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25. AREA UNDER THE CURVE AND LINEAR PROGRAMMING

AREA UNDER THE CURVE

1. INTRODUCTION

In the previous chapters we have studied the process of integration and its physical interpretation. The most important application of integration is finding the area under a curve. In this topic we will discuss different curves and the area bounded by some simple plane curves taken together. In order to find the area, we need to know the basics of plotting a curve and then use integration with appropriate limits to get the answer. The process of finding area of some plane region is called **Quadrature**.

2. CURVE TRACING

Let us now discuss the basics of curve tracing. Curve tracing is a technique which provides a rough idea about the nature and shape of a plane curve. Different techniques are used in order to understand the nature of the curve, but there is no fixed rule which provides all the information to draw the graph of a given function (say f(x)). Sometimes it is also very difficult to draw the exact curve of the given function. However, the following steps can be helpful in trying to understand the nature and the shape of the curve.

Step 1: Check whether the origin lies on the given curve. Also check for other points lying on the curve by putting some values.

Step 2: Check whether the curve is increasing or decreasing by finding the derivative of the function. Also check for the boundary points of the curve.

Step 3: Check whether the curve $f(x, y) = 0$ is symmetric about

(a) X-axis: If the equation remains same on replacing y by –y i.e. $f(x, y) = f(x, -y)$, or, if all the powers of "y" are even, then the graph is symmetric about the X-axis.

(b) Y-axis: If the equation remains same on replacing x by –x i.e. $f(x, y) = f(-x, y)$, or, if all the powers of "x" are even, then the graph is symmetric about the Y-axis.

(c) Origin: If $f(-x, -y) = -f(x, y)$, then the graph is symmetric about the Origin.

For example, the curve given by $x^2 = y+2$ is symmetrical about y-axis, $y^2 = x+2$ is symmetrical about x-axis and the curve $y = x^3$ is symmetrical about the origin.

Step 4: Find out the points of intersection of the curve with the x-axis and y-axis by substituting y = 0 and x = 0 respectively.

For example, the curve $\frac{x^2}{9}+\frac{y^2}{4}=1$ intersects the axes at points (± 3, 0) and (0, ±2).

Step 5: Identify the domain of the given function and the region in which the graph can be drawn. For example, the curve $xy^2 = (8 - 4x)$ or $y = 2\sqrt{\frac{2-x}{x}}$.

Therefore the value of y is defined only when $\frac{2-x}{x} \geq 0$ i.e. $0 < x \leq 2$. Hence, the graph lies between the lines x = 0 and x = 2.

Step 6: Check the behaviour of the graph as $x \to +\infty$ and as $x \to -\infty$. Find all the horizontal, vertical and oblique asymptotes, if any.

Step 7: Determine the critical points, the intervals on which the function (f) is concave up or concave down and the inflection points.

The information obtained from the Steps 1 to 7 are used to trace the curve.

Illustration 1: Trace the curve $y^2 (2a - x) = x^3$ **(JEE MAIN)**

Sol: By using curve tracing method as mentioned above.

Given curve: $y^2 = x^3/(2a - x)$ ...(i)

(a) Origin: The point (0 , 0) satisfies the given equation, therefore, it passes through the origin.

(b) Symmetrical about x-axis: On replacing y by –y, the equation remains same, therefore, the given curve is symmetrical about x-axis.

(c) Tangent at the origin: Equation of the tangent is obtained by equating the lowest degree terms to zero.

$\Rightarrow 2ay^2 = 0 \quad \Rightarrow y^2 = 0 \Rightarrow y=0$

(d) Asymptote parallel to y-axis: Equation of asymptote is obtained by equating the coefficient of lowest degree of y to 0. The given equation can be written as $(2a - x)\, y^2 = x^3$

$\therefore$ Equation of asymptote is $2a - x = 0$ or $x = 2a$

(e) Region of absence of curve: The given equation is

$y^2 (2a - x) = x^3 \quad \Rightarrow \quad y^2 = \frac{x^3}{(2a-x)}$.

For x < 0 and x > 2a, RHS becomes negative, therefore the curve exists only for $0 \leq x < 2a$.

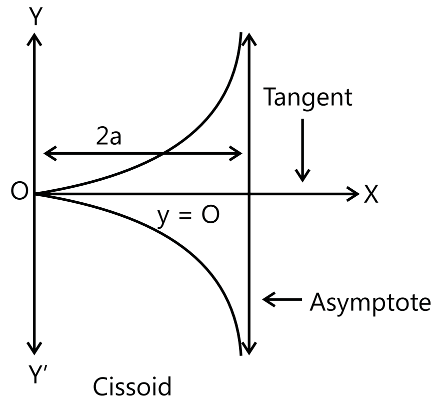


Figure 25.1

Hence the graph of $y^2 (2a - x) = x^3$ is as shown in Fig. 25.1. Such a curve is known as a Cissoid.

Illustration 2: Sketch the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **(JEE MAIN)**

Sol: Same as above illustration.

We have, $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...(i)

(a) Origin: The point (0,0) does not satisfy the equation, hence, the curve does not pass through O.

(b) Symmetry: The equation of the curve contains even powers of x and y so it is symmetric about both x and y axes.

(c) Intercepts: Putting y = 0, we get x = ± 2 i.e. the curve passes through the points (2 , 0) and (-2 , 0). Similarly, on substituting x = 0, we get y = ± 3 i.e. the curve passes through the points (0 , 3) and (0 , -3).

(d) Region where the curve does not exist: If $x^2 > 4$, y becomes imaginary. So the curve does not exist for x > 2 and x < –2. Similarly, if $y^2 > 9$, x becomes imaginary. So, the curve does not exist for y > 3 and y < –3.

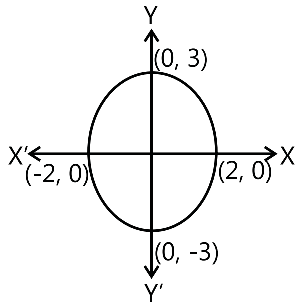


Figure 25.2

(e) Table:

x	– 2	0	1	2
y	0	±3	±2.6	0

Hence the graph of $\frac{x^2}{4}+\frac{y^2}{9}=1$ is as shown in Fig. 25.2.

PLANCESS CONCEPTS

Using the above rules try to trace the Witch of Agnesi

$xy^2 = a^2 (a – x)$.

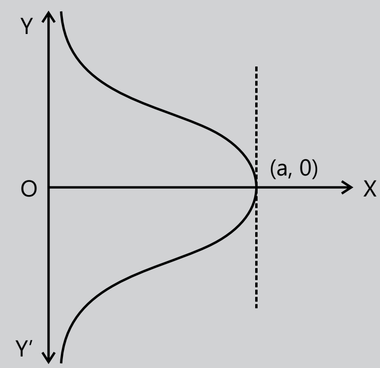


Figure 25.3

Vaibhav Krishnan (JEE 2009 AIR 22)

3. AREA BOUNDED BY A CURVE

3.1 The Area Bounded by a Curve with X-axis

The area bound the curve y=f(x) with the x-axis between the ordinates

x= a and x=b is given by $\text{Area} = \int_a^b y\ dx = \int_a^b f(x)\,dx$

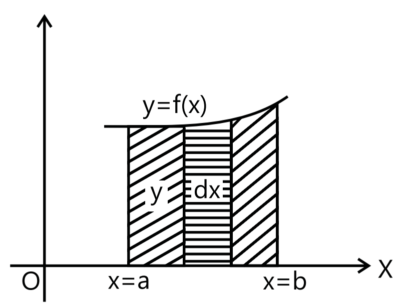


Figure 25.4: Area Bounded By a curve y=f(x) with x-axis

Illustration 3: Find the area bounded by the curve $y = x^3$, x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: By using above formula, we can find out the area under given curve.

Required Area = $\int_1^2 y\,dx = \int_1^2 x^3\,dx = \left(\frac{x^4}{4}\right)_1^2 = \frac{15}{4}$

Illustration 4: Find the area bounded by the curve y = mx x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: Same as above.

Required area $= \int_1^2 y\,dx = \int_1^2 mx\,dx = \left(\frac{mx^2}{2}\right)_1^2 = \frac{m}{2}(4-1) = \frac{3}{2}m$

Illustration 5: Find the area included between the parabola $y^2 = 4ax$ and its latus rectum (x = a).

(JEE ADVANCED)

Sol: Here the curve is $y^2 = 4ax$, latus rectum is x = a, and the curve is symmetrical about the x-axis.

(a) The latus rectum is the line perpendicular to the axis of the parabola and passing through the focus S (a, 0).

(b) The parabola is symmetrical about the x-axis.

∴ The required area AOBSA = 2 × area AOSA

$$= 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{ax}\,dx \quad \left[y^2 = 4ax \Rightarrow y = 2\sqrt{ax}\right]$$

$$= 4\sqrt{a}.\frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{8}{3}\sqrt{a}.a^{3/2} = \frac{8}{3}a^2.$$

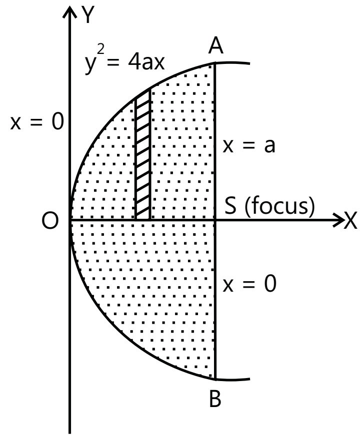


Figure 25.5

Illustration 6: Sketch the region {(x, y): $4x^2 + 9y^2 = 36$} and find its area using integration. **(JEE ADVANCED)**

Sol: The given curve is an ellipse, where a = 3 and b = 2. The X and Y axis divides this ellipse into four equal parts.

Region {(x, y): $4x^2 + 9y^2 = 36$} = Region bounded by $\left(\frac{x^2}{9} + \frac{y^2}{4} = 1\right)$

Limits for the shaded area are x = 0 and x = 3.

∴ The required area of the ellipse

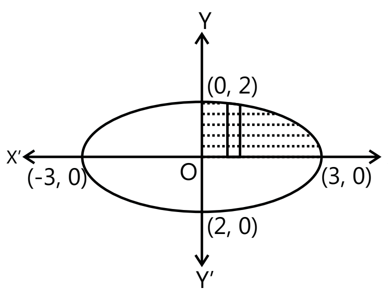


Figure 25.6

$$= 4\int_0^a y\,dx = 4\int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx \quad \left[\because \frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1 - \frac{x^2}{9} \Rightarrow y = 2\sqrt{1-\frac{x^2}{9}}\right]$$

$$= \frac{8}{3}\int_0^3 \sqrt{3^2 - x^2}\,dx = \int_0^3 \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 \quad \left[\text{using} \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{8}{3}\left[0 + \frac{9}{2}\sin^{-1}1 - 0 - 0\right] = \frac{8}{3}\times\frac{9}{2}\times\frac{\pi}{2} = 6\pi \text{ sq. units.}$$

3.2 The Area Bounded by a Curve with y-Axis

The area bound the curve y=f(x) with y-axis between the ordinates

y= a and y=b is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

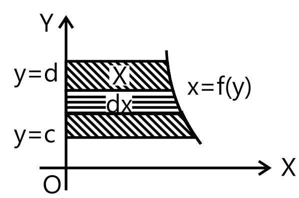


Figure 25.7: Area bounded by a curve with y-axis

Illustration 7: Find the area bounded by the curve $x^2 = \frac{1}{4}y$, y-axis and between the lines y = 1 and y = 4.

(JEE MAIN)

Sol: As we know, area bounded by curve with y – axis is given by $\int_c^d x\,dy = \int_c^d f(y)\,dy$.

Required Area = $\int_1^4 x\,dy = 2\int_1^4 \frac{1}{2}\sqrt{y}\,dy = \frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$ sq. units

Illustration 8: Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line y = 3. **(JEE MAIN)**

Sol: Same as above illustration. $\left(\because y^2 = 4x \quad \frac{y^2}{4} = x\right)$

Area of region is $A = \int_{y=0}^{y=3} x\,dy = \int_0^3 \frac{y^2}{4}dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{4}\left[\frac{3^3}{3} - \frac{0}{3}\right] = \frac{1}{4}[9] = \frac{9}{4}$ sq. units

Hence, the required area is $\frac{9}{4}$ sq. units.

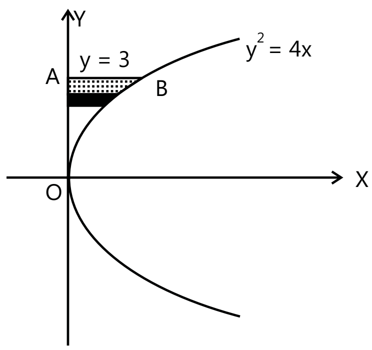


Figure 25.8

PLANCESS CONCEPTS

There is no harm in splitting an integral into multiple components while finding area. If you have any doubt that the integral is changing sign, split the integral at that point.

Vaibhav Gupta (JEE 2009 AIR 54)

3.3 Area of a Curve in Parametric Form

If the given curve is in parametric form say x = f(t), y = g(t), then the area bounded by the curve with x-axis is equal to $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)dt \quad \left[\because dx = d(f(t)) = f'(t)dt\right]$ Where t_1 and t_2 are the values of t corresponding to the values of a and b of x.

Illustration 9: Find the area bounded by the curve x = a cost, y = b sint in the first quadrant. **(JEE MAIN)**

Sol: Solve it using formula of area of a curve in parametric form.

The given equation is the parametric equation of ellipse, on simplifying we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Required area $= \int_0^a y\,dx = \int_{\pi/2}^0 (b\sin t(-a\sin t)dt) = ab\int_0^{\pi/2} \sin^2 t\,dt = \left(\frac{\pi ab}{4}\right)$.

3.4 Symmetrical Area

If the curve is symmetrical about a line or origin, then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

Illustration 10: Find the area bounded by the parabola $y^2 = 4x$ and its latus rectum. **(JEE MAIN)**

Sol: Here the given parabola is symmetrical about x – axis.

Hence required area $= 2\int_0^1 y\,dx$.

Since the curve is symmetrical about x-axis,

$\therefore$ The required Area $= 2\int_0^1 y\,dx = 2\int_0^1 \sqrt{4x}\,dx = 4\cdot\frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{8}{3}$

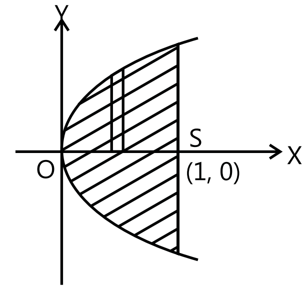


Figure 25.9

3.5 Positive and Negative Area

The area of a plane figure is always taken to be positive. If some part of the area lies above x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add the numerical values to get the desired area.

If the curve crosses the x-axis at c (see Fig. 25.10), then the area bounded by the curve y = f(x) and the ordinates x = a and x = b, (b > a) is given by

$$A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|; \qquad A = \int_a^c f(x)dx - \int_c^b f(x)\,dx$$

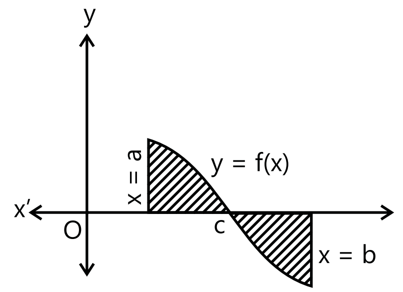


Figure 25.10

PLANCESS CONCEPTS

To reduce confusion of using correct sign for the components, take modulus and add all the absolute values of the components.

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 11: Find the area between the curve y = cos x and x-axis when $\pi/4 < x < \pi$ **(JEE MAIN)**

Sol: Here some part of the required area lies above x-axis and some part lies below x-axis,. Hence by using above mentioned method we can obtain required area.

$\therefore$ Required area $= \int_{\pi/4}^{\pi/2}\cos x\,dx + \left|\int_{\pi/2}^{\pi}\cos x\,dx\right|$

$= [\sin x]_{\pi/4}^{\pi/2} + |[\sin x]_{\pi/2}^{\pi}| = \left(1 - 1/\sqrt{2}\right) + |0-1| = \frac{2\sqrt{2}-1}{\sqrt{2}}$

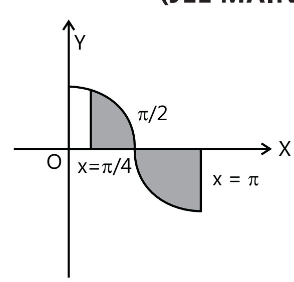


Figure 25.11

Illustration 12: Using integration, find the area of the triangle ABC, whose vertices are A (4, 1), B (6, 6) and C (8, 4) **(JEE ADVANCED)**

Sol: Here by using slope point form we can obtain respective equation of line by which given triangle is made. And after that by using integration method we can obtain required area.

Equation of line AB: $y - 1 = \frac{5}{2}(x-4) \Rightarrow y = \frac{5x}{2} - 9$

Equation of line AC: $y - 1 = \left(\frac{3}{4}\right)(x-4) \Rightarrow y = \frac{3x}{4} - 2$

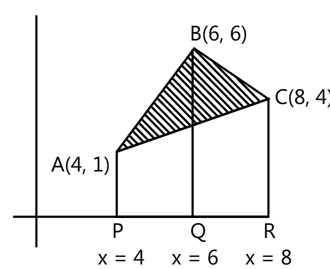


Figure 25.12

Equation of line BC: $(y-6) = \left(\frac{-2}{2}\right)(x-6) \Rightarrow y = -x + 12$

∴ The required area = Area of trapezium ABQP + Area of trapezium BCRQ – Area of trapezium ACRP

$$= \int_4^6 \left(\frac{5}{2}x - 9\right)dx + \int_6^8 (-x+12)dx - \int_4^8 \left(\frac{3}{4}x - 2\right)dx$$

$$= \left(\frac{5}{4}x^2 - 9x\right)_4^6 + \left(12x - \frac{x^2}{2}\right)_6^8 - \left(\frac{3}{8}x^2 - 2x\right)_4^8 = 7 + 10 - 10 = 7 \text{ sq. units.}$$

3.6 Area between Two Curves

(a) Area enclosed between two curves.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves (where $f_1(x) > f_2(x)$), which intersect at two points, A (x = a) and B(x = b), then the area enclosed by the two curves between A and B is

Common area = $\int_a^b (y_1 - y_2)dx = \int_a^b [f_1(x) - f_2(x)]dx$

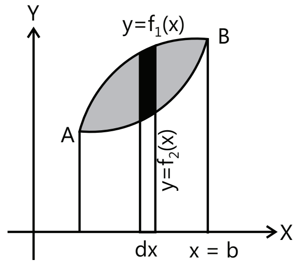


Figure 25.13

Illustration 13: Find the area between two curves $y^2 = 4ax$ and $x^2 = 4ay$. **(JEE MAIN)**

Sol: By using above mentioned formula of finding the area enclosed between two curves, we can obtain required area.

Given, $y^2 = 4ax$ ... (i)

$x^2 = 4ay$ ... (ii)

Solving (i) and (ii), we get x = 4a and y = 4a.

So required area = $\int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx = \left(2\sqrt{a}\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right)_0^{4a}$

$$= \frac{4\sqrt{a}}{3}|4a|^{3/2} - \frac{64a^3}{12a} = \frac{16}{3}a^2$$

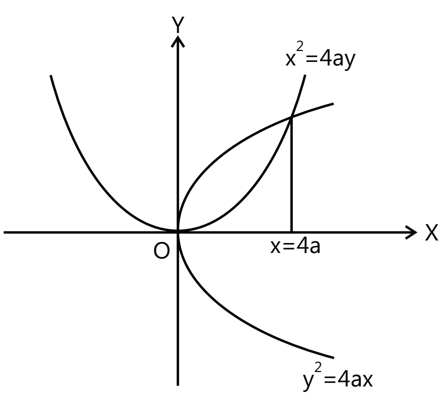


Figure 25.14

(b) Area enclosed by two curves intersecting at one point and the X-axis.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves which intersect at a point P (α, β) and meet x-axis at A (a, 0) and B (b, 0) respectively, then the area enclosed between the curves and x-axis is given by

$$\text{Area} = \int_a^{\alpha} f_1(x)dx + \int_{\alpha}^{b} f_2(x)dx$$

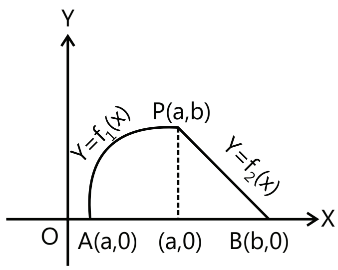


Figure 25.15

(c) Area bounded by two intersecting curves and lines parallel to y-axis.

The area bounded by two curves y = f(x) and y = g(x) (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$, is given by $A = \int_a^b |f(x) - g(x)|\,dx \Rightarrow A = \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

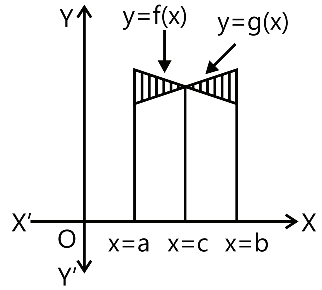


Figure 25.16

Illustration 14: Draw a rough sketch of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Using method of integration, find the area of this enclosed region **(JEE ADVANCED)**

Sol: By solving given equations simultaneously, we will be get intersection points of circles and then by using integration method we can obtain required area.

The figure shown alongside is the sketch of the circles

$x^2 + y^2 = 4$ ... (i)

and, $(x - 2)^2 + y^2 = 4$ ... (ii)

From (i) and (ii), we have $(x - 2)^2 - x^2 = 0$

$\Rightarrow (x - 2 - x)(x - 2 + x) = 0 \quad \Rightarrow x = 1$ ... (iii)

Solving (i) and (iii), we get $y = \pm\sqrt{3}$

Therefore, the circles (i) and (ii) intersect at $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Area of enclosed region = Area OACBO = 2 Area OACO

= 2 [Area OAD + Area ACD]

$$= 2\int_0^1 \sqrt{4 - (x-2)^2}\,dx + 2\int_1^2 \sqrt{4 - x^2}\,dx$$

$$= 2\int_1^2 \sqrt{4 - x^2}\,dx + 2\int_0^1 \sqrt{4 - (x-2)^2}\,dx$$

$$= 2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 + 2\left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 \quad \left[\because \sqrt{a^2 - x^2}\,dx \Rightarrow \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left(\pi - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right)\right) + 2\left(-\frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \pi\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq. units}$$

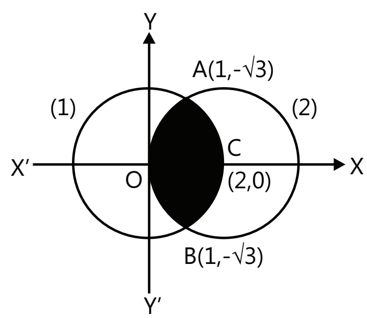


Figure 25.17

Illustration 15: Using integration, find the area of the region given below:

$\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ **(JEE ADVANCED)**

Sol: Same as above illustration, by solving given equation $y = x^2 + 1$ and $y = x + 1$ we will be get their points of intersection and after that using integration method and taking these points as limit we can obtain required area.

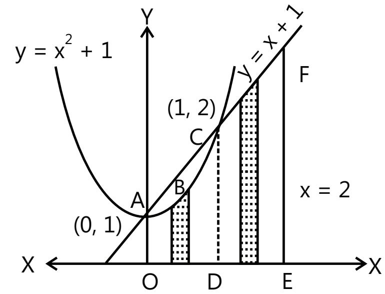


Figure 25.18

The region is shaded as shown in the Fig. 25.18.

Given, $y = x^2 + 1$ ... (i)

$y = x + 1$ ... (ii)

On solving (i) and (ii), we have $x^2 + 1 = x + 1$

$\Rightarrow x = 0, 1$ and $y = 1, 2$

$\therefore$ The shaded region can be divided into two parts OABCDO and CDEFC.

Limits for the area OABEO are $x = 0$ and $x = 1$.

Limits for the area EBDFE are $x = 1$ and $x = 2$.

Area of the shaded region = Area OABEO + Area EBDFE.

$$= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx = \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2 = \left(\frac{1}{3}+1\right)+\left(\frac{4}{2}+2-\frac{1}{2}-1\right) = \frac{23}{6} \text{ sq. units}$$

Illustration 16: Find the area of the following region: $[(x, y): y^2 \le 4x, 4x^2+4y^2 \le 9]$ **(JEE ADVANCED)**

Sol: Similar to above problem, Here the required area is equal to Area AOBA + Area ACBA.

Given $y^2 = 4x$ ... (i)

$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$ ... (ii)

Curves (i) and (ii) intersect at $A\left(\frac{1}{2}, \sqrt{2}\right)$ and $B\left(\frac{1}{2}, -\sqrt{2}\right)$

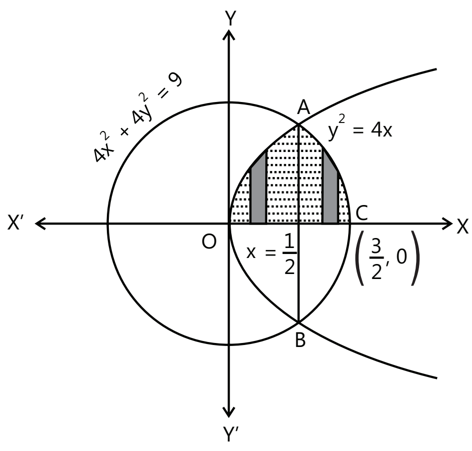


Figure 25.19

Limits for the area OAB are $x = 0,\ x = \frac{1}{2}$

Limits for the area ACB are $x = \frac{1}{2},\ x = \frac{3}{2}$.

The required area = Area AOBA + Area ACBA

$$= 2\left[\int_0^{1/2} y_1\, dx + \int_{1/2}^{3/2} y_2\, dx\right] = 2\left[\int_0^{1/2} \sqrt{4x}\, dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4}-x^2}\, dx\right]$$

$$= 4\left[\frac{2}{3}x^{3/2}\right]_0^{1/2} + 2\left[\frac{x}{2}\cdot\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{x}{3/2}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} + \left[0 - \frac{1}{\sqrt{2}} + \frac{9}{4}\left(\sin^{-1}1 - \sin^{-1}\frac{1}{3}\right)\right] = \frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \frac{1}{3\sqrt{2}} + \frac{9}{4}\cos^{-1}\frac{1}{3}.$$

Illustration 17: Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 8x$ and $x^2 = 8y$, by using method of integration. **(JEE MAIN)**

Sol: As the given two equation is the equation of parabola which intersect at O(0, 0) and A(8, 8), and the required area is equal to Area OBADO – Area OADO.

Given parabolas are $y^2 = 8x$ ... (i)

and, $x^2 = 8y$ ... (ii)

The curves (i) and (ii) intersect at O(0, 0) and A(8, 8).

