CHAPTER 5

BASIC CLASSES OF INTEGRABLE FUNCTIONS

5.1 INTEGRATION OF RATIONAL FUNCTIONS

If the denominator Q (x) of the proper rational fraction $\frac{P(x)}{Q(x)}$ can be represented in the following way :

$Q(x) = (x-a)^k (x-b)^l \ldots (x^2 + \alpha x + \beta)^r (x^2 + \gamma x + \mu)^s \ldots$

where the binomials and trinomials are different and, furthermore, the trinomials have no real roots, then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} + \ldots \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \ldots + \frac{B_l}{(x-b)^l} + \ldots$$

$$\ldots + \frac{M_1 x + N_1}{x^2 + \alpha x + \beta} + \frac{M_2 x + N_2}{(x^2 + \alpha x + \beta)^2} + \ldots + \frac{M_r x + N_r}{(x^2 + \alpha x + \beta)^r} +$$

$$+ \frac{R_1 x + L_1}{x^2 + \gamma x + \mu} + \frac{R_2 x + L_2}{(x^2 + \gamma x + \mu)^2} + \ldots + \frac{R_s x + L_s}{(x^2 + \gamma x + \mu)^s} + \ldots$$

where $A_1, A_2, \ldots, B_1, B_2, \ldots M_1, N_1, M_2, N_2, \ldots, R_1, L_1, R_2, L_2, \ldots$ are some real constants to be determined. They are determined by reducing both sides of the above identity to integral form and then equating the coefficients at equal powers of x, which gives a system of linear equations with respect to the coefficients. (This method is called the method of comparison of coefficients.) A system of equations for the coefficients can also be obtained by substituting suitably chosen numerical values of x into both sides of the identity. (This method is called the method of particular values.) A successful combination of the indicated methods, prompted by experience, often allows us to simplify the process of finding the coefficients.

If the rational fraction $\frac{P(x)}{Q(x)}$ is improper, the integral part should first be singled out.

EXAMPLES

5.1.1. $I = \int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx$

Solution. $\frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} = \frac{A}{(x-3)} + \frac{B}{(x+4)} + \frac{C}{(x-1)} \times$

$$\frac{A(x^2 - 3x - 4) + B(x^2 - 4x + 3) + C(x^2 + x - 12)}{(x-3)(x+4)(x-1)}$$

$$A + B + C = 15$$
$$3A - 4B + C = -4$$
$$-4A + 3B - 12C = -81$$
$$A = 3, B = 5, C = 7$$
$$= 3\int\frac{dx}{x-3} + 5\int\frac{dx}{x+4} + 7\int\frac{dx}{x-1}$$
$$= 3\ln|x-3| + 5\ln|x+4| + 7\ln|x-1| + C$$
$$I = \ln\left|(x-3)^3(x+5)^5(x-1)^7\right| + C$$

5.1.2. $I = \int\frac{x^4\,dx}{(2+x)(x^2-1)}$

Solution.
$$I = \int\frac{(x^4-1+1)}{(2+x)(x^2-1)}dx$$
$$= \int\frac{(x^4-1)}{(x^2-1)(2+x)}dx + \int\frac{dx}{(2+x)(x-1)(x+1)}$$
$$= \int\frac{(x^4-1)}{2+x}dx + \int\frac{dx}{(2+x)(x-1)(x+1)}$$
$$I = \int\frac{x^4-4+4+1}{(2+x)}dx + \int\frac{dx}{(2+x)(x-1)(x+1)}$$
$$= \int(x-2)\,dx + \int\frac{5}{(x+2)} + I_1$$
$$= \left(\frac{x^2}{2} - 2x\right) + 5\ln|x+2| + I_1$$
$$I_1 = \int\frac{dx}{(x+2)(x-1)(x+1)}$$
$$\int\frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x+1)} = \frac{1}{(x+2)(x-1)(x+1)}$$
$$A(x^2-1) + B(x^2+3x+2) + C(x^2+x-2) = 1$$
$$A + B + C = 0$$
$$3B + C = 0$$
$$-A + 2B - 2C = 1$$
$$A + B + C = 0$$
$$-A + 2B - 2C = 1$$
$$3B - C = 1$$
$$3B + C = 0$$
$$6B = 1$$
$$\frac{1}{6} - \frac{1}{2} + A = 0\ ;\ A = \frac{1}{2} - \frac{1}{6}$$
$$A = \frac{1}{3}$$

$$B = \frac{1}{6}$$

$$\frac{1}{2} + C = 0$$

$$C = -\frac{1}{2}$$

$$I_1 = \int \frac{1}{3}\frac{dx}{(x+2)} + \frac{1}{6}\int\frac{dx}{(x-1)} - \frac{1}{2}\int\frac{dx}{(x+1)}$$

$$= \frac{1}{3}\ln|x+2| + \frac{1}{6}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

5.1.3. $I = \int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}\,dx$

Solution.

$$I = \int\left[(x+1) - \frac{x+2}{x\left(x^2-x-2\right)}\right]dx$$

$$= \int (x+1)\,dx - \int\frac{x+2}{\left(x^2-x-2\right)} - \int\frac{dx}{x\left(x^2-x-2\right)}$$

$$= \frac{x^2}{2} + x - \int\frac{dx}{\left(\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right)} - \int\frac{dx}{x\left(x^2-x-2\right)}$$

$$I = \frac{x^2}{2} + x - \frac{1}{2\times\frac{3}{2}}\ln\left[\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}}\right] - \int\frac{dx}{x\left(x^2-x-2\right)}$$

$$I = \frac{x^2}{2} + x - \frac{1}{3}\ln\left|\frac{x-2}{x-1}\right| - \int\frac{dx}{x\left(x^2-x-2\right)}$$

$$\downarrow$$

$$I_1$$

$$I_1 = \int\frac{dx}{x\left(x^2-x-2\right)}$$

$$\frac{1}{x\left(x^2-x-2\right)} = \int\frac{A}{x} + \frac{Bx+C}{\left(x^2-x-2\right)}$$

$$A + B = 0$$

$$-A + C = 0$$

$$-2A = 1$$

$$A = -\frac{1}{2}C \qquad C = -\frac{1}{2} \qquad B = -\frac{1}{2}$$

$$= \frac{1}{2}\left[-\frac{1}{x} + \frac{x-1}{x^2-x-2}\right]$$

$$\int \frac{dx}{x\left(x^2-x-2\right)} = \frac{1}{2}\int -\frac{1}{x}\,dx + \frac{(x-1)}{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left[-\ln x + \frac{1}{2}\int \frac{\left(x-\frac{1}{2}-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx\right]$$

$$= -\frac{1}{2}\ln x + \frac{1}{2}\int \frac{\left(x-\frac{1}{2}\right)dx}{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{1}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

$$= -\frac{1}{2}\ln x + \frac{1}{4}\ln\left|\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right| - \frac{1}{4}\times\frac{1}{2\times 3/2}\ln\left|\frac{x-2}{x+1}\right| + C$$

$$I = \frac{x^2}{2} + x - \frac{1}{2}\ln x + \frac{1}{4}\ln|(x-2)(x+1)| - \frac{1}{12}\ln\left|\frac{x-2}{x+1}\right| + C$$

5.1.4. $I = \int \frac{\left(2x^2-3x+3\right)}{x^3-2x^2+x}\,dx$

Solution.
$$= \int \frac{\left(2x^2-3x+3\right)}{x\left(x^2-2x+1\right)}\,dx$$

$$= \int \frac{\left(2x^2-3x+3\right)}{x(x-1)^2}\,dx$$

$$I = \int \frac{2x^3-3x+3}{x^3-2x^2+x}\,dx$$

$$\frac{2x^3-3x+3}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{D}{(x-1)}$$

$$2x^3-3x+3 = A(x-1)^2 + Bx(x-1) + Bx$$
$$= (A+D)x^2 + (-2A-D+B)x + A$$

Equating the co-efficients

$$A + D = 2$$
$$-2A - D + B = -3$$
$$A = 3;\ B = 2;\ D = -1$$

$$I = 3\int\frac{dx}{x} + 2\int\frac{dx}{(x-1)^2} - \int\frac{dx}{(x-1)}$$

$$= 3\ln|x| - \frac{2}{x-1} - \ln|x-1| + C$$

5.1.5. $I = \int\frac{x^3+1}{x(x-1)^3}$

Solution. $x^3 + 1 = (x+1)(x^2 - x + 1)$

$$= (x+1)\,|x(x-1)+1|$$

$$x^3 + 1 = x(x+1)(x-1) + (x+1)$$

$$I = \int\frac{x(x+1)(x-1)}{x(x-1)^3}dx + \frac{(x+1)}{x(x-1)^3}dx$$

$$= \int\frac{(x+1)}{(x-1)^2}dx + \int\frac{dx}{(x-1)^3} + \frac{dx}{x(x-1)^3}$$

$$= \int\frac{(x-1)}{(x-1)^2} + \frac{2}{(x-1)^2}dx + \frac{-1}{2(x-1)^2} + \int\frac{dx}{x(x-1)^3}$$

$$= \ln|(x-1)|\frac{-2}{(x-1)} - \frac{1}{2(x-1)^2} + \int\frac{dx}{x(x-1)^3}$$

$$\frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}$$

On solving, $A = -1;\ B = 1;\ C = -1;\ D = 1$

$$I = \ln|x-1| - \frac{2}{(x-1)} - \frac{1}{2(x-1)^2} + \int -\frac{dx}{x} + \int\frac{dx}{(x-1)^3}$$

$$-\int\frac{dx}{(x-1)^2} + \int\frac{dx}{(x-1)}$$

$$= 2\ln|x-1| - \frac{2}{(x-1)} - \frac{1}{2(x-1)^2} - \ln x - \frac{1}{2(x-1)^2} + \frac{1}{(x-1)}$$

$$= 2\ln|x-1| - \frac{1}{(x-1)} - \frac{1}{(x-1)^2} - \ln x + C$$

$$I = 2\ln|x-1| - \frac{x}{(x-1)^2} - \ln x + C$$

5.1.6. $I = \int\frac{x\,dx}{x^3+1}$

Solution. $I = \int\frac{x+1-1}{x^3+1}dx$

$$x^3 + 1 = (x+1)(x^2 - x + 1)$$

$$= \int \frac{dx}{(x^2 - x + 1)} - \int \frac{dx}{(x+1)(x^2 - x + 1)}$$

$$= \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} - \int \frac{dx}{(x+1)(x^2 - x + 1)}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{x - \left(\frac{1}{2}\right)}{\sqrt{\frac{3}{2}}} - I_1$$

$$\frac{1}{(x+1)(x^2 - x + 1)} = \frac{A}{(x+1)} + \frac{Bx + C}{x^2 - x + 1}$$

$$A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3}$$

$$= \frac{1}{3}\left[\int \frac{dx}{(x+1)} + \frac{(2-x)}{x^2 - x + 1} dx\right]$$

$$I_1 = \frac{1}{3}\left[\ln(x+1) + \int \frac{-(x-2)}{x^2 - x + 1} dx\right]$$

$$= \frac{1}{3}\left[\ln(1+x) + \frac{1}{2}\int \frac{-(2x-4)}{x^2 - x + 1} dx\right]$$

$$= \frac{1}{3}\left[\ln(1+x) - \frac{1}{2}\left[\int \frac{-(2x - 1 - 3)}{x^2 - x + 1} dx\right]\right]$$

$$= \frac{1}{3}\ln(1+x) - \frac{1}{6}\ln(x^2 - x + 1) + \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(x - \frac{1}{2}\right)}{\sqrt{\frac{3}{2}}}$$

$$-I_1 = \frac{-1}{3}\ln(1+x) + \frac{1}{6}\ln(x^2 - x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{(2x-1)}{\sqrt{3}}$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \frac{(2x-1)}{\sqrt{3}} - \frac{1}{3}\ln(1+x) + \frac{1}{6}\ln(x^2 - x + 1) + C$$

5.1.7. $I = \int \frac{dx}{(x^2 + 1)(x^2 + 4)}$

Solution.

$$I = \frac{1}{3}\int\left(\frac{dx}{x^2 + 1} - \frac{dx}{x^2 + 4}\right)$$

$$= \frac{1}{3}\int\left[\tan^{-1} x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right]$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2}$$

5.1.8. $I = \int \frac{x+1}{(x^2+x+2)(x^2+4x+5)}$

Solution. $$I = \frac{1}{3}\int \frac{dx}{(x^2+x+2)} - \frac{1}{3}\int \frac{dx}{(x^2+4x+5)}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} - \frac{1}{3}\int \frac{dx}{(x+2)^2+1^2}$$

$$= \frac{1}{3} \times \frac{2}{\sqrt{7}} \tan^{-1} \frac{\left(x+\frac{1}{2}\right)}{\sqrt{\frac{7}{2}}} - \frac{1}{3}\tan^{-1}(x+2) + C$$

$$I = \frac{2}{3\sqrt{7}} \tan^{-1} \frac{2x+1}{\sqrt{7}} - \frac{1}{3}\tan^{-1}(x+2) + C$$

5.1.9. $I = \int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2 (x+1)}$

Solution. $$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2 (x+1)} = \frac{Ax+B}{(x^2+2x+3)^2} + \frac{Dx+E}{(x^2+2x+3)} + \frac{F}{x+1}$$

Finding the co-efficients A = 1, B = – 1, D = 0, E = 0, F = 1

$$= \int \frac{(x-1)}{(x^2+2x+3)^2} dx + \int \frac{dx}{x+1}$$

$$= \ln|x+1| + I_1$$

$$I_1 = \int \frac{(x-1)\,dx}{(x^2+2x+3)^2}$$

$$= \int \frac{(x-1)\,dx}{((x+1)^2+2)^2}$$

$$x = 1 = t$$

$$I_2 = \int \frac{(t-2)}{(t^2+2)^2} dt$$

$$= \int \frac{t\,dt}{(t^2+2)^2} - 2\int \frac{dt}{(t^2+2)^2}$$

$$= \frac{1}{2(t^2+2)} - 2I_2$$

$$I_2 = \int \frac{dt}{(t^2+2)^2}$$

Applying Reducing formula

$$I_2 = \frac{1}{4}\frac{1}{(t^2+2)} + \frac{1}{4}\int\frac{dt}{t^2+2}$$

$$= \frac{1}{4}\frac{1}{(t^2+2)} + \frac{1}{4}\times\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C$$

$$I_1 = -\frac{1}{2(x^2+2x+3)} - \frac{x+1}{2(x^2+2x+3)} - \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C$$

$$I = \ln|x+1| - \frac{x+2}{2(x^2+2x+3)} - \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C$$

5.1.10. $I = \int\frac{5x^3+9x^2-2x-8}{x^3-4x}dx$

Solution.

