceed 35 C, $(\theta - \theta_0) \not> 35$ C
- Loss of heat should only be by radiation.
- This law is an extended form of Stefan–Boltzman's law.

For Heating, Newton's law of heating $\frac{\theta_1 - \theta_2}{t} = +H\left[\theta_0 - \frac{\theta_1 + \theta_2}{2}\right]$ where H heating constant.

Derivation of Newton's law from Steafen's Boltzman law

$$\frac{d\theta}{dt} = \frac{\sigma A}{m s J}(T^4 - T_0^4) \quad \left\{\begin{matrix} T - T_0 = \Delta T \\ T = T_0 + \Delta T \end{matrix}\right\}$$

$$\frac{d\theta}{dt} = \frac{\sigma A}{m s J}\left[(T_0 + \Delta T)^4 - T_0^4\right]$$ If $x <<< 1$ then $(1 + x)^n = 1 + nx$

$$\frac{d\theta}{dt} = \frac{\sigma A}{m s J}\left[T_0^4(1 + \frac{\Delta T}{T_0})^4 - T_0^4\right] = \frac{\sigma A}{m s J}T_0^4\left[(1 + \frac{\Delta T}{T_0})^4 - 1\right] = \frac{\sigma A}{m s J}T_0^4\left[1 + 4\frac{\Delta T}{T_0} - 1\right]$$

$$\frac{d\theta}{dt} = \left[4\frac{\sigma A}{m s J}T_0^3\right]\Delta T \Rightarrow \frac{d\theta}{dt} = K\Delta T \text{ constant } K = \frac{4\sigma A T_0^3}{m s J}$$

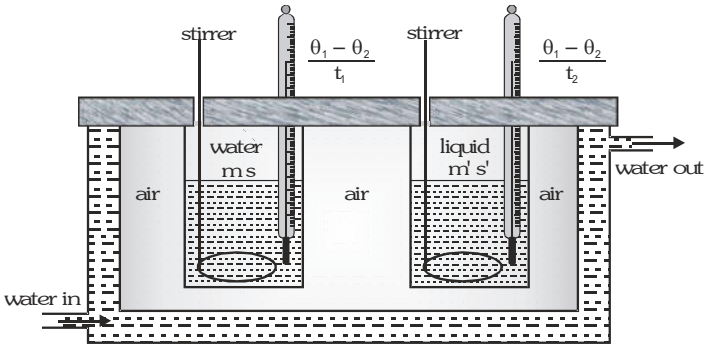
Newton's law of cooling $\frac{d\theta}{dt} \propto \Delta T$ (for small temperature difference)

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

APPLICATION OF NEWTON'S LAW OF COOLING

- **To find out specific heat of a given liquid**

If for the two given liquids their volume, radiating surface area, nature of surface, initial temperature are allowed to cool down in a common environments then rate of loss of heat of these liquids are equal .

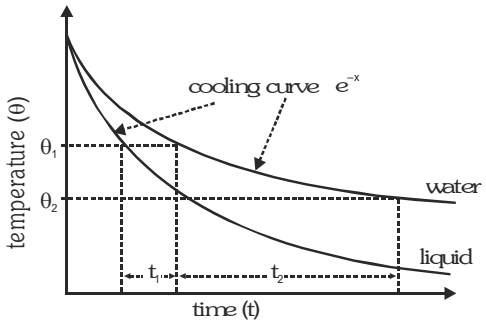


fall of temperature from $\theta_1 \to \theta_2$ in time t_1 for water and t_2 for liquid

$$\because \left[\frac{dQ}{dt}\right]_{Water} = \left[\frac{dQ}{dt}\right]_{Liquid} \quad \therefore (ms+w)\left[\frac{\theta_1-\theta_2}{t_1}\right] = (m's'+w)\left[\frac{\theta_1-\theta_2}{t_2}\right] \Rightarrow \frac{ms+w}{t_1} = \frac{m's'+w}{t_2}$$

where w = water equivalent of calorimeter.

Cooling curve :



Example

When a calorimeter contains 40g of water at 50 C, then the temperature falls to 45 C in 10 minutes. The same calorimeter contains 100g of water at 50 C, it takes 20 minutes for the temperature to become 45 C. Find the water equivalent of the calorimeter.

Solution

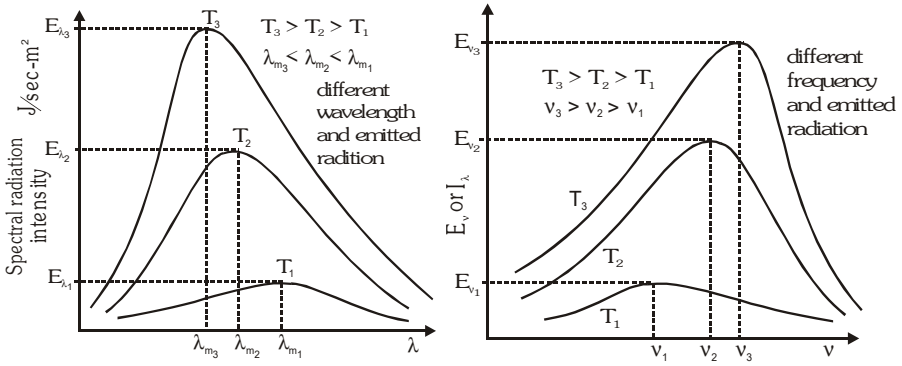
$$\frac{m_1s_1+W}{t_1} = \frac{m_2s_2+W}{t_2}$$ where W is the water equivalent

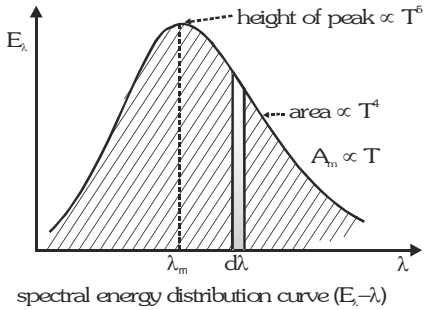
$$\Rightarrow \frac{40\times1+W}{10} = \frac{100\times1+W}{20} \Rightarrow 80 + 2W = 100 + W \Rightarrow W = 20\ g$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

SPECTRAL ENERGY DISTRIBUTION CURVE OF BLACK BODY RADIATIONS

Practically given by : Lumers and Pringshem Mathematically given by : Plank





(i) $\lambda_m \propto \frac{1}{T}$

(ii) $E_{\lambda_m} \propto T^5$

(iii) Area $\int_0^\infty E_\lambda \, d\lambda = E = \sigma T^4$ $\frac{A_1}{A_2} = \left[\frac{T_1}{T_2}\right]^4$

GOLDEN KEY POINTS

- Spectral energy distribution curves are continuous. At any temperature in between possible wavelength (0 – ∞) radiation emitted but for different wavelength quantity of radiations are different.
- As the wave length increases, the amount of radiation emitted first increase, becomes maximum and then decreases.
- At a particular temperature the area enclosed between the spectral energy curve shows the spectral emissive power of the body. $\text{Area} = \int_0^\infty E_\lambda d\lambda = E = \sigma T^4$

WEIN'S DISPLACEMENT LAW

The wavelength corresponding to maximum emission of radiation decrease with increasing temperature $\left[\lambda_m \propto \frac{1}{T}\right]$.

This is known as Wein's displacement law. $\lambda_m T = b$ where b Wein's constant = = 2.89 x 10^{-3} mK.

Dimensions of b : = $M^0 L^1 T^0 \theta^1$

Relation between frequency and temperature $\nu_m = \frac{c}{b} T$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Example

The temperature of furnace is 2000 C, in its spectrum the maximum intensity is obtained at about 4000Å,

If the maximum intensity is at 2000Å calculate the temperature of the furnace in C.

Solution

by using $\lambda_m T = b$, $4000\ (2000+273) = 2000(T) \Rightarrow T = 4546K$

The temperature of furnace = 4546 – 273 = 4273 C

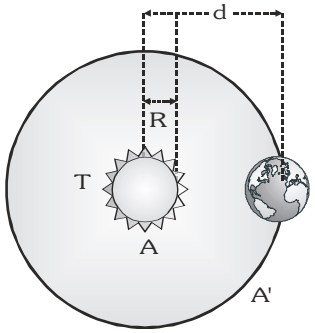
SOLAR CONSTANT 'S'

The Sun emits radiant energy continuously in space of which an in significant part reaches the Earth. The solar radiant energy received per unit area per unit time by a black surface held at right angles to the Sun's rays and placed at the mean distance of the Earth (in the absence of atmosphere) is called solar constant.

The solar constant S is taken to be 1340 watts/m^2 or 1.937 Cal/cm^2–minute

- **Temperature of the Sun**

Let R be the radius of the Sun and 'd' be the radius of Earth's orbit around the Sun. Let E be the energy emitted by the Sun per second per unit area. The total energy emitted by the Sun in one second = E.A = E $4\pi R^2$. (This energy is falling on a sphere of radius equal to the radius of the Earth's orbit around the Sun i.e., on a sphere of surface area $4\pi d^2$)



So, The energy falling per unit area of Earth = $\frac{4\pi R^2 \times E}{4\pi d^2} = \frac{E R^2}{d^2}$

R = 7 10^8m , d = 1.5 10^{11} m, s = 5.7 10^{-8} W m^{-2} K^{-4}

Solar constant $S = \frac{E R^2}{d^2}$

By Stefan's Law $E = \sigma T^4$

$$S = \frac{\sigma T^4 R^2}{d^2} \Rightarrow T = \left[\frac{S \times d^2}{\sigma \times R^2}\right]^{\frac{1}{4}} = \left[\frac{1340 \times (1.5 \times 10^{11})^2}{5.7 \times 10^{-8} \times (7 \times 10^8)^2}\right]^{\frac{1}{4}} = 5732 \text{ K}$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

CALORIMETERY

HEAT

When a hot body is put in contact with a cold one, the former gets colder and the latter warmer. From this observation it is natural to conclude that a certain quantity of heat has passed from the hot body to the cold one. Heat is a form of energy.

Heat is felt by its effects. Some of the effects of heat are :

(a) Change in the degree of hotness (b) Expansion in length, surface area and volume

(c) Change in state of a substance (d) Change in the resistance of a conductor

(e) Thermo e.m.f. effect

SI UNIT : J (joule) Also measured in the unit calorie.

- **Calorie**

It is defined as the amount of heat required to raise the temperature of 1 g water by 1 C.

- **International calorie**

International calorie is the amount of heat required to raise the temperature of 1g water from 14.5 C to 15.5 C rise of temperature.

- **Kilo Calorie**

Kilo calorie is defined as the amount of heat required to raise the temperature of 1 kg water from 14.5 C to 15.5 C. (1 kcal = 1000 calorie).

- **British thermal unit (B. T. U.)**

It is the amount of heat required to raise the temperature of one pound water by 1 F. (1 B.T.U. = 252 calorie).

MECHANICAL EQUIVALENT OF HEAT

According to Joule, work may be converted into heat and vice–versa. The ratio of work done to heat produced is always constant. $\frac{W}{H}$ = constant (J) $\Rightarrow W = J\,H$

W must be in joule, irrespective of nature of energy or work and H must be in calorie.

J is called mechanical equivalent of heat. It is not a physical quantity but simply a conversion factor.

It converts unit of work into that of heat and vice–versa.

J = 4.18 joule/cal or 4.18 10 joule per kilo–cal. For rough calculations we take J = 4.2 joule/cal

SPECIFIC HEAT (s or c)

It is the amount of energy required to raise the temperature of unit mass of that substance by 1 C (or 1K) is called specific heat. It is represented by s or c.

If the temperature of a substance of mass m changes from T to T + dT when it exchanges an amount of heat dQ with its surroundings then its specific heat is $c = \frac{1}{m}\frac{dQ}{dT}$

The specific heat depends on the pressure, volume and temperature of the substance.

For liquids and solids, specific heat measurements are most often made at a constant pressure as functions of temperature, because constant pressure is quite easy to produce experimentally.

SI UNIT : joule/kg–K **CGS UNIT :** cal/g – C

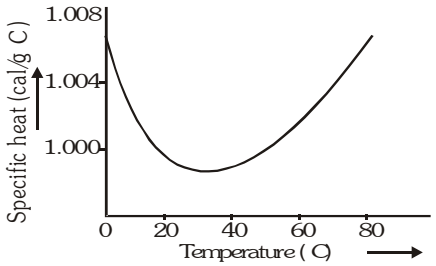
Specific heat of water : c_{water} = 1 cal/g– C = 1 cal/g–K = 1 kcal/kg–K = 4200 joule/kg–K

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

When a substance does not undergo a change of state (i.e., liquid remains liquid or solid remains solid), then the amount of heat required to raise the temperature of mass m of thesubstance by an amount $\Delta\theta$ is $Q = ms\Delta\theta$.

The temperature dependence of the specific heat of water at 1 atmospheric pressure is shown in figure. Its variation is less than 1% over the interval from 0 to 100 C. Such a small variation is typical for most solids and liquids, so their specific heats can generally be taken to be constant over fairly large temperature ranges.

The temperature dependence of the specific heat of water at 1 atm



- There are many processes possible to give heat to a gas.

  A specific heat can be associated to each such process which depends on the nature of process.
- Value of specific heats can vary from zero (0) to infinity.
- Generally two types of specific heat are mentioned for a gas –

  (a) Specific heat at constant volume (C_v) (b) Specific heat at constant pressure (C_p)
- These specific heats can be molar or gram.

MOLAR HEAT CAPACITY

The amount of energy needed to raise the temperature of one mole of a substance by 1 C (or 1K) is called molar heat capacity. The molar heat capacity is the product of molecular weight and specific heat i.e.,

Molar heat capacity C = Molecular weight (M) Specific heat(c) $\Rightarrow$ $C = \frac{1}{\mu}\left(\frac{dQ}{dT}\right)$

If the molecular mass of the substance is M and the mass of the substance is m then number of moles of the substance $\mu = \frac{m}{M} \Rightarrow C = \frac{M}{m}\left(\frac{dQ}{dT}\right)$ **SI UNIT :** J/mol–K

THERMAL CAPACITY

The quantity of heat required to raise the temperature of the whole of that substance through 1 C is called thermal capacity. The thermal capacity of mass m of the whole of substance of specific heat s is = ms

Thermal capacity = mass specific heat

Thermal capacity depends on property of material of the body and mass of the body.

SI UNIT : cal/ C or cal/K, **Dimensions :** $ML^2 T^{-2}K^{-1}$

WATER EQUIVALENT OF A BODY

As the specific heat of water is unity so the thermal capacity of a body (ms) represents its water equivalent also.

- Mass of water having the same thermal capacity as the body is called the water equivalent of the body
- The water equivalent of a body is the amount of water that absorbs or gives out the same amount of heat as is done by the body when heated or cooled through 1 C.

  Water equivalent= mass of body specific heat of the material $\Rightarrow$ (w = ms).

LATENT HEAT OR HIDDEN HEAT

When state of a body changes, change of state takes place at constant temperature [melting point or boiling point] and heat released or absorbed is Q = mL where L is latent heat. Heat is absorbed if solid converts into liquid (at melting point) or liquid converts into vapours (at boiling point) and heat is released if liquid converts into solid or vapours converts into liquid.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

- **Latent heat of fusion**

  It is the quantity of heat (in kilocalories) required to change its 1 kg mass from solid to liquid state at its melting point. Latent heat of fusion for ice : 80 kcal/kg = 80 cal /g.

- **Latent heat of vaporization**

  The quantity of heat required to change its 1 kg mass from liquid to vapour state at its boiling point.

  Latent heat of vaporisation for water : 536 kcal/kg = 536 cal/g

CHANGE OF STATE

- **Melting**

  Conversion of solid into liquid state at constant temperature is known as melting.

- **Boiling**

  Evaporation within the whole mass of the liquid is called boiling. Boiling takes place at a constant temperature known as boiling point. A liquid boils when the saturated vapour pressure on its surface is equal to atmospheric pressure. Boiling point reduces on decreasing pressure.

- **Evaporation**

  Conversion of liquid into vapours at all temperatures is called evaporation. It is a surface phenomenon. Greater the temperature, faster is the evaporation. Smaller the boiling point of liquid, more rapid is the evaporation. Smaller the humidity, more is the evaporation. Evaporation increases on decreasing pressure that is why evaporation is faster in vacuum.

  - **Heat of evaporation**

    Heat required to change unit mass of liquid into vapour at a given temperature is called heat of evaporation at that temperature.

- **Sublimation**

  Direct conversion of solid in to vapour state is called sublimation.

  - **Heat of sublimation**

    Heat required to change unit mass of solid directly into vapours at a given temperature is called heat of sublimation at that temperature.

  - Camphor and ammonium chloride sublimates on heating in normal conditions.
  - A block of ice sublimates into vapours on the surface of moon because of very–very low pressure on its surface

- **Condensation**

  The process of conversion from gaseous or vapour state to liquid state is known as condensation .

  These materials again get converted to vapour or gaseous state on heating.

- **Hoar frost**

  Direct conversion of vapours into solid is called hoar frost. This process is just reverse of the process of sublimation.

  Ex. : Formation of snow by freezing of clouds.

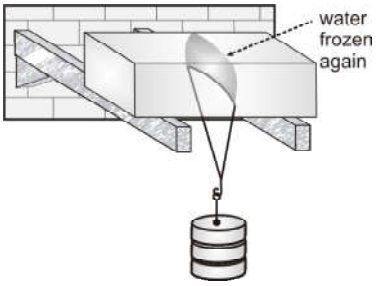
- **Regelation**

  Regelation is the melting of ice caused by pressure and its resolidification when the pressure is removed. Ice shrinks when it melts, and if pressure is applied, deliberately promoting shrinkage, it is found that melting is thereby assisted. In other words, melting of cold ice is ordinarily effected by raising the temperature, but if pressure is present to help with the shrinkage the temperature need not be raised so much.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Ice heals up after being cut through by the wire. Melting takes place under the wire because pressure lowers the melting temperature. Refreezing (regelation) occurs above the wire when the water escapes to normal pressure again.



Increase of pressure lowers the melting (or freezing) point of water. Conversely, if a substance expands on melting, the melting point is raised by pressure.

PHASE OF A SUBSTANCE

The phase of a substance is defined as its form which is homogeneous, physically distinct and mechanically separable from the other forms of that substance.

Phase diagram

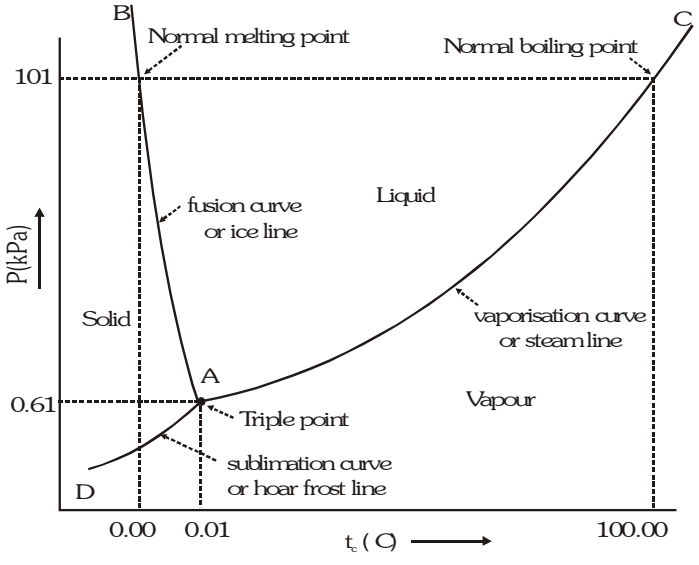
- A phase diagram is a graph in which pressure (P) is represented along the y–axis and temperature (T) is represented along the x–axis.
- **Characteristics of Phase diagram**
  - (i) Different phases of a substances can be shown on a phase diagram.
  - (ii) A region on the phase diagram represents a single phase of the substance, a curve represents equilibrium between two phases and a common point represents equilibrium between three phases.
  - (iii) A phase diagram helps to determine the condition under which the different phases are in equilibrium.
  - (iv) A phase diagram is useful for finding a convenient way in which a desired change of phase can be produced.

PHASE DIAGRAM FOR WATER

The phase diagram for water consists of three curves AB, AC and AD meeting each other at the point A, these curves divide the phase diagram into three regions.

Phase diagram for water
(Scale are not linear)

At the triple point solid, liquid and vapour phases are all in equilibrium



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Region to the left of the curve AB and above the curve AD represents the solid phase of water (ice). The region to the right of the curve AB and above the curve AC represents the liquid phase of water. The region below the curves AC and AD represents the gaseous phase of water (i.e. water vapour). A curve on the phase diagram represents the boundary between two phases of the substance.

Along any curve the two phases can coexist in equilibrium

- Along curve AB, ice and water can remain in equilibrium.This curve is called fusion curve or ice line. This curve shows that the melting point of ice decreases with increase in pressure.
- Along the curve AC, water and water vapour can remain in equilibrium. The curve is called vaporisation curve or steam line. *The curve shows that the boiling point of water increases with increase in pressure.*
- Along the curve AD, ice and water vapour can remain in equilibrium.

  This curve is called sublimation curve or hoar frost line.

TRIPLE POINT OF WATER

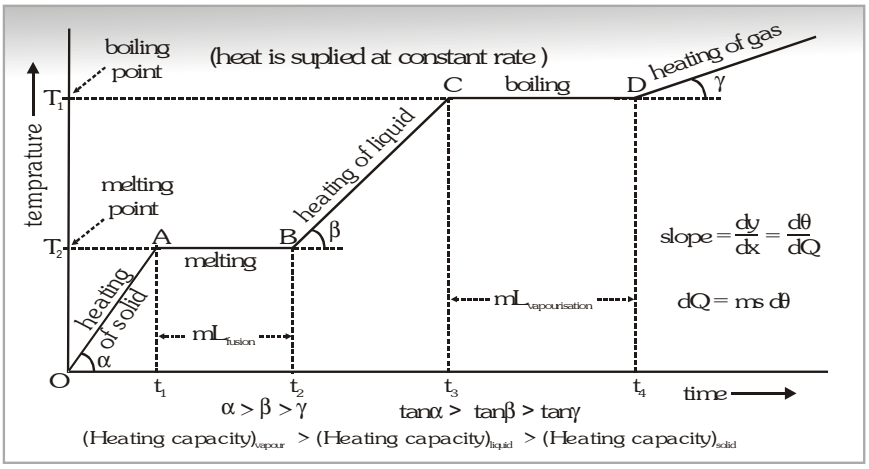
The three curves in the phase diagram of water meet at a single point A, which is called the triple point of water. The triple point of water represents the co–existance of all the three phases of water ice water and water vapour in equilibrium. The pressure corresponding to triple point of water is 6.03 10^{-3} atmosphere or 4.58 mm of Hg and temperature corresponding to it is 273.16K.

- **Significance of triple point of water**

  Triple point of water represents a unique condition and it is used to define the absolute temperature. While making Kelvin's absolute scale, upper fixed point is 273.16 K and lower fixed point is 0 K. One kelvin of temperature is fraction $\frac{1}{273.16}$ of the temperature of triple point of water.

HEATING CURVE

If to a given mass (m) of a solid, heat is supplied at constant rate and a graph is plotted between temperature and time as shown in figure is called heating curve.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

- **In the region OA**

  Rate of heat supply P is constant and temperature of solid is changing with time

  So, $Q = mc_S \Delta T \Rightarrow P\,\Delta t = mc_S\,\Delta T$ $[\because Q = P\,\Delta t]$ $\because \frac{\Delta T}{\Delta t} =$ The slope of temperature–time curve so specific heat of solid $c_S \propto \frac{1}{\text{slope of line OA}}$ specific heat (or thermal capacity) is inversely proportional to the slope of temperature–time curve.

- **In the region AB**

  Temperature is constant, so it represents change of state, i.e., melting of solid with melting point T_1. At point A melting starts and at point B all solid is converted into liquid. So between A and B substance is partly solid and partly liquid. If L_F is the latent heat of fusion then

  $Q = mL_F \Rightarrow L_F = \frac{P(t_2 - t_1)}{m}$ [as $Q = P(t_2 - t_1] \Rightarrow L_F \propto$ length of line AB

  i.e., Latent heat of fusion is proportional to the length of line of zero slope.

  [In this region specific heat $\propto \frac{1}{\tan 0^\circ} = \infty$]

- **In the region BC**

  Temperature of liquid increases so specific heat (or thermal capacity) of liquid will be inversely proportional to the slope of line BC, $c_L \propto \frac{1}{\text{slope of line BC}}$

- **In the region CD**

  Temperature in constant, so it represents change of state, i.e., liquid is boiling with boiling point T_2. At C all substance is in liquid state while at D is vapour state and between C and D partly liquid and partly gas. The length of line CD is proportional to latent heat of vaporisation, i.e., $L_V \propto$ Length of line CD.

  [In this region specific heat $\propto \frac{1}{\tan 0^\circ} = \infty$]

  The line DE represents gaseous state of substance with its temperature increasing linearly with time. The reciprocal of slope of line will be proportional to specific heat or thermal capacity of substance in vapour state.

LAW OF MIXTURES

- When two bodies (one being solid and other liquid or both being liquid) at different temperatures are mixed, heat will be transferred from body at higher temperature to a body at lower temperature till both acquire same temperature. The body at higher temperature released heat while body at lower temperature absorbs it, so that Heat lost = Heat gained. Principle of calorimetry represents the law of conservation of heat energy.
- Temperature of mixture (T) is always ≥ lower temperature (T_L) and ≤ higher temperature (T_H), $T_L \le T \le T_H$

  The temperature of mixture can never be lesser than lower temperature (as a body cannot be cooled below the temperature of cooling body) and greater than higher temperature (as a body cannot be heated above the temperature of heating body). Further more usually rise in temperature of one body is not equal to the fall in temperature of the other body though heat gained by one body is equal to the heat lost by the other.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

5g ice at 0 C is mixed with 5g of steam at 100 C . What is the final temperature?

Solution

Heat required by ice to raise its temperature to 100 C,

$Q_1 = m_1L_1 + m_1c_1\Delta\theta_1 = 5 \quad 80 + 5 \quad 1 \quad 100 = 400 + 500 + 900 = 1800$ cal

Heat given by steam when condensed $Q_2 = m_2L_2 = 5 \quad 536 = 2680$ cal

As $Q_2 > Q_1$. This means that whole steam is not even condensed.

Hence temperature of mixture will remain at 100 C.

Example

A calorimeter of heat capacity 100 J/K is at room temperature of 30 C. 100 g of water at 40 C of specific heat 4200 J/kg–K is poured into the calorimeter. What is the temperature of water in calorimeter?

Solution

Let the temperature of water in calorimeter is t. Then heat lost by water = heat gained by calorimeter

$(0.1) \quad 4200 \quad (40 - t) = 100\ (t - 30) \Rightarrow 42 \quad 40 - 42t = 10t - 300 \Rightarrow t = 38.07$ C

Example

Find the quantity of heat required to convert 40 g of ice at –20 C into water at 20 C.

Given $L_{ice} = 0.336 \quad 10^6$ J/kg. Specific heat of ice = 2100 J/kg–K, specific heat of water = 4200 J/kg–K

Solution

Heat required to raise the temperature of ice from –20 C to 0 C = $0.04 \quad 2100 \quad 20 = 1680$ J

Heat required to convert the ice into water at 0 C = $mL = 0.04 \quad 0.336 \quad 10^6 = 13440$ J

Heat required to heat water from 0 C to 20 C = $0.04 \quad 4200 \quad 20 = 3360$ J

Total heat required = $1680 + 13440 + 3360 = 18480$ J

Example

Steam at 100 C is passed into 1.1 kg of water contained in a calorimeter of water equivalent 0.02 kg at 15 C till the temperature of the calorimeter and its contents rises to 80 C. What is the mass of steam condensed? Latent heat of steam = 536 cal/g.

Solution

Heat required by (calorimeter + water)

$$Q = (m_1c_1 + m_2c_2)\ \Delta\theta = (0.02 + 1.1 \quad 1)\ (80 - 15) = 72.8 \text{ kcal}$$

If m is mass of steam condensed, then heat given by steam

$$Q = mL + mc\ \Delta\theta = m \quad 536 + m \quad 1 \quad (100 - 80) = 556\ m \quad \therefore\ 556\ m = 72.8$$

$\therefore$ Mass of steam condensed $m = \dfrac{72.8}{556} = 0.130$ kg

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

KINETIC THEORY OF GASES

The properties of the gases are entirely different from those of solid and liquid. In case of gases, thermal expansion is very large as compared to solids and liquids .To state the conditions of a gas, its volume, pressure and temperature must be specified.

Intermolecular force Solid > liquid > real gas > ideal gas (zero)

Potential energy Solid < liquid < real gas < ideal gas (zero)

Internal energy, internal kinetic energy, internal potential energy

At a given temperature for solid, liquid and gas:

(i) Internal kinetic energy : Same for all

(ii) Internal potential Energy : Maximum for ideal gas (PE = 0) and Minimum for solids (PE = –ve)

(iii) Internal Energy : Maximum for Ideal gas and Minimum for solid

At a given temperature for rared and compressed gas :

(i) Internal kinetic energy → Same

(ii) Internal potential energy → $(PE)_{Rared} > (PE)_{compressed}$

(iii) Internal Energy → $(U)_{Rared} > (U)_{compressed}$

	N.T.P. (Normal temperature)	**S.T.P. (Standard Temperature and Pressure)**
Temperature	0 C = 273.15 K	0.01 C = 273.16K
Pressure	1 atm = 1.01325 10^5 N/m^2 = 1.01325 10^5 pascal	1 atm
Volume	22.4 litre	22.4 litre

IDEAL GAS CONCEPT

- A gas which follows all gas laws and gas equation at every possible temperature and pressure is known as ideal or perfect gas.
- Volume of gas molecules is negligible as compared to volume of container so volume of gas = volume of container (Except 0 K)
- No intermoleculer force act between gas molecules.
- Potential energy of ideal gas is zero so internal energy of ideal gas is perfectly translational K.E. of gas. It is directly proportional to absolute temperature.

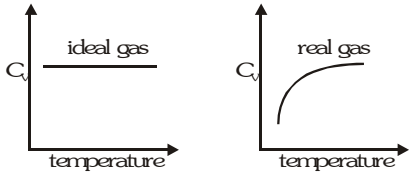
  So, internal energy depends only and only on its temperature.

  $E_{trans} \propto T$

  For a substance $U = U_{KE} + U_{PE}$

  U_{KE} : depends only on T, U_{PE} : depends upon intermolecular forces (Always negative)

- Specific heat of ideal gas is constant quantity and it does not change with temperature
- All real gases behaves as ideal gas at high temperature and low pressure.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory\P65

- Volume expansion coefficient (α) and pressure expansion coefficient (β) is same for a ideal gas and value of each is $\frac{1}{273}$ per C $\quad \alpha = \beta = \frac{1}{273}$ per C
- Gas molecule have point mass and negligible volume and velocity is very high (10^7 cm/s). That's why there is no effect of gravity on them.

EQUATION OF STATE FOR IDEAL GAS

$PV = \mu RT$ where μ = number of moles of gas $\Rightarrow PV = \frac{M}{M_w}RT = \left[\frac{m\,N}{m\,N_0}\right]RT = \left[\frac{R}{N_0}\right]N\,T = NkT$

Example

By increasing temperature of gas by 5 C its pressure increases by 0.5% from its initial value at constant volume then what is initial temperature of gas ?

Solution

$\because$ At constant volume $T \propto P \quad \therefore \frac{\Delta T}{T}\times 100 = \frac{\Delta P}{P}\times 100 = 0.5 \Rightarrow T = \frac{5\times 100}{0.5} = 1000K$

Example

Calculate the value of universal gas constant at STP.

Solution

Universal gas constant is given by $R = \frac{PV}{T}$

One mole of all gases at S.T.P. occupy volume V = 22 4 litre = 22 4 10^{-3} m^3

P = 760 mm of Hg = 760 10^{-3} 13 6 10^3 9.80 N m^{-2} T = 273 K

$\therefore$ R = $\frac{760\times 10^{-3}\times 13.6\times 10^3\times 9.80\times 22.4\times 10^{-3}}{273}$ = 8 31 J mol^{-1} K^{-1}

Example

A closed container of volume 0.02 m^3 contains a mixture of neon and argon gases at a temperature of 27 C and pressure of 1 10^5 Nm^2. The total mass of the mixture is 28 g. If the gram molecular weights of neon and argon are 20 and 40 respectively, find the masses of the individual gases in the container, assuming them to be ideal. Given : R = 8.314 J/mol/K.

Solution

Let m gram be the mass of neon. Then, the mass of argon is (28 – m)g.

Total number of moles of the mixture, $\mu = \frac{m}{20} + \frac{28-m}{40} = \frac{28+m}{40}$ ...(i)

Now, $\mu = \frac{PV}{RT} = \frac{1\times 10^5\times 0.02}{8.314\times 300} = 0.8$ ...(ii)

By (i) and (ii), $\frac{28+m}{40} = 0.8 \Rightarrow 28 + m = 32 \Rightarrow m = 4$ gram or mass of argon = (28 – 4)g = 24 g

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Example

Calculate the temperature of the Sun if density is 1.4 g cm^{-3}, pressure is 1.4 10^9 atmosphere and average molecular weight of gases in the Sun in 2 g/mole. [Given R = 8.4 J $mol^{-1}K^{-1}$]

Solution

$PV = \mu RT \Rightarrow T = \frac{PV}{\mu R}$ ...(i) But $\mu = \frac{M}{M_w}$ and $\rho = \frac{M}{V}$ $\therefore$ $\mu = \frac{\rho V}{M_w}$

From equation (i) $T = \frac{PVM_w}{\rho VR} = \frac{PM_w}{\rho R} = \frac{1.4\times10^9\times1.01\times10^5\times2\times10^{-3}}{1.4\times1000\times8.4} = 2.4$ 10^7 K

Example

At the top of a mountain a thermometer reads 7 C and barometer reads 70 cm of Hg. At the bottom of the mountain they read 27 C and 76 cm of Hg respectively. Compare the density of the air at the top with that at the bottom.

Solution

By gas equation $PV = \frac{M}{M_w}RT \Rightarrow \frac{P}{\rho T} = \frac{R}{M_w}$ $\left[\because \mu = \frac{M}{M_w} \text{ and } \frac{M}{V} = \rho\right]$

Now as M_W and R are same for top and bottom $\left[\frac{P}{\rho T}\right]_T = \left[\frac{P}{\rho T}\right]_B$ So $\frac{\rho_T}{\rho_B} = \frac{P_T}{P_B}\times\frac{T_B}{T_T} = \frac{70}{76}\times\frac{300}{280} = \frac{75}{76} = 0.9868$

Example

During an experiment an ideal gas is found to obey an additional law VP^2 = constant. The gas is initially at temperature T and volume V. What will be the temperature of the gas when it expands to a volume 2V.

Solution

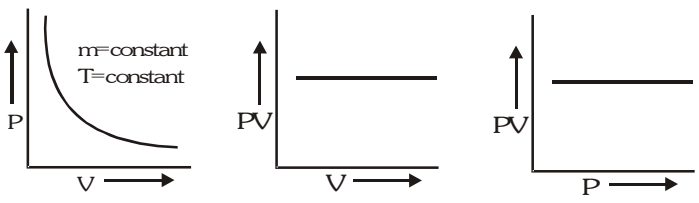
By gas equation PV = μRT and VP^2 = constant on eliminating P

$\left[\frac{A}{\sqrt{V}}\right]V = \mu RT \Rightarrow \sqrt{V} = \frac{\mu R}{A}T$ $\therefore$ $\frac{\sqrt{V_1}}{\sqrt{V_2}} = \left[\frac{T_1}{T_2}\right] \Rightarrow \frac{\sqrt{V}}{\sqrt{2V}} = \frac{T}{T'}$ $\Rightarrow$ $T' = \left(\sqrt{2}\right)T$

GAS LAWS

- **Boyle's Law**

According to it for a given mass of an ideal gas at constant temperature, the volume of a gas is inversely proportional to its pressure, i.e., $V \propto \frac{1}{P}$ if m and T = Constant



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory.P65

Example

A sample of oxygen with volume of 500 cc at a pressure of 2 atm is compressed to a volume of 400 cc. What pressure is needed to do this if the temperature is kept constant ?

Solution

Temperature is constant, so $P_1 V_1 = P_2 V_2 \therefore P_2 = P_1 \frac{V_1}{V_2} = 2\left[\frac{500}{400}\right] = 2.5$ atm

Example

An air bubble doubles in radius on rising from bottom of a lake to its surface. If the atmosphere pressuer is equal to that due to a column of 10 m of water, then what will be the depth of the lake.

(Assuming that surface tension is negligible) ?

Solution

Given that constant temperature, we use $P_1V_1 = P_2V_2$

P_2 = (10) dg (for water column) P_1 = (10+h) dg (where h=depth of lake)

$V_1 = \frac{4\pi}{3}r^3$, $V_2 = \frac{4\pi}{3}(2r)^3 = 8\left(\frac{4\pi}{3}r^3\right) = 8V_1$ Thus for $P_2V_2 = P_1V_1$,

We have 10 dg $(8V_1)$ = (10 + h) dg $V_1 \Rightarrow$ 80 = 10 + h $\Rightarrow$ h = 70 m

Example

A vessel of volume 8.0 10^{-3} m^3 contains an ideal gas at 300 K and 200 k Pa. The gas is allowed to leak till the Pressure falls to 125 kPa. Calculate the amount of the gas leaked assuming that the temperature remains constant.

Solution

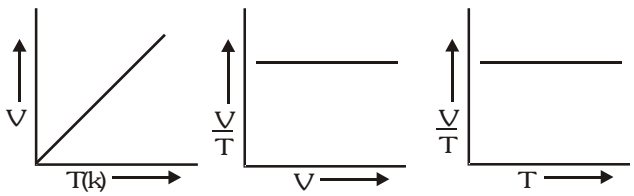
As the gas leaks out, the volume and the temperature of the remaining gas do not change. The number of moles of the gas in the vessel in given by $n = \frac{PV}{RT}$.

The number of moles in the vessel before the leakage is $n_1 = \frac{P_1V}{RT}$ and that after the leakage is $n_2 = \frac{P_2V}{RT}$.

The amount leaked is $n_1 - n_2 = \frac{(P_1 - P_2)V}{RT} = \frac{(200-125)\times10^3\times8.0\times10^{-3}}{8.3\times300} = 0.24$ mole

- **Charle's Law**

According to it for a given mass of an ideal gas at constant pressure, volume of a gas is directly proportional to its absolute temperature, i.e. $V \propto T$ if m and P = Constant



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Example

1500 ml of a gas at a room temperature of 23 C is inhaled by a person whose body temperature is 37 C, if the pressure and mass stay constant, what will be the volume of the gas in the lungs of the person ?

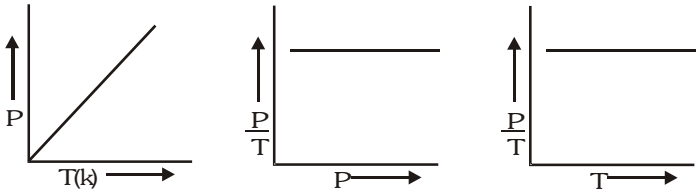
Solution

T_1 = 273 + 37 = 310 K; T_2 = 273 + 23 = 296 K. Pressure and amount of the gas are kept constant,

So $\frac{V_1}{T_1}=\frac{V_2}{T_2} \therefore V_2 = V_1 \times \frac{T_2}{T_1} = 1500 \times \frac{293}{310} = 1417.74$ ml

Gay–Lussac's Law

According to it, for a given mass of an ideal gas at constant volume, pressure of a gas is directly proportional to its absolute temperature, i.e., $P \propto T$ if m and V = constant



Example

A sample of O_2 is at a pressure of 1 atm when the volume is 100 ml and its temperature is 27 C. What will be the temperature of the gas if the pressure becomes 2 atm and volume remains 100 ml.

Solution

T_1 = 273 + 27 = 300 K

For constant volume $\frac{P_1}{T_1}=\frac{P_2}{T_2} \Rightarrow T_2 = T_1 \times \frac{P_2}{P_1} = 300 \times \frac{2}{1}$ = 600 K = 600 – 273 = 327 C

Avogadro's Law

According to it, at same temperature and pressure of equal volumes of all gases contain equal number of molecules, i.e., $N_1 = N_2$ if P,V and T are same.

Dalton's Partial Pressure Mixture Law :

According to it, the pressure exerted by a gasesous mixture is equal to the sum of partial pressure of each component gases present in the mixture, ie., $P = P_1 + P_2 + ...$

Example

The mass percentage in composition of dry air at sea level contains approximately 75.5% of N_2. If the total atmospheric pressure is 1 atm then what will be the partial pressure of nitrogen ?

Solution

The mole fraction of nitrogen $\mu_1 = \left(\frac{M_{\text{nitrogen}}}{\text{Molecular weight}}\right) = 0.755 \times \frac{29}{28}$ = 0.78

The partial pressure of nitrogen $P_1 = \mu_1 \frac{RT}{V} = \frac{\mu_1}{\mu}\frac{\mu RT}{V} = \left(\frac{\mu_1}{\mu}\right)P$ = (0.78) 1 = 0.78 atm

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy \Unit No-5\Thermal physics\Eng\Theory P65

The kinetic theory of gases

- Rudolph Claussius (1822–88) and James Clark Maxwell (1831–75) developed the kinetic theory of gases in order to explain gas laws in terms of the motion of the gas molecules. The theory is based on following assumptions as regards to the motion of molecules and the nature of the gases.

Basic postulates of Kinetic theory of gases

- Every gas consists of extremely small particles known as molecules. The molecules of a given gas are all identical but are different than those another gas.
- The molecules of a gas are identical, spherical, rigid and perfectly elastic point masses.
- The size is negligible in comparision to inter molecular distance (10^{-9} m)

Assumptions regarding motion :

- Molecules of a gas keep on moving randomly in all possible direction with all possible velocities.
- The speed of gas molecules lie between zero and infinity (very high speed).
- The number of molecules moving with most probable speed is maximum.

Assumptions regarding collision:

- The gas molecules keep colliding among themselves as well as with the walls of containing vessel. These collision are perfectly elastic. (ie., the total energy before collision = total energy after the collisions.)

Assumptions regarding force:

- No attractive or repulsive force acts between gas molecules.
- Gravitational attraction among the molecules is ineffective due to extremely small masses and very high speed of molecules.

Assumptions regarding pressure:

- Molecules constantly collide with the walls of container due to which their momentum changes. This change in momentum is transferred to the walls of the container. Consequently pressure is exerted by gas molecules on the walls of container.

Assumptions regarding density:

- The density of gas is constant at all points of the container.

PROPERTIES/ASSUMPTIONS OF IDEAL GAS

- The molecules of a gas are in a state of continuous random motion. They move with all possible velocities in all possible directions. They obey Newton's law of motion.
- Mean momentum = 0; Mean velocity = 0.$< \vec{v} > = 0$; $< v^2 > \neq 0$ (Non zero); $< v^3 > = < v^5 > = 0$
- The average distance travelled by a molecule between two successive collisions is called as mean free path (λ_m) of the molecule.
- The time during which a collision takes place is negligible as compared to time taken by the molecule to cover the mean free path so NTP ratio of time of collision to free time of motion 10^{-8} : 1.
- When a gas taken into a vessel it is uniformly distributed in entire volume of vessel such that its density, moleculer density, motion of molecules etc. all are identical for all direction, therefore root mean velocity

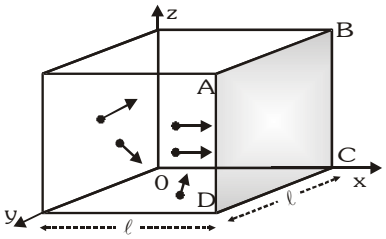
  $\overline{v}_x^2 = \overline{v}_y^2 = \overline{v}_z^2 \rightarrow$ equal Pressure exerted by the gas in all direction $P_x = P_y = P_z = P \rightarrow$ equal
- All those assumptions can be justified, if number of gas molecules are taken very large

  i.e., 10^{23} molecules/cm^3.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

EXPRESSION FOR PRESSURE OF AN IDEAL GAS

Consider an ideal gas enclosed in a cubical vessel of length ℓ. Suppose there are 'N' molecules in a gas which are moving with velocities $\vec{v}_1, \vec{v}_2 \ldots\ldots \vec{v}_N$.



If we consider any single molecule than its instantaneous velocity $\vec{v}$ can be expressed as $\vec{v} = v_x\tilde{i} + v_y\tilde{j} + v_z\tilde{k}$

Due to random motion of the molecule $v_x = v_y = v_z$ $\quad |v| = v_x\sqrt{3} = v_y\sqrt{3} = v_z\sqrt{3} = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Suppose a molecule of mass m is moving with a velocity v_x towards the face ABCD. It strikes the face of the cubical vessel and returns back to strike the opposite face.

Change in momentum of the molecule per collision $\Delta p = - mv_x - mv_x = - 2 mv_x$

Momentum transferred to the wall of the vessel per molecule per collision $\Delta p = 2 mv_x$

The distance travelled by the molecule in going to face ABCD and coming back is 2ℓ.

So, the time between two successive collision is $\Delta t = \dfrac{2\ell}{v_x}$

Number of collision per sec per molecule is $f_c = \dfrac{v_x}{2\ell} = \dfrac{\text{molecule velocity}}{\text{mean free path}}$, $f_c = \dfrac{v_{rms}}{\lambda_m}$ or $f_c = \dfrac{v_m}{\lambda_m}$

Hence momentum transferred in the wall per second by the molecule is = force on the wall

$$\text{force } F = (2 mv_x)\ \frac{v_x}{2\ell} = \frac{mv_x^2}{\ell} = \frac{mv^2}{3\ell}$$

Pressure exerted by gas molecule $P = \dfrac{F}{A} = \dfrac{1}{3}\dfrac{mv^2}{\ell \times A} \Rightarrow P = \dfrac{1}{3}\dfrac{mv^2}{V}$ $\left[\because A \times \ell = V\right]$

Pressure exerted by gas $P = \sum \dfrac{1}{3}\dfrac{mv^2}{V} = \sum \dfrac{1}{3}\dfrac{mv^2}{V} \times \dfrac{N}{N} = \dfrac{1}{3}\dfrac{mN}{V}\dfrac{\sum v^2}{N} = \dfrac{1}{3}\dfrac{mN}{V}v_{rms}^2$

$$v_{rms}^2 = \frac{3PV}{M} = \frac{3\mu RT}{\mu M_w} \Rightarrow v_{rms} = \sqrt{\frac{3RT}{M_w}},\ P = \frac{1}{3}\frac{M}{V}v_{rms}^2 = \frac{1}{3}\rho v_{rms}^2$$

- Average number of molecules for each wall = $\dfrac{N}{6}$. No. of molecules along each axis = $\dfrac{N}{3}$ $(N_x = N_y = N_z)$
- $\bar{v}_x^2 = \bar{v}_y^2 = \bar{v}_z^2 = \dfrac{\bar{v}_{rms}^2}{3}$ Root mean square velocity along any axis for gas molecule is $(v_{rms})_x = (v_{rms})_y = (v_{rms})_z = \dfrac{v_{rms}}{\sqrt{3}}$

All gas laws and gas equation can be obtained by expression of pressure of gas (except Joule's law)

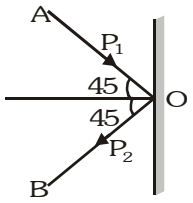
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

The mass of a hydrogen molecule is 3.32 10^{-27} kg. If 10^{23} molecules are colliding per second on a stationary wall of area 2 cm^2 at an angle of 45 to the normal to the wall and reflected elastically with a speed 10^3 m/s. Find the pressure exerted on the wall will be (in N/m^2)

Solution

As the impact is elastic $\therefore |\vec{p}_1| = |\vec{p}_2| = p = mv = 3.32 \quad 10^{-24}$ kg m/s



The change in momentum along the normal $\Delta p = |\vec{p}_2 - \vec{p}_1| = 2p\cos 45° = \sqrt{2}p$

If f is the collision frequency then force applied on the wall $F = \frac{\Delta p}{\Delta t} = \Delta p \times f = \sqrt{2}pf$

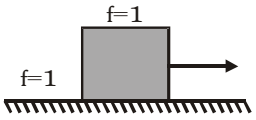
$\therefore$ Pressure $P = \frac{F}{A} = \frac{\sqrt{2}pf}{A} = \frac{\sqrt{2} \times 3.32 \times 10^{-24} \times 10^{23}}{2 \times 10^{-4}} = 2.347 \times 10^3 \, N/m^2$

DEGREE OF FREEDOM (f)

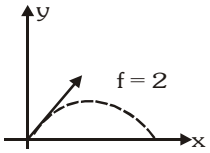
- The number of independent ways in which a molecule or an atom can exhibit motion or have energy is called it's degrees of freedom.
- The number of independent coordinates required to specify the dynamical state of a system is called it's degrees of freedom.

For example

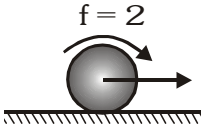
(a) Block has one degree of freedom, because it is confined to move in a straight line and has only one transistional degree of freedom.



(b) The projectile has two degrees of freedom becomes it is confined to move in a plane and so it has two translational degrees of freedom.



(c) The sphere has two degrees of freedom one rotational and another translational. Similarly a particle free to move in space will have three translational degrees of freedom.



Note : In pure rolling sphere has one degree of freedom as KE = $\frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) = \frac{7}{10}mv^2$

- The degrees of freedom are of three types :

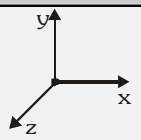
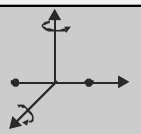
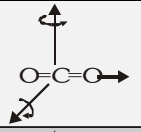
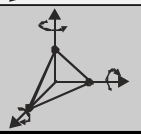
(a) Translational Degree of freedom : Maximum three degree of freedom are there corresponding to translational motion.

(b) Rotational Degree of freedom : The number of degrees of freedom in this case depends on the structure of the molecule.

(c) Vibrational Degree of freedom : It is exhibited at high temperatures.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Degree of freedom for different gases according to atomicity of gas at low temperature

Atomicity of gas	Translational	Rotational	Total	
Monoatomic **Ex.** Ar, Ne, Ideal gas etc	3	0	3	
Diatomic **Ex.** O_2, Cl_2, N_2 etc.	3	2	5	
Triatomic (linear) **Ex.** CO_2, C_2H_2	3	2	5	
Triatomic (Non–linear) or Polyatomic **Ex.** H_2O, NH_3, CH_4	3	3	6	

At high temperatures a diatomic molecule has 7 degrees of freedom. (3 translational, 2 rotational and 2 vibrational)

Example

Calculate the total number of degrees of freedom possessed by the molecules in one cm^3 of H_2 gas at NTP.

Solution

22400 cm^3 of every gas constains 6.02 10^{23} molecules.

$\therefore$ Number of molecules in 1 cm^3 of H_2 gas = $\frac{6.02\times10^{23}}{22400}$ = 0.26875 10^{20}

Number of degrees of freedom of a H_2 gas molecule = 5

$\therefore$ Total number of degrees of freedom of 0.26875 10^{20} 5 = 1.34375 10^{20}.

MAXWELL'S LAW OF EQUIPARTITION OF ENERGY

The total kinetic energy of a gas molecules is equally distributed among its all degree of freedom and the energy associated with each degree of freedom at absolute temperature T is $\frac{1}{2}kT$

For one molecule of gas

Energy related with each degree of freedom = $\frac{1}{2}kT$

Energy related with all degree of freedom = $\frac{f}{2}kT$ $\because \overline{v}_x^2=\overline{v}_y^2=\overline{v}_z^2=\frac{\overline{v}_{rms}^2}{3}\Rightarrow \frac{1}{2}mv_{rms}^2=\frac{3}{2}kT$

So energy related with one degree of freedom = $\frac{1}{2}m\frac{v_{rms}^2}{3}=\frac{3}{2}\frac{kT}{3}=\frac{1}{2}kT$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

A cubical box of side 1 meter contains helium gas (atomic weight 4) at a pressure of 100 N/m^2. During an observation time of 1 second, an atom travelling with the root–mean–square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. Take R = $\frac{25}{3}$ J/mol–K and k = 1.38 10^{-23} J/K.

(a) Evaluate the temperature of the gas.

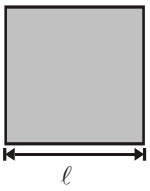
(b) Evaluate the average kinetic energy per atom.

(c) Evaluate the total mass of helium gas in the box.

