CHAPTER 6 THE DEFINITE INTEGRAL

6.1 THE LOWER AND UPPER INTEGRAL SUMS

Let a function $f(x)$ be defined in the closed interval $[a, b]$. The following is called the integral sum :

$$I_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$$

where $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$,

$$\Delta x_i = x_{i+1} - x_i;$$

$$\xi_i \in [x_i, x_{i+1}] \quad (i = 0, 1, \dots, n-1).$$

The sum $S_n = \sum_{i=0}^{n-1} M_i\,\Delta x_i$ is called the **upper (integral)** sum, and $s_n = \sum_{i=0}^{n-1} m_i \Delta x_i$ is called the **lower (integral)** sum, where $M_i = \sup f(x)$ $[m_i = \inf f(x)]$ for $x \in [x_i, x_{i+1}]$.

The definite integral of the function $f(x)$ on the interval $[a, b]$ is the limit of the integral sums

$$\int_a^b f(x)\,dx = \lim \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i \text{ when } \max |\Delta x_i| \to 0.$$

If this limit exists, the function is called **integrable** on the interval $[a, b]$. Any continuous function is integrable.

6.2 EVALUATING DEFINITE INTEGRALS BY THE NEWTON-LEIBNIZ FORMULA

The Newton-Leibniz formula :

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is one of the antiderivatives of the function $f(x)$, *i.e.*

$$F'(x) \equiv f(x) \quad (a \le x \le b)$$

EXAMPLES

6.2.1. ***Evaluate the integral***

$$I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

Solution. Since the function F $(x) = \tan^{-1} x$ is one of the anti-derivatives of the function $f(x) = \dfrac{1}{1+x^2}$ using Newton-Leibniz formula, we get

$$\mathrm{I} = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

$$= \left[\tan^{-1} x\right]_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

6.2.2. ***Compute the integrals***

(a)

$$I = \int_1^{\pi/2} \sin 2x \, dx$$

Solution.

$$\mathrm{I} = \left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{2}[\cos \pi - \cos 0]$$

$$\mathrm{I} = -\frac{1}{2}[-1-1] = 1$$

(b)

$$I = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Solution.

Let $\sin x = t$ $\qquad \sin\frac{\pi}{2} = 1 = t_2$

$\cos x \, dx = dt$ $\qquad \sin\frac{\pi}{6} = \frac{1}{2} = t_1$

$$\mathrm{I} = \int_{1/2}^{1} \frac{dt}{t^3}$$

$$= \int_{1/2}^{1}\left[-\frac{1}{2t^2}\right] = -\frac{1}{2} + \frac{1}{2 \times \frac{1}{4}}$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

(c) $$I = \int_0^2 \frac{dx}{\sqrt{16 - x^2}}$$

Solution. $$I = \int_0^2 \frac{dx}{\sqrt{4^2 - x^2}}$$

$$= \left[\sin^{-1}\frac{x}{4}\right]_0^2$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

6.2.3. *Given the function*

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ \sqrt{x} & 1 \le x \le 2 \end{cases}$$

Solution. $$I = \int_0^1 x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$I = \int_1^2 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}x^{3/2}\right]_1^2 = \frac{2}{3}(2)^{3/2} - \frac{2}{3}$$

$$I = \frac{2}{3}\left[2\sqrt{2} - 1\right]$$

6.2.4. *Evaluate the integral*

$$I = \int_1^2 |1 - x|\, dx$$

Solution. $$|1 - x| = \begin{cases} 1 - x & 0 \le x \le 1 \\ x - 1 & 1 \le x \le 2 \end{cases}$$

$$I = \int_1^2 |1 - x|\, dx$$

$$= \int_0^1 (1 - x)\, dx + \int_1^2 (x - 1)\, dx$$

$$= \left[-\frac{(1 - x)^2}{2}\right]_0^1 + \left[\frac{(x - 1)^2}{2}\right]_1^2$$

$$I = \frac{1}{2} + \frac{1}{2} = 1$$

6.2.5. *Evaluate the integral*

$$I = \int_a^b \frac{|x|}{x}\,dx \quad \text{where } a < b$$

Solution. If $0 \le a \le b$, then $f(x) = \frac{|x|}{x} = 1$

$$\Rightarrow \quad I = \int_a^b dx = b - a$$

If $a \le b \le 0$, then $f(x) = -1$

$$\Rightarrow \quad I = \int_a^b f(x)\,dx$$

$$= -b - (-a) = a - b$$

Finally if $a < 0 < b$, then divide the integral

$$I = \int_a^b f(x)\,dx$$

$$= \int_a^0 f(x)\,dx + \int_0^b f(x)\,dx$$

$$= b - (-a) = b + a$$

$$I = \int_a^b f(x)\,dx = |b| - |a|$$

6.2.6. *Find a mistake in the following evaluation*

$$I = \int_0^{\sqrt{3}} \frac{dx}{1+x^2}$$

Solution.

$$I = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1+x^2}\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left[\tan^{-1}(-\sqrt{3}) - \tan^{-1}0\right] = -\frac{\pi}{6}$$

where $\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\frac{2x}{1+x^2}\right) = \frac{1}{1+x^2} \quad x \ne 1$

The result is a priory wrong

$\frac{1}{2}\tan^{-1}\frac{2x}{1+x^2}$ has a discontinuity of the first kind at the point $x = 1$.

$$\lim_{x \to 0^-} \frac{1}{2}\tan^{-1}\frac{2x}{1+x^2} = \frac{\pi}{4}$$

$$\lim_{x \to 1^+} \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = -\frac{\pi}{4}$$

$$I = \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^{\sqrt{3}}$$

$$I = \tan^{-1}\sqrt{3} - \tan^{-1} 0 = \frac{\pi}{3}$$

$\tan^{-1} x$ is continuous in the interval $\left[0, \frac{\pi}{3}\right]$.

6.2.7. *Find a mistake in the following evaluation of the integral*

$$I = \int_0^{\pi} \frac{dx}{1 + 2\sin^2 x}$$

Solution.

$$= \int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} \quad \because \sin^2 x + \cos^2 x = 1$$

$$= \int_0^{\pi} \frac{\frac{dx}{\cos^2 x}}{1 + 3\tan^2 x}$$

$$I = \int_0^{\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right] = 0$$

The Newton-Leibniz formula is not applicable as the antiderivative $F(x) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$ has a discontinuity of the point $x = \frac{\pi}{2}$. Obviously,

$$\lim_{x \to \pi^-/2} F(x) = \lim_{x \to \pi^-/2} \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = \frac{\pi}{2\sqrt{3}}$$

$$\lim_{x \to \pi^+/2} F(x') = \lim_{x \to \pi^+/2} \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = -\frac{\pi}{2\sqrt{3}}$$

The correct result can be obtained in the following way

$$\int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\pi} \frac{1}{\cot^2 x + 3}\cdot\frac{dx}{\sin^2 x}$$

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\cot x\right)\right]_0^{\pi} = \frac{\pi}{\sqrt{3}}$$

Formula is applicable if we break

$$\int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\pi} F(x)\,dx + \int_{\pi/2}^{\pi} F(x)\,dx = \frac{\pi}{\sqrt{3}}$$

