Class 11

2017-18

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MATHEMATICS

FOR JEE MAIN & ADVANCED

SECOND EDITION

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Formula Sheet

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1. BASIC MATHEMATICS

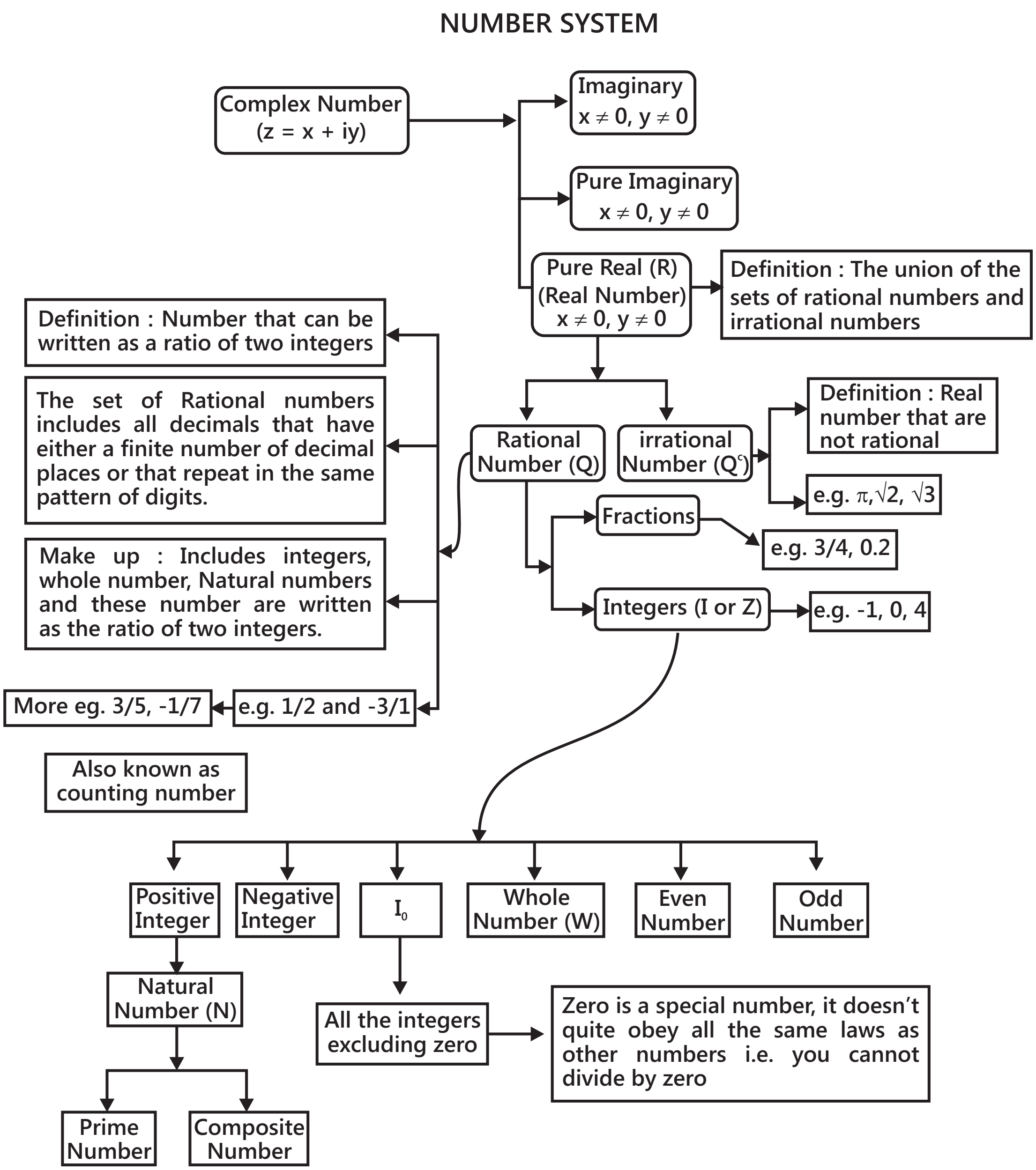


Figure 1.1: The number system

1. NUMBER SYSTEM

(a) Natural Numbers: The counting numbers 1, 2, 3, 4, ... are called natural numbers. The set of natural numbers is denoted by N.

N = {1, 2, 3, 4, ...}   N is also denoted by I′ or Z′

(b) Whole Numbers: Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W.

Thus W = {0, 1, 2, ...}

(c) Integers: The numbers ..... -3, -2, -1, 0, 1, 2, 3 .... are called integers and the set of integers is denoted by I or Z.

Thus I (or Z) = {......-3, -2, -1, 0, 1, 2, 3........}

(a) Set of negative integers is denoted by I^{-} and consists of {......., -3, -2, -1}

(b) Set of non-negative integers is denoted by W.

(c) Set of non-positive integers {......., -3, -2, -1, 0}

(d) Even integers: Integers which are divisible by 2 are called even integers. e.g. 0, ±2, ± 4, ...

(e) Odd integers: Integers which are not divisible by 2 are called odd integers. e.g. ±1, ±3, ±5, ±7

(f) Prime numbers: A natural number (except unity) is said to be a prime number if it is exactly divisible by unity and itself only. e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(g) Composite numbers: Natural numbers which are not prime (except unity) are called composite numbers.

(h) Co-prime numbers: Two natural numbers (not necessarily prime) are said to be co-prime, if their H.C.F. (Highest common factor) is one. e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (4, 9), (5, 6), (7, 8) etc. These numbers are also called as relatively prime numbers.

(i) Twin prime numbers: If the difference between two prime numbers is two, then the numbers are called twin prime numbers. e.g. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

(j) Rational numbers: All the numbers that can be represented in the form p/q, where p and q are integers and q ≠ 0, are called rational numbers and their set is denoted by Q. e.g. $\frac{1}{2}$, 2, 0, -5, $\frac{22}{7}$, 2.5, 0.3333 .... etc. Thus $Q = \left\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\right\}$. It may be noted that every integer is a rational number since it can be written as p/1. The decimal part of rational numbers is either terminating or recurring.

(k) Irrational numbers: There are real numbers which cannot be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^C or $\overline{Q}$ (i.e. complementary set of Q). The decimal part of irrational numbers is neither terminating nor recurring e.g. $\sqrt{2}$, $1 + \sqrt{3}$, π etc.

(l) Real numbers: The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^C$.

(m) Complex numbers: A number of the form a + ib is called a complex number, where $a, b \in R$ and $i = \sqrt{-1}$. A complex number is usually denoted by 'z' and a set of complex numbers is denoted by C.

PLANCESS CONCEPTS

- Zero is neither positive nor negative but zero is non-negative and non-positive.
- '1' is neither prime nor composite
- '2' is the only even prime number
- '4' is the smallest composite number

- Two distinct prime numbers are always co-prime but the converse need not be true.
- Consecutive natural numbers are always co-prime numbers.

  $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$. And both are irrational.

Vaibhav Gupta (JEE 2009, AIR 22)

2. RATIO AND PROPORTION

2.1 Ratio

(a) If A and B are two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (this may be an integer or fraction)

(b) A ratio may be represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb}$ = ... where m, n, ... are non-zero numbers.

(c) To compare two or more ratios, reduce them to their common denominator.

2.2 Proportion

When two ratios are equal, then the four quantities composing them are said to be proportional. If $\frac{a}{b} = \frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

(a) 'a' and 'd' are known as extremes and 'b' and 'c' are known as means.

(b) An important property of proportion; product of extremes = product of means.

(c) If a: b = c : d, then b : a = d : c (invertendo)

(d) If a: b = c : d, then a : c = b : d (alternando)

(e) If a: b = c : d, then $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo)

(f) If a: b = c : d, then $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo)

(g) If a: b = c : d, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo)

3. DEFINITION OF INDICES

If 'a' is any non-zero real or imaginary number and 'm' is a positive integer, then a^m = a.a.a......a (m times). Here 'a' is called the base and m is the index, power, or exponent.

Law of indices:

(a) $a^0 = 1$, $(a \neq 0)$

(b) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(c) $a^{m+n} = a^m.a^n$, where m and n are real numbers

(d) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are real numbers

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

4. SOME IMPORTANT IDENTITIES

(a) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(b) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(c) $a^2 - b^2 = (a + b)(a - b)$

(d) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(e) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(f) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(g) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(h) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

(i) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(j) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(k) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(l) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

5. SURDS

Any root of an arithmetical number which cannot be completely found is called surd. E.g. $\sqrt[3]{2}$, $\sqrt[4]{5}$, $\sqrt[3]{7}$ etc. are all surds.

(a) **Pure Surd:** A surd which consists of purely an irrational number expressed as $\sqrt{n}$ where $a \neq x^n$ $(x \in I)$ is called a pure surd. e.g. $\sqrt[3]{7}$, $\sqrt[5]{5}$ etc.

(b) **Mixed surd:** A pure surd when multiplied with a rational number becomes a mixed surd. e.g. $2\sqrt[3]{3}$, $4\sqrt[5]{5}$, $2\sqrt{3}$ etc.

A mixed surd can be written as a pure surd. e.g. $2 \times \sqrt[3]{3} = \sqrt[3]{3 \times 8} = \sqrt[3]{24}$, $2\sqrt{5} = \sqrt{20}$

(c) **Order of Surd:** The order of a surd is indicated by the number denoting the roots i.e. $\sqrt[4]{2}$, $\sqrt[3]{5}$, $\sqrt[6]{7}$ are surds of the 4th, 3rd and 6th order respectively.

(d) **Simple Surd:** Surds consisting of one term only are called simple surds. E.g. $\sqrt[5]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{a^2bc}$ etc. are simple surds or Monomial surds.

(e) **Compound Surd:** An expression consisting of two or more simple surds connected by (+) or (-) sign is called a compound surd. E.g. $5\sqrt{2} + 4\sqrt{3}$, $\sqrt{3} + \sqrt{2}$, $\sqrt{3} - \sqrt{5}$.

6 LOGARITHM

6.1 Introduction

It is very lengthy and time consuming to find the value of $5\ \sqrt[2]{0.0000165}$, $\sqrt{\dfrac{(45.5)^2}{(3.2)^2(6.5)^2}}$ or finding number of digits in 3^{12}, 2^8. John Napier (1550-1617 AD) invented logarithm (in 1614 AD) to solve such problems. The word “Logarithm” was formed by two Greek words, ‘logos’ which means ‘ratio’, and ‘arithmos’ meaning ‘number’. Henry Briggs (1556-1630 AD) introduced common logarithm. He published logarithm in 1624 AD.

In its simplest form, a logarithm answers the question, “How many of one number do we multiply to get another number?”

Illustration 1: How many 2s do we multiply to get 8? **(JEE MAIN)**

Sol: $2 \times 2 \times 2 = 8$, So we needed to multiply 3 of the 2s to get 8. So the logarithm of 8 to the base 2, written as $\log_2(8)$ is 3.

6.1.1 How to Write it

We would write “the number of 2s you need to multiply to get 8 is 3” as

$$\underbrace{2\times2\times2}_{3} = 8 \leftrightarrow \underset{\substack{\uparrow\\ \text{Base}}}{\log_2}(8) = 3$$

So these two things are the same.

The number we are multiplying is called the “base”, so we would say “The logarithm of 8 with the base 2 is 3”.

Or “log base 2 of 8 is 3” or “the base-2 log of 8 is 3”

6.1.2 Exponents

Exponents and Logarithms are related, let’s find out how...

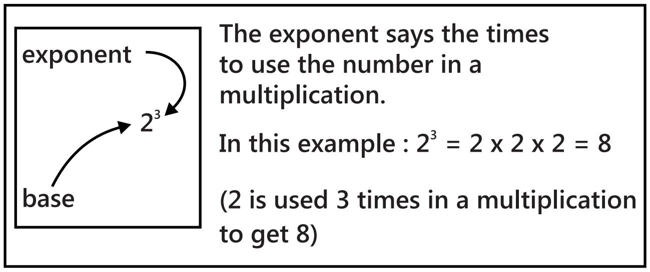


Figure 1.2

So a logarithm answers a question like this: $2^? = 8$

In this way

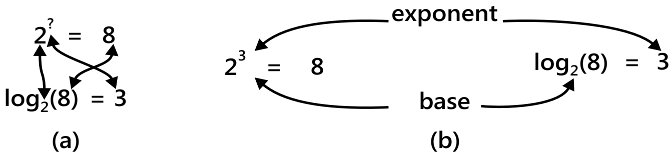


Figure 1.3

So the logarithm answer the question: The general case is

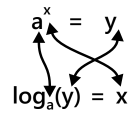


Figure 1.4

6.2 Definition of Logrithm

If $a^x = N$, then x is called the logarithm of N to the base a. It is also designated as $\log_a N$.

So $\log_a N = x$; $a^x = N$, $a > 0$, $a \neq 1$ and $N > 0$

Note:

(a) The logarithm of a number is unique i.e. No number can have two different log to a given base.

(b) From the definition of the logarithm of the number to a given base 'a'. $a^{\log_a N} = N$, $a > 0$, $a \neq 1$ and $N > 0$ is known as the fundamental logarithmic identity.

(c) The base of log can be any positive number other than 1, but basically two bases are mostly used. They are 10 and e (= 2.718 approx.)

Logarithm of a number to the base 10 are named as common logarithm, whereas the logarithm of numbers to the base e are called as Natural or Napierian logarithm.

6.2.1 Common Logarithm: Base 10

Many a times, the logarithm is written without a base, like this, log (100)

This usually means that the base is really 10.

It is called "common logarithm". Engineers love to use it.

6.2.2 Natural Logarithms: Base "e"

Another base that is often used is e (Euler's Number) which is approximately 2.71828.

This is called a "natural logarithm". Mathematicians use this one quite often.

But There is Some Confusion ....!

Mathematicians use "log" (instead of "ln") to mean the natural logarithm. This can lead to confusion:

Example	**Engineer Thinks**	**Mathematician Thinks**	
log(50)	$\log_{10}(50)$	$\log_e(50)$	Confusion
ln (50)	$\log_e(50)$	$\log_e(50)$	No confusion
$\log_{10}$ (50)	$\log_{10}$ (50	$\log_{10}$ (50)	No confusion

So make sure that when you read "log" that you know what base they mean.

Note: Since NCERT assumed log x to be $\log_e x$, for JEE Main and Advanced this convention is to be used.

6.3 Properties of Logarithm

Let M and N be arbitrary positive numbers such that $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and x, y are real numbers. Then,

(a) $\log_a (M \times N) = \log_a M + \log_a N$ (Product rule)

Proof: Let $\log_a M = x$ and $\log_a N = y$

Then from the basic definition of logarithm, $M = a^x$ and $N = a^y$

$\Rightarrow (M \times N) = a^{x+y} \Rightarrow \log_a (M \times N) = x + y$

$\Rightarrow \log_a(M \times N) = \log_a M + \log_a N$

(b) $\log_a (M/N) = \log_a M - \log_a N$ (Division rule)

Proof: Let $\log_a M = x$ and $\log_a N = y$

$M = a^x$ and $N = a^y$

$\Rightarrow M/N = a^{x-y} \Rightarrow \log_a (M/N) = x - y = \log_a M - \log_a N$

(c) $\log_a M^x = x \log_a .M$ (Power rule)

Proof: Let $\log_a M^x = y$ ... (i)

$\Rightarrow M^x = a^y \quad \Rightarrow (M^x)^{1/x} = (a^y)^{1/x}$

$\Rightarrow M = a^{y/x} \quad \Rightarrow \log_a M = y/x$

$\Rightarrow x \log_a M = y$ ... (ii)

From (i) and (ii), we can say that $\log_a M^x = x \log_a M$

(d) $\log_{a^x} M = \frac{1}{x} \log_a M \ (x \neq 0)$ (Power rule for base)

Proof: Let $\log_{a^x} M = y$ ... (i)

$\Rightarrow M = a^{xy} \Rightarrow M^{1/x} = a^y \quad \Rightarrow \log_a M^{1/x} = \log_a a^y$

$\frac{1}{x} \log_a M = y$ ... (ii)

Using (i) and (ii) $\quad \log_{a^x} M = \frac{1}{x} \log_a M$

(e) $\log_b a = \frac{\log_c a}{\log_c b} \ (c > 0, c \neq 1) = \frac{\log a}{\log b}$ (Base changing theorem)

Proof: Let $\log_c a = x$ and $\log_c b = y$

$\Rightarrow a = c^x$ and $b = c^y \Rightarrow a^{1/x} = c$ and $b^{1/y} = c \Rightarrow a^{1/x} = b^{1/y} \Rightarrow (a^{1/x})^x = (b^{1/y})^x \Rightarrow a = b^{x/y}$

$$\Rightarrow \log_b a = \frac{x}{y} = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b}$$

This is the most important property of logarithms and applies to most of the problems. Here, the base can be taken as any positive real number except unity.

E.g. $\log_3 5 = \frac{\log_2 5}{\log_2 3} = \frac{\log_{10} 5}{\log_{10} 3} = \frac{\log_{1/2} 5}{\log_{1/2} 3}$

Note: $\log_3 \pi$ and $\log_\pi 3$ are reciprocals of each other.

The following properties can be deduced using base changing theorem.

(a) $\log_b a = \frac{1}{\log_a b}$; $\quad$ Proof: $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$

(b) $\log_b a \cdot \log_c b \cdot \log_d c = \log_d a$ $\quad$ Proof: $\log_b a \cdot \log_c b \cdot \log_d c = \frac{\log a}{\log b} \cdot \frac{\log b}{\log c} \cdot \frac{\log c}{\log d} = \frac{\log a}{\log d} = \log_d a$

(c) $a^{\log_e c} = c^{\log_e a}$ $\quad$ Proof: $a^{\log_e c} = (a^{\log_e c})^{\log_e a} = c^{\log_e a}$ $\quad (\because \ a^{\log_e N} = N)$

(i) $(\log_b a \cdot \log_a b = 1 \quad \Rightarrow \log_b a = \frac{1}{\log_a b}$ $\quad$ (ii) $e^{x \ln a} = a^x$

Illustration 2: What is logarithm of 32 $\sqrt[5]{4}$ to the base $2\sqrt{2}$ **(JEE MAIN)**

Sol: Here we can write 32 $\sqrt[5]{4}$ as $2^5\,4^{1/5} = (2)^{27/5}$ and $2\sqrt{2}$ as $2^{\frac{3}{2}}$ and then by using the formulae $\log_a M^x = x\log_a.M$ and $\log_{a^x} M = \frac{1}{x}\log_a M$ we can solve it.

$$\log_{2\sqrt{2}} 32\sqrt[5]{4} = \log_{\left(2^{3/2}\right)}\left(2^5\,4^{1/5}\right) = \log_{\left(2^{3/2}\right)}(2)^{27/5} = \frac{2}{3}\frac{27}{5}\log_2 2 = \frac{18}{5} = 3.6$$

Illustration 3: Prove that, $\log_{4/3}(1.\overline{3}) = 1$ **(JEE MAIN)**

Sol: By solving we get $1.\overline{3} = \frac{4}{3}$, and use the formula $\log_a a = 1$.

$\log_{4/3} 1.\overline{3} = 1$

Let $x = 1.333 \ldots$ ... (i)

$10x = 13.3333 \ldots$ ... (ii)

From Equation (i) and (ii), we get

So $9x = 12 \Rightarrow x = 12/9, x = 4/3$;

Now $\log_{4/3} 1/\overline{3} = \log_{4/3}(4/3) = 1$

Illustration 4: If N = n! (n ∈ N, n ≥ 2) then $\lim_{N\to\infty}$ $[(\log_2 N)^{-1} + (\log_3 N)^{-1} + \ldots + (\log_n N)^{-1}]$ is **(JEE ADVANCED)**

Sol: Here by using $\log_a b = \frac{1}{\log_b a}$ we can write given expansion as $\log_N 2 + \log_N 3 + \ldots\ldots + \log_N n$ and then by using $\log_a(M.N) = \log_a M + \log_a N$ and N = n! we can solve this.

$(\log_2 N)^{-1} + (\log_3 N)^{-1} + \ldots\ldots + (\log_n N)^{-1} = \log_N 2 + \log_N 3 + \ldots\ldots + \log_N n = \log_n(2.3 \ldots. N) = \log_N N = 1.$

Illustration 5: If $\log x^2 - \log 2x = 3\log 3 - \log 6$ then x equals **(JEE ADVANCED)**

Sol: By using $\log_a(M.N) = \log_a M + \log_a N$ and $\log_a M^x = x\log_a.M$ we can easily solve above problem.

Clearly x > 0. Then the given equation can be written as $2\log x - \log 2 - \log x = 3\log 3 - \log 2 - \log 3$

$\Rightarrow \log x = 2\log 3 \Rightarrow x = 9$

Illustration 6: Prove that, $\log_{2-\sqrt{3}}(2+\sqrt{3}) = -1$ **(JEE ADVANCED)**

Sol: By multiplying and dividing by $2+\sqrt{3}$ to $2-\sqrt{3}$ we will get $2+\sqrt{3} = \frac{1}{2-\sqrt{3}}$. Therefore by using $\log_{1/N} N = -1$ we can easily prove this.

$$\Rightarrow \log_{2-\sqrt{3}}\frac{1}{2-\sqrt{3}} \quad \Rightarrow \log_{2-\sqrt{3}}\left(2-\sqrt{3}\right)^{-1} \quad \Rightarrow -1.\log_{2-\sqrt{3}}\left(2-\sqrt{3}\right) = -1$$

Illustration 7: Prove that, $\log_5\sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}} = 1$ **(JEE ADVANCED)**

Sol: Here $\sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$ can be represented as $y = \sqrt{5y}$ where $y = \sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$. Hence, by obtaining the value of y we can prove this.

Let $y = \sqrt{5\sqrt{5\sqrt{5\ldots\ldots\infty}}}$

$y = \sqrt{5y}$ $\Rightarrow y^2 = 5y$ or $y^2 - 5y = 0$

$y(y - 5) = 0$ $\Rightarrow y = 0, y = 5$

y = 0 is not possible because log is not defined for zero.

$\therefore$ $\log_5 5 = 1$

Illustration 8: Prove that, $\log_{2.25}\left(0.\bar{4}\right) = -1$ **(JEE MAIN)**

Sol: As similar to illustration 3 we can solve it by using $\log_{1/N} N = -1$.

x = 0.4444..... ... (i)

10x = 4.4444..... ... (ii)

Equ (ii) – Equ (i)

So $9x = 4 \Rightarrow x = 4/9$

Also, $2.25 = \frac{225}{100} = \frac{9}{4}$; $\log_{2.25}\left(0.\bar{4}\right) = \log_{\left(\frac{9}{4}\right)}\left(\frac{4}{9}\right) = -1$

Illustration 9: Find the value of $2^{\log_6 18}.3^{\log_6 3}$ **(JEE MAIN)**

Sol: We can solve above problem by using $\log_a(M.N) = \log_a M + \log_a N$ and $a^{\log_e c} = c^{\log_e a}$ step by step.

$2^{\log_6 18}(3)^{\log_6 3} = 2^{\log_6(6\times3)}.3^{\log_6 3} = 2^{1+\log_6 3}.3^{\log_6 3} = 2\cdot2^{\log_6 3}.3^{\log_6 3}$ $(\because a^{\log_e c} = c^{\log_e a})$

$= 2\cdot(3)^{\log_6 2}\cdot(3)^{\log_6 3} = 2(3)^{\log_6 2+\log_6 3} = 2(3)^{\log_6(6)} = 2\cdot(3) = 6$

Illustration 10: Find the value of, $\log_{\sec\alpha}\left(\cos^3\alpha\right)$ where $\alpha \in (0, \pi/2)$ **(JEE MAIN)**

Sol: Consider $\log_{\sec\alpha}\left(\cos^3\alpha\right) = x$. Therefore by using formula $y = \log_a x \Leftrightarrow a^y = x$ we can write $\cos^3\alpha = (\sec\alpha)^x$. Hence by solving this we will get the value of x.

Let $\log_{\sec\alpha}\cos^3\alpha = x$

$\cos^3\alpha = (\sec\alpha)^x \Rightarrow (\cos\alpha)^3 = \left(\frac{1}{\cos\alpha}\right)^x \Rightarrow (\cos\alpha)^3 = (\cos\alpha)^{-x} \Rightarrow x = -3$

Illustration 11: If $k \in N$, such that $\log_2 x + \log_4 x + \log_8 x = \log_k x$ and $\forall\ x \in R'$ **(JEE ADVANCED)**

If $k = (a)^{1/b}$ then find the value of a + b; $a \in N$, $b \in N$ and b is a prime number.

Sol: By using $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b}$ we can obtain the value of k and then by comparing it to $k = (a)^{1/b}$ we can obtain value of a + b.

Given, $\frac{\log x}{\log 2} + \frac{\log x}{2\log 2} + \frac{\log x}{3\log 2} = \frac{\log x}{\log k} \Rightarrow \frac{\log x}{\log 2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] = \frac{\log x}{\log k} \Rightarrow \frac{\log x}{\log 2}\left(\frac{11}{6}\right) = \frac{\log x}{\log k} \Rightarrow \log x\left[\frac{11}{6}\frac{1}{\log 2} - \frac{1}{\log k}\right] = 0$

Also, $\frac{11}{6}\frac{1}{\log 2} - \frac{1}{\log k} = 0 \Rightarrow \frac{11}{6} = \frac{\log 2}{\log k} \Rightarrow \frac{11}{6} = \log_k 2$

So $2 = k^{\frac{11}{6}}$; $2^{6/11} = k \Rightarrow \left(2^6\right)^{\frac{1}{11}} = k \Rightarrow (64)^{\frac{1}{11}} = k$

Comparing by $k = (a)^{1/b} \Rightarrow a = 64$ and $b = 11 \Rightarrow a + b = 64 + 11 = 75$

6.4 Logarithmic Equation

While solving logarithmic equation, we tend to simplify the equation. Solving the equation after simplification may give some roots which do not define all the terms in the initial equation. Thus, while solving an equation involving logarithmic function, we must take care of all the terms involving logarithm.