Solution

Volume of the box = 1m^3, Pressure of the gas = 100 N/m^2. Let T be the temperature of the gas

(a) Time between two consecutive collisions with one wall = $\frac{1}{500}$ sec



This time should be equal to $\frac{2I}{v_{rms}}$, where ℓ is the side of the cube.

$2\ell v_{rms} = \frac{1}{500} \Rightarrow v_{rms}$ = 1000 m/s $\therefore \sqrt{\frac{3RT}{M}}$ = 1000 $\Rightarrow$ $T = \frac{(1000)^2 M}{3R} = \frac{(10)^6 (3\times10^{-3})}{3\left(\frac{25}{3}\right)} = 160K$

(b) Average kinetic energy per atom = $\frac{3}{2}kT = \frac{3}{2}$ [(1.38 10^{-23}) = 160] J = 3.312 $^{-21}$ J

(c) From PV = nRT = $\frac{m}{M}RT$, Mass of helium gas in the box m= $\frac{PVM}{RT}$

Substituting the values, m = $\frac{(100)(1)(4\times10^{-3})}{\left(\frac{25}{3}\right)(160)}$ = 3.0 10^{-4} kg

DIFFERENT K.E. OF GAS (INTERNAL ENERGY)

- **Translatory kinetic energy (E_T)** $E_T = \frac{1}{2}Mv_{rms}^2 = \frac{3}{2}PV$

Kinetic energy of volume V is = $\frac{1}{2}Mv_{rms}^2$ Note : Total internal energy of ideal gas is kinetic

- **Energy per unit volume or energy density (E_V)**

$E_V = \frac{\text{Total energy}}{\text{Volume}} = \frac{E}{V}$; $E_V = \frac{1}{2}\left[\frac{M}{V}\right]v_{rms}^2 = \frac{1}{2}\rho v_{rms}^2 \because P = \frac{2}{3}\left[\frac{1}{2}\rho v_{rms}^2\right] \therefore E_V = \frac{3}{2}P$

- **Molar K.E. or Mean Molar K.E. (E)**

$E = \frac{1}{2}M_w v_{rms}^2$ for N_0 molecules or M_w (gram) $E = \frac{3}{2}RT = \frac{3}{2}N_0kT$

- **Molecular kinetic energy or mean molecular K.E. ($\bar{E}$)**

$E = \frac{1}{2}M_w v_{rms}^2$, $\bar{E} = \frac{E}{N_0} = \frac{3}{2}\frac{RT}{N_0} = \frac{3}{2}kT$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

GOLDEN KEY POINT

- Except 0 K, at any temperature T , $E > E_m > \overline{E}$
- At a common temperature, for all ideal gas

  E and $\overline{E}$ are same while E_m is different and depends upon nature of gas (M_w or m)
- For thermal equilibrium of gases, temperature of each gas is same and this temperature called as temperature of mixture (T_m) which can be find out on basis of conservation of energy (All gases are of same atomicity). $T_m = \frac{\sum NT}{\sum N} = \frac{N_1T_1 + N_2T_2 + ... + N_nT_n}{N_1 + N_2 ..... N_n}$
- **1 mole gas :** Mean kinetic energy = $\frac{3}{2}RT$; Total kinetic energy = $\frac{f}{2}RT$

  1 molecule of gases : Mean kinetic energy = $\frac{3}{2}kT$; Total kinetic energy = $\frac{f}{2}kT$

  $f \rightarrow$ Degree of freedom

Example

Two ideal gases at temperature T_1 and T_2 are mixed. There is no loss of energy. If the masses of molecules of the two gases are m_1 and m_2 and number of their molecules are n_1 and n_2 respectively. Find the temperature of the mixture.

Solution

Total energy of molecules of first gas = $\frac{3}{2}n_1kT_1$, Total energy of molecules of second gas = $\frac{3}{2}n_2kT_2$

Let temperature of mixture be T then total energy of molecules of mixture = $\frac{3}{2}k(n_1 + n_2)T$

$\therefore \frac{3}{2}(n_1 + n_2)kT = \frac{3}{2}k(n_1T_1 + n_2T_2) \Rightarrow T = \frac{n_1T_1 + n_2T_2}{(n_1 + n_2)}$

Example

The first excited state of hydrogen atom is 10.2 eV above its ground state. What temperature is needed to excite hydrogen atoms to first excited level.

Solution

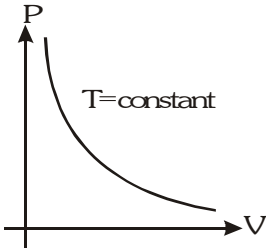
K.E. of the hydrogen atom $\frac{3}{2}kT = 10.2\ eV = 10.2 \times \left(1.6 \times 10^{-19}\right) J$

$\Rightarrow T = \frac{2}{3} \times \frac{10.2 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} = 7.88 \times 10^4\ K$

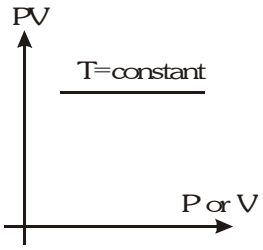
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

EXPLANATION OF GAS LAWS FROM KINETIC THEORY

- **Boyle's Law**

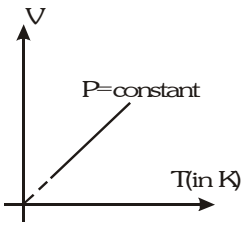


According to this law, the product of the pressure and the volume of a given mass of gas at constant temperature is constant. From the kinetic theory of gases, the pressure of a given mass of an ideal gas is given by $P = \frac{1}{3}\frac{mN}{V}v_{rms}^2$. mN is the mass of the gas which is constant. If the temperature remains constant, the mean–square–velocity of the molecules. (v_{rms}^2) also remains constant. Thus, from the above equation, we have PV = constant.



This is Boyle's law. $V \propto \frac{1}{P}$ (T = constant) or $P_iV_i = P_fV_f$

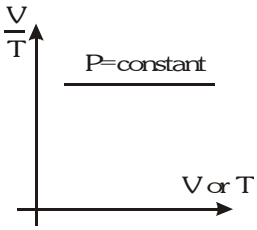
- **Charle's Law**



According to this law, the volume of a given mass of gas at constant pressure is directly proportional to the absolute temperature of the gas.

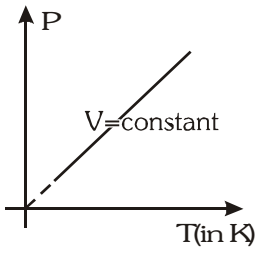
From kinetic theory, we have $V = \frac{2}{3}\frac{N}{P}\left(\frac{1}{2}mv_{rms}^2\right)$

$$V = \frac{2}{3}\frac{N}{P}\left(\frac{3}{2}kT\right) = \frac{N}{P}kT \quad \left[\because \ \frac{1}{2}mv_{rms}^2 = \frac{3}{2}kT\right]$$



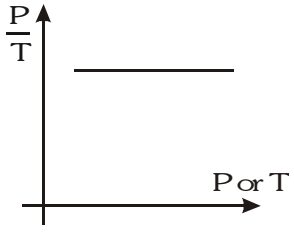
If the pressure P is constant, then for a given mass of the gas, we have $V \propto T$ This is Charle's Law.

- **Gay Lussac's law of Pressure law**



For a given mass of a gas the pressure of a gas at constant volume (called Isochoric process) is directly proportional to its absolute temperature, $P = \frac{1}{3}\frac{mN}{V}v_{rms}^2$ $\because \ v_{rms}^2 \propto T$ $\therefore \ P \propto T$ (V = constant)

or $\frac{P}{T}$ = constant or $\frac{P_i}{T_i} = \frac{P_f}{V_f}$



P–T graph in an isochoric process is a straight line passing through origin or $\frac{P}{T}$ v/s P or T graph is a straight line parallel to P or T axis.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

- **Avogadro's Law**

Equal volumes of 'all gases' under the same conditions of temperature and pressure contain equal number of molecules.

At same pressure equal volumes V of different gases contain respectively N_1 and N_2 molecules of masses m_1 and m_2. $PV = \frac{1}{3}m_1N_1v_{rms_1}^2$ and $PV = \frac{1}{3}m_2N_2v_{rms_2}^2 \Rightarrow m_1N_1v_{rms_1}^2 = m_2N_2v_{rms_2}^2$

Now, if the gases are at the same temperature, their average kinetic energies of translation per molecule are equal. That is $\frac{1}{2}m_1v_{rms_1}^2 = \frac{1}{2}m_2v_{rms_2}^2 \Rightarrow N_1 = N_2$.

- **Dalton's Law of Partial Pressures**

The total pressure exerted by a mixture of non–reacting gases occupying a vessel is equal to the sum of the individual pressures which each gas would exert if it alone occupied the whole vessel.

Let as consider a mixture of gases occupying a volume V. Suppose the first gas contains N_1 molecules, each of mass m_1 having mean–square–speed $v_{rms_1}^2$, the second gas contains N_2 molecules each of mass m_2 and mean–square–speed $v_{rms_2}^2$, and so on. Let P_1, P_2,..... respectively the partial pressures of the gases. Each gas fills the whole volume V. According to kinetic theory, we have

$$P_1V = \frac{1}{3}m_1N_1v_{rms_1}^2, \qquad P_2V = \frac{1}{3}m_2N_2v_{rms_2}^2, \text{ and so on.}$$

Additing, we get $(P_1 + P_2 + ...)V = \frac{1}{3}(m_1N_1v_{rms_1}^2 + m_2N_2v_{rms_2}^2 + ...)$ .....(i)

Now, the whole mixture is at the same temperature. $\therefore \frac{1}{2}m_1v_{rms_1}^2 = \frac{1}{2}m_2v_{rms_2}^2 ..... = \frac{1}{2}mv_{rms}^2$

Substituting this result in eq^n. (i) we have $(P_1 + P_2 + ...)V = \frac{1}{3}(N_1 + N_2 + ...)mv_{rms}^2$

The mixture has a total number of molecules (N_1 + N_2 + ...). Hence the pressure P exerted by the mixture is given by $PV = \frac{1}{3}(N_1 + N_2 + ...)\ mv_{rms}^2$

That P = P_1 + P_2 + ...This is Dalton's law of partial pressures.

DIFFERENT SPEEDS OF GAS MOLECULES

- **Average velocity**

Because molecules are in random motion in all possible direction in all possible velocity. Therefore, the average velocity of the gas in molecules in container is zero. $< \vec{v} > = \frac{\vec{v}_1 + \vec{v}_2 + ... \vec{v}_N}{N} = 0$

RMS speed of molecules $v_{rms} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3RT}{M_w}} = \sqrt{\frac{3kT}{m}} = 1.73\sqrt{\frac{kT}{m}}$

Mean speed of molecules :By maxwell's velocity distribution law v_M or $< |\vec{v}| > = v_{mean}$

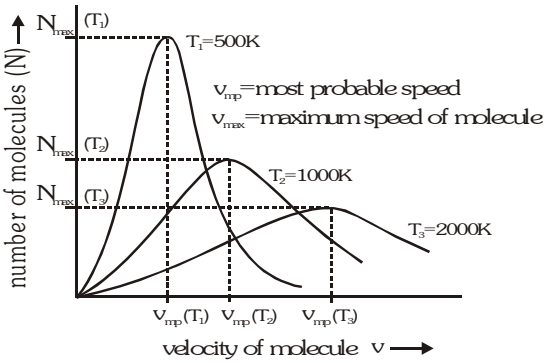
$< |\vec{v}| > = v_{mean} = \frac{|\vec{v}_1| + |\vec{v}_2| + ... |\vec{v}_n|}{N} = \sqrt{\frac{8}{\pi}\frac{P}{\rho}} = \sqrt{\frac{8RT}{\pi M_w}} = \sqrt{\frac{8kT}{\pi m}} = 1.59\sqrt{\frac{kT}{m}}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Most probable speed of molecules (v_{mp})

At a given temperature, the speed to which maximum number of molecules belongs is called as most probable speed (v_{mp}) $v_{mp} = \sqrt{\frac{2P}{\rho}} = \sqrt{\frac{2RT}{M_w}} = \sqrt{\frac{2kT}{m}} = 1.41 \sqrt{\frac{kT}{m}}$

MAXWELL'S LAW OF DISTRIBUTION OF VELOCITIES



GOLDEN KEY POINT

- At any given temperature graph drawn in between molecular velocity and number of molecules is known as velocity distribution curve.
- The velocities of molecules of a gas are in between zero and infinity (0 – ∞)
- With the increase in the temperature, the most probable velocity and maximum molecule velocity both increases.
- The number of molecules within certain velocity range is constant although the velocity of molecule changes continuously at particular temperature.
- The area enclosed between the (N – v) curve and the velocity axis presents the total number of molecules.

On the basis of velocity distribution Maxwell established gives the law of equipartition of energy for gases of any temperature.

Velocity of sound in gas medium (v_s) $v_{sound} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M_w}} = \sqrt{\frac{\gamma kT}{m}}$

- At any temperature $v_{rms} > v_{Mean} > v_{MP} > v_{sound}$ (always)
- For a gas at any temperature (T) $\frac{v_{rms}}{v_{sound}} = \sqrt{\frac{3}{\gamma}}$, $\frac{v_{rms}}{v_{MP}} = \sqrt{\frac{3}{2}}$
- A temperature is not possible at which above order can be changed

$v_{rms} \neq v_{Mean} \neq v_{MP} \neq v_{sound}$ (always)

Example

The velocities of ten particles in ms^{-1} are 0, 2, 3, 4, 4, 4, 5, 5, 6, 9. Calculate

(i) average speed and (ii) rms speed (iii) most probable speed.

Solution

(i) average speed, $v_{av} = \frac{0+2+3+4+4+4+5+5+6+9}{10} = \frac{42}{10} = 4\ 2\ ms^{-1}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory Pt-5

(ii) rms speed, $v_{rms} = \left[\frac{(0)^2+(2)^2+(3)^2+(4)^2+(4)^2+(4)^2+(5)^2+(5)^2+(6)^2+(9)^2}{10}\right]^{1/2} = \left[\frac{228}{10}\right]^{1/2} = 4.77 \text{ ms}^{-1}$

(iii) most probable speed $v_{mp} = 4$ m/s

Example

At what temperature, will the root mean square velocity of hydrogen be double of its value at S.T.P., pressure remaining constant ?

Solution

Let v_1 be the r.m.s. velocity at S.T.P. and v_2 be the r.m.s. velocity at unknown temperature T_2.

$\therefore \quad \frac{v_1^2}{v_2^2} = \frac{T_1}{T_2}$ or $T_2 = T_1\left[\frac{v_2}{v_1}\right]^2 = 273 \quad (2)^2 = 273 \quad 4 = 1092$ K $= (1092 - 273) = 819$ C

Example

Calculate rms velocity of oxygen molecule at 27 C

Solution

Temperature, T = 27 C $\Rightarrow$ 273 + 27 = 300 K,

Molecular weight of oxygen = 32 10^{-3} kg and R = 8 31 J mol^{-1} K^{-1}

rms velocity is $v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8 \cdot 31 \times 300}{32 \times 10^{-3}}} = 483\ 5 \text{ ms}^{-1}$

Example

Calculate the kinetic energy of a gram moelcule of argon at 127 C.

Solution

Temperature, T = 127 C = 273 + 127 = 400 K, R = 8.31 J/mol K

K.E. per gram molecule of argon = $\frac{3}{2}$ R T = $\frac{3}{2}$ 8.31 400 = 4986 J

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-5\Thermal physics\Eng\Theory.P65

THERMODYNAMICS

Branch of physics which deals with the inter–conversion between heat energy and any other form of energy is known as thermodynamics. In this branch of physics we deals with the processes involving heat, work and internal energy. In this branch of science the conversion of heat into mechanical work and vice versa is studied.

- **Thermodynamical System**

  The system which can be represented in of pressure (P), volume (V) and temperature (T), is known thermodynamic system. A specified portion of matter consisting of one or more substances on which the effects of variables such as temperature, volume and pressure are to be studied, is called a system. e.g. A gas enclosed in a cylinder fitted with a piston is a system.

- **Surroundings**

  Anything outside the system, which exchanges energy with the system and which tends to change the properties of the system is called its surroundings.

- **Heterogeneous System**

  A system which is not uniform throughout is said to be heterogeneous. e.g. A system consisting of two or more immiscible liquids.

- **Homogeneous System**

  A system is said to be homogeneous if it is completely uniform throughout. e.g. Pure solid or liquid.

- **Isolated System**

  A system in which there can be no exchange of matter and energy with the surroundings is said to be an isolated system.

- **Universe**

  The system and its surroundings are together known as the universe.

- **Thermodynamic variables of the system**

  (i) Composition (μ) (ii) Temperature (T) (iii) Volume (V) (iv) Pressure (P)

- **Thermodynamic state**

  The state of a system can be described completely by composition, temperature, volume and pressure.

  If a system is homogeneous and has definite mass and composition, then the state of the system can be described by the remaining three variables namely temperature, pressure and volume. These variables are interrelated by equation PV = μRT The thermodynamic state of the system is its condition as identified by two independent thermodynamic variables (P, V or P, T or V, T).

- **Zeroth law of thermodynamics**

  If objects A and B are separately in thermal equilibrium with a third object C (say thermometer), then objects A and B are in thermal equilibrium with each other. Zeroth law of thermodynamics introduce the concept of temperature. Two objects (or systems) are said to be in thermal equilibrium if their temperatures are the same.

  In measuring the temperature of a body, it is important that the thermometer be in the thermal equilibrium with the body whose temperature is to be measured.

- **Thermal equilibrium**

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Thermal equilibrium is a situation in which two objects in thermal contact cease to exchange energy by the process of heat. Heat is the transfer of energy from one object to another object as a result of a difference in temperature between them.

- **Internal Energy**

Internal energy of a system is the energy possessed by the system due to molecular motion and molecular configuration. The energy due to molecular motion is called internal kinetic energy (U_k) and that due to molecular configuration is called internal potential energy (U_p). $dU = dU_k + dU_p$

If there no intermolecular forces, then $dU_p = 0$ and $dU = dU_k = m\ c_v\ dT$

c_v = Specific heat at constant volume and dT = Infinitesimal change in temperature

m = Mass of system M = Molecular weight

Molar heat capacity $C_v = Mc_v$ For μ–moles of ideal gas $dU = \mu C_v dT = \frac{m}{M} C_v dT$

Internal energy in the absence of inter–molecular forces is simply the function of temperature and state only, it is independent of path followed. $\Delta U = U_f - U_i$

U_i = Internal energies in initial state and U_f = Internal energies in final state

- **Thermodynamic Processes**

In the thermodynamic process pressure, volume, temperature and entropy of the system change with time.

Thermodynamic process is said to take place if change occurs in the state of a thermodynamic system.

- **Sign convention used for the study of thermodynamic processes**

Heat gained by a system	Positive
Heat lost by a system	Negative
The work done by a system	Positive
Work done on the system	Negative
Increase in the internal energy of system	Positive
Decrease in the internal energy of system	Negative

- **Indicator Diagram or P–V Diagram**

In the equation of state of a gas $PV = \mu RT$

Two thermodynamic variables are sufficient to describe the behavior of a thermodynamic system.

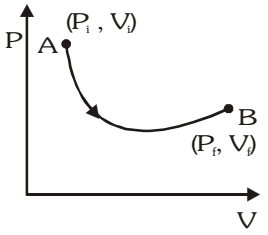
If any two of the three variables P, V and T are known then the third can be calculated.

P–V diagram is a graph between the volume V and the pressure P of the system.

The volume is plotted against X–axis while the pressure is plotted against Y–axis.

The point A represents the initial stage of the system. Initial pressure of the system is P_i and initial volume of the system V_i .

The point B represents the final state of the system. P_f and V_f are the final pressure and final volume respectively of the system. The points between A and B represent the intermediate states of the system. With the help of the indicator diagram we calculate the amount of work done by the gas or on the gas during expansion or compression.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory.P65

- **Cyclic process**

  Cyclic process is that thermodynamic process in which the system returns to its initial stage after undergoing a series of changes.

- **Non–cyclic process**

  Non–cyclic process is that process in which the system does not return to its initial stage.

- **Quasi–static or equilibrium process**

  Quasi–static is a thermodynamic process which proceeds extremely slowly such that at every instant of time, the temperature and pressure are the same in all parts of the system.

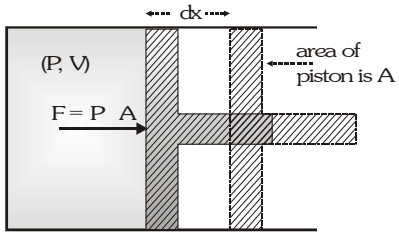
- **Reversible and Irreversible processes**

  A reversible process is one in which the changes in heat and work of direct process from initial to a final state are exactly retraced in opposite sense in the reverse process and the system and surroundings are left in their initial states. The reversibility is an ideal concept and can not be realized in practice.

  The process which is not reversible is the irreversible process. In nature the processes are irreversible.

WORK DONE BY THERMODYNAMIC SYSTEM

One of the simple example of a thermodynamic system is a gas in a cylinder with a movable piston.



- **If the gas expands against the piston**

  Gas exerts a force on the piston and displace it through a distance and does work on the piston.

- **If the piston compresses the gas**

  When piston moved inward, work is done on the gas.

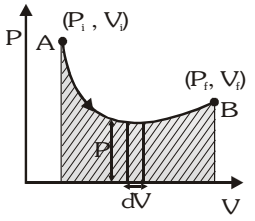
- **The work associated with volume changes**

  If pressure of gas on the piston = P.

  Then the force on the piston due to gas is F = PA

  When the piston is pushed outward an infinitesimal distance dx, the work done by the gas is dW = F dx = PA dx

  The change in volume of the gas is dV = Adx, $\therefore$ dW = PdV



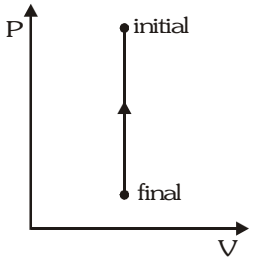
For a finite change in volume from V_i to V_f, this equation is then integrated between V_i to V_f to find the

$$\text{net work done } W = \int_{V_i}^{V_f} dW = \int_{V_i}^{V_f} PdV$$

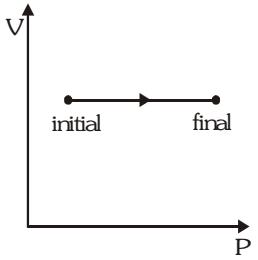
Hence the work done by a gas is equal to the area under P–V graph.

Following different cases are possible.

(i) Volume is constant



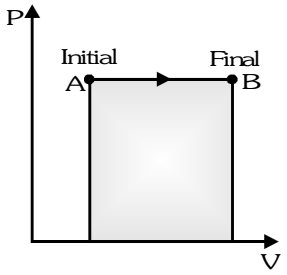
or



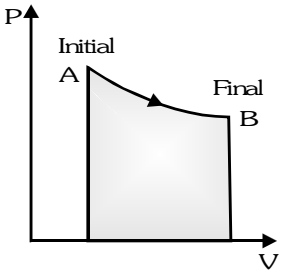
V = constant and $W_{AB} = 0$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

(ii) Volume is increasing

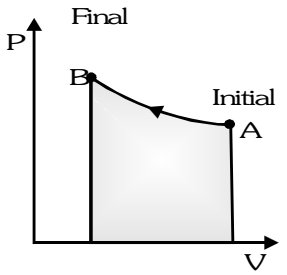


or

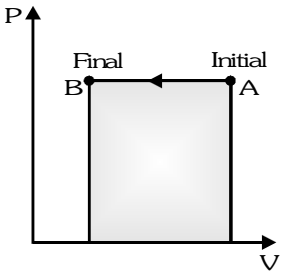


V is increasing $W_{AB} > 0$ W_{AB} = Shaded area

(iii) Volume is decreasing

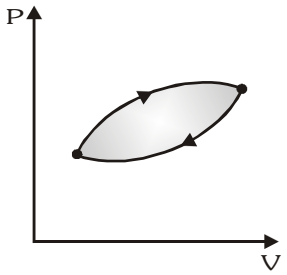


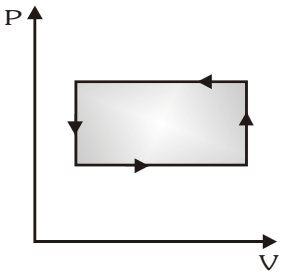
or



V is decreasing $W_{AB} < 0$ W_{AB} = – Shaded area

(iv) Cyclic process

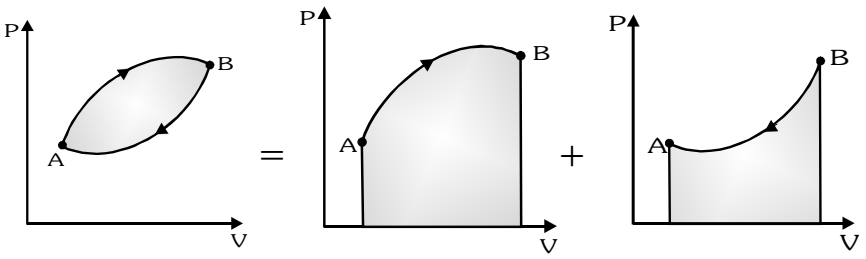




$W_{clockwise\ cycle}$ = + Shaded area

$W_{anticlockwise\ cycle}$ = – Shaded area

WORK DONE IN CLOCKWISE CYCLE



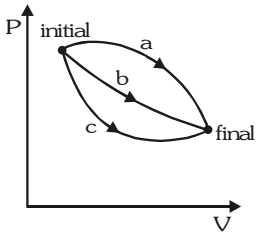
$W_{AB\ I}$ Positive $W_{AB\ II}$ Negative

$W_{cyclic} = W_{AB\ I}$ (positive) + $W_{AB\ II}$ (negative) = area of closed path

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

FIRST LAW OF THERMODYNAMICS

Let a gas in a cylinder with a moveable piston changes from an initial equilibrium state to a final equilibrium state.



System change its state through path **'a'** :

The heat absorbed by the system in this process = δQ_a

The work done by the system = δW_a

Again for path 'b' :

Heat absorbed by the system = δQ_b , Work done by the system = δW_b.

It is experimental fact that the $\delta Q_a - \delta W_a = \delta Q_b - \delta W_b$

Both δQ and δW depend on the thermodynamic path taken between two equilibrium states, but difference ($\delta Q - \delta W$) does not depends on path in between two definite states of the system.

So, there is a function (internal energy) of the thermodynamic coordinates (P, V and T) whose final value (U_f) minus its initial value (U_i) equals the change $\delta Q - \delta W$ in the process.

$dU = \delta Q - \delta W$. This is the first law of thermodynamics.

Heat supplied to the system and work done by the system are path dependent so they are denoted by δQ and δW respectively. Change in internal energy $\Delta U = U_f - U_i$ does not depends on path it depends only on initial and final positions of the system. So, it is denoted by dU (or ΔU)

First Law of Thermodynamics

If some quantity of heat is supplied to a system capable of doing external work, then the quantity of heat absorbed by the system is equal to the sum of the increase in the internal energy of the system and the external work done by the system. $\delta Q = dU + \delta W$ or $Q = W + \Delta U$

* This law is applicable to every process in nature

* The first law of thermodynamics introduces the concept of internal energy.

* The first law of thermodynamics is based on the law of conservation of energy.

* δQ, dU and δW must be expressed in the same units (either in units of work or in units of heat).

* This law is applicable to all the three phases of matter, i.e., solid, liquid and gas.

* dU is a characteristic of the state of a system, it may be any type of internal energy–translational kinetic energy, rotational kinetic energy, binding energy etc.

Example

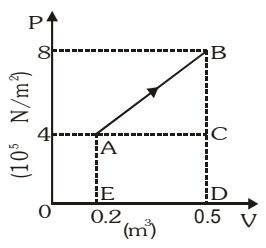
The pressure in monoatomic gas increases linearly from 4 10^5 Nm^{-2} to 8 10^{+5} Nm^{-2} when its volume increases from $0.2m^3$ to 0.5 m^3. Calculate.

(i) Work done by the gas, (ii) Increase in the internal energy,

(iii) Amount of heat supplied, (iv) Molar heat capacity of the gas R = 8.31 J mol^{-1} K^{-1}

Solution

P_1 = 4 10^5 Nm^{-2} P_2 = 8 10^{+5} Nm^{-2}, V_1 = 0.2 m^3, V_2 = 0.5 m^3



(i) Work done by the gas = Area under P–V graph (Area ABCDEA)

$= \frac{1}{2}$ (AE +BD) AC $= \frac{1}{2}$ (4 10^5 + 8 10^5) (0.5–0.2)

$= \frac{1}{2}$ 12 10^5 0.3 = 1.8 10^5 J

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

(ii) Increase in internal energy $\Delta U = C_V (T_2 - T_1) = \frac{C_V}{R} R(T_2 - T_1) = \frac{C_V}{R} (P_2V_2 - P_1V_1)$

For monoatomic gas

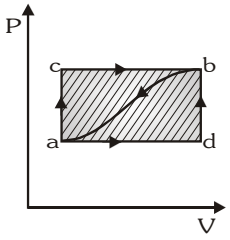
$C_V = \frac{3}{2}R \quad \therefore \Delta U = \frac{3}{2}$ [(8 10^5 0.5) – (4 10^5 0.2)] = $\frac{3}{2}$ [4 10^5 – 0.8 10^5] = 4.8 10^5 J

(iii) $Q = \Delta U + W = 4.8 \ 10^5 + 1.8 \ 10^5 = 6.6 \ 10^5$ J

(iv) $C = \frac{Q}{\eta \Delta T} = \frac{QR}{\eta R \Delta T} = \frac{QR}{\eta(P_2V_2 - P_1V_1)} = \frac{6.6 \times 10^5 \times 8.31}{1 \times 3.2 \times 10^5} = 17.14$ J/mole K

Example

When a system is taken from state a to state b, in figure along the path a → c → b, 60 J of heat flow into the system, and 30 J of work is done :



(i) How much heat flows into the system along the path a → d → b if the work is 10 J.

(ii) When the system is returned from b to a along the curved path, the work done by the system is –20 J. Does the system absorb or liberate heat, and how much?

(iii) If, $U_a = 0$ and $U_d = 22$ J, find the heat absorbed in the process a → d and d → b.

Solution

For the path a, c, b, $\Delta U = Q - W = 60 - 30 = 30$ J or $U_b - U_a = 30$ J

(i) Along the path a, d, b, $Q = \Delta U + W = 30 + 10 = 40$ J

(ii) Along the curved path b, a, $Q = (U_a - U_b) + W = (-30) + (-20) = -50$ J, heat flows out the system

(iii) $Q_{ad} = 32$ J; $Q_{db} = 8$ J

APPLICATION OF FIRST LAW OF THERMODYNAMICS

- **Melting Process :**

When a substance melts, the change in volume (dV) is very small and can, therefore, be neglected. The temperature of a substance remains unchanged during melting process.

Let us consider the melting of a mass m of the solid. Let L be the latent heat of fusion i.e., the heat required L to change a unit mass of a solid to liquid phase at constant temperature.

Heat absorbed during melting process, $\delta Q = mL$

By the first law of thermodynamics , $\delta Q = \Delta U + \delta W \Rightarrow mL = \Delta U$ [$\because \delta W = P\delta V = P \ 0 = 0$]

So, the internal energy increases by mL during the melting process.

- **Boiling Process :**

When a liquid is heated, it changes into vapour at constant temperature (called boiling point) and pressure. When water is heated at normal atmospheric pressure, it boils at 100 C. The temperature remains unchanged during the boiling process.

Let us consider the vaporisation of liquid of mass m. Let V_ℓ and V_v be the volumes of the liquid and vapours respectively.

The work done in expanding at constant temperature and pressure P, $\delta W = P\Delta V = P(V_v - V_\ell)$

Let the latent heat of vaporisation = L $\therefore$ Heat absorbed during boiling process, $\delta Q = mL$

Let U_ℓ and U_v be the internal energies of the liquid and vapours respectively then $\Delta U = U_v - U_\ell$

According to the first law of thermodynamics, $\delta Q = \Delta U + \delta W \ \therefore \ mL = (U_v - U_\ell) + P(V_v - V_\ell)$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

1 kg of water at 373 K is converted into steam at same temperature. Volume of 1 cm^3 of water becomes 1671 cm^3 on boiling. What is the change in the internal energy of the system if the latent heat of vaporisation of water is 5.4 10^5 cal kg^{-1}?

Solution

Volume of 1 kg of water = 1000 cm = 10^{-3} m , Volume of 1 kg of steam = 10 1671 cm = 1.671 m

Change in volume, $\Delta V = (1.671 - 10^{-3})$ m = 1.670 m , Pressure, P = 1 atm. = 1.01 10^5 N m^{-2}

In expansion work done, W = $P\Delta V$ = 1.01 10^5 1.67 J = $\frac{1.691 \times 10^5}{4.2}$ cal = 4.026 10^4 cal

But $\Delta U = Q - W$ (first law of thermodynamics) or ΔU = (5.4 10^5 – 0.4026 10^5) cal = 4.9974 10^5 cal

ISOMETRIC OR ISOCHORIC PROCESS

- Isochoric process is a thermodynamic process that takes place at constant volume of the system, but pressure and temperature varies for change in state of the system.

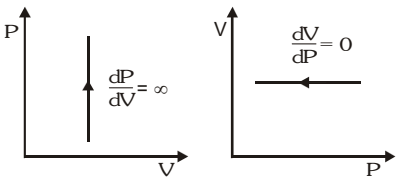
Equation of state P = constant T (P and T are variable, V is constant)

Work done In this process volume remains constant $\Delta V = 0$ or $dV = 0 \Rightarrow W = \int_{V_i}^{V_f} PdV = 0$

Form of first Law $Q = \Delta U$

It means whole of the heat supplied is utilized for change in internal energy of the system. $Q = \Delta U = \mu C_v \Delta T$

Slope of the P–V curve $\frac{dP}{dV} = \infty$



Specific heat at constant volume (C_V)

The quantity of heat required to raise the temperature of 1 gram mole gas through 1 C at constant volume is equal to the specific heat at constant volume.

- A gas enclosed in a cylinder having rigid walls and a fixed piston. When heat is added to the gas, there would be no change in the volume of the gas.
- When a substance melts, the change in volume is negligibly small. So, this may be regarded as a nearly isochoric process.
- Heating process in pressure cooker is an example of isometric process.

Example

An ideal gas has a specific heat at constant pressure $C_P = \frac{5R}{2}$. The gas is kept in a closed vessel of volume 0.0083 m^3 at a temperature of 300K and a pressure of 1.6 10^6 Nm^{-2}. An amount of 2.49 10^4 J of heat energy is supplied to the gas. Calculate the final temperature and pressure of the gas.

Solution

$C_V = C_P - R = \frac{5R}{2} - R = \frac{3R}{2}$, $\Delta V = 0$, T_1 = 300 K, V = 0.0083 m^3, P_1 = 1.6 10^6 Nm^{-2}

From first law of thermodynamics $Q = \Delta U + P\Delta V \Rightarrow \Delta U = Q$ = 2.49 10^4 J

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

From gas equation $n = \frac{PV}{RT} = \frac{1.6\times10^6\times0.0083}{8.3\times300} = \frac{16}{3}$

$\therefore \Delta U = nC_V\Delta T \Rightarrow \Delta T = \frac{\Delta U}{nC_v} = \frac{2.49\times10^4\times6}{3\times8.3\times16} = 375$ K

Final temperature = 300 + 375 = 675K

According to pressure law $P \propto T \Rightarrow \frac{P_2}{P_1} = \frac{T_2}{T_1} \Rightarrow P_2 = \frac{T_2}{T_1}\times P_1 = \frac{1.6\times10^6\times675}{300} = 3.6 \quad 10^6$ Nm^{-2}

Example

5 moles of oxygen is heated at constant volume from 10 C to 20 C. What will be change in the internal energy of the gas? The gram molecular specific heat of oxygen at constant pressure.

C_P = 8 cal/mole and R = 8.36 joule/mole C.

Solution

$\because C_V = C_P - R = 8 - 2 = 6$ cal/mole C

$\therefore$ Heat absorbed by 5 moles of oxygen at constant volume

$Q = nC_V \Delta T = 5 \quad 6\ (20 - 10) = 30 \quad 10 = 300$ cal

At constant volume $\Delta V = 0$. $\therefore \Delta W = 0$

$\therefore$ From first law of thermodynamics $Q = \Delta U + W \Rightarrow 300 = \Delta U + 0 \Rightarrow \Delta U = 300$ cal

ISOBARIC PROCESS

Isobaric process is a thermodynamic process that takes place at constant pressure, but volume and temperature varies for change in state of the system.

- **Equation of state** V = constant T or $V \propto T$
- **Work done** In this process pressure remains constant $\Delta P = 0$

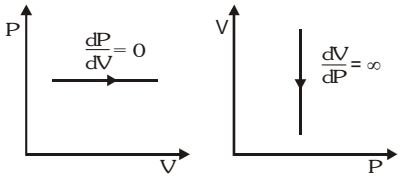
  Work done $W = \int_{V_i}^{V_f} PdV = P(V_f - V_i)$
- **Form of first Law** $Q = \Delta U + P\ (V_f - V_i)$

  $\mu\ C_p\ dT = \mu\ C_v\ dT + P(V_f - V_i)$

It is clear that heat supplied to the system is utilized for :

(i) Increasing internal energy and

(ii) Work done against the surrounding atmosphere.



- **Slope of the PV curve :** $\left(\frac{dP}{dV}\right)_{isobaric} = 0$
- **Specific heat at constant pressure (C_p)**

The quantity of heat required to raise the temperature of 1 gram mole gas through 1 C at constant pressure is equal to the specific heat. Heating of water at atmospheric pressure. • Melting of solids and boiling of liquids at atmospheric pressure.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example

At normal pressure and 0 C temperature the volume of 1 kg of ice is reduced by 91 cm^3 on melting. Latent heat of melting of ice is 3.4 10^5 J/kg. Calculate the change in the internal energy when 2kg of ice melts at normal pressure and 0 C. (P=1.01 10^5 Nm^{-2})

Solution

Heat energy absorbed by 2 kg of ice for melting Q = mL = 2 3.4 10^5 = 6.8 10^5J

Change in volume of 2 kg of ice = 2 91 = 182 cm^3 = 182 10^{-6} m^3

$\therefore$ W = PΔV = 1.01 10^5 (–182 10^{-6}) = –18.4 J

Since, work is done on ice so work W is taken –ve. Now from first law of thermodynamics

$Q = \Delta U + W \quad \Rightarrow \Delta U = Q - W = 6.8 \times 10^5 - (-18.4) = (6.8 \times 10^5 + 18.4)J$

Example

What amount of heat must be supplied to 2.0 10^{-2} kg of nitrogen (at room temperature) to raise the temperature by 45 C at constant pressure. Molecular mass of N_2 = 28, R = 8.3 J mol^{-1} K^{-1}.

Solution

Here m = 2 10^{-2} kg, $\Rightarrow n = \frac{m}{M} = \frac{2\times10^{-2}}{28\times10^{-3}} = \frac{5}{7}$ & $C_P = \frac{7}{2}R$ $\therefore Q = nC_P\Delta T = \frac{5}{7}\times\frac{7}{2}\times8.3\times45 = 933.75$ J

ISOTHERMAL PROCESS

In this process pressure and volume of system change but temperature remains constant.

In an isothermal process, the exchange of heat between the system and the surroundings is allowed.

Isothermal process is carried out by either supplying heat to the substance or by extracting heat from it.

A process has to be extremely slow to be isothermal.

- **Equation of state**

  P V = constant (μRT) (T is constant)

- **Work Done**

Consider μ moles of an ideal gas, enclosed in a cylinder, at absolute temperature T, fitted with a frictionless piston. Suppose that gas undergoes an isothermal expansion from the initial state (P_1, V_1) to the final state (P_2, V_2).

$\therefore$ Work done : $W = \int_{V_1}^{V_2}\frac{\mu RT}{V}dV = \mu RT\int_{V_1}^{V_2}\frac{dV}{V} = \mu RT\left[\log_e V\right]_{V_1}^{V_2} = \mu RT\ [\log_e V_2 - \log_e V_1] = \mu RT\log_e\left[\frac{V_2}{V_1}\right]$

$\Rightarrow$ $W = 2.303\mu RT\ \log_{10}\left[\frac{P_1}{P_2}\right]$ [$\because$ $P_1V_1 = P_2V_2$]

Form of First Law

There is no change in temperature and internal energy of the system depends on temperature only

So $\Delta U = 0,\ Q = 2.303\ \mu RT\ \log_{10}\left[\frac{V_2}{V_1}\right]$

It is clear that Whole of the heat energy supplied to the system is utilized by the system in doing external work. There is no change in the internal energy of the system.

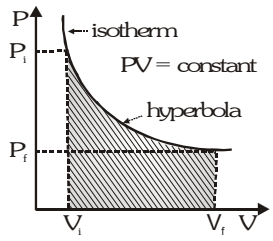
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Slope of the isothermal curve

For an isothermal process, PV = constant

Differentiating, $PdV + VdP = 0 \quad \Rightarrow VdP = -PdV \Rightarrow \quad \frac{dP}{dV} = -\frac{P}{V}$

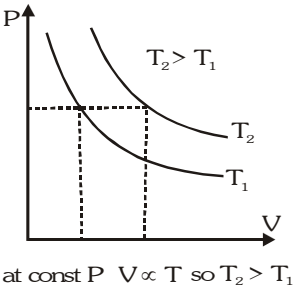
Slope of isothermal curve, $\left[\frac{dP}{dV}\right]_{isothermal} = -\frac{P}{V}$



For a given system :

- The product of the pressure and volume of a given mass of a perfect gas remains constant in an isothermal process.
- Boyle's law is obeyed in an isothermal process.
- A graph between pressure and volume of a given mass of a gas at constant temperature is known as isotherm or isothermal of the gas.
- Two isotherms for a given gas at two different temperatures

  T_1 and T_2 are shown in figure.



at const P $V \propto T$ so $T_2 > T_1$

- The curves drawn for the same gas at different temperatures are mutually parallel and do not cut each other.
- If two isotherms intersect each other at a single point we get same value of P and V at intersection point. $PV = \mu RT_1$ for temperature T_1 and $PV = \mu RT_2$ for temperature T_2.

  It means $T_1 = T_2$ which is not possible.
- An ideal gas enclosed in a conducting cylinder fitted with a conducting piston. Let the gas be allowed to expand very–very slowly.
- This shall cause a very slow cooling of the gas, but heat will be conducted into the cylinder from the surrounding. Hence the temperature of the gas remains constant. If the gas is compressed very–very slowly, heat will be produced, but this heat will be conducted to the surroundings and the temperature of the gas shall remain constant.
- The temperature of a substance remains constant during melting. So, the melting process is an isothermal process.
- Boiling is an isothermal process, when a liquid boils, its temperature remains constant.
- If sudden changes are executed in a vessel of infinite conductivity then they will be isothermal.

Example

Two moles of a gas at 127 C expand isothermally until its volume is doubled. Calculate the amount of work done.

Solution

n = 2, $\quad T = 127 + 273 = 400K, \quad \frac{V_2}{V_1} = 2$

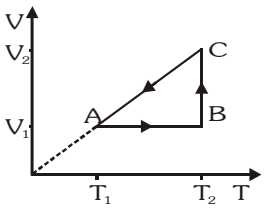
From formula $\quad W = 2.3026\ nRT \log_{10} \frac{V_2}{V_1} = 2.3026 \quad 2 \quad 8.3 \quad 400 \quad \log_{10} 2$

$= 2.3026 \quad 2 \quad 8.3 \quad 400 \quad 0.3010 \approx 4.6 \quad 10^3 J$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy \UnitNo-5\Thermal physics\Eng\Theory P65

Example

Figure shows a process ABCA performed on an ideal gas.



Find the net heat given to the system during the process.

Solution

Since the process is cyclic, hence the change in internal energy is zero.

The heat given to the system is then equal to the work done by it.

The work done in part AB is $W_1 = 0$ (the volume remains constant). The part BC represents an isothermal process so that the work done by the gas during this part is $W_2 = nRT_2 \ell n \frac{V_2}{V_1}$

During the part CA $V \propto T$ So, V/T is constant and hence, $P = \frac{nRT}{V}$ is constant

The work done by the gas during the part CA is $W_3 = P(V_1 - V_2) = nRT_1 - nRT_2 = - nR (T_2 - T_1)$.

The net work done by the gas in the process ABCA is $W = W_1 + W_2 + W_3 = nR\left[T_2 \ell n \frac{V_2}{V_1} - (T_2 - T_1)\right]$

The same amount of heat is given to the gas.

ADIABATIC PROCESS

It is that thermodynamic process in which pressure, volume and temperature of the system do change but there is no exchange of heat between the system and the surroundings.

A sudden and quick process will be adiabatic since there is no sufficient time available for exchange of heat so process adiabatic.

Equation of state : $PV = \mu RT$

Equation for adiabatic process $PV^{\gamma} = \text{constant}$

Work done

Let initial state of system is (P_1, V_1, T_1) and after adiabatic change final state of system is (P_2, V_2, T_2) then we can write $P_1 V_1^{\gamma} = P_2 V_2^{\gamma} = K$ (here K is const.)

So $$W = \int_{V_1}^{V_2} PdV = K\int_{V_1}^{V_2} V^{-\gamma}dV = K\left(\frac{V^{-\gamma+1}}{-\gamma+1}\right)_{V_1}^{V_2} = \frac{K}{(-\gamma+1)}\left[V_2^{-\gamma+1} - V_1^{-\gamma+1}\right] \quad \left(\because K = P_1V_1^{\gamma} = P_2V_2^{\gamma}\right)$$

$$\Rightarrow W = \frac{1}{(\gamma-1)}\left[P_1V_1^{\gamma}V_1^{-\gamma}.V_1 - P_2V_2^{\gamma}V_2^{-\gamma}.V_2\right] = \frac{1}{(\gamma-1)}\ [P_1V_1 - P_2V_2]$$

$$\Rightarrow W = \frac{\mu R}{(\gamma-1)}(T_1 - T_2) \quad (\because PV = \mu RT)$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P6.5

Form of first law : $dU = -\delta W$

It means the work done by an ideal gas during adiabatic expansion (or compression) is on the cost of change in internal energy proportional to the fall (or rise) in the temperature of the gas.

If the gas expands adiabatically, work is done by the gas. So, W_{adia} is positive.

The gas cools during adiabatic expansion and $T_1 > T_2$.

If the gas is compressed adiabatically, work is done on the gas. So, W_{adia} is negative.

The gas heats up during adiabatic compression and $T_1 < T_2$.

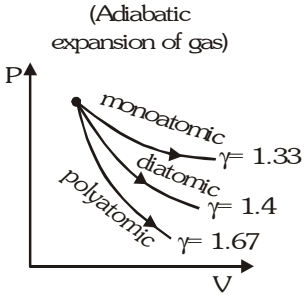
Slope of the adiabatic curve

For an adiabatic process, $PV^{\gamma} = \text{constant}$

Differentiating, $P^{\gamma} V^{\gamma - 1}\, dV + V^{\gamma}\, dP = 0$

$$\Rightarrow V^{\gamma}\, dP = -\gamma\, PV^{\gamma - 1}\, dV \Rightarrow \frac{dP}{dV} = -\frac{\gamma PV^{\gamma-1}}{V^{\gamma}} = -\gamma\frac{P}{V} = \gamma\left(-\frac{P}{V}\right)$$

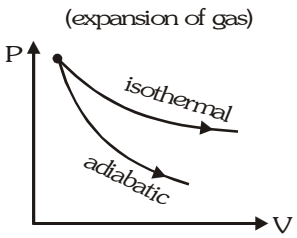
Slope of adiabatic curve, $\left[\frac{dP}{dV}\right]_{adiabatic} = -\frac{\gamma P}{V}$



Slope of adiabatic is greater than the slope of isotherm

$$\left[\frac{dP}{dV}\right]_{adia} = \gamma\left[-\frac{P}{V}\right] = \gamma\left[\frac{dP}{dV}\right]_{iso} \Rightarrow \frac{\text{slope of adiabatic changes}}{\text{slope of isothermal changes}} = \gamma$$

Since γ is always greater than one so an adiabatic is steeper than an isotherm



Examples of adiabatic process

- A gas enclosed in a thermally insulated cylinder fitted with a non–conducting piston. If the gas is compressed suddenly by moving the piston downwards, some heat is produced. This heat cannot escape the cylinder. Consequently, there will be an increase in the temperature of the gas.
- If a gas is suddenly expanded by moving the piston outwards, there will be a decrease in the temperature of the gas.
- Bursting of a cycle tube.
- Propagation of sound waves in a gas.
- In diesel engines burning of diesel without spark plug is done due to adiabatic compression of diesel vapour and air mixture

Example

Why it is cooler at the top of a mountain than at sea level?

Solution

Pressure decreases with height. Therefore if hot air rises, it suffers adiabatic expansion.

From first law of thermodynamics $\Delta Q = \Delta U + \Delta W \Rightarrow \Delta U = -\Delta W$ [$\because \Delta Q = 0$]

This causes a decrease in internal energy and hence a fall of temperature.

Example

$2m^3$ volume of a gas at a pressure of 4 10^5 Nm^{-2} is compressed adiabatically so that its volume becomes $0.5m^3$. Find the new pressure. Compare this with the pressure that would result if the compression was isothermal. Calculate work done in each process. $\gamma = 1.4$

Solution

$V_1 = 2m^3$, $P_1 = 4$ 10^5 Nm^{-2}, $V_2 = 0.5m^3$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

In adiabatic process $P_1V_1^{\gamma} = P_2V_2^{\gamma} \Rightarrow P_2 = 4 \quad 10^5\left[\frac{2}{0.5}\right]^{1.4} = 4 \quad 10^5\ (4)^{1.4} = 2.8 \quad 10^6\ Nm^{-2}$

In isothermal process $P_1V_1 = P_2V_2 \Rightarrow P_2 = \frac{P_1V_1}{V_2} = \frac{4\times10^5\times2}{0.5} = 1.6 \quad 10^6\ Nm^{-2}.$

Now work done in adiabatic process $W = \frac{P_2V_2 - P_1V_1}{\gamma - 1} = \frac{(2.8\times10^6\times0.5) - (4\times10^5\times2)}{1.4-1} = 1.48 \quad 10^6$ J.

Work done in isothermal process $W = 2.3026RT \log\frac{V_2}{V_1} = 2.3026P_1V_1 \log\frac{V_2}{V_1}$

$= 2.3026 \quad 4 \quad 10^5 \quad 2 \quad \log\left[\frac{0.5}{2.0}\right] = 2.3026 \quad 4 \quad 10^5 \quad 2\log\left(\frac{1}{4}\right) = -1.1 \quad 10^6$ J

Example

Two samples of a gas initially at same temperature and pressure are compressed from a volume V to $\frac{V}{2}$. One sample is compressed isothermally and the other adiabatically. In which sample is the pressure greater?

Solution

Let initial volume, $V_1 = V$ and pressure, $P_1 = P$, final volume, $V_2 = \frac{V}{2}$ and final pressure, $P_2 = ?$

For isothermal compression $P_2V_2 = P_1V_1$ or $P_2 = \frac{P_1V_1}{V_2} = \frac{PV}{\frac{V}{2}} = 2P$

For adiabatic compression $P_2' = P_1\left[\frac{V_1}{V_2}\right]^{\gamma} \Rightarrow P_2' = P\left[\frac{V}{V/2}\right]^{\gamma} = 2^{\gamma}P \Rightarrow P_2' = 2^{\gamma}P \quad \gamma > 1 \ \therefore 2^{\gamma} > 2$ and $P_2' > P_2$

Pressure during adiabatic compression is greater than the pressure during isothermal compression.

Example

Calculate the work done when 1 mole of a perfect gas is compressed adiabatically. The initial pressure and volume of the gas are 10^5 N/m^2 and 6 litre respectively. The final volume of the gas is 2 liters. Molar specific heat of the gas at constant volume is $\frac{3R}{2}$ [$(3)^{5/3} = 6.19$]

Solution

For an adiabatic change PV^{γ} = constant $P_1V_1^{\gamma} = P_2V_2^{\gamma}$

As molar specific heat of gas at constant volume $C_v = \frac{3}{2}R$

$C_p = C_V + R = \frac{3}{2}R + R = \frac{5}{2}R \Rightarrow \gamma = \frac{C_P}{C_V} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3}$

$\therefore P_2 = \left[\frac{V_1}{V_2}\right]^{\gamma} P_1 = \left[\frac{6}{2}\right]^{5/3} \times10^5 = (3)^{5/3} \quad 10^5 = 6.19 \quad 10^5\ N/m^2$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P6.5

Work done $W = \frac{1}{1-\gamma}[P_2V_2 - P_1V_1] = \frac{1}{1-(5/3)}[6.19\times10^5\times2\times10^{-3} - 10^{-5}\times6\times10^{-3}]$

$= -\left[\frac{2\times10^2\times3}{2}(6.19-3)\right] = -\ 3\quad 10^2\quad 3.19 = -\ 957$ joules

– ive sign shows external work done on the gas

Example

A motor tyre pumped to a pressure of 3 atm. It suddenly bursts. Calculate the fall in temperature due to adiabatic expansion. The temperature of air before expansion is 27 C. Given γ=1.4.