6.2.8. *Compute the integral*

$$I = \int_0^{\pi} \frac{\sqrt{1 + \cos 2x}}{\sqrt{2}}\,dx = \int_0^{\pi} |\cos x|\,dx$$

Solution.

$$I = \begin{cases} \cos x & 0 \le x \le \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx = \int_0^{\pi}\cos x\,dx + \int_0^{\pi}(-\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} + (-\sin x)]_{\pi/2}^{0} = 2$$

6.2.9. ***Evaluate the integral***

$$I = \int_0^{100\pi}\sqrt{1-\cos 2x\,dx}$$

Solution. $$I = \sqrt{2}\int_0^{100\pi}|\sin x|\,dx$$

Since $|\sin x|$ has a period π, then

$$\int_0^{100\pi}\sqrt{1-\cos 2x}\,dx = \sqrt{2}\int_0^{\pi}|\sin x|\,dx$$

$$= 100\sqrt{2}\int_0^{100\pi}|\sin x|\,dx = 200\sqrt{2}$$

6.2.10. ***Evaluate the integrals :***

(a) $$I = \int_{-2}^{-1}\frac{dx}{(11+5x)^3}$$

Solution. Let $11+5x = t$ $\quad t = 6$ at $x = -1$

$t = 1$ at $x = -2$

$$I = \frac{1}{5}\int_1^6\frac{dt}{t^3} = -\left[\frac{1}{10}t^{-2}\right]_1^6$$

$$= -\frac{1}{10}\times(6)^{-2} + \frac{1}{10}$$

$$= \frac{1}{10}\left[1-\frac{1}{36}\right] = \frac{35}{360} = \frac{7}{72}$$

(b) $$I = \int_{-3}^{-2}\frac{dx}{x^2-1}$$

Solution. $$= \frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_{-3}^{-2}$$

$$I = \frac{1}{2}\left[\ln\left|\frac{-2-1}{-2+1}\right| - \ln\left|\frac{-3-1}{-3+1}\right|\right]$$

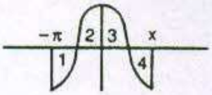
$$= \frac{1}{2}\left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{-4}{-2}\right|\right]$$

$$= \frac{1}{2}\left[\ln 3 - \ln\frac{4}{2}\right]$$

$$= \frac{1}{2}(\ln 3 - \ln 4 + \ln 2)$$

$$I = \frac{1}{2}(\ln 3 - \ln 2) = \frac{1}{2}\ln\frac{3}{2}$$

(c) $$I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{2}$$



Solution.
$$= \int_{-\pi}^{\pi} \frac{1-\cos x}{2}\,dx$$
$$= \int_{-\pi}^{\pi} \frac{1}{2}\,dx - \int_{-\pi}^{\pi} \frac{\cos x}{2}\,dx$$
$$= \frac{1}{2}[\pi+\pi] - \frac{4}{2}\int_0^{\pi/2} \cos x\,dx \text{ [considering area under the curve]}$$
$$= \pi - 2\int_0^{\pi/2} \cos x\,dx$$
$$= \pi - 2\left[\sin x\right]_0^{\pi/2} = \pi - 2$$

(d) $$I = \int_0^{\pi/4} \frac{x^2}{1+x^2}\,dx$$

Solution.
$$I = \int_0^{\pi/4} \frac{x^2+1-1}{1+x^2}\,dx$$
$$= \int_0^{\pi/4} dx - \int_0^{\pi/4} \frac{dx}{1+x^2}$$
$$= [x]_0^{\pi/4} - \left[\tan^{-1} x\right]_0^{\pi/4}$$
$$I = \frac{\pi}{4} - \tan^{-1}\frac{\pi}{4}$$

(e) $$I = \int_e^{e^2} \frac{dx}{x \ln x}$$

Solution. Let $\ln x = t$
$$\frac{1}{x}\,dx = dt$$
$$= \int_e^{e^2} \frac{dt}{t} = [\ln t]_e^{e^2}$$
$$= \ln\,[\ln t]_e^{e^2}$$
$$= \ln(\ln e^2) - \ln(\ln e)$$
$$= \ln 2 - \ln 1$$
$$I = \ln 2 - 0$$

(f) $$I = \int_{1/\pi}^{2/\pi} \frac{\sin\frac{1}{x}}{x^2}\,dx$$

Solution. Let $\frac{1}{x} = t$
$$-\frac{1}{x^2}\,dx = dt$$

$$\frac{1}{2/\pi} = t_1 \qquad t_1 = \frac{\pi}{2}$$

$$t_2 = \pi$$

$$I = -\int_{\pi}^{\pi/2} \sin t\,dt = \int_{\pi/2}^{\pi} \sin t\,dt$$

$$= [\cos t]_{\pi/2}^{\pi} = \cos\pi - \cos\frac{\pi}{2} = -1$$

(g) $$I = \int_0^1 \frac{e^x\,dx}{1+e^{2x}}$$

Solution. Let $e^x = t$

$e^1 = t_1$

$e^0 = t_2 \Rightarrow t_2 = 1$

$e^x\,dx = dt$

$$= \int_1^e \frac{dt}{1+t^2} = [\tan^{-1} t]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1$$

(h) $$I = \int_0^1 \frac{x^3\,dx}{1+x^8}$$

Solution. $$I = \int_0^1 \frac{dt}{1+t^2} \times \frac{1}{4} = \left[\frac{1}{4}\tan^{-1} t\right]_0^1$$ $\quad x^4 = t$, $4x^3\,dx = dt$

$$= \frac{1}{4}\tan^{-1} 1 = \frac{\pi}{16}$$

(i) $$I = \int_0^3 \frac{x\,dx}{\sqrt{1+x}+\sqrt{5x+1}}$$

Solution. $$I = \int_0^3 \frac{x\,dx}{\sqrt{1+x}+\sqrt{5x+1}} \times \frac{\sqrt{5x+1}-\sqrt{1+x}}{\sqrt{5x+1}-\sqrt{1+x}}$$

$$= \int_0^3 \frac{x}{4x} \times \left(\sqrt{5x+1}-\sqrt{1+x}\right) dx$$

$$= \frac{1}{4}\int_0^3 \sqrt{5x+1}\,dx - \frac{1}{4}\int_0^3 \sqrt{x+1}\,dx$$

$$= \frac{1}{20}\left[(5x+1)^{3/2}.\frac{2}{3}\right]_0^3 - \frac{1}{4}\left[\frac{2}{3}(5x+1)^{3/2}\right]_0^3$$

$$= \frac{1}{30}[64-1] - \frac{1}{4}\left[\frac{2}{3}\times(4)^{3/2} - \frac{2}{3}\right]$$

$$= \frac{63}{30} - \frac{1}{4}\times\frac{2}{3}[8-1] = \frac{63}{30} - \frac{14}{12}$$

$$= \frac{126-70}{60} = \frac{56}{60} = \frac{14}{15}$$

(j) $$I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$$

Solution. $$= \int_{-\pi/2}^{\pi/2} \sqrt{\cos x\left(1-\cos^2 x\right)}\, dx$$

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, |\sin x|\, dx$$

$$= \int_{-\pi/2}^{0} \sqrt{\cos x}\, (-\sin x)\, dx$$

$$I = -\int_{-\pi/2}^{0} \sqrt{\cos x}\, \sin x\, dx + \int_{0}^{\pi/2} \sqrt{\cos x}\, \sin x\, dx$$