$$I = \int 5\,dx + \int\frac{9x^2-2x-8}{x(x-2)(x+2)}dx$$

$$= 5x + I_1$$

$$I_1 = \int\frac{9x^2-2x-8}{x(x-2)(x+2)}dx$$

$$\frac{9x^2-2x-8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{D}{x+2}$$

$$= \frac{A(x^2-4)+B(x^2+2x)+D(x^2-2x)}{x(x-2)(x+2)}$$

Equating the co-efficients $= \dfrac{x^2(A+B+D)+x(2B-2D)-4A}{x(x-2)(x+2)}$

$A = 2, B = 3, D = 4$

$$= \frac{2}{x} + \frac{3}{x-2} + \frac{4}{x+2}$$

$$I_1 = 2\ln x + 3\ln|x-2| + 4\ln|x+2|$$

$$= \ln x^2\cdot|x-2|^3\,|x+2|^4$$

$$I = 5x + \ln\left|x^2\,|x-2|^3\,|x+2|^4\right| + C$$

5.1.11. $I = \int\frac{dx}{(x+1)(x+2)^2(x+3)^3}$

Solution.

$$= \int\frac{dt}{t^3(t-1)^2(t-2)} \qquad x+3=t;\ dx=dt$$

$$\frac{1}{t^3(t-1)^2(t-2)} = \frac{At^2+Bt+C}{t^3} + \frac{D}{(t-1)^2} + \frac{E}{(t-1)} + \frac{F}{(t-2)}$$

Solving for A, B, C, D, E, F,

$$t^5(A+E+F)+t^4(-4A+B+D-3E-2F)$$
$$+t^3(5A-4B+C-2D+F+2E)+t^2(-2A+5B-4C)+t(5C-2B)-2C$$

$$A=-\frac{17}{8},\ B=-\frac{5}{4},\ E=2,\ D=-1,\ F=\frac{1}{8}$$

Substituting these values

$$\int\frac{dt}{t^3(t-1)^2(t-2)}=\int-\frac{17t^2-\frac{5}{4}t-\frac{1}{2}}{t^3}dt+\int-\frac{dt}{(t-1)^2}+\int\frac{2dt}{(t-1)}+\int\frac{1}{8}\frac{dt}{(t-2)}$$

$$=\int-\frac{17t^2-10t-4}{8t^3}dt-\int\frac{dt}{(t-1)^2}+2\int\frac{2dt}{t-1}+\frac{1}{8}\int\frac{dt}{t-2}$$

$$=-\frac{17}{8}\ln t+\frac{10}{8}\times\frac{1}{t}+\frac{4}{16t^2}+\frac{1}{t-1}+2\ln(t-1)+\frac{1}{8}\ln(t-2)$$

$$=\ln(x+3)^{-17/8}+\frac{10}{8}\times\frac{1}{x+3}+\frac{4}{16(x+3)^2}+\frac{1}{x+2}$$
$$+2\ln(x+2)+\frac{1}{8}\ln(x+1)+C$$

$$I=\frac{9x^2+50x+68}{4(x+2)(x+3)^2}+\frac{1}{8}\ln\left|\frac{(x+1)(x+2)^{16}}{(x+3)^{17}}\right|+C$$

5.1.12. $I=\int\frac{dx}{(x^2-4x+4)(x^2-4x+5)}$

Solution. $$I=\int\frac{dx}{(x^2-4x+4)}-\frac{dx}{(x^2-4x+5)}$$

$$=\int\frac{dx}{(x-2)^2}-\frac{dx}{(x-2)^2+1^2}$$

$$I=-\frac{1}{(x-2)}-\tan^{-1}(x-2)+C$$

5.1.13. $I=\int\frac{dx}{(1+x)(1+x^2)(1+x^3)}$

Solution. $$\frac{1}{(1+x)^2(1+x^2)(x^2-x+1)}=\frac{A}{1+x}+\frac{B}{(1+x)^2}+\frac{Cx+D}{1+x^2}+\frac{Ex+F}{x^2-x+1}$$

$\therefore$ $$1=A(1+x)(1+x^2)(x^2-x+1)+B(1+x^2)(x^2-x+1)$$
$$+(Cx+D)(1+x)^2(x^2-x+1)+(Ex+F)(1+x)^2(1+$$

Put $x=-1$, $$1=6B \text{ or } B=\frac{1}{6},$$

$$A+E=0$$

Put $x = i$, $1 = (Ci + D)(1+i)^2 (i^2 + 1 - i)$

$$C = 0, D = \frac{1}{2}$$

Put $x = 0$, $1 = A + B + D + F$

$$= A + \frac{1}{6} + \frac{1}{2} + F$$

$$A + F = \frac{1}{3} \quad \ldots (ii)$$

Put $x = 1$, $1 = 4A + 2B + 4(C + D) + 8(E + F)$

$$A + 2E + 2F = -\frac{1}{3} \quad \ldots (iii)$$

From *(ii)* – *(i)*, $F - E = \frac{1}{3}$

From *(iii)* – *(ii)*, $2E + F = -\frac{2}{3}$

$\therefore \quad 3E = -1, E = -1, F = 0, A = \frac{1}{3}$

$$\therefore \quad I = \frac{1}{3}\int \frac{dx}{1+x} + \frac{1}{6}\int \frac{dx}{(1+x)^2} + \frac{1}{2}\int \frac{dx}{1+x^2} - \frac{1}{3}\int \frac{x\,dx}{x^2 - x + 1}$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\frac{1}{(1+x)} + \frac{1}{2}\tan^{-1}x - \frac{1}{6}\int \frac{(2x-1)}{x^2 - x + 1}dx - \frac{1}{6}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$I = \frac{1}{3}\ln|1+x| - \frac{1}{6(1+x)} + \frac{1}{2}\tan^{-1}x - \frac{1}{6}|\ln|\,|x^2 - x + 1| - \frac{1}{3\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

5.1.14. $I = \int \frac{x^3 + 3}{(x+1)(x^2+1)}dx$

Solution. $= \int \left[1 - \frac{x^2 + x - 2}{(x+1)(x^2+1)}\right]dx$

$$= x - \left[-\int \frac{dx}{1+x} + \int \frac{(2x-1)}{x^2+1}dx\right]$$

$$= \frac{x^2 + x - 2}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(x + 1)$$

Put $x = -1$; $-2 = 2A$;

$$A = -1$$

Put $x = i$, $\quad i^2 + i - 2 = (i + 1)(Bi + C)$

$$i - 3 = -B + (B + C)i + C$$

$$B + C = 1$$

$$C - B = -3$$

$$\therefore \quad C = -1, B = 2$$

$$= x + \ln|1 + x| - 2\int \frac{x}{1+x^2} + + \int \frac{dx}{1+x^2}$$

$$I = x + \ln|1 + x| - \ln\left(1 + x^2\right) + \tan^{-1} x + C$$

5.2 INTEGRATION OF CERTAIN IRRATIONAL EXPRESSIONS

Certain type of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by an appropriate change of the variable. Such transformation of an integral is called its **rationalization.**

I. If the integrand is a rational function of fractional powers of an independent variable x, *i.e.*, the function $R\left(x, x^{\frac{p_1}{q_1}}, \dots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x = t^m$, where m is the least common multiple of the numbers $q_1, q_2, \dots, q_k$.

II. If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d} = t^m$, where m has the same sense as above.

EXAMPLES

5.2.1. $I = \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1 + \sqrt[3]{x})} dx$

Solution. $x = t^6 \; dx = 6t^5 \, dt$

$$I = \int \frac{\left(t^6 + t^4 + t\right) t^5}{t^6\left(1 + t^2\right)} dt$$

$$= 6\int \frac{t^5 + t^3 + 1}{1 + t^2} dt$$

$$= 6\int t^3 \, dt + 6\int \frac{dt}{t^2 + 1}$$

$$= \frac{3}{2} t^4 + 6\tan^{-1} t + C$$

$$I = \frac{3}{2} x^{2/3} + 6\tan^{-1} x^{1/6} + C$$

5.2.2. $I = \int \frac{\sqrt{x} + \sqrt[3]{x}}{\sqrt[4]{x^5} - \sqrt[6]{x^7}} dx$

Solution. $x = t^{12}, dx = 12t^{11} dt$

$$I = \int \frac{t^6 + t^4}{t^{15} - t^{14}} dt \cdot 12t^{11}$$

$$= \int \frac{t^4 (t^2 + 1)}{t^{14} (t-1)} dt \cdot 12t^{11}$$

$$= \int \frac{t^3 (t^2 + 1)}{t^{13} (t-1)} dt \cdot 12t^{11}$$

$$= \int \frac{t(t^2 + 1)}{(t-1)} dt$$

$$= \int \frac{t^3}{t-1} dt + \int \frac{t}{t-1} dt$$

$$= \int \frac{t^3 - 1 + 1}{t-1} dt + \int \frac{t-1+1}{t-1} dt$$

$$I = \int (t^2 + t + 1) dt + 2\int \frac{dt}{t-1} + \int dt$$

$$= \frac{t^3}{3} + \frac{t^2}{2} + t + 2 \ln |t-1| + t$$

$$= 12\left(\frac{t^3}{3} + \frac{t^2}{2} + 2t + 2 \ln |t-1|\right) + C$$

$$= 4t^3 + 6t^2 + 24t + 24 \ln |t-1| + C$$

$$= 4 - x^{1/4} + 6x^{1/6} + 24x^{1/12} + 24 \ln \left| x^{1/12} - 1 \right| + C$$

5.2.3. $I = \int \frac{(2x-3)^{1/2} \, dx}{(2x-3)^{1/3} + 1}$

Solution. $2x - 3 = t^6$, $2\,dx = 6t^5 dt$, $dx = 3t^5 dt$

$$= \int \frac{t^3 \cdot 3t^5 \, dt}{1+t^2} = \int \frac{3t^8}{1+t^2} dt$$

$$I = 3\int (t^6 - t^4 + t^2 - 1) dt + 3\int \frac{dt}{1+t^2}$$

$$= \frac{3t^7}{7} - \frac{3t^5}{5} + \frac{3t^3}{3} - 3t + 3\tan^{-1} t$$

$$I = \frac{3}{7}\left[(2x-3)^{7/6}\right] - \frac{3}{5}\left[(2x-3)^{5/6}\right] + (2x-3)^{1/2} - 3(2x-3)^{1/6} + 3\tan^{-1}(2x-3)^{1/6} + C$$

5.2.4. $I = \int \frac{dx}{x\left(2+3\sqrt{\frac{x-1}{x}}\right)}$

Solution. Let $\frac{x-1}{x} = t^3$ or $1-\frac{1}{x} = t^3$

$$+\frac{1}{x^2}dx = 3t^2\,dt \text{ or } dx = 3x^2 t^2\,dt$$

$$I = 3\int \frac{xt^2\,dt}{(t+2)} \qquad 1-\frac{1}{x} = t^3$$

$$= 3\int \frac{t^2\,dt}{(1-t^3)(t+2)} \qquad 1-t^3 = \frac{1}{x} \Rightarrow x = \frac{1}{1-t^3}$$

$$\frac{t^2}{(1-t)(1+t+t^2)(t+2)} = \frac{A}{t+2}+\frac{B}{1-t}+\frac{Ct+D}{1+t+t^2}$$

$$= A(1-t^3) + B(t^3+3t^2+3t+2) + C(-t^3-t^2+2t) + D(-t^2-t+2)$$

$$-A+B-C = 0$$

$$A = B-C \quad B = \frac{1}{9}, C = -\frac{1}{3}$$

$$3B-C-D = 1 \qquad A = \frac{4}{9}, D = -\frac{1}{3}$$

$$3B+2C-D = 0$$

$$A+2B+2D = 0$$

$$I = 3\left[\frac{4}{9}\int\frac{dt}{t+2}+\frac{1}{9}\int\frac{dt}{1-t}-\frac{1}{3\times 2}\int\frac{dt}{1+t+t^2}\right]-\frac{1}{2}\int\frac{2t+1}{1+t+t^2}$$

$$I = \frac{4}{3}\ln|t+2| - \frac{3}{9}\ln|1-t| - \frac{1}{3}\times\frac{3}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{2}\ln\left|1+t+t^2\right|$$

$$= \frac{4}{3}\ln|t+2| - \frac{1}{3}\ln|1-t| - \frac{1}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{4}{3}\ln|t+2| - \frac{1}{3}\ln|1-t| - \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) - \frac{1}{2}\ln|1+t+t^2|$$

$$= \frac{4}{3}\ln|t+2| - \frac{1}{2}\ln|1-t| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) - \frac{1}{2}\ln|1+t+t^2|$$

$$I = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}} + \ln\left|\frac{\sqrt[3]{(t+2)^4}}{\sqrt[3]{t-1}\,\sqrt{t^2+t+1}}\right| + C \qquad t = 3\sqrt{\frac{x-1}{x}}$$

5.2.5. $I = \int \frac{2}{(2-x)} \sqrt[3]{\frac{2-x}{2+x}}\, dx$

Solution.

$$\sqrt[3]{\frac{2-x}{2+x}} = t$$

$$\frac{2-x}{2+x} = t^3$$

$$x = \frac{2-2t^3}{1+t^3}$$

$$2-x = \frac{4t^3}{1+t^3} \qquad dx = \frac{-12t^2}{\left(1+t^3\right)^2} dt$$

$$I = -\int \frac{2\left(1+t^3\right)^2 t \cdot 12t^2}{16t^6\left(1+t^3\right)^2} dt$$

$$= -\frac{3}{2}\int \frac{dt}{t^3} = \frac{3}{4} \times \frac{1}{t^2} + C$$

$$I = \frac{3}{4}\sqrt[3]{\frac{(2+x)^2}{(2-x)^2}} + C$$

5.2.6. $I = \int \frac{dx}{\sqrt[4]{(x-1)^3\,(x+2)^5}}$

Solution.

$$I = \int \frac{dx}{(x-1)\,(x+2)\sqrt[4]{\frac{x+2}{x-1}}}$$

$$\frac{x+2}{x-1} = t^4$$

$$x = \frac{t^4+2}{t^4-1},$$

$$x-1 = \frac{3}{t^4-1},$$

$$x+2 = \frac{3t^4}{t^4-1}$$

$$dx = \frac{-12t^3}{\left(t^4-1\right)^2} dt$$

$$I = -\int \frac{\left(t^4-1\right)\left(t^4-1\right)12t^3}{3 \cdot 3t^4 \cdot t\left(t^4-1\right)^2} dt$$

$$= -\frac{4}{3}\int \frac{dt}{t^2} = \frac{4}{3t} + C$$

$$I = \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + C$$

5.2.7. $I = \int \frac{dx}{(1-x)\sqrt{1-x^2}}$

Solution.