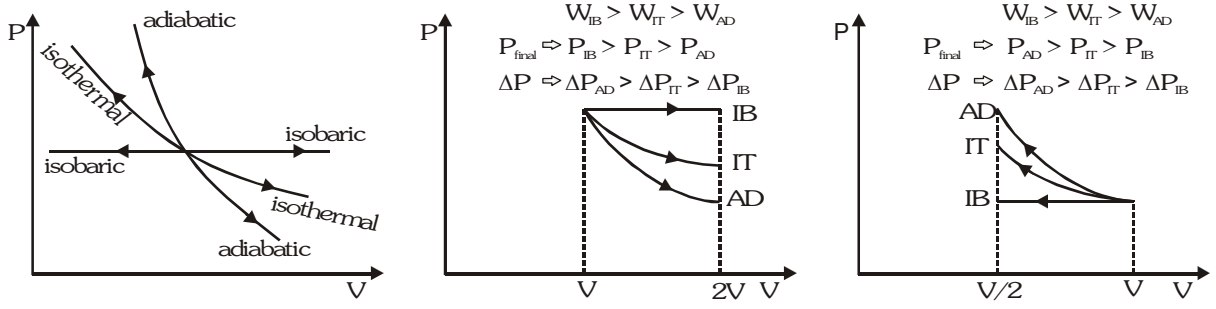
Solution

We know that $T_2^{\gamma}P_2^{1-\gamma} = T_1^{\gamma}P_1^{1-\gamma} \Rightarrow \left[\frac{T_2}{T_1}\right]^{\gamma} = \left[\frac{P_1}{P_2}\right]^{1-\gamma} \Rightarrow \left[\frac{T_2}{300}\right]^{1.4} = \left[\frac{3}{1}\right]^{1-1.4}$

$\Rightarrow \left[\frac{T_2}{300}\right]^{1.4} = \left[\frac{1}{3}\right]^{0.4} \Rightarrow T_2 = 219.2\ K \Rightarrow T_1 - T_2 = (300 - 219.2)\ K = 80.8\ K$

GOLDEN KEY POINT

- When a gas expands its volume increases, then final pressure is less for adiabatic expansion. But, when a gas compresses its volume decreases, then the final pressure is more in case of adiabatic compression.



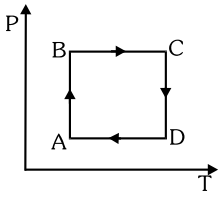
First Law of Thermodynamics Applied to Different Processes

Process	Q	ΔU	W
Cyclic	W	0	Area of the closed curve
Isochoric	ΔU	$\mu C_v\Delta T$ (μ mole of gas)	0
Isothermal	W	0	$\mu RT\log_e\left[\frac{V_f}{V_i}\right] = \mu RT\log_e\left[\frac{P_i}{P_f}\right]$
Adiabatic	0	–W	$\frac{\mu R(T_f - T_i)}{1-\gamma}$
Isobaric	$\mu C_P\Delta T$	$\mu C_V\Delta T$	$P(V_f - V_i) = \mu R(T_f - T_i)$

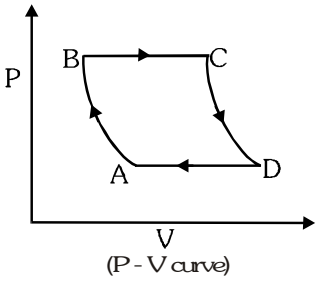
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory\P65

Example

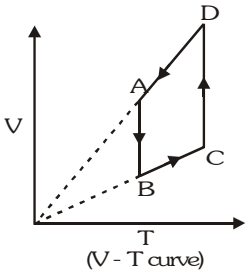
Plot P – V , V – T graph corresponding to the P–T graph for an ideal gas shown in figure. Explain your answers.



Solution



(P - V curve)



(V - T curve)

For process AB T = constant so $P \propto \frac{1}{V}$

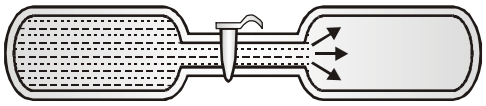
For process BC P = constant so $V \propto T$

For process CD T = constant so $V \propto \frac{1}{P}$

For process DA P = constant so $V \propto T$

FREE EXPANSION

Take a thermally insulated bottle with ideal gas at some temperature T_1 and, by means of a pipe with a stopcock, connect this to another insulated bottle which is evacuated. If we suddenly open the stopcock, the gas will rush from the first bottle into the second until the pressures are equalized.



free expansion of a gas

Experimentally, we find that this process of free expansion does not change the temperature of the gas – when the gas attains equilibrium and stops flowing, the final temperature of both botles are equal to the initial temperature T_1. This process is called a free expansion.

The change in the internal energy of the gas can be calculated by applying the first law of thermodynamics to the free–expansion process.

The process is adiabatic because of the insulation, So Q = 0.

No part of the surroundings moves so the system does no work on its surroundings.

- **For ideal gas** $(\delta W)_{ext.}$ = Work done against external atmosphere
  = P dV = 0 (because P = 0)
  $(\delta W)_{int.}$ = Work done against internal molucular forces = 0
  $\delta Q = dU + \delta W \Rightarrow 0 = dU + 0$

The internal energy does not change dU = 0 So U and T are constant.

The initial and final states of this gas have the same internal energy.

Which implies that the internal energy of an ideal gas does not depend on the volume at all.

The free–expansion process has led us to the following conclusion :

The internal energy U(T) of an ideal gas depends only on the temperature.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

- **For real gas**

In free expansion of real gases, measurements show that the temperature changes slightly in a free expansion. Which implies that the internal energy of a real gas depends on the volume also.

$\delta Q = 0$ $(\delta W)_{ext.} = 0$ $(\because P = 0)$

$(\delta W)_{int.} \neq 0$ (Intermolucular forces are present in real gases)

$\delta Q = dU + \delta W \Rightarrow 0 = dU + (\delta W)_{int.} \Rightarrow dU = -(\delta W)_{int.}$

$\Rightarrow$ U decreases. So T decreases.

RELATION BETWEEN DEGREE OF FREEDOM AND SPECIFIC HEAT OF GAS

Energy related with each degree of freedom $= \frac{1}{2}kT$, Energy related with all degree of freedom $= \frac{f}{2}kT$

Internal energy of one mole of ideal gas (total K.E.) $U = \frac{f}{2}RT$ for Isometric process (volume constant) $\delta W = 0$

By first law of thermodynamics $\delta Q = \delta W + dU \Rightarrow C_V\, dT = dU \Rightarrow C_V = \frac{dU}{dT}$

$C_V = \frac{dU}{dT} = \frac{f}{2}R = \frac{R}{\gamma - 1}$. $C_P = C_V + R = \left[\frac{f}{2} + 1\right]R = \frac{\gamma R}{\gamma - 1}$ and $\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{f}$

$C_V = \frac{R}{\gamma - 1}$, $C_P = \frac{\gamma R}{\gamma - 1}$ and $\gamma = 1 + \frac{2}{f}$

General expression for C (C_P or C_V) in the process PV^x = constant $C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$

For isobaric process P = constant so x = 0 $\therefore C = C_P = \frac{R}{\gamma - 1} + R = C_V + R$

For isothermal process, PV = constant so x = 1 $\therefore$ $C = \infty$

For adiabatic process PV^{γ} = constant so $x = \gamma$ $\therefore$ C = 0

Values of f, U, C_V, C_P and γ for different gases are shown in table below.

Atomicity of gas	f	C_V	C_P	γ
Monoatomic	3	$\frac{3}{2}R$	$\frac{5}{2}R$	$\frac{5}{3} = 1.67$
Diatomic	5	$\frac{5}{2}R$	$\frac{7}{2}R$	$\frac{7}{5} = 1.4$
Triatomic and Triatomic linear (at high temperature)	7	$\frac{7}{2}R$	$\frac{9}{2}R$	$\frac{9}{7} = 1.28$
Poly atomic Triangular Non-linear	6	$\frac{6}{2}R = 3R$	$\frac{8}{2}R = 4R$	$\frac{4}{3} = 1.33$

- $1 < \gamma < 2$
- If atomicity of gases is same U, C_P, C_V and γ is same for gas mixture.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

- If in a gas mixture gases are of different atomicity, then for gas mixture γ changes according to following condition. Diatomic $\gamma_1 \leq$ mixture $\leq \gamma_2$ mono atomic where $\gamma_1 < \gamma_2$
- If 'f' is the degree of freedom per molecule for a gas, then

  Total energy of each molecule $= \frac{fkT}{2}$

  Total energy per mole of gas $= N_0 \frac{f}{2} kT = \frac{f}{2} RT$
- According to kinetic theory of gases, the molecule are not interacting with each other. So potential energy is zero and internal energy of gas molecules is only their kinetic energy.
- For 'μ' mole of a gas : Internal energy at temperature T is $U = \frac{\mu f R T}{2} = \mu C_V T$
- Change in internal energy is given by $dU = \frac{\mu f R}{2}(dT) = \mu C_V dT$

  This change is process independent.

C_P is greater than C_V

If a gas is heated at constant volume, the gas does no work against external pressure. In this case, the whole of the heat energy supplied to the gas is spend in raising the temperature of the gas.

If a gas is heated at constant pressure, its volume increases. In this case, heat energy is required for the following two purpose :

(i) To increase the volume of the gas against external pressure.

(ii) To increase the temperature of 1 mole of gas through 1 K.

Thus, more heat energy is required to raise the temperature of 1 mole of gas through 1 K when it is heated at constant pressure than when it is heated at constant volume. $\therefore C_P > C_V$

The difference between C_P and C_V is equal to thermal equivalent of the work done by the gas in expanding against external pressure.

Mayor's formula : $C_P - C_V = R$

$\because$ At constant pressure $dQ = \mu C_P dT$, $dU = \mu C_V dT$ & $dW = PdV = \mu RdT$

Now from first law of thermodynamics $dQ = dW + dU$

$\Rightarrow \mu C_P dT = \mu RdT + \mu C_V dT \Rightarrow C_P = R + C_V \Rightarrow C_P - C_V = R$

Example

Calculate the difference between two specific heats of 1 g of helium gas at NTP. Molecular weight of helium = 4 and J = 4.186 10^7 erg cal^{-1}.

Solution

Gas constant for 1 g of helium, $r = \frac{R}{M_w} = \frac{PV}{T \times M_w} = \frac{76 \times 13.6 \times 981 \times 22400}{273 \times 4} = 2.08 \quad 10^7$ erg g^{-1} K^{-1}

$C_P - C_V = \frac{r}{J} = \frac{2.08 \times 10^7}{4.186 \times 10^7} = 0.5$ cal g^{-1} K^{-1}

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Example

Calculate the molar specific heat at constant volume. Given : specific heat of hydrogen at constant pressure is 6.85 cal mol^{-1} K^{-1} and density of hydrogen = 0.0899 g cm^{-3}. One mole of gas = 2.016 g, J = 4.2 10^7 erg cal^{-1} and 1 atmosphere = 10 ^{6} dyne cm^{-2}.

Solution

Since the density of hydrogen is 0.0899 g cm^{-3} therefore volume occupied by 0.0899 g of hydrogen at NTP is 1000 cm^3. So, volume of 1 mole (2.016 g) of gas, $V = \frac{1000}{0.0899} \times 2.016$ cm^3

$$C_P - C_V = \frac{R}{J} = \frac{PV}{T \times J} = \frac{10^6 \times 1000 \times 2.016}{0.0899 \times 273 \times 4.2 \times 10^7} = 1.96 \text{ cal mol}^{-1} \text{ K}^{-1}$$

$\therefore$ $C_V = C_P - 1.96 = (6.85 - 1.96) = 4.89$ cal mol^{-1} K^{-1}

Example

The specific heat of argon at constant volume is 0.075 kcal/kg K. Calculate its atomic weight, [R = 2cal/mol K]

Solution

As argon is monoatomic, its molar specific heat at constant volume will be

$$C_V = \frac{3}{2}R = \frac{3}{2} \times 2 = 3 \text{ cal/mol K}, \quad C_V = M_w \, c_V \text{ and } \quad c_V = 0.075 \text{ cal/g K}$$

So $3 = M_w \quad 0.075 \Rightarrow M_w = \frac{3}{0.075} = 40$ gram/mole

Effeciency of a cycle (η) :

$$\eta = \frac{\text{total Mechanical work done by the gas in the whole process}}{\text{Heat absorbed by the gas (only + ve)}}$$

$$= \frac{\text{area under the cycle in P-V curve}}{\text{Heat injected into the system}}$$

Example

n moles of a diatomic gas has undergone a cyclic process ABC as shwon in figure. Temperature at a is T_0. Find

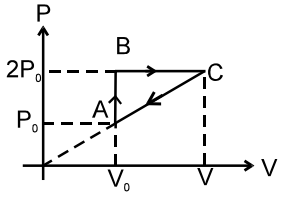
(i) Volume at C.

(ii) Maximum temprature.

(iii) Total heat given to gas.

(iv) Is heat rejected by the gas, if yes how much heat is rejected.

(v) Find out the efficiency.



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-5\Thermal physics\Eng\Theory P65

Solution

(i) Since triangle OAV_0 and OCV are simillar therefore $\frac{2P_0}{V} = \frac{P_0}{V_0} \Rightarrow V = 2V_0$

(ii) Since process AB is isochoric hence $\frac{P_A}{T_A} = \frac{P_B}{T_B} \Rightarrow T_B = 2T_0$

Since process BC is isobaric therefore $\frac{T_B}{V_B} = \frac{T_C}{V_C} \Rightarrow T_C = 2T_B = 4\ T_0$

(iii) Since process is cyclic therefore Q = W = area under the cycle = $\frac{1}{2} P_0V_0$.

(iv) Since U and W both are negative in process CA

$\therefore$ Q is negative in process CA and heat is rejected in process CA

$Q_{CA} = W_{CA} + U_{CA}$

$= -\frac{1}{2} [P_0 + 2P_0] V_0 - \frac{5}{2} nR (T_c - T_a)$

$= -\frac{1}{2} [P_0 + 2P_0] V_0 - \frac{5}{2} nR \left(\frac{4P_0V_0}{nR} - \frac{P_0V_0}{nR}\right)$

$= -9P_0V_0$ = Heat injected.

(v) η = efficiency of the cycle = $\frac{\text{work done by the gas}}{\text{heat injected}} = \frac{P_0V_0 / 2}{Q_{injected}}$ 100

where $Q_{inj} = Q_{AB} + Q_{BC} = \left[\frac{5}{2} nR(2T_0 - T_0)\right] + \left[\frac{5}{2} nR(2T_0) + 2P_0(2V_0 - V_0)\right] = \frac{19}{2} P_0V_0.$

Therefore $\eta = \frac{100}{19}\%$

CARNOT CYCLE

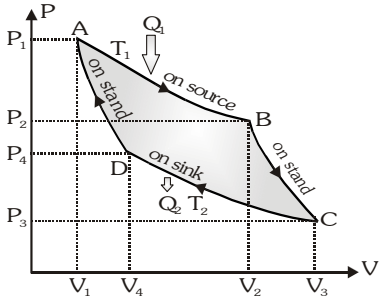
Carnot devised an ideal engine which is based on a reversible cycle of four operations in succession :

(i) Isothermal expansion, $A \rightarrow B$

(ii) Adiabatic expansion, $B \rightarrow C$

(iii) Isothermal compression $C \rightarrow D$

(iv) Adiabatic compression. $D \rightarrow A$



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

MAIN PARTS OF CARNOT'S ENGINE ARE

- **Source of heat** :

  It is a hot body of very large heat capacity kept at a constant high temperature T_1. Its upper surface is perfectly conducting so that working substance can take heat from it.

- **Mechanical arrangements and working substance** :

  It is a hollow cylinder whose walls are perfectly non–conducting and its base is perfectly conducting fitted with non–conducting piston. This piston move without any friction. Ideal gas enclosed in cylinder as a working substance.

- **Heat sink** :

  It is a cold body at low temperature T_2. It is a body of large heat capacity and its upper surface is highly conducting so that working substance can reject heat to it.

- **Stand** :

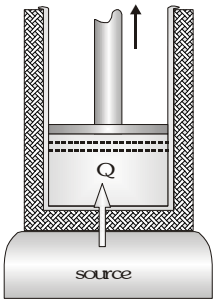
  It is made by perfectly insulating material. When cylinder placed on it working substance can expended or compressed adiabatic.

- **Working** :

  A set of reversible processes through them working substance is taken back to initial condition to get maximum work from this type of ideal engine.

  Processes of Carnot's cycle can be denoted by an indicator diagram.

- **Isothermal expansion A → B**

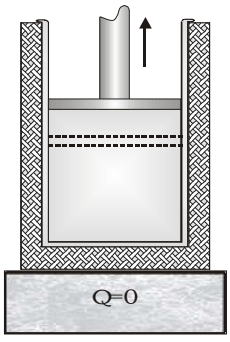


Initially the cylinder is taken to be in thermal equilibrium with the high temperature T_1, this is initial state of working substance denoted by point A (P_1, V_1, T_1). After that the piston is allowed to move outward slowly. With the movement of the piston. The process is very slow so that it is isothermal. Heat from reservoir flows through the base of cylinder into the gas so temperature of the gas remains T_1. Gas expand and receive heat Q_1 from source and gets state B(P_2, V_2, T_1)

This heat input Q_1 to the gas from path A to B is utilized for doing work W_1.

By path A to B the heat input to the gas = the work done against the external pressure. $W_1 = Q_1 = \int_{V_1}^{V_2} PdV = \int_{V_1}^{V_2} \frac{\mu RT_1}{V} dV = \mu RT_1 \ell n \frac{V_2}{V_1}$

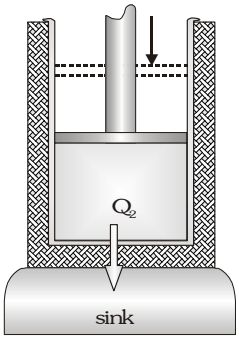
- **Adiabatic expansion B → C**



Now cylinder is put in contact with a non–conducting stand and piston is allowed to move outward, because no heat can enter in or leave out so the expansion of gas is adiabatic. The temperature falls to T_2K and gas describes the adiabatic from B to point C (P_3, V_3, T_2). During this expansion more work is done (W_2) at the expense of the internal energy.

Work done in adiabatic path BC is $W_2 = \frac{\mu R}{\gamma - 1}\left(T_1 - T_2\right)$

- **Isothermal compression C → D**

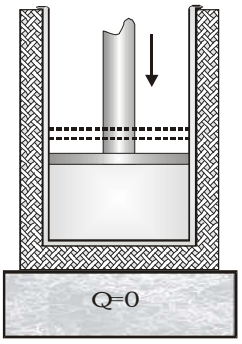


Now the gas cylinder is placed in contact with sink at temperature T_2. The piston is moved slowly inward so that heat produced during compression passes to the sink. The gas is isothermally compressed from C to point D (P_4, V_4, T_2). The heat rejected Q_2 to the cold reservoir (sink) at T_2 occurs over this path. Amount of work done on gas W_3 = amount of heat rejected to the sink

$Q_2 = W_3 = \mu RT_2 \ell n \left(\frac{V_4}{V_3}\right) \Rightarrow Q_2 = \mu RT_2 \ell n \left(\frac{V_4}{V_3}\right)$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

- **Adiabatic compression D → A**



The cylinder is removed from the sink and is put in contact with insulating stand now piston moves inward. Heat is not allowed to go out and it increases the internal energy of the system. Now work is done on the gas during adiabatic compression from state D to initial point A (P_1, V_1, T_1). No heat exchanges occur over the adiabatic path. Work done on the system $W_4 = \frac{\mu R}{\gamma - 1}(T_2 - T_1)$

This cycle of operations is called a Carnot cycle.

In first two steps work is done by engine W_1 and W_2 are positive

In last two steps work is done on gas W_3 and W_4 are negative

The work done in complete cycle W = the area of the closed part of the P–V cycle.

$$W = W_1 + W_2 + W_3 + W_4$$

$$\therefore W = \mu RT_1 \ell n\frac{V_2}{V_1} + \frac{\mu R}{\gamma - 1}(T_1 - T_2) + \mu RT_2 \ell n\frac{V_4}{V_3} + \frac{\mu R}{\gamma - 1}(T_2 - T_1) = \mu RT_1 \ell n\frac{V_2}{V_1} + \mu RT_2 \ell n\frac{V_4}{V_3}$$

Efficiency of Carnot Engine, $\eta = \frac{W}{Q_1} \Rightarrow \eta = \frac{\mu RT_1 \ell n\frac{V_2}{V_1} + \mu RT_2 \ell n\frac{V_4}{V_3}}{\mu RT_1 \ell n\frac{V_2}{V_1}}$

B to C and D to A are adiabatic paths

$$\text{so } T_1V_2^{(\gamma-1)} = T_2V_3^{(\gamma-1)} \text{ and } T_1V_1^{(\gamma-1)} = T_2V_4^{(\gamma-1)} \Rightarrow \frac{V_2}{V_1} = \frac{V_3}{V_4}$$

$$\Rightarrow \eta = \frac{T_1 - T_2}{T_1} = \frac{Q_1 - Q_2}{Q_1} \Rightarrow \eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1} \quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

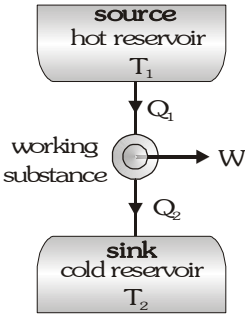
$$\eta = \frac{T_1 - T_2}{T_1} \times 100\% \Rightarrow \eta = \frac{Q_1 - Q_2}{Q_1} \times 100\%$$

The efficiency for the Carnot engine is the best that can be obtained for any heat engine. The efficiency of a Carnot engine is never 100% because it is 100% only if temperature of sink $T_2 = 0$ which is impossible.

CARNOT THEOREM

No irreversible engine (I) can have efficiency greater than Carnot reversible engine (R) working between same hot and cold reservoirs. $\eta_R > \eta_I \Rightarrow \boxed{1 - \frac{T_2}{T_1} > 1 - \frac{Q_2}{Q_1}}$

HEAT ENGINE



Heat engine is a device which converts heat into work.

Three parts of a heat engine:

(i) Source of high temperature reservoir at temperature T_1

(ii) Sink of low temperature reservoir at temperature T_2

(iii) Working substance.

In a cycle of heat engine the working substance extracts heat Q_1 from source, does some work W and rejects remaining heat Q_2 to the sink.

Efficiency of heat engine $\eta = \frac{\text{work done}(W)}{\text{heat taken from source}(Q_1)} \Rightarrow \frac{T_1 - T_2}{T_1} = \frac{Q_1 - Q_2}{Q_1}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

ALLEN Materials Provided By - Material Point Available on Learnaf.com

TYPES OF HEAT ENGINE

- **External combustion engine**

  Steam engine is the example of external combustion engine. Its efficiency is 10 to 20% (small).

- **Internal combustion engine**

  The examples are petrol engine and diesel engine.

  There are four strokes in internal combustion engine.

  (i) Intake strokeor Charging stroke (ii) Compression stroke

  (iii) Working (or power) stroke of Expansion stroke (iv) Exhaust stroke

  The useful work is done in third stroke called work stroke or power stroke.

  The efficiency of internal combustion engine is 40 to 60%.

Example

A carnot engine working between 400 K and 800 K has a work output of 1200 J per cycle. What is the amount of heat energy supplied to the engine from source per cycle?

Solution

$W = 1200$ J, $T_1 = 800$ K, $T_2 = 400$ K $\therefore \eta = 1 - \frac{T_2}{T_1} = \frac{W}{Q_1} \Rightarrow 1 - \frac{400}{800} = \frac{1200}{Q_1} \Rightarrow 0.5 = \frac{1200}{Q_1}$

Heat energy supplied by source $Q_1 = \frac{1200}{0.5}$ = 2400 joule per cycle

Example

The temperatures T_1 and T_2 of the two heat reservoirs in an ideal carnot engine are 1500 C and 500 C respectively. Which of the following : increasing T_1 by 100 C or decreasing T_2 by 100 C would result in a greater improvement in the efficiency of the engine?

Solution

T_1 = 1500 C = 1500 + 273 = 1773 K and T_2 = 500 C = 500 + 273 = 773 K.

The efficiency of a carnot's engine $\eta = 1 - \frac{T_2}{T_1}$

When the temperature of the source is increased by 100 C, keeping T_2 unchanged, the new temperature of the source is T'_1 = 1500 + 100 = 1600 C = 1873 K.

The efficiency becomes $\eta' = 1 - \frac{T_2}{T_1'} = 1 - \frac{773}{1873} = 0.59$

On the other hand, if the temperature of the sink is decreased by 100 C, keeping T_1 unchanged, the new temperature of the sink is T'_2 = 500 – 100 = 400 C = 673 K. The efficiency now becomes

$$\eta'' = 1 - \frac{T'_2}{T_1} = 1 - \frac{673}{1773} = 0.62$$

Since η'' is greater than η', decreasing the temperature of the sink by 100 C results in a greater efficiency than increasing the temperature of the source by 100 C.

Example

A heat engine operates between a cold reservoir at temperature T_2 = 300 K and a hot reservoir at temperature T_1. It takes 200 J of heat from the hot reservoir and delivers 120 J of heat to the cold reservoir in a cycle. What could be the minimum temperature of hot reservoir?

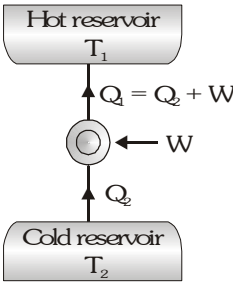
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-5\Thermal physics\Eng\Theory P65

Solution

Work done by the engine in a cycle is $W = 200 - 120 = 80$ J. $\eta = \frac{W}{Q} = \frac{80}{200} = 0.4$

From carnot's Theorem $0.4 \le 1 - \frac{T_2}{T_1} = 1 - \frac{300}{T_1} \Rightarrow \frac{300}{T_1} \le 0.6 \Rightarrow T_1 \ge \frac{300}{0.6} \Rightarrow T_1 \ge 500$

REFRIGERATOR



It is inverse of heat engine. It extracts heat (Q_2) from a cold reservoir, same external work W is done on it and rejects heat (Q_1) to hot reservoir. The coefficient of performance of a refrigerator.

$$\beta = \frac{\text{Heat extracted from cold reservoir}}{\text{Work done on refrigerator}} = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{1}{\frac{Q_1}{Q_2} - 1}$$

For Carnot reversible refrigerator $\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \quad \therefore \beta = \frac{Q_2}{W}\frac{1}{\left[\frac{Q_1}{Q_2} - 1\right]} = \frac{1}{\left[\frac{T_1}{T_2} - 1\right]} \Rightarrow \beta = \frac{T_2}{T_1 - T_2}$

Example

A carnot engine works as a refrigrator between 250 K and 300 K. If it receives 750 cal of heat from the reservoir at the lower temperature. Calculate the amount of heat rejected at the higher temperature.

Solution

$T_1 = 300$ K $\quad T_2 = 250$ K $\quad Q_2 = 750 \quad Q_1 = ?$

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \qquad Q_1 = \frac{300}{250} \times 750 = 900 \text{ cal}$$

Example

The temperature inside and outside of refrigerator are 260 K and 315 K respectively. Assuming that the refrigerator cycle is reversible, calculate the heat delivered to surroundings for every joule of work done.

Solution

$T_2 = 260$ K, $T_1 = 315$ K; $W = 1$ joule

Coefficient of performance of Carnot refrigerator $\beta = \frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$

$$\therefore \frac{Q_2}{1} = \frac{260}{315 - 260} = \frac{260}{42} \Rightarrow Q_2 = \frac{260}{42} = 6.19 \text{ J}$$

Example

A refrigerator takes heat from water at 0 C and transfer it to room at 27 C. If 100 kg of water is converted in ice at 0 C then calculate the work done. (Latent heat of ice is $3.4 \quad 10^5$ J/kg)

Solution

Coefficient of performance (COP) $= \frac{T_2}{T_1 - T_2} = \frac{273}{300 - 273} = \frac{273}{27}$

$$W = \frac{Q_2}{COP} = \frac{mL}{COP} = \frac{100 \times 3.4 \times 10^5}{273/27} = \frac{100 \times 3.4 \times 10^5 \times 27}{273} = 3.36 \quad 10^7 \text{ J}$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

SOME WORKED OUT EXAMPLES

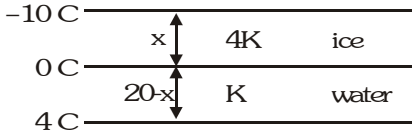
Example#1

In a 20m deep lake, the bottom is at a constant temperature of 4 C. The air temperature is constant at –10 C. The thermal conductivity of ice is 4 times that water. Neglecting the expansion of water on freezing, the maximum thickness of ice will be

(A) $\frac{20}{11}$ m (B) $\frac{200}{11}$ m (C) 20 m (D) 10 m

Solution **Ans. (B)**

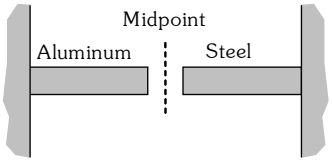
The rate of heat flow is the same through water and ice in the steady state so



$$\frac{KA(4-0)}{20-x}=\frac{4KA[0-(-10)]}{x}\Rightarrow x=\frac{200}{11}m$$

Example#2

The figure shows two thin rods, one made of aluminum [α = 23 10^{-6} (C)$^{-1}$] and the other of steel [α = 12 10^{-6} (C)$^{-1}$]. Each rod has the same length and the same initial temperature. They are attached at one end to two separate immovable walls. Temperature of both the rods is increased by the same amount, until the gap between the rods vanishes. Where do the rods meet when the gap vanishes?



(A) The rods meet exactly at the midpoint.

(B) The rods meet to the right of the midpoint .

(C) The rods meet to the left of the midpoint.

(D) Information insufficient

Solution **Ans. (B)**

As $\alpha_{A\ell} > \alpha_{steel}$ so expansion in aluminum rod is greater.

Example#3

Certain perfect gas is found to obey PV^n = constant during adiabatic process.

The volume expansion coefficient at temperature T is

(A) $\frac{1-n}{T}$ (B) $\frac{1}{(1-n)T}$ (C) $\frac{n}{T}$ (D) $\frac{1}{nT}$

Solution **Ans. (B)**

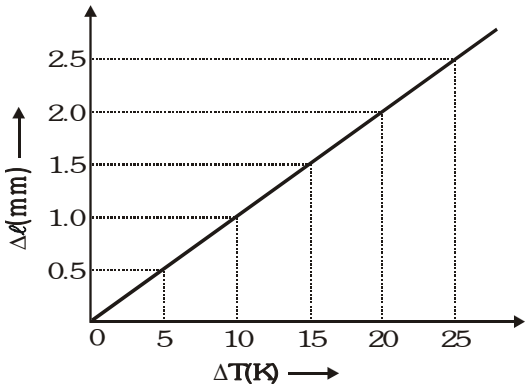
PV^n = constant & PV = μRT $\Rightarrow V \propto T^{\left(\frac{1}{1-n}\right)} \Rightarrow \frac{\Delta V}{V}=\left(\frac{1}{1-n}\right)\frac{\Delta T}{T}$

$\Rightarrow$ volume expansion coefficient $=\frac{\Delta V}{V\Delta T}=\frac{1}{(1-n)T}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example#4

Figures shows the expansion of a 2m long metal rod with temperature. The volume expansion coefficient of the metal is :–



(A) 3 10^{-4} K^{-1} (B) 1.5 10^{-4} K^{-1} (C) 3 10^{-5} K^{-1} (D) 1.5 10^{-5} K^{-1}

Solution **Ans. (B)**

$$\because \alpha = \frac{\Delta \ell}{\ell \Delta T} = \frac{0.5 \times 10^{-3}}{2(5)} = 5 \quad 10^{-5}\ K^{-1} \Rightarrow \gamma = 3\alpha = 1.5 \quad 10^{-4}\ K^{-1}$$

Example#5

The temperature of a body rises by 44 C when a certain amount of heat is given to it. The same heat when supplied to 22 g of ice at – 8 C, raises its temperature by 16 C. The water eqivalent of the body is

[Given : s_{water} = 1 cal/g C & L_f = 80 cal/g, s_{ice} = 0.5 cal/g C]

(A) 25g (B) 50 g (C) 80 g (D) 100 g

Solution **Ans. (B)**

Supplied heat = (22) (0.5) (8) + (22) (80) + (22) (1) (16) = 88 + 1760 + 352 = 2200 cal

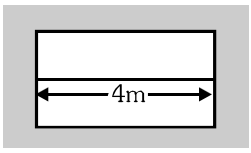
Heat capacityof the body = $\frac{2200 \text{ cal}}{44 \text{ C}}$ = 50 cal/ C

Water equavalent of the body = $\frac{\text{Heat capacity of the body}}{\text{spcific heat capacity of water}} = \frac{50 \text{ cal/ C}}{1 \text{ cal/g C}} = 50\text{g}$

Example#6

A fine steel wire of length 4m is fixed rigidly in a heavy brass frame as shown in figure. It is just taut at 20 C. The tensile stress developed in steel wire if whole system is heated to 120 C is :–

(Given α_{brass} = 1.8 10^{-5} C^{-1}, α_{steel}=1.2 10^{-5} C^{-1},Y_{steel} =2 10^{11} Nm^{-2},Y_{brass}=1.7 10^{7} Nm^{-2})



(A) 1.02 10^{4} Nm^{-2} (B) 1.2 10^{8} Nm^{-2} (C) 1.2 10^{6} Nm^{-2} (D) 6 10^{8} Nm^{-2}

Solution **Ans. (B)**

Stress = Y (strain) = $Y_S(\alpha_b - \alpha_s)\Delta T$= (2 10^{11}) (0.6 10^{-5}) (100) = 1.2 10^{8} Nm^{-2}

Example#7

540 g of ice at 0 C is mixed with 540 g of water at 80 C. The final temperature of the mixture is

(Given latent heat of fusion of ice = 80 cal/g and specific heat capacity of water = 1 cal/g^{0}C)

(A) 0 C (B) 40 C (C) 80 C (D) less than 0 C

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

Solution **Ans. (A)**

Heat taken by ice to melt at 0 C is Q_2 = mL = 540 80 = 43200 cal

Heat given by water to cool upto 0 C is Q_2 = ms$\Delta\theta$ = 540 1 (80–0) = 43200 cal

Hence heat given by water is just sufficient to melt the whole ice and final temperature of mixture is 0 C.

Example#8

A refrigerator converts 100 g of water at 25 C into ice at – 10 C in one hour and 50 minutes. The quantity of heat removed per minute is (specific heat of ice = 0.5 cal/g C, latent heat of fusion = 80 cal/g)

(A) 50 cal (B) 100 cal (C) 200 cal (D) 75 cal

Solution **Ans. (B)**

Heat removed in cooling water from 25 C to 0 C = 100 1 25 = 2500 cal

Heat removed in converting water into ice at 0 C = 100 80 = 8000 cal

Heat removed in cooling ice from 0 to –15 C = 100 0.5 10 = 500 cal

Total heat removed in1 hr 50min = 2500 + 8000 + 500 = 11000 cal

Heat removed per minute $= \frac{11000}{110}$ =100 cal/min

Example#9

5n, n and 5n moles of a monoatomic, diatomic and non-linear polyatomic gases (which do not react chemically with each other) are mixed at room temperature. The equivalent degree of freedom for the mixture is-

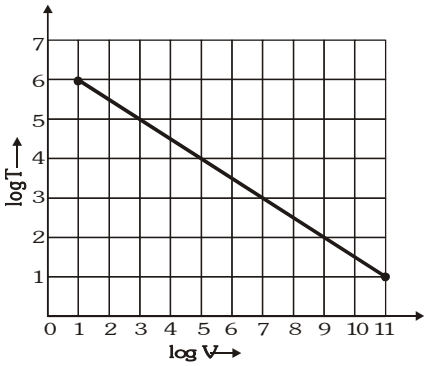
(A) $\frac{25}{7}$ (B) $\frac{48}{11}$ (C) $\frac{52}{11}$ (D) $\frac{50}{11}$

Solution **Ans. (D)**

$$f_{eq} = \frac{f_1n_1 + f_2n_2 + f_3n_3}{n_1 + n_2 + n_3} = \frac{(5n)(3) + (n)(5) + (5n)(6)}{5n + n + 5n} = \frac{50}{11}$$

Example#10

Figure shows the adiabatic curve on log–log scale performed on a ideal gas. The gas must be :–



(A) Monoatomic (B) Diatomic

(C) A mixture of monoatomic and diatomic (D) A mixture of diatomic and polyatomic

Solution **Ans. (C)**

For adiabatic process $TV^{\gamma-1}$ = constant

$\Rightarrow$ log T + (γ–1) log V = constant $\Rightarrow$ slope = – (γ–1) = $-\left(\frac{5}{10}\right) \Rightarrow \gamma = \frac{3}{2}$

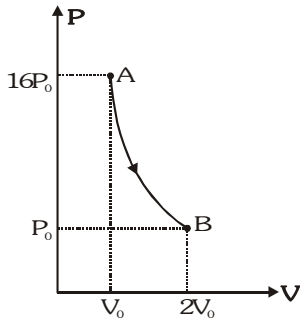
For monoatomic gas $\gamma = \frac{5}{3}$; For diatomic gas $\gamma = \frac{7}{5}$ As $\frac{7}{5} < \gamma = \frac{3}{2} < \frac{5}{3}$

Hence, the gas must be a mixture of monoatomic & diatomic gas.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example#11

Figure demonstrates a polytropic process (i.e. PV^n = constant) for an ideal gas. The work done by the gas will be in process AB is



(A) $\frac{15}{2}P_0V_0$ (B) $\frac{14}{3}P_0V_0$ (C) $8P_0V_0$ (D) Insufficient information

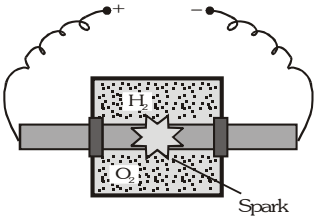
Solution **Ans. (B)**

For a polytropic process PV^n = constant $\Rightarrow 16P_0V_0^{\,n} = P_0(2V_0)^n \Rightarrow n=4$

Work done = $\frac{P_1V_1 - P_2V_2}{n-1} = \frac{16P_0V_0 - P_0(2V_0)}{4-1} = \frac{14}{3}P_0V_0$

Example#12

A vessel contains 14 g (7 moles) of hydrogen and 96 g (3 moles) of oxygen at STP. Chemical reaction is induced by passing electric spark in the vessel till one of the gases is consumed. The temperature is brought back to it's starting value 273 K. The pressure in the vessel is-



(A) 0.1 atm (B) 0.2 atm (C) 0.3 atm (D) 0.4 atm

Solution **Ans. (A)**

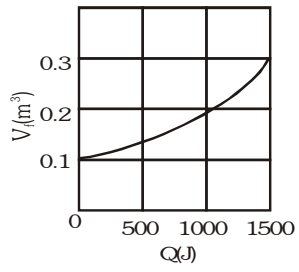
When electric spark is passed, hydrogen reacts with oxygen to form water (H_2O). Each gram of hydrogen reacts with eight grams of oxygen. Thus 96 g of oxygen will be totally consumed together with 12 g of hydrogen.

The gas left in thevessel will be 2g of hydrogen i.e. number of moles $\mu = \frac{2}{2} = 1$.

Using $PV = \mu RT \Rightarrow P \propto \mu \Rightarrow \frac{P_2}{P_1} = \frac{\mu_2}{\mu_1} \Rightarrow \frac{P_2}{1} = \frac{1}{10} \Rightarrow P_2 = 0.1\text{atm}$

Example#13

Suppose 0.5 mole of an ideal gas undergoes an isothermal expansion as energy is added to it as heat Q. Graph shows the final volume V_f versus Q. The temperature of the gas is (use $\ell n\ 9 = 2$ and R= 25/3 J/mol-K)



(A) 293 K (B) 360 K (C) 386 K (D) 412 K

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Solution **Ans. (B)**

$Q = W = nRT\ \ell n\ \frac{V_f}{V_i}$; $T = \frac{Q}{nR\ell nV_f / V_i} = \frac{1500}{0.5\times 25/3\times \ell n3} = \frac{1500}{0.5\times 25/3\times 1} = 360K$

Example#14

The respective speeds of five molecules are 2,1.5,1.6,1.6 and 1.2 km/s. The most probable speed in km/s will be

(A) 2 (B) 1.58 (C) 1.6 (D) 1.31

Solution **Ans. (C)**

Since maximum number of molecules travel with speed 1.6 km/s so v_{mp} = 1.6 km/s

Example#15

When water is boiled at 2 atm pressure, the latent heat of vaporization is 2.2 10^6 J/kg and the boiling point is 120 C. At 2 atm pressure, 1 kg of water has volume of 10^{-3} m^3 and 1 kg of steam has a volume of 0.824 m^3. The increase in internal energy of 1 kg of water when it is converted into steam at 2 atm pressure and 120 C is [1 atm pressure = 1.013 10^5 N/m^2]

(A) 2.033 J (B) 2.033 10^6 J (C) 0.167 10^6 J (D) 2.267 10^6 J

Solution **Ans. (B)**

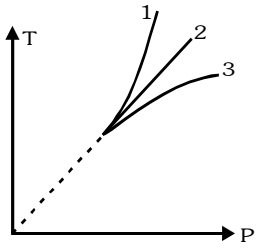
Total heat given to correct convert water into steam at 120 C is Q = mL = 1 2.2 10^6 = 2.2 10^6 J

The work done by the system against the surrounding is

$P\Delta V$ = 2 1.013 10^5 (0.824 0.001) = 0.167 10^6 J $\therefore$ $\Delta U = Q - W$ = 2.033 10^6 J

Example#16

Given T–P curve for three processes. If initial and final pressure are same for all processes then work done in process 1, 2 and 3 is W_1, W_2 & W_3 respectively. Correct order is



(A) $W_1 < W_2 > W_3$ (B) $W_1 > W_2 > W_3$ (C) $W_1 < W_2 < W_3$ (D) $W_1 = W_2 = W_3$

Solution **Ans. (B)**

Here $T \propto P^n$ where for : graph-1, n >1 $\Rightarrow$W > 0 ; graph-2, n =1 $\Rightarrow$W = 0; graph-3, n <1 $\Rightarrow$W < 0

Example#17

The acceleration of a particle moving rectilinearly varies with displacement as a = – 4x. At x =4 m and t =0, particle is at rest. Select the **INCORRECT** alternative

(A) The maximum speed of the particle is 8 m/s.

(B) The distance travelled by the particle in first second is 20 m.

(C) The velocity - acceleration graph of the particle is an ellipse.

(D) The kinetic energy-displacement graph of the particle is a parabola.

Solution **Ans. (B)**

The equation shows a SHM : $a = -\omega^2 x \Rightarrow \omega = \sqrt{4} = 2 \Rightarrow$ Time period = $\frac{2\pi}{\omega} = \pi sec$ $\therefore x = A\sin(\omega t + \phi)$

From the given condition, A =4 & $\phi = \frac{\pi}{2}$; $V_{P_{max}} = A\omega = 8m/s$

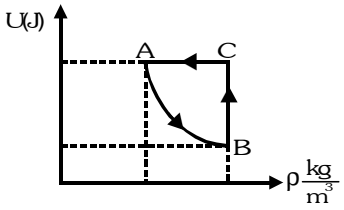
Distance travelled in first second <2A or < 8 m

KE = $\frac{1}{2}m\omega^2(A^2 - x^2)$ which represent a parabola.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Example#18

Figure shows the variation of the internal energy U with density ρ of one mole of an ideal monatomic gas for thermodynamic cycle ABCA. Here process AB is a part of rectangular hyperbola :-



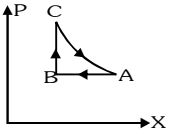
(A) process AB is isothermal & net work in cycle is done by gas.
(B) process AB is isobaric & net work in cycle is done by gas.
(C) process AB is isobaric & net work in cycle is done on the gas.
(D) process AB is adiabatic & net work in cycle is done by gas.

Solution **Ans. (B)**

For the process AB : Uρ = constant (hyperbola)

$U = \frac{3}{2}RT$ (monoatomic ideal gas); $\frac{3}{2}\rho RT$ = constant

Comparing it with ideal gas equation $P = \left(\frac{1}{M}\right)\rho RT \Rightarrow P$ is constant.

P-V graph for the cycle is 

. Thus work done in cycle is positive

Example#19

One mole of an ideal gas undergoes a process whose molar heat capacity is 4R and in which work done by gas for small change in temperature is given by the relation dW = 2RdT, then the ratio $\frac{C_P}{C_V}$ is

(A) 7/5 (B) 5/3 (C) 3/2 (D) 2

Solution **Ans. (C)**

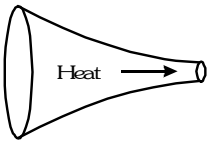
For small change dQ = dU + dW

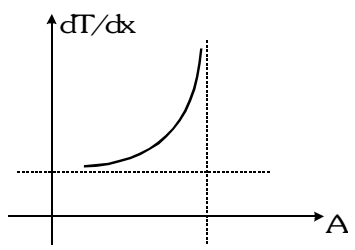
$nCdT = nC_V dT + 2nRdT \therefore C = C_V + 2R$; $4R = C_V + 2R \quad \therefore C_V = 2R$

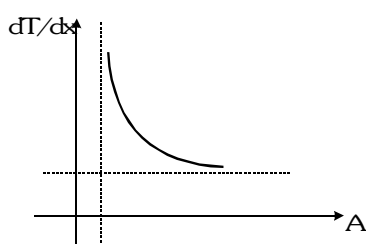
Also $C_V = \frac{R}{\gamma - 1} \therefore \frac{R}{\gamma - 1} = 2R \Rightarrow 2\gamma - 2 = 1 \Rightarrow \gamma = \frac{3}{2}$

Example#20

An irregular rod of same uniform material as shown in figure is conducting heat at a steady rate. The temperature gradient at various sections versus area of cross section graph will be

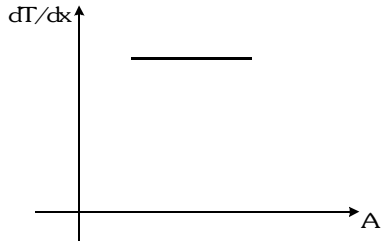


(A) 

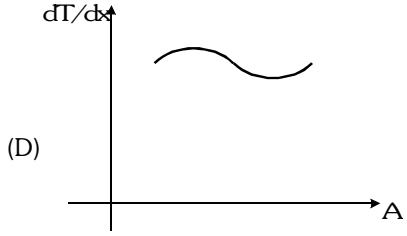
(B) 

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

(C)



(D)



Solution **Ans. (B)**

$H = KA\dfrac{dT}{dx}$ is same in steady state condition, $\therefore A\dfrac{dT}{dx}$ = constant $\therefore$ rectangular hyperbolic graph

Example#21

A gas undergoes an adiabatic process in which pressure becomes $\left(\dfrac{8}{3\sqrt{3}}\right)$ times and volume become $\dfrac{3}{4}$ of initial volume. If initial absolute temperature was T, the final temperature is

(A) $\dfrac{32T}{9\sqrt{3}}$ (B) $\dfrac{2T}{\sqrt{3}}$ (C) $T^{3/2}$ (D) $\dfrac{\sqrt{3}T}{2}$

Solution **Ans. (B)**

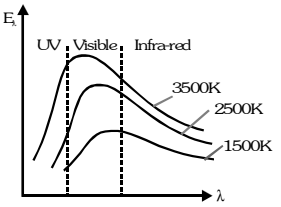
For adiabatic process, $P_1V_1^{\gamma} = P_2V_2^{\gamma} \Rightarrow P_0V_0^{\gamma} = \left(\dfrac{8}{3\sqrt{3}}P_0\right)\left(\dfrac{3}{4}V_0\right)^{\gamma}$ ..(i)

Also, $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow TV_0^{\gamma-1} = T_2\left(\dfrac{3}{4}V_0\right)^{\gamma-1}$ ...(ii) Solving (i) & (ii), $T_2 = \dfrac{2T}{\sqrt{3}}$

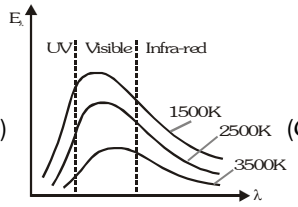
Example#22

Which of the following graph(s) shows the correct variation in intensity of heat radiations by black body and frequency at a fixed temperature–

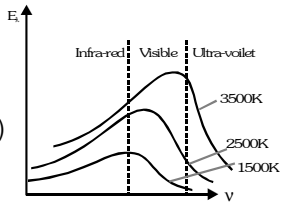
(A)



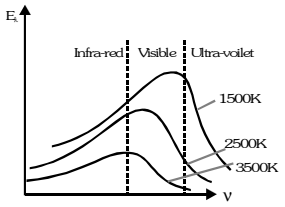
(B)



(C)



(D)



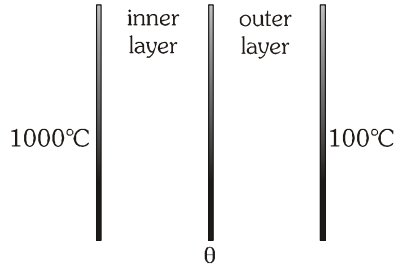
Solution **Ans. (AC)**

According to Wien's law $\lambda_m \propto \dfrac{1}{T} \Rightarrow \nu_m \propto T$. As the temperature of body increases, frequency corresponding to maximum energy in radiation (ν_m) increases.

Also area under the curve $\int E_\lambda d\lambda \propto \int E_\nu d\nu \propto T^4$

Example#23

The temperature drop through a two layer furnace wall is 900 C. Each layer is of equal area of cross–section. Which of the following action(s) will result in lowering the temperature θ of the interface?



\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

(A) By increasing the thermal conductivity of outer layer.
(B) By increasing the thermal conductivity of inner layer.
(C) By increasing thickness of outer layer.
(D) By increasing thickness of inner layer.

Solution **Ans. (AD)**

Rate of heat flow $\frac{K_i A(1000-\theta)}{\ell_i} = \frac{K_0 A(\theta-100)}{\ell_0} \Rightarrow \frac{\theta-100}{900} = \frac{1}{1+\frac{K_0\ell_i}{K_i\ell_0}}$

Now, we can see that θ can be decreased by increasing thermal conductivity of outer layer (K_o) and thickness of inner layer (ℓ_i).

Example#24

5 kg of steam at 100 C is mixed with 10 kg of ice at 0 C. Choose correct alternative/s
(Given s_{water} = 1 cal/g C, L_F = 80 cal/g, L_V = 540 cal/g)
(A) Equilibrium temperature of mixture is 160 C
(B) Equilibrium temperature of mixture is 100 C
(C) At equilibrium, mixture contains $13\frac{1}{3}$ kg of water
(D) At equilibrium, mixture contains $1\frac{2}{3}$ kg of steam

Solution **Ans. (B,C,D)**

Required heat	Available heat
10 kg ice (0 C)	5 kg steam (100 C)
↓ 800 kcal	↓ 2700 cal
10 g water (0 C)	5 g water (100 C)
↓ 1000 kcal	
10 g water (100 C)	

So available heat is more than required heat therefore final temperature will be 100 C.

Mass of heat condensed $= \frac{800+1000}{540} = \frac{10}{3}$ kg. Total mass of water $= 10+\frac{10}{3} = \frac{40}{3} = 13\frac{1}{3}$ kg

Total mass of steam $= 5-\frac{10}{3} = \frac{5}{3} = 1\frac{2}{3}$ kg

Example#25

Water contained in a jar at room temperature (20 C) is intended to be cooled by method-I or method-II given below :
Method -I : By placing ice cubes and allowing it to float.
Method-II : By wrapping ice cubes in a wire mesh and allowing it to sink.
Choose best method(s) to cool the water.
(A) Method-I from 20 C to 4 C
(B) Method-I from 4 C to 0 C
(C) Method-II from 20 C to 4 C
(D) Method-II from 4 C to 0 C

Solution **Ans. (AD)**

Initially (above 4 C), a decrease in temperature, increases the density of water and consequently it descends, replacing the relatively warm water. Convention currents set up in this way demands the location of ice to be on the water surface.
Below 4 C, a decrease in temperature decreases the water density and as a result it ascends up displacing the relatively warm water. To setup convention currents in this way, the position of ice cubes should be at the bottom.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R6.5

Example#26

n moles of an ideal triatomic linear gas undergoes a process in which the temperature changes with volume as $T = k_1V^2$ where k_1 is a constant. Choose correct alternative(s).