Let $\cos x = t$

$-\sin x\, dx = dt$

$$I = \int_0^1 \sqrt{t}\, dt + \int_1^0 -\sqrt{t}\, dt$$

$$= 2\int_0^1 \sqrt{t}\, dt$$

$$= \left[2\,(t)^{3/2} \cdot \frac{2}{3}\right]_0^1 = \frac{4}{3}$$

(k) $$I = \int_1^{\sqrt{3}} \frac{dx}{\left(1+x^2\right)^{3/2}}$$

Solution. Let $x = \tan\theta$ $\quad \sqrt{3} = \tan\theta_1;$

$dx = \sec^2\theta\, d\theta,$

$$\theta_1 = \frac{\pi}{3}$$

$$1 = \tan\theta_2;\quad \theta_2 = \frac{\pi}{4}$$

$$I = \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_{\pi/4}^{\pi/3} \cos\theta\, d\theta$$

$$= \left[\sin\frac{\pi}{3} - \sin\frac{\pi}{4}\right] = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{\sqrt{3}}{\sqrt{2}} - 1\right] = \frac{\sqrt{3}-\sqrt{2}}{2}$$

6.3 CHANGING THE VARIABLE IN A DEFINITE INTEGRAL

If a function $x = \varphi(t)$ satisfies the following conditions :

1. $\varphi(t)$ is a continuous single-valued function defined in $[\alpha, \beta]$ and has in this interval a continuous derivative $\varphi'(t)$;
2. With t varying on $[\alpha, \beta]$ the values of the function $x = \varphi(t)$ do not leave the limits of $[a, b]$;
3. $\varphi(\alpha) = a$ and $\varphi(\beta) = b$,

then the formula for changing the variable (or substitution) in the definite integral is valid for any function $f(x)$ which is continuous on the interval $[a, b]$:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt$$

Instead of the substitution $x = \varphi(t)$ the inverse substitution $t = \psi(x)$ is frequently used. In this case, the limits of integration α and β are determined directly from the equalities $\alpha = \psi(a)$ and $\beta = \psi(b)$. In practice, the substitution is usually performed with the aid of monotonic, continuously differentiable functions. The change in the limits of integration is conveniently expressed in the tabular form :

x	t
a	α
b	β

EXAMPLES

6.3.1. $I = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2}\,dx$

Solution.

$$x = 2\sin t$$
$$\sqrt{3} = 2\sin t$$
$$\frac{\sqrt{3}}{2} = \sin t_1 \quad \text{and} \quad -\frac{\sqrt{3}}{2} = \sin t_2$$
$$I = \int_{-\pi/3}^{\pi/3} \sqrt{4 - 4\sin^2 t}\,(2\cos t)\,dt$$
$$= 2\int_{-\pi/3}^{\pi/3} \cos t\,(2\cos t)\,dt$$
$$= 2\int_{-\pi/3}^{\pi/3} (1 + \cos 2t)\,dt$$
$$I = 2\left[t + \frac{1}{2}\sin 2t\right]_{-\pi/3}^{\pi/3} = \frac{4\pi}{3} + \sqrt{3}$$

6.3.2. $I = \int_2^4 \frac{\sqrt{x^2 - 4}}{x^4}\,dx$

Solution. Let

$$x = 2\sec t$$
$$dx = 2\sec t\tan t\,dt$$
$$4 = 2\sec t_1 \qquad 2 = \sec t_1; \qquad t_1 = \frac{\pi}{3}$$
$$2 = 2\sec t_2; \qquad t_2 = 0$$

$$I = \int_0^{\pi/3} \frac{2\tan t \cdot 2\sec t \tan t\, dt}{2^4 \cdot \sec^4 t}$$

$$= \frac{1}{4}\int_0^{\pi/3} \frac{\tan^2 t}{\sec^3 t} dt$$

$$= \frac{1}{4}\int_0^{\pi/3} \sin^2 t \cdot \cos t\, dt$$

$$= \frac{1}{12}\left[\sin^3 t\right]_0^{\pi/3} = \frac{\sqrt{3}}{32}$$

6.3.3. (a) $I = \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$

Solution. $x = a \sin\theta \qquad a = a\sin\theta_1$

$dx = a\cos\theta\, d\theta \qquad \theta_1 = \frac{\pi}{2}$

$0 = a\sin\theta_2; \qquad \theta_2 = 0$

$$I = \int_0^{\pi/2} a^2 \sin^2\theta \,.\, a\cos\theta \,.\, a\cos\theta\, d\theta$$

$$I = a^4 \int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{a^4}{4}\left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{8}\left[\frac{\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2}\right] = \frac{\pi a^4}{16}$$

(b) $I = \int_1^{\sqrt{3}} \frac{dx}{(1+x^2)^3}$

Solution. Let $x = \tan\theta \qquad \sqrt{3} = \tan\theta_1 ; \theta_1 = \frac{\pi}{3}$

$dx = \sec^2\theta\, d\theta \qquad 1 = \tan\theta_2; \theta_2 = \frac{\pi}{4}$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = [\sin\theta]_{\pi/4}^{\pi/3}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{2}$$

6.3.4. *(a)* $I = \int_0^{\pi/2} \frac{\cos x\, dx}{6 - 5\sin x + \sin^2 x}$

Solution. Let $\sin x = t$ $\qquad \sin\frac{\pi}{2} = 1$

$\cos x\, dx = dt \qquad \sin 0 = 0$

$$I = \int_0^1 \frac{dt}{6 - 5t + t^2}$$

$$= \left[\ln\frac{t-3}{t-2}\right]_0^1 = \ln\frac{4}{3}$$

(b) $I = \int_0^1 \frac{dx}{2 + \cos x}$

Solution. Let $t = \tan\frac{x}{2}, \quad t_1 = \tan\frac{\pi}{4}$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx, \quad t_2 = \tan 0$$

$$= \int_0^1 \frac{2dt}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{1}{1+t^2}$$

$$= 2\int_0^1 \frac{dt}{3 + t^2}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{t}}\right]_0^{-1}$$

$$I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right]$$

$$I = \frac{\pi}{3\sqrt{3}}$$

6.3.5. ***Compute the integral***

$$\int_0^{\pi/4} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \qquad (a > 0, b > 0)$$

Solution. Let $\tan x = t$ $\qquad \tan\frac{\pi}{4} = t_1$

$\sec^2 x\, dx = dt$

$\frac{dx}{\cos^2 x} = dt \qquad 1 = t_1;\ \tan 0 = t_2 \text{ or } 0 = t_2$

$$I = \int_0^{\pi/4} \frac{dx\left(\sec^2 x\right)}{a^2 + b^2\tan^2 x}$$

$$= \int_0^1 \frac{dt}{a^2 + b^2 t^2}$$

$$= \frac{1}{b^2} \int_0^1 \frac{dt}{\frac{a^2}{b^2} + t^2}$$

$$I = \frac{1}{b}\left[\frac{a}{b} \tan^{-1} \frac{bt}{a}\right]_0^1$$

$$I = \frac{1}{ab} \tan^{-1} \frac{b}{a}$$

If $a = b = 1$, then

$$\frac{1}{ab} \tan \frac{b}{a} = \frac{\pi}{4}$$

6.3.6. *Compute the integrals*

(a) $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx$

Solution. Let $x = \tan\theta$ $\qquad \sqrt{3} = \tan\theta_1; \theta_1 = \frac{\pi}{3}$