Let a = log (x) and b = log (x + 2)

In general, a + b = log (x) + log (x + 2) = log [x (x + 2)]

If we take, x = –3, a and b both are not defined, but a + b will be defined.

as a + b = log [(–3) (–3 + 2)] = log (3)

Here, the problem lies in the definition of a and b. a and b is not defined here, so addition of a and b i.e. a + b will not be defined.

Note: A similar situation might arise while solving logarithmic equations. To avoid or to reject extraneous roots we have to define the logarithm.

Illustration 12: Solve $\log_4 8 + \log_4 (x + 3) - \log_4 (x - 1) = 2$ **(JEE MAIN)**

Sol: As we know $\log_a(M.N) = \log_a M + \log_a N$, $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ and $y = \log_a x \Leftrightarrow a^y = x$. By using these formulae we can solve the problem above.

$\log_4 8 + \log_4 (x + 3) - \log_4 (x - 1) = 2$

$\Rightarrow \log_4 \frac{8(x+3)}{x-1} = 2 \Rightarrow \frac{8(x+3)}{x-1} = 4^2 \Rightarrow x + 3 = 2x - 2 \Rightarrow x = 5$

Also for x = 5 all terms of the equation are defined.

Illustration 13: Solve log (–x) = 2 log (x + 1) **(JEE MAIN)**

Sol: Here it's given that log (–x) = 2 log (x + 1). Therefore by using the formula $\log_a M^x = x \log_a .M$. We can evaluate the value of x.

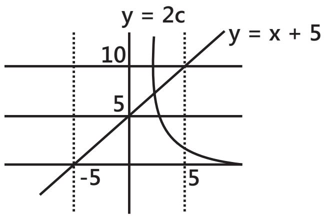
By definition, $x < 0$ and $x + 1 > 0 \Rightarrow -1 < x < 0$

Now $\log(-x) = 2\log(x + 1) \Rightarrow -x = (x + 1)^2 \Rightarrow x^2 + 3x + 1 = 0$

$\Rightarrow x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ (rejected). Hence, $x = \frac{-3+\sqrt{5}}{2}$ is the only solution.

Illustration 14: Find the number of solutions to the equation $\log_2 (x + 5) = 6 - x$. **(JEE MAIN)**

Sol: By using the formula $y = \log_a x \Leftrightarrow a^y = x$, we can write given the equation as $x + 5 = 2^{6-x}$. Hence, by checking the number of intersections made by the graph of $y = x + 5$ and $y = 2^{6-x}$ we will obtain the number of solutions.



Here, $x + 5 = 2^{6-x}$

Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersects only once. Hence, there is only one solution.

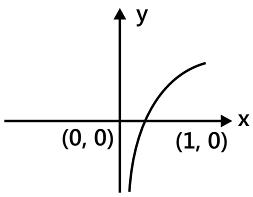
PLANCESS CONCEPTS

Always check your answer by putting it back in the equation; sometimes answer might not be in the domain of logarithm.

Shrikant Nagori (JEE 2009, AIR 7)

6.5 Graph of Logarithmic Function

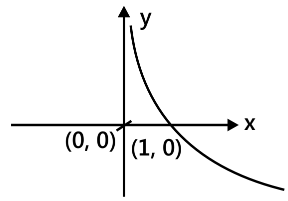
(i)



$y = \log_a x \; \forall \; a > 1$

Figure 1.6

(ii)



$y = \log_a x \; \forall \; 0 < a < 1$

Figure 1.7

If the number and the base are on the same side of unity, then the logarithm is positive, and if the number and the base are on different side of unity then the logarithm is negative.

Illustration 15: Which of the following numbers are positive/negative? **(JEE MAIN)**

(i) $\log_2 7$ (ii) $\log_{1/2} 3$ (iii) $\log_{1/3} (1/5)$ (iv) $\log_4 3$ (v) $\log_2 9$

Sol: By observing whether the Number and Base are on the same side of unity or not we can say whether the numbers are positive or negative.

(i) Let $\log_2 7 = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(ii) Let $\log_{1/2} 3 = x$ (number and base are on the same side of unity) $\Rightarrow x < 0$

(iii) Let $\log_{1/3}(1/5) = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(iv) Let $\log_4 3 = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

(v) Let $(\log_2 9) = x$ (number and base are on the same side of unity) $\Rightarrow x > 0$

6.6 Characteristic and Mantissa

(a) Given a number N, Logarithm can be $\log_{10} N = \text{Integer} + \text{Fraction}$

$$\log_{10} N = \underset{\text{Characterstic}}{\underset{\downarrow}{\text{Integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{Fraction}}}$$

(b) The mantissa part of the log of a number is always kept non-negative, it ranges from [0, 1]

(c) If the characteristic of $\log_{10} N$ is C then the number of digits in N is (C + 1)

(d) If the characteristic of $\log_{10} N$ is (–C) then there exist (C – 1) number of zeros after decimal point of N.

Illustration 16: Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$. **(JEE ADVANCED)**

Sol: Simply by applying log on both sides and using various logarithm formulas we can solve the above illustration.

$\log x = \log(0.15)^{20} = 20 \log\left(\frac{15}{100}\right) = 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2] = 20[\log 3 + 1 - \log 2 - 2] \left(\because \log_{10} 5 = \log_{10} \frac{10}{2}\right)$

$= 20[-1 + \log 3 - \log 2] = -20 \times 0.824 = -16.48 = \overline{17}.52$

Hence, characteristic = – 17 and mantissa = 0.52

Illustration 17: Find the number of digits in the following: (i) 2^{100} (ii) 3^{10} **(JEE ADVANCED)**

Sol: By considering $x = 2^{100}$ and 3^{10} respectively and applying log on both sides we can solve the problems given above.

(i) Let, $x = 2^{100}$

$\log_{10} X = \log_{10} 2^{100} = 100 \log_{10} 2 = 100 \times 0.3010 = 30.10$

Characteristic = 30, Mantissa = 0.10

Number of digits before decimal = C + 1 = 30 + 1 = 31

(ii) Let, $X = 3^{10}$

$\log_{10}x = 10 \log 3 = 10 \times 0.4771 = 4.771$

C = 4, M = 0.771

Number of digits before decimal = C + 1 = 4 + 1 = 5

Note: Let y = log (N) when 0 < N < 1

If N lies between 0 and 1, then the characteristic is negative

$N = 1/10, \log_{10}N = \log_{10}(1/10) = -1, C = -1, M = 0$

$N = 0.01, \log_{11}N = \log_{10}(10)^{-2} = -2,\ C = -2, M = 0$

$N = 0.001, \log_{10}N = \log_{10}10^{-3} = -3,\ C = -3, M = 0$

No. of zeros after decimal = |–3| –1 = 2

$N = 0.002, \log_{10}N = \log(2 \times 10^{-3}) = \log 2 + \log 10^{-3} = 0.03010 + (-3) = -0.3010 = -2.699$

C = – 3, M = 0.3010

Number of zeros after decimal = magnitude of the characteristic –1 = |C| –1 = |–3| –1 = 2

Illustration 18: Find the number of zeros after decimal before a significant figure in

(i) 3^{-50} (ii) 2^{-100} (iii) 7^{-100} **(JEE ADVANCED)**

Sol: Similar to the illustration above, we can solve these too.

(i) $N = 3^{-50}$

$\log_{10} N = \log_{10} 3^{-50} = -50 \log_{10}3 = -50 \times (0.4771) \Rightarrow \log_{10} N = -23.855$

Now to find the characteristic and mantissa many would say that (c = –23, m = –0.855) (which is wrong) because mantissa is always non-negative.

$\log_{10}N = -23.855 = -23 - 1 + 1 - 0.855 = -24 + 0.145$

C = –24, M = 0.145. Number of zeroes after decimal = |–24| –1 = 23 or |–24 + 1| = 23

(ii) $N = 2^{-100}$

$\log_{10} N = -100 \log 2 = -30.10 = -30 - 0.10 = -31 + 0.90$. Number of zeroes after decimal = |–31| –1 = 30 or |–31 + C| = 30

(iii) $N = 7^{-100}$

$\log_{10}N = -100 \log 7 = -100 \times 0.8451 = -84.51 = -84 - 1 + 0.49 = -85 + 0.49$

C = –85, M = 0.49. Number of zeroes after decimal = |–85| –1 = 84 or |–85 + 1| = 84

Illustration 19: Find the number of positive integers which have the characteristic 2, when base of log is 6.

(JEE ADVANCED)

Sol: If any number x has the characteristic a, when base of log is b then $x = b^a$. By using the given condition we can solve the problem above.

$x = 6^2 = 36$; $\log_6 x = \log_6 6^2 = 2 \log_6 6 = 2$

The smallest natural number which has characteristic 3 with base 6 is 6^3

$x = 6^3 = 216$; $\log_6 x = \log_6 6^3 = 3$

Hence x = 215 will give characteristic 2.

Natural numbers ranging from 36 to 215 will give characteristic 2, when taken log with base 6.

Number of positive integers = 215 – 35 = 180

6.7 Algebraic Inequalities

(a) If $a < b$ and $b < c \Rightarrow a < c$

(b) If $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc$, if b and d are of same sign. $\Rightarrow ad > bc$ if b and d are of opposite sign.

(c) If $a > b$ then, $a\lambda > b\lambda$ if $\lambda > 0$; $a\lambda < b\lambda$ if $\lambda < 0$

6.8 Logarithmic Inequalities

If the base is less than one, then the inequality will change. If base is greater than one, then inequality will remain the same.

$$\left.\begin{matrix}\log_a x < \alpha & \Rightarrow 0 < x < a^{\alpha} \\ \log_a x < \log_a y & \Rightarrow 0 < x < y\end{matrix}\right\} \text{ if } a > 1$$

$$\left.\begin{matrix}\log_a x < \alpha & \Rightarrow x > a^{\alpha} \\ \log_a x < \log_a y & \Rightarrow x > y > 0\end{matrix}\right\} \text{ if } 0 < a < 1$$

Illustration 20: Solve $(1/2)^{x^2-2x} < 1/4$ **(JEE MAIN)**

Sol: Here we can write the given equation as $(1/2)^{x^2-2x} < (1/2)^2$ and then by comparing powers on both side we can solve this.

We have $(1/2)^{x^2-2x} < (1/2)^2$. It means $x^2 - 2x > 2$

$\Rightarrow \left(x-\left(1+\sqrt{3}\right)\right)\left(x-\left(1-\sqrt{3}\right)\right) > 0 \Rightarrow x > 1 + \sqrt{3}$ or $x < 1 - \sqrt{3} \Rightarrow x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

Illustration 21: Solve $\frac{1+5^x}{7^{-x}+97} \geq 0$. **(JEE MAIN)**

Sol: Simply by multiplying $\left(7^{-x}-7^2\right)$ on both sides and solving we will get the result.

$g(x) = \frac{1-5^x}{7^{-x}-7} \leq 0$. Now $\left(1-5^x\right)\left(7^{-x}-7\right) \leq 0$; $5^x - 1 = 0 \Rightarrow x = 0$; $7^{-x} - 7 = 0 \Rightarrow x = -1$

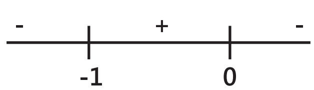


Figure 1.8

g(x) behavior on the number line. Hence, from above, $x \in (-\infty, -1) \cup [0, \infty)$

6.9 Modulus Function

Definition: Modulus of a number. The modulus of a number is denoted by |a|

$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$ Also, $\sqrt{a^2} = |a|$; Eg: y = |x|

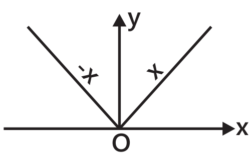


Figure 1.9

Basic properties of modulus

(A) $|ab| = |a|\,|b|$

(B) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $b \neq 0$

(C) $|a + b| \leq |a| + |b|$

(D) $|a - b| \geq |a| - |b|$ equality holds if $ab \geq 0$

Using triangle inequality

If $a > 0$

(i) $|x| = a \Rightarrow x = \pm a$

(ii) $|x| = -a \Rightarrow$ No solution

(iii) $|x| > a \Rightarrow x < -a$ or $x > a$

(iv) $|x| < a \Rightarrow -a < x < a$

(v) $|x| > -a \Rightarrow x \in R$

(vi) $|x| < -a \Rightarrow$ No solution

(vii) $a < |x| < b \Rightarrow x \in (-b, -a) \cup (a, b)$ where $a, b \in R^+$

Illustration 22: Solve for x, $|x - 2| = 3$ **(JEE MAIN)**

Sol: The above illustration can be solved by taking two cases; the first one is by taking $x - 2$ as greater than 0 and second one is by taking x - 2.

Case-I: When $x - 2 \geq 0 \Rightarrow x \geq 2$ ... (i)

Since $x - 2$ is non negative, the modulus can simply be removed. $x - 2 = 3$; $x = 5$

We had taken $x \geq 2$ and we got $x = 5$ hence this result satisfy the initial condition $\Rightarrow x = 5$

Case-II: When $x - 2 < 0 \Rightarrow x < 2$; Since $x - 2$ is negative, the modulus will open with a –ve sign.

$-(x - 2) = 3$; $-x + 2 = 3 \Rightarrow x = -1$ Since $x < 2$ Hence $x = -1, 5$

Illustration 23: Solve for x, $|x + 3| + |x - 2| = 11$ **(JEE ADVANCED)**

Sol: As $x + 3 = 0 \Rightarrow x = -3$ and $x - 2 = 0 \Rightarrow x = 2$. Therefore we can solve it by using the modulus inequality.

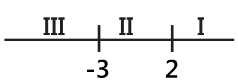


Figure 1.10

Case-I: For $x \geq 2$, $x + 3 > 0$, $x - 2 > 0$; $x + 3 + x - 2 = 11 \Rightarrow 2x = 10 \Rightarrow x = 5$

Case-II: For $-3 \leq x < 2$, $x + 3 \geq 0$, $x - 2 < 0$; $|x + 3| + |x - 2| = 11$

$\Rightarrow x + 3 - x + 2 = 11 \Rightarrow 5 = 11$ is impossible $\Rightarrow$ Hence, No value of x

Case-III: For $x < -3$

$x + 3 < 0$, $x - 2 < 0$; $|x + 3| + |x - 2| = 11 \Rightarrow -(x + 3) - (x - 2) = 11$

$\Rightarrow -x - 3 - x + 2 = 11 \Rightarrow -2x = 12 \Rightarrow x = -6$, since $x < -3$

Hence, to satisfy the initial condition, combining all we get $x = -6, 5$

Illustration 24: Solve for x, $x|x| = 4$ **(JEE MAIN)**

Sol: Here we can solve this problem by using two case, first one for $x > 0$ and the other one is for $x < 0$.

Case-I: For $x > 0$; $x.x = 4$

$x^2 = 4 \Rightarrow x = \pm 2$ but $x > 0$, hence $x = 2$ (-2 rejected)

Case-II: For $x < 0$; $x(-x) = 4$

$x^2 = -4$ no solution ; Hence, the only solution is $x = 2$

Illustration 25: Solve for x, $|x - 3| + 2|x + 1| = 4$ **(JEE ADVANCED)**

Sol: As $x - 3 = 0 \Rightarrow x = 3$; and $x + 1 = 0 \Rightarrow x = -1$.

Therefore by applying the cases $X \geq 3$, $-1 \leq x < 3$ and $x < -1$ we can solve this.

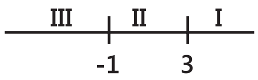


Figure 1.11

Mark the points on number line

Case-I: For $x \geq 3$

$(x - 3)$ is non-negative $(x + 1)$ is also positive

$\Rightarrow (x - 3) + 2(x + 1) = 4 \Rightarrow 3x = 5 \Rightarrow x = 5/3 \Rightarrow x = 5/3$ is discarded, since x should be > 3

Case-II: For $-1 \leq x < 3$; $x - 3$ is –ve, $x + 1$ is positive

$\Rightarrow -(x - 3) + 2(x + 1) = 4 \Rightarrow -x + 3 + 2x + 2 = 4 \Rightarrow x = -1$ satisfies the initial condition

Case-III: $x < -1$

$\Rightarrow -(x - 3) - 2(x + 1) = 4$; $-3x = 3 \Rightarrow x = -1$ $\Rightarrow$ Does not satisfy $x < 1$ Hence, solution is $x = -1$ from case-II.

6.10 Exponential and Logarithm Series

6.10.1 The Number 'e'

The sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ .... $+ \infty$ is denoted by the number e

i.e. $e = \lim\limits_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$

(i) The number e lies between 2 and 3. Approximate value of e = 2.718281828.

(ii) e is an irrational number.

6.10.2 Some Standard Deduction from Exponential Series

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{n!} + ...\infty$

(ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ... \frac{(-1)^n}{n!}x^n + ...\infty$ (Replace x by –x)

(iii) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$ (Substituting x = 1 in (i))

(iv) $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\infty$ (Substituting x = – 1 in (i))

(v) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\infty$

(vi) $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty$

(vii) $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + ...;$ $(a > 0)$, where $\ln a = \log_e(a)$

6.10.3 Logarithmic Series

If $-1 < x \leq 1$

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

(ii) $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

(iii) $\ln(x + 1) - \ln(1 - x) = \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$

(iv) $\ln(1 + x) + \ln(1 - x) = \ln(1 - x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...\right)$

6.11 Antilogarithm

The positive number n is called the antilogarithm of a number m if m = log n. If n is the antilogarithm of m, we write n = antilog m. For example

(i) log (100) = 2 $\Rightarrow$ antilog 2 = 100

(ii) log (431.5) = 2.6350 $\Rightarrow$ antilog (2.6350) = 431.5

(iii) log (0.1257) = $\bar{1}.0993$ $\Rightarrow$ antilog ($\bar{1}.0993$) = 0.1257

6.12 To find the Antilog of a Number

Step I: Determine whether the decimal part of the given number is positive or negative. If it is negative make it positive by adding 1 to the decimal part and by subtracting 1 from the integral part. For, example, in – 2.5983

$-2.5983 = -2 - 0.5983 = -2 - 1 + 1 - 0.5983 = -3 + 0.4017 = \bar{3}.4017$

Step II: In the antilogarithm, look into the row containing the first two digits in the decimal part of the given number.

Step III: In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

Step IV: In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number obtained in step III.

Step V: Obtain the integral part (characteristic) of the given number.

If the characteristic is positive and is equal to n, then insert decimal point after (n + 1) digits in the number obtained in step IV.

Illustration 26: Find the antilogarithm of each of the following: **(JEE MAIN)**

(i) 2.7523 (ii) 0.7523 (iii) $\bar{2}.7523$ (iv) $\bar{3}.7523$

Sol: By using log table and following the above mentioned steps we can find the algorithms of above values.

(i) The mantissa of 2.7523 is positive and is equal to 0.7523.

Now, look into the row starting 0.75. In this row, look at the number in the column headed by 2. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 3. The number there is 4. Add this number to 5649 to get 5653. The characteristic is 2. So, the decimal point is put after 3 digits to get 565.3

(ii) Proceeding as above, we have antilog (0.7523) = 5.653.

(iii) In this case, the characteristic is $\bar{2}$, i.e., – 2. So, we write one zero on the digit side of the decimal point. Hence, antilog ($\bar{2}$.7523) = 0.05653

(iv) Proceeding as above, antilog ($\bar{3}$.7523) = 0.005653

PROBLEM-SOLVING TACTICS

(a) The main thing to remember about surds and working them out is that it is about manipulation. Changing and manipulating the equation so that you get the desired result. Rationalizing the denominator is all about manipulating the algebra expression.

(b) Strategy for Solving Equations containing Logarithmic and Non-Logarithmic Expressions:

- **(i)** Collect all logarithmic expressions on one side of the equation and all constants on the other side.
- **(ii)** Use the Rules of Logarithms to rewrite the logarithmic expressions as the logarithm of a single quantity with coefficient of 1.
- **(iii)** Rewrite the logarithmic equation as an equivalent exponential equation.
- **(iv)** Solve for the variable.
- **(v)** Check each solution in the original equation, rejecting apparent solutions that produce any logarithm of a negative number or the logarithm of 0. Usually, a visual check suffices!

Note: The logarithm of 0 is undefined

(c) Logarithmic series

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$
(ii) $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$
(iii) $\ln(x + 1) - \ln(1 - x) = \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$
(iv) $\ln(1 + x) + \ln(1 - x) = \ln(1 - x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...\right)$

FORMULAE SHEET

(a) Laws of indices

(i) $a^0 = 1$, $(a \neq 0)$	**(ii)** $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$
(iii) $a^{m+n} = a^m.a^n$, where m and n are real numbers	**(iv)** $a^{m-n} = \frac{a^m}{a^n}$

(v) $\left(a^m\right)^n = a^{mn}$	**(vi)** $a^{p/q} = \sqrt[q]{a^p}$
(vii) $\left(ab\right)^n = a^n b^n$	**(viii)** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(b) Some Important Identities

(i) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
(ii) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
(iii) $a^2 - b^2 = (a + b)(a - b)$
(iv) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
(v) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
(vi) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$
(vii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$
(viii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$
(ix) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$
(x) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
(xi) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$
(xii) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

(c) Laws of Surds

(i) $\sqrt[n]{a} = a^{\frac{1}{n}}$	**(ii)** $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$
(iii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	**(iv)** $\left(\sqrt[n]{a}\right)^n = a$
(v) $\left(\sqrt[m]{\sqrt[n]{a}}\right) = \sqrt[mn]{a}$	**(vi)** $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

(d) Logarithm formulas

(i) $\log_a\left(M.N\right) = \log_a M + \log_a N$	**(ii)** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
(iii) $y = \log_a x \Leftrightarrow a^y = x\left(a, x > 0, a \neq 1\right)$	**(iv)** $\log_a M^x = x \log_a .M$

(v) $\log_{a^x} M = \frac{1}{x}\log_a M \quad (x \neq 0)$	**(vi)** $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b} \quad (c > 0, c \neq 1)$
(vii) $\log_a b = \frac{1}{\log_b a}$	**(viii)** $\log_b a \cdot \log_c b \cdot \log_d c = \log_d a$
(ix) $a^{\log_e c} = c^{\log_e a}$	**(x)** $\log_b a \cdot \log_a b = 1$
(xi) $e^{x\ln a} = a^x$	**(xii)** $\log_a 1 = 0$ and $\log_a a = 1$

(e) Exponential series

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...\infty$	**(ii)** $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ... + \frac{(-1)^n}{n!}x^n + ...\infty$
(iii) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$	**(iv)** $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\infty$
(v) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\infty$	**(vi)** $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty$
(vii) $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + ..\ (a > 0)$	

Solved Examples

JEE Main/Boards

Example 1: Evaluate $\sqrt[3]{72.3}$, if $\log_{10} 72.3 = \bar{1}.8591$

Sol: Here consider $x = \sqrt[3]{72.3}$. Now by applying log on both sides and solving using logarithm formula we will get value of $\sqrt[3]{72.3}$.

Let $x = \sqrt[3]{72.3}$, Then, $\log x = \log (72.3)^{1/3}$

$\Rightarrow \log_{10} x = \frac{1}{3} \log_{10} 72.3$ ... (i)

$\log 72.3 = \log(0.723 \times 10^2)$

$\Rightarrow \log 0.723 + \log 10^2 = \bar{1}.8591 + 2 = 1.8591$ ... (ii)

$\Rightarrow \log_{10} x = \frac{1}{3} \times 1.8591$

$\Rightarrow \log_{10} x = 0.6197$; $\Rightarrow x = \text{antilog}\ (0.6197)$

$\Rightarrow x = 4.166$ (using antilog table)

Example 2: Using logarithm, find the value of 6.45×981.4

Sol: Consider $x = 6.45 \times 981.4$ and then apply log on both sides and solve by using $\log_a(M.N) = \log_a M + \log_a N$ and log table.

Then, $\log_{10} x = \log_{10} (6.45 \times 981.4)$

$= \log_{10} 6.45 + \log_{10} 981.4$

$= 0.8096 + 2.9919$ (using log table)

$\therefore x = \text{antilog}\ (3.8015) = 6331$ (using antilog table)

Example 3: Find minimum value of x satisfying

$|x - 3| + 2|x + 1| = 4$

Sol: Similar to illustration 25.

Case-I: When $x < -1$

$-1(x - 3) - 2(x + 1) = 4$

$\Rightarrow -x + 3 - 2x - 2 = 4$; $\Rightarrow -3x + 1 = 4$

$\Rightarrow 3x = -3$; $\Rightarrow x = -1$

$\because$ x < – 1 so, x = –1 is not possible

Case-II: When $-1 \le x < 3$

$\Rightarrow -(x-3) + 2(x+1) = 4 \Rightarrow -x + 3 + 2x + 2 = 4$

$\Rightarrow x + 5 = 4 \quad \Rightarrow x = -1$; So, x = - 1 is a solution.

Case-III: When $x \ge 3$ is taken, $(x-3) + 2(x+1) = 4$

$\Rightarrow 3x - 1 = 4 \quad \Rightarrow x = 5/3 \Rightarrow$ Therefore, no solution

Result x = –1 is the only solution.