$\therefore$ Required Area = Area OBADO – Area OADO

$$= \int_0^8 (y_1 - y_2)dx$$

$$= \int_0^8 \left(\sqrt{8x} - \frac{x^2}{8}\right)dx = \left[2\sqrt{2}.\frac{x^{3/2}}{3/2} - \frac{1}{8}\frac{x^3}{3}\right]_0^8 = \frac{64}{3} \text{ sq. units.}$$

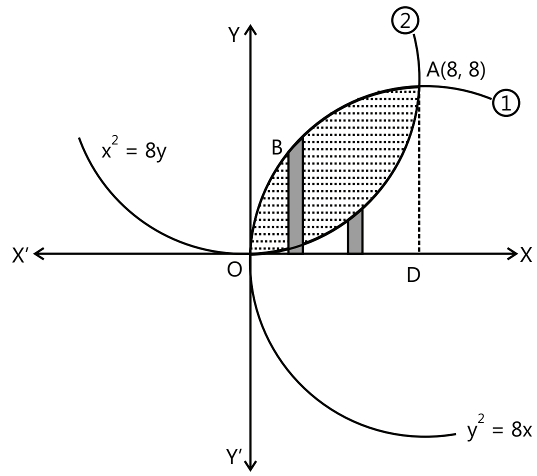


Figure 25.20

Illustration 18: Find the area between the curves y = 2x, x + y = 1 and x-axis. **(JEE MAIN)**

Sol: Here y = 2x and x + y = 1 is a two line intersect at $p\left(\frac{1}{3},\frac{2}{3}\right)$, therefore using integration method we can obtain required area.

Given $y = 2x$ ... (i)

and, $x + y = 1$ ... (ii)

Solving (i) and (ii), we get $x + 2x = 1 \Rightarrow x = 1/3$.

Line (i) intersects with the x – axis at the origin and the line (ii) intersects with the x – axis at x = 1.

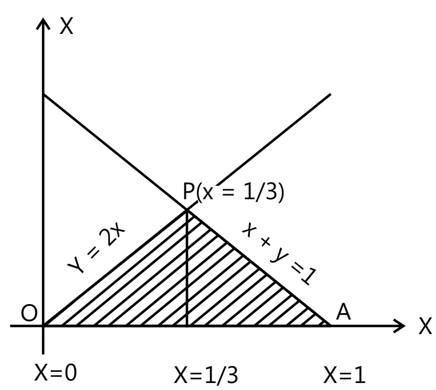


Figure 25.21

So required area = $\int_0^{1/3} 2x\,dx + \int_{1/3}^{1} (1-x)dx = \left[x^2\right]_0^{1/3} + \left(x - \frac{x^2}{2}\right)_{1/3}^{1}$

$$= \frac{1}{9} + \left(\frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{18}\right) = \frac{1}{3} \text{ sq. units}$$

Illustration 19: Using the method of integration, find the area of the region bounded by lines: 2x + y = 4, 3x – 2 y = 6 and x – 3y + 5 = 0 **(JEE ADVANCED)**

Sol: Same as above problem.

Given equation of the lines are $2x + y = 4$ ... (i)

$3x - 2y = 6$ ... (ii)

$x - 3y + 5 = 0$ ... (iii)

Solving (i) and (ii), we get (2, 0)

Solving (ii) and (iii), we get (4, 3)

Solving (i) and (iii), we get (1, 2)

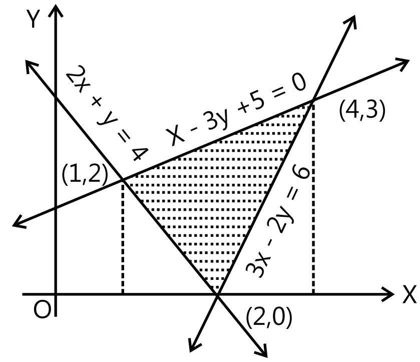


Figure 25.22

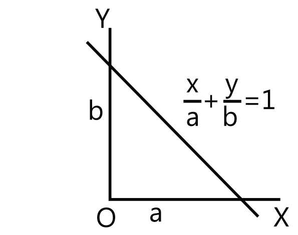
$$\therefore \text{ Required Area } = \int_1^4 \left(\frac{x+5}{3}\right)dx - \int_1^2 (4-2x)\,dx - \int_2^4 \left(\frac{3x-6}{2}\right)dx$$

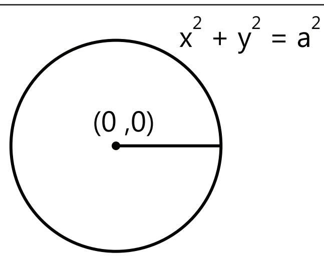
$$= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$$

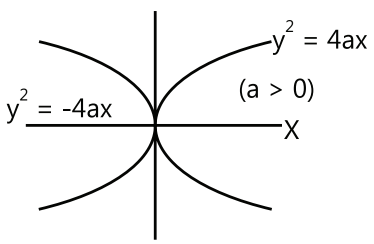
$$= \frac{1}{3}\left[(8+20)-\left(\frac{1}{2}+5\right)\right] - [(8-4)-(4-1)] - \frac{1}{2}[(24-24)-(6-12)]$$

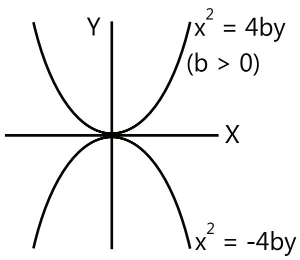
$$= = \frac{7}{2} \text{ sq. units.}$$

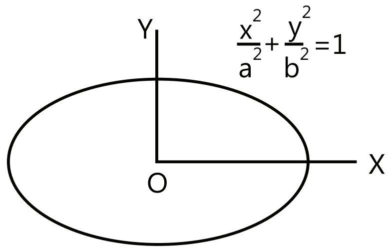
SKETCH OF STANDARD CURVES

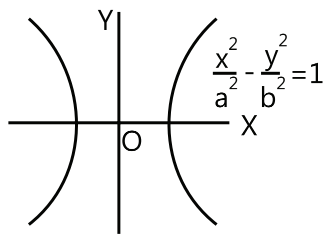


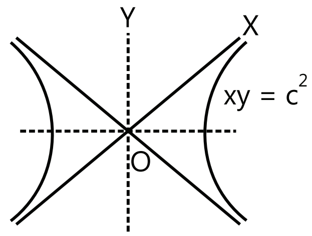


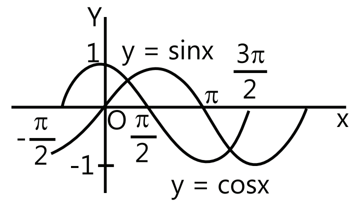


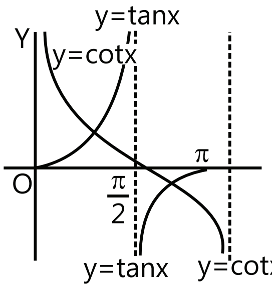


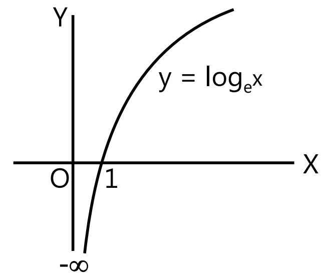


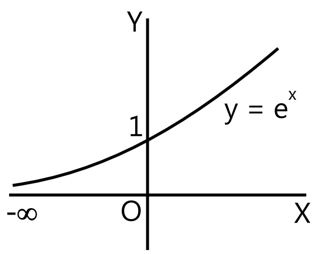












4. STANDARD AREAS

4.1 Area Bounded by Two Parabolas

Area between the parabolas $y^2 = 4ax$ and $x^2 = 4$; $a > 0$, $b > 0$, is

$$|A| = \frac{16ab}{3}$$

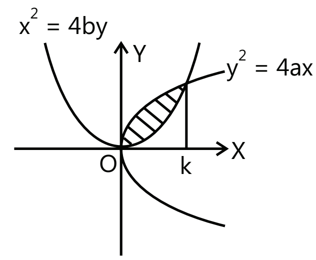


Figure 25.23

Illustration 20: Find the area bounded by $y = \sqrt{x}$ and $x = \sqrt{y}$. **(JEE MAIN)**

Sol: By using above mentioned formula.

Area bounded is shaded in the figure

Here, $a = \frac{1}{4}$ and $b = \frac{1}{4}$

$\therefore$ Using the above formula, Area = (16 ab)/ 3

$$= \frac{16 \times (1/4) \times (1/4)}{3} = \frac{1}{3}$$

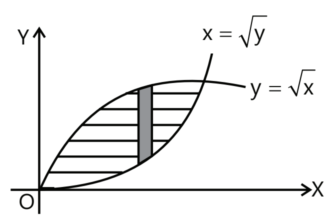


Figure 25.24

4.2 Area Bounded By Parabola and a Line

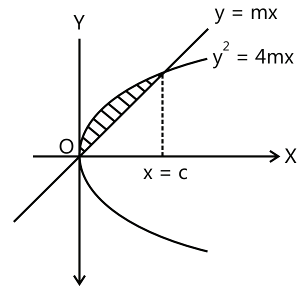


Figure 25.25

Area bounded by $y^2 = 4ax$ and $y = mx$; $a > 0$, $m > 0$ is $A = \frac{8a^2}{3m^3}$

Area bounded by $x^2 = 4ay$ and $y = mx$; $a > m > 0$

is $y = mx$; $a > m > 0$ $A = \frac{8a^2}{3m^3}$

Illustration 21: Find the area bounded by, $x^2 = y$ and $y = |x|$. **(JEE MAIN)**

Sol: Using above formula, i.e. $A = \frac{8a^2}{3m^3}$

Area bounded is shaded in the Fig. 25.26.

Here, $a = 1/4$, $m = 1$

$\therefore$ Using the above formula, Area = $2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times \left(\frac{1}{4}\right)^2}{3 \times (1)^3} = \frac{1}{3}$

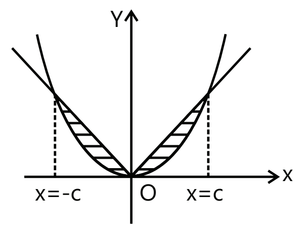


Figure 25.26

Illustration 22: Find the area bounded by $y^2 = x$ and $x = |y|$. **(JEE MAIN)**

Sol: Here, $a = 1/4$, $m = 1$, and required area is divided in to two equal parts at above and below x – axis.

Hence required area will be $2\left(\frac{8a^2}{3m^3}\right)$.

$\therefore$ Using the above formula, Area = $2\left(\frac{8a^2}{3m^3}\right) = \frac{2 \times 8 \times (1/4)^2}{3 \times (1)^3} = \frac{1}{3}$

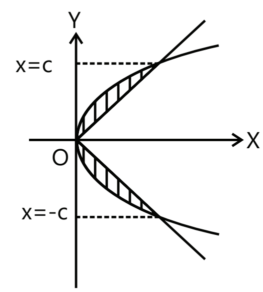


Figure 25.27

4.3 Area Enclosed by Parabola and It's Chord

Area between $y^2 = 4ax$ and its double ordinate at $x = a$ is

Area of AOB = $\frac{2}{3}$ (area ▱ABCD)

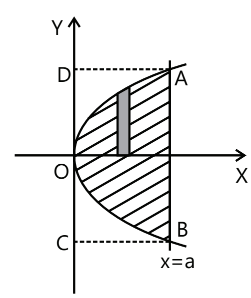


Figure 25.28

Illustration 23: Find the area bounded by $y = 2x - x^2$, $y + 3 = 0$. **(JEE MAIN)**

Sol: Here first obtain area of rectangle ABCD and after that by using above mentioned formula we will be get required area.

Solving $y = 2x - x^2$, $y + 3 = 0$, we get $x = -1$ or 3

Area (ABCD) = 4 × 4 = 16.

$\therefore$ Required area = $\frac{2}{3} \times 16 = \frac{32}{3}$

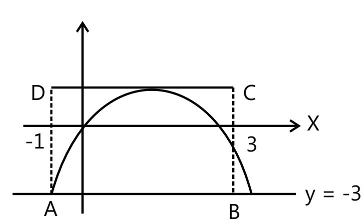


Figure 25.29

4.4 Area of an Ellipse

For an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is A = πab

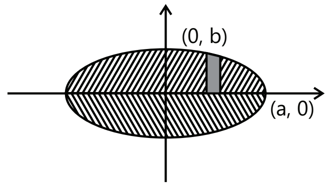


Figure 25.30

PLANCESS CONCEPTS

Try to remember some standard areas like for ellipse, parabola. These results are sometimes very helpful.

Vaibhav Gupta (JEE 2009 AIR 54)

5. SHIFTING OF ORIGIN

Area remains unchanged even if the coordinate axes are shifted or rotated or both. Hence shifting of origin / rotation of axes in many cases proves to be very convenient in finding the area.
For example: If we have a circle whose centre is not origin, we can find its area easily by shifting circle's centre.

Illustration 24: The line 3x + 2y = 13 divides the area enclosed by the curve $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area. **(JEE ADVANCED)**

Sol: Given $9x^2+ 4y^2 - 18x - 16y - 11 = 0$ ... (i)

and, 3x + 2y = 13 ... (ii)

$9(x^2 - 2x) + 4(y^2 - 4y) = 11;$

$\Rightarrow 9[(x-1)^2 - 1] + 4[(y-2)^2 - 4] = 11$

$\Rightarrow 9(x-1)^2 + 4(y-2)^2 = 36$

$\Rightarrow \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1 \Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$ (where X = x – 1 and Y = y – 2)

Hence 3x + 2y = 13

$\Rightarrow 3(X + 1) + 2(Y + 2) = 13$

$\Rightarrow 3X + 2Y = 6$

$\Rightarrow \frac{X}{2}+\frac{Y}{3}=1$

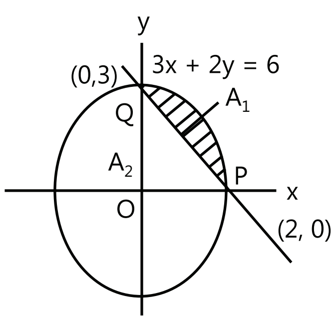


Figure 25.31

$\therefore$ Area of triangle OPQ = 1/2 × 2 × 3 = 3

Also area of ellipse = π (semi major axes) (semi minor axis) = π.3.2 = 6π

$$A_1=\frac{6\pi}{4}-\text{area of }\Delta OPQ=\frac{3\pi}{2}-3$$

$$A_2=3\left(\frac{6\pi}{4}\right)+\text{area of }\Delta OPQ=\frac{9\pi}{2}+3$$

Hence, $$\frac{A_2}{A_1}=\frac{\frac{9\pi}{2}+3}{\frac{3\pi}{2}-3}=\frac{3\pi+2}{\pi-2}$$

6. DETERMINATION OF PARAMETERS

In this type of questions, you will be given area of the curve bounded between some axes or points, and some parameter(s) will be unknown either in equation of curve or a point or an axis. You have to find the value of the parameter by using the methods of evaluating area.

Illustration 25: Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; x = 1 and y = c is equal to $\frac{9}{4}$. **(JEE MAIN)**

Sol: By using method of evaluating area we can find out the value of c.

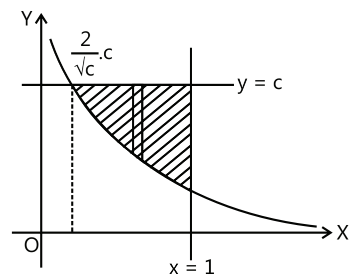


Figure 25.32

$$A = \int_{2/\sqrt{c}}^{1}\left(c - \frac{4}{x^2}\right)dx = \frac{9}{4};\quad \left(cx + \frac{4}{x}\right)_{\frac{2}{\sqrt{c}}}^{1} = \frac{9}{4}$$

$$(c+4) - (2\sqrt{c} + 2\sqrt{c}) = \frac{9}{4};\; c - 4\sqrt{c} + 4 = \frac{9}{4}$$

$$\Rightarrow\; (\sqrt{c}-2)^2 = \frac{9}{4} \Rightarrow (\sqrt{c}-2) = \frac{3}{2} \text{ or } -\frac{3}{2}$$

Hence c = (49/4) or (1/4)

Illustration 26: Consider the two curves:

C_1 : y = 1 + cos x, and C_2 : y = 1 + cos(x – α) for α ∈ (0, π/2) and x ∈ [0, p].

Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and x = 0 is same as that of the area bounded by C_2, y = 1 and x = π. For this value of α, find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 and x = π. **(JEE ADVANCED)**

Sol: Solve C_1 and C_2 to obtain the value of x, after that by following given condition we will be obtain required value of α.

Solving C_1 and C_2 , we get

$$1 + \cos x = 1 + \cos(x-\alpha) \Rightarrow x = \alpha - x \quad \Rightarrow x = \frac{\alpha}{2}$$

According to the question,

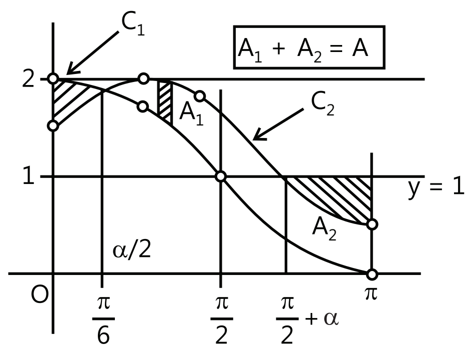


Figure 25.33

$$\int_0^{\alpha/2}(\cos x - \cos(x-\alpha))\,dx = -\int_{\frac{\pi}{2}+\alpha}^{\pi}(\cos(x-\alpha))\,dx$$

$$\Rightarrow\; \left[\sin x - \sin(x-\alpha)\right]_0^{\alpha/2} = \left[\sin(x-\alpha)\right]_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow\; \left[\sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right)\right] - [0 - \sin(-\alpha)] = \sin\left(\frac{\pi}{2}\right) - \sin(\pi - \alpha)$$

$$\Rightarrow\; 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha \text{ . Hence, } 2\sin\frac{\alpha}{2} = 1 \qquad \Rightarrow \alpha = \frac{\pi}{3}$$

7. AREA BOUNDED BY THE INVERSE FUNCTION

The area of the region bounded by the inverse of a given function can also be calculated using this method. The graph of inverse of a function is symmetric about the line y = x. We use this property to calculate the area. Hence, area of the function between x = a to x = b, is equal to the area of inverse function from f(a) to f(b).

Illustration 27: Find the area bounded by the curve g(x), the x-axis and the lines at y = –1 and y = 4, where g(x) is the inverse of the function $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$. **(JEE MAIN)**

Sol: Here f(x) is a strictly increasing function therefore required area will be

$$A=\int_0^2(4-f(x))dx+\int_{-2}^0(f(x)+1)dx$$

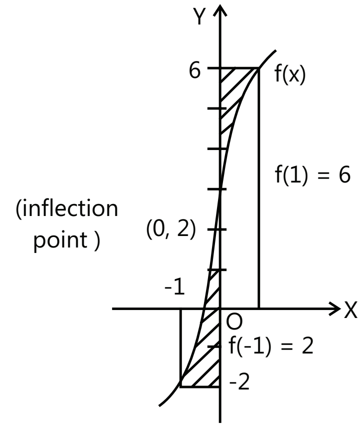


Figure 25.34

Given $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$

$\Rightarrow$ f(0) = 1; f(2) = 4 and f(–2) = –1

Also, $f'(x)=\frac{x^2}{8}+\frac{x}{4}+\frac{13}{12}$,

i.e. f(x) is a strictly increasing function.

$$\therefore A=\int_0^2(4-f(x))dx+\int_{-2}^0(f(x)+1)dx$$

$$A=\int_0^2\left(4-\frac{x^3}{24}-\frac{x^2}{8}-\frac{13x}{12}-1\right)dx+\int_{-2}^0\left(\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1+1\right)dx$$

$$\therefore A=\left[\left(3.2-\frac{2^4}{24.4}-\frac{2^3}{8.3}-\frac{13.2^2}{12.2}\right)-(0)\right]+\left[(0)-\left(\frac{2^4}{24.4}-\frac{2^3}{8.3}+\frac{13.2^2}{12.2}-2.2\right)\right]=\frac{16}{3}$$

Illustration 28: Let f(x) = x^3 + 3x + 2 and g(x) is the inverse of it. Find the area bounded by g(x), the x-axis and the ordinate at x = –2 and x = 6. **(JEE ADVANCED)**

Sol: Let $A=\int_{-2}^{6}\left|f^{-1}(x)\right|dx$

Substitute $x=f(u)$ or $u=f^{-1}(x)$

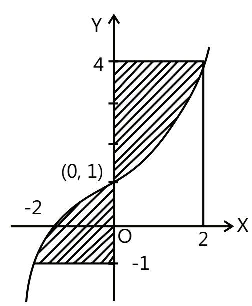


Figure 25.35

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|u|\,f^{-1}(u)\,du$$

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|4|\left(3u^2+3\right)du$$

We have, $f(-1)=2$ and $f(1)=6$

$$=\int_{-1}^{1}|u|\left(3u^2+3\right)du=2\int_0^1\left(3u^3+3u\right)du$$

$$=\left[\frac{3}{2}u^4+3u^2\right]_0^1=\frac{9}{2}\text{ Sq. units.}$$

8. VARIABLE AREA

If y = f(x) is a monotonic function in (a, b), then the area of the function y = f(x) bounded by the lines at x = a, x = b, and the line y = f(c), [where c ∈ (a, b)] is minimum when $c = \frac{a+b}{2}$.

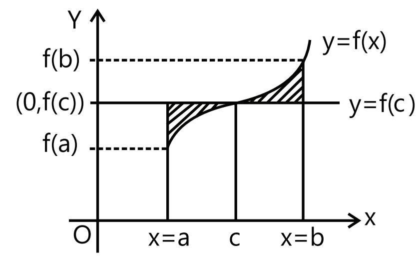


Figure 25.36

Proof: $A = \int_a^c f(c) - f(x)dx + \int_c^b (f(x) - f(c))dx$

$= f(c)\,(c - a) - \int_a^c (f(x))dx + \int_c^b (f(x))dx - f(c)(b - c)$

$= \{(c - a) - (b - c)\}\, f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

$A = [2c - (a + b)]\, f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

For maxima and minima $\frac{dA}{dc} = 0 \Rightarrow$ f ′ (c) = [2c – (a + b)] = 0 (as f ′ (c) = 0) hence $c = \frac{a+b}{2}$ also $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$ Hence A is minimum when $c = \frac{a+b}{2}$

9. AVERAGE VALUE OF A FUNCTION

In this section, we would study the average of a continuous function. This concept of average is frequently applied in physics and chemistry.

Average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

PLANCESS CONCEPTS

(a) Average value can be positive, negative or zero .

(b) If the function is defined in (0, ∞), then $y_{av} = \lim_{b \to \infty} \frac{1}{b}\int_0^b f(x)dx$ provided the limit exists

(c) Root mean square value (RMS) is defined as $\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)\,dx\right]^{\frac{1}{2}}$

(d) If a function is periodic then we need to calculate average of function in particular time period that is its overall mean.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 29: Find the average value of y^2 w.r.t. x for the curve $ay = b\sqrt{a^2 - x^2}$ between x = 0 & x = a. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$. **(JEE MAIN)**

Sol: As average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Let $f(x) = y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ Now $f(x)|_{av} = \frac{b^2}{a^2(a-0)}\int_0^a (a^2 - x^2)dx = \frac{2b^2}{3}$

Again y_{av} w.r.t. x^2 as $f(x)|_{av} = \frac{1}{(a^2-0)}\int_0^{a^2} y\, d(x^2) = \frac{b}{a^2 a}\int_0^{a^2}\sqrt{a^2 - x^2}\, dx^2 = \frac{b}{a^3}\int_0^{a^2} 2t^2\, dt = \frac{2ba^3}{3}$

10. DETERMINATION OF FUNCTION

Sometimes the area enclosed by a curve is given as a variable function and we have to find the function. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$ i.e. derivative of the area function is the function itself. Thus we can easily find f(x) by differentiating area function.

PLANCESS CONCEPTS

If F(x) is integral of f(x) then, $A_a^x = \int f(x)dx = [F(x) + c]$

And since, $A_a^a = 0 = F(a) + c$ $\Rightarrow c = -F(a)$.

$\therefore$ $A_a^x = F(x) - F(a)$. Finally by taking x = b we get, $A_a^b = F(b) - F(a)$

Note that this is true only if the function doesn't have any zeroes between a and b.

If the function has zero at c then area = |F(b) – F(c)| + |F(c) – F(a)|

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 30: The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1$, $x \ge 0$;

(a) Find the average rate of change of A w.r.t. x and x increases from 1 to 8.

(b) Find the instantaneous rate of change of A w.r.t. x at x = 5.

(c) Find the ordinate (height) y of the graph as a function of x.

(d) Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8. **(JEE ADVANCED)**

Sol: Here by differentiating given area function we can obtain the main function.

(a) A(1) = 1, A(8) = 4 ; $\frac{A(8) - A(1)}{8-1} = \frac{3}{7}$

(b) $\left.\frac{dA}{dx}\right|_{x=5} = \left.\frac{1.\,3}{2\sqrt{1+3x}}\right|_{x=5} = \frac{3}{8}$

(c) $y = \frac{3}{2\sqrt{1+3x}}$

(d) $\frac{1}{(8-1)}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{1}{7}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{3}{7}$

Illustration 31: Let C_1 & C_2 be the graphs of the function $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graphs of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P(on C_1), the area of the shaded regions OPQ & ORP are equal, determine the function f(x). **(JEE ADVANCED)**

Sol: Similar to the above mentioned method.

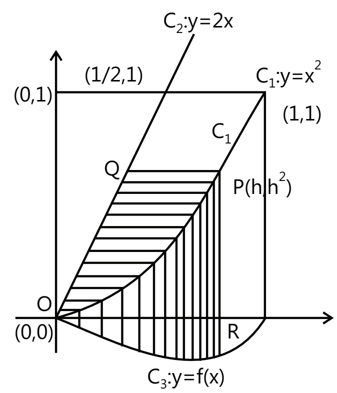


Figure 25.37

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^{h}(x^2 - f(x))\,dx \text{ differentiate both sides w.r.t. h}$$

$$\left(h-\frac{h^2}{2}\right)2h = h^2 - f(h)$$

$$f(h) = h^2 - \left(h-\frac{h^2}{2}\right)2h$$

$$= h^2 - h(2h - h^2) = h^2 - 2h^2 + h^3$$

$$f(h) = h^3 - h^2$$

$$f(x) = x^3 - x^2 = x^2(x - 1)$$

11. AREA ENCLOSED BY A CURVE EXPRESSED IN POLAR FORM

r = a (1 + cosθ) (Cardioid)

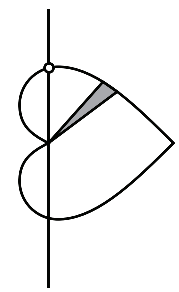


Figure 25.38

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} 4\cos^4\frac{\theta}{2}d\theta$$

Substitute $\frac{\theta}{2} = t$, $d\theta = 2dt$

$$A = a^2\int_0^{\pi} 4\cos^4 t\ dt = 8\times\frac{3\pi a^2}{16}$$

Illustration 32: Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$. **(JEE MAIN)**

Sol:

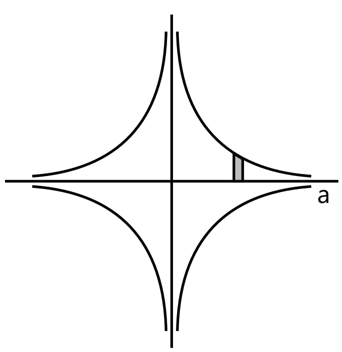


Figure 25.39

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad \text{and } dx = 3a\sin^2 t.\cos t.dt$$

$$A = 4\int_0^{a} y\ dx\ ;\ A = 4a^2\int_0^{\pi/2} 3\cos^3 t\sin^2 t\cos t\,dt$$

$$A = 12a^2\int_0^{\pi/2}\sin^2 t\cos^4 t\ dt = (12a^2)\,.\,\frac{1.3.1}{6.4.2}\,.\,\frac{\pi}{2} = \frac{12a^2\pi}{32} = \frac{3\pi a^2}{8}$$

Linear Programming

1. INTRODUCTION

Linear Programming was developed during World War II, when a system with which to maximize the efficiency of resources was of utmost importance.