$$I = \int \frac{dx}{(1-x)^2\sqrt{\frac{1+x}{1-x}}}$$

Let $$\frac{1+x}{1-x} = t^2$$

$$t^2+1 = \frac{2}{1-x}$$

or $$\frac{1}{t^2+1} = \frac{1-x}{2}$$

$$\frac{2}{t^2+1} = 1-x$$

$$x = 1-\frac{2}{t^2+1}$$

$$x = \frac{t^2-1}{t^2+1}$$

$$dx = \frac{2t\left(t^2+1\right)-2t\left(t^2-1\right)}{\left(t^2+1\right)^2}dt$$

$$dx = \frac{4t}{\left(t^2+1\right)^2}dt$$

$$= \int \frac{4t}{\left(t^2+1\right)^2}\times\frac{dt}{\frac{4}{\left(t^2+1\right)^2}\times t}$$

$$= \int dt + C = t + C$$

$$I = \sqrt{\frac{1+x}{1-x}} + C$$

5.2.8. $I = \int \frac{dx}{\sqrt[3]{(x+1)^2\,(x-1)^4}}$

Solution. $$I = \int \frac{dx}{(x+1)(x-1)}\cdot\sqrt[3]{\frac{x-1}{x+1}}$$

Let $$t^3 = \frac{x-1}{x+1}$$

$$x = \frac{1+t^3}{1-t^3}$$

$$dx = \frac{3t^2\left(1-t^3\right)+3t^2\left(1+t^3\right)}{\left(1-t^3\right)^2}dt$$

$$dx = \frac{6t^2}{\left(1-t^3\right)^2}dt$$

$$\left(\frac{1+t^3}{1-t^3}\right)^2 - 1 = x^2-1$$

$$\frac{4t^3}{(1-t^3)^2} = x^2 - 1$$

$$I = \int \frac{6t^2}{(1-t^3)^2} \times \frac{dt}{\frac{4t^3}{(1-t^3)^2} \times t}$$

$$I = \frac{3}{2}\int \frac{dt}{t^2} - \frac{3}{2}\frac{1}{t} + C - \frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C$$

5.2.9. $I = \int (x-2)\sqrt{\frac{1+x}{1-x}}$

Solution. $= \int (x-1-1)\sqrt{\frac{1+x}{1-x}}$

$$= \int (x-1)\sqrt{\frac{1+x}{1-x}}\,dx - \int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= -\int \sqrt{1-x^2}\,dx - \int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= -\frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x - \int \sqrt{\frac{1+x}{1-x}}\,dx \qquad x = \cos 2\theta \text{ in } \sqrt{\frac{1+x}{1-x}},$$

$$dx = -2\sin 2\theta\, d\theta$$

$$= +2\int \cot\theta \cdot 2\sin 2\theta\, d\theta$$

$$= 4\int \frac{\cos\theta}{\sin\theta} \times \sin\theta\cos\theta\, d\theta$$

$$= 4\int \cos^2\theta\, d\theta$$

$$= 4\int \frac{\cos 2\theta + 1}{2}\, d\theta$$

$$= \int 2\cos 2\theta\, d\theta + \int 2d\theta$$

$$= \int 2d\theta \frac{2\sin 2\theta}{2} + 2\theta$$

$$\therefore \quad \sqrt{\frac{1+x}{1-x}} = \sqrt{1-x^2} + \cos^{-1}x + c$$

$$I = -\frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + \sqrt{1-x^2} + \cos^{-1}x$$

$$\left(1-\frac{1}{2}x\right)\sqrt{1-x^2} - \frac{3}{2}\sin^{-1}x + \sin^{-1}x + \cos^{-1}x + C$$

$$I = \left(1-\frac{1}{2}x\right)\sqrt{1-x^2} - \frac{3}{2}\sin^{-1}x + \frac{x}{2} + C$$

5.3 EULER'S SUBSTITUTIONS

Integrals of the form $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$ are calculated with the aid of one of the three Euler substitutions :

1. $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$;
2. $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$;
3. $\sqrt{ax^2+bx+c} = (x-\alpha)\,t$
   if $ax^2 + bx + c = a\,(x-\alpha)\,(x-\alpha)$,

i.e., if α is a real root of the trinomial $ax^2 + bx + c$.

EXAMPLES

5.3.1. $I = \int \frac{dx}{1+\sqrt{x^2+2x+2}}$

Solution.

$$\sqrt{(x^2+2x+2)} = t - x$$

$$2x + 2tx = t^2 - 2$$

$$x = \frac{t^2-2}{2(1+t)}$$

$$dx = \frac{t^2+2t+2}{2(1+t)^2}\,dt$$

$$1+\sqrt{x^2+2x+2} = 1 + t - \frac{t^2-2}{2(1+t)}$$

$$= \frac{t^4+4t+4}{2(1+t)}$$

$$I = \int \frac{(t^2+2t+2)\,dt}{(1+t)(t+2)^2}$$

$$\frac{t^2+2t+2}{(1+t)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{D}{(t+2)^2}$$

Equating the coefficients, we get A = 1, B = 0, D = – 2

$$= \int \frac{dt}{t+1} - 2\int \frac{dt}{(t+2)^2}$$

$$= \ln|t+1| + \frac{2}{(t+2)} + C$$

$$I = \ln\left(x+1+\sqrt{x^2+2x+2}\right) + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C$$

5.3.2. $I = \dfrac{dx}{x+\sqrt{x^2-x+1}}$

Solution.

$$\sqrt{x^2-x+1} = tx-1$$

$$(2t-1)x = (t^2-1)x^2$$

$$x = \frac{2t-1}{t^2-1}$$

$$dx = \frac{-2\left(t^2-t+1\right)}{\left(t^2-1\right)^2}dt$$

$$x+\sqrt{x^2-x+1} = \frac{t}{t-1}$$

$$I = \int\frac{dx}{x+\sqrt{x^2-x+1}}$$

$$= \int -\frac{2t^2+2t-2}{t(t-1)(t+1)^2}dt$$

$$-\frac{2t^2+2t-2}{t(t-1)(t+1)^2} = \frac{A}{t}+\frac{B}{t-1}+\frac{D}{(t+1)^2}+\frac{E}{t+1}$$

Equating the coefficients, we get

$$A = 2, B = -\frac{1}{2}, D = -3, E = -\frac{3}{2}$$

$$I = 2\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t-1}-3\int\frac{dt}{(t+1)^2}-\frac{3}{2}\int\frac{dt}{t+1}$$

$$= 2\ln|t|-\frac{1}{2}\ln|t-1|+\frac{3}{t+1}-\frac{3}{2}\ln|t+1|+C$$

$$t = \frac{\sqrt{x^2-x+1}+1}{x}$$

5.3.3. $I = \displaystyle\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}$

Solution.

$$\sqrt{\left(1+x-x^2\right)} = tx-1$$

Squaring both sides

$$1+x-x^2 = t^2x^2+1-2tx$$

$$x = \frac{2t+1}{t^2+1}$$

$$x+1 = \frac{t^2+1+2t+1}{t^2+1} \qquad tx = \frac{2t^2+t}{t^2+1}$$

$$= \frac{t^2+2t+2}{t^2+1} \qquad (tx-1) = \frac{2t^2+t-t^2-1}{t^2+1}[t^2+t-1]$$

$$dx = \frac{2(t^2+1)-2t(2t+1)}{(t^2+1)^2}dt$$

$$= \frac{(-2t^2-2t+2)}{(t^2+1)^2}dt$$

$$I = \int \frac{-2(t^2+t-1)}{(t^2+1)^2} \times \frac{(t^2+1)}{(t^2+2t+2)} \frac{dt}{(t^2+t-1)}$$

$$= \int \frac{-2}{(t^2+1)^2} \frac{(t^2+1)^2}{(t^2+2t+2)}dt$$

$$= -2\int \frac{dt}{t^2+2t+2}$$

$$= -2\int \frac{dt}{(t^2+1)^2+1}$$

$$= -2\tan^{-1}\frac{(t+1)}{1}+C$$

$$I = -2\tan^{-1}\left(\frac{\sqrt{1+x-x^2}+1+x}{x}\right)+C$$

5.3.4. $I = \int \frac{x\,dx}{\sqrt{(7x-10-x^2)^3}}$

Solution. $\sqrt{7x-10-x^2} = (x-2)(5-x)$

$$= (x-2)\,t\,(5-x) = (x-2)\,t^2$$

$$x = \frac{5+2t^2}{1+t^2};$$

$$dx = \frac{-6t\,dt}{(1+t^2)^2}$$

$$(x-2)\,t = \left(\frac{5+2t^2}{1+t^2}-2\right)t = \frac{3t}{1+t^2}$$

$$I = \frac{-6}{27}\int \frac{5+2t^2}{t^2}\,dt$$

$$= -\frac{2}{9}\left(\frac{5}{t^2}+2\right)dt$$

$$= -\frac{2}{9}\left(-\frac{5}{t}+2t\right)+C \qquad \text{where, } t = \frac{\sqrt{7x-10-x^2}}{x-2}$$

5.3.5. $I = \int \dfrac{dx}{x-\sqrt{x^2+2x+4}}$

Solution. $\sqrt{(x^2+2x+4)} = x-t$

$$x^2+2x+4 = x^2+t^2-2xt$$

$$2x+2xt = t^2-4$$

$$x = \frac{t^2-4}{2(1+t)}$$

$$dx = \frac{2t(1+t)-t^2+4}{2(1+t)^2}\,dt$$

$$= \frac{t^2+2t+4}{2(1+t)^2}$$

$$dx = \frac{(t+1)^2+3}{2(1+t)^2}\,dt$$

$$I = \frac{1}{2}\int \frac{1}{x-(x-t)} \times \left[\frac{(t+1)^2}{(t+1)^2}+\frac{3}{(t+1)^2}\right]dt$$

$$= \frac{1}{2}\int \frac{1}{t}\cdot\left[1+\frac{3}{(t+1)^2}\right]dt$$

$$= \frac{1}{2}\ln t + \frac{3}{2}\int \frac{dt}{t(t+1)^2}$$

$$\frac{1}{t(t+1)^2} = \frac{A}{t}+\frac{B}{(t+1)}+\frac{C}{(t+1)^2}$$

$$1 = A(t+1)^2 + Bt(t+1) + Ct(t+1)$$

$$A+B = 0$$

$$2A+B+C = 0$$

$$A = 1, B = -1, C = -1$$

$$= \frac{1}{2}\ln t + \frac{3}{2}\left[\int \frac{dt}{t} - \int \frac{dt}{(t+1)} - \int \frac{dt}{(t+1)^2}\right]$$

$$= \frac{1}{2}\ln t + \frac{3}{2}\left[\ln t - \ln(t+1) + \frac{1}{t+1}\right]$$

$$= 2\ln t - \frac{3}{2}\ln(t+1) + \frac{3}{2}\frac{1}{x-\sqrt{x^2+2x+4}} + C$$

$$I = 2\ln\left[x-\sqrt{x^2+2x+4}\right] - \frac{3}{2}\ln\left[x+1-\sqrt{x^2+2x+4}\right] + \frac{3}{2}\times\frac{1}{x-\sqrt{x^2+2x+4}} + C$$

5.3.6. $I = \int \frac{dx}{\sqrt{1-x^2}-1}$

Solution. $x = \sin\theta, \quad dx = \cos\theta\, d\theta$

$$I = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}-1}$$

$$= \int \frac{\cos\theta\, d\theta}{\cos\theta - 1}$$

$$= \int \frac{(\cos\theta - 1 + 1)\, d\theta}{\cos\theta - 1}$$

$$I = \int d\theta + \int \frac{d\theta}{\cos\theta - 1}$$

$$= \theta - \int \frac{d\theta}{\sin^2\theta/2} - \operatorname{cosec}^2\frac{\theta}{2}\, d\theta \qquad \frac{\theta}{2} = t,\ d\theta = 2dt$$

$$I = -2\operatorname{cosec}^2 t\, dt + 2\cot t + C$$

$$\theta + 2\cot\frac{\theta}{2} = C\sin^{-1}x + 2\cot\frac{\sin^{-1}x}{2}$$

$$I = \sin^{-1}x + 2\tan^{-1}\frac{2}{\sin^{-1}x} + C$$

5.3.7. $I = \frac{dx}{\sqrt{(2x-x^2)^3}}$

Solution.

$$2x - x^2 = x(2-x)$$

$$2 - x = xt^2$$

$$x = \frac{2}{t^2+1}$$

$$dx = -\frac{4t}{(t^2+1)^2}dt$$

$$I = \int \frac{-4t}{(t^2+1)^2} \cdot \frac{dt}{\sqrt{x^3(2-x)^3}}$$

$$= \int \frac{-4t}{(t^2+1)^2} \cdot \frac{dt}{\sqrt{x^3 \cdot x^3 t^6}}$$

$$= \int \frac{-4t}{(t^2+1)^2} \cdot \frac{dt}{x^3 t^3}$$

$$I = \int \frac{-4t\, dt}{(t^2+1)^2 \frac{8}{(t^2+1)^2} \cdot t^3}$$

$$= -\frac{1}{2}\int \frac{t^2+1}{t^2}dt$$

$$= -\frac{1}{2}\int dt - \frac{1}{2}\int \frac{dt}{t^2}$$

$$= -\frac{1}{2}\sqrt{\frac{2-x}{x}} + \frac{1}{2}\sqrt{\frac{x}{2-x}} + C$$

$$I = \left[\frac{x-1}{\sqrt{2x-x^2}}\right] + C$$

5.3.8. $I = \int \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{\sqrt{1+x^2}}dx$

Solution.

$$\sqrt{(1+x^2)} = t - x$$

$$1 + x^2 = t^2 + x^2 - 2t^x$$

$$\frac{t^2-1}{2t} = x$$

$$dx = \frac{t^2+1}{2t^2}dt$$

$$\int \frac{(x+t-x)^{15}}{t - \frac{t^2-1}{2t}} \cdot \frac{t^2+1}{2t^2}dt = \int t^{14}\, dt = \frac{t^{15}}{15} + C$$

$$I = \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{15} + C$$

5.4 OTHER METHODS OF INTEGRATING IRRATIONAL EXPRESSIONS

The Euler substitutions often lead to rather cumbersome calculations, therefore they should be applied only when it is difficult to find another method for calculating a given integral. For calculating many integrals of the form

$$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx,$$

simpler methods are used.

I. Integrals of the form

$$I = \int \frac{Mx+N}{\sqrt{ax^2+bx+c}}\, dx$$

are reduced by the substitution $x + \dfrac{b}{2a} = t$ to the form

$$I = M_1 \int \frac{t\,dt}{\sqrt{at^2+K}} + N_1 \int \frac{dt}{\sqrt{at^2+K}},$$

where M_1, N_1, K are new coefficients.