Example 4: Let

$\log_3 N = \alpha_1 + \beta_1, \log_5 N = \alpha_2 + \beta_2, \log_7 N = \alpha_3 + \beta_3$

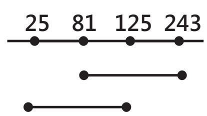
where $\alpha_1, \alpha_2, \alpha_3 \in I$ and $\beta_1, \beta_2, \beta_3 \in [0,1)$ then

(i) Find number of integral values of N if α_1 = 4 and α_1 = 2

(ii) Find the largest integral value of N if α_1 = 5, α_2 = 3, α_3 = 2

(iii) Difference of largest and smallest integral values

Sol: Here by using $y = \log_a x \Leftrightarrow a^y = x$ we can obtain values of N. After that by drawing a number line we will get the required answer.



(i) $N = 3^{4+\beta_1}$ and $N = 5^{2+\beta_2}$

$N = [3^4, 3^5)$ and $N = [5^2, 5^3]$

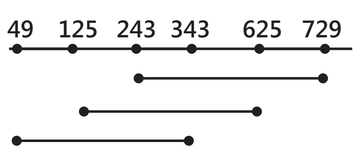
N = [81, 243) and N = [25, 125)

So [81, 125] is the common part hence the no. of integral values of N are 125 – 81 = 44

(i) $N = 3^{5+\beta_1}, N = \left[3^5, 3^6\right), N = \left[243, 729\right)$

$N = 5^{3+\beta_2}, N = \left[5^3, 5^4\right), N = \left[125, 625\right)$

$N = 7^{2+\beta_3}, N = [7^2, 7^3], N = \left[49, 343\right)$



Common part is [243, 343]. So largest integral value = 342

(b) Difference of largest and smallest values

= 342 – 243 = 99

Example 5: Find the number of zeros in, $x = (0.35)^{12}$, Given $\log_{10}(7) = 0.8451$, $\log_{10}(2) = 0.3010$

Sol: By applying $\log_{10}$ on both sides and using logarithm formulae we will get the result.

$$\log_{10} x = 12\log_{10}\left(\frac{35}{100}\right)$$

$$\log_{10} x = 12\log 35 - \log_{10} 100 = 12$$

$$\left[\log_{10} 7 + \log_{10} 5 - 2\right] = 12$$

$$\left[\log_{10} 7 + \log_{10} 10 - \log_{10} 2 - 2\right]$$

= 12[.8451 + 1 – .3010 –2] = 12 [.5441 –1]

$\log_{10} x = -12 + 6.5292$

$$\log_{10} x = -12 + 6 + 0.5292 = -6 + 0.5292 = \bar{6}.5292$$

So $x = 10^{-6}.10^{-5212}$

Hence the number of zeros after the decimal = 5

Example 6: Find the number of zeros in, 2^{-40}

Sol: Consider 2^{-40} = x and solve as in illustration 5.

$$x = \frac{1}{2^{40}} = 2^{-40}$$

$$\log_{10} x = -40\log_{10} 2 = -40\left[.3010\right] = -12.0400$$

$$\log_{10} x = \left(-12 - 0.04\right) + 1 - 1 \Rightarrow \log_{10} x = -13 + 0.96$$

$$\Rightarrow x = 10^{-13}.10^{0.96}$$

Number of zeros = 12

Example 7: Find the number of digits for $x = 3^{12} \times 2^8$

Sol: By applying $\log_{10}$ on both sides and then using a log table we can solve the problem above.

$$\log_{10} x = 12\log_{10} 3 + 8\log_{10} 2$$

$$\left(0.4771\right) + 8\left(0.3010\right) = 5.7252 + 2.4080$$

$$\log_{10} x = 8.1332 \Rightarrow x = \left(10^8\right)10^{0.1332}$$

No. of digits = 8 + 1 = 9

Example 8: Solve $x^{\log_{\sqrt{x}}(x-2)} = 9$

Sol: Here, by using $\log_{a^x} M = \frac{1}{x}\log_a M$ we can solve the problem above.

$x^{\log_{\sqrt{x}}(x-2)} = 9 \quad \Rightarrow x^2\log x\ (x\text{-}2) = 9$

$\Rightarrow x^{\log_x (x-2)^2} = 9$ where x > 0, $x \ne 1$

$\Rightarrow \left(x-2\right)^2 = 9; \quad \Rightarrow x - 2 = \pm 3$

$\Rightarrow$ x = – 1, x = 5

But x = –1 is rejected as x should be greater than 0.

Example 9: $\log_3\left(\log_{1/2}^2 x - 3\log_{1/2} x + 5\right) = 2$

Sol: $\log_3\left(\log_{1/2}^2 x - 3\log_{1/2} x + 5\right) = 2$

$\Rightarrow \log^2_{1/2} x - 3\log_{1/2} x + 5 = 9$;

Let $\log_{1/2}(x) = t \quad \Rightarrow t^2 - 3t - 4 = 0$

(t – 4) (t + 1) = 0 ⇒ t = 4, t = –1

$\Rightarrow \log_{1/2} x = 4, \log_{1/2} x = -1$

x= 1/16, x = 2

Example 10: Solve $\dfrac{1-2\left(\log_{10} x^2\right)^2}{\log_{10} x - 2\left(\log_{10} x\right)^2} = 1$

Sol: Simply by putting $\log_{10} x = t$ we can solve the problem above.

$\dfrac{1-2\left(\log_{10} x^2\right)^2}{\log_{10} x - 2\left(\log_{10} x\right)^2} = 1$, Let $\log_{10} x = t$

$\Rightarrow \dfrac{1-2(2t)^2}{t-2t^2} = 1 \Rightarrow 1 - 8t^2 = t - 2t^2$

$\Rightarrow 6t^2 + 3t - 2t - 1 = 0 \Rightarrow 3t(2t+1) - 1(2t+1) = 0$

t = 1/3, t = –1/2 ⇒ log x = 1/3, log x = $-\dfrac{1}{2}$

$x = 10^{1/3}, x = 10^{-1/2}$

Example 11 $\left(\dfrac{1}{5}\right)^{\log^2_{10} x - \log_{10} x} = \dfrac{1}{125}.5^{\log_{10} x - 1}$

Sol: By using $a^{m-n} = \dfrac{a^m}{a^n}$ we can evaluate the problem above.

$\left(\dfrac{1}{5}\right)^{\log^2_{10} x - \log_{10} x} = 5^{\left(\log_{10} x - 1\right) - 3} \quad \therefore 5^{\log x - \log^2 x} = 5^{\log x - 4}$

$\Rightarrow \log x - \log^2 x = \log x - 4 \quad \Rightarrow \log^2 x = 4$

x = 10^2, x = 10^{-2}

JEE Advanced/Boards

Example 1: Solve,

$\log_{3x+7}\left(9+12x+4x^2\right) + \log_{2x+3}\left(6x^2+23x+21\right) = 4$

Sol: Here $6x^2 + 23x + 21$

$= (2x+3)(3x+7)$ and $\left(9+12x+4x^2\right) = (2x+3)^2$.

Hence substitute it in the above equation and solve using the logarithm formula.

Given that

$\log_{3x+7}\left(9+12x+3x^2\right) + \log_{2x+3}\left(6x^2+23x+21\right) = 4$

$\log_{3x+7}(2x+3)^2 + \log_{2x+3}\left[(2x+3)(3x+7)\right] = 4$ Let

$\log_{3x+7}(2x+3) = A$; $2A + 1 + \dfrac{1}{A} = 4$

$\Rightarrow 2A^2 - 3A + 1 = 0$; $2A^2 - 2A - A + 1 = 0$

⇒ 2A (A –1) –1 (A –1) = 0; A = 1/2, A = 1

$\Rightarrow \log_{3x+7}(2x+3) = 1/2$

For $A = \dfrac{1}{2}, 2x+3 = \sqrt{3x+7}$

$\Rightarrow 4x^2 + 9 + 12x = 3x + 7$; $4x^2 + 9x + 2 = 0$

$\Rightarrow 4x^2 + 8x + x + 2 = 0 \Rightarrow 4x(x+2) + 1(x+2) = 0$

$\Rightarrow x = \dfrac{-1}{4}, x = -2$; For $A = 1, \log_{3x+7} 2x + 3 = 1$

⇒ 2x + 3 = 3x + 7

⇒ x = –4 also 2x + 3 > 0, 3x + 7 > 0

⇒ x > –3/2, x > – 7/3

$\Rightarrow x = \dfrac{-1}{4}$ (–4 and –2 will be rejected)

Example 2: Solve, $(x)^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \left(\frac{5}{4}\right)} = \sqrt{2}$

Sol: By taking $\log_x$ on both sides and solving we will get the result.

Taking log on both sides to the base x

$\log_x\left[(x)\right]^{\frac{3}{4}(\log_2 x)^2 + (\log_2 x) - \frac{5}{4}} = \log_x\left(\sqrt{2}\right)$

$\dfrac{3}{4}(\log_2 x)^2 + (\log_2 x) - \dfrac{5}{4} = \dfrac{1}{2}\log_x 2$

Let $\log_2 x = t$; $\dfrac{3}{4}t^2 + t - \dfrac{5}{4} = \dfrac{1}{2t}$

$3t^3 + 4t^2 - 5t = 2 \Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0$

$\Rightarrow 3t^3 + 3t^2 - 6t + t^2 + t - 2 = 0 \Rightarrow (3t+1)\left(t^2 + t - 2\right) = 0$

$\Rightarrow (3t+1)(t+2)(t-1) = 0; \Rightarrow t = 1, -2, -\dfrac{1}{3}$

Putting t = $\log_2 x$

$\log_2 x = 1 \Rightarrow x = 2$; $\log_2 x = -2 \Rightarrow x = \frac{1}{4}$

$\log_2 x = -1/3 \Rightarrow x = 1/(2)^{1/3}$

Example 3: Solve $|x-1|^{\log_3 x^2 - 2\log_x 9 - 6} = (x-1)$

Sol: As a^x is defined for a > 0 so (x –1) > 0. Therefore by taking log on both side we can solve it.

Now taking log on both sides

$$\left(\log_3 x^2 - 2\log_x 9\right)\log(x-1) = \log(x-1)$$

$$\left(2\log_2 x - \frac{2}{\log_{3^2} x} - 1\right)\log(x-1) = 0$$

Either log (x – 1) = 0 ⇒ x = 2

Let $\log_3 x = t$

(2t – 4/t – 7) = 0 ⇒ $2t^2 - 4 - 7t = 0$

⇒ $2t^2 - 8t + t - 4 = 0 \Rightarrow 2t(t-4) + 1(t-4) = 0$

t = 4, t = –1/2

$\log_3 x = 4$ or $\log_3 x = -1/2$

$x = (3)^4$ or $x = (3)^{-1/2}$

x = 81, x = $1/\sqrt{3}$

For x = $\frac{1}{\sqrt{3}}$ log (x – 1) is not defined, so x = 2 or x = 81.

Example 4: Solve,

$$\log_4\left(x^2-1\right) - \log_4(x-1)^2 = \log_4\sqrt{(4-x)^2}$$

Sol: By using formula $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ and using modulus inequalities we can solve the problem above.

$$\log_4\frac{\left(x^2-1\right)}{(x-1)^2} = \log_4|4-x|\left[\because \sqrt{x^2} = |x|\right]$$

$$\Rightarrow \log_4\frac{(x-1)(x+1)}{(x-1)^2} = \log_4|4-x|$$

So we have $\frac{(x+1)}{(x-1)} = |4-x|$

or (x + 1) = (x – 1) |4 – x|

Case-I: 4 – x > 0 or x < 4 then (x + 1) = (x – 1) (4 – x)

$\Rightarrow x+1 = 4x - x^2 - 4 + x \Rightarrow x^2 - 4x + 5 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16-20}}{2}, x \notin R$

which is not possible

Case-II: (4 – x) < 0 or x > 4 then (x + 1) = (x – 1) (x – 4)

$\Rightarrow x + 1 = x^2 - 5x + 4$ $\Rightarrow x^2 - 6x + 3 = 0$

$\Rightarrow x^2 - 6x + 3 = 0$ $\Rightarrow x = \frac{6 \pm \sqrt{24}}{2}$

$\Rightarrow x = \frac{6 \pm 2\sqrt{6}}{2}$ $\Rightarrow x = 3 \pm \sqrt{6}$

$\because x > 4$ is taken, hence x = 3 + $\sqrt{6}$

Example 5: If the sum of all solutions of the equation $\left[(x)^{\log_{10} 3}\right]^2 - \left(3^{\log_{12} x}\right) - 2 = 0$ is $(a)^{\log b}$ where b and c arerelatively prime and a, b c ∈ N then (a + b + c) = ?

Sol: Here by putting $3^{\log_{10} x}$ = t and solving we will get the result.

$$\left((3)^{\log_{10} x}\right)^2 - \left(m^{\log_{10} x}\right) - 2 = 0$$

Let e $3^{\log_{10} x}$ = t then

$\Rightarrow t^2 - t - 2 = 0$; $\Rightarrow t^2 - 2t + t - 2 = 0$

$\Rightarrow t(t-2) + 1(t-2) = 0$; $\Rightarrow (t+1)(t-2) = 0$

Case-I: $t = -1$; $\Rightarrow t = -1$ & $t = 2$ $\Rightarrow 3^{\log_{10} x} = -1$

Exponential value cannot be negative

Case-II: t = 2; $3^{\log_{10} x} = 2$

Taking $\log_3$ both side

$\log_3 (3)^{\log_{10} x} = \log_3 2 \Rightarrow \log_{10} x = \log_3 2 \Rightarrow x = 10^{\log_3 2}$

Comparing by $(a)^{\log_b c}$ we get

a = 10 , b = 3, c = 2

∴ a + b + c = 10 + 3 + 2 = 15

Example 6: Find the number of zeros after decimal before a significant digit in $(9/8)^{-100}$.

Sol: By putting x = $\left(\frac{9}{8}\right)^{-100}$ and applying $\log_{10}$ on both side we will get the result.

Let x = $\left(\frac{9}{8}\right)^{-100}$

$\Rightarrow \log_{10} x = -100\left[\log_{10} 9 - \log_{10} 8\right]$

$\Rightarrow \log_{10} x = -100\left[2\log_{30} 3 - 3\log_{10} 2\right]$

$\Rightarrow \log_{10} x = -100\left(2\times 0.4771 - 3\times 0.3010\right)$
$= -100\left[0.9542 - 0.9030\right] = -100\left[0.0512\right] = -5.12$

$\log_{30} x = \left(-5 - 0.12\right) + 1 - 1$

$\log_{10} x = \overline{6}.88 \Rightarrow x = 10^{-6} \times 10^{0.88}$

$\therefore$ Number of zeros before any significant digits = 5

Example 7: Solve $\log_4\left(2\log_3\left(1+\log_2\left(1+3\log_2 x\right)\right)\right) = 1/2$

Sol: Here by using $y = \log_a x \Leftrightarrow a^y = x$ we can solve it.

$\log_4\left(2\log_3\left(1+\log_3\left(3\log_3 x\right)\right)\right) = 1/2$

$\Rightarrow 2\log_3\left(1+\log_2\left(1+3\log_3 x\right)\right) = 2$

$\Rightarrow \log_3\left(1+\log_2\left(1+3\log_3 x\right)\right) = 1$

$\Rightarrow 1+\log_2\left(1+3\log_3 x\right) = 3 \Rightarrow \log_2\left(1+3\log_3 x\right) = 2$

$\Rightarrow 1+3\log_3 x = 4 \Rightarrow 3\log_3 x = 3 \Rightarrow \log_3 x = 1 \Rightarrow x = 3$

Example8: Solve $\log_{0.5x} x - 7\log_{16x} x^3 + 40\log_{4x} \sqrt[4]{x} = 0$

Sol: By using $\log_b a = \frac{\log a}{\log b}$ we can reduce the given equation to $\frac{\log_2 x}{\log_2 0.5x} - \frac{7\log_2 x^3}{\log_2 16x} + \frac{40\log_2 \sqrt[4]{x}}{\log_2 4x} = 0$ and then by putting $\log_2 x = t$ we can solve it.

Let $\log_2 x = t$

$$\Rightarrow \frac{t}{-1+t} - \frac{7(3t)}{4+t} + \frac{10t}{2+t} = 0 \Rightarrow \frac{t}{t-1} - \frac{21t}{t+4} + \frac{10t}{t+2} = 0$$

$$\Rightarrow t\left\{\frac{(t+4)(t+2) - 21(t-1)(t+2) + 10(t-1)(t+4)}{(t-1)(t+4)(t+2)}\right\} = 0$$

$$\Rightarrow t\left\{\frac{t^2+6t+8-21t^2-21t+42+10t^2+30t-40}{(t-1)(t+4)(t+2)}\right\} = 0$$

$$\Rightarrow t\left\{\frac{-10t^2+15t+10}{(t-1)(t+4)(t+2)}\right\} = 0$$

$\Rightarrow t = 0, -\frac{1}{2}, 2$ $\quad\therefore \log_2 x = 0 \Rightarrow x = 1$

$\log_2 x = -\frac{1}{2} \Rightarrow x = \frac{1}{\sqrt{2}}$ and $\log_2 x = 2 \Rightarrow x = 4$

Example 9: Solve, $\log_2\left(x/4\right) = \frac{15}{\log_2 \frac{x}{8} - 1}$

Sol: Simply by putting $\log_2$ (x) = t and using basic logarithmic formula we can solve the problem above.

$$\log_2\left(x/4\right) = \frac{15}{\log_2 \frac{x}{8} - 1} \Rightarrow \left(\log_2 x - 2\right) = \frac{15}{\left(\log_2 x - 3\right) - 1}$$

Let $\log_2$ (x) = t

$\Rightarrow t - 2 = \frac{15}{t-4}$ $\quad\Rightarrow$ t2 – 6t +8 = 15

$\Rightarrow t^2 - 6t - 7 = 0 \Rightarrow (t-7)(t+1) = 0$

$\Rightarrow t = 7, t = -1 \Rightarrow \log_2 x = 7$ and $\log_2 x = -1$

$\Rightarrow x = 2^7$ and $x = 2^{-1}$

Example 10: Solve, $\sqrt{\log_2\left(2x^2\right)\log_4\left(16x\right)} = \log_4 x^3$

Sol: By using $\log_a(M.N) = \log_a M + \log_a N$ we can reduce the given equation to $\sqrt{\left(1+2\log_2 x\right)\left(2+\frac{1}{2}\log_2 x\right)} = \frac{3}{2}\log_2 x$.

After that putting $\log_2 x = t$ we will get the result.

$\sqrt{\log_2\left(2x^2\right)\log_4\left(16x\right)} = \log_4 x^3$

$\Rightarrow \sqrt{\left(1+2\log_2 x\right)\left(2+\frac{1}{2}\log_2 x\right)} = \frac{3}{2}\log_2 x$

Let $\log_2 x = t$

$\Rightarrow \sqrt{\left(1+2t\right)\left(2+\frac{t}{2}\right)} = \frac{3}{2}t$

$\Rightarrow \left(1+2t\right)\left(\frac{4+t}{2}\right) = \frac{9t^2}{4} \Rightarrow \left(2t+1\right)\left(t+4\right) = \frac{9t^2}{2}$

$\Rightarrow 2.\left(2t^2+9t+4\right) = 9t^2$

$\Rightarrow 5t^2 - 18t - 8 = 0 \Rightarrow 5t^2 - 20t + 2t - 8 = 0$

$\Rightarrow 5t\left(t-4\right) + 2\left(t-4\right) = 0; t = -2/5, t = 4$

But $t \neq 4 \Rightarrow x = 6$ and logx = –2/5 is Not Possible

$\therefore t = -\frac{2}{5} \Rightarrow \log_2 x = -\frac{2}{5} \quad \therefore x = 2^{-2/5}$

JEE Main/Boards

Exercise 1

Q.1 Solve

(i) $\log_{16} 32$

(ii) $\log_8 16$

(iii) $\log_{1/3}(1/9)$

(iv) $\log_{2\sqrt{3}}(1728)$

(v) $\log_2 \cos 45^\circ$

(vi) $\log_2(\log_2 4)$

(vii) $\log_3(\tan 30^\circ)$

Q.2 Prove the following

(i) $\log_5 \sqrt{5\sqrt{5\sqrt{5-\infty}}} = 1$

(ii) $\log_{0.125}(8) = -1$

(iii) $\log_{1.5}(0.\overline{6}) = -1$

(iv) $\log_{2.25}(0.\overline{4}) = -1$

(v) $\log_{10}(0.\overline{9}) = 0$

Q.3 Find the no. of digits in

(i) 2^{100} (ii) 3^{10}

Q.4 Solve

(i) $\log_{x-1} 3 = 2$

(ii) $\log_3(3^x - 8) = 2 - x$

(iii) $\log_{5-x}(x^2 - 2x + 65) = 2$

(iv) $\log_3(x+1) + \log_3(x+3) = 1$

(v) $x^{2\log_{10} x} = 10.x^2$

(vi) $x^{\frac{\log_{10} x+5}{3}} = 10^{5+\log_{10} x}$

(vii) $x^{\log_3 x} = 9$

Q.5 $1 - \log 5 = \frac{1}{3}\left(\log\frac{1}{2} + \log x + \frac{1}{3}\log 5\right)$

Q.6 $\log x - \frac{1}{2}\log\left(x - \frac{1}{2}\right) = \log\left(x + \frac{1}{2}\right) - \frac{1}{2}\log\left(x + \frac{1}{8}\right)$

Q.7 $x^{\frac{\log_{10} x+7}{4}} = 10^{\log_{10} x+1}$

Q.8 $\left(\frac{\log_{10} x}{2}\right)^{\log^2_{10} x + \log_{10} x^2 - 2} = \log_{10}\sqrt{x}$

Q.9 $\sqrt{\log_2 x} - \log_2 8x + 1 = 0$

Q.10 $\log_{1/3} x - 3\sqrt{\log_{1/3} x} + 2 = 0$

Q.11 $\left(a^{\log_b x}\right)^2 - 5a^{\log_b x} + 6 = 0$

Q.12 $\log_4(x^2 - 1) - \log_4(x-1)^2 = \log_4\left(\sqrt{(4-x)^2}\right)$

Q.13 $2\log_3\frac{x-3}{x-7} + 1 = \log_3\frac{x-3}{x-1}$

Q.14 $\log_x(9x^2)\log_3^2 x = 4$

Q.15 $\log_{0.5x} x^2 + 14\log_{16x} x^2 + 40\log_{4x}\sqrt{x} = 0$

Q.16 $\log_3\left(\log^2_{1/2} x - 3\log_{1/2} x + 5\right) = 2$

Q.17 $\log_2(x/4) = \frac{15}{\log_2\frac{x}{8} - 1}$

Q.18 $\frac{1}{2}\log_{10}(5x-4) + \log_{10}\sqrt{x+1} = 2 + \log_{10} 0.18$

Q.19 $\log_{10} x^2 = \log_{10}(5x-4)$

Q.20 $\frac{1}{6}\log_2(x-2) - \frac{1}{3} = \log_{1/8}\sqrt{3x-5}$

Q.21 $\frac{\log_{10}\left(\sqrt{x+1}+1\right)}{\log_{10}\left(\sqrt[3]{x-40}\right)} = 3$

Q.22 $1-\frac{1}{2}\log_{10}\left(2x-1\right)=\frac{1}{2}\log_{10}\left(x-9\right)$

Q.23 $\log_{10}\left(3x^2+7\right)-\log_{10}\left(3x-2\right)=1$

Q.24 $\left(1+\frac{1}{2x}\right)\log_{10}3+\log_{10}2=\log_{10}\left(27-3^{1/x}\right)$

Q.25 $\frac{1}{2}\log_{10}x+3\log_{10}\sqrt{2+x}=\log_{10}\sqrt{x\left(x+2\right)}+1$

Q.26 $\log_2\left(4^x+1\right)=x+\log_2\left(2^{x+3}-6\right)$

Q.27 $\log_{\sqrt{5}}\left(4^x-6\right)-\log_{\sqrt{5}}\left(2^x-2\right)=2$

Q.28 $\log_{10}\left(3^x-2^{4-x}\right)=2+\frac{1}{4}\log_{10}16-\frac{x\log_{10}4}{2}$

Q.29 $\log_{10}\left(\log_{10}x\right)+\log_{10}\left(\log_{10}x^4-3\right)=0$

Q.30 $\log_3\left(9^x+9\right)=\log_3 3^x\left(28-2.3^x\right)$

Exercise 2

Single Correct Choice Type

Q.1 $\frac{1}{\log_{\sqrt{bc}}abc}+\frac{1}{\log_{\sqrt{ac}}abc}+\frac{1}{\log_{\sqrt{ab}}abc}$

has the value equal to

(A) 1/2 (B) 1 (C) 2 (D) 4

Q.2 The equation, $\log_2\left(2x^2\right)+\log_2 x.x^{\log_x\left(\log_2 x+1\right)}$ $+\frac{1}{2}\log_4 2x^4+2^{-3\log_{1/2}\left(\log_2 x\right)}=1$ has

(A) Exactly one real solution (B) Two real solutions

(C) 3 Real solutions (D) No solution

Q.3 Number of zeros after decimal before a significant figure in $(75)^{-10}$ is:

(Use $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$)

(A) 20 (B) 19 (C) 18 (D) None

Q.4 If $5x^{\log_2 3}+3^{\log_2 x}=162$ then logarithm of x to the base 4 has the value equal to

(A) 2 (B) 1 (C) –1 (D) 3/2

Q.5 $(x)^{\log_{10}^2 x+\log_{10}x^3+3}=\frac{2}{\frac{1}{\sqrt{x+1}-1}-\frac{1}{\sqrt{x+1}+1}}$

where $x_1 > x_2 > x_3$, then

(A) $x_1+x_3=2x_2$ (B) $x_1.x_3=x_2^2$

(C) $x_2=\frac{2x_1x_2}{x_1+x_2}$ (D) $x_1^{-1}+x_1^{-1}=x_3^{-1}$

Q.6 Let $x=2^{\log 3}$ and $y=3^{\log 2}$ where base of the logarithm is 10, then which one of the following holds good?