$dx = \sec^2\theta \, d\theta$ $\qquad 1 = \tan\theta_2; \theta_2 = \frac{\pi}{4}$

$$I = \int_{\pi/4}^{\pi/3} \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta} \sec^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta . \sec\theta \, d\theta}{\tan^2\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sin^2\theta \cdot \cos\theta}$$

Let $\sin\theta = t$ $\qquad \sin\frac{\pi}{3} = t_1; \sin\frac{\pi}{4} = t_2$

$\cos\theta \, d\theta = dt$ $\qquad \frac{\sqrt{3}}{2} = t_1; \frac{1}{\sqrt{2}} = t_2$

$$d\theta = \frac{dt}{\cos\theta}$$

$$I = \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dt}{t^2(1-t^2)} = \int_{1/\sqrt{2}}^{\sqrt{3}/2} \left[\frac{dt}{1-t^2} + \frac{dt}{t^2}\right]$$

$$= \left[\ln\left|\frac{1+t}{1-t}\right|\right]_{1/\sqrt{2}}^{\sqrt{3}/2} - \left[\frac{1}{t}\right]_{1/\sqrt{2}}^{\sqrt{3}/2}$$

$$I = \ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{1}{\sqrt{2}}}\right| - \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

(b) $\int_1^{e^2} \frac{dx}{x\sqrt{1+\ln x}}$

Solution. Let $1 + \ln = t$

$$\left(\frac{1}{x}\right) dx = dt$$

$$1 + \ln e^2 = t_1 \text{ and } 3 = t_1;\ t_2 = 1$$

$$I = \int_1^3 \frac{dt}{\sqrt{t}} = \left[2\sqrt{t}\right]_1^3$$

$$= 2\sqrt{3} - 2 = 2\left(\sqrt{3}-1\right)$$

(c) $I = \int_3^2 \frac{(x-2)^{2/3}}{3+(x-2)^{2/3}}$

Solution. Let $(x-2) = t^3$ $\quad (3-2) = t^3$

$$dx = 3t^3\,dt \qquad 1 = t^3;\ 0 = t^3$$

$$I = \int_1^0 \frac{3t^2 \cdot t^2\,dt}{3+t^2}$$

$$= 3\int_1^0 \frac{\left(t^4 - 9 + 9\right)}{3+t^2}\,dt$$

$$= 3\int_1^0 \left(t^2 - 3\right) dt + 27\int_1^0 \frac{dt}{3+t^2}$$

$$= 3\left[\frac{t^3}{3} - 3t\right]_1^0 + 27 \times \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}}\right]_1^0$$

$$= 3\left[3 - \frac{1}{3}\right] - \frac{27}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{24}{3} - \frac{27}{\sqrt{3}} \times \frac{\pi}{6} = 8 - \frac{9\pi}{2\sqrt{3}}$$

6.3.7. $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Solution.
$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+[\cos(\pi - x)]^2} \qquad \left[\int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$\cos x = t \qquad \cos \pi = t_1, \quad t_1 = -1$$

$$-\sin x\, dx = dt \qquad \cos 0 = t_2, \quad t_2 = 1$$

$$= \frac{\pi}{2}\int_1^{-1} -\frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= 2\cdot\frac{\pi}{2}\int_0^{1} \frac{dt}{1+t^2} \qquad \text{[curve is symmetric]}$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\cdot\frac{\pi}{4} = \frac{\pi^2}{4}$$

6.3.8. *Evaluate the integral*

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

Solution. $x = \tan t$

$$1 = \tan t_1 \qquad t_1 = \frac{\pi}{4}$$

$$0 = \tan t_2 \qquad t_1 = 0$$

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan t)}{1+\tan^2 t}\sec^2 t\, dt$$

$$I = \int_0^{\pi/4} \ln(1+\tan t)\, dt$$

$$1+\tan t = \frac{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}{\cos t}$$

$$I = \int_0^{\pi/4} \frac{1}{2}\ln 2\, dt + \int_0^{\pi/4} \ln\left(\sin t + \frac{\pi}{4}\right) dt - \int_0^{\pi/4} \ln\cos t\, dt$$

$$= \frac{\pi}{8}\ln 2 + \int_0^{\pi/4} \ln\sin\left(t+\frac{\pi}{4}\right) dt - \int_0^{\pi/4} \ln\cos t\, dt$$

Now let us show that $I_1 = I_2$

	I_1	I_2
$t = \left(\frac{\pi}{4}\right) - z$	t	z
$dt = -dz$	0	$\frac{\pi}{4}$
	$\frac{\pi}{4}$	0

$$I_1 = \int_0^{\pi/4} \ln\cos t\, dt$$

$$I_2 = -\int_0^{\pi/4} \ln\cos\left(\frac{\pi}{4} - z\right) dz$$

$$= \int_0^{\pi/4} \ln\sin\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - z\right)\right] dz$$

$$= \int_0^{\pi/4} \ln\sin\left(\frac{\pi}{4} - z\right) dz = I_1$$

$$I = \frac{\pi}{8}\ln 2$$

Note : $\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)t + a]\,dt$

6.3.9. *Compute the sum of two integrals*

$$I = \underbrace{\int_{-4}^{-5} e^{(x+5)^2}\,dx}_{I_1} + \underbrace{3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx}_{I_2}$$

Solution. $I_1 = (b-a)\int_0^1 f[(-5+4)t - 4]\,dt$

$$= -\int_0^1 e^{(-t-4+5)^2}\,dt$$

$$= -\int_0^1 e^{(t-1)^2}\,dt \qquad .....(i)$$

$$I_2 = 3\left(\frac{2}{3} - \frac{1}{3}\right)\int_0^1 e^{9\left(\frac{1}{3}t - \frac{2}{3} + \frac{1}{3}\right)^2}\,dt$$

$$= \int_0^1 e^{(t-1)^2}\,dt \qquad .....(ii)$$

From *(i)* and *(ii)*, we get $I_1 + I_2 = 0$

6.3.10. ***Prove that the integral***

$$\int_0^{\pi} \frac{\sin 2k x}{\sin x} dx$$

Equals to zero if k is an integer

Solution. Let
$$x = \pi - t$$
$$dx = -dt$$
$$I = \int_0^{\pi} \frac{\sin 2k x}{\sin x} dx$$
$$= -\int_0^{\pi} \frac{\sin 2k(\pi - t)}{\sin(\pi - t)} dt$$
$$= -\int_0^{\pi} \frac{\sin 2k t}{\sin t} dt$$
$$I = \int_0^{\pi} \frac{\sin 2k x}{\sin x} = -I$$
$$2I = 0$$
$$I = 0$$

6.3.11. ***Compute the integral***

$$I = \int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}}$$

Solution. $x = \sin t$ (The given function is non-monotonic)

$dx = \cos t\, dt$ $\qquad t_1 = \frac{\pi}{6}, t_2 = \frac{\pi}{3}$

Let us consider, the other values are

$$t_1 = \frac{5\pi}{6}, t_2 = \frac{2\pi}{3}$$

In both the cases, the variable $x = \sin t$ runs throughout the entire interval $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$, the function $\sin t$ being non-monotonic on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $\left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$.