The first integral is reduced to the integral of a power function, while the second, being a tabular one, is reduced to a logarithm (for $a > 0$) or to an arc sine (for $a < 0$, $K > 0$).

II. Integrals of the form

$$\int \frac{P_m(x)}{\sqrt{ax^2+bx+c}}\, dx,$$

where $P_m(x)$ is a polynomial of degree m, are calculated by the reduction formula :

$$\int \frac{P_m(x)\,dx}{\sqrt{ax^2+bx+c}} = P_{m-1}(x)\sqrt{ax^2+bx+c} + K\int \frac{dx}{\sqrt{ax^2+bx+c}},$$

where $P_{m-1}(x)$ is a polynomial $P_{m-1}(x)$ and the constant number K are determined by the method of undetermined coefficients.

III. Integrals of the form

$$\int \frac{dx}{(x-a_1)^m \sqrt{ax^2+bx+c}}$$

are reduced to the preceding type by the substitution

$$x - a_1 = \frac{1}{t}.$$

EXAMPLES

5.4.1. $I = \sqrt{5x+4}$

Solution.

$$I = \frac{1}{4}\int \frac{(t+5)\,dt}{\sqrt{t^2-4}}$$

$2x+1=t$, $x = \dfrac{t-1}{2}$, $dx = \dfrac{1}{2}dt$

$$= \frac{1}{4}\left[\sqrt{t^2-4}\right] + \frac{5}{4}\ln\left|t+\sqrt{t^2-4}\right| + C$$

$$I = \frac{1}{4}\sqrt{4x^2+4x-3} + \frac{5}{4}\ln\left|2x+1+\sqrt{4x^2+4x-3}\right| + C$$

5.4.2. $I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx$ $\sqrt{5x+4}$ $= 5(x+1)-1$

$x^2+2x+5 = (x+1)^2+4$

Solution.

$$I = 5\int \frac{t\,dt}{\sqrt{t^2+4}} - \int \frac{dt}{\sqrt{t^2+4}} \qquad x+1=t, \quad dx = dt$$

$$= \frac{5}{2}\int \frac{2t\,dt}{\sqrt{t^2+4}} - \int \frac{dt}{\sqrt{t^2+4}}$$

$$I = 5\sqrt{x^2+2x+5} - \ln\left\{x+1+\sqrt{x^2+2x+5}\right\} + C$$

5.4.3. $I = \int \frac{x^3-x-1}{\sqrt{x^2+2x+2}}$

Solution.

$$I = \int \frac{x\left(x^2-1\right)}{\sqrt{(x+1)^2+1}}\,dx - \int \frac{dx}{\sqrt{(x+1)^2+1}} \qquad x+1=t, \quad dx = dt$$

$$I = \int \frac{(t-1)(t-2)\,t}{\sqrt{t^2+1}}\,dt - \ln\left[x+1+\sqrt{x^2+2x+2}\right]$$

Let $t^2+1 = X^2$

$2t\,dt = 2X\,dX$

$$I = \int \frac{\left(t^2-3t+2\right)X\,dX}{X} - \ln\left[x+1+\sqrt{x^2+2x+2}\right]$$

$$= \int \left\{\left(X^2-1\right) - 3\left(\sqrt{X^2-1}\right) + 2\right\} dX$$

$$I = \frac{X^3}{3} + X - 3\frac{1}{2}X\sqrt{X^2-1} + \frac{1}{2}\ln\left|X+\sqrt{X^2-1}\right|$$

$$-\ln\left|x+1+\sqrt{x^2+2x+2}\right|$$

$$I = \frac{\left(x^2+2x+2\right)^{3/2}}{3} + \left(x^2+2x+2\right)^{1/2}$$

$$-3\left[\frac{1}{2}\left(x^2+2x+2\right)^{1/2}(x+1) + \frac{1}{2}\ln\left(x^2+2x+2\right)^{1/2} + (x+1)\right]$$

$$= \left[\left(x^2+2x+2\right)^{1/2}\left\{\frac{x^2+2x+2}{3}\right\} + 1 - \frac{3}{2}(x+1)\right.$$

$$\left. -\frac{5}{2}\ln\left|\left(x^2+2x+2\right)^{1/2} + (x+1)\right|\right]$$

$$I = \left(x^2+2x+2\right)^{1/2}\left[\frac{x^2}{3} - \frac{5}{6}x + \frac{1}{6} - \frac{5}{2}\ln\left|\left(x^2+2x+2\right)^{1/2} + (x+1)\right| + C\right]$$

5.4.4. $I = \int \sqrt{(4x^2 - 4x + 3)}\, dx$

Solution.
$$I = \int \sqrt{(2x-1)^2 + 2}\, dx$$
$$2x - 1 = t$$
$$2\, dx = dt$$
$$I = \frac{1}{2}\int \sqrt{t^2 + (\sqrt{2})^2}\, dt$$
$$= \frac{1}{2}\left[\frac{1}{2} t \cdot \sqrt{t^2+2} + \frac{1}{2}\cdot 2 \ln\left|t + \sqrt{t^2+2}\right|\right] + C$$
$$I = \frac{1}{4}(2x-1)\sqrt{4x^2-4x+3} + \frac{1}{2}\ln\left|2x - 1 + \sqrt{4x^2-4x+3}\right| + C$$

5.4.5. $I = \int \frac{9x^3 - 3x^2 + 2}{\sqrt{3x^2 - 2x + 1}}$

Solution.
$$I = \int \frac{9x^3 - 3x^2 + 2}{\sqrt{3x^2 - 2x + 1}}$$
$$I = \left(Ax^2 + Bx + D\right)\left(3x^2 - 2x + 1\right)^{1/2} + k\int \frac{dx}{\sqrt{3x^2 - 2x + 1}}$$

Differentiating this equality, we obtain
$$I' = (2Ax + B)\sqrt{(3x^2 - 2x + 1)}$$
$$+ \left(Ax^2 + Bx + D\right)\cdot\frac{(3x-1)}{(3x^2 - 2x + 1)^{1/2}} + \frac{k}{\sqrt{3x^2 - 2x + 1}}$$

Reducing to common denominator
$$= (2Ax + B)(3x^2 - 2x + 1)^{1/2} + (Ax^2 + Bx + D)(3x - 1) + K$$
$$6A + 3A = 9 \qquad A = 1$$
$$3B - 5A + 3B = -3 \qquad B = \frac{1}{3}$$
$$2A - 3B + 3D = 0 \qquad D = -\frac{1}{3}$$
$$B - D + K = 2 \qquad K = \frac{4}{3}$$
$$I = \left(x^2 + \frac{1}{3}x - \frac{1}{3}\right)\left(3x^2 - 2x + 1\right)^{1/2}$$
$$+ \frac{4}{3}\int \frac{dx}{\left(3x^2 - 2x + 1\right)^{1/2}} + \frac{4}{3}\int \frac{dx}{\sqrt{3}\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right)^{1/2}}$$
$$+ \frac{4}{3\sqrt{3}}\int \frac{dx}{\left(x - \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} + \frac{4}{3\sqrt{3}} \times \frac{3}{\sqrt{2}}\tan^{-1}\frac{\left(x - \frac{1}{3}\right)}{\frac{\sqrt{2}}{3}}$$

$$I = \left(x^2 + \frac{1}{3}x - \frac{1}{3}\right)(3x^2 - 2x + 1)^{1/2} + \frac{2\sqrt{2}}{\sqrt{3}}\tan^{-1}\frac{3x-1}{\sqrt{2}} + C$$

5.4.6. $I = \int \sqrt{x^2 + x + 1}\, dx$

Solution.

$$I = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\times\frac{3}{4}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C$$

5.4.7. $I = \int \frac{(4+x)\,dx}{(x-1)(x+2)^2\sqrt{x^2+x+1}}$

Solution.

$$I = \int \frac{(x+4)}{(x-1)(x+2)^2}\cdot\frac{dx}{\sqrt{x^2+x+1}} \times \frac{(x+4)}{(x-1)(x+2)^2}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x+2)^2} + \frac{D}{(x+2)}$$

Find the coefficients $A = \frac{5}{9}$, $B = \frac{-2}{3}$, $D = -\frac{5}{9}$

Hence,

$$I = \int\left[\frac{5}{9(x-1)} - \frac{2}{3(x+2)^2} - \frac{5}{9(x+2)}\right]\cdot\frac{dx}{\sqrt{x^2+x+1}}$$

$$= \underbrace{\frac{5}{9}\int\frac{dx}{(x-1)\sqrt{x^2+x+1}}}_{I_1} - \underbrace{\frac{2}{3}\int\frac{dx}{(x+2)^2\sqrt{x^2+x+1}}}_{I_2} - \underbrace{\frac{5}{9}\int\frac{dx}{(x+2)\sqrt{x^2+x+1}}}_{I_3}$$

$$I_1 = \frac{5}{9}\int\frac{dx}{(x-1)\sqrt{x^2+x+1}}$$

$$I_1 = \frac{5}{9}\int\frac{-\frac{1}{t^2}\,dx}{\frac{1}{t}\frac{\sqrt{t^2+t+1}}{t}} \qquad x-1 = \frac{1}{t}, \quad dx = -\frac{1}{t^2}\,dt$$

$$= -\frac{5}{9}\int\frac{dt}{\sqrt{t^2+t+1}}$$

$$= -\frac{5}{9}\int\frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\frac{5}{9} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(t + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C_1$$

$$I_1 = C_1 - \frac{10}{9\sqrt{3}} \tan^{-1} \frac{(2t+1)}{\sqrt{3}} + C_1$$

$$I_2 = -\frac{2}{3} \int \frac{dx}{(x+2)^2 \sqrt{x^2 + x + 1}}$$

$$x + 2 = \frac{1}{t}$$

$$I_2 = -\frac{2}{3} \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t^2} \times \frac{1}{t} \sqrt{t^2 + t + 1}}$$

$$= +\frac{2}{3} \int \frac{t\, dt}{\sqrt{t^2 + t + 1}}$$

$$I_2 = \frac{1}{3} \int \frac{2t + 1 - 1}{\sqrt{t^2 + t + 1}}$$

$$I_2 = \frac{2}{3}\left(t^2 + t + 1\right)^{1/2} - \frac{1}{3} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{(2t+1)}{\sqrt{3}} + C_2$$

$$I_3 = -\frac{5}{9} \int \frac{dx}{(x+2)\sqrt{x^2 + x + 1}}$$

$$x + 2 = \frac{1}{t}$$

Let $\frac{5}{9} \int \frac{dt}{\sqrt{t^2 + t + 1}} = \frac{5}{9} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{(2t+1)}{\sqrt{3}} + C_3$

$$dx = -\frac{1}{t^2}\, dt$$

5.4.8. $I = \int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}}$

Solution

$$I = \left(Ax^2 + Bx + D\right)\sqrt{x^2 + 4x + 3} + k \int \frac{dx}{\sqrt{x^2 + 4x + 3}}$$

Differentiating this equality, we obtain

$$I' = \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}}$$

$$I = (2Ax + B)\left(x^2 + 4x + 3\right)^{1/2} + \left(Ax^2 + Bx + D\right)\frac{(x+2)}{\sqrt{x^2 + 4x + 3}} + \frac{k}{\sqrt{x^2 + 4x + 3}}$$

Reduce to a common denominator and equate the numerators

$$x^3 - 6x^2 + 11x - 6 = (2Ax + B)\,(x^2 + 4x + 3)^{1/2} + (Ax^2 + Bx + D)\,(x + 2) + \frac{k}{1}$$

$$2A + A = 1$$

$$A = \frac{1}{3}$$

$$B + 8A + 2A + B = -6$$

$$B + 5A = -3$$

$$B = -\frac{14}{3}$$

$$4B + 6A + D + 2B = 11$$

$$D = \frac{111}{3}$$

$$3B + 2D + k = -6$$

$$-14 + \frac{222}{3} + k = -6$$

$$k = -66$$

$$I = \left(\frac{1}{3}x^2 - \frac{14}{3}x + \frac{111}{3}\right)\left(x^2 + 4x + 3\right)^{1/2}$$

$$+(-66)\int \frac{dx}{\left(x^2 + 4x + 3\right)^{1/2}}$$

$$-66\int \frac{dx}{\sqrt{(x+2)^2 - 1}}$$

$$\left(\frac{1}{3}x^2 - \frac{14}{3}x + \frac{111}{3}\right)(x^2 + 4x + 3)^{1/2} - 66\ln\left|x + 2 + \sqrt{\left(x^2 + 4x + 3\right)}\right|$$

5.4.9. $I = \int \frac{3x^3 + 5x^2 - 7x + 9}{\sqrt{2x^2 + 5x + 7}}\,dx$

Solution. $I = \left(Ax^2 + Bx + D\right)\left(2x^2 + 5x + 7\right)^{1/2} + k\int \frac{dx}{\left(2x^2 + 5x + 7\right)^{1/2}}$

Differentiating this equality, we obtain

$$I' = \frac{3x^2 + 5x^2 - 7x + 9}{\sqrt{2x^2 + 5x + 7}} = (2Ax + B)\left(2x^2 + 5x + 7\right)^{1/2}$$

$$+\left(Ax^2 + Bx + D\right)\frac{2x + \frac{5}{2}}{\sqrt{2x^2 + 5x + 7}} + \frac{k}{\sqrt{2x^2 + 5x + 7}}$$

Reduce to a common denominator and equate the numerators

$$3x^3 + 5x^2 - 7x + 9 = (2Ax + B)\,(2x^2 + 5x + 7) + (Ax^2 + Bx + D)\,(2x + 5/2) + k$$

$$4A + 2A = 3 \qquad 2B + 10A + \frac{5}{2}A + 2B = 5$$

$$6A = 3 \qquad 4B + 5 + \frac{5}{4} = 5$$

$$A = \frac{1}{2} \qquad 4B = 5 - \frac{25}{4}$$

$$B = -\frac{5}{16}$$

$$14A + 5B + \frac{5}{2}B + 2D = -7 \qquad 7B + \frac{5}{2}D + k = 9$$

$$7 + \frac{15}{2} \times \left(-\frac{5}{16}\right) + 2D = -7 \qquad 7 \times -\frac{5}{10} + \frac{5}{2} \times \left(-\frac{373}{64}\right) + k = 9$$

$$2D = -7 - \frac{149}{32} \qquad k = \frac{3297}{128}$$

$$D = -\frac{373}{64}$$

$$I = \frac{1}{64}\left(32x^2 - 20x - 373\right)\left(2x^2 + 5x + 7\right)^{1/2}$$

$$+ \frac{3297}{128\sqrt{2}} \ln\left|4x + 5 + 2\sqrt{4x^2 + 10x + 14}\right| + C$$