(A) 2x < y (B) 2y < x (C) 3x = 2y (D) y = x

Q.7 Number of real solution(s) of the equation

$\left|x-3\right|^{3x^2-10x+3}=1$ is-

(A) Exactly four (B) Exactly three

(C) Exactly two (D) Exactly one

Q.8 If x_1 and x_2 are the roots of the equation $\sqrt{2010x^{\log_{2010}x}}=x^2$, then find the cyphers at the end of the product (x_1x_2)

(A) 1 (B) 3 (C) 2 (D) 4

Q.9 Let x = 2 or x = 3 satisfy the equation, $\log_4(x^2+bx+c)=1$. Then find the value of |bc|.

(A) 50 (B) 60 (C) 40 (D) 55

JEE Advanced/Boards

Exercise 1

Q.1 Let A denotes the value of

$$\log_{10}\left(\frac{ab+\sqrt{(ab)^2-4(a+b)}}{2}\right)+\log_{10}\left(\frac{ab-\sqrt{(ab)^2-4(a+b)}}{2}\right)$$

when a = 43 and b = 57 and B denotes the value of the expression $\left(2^{\log_6 18}\right).\left(3^{\log_6 3}\right)$. Find the value of (A.B).

Q.2 Simplify:

(a) $\log_{1/3}\sqrt[4]{729\sqrt[3]{9^{-1}.27^{-4/3}}}$ (b) $\frac{\log_b(\log_b N)}{\log_b a}$

Q.3 (a) Which is smaller? 2 or $\left(\log_\pi 2+\log_2 \pi\right)$

(b) Prove that $\log_3 5$ and $\log_2 7$ are both irrational.

Q.4 Find the square of the sum of the roots of the equation $\log_3 x\cdot\log_4 x\cdot\log_5 x=\log_3 x\cdot\log_4 x+\log_4 x\cdot\log_5 x+\log_5 x\cdot\log_3 x$.

Q.5 Find the value of the expression

$$\frac{2}{\log_4(2000)^6}+\frac{3}{\log_5(2000)^6}$$

Q.6 Simplify: $\dfrac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}}-(125)^{\log_5 6}\right)$

Q.7 Simplify: $5^{\log_{1/5}(1/2)}+\log_{\sqrt{2}}\frac{4}{\sqrt{7}+\sqrt{3}}+\log_{1/2}\frac{1}{10+2\sqrt{21}}$

Q.8 Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2} 8 = \frac{2}{s^3+1}$

Write $\log_2\frac{a^2 b^5}{c^4}$ as function of 's' (a, b, c > 0) (c ≠ 1).

Q.9 Prove that $\frac{\log_2 24}{\log_{96} 2}-\frac{\log_2 192}{\log_{12} 2}=3$

Q.10 Prove that $a^x - b^y = 0$ wher $x=\sqrt{\log_a b}$ and $y=\sqrt{\log_b a}$, a>0, b>0 & b≠1.

Q.11 (a) Solve for x, $\frac{\log_{10}(x-3)}{\log_{10}(x^2-21)}=\frac{1}{2}$

(b) log (log x) + log (logx^3 –2) = 0; where base of log is 10 everywhere

(c) $\log_x 2.\log_{2x} 2 = \log_{4x} 2$

(d) $5^{\log x} + 5x^{\log 5} = 3$ (a > 0); where base of log is a

Q.12 Solve the system of equations:

$\log_a x \log_a (xyz) = 48$

$\log_a y \log_a (xyz) = 12$

$\log_a z \log_a (xyz) = 84$

Q.13 Let 'L' denotes the antilog of 0.4 to the base 1024. and 'M' denotes the nuber of digits in 6^{10} (Given $\log_{10}2$ = 0.3010, $\log_{10}3$ = 0.4771) and 'N' denotes the number of positive integers which have the characteristic 2, when base of the logarithm is 6. Find the value of LMN.

Q.14 Prove the identity.

$\log_a N . \log_b N + \log_b N . \log_c N + \log_c N . \log_a N = \frac{\log_a N\log_b N\log_c N}{\log_{abc} N}$

Q.15 If x, y > 0, $\log_y x + \log_x y = \frac{10}{3}$ and xy = 144, then $\frac{x+y}{2}=\sqrt{N}$ where N is a natural number, find the vaue of N.

Q.16 If $\log_{10}2$ = 0.0310, $\log_{10}3$ = 0.4771. Find the number of integers in:

(a) 5^{200}

(b) 6^{15}

(c) The number of zeros after the decimal in 3^{-100}.

Q.17 $\log_5 120 + (x-3) - 2\log_5(1-5^{x-3}) = -\log_5(2-5^{x-4})$

Q.18 $\log_{x+1}(x^2+x-6)^2 = 4$

Q.19 $x + \log_{10}(1+2^x) = x\log_{10}5 + \log_{10}6$

Q.20 If 'x' and 'y' are real numbers such that,

$\log_{10}(2x-3x)=\log_{10}x+\log_{10}y$, find $\frac{x}{y}$

Q.21 If a = $\log_{12}18$ and b = $\log_{24}54$ then find the value of ab + 5 (a – b)

Q.22 Find the value of $\log_3 x$ if following is true

$$\sqrt{\log_9\left(9x^4\right)\log_3(3x)}=\log_3 x^3$$

Q.23 Positive numbers x, y and z satisfy xyz = 10^{81} and ($\log_{10}$ x) ($\log_{10}$ yz) + ($\log_{10}$ y) ($\log_{10}$ z) = 468. Find the value of $(\log_{10}x)^2 + (\log_{10}y)^2 + (\log_{10}z)^2$.

Q.24 Find the sum of all solutions of the equation

$$3^{(\log_9 x)^2}-\frac{9}{2}\log_9 x+5=3\sqrt{3}$$

Q.25 Let a, b, c, d are positive integers such that $\log_a b = 3/2$ and $\log_c d = 5/4$. If (a – c) = 9, find the value of (b – d).

Q.26 Find the product of the positive roots of the equation $\sqrt{(2008)}(x)^{\log_{2008}x}=x^2$

Q.27 Find x satisfying the equation

$$\log_{10}^2\left(1+\frac{4}{x}\right)+\log_{10}^2\left(1-\frac{4}{x+4}\right)=2\log_{10}^2\left(\frac{2}{x-1}-1\right)$$

Q.28 Solve: $\log_3\left(\sqrt{x}+\left|\sqrt{x}-1\right|\right)$

$=\log_9\left(4\sqrt{x}-3+4\left|\sqrt{x}-1\right|\right)$

Q.29 Prove that

$$2^{\left(\sqrt{\log_a\sqrt[4]{ab}+\log_b\sqrt[4]{ab}}-\sqrt{\log_a\sqrt[4]{b/a}+\log\sqrt[4]{a/b}}\right)\sqrt{\log_a b}}$$

$$=\begin{cases}2 & \text{if } b\ge a>1\\ 2^{\log_a b} & \text{if} 1<b<a\end{cases}$$

Q.30 Find the value of x satisfying the equation

$$\sqrt{\left[\log_3(3x)^{1/3}+\log_x(3x)^{1/3}\right]\log_3 x^3}$$

$$+\sqrt{\left[\log_3\left(\sqrt{\frac{x}{3}}\right)^{1/3}+\log_x\left(\frac{3}{x}\right)^{1/3}\right]\log_3 x^3}=2$$

Q.31 Let $a=(\log_7 81)(\log_{6561}625)(\log_{125}216)(\log_{1296}2401)$

b denotes the sum of the roots of the equation $x^{\log_2 x}=(2x)^{\log_2}\sqrt{x}$ and c denotes the sum of all natural solution of the equation |x + 1| + |x – 4| = 7.

Find the value of (a + b) ÷ c.

Exercise 2

Single Correct Choice Type

Q.1 Number of ordered pair(s) satisfying simultaneously, the system of equations, $2^{\sqrt{x}+\sqrt{y}}=256$ and $\log_{10}\sqrt{xy}-\log_{10}1.5=1$, is:

(A) Zero (B) Exactly one

(C) Exactly two (D) More than two

Q.2 Let ABC be a triangle right angled at C. The value of $\frac{\log_{b+c}a+\log_{c-b}a}{\log_{b+c}a.\log_{c-b}a}$ (b + c ≠, c – b ≠ 1) equals

(A) 1 (B) 2 (C) 3 (D) ½

Q.3 Let B, C, P and L be positive real number such that log (B · L) + log (B · P) = 2; log (P · L) + log (P · C) = 3; log (C · B) + log (C · L) = 4. The value of the product (BCPL) equals (base of the log is 10)

(A) 10^2 (B) 10^3 (C) 10^4 (D) 10^9

Q.4 If the equation $\frac{\log_{12}(\log_8(\log_4 x))}{\log_5(\log_4(\log_y(\log_2 x)))}=0$ has a solution for 'x' when c < y < b, y ≠ a, where 'b' is as large as possible and 'c' is an small as possible, then the value of (a + b + c) is equal to

(A) 18 (B) 19 (C) 20 (D) 21

Q.5 For N > 1, the product

$\frac{1}{\log_2 N}\cdot\frac{1}{\log_N 8}\cdot\frac{1}{\log_{32} N}\cdot\frac{1}{\log_N 128}$ simplifies to

(A) $\frac{3}{7}$ (B) $\frac{3}{7\ell n2}$ (C) $\frac{3}{5\ell n2}$ (D) $\frac{5}{21}$

Q.6 Let $N = 10^{3\log 2-2\log\left(\log 10^3\right)+\log\left(\left(\log 10^6\right)^2\right)}$ where base of the logarithm is 10. The characteristics of the logarithm of N to the base 3, is equal to

(A) 2 (B) 3 (C) 4 (D) 5

Q.7 If $x=\frac{\sqrt{10}+\sqrt{2}}{2}$ and $y=\frac{\sqrt{10}-\sqrt{2}}{2}$, then the value of $\log_2\left(x^2+xy+y^2\right)$, is equal to

(A) 0 (B) 2 (C) 3 (D) 4

Q.8 The sum $\sqrt{\frac{5}{4}+\sqrt{\frac{3}{2}}}+\sqrt{\frac{5}{4}-\sqrt{\frac{3}{2}}}$ is equal to

(A) $\tan\frac{\pi}{3}$ (B) $\cot\frac{\pi}{3}$ (C) $\sec\frac{\pi}{3}$ (D) $\sin\frac{\pi}{3}$

Q.9 Suppose that x < 0. Which of the following is equal to $\left|2x-\sqrt{(x-2)^2}\right|$

(A) x – 2 (B) 3x – 2 (C) 3x + 2 (D) – 3x + 2

Q.10 Solution set of the inequality

$3^x(0.333.....)^{x-3}\le(1/27)^x$ is:

(A) [3/2, 5] (B) $\left(-\infty,3/2\right]$

(C) $\left(2,\infty\right)$ (D) None of these

Q.11 Solution set of the inequality $\left(\frac{1}{5}\right)^{\frac{2x+1}{1-x}}>\left(\frac{1}{5}\right)^{-3}$ is-

(A) $\left(-\infty,-2\right)\cup\left(1,\infty\right)$ (B) (1, 4)

(C) $\left(-\infty,1\right)\cup\left(2,\infty\right)$ (D) None of these

Q.12 The set of all x satisfying the equation

$x^{\log_3 x^2+(\log_3 x)^2-10}=\frac{1}{x^2}$ is-

(A) {1, 9} (B) $\left\{9,\frac{1}{81}\right\}$ (C) $\left\{1,4,\frac{1}{81}\right\}$ (D) $\left\{1,9,\frac{1}{81}\right\}$

Q.13 If $\frac{(\ln x)^2-3\ln x+3}{\ln x-1}<1$, then x belongs to:

(A) (0, e) (B) (1, e) (C) (1, 2e) (D) (0, 3e)

Multiple Correct Choice Type

Q.14 The number $N=\frac{1+2\log_3 2}{\left(1+\log_3 2\right)^2}+\log_6^2 2$

when simplified reduces to-

(A) A prime number

(B) An irrational number

(C) A real number is less than $\log_3\pi$

(D) A real which is greater than $\log_7 6$

Q.15 The value of x satisfying the equation, $2^{2x}-8.2^x=-12$ is

(A) $1+\frac{\log 3}{\log 2}$ (B) $\frac{1}{2}\log 6$ (C) $1+\log\frac{3}{2}$ (D) 1

Q.16 If $\left(\sqrt{5\sqrt{2}-7}\right)^x+6\left(\sqrt{5\sqrt{2}+7}\right)^x=7$,

then the value of x can be equal to-

(A) 0 (B) $\log_{(5\sqrt{2}-7)}36$

(C) $\frac{-2}{\log_6\left(5\sqrt{2}+7\right)}$ (D) $\log_{\sqrt{5\sqrt{2}-7}}6$

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I.

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false

(D) Statement-I is false, statement-II is true

Q.17 Statement-I: $\sqrt{\log_x\cos(2\pi x)}$ is a meaningful quantity only if x ∈ (0, ¼) ∪ (3/4, 1).

Because

Statement-I: If the number N > 0 and the base of the logarithm b (greater than zero not equal to 1) both lie on the same side of unity then $\log_b N > 0$ and if they lie on different side of unit then $\log_b N < 0$.

Q.18 Statement-I: $\log_2\left(2\sqrt{17-2x}\right) = 1 + \log_2(x-1)$ has a solution.

because

Statement-II: Change of base in logarithms is possible.

Q.19 Consider the following statements

Statement-I: The equation $5^{\log_5\left(x^3+1\right)} - x^2 = 1$ has two distinct real solutions.

Because.

Statement-II: $a^{\log_a N} = N$ when a > 0, a ≠ 1 and N > 0.

Comprehension Type

Paragraph 1: Equations of the form (i) f ($\log_a x$) = 0, a > 0, a ≠ 1 and (ii) g ($\log_x A$) = 0, A > 0, then Eq. (i) is equivalent to f(t) = 0, where t = $\log_a x$. If t_1, t_2, t_3, ... t_k are the roots of f(t) = 0, then $\log_a x = t$, $\log_a x = t_2$, ... , x = t_k and eq. (ii) is equivalent to f(y) = 0, where y $\log_x A$. If y_f , y_2, y_3, ..., y_k are the root of f(y) = 0, then $\log_x A = y_1$, $\log_{x'} A = y_2$, ... , $\log_x A = y_k$.

On the basis of above information, answer the following questions.

Q.20 The number of solution of the equation $\log_x^3 10 - 6\log_x^2 10 + 11\log_x 10 - 6 = 0$ is:

(A) 0 (B) 1 (C) 2 (D) 3

Match the Columns

Q.21

Column-I	Column-II
(A) The value of x for which the radical product $\sqrt{3\sqrt{x}-\sqrt{7x+\sqrt{4x-1}}}\sqrt{2x+\sqrt{4x-1}}\sqrt{3\sqrt{x}+\sqrt{7x+\sqrt{4x-1}}}$ is equal to 13, is not greater than	(p) 4
(B) Let P(x) = $x^7 - 3x^5 + x^3 - 7x^2 + 5$ and Q (x) = x – 2. The remainder of $\frac{P(x)}{Q(x)}$ is not smaller than	(q) 7
(C) Given a right triangle with side of length a, b and c and area equal to $a^2 + b^2 - c^2$. The ratio of the larger to the smaller leg of the triangle is	(r) 10
(D) If a, b and c ∈ N such $\left(\sqrt[3]{4}+\sqrt{2}-2\right)\left(a\sqrt[3]{4}+b\sqrt[3]{2}+c\right) = 20$ Then the value of (a + b – c), is not equal to	(s) 17

Q.22

Column I	Column II
(A) The expression $x = \log_2 \log_9 \sqrt{6+\sqrt{6+\sqrt{6+.........\infty}}}$ simplifies to	(p) An integer
(B) The number $N = 2^{(\log_2 3.\log_3 4.\log_4 5.....\log_{99} 100)}$ simplifies to	(q) A prime
(C) The expression $\frac{1}{\log_5 3}+\frac{1}{\log_6 3}-\frac{1}{\log_{10} 3}$ simplifies to	(r) A natural
(D) The number N = $\sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{14-6\sqrt{5}}}}$ simplifies to	(s) A composite

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.2 Q.3 Q.15
Q.25 Q.26

Exercise 2

Q.3 Q.5 Q.9
Q.10

JEE Advanced/Boards

Exercise 1

Q.6 Q.12 Q.14
Q.16 Q.23

Exercise 2

Q.4 Q.6 Q.11
Q.15 Q.17

Answer Key

JEE Main/Boards

Exercise 1

Q.1 (i) $\frac{5}{4}$ (ii) $\frac{4}{3}$ (iii) 2 (iv) 6 (v) $-\frac{1}{2}$ (vi) 1 (vii) $-\frac{1}{2}$

Q.3 (i) 31 (ii) 5

Q.4 (i) $1+\sqrt{3}$ (ii) 2 (iii) –5 (iv) 0 (v) $10^{\frac{\sqrt{3}+1}{2}}, 10^{\frac{1-\sqrt{3}}{2}}$ (vi) $\frac{1}{10^5}, 1000$

(vii) $3^{\sqrt{2}}, 3^{-\sqrt{2}}$

Q.5 $\frac{2^4}{5^{1/3}}$ **Q.6** 1 **Q.7** 10^{-4}, 10 **Q.8** 10^{-3}, 10, 10^2

Q.9 2, 16 **Q.10** 1/3, $(1/3)^4$ **Q.11** $2^{\log_a b, 3\log_a b}$ **Q.12** $3+\sqrt{6}$

Q.13 – 5 **Q.14.** 3, 1/9 **Q.15** $2^{\left(-1+\sqrt{\frac{17}{5}}\right)}, 2^{\left(-1-\sqrt{\frac{17}{5}}\right)}$ **Q.16** 1/16, 2

Q.17 $2^7, 2^{-1}$ **Q.18** $8, -\frac{41}{5}$ **Q.19** 4, 1 **Q.20** 3

Q.21 48 **Q.22** 13 **Q.23** 1, 9 **Q.24** $\frac{1}{4}, \frac{1}{2}$

Q.25 98 **Q.26** 0 **Q.27** 2 **Q.28** 3

Q.29 $(10)^{-1/4}$, (10) **Q.30** (–1), 2

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** D **Q.3** C **Q.4** D **Q.5** B **Q.6** D

Q.7 B **Q.8** C **Q.9** A

JEE Advanced/Boards

Exercise 1

Q.1 12 **Q.2** (a) 1 (b) $\log_b N$ **Q.3** (a) 2 **Q.4** $(61)^2$

Q.5 1/6 **Q.6** 1 **Q.7** 6 **Q.8** $2s + 10s^2 - 3(s^3 + 1)$

Q.11 (a) 5 (b) 10 (c) $2^{\pm\sqrt{2}}$ (d) $2^{-\log_5 a}$ **Q.12** (a^4, a, a^7) or (a^{-4}, a^{-1}, a^{-7}) **Q.13** 23040

Q.15 507 **Q.16** (a) 140 (b) 12 (c) 47 **Q.17** -0.410 **Q.18** 1

Q.19 1 **Q.20** 4/9 **Q.21** 1 **Q.22** $\frac{5+3\sqrt{5}}{10}$

Q.23 5625 **Q.24** 2196 **Q.25** 93 **Q.26** $(2008)^2$

Q.27 $\sqrt{2}$, $\sqrt{6}$ **Q.28** $[0, 1] \cup \{4\}$ **Q.30** $[1/3, 3] - \{1\}$ **Q.31** 1

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** B **Q.3** B **Q.4** B **Q.5** D **Q.6** B

Q.7 C **Q.8** A **Q.9** D **Q.10** D **Q.11** B **Q.12** D

Q.13 A

Multiple Correct Choice Type

Q.14 C, D **Q.15** A, D **Q.16** A , B, C, D

Assertion Reasoning Type

Q.17 D **Q.18** B **Q.19** B

Comprehenstion Type

Q.20 D

Match the Columns

Q.21 A → q, r, s; B → p, q, r, s; C → p; D → r **Q.22** A → p; B → p, r, s; C → p, r; D → p, q, r

Solutions

JEE Main/Boards

Exercise 1

Sol 1: (i) $\log_{16}32 = \log_{2^4} 2^5$

we know $\log_{x^n} y^m = \frac{m}{n}\log_x y$

$\Rightarrow \log_{2^4} 2^5 = \frac{5}{4}\log_2 2 = \frac{5}{4}$

(ii) $\log_8 16 = \log_{2^3} 2^4 = \frac{4}{3}\log_2 2 = \frac{4}{3}(1) = \frac{4}{3}$

(iii) $\log_{1/3}(1/9) = \log_{1/3}(1/3)^2 = 2\log_{1/3}(1/3) = 2.(1) = 2$

(iv) $\log_{2\sqrt{3}}(1728) = \log_{2\sqrt{3}}(2\sqrt{3})^6$

$= 6\log_{2\sqrt{3}} 2\sqrt{3} = 6(1) = 6$

(v) $\log_2 \cos 45° = \log_2 \frac{1}{\sqrt{2}} = \log_2 (2)^{-\frac{1}{2}} = -\frac{1}{2}\log_2 2 = -\frac{1}{2}$

(vi) $\log_2 (\log_2^4)$

$\Rightarrow \log_2\left(\log_2 2^2\right)$

$\Rightarrow \log_2\left(2\log_2^2\right)$

$\Rightarrow \log_2^2 \quad \because \log_a^a = 1$

$\Rightarrow 1$

(vii) $\log_3 (\tan 30°)$

$\because \tan 30° = \frac{1}{\sqrt{3}}$

$\Rightarrow \log_3\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \log_3\left(3^{\frac{-1}{2}}\right)$

$\Rightarrow \frac{-1}{2}\log_3^3$

$\Rightarrow \frac{-1}{2}$

Sol 2: (i) $\log_5 \sqrt{5\sqrt{5\sqrt{5..........\infty}}}$

Let $5\sqrt{5\sqrt{5..........\infty}} = x$

$\Rightarrow 5\sqrt{x} = x \Rightarrow 5 = x^{1/2} \Rightarrow x = 25$

$\log_5 \sqrt{x} = \log_5 \sqrt{25} = \log_5 5 = 1$

(ii) $\log_{0.125} 8 = \log_{0.5}^2 \Rightarrow \frac{3}{3}\log_{0.5}^2$

$\Rightarrow \log_{1/2}^2 \Rightarrow \log_{(2)^{-1}}^2 \Rightarrow -1.\log_2^2 = -1$

(iii) $\log\left(0.\bar{6}\right)$

$\because 0.\bar{6} = 0.6666............ = \frac{2}{3}$

$\Rightarrow \log_{3/2}\left(3/2\right) = \log_{3/2}\left(3/2\right)^{-1} = -1$

(iv) $\log_{2.25}\left(0.\bar{4}\right)$

$\Rightarrow 0.\bar{4} = 0.444....... = \frac{4}{9} \Rightarrow 2.25 = \frac{225}{100} = \frac{9}{4}$

$\Rightarrow \log_{9/4}\left(4/9\right) = \log_{9/4}\left(9/4\right)^{-1} = 1$

(v) $\log_{10}\left(0.\bar{9}\right)$

$0.\bar{9} = 0.99999 = 1 \Rightarrow \log_{10} 1 = 0$

Sol 3: We have to find out no. of digits in

(i) $2^{100} = x$ (Assume)

$\Rightarrow \log_{10}x = \log_{10} 2^{100} = 100\log_{10} 2 = 100(0.3010) = 30.10$

$\Rightarrow x = 10^{30.103} = 10^{30}(10)^{0.10}$

Total no. of digit = 30 + 1 = 31

(ii) $x = 3^{10}$

$\Rightarrow \log_{10}x = \log_{10}3^{10} = 10\log_{10}3 = 10(0.47712) = 4.7712$

$\Rightarrow x = 10^{4.7712} = 10^4 \times 10^{0.7712}$

Total no. of digits = 4 + 1 = 5

Sol 4: (i) $\log_{x-1} 3 = 2$ ($x \ne 1, 2$)

$\frac{1}{2}\log_{x-1}3 = 1 \Rightarrow \log_{x-1}3^{1/2} = 1$

$3^{1/2} = x - 1 \Rightarrow x = 1 + \sqrt{3}$

(ii) $\log_3(3^x - 8) = 2 - x$

$\Rightarrow (3^x - 8) = (3)^{2-x} = 3^2 . 3^{-x} = 9.3^{-x} \Rightarrow 3^x - 9.3^{-x} = 8$

Assume $3^x = y$

$\Rightarrow y - \dfrac{9}{y} = 8 \Rightarrow y^2 - 9 = 8y \Rightarrow y^2 - 8y - 9 = 0$

$\Rightarrow y = \dfrac{8 \pm \sqrt{8^2 + 4(1)(9)}}{2(1)} \Rightarrow y = \dfrac{8 \pm \sqrt{64+36}}{2} = \dfrac{8 \pm \sqrt{100}}{2}$