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = \int_{\pi/6}^{\pi/3} \frac{\cos t\, dt}{\sin t \cdot \cos t}$$
$$= \int_{\pi/6}^{\pi/3} \frac{dt}{\sin t}$$
$$= \ln\left[\tan\frac{t}{2}\right]_{\pi/6}^{\pi/3}$$
$$= \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$$

Similarly, taking into consideration that cos t is negative on the interval $\left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$, we get

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = \int_{5\pi/6}^{2\pi/3} \frac{\cos t\, dt}{\sin t \cdot (-\cos t)}$$

$$= \int_{2\pi/3}^{5\pi/6} \frac{dt}{\sin t}$$

$$= \ln\left[\tan\frac{t}{2}\right]_{2\pi/3}^{5\pi/6} = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$$

Note : Do not take $t_1 = \frac{5\pi}{6}$, $t_2 = \frac{\pi}{3}$ as with t varying on the interval $\left[\frac{\pi}{3}, \frac{5\pi}{6}\right]$ the values of the function $x = \sin t$ lie beyond the limits of the interval $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$.

6.3.12. ***Prove that the function L (x) defined on the interval (0, ∞) by the integral*** $L(x) = \int_1^x \frac{dt}{t}$ ***possesses the following properties.***

Solution. As we know that $L(x_1 x_2) = L(x_1) + L(x_2)$

$$L\left(\frac{x_1}{x_2}\right) = L(x_1) - L(x_2)$$

$$L(x_1 x_2) = \int_1^{x_1 x_2} \frac{dt}{t}$$

$$(x_1 x_2) = \int_1^{x_1} \frac{dt}{t} + \int_{x_1}^{x_1 x_2} \frac{dt}{t}$$

$$t = x_1 z \qquad x_2 x_1 = x_1 z_1$$

$$dt = x_1 dz \qquad z_1 = x_2;\ \ z_2 = 1$$

$$L(x_1 x_2) = \int_1^{x_1} \frac{dt}{t} + \int_1^{x_2} \frac{dz}{z}$$

$$= L(x_1) + L(x_2)$$

Putting $x_1 x_2 = x_3$, $\qquad x_2 = \frac{x_3}{x_1}$

$$L(x_3) = L(x_1) + L\left(\frac{x_3}{x_1}\right)$$

i.e., $\qquad L\left(\frac{x_3}{x_1}\right) = L(x_3) - L(x_1)$

Similarly, $\qquad L\left(x^{m/n}\right) = \frac{m}{n} L(x) = mL(x^{1/n})$

$$L(x) = nL(x^{1/n})$$

6.3.13. *Transform the integral $\int_0^3 (x-2)^2\,dx$ by the substitution $(x-2)^2 = t$.*

Solution. $\int_0^3 (x-2)^2\,dx = \int_0^2 (x-2)^2\,dx + \int_2^3 (x-2)^2\,dx$

Integral is broken into parts because

$$x_1 = 2-\sqrt{t},\quad x_2 = 2+\sqrt{t}$$
$$x_1 < 2 \qquad x_2 > 2$$

So, $$\int_0^3 (x-2)^2\,dx = \underbrace{\int_0^2 (x-2)^2\,dx}_{I_1} + \underbrace{\int_2^3 (x-2)^2\,dx}_{I_2}$$

$$I_1 = \int_0^2 (x-2)^2\,dx$$
$$= -\int_4^0 \frac{t\,dt}{2\sqrt{t}}$$
$$I_1 = \frac{1}{2}\int_0^4 \sqrt{t}\,dt = \frac{8}{3}$$
$$I_2 = \int_2^3 (x-2)^2\,dx$$
$$= \int_0^1 \frac{t\,dt}{2\sqrt{t}}$$
$$I_2 = \frac{1}{2}\int_0^1 \sqrt{t}\,dt = \frac{1}{3}$$

Hence, $$I = \frac{8}{3}+\frac{1}{3} = 3$$

This can be verified by directly computing the initial integral as

$$\int_0^3 (x-2)^2\,dx = \left[\frac{(x-2)^3}{3}\right]_0^3 = 3$$

6.3.14. *Compute the integrals*

(a) $$I = \int_0^1 \frac{dx}{1+\sqrt{x}}$$

Solution. Let $$x = t^2$$
$$dx = 2t\,dt$$
$$I = \int_0^1 \frac{2t\,dt}{1+t}$$
$$= 2\int_0^1 \frac{t+1-1}{1+t}\,dt$$
$$I = 2\int_0^1 dt - 2\int_0^1\left(\frac{1}{1+t}\right)dt$$
$$= 2[t]_0^1 - 2\,[\log(1+t)]_0^1$$
$$= 2 - 2\ln 2$$

(b) $$I = \int_0^5 \frac{dx}{2x + \sqrt{3x+1}}$$

Solution. Let $3x + 1 = t^2 \qquad 3\,dx = 2t\,dt$

$$3x = t^2 - 1; \quad x = \frac{t^2-1}{3}$$

$$I = \frac{2}{3}\int_1^4 \frac{t\,dt}{2\left(\frac{t^2-1}{3}\right)+t}$$

$$I = \frac{2}{3}\int_1^4 \frac{3t\,dt}{2t^2-2+3t}$$

$$= \frac{2}{3}\int_1^4 \frac{3t\,dt}{2t^2+3t-2}$$

$$= \frac{2}{4}\left[\int_1^4 \frac{(4t+3-3)\,dt}{(2t^2+3t-2)}\right]$$

$$I = \frac{1}{2}\int_1^4 \frac{(4t+3)\,dt}{(2t^2+3t-2)} + \frac{1}{2}\int_1^4 \frac{-3\,dt}{(2t^2+3t-2)}$$

$$I = \frac{1}{2}[\ln 42] - \frac{3}{4}\int_1^4 \frac{dt}{\left(t^2+\frac{3}{2}t-1\right)}$$

$$= \frac{1}{2}[\ln 42] - \frac{3}{4}\int_1^4 \frac{dt}{\left(t+\frac{3}{4}\right)^2 - \frac{25}{16}}$$

$$I = \frac{1}{2}[\ln 42] - \frac{3}{4}\left[\ln\frac{\left(t+\frac{3}{4}-\frac{5}{4}\right)}{\left(t+\frac{3}{4}+\frac{5}{4}\right)}\right]_1^4$$

$$= \frac{1}{2}[\ln 42] - \frac{3}{4}\left[\ln\frac{\left(t+\frac{1}{2}\right)}{(t+2)}\right]_1^4$$

$$I = \frac{1}{2}[\ln 42] - \frac{3}{4}\left[\ln\frac{7}{12} - \ln\frac{1}{6}\right]$$

$$= \frac{1}{2}\ln 42 - \frac{3}{4}\ln\frac{7}{2}$$

$$I = \frac{1}{2}\left[\ln\frac{42}{(7/2)^{3/2}}\right]$$

(c) $$I = \int_{\pi/4}^{\pi/3} \frac{dx}{1-\sin x}$$

Solution. $$I = \int_{\pi/4}^{\pi/2} \frac{dx}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$= \int_{\pi/4}^{\pi/3} \frac{dx}{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}$$