(A) At normal temperature $C_v = \frac{5}{2}R$

(B) At any temperature $C_p - C_v = R$

(C) At normal temperature molar heat capacity C=3R

(D) At any temperature molar heat capacity C=3R

Solution **Ans. (ABC)**

At normal temperature $C_v = \frac{f}{2}R = \frac{5}{2}R$; At any temperature $C_p - C_v = \left(\frac{f}{2}+1\right)-\frac{f}{2}R = R$

from process $T = k_1V^2$ & ideal gas equation $PV = nRT$ we have PV^{-1} = constant $\Rightarrow$ x =–1

$\Rightarrow C = C_V + \frac{R}{1-x} = C_V + \frac{R}{1+1} = C_V + \frac{R}{2}$ At normal temperature $C = \frac{5}{2}R + \frac{R}{2} = 3R$

Example#27

An ideal gas expands in such a way that PV^2 = constant throughout the process. Select correct alternative

(A) This expansion is not possible without heating

(B) This expansion is not possible without cooling

(C) Internal energy remains constant in this expansion

(D) Internal energy increases in this expansion

Solution **Ans. (B)**

$PV = nRT$ & PV^2 = constant $\Rightarrow V \propto \frac{1}{T} \Rightarrow$ gas can expand only if it cools

As tempeature decreases during expansion so internal energy will decrease.

Example#28

Which of the following processes must violate the first law of thermodynamics ($Q = W + \Delta E_{int}$)?

(A) $W > 0$, $Q < 0$ and $\Delta E_{int} > 0$ (B) $W > 0$, $Q < 0$ and $\Delta E_{int} < 0$

(C) $W < 0$, $Q > 0$ and $\Delta E_{int} < 0$ (D) $W > 0$, $Q > 0$ and $\Delta E_{int} < 0$

Solution **Ans. (AC)**

For (A) : $W > 0$ & $\Delta E_{int} > 0 \Rightarrow Q > 0$ **For (B) :** $W > 0$ & $\Delta E_{int} < 0 \Rightarrow Q > 0$ or $Q < 0$

For (C) : $W < 0$ & $\Delta E_{int} < 0 \Rightarrow Q < 0$ **For (D) :** $W > 0$ & $\Delta E_{int} < 0 \Rightarrow Q > 0$ or $Q < 0$

Example#29

In a process on a closed system of ideal gas, the initial pressure and volume is equal to the final pressure and volume

(A) initial internal energy must be equal to the final internal energy

(B) the work done on the system is zero

(C) the work done by the system is zero

(D) the initial temperature must be equal to final temperature

Solution **Ans. (AD)**

Here n = constant so if $P_2 = P_1$ and $V_2 = V_1$ then $T_2 = T_1$

Also work done by the system may be zero or may not be zero.

Example#30

In Newton's law of cooling, $\frac{d\theta}{dt} = -k(\theta-\theta_0)$, the constant k is proportional to

(A) A, surface area of the body (B) S, specific heat of the body

(C) $\frac{1}{m}$, m being mass of the body (D) e, emissivity of the body

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

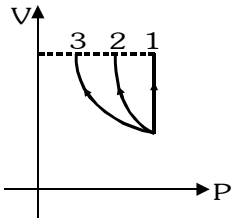
Solution **Ans. (AC)**

$$\frac{dQ}{dt} = e\sigma A\left(\theta^4 - \theta_0^4\right) \approx \left(3e\sigma A\theta_0^3\right)\Delta\theta$$

$$\frac{dQ}{dt} = e\sigma A\left(\theta^4 - \theta_0^4\right) \approx \left(3e\sigma A\theta_0^3\right)\Delta\theta$$

Example#31

The graph below shows V-P curve for three processes.



Choose the correct statement(s)

(A) Work done is maximum in process 1.

(B) Temperature must increase in process 2 & 3.

(C) Heat must be supplied in process 1.

(D) If final volume of gas in process 1, 2 and 3 are same then temperature must be same.

Solution **Ans. (AC)**

Area under P-V curve and volume axis represent work.
Internal energy in 1 increases (expansion at constant pressure)
In process 2 and 3 internal energy may decrease depending on the process.
Since final pressure of 1, 2 & 3 are different, final temperature must also be different.

Example#32

During an experiment, an ideal gas is found to obey a condition $\frac{P^2}{\rho}$ = constant [ρ=density of the gas]. The gas is initially at temperature T, pressure P and density ρ. The gas expands such that density changes to $\frac{\rho}{2}$

(A) The pressure of the gas changes to $\sqrt{2}P$
(B) The temperature of the gas changes to $\sqrt{2}T$.
(C) The graph of the above process on the P-T diagram is parabola.
(D) The graph of the above process on the P-T diagram is hyperbola.

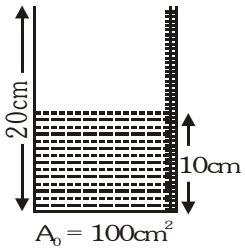
Solution **Ans. (B,D)**

$PV = nRT$

$P \propto \rho T$ .....(i) $\qquad T \propto \frac{1}{\sqrt{\rho}}$ ...(ii) $\qquad T \propto \frac{1}{\sqrt{\rho}}$ ...(iii)

Example#33 to 35

At 20 C a liquid is filled upto 10 cm height in a container of glass of length 20 cm and cross-sectional area 100 cm^2 Scale is marked on the surface of container. This scale gives correct reading at 20 C. Given $\gamma_L = 5 \quad 10^{-5}\ k^{-1}$, $\alpha_g = 1 \quad 10^{-5}\ C^{-1}$



33. The volume of liquid at 40 C is :–
(A) 1002 cc (B) 1001 cc (C) 1003 cc (D) 1000.5 cc

34. The actual height of liquid at 40 C is-
(A) 10.01 cm (B) 10.006 cm (C) 10.6 cm (D) 10.1 cm

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

35. The reading of scale at 40 C is-

(A) 10.01 cm (B) 10.004 cm (C) 10.006cm (D) 10.04 cm

Solution

33. Ans. (B)

$V = V_0 (1+\gamma_L \Delta T) = (10)(100)[1 + 5 \quad 10^{-5} \quad 20] = 1000 (1 + 0.001) = 1001 \text{ cm}^3 = 1001 \text{ cc}$

34. Ans. (B)

Cross sectional area of vessel at 40 C

$A = A_0(1 + 2\alpha_g \Delta T) = 100 (1 + 2 \quad 10^{-5} \quad 20) = 100.04 \text{ cm}^2$

$$\text{Actual height of liquid} = \frac{\text{Actual volume of liquid}}{\text{cross} - \text{sec tional area of vessel}} = \frac{1001}{100.04} = (1001)(100 + 0.04)^{-1}$$

$$= \left(\frac{1001}{100}\right)\left(1+\frac{0.04}{100}\right)^{-1} = \frac{(1001)}{100}\left(1-\frac{0.04}{100}\right) = \frac{1}{100}(1001 - 0.4) = 10.006 \text{ cm}$$

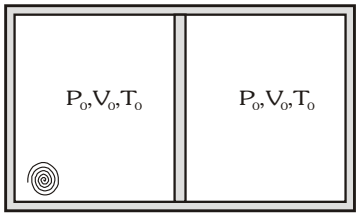
35. Ans. (B)

$\because$ TV = SR $(1 + \alpha_g \Delta T) = (TV)(1 - \alpha_g \Delta T)$

$\therefore$ SR = $(TV)(1 + \alpha g\ \Delta T)^{-1} = (10.006)(1 - 10^{-5} \quad 20) = 10.006 - 0.002 = 10.004$ c

Example#36 to 38

One mole of a monoatomic ideal gas occupies two chambers of a cylinder partitioned by means of a movable piston. The walls of the cylinder as well as the piston are thermal insulators. Initially equal amounts of gas fill both the chambers at (P_0, V_0, T_0). A coil is burnt in the left chamber which absorbs heat and expands, pushing the piston to the right. The gas on the right chamber is compressed until to pressure becomes 32 P_0.



36. The final volume of left chamber is

(A) $\frac{V_0}{8}$ (B) $\frac{15}{8}V_0$ (C) $\frac{7}{8}V_0$ (D) $\frac{9}{8}V_0$

37. The work done on the gas in the right chamber is

(A) $\frac{9}{2}P_0V_0$ (B) $-\frac{9}{2}P_0V_0$ (C) $\frac{13}{2}P_0V_0$ (D) $\frac{17}{2}P_0V_0$

38. The change in internal energy of the gas in the left chamber is

(A) $\frac{186}{4}RT_0$ (B) $\frac{177}{4}RT_0$ (C) $\frac{59}{2}RT_0$ (D) $\frac{131}{4}RT_0$

Solution

36. Ans. (B)

Since the compression on the right is adiabatic so $P_0V_0^{\gamma} = P_RV_R^{\gamma}$

$$\Rightarrow P_0V_0^{5/3} = 32P_0V_R^{5/3} \Rightarrow V_R = \frac{V_0}{8} \Rightarrow V_L = V_0 + \frac{7}{8}V_0 = \frac{15}{8}V_0$$

37. Ans. (A)

$$\text{Work done on the gas} = \frac{P_RV_R - P_0V_0}{\gamma - 1} = \frac{4P_0V_0 - P_0V_0}{5/3 - 1} = \frac{9}{2}P_0V_0$$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

38. Ans. (B)

For mechanical equilibrium $P_L = P_R = 32\ P_0$

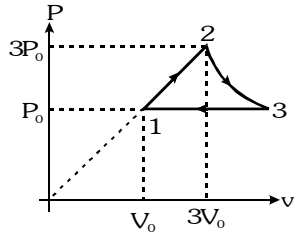
So $P_LV_L = (32P_0)\left(\frac{15V_0}{8}\right) = 60P_0V_0 = 60nRT_0 = nRT_L \Rightarrow T_L = 60T_0$

The change in the internal energy of the gas in the left chamber

$$\Delta U = nC_V\Delta T = \frac{1}{2}\times\frac{3}{2}R\times 59T_0 = \frac{177}{4}RT_0$$

Example#39 to 41

One mole of an ideal monoatomic gas undergoes a cyclic process as shown in figure. Temperature at point 1 = 300 K and process 2-3 is isothermal.



39. Net work done by gas in complete cycle is

(A) $(9\ell n3+12)P_0V_0$ (B) $(9\ell n3+4)P_0V_0$ (C) $(9\ell n3-4)P_0V_0$ (D) $(9\ell n3-8)P_0V_0$

40. Heat capacity of process $1 \to 2$ is

(A) $\frac{R}{2}$ (B) $\frac{3R}{2}$ (C) $\frac{5R}{2}$ (D) 2R

41. The efficiency of cycle is

(A) $\left(\frac{9\ell n3+4}{9\ell n3+12}\right)$ (B) $\left(\frac{9\ell n3-4}{9\ell n3+12}\right)$ (C) $\left(\frac{9\ell n3-4}{9\ell n3+16}\right)$ (D) $\left(\frac{9\ell n3+12}{9\ell n3+16}\right)$

Solution

39. Ans. (C)

$$W = W_{12}+W_{23}+W_{31} = \frac{1}{2}(4P_0)(2V_0)+nRT\ell n(3)-P_0(8V_0) = -4P_0V_0+9P_0V_0\ell n3 = (9\ell n3-4)P_0V_0$$

40. Ans. (D)

$$C = \frac{\Delta Q}{n\Delta T} = \frac{n\left(\frac{3R}{2}.\Delta T\right)+\Delta W}{n\Delta T} = \frac{3R}{2}+\frac{4P_0V_0}{n\Delta T} = \frac{3R}{2}+\frac{P_0V_0}{n(2T_0)} = \frac{3R}{2}+\frac{R}{2} = 2R$$

$$\Delta Q_{123} = n(2R)(\Delta T_{12})+(W_{23}) = n(2R)(8T_0)+nR(9T_0)\ell n(3) = nRT_0(16+9\ell n3)$$

41. Ans. (C)

$$\eta = \frac{W}{\Delta Q_{123}} = \left(\frac{9\ell n3-4}{9\ell n3+16}\right)$$

Example#42 to 44

A body cools in a surrounding of constant temperature 30 C. Its heat capacity is 2J/ C. Initial temperature of the body is 40 C. Assume Newton's law of cooling is valid. The body cools to 36 C in 10 minutes.

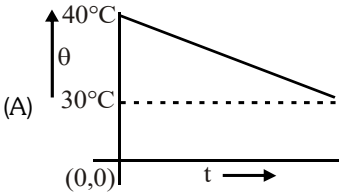
42. In further 10 minutes it will cool from 36 C to :

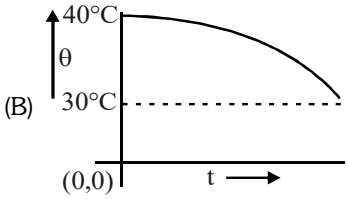
(A) 34.8 C (B) 32.1 C (C) 32.8 C (D) 33.6 C

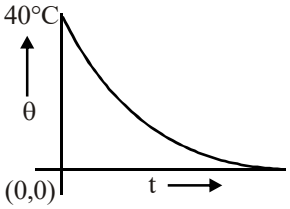
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory Pb 5

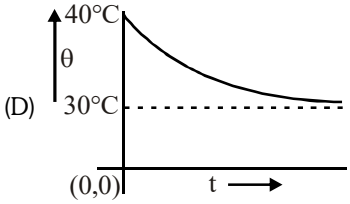
ALLEN Materials Provided By - Material Point Available on Learnaf.com

43. The temperature of the body in C denoted by θ the variation of θ versus time t is best denoted as

(A) 

(B) 

(C) 

(D) 

44. When the body temperature has reached 36 C, it is heated again so that it reaches to 40 C in 10 minutes. Assume that the rate of loss of heat at 38 C is the average rate of loss for the given time. The total heat required from a heater by the body is :

(A) 7.2 J (B) 0.728 J (C) 16 J (D) 32 J

Solution

42. Ans. (D)

$\frac{40-36}{10}$ = k(38 – 30) and $\frac{36-x}{10} = k\left(\frac{36+x}{2} - 30\right) \Rightarrow x = 33.6$

43. Ans. (D)

$\frac{d\theta}{dt} = -\frac{kA}{ms}(T - T_0)$ Magnitude of slope will decrease with time.

44. Ans. (C)

$\because \frac{40-36}{10} = k\ (38 - 30) \Rightarrow k = \frac{4}{10 \times 8} = \frac{1}{20}$

when the block is at 38 C and room temperature is at 30 C the rate of heat loss ms $\frac{d\theta}{dt}$ = ms k (38 – 30)

Total heat loss in 10 minute $\Rightarrow \Delta Q$ = ms k (38 – 30) 10 = 2 $\frac{1}{20}$ 8 10 = 8 J

Now heat agained by the object in the said 10 minutes. Q = ms $\Delta\theta$ = 2 4 = 8 J

Total heat required = 8 + 8 = 16 J

Example#44

	Column I (Questions)	Column II (Answers)
(A)	The temperature of an iron piece is increased from 20 to 70 . What is change in its temperature on the Fahrenheit scale (in F)?	(P) 20
(B)	At what temperature (in C) do the Celsius and Fahrenheit readings have the same numerical value?	(Q) 40
(C)	100 g ice at 0 C is converted into steam at 100 C. Find total heat required (in kcal) (L_f = 80 cal/g, s_w = 1 cal/g C, L_v = 540 cal/g)	(R) – 40
(D)	A ball is dropped on a floor from a height of 5 m. After the collision it rises upto a height of 3m. Assume that 50% of the mechanical energy lost goes as thermal energy into the ball. Calculate the rise in temperature (in milli centigrade) of the ball in the collision. (s_{ball} = 500 J/K, g = 10 m/s^2)	(S) 72 (T) 90

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P65

Solution **Ans.(A) → (S); (B) → (R); (C) → (T); (D) → (P)**

(A) $\frac{C-0}{100}=\frac{F-32}{180}\Rightarrow \Delta F=\frac{9}{5}\Delta C=\frac{9}{5}\times 50=90°F$

(B) $\frac{x-0}{100}=\frac{x-32}{180}\Rightarrow x=-40$

(C) Total heat required = $mL_f + ms\Delta\theta + mL_v$ = 72000 cal = 72 kcal

(D) $ms\Delta\theta=\frac{50}{100}\times mg(h_1-h_2)\Rightarrow \Delta\theta=\frac{1}{50}°C=20\times10^{-3}°C$

Example#46

An ideal gas whose adiabatic exponent equals to $\gamma=\frac{7}{5}$ is expanded according to the law P=2V. The initial volume of the gas is equal to V_0= 1 unit. As a result of expansion the volume increases 4 times. (Take R = $\frac{25}{3}$ units)

	Column - I	Column - II
(A)	Work done by the gas	(P) 25 units
(B)	Increment in internal energy of the gas	(Q) 45 units
(C)	Heat supplied to the gas	(R) 75 units
(D)	Molar heat capacity of the gas in the process	(S) 15 units
		(T) 55 units

Solution **Ans. (A) → (S); (B) → (R); (C) → (Q); (D) → (P)**

$W=\int PdV=\int_{V_0}^{4V_0}2VdV=(V^2)_{V_0}^{4V_0}=15V_0^2=$ 15 units

From PV = nRT, $2V^2 = nRT \Rightarrow 2(V_2^2-V_1^2)=nR(\Delta T)\Rightarrow nR\Delta T = 30V_0^2$

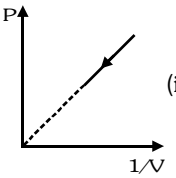
$\Delta U=nC_V\Delta T=\frac{nR}{\gamma-1}\Delta T=\frac{30V_0^2}{\gamma-1}=\frac{30(1)^2}{\frac{7}{5}-1}=\frac{30}{2}(5)$ = 75 units

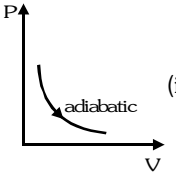
Q = W + ΔU = 15 + 30 = 45 units

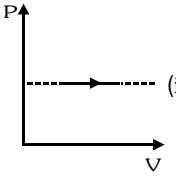
Molar heat capacity : $C=C_V+\frac{R}{1-x}=\frac{5}{2}R+\frac{R}{1-(-1)}=\frac{5}{2}R+\frac{R}{2}=3R=3\times\frac{25}{3}=$ 25 units

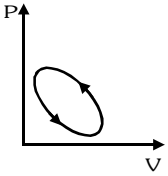
Example#47

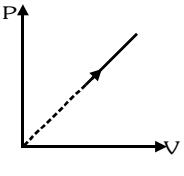
The figure given below show different process for a given amount for an ideal gas. W is work done by the system and ΔQ is heat absorbed by the system.

(i) 

(ii) 

(iii) 

(iv) 

(v) 

	Column-I	Column-II
(A)	ΔQ > 0	(P) In figure (i)
(B)	W < 0	(Q) In figure (ii)
(C)	ΔQ < 0	(R) In figure (iii)
(D)	W > 0	(S) In figure (iv)
		(T) In figure (v)

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

Solution **Ans. (A) →(PRT) ; (B) →(S) ; (C) →(S) ; (D) →(PQRT)**

figure (i) $P \propto \frac{1}{V} \Rightarrow PV$ = constant

Isothermal (T = constant), so $\Delta U = 0$

$\frac{1}{V}$ is decreasing; So V is increasing hence, $\Delta W > 0$

$\Delta Q = \Delta U + \Delta W = \Delta W > 0$

Example#48

In an industrial process 10 kg of water per hour is to be heated from 20 C to 80 C . To do this steam at 150 C is passed from a boiler into a copper coil immersed in water. The steam condenses in the coil and is returned to the boiler as water at 90 C. How many kg of steam is required per hour.
(Specific heat of steam = specific heat of water = 1 cal/g C, Latent heat of vaporisation = 540 cal/g)

Solution **Ans. 1**

Suppose m kg steam is required per hour
Heat is released by steam in following three steps

(i) When 150 C steam $\xrightarrow{Q_1}$ 100 C steam $Q_1 = mc_{steam}\Delta\theta = m \quad 1\ (150-100) = 50\ m$ cal

(ii) When 100 C steam $\xrightarrow{Q_2}$ 100 C water $Q_2 = mL_v = m \quad 540 = 540\ m$ cal

(iii) When 100 C water $\xrightarrow{Q_2}$ 90 C water $Q_3 = mc_W\Delta\theta = m \quad 1 \quad (100-90) = 10\ m$ cal

Hence total heat given by the steam $Q = Q_1 + Q_2 + Q_3 = 600\ m$ cal ....(i)

Heat taken by 10 kg water $Q' = mc_W\Delta\theta = 10 \quad 10^3 \quad 1 \quad (80-20) = 600 \quad 10^3$ cal

Hence $Q = Q' \Rightarrow 600\ m = 600 \quad 10^3 \Rightarrow m = 10^3$ gm = 1 kg

Example#49

The specific heat of a metal at low temperatures varies according to $S = (4/5)T^3$ where T is the absolute temperature. Find the heat energy needed to raise unit mass of the metal from T = 1 K to T = 2 K.

Solution **Ans. 3**

$$Q = \int mSdT = \frac{mT^4}{5} \Rightarrow \frac{Q}{m} = \frac{15}{5} = 3$$

Example#50

A rod has variable co-efficient of linear expansion $\alpha = \frac{x}{5000}$. If length of the rod is 1m. Determine increase in length of the rod in (cm) on increasing temperature of the rod by 100 C.



Solution **Ans. 1**

Increase in length of dx = $\alpha\ \ell_0\Delta T = \left(\frac{x}{5000}\right)(dx)(100) = \frac{x}{50}dx$

$\Rightarrow$ Total thermal expansion = $\int_0^1 \frac{x}{50}dx = \left(\frac{x^2}{100}\right)_0^1 = \frac{1}{100}m = 1cm$

Example#51

A clock pendulum made of invar has a period of 2 s at 20 C. If the clock is used in a climate where average temperature is 40 C, what correction (in seconds) may be necessary at the end of 10 days to the time given by clock ? (α_{invar} = 7 10^{-7} C^{-1}, 1 day = 8.64 10^4 s)

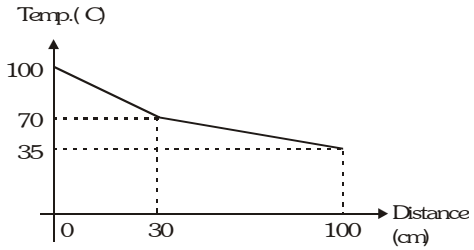
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory\P65

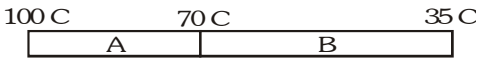
Solution **Ans. 6**

$$T = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow \frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta \ell}{\ell} = \frac{1}{2}\alpha\Delta\theta \Rightarrow \Delta T = \frac{T}{2}(\alpha\Delta\theta) = \left(\frac{10\times 8.64\times 10^4}{2}\right)(7\times 10^{-7})(20) = 6 \text{ s}$$

Example#52

Two different rods A and B are kept as shown in figure.





The variation of temperature of different cross sections is plotted in a graph shown in figure. Find the ratio of thermal conductivities of B to A.

Solution **Ans. 2**

$$\because \left(\frac{\Delta Q}{\Delta t}\right)_A = \left(\frac{\Delta Q}{\Delta t}\right)_B \therefore \frac{K_A A(100-70)}{30} = \frac{K_B A(70-35)}{70} \Rightarrow K_A = \frac{K_B}{2} \Rightarrow \frac{K_B}{K_A} = 2$$

Example#53

A body cools from 50 C to 49.9 C in 5 sec. It cools from 40 C to 39.9 C in t sec. Assuming Newtons law of cooling to be valid and temperature of surrounds at 30 C, value of t/5 will be?

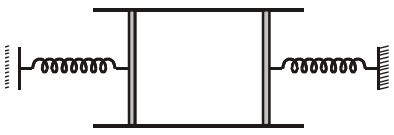
Solution **Ans. 2**

From $\frac{\theta_1 - \theta_2}{t} = k\left(\frac{\theta_1 + \theta_2}{2} - \theta_0\right)$

We have $\frac{0.1}{5} = k\left(\frac{39.5}{2}\right)$ and $\frac{0.1}{t} = k\left(\frac{19.5}{2}\right) \Rightarrow \frac{t}{5} = \frac{39.5}{19.5} = 2 \Rightarrow t = 10 \Rightarrow \frac{t}{5} = 2$

Example#54

A cylinder of cross-section area A has two pistons of negligible mass separated by distances ℓ loaded with spring of negligible mass. An ideal gas at temperature T_1 is in the cylinder where the springs are relaxed. When the gas is heated by some means its temperature becomes T_2 and the springs get compressed by $\frac{\ell}{2}$ each. If P_0 is atmospheric pressure and spring constant $k = \frac{2P_0A}{\ell}$, then find the ratio of T_2 and T_1.



Solution **Ans. 4**

$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ where $V_1 = \ell A$ and $V_2 = 2\ell A$ and $P_1 = P_0$ and $P_2 = \frac{kx}{A} + P_0 = 2P_0 \Rightarrow \frac{T_2}{T_1} = \frac{P_2V_2}{P_1V_1} = 4$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P6.5

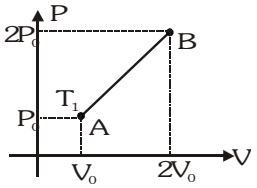
Example#55

A vessel contains 1 mole of O_2 gas (molar mass 32) at a temperature T. The pressure of the gas is P. An identical vessel containing one mole of He gas (molar mass 4) at a temperature 2T has a pressure of xP. Find the value of x.

Solution **Ans. 2**

PV = nRT, V is constant

Example#56



One mole of a gas is taken from state A to state B as shown in figure.

Work done by the gas is $\alpha \quad 10^{\beta}$ J. Find the value of $\alpha + \beta$.

(Given : T_1=320 K, R= $\frac{25}{3}$)

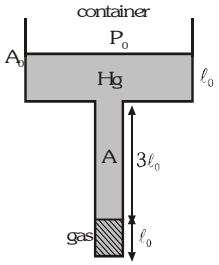
Solution **Ans. 7**

Work done = $\frac{1}{2}[P_0 + 2P_0][2V_0 - V_0] = \frac{3}{2}P_0V_0$

and P_0V_0 = R 320

So work done $\frac{3}{2} \times R \times 320 = \frac{3}{2} \times \frac{25}{3} \times 320 = 4000J = 4 \times 10^3 J \Rightarrow \alpha + \beta = 4 + 3 = 7$

Example#57



A container having base area A_0. Contains mercury upto a height ℓ_0. At its bottom a thin tube of length $4\ell_0$ and cross-section area A ($A << A_0$) having lower end closed is attached. Initially the length of mercury in tube is $3\ell_0$. In remaining part 2 mole of a gas at temperature T is closed as shown in figure. Determine the work done (in joule) by gas if all mercury is displaced from tube by heating slowly the gas in the rear end of the tube by means of a heater. (Given : density of mercury = ρ, atmospheric pressure P_0 = $2\ell_0\rho g$, C_V of gas = 3/2 R, A= $(3/\rho)m^2$, ℓ_0 = (1/9) m, all units in S.I.)

Solution **Ans. 5**

If x is length of mercury in tube then pressure of gas

$P' = P_0 + \rho g\ell_0 + \rho gx = 3\rho g\ell_0 + \rho gx \Rightarrow W = -\int_{3\ell_0}^{0} (3\rho g\ell_0 + \rho gx)A\,dx = 13.5\ \rho gA\ell_0^2 = 5$

Example#58

The relation between internal energy U, pressure P and volume V of a gas in an adiabatic process is: U=a+bPV where a = b = 3. Calculate the greatest integer of the ratio of specific heats [γ].

Solution **Ans. 1**

For an adiabatic process, dQ=dU+PdV =0

$\Rightarrow$ d[a+bPV] + PdV=0 $\Rightarrow$ bPdV + bV dP + PdV =0 $\Rightarrow$ (b+1)PdV+bV dP=0 $\Rightarrow$ $(b+1)\frac{dV}{V} + b\frac{dP}{P} = 0$

$\Rightarrow$ (b+1)logV + b logP= constant; $V^{b+1}P^b$ =constant $\Rightarrow$ $PV^{\frac{b+1}{b}}$ = constant $\therefore \gamma = \frac{b+1}{b} = \frac{4}{3} = 1.33$

\\node6\E_NOD6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory P&S

Example#59

One mole of an ideal monatomic gas undergoes the process $P = \alpha T^{1/2}$, where α is constant. If molar heat capacity of the gas is βR when R = gas constant then find the value of β.

Solution **Ans.2**

$$P \propto T^{1/2} \text{ and } PV = nRT \Rightarrow PV^{-1} = \text{constant} \quad \therefore C = C_V + \frac{R}{1-(-1)} = \frac{3R}{2} + \frac{R}{2} = 2R$$

Example#60

One mole of a monoatomic gas is enclosed in a cylinder and occupies a volume of 4 liter at a pressure 100 N/m^2. It is subjected to a process $T = \alpha V^2$, where α is a positive constant, V is volume of the gas and T is Kelvin temperature. Find the work-done by gas (in joule) in increasing the volume of gas to six times initial volume.

Solution **Ans. 7**

$$W = \int PdV \text{ where } P = \frac{nRT}{V} = \alpha nRV \Rightarrow W = n\alpha R \int_{V_0}^{6V_0} VdV = \frac{n\alpha R}{2}\left[(6V_0)^2 - (V_0)^2\right]$$

$$\Rightarrow W = \frac{h\alpha R}{2}(35)V_0^2 = \frac{P_0V_0 \times 35}{2}\left(\because \alpha = \frac{P_0}{nRV_0}\right)$$

Example#61

Two taps A and B supply water at temperatures 10° and 50 C respectively. Tap A alone fills the tank in 1 hour and tap B alone fills the tank in 3 hour. If we open both the taps together in the empty tank, if the final temperature of the water in the completely filled tank is found to be 5α (in C). Find the value of α. Neglect loss of heat to surrounding and heat capacity of the tank.

Solution **Ans. 4**

$$m\,(T-10)\,s = \frac{m}{3}S(50-T) \Rightarrow T = 20^\circ$$

Example#62

Two identical metal plates are welded end to end as shown in figure-(i). 20 cal of heat flows through it in 4 minutes. if the plates are welded as shown in figure-(ii), find the time (in minutes) taken by the same amount of heat to flow through the plates.

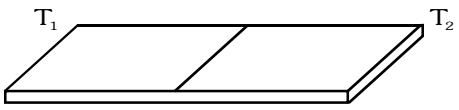


Fig. (i)

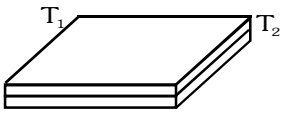


Fig. (ii)

Solution **Ans. 1**

Rate of heat flow $\frac{\Delta Q}{\Delta t} = \frac{kA(T_1 - T_2)}{\ell} \Rightarrow \Delta t \propto \frac{\ell}{A}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-5\Thermal physics\Eng\Theory R5

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 12
SIMPLE HARMONIC MOTION - Theory

Index

SIMPLE HARMONIC MOTION (S.H.M.)

Serial No.	CONTENTS	Page No.
1.	Periodic & Oscillatory Motion	1
2.	Linear and Angular SHM	1
3.	Geometrical meaning of S.H.M.	6
4.	Energy in SHM	12
5.	Spring system &spring pendulum	15
6.	Simple Pendulum	20
7.	Compound Pendulum	22
8.	Examples of SHM	28
9.	Exercise-I Check your Grasp (Objective)	42
10.	Exercise-2 Brain Teasers (MCQ's with one or more than one correct answer)	48
11.	Exercise-3 Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	52
12.	Exercise-4A Conceptual Subjective Exercise	57
13.	Exercise-4B Brain Storming Exercise	59
14.	Exercise-5A Previous years questions	62
15.	Exercise-5B Previous years questions	65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Mains : Periodic motion - period, frequency, displacement as a function of time. Periodic functions. Simple harmonic motion (S.H.M.) and its equation; phase; oscillations of a spring -restoring force and force constant; energy in S.H.M. - kinetic and potential energies; Simple pendulum - derivation of expression for its time period; Free, forced and damped oscillations, resonance.

JEE Advanced : Linear and angular simple harmonic motions.

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SIMPLE HARMONIC MOTION (S.H.M.)

PERIODIC MOTION

- Any motion which repeats itself after regular interval of time is called periodic motion or harmonic motion.
- The constant interval of time after which the motion is repeated is called time period.

Examples **:**(i) Motion of planets around the sun. (ii) Motion of the pendulum of wall clock.

OSCILLATORY MOTION

- The motion of body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after regular interval of time.
- The fixed point about which the body oscillates is called mean position or equilibrium position.

Examples : (i) Vibration of the wire of 'Sitar'. (ii) Oscillation of the mass suspended from spring.

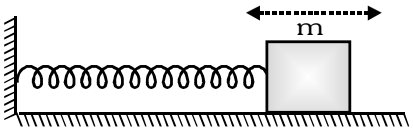
Note : Every oscillatory motion is periodic but every periodic motion is not oscillatory.

SIMPLE HARMONIC MOTION (S.H.M.)

Simple harmonic motion is the simplest form of vibratory or oscillatory motion.

(i) S.H.M. are of two types

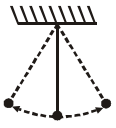
- **Linear S.H.M. :** When a particle moves to and fro about a fixed point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.



Example : Motion of a mass connected to spring.

- **Angular S.H.M.**

When a system oscillates angularly with respect to a fixed axis then its motion is called angular simple harmonic motion.
Example :- Motion of a bob of simple pendulum.



(ii) Necessary Condition to execute S.H.M.

- **In linear S.H.M. :** The restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position $\therefore F \propto -x$ or $a \propto -x$

Negative sign shows that direction of force and acceleration is towards equilibrium position and x is displacement of particle from equilibrium position.

- **In angular S.H.M. :** The restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position

$\therefore \quad \tau \propto -\theta$ or $\alpha \propto -\theta$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

EQUATION OF SIMPLE HARMONIC MOTION

- **In linear S.H.M.**

Restoring force acting on the particle,

$$ma = -kx \Rightarrow a = -\frac{kx}{m} \Rightarrow \frac{dv}{dt} = -\frac{kx}{m} \Rightarrow v\frac{dv}{dt} = -\frac{kx}{m}\frac{dx}{dt} [\because v = dx/dt]$$

$$\int v\,dv = -\int \frac{kx}{m}dx \Rightarrow \frac{v^2}{2} = -\frac{kx^2}{2m} + C$$

$$\text{At } x = 0, v = v_0 \therefore C = \frac{v_0^2}{2} \Rightarrow \frac{v^2}{2} = -\frac{\omega^2x^2}{2} + \frac{v_0^2}{2} [\because \omega^2 = k/m] \Rightarrow v = \sqrt{v_0^2 - \omega^2x^2} \Rightarrow \frac{dx}{dt} = \sqrt{v_0^2 - \omega^2x^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{v_0^2 - \omega^2x^2}} = \int dt \Rightarrow \frac{1}{\omega}\sin^{-1}\left(\frac{\omega x}{v_0}\right) = t + C_1 [\because v_0 = \omega A] \Rightarrow x = A\sin(\omega t + \omega C_1)$$

$\because$ At t = 0, x = 0 and if velocity is in +x direction $x = A\sin\omega t$

if velocity is in -x direction, $x = -A\sin\omega t$

- **In Angular SHM**

Restoring torque acting on the particle $\tau = -C\theta$ where C is a constant which can be defined as torque per unit angular displacement.

Mathematically, $I\alpha = -C\theta$. Where I is the moment of inertia of the system about the axis of rotation.

$$\Rightarrow I\frac{d^2\theta}{dt^2} + C\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0 \text{ . Since, } \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \Rightarrow \omega = \sqrt{\left(\frac{C}{I}\right)} \quad \text{.....(i)}$$

Example

If equation of displacement of a particle is y = A sinQt + B cosQt then find the nature of the motion of particle.

Solution

y = AsinQt + BcosQt

Differentiate with respect to t $\quad \frac{dy}{dt} = AQ\cos Qt - BQ\sin Qt$

Again differentiating with respect to t $\quad \frac{d^2y}{dt^2} = -Q^2A\sin Qt - Q^2B\cos Qt$

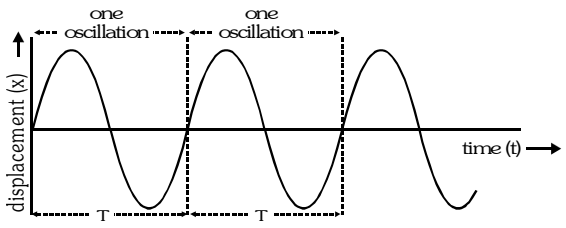
$$\frac{d^2y}{dt^2} = -Q^2(A\sin Qt + B\cos Qt) \qquad \Rightarrow \frac{d^2y}{dt^2} = -Q^2y \Rightarrow \frac{d^2y}{dt^2} + Q^2y = 0$$

It is a differential equation of linear S.H.M. So motion of the particle is simple harmonic

\Nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

SOME BASIC TERMS

- **Mean Position :** The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.
- **Restoring Force**
  - The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.
  - Restoring force always acts in a direction opposite to that of displacement. Displacement is measured from the mean position.
- **Amplitude :** The maximum (positive or negative) value of displacement of particle from mean position is define as amplitude. Since maximum and minimum value of sin or cos function is +1& -1, therefore here amplitude is A.
- **Time period (T)**
  - The minimum time after which the particle keeps on repeating its motion is known as time period.
  - The smallest time taken to complete one oscillation or vibration is also define as time period.
  - It is given by $T = \frac{2\pi}{\omega} = \frac{1}{n}$ where ω is angular frequency and n is frequency.
- **Oscillation or Vibration :** When a particle goes on one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.



- **Frequency (n or f)**
  - The number of oscillations per second is define as frequency.
  - It is given by $n = \frac{1}{T}$, $n = \frac{\omega}{2\pi}$
  - **SI UNIT :** hertz (Hz), 1 hertz = 1 cycle per second (cycle is a number not a dimensional quantity).
  - **Dimensions :** $M^0 L^0 T^{-1}$.
- **Phase** :
  - Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.
  - In the equation $x = A \sin(\omega t + \theta)$, $(\omega t + \theta)$ is the phase of the particle.
  - The phase angle at time t = 0 is known as initial phase or epoch.
  - The difference of total phase angles of two particles executing S.H.M. with respect to the mean position is known as phase difference.
  - Two vibrating particles are said to be in same phase if the phase difference between them is an even multiple of π, i.e., $\Delta\phi = 2n\pi$ Where n = 0, 1, 2, 3,....
  - Two vibrating particle are said to be in opposite phase if the phase difference between them is an odd multiple of π i.e., $\Delta\phi = (2n + 1)\pi$ Where n = 0, 1, 2, 3,....

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

- **Angular frequency (ω)** :

The rate of change of phase angle of a particle with respect to time is define as its angular frequency.

SI unit : radian/second, *Dimensions* : $M^0\ L^0\ T^{-1}$, $\omega = \sqrt{\frac{k}{m}}$

Example

If two S.H.M.'s are represented by equations $y_1 = 10\sin\left[3\pi t + \frac{\pi}{4}\right]$ and $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ then find the ratio of their amplitudes and phase difference in between them.

Solution

As $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ ...(i)

So if $5 = A\cos\phi$ and $5\sqrt{3} = A\sin\phi$

Then $A = \sqrt{5^2 + (5\sqrt{3})^2} = 10$ and $\tan\phi = \frac{5\sqrt{3}}{5} = \sqrt{3}$ so $\phi = \frac{\pi}{3}$

The above equation (i) becomes $y_2 = A\cos\phi \sin(3\pi t) + A\sin\phi \cos(3\pi t) \Rightarrow y_2 = A\sin(3\pi t + \phi) = 10 \sin[3\pi t + (\frac{\pi}{3})]$

so, $\frac{A_1}{A_2} = \frac{10}{10} \Rightarrow A_1 : A_2 = 1 : 1$, Phase difference = $\frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

DISPLACEMENT IN S.H.M.

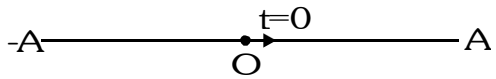
(i) The displacement of a particle executing linear S.H.M. at any instant is defined as the distance of the particle from the mean position at that instant.

(ii) It can be given by relation $x = A\sin\omega t$ or $x = A\cos\omega t$.

The first relation is valid when the time is measured from the mean position and the second relation is valid when the time is measured from the extreme position of the particle executing S.H.M. along a straight line path.

Example

A particle starts from mean position and moves towards positive extreme as shown below. Find the equation of the SHM. Amplitude of SHM is A.



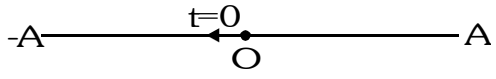
Solution

General equation of SHM can be written as $x = A\sin(\omega t+\phi)$

At $t = 0$, $x = 0$ $\therefore\ 0 = A\sin\phi$ $\therefore\ \phi = 0, \pi$ $\phi \in [0, 2\pi]$

Also, at $t = 0$, $v = +ve$ $\therefore A\omega\cos\phi$ = maximum +ve or $\phi = 0$

Hence, if the particle is at mean position at $t = 0$ and is moving towards +ve extrme, then the euqation of SHM is given by $x = A\sin\omega t$



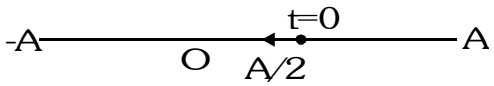
Similarly, for particle moving towards –ve extreme then $\phi = \pi$

$\therefore$ equation of SHM is $x = A\sin(\omega t + \pi)$ or, $x = -A\sin\omega t$

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

Write the equation of SHM for the situation show below :



Solution

General equation of SHM can be written as $x=A\sin(\omega t+\phi)$

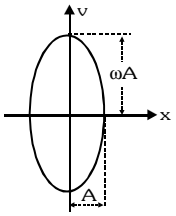
At t=0, $x=A/2 \Rightarrow \frac{A}{2}=A\sin\phi \Rightarrow \phi = 30^\circ, 150^\circ$

Also at t=0, v = –ve; $A\omega\cos\phi$ = –ve $\Rightarrow \phi = 150^\circ$

VELOCITY IN SHM

(i) It is define as the time rate of change of the displacement of the particle at the given instant.

(ii) Velocity in S.H.M. is given by $v=\frac{dx}{dt}=\frac{d}{dt}(A\sin\omega t) \Rightarrow v = A\omega\cos\omega t$



$v=\pm A\omega\sqrt{1-\sin^2\omega t} \Rightarrow v=\pm A\omega\sqrt{1-\frac{x^2}{A^2}}=\pm\omega\sqrt{(A^2-x^2)}$ $[\because x = A\sin\omega t]$

Squaring both the sides $v^2 = \omega^2(A^2 - x^2) \Rightarrow \frac{v^2}{\omega^2}=A^2-x^2 \Rightarrow \frac{v^2}{\omega^2A^2}=1-\frac{x^2}{A^2} \Rightarrow \frac{x^2}{A^2}+\frac{v^2}{A^2\omega^2}=1$

This is equation of ellipse. So curve between displacement and velocity of particle executing S.H.M. is ellipse.

(iii) The graph between velocity and displacement is shown in figure. If particle oscillates with unit angular frequency (ω = 1) then curve between v and x will be circular.

Note :

(i) The direction of velocity of a particle in S.H.M. is either towards or away from the mean position.

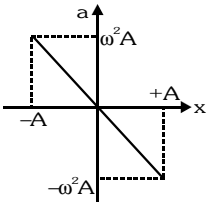
(ii) At mean position (x = 0), velocity is maximum (=$A\omega$) and at extreme position (x = ±A), the velocity of particle executing S.H.M. is zero (minimum).

ACCELERATION IN SHM

(i) It is define as the time rate of change of the velocity of the particle at given instant.

(ii) Acceleration in S.H.M. is given by $a=\frac{dv}{dt}=\frac{d}{dt}(A\omega\cos\omega t)$

$a = -\omega^2 A\sin\omega t \Rightarrow a = -\omega^2 x$



(iii) The graph between acceleration and displacement as shown in figure

Note

(i) The acceleration of a particle executing S.H.M. is always directed towards the mean position.

(ii) The acceleration of the particle executing S.H.M. is maximum at extreme position (= ω^2A) and minimum at mean position (= zero)

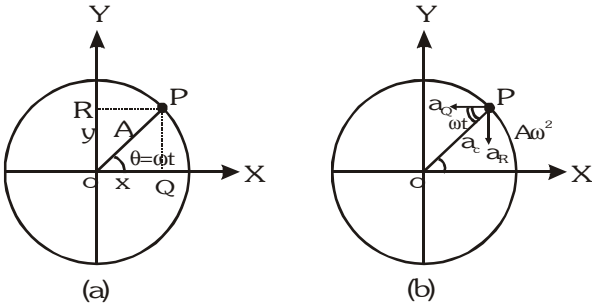
\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

GEOMETRICAL MEANING OF S.H.M.

If a particle is moving with uniform speed along the circumference of a circle then the straight line motion of the foot of perpendicular drawn from the particle on the diameter of the circle is called 'S.H.M.'

S.H.M. AS A PROJECTION OF CIRCULAR MOTION

Suppose a particle P is moving uniformly on a circle of radius A with angular speed ω . Q and R be the two feet of the perpendicular drawn from P on two diameters one along X-axis and the other along Y-axis.



Suppose the particle P is on the X-axis at t=0. Radius OP makes an angle $\theta = \omega t$ with the X-axis at time t then

$$x = A\cos\omega t\ ,\ \ y = A\sin\omega t$$

Here, x and y are the displacement of Q and R from the origin at time t, which are the displacement equations of SHM. It implies that although P is under uniform circular motion, Q and R are performing SHM about O with the same angular speed ω as that of P. From figure (b) centripetal acceleration of P = $a_c = A\omega^2$ (towards the centre) a_c can have two components as shown in the figure $a_R = A\omega^2\sin\omega t = \omega^2 x$, $a_Q = A\omega^2\cos\omega t = \omega^2 y$

a_R and a_Q are actually the acceleration corresponding to the points R and Q respectively.

GOLDEN KEY POINTS

- Every periodic motion can be resolved into a number of simple harmonic motions.
- Oscillatory motion can be treated as simple harmonic motion only in the limit of small amplitudes because in this limit the restoring force (or torque) becomes linear.
- Harmonic oscillations is that oscillations which can be expressed in terms of single harmonic function (i.e. sine functions or cosine function)
- The motion of the molecules of a solid, the vibration of the air columns and the vibration of string of music instruments are either simple harmonic or superposition of simple harmonic motions.

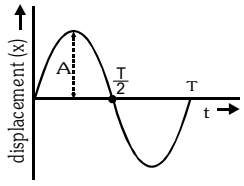
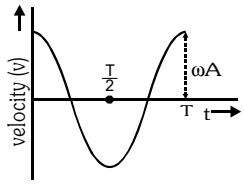
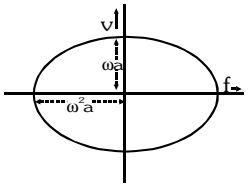
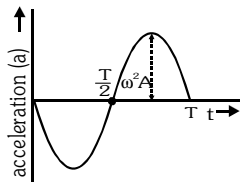
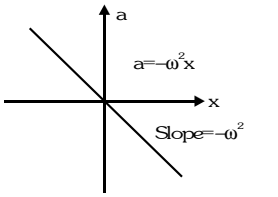
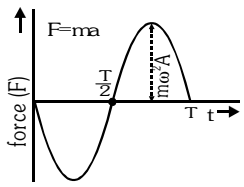
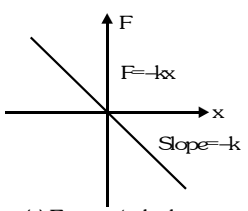
\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Comparison between linear and angular S.H.M.

Linear S.H.M.	Angular S.H.M.
$F \propto -x \Rightarrow F = -kx$	$\tau \propto -\theta \Rightarrow \tau = -C\theta$
where k is the restoring force constant	where C is the restoring torque constant
$a = -\frac{k}{m}x \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$	$\alpha = -\frac{C}{I}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$
It is known as differential equation of linear SHM	It is known as differential equation of angular SHM
$x = A\sin\omega t$; $a = -\omega^2 x$	$\theta = \theta_0 \sin\omega t$; $\alpha = -\omega^2\theta$
where ω is the angular frequency	
$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$	$\omega^2 = \frac{C}{I} \Rightarrow \omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$
where T is time period and n is frequency	where T is time period and n is frequency
$T = 2\pi\sqrt{\frac{m}{k}}$, $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	$T = 2\pi\sqrt{\frac{I}{C}}$, $n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$
This concept is valid for all types of linear S.H.M.	This concept is valid for all types of angular SHM.

GRAPHICAL REPRESENTATION

Graphical study of displacement, velocity, acceleration and force in S.H.M.

S. No.	Graph	In form of t	In from of x	Maximum value
1.	Displacement 	$x = A\sin\omega t$	$x = x$	$x = \pm A$
2.	Velocity 	$v = A\omega\cos\omega t$	$v = \pm\omega\sqrt{A^2 - x^2}$ 	$v = \pm\omega A$
3.	Acceleration 	$a = -\omega^2 A\sin\omega t$	$a = -\omega^2 x$  (a) Acceleration v/s displacement	$a = \pm\omega^2 A$
4.	Force (F = ma) 	$F = -m\omega^2 A\sin\omega t$	$F = -m\omega^2 x$  (a) Force v/s displacement	$F = \pm m\omega^2 A$

\\node6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e., x = 0 then calculate.

(i) Frequency of oscillation (ii) The displacement at 0.5 s

(iii) The maximum acceleration of the object. (iv) The velocity at a displacement of 3 cm.

Solution

(i) Frequency $f = \frac{1}{T} = \frac{1}{4} = 0.25$ Hz

(ii) The displacement equation of object x = Asinωt at t = 0.5 s, x = 5sin(2π 0.25 0.5) = $5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$ cm

(iii) Maximum acceleration $a_{max} = \omega^2 A = (0.5\ \pi)^2$ 5 = 12.3 cm/s^2

(iv) Velocity at x = 3 cm is $v = \pm\omega\sqrt{A^2 - x^2} = \pm 0.5\pi\sqrt{5^2 - 3^2} = \pm 6.28$ cm/s

Example

Amplitude of a harmonic oscillator is A, when velocity of particle is half of maximum velocity, then determine position of particle.

Solution

$v = \omega\sqrt{A^2 - x^2}$ but $v = \frac{v_{max}}{2} = \frac{A\omega}{2} = \omega\sqrt{A^2 - x^2} \Rightarrow A^2 = 4[A^2 - x^2] \Rightarrow x^2 = \frac{4A^2 - A^2}{4} \Rightarrow x = \pm\frac{\sqrt{3}A}{2}$

Example

Which of the following functions represent SHM :–

(i) sin^{2}ωt (ii) sin 2ωt (iii) sin ωt + 2cos ωt (iv) sin ωt + cos 2ωt

Solution

A motion will be S.H.M. if acceleration ∝ – y

(i) $y = \sin\ \omega t \Rightarrow \frac{dy}{dt} = 2(\sin\omega t)(\omega\cos\omega t) = \omega\sin 2\omega t$, $\frac{d^2y}{dt^2} = 2\omega^2\cos 2\omega t \Rightarrow \frac{d^2y}{dt^2} \propto y = 2\omega^2(1 - 2y)$

(Oscillatory but S.H.M. not possible)

(ii) As $y = \sin 2\omega t \Rightarrow v = \frac{dy}{dt} = 2\omega\cos 2\omega t \Rightarrow$ Acceleration $= \frac{d^2y}{dt^2} = -4\omega^2\sin 2\omega t = -4\omega^2 y$

so y = sin 2ωt represents S.H.M.