$\Rightarrow y = 4 \pm 5 = 9, -1$

Terefore, $3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x = 2$

$3^x = -1 \Rightarrow$ no solution

Hence x = 2

(iii) $\log_{5-x}(x^2 - 2x + 65) = 2$

$\Rightarrow x^2 - 2x + 65 = (5 - x)^2 = x^2 + 5^2 - 2(5)x$

$\Rightarrow -2x + 65 = 25 - 10x \Rightarrow 10x - 2x = 25 - 65 = -40$

$\Rightarrow 8x = -40 \Rightarrow x = -\dfrac{40}{8} = -5$

(iv) $\log_3(x + 1) + \log_3(x + 3) = 1$

$\Rightarrow \log_3[(x + 1) . (x + 3)] = 1$

$\Rightarrow (x+1)(x+3) = 3 \Rightarrow x^2 + x + 3x + 3(1) = 3$

$\Rightarrow x^2 + 4x = 0 \Rightarrow x(x + 4) = 0 \Rightarrow x = 0, -4$

But at x = – 4, equation is

$\log_3(-4 + 1) + \log_3(-4 + 3) = 1$

It can't be –ve so $x \neq -4 \Rightarrow x = 0$

(v) $x^{2\log x} = 10x^2$

Take logarithms is both sides

$\log_{10}(x^{2\log x}) = \log_{10} 10x^2$

$2\log_{10}x(\log_{10}x) = \log_{10}10 + \log_{10}x^2$

$2\log_{10}x(\log_{10}x) = 1 + 2\log_{10}x$

Assume $\log_{10} x = y$ ... (i)

$\Rightarrow 2y(y) = 1 + 2y \Rightarrow 2y^2 = 1 + 2y \Rightarrow 2y^2 - 2y - 1 = 0$

$\Rightarrow y = \dfrac{2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$

$y = \dfrac{2 \pm \sqrt{4+8}}{4} = \dfrac{2 \pm 2\sqrt{3}}{4} = \dfrac{1 \pm \sqrt{3}}{2}$

So from equation (i)

$\log_{10}x = \dfrac{1 \pm \sqrt{3}}{2} \Rightarrow x = 10^{\frac{(1+\sqrt{3})}{2}}$ and $10^{\frac{(1-\sqrt{3})}{2}}$

(vi) $x^{\frac{\log_{10} x+5}{3}} = 10^{5+\log_{10}x}$

take logarithm (base 10) on both side

$\log_{10}\left[x^{\frac{\log_{10} x+5}{3}}\right] = \log_{10} 10^{5+\log x}$

$\Rightarrow \left(\dfrac{\log_{10} x+5}{3}\right)\log_{10} x = (5 + \log_{10} x)\log_{10} 10$

$\Rightarrow \left(\dfrac{5 + \log_{10} x}{3}\right)\log_{10} x = (5 + \log_{10} x)$

$\Rightarrow \log_{10}x = 1(3) = 3 \Rightarrow x = 10^3 = 1000$

2^{nd} solution $\Rightarrow 5 + \log x = 0$

$\Rightarrow \log_{10}x = -5 \Rightarrow x = 10^{-5}$

(vii) $x^{\log_3 x} = 9$

Take logarithm (base 3) in both sides

$\log_3[x^{\log_3^x}] = \log_3 9 = \log_3 3^2 = 2\log_3 3$

$\Rightarrow (\log_3 x)^2 = 2 \Rightarrow |\log_3 x| = 2^{1/2} \Rightarrow \log_3 x = \pm\sqrt{2}$

$\Rightarrow x = 3^{\sqrt{2}}, 3^{-\sqrt{2}}$

Sol 5: $1 - \log_{10} 5 = \dfrac{1}{3}\left(\log_{10}\dfrac{1}{2} + \log_{10} x + \dfrac{1}{3}\log_{10} 5\right)$

$3(1 - \log_{10} 5) = \log_{10}\dfrac{1}{2} + \log_{10} x + \dfrac{1}{3}\log_{10} 5$

$3 - \log_{10} 5 = \log_{10}\dfrac{1}{2} + \log_{10} 5^{1/3} + \log_{10} x$

$\Rightarrow 3 = \log_{10} 5^3 + \log_{10}\dfrac{1}{2} + \log_{10} 5^{1/3} + \log_{10} x$

$\Rightarrow 3 = \log_{10}\left[5^3 \times \dfrac{1}{2} \times 5^{1/3}\right] + \log_{10} x$

$\Rightarrow \log_{10} x = 3 - \log_{10}\left[5^{3+\frac{1}{3}} \times \left(\dfrac{1}{2}\right)\right]$

$\log_{10} x = \log_{10} 10^3 - \log_{10}\left(5^{10/3} \times 2^{-1}\right)$

$$= \log_{10}\left(\frac{10^3}{5^{10/3}2^{-1}}\right) = \log_{10}\frac{5^3\times 2^3}{5^{10/3}\times 2^{-1}}$$

$$= \log_{10}[5^{\frac{9-10}{3}}\, 2^{3+1}] = \log_{10}[5^{-1/3}\, 2^4]$$

$$\log_{10} x = \log_{10}\frac{2^4}{5^{1/3}} \Rightarrow x = \frac{2^4}{5^{1/3}}$$

Sol 6: $\log_{10} x - \frac{1}{2}\log_{10}\left(x-\frac{1}{2}\right) = \log_{10}\left(x+\frac{1}{2}\right)$

$$-\frac{1}{2}\log_{10}\left(x+\frac{1}{8}\right)$$

$$2\log_{10} x - \log_{10}\left(x-\frac{1}{2}\right) = 2\log_{10}\left(x+\frac{1}{2}\right) - \log_{10}\left(x+\frac{1}{8}\right)$$

$$\log_{10} x^2 - \log_{10}\left(x-\frac{1}{2}\right) = \log_{10}\left(x+\frac{1}{2}\right)^2 - \log_{10}\left(x+\frac{1}{8}\right)$$

$$\Rightarrow \log_{10}\left(\frac{x^2}{x-\frac{1}{2}}\right) = \log\left[\frac{\left(x+\frac{1}{2}\right)^2}{\left(x+\frac{1}{8}\right)}\right]$$

$$\Rightarrow \log_{10}\left(\frac{x^2}{x-\frac{1}{2}}\right) - \log_{10}\left[\frac{\left(x+\frac{1}{2}\right)^2}{\left(x+\frac{1}{8}\right)}\right] = 0$$

$$\Rightarrow \log_{10}\left[\frac{x^2}{\left(x-\frac{1}{2}\right)}\times\frac{x+\frac{1}{8}}{\left(x+\frac{1}{2}\right)^2}\right] = 0$$

$$\Rightarrow \left(\frac{x^2}{x-\frac{1}{2}}\right)\left(\frac{x+\frac{1}{8}}{\left(x+\frac{1}{2}\right)^2}\right) = 1 \Rightarrow \frac{x^2\left(x+\frac{1}{8}\right)}{\left(x^2-\frac{1}{4}\right)\left(x+\frac{1}{2}\right)} = 1$$

$$\Rightarrow x^2\left(x+\frac{1}{8}\right) = \left(x^2-\frac{1}{4}\right)\left(x+\frac{1}{2}\right)$$

$$\Rightarrow x^3 + \frac{x}{8} = x^3 + \frac{x^2}{2} - \frac{x}{4} - \frac{1}{4}\left(\frac{1}{2}\right)$$

$$\Rightarrow x^3 + \frac{x^2}{8} = x^3 + \frac{x^2}{2} - \frac{x}{4} - \frac{1}{8}$$

$$\Rightarrow x^2 = 4x^2 - 2x - 1 \Rightarrow 3x^2 - 2x - 1 = 0$$

$$\Rightarrow x = \frac{2\pm\sqrt{4+12}}{6} \Rightarrow x = \frac{2\pm 4}{6}$$

$x = 1, -\frac{1}{3}$ at $x = 1, -\frac{1}{3}$

$$2\log(-2) = \log(4)$$

Which is not possible $\Rightarrow x = 1$

Sol 7: $x^{\frac{\log_{10} x+7}{4}} = 10^{\log_{10} x+1}$

Take logarithm on both sides

$$\log_{10}\left(x^{\frac{\log_{10} x+7}{4}}\right) = \log_{10}\left(10^{\log_{10} x+1}\right)$$

$$\Rightarrow \left(\frac{\log_{10} x+7}{4}\right)(\log_{10} x) = (\log_{10} x+1)\log_{10} 10$$

$\Rightarrow$ Assume $\log_{10} x = y \Rightarrow \left(\frac{y+7}{4}\right)(y) = y + 1$

$\Rightarrow y^2 + 7y = 4(y + 1) = 4y + 4$

$\Rightarrow y^2 + 7y - 4y - 4 = 0 \Rightarrow y^2 + 3y - 4 = 0$

$\Rightarrow (y + 4)(y - 1) = 0 \Rightarrow y = -4$ and $+1$

$\Rightarrow \log_{10} x = -4$ or 1

Hence $x = 10^{-4}$ or 10

Sol 8: $\left(\frac{\log_{10} x}{2}\right)^{\log_{10}^2 x+\log_{10} x^2-2} = \log_{10}\sqrt{x}$

$\Rightarrow \left(\log_{10} x^{1/2}\right)^{\log_{10}^2 x+\log_{10} x^2-2} = \log_{10} x^{1/2}$

$\Rightarrow \log_{10}^2 x + \log_{10} x^2 - 2 = 1$ or $\log_{10} x^{1/2} = 1$

$\Rightarrow \log_{10}^2 x + 2\log_{10} x - 2 = 1$; $\log_{10} x = 2 \Rightarrow x = 10^2$

$\Rightarrow \log^2 x + 2\log x - 2 = 0$

Assume that $\log x = y$

$\Rightarrow y^2 + 2y - 2 = 1 \Rightarrow y^2 + 2y - 2 - 1 = 0$

$\Rightarrow (y + 3)(y - 1) = 0$

$y = -3$ or $y = 1$

$\log_{10} x = -3$ or $\log_{10} x = 1$

$\Rightarrow x = 10^{-3}$ or $10^1 \Rightarrow x = 10^{-3}, 10, 10^2$

Sol 9: $3\sqrt{\log_2 x} - \log_2 8x + 1 = 0$

$\Rightarrow 3\sqrt{\log_2 x} = \log_2 2^3 x - 1 \Rightarrow 3\sqrt{\log_2 x} = 2 + \log_2 x$

Assume that $\log_2 x = y$

$\Rightarrow 3\sqrt{y} = 2 + y$

Square on both sides

$\Rightarrow (3\sqrt{y})^2 = (2 + y)^2 \Rightarrow 9y = 2^2 + y^2 + 2(2)(y)$

$\Rightarrow 9y = 4 + y^2 + 4y \Rightarrow y^2 - 5y + 4 = 0$

$\Rightarrow (y - 4)(y - 1) = 0 \Rightarrow y = 4$ or $y = 1$

$\Rightarrow \log_2 x = 4$ or $\log_2 x = 1$

$\Rightarrow x = 2^4$ or $x = 2^1 \Rightarrow x = 16$ or 2

Sol 10: $\log_{1/3} x - 3\sqrt{\log_{1/3} x} + 2 = 0$

$\log_{1/3} x + 2 = 3\sqrt{\log_{1/3} x}$

Assume that $\log_{1/3} x = y$ .... (i)

$\Rightarrow y + 2 = 3\sqrt{y} \Rightarrow y = 4$ or $y = 1$ [Refer above solution]

$\log_{1/3} x = 4$ or $\log_{1/3} x = 1$

$\Rightarrow x = \left(\frac{1}{3}\right)^4$ or $x = \left(\frac{1}{3}\right)^1 \Rightarrow x = \frac{1}{81}$ or $\frac{1}{3}$

Sol 11: $\left(a^{\log_b x}\right)^2 - 5x^{\log_b x} + 6 = 0$

Assume that $x = b^y$

$\Rightarrow (a^y)^2 - 5(a^{\log_b x}) + 6 = 0 \Rightarrow a^{2y} - 5a^y + 6 = 0$

$\Rightarrow (a^y - 3)(a^y - 2) = 0 \Rightarrow a^y = 2, 3$

$y = \log_a 2, \log_a 3 \quad \therefore x = 2^{\log_a b}, 3^{\log_a b}$

Sol 12: $\log_4(x^2 - 1) - \log_4(x - 1)^2 = \log_4(\sqrt{(4-x)^2})$

$\log_4\left(\frac{x^2-1}{(x-1)^2}\right) = \log_4(\sqrt{(4-x)^2})$

$\frac{x^2-1}{(x-1)^2} = \sqrt{(4-x)^2}$

$\Rightarrow \frac{(x-1)(x+1)}{(x-1)^2} = \sqrt{(4-x)^2} \quad ; \quad x \neq 1,$

$\Rightarrow \frac{x+1}{(x-1)} = \sqrt{(4-x)^2} \quad \Rightarrow \frac{x+1}{(x-1)} = |4 - x|$

Case-I: When $4 - x \geq 0$

$\Rightarrow \frac{x+1}{x-1} = 4 - x \Rightarrow (x + 1) = (4 - x)(x - 1)$

$\Rightarrow x + 1 = 4x - 4 - x^2 + x \Rightarrow x^2 - 4x - x + x + 1 + 4 = 0$

$\Rightarrow x^2 - 4x + 5 = 0 \Rightarrow x = \frac{4 \pm \sqrt{4^2 - 4(5)(1)}}{2(1)}$

$\Rightarrow x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$ (no solution)

Case-II: When $4 - x < 0$

$\Rightarrow \frac{x+1}{x-1} = x - 4 \Rightarrow x + 1 = (x - 1)(x - 4) = x^2 + 4 - x - 4x$

$\Rightarrow x^2 - 4x - x - x + 4 - 1 = 0 \Rightarrow x^2 - 6x + 3 = 0$

$\Rightarrow x = \frac{6 \pm \sqrt{6^2 - 4(3)(1)}}{2(1)} \Rightarrow x = \frac{6 \pm \sqrt{36-12}}{2}$

$= \frac{6 \pm \sqrt{24}}{2} = 3 \pm \sqrt{6}$

But $x > 4$

So, $x = 3 + \sqrt{6}$

Sol 13: $2\log_3 \frac{x-3}{x-7} + 1 = \log_3 \frac{x-3}{x-1}$

$\log_3\left(\frac{x-3}{x-7}\right)^2 + \log_3 3 = \log_3 \frac{x-3}{x-1}$

$\log_3\left[\frac{(x-3)^2}{(x-7)^2} \times 3\right] = \log_3 \frac{x-3}{x-1}$

$\Rightarrow \frac{3(x-3)^2}{(x-7)^2} = \frac{(x-3)}{(x-1)} \Rightarrow 3(x - 3)(x - 1) = (x - 7)^2$

$\Rightarrow 3x^2 + 9 - 3x - 9x = x^2 - 14x + 49$

$\Rightarrow 2x^2 + 2x - 40 = 0 \Rightarrow x^2 + x - 20 = 0$

$\Rightarrow (x + 5)(x - 4) = 0$

$x = -5$ or $x = 4$

At $x = 4$, equation is $2\log_3\left(\frac{4-3}{4-7}\right) + 1 = \log \frac{4-3}{4-7}$

$\frac{4-3}{4-7} = \frac{+1}{-3} \Rightarrow$ –ve which is not possible

Hence $x \neq 4$, $x = -5$

Sol 14: $\log_x(9x^2)\log_3^2 x = 4 \Rightarrow (\log_x 3^2 x^2)(\log_3 x)^2 = 4$

$$\Rightarrow 2[\log_x 3x]\left[\frac{\log_e x}{\log_e 3}\right]^2 = 4$$

We know that $\log_m n = \dfrac{\log_e n}{\log_e m}$

$$\Rightarrow 2[\log_x 3 + \log_x x]\left[\frac{\log_e x}{\log_e 3}\right]^2 = 4$$

$$\Rightarrow \left[\frac{\log_e 3}{\log_e x}+1\right]\left[\frac{\log_e x}{\log_e 3}\right]^2 = 2$$

$$\Rightarrow \frac{\log_e 3}{\log_e x} \times \frac{(\log_e x)^2}{(\log_e 3)^2} + (\log_3 x)^2 = 2$$

$\Rightarrow \log_3 x + (\log_3 x)^2 = 2$

Assume that $\log_3 x = y$

$\Rightarrow y^2 + y = 2 \Rightarrow y^2 + y - 2 = 0 \Rightarrow (y + 2)(y - 1) = 0$

$\Rightarrow y = -2$ or $y = 1$

Now, we have $\log_2 x = -2$ or $\log_3 x = 1$

$\Rightarrow x = 3^{-2}$ or $x = 3^{+1}$

Hence, $x = \dfrac{1}{9}$ or $x = 3$

Sol 15: $\log_{0.5x} x^2 + 14\log_{16x} x^2 + 40\log_{4x}\sqrt{x} = 0$

$$\frac{\log_2 x^2}{\log_2(0.5x)} + \frac{14\log_2 x^2}{\log_2(16x)} + \frac{40\log_2\sqrt{x}}{\log_2(4x)} = 0$$

Assume that $\log_2 x = y$

$$\Rightarrow \frac{2y}{\log_2 .2^{-1}+y} + \frac{28y}{\log_2 2^4+y} + \frac{20y}{\log_2 2^2+y} = 0$$

$$\Rightarrow \frac{y}{y-1} + \frac{14y}{y+4} + \frac{10y}{y+2} = 0$$

$$y = 0 \text{ or } \left(\frac{1}{y-1}+\frac{14}{y+4}+\frac{10}{y+2}\right) = 0$$

$\Rightarrow \log_2 x = y \Rightarrow x = 2^y = 2^\circ = 1$

or $(y + 4)(y + 2) + 14(y - 1)(y + 2) + 10(y - 1)(y + 4) = 0$

$\Rightarrow y^2 + 8 + 6y + 14y^2 - 28 + 14y + 10y^2 - 40 + 30y = 0$

$\Rightarrow 25y^2 + 50y - 60 = 0$

$\Rightarrow y^2 + 2y - \dfrac{60}{25} = 0 \Rightarrow y^2 + 2y - \dfrac{12}{5} = 0$

$$\Rightarrow y = -\frac{2-\sqrt{(2)^2-41\left(-\frac{12}{5}\right)}}{2(1)} \Rightarrow y = \frac{2-\sqrt{2\left(1+\frac{12}{5}\right)}}{2}$$

$$\Rightarrow y = \frac{2-2\sqrt{\frac{5+12}{5}}}{2} \Rightarrow y = -1 \pm \sqrt{\frac{17}{5}}$$

Now, we have $\log_2 x = y$

$$\Rightarrow x = 2^{\left(-1+\sqrt{17/5}\right)} \text{ or } 2^{\left(-1-\sqrt{17/5}\right)}$$

Sol 16: $\log_3[\log_{1/2}^2 x - 3\log_{1/2}x + 5] = 2$

Assume that $\log_{1/2} x = y$

$\Rightarrow \log_3[y^2 - 3y + 5] = 2 \Rightarrow y^2 - 3y + 5 = 9$

$\Rightarrow y^2 - 3y - 4 = 0 \Rightarrow (y - 4)(y + 1) = 0$

$\Rightarrow y = 4$ or $y = -1 \Rightarrow \log_{1/2} x = 4$ or $\log_{1/2}x = -1$

$$\Rightarrow x = \left(\frac{1}{2}\right)^4 \text{ or } x = \left(\frac{1}{2}\right)^{-1}$$

$\Rightarrow x = \dfrac{1}{16}$ or $x = 2.$

Sol 17: $\log_2(x/4) = \dfrac{15}{\log_2\frac{x}{8}-1}$

$$\Rightarrow \log_2 x - \log_2 4 = \frac{15}{\log_2 x - \log_2 8 - 1}$$

$\Rightarrow$ Assume that $\log_2 x = y$

$$y - 2 = \frac{15}{y-3-1} = \frac{15}{y-4}$$

$\Rightarrow (y - 2)(y - 4) = 15 \Rightarrow y^2 - 6y + 8 = 15$

$\Rightarrow y^2 - 6y - 7 = 0 \Rightarrow (y - 7)(y + 1) = 0$

$y = 7$ or $y = -1$

Now, we have $\log_2 x = 7$ or $\log_2 x = -1$

Hence $x = 2^7$ or $x = 2^{-1}$

Sol 18: $\dfrac{1}{2}\log_{10}(5x - 4) + \log_{10}\sqrt{x+1} = 2 + \log_{10}0.18$

$\Rightarrow \log_{10}(5x - 4) + 2\log_{10}\sqrt{x+1} = 2[2 + \log_{10}0.18]$

$\Rightarrow \log_{10}(5x - 4) + \log_{10}(x + 1) = 4 + 2\log_{10}0.18$

$\Rightarrow \log_{10}[(5x - 4)(x + 1)] = 4 + \log_{10}(0.18)^2$

$\Rightarrow \log_{10}[(5x-4)(x+1)] = \log_{10}[10^4 \times (0.18)^2]$

$\Rightarrow (5x-4)(x+1) = 10^4(0.0324) = 324$

$\Rightarrow 5x^2 + x - 4 = 324 \Rightarrow 5x^2 + x - 328 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(5)(-328)}}{(10)}$$

$$x = -\frac{-1 \pm \sqrt{1+20(328)}}{10} = \frac{-1 \pm \sqrt{6561}}{10}$$

$$x = \frac{-1 \pm 81}{10} = 8, -\frac{41}{5},$$

Also it is clear that $x > 4/5$

$\therefore x = \frac{-41}{5}$ is rejected

Sol 19: $\log_{10}x^2 = \log_{10}(5x-4)$

$\Rightarrow x^2 = 5x - 4 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x - 4 = 0$or $x - 1 = 0$

Hence, x = 4, 1

Sol 20: $\frac{1}{6}\log_2(x-2) - \frac{1}{3} = \log_{1/8}\sqrt{3x-5}$

$\Rightarrow \frac{1}{6}\log_2(x-2) - \frac{1}{3} = \log_{2^{-3}}\sqrt{3x-5}$

$\Rightarrow \frac{1}{6}\log_2(x-2) - \frac{1}{3} = -\frac{1}{3}\log_2\sqrt{3x-5}$

$\Rightarrow \frac{1}{2}\log_2(x-2) - 1 = -\log_2\sqrt{3x-5}$

$\Rightarrow \log_2(x-2) + 2\log_2\sqrt{3x-5} = 2$

$\Rightarrow \log_2(x-2) + \log_2(3x-5) = \log_2 2^2$

$\Rightarrow (x-2)(3x-5) = 4 \Rightarrow 3x^2 + 10 - 6x - 5x = 4$

$\Rightarrow 3x^2 - 11x + 6 = 0$

$$x = \frac{11 \pm \sqrt{121 - 4(3)(6)}}{2(3)} = \frac{11 \pm \sqrt{121-72}}{6}$$

$$\Rightarrow x = \frac{11 \pm \sqrt{49}}{6} = 3, \frac{2}{3}$$

At $x = \frac{2}{3}$, eq. $\Rightarrow \frac{1}{6}\log_2\left(\frac{1}{3} - 2\right) - \frac{1}{3}$

$= \log_{1/8}\sqrt{3\frac{(2)}{3} - 5} = \sqrt{-3}$ (Not a possible solution)

So x = 3

Sol 21: $\frac{\log_{10}(\sqrt{x+1}+1)}{\log_{10}(x-40)^{1/3}} = 3 \Rightarrow \frac{\log_{10}(\sqrt{x+1}+1)}{\frac{1}{3}\log_{10}(x-40)} = 3$

$\log_{10}(\sqrt{x+1} + 1) = \log_{10}(x-40)$

$\sqrt{x+1} + 1 = x - 40 \Rightarrow \sqrt{x+1} = x - 41$

On squaring both sides

$x + 1 = (x-41)^2 = x^2 + 41^2 - 2(41)x$

$\Rightarrow x^2 - 82x - x + 41^2 - 1 = 0 \Rightarrow x^2 - 83x + 1680 = 0$

$$x = 83 \pm \frac{\sqrt{(83)^2 - 4(1680)(1)}}{2(1)} = \frac{83 \pm \sqrt{169}}{2} = \frac{83 \pm 13}{2} = 48, 35$$

Now, for x = 35

The given equations yields $\frac{\log_{10}\sqrt{35+1}+1}{\log_{10}3\sqrt{35-40}} = 3\sqrt{-5}$

Wich is not a possible solutions

Hence $x \neq 35$ and $x = 48$

Sol 22: $1 - \frac{1}{2}\log_{10}(2x-1) = \frac{1}{2}\log_{10}(x-9)$

$\Rightarrow 2 - \log_{10}(2x-1) = \log_{10}(x-9)$

$\Rightarrow \log_{10}(x-9) + \log_{10}(2x-1) = 2$

$\Rightarrow \log_{10}(x-9)(2x-1) = \log_{10}10^2$

$\Rightarrow (x-9)(2x-1) = 100$

$\Rightarrow 2x^2 - 18x - x + 9 = 100 \Rightarrow 2x^2 - 19x - 91 = 0$

$$x = \frac{19 \pm \sqrt{19^2 - 4(2)(-91)}}{2(2)} = \frac{19 \pm \sqrt{1089}}{4} = 13, -\frac{7}{2}$$

But $x = -\frac{7}{2}$ is not in the domain hence x = 13

Sol 23: $\log_{10}(3x^2 + 7) - \log_{10}(3x-2) = 1$

$\log_{10}\left(\frac{3x^2+7}{3x-2}\right) = 1 = \log_{10}10$

$\frac{3x^2+7}{3x-2} = 10$ and $3x^2 + 7 = 10(3x-2)$

$\Rightarrow 3x^2 + 7 = 30x - 20 \Rightarrow 3x^2 - 30x + 27 = 0$

$\Rightarrow x^2 - 10x + 9 = 0 \Rightarrow (x-1)(x-9) = 0$

Hence x = 9, 1

Sol 24: $\left(1+\frac{1}{2x}\right)\log_{10}3 + \log_{10}2 = \log_{10}(27 - 3^{1/x})$

$\Rightarrow \log_{10} 3^{(1+1/2x)} + \log_{10}2 = \log_{10}(27 - 3^{1/x})$

$\Rightarrow \log_{10}2\times(3)^{1+1/2x} = \log_{10}(27-3^{1/x})$

$\Rightarrow 2 \times 3^{1+1/2x} = 27 - 3^{1/x}$

Assume that $3^{1/x} = y$

$\Rightarrow 2\times3\times\sqrt{y} = 27 - y$

On squaring both sides, we get

$\Rightarrow 2^2 \times 3^2 \times y = (27 - y)^2 \Rightarrow 36y = 27^2 + y^2 - 2(27)y$

$\Rightarrow y^2 - 54y - 36y + 27^2 = 0 \Rightarrow y^2 - 90y + 27^2 = 0$

$(y - 81)(y - 9) = 0 \Rightarrow y = 81, 9$

$\therefore\ x = \frac{1}{\log_3 y} \Rightarrow x = \frac{1}{\log_3 81}$ or $\frac{1}{\log_3 9} = \frac{1}{4}, \frac{1}{2}$

Clearly $3^{1/x} < 27\ \therefore x > \frac{1}{3}$

So x = 1/4 is not valid

Sol 25: $\frac{1}{2}\log_{10} x + 3\log_{10}\sqrt{2+x} = \log_{10}\sqrt{x(x+2)} + 1$

$\log_{10} x + 6\log_{10}\sqrt{2+x} = 2\log_{10}\sqrt{x(x+2)} + 2$

$\Rightarrow \log_{10} x + \log_{10}(2 + x)^3 - \log_{10}[x(x + 2)] = 2$

$\Rightarrow \log_{10}\left[\frac{x(2+x)^3}{x(x+2)}\right] = \log_{10}{}^{2}$

$\Rightarrow (2 + x)^2 = 100 \Rightarrow 2 + x = \pm 100$

$x\left[\begin{array}{l} 100-2=98 \\ -100-2=-102 \end{array}\right.$

Here, x = – 102 does not satisfy the equation

Hence x = 98

Sol 26: $\log_2(4^x + 1) = x + \log_2(2^{x+3} - 6)$

$\log_2(4^x + 1) = \log_2 2^x + \log_2(2^{x+3} - 6)$

$\Rightarrow \log_2(4^x + 1) = \log_2[2^x[2^{x+3} - 6]]$

$\Rightarrow 4^x + 1 = 2^x[2^x 2^3 - 6]$

Assume that $2^x = y$

$\Rightarrow y^2 + 1 = y(8y - 6) \Rightarrow y^2 + 1 = 8y^2 - 6y$

$\Rightarrow 7y^2 - 6y - 1 = 0 \Rightarrow (y - 1)(7y + 1) = 0$

$y = 1$or $y = -\frac{1}{7}$

$2^x = 1$ or $2^x = -\frac{1}{7}$ (not valid)

$\Rightarrow x = 0$ and so, x = 0 is only solution.