5.4.10. $I = \int \frac{dx}{(x+1)^5 \sqrt{x^2 + 2x + 0}}$

Solution.

$$= \int \frac{t\,dt}{(x+1)^5\,(x+1)\,t} = \int \frac{dt}{\left(t^2+1\right)^3}$$

Let $x^2 + 2x = t^2$

$(2x+2)\,dx = 2t\,dt$

$(x+1)\,dx = t\,dt$

Let $t = \tan\theta$

$dt = \sec^2\theta\,d\theta$

$$I = \int \frac{\sec^2\theta\,d\theta}{\sec^6\theta}$$

$$= \int \cos^4\theta\,d\theta$$

$$= \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$I = \int \left(\frac{1}{4} + \frac{1}{4}\cos^2 2\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int \frac{1}{4}\,d\theta + \int \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) d\theta + \int \frac{1}{2}\cos 2\theta\,d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{8}\theta + \frac{1}{8} \times \frac{1}{4}\sin 4\theta + \frac{1}{4}\sin 2\theta + C$$

$$I = \frac{3}{8}\theta + \frac{1}{4} \times \sin 2\theta \left[\frac{1}{4}\cos 2\theta + 1\right] + C$$

$$= \frac{3}{8}\tan^{-1} t + \frac{1}{4} \times \frac{2\tan\theta}{1+\tan^2\theta}\left[\frac{1}{4} \times \frac{1-\tan^2\theta}{1+\tan^2\theta} + 1\right] + C$$

$$= \frac{3}{8}\tan^{-1}\sqrt{x^2+2x} + \frac{1}{2}\left(\frac{t}{1+t^2}\right)\left[\frac{1}{4}\frac{(1-t^2)}{1+t^2} + 1\right] + C$$

$$I = \frac{3}{8}\tan^{-1}\sqrt{x^2+2x} + \frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2}\left[\frac{1}{4}\frac{(1-x^2-2x)}{(1+x)^2} + 1\right] + C$$

5.4.11. $I = \int \frac{x\,dx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$

Solution.

$$I = \int \frac{x\,dx}{(x-1)(x-2)\sqrt{(x-1)(x-3)}}$$

$x = t^2 + 1$

$dx = 2t\,dt$

$$= \int \frac{(t^2+1)\,2t\,dt}{t^2\cdot(t^2-1)\sqrt{(t^2-2)\,t^2}}$$

$$= \int \frac{2(t^2+1)\,dt}{t^2\cdot(t^2-1)\sqrt{t^2-2}}$$

$$= 2\int \frac{(t^2+1)\,dt}{t^2\cdot(t^2-1)\sqrt{t^2-2}}$$

$$= \underbrace{2\int \frac{dt}{(t^2-1)\sqrt{t^2-2}}}_{I_1} + \underbrace{2\int \frac{dt}{t^2(t^2-1)\sqrt{t^2-2}}}_{I_2}$$

$$I_1 = 2\int \frac{dt}{(t^2-1)\sqrt{t^2-2}}$$

$t = \frac{1}{u}$

$dt = -\frac{1}{u^2}\,du$

$$I_1 = -2\int \frac{u\,du}{(1-u^2)\sqrt{1-2u^2}}$$

$1 - 2u^2 = t^2$

$-4u\,du = 2t\,dt$

$-2u\,du = t\,dt$

$$I_1 = -\int \frac{t\,dt}{\left(\frac{1-t^2}{2}-1\right)t} \qquad \frac{1-t^2}{2} = u^2$$

$$I_1 = \int \frac{2\,dt}{1+t^2} = 2\tan^{-1} t \qquad 1-u^2 = 1-\left(\frac{1-t^2}{2}\right)$$

$$I_1 = 2\tan^{-1}\sqrt{1-2u^2} \qquad = \frac{2-1+t^2}{2} = \frac{1+t^2}{2}$$

$$= 2\tan^{-1}\sqrt{1-\frac{2}{t^2}}$$

$$= 2\tan^{-1}\frac{\sqrt{t^2-2}}{t}$$

$$I_1 = 2\tan^{-1}\sqrt{\frac{(x-3)}{(x-1)}}$$

$$I_2 = 2\int \frac{dt}{t^2\left(t^2-1\right)\sqrt{t^2-2}}$$

$$= 2\int \frac{dt}{\left(t^2-1\right)\sqrt{t^2-2}} - 2\int \frac{dt}{t^2\sqrt{t^2-2}}$$

$$= 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - 2\int \frac{dt}{t^2\sqrt{t^2-2}} \qquad t = \frac{1}{u}, \quad dt = \frac{1}{u^2}du$$

$$= 2\tan^{-1}\left(\frac{\sqrt{x-3}}{x-1}\right) - 2\int \frac{-\frac{1}{u^2}du}{u^2 \times \frac{1}{4}\sqrt{1-2u^2}}$$

$$I_2 = 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} + \int \frac{2u\,du}{\sqrt{1-2u^2}}$$

$$1-2u^2 = X^2$$

$$-4u\,du = 2X\,dx$$

$$-2u\,du = X\,dx$$

$$I_2 = 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - \int \frac{X\,dx}{X} - \int dx$$

$$= 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - X + C$$

$$= 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - \sqrt{1-2u^2} + C$$

$$= 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - \frac{\sqrt{t^2-2}}{t} + C$$

$$I_2 = 2\tan^{-1}\frac{\sqrt{x-3}}{x-1} - \frac{\sqrt{(x-3)}}{\sqrt{(x-1)}} + C \qquad \left[\because\ u = \frac{1}{t}\right]$$

$$I = I_1 + I_2$$

$$= 4\tan^{-1}\frac{\sqrt{x-3}}{x-1} - \frac{\sqrt{(x-3)}}{\sqrt{(x-1)}} + C \qquad \because\ t = \sqrt{x-1}$$

5.4.12. $I = \int \frac{dx}{(x+1)^3\sqrt{x^2+3x+2}}$

Solution. $x + 1 = t$

$dx = dt$

$$I = \int \frac{dt}{t^3\sqrt{(t)^2+t}} \qquad \text{Let } \frac{1}{u} = t$$

$$dt = -\frac{1}{u^2}du$$

$$I = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^4}\sqrt{u^2+1}}$$

$$= \int \frac{-u^2\,du}{\sqrt{u^2+1}}$$

$$= -\int \frac{(u^2+1-1)\,du}{\sqrt{u^2+1}}$$

$$= -\int \sqrt{u^2+1}\,du + \int \frac{du}{\sqrt{u^2+1}} + C$$

$$I = -\frac{1}{2}u\cdot\sqrt{u^2+1} - \frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right| + \frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right| + C$$

$$I = -\frac{1}{2}u\sqrt{u^2+1} + C$$

$$= -\frac{1}{2t}\sqrt{\frac{1}{t^2}+1} + C$$

$$= -\frac{1}{2t^2}\sqrt{1+t^2} + C$$

$$I = \frac{1}{2(x+1)^2}\times\sqrt{(x+1)^2+1} + C$$

5.4.13. $I = \int \frac{(x^2+1)\,dx}{x\sqrt{x^4+3x^2+1}}$

Solution. Let, $x^2 = t$

$$2x\,dx = dt$$

$$I = \frac{1}{2}\int \frac{(t-1)\,dt}{t\sqrt{t^2+3t+1}}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t^2+3t+1}} - \frac{1}{2}\int \frac{dt}{t\sqrt{t^2+3t+1}}$$

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{\left(t+\frac{3}{2}\right)^2 - \frac{5}{4}}} - \frac{1}{2}\int \frac{dt}{t\cdot\sqrt{t^2+3t+1}}$$

$$I = \frac{1}{2}\log\left\{t+\frac{3}{2}+\sqrt{t^2+3t+1}\right\} - \underbrace{\frac{1}{2}\int \frac{dt}{t\cdot\sqrt{t^2+3t+1}}}_{I_1}$$

$$I_1 = \frac{1}{2}\int \frac{dt}{t\cdot\sqrt{t^2+3t+1}}$$

$$t = \frac{1}{u},\ dt = -\frac{1}{u^2}\,du$$

$$I = \frac{1}{2}\int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u}\cdot\frac{1}{u}\sqrt{1+3u+u^2}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{\left(u+\frac{3}{2}\right)^2 - \left(\frac{5}{4}\right)}}$$

$$= -\frac{1}{2}\ln\left\{u+\frac{3}{2}+\sqrt{1+3u+u^2}\right\} + C$$

$$I = \frac{1}{2}\log\left\{x^2+\frac{3}{2}+\sqrt{x^4+3x^2+1}\right\} - \frac{1}{2}\log\left\{\frac{1}{x^2}+\frac{3}{2}+\frac{1}{x^2}\sqrt{x^4+3x^2+1}\right\}$$

5.5 INTEGRATION OF A BINOMIAL DIFFERENTIAL

The integral $\int x^m \left(a + bx^n\right)^p dx$, where m, n, p are rational numbers, is expressed through elementary functions only in the following three cases :

Case I. p is an integer. Then, if $p > 0$, the integrand is expanded by the formula of the Newton binomial; but if $p < 0$, then we put $x = t^k$, where k is the common denominator of the fractions m and n.

Case II. $\dfrac{m+1}{n}$ is an integer. We put $a + bx^n = t^\alpha$, where α is the denominator of the fraction p.

Case III. $\dfrac{m+1}{n} + p$ is an integer. We put $a + bx^n = t^\alpha x^n$, where α is the denominator of the fraction p.

EXAMPLES

5.5.1. $I = \int \sqrt[3]{x}\left(2 + \sqrt{x}\right)^2 dx$

Solution.

$$I = \int x^{1/3}\left[4 + 4\sqrt{x} + x\right] dx$$

$$= \int 4x^{1/3}\, dx + \int 4 \cdot x^{1/3} \cdot x^{1/2}\, dx + \int x^{1/3} \cdot x^1\, dx$$

$$= \frac{4x^{4/3}}{4/3} + \frac{4x^{11/3}}{11/6} + \frac{x^{7/3}}{7/3} + C$$

$$= 3x^{4/3} + \frac{24}{11}x^{11/6} + \frac{3}{7}x^{7/3} + C$$

5.5.2. $I = \int x^{-2/3}\left(1 + x^{2/3}\right)^{-1} dx$

Solution.

$$1 + x^{2/3} = t$$

$$x^{2/3} = t - 1$$

$$\frac{2}{3}x^{-1/3}\, dx = dt$$

$$I = \frac{3}{2}\int \frac{dt}{t\sqrt{t-1}}$$

$$t - 1 = u^2$$

$$dt = 2u\, du$$

$$I = \frac{3}{2}\int \frac{du}{u^2 + 1}$$

$$= \frac{3}{2}\tan^{-1} u + C$$

$$I = \frac{3}{2}\tan^{-1}(x^{1/3}) + C$$

5.5.3. $I = \int \frac{\sqrt{1+x^{1/3}}}{x^{2/3}}\,dx$

Solution. Let $1 + x^{1/3} = t^2$

$$\frac{1}{3}x^{-2/3}\,dx = 2t\,dt$$

$$I = \int 6t^2\,dt = 2t^3 + C$$

$$I = 2\left(1+x^{1/3}\right)^{3/2} + C$$

5.5.4. $I = \int x^{1/3}\left(2+x^{2/3}\right)^{1/4} dx$

Solution. Let, $\left(2+x^{2/3}\right)^{1/4} = t \quad \Rightarrow x^{2/3} = t^4 - 2$

$$\frac{2}{3}x^{-1/3}\,dx = 4t^3\,dt$$

$$dx = 6t^3 \cdot x^{1/3}\,dt$$

$$I = \int x^{2/3} \cdot t \cdot 6t^3\,dt$$

$$= 6\int\left(t^4-2\right)t^4\,dt$$

$$I = \int 6t^8\,dt - \int 12t^4\,dt$$

$$= \int \frac{6t^9}{9} - \frac{12t^5}{5} + C$$

$$= \frac{2}{3}t^9 - \frac{12}{5}t^5 + C$$

$$= \frac{2}{3}\left(2+x^{2/3}\right)^{9/4} - \frac{12}{5}\left(2+x^{2/3}\right)^{5/4} + C$$

5.5.5. $I = \int x^6\left(1+x^2\right)^{2/3} dx$

Solution. Let $1 + x^2 = t^3 \quad \Rightarrow x^2 = t^3 - 1$

$$2x\,dx = 3t^2\,dt$$

$$= \frac{3}{2}\int\left(t^6+1-2t^3\right)t^4\,dt$$

$$= \frac{3}{2}\frac{t^{11}}{11} + \frac{3}{2}\frac{t^5}{5} - \frac{3t^8}{8} + C$$

$$= \frac{3}{22}\left(1+x^2\right)^{11/3} + \frac{3}{10}\left(1+x^2\right)^{5/3} - \frac{3}{8}\left(1+x^2\right)^{8/3} + C$$

5.5.6. $I = \int x^{-11}\left(1+x^4\right)^{-1/2} dx$

Solution. $I = \int \frac{dx}{x^{13}\left(1+\frac{1}{x^4}\right)^{1/2}}$ Let $1+\frac{1}{x^4} = t^2$

$-\frac{4}{x^5}dx = 2t\,dt$

$dx = \frac{x^5}{2}t\,dt$

$$I = -\frac{1}{2}\int \frac{x^5 t\,dt}{x^{13}t} = -\frac{1}{2}\int \frac{dt}{x^8}$$

$$= -\frac{1}{2}\int \left(t^2-1\right)^2 dt$$

$$= -\frac{1}{2}\left[\frac{t^5}{5}+t-\frac{2t^3}{3}\right]+C$$

$$I = -\frac{1}{2}\times\frac{1}{5}\left(1+\frac{1}{x^4}\right)^{5/2} - \frac{1}{2}\left(1+\frac{1}{x^4}\right)^{1/2} + \frac{2}{3}\times\frac{1}{2}\left(1+\frac{1}{x^4}\right)^{3/2} + C$$

$$I = -\frac{1}{10x^{10}}\sqrt{\left(1+x^4\right)^5} + \frac{1}{3x^6}\sqrt{\left(1+x^4\right)^3} - \frac{1}{2x^2}\sqrt{1+x^4} + C$$

5.5.7. $I = \int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$

Solution. $x = t^4$

$dx = 4t^3\,dt$

$$I = 4\int (1+t)^{1/3}\,t\,dt$$

$1+t = X^3$

$dt = 3X^2\,dX$

$$= 4\int \left(X^3-1\right)3X^3\,dX$$

$$I = 4\left(\frac{3X^7}{7}-\frac{3}{4}\left(X^4\right)\right)+C$$

$$I = \frac{12}{7}\left(1+x^{1/4}\right)^{7/3} - 3\left(1+x^{1/4}\right)^{4/3} + C$$