2. LINEAR PROGRAMMING

Linear programming may be defined as the problem of maximising or minimising a linear function subject to linear constraints. The constraints may be equalities or inequalities. Here is an example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$x_1 + 2x_2 \leq 4$$

$$4x_1 + 2x_2 \leq 12$$

$$-x_1 + x_2 \leq 1$$

Here we have two unknowns and five inequalities (constraints). Notice that these constraints are all linear functions of the variables. The first two constraints, $x_1 \geq 0$ and $x_2 \geq 0$, are special. These are called no negativity constraints and are often found in linear programming problems. The other constraints are called the main constraints. The function to be maximised (or minimized) is called the objective function. In the above example the objective function is $x_1 + x_2$.

3. GRAPHICAL METHOD

As we have only two variables, we can solve this problem by plotting the constraints with x_1 and x_2 as axes. The intersection region of these inequalities is called feasible region for the objective function. This is the region which satisfies all the constraints. Now from this feasible region we have to select point(s) such that objective function is maximized or minimized.

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let Z=ax + by be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem and let Z=ax + by be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at corner point (vertex) of R.

Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists it must occur at a corner point of R. (By Theorem 1).

So for the above example

Corner point (x_1, x_2)	Z (= x_1 + x_2) value
0,1	1
3,0	3
8/3,2/3	10/3
2/3,5/3	7/3

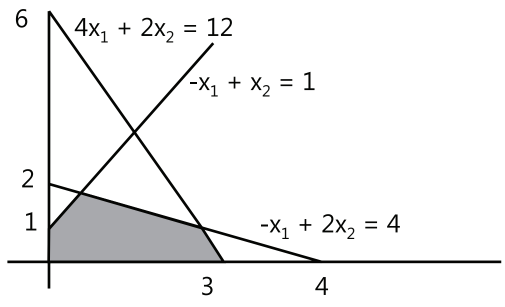


Figure 25.40

Hence (8/3, 2/3) is the optimal solution.

Note that z has also minimum value in the feasible region at (0, 1).

This method of solving is generally called as corner point method. Note that a function can have more than one optimal points.

4. MODELS

There are few important linear programming models which are more frequently used and some of them we encounter in our daily lives.

(a) Manufacturing/Assignment problems: In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of product, warehouse space per unit of the output. In order to make maximum profit.

Example: There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij}, for i = 1,......,I, and j = 1,.....,J. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers, x_{ij}, for i = 1,......,I, and j=1,......,J, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \le 1 \quad \text{For } i = 1,......, I \qquad ... (i)$$

$$\sum_{i=1}^{I} x_{ij} \le 1 \quad \text{For } j=1,......, J \qquad ... (ii)$$

And $x_{ij} \ge 0$ for i = 1,......,I, and j=1,......,J ... (iii)

Equation (i) reflects the fact that a person cannot spend more than 100% of his time working, (ii) means that only one person is allowed on a job at a time, and (iii) says that no one can work a negative amount of time on any job, Subject to (i), (ii) and (iii) , we wish to maximize the total value of $\sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}x_{ij}$

(b) Diet problems: In these problems, we determine the amount of different kinds of nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each nutrients.

Example: There are m different types of food, $F_1,........,F_m$, that supply varying quantities of the n nutrients , $N_1,........,N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1y_1 + b_2y_2 + ... + b_my_m \qquad ... (i)$$

The amount of nutrient N_j contained in this diet is

$$a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m$$

For j = 1,........, n. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j}y_1 + a_{2j}y_2 + ... + a_{mj}y_m \ge c_j \quad \text{For } j = 1,........, n \qquad ... (ii)$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \ge 0,\ y_2 \ge 0, ..., y_m \ge 0 \qquad ... (iii)$$

Our problem is: minimize (i) subject to (ii) and (iii). This is exactly the standard minimum problem.

(c) Transportation problems: In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Example: There are I ports, or production plants, $P_1,.........P_I$, that supply a certain commodity, and there are J markets, $M_1,.........M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity (i=1,2,......I), and market M_j must receive the amount r_j of the commodity (j = 1,........J). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost is

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}b_{ij} \quad \text{... (i)}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij} \le y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \text{ for i = 1,......,I} \quad \text{... (ii)}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \le r_j \text{ for j = 1,......,I} \quad \text{... (iii)}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \quad \text{for I = 1,........I and j =1,........J.} \quad \text{... (iv)}$$

Our problem is minimize (i) subject to (ii), (iii) and (iv).

FORMULAE SHEET

(a) Area bounded by a curve with x – axis: Area = $\int_a^b y\,dx = \int_a^b f(x)\,dx$

(b) Area bounded by a curve with y – axis: Area = $\int_c^d x\,dy = \int_c^d f(y)\,dy$

(c) Area of a curve in parametric form: Area = $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)\,dt$

(d) Positive and Negative Area: $A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$;

(e) Area between two curves:

(i) Area enclosed between two curves intersecting at two different points.

Area = $\int_a^b (y_1 - y_2)\,dx = \int_a^b [f_1(x) - f_2(x)]\,dx$

(ii) Area enclosed between two curves intersecting at one point and the x – axis.

Area = $\int_a^{\alpha} f_1(x)\,dx + \int_{\alpha}^{b} f_2(x)\,dx$

(iii) Area bounded by two intersecting curves and lines parallel to y – axis.

Area = $\int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

(a) Standard Areas:

(i) Area bounded by two parabolas $y^2 = 4ax$ and $x^2 = 4by$; $a > 0$, $b > 0$: Area $= \frac{16ab}{3}$

(ii) Area bounded by Parabola $y^2 = 4ax$ and Line $y = mx$: Area $= \frac{8a^2}{3m^3}$

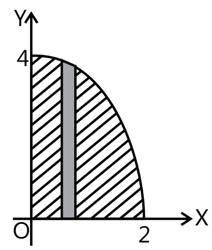
(iii) Area of an Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Area $= \pi ab$

Solved Examples

JEE Main/Boards

Example 1: Find area bounded by $y = 4 - x^2$, x-axis and the lines $x = 0$ and $x = 2$.

Sol: By using the formula of Area Bounded by the x – axis, we can obtain

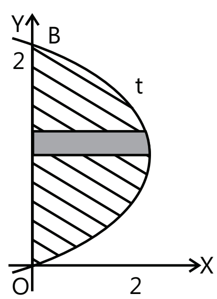


Required Area.

$$= \int_0^2 y\,dx = \int_0^2 (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \text{ sq. units}$$

Example 2: Find the area bounded by the curve $y^2 = 2y - x$ and the y-axis.



Sol: Here given equation is the equation of parabola with vertex (1, 1) and curve passes through the origin.

Curve is $y^2 - 2y = -x$ or $(y - 1)^2 = -(x - 1)$

It is a parabola with

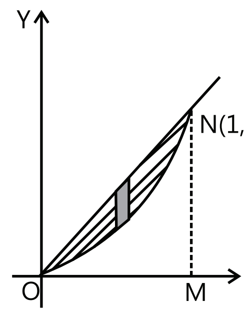
Vertex at (1, 1) and the curve passes through the origin. At B, $x = 0$ and $y = 2$

Area

$$= \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = \left(y^2 - \frac{y^3}{3}\right)_0^2 = \frac{4}{3} \text{ sq. units}$$

Example 3: Find the area of the region $\{(x, y): x^2 \le y \le x\}$

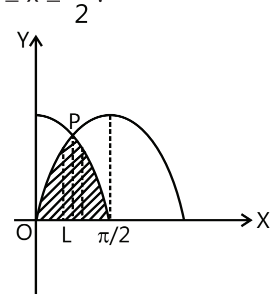
Sol: Consider the function $y = x^2$ and $y = x$ Solving them, we get $x = 0$, $y = 0$ and $x = 1$, $y = 1$; $x^2 \le y \Rightarrow$ area is above the curve $y = x^2$ $y \le x \Rightarrow$ area is below the line $y = x$



$$\text{Area} = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6} \text{ sq. units}$$

Example 4: Find the area of the region enclosed by $y = \sin x$, $y = \cos x$ and x-axis, $0 \le x \le \frac{\pi}{2}$.

Sol: Find point of intersection is P. Therefore after obtaining the co-ordinates of P and then integrating with appropriate limits, we can obtain required Area.



At point of intersection P, $x = \frac{\pi}{4}$ as ordinates of $y = \sin x$ and ; $y = \cos x$ are equal

Hence, P is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ Required area

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = \left(-\cos x\right)_0^{\pi/4} + \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units}$$

Example 5: The area bounded by the continuous curve y = f(x), (lying above the x-axis), x-axis and the ordinates x = 1 and x = b is (b – 1) sin (3b + 4). Find f(x)

Sol: Using Leibniz rule, we can solve given problem.

$$\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$$

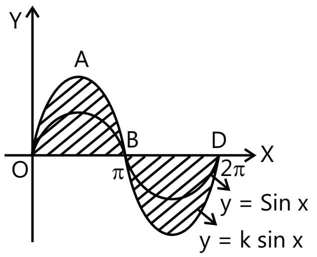
Apply Leibniz Rule: differentiate both sides w.r.t. "b",

f(b) = sin (3b + 4) + 3(b – 1) cos (3b + 4)

$\Rightarrow$ f(x) = sin (3x + 4) + 3(x – 1) cos (3x + 4)

Example 6: Find the area bounded by the curve y = k sin x and y = 0 from x = 0 to x = 2π.

Sol: Here the area of OAB is above the x-axis (y = 0) and thus it is positive while the area BCD is below x-axis (y = 0) and in negative but equal in quantity.



Area OAB = $\int_0^{\pi} y\,dx = \int_0^{\pi} k\sin x\,dx = k[-\cos x]_0^{\pi}$

= k[– cos π] – k [– cos 0]

= k[– (–1)] – k [–(1)] = k + k = 2k

∴ Total area = 4k sq. units.

Example 7: Find the area bounded by the curve

x = a (θ – sin θ), y = a (1 – cos θ), 0 ≤ θ ≤ 2π, with x-axis.

Sol: Substitute the value of y and dx and integrate.

Area = $\int_{\theta=0}^{\theta=2\pi} y\,dx = \int_{\theta=0}^{\theta=2\pi} y\frac{dx}{d\theta}d\theta$

$= \int_0^{2\pi} a(1-\cos\theta)a(1-\cos\theta)d\theta$

$= a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta$

$= a^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]$

$= a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ = $3\pi a^2$ sq. units.

Example 8: Find the area bounded by the curves {(x, y) : $y \ge x^2$, $y \le |x|$}

Sol: Here the region is symmetric about y-axis, the required area is 2[area of shaded region in first quadrant].

The curves intersect each other at x = 0 and x = ±1 as shown in figure. The points of intersection are (–1, 1), (0, 0) and (1, 1).

Since, the region is symmetric about y-axis, the required area is 2[area of shaded region]

Hence, Area = $2\int_0^1 (x-x^2)dx = 2\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$ sq. units.

Example 9: Draw a rough sketch of the curve y = $\sin^2 x$, $x\in\left[0,\frac{\pi}{2}\right]$. Find the area enclosed between the curve, x-axis and the line $x=\frac{\pi}{2}$.

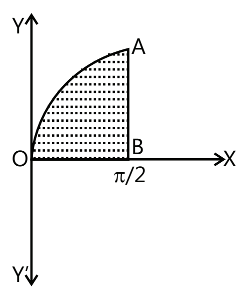
Sol: Here by substituting x = 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ we will get respective values of y. hence by plotting these values we can draw the given curve.

Some points on the $\sin^2 x$ graph are :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	0	0.25	0.5	0.75	1

By plotting points and joining them, we trace the curve.

Area bounded by curve y = $\sin^2 x$ between x = 0 and $x=\frac{\pi}{2}$



$= \int_0^{\pi/2} y\,dx = \int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)dx$

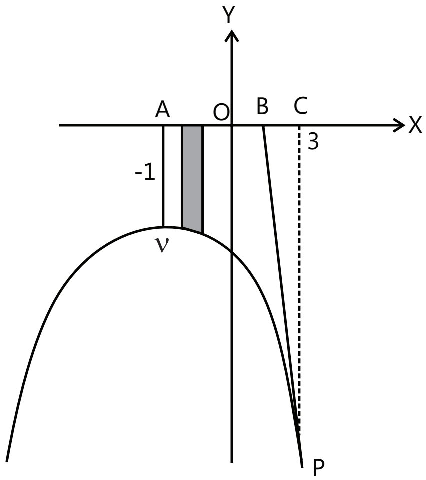
$= \frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right]$

$= \frac{\pi}{4}$ sq.units

JEE Advanced/Boards

Example 1: A tangent is drawn to $x^2 + 2x - 4ky + 3 = 0$ at a point whose abscissa is 3. The tangent is perpendicular to $x + 3 = 2y$. Find the area bounded by the curve, this tangent, x-axis and line $x = -1$

Sol: As we know multiplication of slopes of two perpendicular line is – 1, by using this, we can obtain the value of k and will get standard equation. After that using integration with respective limit, we will be get required area.



$x^2 + 2x - 4ky + 3 = 0$; $\frac{dy}{dx} = \frac{x+1}{2k}$ Tangent is perpendicular to $x + 3 = 2y$

$\therefore \frac{x+1}{2k}\left(\frac{1}{2}\right) = -1$ at $x = 3$

$\Rightarrow 1/k = -1 \Rightarrow k = -1$

$\therefore$ Curve becomes $(x + 1)^2 = -4(y + 1/2)$ which is a parabola with vertex at $V(-1, -1/2)$.

Coordinates of P are $(3, -9/2)$.

Equation of tangent at P is $y + 9/2 = -2(x - 3)$

B is $(3/4, 0)$, C is $(3, 0)$

Required Area = Area (ACPV) – Area of triangle BPC.

$$= \left|\int_{-1}^{3} \frac{x^2 + 2x + 3}{-4} dx\right| - \frac{1}{2}(BC)(CP)$$

$$= \frac{1}{4}\left|\left(\frac{x^3}{3} + x^2 + 3x\right)_{-1}^{3}\right| - \frac{1}{2}\left(3 - \frac{3}{4}\right)\left(\frac{9}{2}\right)$$

$$= \frac{1}{4}\left(27 + \frac{1}{3} - (1-3)\right) - \frac{81}{16} = \frac{109}{48} \text{ sq. units.}$$

Example 2: Let A_n be the area bounded by the curve $y = (\tan x)^n : n \in N$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

Sol: We can write $(\tan x)^n$ as $\tan^{n-2} x\,(\sec^2 x - 1)$. Therefore by solving $A_n = \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)dx$ we can prove given equation.

$$A_n = \int_0^{\pi/4} \tan^n x\, dx : n > 2$$

$$= \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)dx$$

$$\text{or } A_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/2} - A_{n-2}$$

$$\therefore A_n + A_{n-2} = \frac{1}{n-1} \quad \text{... (i)}$$

$\tan^n x \le \tan^{n-2} x$

(as $0 \le \tan x \le 1$ for $0 \le x \le \frac{\pi}{4}$)

$\Rightarrow A_n < A_{n-2}$

$\therefore A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$ by (1)

$$\therefore A_n < \frac{1}{2(n-1)} \quad \text{... (ii)}$$

Similarly $A_{n+2} < A_n$

$\Rightarrow A_{n+2} + A_n < A_n + A_n$

or $\frac{1}{(n+2)-1} < 2A_n$ by (1)

$$\Rightarrow \frac{1}{2n+2} < A_n \quad \text{... (iii)}$$

$$\Rightarrow \frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$

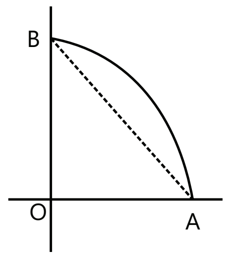
Example 3: $A(a, 0)$ and $B(0, b)$ are points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the area between the arc AB and chord AB of the ellipse is $\frac{1}{4}$ ab $(\pi - 2)$.

Sol: Area between the chord and ellipse = Area bounded by curve AB - Area of Δ OAB.

Equation of line AB is : $y = -\frac{b}{a}(x - a)$

Equation of curve AB is $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Area of bounded region is



$$\int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - \left(-\frac{b}{a}(x-a)\right)\right] dx$$

$$= \frac{b}{a}\left[0 + \frac{a^2\pi}{4} - \frac{a^2}{2}\right] = \frac{(\pi - 2)}{4}ab$$

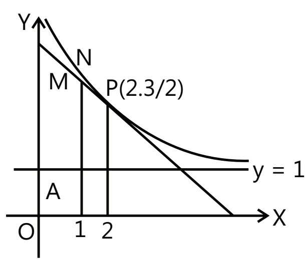
Alternate method:

Area between the chord and ellipse = Area bounded by curve AB - Area of Δ OAB

$$= \frac{1}{4}\pi ab - \frac{1}{2}ab = \frac{(\pi - 2)ab}{4} \text{ sq. units}$$

Example 4: Find the area of the region bounded $y = \frac{1}{x} + 1, x = 1$ and tangent drawn at the point P(2, 3/2) to the curve $y = \frac{1}{x} + 1$.

Sol: Here first obtain equation of tangent and then use the formula for area.



Equation of tangent at P(2, 3/2) to $y = \frac{1}{x} + 1$ is $y - \frac{3}{2} = -\frac{1}{4}(x-2)$ or x + 4y = 8.

Required area is area of region PMN

$$\text{Area} = \int_1^2 \left(\left(\frac{1}{x} + 1\right) - \frac{8-x}{4}\right) dx$$

$$= \left(\ln x - x + \frac{1}{4}\frac{x^2}{2}\right)_1^2 = \ln 2 - \frac{5}{8} \text{ sq. units}$$

Example 5: Find the area of the region bounded by the x-axis and the curve $y = \frac{1}{2}(2 - 3x - 2x^2)$.

Sol: Here the curve will intersect the x-axis when y = 0, therefore by substituting y = 0 in the above equation we will get the points of intersection of curve and x – axis.

$\Rightarrow 2 - 3x - 2x^2 = 0$ or $(2 + x)(1 - 2x) = 0$ or $x = -2,\ x = \frac{1}{2}$

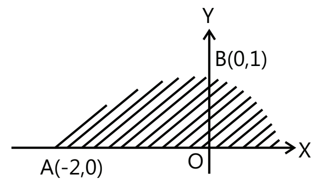
Thus, the curve passes through the points (–2, 0) and $\left(\frac{1}{2}, 0\right)$ on the x-axis.

It will have a turning points where $\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{1}{2}(-3 - 4x) = 0$ Þ $x = -\frac{3}{4}$

Also $\frac{d^2y}{dx^2} = -4$. That is, it is a max. at $x = \frac{3}{4}$

Also it cuts y-axis where x = 0, then y = 1. Thus the shape of the curve is as shown in the figure.



The required area is ABC. It is given by

$$\int_{-2}^{1/2} y\, dx = \int_{-2}^{1/2} \frac{1}{2}(2 - 3x - 2x^2)\, dx$$

$$= \frac{1}{2}\left[2x - \frac{3}{2}x^2 - \frac{2x^3}{3}\right]_{-2}^{1/2}$$

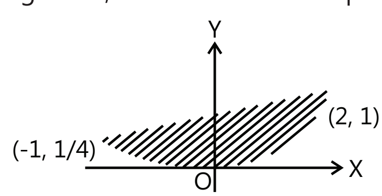
$$= \frac{1}{2}\left[2\left(\frac{1}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right)^2 - \frac{2}{3}\left(\frac{1}{2}\right)^3\right] -$$

$$\frac{1}{2}\left[2(-2) - \frac{3}{2}(-2)^2 - \frac{2}{3}(-2^3)\right]$$

$$= \frac{1}{2}\left(\frac{13}{24}\right) - \frac{1}{2}\left(-\frac{14}{3}\right) = \frac{125}{48} \text{ sq. units.}$$

Example 6: Find the area of the region bounded by the curve $x^2 = 4y$ and $x = 4y - 2$.

Sol: Solving given equation simultaneously, we will get the point of intersection. Using these points as the limits of integration, we calculate the required area.



The curve intersect each other, where $\frac{x^2}{4} = \frac{x+2}{4}$, or $x^2 - x - 2 = 0$, or x = –1, 2

Hence, the points of intersection are (–1, 1/4) and (2, 1). The region is plotted in figure. Since, the straight line x = 4y – 2 is always above the parabola $x^2 = 4y$ in the interval [–1, 2], the required area is given by

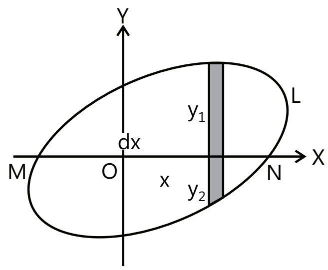
$$\text{Area} = \int_{-1}^{2} [f(x) - g(x)]\, dx$$

$$\text{Area} = \int_{-1}^{2}\left[\frac{x+2}{4}-\frac{x^2}{4}\right]dx = \frac{1}{4}\left[\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right] = \frac{9}{8} \text{ sq. units.}$$

Example 7: Find by using integration, the area of the ellipse $ax^2 + 2hxy + by^2 = 1$.

Sol: The equation can be put in the form $by^2 + 2hxy + (ax^2 - 1) = 0$



Cut an elementary strip.

Let the thickness of strip = dx

If y_1, y_2 be the values of y corresponding to any value at x.

Length of strip = $y_1 - y_2$

$$= \frac{2}{b}\sqrt{h^2x^2 - b(ax^2-1)} = \frac{2}{b}\sqrt{b-(ab-h^2)x^2}$$

$ab - h^2$ being positive here, since the conic is an ellipse.

The extreme values of x, are given by

$$y_1 - y_2 = 0, \text{ i.e., } x = \pm\sqrt{\frac{b}{ab-h^2}}$$

$$\text{Hence, the area required} = \int_{-\sqrt{b/ab-h^2)}}^{\sqrt{b/ab-h^2)}} (y_1 - y_2)dx$$

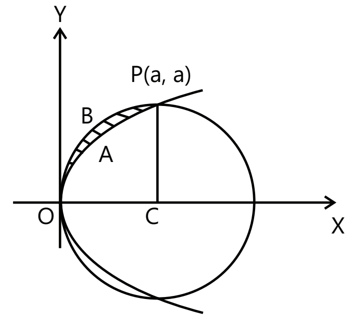
$$= \int_{-\sqrt{b/(ab-h^2)}}^{\sqrt{b/(ab-h^2)}} \frac{2}{b}\sqrt{b-(ab-h^2)x^2}\,dx$$

and putting $\sqrt{(ab-h^2)}\,x = \sqrt{b}\sin\theta$, this becomes

$$\frac{2}{\sqrt{(ab-h^2)}}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{(ab-h^2)} \text{ sq. units.}$$

Example 8: Find the area of region lying above x-axis, and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

Sol: By solving these two equation simultaneously, we can obtain their intersection points and then by subtracting area of parabola from area of circle we will be get the result.



Solving the two equation, simultaneously we see that the two curves intersect at (0, 0), (a, a) and (a, –a). We have to find the area of the region OAPBO, where P is the point of intersection (a, a)

$$\text{Required area} = \int_0^a [f(x) - g(x)]dx$$

$$= \int_0^a \sqrt{(2ax-x^2)}\,dx - \int_0^a \sqrt{ax}\,dx$$

$$\text{Now, } \int_0^a \sqrt{(2ax-x^2)}\,dx = \int_0^a \sqrt{[a^2-(a-x)^2]}\,dx$$

To evaluate this integral, we substitute $a - x = a \sin\theta$ and obtain

$$\int_0^a \sqrt{(2ax-x^2)}dx = \int_{\pi/2}^{0} (a\cos\theta)(-a\cos\theta)d\theta$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\,d\theta = a^2\frac{1}{2}\frac{\pi}{2} = \frac{\pi a^2}{4}$$

$$\text{Also } \int_0^a \sqrt{(ax)}\,dx = \left|\sqrt{a}\frac{2}{3}x^{3/2}\right|_0^a = \frac{2a^2}{3}$$

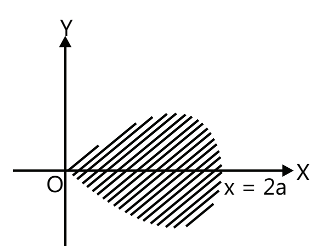
$$\therefore \text{ Required area } = a^2\left(\frac{\pi}{4}-\frac{2}{3}\right) \text{ sq. units}$$

Example 9: Prove that the area of the region bounded by the curve $a^4y^2 = x^5(2a - x)$, is $\frac{5}{4}$ times to that of the circle whose radius is a.

Sol: The curve is a loop lying between the line x = 0 and x = 2a and is symmetrical about the x-axis. Therefore the required area

$$= 2\int_0^{2a} y\,dx$$

$$= \frac{2}{a^2}\int_0^{2a} x^{5/2}\sqrt{2a-x}dx$$



To evaluate this integral, we put x = 2a $\sin^2 q$. When, x = 0, $\theta = 0$ and when x = 2a, $\theta = \frac{1}{2}\pi$ $\therefore$ Required area

$$= \frac{2}{a^2}\int_0^{\pi/2} (2a)^{5/2} \sin^5\theta.\sqrt{2a}\cos\theta.4a\sin\theta\cos\theta\, d\theta$$

$$= 64a^2 \int_0^{\pi/2} \sin^6\theta\cos^2\theta d\theta = 64a^2 \frac{5.3.1.1}{8.6.4.2}\cdot\frac{\pi}{2} = \frac{5a^2\pi}{4}$$

$= \frac{5}{4}$x area of the circle whose radius is a.

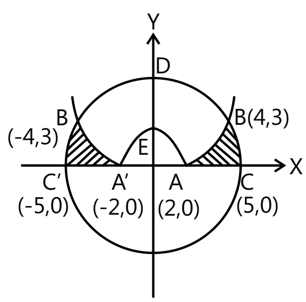
Example 10: Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and y = 0.