Sol 27: $\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) = 2$

$\log_{\sqrt{5}}\left(\frac{4^x-6}{2^x-2}\right) = 2 \Rightarrow \frac{4^x-6}{2^x-2} = 5$

Assume that $2^x = y$

$\Rightarrow \frac{y^2-6}{y-2} = 5 \Rightarrow y^2 - 6 = 5(y - 2) = 5y - 10$

$\Rightarrow y^2 - 5y - 6 + 10 = y^2 - 5y + 4 = 0$

$\Rightarrow (y - 4)(y - 1) = 0$

$y = 4$ or $y = 1$

$\Rightarrow 2^x = 4$ or $2^x = 1$

$\Rightarrow x = 2$ or $x = 0$

x = 0 does not satisfy the equation, hence x = 2

Sol 28: $\log_{10}(3^x - 2^{4-x}) = 2 + \frac{1}{4}\log_{10}16 - \frac{x\log_{10}4}{2}$

$\Rightarrow \log_{10}(3^x-2^{4-x}) = \log_{10}10^2 + \frac{1}{4}\log_{10}2^4 - \frac{x\log 2^2}{2}$

$\Rightarrow \log_{10}(3^x-2^{4-x}) = \log_{10}100 + \frac{4}{4}\log_{10}2 - \frac{x\times2\log_{10}2}{2}$

$\Rightarrow \log_{10}(3^x - 2^{4-x}) = \log_{10}[100 \times 2] - \log 2^x$

$\Rightarrow \log_{10}(3^x - 2^{4-x}) = \log_{10}\frac{(200)}{2^x}$

$\Rightarrow 3^x - \frac{2^4}{2^x} = \frac{200}{2^x} \Rightarrow 3^x.2^x - 2^4 = 200$

$\Rightarrow 6^x = 200 + 2^4 \Rightarrow 216 = 6^3 \Rightarrow x = 3$

Sol 29: $\log_{10}(\log_{10} x) + \log_{10}(\log_{10} x^4 - 3) = 0$

$\log_{10}[(\log_{10} x)(\log_{10} x^4 - 3)] = 0$

$\Rightarrow (\log_{10} x)(\log_{10}x^4 - 3) = 1$

$(\log_{10} x)(4\log_{10} x - 3) = 1$

Assume that $\log_{10} x = y$

$\Rightarrow y(4y - 3) = 1;\ 4y^2 - 3y = 1$

$\Rightarrow 4y^2 - 3y - 1 = 0 \Rightarrow (y - 1)(4y + 1) = 0$

$\Rightarrow y = 1$ or $y = -\frac{1}{4}$

$\log_{10} x = 1$ or $\log_{10} x = -\frac{1}{4}$

$\Rightarrow x = 10$ or $x = 10^{-\frac{1}{4}}$

for $x \pm 10^{-\frac{1}{4}}$ given log function is not defined.

Hence, x = 10

Sol 30: $\log_3(9^x + 9) = \log_3 3^x(28 - 2.3^x)$

$\Rightarrow 9^x + 9 = 3^x(28 - 2 . 3^x)$

Assume that $3^x = y$

So $9^x = (3^2)^x = (3^x)^2 = y^2$

$\Rightarrow y^2 + 9 = y(28 - 2y) \Rightarrow y^2 + 9 = 28y - 2y^2$

$\Rightarrow 3y^2 - 28y + 9 = 0 \Rightarrow (3y - 1)(y - 9) = 0$

This gives $y = 9, \frac{1}{3}$

Hence, x = 2, –1

Exercise 2

Single Correct Choice Type

Sol 1: (B) $\frac{1}{\log_{\sqrt{bc}} abc} + \frac{1}{\log_{\sqrt{ac}} abc} + \frac{1}{\log_{\sqrt{ab}} abc}$

$= \frac{\log_{10}\sqrt{bc}}{\log_{10} abc} + \frac{\log_{10}\sqrt{ac}}{\log_{10} abc} + \frac{\log_{10}\sqrt{ab}}{\log_{10} abc}$

$= \frac{\log_{10}\sqrt{bc} + \log_{10}\sqrt{ac} + \log_{10}\sqrt{ab}}{\log_{10} abc}$

$= \frac{\log_{10}\sqrt{bc}\sqrt{ac}\sqrt{ab}}{\log_{10} abc} = \frac{\log_{10} abc}{\log_{10} abc} = 1$

Sol 2: (D) $\log_2(2x^2) + \log_2 x.\ x^{\log_x(\log_2 x+1)}$

$+ \frac{1}{2}\log_4 2x^4 + 2^{-3\log_{1/2}(\log_2 x)} = 1$

$\Rightarrow \log_2(2x^2) + (\log_2 x)\ (x)^{\log_x(\log_2 x+1)}$

$+ \frac{1}{2}\log_4 4^{1/2}x^4 + 2^{-3\log_{1/2}(\log_2 x)} = 1$

$\Rightarrow 1 + 2\log_2 x + (\log_2 x)\,(x)^{\log_x(\log_2 x+1)}$

$+ \frac{1}{4}\log_4 4 + \frac{4}{2}\log_4 x + 2^{3\log_2(\log_2 x)} = 1$

$\Rightarrow 1 + 2\log_2 x + (\log_2 x + 1)(\log_2 x) + \frac{1}{4}$

$+ \log_2 x + (2)^{\log_2(\log_2 x)^3} = 1$

$\Rightarrow 1 + 2\log_2 x + (\log_2 x)(\log_2 x + 1) + \frac{1}{4}$

$+ \log_2 x + (\log_2 x)^3 = 1$

Assume $\log_2 x = y$

$\Rightarrow 2y + y(y + 1) + \frac{1}{4} + y + y^3 = 0$

$\Rightarrow y^3 + 4y + y^2 + \frac{1}{4} = 0$

Differential of equation is $\frac{d}{dy}[y^3 + 4y + y^2 + \frac{1}{4}] = 0$

$\Rightarrow 3y^2 + 4 + 2y = 0 \Rightarrow y = -\frac{-2 \pm \sqrt{2^2 - 4(4)(3)}}{2(3)}$

$y = \frac{-2 \pm \sqrt{-48 + 4}}{6}$

No solution so there is no minima and maximum

At $y = 0 \Rightarrow f(y) = 0 + 0 + 0 + \frac{1}{4} > 0$

$y = -1$, $f(y) = (-1)^3 + 4(-1) + (-1)^2 + \frac{1}{4}$

$\Rightarrow -1 - 4 + 1 + \frac{1}{4} = -4 + \frac{1}{4} = -\frac{15}{4} < 0$

It mean f(y) is zero some where $-1 < y < 0$

So $\log_2 x < 0$

But in equation (original) $\log_2 x$ should be positive so there is no solution

Sol 3: (C) $x = (75)^{-10}$

$\log_{10} x = \log_{10}(75)^{-10} = -10\log_{10} 75 = -10\log_{10}\left(100 \times \frac{3}{4}\right)$

$= -10[\log_{10} 10^2 + \log_{10} 3 - \log_{10} 2^2]$

$= -10[2 + 0.477 - 2(0.301)] = -18\ 75$

$\Rightarrow x = 10^{-18.75} = 10^{-19} \times 10^{-0.25}$

Number of zeros = 18

Sol 4: (D) $5x^{\log_2 3} + 3^{\log_2 x} = 162$

Assume $x = 2^y \Rightarrow 5.2^{y\log_2 3} + 3^{\log_2 2^y} = 162$

$\Rightarrow 5.2^{\log_2 3^y} + 3^{y\log_2 2} = 162 \Rightarrow 5.3^y + 3^y = 6.3^y = 162$

$3^y = \frac{162}{6} = 27 = 3^3$

$y = 3;\ x = 2^y = 2^3 = 8$

$\log_4 x = \log_4 8 = \log_4 (4)^{3/2} = \dfrac{3}{2}$

Sol 5: (B) $(x)^{\log_{10}^2 x + \log_{10} x^3 + 3}$

$= \dfrac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}} = B$ (Assume)

$B = \dfrac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}} = \dfrac{2}{\dfrac{\sqrt{x+1}+1-\sqrt{x+1}+1}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}}$

$B = ((\sqrt{x+1})^2 - (1)^2 = x + 1 - 1 = x$

So $(x)^{\log_{10}^2 x + 3\log_{10} x + 3} = x \Rightarrow x = 1$

Or $\Rightarrow$ Assume $\log_{10} x = y$

$\Rightarrow y^2 + 3y + 3 = 1 \Rightarrow y^2 + 3y + 2 = 0$

$\Rightarrow (y + 2)(y + 1) = 0$

$y = -2$ or $y = -1$

$\log_{10} x = -2$ or $\log_{10} x = -1$

$x = 10^{-2}, 10^{-1}$

$x_1, x_2, x_3 = 1, 10^{-1}, 10^{-2}$

$x_1 . x_3 = 1.10^{-2} = (10^{-1})^2 = (x_2)^2$

Sol 6: (D) $x = 2^{\log 3}, y = 3^{\log 2}$

$x = 2^{\log 3} = 3^{\log 2} = y$

As $a^{\log_n m} = m^{\log_n a}$

Sol 7: (B) $|x-3|^{3x^2-10x+3} = 1 ; x \neq 3$

Or if $|x - 3| = 1$

$\Rightarrow x = 2$ or 4 is solution

If $x - 3 \neq 0$ then $3x^2 - 10x + 3 = 0$ is another sol^n

$3x^2 - 10x + 3 = 0 \Rightarrow (3x\ \ 1)(x\ \ 3) = 0$

$x = +3$ or $= \dfrac{+1}{3}$

But $x \neq 3$; so, $x = \dfrac{1}{3}$

total solution $\Rightarrow x = \dfrac{1}{3}, 2, 4$

Sol 8: (C) x_1 and x_2 are roots of the equation

$\sqrt{2010}\ x^{\log_{2010} x} = x^2$

Assume that $x = (2010)^y$

$\Rightarrow (2010)^{1/2}\ (2010)^{y \log_{2010} (2010)^y} = (2010)^{2y}$

$\Rightarrow (2010)^{1/2}\ (2010)^{y^2} = (2010)^{2y}$

$\Rightarrow y^2 + \dfrac{1}{2} = 2y \Rightarrow y^2 - 2y + \dfrac{1}{2} = 0$

$\Rightarrow y = \dfrac{2 \pm \sqrt{2^2 - 4(1)(1/2)}}{2} = \dfrac{2 \pm \sqrt{2}}{2} = 1 \pm \dfrac{1}{\sqrt{2}}$

$x_1 x_2 = (2010)^{1-\frac{1}{\sqrt{2}}}\ (2010)^{1+\frac{1}{\sqrt{2}}} = (2010)^2 = (201 \times 10)^2$

No. of zeros in $x_1 x_2 = 2$

Sol 9: (A) Given that $x = 2$ or $x = 3$ satisfy the equation

$\log_4(x^2 + bx + c) = 1 = \log_4 4$

$\Rightarrow x^2 + bx + c - 4 = 0$

$\Rightarrow b = 2 + 3 = 5$ and $c - 4 = 2 . 3 \Rightarrow c = 10$

$bc = 10(-5) = -50$

$|bc| = 50$

JEE Advanced/Boards

Exercise 1

Sol 1: $B = (2^{\log_6 18}).(3^{\log_6 3})$

$\Rightarrow B = 2^{\log_6 (6\times 3)} . 3^{\log_6 3} \Rightarrow B = 2^{\log_6 6 + \log_6 3} . 3^{\log_6 3}$

$\Rightarrow B = 2^{1+\log_6 3}\ 3^{\log_6 3} = 2 \times 2^{\log_6 3} . 3^{\log_6 3}$

$\Rightarrow B = 2\{6\}^{\log_6 3} = 2.3 = 6$

$A = \log_{10} \dfrac{ab + \sqrt{(ab)^2 - 4(a+b)}}{2} + \log_{10} \dfrac{ab - \sqrt{(ab)^2 - 4(a+b)}}{2}$

$A = \log_{10} \left[\dfrac{ab + \sqrt{(ab)^2 - 4(a+b)}}{2} \times \dfrac{ab - \sqrt{(ab)^2 - 4(a+b)}}{2} \right]$

$= \log_{10} \left[\dfrac{(ab)^2 - ((ab)^2 - 4(a+b))^{2/2}}{4} \right]$

$= \log_{10} \left[\dfrac{(ab)^2 - (ab)^2 + 4(a+b)}{4} \right] = \log_{10} \dfrac{4(a+b)}{4}$

$= \log_{10}(a + b) = \log_{10}(43 + 57) = \log_{10}100 = 2$

$\Rightarrow$ A = 2 and B = 6

Hence, AB = 12

Sol 2: (a) $\log_{1/3} \sqrt[4]{729\sqrt[3]{9^{-1}.27^{-4/3}}}$

$= \log_{1/3} \sqrt[4]{729\sqrt[3]{3^{-2}.3^{-4}}}$

$= \log_{1/3} \sqrt[4]{729.3^{-2}} = \log_{1/3} \sqrt[4]{81} = \log_{1/3}3 = 1$

(b) $a^{\frac{\log_b(\log_b N)}{\log_b a}} = a^x$ say

$x = \frac{\log_b(\log_b N)}{\log_b a} = \log_a(\log_b N)$

So $a^x = a^{\log_a(\log_b N)} = \log_b N$

Sol 3: (a) $\log_{\pi}2 + \log_2 p$

$\Rightarrow \frac{\log 2}{\log\pi} + \frac{\log\pi}{\log 2}$ Assume that $\frac{\log 2}{\log\pi}$ = x (+ve always)

$(2 < \pi < 10) \Rightarrow x + \frac{1}{x} = c$ (Assume)

$x^2 - cx + 1 = 0 \Rightarrow x = \frac{c \pm \sqrt{c^2 - 4}}{2}$

For x to be real $c^2 - 4 \geq 0$

$c^2 \geq 4 \Rightarrow c \geq 2 \Rightarrow c = 2 \Rightarrow x = 1$

For all other value c > 2 (Not Possible)

Here, $\log_{\pi}2 + \log_2\pi$ is greater than 2

(b) For $\log_3 5$ and $\log_2 7$

Assume that $\log_3 5$ is rational $\therefore \log_3 5 = a \Rightarrow 5 = 3^a$

This is not possible when a is rational $\therefore$ a is irrational

Similarly, $\log_2 7 = b$ assuming b is rational gives $7 = 2^b$

Which is not possible, so b is irrational.

Sol 4: $\log_3 x . \log_4 x . \log_5 x = \log_3 x.\log_4 x + \log_4 x. \log_5 x + \log_5 x \log_3 x$

Assume that $\log_{10}x = y$

$\Rightarrow \frac{\log_{10} x.\log_{10} x.\log_{10} x}{\log_{10} 3 \log_{10} 4 \log_{10} 5}$

$= \frac{\log_{10} x\log_{10} x}{\log_{10} 3\log_{10} 4} + \frac{\log_{10} x\log_{10} x}{\log_{10} 4.\log_{10} 5} + \frac{\log_{10} x.\log_{10} x}{\log_{10} 5.\log_{10} 3} \Rightarrow y^3$

$= (\log_{10}5)y^2 + (\log_{10}3)y^2 + (\log_{10}4)y^2$

$y^3 = y^2[\log_{10}5 + \log_{10}3 + \log_{10}4]$

$\Rightarrow y^3 = y^2[\log_{10}(3.4.5)] = y^2 \log_{10} 60$

$\Rightarrow y = 0$ or $y = \log_{10}60$

$\Rightarrow \log_{10} x = 0$ or $y = \log_{10}x = \log_{10}60$

$\Rightarrow x = 1$ or $x = 60$

Sum of roots = 1 + 60 = 61

Square of sum of roots = $(61)^2 = 3721$

Sol 5: $\frac{2}{\log_4(2000)^6} + \frac{3}{\log_5(2000)^6}$

$\frac{2}{6\log_4(2000)} + \frac{3}{6\log_5(2000)}$

$= \frac{1}{6}\left[\frac{2}{\log_4(4^2 \times 5^3)} + \frac{3}{\log_5(5^3 \times 4^2)}\right]$

$= \frac{1}{6}\left[\frac{2}{\log_4 4^2 + \log_4 5^3} + \frac{3}{\log_5 5^3 + \log_5 4^2}\right]$

$= \frac{1}{6}\left[\frac{2}{2 + 3\log_4 5} + \frac{3}{3 + 2\log_5 4}\right]$

$= \frac{1}{6}\left[\frac{2}{2 + \frac{3\log_{10} 5}{\log_{10} 4}} + \frac{3}{3 + \frac{2\log_{10} 4}{\log_{10} 5}}\right]$

$= \frac{1}{6}\left[\frac{2\log_{10} 4}{2\log_{10} 4 + 3\log_{10} 5} + \frac{3\log_{10} 5}{3\log_{10} 5 + 2\log_{10} 4}\right]$

$= \frac{1}{6}\left[\frac{2\log_{10} 4 + 3\log_{10} 5}{2\log_{10} 4 + 3\log_{10} 5}\right] = \frac{1}{6}$

Sol.6 $\frac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left((\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right)$

$= \frac{9^{2\log_9 5} + 3^{3\log_3 \sqrt{6}}}{409}\left((\sqrt{7})^{2\log_7 25} - (25)^{\frac{3}{2}\log_{25} 6}\right)$

$= \frac{9^{\log_9 5^2} + 3^{\log_3 (\sqrt{6})^3}}{409}[7^{\log_7 25} - 25^{\log_{25} 6^{3/2}}]$

$= \frac{5^2 + (\sqrt{6})^3}{409}[25 - 6^{3/2}] = \frac{(5^2)^2 - (6^{3/2})^2}{409}$

$= \dfrac{(25)^2 - 6^3}{409} = \dfrac{409}{409} = 1$

Sol 7: $(5)^{\log_{1/5}\left(\frac{1}{2}\right)} + \log_{\sqrt{2}} \dfrac{4}{\sqrt{7}+\sqrt{3}} + \log_{1/2} \dfrac{1}{10+2\sqrt{21}}$

$= 5^{\log_5 2} + \log_{2^{\frac{1}{2}}} \left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right) + \log_{2^{-1}} \dfrac{1}{10+2\sqrt{21}}$

$= 2 + \log_2 \left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right)^2 + \log_2 10+2\sqrt{21}$

$= \left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right)^2 = \dfrac{16}{7+3+2\sqrt{7}\sqrt{3}} = \dfrac{16}{10+2\sqrt{21}}$

$= 2 + \log_2 \dfrac{16}{10+2\sqrt{21}} \left(10+2\sqrt{21}\right)$

$= 2 + \log_2 2^4 = 2 + 4 = 6$

Sol 8: $\log_2 a = s \Rightarrow a = 2^s$

$\log_4 b = s^2 \Rightarrow b = 4^{s^2} = (2)^{2s^2}$

and $\log_{c^2} 8 = \dfrac{2}{s^3+1} \Rightarrow 8^{\frac{1}{2}} = c^{\frac{2}{s^3+1}}$

$\Rightarrow c = (2^{3/2})^{\frac{s^3+1}{2}}$; $c = 2^{\frac{3(s^3+1)}{4}}$

Then $\dfrac{a^2b^5}{c^4} = \dfrac{(2^s)^2(2^{2s^2})^5}{\left(2^{3\frac{(s^3+1)}{4}}\right)^4} = \dfrac{2^{2s}\ 2^{10s^2}}{2^{3(s^3+1)}} = (2)^{(2s+10s^2-3(s^3+1))}$

$\Rightarrow \log_2 \dfrac{a^2b^5}{c^4} = (2s+10s^2-3(s^3+1))$

Sol 9: $\dfrac{\log_2 24}{\log_{96} 2} - \dfrac{\log_2 192}{\log_{12} 2}$

$\Rightarrow$ We know that $\log_m n = \dfrac{1}{\log_n m}$

$\Rightarrow (\log_2 96)(\log_2 24) - (\log_2 192)(\log_2 12)$

Where, $\log_2 24 = \log_2 12 \times 2 = \log_2 12 + \log_2 2$

$\Rightarrow \log_2 96(\log_2 12 + \log_2 2) - \log_2(96\times 2) \log_2 12$

$\Rightarrow \cancel{\log_2 96.\log_2 12} + \log_2 96 - \cancel{\log_2 96.\log_2 12} - \log_2 12$

$\Rightarrow \log_2\left(2^3 \times 12\right) - \log_2 12 \Rightarrow 3 + \log_2 12 - \log_2 12 = 13$

Sol 10: We have to prove that

$a^x - b^y = 0$, where $x = \sqrt{\log_a b}$

and $y = \sqrt{\log_b a} \Rightarrow x^2 = \log_a b$

$y^2 = \log_b a \Rightarrow y^2 = \dfrac{1}{x^2} \Rightarrow x^2y^2 = 1$

$xy = 1$ $(x, y > 0)$ now $a^x - b^y = (b^{y^2})^x - (a^{x^2})^y$

$\Rightarrow (b^{xy})^y - (a^{xy})^x \Rightarrow b^y - a^x \Rightarrow a^x - b^y = b^y - a^x = -(a^x - b^y)$

$\Rightarrow a^x - b^y + a^x - b^y = 0 \Rightarrow 2(a^x - b^y) = 0 \Rightarrow a^x - b^y = 0$

Sol 11: (a) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \dfrac{1}{2}$

$\Rightarrow 2\log_{10}(x-3) = \log_{10}(x^2-21)$

$\Rightarrow \log_{10}(x-3)^2\ \ \log_{10}(x^2-21) = 0 \Rightarrow \log_{10} \dfrac{(x-3)^2}{(x^2-21)} = 0$

$\Rightarrow \dfrac{(x-3)^2}{x^2-21} = 1 \Rightarrow x^2 + 3^2 - 2(3)x = x^2 - 21$

$\Rightarrow 9 - 6x = -21 \Rightarrow 6x = 9 + 21 \Rightarrow x = \dfrac{30}{6} = 5$

(b) $\log(\log x) + \log(\log x^3 - 2) = 0$

$\Rightarrow \log[\log x(\log x^3 - 2)] = 0 \Rightarrow (\log x)(\log x^3 - 2) = 1$

$\Rightarrow (\log x)(3 \log x - 2) = 1$ Assume that $\log x = y$

$\Rightarrow y(3y - 2) = 1 \Rightarrow 3y^2 - 2y - 1 = 0$

$\Rightarrow 3y(y-1) + 1(y-1) = 0 \Rightarrow y = -\dfrac{1}{3}$ or $y = 1$

$\Rightarrow \log_{10} x = -\dfrac{1}{3}$ or $\log_{10} x = 1 \Rightarrow x = (10)^{-\frac{1}{3}}$ or $x = 10^1$