(iii) $y = \sin\omega t + 2\cos\omega t \Rightarrow v = \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin\omega t$,

Acceleration $= \frac{dv}{dt} = -\omega^2\sin\omega t - 2\omega^2\cos\omega t = -\omega^2(\sin\omega t + 2\cos\omega t) = -\omega^2 y$

∴ The given function represents SHM

(iv) $y = \sin\omega t + \cos 2\omega t \Rightarrow \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin 2\omega t$, $\frac{d^2y}{dt^2} = -\omega\sin\omega t - 4\omega\cos 2\omega t = -\omega(\sin\omega t + 4\cos 2\omega t)$

$\frac{d^2y}{dt^2} \not\propto (-y)$ (Oscillatory but S.H.M. not possible)

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

Periodic time of a simple pendulum is 2 second and it can travel to and fro from equilibrium position upto maximum 5cm. At start the pendulum is at maximum displacement on right side of equilibrium position. Find displacement and time relation.

Solution

Displacement expression for S.H.M. , $x = A \sin(\omega t + \phi)$

Time period of simple pendulum $T = \frac{2\pi}{\omega} = 2$ s $\quad \therefore \omega = \pi$ rad/s

Amplitude of pendulum $A = 5$ cm $\quad \therefore x = 5 \sin(\pi t + \phi)$

Now at $t = 0$, displacement $x = 5$ cm $\quad \therefore 5 = 5 \sin(\pi \ 0 + \phi) \Rightarrow \sin\phi = 1 \Rightarrow \phi = \frac{\pi}{2}$

Therefore, $x = 5 \sin \pi (t + \frac{1}{2}) \Rightarrow x = 5\sin(\pi t + \frac{\pi}{2}) \Rightarrow x = 5\cos \pi t$

Example

The velocity of a particle in S.H.M. at positions x_1 and x_2 are v_1 and v_2 respectively. Determine value of time period and amplitude.

Solution

$v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2 (A^2 - x^2)$

At position x_1 , $v_1^2 = \omega^2 (A^2 - x_1^2)$ ...(i)

At position x_2 , $v_2^2 = \omega^2 (A^2 - x_2^2)$ ...(ii)

Subtracting (ii) from (i) $v_1^2 - v_2^2 = \omega^2(x_2^2 - x_1^2) \Rightarrow \omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

Time period $T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

Dividing (i) by (ii) $\frac{v_1^2}{v_2^2} = \frac{A^2 - x_1^2}{A^2 - x_2^2} \Rightarrow v_1^2A^2 - v_1^2x_2^2 = v_2^2A^2 - v_2^2x_1^2$

So $A^2(v_1^2 - v_2^2) = v_1^2x_2^2 - v_2^2x_1^2 \Rightarrow A = \sqrt{\frac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$

Example

A particle executing S.H.M. having amplitude 0.01 m and frequency 60 Hz. Determine maximum acceleration of particle.

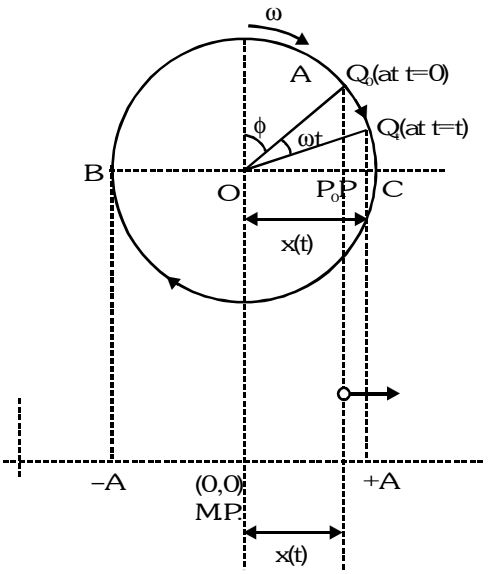
Solution

Maximum acceleration $a_{max.} = \omega^2A = 4\pi^2n^2A = 4\pi^2(60)^2 \ (0.01) = 144 \ \pi^2$ m/s^2

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P executes a simple harmonic motion on the x-axis between B and C. The angle that the radius OQ makes with the +ve vertical in clockwise direction in at t = 0 is equal to phase constant (ϕ).

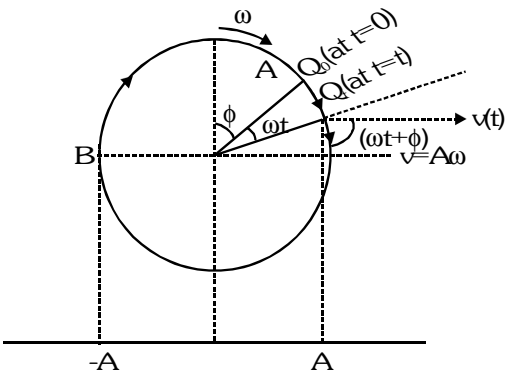


Let the radius OQ_0 makes an angle ωt with the OQ_t at time t. Then x(t)=Asin(ωt+ϕ)

In the above discussion the foot of projection is x-axis so it is called horizontal phasor. Similarly the foot of perpendiuclar on y-axis will also executes SHM of amplitude A and angular frequency ω[y(t)=Acosωt]. This is called vertical phasor. The phaser of the two SHM differ by π/2.

Problem solving stretegy in horiozntal phasor.

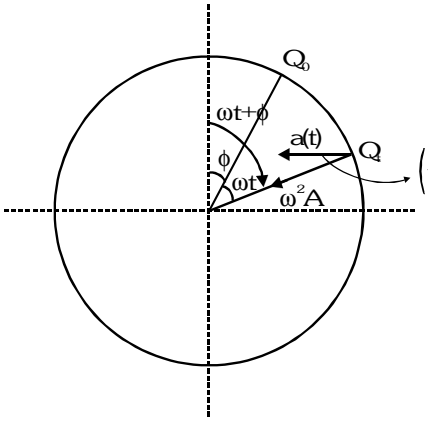
1. First assume circle of radius equal to amplitude of SHM.
2. Assume a particle rotating in a circular path moving with cosntant ω same as that of SHM in clockwise direction.
3. Angle made by the particle at t=0 with the upper vertical is equal to phase constant.
4. Horizontal component of velocity of particle gives you the velocity of particle performing SHM for example



from figure v(t) =Aωcos(ωt+ϕ)

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

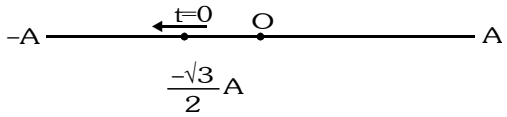
5. Component of acceleration of particle in horiozntal direction is equals to the acceleration of particle performing SHM. The acceleration of a particle in uniform circular motion is only centripetal and has a magnitude $a = \omega^2 A$



from figure $a(t) = -\omega^2 A\sin(\omega t+\phi)$

Example

A particle starts from point $x = \frac{-\sqrt{3}}{2}A$ and move towards negative extreme as shown following



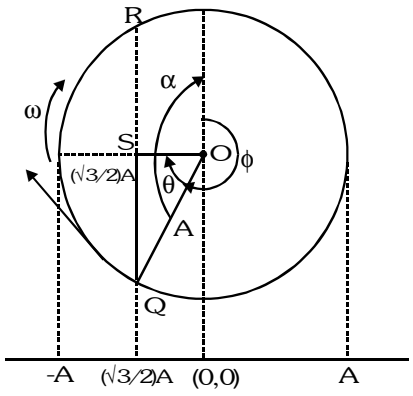
(a) Find the equation of the SHM

(b) Find the time taken by the particle to go directly from its initial position to negative extreme.

(c) Find the time taken by the particle to reach at mean position.

Solution

Figure shows the solution of the problem with the help of phasor. Horizontal component of velocity at Q gives the required direction of velocity at t = 0 so we will choose it.



In ΔOSQ : $\cos\theta = \frac{\sqrt{3}/2A}{A} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

Now $\phi = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

So equation of SHM is $x = A\sin\left(\omega t + \frac{4\pi}{3}\right)$

(b) Now to reach the particle at left extreme point it will travel angle θ along the circle. So time taken.

$t = \frac{\pi}{\omega} = \frac{\pi}{6\omega} \Rightarrow t = \frac{T}{12}\text{sec}$

(c) To reach the particle at mean position it will travel angle $\alpha = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

So, tike taken = $\frac{\alpha}{\omega} = \frac{T}{3}\text{sec}$

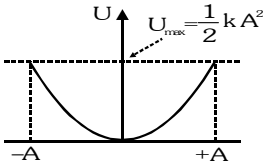
\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ENERGY OF PARTICLE IN S.H.M.

- **Potential Energy (U or P.E.)**

(i) In terms of displacement

The potential energy is related to force by the relation $F = -\frac{dU}{dx} \Rightarrow \int dU = -\int F dx$

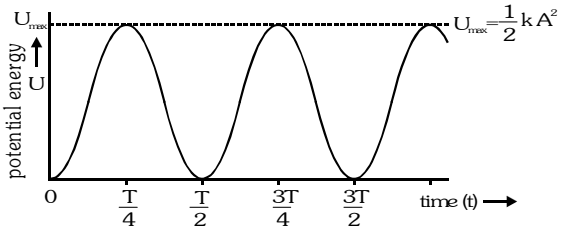


For S.H.M. $F = -kx$ so $\int dU = -\int(-kx)dx = \int kx\,dx \Rightarrow U = \frac{1}{2}kx^2 + C$

At $x = 0$, $U = U_0 \Rightarrow C = U_0$ So $U = \frac{1}{2}kx^2 + U_0$

Where the potenital energy at equilibrium position = U_0 when $U_0 = 0$ then $U = \frac{1}{2}kx^2$

(ii) In terms of time



Since $x = A\sin(\omega t + \phi)$, $U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$

If initial phase (ϕ) is zero then $U = \frac{1}{2}kA^2\sin^2\omega t = \frac{1}{2}m\omega^2A^2\sin^2\omega t$

Note :

(i) In S.H.M. the potential energy is a parabolic function of displacement, the potential energy is minimum at the mean position ($x = 0$) and maximum at extreme position ($x = \pm A$)

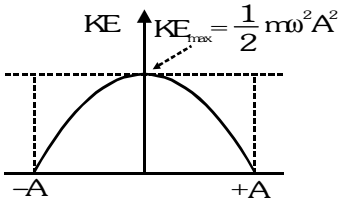
(ii) The potential energy is the periodic function of time.

It is minimum at $t = 0, \frac{T}{2}, T, \frac{3T}{2}\ldots$ and maximum at $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}\ldots$

- **Kinetic Energy (K)**

(i) In terms of displacement

If mass of the particle executing S.H.M. is m and Its velocity is v then kinetic energy at any instant.



$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

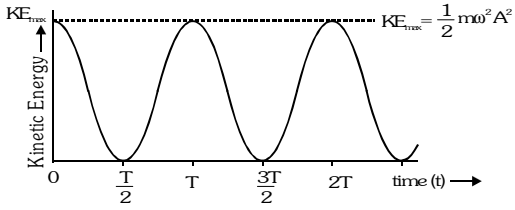
(ii) In terms of time

$v = A\omega\cos(\omega t + \phi)$

$K = \frac{1}{2} m\omega^2A^2 \cos^2(\omega t + \phi)$

If initial phase ϕ is zero

$K = \frac{1}{2} m\omega^2A^2 \cos^2\omega t$



Note :

(i) In S.H.M. the kinetic energy is a inverted parabolic function of displacement. The kinetic energy is maximum $(\frac{1}{2} kA^2)$ at mean position (x = 0) and minimum (zero) at extreme position ($x = \pm A$)

(ii) The kinetic energy is the periodic function of time. It is maximum at t = 0, T, 2T, 3T.....and minimum at $t = \frac{T}{2}, \frac{3T}{2}, \frac{5T}{2} \ldots$

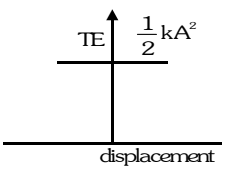
- **Total energy (E)**

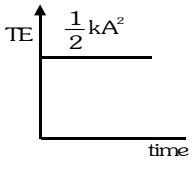
Total energy in S.H.M. is given by ; E = potential energy + kinetic energy = U + K

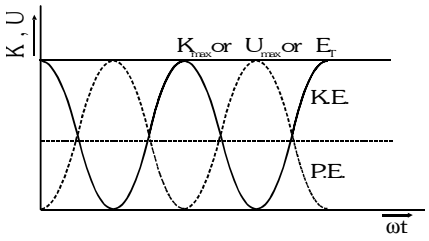
(i) w.r.t. position $E = \frac{1}{2} kx^2 + \frac{1}{2} k(A^2 - x^2) \quad \Rightarrow \quad E = \frac{1}{2} kA^2 = \text{constant}$

(ii) w.r.t. time

$$E = \frac{1}{2} m\omega^2A^2 \sin^2\omega t + \frac{1}{2} m\omega^2A^2 \cos^2\omega t = \frac{1}{2} m\omega^2A^2 \left(\sin^2\omega t + \cos^2\omega t\right) = \frac{1}{2} m\omega^2A^2 = \frac{1}{2} kA^2 = \text{constant}$$







Note :

(i) Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

- **Average energy in S.H.M.**

(i) The time average of P.E. and K.E. over one cycle is

(a) $\langle KE\rangle_t = \langle \frac{1}{2} m\omega^2A^2 \cos^2\omega t \rangle = \frac{1}{2} m\omega^2A^2 \langle \cos^2\omega t \rangle = \frac{1}{2} m\omega^2A^2 \left(\frac{1}{2}\right) = \frac{1}{4} m\omega^2A^2 = \frac{1}{4} kA^2$

(b) $\langle PE\rangle_t = \langle \frac{1}{2} m\omega^2A^2 \cos^2\omega t \rangle = \frac{1}{2} m\omega^2A^2 \langle \sin^2\omega t \rangle = \frac{1}{2} m\omega^2A^2 \left(\frac{1}{2}\right) = \frac{1}{4} m\omega^2A^2 = \frac{1}{4} kA^2$

(c) $\langle TE\rangle_t = \langle \frac{1}{2} m\omega^2A^2 + U_0 \rangle = \frac{1}{2} m\omega^2A^2 + U_0 = \frac{1}{2} kA^2 + U_0$

(ii) The position average of P.E. and K.E. between x = -A to x=A

\\node6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

(a) $\langle KE\rangle_x = \dfrac{\int_{-A}^{A}\frac{1}{2}m\omega^2\left(A^2-x^2\right)dx}{\int_{-A}^{A}dx} = \dfrac{\frac{1}{2}m\omega^2\left(A^2x-\frac{x^3}{3}\right)_{-A}^{A}}{2A} = \dfrac{1}{3}kA^2$

(b) $\langle PE\rangle_x = \dfrac{\int_{-A}^{A}(PE)dx}{\int_{-A}^{A}dx} = \dfrac{\int_{-A}^{A}\left(U_0+\frac{1}{2}kA^2\right)}{\int_{-A}^{A}dx} = \dfrac{U_0(2A)+\frac{1}{2}k\left(\frac{x^3}{3}\right)_{-A}^{A}}{2A} = U_0 = \dfrac{1}{6}kA^2$

(c) $\langle TE\rangle_x = \dfrac{\int_{-A}^{A}(TE)dx}{\int_{-A}^{A}dx} = \dfrac{\int_{-A}^{A}\left(\frac{1}{2}kA^2+U_0\right)}{\int_{-A}^{A}dx} = \dfrac{1}{2}kA^2+U_0$

GOLDEN KEY POINTS

- Both kinetic energy and potential energy varies periodically but the variation is not simple harmonic.
- The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.
- Frequency of total energy is zero because it remains constant.

Example

In case of simple harmonic motion –

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude.

(b) At what displacement the kinetic and potential energies are equal.

Solution

In S.H.M. : Kinetic Energy $K = \frac{1}{2}k(A^2 - x^2)$,Potential Energy $U = \frac{1}{2}kx^2$, Total Energy (TE) $= \frac{1}{2}KA^2$

(a) Fraction of Kinetic Energy $f_{K.E.} = \dfrac{K}{T.E.} = \dfrac{A^2-x^2}{A^2}$, Fraction of Potential Energy $f_{P.E.} = \dfrac{U}{T.E} = \dfrac{x^2}{A^2}$

at $x = \dfrac{A}{2}$ $\quad f_K = \dfrac{A^2-A^2/4}{A^2} = \dfrac{3}{4}$ and $f_U = \dfrac{A^2/4}{A^2} = \dfrac{1}{4}$

(b) $K = U \Rightarrow \frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2 \Rightarrow 2x^2 = A^2 \Rightarrow x = \pm\dfrac{A}{\sqrt{2}}$

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

The potential energy of a particle oscillating on x-axis is $U = 20 + (x - 2)^2$. Here U is in joules and x in meters. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Solution

(a) $F = -\frac{dU}{dx} = -2(x - 2)$ By assuming $x - 2 = X$, we have $F = -2X$

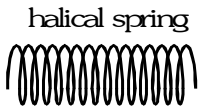
Since, $F \propto -X$ The motion of the particle is simple harmonic

(b) The mean position of the particle is $X = 0 \Rightarrow x - 2 = 0$, which gives $x = 2m$

(c) Maximum kinetic energy of the particle is, $K_{max} = E - U_{min} = 36 - 20 = 16$ J

Note : U_{min} is 20 J at mean position or at x = 2m.

SPRING SYSTEM



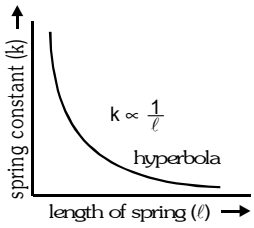


(i) When spring is given small displacement by stretching or compressing it, then restoring elastic force is developed in it because it obeys Hook's law.

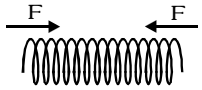
$F \propto -x \quad \Rightarrow F = -kx$ Here k is spring constant

(ii) Spring is assumed massless, so restoring elastic force in spring is assumed same everywhere.

(iii) Spring constant (k) depends on length (ℓ), radius and material of wire used in spring. for spring $k\ell$ = constant



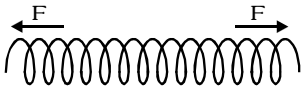
(iv) When spring is compressed or stretched then work done on it is stored as elastic potential energy.



compressing a spring

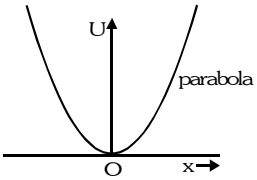


spring without deformation



stretching a spring

$W = \int F dx = \int kx\, dx$ and $U = W = \frac{1}{2}kx^2$

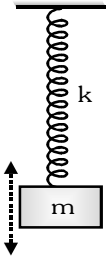


When spring is stretched from ℓ_1 to ℓ_2 then Work done $W = \frac{1}{2}k(\ell_2^2 - \ell_1^2)$

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

SPRING PENDULUM

(i) When a small mass is suspended from a mass-less spring then this arrangement is known as spring pendulum. For small linear displacement the motion of spring pendulum is simple harmonic.



(ii) For a spring pendulum

$$F = -kx \Rightarrow m\frac{d^2x}{dt^2} = -kx \quad [\because F = ma = m\frac{d^2x}{dt^2}]$$

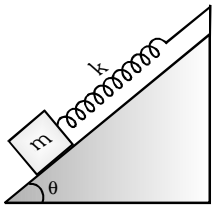
$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \because \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega^2 = \frac{k}{m}$$

This is standard equation of linear S.H.M.

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$, Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

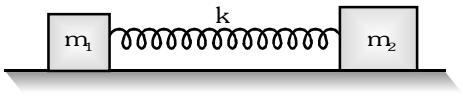
(iii) Time period of a spring pendulum is independent of acceleration due to gravity. This is why a clock based on oscillation of spring pendulum will keep proper time everywhere on a hill or moon or in a satellite or different places of earth.

(iv) If a spring pendulum oscillates in a vertical plane is made to oscillate on a horizontal surface or on an inclined plane then time period will remain unchanged.



(v) By increasing the mass, time period of spring pendulum increases $(T \propto \sqrt{m})$, but by increasing the force constant of spring (k), Its time period decreases $\left[T \propto \frac{1}{\sqrt{k}}\right]$ whereas frequency increases $(n \propto \sqrt{k})$

(vi) If two masses m_1 and m_2 are connected by a spring and made to oscillate then time period $T = 2\pi\sqrt{\frac{\mu}{k}}$



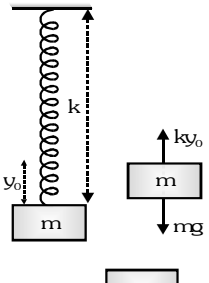
Here, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass

(vii) If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass m.

$ky_0 = mg$ i.e., $\frac{m}{k} = \frac{y_0}{g}$

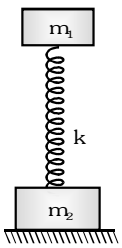
So, time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$

But remember time period of spring pendulum is independent of acceleration due to gravity.



(viii) If two particles are attached with spring in which only one is oscillating

Time period $= 2\pi\sqrt{\frac{\text{mass of oscillating particle}}{\text{force constant}}} = 2\pi\sqrt{\frac{m_1}{k}}$

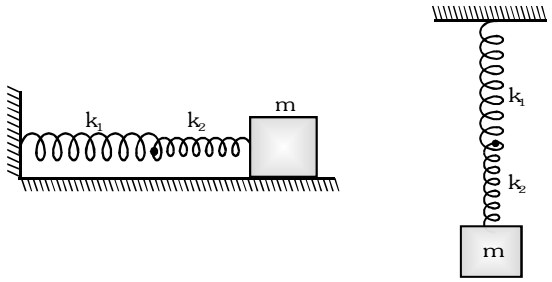


\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory, pg5

VARIOUS SPRING ARRANGEMENTS

- **Series combination of springs**

In series combination same restoring force exerts in all springs but extension will be different.



Total displacement $x = x_1 + x_2$

Force acting on both springs $F = -k_1x_1 = -k_2x_2$

$\because x_1 = -\frac{F}{k_1}$ and $x_2 = -\frac{F}{k_2}$ $\quad\therefore\quad x = -\left[\frac{F}{k_1}+\frac{F}{k_2}\right]$ ....(i)

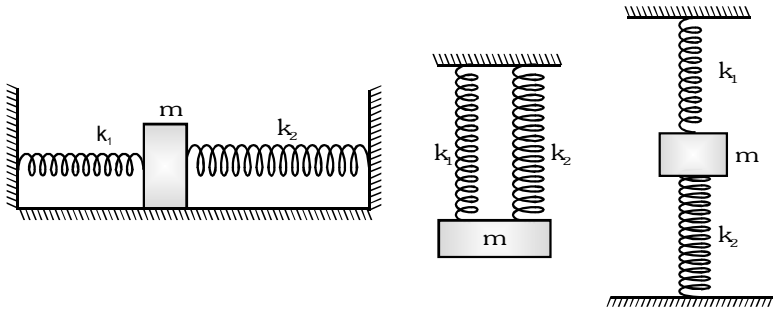
If equivalent force constant is k_s then $F = -k_sx$

so by equation (i) $-\frac{F}{k_s} = -\frac{F}{k_1} - \frac{F}{k_2} \Rightarrow \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k_s = \frac{k_1k_2}{k_1+k_2}$

Time period $T = 2\pi\sqrt{\frac{m}{k_s}} = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$

Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k_s}{m}}$, Angular frequency $\omega = \sqrt{\frac{k_s}{m}}$

- **Parallel Combination of springs**



In parallel combination displacement on each spring is same but restoring force is different.

Force acting on the system $F = F_1 + F_2 \Rightarrow F = -k_1x - k_2x$ ...(i)

If equivalent force constant is k_P then, $F = -k_Px$, so by equation (i) $-k_Px = -k_1x - k_2x \Rightarrow k_P = k_1 + k_2$

Time period $T = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$; Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k_P}{m}}$; Angular frequency $\omega = \sqrt{\frac{k_1+k_2}{m}}$

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example

A body of mass m attached to a spring which is oscillating with time period 4 seconds. If the mass of the body is increased by 4 kg, its timer period increases by 2 sec. Determine value of initial mass m.

Solution

In *I^st case* : $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow 4 = 2\pi\sqrt{\frac{m}{k}}$ ...(i) and in *II^nd case:* $6 = 2\pi\sqrt{\frac{m+4}{k}}$ ...(ii)

Divide (i) by (ii) $\frac{4}{6} = \sqrt{\frac{m}{m+4}} \Rightarrow \frac{16}{36} = \frac{m}{m+4} \Rightarrow m = 3.2$ kg

Example

One body is suspended from a spring of length ℓ, spring constant k and has time period T. Now if spring is divided in two equal parts which are joined in parallel and the same body is suspended from this arrangement then determine new time period.

Solution

Spring constant in parallel combination k' = 2k + 2k = 4k

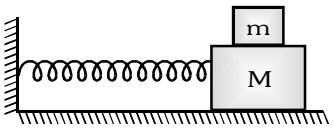
$$\therefore T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{m}{k}} \times \frac{1}{\sqrt{4}} = \frac{T}{\sqrt{4}} = \frac{T}{2}$$

Example

A block is on a horizontal slab which is moving horizontally and executing S.H.M. The coefficient of static friction between block and slab is μ. If block is not separated from slab then determine angular frequency of oscillation.

Solution

If block is not separated from slab then restoring force due to S.H.M. should be less than frictional force between slab and block.



$$F_{restoring} \le F_{friction} \Rightarrow m\,a_{max.} \le \mu mg \Rightarrow a_{max.} \le \mu g \Rightarrow \omega^2 A \le \mu g \Rightarrow \omega \le \sqrt{\frac{\mu g}{A}}$$

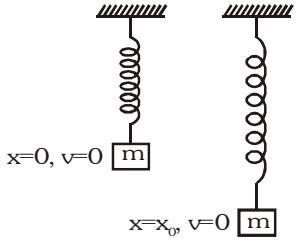
Example

A block of mass m is suspended from a spring of spring constant k. Find the amplitude of S.H.M.

Solution

Let amplitude of S.H.M. be x_0 then by work energy theorem W = ΔKE

$$mgx_0 - \frac{1}{2}kx_0^2 = 0 \Rightarrow x_0 = \frac{2mg}{k}$$



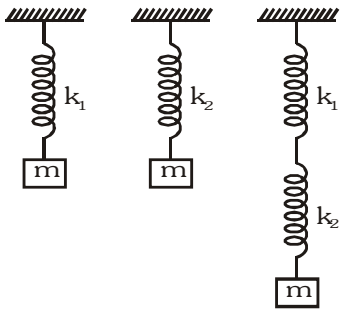
\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

Periodic time of oscillation T_1 is obtained when a mass is suspended from a spring if another spring is used with same mass then periodic time of oscillation is T_2. Now if this mass is suspended from series combination of above springs then calculate the time period.

Solution

$$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \Rightarrow T_1^2 = 4\pi^2\frac{m}{k_1} \Rightarrow k_1 = \frac{4\pi^2 m}{T_1^2} \text{ and } T_2 = 2\pi\sqrt{\frac{m}{k_2}} \Rightarrow T_2^2 = 4\pi^2\frac{m}{k_2} \Rightarrow k_2 = \frac{4\pi^2 m}{T_2^2}$$



Now $T = 2\pi\sqrt{\frac{m}{k'}}$ where $\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k' = \frac{k_1 k_2}{k_1 + k_2} = \frac{\left(\frac{4\pi^2 m}{T_1^2}\right)\left(\frac{4\pi^2 m}{T_2^2}\right)}{\frac{4\pi^2 m}{T_1^2} + \frac{4\pi^2 m}{T_2^2}}$; $k' = \frac{4\pi^2 m \left[\frac{4\pi^2 m}{T_1^2 T_2^2}\right]}{4\pi^2 m\left[\frac{1}{T_1^2} + \frac{1}{T_2^2}\right]} = \frac{4\pi^2 m}{T_1^2 + T_2^2}$

$$\therefore T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{\frac{4\pi^2 m}{T_1^2 + T_2^2}}} = \sqrt{T_1^2 + T_2^2}$$

Example

Infinite spring with force constants k, 2k, 4k, 8k, ... respectively are connected in series. Calculate the effective force constant of the spring.

Solution

$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \ldots\ldots\ldots\ldots\infty$ (For infinite G.P. $S_\infty = \frac{a}{1-r}$ where a = First term, r = common ratio)

$$\frac{1}{k_{eff}} = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\right] = \frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{k} \text{ so } k_{eff} = k/2$$

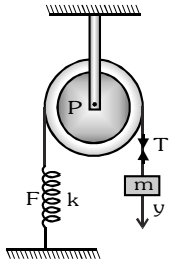
Example

Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass m = 1 kg. If the block is slightly displaced vertically down from its equilibrium position and released find the frequency of its vertical oscillation in given cases.

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Solution

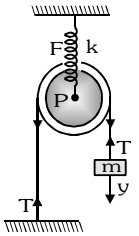
Case (A) :



As the pully is fixed and string is inextensible, if mass m is displaced by y the spring will stretch by y, and as there is no mass between string and spring (as pully is massless) F = T = ky i.e., restoring force is linear and so motion of mass m will be linear simple harmonic with frequency

$$n_A = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4000}{1}} \approx 10 \text{ Hz}$$

Case (B) :

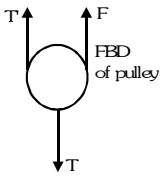


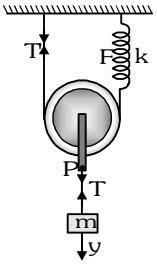
The pulley is movable and string inextensible, so if mass m moves down a distance y, the pulley will move down by (y/2). So the force in the spring F = k(y/2).

Now as pully is massless F = 2T, $\Rightarrow$ T = F/2 = (k/4)y. So the restoring force on the mass m $T = \frac{1}{4}ky = k'y \Rightarrow k' = \frac{1}{4}k$

So $n_B = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{4\pi}} = \frac{n_A}{2} = 5$ Hz

Case (C) :





In this situation if the mass m moves by y the pully will also move by y and so the spring will stretch by 2y (as string is inextensible) and so T' = F = 2ky. Now as pulley is massless so T = F + T'= 4ky, i.e., the restoring force on the mass m

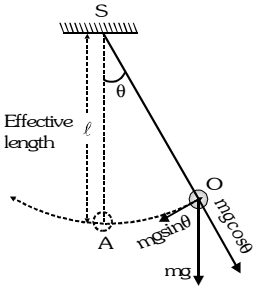
$$T = 4ky = k'y \Rightarrow k' = 4k$$

so $n_C = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}} = 2n_A = 20\text{Hz}$

SIMPLE PENDULUM

If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum

- **Expression for time period**



Restoring force acting on pendulum F = – mg sinθ

For small angle $\sin\theta \approx \frac{OA}{SA} = \frac{y}{\ell}$ $\therefore ma = -mg \times \frac{y}{\ell} \Rightarrow a = -\frac{g}{\ell}y$

It proves that if displacement is small then simple pendulum performs S.H.M.

$$\because |a| = \omega^2 y \Rightarrow \omega^2 = \frac{g}{\ell} \Rightarrow \omega = \sqrt{\frac{g}{\ell}} \therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

- $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum (ℓ) is negligible as compare to radius of earth ($\ell << R_e$) but

  if ℓ is comparable to radius of earth then time period $T = 2\pi\sqrt{\frac{R_e}{\left[1+\frac{R_e}{\ell}\right]g}}$

- The time period of oscillation of simple pendulum of infinite length ($\ell \to \infty$)

  $$T = 2\pi\sqrt{\frac{R}{g}} \approx 84.6 \text{ minute} \approx 1\frac{1}{2} \text{ hour (It is maximum time period)}$$

- If angular amplitude (θ_0) is large ($\theta_0 > 15$) then time period is given by $T = 2\pi\sqrt{\frac{\ell}{g}}\left[1+\frac{\theta_0^2}{16}\right]$ here θ_0 is in radian.

- If a simple pendulum of density ρ is made to oscillate in a liquid of density σ then its time period will increase as compare to that of air and is given by $T = 2\pi\sqrt{\frac{\ell}{\left[1-\frac{\sigma}{\rho}\right]g}}$

- **Second's pendulum**

  If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

  For second's pendulum, time period $T = 2 = 2\pi\sqrt{\frac{\ell}{g}}$. At the surface of earth $g = 9.8\ m/s^2 \approx \pi^2\ m/s^2$.

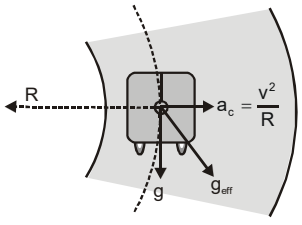
  So length of second pendulum at the surface of earth $\ell \approx 1$ meter

Example

A simple pendulum of length L and mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes oscillation in a radial direction about its equilibrium position, then calculate its time period .

Solution

Centripetal acceleration $a_c = \frac{v^2}{R}$ & Acceleration due to gravity = g



So $g_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2} \Rightarrow$ Time period $T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2+\frac{v^4}{R^2}}}}$

\\node6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

A simple pendulum is suspended from the ceiling of a lift. When the lift is at rest, its time period is T. With what acceleration should lift be accelerated upwards in order to reduce its time period to $\frac{T}{2}$.

Solution

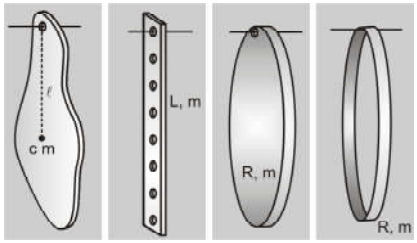
In stationary lift $T = 2\pi\sqrt{\frac{\ell}{g}}$ ...(i) In accelerated lift $\frac{T}{2} = T' = 2\pi\sqrt{\frac{\ell}{g+a}}$ ...(ii)

Divide (i) by (ii) $2 = \sqrt{\frac{g+a}{g}} \Rightarrow g + a = 4g \Rightarrow a = 3\ g$

COMPOUND PENDULUM

Any rigid body which is free to oscillate in a vertical plane about a horizontal axis passing through a point, is define compound pendulum

- **Expression for time period**



Torque acting on a body $\tau = -mg\ell\sin\theta$ if angle is very small $\sin\theta \approx \theta$ then $\tau = -mg\ell\theta$ ...(i) and $\tau = I_s\,\alpha$ ...(ii)

Here m = mass of the body

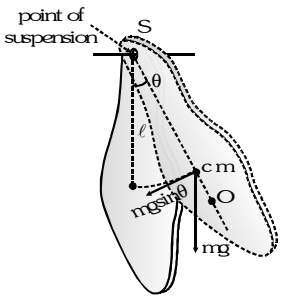
ℓ = distance between point of suspension and centre of mass

I_s = moment of inertia about horizontal axis passes through point of suspension

From equation (i) and (ii) $I_s\,\alpha = -mg\ell\theta$

$$I_s\frac{d^2\theta}{dt^2} + mg\ell\theta = 0\ ,\ \frac{d^2\theta}{dt^2} + \frac{mg\ell}{I_s}\ \theta = 0 \quad ...(iii)$$

$$\because\ \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad ...(iv)$$



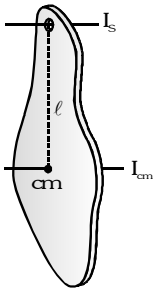
Compare equation (iii) and (iv) $\omega^2 = \frac{mg\ell}{I_s} \Rightarrow \omega = \sqrt{\frac{mg\ell}{I_s}}$

Time period of compound pendulum $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_s}{mg\ell}}$

Applying parallel axis theorem $I_s = I_{CM} + m\ell^2 \Rightarrow I_s = mK^2 + m\ell^2$

\Nodes\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory p65

$\because \quad T = 2\pi\sqrt{\frac{I_s}{mg\ell}} = 2\pi\sqrt{\frac{mK^2 + m\ell^2}{mg\ell}} \Rightarrow T = 2\pi\sqrt{\frac{\frac{K^2}{\ell} + \ell}{g}}$



Here S = point of suspension ; O = point of oscillation ; K = radius of gyration

$L = \frac{k^2}{\ell} + \ell$ = equivalent length of simple pendulum

ℓ = distance between point of suspension and point of oscillation

Time Period $T = 2\pi\sqrt{\frac{\frac{K^2}{\ell} + \ell}{g}}$

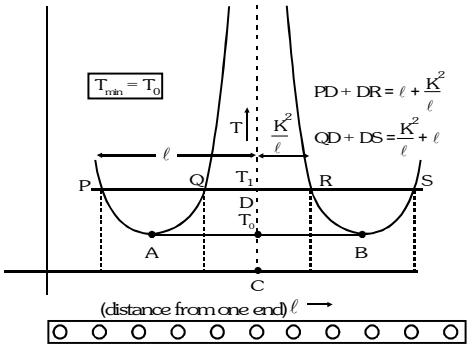
For maximum time period $\ell = 0$ Maximum time period $T_{max} = \infty$

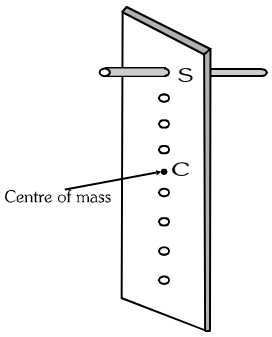
For minimum time period $\frac{dT}{d\ell} = 0$ then $K = \ell \Rightarrow T_{min} = T = 2\pi\sqrt{\frac{\frac{K^2}{K} + K}{g}} = 2\pi\sqrt{\frac{2K}{g}}$

Note : $\frac{k^2}{\ell}$ = Distance between centre of mass and point of oscillation. It has uniform cross section area and width at any point.

- **Bar pendulum**

A bar pendulum is a steel bar of 1 meter length with holes at regular intervals for suspension. The time period is measured for different values of ℓ (distance between S and C). The graph between T and length from one end ℓ is as shown in fig below. The time period is infinite when $\ell = 0$, i.e., when it is suspended from the centre of gravity (centre of mass).





At four points P, Q, R and S, the time period is the same T_1.
The distance are such that

$\frac{PR + QS}{2} = \ell_{eq} = \ell + K^2/\ell$ Also $PD = \ell$ $DR = K^2/\ell$

The time period is minimum when $\ell = K$

In Fig. AB= 2K. The minimum period is $T_0 = 2\pi\sqrt{\frac{2K}{g}}$

Condition for T minimum

$T^2 = \frac{4\pi^2}{g}\left(\frac{K^2}{\ell} + \ell\right)$

diff. w.r. to ℓ

$2T\frac{dT}{d\ell} = \left(\frac{4\pi^2}{g}\right)\left[-\frac{K^2}{\ell^2} + 1\right]$

$\because T \neq 0$, and with $\frac{dT}{d\ell} = 0$

$-\frac{K^2}{\ell^2} + 1 = 0$ or $K^2 = \ell^2$

$K = \pm\ell$ then $T_{min} = T_0$

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

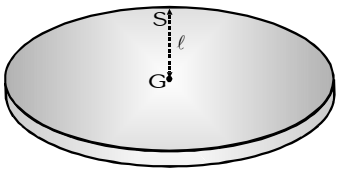
GOLDEN KEY POINTS

- There are maximum four points for which time period of compound pendulum is same.
- Minimum time period is obtained at two points
- The point of suspension and point of oscillation are mutually interchangeable.
- Maximum time period will obtain at centre of gravity, which is infinite means compound pendulum will not oscillate at this point.
- Compound pendulum executes angular S.H.M. about its mean position. Here restoring torque is provided by gravitational force.

Example

A disc is made to oscillate about a horizontal axis passing through mid point of its radius. Determine time period.

Solution



For disc $I = MK^2 = \frac{MR^2}{2} \Rightarrow K = \frac{R}{\sqrt{2}}, \ell = \frac{R}{2}$

$$L = \ell + \frac{K^2}{\ell} = \frac{R}{2} + \frac{R^2}{2\left(\frac{R}{2}\right)} = \frac{R}{2} + R = \frac{3R}{2} \Rightarrow T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Example

A rod with rectangular cross section oscillates about a horizontal axis passing through one of its ends and it behaves like a second's pendulum. Determine its length.

Solution

Because oscillating rod behaves as a second's pendulum so its time period will be 2 second.

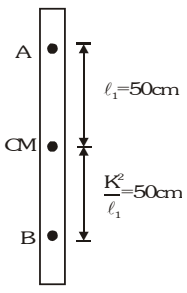
$$T = 2\pi\sqrt{\frac{\ell + \frac{K^2}{\ell}}{g}} = 2s \Rightarrow \ell + \frac{K^2}{\ell} = 1 \ldots(i) \ [\because \pi^2 = g]$$

Assume length of rod is L, because axis passes through one end So $\ell = \frac{L}{2}$ and $K^2 = \frac{L^2}{12}$

Putting this values in equation we get $\frac{L}{2} + \frac{L^2}{12} \times \frac{2}{L} = 1 \quad \Rightarrow L = 1.5$ m

Example

The time period of a bar pendulum when suspended at distances 30 cm and 50 cm from its centre of gravity comes out to be the same. If the mass of the body is 2kg. Find out its moment of inertia about an axis passing through first point.



Solution

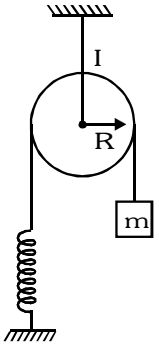
$I_A = MK^2 + M\ell_1^2$ but $\ell_1 = 50$ cm & $\frac{K^2}{\ell_1} = 30$ cm $\Rightarrow K^2 = 30 \quad 50 \ cm^2$

$I_A = 2 \quad (30 \quad 50 \quad 10^{-4}) + 2 \quad (50 \quad 10^{-2})^2 = 0.3 + 0.18 = 0.48 \ kg\text{-}m^2$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

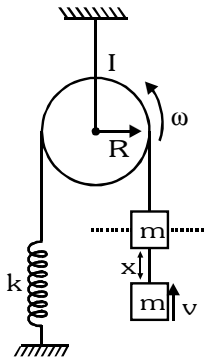
Example

Figure shows a pulley block system in equilibrium. If the block is displaced down slightly from its equilibrium position and released. Find the time period of oscillation of the system. Assume there is sufficient friction present between pulley and string so that string will not slip over pulley surface.



Solution

If m is in equilibrium tension in string must be mg and spring is stretched by h so that mg=kh. If we displace the block downward by a distance A and releaseased, it starts executing SHM with amplitude A. During its oscillation we consider the block at a displacement x below the equilirbium position, if it is moving at a speed v at this position, the pulley will be rotating at an angular speed ω given as $\omega = \frac{v}{r}$



Thus at this position the total energy of oscillating system is $E_T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x+h)^2 - mgx$

Differentiating with respect to time,

we get $\frac{dE_T}{dt} = \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + \frac{1}{2}I\left(\frac{1}{r^2}\right)\left(2v\frac{dv}{dt}\right) + \frac{1}{2}k\left[2(x+h)\frac{dx}{dt}\right] - mg\left(\frac{dx}{dt}\right) = 0$

or $mva + \frac{1}{r^2}va + k(x+h)v - mgv = 0$ or $a + \left(\frac{k}{m + \frac{1}{r^2}}\right)x = 0$ [as mg = kg]

Comparing above equation with standard differential equation of SHM we get $\omega = \sqrt{\left(\frac{k}{m + \frac{I}{1/r^2}}\right)}$

Thus time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m + \frac{I}{r^2}}{k}}$

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example

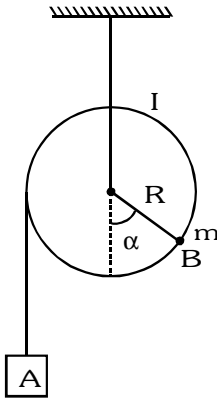


Figure shows a pulley block system in which a block A is hanging on one side of pulley and an other side a small bead B of mass m is welded on pulley. The moment of inertia of pulley is I and the system is in equilibrium when bead is at an angle α from the vertical. If the system is slightly disturbed from its equilibrium position, find the time period of its oscillations.

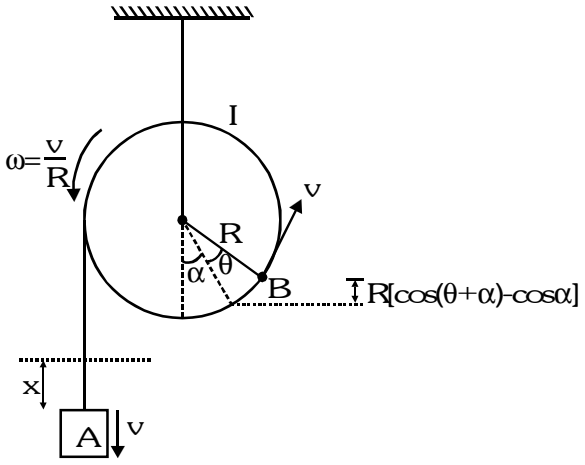
Solution

In equilibrium the net torque on pulley must be zero, thus we have

$MgR=mgR\sin\alpha$ or $M=m\sin\alpha$ [if mass of block A is assumed to be M]

Now if block is displace down by distance A and released, it starts oscillating with amplitude A. Now consider the block at a distance x below the equilibrium position when it is going down at speed v. Figure shows the corresponding situation at this instant and the total energy of oscillating system can be written as

$$E_T=\frac{1}{2}Mv^2+\frac{1}{2}mv^2+\frac{1}{2}I\left(\frac{v}{R}\right)^2-Mgx+mgR\left[\cos\alpha-\cos(\theta+\alpha)\right]$$



Differentiating the above equation w.r. to time, we get

$$\frac{dE_T}{dt}=\frac{1}{2}M\left(2v\frac{dv}{dt}\right)+\frac{1}{2}m\left(2v\frac{dv}{dt}\right)+\frac{1}{2}\frac{I}{R^2}\left(2v\frac{dv}{dt}\right)-Mg\left(\frac{dx}{dt}\right)+mgR\left[-\sin(\theta+\alpha)\frac{d\theta}{dt}\right]=0$$

$$\Rightarrow Mva+mva+\frac{I}{R^2}va-Mgv+mgR\sin(\theta+\alpha)\left(\frac{v}{R}\right)=0 \quad \left[\text{as}\frac{d\theta}{dt}=\omega=\frac{v}{R}\right]$$

$$\Rightarrow \left(M+m+\frac{I}{R^2}\right)a-Mg-mg\left[\alpha\cos\alpha+\sin\alpha\right]=0$$

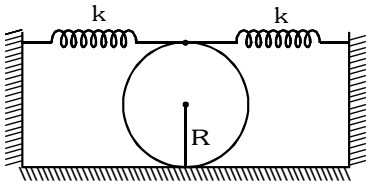
$$\Rightarrow\left(M+m+\frac{I}{R^2}\right)a+mg\cos\alpha\cdot\frac{x}{R}=0 \quad [\text{as } M=m\sin\alpha \text{ and } \theta=\frac{x}{R}] \Rightarrow a=-\frac{mg\cos\alpha}{R\left(M+m+\frac{I}{R^2}\right)}x$$

Comparing equation with basic differential equation of SHM, we get the angular frequency of SHM of system as $\omega=\sqrt{\dfrac{mg\cos\alpha}{R\left(M+m+\dfrac{I}{R^2}\right)}}$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

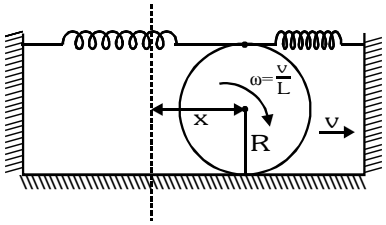
Example

A solid uniform cylinder of mass M performs small oscillations in horizontal plane if slightly displaced from its mean position shown in figure. If it is given that initially springs are in natural lengths and cylinder does not slip on ground during oscillations due to friction between ground and cylinder. Force constant of each spring is k. Find time period of these oscillation.



Solution

In the situation given in problem, the cylinder is in its equilibrium position when springs are unstrained. When it slightly rolled and released. It starts executing SHM and due to friction, the cylinder is in pure rolling motion. Now during oscillations we consider the cylinder whe it is at a distance x from the mean position and moving with a speed v as shown in figure. As cylinder is in pure rolling, its angular speed of rotation can be given as



As centre of cylinder is at a distance x from the initial position, the springs which are connect at a point on its rim must be compressed and stretched by a distance 2x. Thus at this intermediate position total energy of the oscillating system can be given as

$$E_T = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 + \frac{1}{2}k(2x)^2 \times 2$$

Differentiating with respect to time, we get

$$\frac{dE_T}{dt} = \frac{1}{2}M\left(2v\frac{dv}{dt}\right) + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{1}{R^2}\left(2v\frac{dv}{dt}\right) + 4k\left(2x\frac{dx}{dt}\right) = 0$$

$$\Rightarrow Mva + \frac{1}{2}Mva + 8kxv = 0 \quad \Rightarrow \quad a = -\frac{16}{3}\frac{k}{M}x$$

Comparing equation with basic differential equation of SHM.

We get, the angular frequency of SHM as $\omega = \sqrt{\frac{16k}{3M}}$

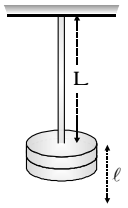
Thus time period of these oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{16k}} = \frac{\pi}{2}\sqrt{\frac{3M}{k}}$

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

EXAMPLES OF SIMPLE HARMONIC MOTION

- If a mass m is suspended from a wire of length L, cross section A and young's modulus Y and is pulled along the length of the wire then restoring force will be developed by the elasticity of the wire.

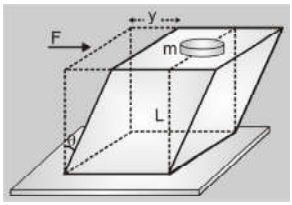
$$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow \frac{F/A}{\ell/L} \Rightarrow = \frac{FL}{\ell A} \Rightarrow F = -\frac{YA}{L}\ell$$



Restoring force is linear so motion is linear simple harmonic with force constant

$$k = \frac{YA}{L} \quad \text{i.e.,} \quad n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{YA}{mL}}$$

- If the lower surface of a cube of side L and of modulus of rigidity η fixed while fixing a particle of mass m on the upper face, a force parallel to upper face is applied and withdrawn; Here restoring force will be developed due to elasticity of block.



Modulus of rigidity of the block $\eta = \frac{\text{shear stress}}{\text{shear strain}}$

$$\eta = \frac{F}{A\theta} \Rightarrow F = \eta\frac{A}{L}y \quad [\text{as } \theta = \frac{y}{L}]$$

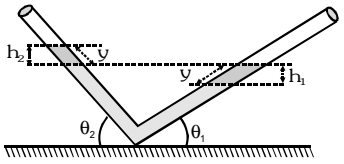
Restoring force is linear so motion will be linear S.H.M.

$$\text{Force constant } (k) = \eta\frac{A}{L} = \eta L \quad [\text{as } A = L^2] \quad \text{So } T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\eta L}}$$

- **Motion of a liquid in a V-shape tube when it is slightly depressed and released**

Here cross-section of the tube is uniform and the liquid is incompressible and non viscous. Initially the level of liquid in the two limbs will be at the same height. If the liquid is pressed by y in one limb, it will rise by y along the length of the tube in the other limb so the restoring force will developed by hydrostatic pressure difference, i.e.,

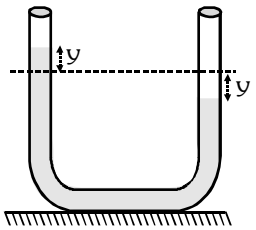
$$F = -\Delta P \quad A = -(h_1 + h_2)gdA \Rightarrow \quad F = -Agd(\sin\theta_1 + \sin\theta_2)y$$



As the restoring force is linear, motion will be linear simple harmonic.

Force constant (k) = $Agd(\sin\theta_1 + \sin\theta_2)$ So $T = 2\pi\sqrt{\frac{m}{Adg(\sin\theta_1 + \sin\theta_2)}}$

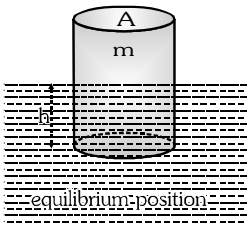
Note : If the tube is a U-tube and liquid is filled to a height h



$\theta_1 = \theta_2 = 90$ and $m = hAd \quad 2$ So time period $T = 2\pi\sqrt{\frac{h}{g}}$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

- **When a partially submerged floating body is slightly pressed and released :**

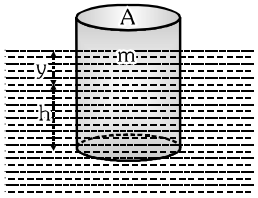


If a body of mass m and cross section A is floating in a liquid of density σ with height h inside the liquid then

$mg = \text{Thrust} = Ah\sigma g$, i.e., $m = Ah\sigma$ ...(i)

Now from this equilibrium position if it is pressed by y, restoring force will developed due to extra thrust i.e.

$F = -A\sigma gy$



As restoring force is linear, motion will be linear simple harmonic with force constant $k = A\sigma g$,

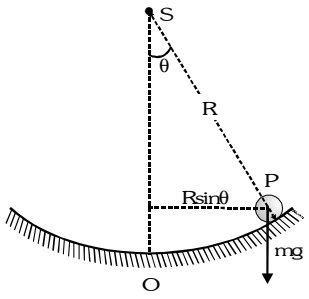
So $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{A\sigma g}}$

From this expression it is clear that if density of liquid decreases, time period will increase and vice-versa.