Sol: Here $x^2 + y^2 = 25$ represent circle with centre at origin and radius 5 unit. Therefore the required area = 2 area ABC

$$= 2\left[\int_2^4 \frac{1}{4}(4 - x^2)dx + \int_4^5 \sqrt{25^2 - x^2}\, dx\right]$$

Note: Here the portion is also bounded by two curves but we do not apply $A = \int [f(x) - g(x)]dx$ rule.

Reason: Range of integration of both the curves is not same.



$$= 2\left[\int_2^4 \frac{1}{4}(x^2 - 4)dx + \int_4^5 \sqrt{(5^2 - x^2)}\, dx\right]$$

$$= \frac{2}{4}\left[\left(\frac{x^3}{3} - 4x\right)\right]_2^4 + 2\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_4^5$$

$$= \frac{1}{2}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] +$$

$$2\left[\left(0 + \frac{25}{2}\sin^{-1}1\right) - \left(6 + \frac{25}{2}\sin^{-1}\frac{4}{5}\right)\right]$$

$$= \frac{1}{2}\left[\frac{32}{3}\right] + 25\sin^{-1}1 - 12 - 25\sin^{-1}\frac{4}{5}$$

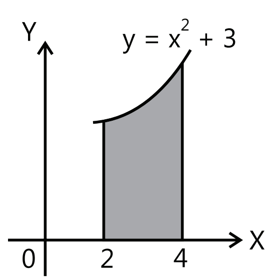
$$= \left(\frac{16}{3} - 12\right) + 25\frac{\pi}{2} - 25\sin^{-1}\frac{4}{5}$$

$$= -\frac{20}{3} + 25\left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5}\right) = \left(25\cos^{-1}\frac{4}{5} - \frac{20}{3}\right) \text{ sq. units.}$$

JEE Main/Boards

Exercise 1

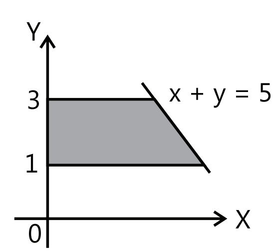
Q.1 Write an expression for finding the area of the shaded portion.



Q.2 Find the area bounded by the curve y = cos x, x-axis and between x = 0, x = π.

Q.3 Find the area bounded by the curve y = sinx, x-axis and between x = 0, x = π.

Q.4 Write an expression for finding the area of the shaded portion.



Q.5 Write an expression for finding the area bounded by the curve $x^2 = y$ and the line y = 2.

Q.6 Write an expression for finding the area of a circle $x^2 + y^2 = a^2$, above x-axis.

Q.7 On sketching the graph of $y = |x - 2|$ and evaluating $\int_{-1}^{3}|x-2|\,dx$, what does $\int_{-1}^{3}|x-2|\,dx$, represent on the graph?

Q.8 Draw the rough sketch of the curve $y = \sqrt{3x+4}$ and find the area under the curve above x-axis and between x = 0 and x = 4.

Q.9 Find the area under the curve $y = \frac{3}{(1-2x)^3}$ above x-axis and between x = –4 an x = –1.

Q.10 Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.

Q.11 Draw a rough graph of $f(x) = \sqrt{x} + 1$ in the interval [0, 4] and find the area of the region enclosed by the curve, x-axis and the lines x = 0 and x = 4.

Q.12 Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$; x-axis and the lines x = 3, x = 4.

Q.13 Find the area bounded by the curve $y = x \sin x^2$, x-axis and between x = 0 and $x = \sqrt{\frac{\pi}{2}}$.

Q.14 Using integration, find the area of the region bounded by the following curves, after making a rough sketch:

$y = |x + 1| + 1$, $x = -2$, $x = 3$, $y = 0$.

Q.15 Draw the rough sketch of y = sin 2x and determine the area enclosed by the curve, the x-axis and the lines $x = \pi/4$ and $x = 3\pi/4$.

Q.16 Find the area of the following region: $\{(x, y) : x^2 + y^2 \le 2ax, y^2 \ge ax, x \ge 0, y \ge 0\}$.

Q.17 Find the area bounded by the curve $y^2 = 4a^2(x - 3)$ and the lines x = 3, y = 4a.

Q.18 Make a rough sketch of the region given below and find its area using integration. $\{(x,y): 0 \le y \le x^2 + 3 ;\ 0 \le y \le 2x + 3,\ 0 \le x \le 3\}$.

Q.19 Determine the area enclosed between the curve $y = 4x - x^2$ and the x-axis.

Exercise 2

Single Correct Choice Type

Q.1 The area of the figure bounded by the curve $y = e^x$, $y = e^{-x}$ and the straight line x = 1 is

(A) $e + \frac{1}{e}$ (B) $e - \frac{1}{e}$

(C) $e + \frac{1}{e} - 2$ (D) None of these

Q.2 The area bounded in the first quadrant by the normal at (1, 2) on the curve y = 4x, x-axis & the curve is given by

(A) $\frac{10}{3}$ (B) $\frac{7}{3}$ (C) $\frac{4}{3}$ (D) $\frac{9}{2}$

Q.3 The area of the figure bounded by the curves $y = \ln x$ and $y = (\ln x)^2$ is

(A) e + 1 (B) e – 1 (C) 3 – e (D) 1

Q.4 The area bounded by the curves $y = x^2 + 1$ & the tangents to it drawn from the origin is:

(A) 2/3 (B) 4/3 (C) 1/3 (D) 1

Q.5 The area bounded by $x^2 + y^2 - 2x = 0$ & $y = \sin\frac{\pi x}{2}$ in the upper half of the circle is

(A) $\frac{\pi}{2} - \frac{4}{\pi}$ (B) $\frac{\pi}{4} - \frac{2}{\pi}$ (C) $\pi - \frac{8}{\pi}$ (D) $\frac{\pi}{2} - \frac{2}{\pi}$

Q.6 Consider the region formed by the lines x = 0, y = 0, x = 2, y = 2. Area enclosed by the curves $y = e^x$ and $y = \ln x$, within this region is removed, then the area of the remaining region is

(A) $2(1 + 2\ell n^2)$ (B) $2(2\ell n2 - 1)$

(C) $(2\ell n2 - 1)$ (D) $1 + 2\ell n^2$

Q.7 The area bounded by the curves $y = x(1 - \ln x)$; $x = e^{-1}$ and positive x-axis between $x = e^{-1}$ and $x = e$ is

(A) $\left(\frac{e^2 - 4e^{-2}}{5}\right)$ (B) $\left(\frac{e^2 - 5e^{-2}}{4}\right)$

(C) $\left(\frac{4e^2 - e^{-2}}{5}\right)$ (D) $\left(\frac{5e^2 - e^{-2}}{4}\right)$

Q.8 The positive values of the parameter 'a' for which the area of the figure bounded by the curve y = cos ax, $y = 0,\ x = \frac{\pi}{6a},\ x = \frac{x\pi}{2a}$ is greater than 3 are

(A) f (B) (0, 1/3)

(C) (3, ∞) (D) None of these

Q.9 The value of 'a' (a > 0) for which the area bounded by the curves $y = \frac{x}{6} + \frac{1}{x^2}, y = 0,\ x = a$ and $x = 2a$ has the least value, is

(A) 2 (B) $\sqrt{2}$ (C) $2^{1/3}$ (D) 1

Q.10 The ratio in which the area enclosed by the curve $y = \cos x \left(0 \le 0 \le \frac{\pi}{2}\right)$ in the first quadrant is divided by the curve y = sin x, is

(A) $(\sqrt{2}-1):1$ (B) $(\sqrt{2}+1):1$

(C) $\sqrt{2}:1$ (D) $\sqrt{2}+1:\sqrt{2}$

Q.11 The area bounded by the curve y = f(x), the co-ordinate axes & the line $x = x_1$ is given by $x_1.e^{x_1}$. Therefore f(x) equals

(A) e^x (B) $x\,e^x$ (C) $xe^x - e^x$ (D) $xe^x + e^x$

Q.12 The area bounded by the curves $y = -\sqrt{-x}$ and $x = -\sqrt{-y}$ where x, y ≤ 0

(A) Cannot be determined

(B) Is 1/3

(C) Is 2/3

(D) Is same as that of the figure bounded by the curves $y = \sqrt{-x}$; x ≤ 0 and $x = \sqrt{-y}$; y ≤ 0

Q.13 The area from 1 to x under a certain graph is given by $A = (1 + 3x)^{1/2} - 1,\ x \ge 0$.. The average value of y w.r.t. x as x increases from 1 to 8 is

(A) 3/7 (B) 1/2 (C) 3/8 (D) 4/7

Q.14 The slope of the tangent to a curve y = f(x) at (x, f(x)) is 2x + 1. If the curve passes through the point (1, 2) then the area of the region by the curve, the x-axis and the line x = 1 is

(A) 5/6 (B) 6/5 (C) 1/6 (D) 1

Q.15 The area of the region for which $0 < y < 3 - 2x - x^2$ & x > 0 is

(A) $\int_1^3 (3-2x-x^2)dx$ (B) $\int_0^3 (3-2x-x^2)dx$

(C) $\int_0^1 (3-2x-x^2)dx$ (D) None of these

Q.16 The graphs of $f(x) = x^2$ and $g(x) = cx^3$ (c > 0) intersect at the points (0, 0) & $\left(\frac{1}{c}, \frac{1}{c^2}\right)$. If the region which lies between these graphs & over the interval [0, 1/c] has the area equal to 2/3 then the value of c is

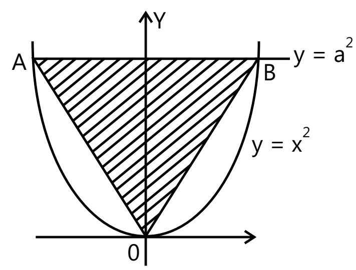
(A) 1 (B) 1/3 (C) 1/2 (D) 2

Q.17 The curvilinear trapezoid is bounded by the curve $y = x^2 + 1$ and the straight lines x = 1 and x = 2. The co-ordinates of the point (on the given curve) with abscissa xÎ[1, 2] where tangent drawn cut off from the curvilinear trapezoid are ordinary trapezium of the greatest area, is

(A) (1, 2) (B) (2, 5)

(C) $\left(\frac{3}{2}, \frac{13}{4}\right)$ (D) None of these

Q.18 In the given figure, if A_1 is the area of the ΔAOB and A_2 is the area of the parabolic region AOB then the ratio $\frac{A_1}{A_2}$ as a → 0 is



(A) 1 (B) 8/9 (C) 3/4 (D) 2/3

Previous Years' questions

Q.1 The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is ***(2003)***

(A) 27/4 sq. unit (B) 9 sq. unit

(C) 27/2 sq. unit (D) 27 sq. unit

Q.2 The area bounded by the curves $y=\sqrt{x}, 2y+3=x$ and x-axis in the 1st quadrant is ***(2003)***

(A) 9 sq. unit (B) 27/4 sq. unit

(C) 36 sq. unit (D) 18 sq. unit

Q.3 The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq unit. Then, the value of a is ***(2004)***

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1}{3}$

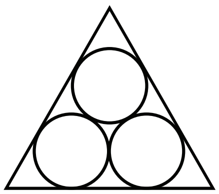
Q.4 The area of the equilateral triangle, in which three coins of radius 1 cm are placed, as shown in the figure, is ***(2005)***

(A) $(6+4\sqrt{3})$ sq. cm

(B) $(4\sqrt{3}-6)$ sq. cm

(C) $(7+4\sqrt{3})$ sq. cm

(D) $4\sqrt{3}$ sq. cm



Q.5 The area enclosed within the curve $|x| + |y| = 1$ is …….. ***(1981)***

Q.6 The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is …….. **(1989)**

Q.7 The area bounded by the curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y=\frac{1}{4}$ is ***(2005)***

(A) $\frac{1}{3}$ sq. unit (B) $\frac{2}{3}$ sq. unit

(C) $\frac{1}{4}$ sq. unit (D) $\frac{1}{5}$ sq. unit

Q.8 The area (in sq. units) of the region $\left\{(x,y): y^2 \ge 2x \text{ and } x^2+y^2 \le 4x, x \ge 0, y \ge 0\right\}$ is: ***(2016)***

(A) $\pi-\frac{8}{3}$ (B) $\pi-\frac{4\sqrt{2}}{3}$

(C) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$ (D) $\pi-\frac{4}{3}$

Q.9 The area (in sq. units) of the region described by $\left\{(x,y): y^2 \le 2x \text{ and } y \ge 4x-1\right\}$ is : ***(2015)***

(A) $\frac{5}{64}$ (B) $\frac{15}{64}$ (C) $\frac{9}{32}$ (D) $\frac{7}{32}$

Q.10 The area (in square units) bounded by the curves $y=\sqrt{x}, 2y-x+3=0$, x-axis, and lying in the first quadrant is ***(2013)***

(A) 36 (B) 18 (C) $\frac{27}{4}$ (D) 9

Q.11 The area of the region described by A $= \left\{(x,y): x^2+y^2 \le 1 \text{ and } y^2 \le 1-x\right\}$ is ***(2014)***

(A) $\frac{\pi}{2}+\frac{4}{3}$ (B) $\frac{\pi}{2}-\frac{4}{3}$

(C) $\frac{\pi}{2}-\frac{2}{3}$ (D) $\frac{\pi}{2}+\frac{2}{3}$

Q.12 If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to ***(2012)***

(A) -1 (B) $\frac{2}{9}$ (C) $\frac{9}{2}$ (D) 0

Q.13 The area bounded between the parabolas $x^2=\frac{y}{4}$ and $x^2=9y$ and the straight line $y=2$ is **(2012)**

(A) $20\sqrt{2}$ (B) $\frac{10\sqrt{2}}{3}$ (C) $\frac{20\sqrt{2}}{3}$ (D) $10\sqrt{2}$

Q.14 The area bounded by the curves $y=\cos x$ and $y=\sin x$ between the ordinates $x=0$ and $x=\frac{3\pi}{2}$ ***(2010)***

(A) $4\sqrt{2}+2$ (B) $4\sqrt{2}-1$

(C) $4\sqrt{2}+1$ (D) $4\sqrt{2}-2$

Q.15 The area of the region enclosed by the curves $y=x, x=e, y=\frac{1}{x}$ and the positive x-axis is ***(2011)***

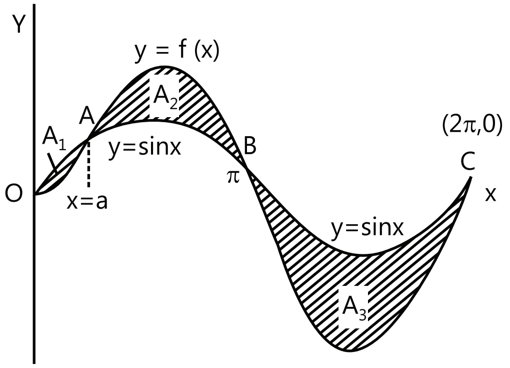
(A) 1 sq. unit (B) $\frac{3}{2}$ sq. units

(C) $\frac{5}{2}$ sq. units (D) $\frac{1}{2}$ sq. units

JEE Advanced/Boards

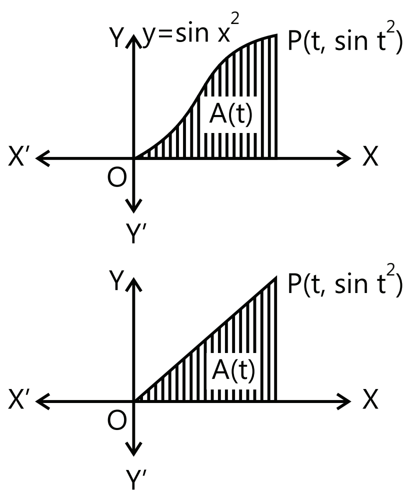
Exercise 1

Q.1 In the adjacent figure, graphs of two functions $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at $A(a, f(a))$; $B(\pi, 0)$ and $C(2\pi, 0)$. A_i ($i = 1, 2, 3$) is the area bounded by the curve $y = f(x)$ and $y = \sin x$ between $x = 0$ and $x = a$; $i = 1$, between $x = a$ and $x = \pi$; $i = 2$, between $x = \pi$ and $x = 2\pi$; $i = 3$.



If $A_i = 1 - \sin a + (a - 1)\cos a$, determine the function $f(x)$. Hence determine 'a' and A_1. Also calculate A_2 and A_3.

Q.2 The figure shows two regions in the first quadrant.



A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim\limits_{t\to 0}\frac{A(t)}{B(t)}$.

Q.3 A polynomial function f(x) satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find f(x) if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

Q.4 Show that the area bounded by the curve $y = \frac{\log x - c}{x}$, the x-axis and the vertical line through the maxima point of the curve is independent of the constant c.

Q.5 Consider the curve $y = x^n$ where $n > 1$ in the 1st quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.

Q.6 For what value of 'a' is the area of the figure bounded by the lines $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ln\frac{4}{\sqrt{5}}$?

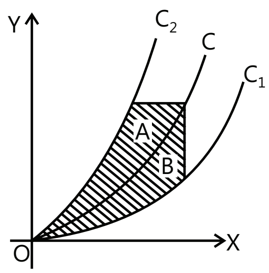
Q.7 For the curve $f(x)\frac{1}{1+x^2}$ let two points on it are $A(\alpha, f(\alpha)), B\left(-\frac{1}{\alpha}, f'\left(-\frac{1}{\alpha}\right)\right)$ ($\alpha > 0$). Find the minimum area bounded by the line segments OA, OB and f(x), where 'O' is the origin.

Q.8 Find the area bounded by the curve $y = \sin^{-1} x$ and the lines $x = 0$, $|y| = \frac{\pi}{2}$.

Q.9 If f(x) is monotonic in (a, b) then prove that the area bounded by the ordinates at $x = a$; $x = b$; $y = f(x)$ and $y = f(c)$, $c \in (a, b)$ is minimum when $c = \frac{a+b}{2}$. Hence if the area bounded by the graph of $f(x) = \frac{x^3}{3} - x^2 + a$, the straight lines $x = 0$, $x = 2$ and the x-axis is minimum then find the value of 'a'.

Q.10 Let 'c' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point (1, c) with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

Q.11 Let C_1 & C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 & C_2, if for each point P of C, the two shaded regions A & B shown in the figure have equal area. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ & that the lower curve C_1 has the equation $y = x^2/2$.



Q.12 Consider one side AB of a square ABCD, (read in order) on the line y = 2x – 17 and the other two vertices C, D on the parabola $y = x^2$.

(i) Find the minimum intercept of the line CD on y-axis.

(ii) Find the maximum possible area of the square ABCD.

(iii) Find the area enclosed by the line CD with minimum y-intercept and the parabola $y = x^2$. Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos (x - \alpha)$ for $\alpha \in (0, \pi/2)$; $x \in [0, p]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & x = 0 is same as that of the figure bounded by C_2, y = 1 & x = π. For this value of α, find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 & x = π.

Q.13 Draw the rough sketch of $y^2 = x + 1$ and $y^2 = -x + 1$ and determine area enclosed by the two curves.

Q.14 Let f : (0, ∞) → R be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t f^2(t)dt$, ∀ x > 0. Find the area enclosed by y = f(x), the x-axis and the ordinate at x = 3.

Q.15 For what values of a Î[0, 1] does the area of the figure bounded by the graph of the function y = f(x) and the straight lines x = 0, x = 1 & y = f(a) is at a minimum & for what values it is at a maximum if f(x) = $\sqrt{1-x^2}$. Find also the maximum & the minimum area.

Q.16 Let f(x) = sin x ∀ x ∈ $\left[0, \frac{\pi}{2}\right]$

$f(x) + f(\pi - x) = 2$ ∀ x ∈ $\left(\frac{\pi}{2}, \pi\right]$ and $f(x) = f(2\pi - x)$ ∀ x ∈ $(\pi, 2\pi)$.

If the area enclosed by y = f(x) and x-axis is $a\pi + b$, then find the value of $(a^2 + b^2)$.

Q.17 Find the values of m(m > 0) for which the area bounded by the line y = mx + 2 and $x = 2y - y^2$ is, (i) 9/2 square units & (ii) minimum. Also find the minimum area.

Q.18 Find the area bounded by the curve $y = x e^{-x}$; xy = 0 and x = c where c is the x-coordinate of the curve's inflection point.

Q.19 Find the area of the region

$\left\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\right\}$

Q.20 For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line y = 0, x = 0 & x = 1 the least?

Q.21 Consider two curves C_1: $y = \frac{1}{x}$ and C_2: y = ln x on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line x = 1 and D_2 denotes the region surrounded by C_1, C_2 and the line x = a. If $D_1 = D_2$. Find the value of 'a'.

Exercise 2

Single Correct Choice Type

Q.1 The area bounded by the curve $y = x^2 - 1$ & the straight line x + y = 3 is

(A) $\frac{9}{2}$ (B) 4 (C) $\frac{7\sqrt{17}}{2}$ (D) $\frac{17\sqrt{17}}{6}$

Q.2 The area bounded by the curve $y = e^{-x}$ & the lines $y = e^{-4}$ & x = 1 is given by

(A) $\frac{e^3 - 4}{e^4}$ (B) $\frac{e^3 + 4}{e^4}$

(C) $\frac{e^3 + 1}{e^4}$ (D) None of these

Q.3 Area common to the curve $y = \sqrt{9 - x^2}$ & $x^2 + y^2 = 6x$ is

(A) $\frac{\pi + \sqrt{3}}{4}$ (B) $\frac{\pi - \sqrt{3}}{4}$

(C) $3\left(\pi + \frac{\sqrt{3}}{4}\right)$ (D) $3\left(\pi + \frac{3\sqrt{3}}{4}\right)$

Q.4 The area bounded by y = 2 – |2 – x| & $y = \frac{3}{|x|}$ is

(A) $\frac{4+3\ell n3}{2}$ (B) $\frac{4-3\ell n3}{2}$

(C) $\frac{3}{2}+\ell n3$ (D) $\frac{1}{2}+\ell n3$

Q.5 The area bounded by the curves y = sin x & y = cos x between x = 0 & x = 2 π is

(A) $\int_0^{2\pi}(\sin x-\cos x)dx$ (B) $2\sqrt{2}$ sq. unit

(C) 0 (D) $4\sqrt{2}$ sq. unit

Q.6 If f(x) = –1 + |x – 2|, 0 ≤ x ≤ 4 ; g(x) = 2 – |x|, –1 ≤ x ≤ 3.

Then the area bounded by y = gof(x); x = 1, x = 4 and x-axis is

(A) 7/2 sq. units (B) 9/4 sq. units

(C) 9/2 sq. units (D) None of these

Q.7 Area enclosed between the curve y = $\sec^{-1}x$, y = $\operatorname{cosec}^{-1}x$ and the line x = 1 is

(A) $\ln(3+2\sqrt{2})$ (B) $\ln(3+2\sqrt{2})-1$

(C) $\ln(3+2\sqrt{2})-\pi/2$ (D) None of these

Q.8 The area of the closed figure bounded by y = x, y = –x the tangent to the curve $y=\sqrt{x^2-5}$ at the point (3, 2) is

(A) 5 (B) $\frac{15}{2}$

(C) 10 (D) $\frac{35}{2}$

Q.9 The line y = mx bisects the area enclosed by the curve y = 1 + 4x – x^2 & the lines x = 0, $x=\frac{3}{2}$ & y = 0, then m is equal to

(A) $\frac{13}{6}$ (B) $\frac{6}{13}$ (C) $\frac{3}{2}$ (D) 4

Q.10 The area common to $y \ge \sqrt{x}$ & $x > -\sqrt{y}$ and the curve $x^2 + y^2 = 2$ is

(A) $\frac{\pi}{4}+\frac{1}{3}$ (B) $\frac{3\pi}{2}$ (C) $\frac{1}{3}$ (D) $\frac{\pi}{2}$

Q.11 The ratio in which the curve y = x^2 divides the region bounded by the curve ; y = $\sin\left(\frac{\pi x}{2}\right)$ & the x-axis as x varies from 0 to 1, is

(A) 2 : π (B) 1 : 3

(C) 3 : π (D) (6 – π) : π

Q.12 Area of the region enclosed between the curves x = y^2 – 1 and $x = |y|\sqrt{1-y^2}$ is

(A) 1 (B) 4/3 (C) 2/3 (D) 2

Q.13 Let y = g(x) be the inverse of a bijective mapping f : R → R, f(x) = $3x^3$ + 2x. The area bounded by the graph of g(x). The x-axis and the ordinate at x = 5 is

(A) 5/4 (B) 7/4 (C) 9/4 (D) 13/4

Previous Years' Questions

Q.1 The area of the region between the curves $y=\sqrt{\frac{1+\sin x}{\cos x}}$ and $y=\sqrt{\frac{1-\sin x}{\cos x}}$ and bounded by the lines x = 0 and x = $\frac{\pi}{4}$ is ***(2008)***

(A) $\int_0^{\sqrt{2}-1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

(B) $\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(C) $\int_0^{\sqrt{2}+1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(D) $\int_0^{\sqrt{2}+1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

Q.2 Let the straight line x = b divide the area enclosed by y = $(1 – x)^2$, y = 0 and x = 0 into two parts R_1(0 ≤ x ≤ b) and R_2(b ≤ x ≤ 1) such $R_1 - R_2 = \frac{1}{4}$. Then, b equals ***(2011)***

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Q.3 Let f : [–1, 2] → [0, ∞) be a continuous function such that f(x) = f(1 – x) for all xÎ[–1, 2], Let $R_1 = \int_{-1}^{2} x f(x)dx$ and R_2 be the area of the region bounded by y = f(x), x = –1, x = 2 and the x-axis. Then, ***(2011)***

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$

(C) $2R_1 = R_2$ (D) $3R_1 = R_2$

Q.4 Find the area bounded by the curve $x^2 = 4y$ and the straight line x = 4y – 2. ***(1981)***

Q.5 Find the area bounded by the x-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at x = 2 and x = 4. If the ordinate at x = a divides the area into two equal parts, find a. ***(1983)***

Q.6 Find the area of the region bounded by the x-axis and the curves defined by y = tan x, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$ and y = cot x, $\frac{\pi}{6} \le x \le \frac{\pi}{2}$ ***(1984)***

Q.7 Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and y = |x – 1| and find its area. ***(1985)***

Q.8 Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$ and x = y. **(1986)**

Q.9 Find the area of the region bounded by the curve C : y = tan x, tangent drawn to C at x = $\frac{\pi}{4}$ and the x-axis. ***(1988)***

Q.10 Find all maxima and minima of the function $y = x(x - 1)^2$, $0 \le x \le 2$.