At $x = 10^{-1/3}$ equation does not satisfy

Hence, $x = 10$

(c) $\log_x 2 . \log_{2x} 2 = \log_{4x} 2$

$\Rightarrow \dfrac{1}{\log_2 x} . \dfrac{1}{\log_2 2x} = \dfrac{1}{\log_2 4x}$

$\Rightarrow \log_2 2^2 + \log x = (\log_2 x)(\log_2 2 + \log_2 x)$

Assume $\log_2 x = y$

$\Rightarrow 2 + y = y(1 + y) \Rightarrow 2 + y = y^2 + y$

$\Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

$\log x = \pm\sqrt{2}$

$\log_2 x = +\sqrt{2}$ or $\log_2 x = -\sqrt{2}$

$x = (2)^{\sqrt{2}}$ or $x = 2^{-\sqrt{2}}$

(d) $5^{\log_a x} + 5x^{\log_a 5} = 3$, $(a > 0)$

Assume $x = a^y$

$\Rightarrow 5^{\log_a a^y} + 5a^{y\log_a 5} = 3 \Rightarrow 5^y + 5a^{\log_a 5^y}$

$= 5^y + 5.5^y = 6.5^y = 3 \Rightarrow 5^y = \frac{3}{6} = \frac{1}{2} = 2^{-1}$

Take logarithm (base 5) both side

$\Rightarrow \log_5 5^y = \log_5 2^{-1} \Rightarrow y = \log_5 2^{-1}$

So $x = a^y = a^{\log_5 2^{-1}}$

$\Rightarrow x = 2^{-\log_5 a}$

Sol 12: $\log_a x \log_a(xyz) = 48$ ... (i)

$\log_a y \log_a(xyz) = 12$ ... (ii)

$\log_a z \log_a(xyz) = 84$ ... (iii)

When sum of all equation is taken

$\log_a(xyz)[\log_a x + \log_a y + \log_a z]$

$= 48 + 12 + 84 = 144 = 12^2$

$(\log_a(xyz))(\log_a(xyz)) = 12^2$

$(\log_a xyz)^2 = 12^2 \Rightarrow \log_a xyz = 12\ (\pm 1)$

In equation

(i) $\log_a x\ (\pm 12) = 48$

$\log_a x = \pm 4 \Rightarrow x = a^4, a^{-4}$

(ii) $\log_a y(\pm 12) = 12$

$\log_a y = \pm 1 \Rightarrow y = a, a^{-1}$

(iii) $\log_a z(\pm 12) = 84$

$\log_a z = \pm 7 \Rightarrow z = a^7, a^{-7}$

$(x, y, z) = (a^4, a, a^7)$ or (a^{-4}, a^{-1}, a^{-7})

Sol 13: Given

L = antilog of 0.4 to the base 1024

$\Rightarrow L = (1024)^{0.4} = (2^{10})^{0.4} = 2^4 = 16$

L = 16

And M is the number of digits in 6^{10}

$\Rightarrow \log_{10} 6^{10} = 10\log_{10} 6 \Rightarrow 10[0.7761] = 7.761$

$\Rightarrow 6^{10} = 10^{7.761} = 10^7.10^{0.761}$

No. of digits = 7 + 1 = 8 $\therefore$ M = 8

$\Rightarrow \log_6 6^2 = 2$ (characteristic 2)

$\Rightarrow \log_6 6^3 = 3$ (characteristic 3)

Total no. of positive integers which have the characteristic 2(between 6^2 and 6^3) = $6^3 - 6^2$

= 216 – 36 = 180

LMN = 16 × 8 × 180 = 23040

Sol 14: $\log_a N.\log_b N + \log_b N.\log_c N + \log_c N.\log_a N$ ... (i)

$$= \frac{\log_a N.\log_b N.\log_c N}{\log_{abc} N}$$

We know that $\log_x y = \frac{\log y}{\log x}$

So, in equation (i) at R.H.S, we have

$$= \frac{\frac{\log N}{\log a}\cdot\frac{\log N}{\log b}\cdot\frac{\log N}{\log c}}{\frac{\log N}{\log abc}} = \frac{(\log N)^2 \log abc}{(\log a)\log b(\log c)}$$

$$= \frac{\log N^2(\log a + \log b + \log c)}{\log a \log b \log c}$$

$$= \frac{(\log N)(\log N)}{\log b \log c} + \frac{\log N \log N}{\log a \log c} + \frac{\log N \log N}{\log a \log b}$$

$= \log_a N \log_b N + \log_a N \log_c N + \log_b N \log_c N$

R.H.S. = L.H.S.

Sol 15: $x, y > 0$ and $\log_y x + \log_x y = \frac{10}{3}$

$$\Rightarrow \frac{\log_{12} x}{\log_{12} y} + \frac{\log_{12} y}{\log_{12} x} = \frac{10}{3}$$

Assume that $\frac{\log_{12} x}{\log_{12} y} = a$

$\Rightarrow a + \frac{1}{a} = \frac{10}{3} \Rightarrow 3a^2 - 10a + 3 = 0$

$\Rightarrow (3a\ \ 1)(a\ \ 3) = 0 \Rightarrow a = 3, \left(\frac{1}{3}\right)$

So $\frac{\log_{12} x}{\log_{12} y} = 3 \Rightarrow$ add + 1 both side

$\frac{\log_{12} x}{\log_{12} y} + 1 = 3 + 1 = 4 \Rightarrow \frac{\log x + \log y}{\log_{12} y} = 4$

$\Rightarrow \dfrac{\log_{12}(xy)}{\log_{12} y} = \dfrac{\log_{12} 12^2}{\log y} = 4 \Rightarrow \dfrac{2}{\log_{12} y} = 4$

$\log_{12} y = \dfrac{2}{4} = \dfrac{1}{2} \Rightarrow y = 12^{1/2}$

So $x = \dfrac{144}{y} = 144 \times 12^{-\frac{1}{2}} = 12^{2-\frac{1}{2}} = 12^{\frac{3}{2}}$

$\dfrac{x+y}{2} = \sqrt{N}$

$\Rightarrow \dfrac{(x+y)^2}{2^2} = N \Rightarrow x^2 + y^2 + 2xy = 4N$

$\Rightarrow (12^{3/2})^2 + (12^{1/2}) + 2(144) = 4\ N$

$\Rightarrow 12^3 + 12 + 2 \times 144 = 4\ N$

$4N = 2028 \Rightarrow N = \dfrac{2028}{4} \Rightarrow N = 507$

Sol 16: (a) $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$

$\Rightarrow 5^{200} = x$ (Assume)

$\log_{10} x = \log_{10} 5^{200} = 200\log_{10}5$

$= 200\log_{10} \dfrac{10}{2} = 200(\log_{10}10 - \log_{10}2)$

$= 200(1 - 0.3010) = 200(0.699) = 139.8$

$\Rightarrow x = 10^{139} \times 10^{0.8}$

no. of digits in x = 139 + 1 = 140

(b) $x = 6^{15} \Rightarrow \log_{10}x = \log_{10} 6^{15} = 15 \log_{10} 6$

$= 15(\log 2 + \log 3) = 15 \times (0.778) = 11.67$

$\therefore x = 10^{11.67} = 10^{11}\ 10^{0.67}$

No. of digits in x = 11 + 1 = 12

(c) Number of zeros after the decimal in $3^{-100} = (x)$ (Assume)

$\log x = \log 3^{-100} = -100 \log_{10}3 = -100(0.4771) = -47.71$

So $x = 10^{-47.71} = 10^{-47} \times 10^{-0.71}$

$\therefore$ No. of zeros = 47

Sol 17: $\log_5 120 + (x-3) - 2\log_5(1-5^{x-3}) = -\log_5(2-5^{x-4})$

$\Rightarrow \log_5 120 + (x-3) - \log_5(1-5^{x-3})^2 + \log_5(2-5^{x-4}) = 0$

$\Rightarrow \log_5 \dfrac{120 \times 5^{x-3} \times (2-5^{x-4})}{(1-5^{x-3})^2} = 0$

$\Rightarrow \dfrac{120 \times 5^{x-3} \times (2-5^{x-4})}{(1-5^{x-3})^2} = 1$

$\Rightarrow \dfrac{120}{5^3}\ 5^x\left[2 - \dfrac{5^x}{5^4}\right] = 1^2 + 5^{2(x-3)} - 2(5^{x-3})$

Assume that $5^x = y$

$\Rightarrow \dfrac{120}{5 \times 5 \times 5}\ y\left[2 - \dfrac{y}{25 \times 25}\right] = 1 + y^2\ 5^{-6} - \dfrac{2 \times y}{5^3}$

Multiply by 5^6

$\Rightarrow 5^3 \times 120y[2 - y\ 5^{-4}] = 5^6 + y^2 - 2x5^3y$

$\Rightarrow 5^3 \times 240y - \dfrac{120y^2}{5} = 5^6 + y^2 - 2 \times 5^3y$

$\Rightarrow 5^3 \times 240y - 24y^2 = 5^6 + y^2 - 2 \times 5^3y$

$5^4 \times 48y - 25y^2 = 5^6 - 10 \times 5^2y$

Divide by 5^2 on the both side

$5^2 \times 48y - y^2 = 5^4 - 10y$

$\Rightarrow y^2 - y(10 + 5^2 \times 48) + 5^4 = 0$

$\Rightarrow y^2 - 1210y + 625 = 0$

$\Rightarrow y = \dfrac{1210 \pm \sqrt{(1210)^2 - 4(1)(625)}}{2}$

$\Rightarrow y = \dfrac{1210 \pm 1208.96}{2}$

y = 0.51675 or y = 1209. 48 (Rejected)

$5^x = y = 0.51675 \Rightarrow x = \log_5 y$

Hence, x = – 0.410

Sol 18: Given that $\log_{x+1} (x^2 + x - 6)^2 = 4$

$\Rightarrow (x^2 + x - 6)^2 = (x + 1)^4 \Rightarrow (x^2 + x - 6) = \pm (x + 1)^2$

When +ve case is taken $\rightarrow x^2 + x - 6 = (x + 1)^2$

(and $x^2 + x - 6 \geq 0$)

$x^2 + x - 6 = x^2 + 1 + 2x$

$x = -6 - 1 = -7$

In the given equation, base is $x + 1 = -7 + 1 = -6$ which is negative

So $x \neq -7$

When -ve case is taken $\rightarrow x^2 + x - 6 < 0$

$\Rightarrow x^2 + x - 6 = -(x + 1)^2 \Rightarrow x^2 + x - 6 = -x^2 - 1 - 2x$

$\Rightarrow 2x^2 + 3x - 5 = 0 \Rightarrow (2x + 5)(x - 1) = 0$

$x = -\frac{5}{2}$ or $x = 1$

$x = -\frac{5}{2}$ also does not satisfy equation

So $x = 1$

Sol 19: Given that $x + \log_{10}(1 + 2^x) = x\log_{10}5 + \log_{10}6$

$\Rightarrow \log_{10}10^x + \log_{10}(1+2^x) = \log_{10}5^x + \log_{10}6$

$\Rightarrow \log_{10}[10^x(1 + 2^x)] = \log_{10}[5^x\ 6]$

$\Rightarrow 10^x\ (1 + 2^x) = 6\ 5^x \Rightarrow 10^x + 20^x = 5^x\ 6$

Divide by 5^x on the both the sides

$\Rightarrow \frac{10^x}{5^x} + \frac{20^x}{5^x} = \frac{6\times 5^x}{5^x} = 6$

$\Rightarrow 2^x + 4^x = 6 \Rightarrow 2^x + 2^{2x} = 6$

Assume that $2^x = y$

$\Rightarrow y + y^2 = 6 \Rightarrow y^2 + y - 6 = 0$

$\Rightarrow (y - 2)(y + 3) = 0 \Rightarrow y = -3$ or $y = 2$

$\Rightarrow 2^x = -3$ or $2^x = 2 \Rightarrow 2^x = -3$ is not possible so, $2^x = 2$

Therefore, the real solution $\Rightarrow x = 1$

Sol 20: $2\log_{10}(2y - 3x) = \log_{10}x + \log_{10}y$

We have to find $\left(\frac{x}{y}\right)$

$\Rightarrow \log_{10}(2y - 3x)^2 = \log_{10}(xy) \Rightarrow 4y^2 - 12xy + 9x^2 = xy$

Let $x = ky$

$\Rightarrow 4y^2 - 12ky^2 + 9k^2 y^2 = ky^2 \Rightarrow 9k^2 - 13k + 4 = 0$

$\Rightarrow (9k - 4)(k\ \ 1) = 0 \Rightarrow k = 1, \frac{4}{9}$

If $k = 1 \Rightarrow x = y \Rightarrow 2y - 3x$ is –ve

$\therefore \frac{x}{y} = \frac{4}{9}$

Sol 21: We have $a = \log_{12}18$ and $b = \log_{24}54$

$\Rightarrow a = \frac{\log_2 18}{\log_2 12} = \frac{2\log_2 3 + 1}{2 + \log_2 3}$

$\Rightarrow (a - 2)\log_2 3 = 1 - 2a$ ... (i)

Similarly $b = \frac{\log_2 54}{\log_2 24} = \frac{3\log_2 3 + 1}{3 + \log_2 3}$

$\Rightarrow (b - 3)\log_2 3 = 1 - 3b$ ... (ii)

Dividing E.q. (i) and (ii), we get

$(a - 2)(1 - 3b) = (1 - 2a)(b - 3)$

$\Rightarrow 2a(b - 3) + (a - 2)(1 - 3b) = b - 3$

$\Rightarrow 2ab - 6a + a - 3ab - 2 + 6b = b - 3$

$\Rightarrow -ab - 5a + 5b + 1 = 0 \Rightarrow 5(b - a) - ab + 1 = 0$

$\Rightarrow 5(a - b) + ab = 1$

Sol 22: $\sqrt{\log_9(9x^4)\log_3(3x)} = \log_3 x^3$

$\Rightarrow \sqrt{(1 + 4\log_3 x)[1 + \log_3 x]} = 3\log x$

Assume that $\log_3 x = y$

$\Rightarrow (1 + 4y)(1 + y) = (3y)^2 = 9y^2$

$\Rightarrow 1 + 4y^2 + 4y + y = 9y^2 \Rightarrow 5y^2 - 5y - 1 = 0$

$\Rightarrow y = \frac{5 \pm \sqrt{5^2 - 4(-1)(5)}}{2(5)} = \frac{5 \pm \sqrt{25 + 20}}{10}$

$y = \frac{5 \pm \sqrt{45}}{10} = \frac{5 \pm \sqrt{3^2 \times 5}}{10} = \frac{5 \pm 3\sqrt{5}}{10}$

In equation (i) $\log_3 x > 0$

Hence, $y = \frac{5 + 3\sqrt{5}}{10}$

Sol 23: Given that $xyz = 10^{81}$

$(\log_{10}x)(\log_{10}yz) + (\log_{10}y)(\log_{10}z) = 468$

We know that $(a + b + c)^2$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$= a^2 + b^2 + c^2 + 2a(b + c) + 2bc$ ... (i)

$\Rightarrow \log_{10}x(\log_{10} y + \log z) + (\log_{10} y)\ \ (\log_{10} z) = 468$

Assume that $\log 10x = a$, $\log_{10} y = b$ and $\log_{10} z = c$

$\Rightarrow a(b + c) + bc = 468$

From equation (i)

$2a(b+c) + 2bc = (a+b+c)^2 - (a^2+b^2+c^2)$

$\Rightarrow 2a(b + c) + 2bc = 2 \times 468 = 936$

$\Rightarrow (a + b + c)^2 - (a^2 + b^2 + c^2) = 936$

$\Rightarrow a + b + c = \log_{10}x + \log_{10}y + \log_{10}z$

$= \log_{10}xyz = \log_{10}10^{81} = 81$

$\Rightarrow 81^2 - (a^2 + b^2 + c^2) = 936$

$a^2 + b^2 + c^2 = 81^2 - 936 = 5625$

$\Rightarrow (\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2 = 5625$

Sol 24: Sum of all solution of equation

$\Rightarrow [3]^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = 3\sqrt{3}$

$\Rightarrow (3)^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = (3)^{3/2}$

$\Rightarrow (\log_9 x)^2 - \frac{9}{2}\log_9 x + 5 = \frac{3}{2}$

Assume that $\log_9 x = y$

$\Rightarrow y^2 - \frac{9}{2}y + 5 = \frac{3}{2} \Rightarrow y^2 - \frac{9}{2}y + 5 - \frac{3}{2} = y^2 - \frac{9}{2}y + \frac{7}{2} = 0$

$\Rightarrow 2y^2 - 9y + 7 = 0$

$\Rightarrow (2y - 7)(y - 1) = 0 \ \Rightarrow y = \frac{7}{2};\ y = 1$

$\log_9 x = \frac{7}{2} \log_9 x = 1$

$\Rightarrow x = (9)^{7/2} = 3^7; x = 9$

Sum of solution $= 3^7 + 9 = 2196$

Sol 25: a, b, c, d > 0

$\because \log_a b = \frac{3}{2}$ and $\log_c d = \frac{5}{4}$, $a - c = 9$

$\frac{\log_{10} b}{\log_{10} a} = \frac{3}{2};\ \frac{\log_{10} d}{\log_{10} c} = \frac{5}{4}$

$2\log_{10} b = 3 \log_{10} a$

$4 \log_{10} d = 5 \log_{10} c$

$b = a^{\frac{3}{2}}, d = c^{\frac{5}{4}}$

$\therefore$ a should be perfect square and c should be perfect power of 4

Let a = 25, c = 16

$\therefore b = (5)^3 = 125 \Rightarrow \quad d = (16)^{5/4} = 32 \quad \therefore b - d = 93$

Sol 26: Refer Sol 11 of Ex 2 JEE Main

Sol 27:

$\log_{10}^2 \left[1 + \frac{4}{x}\right] + \log_{10}^2 \left[1 - \frac{4}{x+4}\right] = 2 \log_{10}^2 \left[\frac{2}{x-1} - 1\right]$

$\log_{10}^2 \left[\frac{x+4}{x}\right] + \log_{10}^2 \left[\frac{x+4-4}{x+4}\right] = 2\log_{10}^2 \left[\frac{2-(x-1)}{x-1}\right]$

$\log_{10}^2 \left(\frac{x+4}{x}\right) + \log_{10}^2 \left(\frac{x}{x+4}\right) = 2\log_{10}^2 \left(\frac{2-x+1}{x-1}\right)$

We know $\log_{10} \frac{1}{x} = -\log_{10} x$. So $\left(\log_{10} \frac{1}{x}\right)^2 = (\log_{10} x)^2$

$\Rightarrow \log_{10}^2 \left(\frac{x+4}{x}\right) + \log_{10}^2 \left(\frac{x+4}{x}\right) = 2\log_{10}^2 \left(\frac{3-x}{x-1}\right)$

$\log_{10}^2 \left(\frac{x+4}{x}\right) = \log_{10}^2 \left(\frac{3-x}{x-1}\right)$

So $\frac{x+4}{x} = \frac{3-x}{x-1}$ or $\frac{x}{x+4} = \left(\frac{3-x}{x-1}\right)$

$x^2 + 4x - x - 4 = 3x - x^2$ or $x^2 - x = 3x + 12 - x^2 - 4x$

$\Rightarrow 2x^2 - 4 = 0$ or $2x^2 = 12 \ \Rightarrow x^2 = 2$ or $x^2 = 6$

$x = \pm\sqrt{2}$ or $x = \pm\sqrt{6}$

$x = \sqrt{2}$ and $-\sqrt{6}$ do not satisfy equation

So $x = \sqrt{2}, \sqrt{6}$

Sol 28: $\log_3(\sqrt{x} + |\sqrt{x} - 1|) = \log_9(4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

$\log_3\left(\sqrt{x} + |\sqrt{x} - 1|\right) = \frac{1}{2}\log_3\left(4\sqrt{x} + -3 + 4|\sqrt{x} - 1|\right)$

$\Rightarrow 2\log_3(\sqrt{x} + |\sqrt{x} - 1|) = \log_3(4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

$\Rightarrow \log_3(\sqrt{x} + |\sqrt{x} - 1|)^2 = \log_3(4\sqrt{x} - 3 + 4|(\sqrt{x} - 1|)$

$\Rightarrow (\sqrt{x} + |\sqrt{x} - 1|)^2 = (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

$x + (\sqrt{x} - 1)^2 + 2\sqrt{x}|\sqrt{x} - 1| = 4\sqrt{x} - 3 + 4|\sqrt{x} - 1|$

(i) Assume $(\sqrt{x} - 1) < 0$

$\Rightarrow |\sqrt{x} - 1| = 1 - \sqrt{x}$

$\Rightarrow x + x + 1 - 2\sqrt{x} + 2\sqrt{x}(1 - \sqrt{x}) = 4\sqrt{x} - 3 + 4(1 - \sqrt{x})$

$\Rightarrow 1 + 2x - 2\sqrt{x} + 2\sqrt{x} - 2x = 4\sqrt{x} - 3 + 4 - 4\sqrt{x}$

1 = 1 always correct

So $\sqrt{x} - 1 < 0$ and $x > 0$

$\sqrt{x} < 1$

$\Rightarrow x \in (0, 1)$ and if $\sqrt{x} - 1 \geq 0$, $\sqrt{x} > 0$

$x + x + 1 - 2\sqrt{x} + 2\sqrt{x}(\sqrt{x} - 1) = 4\sqrt{x} - 3 + 4(\sqrt{x} - 1)$

$\Rightarrow 2x + 1 - 2\sqrt{x} + 2x - 2\sqrt{x} = 4\sqrt{x} - 3 - 4 + 4\sqrt{x}$

$\Rightarrow 4x + 1 + 7 - 4\sqrt{x} = 8\sqrt{x} \Rightarrow 4x - 12\sqrt{x} + 8 = 0$

$\Rightarrow x - 3\sqrt{x} + 2 = 0 \Rightarrow (\sqrt{x} - 2)(\sqrt{x} - 1) = 0$

$\Rightarrow \sqrt{x} - 2 = 0$ or $\sqrt{x} - 1 = 0 \Rightarrow x = 4$ or $x = 1$

Put condition was $\Rightarrow \sqrt{x} - 1 \geq 0$

So $x = [0, 1] \cup \{4\}$

Sol 29:

$$2^{\left(\sqrt{\log_a \sqrt[4]{ab}+\log_b \sqrt[4]{ab}}-\sqrt{\log_a \sqrt[4]{b/a}+\log_b \sqrt[4]{a/b}}\right)\cdot\sqrt{\log_a b}} = 2^x$$

$$\Rightarrow x = \left(\sqrt{\frac{1}{4}(\log_a(a\times b)+\log_b(a\times b))} - \sqrt{(\log_a ba^{-1}+\log_b ab^{-1})\frac{1}{4}}\right)\sqrt{\log_a b}$$

$$x = \frac{1}{2}\left[\sqrt{1+\log_a b+1+\log_b a} - \sqrt{-1+\log_a b-1+\log_b a}\right]\sqrt{\log_a b}$$

$$x = \frac{1}{2}\left[\sqrt{2\log_a b+1+\left(\log_a b\right)^2} - \sqrt{-2\log_a b+\left(\log_a b\right)^2+1}\right]$$

We know $\log_a b = \dfrac{1}{\log_b a}$

$$x = \frac{1}{2}\left(\sqrt{(1+\log_a b)^2} - \sqrt{\left(\log_a b-1\right)^2}\right)$$

$$x = \frac{1}{2}(|1+\log_a b| - |\log_a b - 1|)$$

When $\log_a b \geq 1 \Rightarrow b \geq a > 1$

$$x = \frac{1}{2}(1+\log_a b - \log_a b + 1) = \frac{1}{2}\times 2 = 1$$

so $2^x = 2^1 = 2$ (when $b \geq a > 1$)

When $\log_a b < 1$

$\Rightarrow b < a,\ a, b > 1$

$$\Rightarrow x = \frac{1}{2}[1 + \log_a b - (1 - \log_a b)]$$

$$x = \frac{1}{2}[1 + \log_a b + \log_a b] = \frac{1}{2}2\log_a b$$

$x = \log_a b$

$2^x = 2^{\log_a b}$ (if $1 < b < a$)

Sol 30: $\sqrt{[\log_3(3x)^{1/3} + \log_x(3x)^{1/3}]\log_3 x^3} +$

$$\sqrt{\left[\log_3\left(\frac{x}{3}\right)^{\frac{1}{3}} + \log_x\left(\frac{3}{x}\right)^{\frac{1}{3}}\right]\log_3 x^3}$$

Assume that

$$A = \sqrt{\left[\frac{1}{3}\log_3(3x) + \frac{1}{3}\log_x(3x)\right]\log_3 x^3}$$

$$\Rightarrow \sqrt{\frac{3}{3}\left[(\log_3 x+1)+(\log_x 3+1)\right]\log_3 x}$$

$$A = \sqrt{(2\log_3 x + (\log_3 x)^2 + 1)}$$

$A = |\log_3 x + 1|$

$$\text{And } B = \sqrt{\left(\left(\log_3\frac{x}{3}\right)\frac{1}{3} + \frac{1}{3}\left(\log_x\frac{3}{x}\right)\right)\log_3 x^3}$$

$$\Rightarrow \sqrt{\frac{3}{3}\left[\log_3 x - 1 + \log_x 3 - 1\right]\log_3 x}$$

$$B = \sqrt{((\log_3 x)^2 - 2\log_3 x + 1)}$$

$$B = \sqrt{(\log_3 x - 1)^2} = |\log_3 x - 1|$$

$A + B = 2 \Rightarrow |\log_3 x + 1| + |\log_3 x - 1| = 2$

$\log_3 x \geq 1 \Rightarrow x \geq 3$

$A + B \Rightarrow \log_3 x + 1 + \log_3 x - 1 = 2\log_3 x = 2$

$\log_3 x = 1 \Rightarrow x = 3$

$x \geq 3$ and $x = 3 \Rightarrow x = 3$

If $\log_3 x < 1$ and $\log_3 x + 1 > 0 \Rightarrow x < 3$ and $x > \dfrac{1}{3}$

$A + B \Rightarrow \log_3 x + 1 - (\log_3 x - 1)$

$= \log_3 x + 1 - \log_3 x + 1 = 2 = 2$(always)

So $x \in \left(\dfrac{1}{3}, 3\right)$

$\log_3 x \leq -1 \Rightarrow x \leq \dfrac{1}{3}$

$A + B = -(\log_3 x + 1) - (\log_3 x - 1)$

$= -\log_3 x - 1 - \log_3 x + 1 = -2\log_3 x = 2$

$\Rightarrow \log_3 x = -1 \Rightarrow x = 3^{-1} = \dfrac{1}{3}$

$x \geq \dfrac{1}{3}$ and $x = \dfrac{1}{3} \Rightarrow x = \dfrac{1}{3}$

So $x = \left[\frac{1}{3}, 3\right] - \{1\}$

$x \neq 1$ because base can't be 1

Sol 31: $a = (\log_7 81)(\log_{6561} 625)(\log_{125} 216)(\log_{1296} 2401)$

$\Rightarrow a = (\log_7 3^4)\ (\log_{3^8} 5^4)\ (\log_{5^3} 6^3)\ (\log_{6^4} 7^4)$

$\Rightarrow a = 4(\log_7 3)\frac{4}{8}(\log_3 5)(\log_5 6)\left(\frac{3}{3}\right)\left(\frac{4}{4}\right)\log_6 7$

$\Rightarrow\ a = \frac{2\log_{10} 3}{\log_{10} 7}\,\frac{\log_{10} 5}{\log_{10} 3}\,\frac{\log_{10} 6}{\log_{10} 5}\,\frac{\log_{10} 7}{\log_{10} 6} = 2$

$\Rightarrow$ and b = sum of roots of the equation

$x^{\log_2 x} = (2x)^{\log_2 \sqrt{x}}$

$x^{\log_2 x} = (2x)^{\log_2 x^{1/2}}$

Take logarithm (base x) both sides

$\log_x x^{\log_2 x} = \log_x (2x)^{\log_2 x^{1/2}}$

$(\log_2 x)(1) = \log_2 x^{1/2}[\log_x (2x)]$

$\log_2 x = \frac{1}{2}\log_2 x(\log_x 2 + 1)$

$\log_2 x = 0\ \Rightarrow x = 1$ or $2 = \log_x 2 + 1$

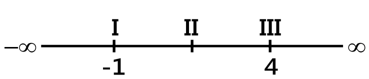
$\log_x 2 = 1\ \Rightarrow x = 2$

$x_1 + x_2 = 1 + 2 = 3$

$b = 3$

and c = sum of all natural solution of equation

$|x + 1| + |x - 4| = 7$



If $x < -1 \rightarrow |x + 1| = -1 - x$

$|x - 4| = 4 - x$

$\Rightarrow$ eq.$\rightarrow -1 - x + 4\ - x = 3 - 2x = 7$

$\Rightarrow 2x = 3 - 7 = -4 \Rightarrow x = -\frac{4}{2} = -2$

If $x > 4 \rightarrow |x + 1| = x + 1$

$|x - 4| = x - 4$

Eq. $\rightarrow x + 1 + x - 4 = 2x - 3 = 7$

$\Rightarrow 2x = \frac{7+3}{1} = 10?2x = 10 \Rightarrow x = \frac{10}{2} = 5$

If $-1 < x < 4$

$\Rightarrow |x + 1| \rightarrow 1 + x$

$|x - 4| \rightarrow 4 - x$

$\Rightarrow 1 + x + 4 - x = 5 \neq 7$

So no solution for this region $\rightarrow x = 5$ and -2

But -2 is not natural no.