$\because \sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\frac{x}{2}}{\left(\tan\frac{x}{2}-1\right)^2}dx$$

$$= 2\left[\frac{1}{\tan\frac{x}{2}-1}\right]_{\pi/4}^{\pi/3}$$

$\tan\frac{x}{2}-1 = t$

$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= \frac{2}{\tan\frac{\pi}{6}-1} - \frac{2}{\tan\frac{\pi}{6}-1}$$

$\sec^2\frac{x}{2}dx = 2dt$

$$= \frac{2}{\frac{1}{\sqrt{3}}-1} - \frac{2}{\tan\frac{\pi}{8}-1}$$

$$= 2\left[\frac{\sqrt{3}}{1-\sqrt{3}} - \frac{2}{\tan\frac{\pi}{8}-1}\right]$$

(d) $$I = \int_0^1 \sqrt{2x-x^2}\,dx$$

Solution. $$I = \int_0^1 \sqrt{1-(x-1)^2}\,dx$$

$$I = \left[\frac{1}{2}(x-1)\sqrt{2x-x^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^1$$

$$\left\{\int_0^a \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right\}$$

$$I = \frac{1}{2}\sin^{-1}(-1) = +\frac{\pi}{4}$$

(e) $$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x}\,dx$$

Solution. $$I = -\int_0^{\pi/4} \frac{(\sin x + \cos x)\,dx}{(\sin x - \cos x)^2 - 4}$$

Let, $\sin x - \cos x = t$

$(\cos x + \sin x)\,dx = dt$

$$I = -\int_{-1}^{0} \frac{dt}{t^2 - 2^2}$$

$$= -\left[\frac{1}{2\times 2}\ln\left|\frac{t-2}{t+2}\right|\right]_{-1}^{0}$$

$$= -\left[\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right|\right]_{-1}^{0} = \frac{1}{4}\ln 3$$

(f) $$I = \int_0^a x^2 \sqrt{\frac{a-x}{a+x}}\,dx$$

Solution. Let $x = a\cos t$

$dx = -a\sin t\,dt$

$$I = \int_{\pi/2}^{a} a^2\cos^2 t\sqrt{\frac{a(1-\cos t)}{a(1+\cos t)}}(-a\sin t)\,dt$$

$$= \int_0^{\pi/2} a^3\cos^2 t\cdot\sin t\cdot\tan\frac{t}{2}\,dt$$

$$= a^3\int_0^{\pi/2} \cos^2 t\cdot\sin t\cdot\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\,dt$$

$$= a^3\int_0^{\pi/2} \cos^2 t\cdot\sin t\cdot\frac{2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}\,dt$$

$$= a^3\int_0^{\pi/2} \cos^2 t\cdot\frac{\sin t\cdot\sin t\cdot}{1+\cos t}\,dt$$

$$= a^3\int_0^{\pi/2} \cos^2 t\cdot(1-\cos t)\,dt$$

$$= a^3\int_0^{\pi/2} \cos^2 t - a^3\int_0^{\pi/2} \cos^3 t\,dt$$

$$= a^3\left[\frac{\pi}{4} - \frac{2}{3}\right]$$

(g) $$I = \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

Solution. $$= \left[\frac{1}{2}(x-a)\cdot\sqrt{2ax - x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{(x-a)}{a}\right]_0^{2a}$$

$$I = \frac{\pi a^2}{4}$$

(h) $$I = \int_{-1}^{1} \frac{dx}{(1+x^2)^2}$$

Solution. $$= 2\int_{-1}^{1} \frac{dx}{(1+x^2)^2}$$ [$\because f(x)$ is an even function]

Applying Reduction formula

$$I_n = \int \frac{dx}{(x^2+a^2)^n}$$

$$I_{n+1} = \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\cdot\frac{1}{a^2}\cdot I_n$$

$$I_2 = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot I_1$$

Here, $a = 1, n = 2$

$$= 2\left[\frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x\right]_0^1$$

$$= 2\left[\frac{1}{4} + \frac{1}{2}\times\frac{\pi}{4} - 0\right] = \frac{1}{2} + \frac{\pi}{4}$$

6.3.15. ***Applying a suitable change of the variables, find the following definite integral :***

(a) $$I = \int_0^2 \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}}$$

Solution. Let $x + 1 = t^2$ $\quad x = 2;\ t = \sqrt{3}$

$dx = 2t\,dt$ $\quad x = 0;\ t = 1$

$$I = \int_1^{\sqrt{3}} \frac{2t\,dt}{t + t^3}$$

$$= \int_1^{\sqrt{3}} \frac{2t\,dt}{t(1+t^2)}$$

$$= 2\int_1^{\sqrt{3}} \frac{dt}{1+t^2}$$

$$= 2\left[\tan^{-1} t\right]_1^{\sqrt{3}}$$

$$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right]$$

$$= 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{6}$$

(b) $$I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$$

Solution. Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$x = 0;\quad t = 0$

$x = a;\quad t = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta + a\cos\theta}$$

$$I = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} \quad .....(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\sin\theta + \cos\theta} \quad .....(ii)$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

From *(i)* and *(ii)*, we get, $$2I = \int_0^{\pi/2} \frac{(\cos\theta + \sin\theta)\, d\theta}{(\cos\theta + \sin\theta)}$$

$$= \int_0^{\pi/2} d\theta$$

$$I = \frac{\pi}{4}$$

(c) $$I = \int_1^2 \frac{dx}{x + \left(1 + x^4\right)}$$

Solution. Let $x^4 = t$

$$4x^3\, dx = dt$$

$$dx = \frac{dt}{4x^3}$$

$$I = \frac{1}{4}\int_1^{16} \frac{1}{t(1+t)}$$

$$= \frac{1}{4}\int_1^{16}\left[\frac{dt}{t} - \frac{dt}{(1+t)}\right]$$

$$= \frac{1}{4}\left[\ln\frac{t}{t+1}\right]_1^{16}$$

$$= \frac{1}{4}\left[\ln\frac{16}{17} - \ln\frac{1}{2}\right] = \frac{1}{4}\ln\frac{32}{17}$$

(d) $I = \int_{\sqrt{(3a^2+b^2)/2}}^{\sqrt{(a^2+b^2)/2}} \frac{x\,dx}{(a^2-b^2)(b^2-x^2)}$

Solution. $I = \frac{\sin t\cos t\,(b^2-x^2)\,dt}{\sin t\cos t\,(b^2-x^2)}$

Let $x^2 = a^2\cos^2 t + b^2\sin^2 t$

$= \frac{a^2 + b^2\tan^2 t}{1+\tan^2 t}$

$I = \int_{\sqrt{(3a^2+b^2)/2}}^{\sqrt{(a^2+b^2)/2}}$

$x^2 - a^2 = b^2\sin^2 t - a^2\sin^2 t$

$= (b^2 - a^2)\sin^2 t$

$I = \int_{\sqrt{(3a^2+b^2)/2}}^{\sqrt{(a^2+b^2)/2}} \tan^{-1}\frac{x^2-a^2}{b^2-x^2}$

$b^2 - x^2 = b^2 - a^2\cos^2 t - b^2\sin^2 t$

$= b^2\cos^2 t - a^2\cos^2 t$

$= (b^2 - a^2)\cos^2 t$

$I = \frac{\pi}{12}$

$2x\,dx = (-2a^2\cos t\sin t + b^2\,2\sin t\cos t)\,dt$

$2x\,dx = 2\sin t\cos t\,[b^2 - a^2]$

6.3.16. ***Consider the integral*** $\int_{-2}^{2}\frac{dx}{4+x^2}$. ***It is easy to conclude that it is equal*** $\frac{\pi}{4}$.