Also, determine the area bounded by the curve $y=x(x - 1)^2$, the y-axis and the line x = 2. ***(1989)***

Q.11 Compute the area of the region bounded by the curves y = ex log x and y = $\frac{\log x}{ex}$ where log e = 1 ***(1990)***

Q.12 If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$,

f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angled at V. Find the area enclosed by f(x) and chord AB. ***(2005)***

Q.13 A curve passes through (2, 0) and the slope of tangent at point P(x, y) equals $\frac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant. ***(2004)***

Q.14 Area of the region $\left\{(x, y) \in R^2 : y \ge \sqrt{|x+3|},\ 5y \le x + 9 \le 15\right\}$ is equal to ***(2016)***

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

Q.15 Let S be the area of the region enclosed by $y = e^{-x^2}, y = 0, x = 0$, and $x = 1$. Then ***(2012)***

(A) $S \ge \frac{1}{e}$ (B) $S \ge -\frac{1}{e}$

(C) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (D) $S \ge \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.4 Q.10 Q.17

Exercise 2

Q.2 Q.6 Q.10
Q.12 Q.15 Q.17
Q.18

Previous Years' Questions

Q.2 Q.4 Q.7

JEE Advanced/Boards

Exercise 1

Q.1 Q.5 Q.12
Q.14 Q.16 Q.20
Q.21

Exercise 2

Q.2 Q.3 Q.7
Q.11 Q.13

Previous Years' Questions

Q.1 Q.3 Q.7
Q.10 Q.11 `Q.13

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\frac{74}{3}$ sq.units

Q.2 2 sq. units

Q.3 2 sq. units

Q.4 $\int_1^3 (5-y)\,dy$

Q.5 $2\int_0^2 \sqrt{y}\,dy$

Q.6 $\int_{-2}^{a} \sqrt{a^2-x^2}\,dx$

Q.7 5 sq. units

Q.8 $\frac{112}{9}$ sq. units

Q.9 $\frac{2}{27}$ sq. units

Q.10 $\frac{64}{3}$ sq. units

Q.11 $\frac{28}{3}$ sq. units

Q.12 (3 + 16 log 2) sq. units

Q.13 $\frac{1}{2}$ sq. units

Q.14 13.5 sq. units

Q.15 1 sq. units

Q.16 $\frac{a^2}{12}(3\pi-8)$ sq. units

Q.17 $\frac{16a}{3}$ sq. units

Q.18 $\frac{50}{3}$ sq. units

Q.19 $\frac{32}{3}$ sq. units

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** A	**Q.3** C	**Q.4** A	**Q.5** A	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** D	**Q.10** C	**Q.11** D	**Q.12** B
Q.13 A	**Q.14** A	**Q.15** C	**Q.16** C	**Q.17** C	**Q.18** C

Previous Years' Questions

Q.1 D	**Q.2** A	**Q.3** A	**Q.4** A	**Q.5** 2 sq. units	
Q.6 $2\sqrt{3}$ sq. units	**Q.7** A	**Q.8** A	**Q.9** C	**Q.10** D	**Q.11** A
Q.12 C	**Q.13** C	**Q.14** D	**Q.15** B		

JEE Advanced/Boards

Exercise 1

Q.1 f(x) = x sin x, a = 1 ; A_1 = 1 – sina ; A_2 = π – 1 – sina ; A_3 = (3π – 2) sq. units

Q.2 2/3

Q.3 f(x) = x^2 + 1;y = ± 2x ; A = $\frac{2}{3}$sq. units

Q.4 $\frac{1}{2}$

Q.5 $\sqrt{2}+1$

Q.6 a = 8

Q.7 $\frac{(\pi-1)}{2}$

Q.8 2

Q.9 $a=\frac{2}{3}$

Q.10 104

Q.11 $(16/9)x^2$

Q.12 (i) 3 ; (ii) 1280 sq. units ; (iii) $\frac{32}{3}$ sq. units

Q.13 $\sqrt{3}$

Q.14 $\frac{3}{2}$

Q.15 $\frac{8}{3}$ sq. units

Q.16 4

Q.17 A $\frac{3m+2m^2+\frac{7}{6}}{m^3}$

Q.18 $1-3e^{-2}$

Q.19 $\left(\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}\right)$ sq. units

Q.20 a = –3/4

Q.21 e

Exercise 2

Single Correct Choice Type

Q.1 D	**Q.2** A	**Q.3** D	**Q.4** B	**Q.5** D	**Q.6** C	**Q.7** C
Q.8 A	**Q.9** A	**Q.10** A	**Q.11** D	**Q.12** D	**Q.13** B	

Previous Years' Questions

Q.1 B

Q.2 B

Q.3 C

Q.4 9/8 sq. units

Q.5 $2\sqrt{2}$

Q.6 $\frac{1}{2}\log_e 3$ sq. units

Q.7 $\frac{5\pi}{4}-\frac{1}{2}$ sq. units

Q.8 $\frac{1}{3}-\pi$ sq. units

Q.9 $\left(\log\sqrt{2}-\frac{1}{4}\right)$sq. units

Q.10 10/3 sq. units

Q.11 $\frac{e^2-5}{4e}$ sq. units

Q.12 125/3 sq. units

Q.13 4/3 sq. units

Q.14 3/2 sq. units

Q.15 A, B, D

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Area = $\int_2^4 x^2 + 3 = \frac{x^3}{3} + 3x\Big|_2^4$

$= \frac{64}{3} + 12 - \frac{8}{3} - 6 \Rightarrow$ Area = $\frac{56}{3} + 6 = \frac{74}{3}$ sq. units

Sol 2: $2\int_0^{\pi/2} \cos x dx = 2\left[\sin x\right]_0^{\pi/2}$ = 2[1 – 0] = 2 sq. units

Sol 3: $\int_0^{\pi} \sin x dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi + \cos 0$ = 1 + 1

= 2 sq. units

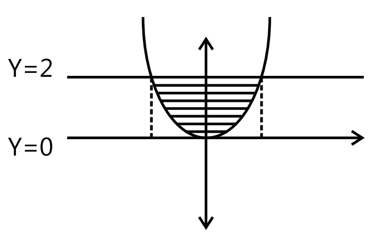
Sol 4: x + y = 5 ⇒ x = –y + 5

$\Rightarrow \int_4^2 x = \int_1^3 (-y+5) dy = \left[-\frac{y^2}{2} + 5y\right]_1^3 = \left|\frac{9}{2} - 15 - \frac{1}{2} + 5\right|$

= 6 sq. units

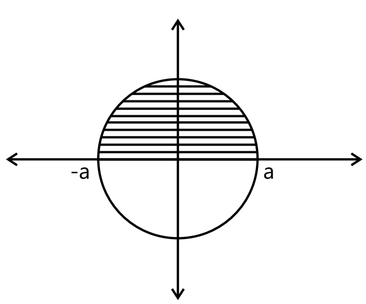
Expression = $\left|\int_1^3 (5-y)\right| dy$

Sol 5: $y = x^2$;



$x = \sqrt{y}$ $A = 2\int_0^2 \sqrt{y} dy$

Sol 6:



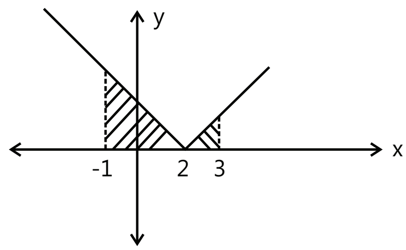
Area = $\int_{-a}^{a} \sqrt{a^2 - x^2} dx$

Sol 7: $\int_{-1}^{3} |x-2| dx$ is the area under curve |x – 2|

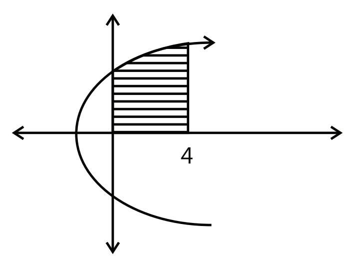
where x ∈ [–1, 3]

A = $\int_{-1}^{2} 2 - x + \int_2^3 x - 2 = 2x - \frac{x^2}{2}\Big|_{-1}^{2} + \frac{x^2}{2} - 2x\Big|_2^3$

⇒ A = 2 + 2 + $\frac{1}{2} + \frac{9}{2}$ – 6 – 2 + 4 = 5 sq. units



Sol 8: y = $\sqrt{3x+4}$



A = $\int_0^4 y dx = \int_0^4 \sqrt{3x+4} dx = \frac{2\left[(3x+4)^{3/2}\right]_0^4}{9x^3}$

= $\frac{2}{9}\left[16^{3/2} - 4^{3/2}\right] = \frac{2}{9}\left[4^3 - 2^3\right]$

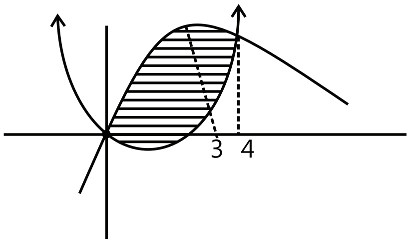
= $\frac{2}{9}\left[64 - 8\right] = \frac{56 \times 2}{9} = \frac{112}{9}$ sq. units

Sol 9: y = $\frac{3}{(1-2x)^3}$ above x axis & x ∈ [–4, –1]

$\int_{-4}^{-1} \frac{3dx}{(1-2x)^3} = \left[\frac{3}{(1-2x)^{+2}} \times \frac{1}{(-2)(-2)}\right]_{-4}^{-1}$

$= \frac{3}{4}\left[\frac{1}{(1-2x)^2}\right]_{-4}^{-1} = \frac{3}{4}\left[\frac{1}{3^2} - \frac{1}{9^2}\right] = \frac{3 \times 8}{4 \times 81} = \frac{2}{27}$ sq. units

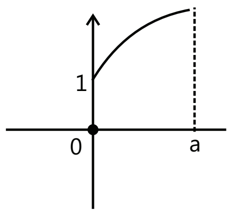
Sol 10:



Area = $\int_0^4 \left(6x - x^2 - x^2 + 2x\right)dx$

$= \left[8x - 2x^2\right]_0^4 = \left[4x^2 - \frac{2x^3}{3}\right]_0^4 = 64 - \frac{2}{3} \times 64 = \frac{64}{3}$ sq. units

Sol 11: $f(x) = 1 + \sqrt{x}$

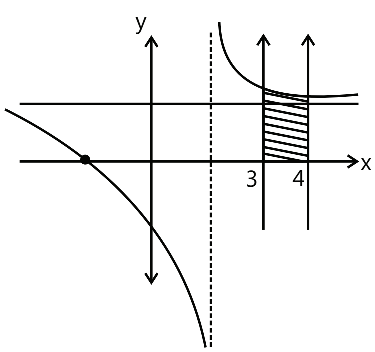


$\int_0^4 y\,dx = \int_0^4 \sqrt{x} + 1 = \left[\frac{2x^{3/2}}{3} + x\right]_0^4$

$= \frac{2}{3} \times 2^3 + 4 = \frac{16}{3} + 4 = \frac{28}{3}$ sq. units

Sol 12: $xy - 3x - 2y - 10 = 0$

$y = \frac{3x + 10}{x - 2}$



$A = \int_3^4 \frac{3x + 10}{x - 2}dx$ $A = \frac{3x - 6 + 16}{x - 2} = \int_3^4 \left(3 + \frac{16}{x - 2}\right)dx$

$= \left[3x + 16\ell n(x - 2)\right]_3^4$

$\left[12 + 16\ \ell\ n2 - 9 - \ln 1\right]$

= (3 + 16log 2) sq units.

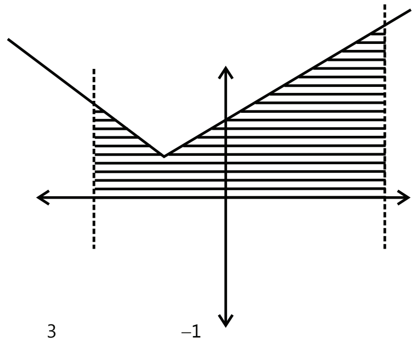
Sol 13: $\int y\,dx = \int_0^{\sqrt{\pi/2}} x \sin x^2 dx$

Substituting $x^2 = t$

$xdx = \frac{dt}{2}$

$\int_0^{\pi/2} \sin t \frac{dt}{2} = \left[-\frac{\cos t}{2}\right]_0^{\pi/2} = \frac{-0 + 1}{2} = \frac{1}{2}$ sq. units

Sol 14:

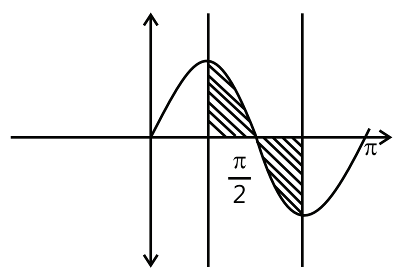


$\int_{-2}^{3} |x + 1| + 1 = \int_{-1}^{3} x + 1 + 1 + \int_{-2}^{-1} -1 - x + 1$

$= \left.\frac{x^2}{2} + 2x\right|_{-1}^{3} - \left[\frac{x^2}{2}\right]_{-2}^{-1}$

$= \frac{9}{2} + 6 - \frac{1}{2} + 2 - \frac{1}{2} + 2 = \frac{7}{2} + 10 = \frac{27}{2}$ sq. units

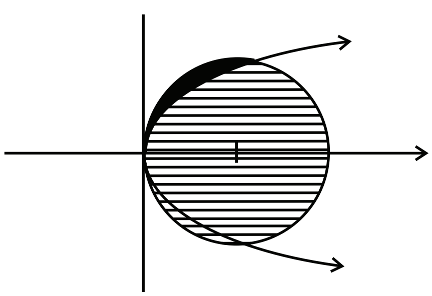
Sol 15:



Area = $2\int_{\pi/4}^{\pi/2} \sin 2x\,dx = \frac{2}{2}\left[-\cos 2x\right]_{\pi/4}^{\pi/2}$

= |+1 – 2| = 1 sq. units

Sol 16:



$x^2 - 2ax + y^2 \le 0$

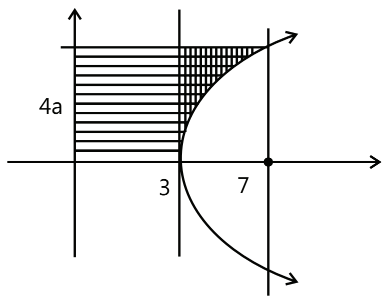
$y^2 - ax \ge 0$

$$A = \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax} \right) dx \Rightarrow A = \int_0^a \sqrt{2ax - x^2}\, dx - \int_0^a \sqrt{ax}\, dx$$

$$= \pi \frac{a^2}{4} - \frac{\sqrt{a}\, 2x^{3/2}}{3} \Bigg|_0^a$$

$$= \frac{\pi a^2}{4} - \frac{2\sqrt{a}}{3} a^{3/2} = \frac{\pi a^2}{4} - \frac{2a^2}{3} = \frac{a^2}{12}(3\pi - 8) \text{ sq. units}$$

Sol 17:



$y = 4a \Rightarrow 16a^2 = 4a^2(x - 3) \Rightarrow x = 7$

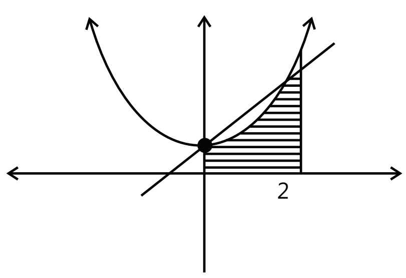
$$= \int_0^{4a} x\, dy = \int_0^{4a} \left(\frac{y^2}{4a^2} + 3 \right) dy = \left[\frac{y^3}{12a^2} + 3y \right]_0^{4a}$$

$$= \frac{(4a)^3}{12a^2} + 12a = \frac{64}{12} a + 12a$$

$$\Rightarrow \left(\frac{16 + 36}{3} \right) a = \frac{52a}{3}$$

$$A = \frac{52a}{3} - 3 \times 4a = \frac{16a}{3} \text{ sq. units}$$

Sol 18: The points of intersection of $y = x^2 + 3$ and $y = 2x + 3$ are (0, 3) and (2, 7).

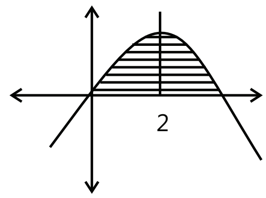


$$A_1 = \int_2^3 2x + 3 = x^2 + 3x \Big|_2^3 = \frac{27}{3} + 9 - 4 - 6 = 18 - 10 = 8$$

$$A_2 = \int_0^2 x^2 + 3 = \frac{x^3}{3} + 3x \Big|_0^2 = \frac{8}{3} + 6$$

$$A_1 + A_2 = \frac{8}{3} + 6 + 8 = \frac{50}{3} \text{ sq. units}$$

Sol 19: $y = 4x - x^2$



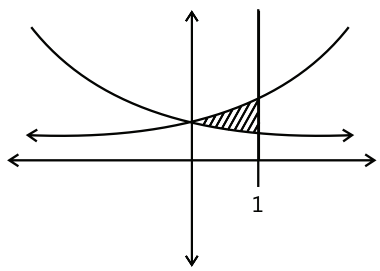
$$\int (4x - x^2) dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4$$

$$= 32 - \frac{64}{3} = \frac{32}{3} \text{ sq. units}$$

Exercise 2

Single Correct Choice Type

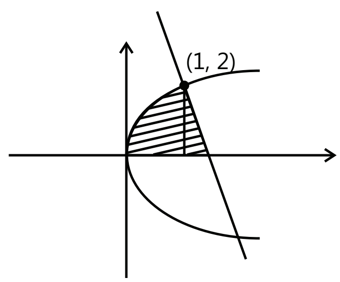
Sol 1: (C)



$$A = \int_0^1 \left(e^x - e^{-x} \right) dx = \left[e^x + e^{-x} \right]_0^1$$

$$= e + \frac{1}{e} - 1 - 1 = e + \frac{1}{e} - 2$$

Sol 2: (A)



Area = $A_1 + A_2$

$$A_1 = \int_0^1 \sqrt{4x}\, dx = 2 \frac{\left[x^{3/2} \right]_0^1}{3/2} = \frac{4}{3}$$

Equation normal $\frac{y - 2}{x - 1} = -1$

$\Rightarrow y - 2 = 1 - x$

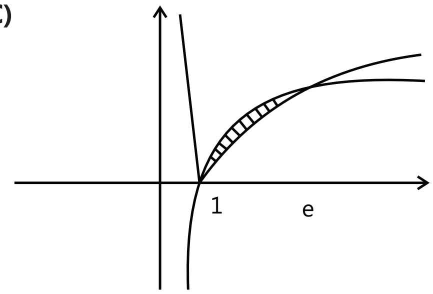
$x + y = 3$ ...... (i)

$$A_2 = \int_1^3 (3 - x) dx = \left[3x - \frac{x^2}{2} \right]_1^3 = 9 - \frac{9}{2} - 3 + \frac{1}{2}$$

$= 6 - 4 = 2$

$$\text{Area} = 2 + \frac{4}{3} = \frac{10}{3}$$

Sol 3: (C)



$$\mathbf{A} = \int_1^e \left(\ln x - (\ln x)^2 \right) dx \quad \int \ln dx = x \ln x - x$$

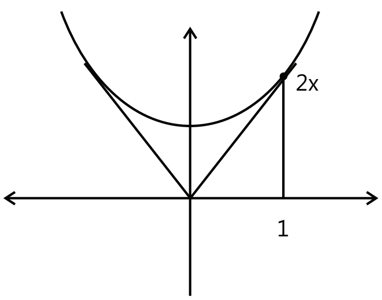
$$A = (x\ln x - x) - \left[\ln x \left(x\ln x - x \right) - \int (\ln x - 1) \right]$$

$$= x\ln x - x - x(\ln x)^2 + x\ln x + x\ln x - x - x$$

$$\left[-x(\ln x)^2 + 3x\ln x - 3x \right]_1^e$$

$$= -e + 3e - 3e + 3 = 3 - e$$

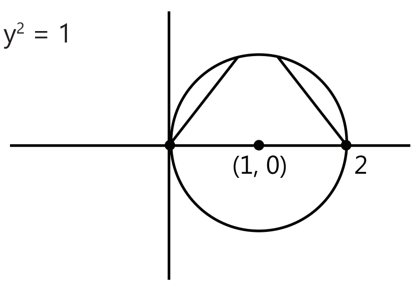
Sol 4: (A)



$$A = 2\int_0^1 \left(x^2 + 1 - 2x \right) dx = 2\int_0^1 (x-1)^2 dx = 2 \Rightarrow \left. \frac{(x-1)^3}{3} \right|_0^1 = \frac{2}{3}$$

Sol 5: (A) $(x - 1)^2 + y^2 = 1$

If $\sin\left(\frac{\pi x}{2} \right)$



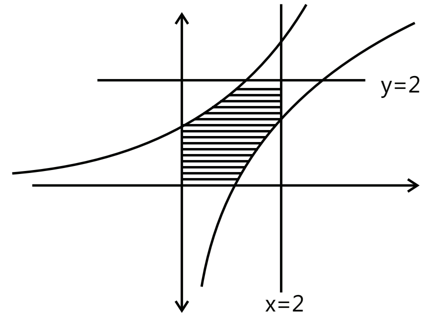
$$A = \int_0^2 \left(\sqrt{1 - (x-1)^2} - \sin\frac{\pi x}{2} \right) dx$$

$$= \left[\sqrt{2x - x^2} - \sin\frac{\pi x}{2} \right]_0^2$$

$$= \left[\frac{\sin^{-1}(x-1) + (x-1)\sqrt{2x - x^2}}{2} + \frac{2}{\pi}\cos\frac{\pi x}{2} \right]_0^2$$

$$= \frac{\pi}{2 \times 2} + \frac{2}{\pi}(-1) + \frac{\pi}{2.2} - \frac{2}{\pi} = \frac{\pi}{2} - \frac{4}{\pi}$$

Sol 6: (B)



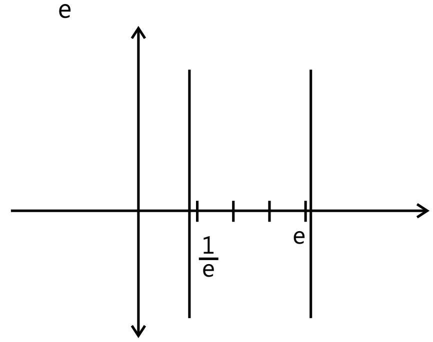
$$\text{Area} = \int_1^2 \ln x\, dx + \int_1^2 \ln y\, dy$$

$$= 2\int_1^2 \ln x = 2\left[x\ln x - x \right]_1^2 = 2[2\ln 2 - 2 + 1]$$

$$= 4\ln 2 - 2 = 2(2\ln 2 - 1)$$

Sol 7: (B) $y = x(1 - \ln x)$ & $x = \frac{1}{e}$

Between $x = \frac{1}{e}$ & e



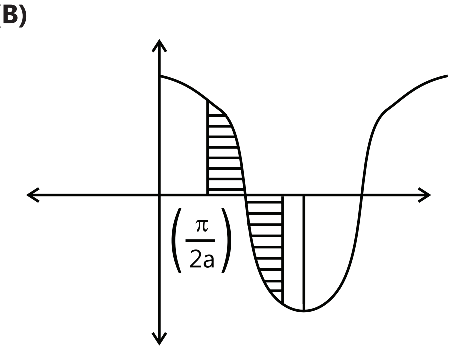
$$I = \int \left(x - x\ln x \right) dx = \frac{x^2}{2} - \frac{1}{2}\left[x^2\ln x - \frac{x^2}{2} \right]$$

$$I = \frac{3x^2}{2} - \frac{x^2}{2}\ln x$$

$$\Rightarrow A = \left[I \right]_{\frac{1}{e}}^{e}$$

$$A = \frac{3e^2}{4} - \frac{e^2}{2} - \left(\frac{3}{4e^2} + \frac{1}{2e^2} \right) = \frac{1}{4}e^2 - \frac{5}{4e^2}$$

Sol 8: (B)



$$\text{Area} = \int_{\pi/6a}^{\pi/2a} (\cos ax)\,dx = \frac{2}{a}[\sin ax]_{\pi/6a}^{\pi/2a}$$

$$= \frac{2}{a}\left[1 - \frac{1}{2}\right] = \frac{1}{a} > 3 \;= a \in \left(0, \frac{1}{3}\right)$$

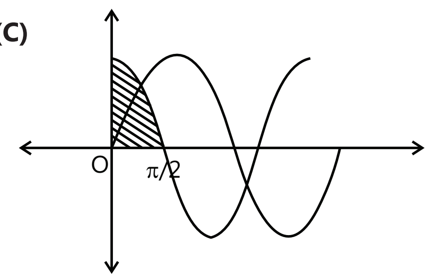
Sol 9: (D) $y = \frac{x}{6} + \frac{1}{x^2}; y = 0$

$$A = \int_a^{2a} \left(\frac{x}{6} + \frac{1}{x^2}\right) dx = \left[\frac{x^2}{12} - \frac{1}{x}\right]_a^{2a}$$

$$= \frac{a^2}{3} - \frac{1}{2a} - \frac{a^2}{12} + \frac{1}{a} \;=\; \frac{a^2}{4} + \frac{1}{2a}$$

A_{least} when $A' = \frac{a}{2} - \frac{1}{2a^2} = 0 \Rightarrow a = 1$

Sol 10: (C)



$$\text{Area}_1 = \left[\int_0^{\pi/4} \sin x\right] 2 \;=\; 2[-\cos x]_0^{\pi/4}$$

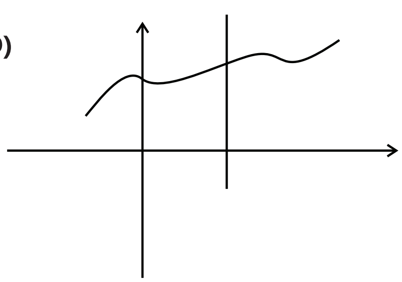
$$= 2\left[-\frac{1}{\sqrt{2}} + 1\right] \;=\; \left(\sqrt{2} - 1\right)\sqrt{2} = 2 - \sqrt{2}$$

$$\text{Area}_2 = \int_0^{\pi/2} \cos x - \text{area}_1 = [\sin x]_0^{\pi/2} - 2 + \sqrt{2}$$

$$= 1 - 2 + \sqrt{2} \;=\; \sqrt{2} - 1$$

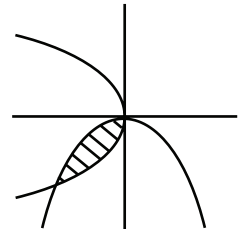
ratio is $\frac{\sqrt{2}\left(\sqrt{2} - 1\right)}{\sqrt{2} - 1} = \sqrt{2}$

Sol 11: (D)



$$\int_0^{x_1} f(x) = x_1 e^{x_1} \Rightarrow f(x) = xe^x + e^x$$

Sol 12: (B)



$$\text{Area} = \int_{-1}^{0} -\sqrt{-x} - \left(-x^2\right) \;=\; \int_{-1}^{0} -\sqrt{-x} + x^2$$

$$= \left[\frac{x^3}{3} + \frac{2\left(-x\right)^{3/2}}{3}\right]_{-1}^{0} \;=\; \left|\frac{1}{3} - \frac{2}{3}\right| = \frac{1}{3}$$