So $c = 5$

$a + b = 2 + 3 = 5$

$(a + b) \div c = \frac{5}{5} = 1$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $2^{\sqrt{x}+\sqrt{y}} = 256$ and $\log_{10}\sqrt{xy} - \log_{10} 1.5 = 1$

$\Rightarrow 2^{\sqrt{x}+\sqrt{y}} = 256 = 2^8$

$\Rightarrow \sqrt{x} + \sqrt{y} = 8$ ... (i)

and $\log_{10}\sqrt{xy} = 1 + \log_{10} 1.5 = \log_{10} 10 + \log_{10} 1.5$

$\log_{10}\sqrt{xy} = \log_{10}(10 \times 1.5) = \log_{10} 15$

$\Rightarrow \sqrt{xy} = 15 \Rightarrow xy = 15^2 = 225$

$|\sqrt{x} - \sqrt{y}| = \sqrt{(\sqrt{x}+\sqrt{y})^2 - 4\sqrt{xy}}$

$= \sqrt{8^2 - 4\times 15} = \sqrt{64 - 60}$

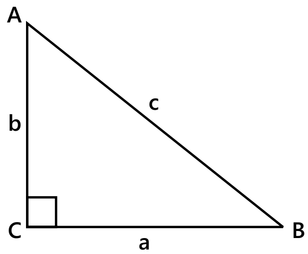
$|\sqrt{x} - \sqrt{y}| = \sqrt{4} = 2$

$\sqrt{x} + \sqrt{y} = 8$

$\Rightarrow$ If $\sqrt{x} > \sqrt{y} \Rightarrow (x, y) = (25, 9)$

$\Rightarrow$ If $\sqrt{x} < \sqrt{y} \Rightarrow (x, y) = (9, 25)$

Sol 2: (B)



$\Rightarrow c^2 = a^2 + b^2 \Rightarrow c^2 - b^2 = a^2$

$$\frac{\log_{b+c} a + \log_{c-b} a}{\log_{b+c} a \cdot \log_{c-b} a} = \frac{\frac{\log_a a}{\log_a(b+c)} + \frac{\log_a a}{\log_a(c-b)}}{\frac{\log_a a}{\log_a(b+c)}\frac{\log_a a}{\log_a(c-b)}}$$

$= \left(\log_a(c-b) + \log_a(b+c)\right) = \log_a(c^2 - b^2) = 2$

Sol 3: (B) B, C, P, and L are positive number

$\therefore$ log(B.L) + log(B.P) = 2; log(P.L) + log(P.C) = 3

and log(C.B) + log(C.L) = 4

Adding all the above equations, we have

log[B.L.B.P.P.L.P.C.C.B.C.L] = 2 + 3 + 4 = 9

$\log(BCPL)^3 = 9 \Rightarrow 3\log BCPL = 9$

$\Rightarrow \log BCPL = \frac{9}{3} = 3$

$\therefore BCPL = 10^3$

Sol 4: (B) $\frac{\log_{12}(\log_8(\log_4 x))}{\log_5(\log_4(\log_y(\log_2 x)))} = 0$

$c < y < b,\ y \neq a$

where 'b' is as large as possible and 'c' is as small as possible.

$\Rightarrow \log_{12}(\log_8(\log_4 x)) = 0 \Rightarrow \log_8(\log_4 x) = 1 = \log_8 8$

$\log_4 x = 8 \Rightarrow x = 4^8 = 2^{2\times 8} = 2^{16}$

and $\log_5(\log_4(\log_y(\log_2 x))) \neq 0$

$\Rightarrow \log_5(\log_4(\log_y(\log_2 2^{16}))) \neq 0$

$\Rightarrow \log_5(\log_4(\log_y 16)) \neq 0,\ y \neq 1$

$\Rightarrow \log_4(\log_y 16) \neq 1 \Rightarrow \log_y 16 \neq 4$

$\Rightarrow \log_{2^4} y \neq \frac{1}{4} \Rightarrow \frac{1}{4}\log_2 y \neq \frac{1}{4} \Rightarrow \log_2 y \neq 1 \Rightarrow y \neq 2$

$\log_4(\log_y 16) \neq 0 \Rightarrow \log_y 16 \neq 1$

$\log_{16} y \neq 1 \Rightarrow y \neq 16$

$\log_4(\log_y 16) > 0$

$\log_y 16 > 1 \Rightarrow y < 16$

$\log_y 16 > 0$

$\Rightarrow a = 2, b = 16, c = 1$

$a + b + c = 2 + 16 + 1 = 19$

Sol 5: (D) $\frac{1}{\log_2 N}\cdot\frac{1}{\log_N 8}\cdot\frac{1}{\log_{32} N}\cdot\frac{1}{\log_N 128}$

$= \frac{\log_2}{\log N}\cdot\frac{\log N}{3\log_2}\cdot\frac{5\log_2}{\log N}\cdot\frac{\log N}{7\log_2} = \frac{5}{21}$

Sol 6: (B) $N = 10^p;\ p = \log_{10} 8 - \log_{10} 9 + 2\log_{10} 6$

$p = \log\left(\frac{8.36}{9}\right) = \log_{10} 32$

$\therefore N = 10^{\log_{10} 32} = 32$

Hence characteristics of $\log_3 32$ is 3

Sol 7: (C) $\log 2\left((x+y)^2 - xy\right)$

But $x + y = \sqrt{2}$; $\quad xy = \frac{10-2}{4} = 2$

$\log_2(10-2) = \log_2 8 = 3$

Sol 8: (A) Let $x = \sqrt{\frac{5}{4} + \sqrt{\frac{3}{2}}} + \sqrt{\frac{5}{4} - \sqrt{\frac{3}{2}}}$

$\Rightarrow x^2 = \frac{5}{2} + 2\sqrt{\frac{25}{16} - \frac{3}{2}} = \frac{5}{2} + 2.\frac{1}{4} = 3$

$\Rightarrow x = \sqrt{3} = \tan\frac{\pi}{3}$

Sol 9: (D) $y = |2x - |x - 2|| = |2x - (2 - x)| = |3x - 2|$ as $x < 0$
Hence $y = 2 - 3x$

Sol 10: (D) $3^x(0.333\ \ldots)^{(x-3)} \leq \left(\frac{1}{27}\right)^x$

$\Rightarrow 3^x\left(\frac{1}{3}\right)^{x-3} \leq \left(\frac{1}{3^3}\right)^x = \left(\frac{1}{3}\right)^{3x}$

$\Rightarrow 3^x\, 3^{-(x-3)} = 3^x \,.\, 3^{3-x} \leq \left(\frac{1}{3}\right)^{3x}$

$3^3 = 27 \le \left(\frac{1}{3}\right)^{3x} = 3^{-3x}$

$3 \le -3x \Rightarrow -x \ge 1 \Rightarrow x \le -1$

$x \in [-\infty, -1]$

Sol 11: (B) $\left(\frac{1}{5}\right)^{\frac{2x+1}{1-x}} > \left(\frac{1}{5}\right)^{-3}$

$\frac{2x+1}{1-x} < -3$

$2x + 1 < -3(1 - x) = -3 + 3x$ (if $(1 - x) > 0$)

$\Rightarrow 2x + 1 < -3 + 3x \Rightarrow 3x - 2x > 1 + 3 = 4$

$\Rightarrow x > 4 \Rightarrow x > 4$ and $x < 1$ which implies no solution

If $x > 1 \Rightarrow 1 - x < 0 \Rightarrow \frac{2x+1}{1-x} < -3$

$\Rightarrow \frac{2x+1}{1} > -3(1-x) = 3x - 3$

$\Rightarrow 3x - 2x < 1 + 3 = 4 \Rightarrow x < 4$ and $x > 1 \Rightarrow x \in (1, 4)$

Sol 12: (D) $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = \frac{1}{x^2} = x^{-2}$

$\Rightarrow \log_3 x^2 + (\log_3 x)^2 - 10 = -2$

Assume $\log_3 x = y \rightarrow 2y + y^2 - 10 = -2$

$\Rightarrow y^2 + 2y - 10 + 2 = y^2 + 2y - 8 = 0$

$\Rightarrow (y + 4)(y - 2) = 0 \Rightarrow y = -4$ or $y = 2$

$x = 3^{-4} = \frac{1}{81}$; $x = 9$

$x = \left\{1, 9, \frac{1}{81}\right\}$

Sol 13: (A) $\frac{(l\text{n})^2 - 3l\text{n}x + 3}{l\text{n}x - 1} < 1$

If $ln\,x - 1 > 0 \Rightarrow ln\,x > 1 \Rightarrow x > e$

$\Rightarrow (ln\,x)^2 - 3ln\,x + 3 < 1[(ln(x)) - 1]$

Assume $ln\,x = y$

$\Rightarrow y^2 - 3y + 3 < y - 1 \Rightarrow y^2 - 3y - y + 3 + 1 < 0$

$\Rightarrow y^2 - 4y + 4 < 0 \Rightarrow (y - 2)^2 < 0$ always false

So if $\ln x < 1 \Rightarrow x < e$ and $x > 0$

$y^2 - 3y + 3 > (y - 1) \Rightarrow y^2 - 3y - y + 3 + 1 > 0$

$y^2 - 4y + 4 > 0 \Rightarrow (y - 2)^2 > 0$ always true

So, $x \in (0, e)$

Multiple Correct Choice Type

Sol 14: (C, D) $N = \frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \log_6^2 2$

$N = \frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \left(\frac{\log_3 2}{\log_3 6}\right)^2$

Assume that $\log_3 2 = y$

$\Rightarrow N = \frac{1+2y}{(1+y)^2} + \frac{y^2}{(\log_3 2 + \log_3 3)^2}$

$\Rightarrow N = \frac{1+2y}{(1+y)^2} + \frac{y^2}{(1+y)^2} = \frac{y^2 + 2y + 1}{(1+y)^2}$

$\Rightarrow N = \frac{(1+y)^2}{(1+y)^2} = 1$

And $\pi = 3.147 > 3$ and $7 > 6$

So, $\log_3 \pi > 1$ and $\log_7 6 < 1$

Sol 15: (A, D) $2^{2x} - 8.2^x = -12$

Assume that $2^x = y$

$\Rightarrow y^2 - 8y = -12 \Rightarrow (y - 6)(y - 2) = 0 \Rightarrow y = 6$ or $y = 2$

$2^x = 6$; $2^x = 2^1$

$x \log_{10} 2 = \log_{10} 6 = \log_{10}(2 \times 3)$

$x = \frac{\log_{10} 2 + \log_{10} 3}{\log_{10} 2} = 1 + \frac{\log_{10} 3}{\log_{10} 2}; x = 1$

Sol 16: (A, B, C, D) $\left(\sqrt{5\sqrt{2} - 7}\right)^x + 6\left(\sqrt{5\sqrt{2} + 7}\right)^x = 7$

Assume $x = \log_{\sqrt{5\sqrt{2}-7}} y$

$\Rightarrow \left(\sqrt{5\sqrt{2} - 7}\right)^{\log_{\sqrt{5\sqrt{2}-7}} y} + 6\left(\sqrt{5\sqrt{2} + 7}\right)^{\log_{\sqrt{5\sqrt{2}-7}} y} = 7$

$\sqrt{5\sqrt{2} - 7} = \sqrt{5\sqrt{2} - 7} \times \frac{\sqrt{5\sqrt{2}+7}}{\sqrt{5\sqrt{2}+7}}$

$= \frac{\sqrt{50 - 49}}{\sqrt{5\sqrt{2}+7}} = \left(\sqrt{5\sqrt{2}+7}\right)^{-1}$

$\Rightarrow y + 6\left(\sqrt{5\sqrt{2}+7}\right)^{-\log_{\sqrt{5\sqrt{2}+7}} y} = 7 \Rightarrow y + 6y^{-1} = 7$

$\Rightarrow y^2 + 6 = 7y \Rightarrow y^2 - 7y + 6 = 0$

$\Rightarrow (y-6)(y-1) = 0$ which gives $y = 6$ or $y = 1$

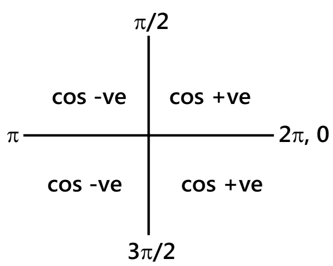
$x = \log_{\sqrt{5\sqrt{2}-7}} 6$ or $x = \log_{\sqrt{5\sqrt{2}-7}} 1 = 0$

$\Rightarrow x = \log_{(5\sqrt{2}-7)^{1/2}} 6 = 2\log_{(5\sqrt{2}-7)} 6 = \log_{(5\sqrt{2}-7)} 36$

$$x = \frac{2}{\log_6(5\sqrt{2}-7)} = \frac{-2}{\log_6(5\sqrt{2}+7)}$$

Assertion Reasoning Type

Sol 17: (D) Statement-I



$\sqrt{\log_x \cos(2\pi x)}$ is a meaningful quantity only if

$x \in \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$

$\cos 2\pi x > 0 \Rightarrow \frac{\pi}{2} > 2\pi x > 0$

$\frac{1}{4} > x > 0$ and $x \neq 1$, $x > 0$

$\frac{3\pi}{2} < 2\pi x < 2\pi \Rightarrow \frac{3}{4} < x < 1$

So $x \in \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$

But also $\log_x \cos(2\pi x) > 0 = \log_x 1$

$\cos 2\pi x > 1$ which is never possible

So statement-I is false

Statement-II If the number N > O and the base of the logarithm b(greater than zero not equal to)

Both lie on the same side of unity than $\log_b N > 0$ and if they lie on the different side of unity then $\log_b N < 0$ statement-II is true

Sol 18: (B) Statement-I

$\log_2(2\sqrt{17-2x}) = 1 + \log_2(x-1)$ has a solution

$\Rightarrow 1 + \log_2(\sqrt{17-2x}) = 1 + \log_2(x-1)$

$\Rightarrow \sqrt{17-2x} = (x-1)$

Squaring both sides

$\Rightarrow 17 - 2x = (x-1)^2 = x^2 - 2x + 1$

$\Rightarrow 17 = x^2 + 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

$\Rightarrow x \neq -4$ does not satisfy equation in statement-I

So x = 4. x has a solution

Statement-II

"Change of base in logarithm is possible" which is true but not the correct explanation for statement-I.

Sol 19: (B) Statement-I: $5^{\log_5(x^3+1)} - x^2 = 1$ have two distinct real solutions.

Statement-II: $a^{\log_a N} = N$ when $a > 0$ $a \neq 1$, $N > 0$

$\Rightarrow 5^{\log_5 x^3+1} - x^2 = 1$

$[5^{\log_5(x^3+1)} = x^3 + 1]$ from statement-II

$\Rightarrow x^3 + 1 - x^2 = 1 \Rightarrow x^3 - x^2 = 0$

$\Rightarrow x^3 = x^2 \Rightarrow x = 0$ or 1

Statement-I is true and II is true and II is not the correct explanation for statement -I.

Comprehension Type

Paragraph 1:

Sol 20: (D) $\log_x^3 10 - 6\log_x^2 10 + 11 \log_x 10 - 6 = 0$

Assume that $\log_x 10 = y$

$\Rightarrow y^3 - 6y^2 + 11y - 6 = 0$

$f(y) = y^3 - 6y^2 + 11y - 6$

$\frac{df(y)}{dy} = 3y^2 - 12y + 11 \to 0$

$\Rightarrow y = \frac{12 \pm \sqrt{12^2 - 4 \times 3 \times 11}}{2(3)} = \frac{12 \pm \sqrt{12}}{6}$

There is maxima and minima at

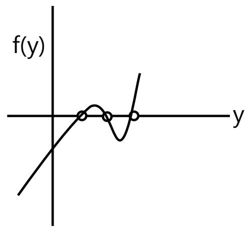
$y = \frac{12 \pm \sqrt{12}}{6} = 2 \pm \frac{\sqrt{6}\sqrt{2}}{6} = 2 \pm \frac{\sqrt{2}}{6} = 2 \pm \frac{1}{\sqrt{3}}$

At $y = 2 + \frac{1}{\sqrt{3}}$

$y^3 - 6y^2 + 11y - 6$ is negative and at $y = 2 - \frac{1}{\sqrt{3}}$,

Equation $y^3 - 6y^2 + 11y - 6$ is positive

So there is total 3 solutions for this equation



Match the Columns

Sol 21: A $\rightarrow$ q, r, s; B $\rightarrow$ p, q, r, s; C $\rightarrow$ p; D $\rightarrow$ r

(A)

$$\sqrt{3\sqrt{x}-\sqrt{7x+\sqrt{4x-1}}}\ \sqrt{2x+\sqrt{4x-1}}\ \sqrt{3\sqrt{x}+\sqrt{7x+\sqrt{4x-1}}} = 13$$

$$\sqrt{(3\sqrt{x}-\sqrt{7x+\sqrt{4x-1}})(3\sqrt{x}+\sqrt{7x+\sqrt{4x-1}})(\sqrt{2x+\sqrt{4x-1}})}$$

$$= \sqrt{\left(3\sqrt{x}\right)^2-\left(\sqrt{7x+\sqrt{4x-1}}\right)^2\left(2x+\sqrt{4x-1}\right)}$$

$$= \sqrt{\left(9x-7x-\sqrt{4x-1}\right)\left(2x+\sqrt{4x-1}\right)}$$

$$= \sqrt{\left(2x-\sqrt{4x-1}\right)\left(2x+\sqrt{4x-1}\right)}$$

$$= \sqrt{\left(2x\right)^2-\left(4x-1\right)} = 13 \Rightarrow \sqrt{\left(4x\right)^2-4x-1} = 13$$

$$\Rightarrow \left(2x-1\right) = 13 \Rightarrow x = \frac{14}{2} = 7$$

(B) $P(x) = x^7 - 3x^5 + x^3 - 7x^2 + 5$

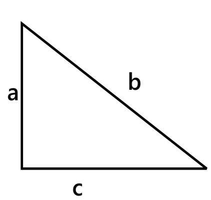
$Q(x) = x - 2$

Remainder $\frac{P(x)}{Q(x)}$

$Q(x) = 0$ at $x = 2$

So $P(2) = 2^7 - 3(2)^5 + 2^3 - 7(2)^2 + 5 = 17$

(C)



Area of triangle is

Area $= a^2 + b^2 - c^2$

Also, we have $b^2 = a^2 + c^2$

So area $= a^2 + (a^2 + c^2) - c^2 = \frac{1}{2} \times a \times c = \frac{ac}{2}$

$\Rightarrow 2a^2 = \frac{ac}{2} \Rightarrow 4 = \frac{ac}{a^2} = \frac{a}{c}$

$\Rightarrow$ ratio $= \frac{c}{a} = 4$

(D) a, b, c $\in$ N

$\therefore ((4)^{1/3} + (2)^{1/3} - 2)(a(4)^{1/3}+b(2)^{1/3}+c)=20$

$= (2^{2/3}+2^{1/3}-2)(a2^{2/3} + b2^{1/3}+c)=20$

$\Rightarrow a(2^{4/3} + 2 - 2.2^{2/3}) + b[2^{3/3} + 2^{2/3} - 2.2^{1/3}] + c(2^{2/3}+2^{1/3}-2^{3/3})=20$

$\Rightarrow 2^{1/3}(2a - 2b + c) + 2^{3/3}(a + b - c) + 2^{2/3}(-2a + b + c)= 20$

$\Rightarrow a + b - c = \frac{20}{2} = 10$

Sol 22: A $\rightarrow$ p; B $\rightarrow$ p, r, s; C $\rightarrow$ p, r; D $\rightarrow$ p, q, r

(A) $x = \log_2 \log_9 \sqrt{6+\sqrt{6+\sqrt{6+......\infty}}}$

Assume that $x = \log_2 \log_9 y$

$\Rightarrow y = \sqrt{6+\sqrt{6+\sqrt{6+......\infty}}} = \sqrt{6+y}$

$\Rightarrow y^2 = 6 + y \Rightarrow y^2 - 6 - y = 0$

$\Rightarrow (y - 3)(y + 2) = 0 \Rightarrow y = 3$ or $y = -2$, $y \neq -2$

$\therefore$ y = 3

$x = \log_2 \log_9 3 = \log_2 \log_9 (9)^{1/2}$

$\Rightarrow x = \log_2\left(\frac{1}{2}\right) = \log_2 2^{-1} = -1$

$\Rightarrow x = -1$ is an integer

(B) $N = 2^{(\log_2 3.\log_3 4.\log_4 5......\log_{99} 100)}$

$N = 2^x$ (Assume)

$\Rightarrow x = \frac{\log 3}{\log 2}.\frac{\log 4}{\log 3}...\frac{\log 100}{\log 99} = \frac{\log 100}{\log 2} = \log_2 100$

$N = 2^{\log_2 100} = 100$

N = 100 which is a composite, integer, natural number

(C) $\frac{1}{\log_5 3} + \frac{1}{\log_6 3} \quad \frac{1}{\log_{10} 3}$

$$\Rightarrow \frac{\log 5}{\log 3} + \frac{\log 6}{\log 3} - \frac{\log 10}{\log 3} = \left(\frac{\log 5 + \log 6 - \log 10}{\log 3}\right)$$

$$\Rightarrow \frac{\log(5 \times 6 \div 10)}{\log 3} = \frac{\log 3}{\log 3} = 1$$

$\Rightarrow$ 1 is natural and integer number

(D) $N = \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{14-6\sqrt{5}}}}$

$$N = \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{(3-\sqrt{5})^2}}}$$

$$N = \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+(-\sqrt{5}+3)}} = \sqrt{2+\sqrt{5}-\sqrt{9-4\sqrt{5}}}$$

$$N = \sqrt{2+\sqrt{5}-\sqrt{(\sqrt{5})^2+(2)^2-2(2)\sqrt{5}}}$$

$$N = \sqrt{2+\sqrt{5}-\sqrt{(\sqrt{5}-2)^2}} = \sqrt{2+\sqrt{5}-\sqrt{5}+2} = \sqrt{4} = 2$$

2 is natural prime and an integer.

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