And also as from eq^{n}. (i) $m = Ah\sigma$, $T = 2\pi\sqrt{\frac{h}{g}}$ where h is the height of the body inside the liquid.

- **Motion of a ball in a bowl**

If a small steel ball of mass m is placed at a small distance from O inside a smooth concave surface of radius R and released, it will oscillate about O. The restoring torque here will be due to the force of gravity mg on the ball i.e., $\tau = -\text{mg}(R\sin\theta) = -\text{mgR}\theta$ [As θ is small]



Now as restoring torque is angular so motion will be angular simple harmonic. And as by definition.

$$\tau = I\alpha = mR^2\left[\frac{d^2\theta}{dt^2}\right] \quad [\text{as } I = mR^2 \text{ and } \alpha = \frac{d^2\theta}{dt^2}]$$

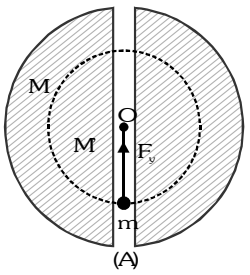
so, $mR^2\frac{d^2\theta}{dt^2} = -\text{mgR}\theta$ i.e., $\frac{d^2\theta}{dt^2} = -\omega^2\theta \Rightarrow \omega^2 = \frac{g}{R}$ so $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

- **Motion of a ball in a tunnel through the earth**

Case I :

If the tunnel is along a diameter and a ball is released from the surface. If the ball at any time is at a distance y from the centre of earth, then the restoring force will act on the ball due to gravitation between ball and earth. But from theory of gravitation we know that force that acts on a particle inside the earth at a distance y from its centre is only due to mass M' of the earth that lies within sphere of radius y. (the portion of the earth that lies out side this sphere does not exert any net force on the particle) so $F = \frac{-GmM'}{y^2}$



But as $M = \frac{4}{3}\pi R^3\rho$ and $M' = \frac{4}{3}\pi y^3\rho$, i.e., $M' = M\left[\frac{y}{R}\right]^3$

$$F = \frac{-Gm}{y^2} \quad M\left[\frac{y^3}{R^3}\right] = -\frac{GMm}{R^3}y$$

Restoring force is linear so the motion is linear SHM with force constant .

$$k = \frac{GMm}{R^3} \quad \text{so} \quad T = 2\pi\sqrt{\frac{m}{k}} \qquad = 2\pi\sqrt{\frac{R^3}{GM}}$$

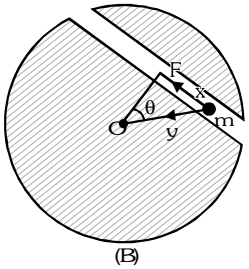
Further more as $g = \frac{GM}{R^2} \Rightarrow T = 2\pi\sqrt{\frac{R}{g}}$

Which is same as that of a simple pendulum of infinite length and is equal to 84.6 minutes.

Case II :

If the tunnel is along a chord and ball is released from the surface and if the ball at any time is at a distance x from the centre of the tunnel. The restoring force will be :

$$F' = F\sin\theta = \left[-\frac{GMm}{R^3}y\right]\left[\frac{x}{y}\right] = -\frac{GMm}{R^3}x$$



Which is again linear with same force constant $k = \frac{GMm}{R^3}$

So that motion is linear simple harmonic with same time period -

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minutes}$$

Note : In SHM $v_{max} = \omega A$

(i) In I case and IInd case time period will be same but v_{max} will be different.

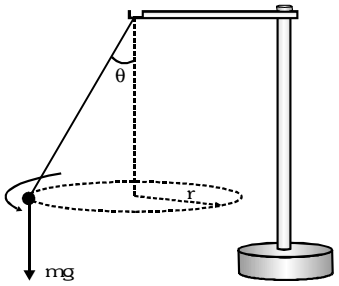
(ii) If ball is dropped from height h it will perform oscillatory motion not SHM [$F \propto \frac{1}{r^2}$ and not $F \propto (-r)$]

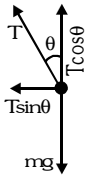
\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

- **Conical Pendulum**

It is not example of S.H.M. but example of periodic motion.

$T = 2\pi\sqrt{\frac{h}{g}}$ where $h = L\cos\theta$



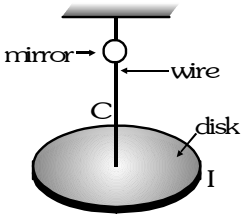


$h = \sqrt{L^2 - r^2}$

$\omega = \sqrt{\frac{g}{L\cos\theta}}$

- **Torsional Oscillator :** (Angular SHM)

$T = 2\pi\sqrt{\frac{I}{C}}$ where $C = \frac{\eta\pi r^4}{2\ell}$



η = modulus of elasticity of the wire ; r = radius of the wire

L = length of the wire ; I = Moment of inertia of the disc

- **Oscillation of piston in a frictionless gas chamber piston :**

$T = 2\pi\sqrt{\frac{Vm}{A^2E}}$ where V= volume of cylinder

m = mass of piston , A = area of cylinder ball , E = bulk modulus = $\frac{\Delta P}{-\Delta V/V}$

For Isothermal process : $E = P$, so $T = 2\pi\sqrt{Vm/PA^2}$

For Adiabatic process : $E = \gamma P$, so $T = 2\pi\sqrt{Vm/\gamma PA^2}$

Example

A liquid of mass m is set into oscillations in a U-tube of cross section A. Its time period recorded is T, where $T = 2\pi\sqrt{\frac{\ell}{2g}}$, here ℓ is the length of liquid column. If the liquid of same mass is set into oscillations in U-tube of cross section $\frac{A}{16}$ then determine time period of oscillation.

Solution

Mass is constant $\Rightarrow$ volume density = constant $\Rightarrow$ $V_1d = V_2d$

$(A\ell)d = \left[\frac{A}{16}\ell'\right]d \Rightarrow \ell' = 16\,\ell \quad \because T = 2\pi\sqrt{\frac{\ell}{2g}} \quad \therefore \frac{T'}{T} = \sqrt{\frac{\ell'}{\ell}} = \sqrt{\frac{16\ell}{\ell}} = 4 \Rightarrow \quad T' = 4T$

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

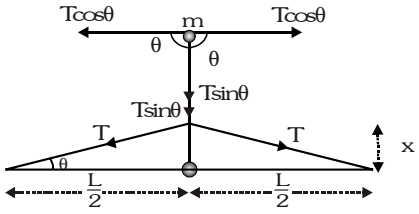
Example

A ball of mass m kept at the centre of a string of length L is pulled from center in perpendicular direction and released. Prove that motion of ball is simple harmonic and determine time period of oscillation

Solution

Restoring force $F = -2T\sin\theta$

When θ is small $\sin\theta \approx \tan\theta \approx \theta = \dfrac{x}{L/2}$



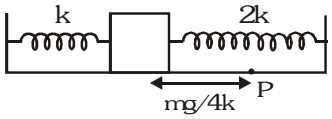
$$m\frac{d^2x}{dt^2} = -2T\sin\theta = -2T\theta = -2T\frac{x}{L/2}$$

$$\frac{d^2x}{dt^2} = -\frac{4T}{mL}x \Rightarrow \frac{d^2x}{dt^2} \propto -x$$

So motion is simple harmonic $\omega = \dfrac{2\pi}{T} = \sqrt{\dfrac{4T}{mL}} \Rightarrow T = 2\pi\sqrt{\dfrac{mL}{4T}}$

Example

In the spring mass system shown in the figure, the spring is compressed by $x_0 = \dfrac{mg}{2k}$ from its natural length and block is released from rest. Find the speed of the block when it passes through P (mg/4k distance from mean position)



Solution

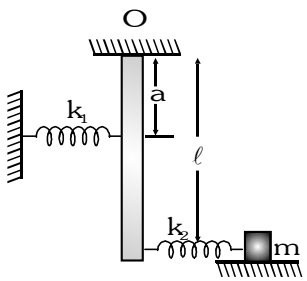
$\omega = \sqrt{\dfrac{3k}{m}}$, $x = A\sin(\omega+\phi)$, $v = A\omega\cos(\omega t+\phi)$ at $t=0$, $x=0 \Rightarrow \phi = 0$

$$x = A\sin\omega t \Rightarrow \frac{mg}{4k} = \frac{mg}{2k}\sin\omega t \Rightarrow \sin\omega t = \frac{1}{2} \Rightarrow \omega t = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{6\omega} = \frac{T}{12}$$

$$v = A\omega\cos\omega t, \quad v = \frac{mg}{2k}\sqrt{\frac{3k}{m}}\cos\left[\sqrt{\frac{3k}{m}}\left(2\pi\sqrt{\frac{m}{3k}}/12\right)\right] = g\sqrt{\frac{9m}{16k}}$$

Example

A very light rod of length ℓ pivoted at O is connected with two springs of stiffness k_1 & k_2 at a distance of a & ℓ from the pivot respectively. A block of mass m attached with the spring k_2 is kept on a smooth horizontal surface. Find the angular frequency of small oscillation of the block m.



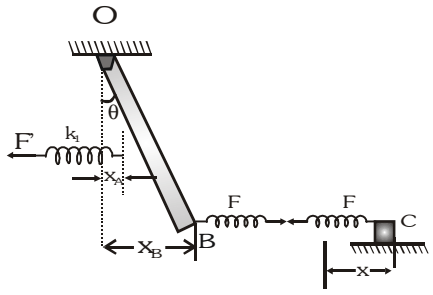
\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Solution

Let the block be pulled towards right through a distance x, then $x=x_B + x_{CB}$ ...(i)

where, x_{CB} = displacement of C (the block) relative to B

Thus $x_{CB} = \frac{F}{k_2}$ ...(ii) and $x_B = \left(\frac{F'}{k_1}\right)\frac{\ell}{a}$ ...(iii)



Torque acting on the rod about point O,

$$\tau_0 = F'a - F\ell \Rightarrow I_0\frac{d^2\theta}{dt^2} = F'a - F\ell$$

Since the rod is very light its moment of inertia

I_0 about O is approximately equal to zero $\Rightarrow F'=F\left(\frac{\ell}{a}\right)$ ...(iv)

Using (iii) & (iv) $\Rightarrow x_B = \frac{F}{k_1}\left(\frac{\ell}{a}\right)^2$ ....(v)

Using (i), (ii) & (v) $\Rightarrow x = \frac{F}{k_1}\left(\frac{\ell}{a}\right)^2 + \frac{F}{k_2}$

As force F is opposite to displacement x, then

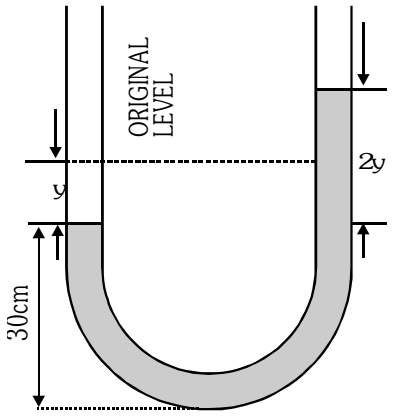
$$\Rightarrow F= -\frac{k_1k_2}{k_2\left(\frac{\ell}{a}\right)^2 + k_1}x \Rightarrow m\omega^2x = \frac{k_1k_2}{k_2\left(\frac{\ell}{a}\right)^2 + k_1}x \Rightarrow \omega = \sqrt{\frac{k_1k_2a^2}{m\left(k_1a^2 + k_2\ell^2\right)}}$$

Example

A vertical U-tube of uniform cross-section contains water upto a height of 30 cm. Show that if the water on one side is depressed and then released, its motion up and down the two sides of the tube is simple harmonic. Calculate its period.

Solution

Figure shows a U-tube of uniform cross-sectional area A. Let the liquid be depressed through the distance y in a limb, the difference of levels between two limbs will be 2 y as shown in figure.



The liquid now oscillates about the initial positions.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Excess pressure on whole liquid = (excess height of the liquid column) (density) (g)

= 2y 1 g (as density of water =1)

Restoring Force on the liquid = Pressure area of cross-section = 2ygA

Due to this force the liquid accelerates and if its acceleration is a, we have ma = –2ygA

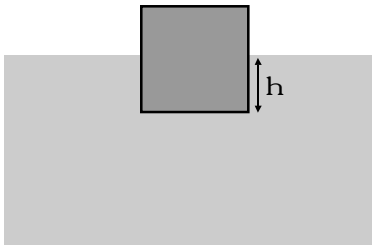
$$\Rightarrow (2 \quad 30 \quad A)\ a = -\ 2ygA \Rightarrow a = -\ \frac{g}{30}y$$

Hence acceleration is directly proportional to displacement, so the motion is simple harmonic motion. Thus the time period T is given by

$$T = \frac{2\pi}{\omega} = 2\omega\sqrt{\left(\frac{30}{g}\right)} = 2\pi\sqrt{\left(\frac{30}{980}\right)}\left[\text{as } \omega=\sqrt{\frac{g}{30}}\right] = 1.098 \text{ second}$$

Example

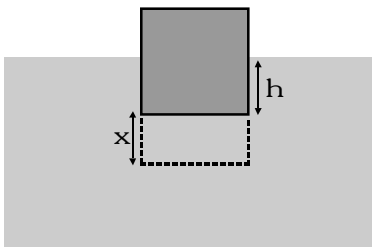
Calculate the period of small oscillations of a floating box as shown in figure, which was slightly pushed down in vertical direction. The mass of box is m, area of its base is A and the density of liquid is ρ. The resistance of the liquid is assumed to be negligible.



Solution

Initially when box is floating in liquid, if its h depth is sumerged in liquid then buoyancy force on it is

F_B = weight of liquid displaced = Ahρg



As the box is in equilibrium, we have Ahρg = mg

Now if box is further pushed down by a distance x, net restoring force on it in upward (toward mean position) direction is

F_R = – [A(h+x)ρg–mg] = –Axρg [as mg = Ahρg]

If a is the acceleration of box in upward direction we have $a = -\left(\frac{A\rho g}{m}\right)x$

Equation shows that the box executes SHM with angular frequency ω given as $\omega = \sqrt{\frac{A\rho g}{m}}$

Thus time period of its oscillation can be given as $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{A\rho g}}$

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

SOME WORKED OUT EXAMPLES

Example#1

A simple pendulum has time period 2s. The point of suspension is now moved upward according to relation $y=(6t - 3.75\ t^2)$m where t is in second and y is the vertical displacement in upward direction. The new time period of simple pendulum will be

(A) 2s (B) 1s (C) 4s (D) None of these

Solution **Ans. (C)**

$T_1 = 2s = 2\pi\sqrt{\frac{\ell}{g}}$, $T_2 = 2\pi\sqrt{\frac{\ell}{g'}}$ where $g' = g+\frac{d^2y}{dt^2} = 10-7.5 = 2.5 = \frac{g}{4} \Rightarrow T_2 = 2\pi\sqrt{\frac{\ell}{g/4}} = 2 \quad 2 = 4s$

Example#2

Pendulum A is a physical pendulum made from a thin rigid and uniform rod whose length is ℓ. One end of this rod is attached to the ceiling by a frictionless hinge so that rod is free to swing back and forth. Pendulum B is a simple pendulum whose length is also ℓ. The ratio $\frac{T_A}{T_B}$ for small angular oscillations-

(A) $\sqrt{\frac{3}{2}}$ (B) $\sqrt{\frac{2}{3}}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Solution **Ans. (B)**

$T_A = 2\pi\sqrt{\frac{I}{mg\ell}} = 2\pi\sqrt{\frac{m\ell^2/3}{mg\ell/2}} = 2\pi\sqrt{\frac{2\ell}{3g}}$ & $T_B = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow \frac{T_A}{T_B} = \sqrt{\frac{2}{3}}$

Example#3

A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its KE is 8 10^{-3} J. Find the equation of motion of this particle if the initial phase of oscillation is 45 .

(A) $y = 0.1 \cos (3t + \pi/4)$ (B) $y = 0.1 \sin (6t + \pi/4)$

(C) $y = 0.1 \sin (4t + \pi/4)$ (D) $y = 0.1 \cos (4t + \pi/4)$

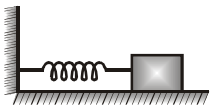
Solution **Ans. (C)**

At mean position KE = $\frac{1}{2}m\omega^2a^2 = 8\times10^{-3} \Rightarrow \omega = 4$ rad/sec

Now let equation of SHM be $y = 0.1 \sin(4t+\phi)$. At t=0, $\phi = 45 = \frac{\pi}{4}$. Therefore $y = 0.1\sin\left(4t+\frac{\pi}{4}\right)$

Example#4

Spring of spring constant 1200 Nm^{-1} is mounted on a smooth frictionless surface and attached to a block of mass 3 kg. Block is pulled 2 cm to the right and released. The angular frequency of oscillation is



(A) 5 rad/sec (B) 30 rad/sec (C) 10 rad/sec (D) 20 rad/sec

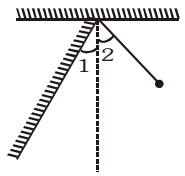
Solution **Ans. (D)**

Angular frequency $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{1200}{3}} = 20$ rad / sec

\\node6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example#5

A simple pendulum of length 1m is allowed to oscillate with amplitude 2 . It collides elastically with a wall inclined at 1 to the vertical. Its time period will be : (use g = π^2)



(A) 2/3 sec (B) 4/3 sec (C) 2 sec (D) None of these

Solution **Ans. (B)**

Time period for half part : $T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{1}{g}} = \frac{2\pi}{\pi} = 2\,\text{sec}.$

So 2 part will be covered in a time t = $\frac{T}{2} = 1$ sec.

For the left 1 part : $\theta = \theta_0 \sin(\omega t) \Rightarrow 1 = 2 \sin\left(\frac{2\pi}{T}\times t\right) \Rightarrow \frac{1}{2} = \sin\left(\frac{2\pi}{2}\times t\right) \Rightarrow \frac{\pi}{6} = \pi \quad t \Rightarrow t = 1/6$ sec.

Total time = $\frac{T}{2} + 2t \Rightarrow 1+2\ \frac{1}{6} = 1 + \frac{1}{3} = \frac{4}{3}$ sec.

Example#6

Values of the acceleration $\ddot{x}$ of a particle moving in simple harmonic motion as a function of its displacement x are given in the table below.

$\ddot{x}\left(mm/s^2\right)$	16	8	0	–8	–16
x (mm)	–4	–2	0	2	4

The period of the motion is

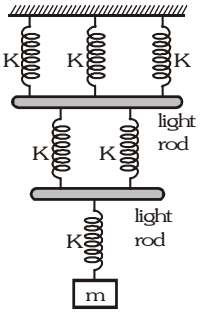
(A) $\frac{1}{\pi}$s (B) $\frac{2}{\pi}$s (C) $\frac{\pi}{2}$s (D) π s

Solution **Ans. (D)**

In SHM a = $-\omega^2 x$. So $16 = -\omega^2(-4) \Rightarrow \omega = 2 \Rightarrow$ Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi s$

Example#7

The time period of small oscillations of mass m :-



(A) $2\pi\sqrt{\frac{m}{6K}}$ (B) $2\pi\sqrt{\frac{11m}{6K}}$ (C) $2\pi\sqrt{\frac{6m}{11K}}$ (D) $2\pi\sqrt{\frac{m}{K}}$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Solution **Ans. (B)**

Equivalent spring constant $\frac{1}{K_{eq}}=\frac{1}{3K}+\frac{1}{2K}+\frac{1}{K}\Rightarrow K_{eq}=\frac{6K}{11}\therefore T=2\pi\sqrt{\frac{m}{K_{eq}}}=2\pi\sqrt{\frac{11m}{6K}}$

Example#8

The time period of oscillation of a mass m suspended from a spring is 2 seconds. If another 2 kg mass is also suspended along with m the time period increases by 1 second. The value of mass m is

(A) 2 kg (B) 1 kg (C) 1.6 kg (D) 2.6 kg

Solution **Ans. (C)**

Here $2\pi\sqrt{\frac{m}{k}}=2s$ and $2\pi\sqrt{\frac{m+2}{k}}=3s\Rightarrow\frac{3}{2}=\sqrt{\frac{m+2}{m}}\Rightarrow\frac{9}{4}=\frac{m+2}{m}$ which yields m = 1.6 kg

Example#9

The angle made by the string of a simple pendulum with the vertical depends upon time as $\theta=\frac{\pi}{90}\sin\pi t$. Find the length (in m) of the pendulum if $g=\pi^2\,m/s^2$

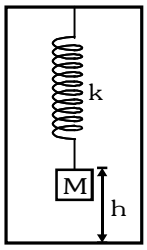
(A) 1 (B) 2 (C) 3 (D) 4

Solution **Ans. (A)**

Here $T=\frac{2\pi}{\omega}=\frac{2\pi}{\pi}=2\pi\sqrt{\frac{\ell}{g}}\Rightarrow\frac{\ell}{g}=\frac{1}{\pi^2}$ But $g=\pi^2$ therefore ℓ = 1 m

Example#10

A block is in equilibrium at rest in a lift as shown in figure. Now lift starts accelerating. The minimum magnitude of acceleration of lift for which block will touch the floor of lift, is



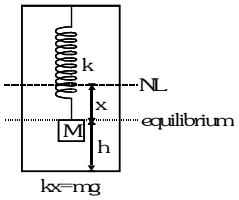
(A) $\frac{kh}{M}$ (B) $\frac{2kh}{M}$ (C) $\frac{kh}{2M}$ (D) $\frac{3}{2}\frac{kh}{M}$

Solution **Ans. (C)**

From work energy theorem $W_g+W_{pseudo}+W_{spring}=\Delta KE$

$\Rightarrow Mgh+Mah+\frac{1}{2}kx^2-\frac{1}{2}k(h+x)^2=0$ where $Mg=kx$

$\Rightarrow Mgh+Mah-\frac{1}{2}kh^2-khx=0\quad\therefore a=\frac{kh}{2M}$



Example#11

Time period of a spring mass system can be changed by

(A) changing the mass

(B) accelerating the point of suspension of the block

(C) cutting the spring (i.e. changing the length of the spring)

(D) immersing the mass in a liquid

Solution **Ans. (A,C,D)**

Time period of spring mass system $T=2\pi\sqrt{\frac{M}{K}}$ Which can be changed by (A), (C) & (D)

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example#12

The time period of a particle executing SHM is T. After a time T/6 after it passes its mean position, its :
(A) displacement will be one-half of its amplitude
(B) velocity will be one-half of its maximum velocity
(C) kinetic energy = 1/3 (potential energy)
(D) acceleration will be $\frac{\sqrt{3}}{2}$ times of its maximum acceleration

Solution **Ans. (B,C,D)**

Let x=Asinωt where $\omega = \frac{2\pi}{T}$

For (A) : At t = $x = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{6}\right) = A\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}A$, $x = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{6}\right) = A\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}A$

For (B) : $v = \frac{dx}{dt} = A\omega\cos\omega t$ At $t = \frac{T}{6}$, $v = A\omega\cos\left(\frac{2\pi}{T}\right)\left(\frac{T}{6}\right) = A\omega\cos\frac{\pi}{3} = \frac{A}{2}$

For (C) : KE = $\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v_{max}}{2}\right)^2 = \frac{1}{4}(KE)_{max} = \frac{1}{4}(TE)$ & PE = TE – KE = $\frac{3}{4}TE \Rightarrow KE = \frac{1}{3}(PE)$

For (D) : Acceleration a = $\frac{dv}{dt} = -A\omega^2\sin\omega t = -\omega^2 x$

Example#13

The potential energy of a particle of mass 100 g, moving along the x-axis, is given by U = 5x(x–4) J, where x is in metre. Select correct alternative(s).
(A) The particle execute SHM with mean position at x=4 m
(B) The particle execute SHM with mean position at x=2 m
(C) The particle execute SHM with time period π/5 second
(D) The particle execute SHM with time period π/10 second

Solution **Ans. (B,C)**

$U = 5x^2 - 20x \Rightarrow F = -\frac{dU}{dx} = -10x + 20 = -10(x-2)$

Acceleration $a = \frac{F}{m} = -100(x-2)$ so $\omega^2 = 100 \Rightarrow \omega = 10$

Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}$ sec

Example#14

A particle moves along the Z-axis according to the equation z = 5+12 cos $\left(2\pi t + \frac{\pi}{2}\right)$, where z is in cm and t is in seconds. Select the correct alternative (s)-
(A) The motion of the particle is SHM with mean position at z = 5cm
(B) The motion of the particle is SHM with extreme position at z = –7cm and z = +17 cm.
(C) Amplitude of SHM is 13 cm
(D) Amplitude of SHM is 12 cm

Solution **Ans. (A,B,D)**

z =5 + 12 cos $\left(2\pi t + \frac{\pi}{2}\right)$ = 5–12 sin(2πt) ⇒ (z–5) = –12 sin 2πt ⇒ SHM with mean position z = 5cm

Extreme position z_{max} = 5 + 12 = 17 cm, z_{min} = 5 – 12 = –7 cm Amplitude of SHM = 12 cm

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Example#15

For a SHM with given angular frequency, two arbitrary initial conditions are necessary and sufficient to determine the motion completely. These initial conditions may be

(A) Amplitude and initial phase
(B) Amplitude and total energy of oscillation
(C) Initial phase and total energy of oscillation
(D) Initial position and initial velocity

Solution **Ans. (A,C,D)**

$x = A \sin(\omega t + \phi)$ As ω is given, to describe the motion completely we need A and ϕ.

Example#16

The motion of a particle is represented by the function "$A\sin^2\omega t + B\cos^2\omega t + C\sin\omega t\cos\omega t$" where x, ω and t represent displacement, angular frequency and time respectively. The constants A, B and C are related to the amplitudes of displacement. Select **CORRECT** alternative(s).

(A) The motion of particle is SHM when A=B.

(B) The motion of particle is SHM when A=B and C=0.

(C) If $B = \frac{C}{2} = -A$, then the amplitude of SHM is $B\sqrt{2}$.

(D) If $A=B = \frac{C}{2}$, then the axis of vibration of SHM shifts by a distance B towards +x axis.

Solution **Ans. (ACD)**

If A=B then $x = A + C\sin\omega t\cos\omega t = A + \frac{C}{2}\sin 2\omega t \Rightarrow$SHM

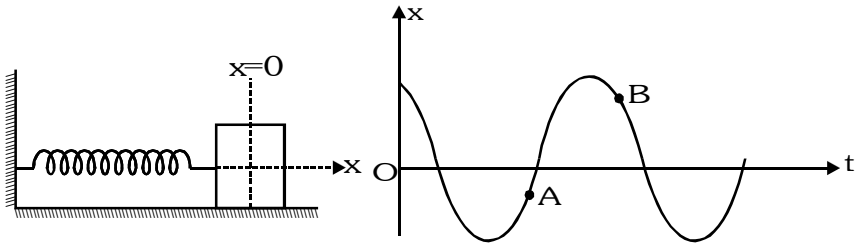
If A = B & C = 0 then $x = A \Rightarrow$ along a straight line.

If $B = \frac{C}{2} = -A$ then $x = B\cos 2\omega t + B\sin 2\omega t \Rightarrow$ amplitude $= B\sqrt{2}$

If $A = B = \frac{C}{2}$ then $x = B + B\sin 2\omega t \Rightarrow$Axis of vibration of SHM shifts by a distance B towards + x-axis.

Example#17 to 19

A 100 g block is connected to a horizontal massless spring of force constant 25.6 N/m. The block is free to oscillate on a horizontal frictionless surface. The block is displaced 3 cm from the equilibrium position, and at t =0, it is released from rest at x=0. The position-time graph of motion of the block is shown in figure.



17. When the block is at position A on the graph, its

(A) position and velocity both are negative.
(B) position and velocity both are positive.
(C) position is negative and velocity is positive.
(D) position is positive and velocity is negative.

18. When the block is at position B on the graph, its

(A) position and velocity both are negative.
(B) position and velocity both are positive.
(C) position is negative and velocity is positive.
(D) position is positive and velocity is negative.

19. Let us now make a slight change to the initial conditions. At t=0, let the block be released from the same position with an initial velocity v_i = 64 cm/s. Position of the block as a function of time can be expressed as

(A) x = 5 sin16t (B) x =5sin(16t+37) (C) x =5sin(16t–37) (D) x =5cos(16t+37)

\\node6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

Solution

17. Ans. (C)

At position A, x is negative and $\frac{dx}{dt}$ is positive.

18. Ans. (D)

At position B, x is positive and $\frac{dx}{dt}$ is negative.

19. Ans. (B)

Let x = A sin(16t+ϕ) v = Aωcos(16t+ϕ) where $\omega^2 = \frac{K}{m} = \frac{25.6}{0.1} = 256 \Rightarrow \omega = 16$ rad/sec

At t=0, 3 = Asinϕ & 64 = 16Acosϕ $\Rightarrow \tan\phi = \frac{3}{4} \Rightarrow \phi = 37°$ Also A = 5 cm

Therefore equation of SHM x =5 sin(16t + 37)

Example#20 to 22

X_1 and X_2 are two points on the path of a particle executing SHM in a straight line, at which its velocity is zero. Starting from a certain point X ($X_1X < X_2X$) the particle crosses this point again at successive intervals of 2s and 4s with a speed of 5 m/s.

20. The time period of SHM is

(A) 6 s (B) 3s (C) 4s (D) 2s

21. The amplitude of SHM is

(A) $10\sqrt{3}$ m (B) $\frac{10}{\sqrt{3}\pi}$ m (C) $\frac{10\sqrt{3}}{\pi}$ m (D) can't be determined

22. The maximum speed of the particle is

(A) $\frac{10\sqrt{3}}{\pi}$ m/s (B) $10\sqrt{3}$ m/s (C) $\frac{\sqrt{3}}{10}$ m/s (D) $\frac{10}{\sqrt{3}}$ m/s

Solution

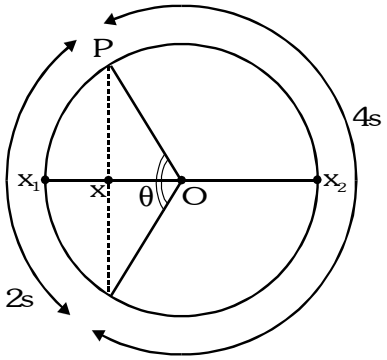
20. Ans. (A)



Time period = 2 + 4 = 6s

21. Ans. (C)

The given SHM can be represented in circular motion



In shown diagram $\theta = \left(\frac{2\pi}{6}\right)(2) = \frac{2\pi}{3}$

Let OX = x_0 and amplitude = OP=A

Now $\frac{x_0}{A} = \cos\frac{\theta}{2}$ so $x_0 = \frac{A}{2}$

As $v = \omega\sqrt{A^2 - X_0^2}$ so $5 = \frac{2\pi}{6}\sqrt{A^2 - \frac{A^2}{4}} \Rightarrow A = \frac{10\sqrt{3}}{\pi}$ m

22. Ans. (D)

Maximum speed = $A\omega = \left(\frac{10\sqrt{3}}{\pi}\right)\left(\frac{2\pi}{6}\right) = \frac{10}{\sqrt{3}}$ m/s

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory p65

Example#23

The speed (v) of a particle moving along a straight line, when it is at a distance (x) from a fixed point on the line, is given by : $v^2 = 144 - 9x^2$.

	Column I		Column II
(A)	Motion is simple harmonic of period	(P)	$\frac{2\pi}{3}$ units
(B)	Maximum displacement from the fixed point is	(Q)	12 units
(C)	Maximum velocity of the particle	(R)	27 units
(D)	Magnitude of acceleration at a distance 3 units from the fixed point is	(S)	4 unit
		(T)	16 units

Solution **(A) P, (B) S (C) Q (D) R**

For (A) : $v^2 = 144 - 9x^2 \Rightarrow 2v\frac{dv}{dx} = 0 - 18x \Rightarrow a = -9x \Rightarrow \omega^2 = 9 \Rightarrow \omega = 3$

Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$ units

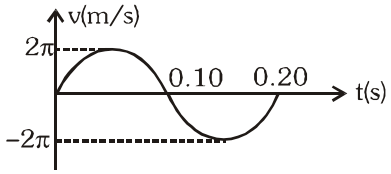
For (B) : $\because v^2 \geq 0 \ \therefore 144 - 9x^2 \geq 0 \Rightarrow x^2 \leq 16 \Rightarrow x \leq 4 \Rightarrow$ Amplitude = 4 units

For (C) : Maximum velocity = $A\omega$ = (4) (3) = 12 units

For (D) : At x = 3 units, a = – 9x = – 27 units

Example#24

A simple harmonic oscillator consists of a block attached to a spring with k = 200 N/m. The block slides on a frictionless horizontal surface, with equilibrium point x = 0. A graph of the block's velocity v as a function of time t is shown (use $\pi^2 = 10$)



	Column I		Column II
(A)	The block's mass in kg	(P)	–0.20
(B)	The block's displacement at t = 0 in meters	(Q)	–200
(C)	The block's acceleration at t = 0.10 s in m/s^2	(R)	+0.20
(D)	The block's maximum kinetic energy in joule	(S)	+4.0
		(T)	+200

Solution **(A) R, (B) P, (C) Q, (D) S**

$v_m = A\omega \Rightarrow A = \frac{v_m}{\omega} = \frac{2\pi}{2\pi}$ (0.2) = 0.20 m; $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow m = \frac{T^2k}{4\pi^2}$ = 0.2 kg

At t = 0.1, acceleration is maximum $\Rightarrow -\omega^2 A = -200$ m/s^2

Maximum energy = $\frac{1}{2}mv_m^2$ = 4J or $\frac{1}{2}kA^2 = E_{max} = \frac{1}{2}$ 200 0.04 = 4 J

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\SHM\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.

Material Point
Materials For all Exams

ALLEN
Materials
Provided By -

Material Point

Powered By -

Learnaf
Educational Revolution

Learnaf
Educational Revolution

PHYSICS

ALLEN

Study Package - 1

For – JEE (Advanced)

CHAPTER – 13

WAVE MOTION - Theory

Index

WAVE THEORY, SOUND WAVES & DOPPLER'S EFFECTS

Serial No.	CONTENTS	Page No.
1.	Introduction to wave motion	1
2.	Plane progressive wave	6
3.	Intensity of wave	7
4.	Superposition principle	7
5.	Interference of waves	8
6.	Velocity of transverse waves	9
7.	Reflection from rigid & free end	9
8.	Stationary waves	9
9.	Speed of longitudinal (sound) waves	13
10.	Vibrations of organ pipes (closed & open)	16
11.	Apparatus for detecting speed of sound	18
12.	Beats	20
13.	Doppler effects in sound	23
14.	**Exercise-1** Check your Grasp (Objective)	41
15.	**Exercise-2** Brain Teasers (MCQ's with one or more than one correct answer)	46
16.	**Exercise-3** Miscellaneous Type Questions (True / False, Fill in the Blanks, Match the Column, Assertion & Reason, Comprehension Based Question)	49
17.	**Exercise-4A** Conceptual Subjective Exercise	55
18.	**Exercise-4B** Brain Storming Exercise	58
19.	**Exercise-5A** Previous years questions	60
20.	**Exercise-5B** Previous years questions	62

ALLEN Materials Provided By - Material Point Available on Learnaf.com

SYLLABUS :

JEE Mains : Wave motion. Longitudinal and transverse waves, speed of a wave. Displacement relation for a progressive wave. Principle of superposition of waves, reflection of waves, Standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler effect in sound

JEE Advanced : Wave motion (plane waves only), longitudinal and transverse waves, superposition of waves; Progressive and stationary waves; Vibration of strings and air columns;Resonance; Beats; Speed of sound in gases; Doppler effect (in sound).

ALLEN Materials Provided By - Material Point Available on Learnaf.com

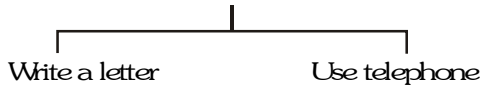
WAVE THEORY, SOUND WAVES & DOPPLER'S EFFECTS

INTRODUCTION OF WAVES

What is wave motion ?

- When a particle moves through space, it carries KE with itself. Wherever the particle goes, the energy goes with it. (One way of transport energy from one place to another place)
- There is another way (wave motion) to transport energy from one part of space to other without any bulk motion of material together with it. Sound is transmitted in air in this manner.

Ex. You (Kota) want to communicate your friend (Delhi)



1st option involves the concept of particle & the second choice involves the concept of wave.

Ex. When you say "Namaste" to your friend no material particle is ejected from your lips to fall on your friends ear. Basically you create some disturbance in the part of the air close to your lips. Energy is transferred to these air particles either by pushing them ahead or pulling them back. The density of the air in this part temporarily increases or decreases. These disturbed particles exert force on the next layer of air, transferring the disturbance to that layer. In this way, the disturbance proceeds in air and finally the air near the ear of the listener gets disturbed.

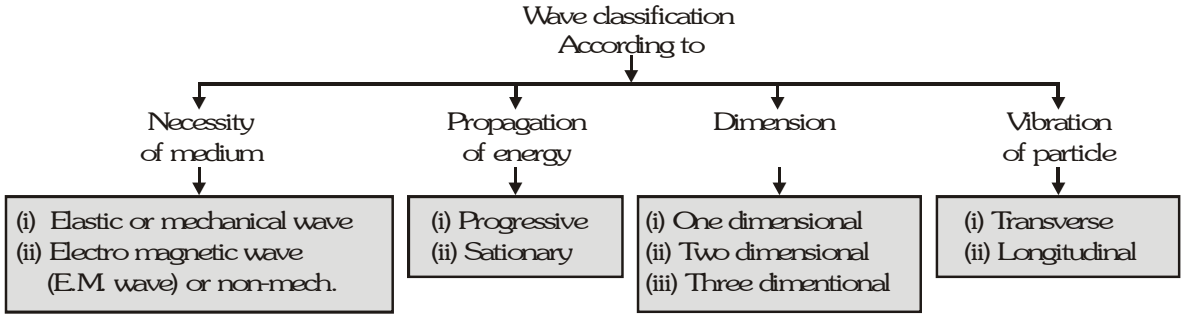
Note :- In the above example air itself does not move.

A **wave** is a disturbance that propagates in space, transports energy and momentum from one point to another without the transport of matter.

Few examples of waves :

The ripples on a pond (water waves), the sound we hear, visible light, radio and TV signals etc.

CLASSIFICATION OF WAVES



1. ***Based on medium necessity*** :- A wave may or may not require a medium for its propagation. The waves which do not require medium for their propagation are called non-mechanical, e.g. light, heat (infrared), radio waves etc. On the other hand the waves which require medium for their propagation are called mechanical waves. In the propagation of mechanical waves elasticity and density of the medium play an important role therefore mechanical waves are also known as **elastic waves**.

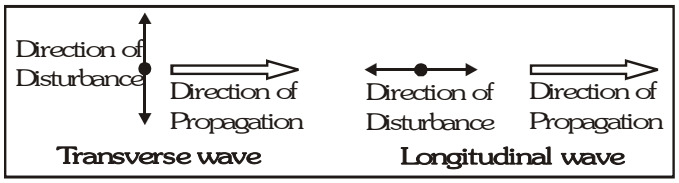
Example : Sound waves in water, seismic waves in earth's crust.

2. **Based on energy propagation :-** Waves can be divided into two parts on the basis of energy propagation (i) Progressive wave (ii) Stationary waves. The progressive wave propagates with constant velocity in a medium. In stationary waves particles of the medium vibrate with different amplitude but energy does not propagate.

\\nodes6\E_NODES (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

3. ***Based on direction of propagation*** :- Waves can be one, two or three dimensional according to the number of dimensions in which they propagate energy. Waves moving along strings are one-dimensional. Surface waves or ripples on water are two dimensional , while sound or light waves from a point source are three dimensional.

4. ***Based on the motion of particles of medium*** :



Waves are of two types on the basis of motion of particles of the medium.

(i) Longitudinal waves

(ii) Transverse waves

In the transverse wave the direction associated with the disturbance (i.e. motion of particles of the medium) is at right angle to the direction of propagation of wave while in the longitudinal wave the direction of disturbance is along the direction of propagation.

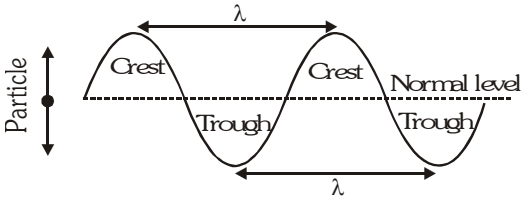
TRANSVERSE WAVE MOTION

Mechanical transverse waves produce in such type of medium which have shearing property, so they are known as shear wave or S-wave

Note :- Shearing is the property of a body by which it changes its shape on application of force.

$\Rightarrow$ Mechanical transverse waves are generated only in solids & surface of liquid.

In this individual particles of the medium execute SHM about their mean position in direction $\perp^r$ to the direction of propagation of wave motion.

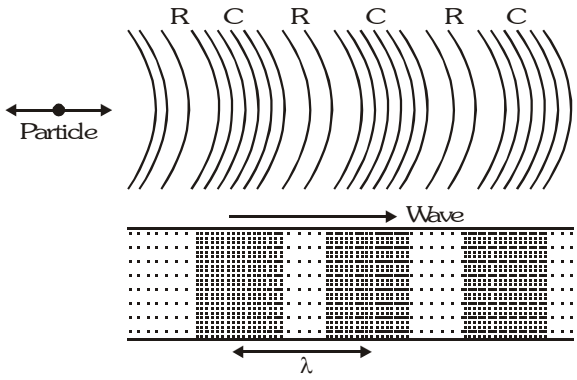


A **crest** is a portion of the medium, which is raised temporarily above the normal position of rest of particles of the medium , when a transverse wave passes.

A **trough** is a portion of the medium, which is depressed temporarily below the normal position of rest of particles of the medium , when a transverse wave passes.

LONGITUDINAL WAVE MOTION

In this type of waves, oscillatory motion of the medium particles produces regions of compression (high pressure) and rarefaction (low pressure) which propagated in space with time (see figure).



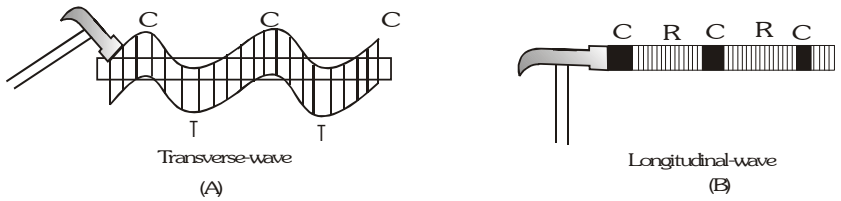
Note : The regions of high particle density are called compressions and regions of low particle density are called rarefactions.

The propagation of sound waves in air is visualized as the propagation of pressure or density fluctuations. The pressure fluctuations are of the order of 1 Pa, whereas atmospheric pressure is 10^5 Pa.

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Mechanical Waves in Different Media

- A mechanical wave will be transverse or longitudinal depends on the nature of medium and mode of excitation.
- In strings mechanical waves are always transverse when string is under a tension. In gases and liquids mechanical waves are always longitudinal e.g. sound waves in air or water. This is because fluids cannot sustain shear.
- In solids, mechanical waves (may be sound) can be either transverse or longitudinal depending on the mode of excitation. The speed of the two waves in the same solid are different. (Longitudinal waves travels faster than transverse waves). e.g., if we struck a rod at an angle as shown in fig. (A) the waves in the rod will be transverse while if the rod is struck at the side as shown in fig. (B) or is rubbed with a cloth the waves in the rod will be longitudinal. In case of vibrating tuning fork waves in the prongs are transverse while in the stem are longitudinal.

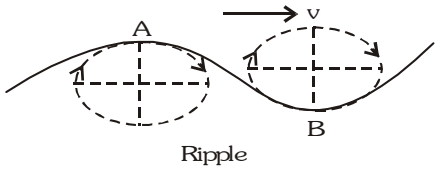


Transverse-wave (A) Longitudinal-wave (B)

Further more in case of seismic waves produced by Earthquakes both S (shear) and P (pressure) waves are produced simultaneously which travel through the rock in the crust at different speeds

[$v_S \cong 5$ km/s while $v_P \cong 9$ km/s] S–waves are transverse while P–waves longitudinal.

Some waves in nature are neither transverse nor longitudinal but a combination of the two. These waves are called 'ripple' and waves on the surface of a liquid are of this type. In these waves particles of the medium vibrate up and down and back and forth simultaneously describing ellipses in a vertical plane.



Ripple

CHARACTERISTICS OF WAVE MOTION

Some of the important characteristics of wave motion are as follows :

- In a wave motion, the disturbance travels through the medium due to repeated periodic oscillations of the particles of the medium about their mean positions.
- The energy is transferred from place to another without any actual transfer of the particles of the medium.
- Each particle receives disturbance a little later than its preceding particle i.e., there is a regular phase difference between one particle and the next.
- The velocity with which a wave travels is different from the velocity of the particles with which they vibrate about their mean positions.
- The wave velocity remains constant in a given medium while the particle velocity changes continuously during its vibration about the mean position. It is maximum at the mean position and zero at the extreme position.
- For the propagation of a mechanical wave, the medium must possess the properties of inertia, elasticity and minimum friction amongst its particles.

SOME IMPORTANT TERMS CONNECTED WITH WAVE MOTION

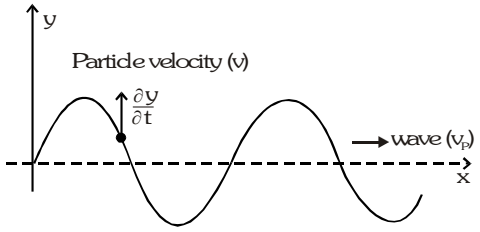
- **Wavelength (λ) [length of one wave]**

  Distance travelled by the wave during the time, any one particle of the medium completes one vibration about its mean position. We may also define wavelength as the distance between any two nearest particles of the medium, vibrating in the same phase.
- **Frequency (n)** :Number of vibrations (Number of complete wavelengths) complete by a particle in one second.

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

- **Time period (T)** : Time taken by wave to travel a distance equal to one wavelength.
- **Amplitude (A)** : Maximum displacement of vibrating particle from its equilibrium position.
- **Angular frequency (ω) :** It is defined as $\omega = \frac{2\pi}{T} = 2\pi n$
- **Phase :** Phase is a quantity which contains all information related to any vibrating particle in a wave. For equation y = A sin (ωt – kx); (ωt – kx) = phase.
- **Angular wave number (k) :** It is defined as $k = \frac{2\pi}{\lambda}$
- **Wave number ($\vec{\nu}$) :** It is defined as $\vec{\nu} = \frac{1}{\lambda} = \frac{k}{2\pi}$ = number of waves in a unit length of the wave pattern.
- **Particle velocity, wave velocity and particle's acceleration :** In plane progressive harmonic wave particles of the medium oscillate simple harmonically about their mean position. Therefore, all the formulae what we have read in SHM apply to the particles here also. For example, maximum particle velocity is ± Aω at mean position and it is zero at extreme positions etc. Similarly maximum particle acceleration is $\pm\omega^2 A$ at extreme positions and zero at mean position. However the wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between + Aω and – Aω) the wave velocity is constant for given characteristics of the medium.
- **Particle velocity in wave motion :**

  The individual particles which make up the medium do not travel through the medium with the waves. They simply oscillate about their equilibrium positions. The instantaneous velocity of an oscillating particle of the medium, through which a wave is travelling, is known as "Particle velocity".



- **Wave velocity :** The velocity with which the disturbance, or planes of equal phase (wave front), travel through the medium is called wave (or phase) velocity.
- **Relation between particle velocity and wave velocity :**

  Wave equation :- y = A sin (ωt – kx), Particle velocity $v = \frac{\partial y}{\partial t}$ = Aω cos (ωt – kx).

  Wave velocity = $v_p = \frac{\lambda}{T} = \lambda\frac{\omega}{2\pi} = \frac{\omega}{k}$, $\frac{\partial y}{\partial x}$ = – Ak cos (ωt–kx) = $-\frac{A}{\omega}\omega k$ cos (ωt – kx) = $-\frac{1}{v_p}\frac{\partial y}{\partial t} \Rightarrow \frac{\partial y}{\partial x} = -\frac{1}{v_p}\frac{\partial y}{\partial t}$

  Note : $\frac{\partial y}{\partial x}$ represent the slope of the string (wave) at the point x.

  Particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.
- **Differential equation of harmonic progressive waves :**

  $\frac{\partial^2 y}{\partial t^2} = -A\omega^2 \sin(\omega t - kx) \Rightarrow \frac{\partial^2 y}{\partial x^2} = -Ak^2 \sin(\omega t - kx) \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v_p^2}\frac{\partial^2 y}{\partial t^2}$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

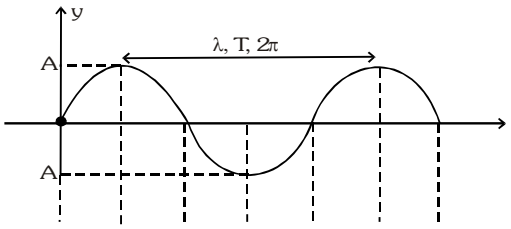
- **Particle velocity (v_p) and acceleration (a_p) in a sinusoidal wave :**

The acceleration of the particle is the second particle is the second partial derivative of y (x, t) with respect to t,

$$\therefore\ a_P = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) = -\omega^2 y(x,t)$$

i.e., the acceleration of the particle equals $-\omega^2$ times its displacement, which is the result we obtained for SHM. Thus, $a_P = -\omega^2$ (displacement)

- **Relation between Phase difference, Path difference & Time difference**



Phase (ϕ)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
Wave length (λ)	0	$\frac{\lambda}{4}$	$\frac{\lambda}{2}$	$\frac{3\lambda}{4}$	λ	$\frac{5\lambda}{4}$	$\frac{3}{2}\lambda$
Time-period (T)	0	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	T	$\frac{5T}{4}$	$\frac{3T}{2}$

$\Rightarrow \frac{\Delta\phi}{2\pi} = \frac{\Delta\lambda}{\lambda} = \frac{\Delta T}{T} \Rightarrow$ Path difference = $\left(\frac{\lambda}{2\pi}\right)$ Phase difference

Example

A progressive wave of frequency 500 Hz is travelling with a velocity of 360 m/s. How far apart are two points 60° out of phase.

Solution

We know that for a wave $v = f\lambda$ So $\lambda = \frac{v}{f} = \frac{360}{500} = 0.72$ m

Phase difference $\Delta\phi = 60° = (\pi/180) \times 60 = (\pi/3)$ rad, so path difference $\Delta x = \frac{\lambda}{2\pi}(\Delta\phi) = \frac{0.72}{2\pi} \times \frac{\pi}{3} = 0.12$ m

THE GENERAL EQUATION OF WAVE MOTION

Some physical quantity (say y) is made to oscillate at one place and these oscillations of y propagate to other places. The y may be,

(i) displacement of particles from their mean position in case of transverse wave in a rope or longitudinal sound wave in a gas.

(ii) pressure difference (dP) or density difference ($d\rho$) in case of sound wave or

(iii) electric and magnetic fields in case of electromagnetic waves.

The oscillations of y may or may not be simple harmonic in nature. Consider one-dimensional wave travelling along x-axis. In this case y is a function of x and t. i.e. y = f(x, t) But only those function of x & t, represent a wave motion which satisfy the differential equation. $\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ...(i)

The general solution of this equation is of the form $y(x, t) = f(ax \pm bt)$ ...(ii)

Thus, any function of x and t and which satisfies equation (i) or which can be written as equation (ii) represents a wave. The only condition is that it should be finite everywhere and at all times, Further, if these conditions are satisfied, then speed of wave (v) is given by $v = \frac{\text{coefficient of t}}{\text{coefficient of x}} = \frac{b}{a}$

\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Example

Which of the following functions represent a travelling wave ?

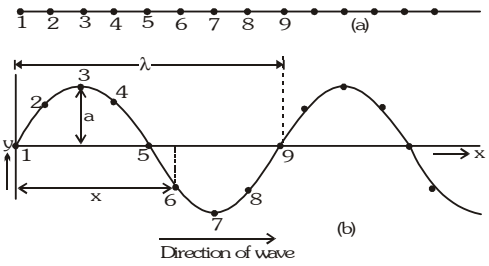
(a) $(x - vt)^2$ (b) $\ell n(x + vt)$ (c) $e^{-(x-vt)^2}$ (d) $\frac{1}{x + vt}$

Solution

Although all the four functions are written in the form f (ax $\pm$ bt), only (c) among the four functions is finite everywhere at all times. Hence only (c) represents a travelling wave.