Sol 13: (A) $\int_1^x f(x)\,dx = \sqrt{1 + 3x} - 2$

$$f(x) = \frac{3}{2\sqrt{1 + 3x}}$$

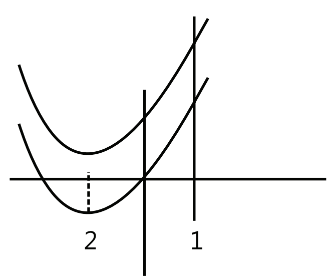
$x = 1, y = \frac{3}{4}$ and $x = 8, y = \frac{3}{10}$

$$A = \frac{\int_1^8 y\,dx}{\Delta x} = \frac{\sqrt{1 + 3.8} - 2}{8 - 1} = \frac{3}{7}$$

Sol 14: (A) $\frac{dy}{dx} = 2x + 1$

$y = x^2 + x + c$

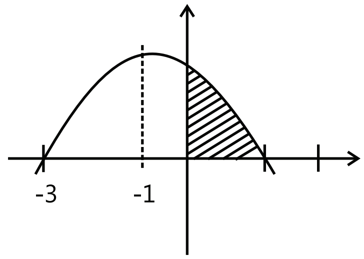
$(1, 2)$; $c = 0$



$y = x^2 + x + 1$

$$A = \int_0^1 x^2 + x = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{6}$$

Sol 15: (C)

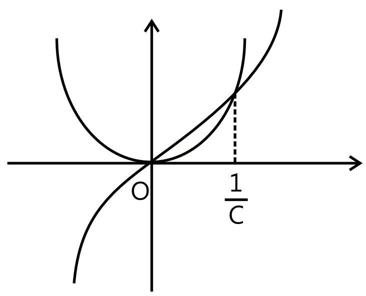


$0 \le y < 3 - 2x - x^2$

$0 < y < 4 - (x + 1)^2$

$$= \int_0^1 (3 - 2x - x^2)\,dx$$

Sol 16: (C)

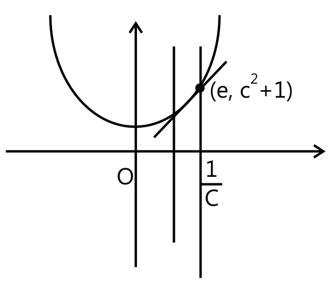


$$\int_0^{\frac{1}{c}} \left(x^2 - cx^3\right)dx = \frac{2}{3}$$

$$\Rightarrow \left[\frac{x^3}{3} - \frac{cx^4}{4}\right]_0^{1/c} = \frac{1}{3c^3} - \frac{1}{4c^3} = \frac{2}{3}$$

$$\Rightarrow \frac{1}{12c^3} = \frac{2}{3} \Rightarrow c = \frac{1}{2}$$

Sol 17: (C)



$$\frac{y - c^2 - 1}{x - c} = 2c$$

$\Rightarrow y = 2cx - c^2 + 1$

$x = 1, \quad y = 2c - c^2 + 1$

$x = 2, \quad y = 4c - c^2 + 1$

$\Rightarrow$ Area of trapezoid $= \frac{1}{2}\left[6c - 2c^2 + 2\right] \times 1 = 3c - c^2 + 1$

For $\text{area}_{max} = A' = 3 - 2c = 0 \Rightarrow C = \frac{3}{2}$

$$\left(\frac{3}{2}, \frac{13}{4}\right)$$

Sol 18: (C) $A_1 = \frac{1}{2} \times a^2 \times 2a = a^3$

$$\Rightarrow = \int_{-a}^{a} x^2 = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

$$\Rightarrow_2 = 2a \times a^2 - \frac{2a^3}{3} = \frac{4a^3}{3}$$

$$\Rightarrow \frac{A_1}{A_2} = \frac{3}{4}$$

Previous Years' Questions

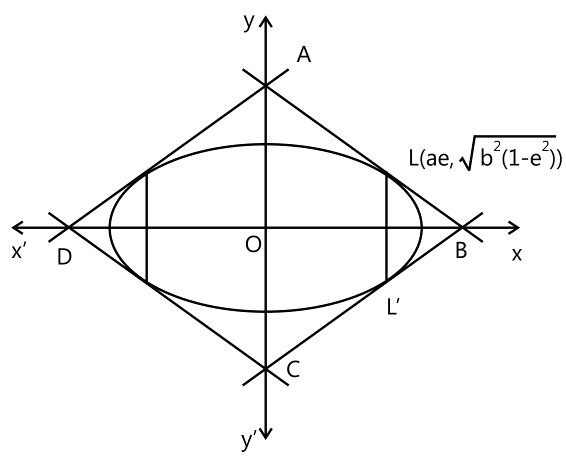
Sol 1: (D) Given, $\frac{x^2}{9} + \frac{y^2}{5} = 1$

To find tangents at the end points of latus rectum, we find ae.

i.e. $ae = \sqrt{a^2 - b^2} = \sqrt{4} = 2$

and $\sqrt{b^2(1 - e^2)} = \sqrt{5\left(1 - \frac{4}{9}\right)} = \frac{5}{3}$

By symmetry, the quadrilateral is a rhombus.



So, area is four times the area of the right angled triangle formed by the tangent and axes in the 1st quadrant.

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is

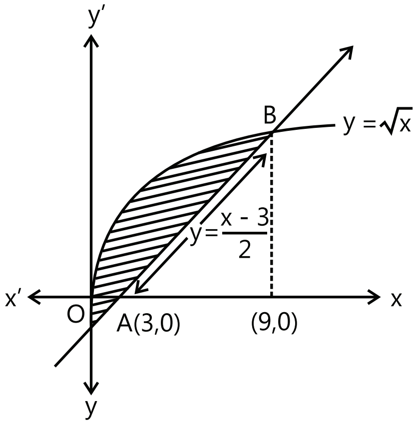
$$\frac{2}{9}x + \frac{5}{3}.\frac{y}{5} = 1 \Rightarrow \frac{x}{9/2} + \frac{y}{3} = 1$$

$\therefore \quad \frac{x}{9/2}+\frac{y}{3}=1$

$\therefore$ Area of quadrilateral ABCD

$= 4$ (area of ΔAOB)$= 4\left(\frac{1}{2}.\frac{9}{2}.3\right)$ = 27 sq. units

Sol 2: (A) To find the area between the curves, $y=\sqrt{x}, 2y+3=x$ and x-axis in the 1st quadrant (we can plot the above condition as) ;



Area of shaded portion OABO

$= \int_0^9 \sqrt{x}\,dx - \int_3^9 \left(\frac{x-3}{2}\right)dx$

$= \left(\frac{x^{3/2}}{3/2}\right)_0^9 - \frac{1}{2}\left(\frac{x^2}{2}-3x\right)_3^9$

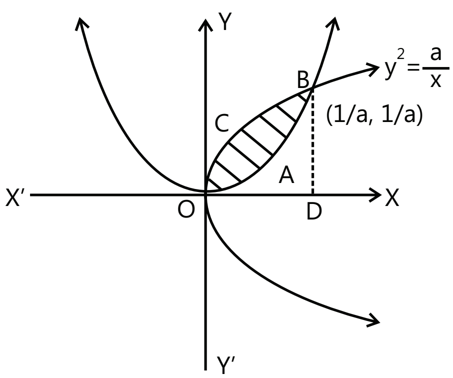
$= \left(\frac{2}{3}.27\right) - \frac{1}{2}\left\{\left(\frac{81}{2}-27\right)-\left(\frac{9}{2}-9\right)\right\}$

$= 18 - \frac{1}{2}(18) = 9$ sq. unit

Sol 3: (A) As from the figure, area enclosed between the curves is OABCO.

Thus, the point of intersection of

$y = ax^2$ and $x = ay^2$



$\Rightarrow x = a\,(ax^2)^2$

$\Rightarrow x = 0, \frac{1}{a} \Rightarrow y = 0, \frac{1}{a}$

$\therefore$ Point of intersection are (0, 0) and $\left(\frac{1}{a},\frac{1}{a}\right)$

Thus, required area OABCO = Area of curve OCBDO – area of curve OABDO

$\Rightarrow \int_0^{1/a}\left(\sqrt{\frac{x}{a}}-x^2\right)dx = 1$ (given)

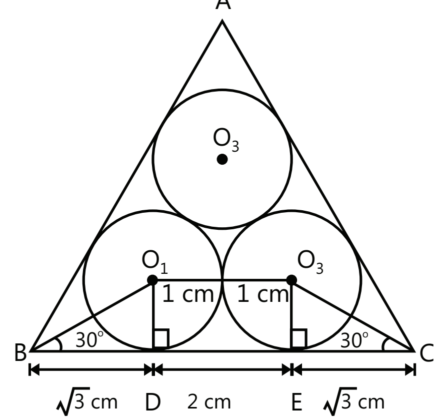
$\Rightarrow \left[\frac{1}{\sqrt{a}}.\frac{x^{3/2}}{3/2}-\frac{ax^3}{3}\right]_0^{1/a} = 1$

$\Rightarrow \frac{2}{3a^2}-\frac{1}{3a^2}=1$

$\Rightarrow a^2 = \frac{1}{3} \Rightarrow d = \frac{1}{\sqrt{3}}$ ($\because$ a > 0)

Sol 4: (A) Since, tangents drawn from external point to the circle subtends equal angle at the centre

$\therefore \angle O_1BD = 30°$



In ΔO_1BD, $\tan 30° = \frac{O_1D}{BD}$

$\Rightarrow BD = \sqrt{3}$ cm

Also, DE = O_1O_2 = 2 cm and EC = $\sqrt{3}$ cm

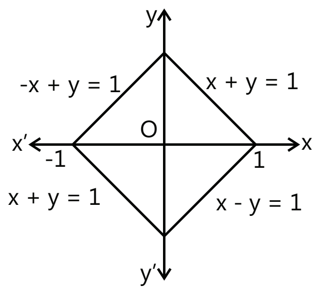
Now, BC = BD + DE + EC = $2+2\sqrt{3}$

$\Rightarrow$ Area of ΔABC

$= \frac{\sqrt{3}}{4}(BC)^2 = \frac{\sqrt{3}}{4}.\,4(1+\sqrt{3})^2$

$= (6+4\sqrt{3})$ sq. cm.

Sol 5: The area formed by |x| + |y| = 1 is square shown as below,



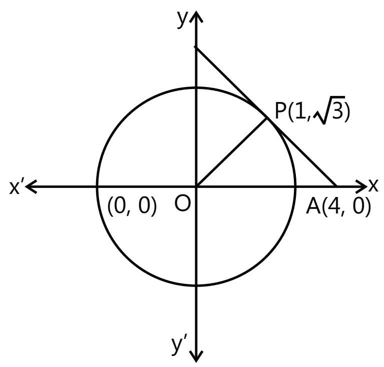
$\therefore$ Area of square = $(\sqrt{2})^2$ = 2 sq. units

Sol 6: Equation of tangent at the point $(1, \sqrt{3})$ to the curve

$x^2 + y^2 = 4$

is $x + \sqrt{3}y = 4$,

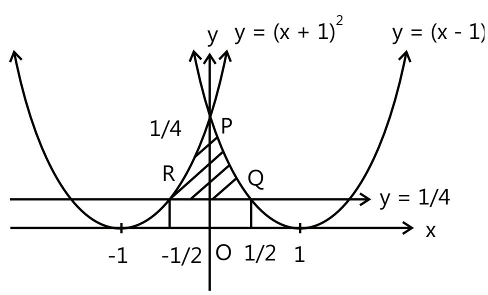
whose x-axis intercept (4, 0)



Thus, area of Δ formed by (0, 0) $(1, \sqrt{3})$ and (4, 0)

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & \sqrt{3} & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2}|(0 - 4\sqrt{3})| = 2\sqrt{3} \text{ sq. units}$$

Sol 7: (A) The curves y = $(x - 1)^2$, y = $(x + 1)^2$ and y = 1/4 are shown are



Where points of intersection are

$(x - 1)^2 = \frac{1}{4} \Rightarrow x = \frac{1}{2}$

and $(x + 1)^2 = \frac{1}{4} \Rightarrow -\frac{1}{2}$

$\therefore Q\left(\frac{1}{2}, \frac{1}{4}\right)$ and $R\left(-\frac{1}{2}, \frac{1}{4}\right)$

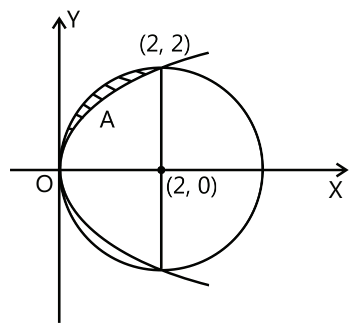
$\therefore$ Required area

$$= 2\int_0^{1/2}\left[(x-1)^2 - \frac{1}{4}\right]dx = 2\left[\frac{(x-1)^3}{3} - \frac{1}{4}x\right]_0^{1/2}$$

$$= 2\left[-\frac{1}{8.3} - \frac{1}{8} - \left(-\frac{1}{3} - 0\right)\right] = \frac{8}{24} = \frac{1}{3} \text{ sq. unit}$$

Sol 8: (A) Region $(x, y): y^2 \geq 2x$

$x^2 + y^2 < 4x$ and $x \geq 0$, $y \geq 0$



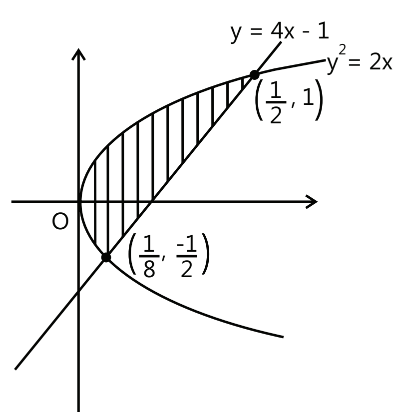
$$\text{Area } = \frac{1}{4}\pi(2)^2 - \int_0^2 \sqrt{2x}\, dx$$

$$= \pi - \left[\sqrt{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \pi - \frac{2\sqrt{2}}{3}\left[2^{\frac{3}{2}}\right]$$

$$= \pi - \frac{2\sqrt{2}}{3} \times 2\sqrt{2}$$

$$= \pi - \frac{8}{3} \text{ sq. units}$$

Sol 9: (C) Region $(x, y): y^2 < 2x$ and $y \geq 4x - 1$



$$\text{Area} = \int_{-1/2}^{1}\left(\frac{y+1}{4}-\frac{y^2}{2}\right)dy$$

$$=\left[\frac{y^2}{8}+\frac{y}{4}-\frac{y^3}{6}\right]_{-1/2}^{1} = \left[\frac{1}{3}+\frac{1}{4}-\frac{1}{6}-\frac{1}{32}+\frac{1}{8}-\frac{1}{48}\right] = \frac{9}{32}$$

Sol 10: (D) The point of intersection $x-3=2\sqrt{x}$

$\Rightarrow (x-3)^2 = 4x$

$\Rightarrow x^2-10x+9=0$

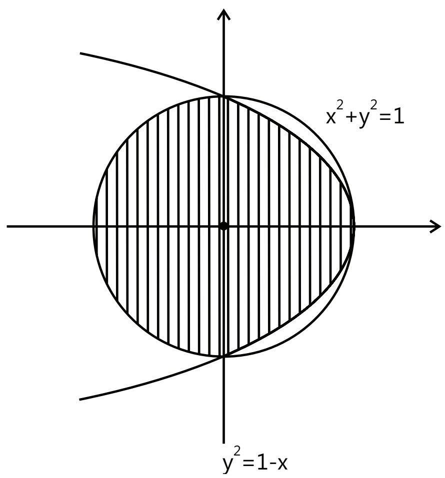
$\Rightarrow x=1,9$

$\Rightarrow y=1,3$

$$\text{Area} \int_0^3\left[2y+3-y^2\right]dy = \left[y^2+3y-\frac{y^3}{3}\right]_0^3$$

$=\left[9+9-9\right]=9$ sq. units

Sol 11: (A)



Area = Area of half circle

$$+2\int_0^1\sqrt{1-x}\,dx = \frac{1}{2}\pi(1)^2+2\left[-\frac{(1-x)^{3/2}}{3/2}\right]_0^1$$

$$=\frac{\pi}{2}+2\left[\frac{2}{3}\right]=\frac{\pi}{2}+\frac{4}{3}$$

Sol 12: (C) $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=m$

$\Rightarrow$ Any point on this line is $(2m+1, 3m-1, 4m+1)$

Similarly for $\frac{x-3}{1}=\frac{y-k}{2}=\frac{2}{1}=n$

$(n+3, 2n+k, n)$

It they intersect, then

$2m+1=n+3 \Rightarrow 2m-n=2$

$3m-1=2n+k \Rightarrow 3m-2n=k+1$

$4m+1=n \Rightarrow 4m-n=-1$

On solving these equations, we get

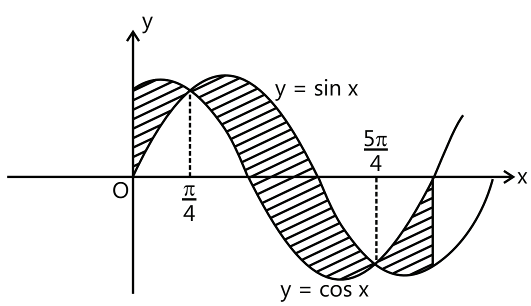
$m=-\frac{3}{2}, n=-5 \Rightarrow K=\frac{9}{2}$

Sol 13: (C) The required area

$$=2\left[\int_0^2\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy\right] = 2\left[\int_0^2\frac{5\sqrt{y}}{2}dy\right] = 5\int_0^2\sqrt{y}\,dy$$

$$=5\left[\frac{y^{3/2}}{3/2}\right]_0^2 = \frac{10}{3}\left[2^{3/2}\right] = \frac{10\times 2\sqrt{2}}{3} = \frac{20\sqrt{2}}{3}$$

Sol 14: (D)



$$\text{Area} = \int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{5/4}(\sin x-\cos x)dx$$

$$+\int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}(\cos x-\sin x)dx$$

$$=\left[\sin x+\cos x\right]_0^{\pi/4}+\left[-\cos x-\sin x\right]_{\pi/4}^{5\pi/4}$$

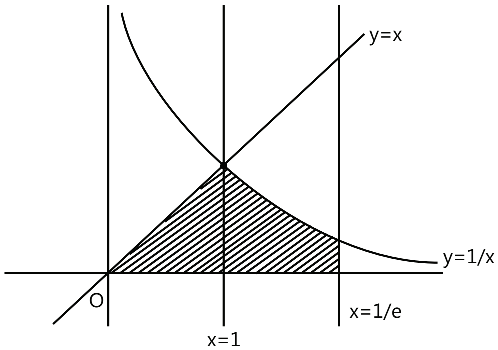
$$+\left[\sin x+\cos x\right]_{5\pi/4}^{3\pi/2}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1+\left[\frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$+\left[-1+0+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right]$$

$$=4\sqrt{2}-2$$

Sol 15: (B)



Area $\left|\int_0^1 xdx\right| + \left|\int_1^{1/e}\frac{1}{x}dx\right|$

$= \left[\frac{x^2}{2}\right]_0^1 + \left|[\ln x]_1^{1/e}\right| = \frac{1}{2} + |\ln 1/e|$

$= \frac{1}{2} + 1 = \frac{3}{2}$ sq. units

JEE Advanced/Boards

Exercise 1

Sol 1: $A_1 = \int_0^a \sin x - f(x)$

$= -\cos x|_0^a - \int_0^a f(x) = 1 - \cos a - \int_0^a f(x)$

$\Rightarrow A_2 = \int_a^\pi f(x) - \sin x = \int_a^\pi fx + \cos x|_a^\pi = \int_a^\pi f(x) - 1 - \cos a$

$\Rightarrow A_3 = \int_\pi^{2\pi} \sin x - f(x) = -\cos x|_\pi^{2\pi} - \int_\pi^{2\pi} f(x) = -2 - \int_\pi^{2\pi} f(x)$

$\Rightarrow A_1 = 1 - \sin a + a\cos a - \cos a$

$= 1 - \cos a - \int_0^a f(x)\,dx$

$\Rightarrow -a\cos a + \sin a = \int_0^a f(x)\,dx = \sin a - a\cos a$

$\Rightarrow f(x) = x\sin x$

$\int_a^\pi f(x)dx = \int_a^\pi x\sin x\,dx = [-x\cos x + \sin x]_a^\pi$

$= \pi - [-a\cos a + \sin a] = \pi + a\cos a - \sin a$

$\Rightarrow A_2 = \pi + a\cos a - \sin a - 1 - \cos a$

$\int_\pi^{2\pi} f(x)dx = \int_a^\pi x\sin x\,dx = [-x\cos x + \sin x]_\pi^{2\pi}$

$= -2\pi - (\pi) = -3p$

$\Rightarrow A_3 = (3\pi - 2)$ sq units.

$x\sin x = \sin x \Rightarrow x = 1$

$\Rightarrow a = 1 \Rightarrow A_2 = \pi - 1 - \sin a$

$\Rightarrow A_1 = 1 - \sin a$

Sol 2: $A = \int_0^t \sin x^2 dx$, $B = \frac{1}{2}\times t\sin t^2$

Now, $\lim_{t\to 0}\frac{A}{B} = \frac{\int_0^t \sin x^2 dx}{\frac{1}{2}\times t\sin t^2}$

$\frac{0}{0}$ form $\Rightarrow \lim_{t\to 0} = \frac{\sin t^2}{\frac{t}{2}(2t)\cos t^2 + \frac{1}{2}\sin t^2}$

[L' Hospital's rule]

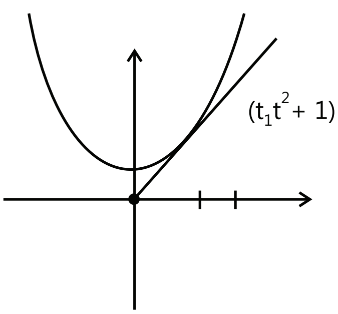
$= \frac{2\sin t^2}{2t^2\cos t^2 + \sin t^2} = \frac{2\tan t^2}{2t^2 + \tan t^2}$

$= \frac{\frac{2\tan t^2}{t^2}}{2 + \frac{\tan t^2}{t^2}} = \frac{2}{2+1} = \frac{2}{3}$

Sol 3: $f(x + 1) = f(x) + 2x + 1$

$f(0) = 1$

$f(1) = 2 \Rightarrow f(x) = x^2 + 1$



$f(-1) = 2; f(2) = 5; f(-2) = 5$

$y = x^2 + 1$

Equation of tangent at $P(t, t^2 + 1)$ $y - (t^2 + 1) = 2t(x - t)$

Since, it passes through (0, 0)

$-t^2 - 1 = 2t(-t)$

$\Rightarrow t^2 = t \qquad \Rightarrow t = \pm 1$

$\Rightarrow y = \pm 2x$

$$\text{Area} = 2\int_0^1 \left(x^2 + 1 - 2x\right) dx$$

$$= 2\int_0^1 (x-1)^2 dx = \frac{2}{3}(x-1)^3\Big|_0^1 = \frac{2}{3}[0+1] = \frac{2}{3} \text{ sq. units}$$

Sol 4: $y = \dfrac{\ln x - c}{x} = 0 \Rightarrow x = e^c$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - (\ln x - c)}{x^2} = 0$$

$\Rightarrow 1 - \ln x + c = 0$

$\Rightarrow \ln x = c + 1$

$\Rightarrow x = e^{c+1}$

$$\Rightarrow \int_{e^c}^{e^{c+1}} \frac{\ln x - c}{x} dx = \int_{e^c}^{e^{c+1}} \frac{\ln x}{x} dx - \int_{e^c}^{e^{c+1}} \frac{c}{x}\, dx \sqrt{b^2 - 4ac}$$

$$= \left[\frac{(\ln x)^2}{2}\right]_{e^c}^{e^{c+1}} - c\left[\ln x\right]_{e^c}^{e^{c+1}}$$

$$= \frac{(c+1)^2 - c^2}{2} - c[c+1-c] = \frac{2c+1}{2} - [c] = \frac{1}{2}$$

Sol 5: $y = x^n$

$y - 1 = n(x - 1)$

$$A = \int_0^1 x^n dx - \int_{1-\frac{1}{n}}^{1} [n(x-1)+1] dx$$

$$= \frac{\left[x^{n+1}\right]_0^1}{n+1} - \left[n\frac{(x-1)^2}{2} + x\right]_{1-\frac{1}{n}}^{1}$$

$$= \frac{1}{n+1} - \left[1 - \frac{n}{2n^2} - 1 + \frac{1}{n}\right] = \frac{1}{n+1} - \frac{1}{2n}$$

$$\Rightarrow = \frac{n-1}{2n(n+1)}$$

$\dfrac{dA}{dn} = 0 \Rightarrow 2n(n + 1) - (n - 1)(4n + 2) = 0$

$\Rightarrow 2n^2 + 2n - 4n^2 - 2n + 4n + 2 = 0$

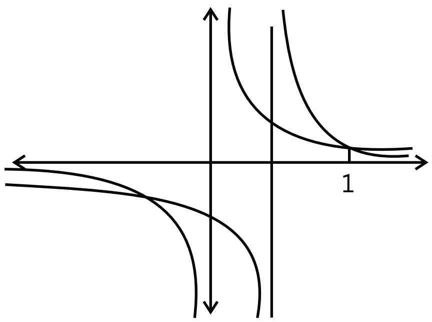
$\Rightarrow -2n^2 + 4n + 2 = 0$

$\Rightarrow n^2 - 2n - 1 = 0$

$$n = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$\Rightarrow > 1$ i. e. $n = \sqrt{2} + 1$

Sol 6:



$$A = \int_2^a \frac{1}{x} - \frac{1}{2x-1} = \left[\ln x - \frac{\ln(2x-1)}{2}\right]_2^a$$

$$= \ln a - \frac{\ln(2a-1)}{2} - \ln 2 + \frac{\ln 3}{2}$$

$$= \ln \frac{3a}{2(2)(2a-1)} \times 2 = \ln \frac{3a}{4a-2} - \ln \frac{4}{5}$$

$\Rightarrow 15a = 16a - 8 \qquad \Rightarrow a = 8$

Sol 7: $f(x) = \dfrac{1}{1+x^2}$

We have, $A(\alpha, f(\alpha))$ and $B\left(-\frac{1}{\alpha}, f\left(-\frac{1}{\alpha}\right)\right)$

$OA \to y = \dfrac{x}{\alpha(1+\alpha^2)}$ .... (i)

$OB \to y = \dfrac{-\alpha^3}{1+\alpha^2}(x)$

$$\int_0^\alpha \left(\frac{1}{1+x^2} - \frac{x}{\alpha(1+\alpha^2)}\right) dx + \int_{-\frac{1}{\alpha}}^{0} \left(\frac{1}{1+x^2} + \frac{\alpha^3}{1+\alpha^2}x\right) dx$$

$$= \left[\tan^{-1} x - \frac{x^2}{2\alpha(1+\alpha^2)}\right]_0^\alpha + \left[\tan^{-1} x + \frac{\alpha^3 x^2}{2(1+\alpha^2)}\right]_{-\frac{1}{\alpha}}^{0}$$

$$= \tan^{-1}\alpha - \frac{\alpha}{2(1+\alpha^2)}$$

$$+ \left[0 + 0 - \tan^{-1}\left(-\frac{1}{\alpha}\right) - \frac{\alpha}{2(1+\alpha^2)}\right]$$

$$= \tan^{-1}\alpha - \tan^{-1}\left(-\frac{1}{\alpha}\right) - \frac{(\alpha+\alpha)}{2(1+\alpha^2)}$$

$$= \tan^{-1}\frac{\alpha^2 - 1}{1-1} - \frac{(\alpha+\alpha)}{2(1+\alpha^2)}$$

$$= \frac{\pi}{2} - \frac{(\alpha+\alpha)}{2(1+\alpha^2)} = \frac{\pi}{2} - \frac{2\alpha}{2(1+\alpha^2)}$$

$A' = 0 \Rightarrow \alpha = 1$

$A = \frac{\pi}{2} - \frac{1}{2}$

Sol 8: The curve is $y = \sin^{-1} x$, i.e., x = sin y. This is a standard curve. Lines x = 0, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are the y-axis and two lines parallel to the x-axis at a distance $\frac{\pi}{2}$, one above and the other below the x-axis respectively.