Solution. $\int_{-2}^{2}\frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{\pi}{4}$

Substituting $x = 1/t$, and $dx = \frac{dt}{t^2}$, we get

$$\int_{-2}^{2}\frac{dx}{4+x^2} = -\int_{-1/2}^{1/2}\frac{dx}{\left(4+\frac{1}{t^2}\right)t^2} = -\int_{-1/2}^{1/2}\frac{dt}{4t^2+1} = -\frac{\pi}{4}$$

This result is obviously wrong as the integrand $\frac{1}{(4+x^2)} > 0$ the definite integral of this function can not be equal to a negative number $-\pi/4$.

Substitution of $x = \frac{1}{t}$ is not valid because at $t = 0$,

$\frac{1}{t}$ does not exist or

$$-2 < 0 < 2$$

6.3.17. ***Consider the integral*** $I = \int_0^{2\pi} \frac{dx}{5 - 2\cos x}$.

Solution. Making the substitution $\tan\frac{x}{2} = t$, we have

$$\int_0^{2\pi} \frac{dx}{5 - 2\cos x} = \int_0^0 \frac{2\,dt}{(1+t^2)\left(5 - 2\frac{1-t^2}{1+t^2}\right)} = 0$$

The result is obviously wrong as the integrand is positive and consequently, the integral of this function can not be equal to zero.

As $t = \tan\frac{x}{2}$ will not do that the function is discontinuous at $t = 0$.

6.3.18. ***Make sure that a formal change of the variable*** $t = \frac{x^2}{5}$ ***leads to the wrong result in the integral*** $\int_{-2}^{2} \sqrt[5]{x^2}\,dx$. ***Find the mistake and explain it.***

Solution. Since $x = \pm\sqrt{t^5}$ is double-valued to obtain the correct result. It is necessary to divide the initial interval of integration into two parts

$$\int_{-2}^{2} \sqrt[5]{x^2}\,dx = \int_{-2}^{0} \sqrt[5]{x^2}\,dx + \int_0^2 \sqrt[5]{x^2}\,dx$$

$$x = -\sqrt{t^5} \text{ in } -2 < x < 0$$

$$x = +\sqrt{t^5} \text{ in } 0 < x < 2$$

6.3.19. ***Is it possible to make the substitution*** $x = \sec t$ ***in the integral***

$$I = \int_0^1 \sqrt{x^2 + 1}\,dx$$

Solution. It is impossible as $\sec t \geq 1$ and interval of integration is $[0, 1]$.

6.3.20. ***Given the integral*** $\int_0^1 \sqrt{1 - x^2}\,dx$. ***Make the substitution*** $x = \sin t$. ***Is it possible to make the numbers*** π ***and*** $\frac{\pi}{2}$ ***as the limits for*** t.

Solution. Let us consider with

$$x = \sin t \qquad 1 = \sin t \qquad t = \frac{\pi}{2}$$

$$dx = \cos t\,dt \qquad 0 = \sin t \qquad t = 0 \text{ or } \pi$$

$$= \int_\pi^{\pi/2} \sqrt{1 - \sin^2 t} \cdot \cos t\,dt$$

$$= \int_\pi^{\pi/2} \cos^2 t\,dt$$

$$= \int_{\pi}^{\pi/2} \frac{1+\cos 2t}{2} dt$$

$$= \int_{\pi}^{\pi/2} \frac{1}{2} dt + \int_{\pi}^{\pi/2} \frac{\cos 2t}{2} dt$$

$$= \left[\frac{1}{2}\right]_{\pi}^{\pi/2} + \left[\frac{\sin 2t}{4}\right]_{\pi}^{\pi/2}$$

$$= \frac{\pi}{2} - \pi + \frac{1}{4}[\sin \pi - \sin 2\pi] = -\frac{\pi}{2}$$

6.3.21. ***Prove the equality***

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]\,dx$$

Solution. for any continuous function $f(x)$

$$\int_{-a}^{a} f(x)\,dx = \underset{I_1}{\int_{-a}^{0} f(x)\,dx} + \underset{I_2}{\int_{0}^{a} f(x)\,dx}$$

$$I = \int_{-a}^{a} f(x)\,dx$$

Let $x = -t$ $\qquad -a = -t_1 \qquad t_1 = a$

$dx = -dt \qquad 0 = t_2$

$$I = \int_{0}^{a} f(t)\,dt = \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$$

6.3.22. ***Transform the definite integral*** $\int_{0}^{2\pi} f(x) \cos x\,dx$ ***by the substitution*** $\sin x = t$.

Solution. Let $\sin x = t$

$\cos x\,dx = dt$

$$I = \int_{0}^{\pi/2} f(x)\cos x\,dx + \int_{\pi/2}^{3\pi/2} f(x)\cos x\,dx + \int_{3\pi/2}^{2\pi} f(-x)\cos x\,dx$$

$\sin x = t$

$x = \sin^{-1} t$

$0 \le x \le \frac{\pi}{2} \qquad 0 \le t \le 1$

$\frac{\pi}{2} \le x \le \frac{3\pi}{2} \qquad -1 \le t \le 1$

$\frac{3\pi}{2} \le x \le 2\pi \qquad 0 \le t \le 1$

$$I = \int_{0}^{1} f\left(\sin^{-1} t\right) dt + \int_{-1}^{1} f\left(\pi - \sin^{-1} t\right) dt + \int_{-1}^{0} f\left(2\pi + \sin^{-1} t\right) dt$$

$\downarrow \; x = \sin^{-1} t \qquad \downarrow \; x = \pi - \sin^{-1} t \qquad \downarrow \; x = 2\pi + \sin^{-1} t$

6.4 SIMPLIFICATION OF INTEGRALS BASED ON THE PROPERTIES OF SYMMETRY OF INTEGRANDS

1. If the function $f(x)$ is even on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

2. If the function $f(x)$ is odd on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

3. If the function $f(x)$ is periodic with period T, then

$$\int_{a}^{b} f(x)\,dx = \int_{a+nT}^{b+nT} f(x)\,dx$$

where n is an integer.

EXAMPLES

6.4.1. ***Compute the integral*** $\int_{-1}^{1} |x|\,dx$

Solution. $f(x)$ is an even function

$$\int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} |x|\,dx$$

$$= 2\int_{0}^{1} x\,dx = \left[x^2\right]_0^1 = 1$$

6.4.2. ***Compute the integral***

$$I = \int_{-7}^{7} \frac{x^4 \sin x}{x^6 + 2}\,dx$$

Solution. Since the integrand is odd, we conclude at once that the integral equals to zero.

6.4.3. **(a)** $\int_{-\pi}^{\pi} f(x) \cos nx\,dx$

(b) $\int_{-\pi}^{\pi} f(x) \sin nx\,dx$.