5.5.8. $I = \int \frac{dx}{x\left(1+x^{1/3}\right)^2}$

Solution. $I = \int \frac{3t^2\,dt}{t^3(1+t)^2}$ $x^{1/3} = t$

$s = t^3$

$dx = 3t^2 dt$

$$= 3\int \frac{dt}{t\,(1+t)^2}$$

$$I = 3\left[\int \frac{dt}{t} - \int \frac{(t+2)}{(t+1)^2}\,dt\right]$$

$$= 3\ln t - \int \frac{3}{t+1}\,dt - \int \frac{3}{(t+1)^2}\,dt$$

$$= 3\left[\ln \frac{t}{t+1} + \frac{1}{t+1}\right] + C$$

$$I = 3\ln\left[\frac{x^{1/3}}{x^{1/3}+1}\right] + \left[\frac{3}{x^{1/3}+1}\right] + C$$

5.5.9. $I = \int x^3 \left(1+x^2\right)^{1/2} dx$

Solution.

$$1 + x^2 = t^2$$

$$2x\,dx = 2t\,dt$$

$$x\,dx = t\,dt$$

$$I = \int \left(t^2 - 1\right) t^2\,dt$$

$$= \int t^4\,dt - \int t^2\,dt$$

$$I = \frac{t^5}{5} - \frac{t^3}{3} + C$$

$$I = \frac{\left(1+x^2\right)^{5/2}}{5} - \frac{\left(1+x^2\right)^{3/2}}{3} + C$$

5.5.10. $I = \int \frac{dx}{x^4 \left(1+x^2\right)^{1/2}}$

Solution.

$$I = \int \frac{dx}{x^5\left(1+\frac{1}{x^2}\right)^{1/2}}$$

$$I = \int -\frac{x^3\,t\,dt}{x^5\,t} \qquad \text{Let } 1 + \frac{1}{x^2} = t^2$$

$$= -\int \frac{dt}{x^2} = -\int \left(t^2 - 1\right) dt \qquad -\frac{2}{x^3}\,dx = 2t\,dt$$

$$-\frac{dx}{x^3} = t\,dt$$

$$= \frac{-t^3}{3} + t + C$$

$$= -\frac{1}{3}\frac{\left(1+x^2\right)^{3/2}}{x^3} + \frac{\left(x^2+1\right)^{1/2}}{x} + C$$

5.5.11. $I = \int x^{1/3} \sqrt[7]{1 + x^{4/3}}\, dx$

Solution.

$$x = t^3$$
$$dx = 3t^2\, dt$$
$$= \int t\left(1+t^4\right)^{1/7} \cdot 3t^2\, dt$$
$$= 3\int t^3\left(1+t^4\right)^{1/7} dt$$
$$1 + t^4 = X^7$$
$$4t^3\, dt = 7X^6\, dX$$
$$= \frac{3}{4}\int 7X^7\, dX$$
$$= \frac{21}{32}X^8 + C$$
$$= \frac{21}{32}\left(1+x^{4/3}\right)^{8/7} + C$$

5.5.12. $I = \int \dfrac{dx}{x^3 \cdot \left(1 + \dfrac{1}{x}\right)^{1/5}}$

Solution. Let $1 + \dfrac{1}{x} = t^5$

$$-\frac{1}{x^2}\, dx = 5t^4\, dt$$
$$I = -\int 5t^3\left(t^5 - 1\right) dt$$
$$= \frac{-5}{9}t^9 + \frac{5}{4}t^4 + C$$
$$= \frac{-5}{9}\left(1+\frac{1}{x}\right)^{9/5} + \frac{5}{4}\left(1+\frac{1}{x}\right)^{4/5} + C$$

5.6 INTEGRATION OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

I. Integrals of the form

$$I = \int \sin^m x \cos^n x\, dx$$

where m and n are rational numbers, are reduced to the integral of the binomial differential

$$I = \int t^m \left(1 - t^2\right)^{\frac{n-1}{2}} dt,\ t = \sin x$$

and are, therefore, integrated in elementary functions only in the following three cases :

1. n is odd $\left(\frac{n-1}{2} \text{ an integer}\right)$,

2. m is odd $\left(\frac{m+1}{2} \text{ an integer}\right)$,

3. $m + n$ is even $\left(\frac{m+1}{2} + \frac{n-1}{2} \text{ an integer}\right)$

If n is an odd number, the substitution $\sin x = t$ is applied.

If m is an odd number, the substitution $\cos x = t$ is applied.

If the sum $m + n$ is an even number, use the substitution $\tan x = t$ (or $\cot x = t$).

In particular, this kind of substitution is convenient for integrals of the form

$$\int \tan^n x\, dx \left(\text{or} \int \cot^n x\, dx\right)$$

where n is a positive integer. But the last substitution is inconvenient if both m and n are positive numbers. If m and n are non-negative even numbers, then it appears more convenient to use the method of reducing the power with the aid of trigonometric transformations :

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x),\ \sin^2 x = \frac{1}{2}(1 - \cos 2x) \text{ or } \sin x \cos x = \frac{1}{2}\sin 2x.$$

EXAMPLES

5.6.1. $I = \int \frac{\sin^3 x}{(\cos^2 x)^{1/3}}$

Solution. Let $\cos x = t^3$

$$-\sin x\, dx = 3t^2\, dt$$

$$I = \int -\frac{(1-t^6)\,3t^2\, dt}{t^2} = -3t + \frac{3t^7}{7} + C$$

$$= -3(\cos x)^{1/3} + \frac{3}{7}(\cos x)^{7/3} + C$$

5.6.2. $I = \int \frac{\cos^3 x}{\sin^6 x}\, dx$

Solution. Let $\sin x = t$

$$\cos x\, dx = dt$$

$$I = \int \frac{(1-t^2)}{t^6}\, dt$$

$$I = \int \frac{dt}{t^6} - \int \frac{dt}{t^4} = -\frac{1}{5t^5} + \frac{1}{3t^3} + C$$

$$I = -\frac{1}{5} \times \frac{1}{\sin^5 x} + \frac{1}{3} \times \frac{1}{\sin x} + C$$

5.6.3. $I = \int \sin^4 x \cdot \cos^6 x \, dx$

Solution.
$$I = \frac{1}{16}\int (2\sin x \cos x)^4 \cos^2 x \, dx$$
$$I = \frac{1}{32}\int \sin^4 2x\,(1+\cos 2x)\,dx$$
$$= \frac{1}{32}\int \sin^4 2x\,dx + \frac{1}{32}\int \sin^4 2x \cdot \cos 2x\,dx$$
$$= \frac{1}{128}\int (1-\cos 4x)^2\,dx + \frac{1}{320}\sin^5 2x + C$$
$$= \frac{1}{128}\int\left(x - \frac{1}{2}\sin 4x\right) + \frac{1}{256}\int (1+\cos 8x)\,dx + \frac{1}{320}\sin^5 2x + C$$
$$I = \frac{3}{256}x - \frac{1}{256}\sin 4x + \frac{1}{2048}\sin 8x + \frac{1}{320}\sin^5 2x + C$$

5.6.4. $I = \int \frac{\sin^2 x\,dx}{\cos^6 x}$

Solution. Let $\tan x = t, \quad \tan^2 x = t^2$

$\Rightarrow \quad \sec^2 x - 1 = t^2 \quad$ or $\quad \sec^2 x = 1 + t^2$

or $\quad \frac{1}{\cos^2 x} = 1 + t^2$

$$\frac{dx}{\cos^2 x} = dt$$
$$I = \int t^2\left(1+t^2\right)dt = \frac{t^3}{3} + \frac{t^5}{5} + C$$
$$I = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

5.6.5. $I = \int \frac{\cos^4 x}{\sin^2 x}\,dx$

Solution.
$$I = \int \frac{\left(1-\sin^2 x\right)^2}{\sin^2 x}\,dx$$
$$= \int\left(\frac{1}{\sin^2 x} - 2 + \sin^2 x\right)dx$$
$$= -\cot x - 2x + \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$I = -\left(\cot x + \frac{\sin 2x}{4} + \frac{3x}{2}\right) + C$$

5.6.6. $I = \int \frac{dx}{\cos^4 x}$

Solution.

$$I = \int \sec^4 x \, dx$$

$$I = \int \sec^2 x \left(1 + \tan^2 x\right) dx$$

$$\tan x = t \Rightarrow \sec^2 x \, dx = dt$$

$$I = \int \left(1 + t^2\right) dt$$

$$I = t + \frac{t^3}{3} + C$$

$$I = \tan x + \frac{\tan^3 x}{3} + C$$

5.6.7. $I = \int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}}$

Solution. Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\frac{dx}{\cos^2 x} = dt \qquad \text{and} \qquad 1 + \tan^2 x = 1 + t^2 \Rightarrow \sec^2 x = 1 + t^2$$

$$= \int \frac{dx}{\cos^4 x \cdot \sqrt[3]{\tan^{11} x}} \qquad \frac{1}{\cos^2 x} = 1 + t^2$$

$$I = \int \frac{\left(1 + t^2\right) dt}{\left(t^{11}\right)^{1/3}}$$

$$= \int \left(t^{-11/3} + t^{-5/3}\right) dt$$

$$= -\frac{3}{8} t^{-8/3} - \frac{3}{2} t^{-2/3} + C$$

$$= -\frac{3\left(1 + 4\tan^2 x\right)}{8\left(\tan^2 x\right)\sqrt[3]{\tan^2 x}} + C$$

5.6.8. ***(a)*** $I = \int \tan x \, dx$

Solution. Let

$$I = \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + C$$

(b) $I = \int \cot x \, dx$

$$I = \int \frac{\cos x}{\sin x} dx = -\ln |\sin x| + C$$

5.6.9. $I = \int \tan^7 x\, dx$

Solution. Let

$$t = \tan x$$
$$x = \tan^{-1} t$$
$$dx = \frac{1}{1+t^2}\, dt$$
$$I = \int \frac{t^7 \cdot dt}{1+t^2}$$
$$= \int \left(t^5 - t^3 + t - \frac{t}{1+t^2} \right) dt$$
$$= \frac{t^6}{6} - \frac{t^4}{4} + \frac{t^2}{2} - \frac{1}{2}\ln\left(1+t^2\right) + C$$
$$= \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \frac{1}{2}\ln\left(1+\tan^2 x\right) + C$$

5.6.10. (a) $I = \int \cot^6 x\, dx$

Solution. Let

$$\cot x = t$$
$$x = \cot^{-1} t$$
$$dx = \frac{1}{1+t^2}\, dt$$
$$I = -\int \frac{t^6}{1+t^2}\, dt$$
$$= -\int \left(\left(t^4 - t^2 - 1\right) - \frac{1}{1+t^2} \right) dt$$
$$= -\frac{t^5}{5} + \frac{t^3}{3} - t + \tan^{-1} t + C$$
$$I = -\frac{(\cot x)^5}{5} + \frac{(\cot x)^3}{3} - \cot x + \tan^{-1}(\cot x) + C$$

(b) $I = \int \tan^3 x\, dx$

$$= \tan^2 x \cdot \tan x\, dx$$

Solution.

$$I = \int \left(\sec^2 x - 1\right) \tan x\, dx$$
$$= \int \tan x \cdot \sec^2 x\, dx - \int \tan x\, dx$$
$$= \int \frac{(\tan x)^2}{2} + \ln|\cos x| + C$$

5.6.11. $I = \int \frac{\cos^4 x}{\sin^3 x} dx$

Solution. $\cos x = t, \qquad -\sin x\, dx = dt$

$$I = -\int \frac{t^4}{(1-t^2)} dt$$

$$I = -\frac{t^3}{2(1-t^2)} + \frac{3}{2}\int \frac{t^2\, dt}{1-t^2}$$

$$= -\frac{t^3}{2(1-t^2)} + \frac{3}{2}\int \frac{t^2-1+1}{1-t^2} dt$$

$$= -\frac{t^3}{2(1-t^2)} - \frac{3}{2}t + \frac{3}{4}\ln\left(\frac{1+t}{1-t}\right) + C$$

$$= \frac{-\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x + \frac{3}{4}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

II. Integral of the form $\int R(\sin x, \cos x) dx$, where R is a rational function of $\sin x$ and $\cos x$ are transformed into integrals of a rational function by the substitution:

$$\tan\left(\frac{x}{2}\right) = t \qquad (-\pi < x < \pi)$$

This is called **universal** subsitution. We can substitute

$$\sin x = \frac{2t}{1+t^2}; \quad \cos x\ \frac{1-t^2}{1+t^2}$$

$$x = 2\tan^{-1} t$$

$$dx = \frac{2dt}{1+t^2}$$

Sometimes, instead of $\tan \frac{x}{2} = t$, it is more advantageous to substitute

$$\cot \frac{x}{2} = t \ (0 < x < 2\pi)$$

The following cases are indicated as

(a) If the equality

$R(-\sin x, \cos x) \equiv -R(\sin x, \cos x)$

$R(\sin x, -\cos x) \equiv -R(\sin x, \cos x)$

is satified, then substituting $\cos x = t$ to the former equality and $\sin x = t$ to the latter.

(b) If the equality

R (–sin x,– cos x) ≡ – R (sin x, cos x) is fulfilled, then a better effect is gained by substituting tan x = t or cot x = t.