Hence, the shaded part is the required area Δ. By symmetry of the curve and the lines,

ar (OABO) = ar (OCDO)

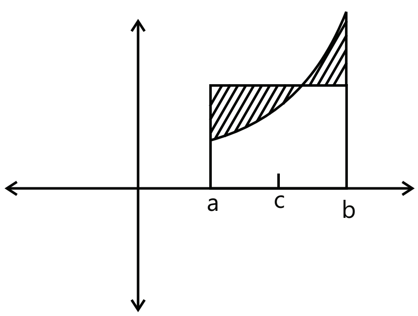
$\therefore \Delta = 2 \times \text{ar}(\text{OABO})$

$= 2\int_0^{\frac{\pi}{2}} (x)_{\text{curve}} dy = 2\int_0^{\frac{\pi}{2}} \sin y \, dy,$

($\because$ The equation of the curve is x = sin y)

$\therefore \Delta = 2\left[-\cos y\right]_0^{\frac{\pi}{2}} = 2\left[0+1\right] = 2$

Sol 9:



$A = \left(\int_a^c f(c) - f(x)\right) dx + \int_c^b \left(f(x) - f(c)\right) dx \; A$

$= f(c)(c-a) - \int_a^c f(x)\, dx + \int_c^b f(x)\, dx - f(c)(b-c)$

$A = f(c)\,[c - a - b + c] - \int_a^c f(x)\, dx + \int_c^b \left(f(x)\right) dx$

This is minimum when 2c – a – b = 0

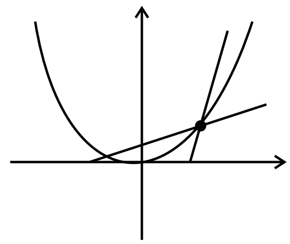
$\Rightarrow c = \frac{a+b}{2}$

$\int f(x) = \int_0^2 \left(\frac{x^3}{3} - x^2 + a\right) dx = \left[\frac{x^4}{12} - \frac{x^3}{3} + ax\right]_0^2$

$= \frac{16}{12} - \frac{8}{3} + 2a = \left(2a - \frac{4}{3}\right)$

Is minimum $\Rightarrow 2a = \frac{4}{3} \Rightarrow a = \frac{2}{3}$

Sol 10:



$y - c = m(x - 1)$

$A = \int \left(x^2 - m(x-1) - c\right) dx$

$= \left[\frac{x^3}{3} - \frac{m(x-1)^2}{2} - cx\right]_a^b$

$= \left[\frac{b^3 - a^3}{3} - \frac{m}{2}\left[(b-1)^2 - (a-1)^2\right] - c(b-a)\right]$

$= \frac{(b-a)(b^2 + a^2 + ab)}{3}$

$-\frac{m}{2}\left[b^2 - a^2 - 2b + 2a\right] - c(b-a)$

$c + mx - m = x^2$

$\Rightarrow x^2 - mx + m - c = 0$

$x_1 + x_2 = m$

$x_1 x_2 = m - c$

$\Rightarrow x_1 - x_2 = \sqrt{m^2 - 4m + 4c}$

$\Rightarrow (b - a)\left[\frac{m^2 - m + c}{3} - \frac{m}{2}[m-2] - c\right]$

$\Rightarrow (b - a)\left[\frac{m^2 - m + c}{3} - \frac{m^2}{2} - \frac{3c}{3} + \frac{3m}{3}\right]$

$\Rightarrow (b - a)\left[-\frac{m^2}{6} - \frac{2c}{3} + \frac{2m}{3}\right]$

$A = +\frac{1}{6}\,[m^2 + 4c - 4m]^{3/2}$

$\boxed{m^2 - 4m + 4c = 36}$

$B^2 - 4AC = 0$

$16-4(4c-36)=0$

$\Rightarrow c = 10 \Rightarrow m = 2$

$\Rightarrow c^2 + m^2 = 104$

Sol 11: $A_1 + A_2$ for any point $P(t, t^2)$

$$\int_0^k C_2 dx - C + \int_k^t t^2 dx - C = \int_0^t (C - C_1) dx$$

$$\int_0^k C_2 dx - \left[\frac{x^3}{3}\right]_0^k + \left[\frac{-x^3}{3}\right]_k^t + t^2(t-k) = \frac{t^3}{6}$$

$$\int_0^k C_2 dx - \frac{k^3}{3} + \frac{k^3}{3} - \frac{t^3}{3} + t^3 - t^2 k = \frac{t^3}{6}$$

$$\int_0^k C_2 dx = -\frac{t^3}{2} + t^2 k$$

Let $C_2 = f(x)\ l^2x^2$

$$\int_0^{t/\lambda} (\lambda^2 x^2) dx = \frac{t^3}{2} + t^2 k$$

$$\frac{\lambda^2 \left[x^3\right]_0^{t/\lambda}}{3} = -\frac{t^3}{2} + \frac{t^3}{\lambda} \Rightarrow \frac{\lambda^2}{3}\frac{t^3}{\lambda^3} = -\frac{t^3}{2} + \frac{t^3}{\lambda}$$

$$\Rightarrow \frac{10}{2} = \frac{1}{\lambda} - \frac{1}{3\lambda} = \frac{2}{3\lambda} \Rightarrow \lambda = \frac{4}{3}$$

$$\Rightarrow c_2 = \frac{16x^2}{9}$$

Sol 12: (i) Let equation of CD be $y = 2x+c$

For intersection with $y = x^2$

$\Rightarrow x^2 = 2x + c;\ x^2 - 2x - c = 0$

$\Rightarrow x_1 + x_2 = 2$ and $x_1 x_2 = -c$

Length of CD $= |x_1 + x_2|\sqrt{5}$

$= 2\sqrt{5}\sqrt{1+c}$

Length of $AC = BD = \dfrac{c+17}{\sqrt{5}}$

Given ABCD is square than,

$$2\sqrt{5}\sqrt{1+c} = \frac{c+17}{\sqrt{5}}$$

$\Rightarrow c^2 - 66c + 189 = 0$

$\Rightarrow c = 3, 63$

Therefore, least value of c is 3.

(ii) For maximum Area of sq. ABCD

Length $2\sqrt{5}\sqrt{1+c}$ must be maximum

For c = 63 (From Previous questions)

Area $= \left(2\sqrt{5}\sqrt{1+c}\right)^2$

$= \left(2\sqrt{5}\sqrt{1+63}\right)^2 = 4 \times 5 \times 64$

= 1280 sq. units.

(iii) Area bounded by $y = 2x + 3$ and

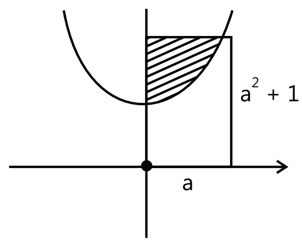
$$\text{Area} = \int_{-1}^{3} \left(2x + 3 - x^2\right) dx$$

$$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

$$= 9 + 9 - 9 - 1 + 3 - \frac{1}{3}$$

$$= 11 - \frac{1}{3} = \frac{32}{3} \text{ sq. units}$$

Sol 13:



Area of rectangle = $a(a^2 + 1)$

$$A = \int_1^{a^2+1} \sqrt{y-1}\, dy = \left[\frac{2(y-1)^{3/2}}{3}\right]_1^{a^2+1}$$

$$= \frac{2}{3}a^3 - 0 = \frac{2}{3}a^3$$

$$= \frac{2}{3}a^3 = \frac{1}{2}a(a^2+1) = \frac{4}{3}a^3 = a^3 + a$$

$a^3 = 3a \Rightarrow a = \sqrt{3}, -\sqrt{3}, 0$

$\Rightarrow a = \sqrt{3}$

Sol 14: $f^3(x) = \int_0^x t f^2(t)\, dt$

$\Rightarrow f'(x)\, 3f^2(x) = x f^2(x)$

$\Rightarrow f^2(x)[3f'(x) - x] = 0$

$\Rightarrow f'(x) = \dfrac{x}{3} \Rightarrow f(x) = \dfrac{x^2}{6} + c$

$\Rightarrow f(x) = \dfrac{x^2}{6} + c$

$$\left[\frac{x^2}{6}+c\right]^3 = \int_0^x t\left(\frac{t^2}{6}+c\right)^2 dx$$

$$\frac{x^6}{216}+c^3+\frac{3cx^4}{36}+\frac{3x^2c^2}{6}$$

$$= \int_0^x \left[\frac{t^4}{36}+c^2+\frac{ct^2}{3}\right] t\ dx = \int_0^x \left[\frac{t^5}{36}+c^2 t+\frac{ct^3}{3}\right] dx$$

$$= \left[\frac{t^6}{216}+\frac{c^2t^2}{2}+\frac{ct^4}{12}\right]_0^x = \frac{x^6}{216}+\frac{c^2x^2}{2}+\frac{cx^4}{12} \Rightarrow c = 0$$

$$\Rightarrow f(x) = \frac{x^2}{6}$$

Hence, $\int_0^3 f(x)dx = \left[\frac{x^3}{18}\right]_0^3 = \frac{3}{2}$

Sol 15: Given curves are $y^2 = x+1$ .... (i)

and $y^2 = -x+1$ or $y^2 = -(x-1)$ .... (ii)

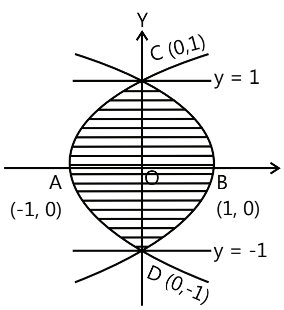
Curve (i) is the parabola having axis y = 0 and vertex (-1, 0).

Curve (ii) is the parabola having axis y = 0 and vertex (1, 0)

(1) - (2) 2x = 0 x = 0

From (i), x = 0 $\Rightarrow y = \pm 1$

Required area = area ACBDA



$$= \int_{-1}^{1} (x_1 - x_2)\ dy$$

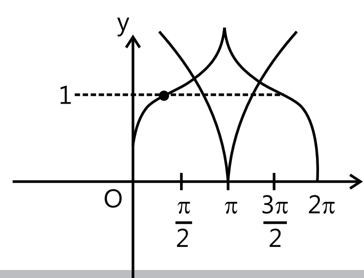
$$= \int_{-1}^{1} \left[(1-y^2)-(y^2-1)\right] dy$$

$$= 2\int_{-1}^{1} (1-y^2)\ dy$$

$$= 2\left[y-\frac{y^3}{3}\right]_{-1}^{1} = 2\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= \frac{8}{3} \text{ sq. units}$$

Sol 16:



$$A = 2\int_0^{\pi/2} \sin x + 2\int_{\pi/2}^{\pi} 2-\sin x$$

$$= 2\left[-\cos x\right]_0^{\pi/2} + 2\left[2x+\cos x\right]_{\pi/2}^{\pi}$$

$= 2[+1] + 2[2\pi - 1 - p] = 2\pi = a\pi + b$

a = 2, b = 0

$a^2 + b^2 = 4$

Sol 17: $-1 + x = 2y - y^2 - 1$

$x - 1 = -(y-1)^2$

y = $\sqrt{1-x} + 1 = mx + 2$

$1 - x = (mx + 1)^2$

$1 - x = m^2x^2 + 1 + 2mx$

$\Rightarrow m^2x^2 + (2m + 1)x = 0$

$$\Rightarrow x_1 = 0,\ x_2 = \frac{-(2m+1)}{m^2}$$

$$x_1x_2 = 0,\ x_1 + x_2 = \frac{-(2m+1)}{m^2}$$

$$A = \int_{x_1}^{x_2} \left(1+\sqrt{1-x}-mx-2\right) dx$$

$$A = \left[x-\frac{(1-x)^{3/2}}{3/2}-\frac{mx^2}{2}-2x\right]_{x_1}^{x_2}$$

$$= \left[\frac{-2(1-x)\sqrt{1-x}}{3}-\frac{mx^2}{2}-x\right]_{x_1}^{x_2}$$

$$= -\frac{2}{3}\left[(1-x_2)^{3/2}-(1-x_1)^{3/2}\right] - \left[\frac{m}{2}\left(x_2^2-x_1^2\right)\right]-\left[(x_2-x_1)\right]$$

$$= -\frac{2}{3}\left[(1-x)^{3/2}-1\right]-\frac{m}{2}x^2-x$$

$$= \frac{2}{3}-\frac{2}{3}(1-x)^{3/2}-\frac{mx^2}{2}-x$$

$$= \frac{2}{3}-\frac{2}{3}\left(1+\frac{(2m+1)}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{2m+1}{m^2}\right)^2+\left(\frac{2m+1}{m^2}\right)$$

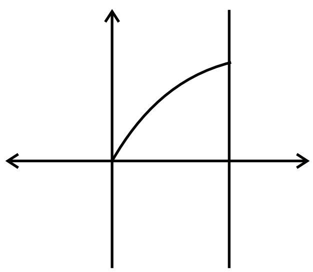
$$= \frac{2}{3}-\frac{2}{3m^3}(m+1)^3+\frac{2}{m}+\frac{1}{m^2}$$

$$-\frac{m}{2}\left(\frac{4}{m^2}+\frac{1}{m^4}+\frac{4}{m^3}\right) = \frac{2}{3}+\frac{2}{m}+\frac{1}{m^2}-\frac{2}{m}-\frac{1}{2m^3}$$

$$-\frac{2}{m^2}-\frac{2}{3}-\frac{2}{3m^3}-\frac{2}{m}-\frac{2}{m^2}=-\frac{3}{m^2}-\frac{2}{m}-\frac{7}{6m^3}$$

$$A=\frac{3m+2m^2+\frac{7}{6}}{m^3}$$

Sol 18:



$f'(x) = -xe^{-x} + e^{-x}$

$f''(x) = xe^{-x} - e^{-x} - e^{-x}$

$= (x - 2)e^{-x} = 0$

$\Rightarrow x = 2$ inflection point

$$\int_0^2 xe^{-x} = \left[-xe^{-x}-e^{-x}\right]_0^2$$

$= -2e^{-2} - e^{-2} + 1 = 1 - 3e^{-2}$

Sol 19: Let $R=\{(x,y):y^2\le 4x,\ 4x^2+4y^2\le 9\}$

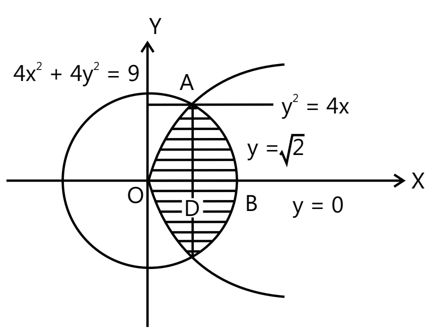
$=\{(x,y):y^2\le 4x\}\cap\{(x,y):4x^2+4y^2\le 9\}=R_1\cap R_1$

Where $R_1=\{(x,y):y^2\le 4x\}$ and $R_2=\{(x,y):4x^2+4y^2\le 9\}$

Equation of the given curves are

$y^2=4x$ ... (i)

and $4x^2+4y^2=9$ ... (ii)



Curve (i) is a parabola having axis y = 0 and vertex (0, 0).

Curve (ii) is a circle having centre at (0, 0) and radius $\frac{3}{2}$.

Clearly region R_1 is the interior of the parabola (i) and region R_2 is the interior of the circle (ii). Therefore, $R_1\cap R_2$ is the shaded region.

Putting the value of y^2 from (i) in (ii), we get

$4x^2+16x-9=0\Rightarrow x=\frac{1}{2}-\frac{9}{2}$

From (i), $x=\frac{1}{2}\Rightarrow y=\pm\sqrt{2}$

And $x=-\frac{9}{2}$ is not possible

Thus, $A=\left(\frac{1}{2},\sqrt{2}\right)$ and $C=\left(\frac{1}{2},-\sqrt{2}\right)$

Required area = 2 area OABD

$$=2\int_0^{\sqrt{2}}(x_1-x_2)dy$$

$$=2\int_0^{\sqrt{2}}\frac{1}{2}\sqrt{3^2-(2y)^2}\,dy-2\int_0^{\sqrt{2}}\frac{y^2}{4}\,dy$$

$$=\frac{1}{2}\int_0^{2\sqrt{2}}\sqrt{3^2-z^2}\,dz-\frac{1}{2}\left[\frac{y^3}{3}\right]_0^{\sqrt{2}}$$

[Putting z = 2y in first integral]

$$=\frac{1}{2}\left[\frac{2\sqrt{9-z^2}}{2}+\frac{9}{2}\sin^{-1}\frac{z}{3}\right]_0^{2\sqrt{2}}-\frac{1}{6}.2\sqrt{2}$$

$$=\frac{1}{2}\left[\sqrt{2}+\frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right]-\frac{2\sqrt{2}}{6}$$

$$=\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}=\frac{\sqrt{2}}{6}+\frac{9}{4}\cos^{-1}\frac{1}{3}$$

$\sin^{-1}\frac{2\sqrt{3}}{3}=\theta$, then $\sin\theta=\frac{2\sqrt{2}}{3}$ and

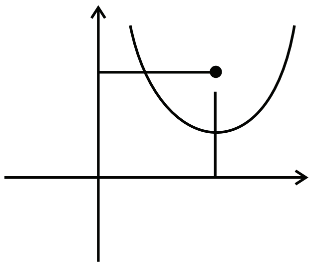
$\cos\theta=\sqrt{1-\sin^2\theta}=\frac{1}{3}$

Required area = $\frac{\sqrt{2}}{6}+\frac{9}{4}\left[\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right]$

$\left[\because \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x\right]$

$=\left(\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}\right)$ sq. units.

Sol 20:



$y = a^2x^2 + ax + 1$

$y = \left(ax + \frac{1}{2}\right)^2 + \frac{3}{4}$

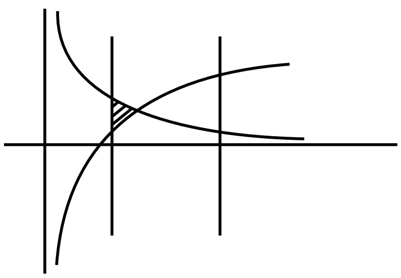
$\Rightarrow A = \int_0^1 \left[\left(ax + \frac{1}{2}\right)^2 + \frac{3}{4}\right] dx$

$\Rightarrow A = \left[\frac{a^2x^3}{3} + \frac{ax^2}{2} + x\right]_0^1 = -\frac{a^2}{3} + \frac{a}{2} + 1$

A least if A′=0 ie $\frac{2a}{3} + \frac{1}{2} = 0$

$a = -\frac{3}{4}$

Sol 21:



$\int_1^K \left(\frac{1}{x} - \ell nx\right) dx = \int_K^a \left(\ell nx - \frac{1}{x}\right) dx$

$1 = K\ell nK$

$K^K = e$

$\left[\ell nx - x\ell nx + x\right]_1^K = \left[x\ell nx - x - \ell nx\right]_K^a$

$\ell nK - 1 + K - 1$

$= a\ell na - a - \ell na - 1 + K + \ell nK$

$(a - 1) = (a - 1)(\ell na)$

$(a - 1)[1 - \ell na] = 0$

Either a = 1 or a = e

$\therefore a = e$

Exercise 2

Single Correct Choice Type

Sol 1: (D)

$A = \int_a^b \left(x^2 - 1 - 3 + x\right) dx = \int_a^b \left(x^2 + x - 4\right) dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2} - 4x\right]_a^b$

$= \frac{b^3 - a^3}{3} + \frac{b^2 - a^2}{2} - 4(b - a)$

$= \sqrt{17}\left|\frac{1+4}{3} + \frac{(-1)}{2} - 4\right| = \sqrt{17}\left|\frac{5}{3} - \frac{9}{2}\right| = \frac{17\sqrt{17}}{6}$

For point of intersection

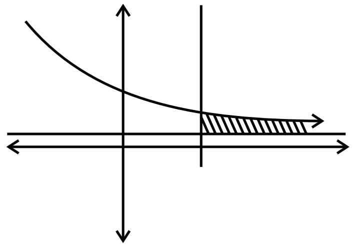
$x^2 - 1 = 3 - x$

$\Rightarrow x^2 + x - 4 = 0$

$\Rightarrow b + a = -1 \Rightarrow ab = -4$

$\Rightarrow b - a = \sqrt{1+16} \Rightarrow b - a = \sqrt{17}$

Sol 2: (A) e^{-x} and linearly = e^{-4} & x = 1



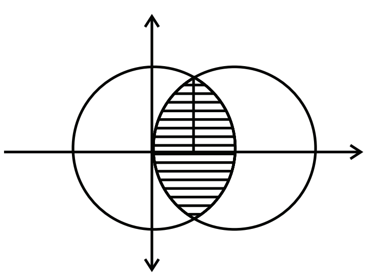
$\int_1^4 \left[e^{-x} - e^{-4}\right] dx$

$= -e^{-4} - 4e^{-4} - (-e^{-1} - e^{-4})$

$= -5e^{-4} + e^{-4} + e^{-1}$

$= e^{-1} - 4e^{-4} = \frac{1}{e} - \frac{4}{e^4} = \frac{e^3 - 4}{e^4}$

Sol 3: (D) $y = \sqrt{9 - x^2}$ $x^2 - 6x + y^2 = 0$



$$\text{Area} = 2\int_0^{3/2}\left(\sqrt{9-x^2}-\sqrt{6x-x^2}\right)dx$$

$$= 2\int_0^{3/2}\left[\sqrt{9-x^2}-\sqrt{9-(x-3)^2}\right]dx$$

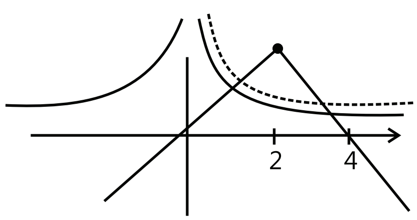
$$= 2\left[\frac{9\sin^{-1}\frac{x}{3}+x\sqrt{9-x^2}}{2}\right]_0^{3/2} -$$

$$2\left[\frac{9\sin^{-1}\frac{x-3}{3}+(x-3)\sqrt{6x-x^2}}{2}\right]_0^{3/2}$$

$$= 9.\frac{\pi}{6}+\frac{3}{2}\sqrt{9-\frac{9}{4}}+9.\frac{\pi}{6}+\frac{3}{2}\sqrt{9-\frac{9}{4}}$$

$$= 9.\frac{\pi}{3}+3\times 3\frac{x\sqrt{3}}{2} = 3\frac{\pi}{6}+\frac{9\sqrt{3}}{2} = 3\left[\pi+\frac{3\sqrt{3}}{2}\right]$$

Sol 4: (B)



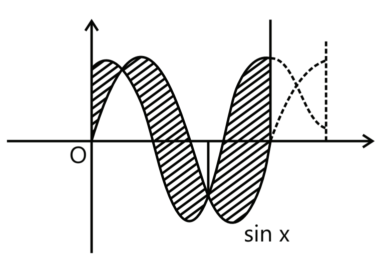
$$\text{Area} = \int_{\sqrt{3}}^{2}\left(x-\frac{3}{x}\right)dx+\int_{2}^{3}\left(4-x-\frac{3}{x}\right)dx$$

$$= \left[\frac{x^2}{2}-3\ell nx\right]_{\sqrt{3}}^{2}+\left[4x-\frac{x^2}{2}-3\ell nx\right]_{2}^{3}$$

$$= 2-3\,\ell n2-\frac{3}{2}+3\,\ell n\sqrt{3}+12-\frac{9}{2}-3\,\ell n3-8$$
$$+ 2 + 3\,\ell n2$$

$$= \frac{1}{2}+\frac{15}{2}-6+3\ell n\frac{1}{\sqrt{3}} = 2-\frac{3}{2}\ell n3 = \frac{4-3\ell n3}{2}$$

Sol 5: (D)



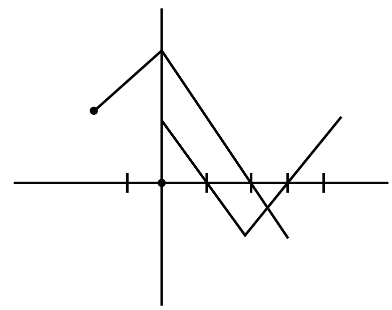
Area =

$$\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)\,dx$$
$$+\int_{5\pi/4}^{2\pi}(\cos x-\sin x)\,dx$$

$$= \left[\sin+\cos\right]_0^{\pi/4}+\left[\sin+\cos\right]_{5\pi/4}^{2\pi}+$$
$$\left[-\sin-\cos\right]_{\pi/4}^{5\pi/4}$$

$$= \sqrt{2}-1+1+\sqrt{2}-[-\sqrt{2}-\sqrt{2}] = 4\sqrt{2}\text{ sq. units}$$

Sol 6: (C)



$f(x) = -1 + |x-2|$

$g(x) = 2 - |x|$

$f(x) = 1 - x \in (0, 2)$

$x - 3 \in (2, 4)$

$g(x) = 2 - x \in (0, 3)$

$2 + x \in (-1, 0)$

$g(f(x)) = 2 - f(x)\ f(x) \in (0, 3)$

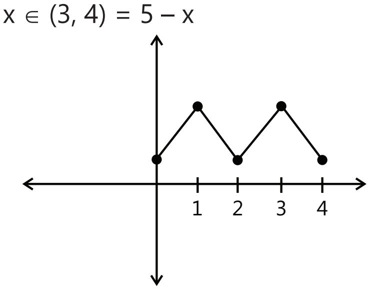
$= 2 + f(x) f(x) \in (-1, 0)$

$= 2 - (1-x),\ x \in (0, 1) = 1 + x$

$= 2 + (1-x),\ x \in (1, 2) = 3 - x$

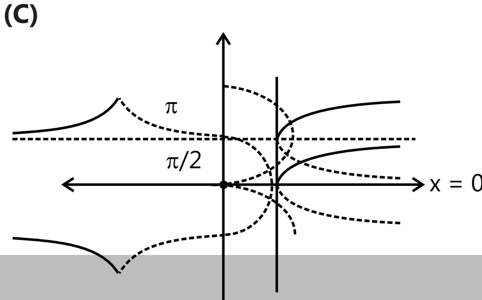
$= 2 + (x-3),\ x \in (2, 3) = x - 1$

$= 2 - (x-3),\ x \in (3, 4) = 5 - x$



$$\text{Area} = \frac{3}{2}\left[1+2\right]\times 1 = \frac{9}{2}$$

Sol 7: (C)