Equation of a Plane Progressive Wave

If, on the propagation of wave in a medium, the particles of the medium perform simple harmonic motion then the wave is called a 'simple harmonic progressive wave'. Suppose, a simple harmonic progressive wave is propagating in a medium along the positive direction of the x-axis (from left to right). In fig. (a) are shown the equilibrium positions of the particles 1, 2, 3 .....



When the wave propagates, these particles oscillate about their equilibrium positions. In Fig. (b) are shown the instantaneous positions of these particles at a particular instant. The curve joining these positions represents the wave. Let the time be counted from the instant when the particle 1 situated at the origin starts oscillating. If y be the displacement of this particle after t seconds, then $y = a \sin \omega t$...(i)

where a is the amplitude of oscillation and $\omega = 2\pi n$, where n is the frequency. As the wave reaches the particles beyond the particle 1, the particles start oscillating. If the speed of the wave be v, then it will reach particle 6, distant x from the particle 1, in x/v sec. Therefore, the particle 6 will start oscillating x/v sec after the particle 1. It means that the displacement of the particle 6 at a time t will be the same as that of the particle 1 at a time x/v sec earlier i.e. at time t – (x/v). The displacement of particle 1 at time t – (x/v) can be the particle 6, distant x from the origin (particle 1), at time t is given by

$$y = a \sin \omega \left(t - \frac{x}{v}\right) \qquad \text{But } \omega = 2\pi n,\ y = a \sin (\omega t - kx) \qquad \left(k = \frac{\omega}{v}\right) \quad ...(ii)$$

$$y = a \sin \left[\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right] \qquad \text{Also } k = \frac{2\pi}{\lambda} \quad ...(iii) \qquad y = a \sin 2\pi \left[\frac{t}{T} - \frac{x}{\lambda}\right] \quad ...(iv)$$

This is the equation of a simple harmonic wave travelling along +x direction. If the wave is travelling along the –x direction then inside the brackets in the above equations, instead of minus sign there will be plus sign. For example, equation (iv) will be of the following form : $y = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$. If ϕ be the phase difference between the above wave travelling along the +x direction and an other wave, then the equation of that wave will be

$$y = a \sin \left\{2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \pm \phi\right\}$$

Example

The equation of a wave is, $y(x,t) = 0.05 \sin\left[\frac{\pi}{2}(10x - 40t) - \frac{\pi}{4}\right]$m

Find : (a) The wavelength, the frequency and the wave velocity

(b) The particle velocity and acceleration at x=0.5 m and t=0.05 s.

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Solution

(a) The equation may be rewritten as, $y(x,t) = 0.05\sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right)m$

Comparing this with equation of plane progressive harmonic wave,

$y(x,t) = A\sin(kx - \omega t + \phi)$ we have, wave number $k = \frac{2\pi}{\lambda} = 5\pi rad/m$ $\therefore \lambda = 0.4m$

The angular frequency is, $\omega = 2\pi f = 20\pi rad/s$ $\therefore$ f = 10Hz

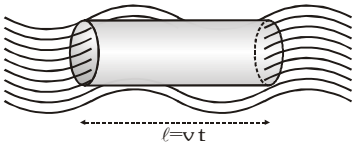
The wave velocity is, $v = f\lambda = \frac{\omega}{k} = 4ms^{-1}$ in + x direction

(b) The particle velocity and acceleration are, $v_p = \frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$ =2.22m/s

$a_p = \frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$ =140 m/s^2

INTENSITY OF WAVE

The amount of energy flowing per unit area and per unit time is called the intensity of wave. It is represented by I. Its units are J/m^{2}s or watt/metre^2. $I = 2\pi^2 f^2 A^2 \rho v$ i.e. $I \propto A^2$ and $I \propto A^2$.



If P is the power of an isotropic point source, then intensity at a distance r is given by,

$$I = \frac{P}{4\pi r^2} \text{ or } I \propto \frac{1}{r^2} \text{ (for a point source)}$$

If P is the power of a line source, then intensity at a distance r is given by,

$$I = \frac{P}{2\pi r\ell} \text{ or } I \propto \frac{1}{r} \text{ (for a line source) As, } I \propto A^2$$

Therefore, $A \propto \frac{1}{r}$ (for a point source) and $A \propto \frac{1}{\sqrt{r}}$ (for a line source)

SUPERPOSITION PRINCIPLE

Two or more waves can propagate in the same medium without affecting the motion of one another. If several waves propagate in a medium simultaneously, then the resultant displacement of any particle of the medium at any instant is equal to the vector sum of the displacements produced by individual wave. The phenomenon of intermixing of two or more waves to produce a new wave is called Superposition of waves. Therefore according to superposition principle.

> The resultant displacement of a particle at any point of the medium, at any instant of time is the vector sum of the displacements caused to the particle by the individual waves.

If $\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots$ are the displacement of particle at a particular time due to individual waves, then the resultant displacement is given by $\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \ldots$

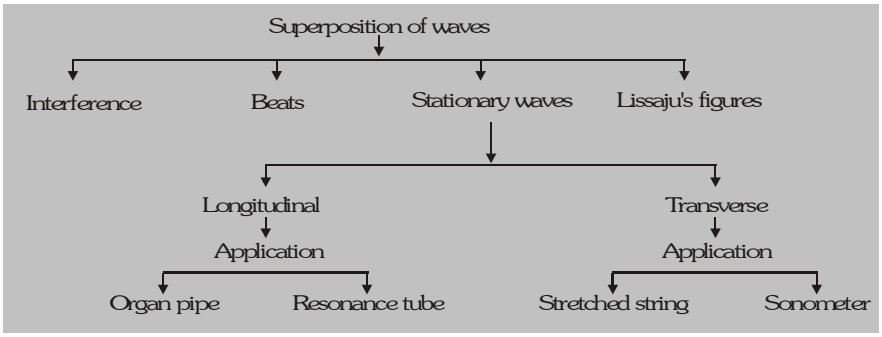
Principle of superposition holds for all types of waves, i.e., mechanical as well as electromagnetic waves. But this principle is not applicable to the waves of very large amplitude.

Due to superposition of waves the following phenomenon can be seen

- **Interference** : Superposition of two waves having equal frequency and nearly equal amplitude.
- **Beats** : Superposition of two waves of nearly equal frequency in same direction.

\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

- **Stationary waves :** Superposition of equal wave from opposite direction.
- **Lissajous' figure :** Superposition of perpendicular waves.



INTERFERENCE OF WAVES :

When two waves of equal frequency and nearly equal amplitude travelling in same direction having same state of polarisation in medium superimpose, then intensity is different at different points. At some points intensity is large, whereas at other points it is nearly zero.

Consider two waves $y_1 = A_1 \sin(\omega t - kx)$ and $y_2 = A_2 \sin(\omega t - kx + \phi)$

By principle of superposition $y = y_1 + y_2 = A \sin(\omega t - kx + \delta)$

where $A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi$ and $\tan\delta = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi}$

As intensity $I \propto A^2$ so $I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi$

- **Constructive interference (maximum intensity) :**

Phase difference $\phi = 2n\pi$ or path difference $= n\lambda$ where n = 0, 1, 2, 3, ...

$\Rightarrow A_{max} = A_1 + A_2$ and $I_{max} = I_1 + I_2 + 2\sqrt{I_1I_2} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$

- **Destructive interference (minimum intensity) :**

Phase difference $\phi = (2n+1)\pi$, or path difference $= (2n-1)\dfrac{\lambda}{2}$ where n = 0, 1, 2, 3, ...

$\Rightarrow A_{min} = A_1 - A_2$ and $I_{min} = I_1 + I_2 - 2\sqrt{I_1I_2} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

GOLDEN KEY POINTS

- Maximum and minimum intensities in any interference wave form. $\dfrac{I_{Max}}{I_{Min}} = \left(\dfrac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\dfrac{A_1 + A_2}{A_1 - A_2}\right)^2$
- Average intensity of interference wave form $\&$ $< I >$ or $I_{av} = \dfrac{I_{max} + I_{min}}{2} = I_1 + I_2$

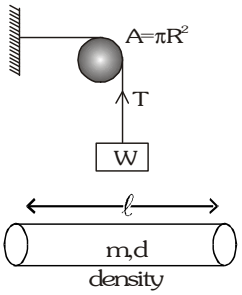
  if $A = A_1 = A_2$ and $I_1 = I_2 = I$ then $I_{max} = 4I$, $I_{min} = 0$ and $I_{AV} = 2I$
- Degree of interference Pattern (f) : Degree of hearing (Sound Wave) or

  Degree of visibility (Light Wave) $f = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$ 100

  In condition of perfect interference degree of interference pattern is maximum $f_{max} = 1$ or 100%
- Condition of maximum contrast in interference wave form $a_1 = a_2$ and $I_1 = I_2$ then $I_{max} = 4I$ and $I_{min} = 0$

  For perfect destructive interference we have a maximum contrast in interference wave form.

VELOCITY OF TRANSVERSE WAVE



Mass of per unit length $m = \frac{\pi r^2 \ell \times d}{\ell}$, $m = \pi r^2 d$, where d = Density of matter

Velocity of transverse wave in any wire $v = \sqrt{\frac{T}{m}}$ or $\sqrt{\frac{T}{\pi r^2 d}} = \sqrt{\frac{T}{Ad}}$ $\because \pi r^2 = A$

- If m is constant then, $v \propto \sqrt{T}$ it is called tension law.
- If tension is constant then $v \propto \sqrt{\frac{1}{m}}$ ← it is called law of mass
- If T is constant & take wire of different radius for same material then $v \propto \frac{1}{r}$ ← it is called law of radius
- If T is constant & take wire of same radius for different material. Then $v \propto \sqrt{\frac{1}{d}}$ ← law of density

REFLECTION FROM RIGID END

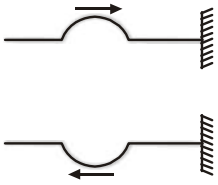
When the pulse reaches the right end which is clamped at the wall, the element at the right end exerts a force on the clamp and the clamp exerts equal and opposite force on the element. The element at the right end is thus acted upon by the force from the clamp. As this end remains fixed, the two forces are opposite to each other. The force from the left part of the string transmits the forward wave pulse and hence, the force exerted by the clamp sends a return pulse on the string whose shape is similar to a return pulse but is inverted. The original pulse tries to pull the element at the fixed end up and the return pulse sent by the clamp tries to pull it down, so the resultant displacement is zero. Thus, the wave is reflected from the fixed end and the reflected wave is inverted with respect to the original wave. The shape of the string at any time, while the pulse is being reflected, can be found by adding an inverted image pulse to the incident pulse.

Equation of wave propagating in +ve x-axis

Incident wave $y_1 = a \sin(\omega t - kx)$

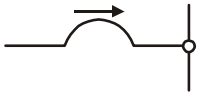
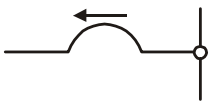
Reflected wave $y_2 = a \sin(\omega t + kx + \pi)$

or $y_2 = -a \sin(\omega t + kx)$



REFLECTION FROM FREE END

The right end of the string is attached to a light frictionless ring which can freely move on a vertical rod. A wave pulse is sent on the string from left. When the wave reaches the right end, the element at this end is acted on by the force from the left to go up. However, there is no corresponding restoring force from the right as the rod does not exert a vertical force on the ring. As a result, the right end is displaced in upward direction more than the height of the pulse i.e., it overshoots the normal maximum displacement. The lack of restoring force from right can be equivalent described in the following way. An extra force acts from right which sends a wave from right to left with its shape identical to the original one. The element at the end is acted upon by both the incident and the reflected wave and the displacements add. Thus, a wave is reflected by the free end without inversion.

Incident wave $y_1 = a \sin(\omega t - kx)$  Reflected wave $y_2 = a \sin(\omega t + kx)$ 

STATIONARY WAVES

* **Definition :** The wave propagating in such a medium will be reflected at the boundary and produce a wave of the same kind travelling in the opposite direction. The superposition of the two waves will give rise to a stationary wave. Formation of stationary wave is possible only and only in bounded medium.

\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

ANALYTICAL METHOD FOR STATIONARY WAVES

- **From rigid end** : We know equation for progressive wave in positive x-direction $y_1 = a \sin(\omega t - kx)$
  After reflection from rigid end $y_2 = a \sin(\omega t + kx + \pi) = -a \sin(\omega t + kx)$
  By principle of super position. $y = y_1 + y_2 = a \sin(\omega t - kx) - a \sin(\omega t + kx) = -2a \sin kx \cos \omega t$
  This is equation of stationary wave reflected from rigid end

  Amplitude $=2a \sin kx$ **Velocity of particle** $v_{pa} = \frac{dy}{dt} = 2a\,\omega \sin kx \sin \omega t$

  Strain $\frac{dy}{dx} = -2ak \cos kx \cos \omega t$ **Elasticity** $E = \frac{\text{stress}}{\text{strain}} = \frac{dp}{\frac{dy}{dx}}$ **Change in pressure** $dp = E\frac{dy}{dx}$

  • **Node** $x = 0, \frac{\lambda}{2}, \lambda$ ..... $A = 0$, $V_{pa} = 0$, strain $\rightarrow$ max. **Change in pressure** $\rightarrow$ max

  • **Antinode** $x = \frac{\lambda}{4}, \frac{3\lambda}{4}$ ..... $A \rightarrow$ max, $-V_{pa} \rightarrow$ max. strain = 0 **Change in pressure** = 0

- **From free end** : we know equation for progressive wave in positive x-direction $y_1 = a \sin(\omega t - kx)$
  After reflection from free end $y_2 = a \sin(\omega t + kx)$
  By Principle of superposition $y = y_1 + y_2 = a \sin(\omega t - kx) + a \sin(\omega t + kx) = 2\ a \sin \omega t \cos kx$

  Amplitude $= 2a \cos kx$, **Velocity of particle** $= v_{Pa} = \frac{dy}{dt} = 2a\,\omega \cos \omega t \cos kx$

  Strain $\frac{dy}{dx} = -2ak \sin \omega t \sin kx$ **Change in pressure** $dp = E\frac{dy}{dx}$

  • **Antinode :** $x = 0, \frac{\lambda}{2}, \lambda$ ..... $A \rightarrow$ Max, $V_{pa} = \frac{dy}{dt} \rightarrow$ max. Strain = 0, dp = 0

  • **Node :** $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ..... $A = 0$, $V_{pa} = \frac{dy}{dt} = 0$, strain $\rightarrow$ max , dp $\rightarrow$ max

PROPERTIES OF STATIONARY WAVES

The stationary waves are formed due to the superposition of two identical simple harmonic waves travelling in opposite direction with the same speed.

Important characteristics of stationary waves are:–

(i) Stationary waves are produced in the bounded medium and the boundaries of bounded medium may be rigid or free.

(ii) In stationary waves nodes and antinodes are formed alternately. Nodes are the points which are always in rest having maximum strain. Antinodes are the points where the particles vibrate with maximum amplitude having minimum strain.

(iii) All the particles except at the nodes vibrate simple harmonically with the same period.

(iv) The distance between any two successive nodes or antinodes is $\lambda/2$.

(v) The amplitude of vibration gradually increases from zero to maximum value from node to antinode.

(vi) All the particles in one particular segment vibrate in the same phase, but the particle of two adjacent segments differ in phase by 180

(vii) All points of the medium pass through their mean position simultaneously twice in each period.

(viii) Velocity of the particles while crossing mean position varies from maximum at antinodes to zero at nodes.

(ix) In a stationary wave the medium is splited into segments and each segment is vibrating up and down as a whole.

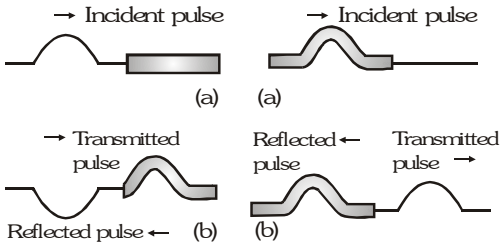
\\nodes\E_NODES (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion \Eng \Theory.p65

(x) In longitudinal stationary waves, condensation (compression) and refraction do not travel forward as in progressive waves but they appear and disappear alternately at the same place.

(xi) These waves do not transfer energy in the medium. Transmission of energy is not possible in a stationary wave.

TRANSMISSION OF WAVES

We may have a situation in which the boundary is intermediate between these two extreme cases, that is, one in which the boundary is neither rigid nor free. In this case, part of the incident energy is transmitted and part is reflected. For instance, suppose a light string is attached to a heavier string as in (figure). When a pulse travelling on the light reaches the knot, same part of it is reflected and inverted and same part of it is transmitted to the heavier string.



As one would expect, the reflected pulse has a smaller amplitude than the incident pulse, since part of the incident energy is transferred to the pulse in the heavier string. The inversion in the reflected wave is similar to the behaviour of a pulse meeting a rigid boundary, when it is totally reflected. When a pulse travelling on a heavy string strikes the boundary of a lighter string, as in (figure), again part is reflected and part is transmitted. However, in this case the reflected pulse is not inverted. In either case, the relative height of the reflected and transmitted pulses depend on the relative densities of the two string. In the previous section, we found that the speed of a wave on a string increases as the density of the string decreases. That is, a pulse travels more slowly on a heavy string than on a light string, if both are under the same tension. The following general rules apply to reflected waves. When a wave pulse travels from medium A to medium B and $v_A > v_B$ (that is, when B is denser than A), the pulse will be inverted upon reflection. When a wave pulse travels from medium A to medium B and $v_A < v_B$ (A is denser than B), it will not be inverted upon reflection.

GOLDEN KEY POINTS

Phenomenon of reflection and transmission of waves obeys the laws of reflection and refraction. The frequency of these wave remains constant i.e. does not change. $\omega_i = \omega_r = \omega_t = \omega$

From rarer to denser medium $y_i = a_i \sin(\omega t - k_1 x)$ $y_r = -a_i \sin(\omega t + k_1 x)$ $y_t = a_t \sin(\omega t - k_2 x)$

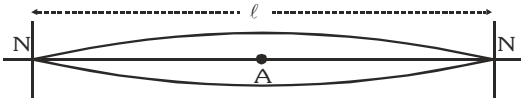
From denser to rarer medium $y_i = a_i \sin(\omega t - k_1 x)$ $y_r = a_i \sin(\omega t + k_1 x)$ $y_t = a_t \sin(\omega t - k_2 x)$

STATIONARY WAVE ARE OF TWO TYPES :

(i) Transverse st. wave (stretched string) **(ii)** Longitudinal st. wave (organ pipes)

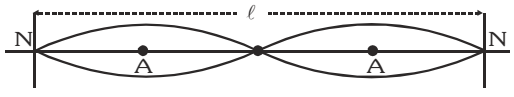
(i) Transverse Stationary wave (Fixed at Both ends)

Fundamental Harmonic



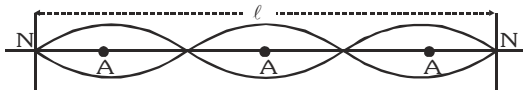
$$\frac{\lambda}{2} = \ell \Rightarrow \lambda = 2\ell \Rightarrow f = \frac{v}{2\ell}$$

Second Harmonic



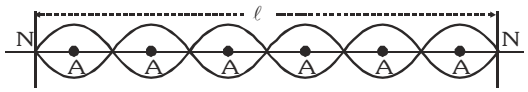
$$\frac{2\lambda}{2} = \ell \Rightarrow \lambda = \ell \Rightarrow f = \frac{2v}{2\ell}$$

Third Harmonic



$$\frac{3\lambda}{2} = \ell \Rightarrow \lambda = \frac{2\ell}{3} \Rightarrow f = \frac{3v}{2\ell}$$

\\nodes6\E_NODES\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

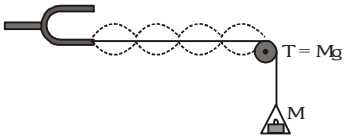
p^{th} harmonic 

$\frac{p\lambda}{2} = \ell \Rightarrow \lambda = \frac{2\ell}{p} \Rightarrow f = \frac{pv}{2\ell}$

- **Law of length :** For a given string, under a given tension, the fundamental frequency of vibration is inversely proportional to the length of the string, i.e, $n \propto \frac{1}{\ell}$ (T and m are constant)
- **Law of tension :** The fundamental frequency of vibration of stretched string is directly proportional to the square root of the tension in the string, provided that length and mass per unit length of the string are kept constant. $n \propto \sqrt{T}$ (ℓ and m are constant)
- **Law of mass :** The fundamental frequency of vibration of a stretched string is inversely proportional to the square root of its mass per unit length provided that length of the string and tension in the string are kept constant, i.e., $n \propto \frac{1}{\sqrt{m}}$ (ℓ and T are constant)
- **Melde's experiment :** In Melde's experiment, one end of a flexible piece of thread is tied to the end of a tuning fork. The other end passed over a smooth pulley carries a pan which can be loaded. There are two arrangements to vibrate the tied fork with thread.

Transverse arrangement :

Case 1. In a vibrating string of fixed length, the product of number of loops and square root of tension are constant or $p\sqrt{T}$ = constant.

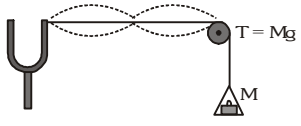


Case 2. When the tuning fork is set vibrating as shown in fig. then the prong vibrates at right angles to the thread. As a result the thread is set into motion. The frequency of vibration of the thread (string) is equal to the frequency of the tuning fork. If length and tension are properly adjusted then, standing waves are formed in the string. (This happens when frequency of one of the normal modes of the string matched with the frequency of the tuning fork). Then, if p loops are formed in the thread, then the frequency of the tuning fork is given by $n = \frac{p}{2\ell}\sqrt{\frac{T}{m}}$

Case 3. If the tuning fork is turned through a right angle, so that the prong vibrates along the length of the thread, then the string performs only a half oscillation for each complete vibrations of the prong. This is because the thread only makes node at the midpoint when the prong moves towards the pulley i.e. only once in a vibration.

Longitudinal arrangement :

The thread performs sustained oscillations when the natural frequency of the given length of the thread under tension is half that of the fork.

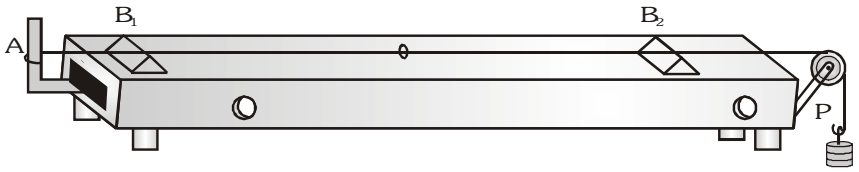


Thus if p loops are formed in the thread, then the frequency of the tuning fork is $n = \frac{2p}{2\ell}\sqrt{\frac{T}{m}}$

\Nodes\E_NODES\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

SONOMETER :

Sonometer consists of a hollow rectangular box of light wood. One end of the experimental wire is fastened to one end of the box. The wire passes over a frictionless pulley P at the other end of the box. The wire is stretched by a tension T.



The box serves the purpose of increasing the loudness of the sound produced by the vibrating wire. If the length the wire between the two bridges is ℓ, then the frequency of vibration is $n = \frac{1}{2\ell}\sqrt{\frac{T}{m}}$

To test the tension of a tuning fork and string, a small paper rider is placed on the string. When a vibrating tuning fork is placed on the box, and if the length between the bridges is properly adjusted, then when the two frequencies are exactly equal, the string quickly picks up the vibrations of the fork and the rider is thrown off the wire.

COMPARISON OF PROGRESSIVE AND STATIONARY WAVES

Progressive waves	Stationary waves
1. These waves travels in a medium with definite velocity.	These waves do not travel and remain confined between two boundaries in the medium.
2. These waves transmit energy in the medium.	These waves do not transmit energy in the medium.
3. The phase of vibration varies continuously from particle to particle.	The phase of all the particles in between two nodes is always same. But particles of two Adjacent nodes differ in phase by 180
4. No particle of medium is Permanently at rest.	Particles at nodes are permanently at rest.
5. All particles of the medium vibrate and amplitude of vibration is same.	The amplitude of vibration changes from particle to particle. The amplitude is zero for all at nodes and maximum at antinodes.
6. All the particles do not attain the maximum displacement position simultaneously.	All the particles attain the maximum displacement

SPEED OF LONGITUDINAL (SOUND) WAVES

Newton Formula $v_{medium} = \sqrt{\frac{E}{\rho}}$ (Use for every medium)

Where E = Elasticity coefficient of medium & ρ = Density of medium

- **For solid medium** $v_{solid} = \sqrt{\frac{Y}{\rho}}$ Where E = Y = Young's modulas
- **For liquid Medium** $v_{liquid} = \sqrt{\frac{B}{\rho}}$ Where E = B, where B = volume elasticity coefficient of liquid

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

- **For gas medium**

The formula for velocity of sound in air was first obtained by Newton. He assumed that sound propagates through air and temperature remains constant. (i.e. the process is isothermal) so Isothermal Elasticity = P $\therefore$ $v_{air} = \sqrt{(P/\rho}$

At NTP for air $P = 1.01 \times 10^5$ N/m^2 and $\rho = 1.3$ kg/m^3 so $v_{air} = \sqrt{\frac{1.01\times10^5}{1.3}} = 279$ m/s

However, the experimental value of sound in air is 332 m/s which is much higher than given by Newton's formula.

- **Laplace Correction**

In order to remove the discrepancy between theoretical and experimental values of velocity of sound, Laplace modified Newton's formula assuming that propagation of sound in air is adiabatic process, i.e.

Adiabatic Elasticity = γp so that $v = \sqrt{\frac{\gamma P}{\rho}}$ i.e. $v = \sqrt{1.41} \times 279 = 331.3$ m/s [as $\gamma_{air} = 1.41$]

Which is in good agreement with the experimental value (332 m/s). This in turn establishes that sound propagates adiabatically through gases.

The velocity of sound in air at NTP is 332 m/s which is much lesser than that of light and radio–waves (= 3 x 10^8 m/s). This implies that –

(a) If we set our watch by the sound of a distant siren it will be slow.

(b) If we record the time in a race by hearing sound from starting point it will be lesser than actual.

(c) In a cloud–lightening, though light and sound are produced simultaneously but as c > v, light proceeds thunder. An in case of gases –

$$v_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\text{mass}}} \left[\text{as } \rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{V}\right] \text{ or } v_s = \sqrt{\frac{\gamma \mu RT}{M}} \text{ [as } PV = \mu RT] \text{ or } v_s = \sqrt{\frac{\gamma RT}{M_w}}$$

$$\left[\text{as } \mu = \frac{\text{mass}}{M_w} = \frac{M}{M_w} \text{ where } M_w = \text{Molecular weight}\right]$$

And from kinetic-theory of gases $v_{rms} = \sqrt{(3RT/M_w)}$ So $\frac{v_s}{v_{rms}} = \sqrt{\frac{\gamma}{3}}$

EFFECT OF VARIOUS QUANTITIES

(1) Effect of temperature

For a gas γ & M_W is constant $v \propto \sqrt{T} \Rightarrow \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{t+273}{273}} \Rightarrow v_t = v_0\left[1+\frac{t}{273}\right]^{\frac{1}{2}}$

By applying Binomial theorem.

(i) For any gas medium $v_t = v_0\left[1+\frac{t}{546}\right]$ **(ii)** For air : $v_t = v_0 + 0.61\ t$ m/sec. ($v_0 = 332$ m/sec.)

(2) Effect of Relative Humidity

With increase in humidity, density decreases so in the light of $v = \sqrt{\gamma P/\rho)}$ We conclude that with rise in humidity velocity of sound increases. This is why sound travels faster in humid air (rainy season) than in dry air (summer) at same temperature. Due to this in rainy season the sound of factories siren and whistle of train can be heard more than summer.

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion \Eng\Theory.p65

(3) Effect of Pressure

As velocity of sound $v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$

So pressure has no effect on velocity of sound in a gas as long as temperature remain constant. This is why in going up in the atmosphere, though both pressure and density decreases, velocity of sound remains constant as long as temperature remains constant. Further more it has also been established that all other factors such as amplitude, frequency, phase, loudness pitch, quality etc. has partially no effect on velocity of sound.

Velocity of sound in air is measured by resonance tube or Hebb's method while in gases by Quinke's tube. Kundt's tube is used to determine velocity of sound in any medium solid, liquid or gas.

(4) Effect of Motion of Air

If air is blowing then the speed of sound changes. If the actual speed of sound is v and the speed of air is w, then the speed of sound in the direction in which air is blowing will be (v+ w), and in the opposite direction it will be (v – w).

(5) Effect of Frequency

There is no effect of frequency on the speed of sound. Sound waves of different frequencies travel with the same speed in air although their wavelength in air are different. If the speed of sound were dependent on the frequency, then we could not have enjoyed orchestra.

Example

A piezo electric quartz plate of thickness 0.005 m is vibrating in resonant conditions. Calculate its fundamental frequency if for quartz $Y = 8 \times 10^{10}$ N/m^2 and $\rho = 2.65 \times 10^3$ kg/m^3

Solution

We known that for longitudinal waves in solids $v = \sqrt{\frac{Y}{\rho}}$, So $v = \sqrt{\frac{8\times10^{10}}{2.65\times10^3}} = 5.5 \times 10^3$ m/s

Further more for fundamental mode of plate – $(\lambda/2) = L$ So $\lambda = 2 \times 5 \times 10^{-3} = 10^{-2}$ m

But as $v = f\lambda$, i.e., $f = (v/\lambda)$ so $f = [5.5 \times 10^3/10^{-2}] = 5.5 \times 10^5$ Hz = 550 kHz

Example

Determine the change in volume of 6 liters of alcohol if the pressure is decreased from 200 cm of Hg to 75 cm. [velocity of sound in alcohol is 1280 m/s, density of alcohol = 0.81 gm/cc, density of Hg = 13.6 gm/cc and g = 9.81 m/s^2]

Solution

For propagation of sound in liquid $v = \sqrt{(B/\rho)}$ i.e., $B = v^2\rho$

But by definition $B = -V\frac{\Delta P}{\Delta V}$ So $-V\frac{\Delta P}{\Delta V} = v^2\rho$, i.e. $\Delta V = \frac{V(-\Delta P)}{\rho v^2}$

Here $\Delta P = H_2\rho g - H_1\rho g = (75 - 200) \times 13.6 \times 981 = -1.667 \times 10^6$ dynes/cm^2

So $\Delta V = \frac{(6\times10^3)(1.667\times10^6)}{0.81\times(1.280\times10^5)^2} = 0.75$ cc

Example

(a) Speed of sound in air is 332 m/s at NTP. What will the speed of sound in hydrogen at NTP if the density of hydrogen at NTP is (1/16) that of air.

(b) Calculate the ratio of the speed of sound in neon to that in water vapour at any temperature. [Molecular weight of neon = 2.02×10^{-2} kg/mol and for water vapours = 1.8×10^{-2} kg/mol]

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion \Eng\Theory.p65

Solution

The velocity of sound in air is given by $v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$

(a) In terms of density and pressure $\frac{v_H}{v_{air}} = \sqrt{\frac{P_H}{\rho_H} \times \frac{\rho_{air}}{P_{air}}} = \sqrt{\frac{\rho_{air}}{\rho_H}}$ [as $P_{air} = P_H$]

$$\Rightarrow \quad v_H = v_{air} \times \sqrt{\frac{\rho_{air}}{\rho_H}} = 332 \times \sqrt{\frac{16}{1}} = 1328 \text{ m/s}$$

(b) In terms of temperature and molecular weight $\frac{v_{Ne}}{v_W} = \sqrt{\frac{\gamma_{Ne}}{M_{Ne}} \times \frac{M_W}{\gamma_W}}$ [as $T_N = T_W$]

Now as neon is mono atomic ($\gamma = 5/3$) while water vapours poly atomic ($\gamma = 4/3$) so

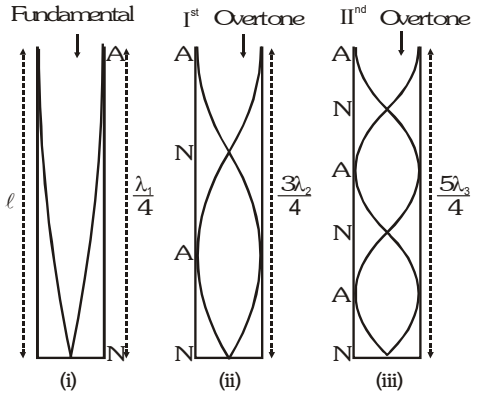
$$\frac{v_{Ne}}{v_W} = \sqrt{\frac{(5/3) \times 1.8 \times 10^{-2}}{(4/3) \times 2.02 \times 10^{-2}}} = \sqrt{\frac{5}{4} \times \frac{1.8}{2.02}} = 1.055$$

VIBRATION IN ORGAN PIPES

When two longitudinal waves of same frequency and amplitude travel in a medium in opposite directions then by superposition, standing waves are produced. These waves are produced in air columns in cylindrical tube of uniform diameter. These sound producing tubes are called organ pipes.

1. Vibration of air column in closed organ pipe :

The tube which is closed at one end and open at the other end is called close organ pipe. On blowing air at the open end, a wave travels towards closed end from where it is reflected towards open end. As the wave reaches open end, it is reflected again. So two longitudinal waves travel in opposite directions to superpose and produce stationary waves. At the closed end there is a node since particles does not have freedom to vibrate whereas at open end there is an antinode because particles have greatest freedom to vibrate.



Hence on blowing air at the open end, the column vibrates forming antinode at free end and node at closed end. If ℓ is length of pipe and λ be the wavelength and v be the velocity of sound in organ pipe then

Case (i) $\ell = \frac{\lambda}{4} \quad \Rightarrow \quad \lambda = 4\ell \quad \Rightarrow \quad n_1 = \frac{v}{\lambda} = \frac{v}{4\ell}$ fundamental frequency.

Case (ii) $\ell = \frac{3\lambda}{4} \quad \Rightarrow \quad \lambda = \frac{4\ell}{3} \quad \Rightarrow \quad n_2 = \frac{v}{\lambda} = \frac{3v}{4\ell}$ First overtone / IIIrd Harmonic

Case (iii) $\ell = \frac{5\lambda}{4} \quad \Rightarrow \quad \lambda = \frac{4\ell}{5} \quad \Rightarrow \quad n_3 = \frac{v}{\lambda} = \frac{5v}{4\ell}$ Second overtone / Vth Harmonic

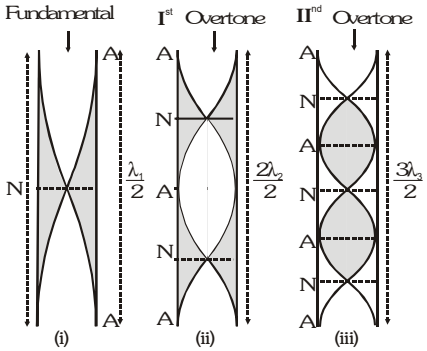
When closed organ pipe vibrate in mth overtone then $\ell = (2m+1)\frac{\lambda}{4}$

So $$\lambda = \frac{4\ell}{(2m+1)} \Rightarrow n = (2m+1)\frac{v}{4\ell}$$

Hence frequency of overtones is given by $n_1 : n_2 : n_3$ ..... = 1 : 3 : 5 ....

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

2. Vibration of air columns in open organ pipe :



The tube which is open at both ends is called an open organ pipe. On blowing air at the open end, a wave travel towards the other end after reflection from open end waves travel in opposite direction to superpose and produce stationary wave. Now the pipe is open at both ends by which an antinode is formed at open end. Hence on blowing air at the open end antinodes are formed at each end and nodes in the middle. If ℓ is length of the pipe and λ be the wavelength and v is velocity of sound in organ pipe.

Case (i) $\ell = \frac{\lambda}{2} \Rightarrow \lambda = 2\ell \Rightarrow n_1 = \frac{v}{\lambda} = \frac{v}{2\ell}$ Fundamental frequency.

Case (ii) $\ell = \frac{2\lambda}{2} \Rightarrow \lambda = \frac{2\ell}{2} \Rightarrow n_2 = \frac{v}{\lambda} = \frac{2v}{2\ell}$ First overtone frequency.

Case (iii) $\ell = \frac{3\lambda}{2} \Rightarrow \lambda = \frac{2\ell}{3} \Rightarrow n_3 = \frac{v}{\lambda} = \frac{3v}{2\ell}$ Second overtone frequency.

Hence frequency of overtones are given by the relation $n_1 : n_2 : n_3$ ..... = 1 : 2 : 3 ....

When open organ pipe vibrate in m^{th} overtone then $\ell = (m+1)\frac{\lambda}{4}$ so $\lambda = \frac{4\ell}{m+1} \Rightarrow n = (m+1)\frac{v}{2\ell}$

GOLDEN KEY POINTS

- A rod clamped at one end or a string fixed at one end is similar to vibration of closed end organ pipe.
- A rod clamped in the middle is similar to the vibration of open end organ pipe.
- If an open pipe is half submerged in water, it becomes a closed organ pipe of length half that of open pipe i.e. frequency remains same.
- Due to finite motion of air molecular in organ pipes reflection takes place not exactly at open end but some what above it so in an organ pipe antinode is not formed exactly at free–end but above it at a distance e = 0.6r (called end correction or Rayleigh's correction) with r being the radius of pipe. So for closed organ pipe $L \rightarrow L + 0.6r$ while for open $L \rightarrow L + 2 \times 0.6r$ (as both ends are open)

  so that – $f_C = \frac{v}{4(L+0.6r)}$ while $f_0 = \frac{v}{2(L+1.2r)}$

  This is why for a given v and L narrower the pipe higher will the frequency or pitch and shriller will be the sound.
- For an organ pipe (closed or open) if v = constant. $f \propto (1/L)$

  So with decrease in length of vibrating air column, i.e., wavelength ($\lambda \propto L$), frequency or pitch will increase and vice–versa. This is why the pitch increases gradually as an empty vessel fills slowly.
- For an organ pipe if f = constant. $v \propto \lambda$ or $v \propto L$, $f = \frac{v}{\lambda}$ = constant i.e. the frequency of an organ pipe will remain unchanged if the ratio of speed of sound in to its wave length remains constant.
- As for a given length of organ pipe ℓ = constant. $f \propto v$ So
  - **(a)** With rise in temperature as velocity will increase ($v \propto \sqrt{T}$), the pitch will increase.
    (Change in length with temperature is not considered unless stated)
  - **(b)** With change in gas in the pipe as v will change and so f will change ($v \propto \sqrt{\gamma / M}$)
  - **(c)** With increase in moisture as v will increase and so the pitch will also.

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example

For a certain organ pipe, three successive resonant frequencies are observed at 425, 595 and 765 Hz respectively. Taking the speed for sound in air to be 340 m/s (a) Explain whether the pipe is closed at one end or open at both ends (b) determine the fundamental frequency and length of the pipe.

Solution

(a) The given frequencies are in the ratio

425 : 595 : 765, i.e., 5 : 7 : 9

And as clearly these are odd integers so the given pipe is closed pipe.

(b) From part (b) it is clear that the frequency of 5th harmonic (which is third overtone) is 425 Hz

so $425 = 5f_c \Rightarrow f_C = 85$ Hz Further as $f_C = \frac{v}{4L}$, $L = \frac{v}{4f_C} = \frac{340}{4\times 85} = 1$ m

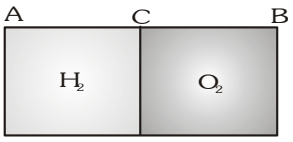
Example

AB is a cylinder of length 1 m fitted with a thin flexible diaphram C at middle and two other thin flexible diaphram A and B at the ends. The portions AC and BC contain hydrogen and oxygen gases respectively. The diaphragms A and B are set into vibrations of the same frequency. What is the minimum frequency of these vibrations for which diaphram C is a node ? Under the condition of the experiment the velocity of sound in hydrogen is 1100 m/s and oxygen 300 m/s.

Solution

As diaphragm C is a node, A and B will be antinode (as in a organ pipe either both ends are antinode or one end node and the other antinode), i.e., each part will behave as closed end organ pipe so that

$f_H = \frac{v_H}{4L_H} = \frac{1100}{4\times 0.5} = 550$ Hz And $f_0 = \frac{v_0}{4L_0} = \frac{300}{4\times 0.5} = 150$Hz



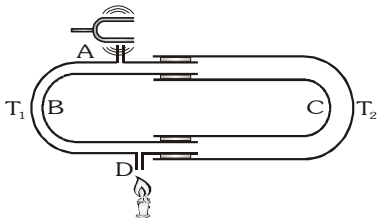
As the two fundamental frequencies are different, the system will vibrate with a common frequency f such that $f = n_H f_H = n_0 f_0$ i.e. $\frac{n_H}{n_0} = \frac{f_0}{f_H} = \frac{150}{550} = \frac{3}{11}$

i.e., the third harmonic of hydrogen and 11th harmonic of oxygen or 6th harmonic of hydrogen and 22nd harmonic of oxygen will have same frequency. So the minimum common frequency

$f = 3 \times 550$ or $11 \times 150 = 1650$ Hz

APPARATUS FOR DETERMINING SPEED OF SOUND

1. Quinck's Tube :



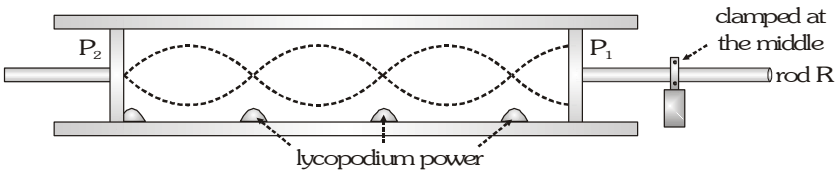
It consists of two U shaped metal tubes. Sound waves with the help of tuning fork are produced at A which travel through B & C and comes out at D where a sensitive flame is present. Now the two waves coming through different path interfere and flame flares up. But if they are not in phase, destructive interference occurs and flame remains undisturbed.

Suppose destructive interference occurs at D for some position of C. If now the tube C is moved so that interference condition is disturbed and again by moving a distance x, destructive interference occurs so that $2x = \lambda$. Similarly if the distance moved between successive constructive and destructive interference is x then $2x = \frac{\lambda}{2}$ Now by having value of x, speed of sound is given by $v = n\lambda$.

2. Kundt's tube : It is the used to determine speed of sound in different gases. It consists of a glass tube in which a small quantity of lycopodium powder is spread. The tube is rotated so that powder starts slipping. Now rod CD is rubbed at end D so that stationary waves form. The disc C vibrates so that air column also vibrates with the frequency of the rod. The piston P is adjusted so that frequency of air column become same as that of rod.

\\nodes\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

So resonance occurs and column is thrown into stationary waves. The powder sets into oscillations at antinodes while heaps of powder are formed at nodes.



Let n is the frequency of vibration of the rod then, this is also the frequency of sound wave in the air column in the tube.

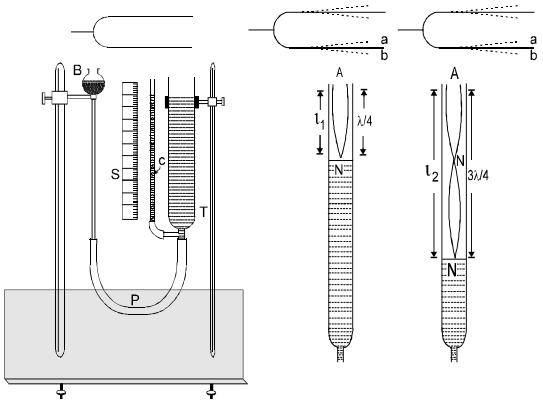
For rod : $\frac{\lambda_{rod}}{2} = \ell_{rod}$ **For air :** $\frac{\lambda_{air}}{2} = \ell_{air}$

Where ℓ_{air} is the distance between two heaps of powder in the tube (i.e. distance between two nodes). If v_{air} and v_{rod} are velocity of sound waves in the air and rod respectively, then $n = \frac{v_{air}}{\lambda_{air}} = \frac{v_{rod}}{\lambda_{rod}}$

Therefore, $\frac{v_{air}}{v_{rod}} = \frac{\lambda_{air}}{\lambda_{rod}} = \frac{\ell_{air}}{\ell_{rod}}$ Thus knowledge of v_{rod} determines v_{air}.

(ii) RESONANCE TUBE

Construction : The resonance tube is a tube T (figure) made of brass or glass, about 1 meter long and 5 cm in diameter and fixed on a vertical stand. Its lower end is connected to a water reservoir B by means of a flexible rubber tube. The rubber tube carries a pinch-cock P. The level of water in T can be raised or lowered by water adjusting the height of the reservoir B and controlling the flow of water from B to T or from T to B by means of the pinch-cock P. Thus the length of the air–column in T can be changed. The position of the water level in T can be read by means of a side tube C and a scale S.



Determination of the speed of sound in air by resonance tube

First of all the water reservoir B is raised until the water level in the tube T rises almost to the top of the tube. Then the pinch–cock P is tightened and the reservoir B is lowered. The water level in T stays at the top. Now a tuning fork is sounded and held over the mouth of tube .The pinch–cock P is opened slowly so that the water level in T falls and the length of the air–column increases. At a particular length of air–column in T, a loud sound is heard. This is the first state of resonance. In this position the following phenomenon takes place inside the tube.

(i) For first resonance $\ell_1 = \lambda/4$

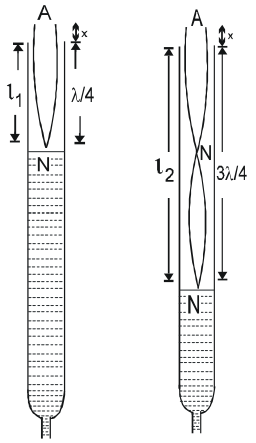
(ii) For second resonance $\ell_2 = 3\lambda/4 \Rightarrow \ell_2 - \ell_1 = \lambda/2 \Rightarrow \lambda = 2(\ell_2 - \ell_1)$

If the frequency of the fork be n and the temperature of the air-column be t°C, then the speed of sound at t°C is given by $v_t = n\lambda = 2n(\ell_2 - \ell_1)$

The speed of sound wave at 0°C $v_0 = (v_t - 0.61\ t)$ m/s.

\\nodes6\E_NODES (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

End Correction : In the resonance tube, the antinode is not formed exactly at the open but slightly outside at a distance x. Hence the length of the air -column in the first and second states of resonance are (l_1 + x) and (ℓ_2 + x) then



(i) For first resonance $\ell_1 + x = \lambda/4$ .....(i)

(ii) For second resonance $\ell_2 + x = 3\lambda/4$ .....(ii))

Subtract Equation (ii) from Equation (i)

$$\ell_2 - \ell_1 = \lambda/2$$

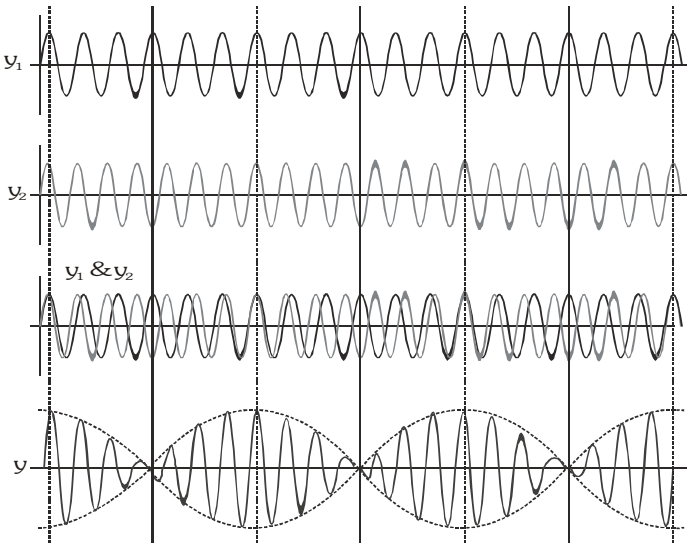
$$\lambda = 2(\ell_2 - \ell_1)$$

Put the value of λ in Equation (i) $\ell_1 + x = \dfrac{2(\ell_2 - \ell_1)}{4}$

$$\Rightarrow \ell_1 + x = \frac{\ell_2 - \ell_1}{2} \Rightarrow x = \frac{\ell_2 - 3\ell_1}{2}$$

BEATS

When two sound waves of same amplitude travelling in same direction with slightly different frequency superimpose, then intensity varies periodically with time. This effect is called Beats. Suppose two waves of frequencies f_1 and f_2 ($<f_1$) are meeting at some point in space. The corresponding periods are T_1 and T_2 ($>T_1$). If the two waves are in phase at t=0, they will again be in phase when the first wave has gone through exactly one more cycle than the second. This will happen at a time t=T, the period of the beat. Let *n* be the number of cycles of the first wave in time T, then the number of cycles of the second wave in the same time is (n–1). Hence, $T = nT_1 = (n-1)\,T_2$



Eliminating n we have $T = \dfrac{T_1T_2}{T_2 - T_1} = \dfrac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \dfrac{1}{f_1 - f_2}$

The reciprocal of the beat period is the beat frequency $f = \dfrac{1}{T} = f_1 - f_2$

Waves Interference On The Bases Of Beats :

Conditions : Two equal frequency waves travel in same direction.

Mathematical analysis

If displacement of first wave $y_1 = a \sin \omega_1 t \longrightarrow (N_1, a)$ $\quad I \propto N^2a^2$

Displacement of second wave $y_2 = a \sin \omega_2 t \longrightarrow (N_2, a)$

By superposition $y = y_1 + y_2$

Equation of resulting wave $y = a\{\sin 2\pi N_1 t + \sin 2\pi N_2 t\}$

\\nodes\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

$$y = a\left\{2\sin 2\pi t\frac{(N_1+N_2)}{2}\cos 2\pi t\frac{(N_1-N_2)}{2}\right\} = \left\{2a\cos 2\pi t\ \frac{(N_1-N_2)}{2}\right\}\ \sin\ 2\pi t\ \frac{(N_1+N_2)}{2} = A\ \sin\ 2\pi N't$$

Amplitude $A = 2a\ \text{Cos}\ 2\pi t\ \left(\frac{N_1-N_2}{2}\right) = 2a\ \cos\ \pi t\ (N_1 - N_2)$ **Frequency** $N' = \frac{N_1+N_2}{2}$

- **For max Intensity (A = ± 2a): –**

If $\cos\ \pi\ (N_1 - N_2)\ t = \pm 1 \Rightarrow \cos\ \pi\ (N_1 - N_2)\ t = \cos\ n\pi$, n = 0, 1, 2, ...

$$\Rightarrow \quad \pi\ (N_1 - N_2)t = n\pi \Rightarrow t = \frac{n}{N_1 - N_2} = 0, \frac{1}{\Delta N}, \frac{2}{\Delta N}, \frac{3}{\Delta N} \quad .....$$

- **For Minimum Intensity (A = 0) :**

$$\Rightarrow \quad \cos\ \pi\ (N_1 - N_2)\ t = 0 \Rightarrow \cos\ \pi\ (N_1 - N_2)\ t = \cos\ (2n + 1)\ \frac{\pi}{2} \qquad n = 0, 1, 2 \quad .....$$

$$\Rightarrow \quad \pi\ (N_1 - N_2)\ t = (2n + 1)\ \frac{\pi}{2} \Rightarrow t = \frac{2n+1}{2(N_1 - N_2)} = \frac{1}{2\Delta N}, \frac{3}{2\Delta N}, \frac{5}{2\Delta N} \quad .....$$

GOLDEN KEY POINTS

- When we added wax on tuning fork then the frequency of fork decreases.
- When we file the tuning fork then the frequency of fork increases.

Example

A tuning fork having n = 300 Hz produces 5 beats/s with another tuning fork. If impurity (wax) is added on the arm of known tuning fork, the number of beats decreases then calculate the frequency of unknown tuning fork.

Solution

The frequency of unknown tuning fork should be 300 $\pm$ 5 = 295 Hz or 305 Hz.

When wax is added, if it would be 305 Hz, beats would have increases but with 295 Hz beats is decreases so frequency of unknown tuning fork is 295 Hz.

Example

A tuning fork having n = 158 Hz, produce 3 beats/s with another. As we file the arm of unknown, beats become 7 then calculate the frequency of unknown.

Solution

The frequency of unknown tuning fork should be 158 ± 3 = 155 Hz or 161 Hz.

After filling the number of beats = 7 so frequency of unknown tuning fork should be

158 ± 7 = 165 Hz or 151 Hz.

As both above frequency can be obtain by filing so frequency of unknown = 155/161 Hz.