EXAMPLES

5.6.12. $I = \int \frac{\sin^4 x}{\cos x} dx$

Solution. Let $\sin x = t$

$$\cos x\, dx = dt$$

$$dx = \frac{dt}{\cos x}$$

$$I = \int \frac{t^4}{(1-t^2)} dt$$

$$= -\int \frac{(t^4 - 1 + 1)}{t^2 - 1} dt$$

$$I = -\int (t^2+1)\, dt - \int \frac{dt}{t^2-1}$$

$$= -\frac{t^3}{3} - t - \ln\left|\frac{t-1}{t+1}\right|$$

$$I = -\frac{\sin^3 x}{3} - \sin x - \ln\left|\frac{\sin x - 1}{\sin x + 1}\right| + C$$

5.6.13. $I = \int \frac{dx}{\sin x\,(2 + \cos x - 2 \sin x)}$

Solution. Let $\tan\frac{x}{2} = t$

$$\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$$

$$dx = \frac{2dt}{\sec^2\frac{x}{2}}$$

$$= \frac{2dt}{(1+t^2)}$$

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}\left(2 + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2}\right)}$$

$$= \int \frac{(1+t^2)\, dt}{t\,(t^2 - 4t + 3)}$$

Expanding into simple fractions

$$\frac{1+t^2}{t(t-3)(t-1)} = \frac{A}{t}+\frac{B}{t-3}+\frac{D}{t-1}$$

Finding the coefficients, we get

$$A = \frac{1}{3}, B = \frac{5}{3}, D = -1$$

$$I = \frac{1}{3}\int\frac{dt}{t}+\frac{5}{3}\int\frac{dt}{t-3}-\int\frac{dt}{t-1}$$

$$= \frac{1}{3}\ln|t|+\frac{5}{3}\ln|t-3|-\ln|t-1|+C$$

$$= \frac{1}{3}\ln\left|\tan\frac{x}{2}\right|+\frac{5}{3}\left[\ln\left(\tan\frac{x}{2}-3\right)\right]-\ln\left[\left|\tan\frac{x}{2}-1\right|\right]+C$$

5.6.14. $I = \int\frac{dx}{5+\sin x+3\cos x}$

Solution. Let $\tan\frac{x}{2} = t;\ \frac{x}{2} = \tan^{-1}t$

$$x = 2\tan^{-1}t;\ dx = \frac{2}{1+t^2}dt$$

$$I = \int\frac{\frac{2}{1+t^2}dt}{5+\frac{2t}{1+t^2}+\frac{3(1-t^2)}{1+t^2}}$$

$$= \int\frac{2\,dt}{5+5t^2+2t+3-3t^2}$$

$$= \int\frac{2\,dt}{8+2t^2+2t} = \int\frac{dt}{t^2+t+4}$$

$$= \int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{15}}\tan^{-1}\frac{\left(t+\frac{1}{2}\right)}{\frac{\sqrt{15}}{2}}$$

$$I = \frac{2}{\sqrt{15}}\tan^{-1}\frac{\left(2\tan\frac{x}{2}+1\right)}{\sqrt{15}}+C$$

5.6.15. $I = \int \frac{dx}{\sin x\left(2\cos^2 x - 1\right)}$

Solution. Let $\cos x = t$

$$-\sin x\, dx = dt$$

$$I = \int \frac{-dt}{\sin^2 x\left(2\cos^2 x - 1\right)}$$

$$= -\int \frac{dt}{\left(1-t^2\right)\left(2t^2-1\right)}$$

$$= \int \frac{dt}{\left(1-t^2\right)\left(1-2t^2\right)}$$

$$I = 2\int \frac{dt}{1-2t^2} - \int \frac{dt}{1-t^2}$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{1+t\sqrt{2}}{1-t\sqrt{2}}\right| - \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| - \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

5.6.16. $I = \int \frac{\sin^2 x \cos x}{\sin x + \cos x}$

Solution. Let $\tan x = t$

$$dt = \frac{dx}{\cos^2 x}$$

$$I = \int \frac{\tan^2 x \cdot \cos x}{(1+\tan x)} \cdot \frac{dx}{\cos^2 x}$$

$$= \int \frac{t^2}{(1+t)\left(t^2+1\right)^2}\, dt$$

Expand into partial fractions

$$\frac{t^2\, dt}{(1+t)\left(t^2+1\right)^2} = \frac{A}{t+1} + \frac{Bt+D}{t^2+1} + \frac{Et+F}{\left(t^2+1\right)^2}$$

Solving for co-efficients, we get

$$A = \frac{1}{4}, B = -\frac{1}{4}, D = \frac{1}{4}, E = \frac{1}{2}, F = -\frac{1}{2}$$

$$I = \frac{1}{4}\int \frac{dt}{t+1} - \frac{1}{4}\int \frac{t-1}{t^2+1}\, dt + \frac{1}{2}\int \frac{t-1}{\left(t^2+1\right)^2}\, dt$$

$$I = \frac{1}{4}\ln\left[\frac{1+t}{\sqrt{1+t^2}}\right] - \frac{1}{4}\left[\frac{1+t}{1+t^2}\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{1+\dfrac{\sin x}{\cos x}}{\sqrt{1+\tan^2 x}}\right| - \frac{1}{4}\cos x(\sin x + \cos x) + C$$

$$= \frac{1}{4}\ln|\sin x + \cos x| - \frac{1}{4}\cos x(\sin x + \cos x) + C$$

5.6.17. $I = \int \frac{2\tan x + 3}{\sin^2 x + 2\cos^2 x}\,dx$

Solution. Let $\tan x = t$

$$\frac{dx}{\cos^2 x} = dt$$

$$I = \int \frac{2t+3}{t^2+2}\,dt$$

$$I = \ln(t^2+2) + \frac{3}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C$$

$$= \ln(\tan^2 x + 2) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

5.6.18. $I = \int \frac{\sin x}{1+\sin x}\,dx$

Solution.

$$I = \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$

$$I = \int \frac{\sin x}{1-\sin^2 x}\,dx - \int \frac{\sin^2 x}{1-\sin^2 x}\,dx$$

$$I = \int \frac{\sin x}{\cos^2 x}\,dx - \int \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$I = \frac{1}{\cos x} - \tan x + x + C$$

5.6.19. $I = \int \frac{dx}{\cos^4 x \sin^2 x}$

Solution. Let $\tan x = t$, but simpler method can be applied.

$$I = \int \frac{(\sin^2 x + \cos^2 x)^2}{\cos^4 x \cdot \sin^2 x}\,dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x)}{\cos^4 x \cdot \sin^2 x}\,dx$$

$$= \int \frac{\sin^2 x}{\cos^4 x} + 2\int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x}$$

$$= \frac{1}{3}\tan^3 x + 2\tan x - \cot x + C$$

5.6.20. $I = \int \frac{dx}{\sqrt{(5+2x+x^2)^3}}$

Solution. $5 + 2x + x^2 = 4 + (x+1)^2$

Let $x + 1 = t$

$$I = \int \frac{dt}{\sqrt{(4+t^2)^3}}$$

Let $t = 2\tan z$

$$dt = 2\sec^2 z\, dz$$

$$I = \int \frac{2\sec^2 z\, dz}{\sqrt{(4+4\tan^2 z)^3}}$$

$$I = \frac{1}{4}\int \cos z\, dz$$

$$= \frac{1}{4}\int [\sin z] + C$$

$$I = \frac{1}{4}\frac{\tan z}{\sqrt{1+\tan^2 z}} + C$$

$$I = \frac{1}{4}\frac{t/2}{\sqrt{1+t^2/4}} + C$$

$$= \frac{1}{4}\frac{x+1}{\sqrt{5+2x+x^2}} + C$$

III. Integration of hyperbolic functions : Functions rationally depending on hyperbolic functions are integrated in the same way as trigonometric functions.
Keep in mind the following basic formulae :

$$\cosh^2 x - \sinh^2 x = 1;$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1);$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1);$$

$$\sinh x \cosh x = \frac{1}{2}\sinh 2x.$$

If $\tanh \frac{x}{2} = t$, then

$$\sinh x = \frac{2t}{1-t^2};$$

$$\cosh x = \frac{1+t^2}{1-t^2};$$

$$x = 2\tanh^{-1} t = \ln\left(\frac{1+t}{1-t}\right) \quad (-1 < t < 1);$$

$$dx = \frac{2\,dt}{1-t^2}$$

EXAMPLES

5.6.21. $I = \int \cosh^2 x\, dx$

Solution.

$$I = \int \frac{1}{2}(\cosh 2x + 1)\, dx$$

$$= \frac{1}{4}\sinh 2x + \frac{1}{2}x + C$$

5.6.22. $I = \int \cosh^3 x\, dx$

Solution. Since $\cosh x$ is raised to an odd power, we put $\sinh x = t$;

$$\cosh x\, dx = dt.$$

We obtain

$$I = \int \cosh^2 x \cosh x\, dx$$

$$= \int (1+t^2)\, dt$$

$$= t + \frac{t^3}{3} + C$$

$$= \sinh x + \frac{1}{3}\sinh^3 x + C$$

5.7 INTEGRATION OF CERTAIN IRRATIONAL FUNCTIONS WITH THE AID OF TRIGONOMETRIC OR HYPERBOLIC SUBSTITUTIONS

Integration of functions rationally depending on x and $\sqrt{ax^2+bx+c}$ can be reduced to finding integrals of one of the following forms :

I. $\int R\left(t, \sqrt{p^2t^2+q^2}\right) dt$;

II. $\int R\left(t, \sqrt{p^2t^2-q^2}\right) dt$;

III. $\int R\left(t, \sqrt{q^2-p^2t^2}\right) dt$;

where $t = x + \frac{b}{2a}$; $ax^2+bx+c = \pm p^2t^2 \pm q^2$ (singling out a perfect square).

Integrals of the forms I to III can be reduced to integrals of expressions rational with respect to sine or cosine (ordinary or hyperbolic) by means of the following substitutions :

I. $$t = \frac{q}{p}\tan z$$

or $$t = \frac{q}{p}\sinh z$$

II. $$t = \frac{q}{p}\sec z$$

or $$t = \frac{q}{p}\cosh z$$

III. $$t = \frac{q}{p}\sin z$$

or $$t = \frac{q}{p}\tanh z$$

EXAMPLES

5.7.1. $I = \int \frac{dx}{\sqrt{(5+2x+x^2)^3}}$

Solution. $$5 + 2x + x^2 = 4 + (x+1)^2$$

Let $$x + 1 = t$$

$$I = \int \frac{dx}{\sqrt{(5+2x+x^2)^3}} = \int \frac{dt}{\sqrt{(4+t^2)^3}}$$

We have obtained an integral of the form I. Let us introduce the substitution :

$$t = 2\tan z;$$

$$dt = \frac{2\,dz}{\cos^2 z};$$

$$\sqrt{4+t^2} = 2\sqrt{1+\tan^2 z}$$

$$= 2\sec z$$

$$= \frac{2}{\cos z}$$

We get, $$I = \frac{1}{4}\int \cos z\,dz$$

$$= \frac{1}{4}\sin z + C$$

$$= \frac{1}{4}\frac{\tan z}{\sqrt{1+\tan^2 z}} + C$$

$$= \frac{1}{4}\frac{\frac{t}{2}}{\sqrt{1+\frac{t^2}{4}}} + C$$

$$= \frac{x+1}{4\sqrt{5+2x+x^2}} + C$$

5.7.2. $I = \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$

Solution. $x^2 + 2x + 2 = (x+1)^2 + 1$

Let, $x + 1 = t;$

$$dx = dt$$

$$I = \int \frac{dt}{t^2\sqrt{t^2+1}}$$

Again we have an integral of the form I. Make the substitution

$$t = \sinh z.$$

Then $dt = \cosh z\, dz;$

$$\sqrt{t^2+1} = \sqrt{1+\sinh^2 z} = \cosh z$$

Hence, $I = \int \frac{\cosh z\, dz}{\sinh^2 z \cosh z}$

$$= \int \frac{dz}{\sinh^2 z}$$

$$I = -\coth z + C$$

$$= -\frac{\sqrt{1+\sinh^2 z}}{\sinh z} + C$$

$$I = -\frac{\sqrt{1+t^2}}{t} + C$$

$$I = -\frac{\sqrt{x^2+2x+2}}{x+1} + C.$$

5.7.3. $I = \int \frac{\sqrt{x^2-1}}{x^2}\, dx$

Solution. Let $x = \tan\theta \quad dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec^3\theta\, d\theta}{\tan^2\theta}$$

$$= \int \frac{d\theta}{\cos\theta \cdot \sin^2\theta}$$

$$= \int \frac{(\sin^2\theta + \cos^2\theta)d\theta}{\cos\theta \cdot \sin^2\theta}$$

$$= \int \sec\theta\, d\theta + \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \ln(\sec\theta + \tan\theta) - \operatorname{cosec}\theta$$

$$= \ln\left(x + \sqrt{x^2 - 1}\right) - \left(\frac{1}{\tan^2\theta} + 1\right)^{1/2}$$

$$= \ln\left(x + \sqrt{x^2 + 1}\right) - \frac{(1 + x^2)^{1/2}}{x} + C$$

5.7.4. $I = \int \sqrt{(x^2 - 1)^3}\, dx$

Solution. Let, $x = \cosh t;$

$dx = \sinh t\, dt$

Hence, $I = \int \sqrt{(\cosh^2 t - 1)^3}\, \sinh t\, dt = \int \sinh^4 t\, dt$

$$= \int \left(\frac{\cosh 2t - 1}{2}\right)^2 dt$$

$$I = \frac{1}{4}\int \cosh^2 2t\, dt - \frac{1}{2}\int \cosh 2t\, dt + \frac{1}{4}\int dt$$

$$= \frac{1}{8}\int (\cosh 4t + 1)\, dt - \frac{1}{4}\sinh 2t + \frac{1}{4}t$$

$$I = \frac{1}{32}\sinh 4t - \frac{1}{4}\sinh 2t + \frac{3}{8}t + C$$

Let us return to x :

$$t = \cosh^{-1} x$$

$$= \ln\left(x + \sqrt{x^2 - 1}\right);$$

$$\sinh 2t = 2\sinh t \cosh t$$

$$= 2x\sqrt{x^2 - 1};$$

$$\sinh 4t = 2\sinh 2t \cosh 2t$$

$$= 4x\sqrt{x^2 - 1}\,(2x^2 - 1).$$

Hence, $I = \frac{1}{8}x(2x^2 - 1)\sqrt{x^2 - 1} - \frac{1}{2}x\sqrt{x^2 - 1} + \frac{3}{8}\ln\left(x + \sqrt{x^2 - 1}\right) + C$

5.7.5. $I = \int \frac{dx}{(1 + \sqrt{x})\left(\sqrt{x - x^2}\right)}$

Solution. Let, $x = \sin^2 t$

$$dx = 2\sin t\cos t$$

$$I = \int \frac{2\sin t\cdot\cos t\,dt}{(1+\sin t)\sqrt{\sin^2 t-\sin^4 t}}\,dt$$

$$= \int \frac{2dt}{1+\sin t}$$

$$= 2\int \frac{(1-\sin t)}{\left(1-\sin^2 t\right)}\,dt$$

$$= 2\tan t - \frac{2}{\cos t} + C$$

$$= \frac{2\sqrt{x}}{\sqrt{1-x}} - \frac{2}{\sqrt{1-x}} + C$$

$$= \frac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}} + C$$

5.7.6. $I = \int \sqrt{3-2x-x^2}\;dx$

Solution. $I = \int \sqrt{4-(x+1)^2}\;dx$

Let $x + 1 = t$

$$dx = dt$$

$$I' = \int \sqrt{2^2 - t^2}\;dt$$

$$= \frac{1}{2}t\sqrt{4-t^2} + \frac{1}{2}\times 4\ln\left|t+\sqrt{4-t^2}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\ln\left|x+1+\sqrt{3-2x-x^2}\right| + C$$

5.7.7. $I = \int \frac{dx}{\left(x^2-2x+5\right)^{3/2}}$

Solution. $I = \int \frac{dx}{\left((x-1)^2+2^2\right)^{3/2}}$

Let $x - 1 = t$

$$dx = dt$$

$$I = \int \frac{dt}{\left[t^2+2^2\right]^{3/2}}$$

$$t = 4\tan\theta$$

$$dt = 4\sec^2\theta\,d\theta$$

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$$I = \int \frac{4\sec^2\theta}{8\sec^3\theta}\, d\theta$$

$$I = \frac{1}{2}\int \cos\theta\, d\theta = \frac{1}{2}\sin\theta + C$$

$$= \frac{1}{2}\frac{\frac{t}{4}}{\left(\frac{t^2}{16}+1\right)^{1/2}} + C$$

$$I = \frac{1}{8}\frac{4(x-1)}{\left((x-1)^2+16\right)^{1/2}}$$

$$I = \frac{1}{2}\frac{(x-1)}{\left[(x-1)^2+16\right]^{1/2}} + C$$

5.8 INTEGRATION OF OTHER TRANSENDENTAL FUNCTIONS

EXAMPLES

5.8.1. $I = \int \frac{\ln x}{x^2}\, dx$.