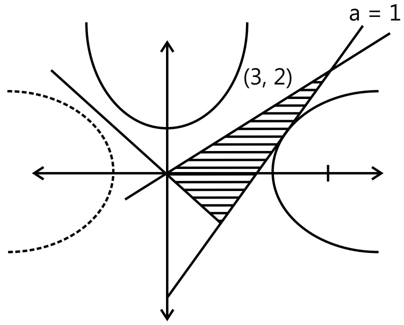


$$\text{Area} = 2\int_0^{\pi/4}(\sec x - 1)dx$$

$$= 2\left[\ell n(\sec x + \tan x)\right]_0^{\pi/4} - \frac{\pi}{2}$$

$$= 2\ln\left(\sqrt{2}+1\right) - \frac{\pi}{2} \quad = \ln\left(3+2\sqrt{2}\right) - \frac{\pi}{2}$$

Sol 8: (A)



Eq. of tangent

$$\frac{y-2}{x-3} = \frac{2\times 3}{2\sqrt{4}} = \frac{3}{2} \Rightarrow 2y - 4 = 3x - 9$$

$$3x - 2y = 5 \qquad \text{... (i)}$$

$$A = \frac{1}{2}\times 2\times 1 + \int_1^5\left(x - \left(\frac{3x-5}{2}\right)\right)dx$$

$$= 1 + \left[\frac{-x^2+10x}{4}\right]_1^5 = 1 + \frac{1}{4}(16) = 5$$

Sol 9: (A) $y = 1 + 4x - x^2$

$\Rightarrow y - 5 = -(x-2)^2$

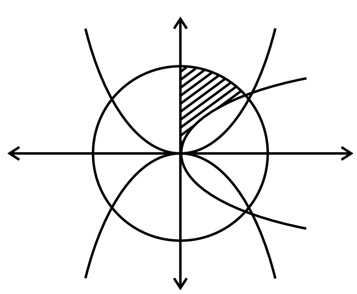
$$\text{Area} = \int_0^{3/2} 1 + 4x - x^2$$

$$= \left[x + 2x^2 - \frac{x^3}{3}\right]_0^{3/2} = \frac{3}{2} + 2\times\frac{9}{4} - \frac{27}{24} = 6 - \frac{9}{8} = \frac{39}{8}$$

$$\text{Area of } \Delta = \frac{1}{2}\times\frac{3}{2}\times\frac{3m}{2} = \frac{1}{2}\times\frac{39}{8}$$

$$9m = \frac{39}{2} \Rightarrow m = \frac{13}{6}$$

Sol 10: (A)



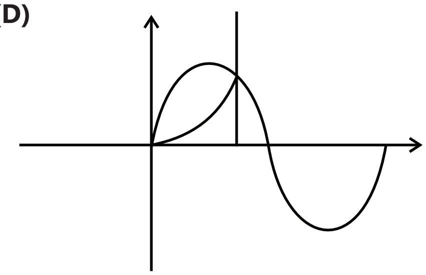
$$\text{Area}_1 = \int_0^1 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3}$$

$$\text{Area}_2 = \int_0^1\left(\sqrt{2-x^2}\right)dx$$

$$= 2\left[\frac{\sin^{-1}\frac{x}{\sqrt{2}} + x\sqrt{2-x^2}}{2}\right]_0^1 = \frac{\pi}{4}+1$$

$$\text{Area} = \frac{\pi}{4} + 1 - \frac{2}{3} = \left(\frac{\pi}{4}+\frac{1}{3}\right)$$

Sol 11: (D)



$$A_1 = \int_0^1 \sin\frac{\pi x}{2}dx = -\frac{2}{\pi}\left[\cos\frac{\pi x}{2}\right]_0^1 = -\frac{2}{\pi}(-1) = \frac{2}{\pi}$$

$$A_2 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\text{Ratio} = \frac{\frac{1}{3}}{\frac{2}{\pi}-\frac{1}{3}} = \frac{6-\pi}{\pi}$$

Sol 12: (D) $A = \int\left(y^2 - 1 - |y|\sqrt{1-y^2}\right)dy$

$$= \int_0^1\left(y^2 - 1 - y\sqrt{1-y^2}\right)dy + \int_{-1}^0\left(y^2 - 1 + y\sqrt{1-y^2}\right)dy$$

$$= \left[\frac{y^3}{3} - y + \frac{\left(\sqrt{1-y^2}\right)^{3/2}}{2\left(\frac{3}{2}\right)}\right]_0^1 + \left[\frac{y^3}{3} - y - \frac{\left(\sqrt{1-y^2}\right)^{3/2}}{3}\right]_{-1}^0$$

$$= \frac{1}{3} - 1 - \left(0 - 0 + \frac{1}{3}\right) + \left(0 - 0 - \frac{1}{3} - \left(-\frac{1}{3}+1\right)\right)$$

$$= -1 + \frac{1}{3} - \frac{1}{3} - 1 = -2; \qquad \text{Area} = 2$$

Sol 13: (B) $f(x) = 3x^3 + 2x$

$g(x) = f^{-1}(x)$

$$\int_0^5 g(x) = \int_0^1 f(x) = \left[\frac{3x^4}{4} + x^2\right]_0^1 = \left[\frac{3}{4} + 1\right] = \frac{7}{4}$$

Previous Years' Questions

Sol 1: (B) Required area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$= \left(\because \frac{1+\sin x}{\cos x} > \frac{1-\sin x}{\cos x} > 0\right)$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}} - \sqrt{\frac{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$= \int_0^{\pi/4}\frac{1+\tan\frac{x}{2} - 1 + \tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx = \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Substitute $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= \int_0^{\tan\frac{\pi}{8}}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$$

As $\tan\frac{\pi}{8} = \sqrt{2} - 1$

So, $\int_0^{\sqrt{2}-1}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$

Sol 2: (B) Here, area between 0 to b is R_1 and b to 1 to R_2.

$$\therefore \int_0^b (1-x)^2dx - \int_b^1 (1-x)^2dx = \frac{1}{4}$$

$$\Rightarrow \left(\frac{(1-x)^3}{-3}\right)_0^b - \left(\frac{(1-x)^3}{-3}\right)_b^1 = \frac{1}{4}$$

$$\Rightarrow -\frac{1}{3}\{(1-b)^3 - 1\} + \frac{1}{3}\{0 - (1-b)^3\} = \frac{1}{4}$$

$$\Rightarrow -\frac{2}{3}(1-b)^3 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$$

$$\Rightarrow (1-b)^3 = \frac{1}{8} \Rightarrow (1-b) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$$

Sol 3: (C) $R_1 = \int_{-1}^2 x\, f(x)dx$ ... (i)

Using, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$R_1 = \int_{-1}^2 (1-x)f(1-x)dx,$

[given, f(x) = f(1 – x)]

$\therefore R_1 = \int_{-1}^2 (1-x)f(x)dx$ ... (ii)

Given, R_2 is area bounded by

f(x), x = –1 and x = 2

$\therefore R_2 = \int_{-1}^2 f(x)dx$ ... (iii)

Adding Eqs. (i) and (ii), we get

$2R_1 = \int_{-1}^2 f(x)dx$ ... (iv)

$\therefore$ From Eqs. (iii) and (iv), we get

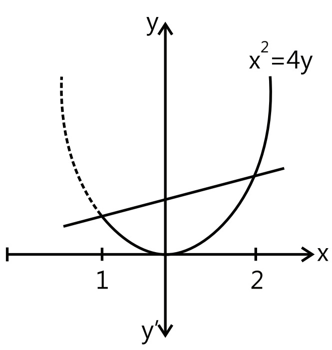
$2R_1 = R_2$

Sol 4: The point of intersection of the curves $x^2 = 4y$ and $x = 4y - 2$ could be sketched as, are $x = -1$ and $x = 2$.

$\therefore$ Required area

$$= \int_{-1}^2\left\{\left(\frac{x+2}{4}\right) - \left(\frac{x^2}{4}\right)\right\}dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

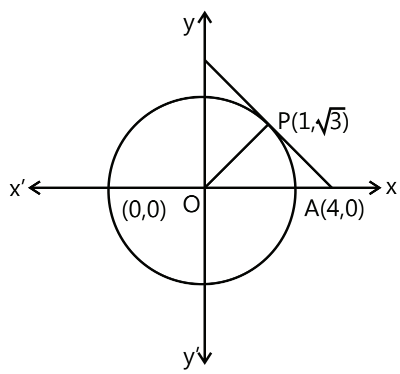


$$= \frac{1}{4}\left[\frac{10}{3} - \left(\frac{-7}{6}\right)\right] = \frac{1}{4}.\frac{9}{2} = \frac{9}{8} \text{ sq unit.}$$

Sol 5: Here, $\int_2^a \left(1 + \frac{8}{x^2}\right)dx = \int_a^4 \left(1 + \frac{8}{x^2}\right)dx$

$$\Rightarrow \left[x - \frac{8}{x}\right]_2^a = \left[x - \frac{8}{x}\right]_a^4$$

$$\Rightarrow \left(a - \frac{8}{a}\right) - (2 - 4) = (4 - 2) - \left(a - \frac{8}{a}\right)$$



Thus, area of Δ formed by (0, 0) $(1, \sqrt{3})$ and (4, 0)

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & \sqrt{3} & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2}(0 - 4\sqrt{3}) = 2\sqrt{3} \text{ sq unit}$$

$$\Rightarrow a - \frac{8}{a} + 2 = 2 - a + \frac{8}{a}$$

$$\Rightarrow 2a - \frac{16}{a} = 0$$

$\Rightarrow 2(a^2 - 8) = 0$

$\therefore a = \pm 2\sqrt{2}$, (neglecting –ve sign)

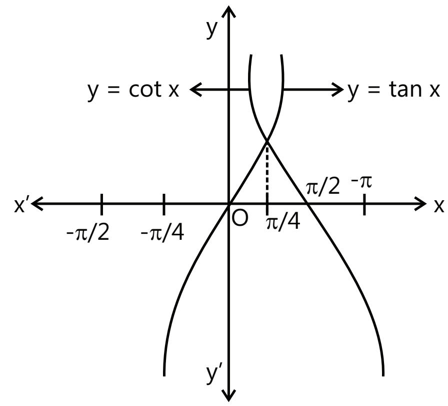
$a = 2\sqrt{2}$.

Sol 6: Given, $y = \begin{cases} \tan x, & -\frac{\pi}{3} \le x \le \frac{\pi}{3} \\ \cot x, & \frac{\pi}{6} \le x \le \frac{\pi}{2} \end{cases}$

which could be plotted as, y-axis.

$\therefore$ Required area $= \int_0^{\pi/4} (\tan x)dx + \int_{\pi/4}^{\pi/3} (\cot x)dx$

$$= \left[-\log|\cos x|\right]_0^{\pi/4} + \left[\log \sin x\right]_{\pi/4}^{\pi/3}$$

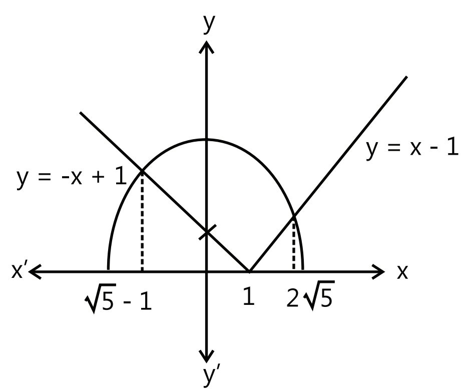


$$= \left(\log\frac{1}{\sqrt{2}} - 0\right) + \left(\log\frac{\sqrt{3}}{2} - \log\frac{1}{\sqrt{2}}\right)$$

$$= \log\frac{\sqrt{3}}{2} - 2\log\frac{1}{\sqrt{2}} = \log\frac{\sqrt{3}}{2}$$

$$\Rightarrow -\log\frac{1}{2} = \frac{1}{2}\log_e 3 \text{ sq. units}$$

Sol 7: Given curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ could be sketche as shown whose point of intersection are.



$5 - x^2 = (x - 1)^2$

$\Rightarrow 5 - x^2 = x^2 - 2x + 1$

$\Rightarrow 2x^2 - 2x - 4 = 0$

$\Rightarrow x = 2, -1$

$\therefore$ Required area

$$= \int_{-1}^{2} \sqrt{5 - x^2}\,dx - \int_{-1}^{1} (-x + 1)dx - \int_{1}^{2} (x - 1)dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \left[\frac{-x^2}{2} + x\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(-1 + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

$$-\left(-\frac{1}{2}+1+\frac{1}{2}+1\right)-\left(2-2-\frac{1}{2}+1\right)$$

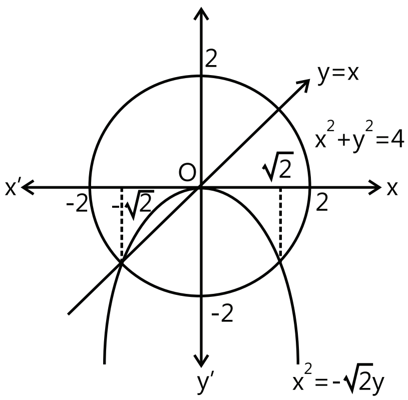
$$=\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)-\frac{1}{2}$$

$$=\frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}}+\frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}}\right)-\frac{1}{2}$$

$$=\frac{5}{2}\sin^{-1}(1)-\frac{1}{2}=\frac{5\pi}{4}-\frac{1}{2} \text{ sq. units}$$

Sol 8: Given curves are $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$

Thus, the required area



$$=\left|\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{4-x^2}\,dx\right|-\left|\int_{-\sqrt{2}}^{0}x\,dx\right|-\left|\int_{0}^{\sqrt{2}}\frac{-x^2}{\sqrt{2}}dx\right|$$

$$=2\int_0^{\sqrt{2}}\sqrt{4-x^2}\,dx-\left|\left(\frac{x^2}{2}\right)_{-\sqrt{2}}^{0}\right|-\left|\frac{x^3}{3\sqrt{2}}\right|_0^{\sqrt{2}}$$

$$=2\left\{\frac{x}{2}\sqrt{4-x^2}-\frac{4}{2}\sin^{-1}\frac{x}{2}\right\}_0^{\sqrt{2}}-1-\frac{2}{3}$$

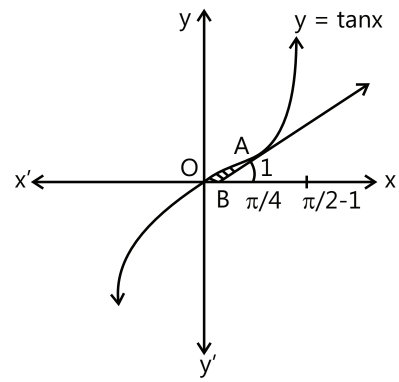
$$=(2-\pi)-\frac{5}{3}=\frac{1}{3}-\pi \text{ sq. units}$$

Sol 9: $y = \tan x \Rightarrow \frac{dy}{dx}=\sec^2 x$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}}=2$$

Hence, equation of tangent at $A\left(\frac{\pi}{4},1\right)$ is

$$\frac{y-1}{x-\pi/4}=2 \Rightarrow y-1=2x-\frac{\pi}{2}$$



$$\Rightarrow (2x-y)=\left(\frac{\pi}{2}-1\right)$$

$\therefore$ Required area is OABO

$$=\int_0^{\pi/4}(\tan x)dx-\text{area of }\Delta ALM$$

$$=\left[\log|\sec x|\right]_0^{\pi/4}-\frac{1}{2}BL\,.\,AL$$

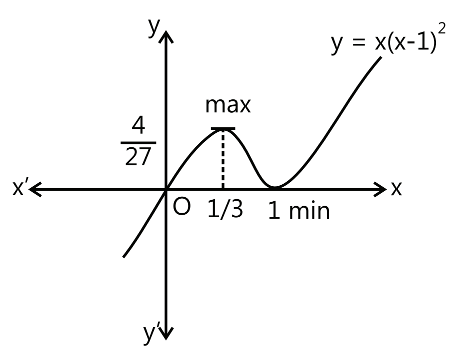
$$=\log\sqrt{2}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi-2}{4}\right).1$$

$$=\left(\log\sqrt{2}-\frac{1}{4}\right)\text{ sq. unit}$$

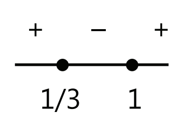
Sol 10: $y = x(x-1)^2 \Rightarrow \frac{dy}{dx}=x\,.\,2(x-1)+(x-1)^2$

$\therefore$ Maximum at $x = 1/3 \Rightarrow y_{max}=\frac{1}{3}\left(-\frac{2}{3}\right)^2=\frac{4}{27}$

Minimum at x = 1

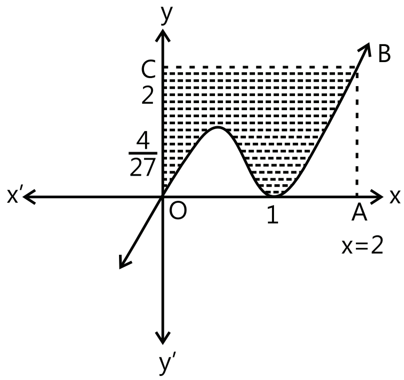


$$=(x-1)(2x+x-1)$$
$$=(x-1)(3x-1)$$



$\Rightarrow \qquad y_{min} = 0$

Now, to find the area bounded by the curve y = x(x – 1)^2, the y-axis and line x = 2



$\therefore$ Required area

$= \text{Area of square OABC} - \int_0^2 y\,dx$

$= 2\times 2 - \int_0^2 x(x-1)^2 dx = 4 - \int_0^2 \left(x^3 - 2x^2 + x\right) dx$

$= 4 - \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^2 = 4 - \left[\frac{16}{4} - \frac{16}{3} + \frac{y}{2}\right]$

$= \frac{10}{3}$ sq. units

Sol 11: Both the curves are defined for x > 0. Both are positive when x > 1 and negative when 0 < x < 1

We know, $\lim_{x\to 0^+} (\log x) \to -\infty$

Hence, $\lim_{x\to 0^+} \frac{\log x}{ex} \to -\infty$, This, y-axis is asymptote of second curve.

And $\lim_{x\to 0^+}$ ex log x [(0) x ∞ form]

$= \lim_{x\to 0^+} \frac{e\log x}{1/x}\left(-\frac{\infty}{\infty}\text{form}\right) = \lim_{x\to 0^+} \frac{e\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0$

(using L'Hospital's rule)

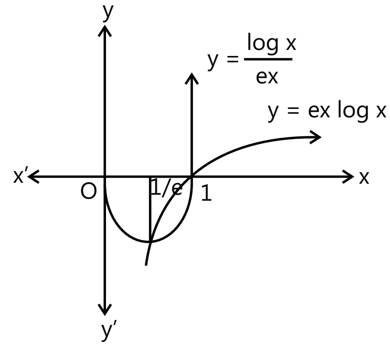
Thus, the first curve starts from (0, 0) but does not include (0, 0).

Now, the given curves intersect, therefore

$ex\log x = \frac{\log x}{ex}$

i.e. $(e^2x^2 - 1)\log x = 0$

$\Rightarrow x = 1, \frac{1}{e}$ (since x > 0)



$\therefore$ The required area $= \int_{1/e}^{1}\left(\frac{\log x}{ex} - ex\log x\right)dx$

$= \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1} - e\left[\frac{x^2}{4}(2\log x - 1)\right]_{1/e}^{1} = \frac{e^2-5}{4e}$ sq. units

Sol 12: Given, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$

$\Rightarrow 4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a,$ ... (i)

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$ ... (ii)

and $4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$ ... (iii)

Where f(x) is quadratic expression given by,

$f(x) = ax^2 + bc + c$ and (i), (ii) and (iii)

$\Rightarrow 4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$

or $\{4 f(-1) - 3\}x^2 + \{4 f(1) - 3\} x + f\{2\} = 0$ ... (iv)

As above equation has 3 roots a, b an c

$\therefore$ above equation is identity in x.

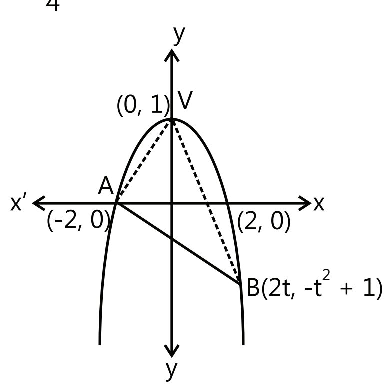
i.e., Coefficients must be zero.

$\Rightarrow f(-1) = 3/4, f(1) = 3/4, f(2) = 0$ ... (v)

$\because f(x) = ax^2 + bx + c$

$\therefore a = -1/4$, b = 0 and c = 1, using Eq. (v)

Thus, $f(x) = \frac{4-x^2}{4}$ shown as,



Let $A(-2,0), B = (2t, -t^2+1)$ v

Since, AB subtends right angle at vertex V(0, 1)

$$\Rightarrow \frac{1}{2}.\frac{-t^2}{2t} = -1 \Rightarrow t = 4$$

$\therefore B(8, -15)$

Equation of chord AB is

$$y = \frac{-(3x+6)}{2}$$

$\therefore$ Required area

$$= \left|\int_{-2}^{8}\left(\frac{4-x^2}{4}+\frac{3x+6}{2}\right)dx\right|$$

$$= \left|\left(x-\frac{x^3}{12}+\frac{3x^2}{4}+3x\right)_{-2}^{8}\right|$$

$$= \left[\left[8-\frac{128}{3}+48+24-\left(-2+\frac{2}{3}+3-6\right)\right]\right]$$

$$= \frac{125}{3} \text{ sq. units}$$

Sol 13: Here, slope of tangent

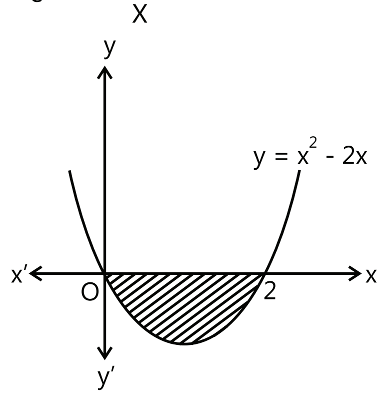
$$\frac{dy}{dx} = \frac{(x+1)^2+y-3}{(x+1)} \Rightarrow \frac{dy}{dx} = (x+1)+\frac{(y-3)}{(x+1)}, X$$

Substitute x + 1 = X and y – 3 = Y

$$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \frac{dY}{dX} = X + \frac{Y}{X} \Rightarrow \frac{dY}{dX} - \frac{1}{X}Y = X$$

$$\text{I.F} = e^{\int -\frac{1}{x}dX} = e^{-\log X} = \frac{1}{X}$$



$\therefore$ Solution is,

$$Y.\frac{1}{X} = \int X.\frac{1}{X}dX + c \Rightarrow \frac{Y}{X} = X + c$$

y – 3 = (x + 1)^2 + c(x + 1), which passes through (2, 0).

$\Rightarrow -3 = (3)^2 + 3c$

$\Rightarrow c = -4$

$\therefore$ Required curve y = (x + 1)^2 – 4(x + 1) + 3

$\Rightarrow y = x^2 - 2x$

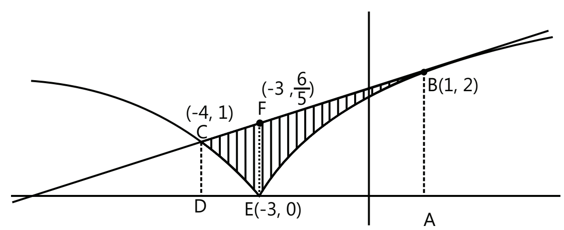
$$\therefore \text{ Required area } = \left|\int_0^2 (x^2-2x)dx\right| = \left|\left(\frac{x^3}{3}-x^2\right)_0^2\right|$$

$$= \frac{8}{3} - 4 = \frac{4}{3} \text{ sq. units}$$

Sol 14:

$$\text{Region } = \left\{(x,y) \in R^2 : y \ge \sqrt{|x+3|}, 5y \le x+9 \le 15\right\}$$

Plotting all the curves

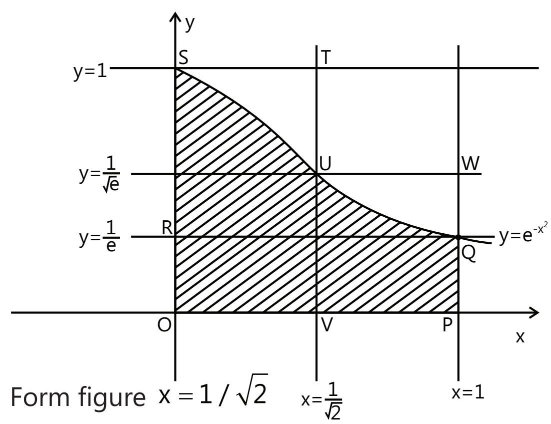


$$\text{Area} = \text{Area}(ABFE) - \text{Area}(AEB)$$

$$+ \text{Area}(DEFC) - \text{Area}(D\,E\,C)$$

$$= \frac{32}{5} - \int_{-3}^{1}\left(\sqrt{-x-3}\right)dx + \frac{11}{10} - \int_{-4}^{-3}\left(\sqrt{x+3}\right)dx = \frac{3}{2} \text{ sq. units}$$

Sol 15: (A, B, D)



Form figure $x = 1/\sqrt{2}$

S > Area $(OPQR)$

$$\Rightarrow S > 1 \times \frac{1}{e} \Rightarrow S > \frac{1}{e}$$

S > Area $(PVUW)$ + Area $(OSTV)$

$$< \left(1-\frac{1}{12}\right)\frac{1}{\sqrt{e}} + 1 \times \frac{1}{\sqrt{2}}$$

$$< \frac{1}{12} + \frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$$ D is correct

Now, $e^{-x} \le e^{-x^2}$ if $x \in (0, 1)$

$$\int_0^1 e^{-x}dx \le \int_0^1 e^{-x^2}dx \Rightarrow \left(1-\frac{1}{e}\right) \le S$$ B is correct

$$1-\frac{1}{e} > \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$$ and $S > 1-\frac{1}{e}$

$$\Rightarrow S > \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$$

(B) and (D) Correct

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