SPECIAL POINTS

1. Displacement and pressure waves

A sound wave (i.e. longitudinal mechanical wave) can be described either in terms of the longitudinal displacement suffered by the particles of the medium (called displacement-wave) or in terms of the excess pressure generated due to compression and rarefaction (called pressure-wave). Consider a sound wave travelling in the x-direction in a medium. Suppose at time t, the particle at the undisturbed position x suffers a displacement y in the x-direction. The displacement wave then will be described by $y = A \sin (\omega t - kx)$ ...(i)

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Now consider the element of medium which is confined with in x and x+Δx in the undisturbed state. If S is the cross-section, the volume element in undisturbed state will be V = S Δx. As the wave passes the ends at x and x +Δx are displaced by amount y and y + Δy so that increase in volume of the element will be ΔV = S Δy.

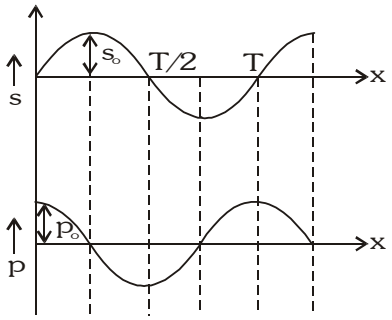
This in turn implies that volume strain for the element under consideration $\frac{\Delta V}{V} = \frac{S}{S}\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x}$ ... (ii)

So corresponding stress, i.e. excess pressure

$P = B\left[\frac{-\Delta V}{V}\right]\left[\text{as } B = -V\frac{\Delta P}{\Delta V} = -V\frac{P}{\Delta V}\right]$ or $P = -B\frac{\Delta y}{\Delta x}$ [From equation (ii)) ... (iii)

Note: For a harmonic -progressive - wave from $\frac{dy}{dx} = \frac{v_{pa}}{v}$, $P = -B\frac{dy}{dx} = -B\left(\frac{v_{pa}}{v}\right)$

i.e. pressure in a sound wave is equal to the product of elasticity of gas with the ratio of particle speed to wave speed.



But from equation (i) $\frac{\Delta y}{\Delta x} = -Ak\cos(\omega t - kx)$

So $P = AkB\cos(\omega t - kx)$

i.e. $P = P_0\cos(\omega t - kx)$ ...(iv)

with $P_0 = ABk$ ...(iv)

From equation (i) and (iv) it is clear that

- A sound wave may be considered as either a displacement wave $y = A\sin(\omega t - kx)$ or a pressure wave $P = P_0\cos(\omega t - kx)$.
- The pressure wave is 90^0 out of phase with respect to displacement wave, i.e, displacement will be maximum when pressure is minimum and vice-versa. This is shown in figure
- The amplitude of pressure wave:– $P_0 = ABk = Ak\rho v^2 = \rho v A\omega$ [as $v = \sqrt{B/\rho}$, $v = \omega/k$] ...(v)
- As sound-sensors (e.g. ear or mic) detects pressure changes, description of sound as pressure-wave is preferred over displacement wave.

2. Ultrasonic, Infrasonic and Audible (sonic) Sound :

Sound waves can be classified in three groups according to their range of frequencies.

- **Infrasonic Waves**

  Longitudinal waves having frequencies below 20 Hz are called infrasonic waves. They cannot be heard by human beings. They are produced during earthquakes. Infrasonic waves can be heard by snakes.

- **Audible Waves**

  Longitudinal waves having frequencies lying between 20-20,000 Hz are called audible waves.

- **Ultrasonic Waves**

  Longitudinal waves having frequencies above 20,000 Hz are called ultrasonic waves. They are produced and heard by bats. They have a large energy content.

- **Applications of Ultrasonic Waves**

  Ultrasonic waves have a large range of application. Some of them are as follows:

  (i) The fine internal cracks in metal can be detected by ultrasonic waves.

  (ii) Ultrasonic waves can be used for determining the depth of the sea, lakes etc.

  (iii) Ultrasonic waves can be used to detect submarines, icebergs etc.

  (iv) Ultrasonic waves can be used to clean clothes, fine machinery parts etc.

  (v) Ultrasonic waves can be used to kill smaller animals like rates, fish and frogs etc.

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

- **Shock Waves**

  If the speed of the body in air is greater than the speed of the sound, then it is called supersonic speed. Such a body leaves behind a conical region of disturbance which spreads continuously. Such a disturbance is called a **'Shock Wave'**. This wave carries huge energy. If it strikes a building, then the building may be damaged.

3. Sound intensity in decibels

The physiological sensation of loudness is closely related to the intensity of wave producing the sound. At a frequency of 1 kHz people are able to detect sounds with intensities as low as 10^{-12}W/m^2. On the other hand an intensity of 1 W/m^2 can cause pain, and prolonged exposure to sound at this level will damage a person's ears. Because the range in intensities over which people hear is so large, it is convenient to use a logarithmic scale to specify intensities. If the intensity of sound in watts per square meter is I, then the intensity level β in decibels (dB) is given by $\beta = 10\log\frac{I}{I_0}$

where the base of the logarithm is 10, and $I_0 = 10^{-12}$ W/m^2 (roughly the minimum intensity that can be heard).

On the decibel scale, the pain threshold of 1 W/m^2 is then $\beta = 10\log\frac{1}{10^{-12}} = 120\text{dB}$

Example

Calculate the change in intensity level when the intensity of sound increases by 10^6 times its original intensity.

Solution

Here $\frac{I}{I_0} = \frac{\text{Final intensity}}{\text{Initial intensity}} = 10^6$

Increase in intensity level, L = 10 $\log_{10}\left(\frac{I}{I_0}\right)$ = 10 $\log_{10}$ (10^6) = 10 $\log_{10}10$ = 10 6 1 = 60 decibels.

ECHO

Multiple reflection of sound is called an echo. If the distance of reflector from the source is d then, $2d = vt$

Hence, v = speed of sound and t, the time of echo. $\therefore d = \frac{vt}{2}$

Since, the effect of ordinary sound remains on our ear for 1\10 second, therefore, if the sound returns to the starting point before 1/10 second, then it will not be distinguished from the original sound and no echo will be heard. Therefore, the minimum distance of the reflector is, $d_{min} = \frac{v \times t}{2} = \left(\frac{330}{2}\right)\left(\frac{1}{10}\right) = 16.5\text{m}$

ACOUSTIC DOPPLER EFFECT (DOPPLER EFFECT FOR SOUND WAVES)

The apparent change in the frequency of sound when the source of sound, the observer and the medium are in relative motion is called Doppler effect. While deriving these expressions, we make the following assumptions

(i) The velocity of the source, the observer and the medium are along the line joining the positions of the source and the observer.

(ii) The velocity of the source and the observer is less than velocity of sound.

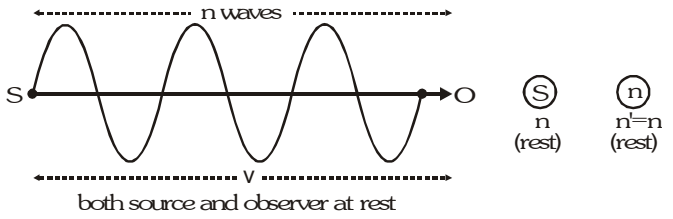
Doppler effect takes place both in sound and light. In sound it depends on whether the source or observer or both are in motion while in light it depends on whether the distance between source and observer is increasing or decreasing.

NOTATIONS

n $\rightarrow$ actual frequency	n' $\rightarrow$ observed frequency (apparent frequency)
λ $\rightarrow$ actual wave length	λ' $\rightarrow$ observed (apparent) wave length
v $\rightarrow$ velocity of sound	v_s $\rightarrow$ velocity of source
v_0 $\rightarrow$ velocity of observer	v_w $\rightarrow$ wind velocity

\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Case I : Source in motion, observer at rest, medium at rest :

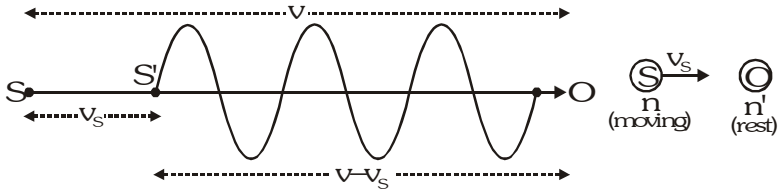


Suppose the source S and observer O are separated by distance v. Where v is the velocity of sound. Let n be the frequency of sound emitted by the source. Then n waves will be emitted by the source in one second. These n waves will be accommodated in distance v.

So, wave length $\lambda = \frac{\text{total distance}}{\text{total number of waves}} = \frac{v}{n}$

(i) Source moving towards stationary observer :

Let the sources start moving towards the observer with velocity v_s. After one second, the n waves will be crowded in distance $(v - v_s)$. Now the observer shall feel that he is listening to sound of wavelength λ' and frequency n'



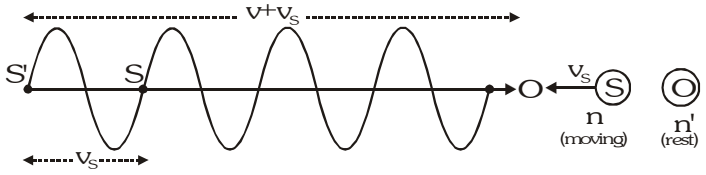
Now apparent wavelength $\lambda' = \frac{\text{total distance}}{\text{total number of waves}} = \frac{v - v_s}{n}$

and changed frequency, $n' = \frac{v}{\lambda'} = \frac{v}{\left(\frac{v - v_s}{n}\right)} = n\left(\frac{v}{v - v_s}\right)$

So, as the source of sound approaches the observer the apparent frequency n' becomes greater than the true frequency n.

(ii) When source move away from stationary observer :-

For this situation n waves will be crowded in distance $v + v_s$.



So, apparent wavelength $\lambda' = \frac{v + v_s}{n}$

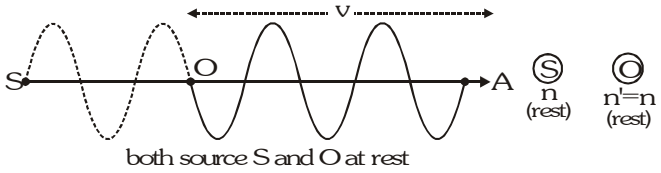
and

Apparent frequency $n' = \frac{v}{\lambda'} = \frac{v}{\left(\frac{v + v_s}{n}\right)} = n\left(\frac{v_s}{v + v_s}\right)$ So $n' < n$

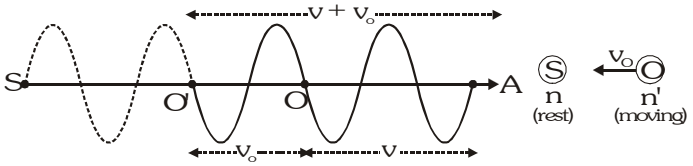
\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Case II : Observer in motion, source at rest, medium at rest

Let the source (S) and observer (O) are in rest at their respective places. Then n waves given by source 'S' would be crossing observer 'O' in one second and fill the space OA (=v)



(i) Observer moves towards stationary source



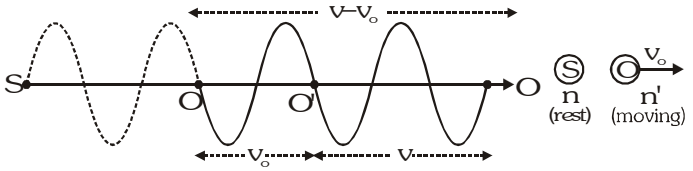
When observer 'O' moves towards 'S' with velocity v_o, it will cover v_o distance in one second. So the observer has received not only the n waves occupying OA but also received additional number of Δn waves occupying the distance OO' (= v_o).

So, total waves received by observer in one second i.e., apparent frequency (n') = Actual waves (n) + Additional waves (Δn)

$$n' = \frac{v}{\lambda} + \frac{v_o}{\lambda} = \frac{v + v_o}{(v/n)} = n\left(\frac{v + v_o}{v}\right)\left(\because \lambda = \frac{v}{n}\right) \quad \text{(so, } n' > n\text{)}$$

(ii) Observer moves away from stationary source :-

For this situation n waves will be crowded in distance $v - v_o$.



When observer move away from source with v_o velocity then he will get Δn waves less than real number of waves. So, total number of waves received by observer i.e.

Apparent frequency (n') = Actual waves (n) – reduction in number of waves (Δn)

$$n' = \frac{v}{\lambda} - \frac{v_o}{\lambda} = \frac{v - v_o}{\lambda} = \frac{v - v_o}{(v/n)} = \left(\frac{v - v_o}{v}\right)n \quad \left(\because \lambda = \frac{v}{n}\right) \text{ (so } n' < n\text{)}$$

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

GOLDEN KEY POINT

- If medium (air) is also moving with v_m velocity in direction of source to observer. Then velocity of sound relative to observer will be $v \pm v_m$ (–ve sign, if v_m is opposite to sound velocity). So, $n' = n\left(\frac{v \pm v_m \pm v_o}{v \pm v_m \mp v_s}\right)$
- If medium moves in a direction opposite to the direction of propagation of sound, then $n' = \left(\frac{v - v_m - v_o}{v - v_m - v_S}\right)n$
- Source in motion towards the observer. Both medium and observer are at rest. $n' = \left(\frac{v}{v - v_S}\right)n$

  So, when a source of sound approaches a stationary observer, the apparent frequency is more than the actual frequency.
- Source in motion away from the observer. Both medium and observer are at rest. $n' = \left(\frac{v}{v + v_S}\right)n$. So, when a source of sound moves away from a stationary observer, the apparent frequency is less than actual frequency.
- Observer in motion towards the source. Both medium and source are at rest. $n' = \left(\frac{v + v_o}{v}\right)n$. So, when observer is in motion towards the source, the apparent frequency is more than the actual frequency.
- Observer in motion away from the source. Both medium and source are at rest. $n' = \left(\frac{v - v_o}{v}\right)n$. So, when observer is in motion away from the source, the apparent frequency is less than the actual frequency.
- Both source and observer are moving away from each other. Medium at rest. $n' = \left(\frac{v - v_o}{v + v_S}\right)n$

DOPPLER'S EFFECT IN REFLECTION OF SOUND (ECHO)

When the sound is reflected from the reflector the observer receives two notes one directly from the source and other from the reflector. If the two frequencies are different then superposition of these waves result in beats and by the beat frequency we can calculate speed of the source.

If the source is at rest and reflector is moving towards the source with speed u,

then apparent frequency heard by reflector $n_1 = \left(\frac{v + u}{v}\right)n$

Now this frequency n_1 acts as a source so that apparent frequency received by observer is

$$n_2 = \left(\frac{v}{v - u}\right)n_1 = \left(\frac{v}{v - u}\right) \times \left(\frac{v + u}{v}\right)n = \left(\frac{v + u}{v - u}\right)n$$

If $u << v$ then $n_2 = n\left(1 + \frac{u}{v}\right)\left(1 - \frac{u}{v}\right)^{-1} \approx n\left(1 + \frac{u}{v}\right)^2 \approx n\left(1 + \frac{2u}{v}\right)$

Beat frequency $\Delta n = n_2 - n = \left(\frac{2u}{v}\right)n$ So speed of the source $u = \frac{v}{2}\left(\frac{\Delta n}{n}\right)$

\\nodes\E_NODE6 (E:)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

CONDITIONS WHEN DOPPLER'S EFFECT IS NOT OBSERVED FOR SOUND WAVES

- When the source of sound and observer both are at rest then Doppler effect is not observed.
- When the source and observer both are moving with same velocity in same direction.
- When the source and observer are moving mutually in perpendicular directions.
- When the medium only is moving.
- When the distance between the source and observer is constant.

Example

When both source and observer approach each other with a speed equal to the half the speed of sound, then determine the percentage change in frequency of sound as detected by the listener.

Solution

Source $\xrightarrow{\frac{v}{2}}$ $\xleftarrow{\frac{v}{2}}$ Observer $\qquad n' = \left(\frac{v+\frac{v}{2}}{v-\frac{v}{2}}\right) n = \left(\frac{\frac{3}{2}v}{\frac{1}{2}v}\right) n = 3n$

$$\% \text{ change } = \frac{n'-n}{n} \times 100 = \frac{3n-n}{n} \times 100 = \frac{2n}{n} \times 100 = 200\ \%$$

Example

Two trains travelling in opposite directions at 126 km/hr each, cross each other while one of them is whistling. If the frequency of the node is 2.22 kHz find the apparent frequency as heard by an observer in the other train :

(a) Before the trains cross each other

(b) After the trains have crossed each other. (v_{sound} = 335 m/sec)

Solution

Here $v_1 = 126 \ \frac{5}{18} = 35$ m/s

(i) In this situation $\circ \longrightarrow v_1 \qquad v_1 \longleftarrow \circ$

Observed frequency $\qquad n' = \left(\frac{v+v_1}{v-v_1}\right) \times n = \left(\frac{335+35}{335-35}\right) \times 2220 = 2738$ Hz

(ii) In this situation $v_1 \longleftarrow \circ \qquad \circ \longrightarrow v_1$

Observed frequency $\qquad n' = \left(\frac{v-v_1}{v+v_1}\right) \times n = \left(\frac{335-35}{335+35}\right) \times 2220 = 1800$ Hz

Example

A stationary source emits sound of frequency 1200 Hz. If wind blows at the speed of 0.1v, deduce

(a) The change in the frequency for a stationary observer on the wind side of the source.

(b) Report the calculations for the case when there is no wind but the observer moves at 0.1v speed towards the source. (Given : velocity of sound = v)

\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Solution

(a) Medium moves in the direction of sound propagation i.e. from source to observer so effective velocity of sound $v_{eff} = v + v_m$

since both source and observer are at rest $n' = \left(\frac{v + v_m + 0}{v + v_m + 0}\right) n = \left(\frac{v + 0.1v}{v + 0.1v}\right) n = n$

so there is no change in frequency

(b) When observer move towards source $n' = \left(\frac{v + v_0}{v}\right) n = \left(\frac{v + 0.1v}{v}\right) n$

$= 1.1\ n = 1.1 \quad 1200\ Hz = 1320\ Hz$

Example

A bat is flitting about in a cave, navigating via ultrasonic beeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall ?

Solution

The apparent frequency heard by the bat of reflected sound

$$n' = \left(\frac{v + v_0}{v - v_s}\right) n = \left(\frac{v + 0.03v}{v - 0.03v}\right) \times 40 = \frac{1.03v}{0.97v} \times 40 = 42.47\ kHz$$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

SOME WORKED OUT EXAMPLES

Example#1

A particle of mass 50 g participates in two simple harmonic oscillations, simultaneously as given by x_1 = 10(cm) cos[80π(s^{-1}) t] and x_2 = 5(cm) sin[(80π(s^{-1}) t + π/6]. The amplitude of particle's oscillations is given by 'A'. Find the value of A^2 (in cm^2).

(A) 175 (B) 165 (C) 275 (D) 375

Solution **Ans. (A)**

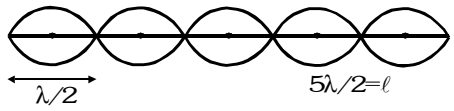
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} = \sqrt{10^2 + 5^2 + 2\times5\times10\times\frac{1}{2}} = \sqrt{175} \Rightarrow A^2 = 175$$

Example#2

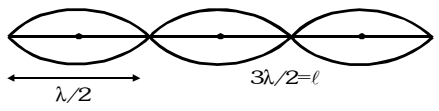
A sonometer wire resonates with a given tuning fork forming a standing wave with five antinodes between the two bridges when a mass of 9 kg is suspended from the wire . When this mass is replaced by a mass 'M' kg, the wire resonates with the same tuning fork forming three antinodes for the same positions of the bridges. Find the value of M.

(A) 25 (B) 20 (C) 15 (D) 10

Solution **Ans. (A)**



$$f = \frac{\sqrt{\frac{9g}{\mu}}}{\frac{2\ell}{5}}$$



$$f = \frac{\sqrt{\frac{Mg}{\mu}}}{\frac{2\ell}{3}} \Rightarrow f = \frac{\sqrt{\frac{9g}{\mu}}}{\frac{2\ell}{5}} = \frac{\sqrt{\frac{Mg}{\mu}}}{\frac{2\ell}{3}} \Rightarrow \sqrt{M} = 5$$

Example#3

A steel wire of length 1 m and mass 0.1 kg and having a uniform cross-sectional area of 10^{-6} m^2 is rigidly fixed at both ends. The temperature of the wire is lowered by 20 C. If the wire is vibrating in fundamental mode, find the frequency (in Hz).(Y_{steel} = 2 10^{11} N/m^2, α_{steel} = 1.21 10^{-5}/ C)

(A) 11 (B) 20 (C) 15 (D) 10

Solution **Ans. (A)**

$$\Delta\ell = \alpha\ell\Delta\theta \Rightarrow Y = \frac{T/A}{\Delta\ell/\ell} \Rightarrow T = YA\frac{\Delta\ell}{\ell} \Rightarrow T = \alpha YA\Delta\theta = 48.4N \; ; \; v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{48.4}{\left(\frac{0.1}{1}\right)}} = 22m/s$$

$\because$ for fundamental note $\ell = \frac{\lambda}{2} \Rightarrow \lambda = 2m \Rightarrow f = \frac{v}{\lambda} = \frac{22}{2} = 11Hz$

Example#4

A progressive wave on a string having linear mass density ρ is represented by $y = A\sin\left(\frac{2\pi}{\lambda}x - \omega t\right)$ where y is in mm. Find the total energy (in μJ) passing through origin from t = 0 to $t = \frac{\pi}{2\omega}$.

[Take : ρ = 3 10^{-2} kg/m; A = 1mm; ω = 100 rad/sec; λ = 16 cm]

(A) 6 (B) 7 (C) 8 (D) 9

\\nodes6\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Solution **Ans. (A)**

Total energy $\frac{1}{2}\rho A^2\omega^2 \times \frac{\lambda}{4}$

Example#5

Two tuning forks A and B lying on opposite sides of observer 'O' and of natural frequency 85 Hz move with velocity 10 m/s relative to stationary observer O. Fork A moves away from the observer while the fork B moves towards him. A wind with a speed 10 m/s is blowing in the direction of motion of fork A. Find the beat frequency measured by the observer in Hz. [Take speed of sound in air as 340 m/s]

(A) 5 (B) 6 (C) 7 (D) 8

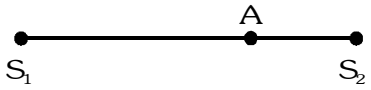
Solution **Ans. (A)**

$$f_{\text{observer for source 'A'}} = f_0\left[\frac{v_{sound} - v_{medium}}{v_{sound} - v_{medium} + v_{source}}\right] = \frac{33}{34}f_0 \; ; \; f_{\text{observer for source 'B'}} = f_0\left[\frac{v_{sound} + v_{medium}}{v_{sound} + v_{medium} - v_{source}}\right] = \frac{35}{34}f_0$$

$$\therefore \text{ Beat frequency} = f_1 - f_2 = \left(\frac{35-33}{34}\right)f_0 = 5$$

Example#6

If y_1 = 5 (mm) sinπt is equation of oscillation of source S_1 and y_2 = 5 (mm) sin(πt + π/6) be that of S_2 and it takes 1 sec and ½ sec for the transverse waves to reach point A from sources S_1 and S_2 respectively then the resulting amplitude at point A, is



(A) $5\sqrt{2+\sqrt{3}}$ mm (B) $5\sqrt{3}$ mm (C) 5 mm (D) $5\sqrt{2}$ mm

Solution **Ans. (C)**

Wave originating at t =0 from S_1 reaches point A at t = 1.

Wave originating at t = $\frac{1}{2}$ from S_2 reaches point A at t = 1.

So phase difference in these waves = $\frac{\pi}{2}+\frac{\pi}{6}$; A = $\sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} = 5$

Example#7

A transverse wave, travelling along the positive x-axis, given by y=Asin(kx−ωt) is superposed with another wave travelling along the negative x-axis given by y=−Asin(kx+ωt). The point x=0 is

(A) a node (B) an anitnode

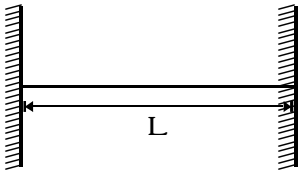
(C) neither a node nor an antinode (D) a node or antinode depending on t.

Solution **Ans. (B)**

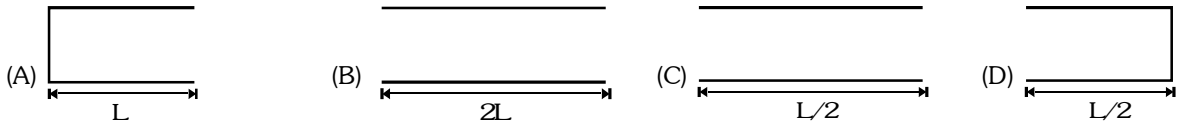
At x =0, y_1 = Asin (−ωt) and y_2 = −Asinωt; $y_1 + y_2 = -2A\sin\omega t$ (antinode)

Example#8

Figure shows a streched string of length L and pipes of length L, 2L, L/2 and L/2 in options (A), (B), (C) and (D) respectively. The string's tension is adjusted until the speed of waves on the string equals the speed of sound waves in air. The fundamental mode of oscillation is then set up on the string. In which pipe will the sound produced by the string cause resonance?



\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

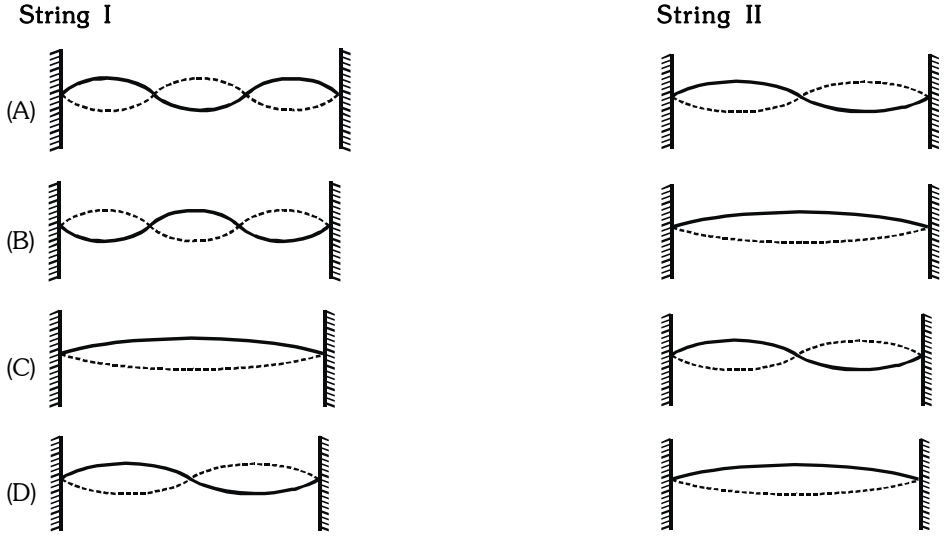


Solution

Ans. (B)

Example#9

String I and II have identical lengths and linear mass densities, but string I is under greater tension than string II. The accompanying figure shows four different situations, A to D, in which standing wave patterns exist on the two strings. In which situation it is possible that strings I and II are oscillating at the same resonant frequency?



Solution **Ans. (C)**

Since tension in I > tension in II $\Rightarrow V_I > V_{II}$ Thus, for same frequency, $\lambda_I > \lambda_{II}$

Example#10

A standing wave is created on a string of length 120 m and it is vibrating in 6th harmonic. Maximum possible amplititude of any particle is 10 cm and maximum possible velocity will be 10 cm/s. Choose the correct statement.

(A) Angular wave number of two waves will be $\frac{\pi}{20}$.

(B) Time period of any particle's SHM will be 4π sec.

(C) Any particle will have same kinetic energy as potential energy.

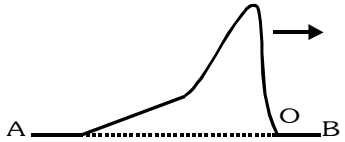
(D) Amplitude of interfering waves are 10 cm each.

Solution **Ans. (A)**

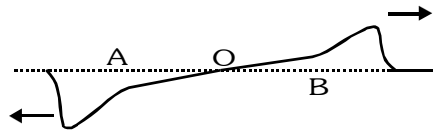
$$6\left(\frac{\lambda}{2}\right) = 120 \Rightarrow \lambda = 40 \Rightarrow k = \frac{\pi}{20} \Rightarrow A\omega = v_{max} \Rightarrow \omega = 1 \Rightarrow T = 2\pi$$

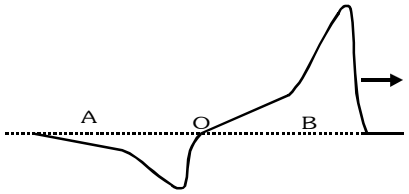
Example#11

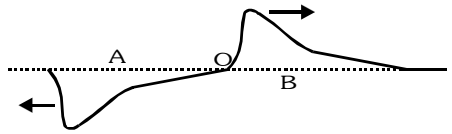
Two strings, A and B, of lengths 4L and L respectively and same mass M each, are tied together to form a knot 'O' and stretched under the same tension. A transverse wave pulse is sent along the composite string from the side A, as shown to the right. Which of the following diagrams correctly shows the reflected and transmitted wave pulses near the knot 'O'?

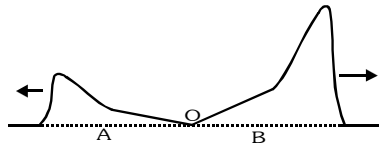


\\node6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion \Eng\Theory.p65

(A) 

(B) 

(C) 

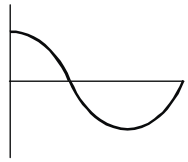
(D) 

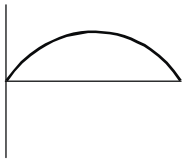
Solution **Ans. (A)**

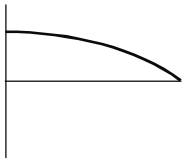
The wave suffers a phase difference of π when reflected by denser medium.

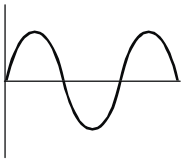
Example#12

Which of the figures, shows the pressure difference from regular atmospheric pressure for an organ pipe of length L closed at one end, corresponds to the 1st overtone for the pipe?

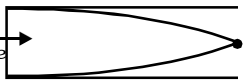
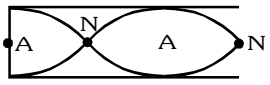
(A) 

(B) 

(C) 

(D) 

Solution **Ans. (A)**

For pressure standing wave antinode  Note 

fundamental frequency first overtone

Example#13

A man generates a symmetrical pulse in a string by moving his hand up and down. At t = 0 the point in his hand moves downward. The pulse travels with speed of 3 m/s on the string & his hands passes 6 times in each second from the mean position. Then the point on the string at a distance 3m will reach its upper extreme first time at time t =

(A) 1.25 sec. (B) 1 sec (C) $\frac{13}{12}$ sec (D) none

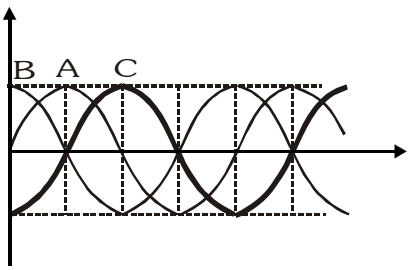
Solution **Ans. (A)**

Frequency of wave $= \frac{6}{2} = 3 \Rightarrow T = \frac{1}{3}s\ ;\ \lambda = vT = (3)\left(\frac{1}{3}\right) = 1m$

Total time taken = $\frac{3}{3} + \frac{3T}{4} = 1.25$ sec

Example#14

Three progressive waves A, B and C are shown in figure.



With respect to wave A

(A) The wave C lags behind in phase by $\pi/2$ and B leads by $\pi/2$.

(B) The wave C leads in phase by π and B lags behind by π.

(C) The wave C leads in phase by $\pi/2$ and B lags behind by $\pi/2$.

(D) The wave C lags behind in phase by π and B leads by π.

Ans. (A)

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Example#15

Following are equations of four waves :

(i) $y_1 = a \sin \omega\left(t - \frac{x}{v}\right)$ (ii) $y_2 = a \cos \omega\left(t + \frac{x}{v}\right)$

(iii) $z_1 = a \sin \omega\left(t - \frac{x}{v}\right)$ (iv) $z_2 = a \cos \omega\left(t + \frac{x}{v}\right)$

Which of the following statements is/are **CORRECT**?

(A) On superposition of waves (i) and (iii), a travelling wave having amplitude $a\sqrt{2}$ will be formed.

(B) Superposition of waves (ii) and (iii) is not possible.

(C) On superposition of waves (i) and (ii), a transverse stationary wave having maximum amplitude $a\sqrt{2}$ will be formed.

(D) On superposition of waves (iii) and (iv), a transverse stationary wave will be formed.

Solution **Ans. (AD)**

Superposition of waves (i) & (iii) will give travelling wave having amplitude of $a\sqrt{2}$

{waves are along x-axis but particle displacements are along y & z-axis respectively}

$$z_1 + z_2 = a\left[\sin \omega\left(t - \frac{x}{v}\right) + \sin\left\{\omega\left(t + \frac{x}{v}\right) + \frac{\pi}{2}\right\}\right]$$

Example#16

Two mechanical waves, $y_1 = 2 \sin 2\pi (50\, t - 2x)$ & $y_2 = 4 \sin 2\pi (ax + 100\, t)$ propagate in a medium with same speed.

(A) The ratio of their intensities is 1: 16 (B) The ratio of their intensities is 1: 4

(C) The value of 'a' is 4 units (D) The value of 'a' is 2 units

Solution **Ans. (AC)**

$I = \frac{1}{2}\rho v \omega^2 A^2$ and velocity $= \frac{\omega}{k}$

Example#17

Three simple harmonic waves, identical in frequency n and amplitude A moving in the same direction are superimposed in air in such a way, that the first, second and the third wave have the phase angles $\phi, \phi + \frac{\pi}{2}$ and $(\phi+\pi)$ respectively at a given point P in the superpositon.

Then as the waves progress, the superposition will result in

(A) a periodic, non-simple harmonic wave of amplitude 3A

(B) a stationary simple harmonic wave of amplitude 3A

(C) a simple harmonic progressive wave of amplitude A

(D) the velocity of the superposed resultant wave will be the same as the velocity of each wave

Solution **Ans. (CD)**

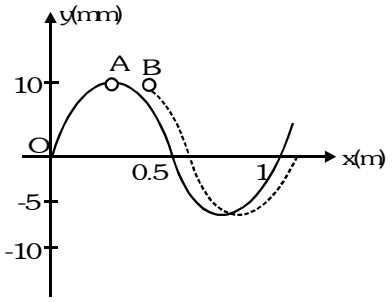
Since the first wave and the third wave moving in the same direction have the phase angles ϕ and $(\phi+\pi)$, they superpose with opposite phase at every point of the vibrating medium and thus cancel out each other, in displacement, velocity and acceleration. They, in effect, destroy each other out. Hence we are left with only the second wave which progresses as a simple harmonic wave of amplitude A. The velocity of this wave is the same as if it were moving alone.

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

ALLEN Materials Provided By - Material Point Available on Learnaf.com

Example#18

Two idetncial waves A and B are produced from the origin at different instants t_A and t_B along the positive x-axis, as shown in the figure. If the speed of wave is 5m/s then



(A) the wavelength of the waves is 1m
(B) the amplitude of the waves is 10 mm
(C) the wave A leads B by 0.0167 s
(D) the wave B leads A by 1.67 s

Solution **Ans. (AB)**

Wavelength of the waves = 1m; Amplitude of the waves = 10 mm

Example#19

A progressive wave having amplitude 5 m and wavelength 3 m. If the maximum average velocity of particle in half time period is 5 m/s and wave is moving in the positive x-direction then find which may be the **correct** equation(s) of the wave? [where x in meter]

(A) $5\sin\left(\frac{2\pi}{5}t-\frac{2\pi}{3}x\right)$
(B) $4\sin\left(\frac{\pi t}{2}-\frac{2\pi}{3}x\right)+3\cos\left(\frac{\pi t}{2}-\frac{2\pi}{3}x\right)$
(C) $5\sin\left(\frac{\pi t}{2}-\frac{2\pi}{3}x\right)$
(D) $3\cos\left(\frac{2\pi}{5}t-\frac{2\pi}{3}x\right)-4\sin\left(\frac{2\pi}{5}t-\frac{2\pi}{3}x\right)$

Solution **Ans. (BC)**

$\because \lambda = 3m \quad \therefore k = \frac{2\pi}{\lambda} = \frac{2\pi}{3}$

Maximum displacement in half time period = 2a = 10 m

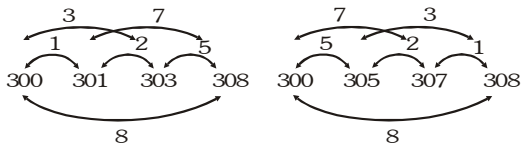
So maximum average velocity = $\frac{10}{T/2} = 5 \Rightarrow T = 4\ s \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

Example#20

You are given four tuning forks, the lowest frequency of the forks is 300 Hz. By striking two tuning forks at a time any of 1, 2, 3, 5, 7 & 8 Hz beat frequencies are heard. The possible frequencies of the other three forks are-

(A) 301,302 & 307 (B) 300,304 & 307 (C) 301, 303 & 308 (D) 305, 307 & 308

Solution **Ans. (CD)**



\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Example#21

A standing wave of time period T is set up in a string clamped between two rigid supports. At t = 0 antinode is at its maximum displacement 2A.

(A) The energy density of a node is equal to energy density of an antinode for the first time at t = T/4.

(B) The energy density of node and antinode becomes equal after T/2 second.

(C) The displacement of the particle at antinode at $t = \frac{T}{8}$ is $\sqrt{2}A$

(D) The displacement of the particle at node is zero

Solution **Ans. (CD)**

Equation of SHM of particle who is at antinode is $y=2A\sin\left(\frac{2\pi}{T}\right)t$ at time $t = \frac{T}{8}$

$y= 2A\sin\frac{\pi}{4} = \sqrt{2}A$; Displacement of particle at note is always zero.

Example#22

Two notes A and B, sounded together, produce 2 beats per sec. Notes B and C sounded together produce 3 beats per sec. The notes A and C separately produce the same number of beats with a standard tuning fork of 456 Hz. The possible frequency of the note B is

(A) 453.5 Hz (B) 455.5 Hz (C) 456.5 Hz (D) 458.5 Hz

Solution **Ans. (ABCD)**

Let frequency of note B be n then according to question

n_A = n–2 or n +2

n_C = n–3 or n +3

As A & C produce same number of beats with T.F. of frequency 456 Hz so

(n–2) – 456 = 456 – (n–3) $\Rightarrow$ n = 458.5 Hz

(n+3) – 456 = 456 – (n–2) $\Rightarrow$ n = 455.5 Hz

(n+2) – 456 = 456 – (n–3) $\Rightarrow$ n = 456.5 Hz

(n+3) – 456 = 456 – (n+2) $\Rightarrow$ n = 453.5 Hz

Example#23 to 25

A metallic rod of length 1m has one end free and other end rigidly clamped. Longitudinal stationary waves are set up in the rod in such a way that there are total six antinodes present along the rod. The amplitude of an antinode is 4 10^{-6} m. Young's modulus and density of the rod are 6.4 10^{10} N/m^2 and 4 10^3 Kg/m^3 respectively. Consider the free end to be at origin and at t=0 particles at free end are at positive extreme.

23. The equation describing displacements of particles about their mean positions is

(A) $s = 4\times10^{-6}\cos\left(\frac{11\pi}{2}x\right)\cos(22\pi\times10^3 t)$ (B) $s = 4\times10^{-6}\cos\left(\frac{11\pi}{2}x\right)\sin(22\pi\times10^3 t)$

(C) $s = 4\times10^{-6}\cos(5\pi x)\cos(20\pi\times10^3 t)$ (D) $s = 4\times10^{-6}\cos(5\pi x)\sin(20\pi\times10^3 t)$

24. The equation describing stress developed in the rod is

(A) $140.8\pi\times10^4\cos\left(\frac{11}{2}\pi x+\pi\right)\cos(22\pi\times10^3 t)$ (B) $140.8\pi\times10^4\sin\left(\frac{11}{2}\pi x+\pi\right)\cos(22\pi\times10^3 t)$

(C) $128\pi\times10^4\cos(5\pi x+\pi)\cos(20\pi\times10^3 t)$ (D) $128\pi\times10^4\sin(5\pi x+\pi)\cos(20\pi\times10^3 t)$

25. The magnitude of strain at midpoint of the rod at t= 1 sec is

(A) $11\sqrt{3}\pi\times10^{-6}$ (B) $11\sqrt{2}\pi\times10^{-6}$ (C) $10\sqrt{3}\pi\times10^{-6}$ (D) $10\sqrt{2}\pi\times10^{-6}$

\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

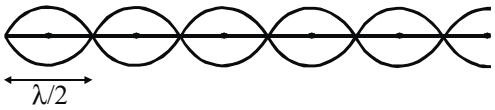
Solution

23. Ans. (A)

24. Ans. (B)

25. Ans. (B)

Solution (23 to 25)

Speed of wave $v=\sqrt{\frac{y}{\rho}}=4\times10^3$ 

$\lambda=\frac{5\lambda}{2}+\frac{\lambda}{4}\Rightarrow\lambda=\frac{4\ell}{11}$

Frequency $\nu=\frac{v}{\lambda}=\frac{4\times10^3}{\frac{4}{11}\times1}=11\times10^3\text{Hz}$; Wave Number $K=\frac{2\pi}{\lambda}=\frac{11\pi}{2}$

(i) Equation of standing wave in the rod $S = A\cos kx \sin(\omega t+\phi)$ where $A = 4 \quad 10^{-6}$ m

$\because$ at x =0, t =0 $\Rightarrow S=A \Rightarrow A=A \Rightarrow \cos k(0)\sin\phi \Rightarrow \sin\phi=1 \Rightarrow \phi=\frac{\pi}{2}$

$$S=4\times10^{-6}\cos\left(\frac{11\pi}{2}x\right)\cos(22\pi\times10^3t)$$

(ii) Strain $=\frac{ds}{dx}=-22\pi\times10^{-6}\sin\left(\frac{11\pi}{2}x\right)\cos(22\pi\times10^3t)$ $\because$ stress = Y strain

$$\Rightarrow \text{stress}=140.8\times10^4\cos(22\pi\times10^3t)\sin\left(\frac{11\pi}{2}x+\pi\right)$$

(iii) Strain at t = 1s and $x=\frac{\ell}{2}=\frac{1}{2}\text{m}$; $\left|\frac{ds}{dx}\right|_{x=\frac{\ell}{2}}^{t=1}=22\pi\times10^{-6}\times\sin\left(\frac{11\pi}{4}\right)=11\sqrt{2}\pi\times10^{-6}$

Example#26 to 28

A detector at x =0 receives waves from three sources each of amplitude A and frequencies f +2, f and f–2.

26. The equations of waves are ; $y_1=A\sin[2\pi(f+2)t]$, $y_2=A\sin2\pi ft$ and $y_3=A\sin[2\pi(f-2)t]$. The time at which intensity is minimum, is

(A) t=0, 1/4, 1/2, 3/4,... sec
(B) t=1/6, 1/3, 2/3, 5/6, ...sec
(C) t=0, 1/2, 3/2, 5/2, ...sec
(D) t=1/2, 1/4, 1/6, 1/8, ...sec

27. The time at which intensity is maximum, is

(A) t=0, 1/4, 1/2, 3/4, ...sec
(B) t=1/6, 1/3, 2/3, 5/6...sec
(C) t=0, 1/2, 3/2, 5/2 ...sec
(D) t=1/2, 1/4, 1/6, 1/8...sec

28. If $I_0 \propto A^2$, then the value of maximum intensity, is

(A) $2I_0$
(B) $3I_0$
(C) $4I_0$
(D) $9I_0$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Solution

26. Ans. (B)

$y = y_1 + y_2 + y_3 = A\sin 2\pi ft + A\sin[2\pi(f-2)t] + A\sin[2\pi(f+2)t] = A\sin 2\pi ft + 2A\sin 2\pi ft\cos 4\pi t$

$= A[1+2\cos 4\pi t]\sin 3\pi ft = A_0 \sin 2\pi ft$

[where A_0 = Amplitude of the resultant oscillation = A [1+2cos4πt]

Intensity $\propto A_0^2$ $\therefore I \propto (1+2\cos 4\pi t)^2$

For maxima or minima of the intensity.

$\frac{dI}{dt} = 0 \Rightarrow 2(1+2\cos 4\pi t)(2)(-\sin 4\pi t)4\pi = 0 \Rightarrow 1+2\cos 4\pi t = 0$ or $\sin 4\pi t = 0$

$\therefore \cos 4\pi t = -\frac{1}{2} \Rightarrow 4\pi t = 2\pi n \pm \frac{2\pi}{3}$ $\therefore t = \frac{n}{2} + \frac{1}{6} \Rightarrow t = \frac{1}{6}, \frac{1}{3}, \frac{2}{5}, \frac{5}{6}$ ... (point of minimum intensity)

27. Ans. (A)

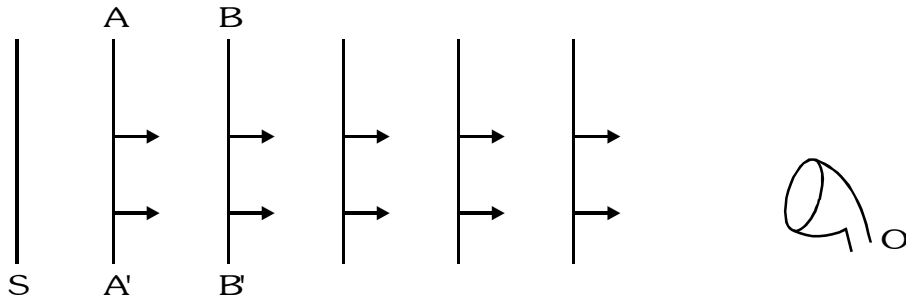
$\sin 4\pi t = 0 \Rightarrow t = \frac{n}{4}$ $\therefore t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ ... (point of maximum intensity)

28. Ans. (D)

At t=0, $I_{max} \propto (1+2)^2A^2 = 9A^2$ $\therefore$ $I_{max} = 9I_0$

Example#29

Consider a large plane diaphragm 'S' emitting sound and a detector 'O'. The diagram shows plane wavefronts for the sound wave travelling in air towards right when source, observer and medium are at rest. AA' and BB' are fixed imaginary planes. Column-I describes about the motion of source, observer or medium and column-II describes various effects. Match them correctly.



Column I		Column II	
(A)	Source starts moving towards right	(P)	Distance between any two wavefronts will increase.
(B)	Air starts moving towards right	(Q)	Distance between any two wavefronts will decrease.
(C)	Observer and source both move towards left with same speed.	(R)	The time needed by sound to move from plane AA' to BB' will increase.
(D)	Source and medium (air) both move towards right with same speed.	(S)	The time needed by sound to move from plane AA' to BB' will decrease.
		(T)	Frequency received by observer increases.

\\nodes6\E_NODE6\(E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Solution **Ans. (A) →(Q,T); (B) →(P,S); (C) →(P); (D) →(S,T)**

Velocity of sound in a medium is always given in the reference frame of medium.

Example#30

Column I represents the standing waves in air columns and string. Column II represents frequency of the note. Match the column-I with column-II. [v = velocity of the sound in the medium]

	Column -I		**Column-II**
(A)	Second harmonic for the tube open at both ends	(P)	$\frac{v}{4\ell}$
(B)	Fundamental frequency for the tube closed at one end	(Q)	$\frac{v}{2\ell}$
(C)	First overtone for the tube closed at one end	(R)	$\frac{3v}{4\ell}$
(D)	Fundamental frequency for the string fixed at both ends	(S)	$\frac{v}{\ell}$
		(T)	$\frac{5v}{4\ell}$

Solution **Ans. (A) S (B) P (C) R (D) Q**

For (A) : For open organ pipe 2nd harmonic = $2\left(\frac{v}{2\ell}\right)$

For (B) : For closed organ pipe fundamental frequency = $\frac{v}{4\ell}$

For (C) : For closed organ pipe, first overtone frequency = $\frac{3v}{4\ell}$

For (D) : For string fixed at both ends, fundamental frequency = $\frac{v}{2\ell}$

Example#31

Two vibrating tuning forks produce progressive waves given by $y_1 = 4\sin(500\pi t)$ and $y_2 = 2\sin(506\pi t)$. These tuning forks are held near the ear of a person. The person will hear α beats/s with intensity ratio between maxima and minima equal to β. Find the value of $\beta-\alpha$.

Solution **Ans. 6**

$y_1 = 4\sin(500\ \pi t)$ $\quad y_2 = 2\ \sin(506\ \pi t)$

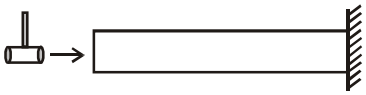
Number of beats $= \frac{n_1 - n_2}{2} = \frac{506-500}{2} = 3$ beat/sec.

As $I_1 \propto (16)$ and $I_2 \propto 4 \Rightarrow \frac{I_{max}}{I_{min}} = \frac{\left(\sqrt{I_1}+\sqrt{I_2}\right)^2}{\left(\sqrt{I_1}-\sqrt{I_2}\right)^2} \Rightarrow \left(\frac{4+2}{4-2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$

\\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Example#32

A 1000 m long rod of density 10.0 10^4 kg/m^3 and having young's modulus Y = 10^{11} Pa, is clamped at one end. It is hammered at the other free end as shown in the figure. The longitudinal pulse goes to right end, gets reflected and again returns to the left end. How much time (in sec) the pulse take to go back to initial point?



Solution **Ans. 2**

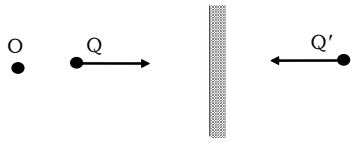
Velocity of longitudinal $u = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{10^{11}}{10 \times 10^4}} = 10^3 \text{ ms}^{-1}$

Required time $\frac{2\ell}{v} = \frac{2 \times 1000}{10^3} = 2 \text{ s}$

Example#33

A tuning fork P of unknown frequency gives 7 beats in 2 seconds with another tuning fork Q. When Q is moved towards a wall with a speed of 5 m/s, it gives 5 beats per second for an observer located left to it. On filing, P gives 6 beats per second with Q. The frequency (in Hz) of P is given by (80 α) + β. ($\alpha, \beta \in I$, $0 \le \alpha, \beta \le 9$) then find the value of $\alpha + \beta$. Assume speed of sound = 332 m/s.

Solution **Ans. 9**



Let f_1 and f_2 be the frequencies of tuning forks P and Q,Then $| f_1 - f_2 | = 7/2$

Apparent frequency for O corresponding to signal directly coming from Q = $f_2 \left(\frac{v}{v + v_q} \right)$

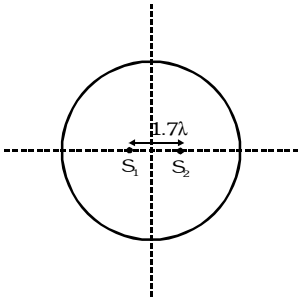
Apparent frequency of the echo = $f_2 \left(\frac{v}{v - v_q} \right)$ $\therefore \Delta f_2 = f_2 \left[\frac{2 v_q v}{v^2 - v_q^2} \right]$

Since, $\Delta f_2 = 5$ (given) $\therefore f_2 = 163.5$ Hz. Now, $f_1 = 163.5 \pm 3.5 = 167$ or 160 Hz, when P is filed, its frequency will increase, since it is given that filed P gives greater number of beats with Q. It implies that f_1 must be 167 Hz.

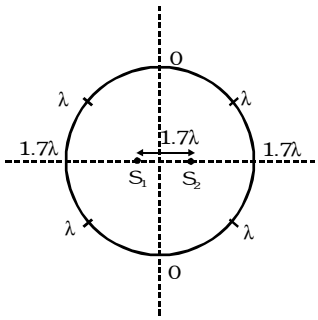
\\nodes6\E_NODE6 (E)\Data\2014\Kota\JEE-Advanced\SMP\Phy\Unit No-6\Wave Motion\Eng\Theory.p65

Example#34

Find the number of maxima attend on circular perimeter as shown in the figure. Assume radius of circle >>>λ.



Solution **Ans. 6**



1 in each quadrant, 1 top point, 1 bottom point

\nodes\E_NODE6\E\Data\2014\Kota\JEE-Advanced\SMP\Phy\UnitNo-6\Wave Motion\Eng\Theory\p65

Learnaf

Educational Revolution

Thank you for your love and support, we hope you are always being happy and get success in your life, we are happy to see you again.

Regards from Learnaf team

Click on very top right corner for download and other more options.