Solution.

$$I = \ln x . \int \frac{dx}{x^2} - \int\left(\frac{d}{dx}\ln x . \int \frac{dx}{x^2}\right)dx$$

$$= -\frac{\ln x}{x} + \int \frac{dx}{x^2}$$

$$I = -\frac{\ln x}{x} - \frac{1}{x} + C$$

5.8.2. $I = \int \frac{\ln x\, dx}{\sqrt{1-x}}$

Solution.

$$I = \ln x . \int \frac{dx}{\sqrt{1-x}} - \int\left(\frac{d}{dx}(\ln x) . \int \frac{dx}{\sqrt{1-x}}\right)dx$$

$$= -2\ln x . \sqrt{1-x} + 2\int \frac{1}{x}\sqrt{1-x}\, dx$$

$$= -2\ln x . \sqrt{1-x} + 2\int \frac{\sqrt{1-x}}{x}\, dx$$

Let $1 - x = t^2$

$$-dx = 2t\, dt$$

$$I = -2\int \frac{2t^2}{1-t^2}\, dt + 4\int \frac{1-t^2-1}{1-t^2}\, dt$$

$$= +4t + 4\ln\left|\frac{t-1}{t+1}\right| + C$$

$$I = 4\sqrt{1-x} + 4\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$$

5.8.3. $I = \int \frac{e^x dx}{(1+e^{2x})^2}$

Solution. Let $e^x = t$

$$e^x\,dx = dt$$

$$I = \int \frac{dt}{(1+t^2)^2}$$

Applying Reduction formula

$$I = \frac{t}{2(t^2+1)} + \frac{1}{2}\int \frac{dt}{1+t^2}$$

$$I = \frac{t}{2(t^2+1)} + \tan^{-1} t + C$$

$$I = \frac{e^x}{2(e^{2x}+1)} + \frac{1}{2}\tan^{-1} e^x + C.$$

5.8.4. $I = \int e^x \ln(e^x + 1)\, dx$

Solution. Let $I = \ln(e^x+1).\int e^{-x}\,dx - \int\left(\frac{d}{dx}(\ln(e^x+1)).\int e^{-x}dx\right)dx$

$$I = -e^{-x}.\ln(e^x+1) + \int \frac{e^x.e^{-x}}{e^x+1}dx$$

$$= -e^{-x}\ln(e^x+1) + \int \frac{-e^x+e^x+1}{1+e^x}dx$$

$$= -e^{-x}\ln(e^x+1) + \int dx - \int \frac{e^x}{1+e^x}dx$$

$$I = -e^{-x}\ln(e^x+1) + x - \ln(1+e^x) + C$$

5.8.5. $I = \int \frac{e^{\alpha\tan^{-1}x}}{(1+x^2)^{3/2}}dx$

Solution. Let $x = \tan\theta$

$$dx = \sec^2\theta\, d\theta$$

$$I = \int \frac{e^{\alpha\theta}}{\sec^3\theta}.\sec^2\theta\, d\theta$$

$$= \int e^{\alpha\theta}.\cos\theta\, d\theta$$

$$= \frac{\cos\theta.e^{\alpha\theta}}{\alpha} - \int \frac{-\sin\theta.e^{\alpha\theta}}{\alpha}d\theta$$

$$\frac{\cos\theta.e^{\alpha\theta}}{\alpha} + \left[\frac{\sin\theta.e^{\alpha\theta}}{\alpha^2} - \int \frac{\cos\theta.e^{\alpha\theta}}{\alpha^2}d\theta\right]$$

$$I\left(1+\frac{1}{\alpha^2}\right) = \frac{(\alpha\cos\theta+\sin\theta)}{\alpha^2}e^{\alpha\theta}$$

$$I = \frac{(\alpha\cos\theta+\sin\theta)}{1+\alpha^2}e^{\alpha\theta}$$ where, $\theta = \tan^{-1}x$.

5.8.6. $I = \int \frac{x\tan^{-1}x}{(1+x^2)^{1/2}}\,dx$

Solution.

$$I = \tan^{-1}x\,.\int\frac{x\,dx}{(1+x^2)^{1/2}} - \int\left(\frac{d}{dx}(\tan^{-1}x)\right).\int\frac{x\,dx}{(1+x^2)^{1/2}}$$

$$I = \sqrt{1+x^2}\,.\tan^{-1}x - \int\sqrt{1+x^2}\,.\frac{dx}{1+x^2}$$

$$I = \sqrt{1+x^2}\,.\tan^{-1}x - \ln(x+x^2+1) + C.$$

5.9 METHODS OF INTEGRATION

List of Basic Forms of Integrals

Integral	*Method of Integration*
1. $\int F[\varphi(x)]\,\varphi'(x)\,dx$	1. Substitution $\varphi(x) = t$
2. $\int f(x)\,\varphi'(x)\,dx$	2. Integration by parts $\int f(x)\,\varphi'(x)\,dx = f(x)\,\varphi(x) - \int\varphi(x)\,f'(x)\,dx$. This method is applied, for example, to integrals of the form $\int p(x)\,f(x)\,dx$, where $p(x)$ is a polynomial, and $f(x)$ is one of the following functions : $e^{\alpha x}$; $\cos\alpha x$, $\sin\alpha x$; $\ln x$; $\tan^{-1}x$, $\sin^{-1}x$ and also to integrals of products of an exponential function by cosine or sine.
3. $\int f(x)\,\varphi^{(n)}(x)\,dx$	3. Reduced to integration of the product $f^{(n)}(x)\,\varphi(x)$ by the formula for multiple integration by parts $\int f(x)\,\varphi^{(n)}(x)\,dx = f(x)\,\varphi^{(n-1)}(x) - f'(x)\,\varphi^{(n-2)}(x) + f''(x)\,\varphi^{(n-3)}(x) - \ldots + (-1)^{n-1}f^{(n-1)}(x)\,\varphi(x) + (-1)_n\int f^n(x)\,\varphi(x)\,dx$

4. $\int e^{\alpha x} p_n(x)\, dx$ where $p_n(x)$ = polynomial of degree n.	**4.** Applying the formula for multiple integration by parts, we get $\int e^{\alpha x} p_n(x)\, dx = e^{\alpha x}\left[\frac{p_n(x)}{\alpha} - \frac{p_n'(x)}{\alpha^2} + \frac{p_n''(x)}{\alpha^3} - \ldots + \ldots + (-1)^n \frac{p_n^{(n)}(x)}{\alpha^{n+1}}\right] + C$
5. $\int \frac{Mx+N}{x^2+px+q}\, dx$, $p^2-4q<0$	**5.** Substitution $x + \frac{p}{2} = t$
6. $I_n = \int \frac{dx}{(x^2+1)^n}$	**6.** Reduction formula is used $I_n = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} I_{n-1}$
7. $\int \frac{P(x)}{Q(x)}\, dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational fraction. $Q(x) = (x-x_1)^l (x-x_2)^m \ldots (x^2+px+q)^k \ldots$	**7.** Integrand is expressed in the form of a sum of partial fractions $\frac{P(x)}{Q(x)} = \frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_1)^2} + \ldots + \frac{A_l}{(x-x_1)^l} + \ldots + \frac{B_1}{(x-x_2)} + \frac{B_2}{(x-x_2)^2} + \ldots + \frac{B_m}{(x-x_2)^m} + \ldots + \frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \ldots + \frac{M_kx+N_k}{(x^2+px+q)^k} + \ldots$
8. $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right) dx$, where R is a rational function of its arguments.	**8.** Reduced to the integral of a rational fraction by the substitution $x = t^k$, where k is a common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$
9. $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}\right] dx$ where R is a rational function of its arguments.	**9.** Reduced to the integral of a rational fraction by the substitution $\frac{ax+b}{cx+d} = t^n$
10. $\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}\, dx$	**10.** By the substitution $x + \frac{b}{2a} = t$ the integral is reduced to a sum of two integrals : $\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}\, dx$

$$= M_1 \int \frac{t\,dt}{\sqrt{at^2+m}} + N_1 \int \frac{dt}{\sqrt{at^2+m}}$$

The first integral is reduced to the integral of a power function and the second one is a tabular integral.

11. $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$, where R is a rational function of x and $\sqrt{ax^2+bx+c}$

11. Reduced to an integral of rational fraction by the Euler substitutions :

$$\sqrt{ax^2+bx+c} = t \pm \sqrt{a} \quad (a > 0),$$

$$\sqrt{ax^2+bx+c} = tx \pm \sqrt{c} \quad (c > 0),$$

$$\sqrt{ax^2+bx+c} = t\,(x - x_1)\ (4ac - b^2 < 0).$$

where x_1 is the root of the trinomial $ax^2 + bx + c$.

The indicated integral can also be evaluated by the trigonometric substitutions :

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{b^2-4ac}}{2a} \sin t \\ \dfrac{\sqrt{b^2-4ac}}{2a} \cos t \end{cases} \quad (a < 0,\ 4ac - b^2 < 0)$$

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{b^2-4ac}}{2a} \sec t \\ \dfrac{\sqrt{b^2-4ac}}{2a} \operatorname{cosec} t \end{cases} \quad (a > 0,\ 4ac - b^2 < 0)$$

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{4ac-b^2}}{2a} \tan t \\ \dfrac{\sqrt{4ac-b^2}}{2a} \cot t \end{cases} \quad (a > 0,\ 4ac - b^2 > 0)$$

12. $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx$, where $P_n(x)$ = polynomial of degree n.

12. Write the equality

$$\int \frac{P_n(x)\,dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + k \int \frac{dx}{\sqrt{ax^2+bx+c}},$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$. Differentiating both parts of this equality and multiplying by $\sqrt{ax^2+bx+c}$, we get the identity

$$P_{(n)}(x) \equiv Q'_{n-1}(x)\left(ax^2+bx+c\right) + \frac{1}{2} Q_{n-1}(x)(2ax+b) + k,$$

which gives a system of $n + 1$ linear equations for

	determining the coefficients of the polynomial $Q_{n-1}(x)$ and factor k. And the integral $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ is taken by the method considered in the form $\int \frac{M_x+N}{\sqrt{ax^2+bx+c}}\,dx$ (M = 0; N = 1).
13. $\int \frac{dx}{(x-x_1)^m\sqrt{ax^2+bx+c}}$	**13.** This integral is reduced to the above-considered integral by the substitution $x - x_1 = \frac{1}{t}$
14. $\int x^m(a+bx^n)^p\,dx$ where m, n, p are rational numbers (an integral of a binomial differential).	**14.** This integral is expressed through elementary functions only if one of the following conditions is fulfilled : 1. if p is an integer, 2. if $\frac{m+1}{n}$ is an integer, 3. if $\frac{m+1}{n}+p$ is an integer. ***1st Case :*** *(a)* If p is a positive integer, remove the brackets $(a+bx^n)^p$ according to the Newton binomial and calculate the integrals of powers; *(b)* If p is a negative integer, then the substitution $x = t^k$, where k is the common denominator of the fractions m and n, leads to the integral of a rational fraction; ***2nd Case :*** if $\frac{m+1}{n}$ is an integer, then the substitution $a + bx^n = t^k$ is applied, where k is the denominator of the fraction p; ***3rd Case :*** if $\frac{m+1}{n}+p$ is an integer, then the substitution $a + bx^n = x^n t^k$ is applied, where k is the denominator of the fraction p.
15. $\int R(\sin x, \cos x)\,dx$	**15.** Universal substitution $\tan\frac{x}{2} = t$. If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, then the substitution $\cos x = t$ is applied.

	If R (sin x, − cos x) = − R (sin x, cos x), then the substitution $\sin x = t$ is applied. If R (− sin x, − cos x) = R (sin x, cos x), then the substitution $\tan x = t$ is applied.
16. $\int R(\sinh x, \cosh x)\, dx$	**16.** The substitution $\tanh \frac{x}{2} = t$ is used. In this case, $\sinh x = \frac{2t}{1-t^2}$; $\cosh x = \frac{1+t^2}{1-t^2}$; $dx = \frac{2\,dt}{1-t^2}$.
17. $\int \sin ax \sin bx\, dx$ $\int \sin ax \cos bx\, dx$ $\int \cos ax \cos bx\, dx$	**17.** Transform the product of trigonometric function into a sum or difference, using one of the following formulas : $\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$ $\cos ax \cos bx = \frac{1}{2}[\cos(a-b)x + \cos(a+b)x]$ $\sin ax \cos bx = \frac{1}{2}[\sin(a-b)x + \sin(a+b)x]$
18. $\int \sin^m x \cos^n x\, dx$ where m and n are integers.	**18.** If m is an odd positive number, then apply the substitution $\cos x = t$. If n is an odd positive number, then apply the substitution $\sin x = t$. If $m + n$ is an even negative number, then apply the substitution $\tan x = t$. If m and n are even non-negative numbers, then use the formulae $\sin^2 x = \frac{1-\cos 2x}{2}$; $\cos^2 x = \frac{1+\cos 2x}{2}$
19. $\int \sin^p x \cos^q x\, dx \left(0 < x < \frac{\pi}{2}\right)$ where p and q are rational numbers.	**19.** Reduce to the integral of the binomial differential by the substitution $\sin x = t$ $\int \sin^p x \cos^q x\, dx = \int t^p \left(1-t^2\right)^{\frac{q-1}{2}} dt$
20. $\int R\left(e^{ax}\right) dx$	**20.** Transform into an integral of a rational function by the substitution $e^{ax} = t$.