Solution.

(a) If $f(x)$ is an even function, then

$f(x) \cos nx$ will also be even function.

$$\int_{-\pi}^{\pi} f(x) \cos nx\,dx = 2\int_{0}^{\pi} f(x) \cos nx\,dx$$

Hence, $\int_{-\pi}^{\pi} f(x) \sin nx\,dx = 0$

(b) If $f(x)$ is an odd function, then

$f(x) \cos nx$ will be odd function.

$f(x) \sin nx$ will be even function.

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Hence,
$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\int_0^{\pi} f(x)\sin nx\,dx$$

6.4.4. *Compute the integral* $\displaystyle\int_{-5}^{5} \frac{x^5 \sin^2 x}{x^4 + 2x^2 + 1}\,dx$

Solution. $\displaystyle f(x) = \frac{x^5 \sin^2 x}{x^4 + 2x^2 + 1}$ is a odd function

Hence, $\displaystyle\int_{-5}^{5} f(x)\,dx = 0$

6.4.5. *Compute the integral* $\displaystyle\int_{\pi}^{5/4\pi} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\,dx$

Solution. This integrand is a periodic function with periodic π, as

$$f(x+\pi) = \frac{\sin 2(x+\pi)}{\cos^4(x+\pi) + \sin^4(x+\pi)}$$

$$= \frac{\sin 2x}{\cos^4 x + \sin^4 x} = f(x)$$

Therefore it is possible to subtract the number π from the upper and lower limits.

$$\int_{\pi}^{5/4\pi} \frac{\sin 2x\,dx}{\cos^4 x + \sin^4 x} = \int_0^{\pi/4} \frac{\sin 2x\,dx}{\cos^4 x + \sin^4 x}$$

$$= 2\int_0^{\pi/4} \frac{\tan x\,dx}{\cos^2 x\left(1 + \tan^4 x\right)}$$

$\tan x = t$

$\displaystyle dt = \frac{dx}{\cos^2 x}$

x	t
0	0
$\frac{\pi}{4}$	1

$$2\int_0^{\pi/4} \frac{\tan x\,dx}{\cos^2 x\left(1 + \tan^4 x\right)} = \int_0^1 \frac{2t\,dt}{1+t^4}$$

$$= \left[\tan^{-1} t^2\right]_0^1 = \frac{\pi}{4}$$

6.4.6. *Prove the equality*

$$\int_{-a}^{a} \cos x \cdot f\left(x^2\right) dx = 2\int_0^{a} \cos x \cdot f\left(x^2\right) dx$$

Solution. It is sufficient to show that the integrand is even

$$\cos(-x) f(-x)^2 = \cos x \cdot f(x^2)$$

$$\int_{-a}^{a} \cos x \cdot f\left(x^2\right) dx = \int_{-a}^{0} \cos x \cdot f\left(x^2\right) dx + \int_{0}^{a} \cos x \cdot f\left(x^2\right) dx$$

Let $x = -t$

$dx = -dt$

$$\int_{-a}^{a} \cos x \, f\left(x^2\right) dx = \int_{0}^{a} \cos t \cdot f\left(t^2\right) dt + \int_{0}^{a} \cos x \cdot f\left(x^2\right) dx$$

$$= 2\int_{0}^{a} \cos x \cdot f\left(x^2\right) dx$$

6.4.7. ***Compute the integral***

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7 + 3x^6 - 10x^5 - 7x^3 - 12x^2 + x + 1}{x^2 + 2} dx$$

Solution.
$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7 - 10x^5 - 7x^3 + x}{x^2 + 2} dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^2\left(x^4 - 4\right) + 1}{x^2 + 2} dx$$

$$= 0 + 2\int_{0}^{\sqrt{2}} \left[3\left(x^4 - 2x^2\right) + \frac{1}{x^2 + 1}\right] dx$$

$$= \left[\frac{6}{5}x^5 - 4x^3 + \frac{2}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$I = -\frac{16}{5}\sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

Hence, we expanded the given integral into the sum of two integrals so as to obtain an odd integrand in the first integral and an even integrand in the second.

6.4.8. ***Compute the integral***

$$\int_{-1/2}^{1/2} \cos x \cdot \ln\frac{1 + x}{1 - x} dx$$

Solution. $f(x) = \cos x$ is an even function.

$\ln\left(\frac{1 + x}{1 - x}\right)$ is a odd function.

Product of even and odd function is a odd function. Hence,

$$\int_{-1/2}^{1/2} \cos x \cdot \ln\frac{1 + x}{1 - x} dx = 0$$

6.4.9. ***Prove the validity of the following equalities.***

Solution.

(a) $\int_{-\pi/8}^{\pi/8} x^8 \sin^9 x \, dx = 0$

As $\sin^9 x$ is an odd function.

x^8 is an even function.

So, $x^8 \sin^9 x$ is a odd function.

(b) $\int_{-1/2}^{1/2} e^{\cos x}\, dx = 2\int_0^{1/2} e^{\cos x}\, dx$

cos x is even function.

So, $e^{\cos x}$ is also even function.

(c) $\int_{-\pi}^{\pi} \sin mx \cdot \cos nx\, dx = 0$ *(m and n are natural numbers)*

sin mx is an odd function

cos nx is an even function.

So, product sin mx · cos nx is an odd function.

(d) $\int_{-a}^{a} \sin x\, f(\cos x)\, dx = 0$

sin x is an odd function

$f(\cos x)$ is an even function.

So, product sin $x \cdot f(\cos x)$ is an odd function.

6.4.10. ***Prove the equality***

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Solution. Let

$$t = a+b-x \quad b_1 = a+b-b = a$$
$$x = a+b-t\,; \quad t_2 = a+b-a = b$$
$$dx = -dt$$
$$\int_a^b f(x) = -\int_a^b f(t)\,dt$$
$$= \int_a^b f(t)\,dt = \int_a^b f(a+b-x)\,dx$$

6.4.11. ***Prove the equality***

$$\int_0^t f(x)\cdot g(t-x)\,dx = \int_0^t g(x)\cdot f(t-x)\,dx$$

Solution. Applying the substitution $t - x = z$ in the right hand integral

$$\int_0^t f(x)\cdot g(t-x)\,dx = -\int_0^t g(t-z)\cdot f(z)\,dz$$
$$= \int_0^t f(z)\cdot g(t-z)\,dz$$
$$= \int_0^t f(x)\cdot g(t-x)\,dx$$

6.4.12. ***Prove the equality*** $\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,dx$ ***and applying the obtained result in computing the following integrals.***

$\int_0^{\pi/2} \cos^2 x\,dx$ ***and*** $\int_0^{\pi/2} \sin^2 x\,dx$

Solution. $\quad \mathrm{I} = \int_0^{\pi/2} \sin^m x$

$$I = \int_0^{\pi/2} \sin^m\left(\frac{\pi}{2} - x\right) dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/2} \cos^m x\,dx$$

$$I = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin^2 x\,dx$$

$$I + I = \int_0^{\pi/2} \left(\cos^2 x + \sin^2 x\right) dx$$

$$2I = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$$

Hence, $$I = \frac{\pi}{4}$$

6.4.13. *Prove the equality*

$$\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$$

Solution. Since $\int_0^{\pi} f(\sin x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx + \int_{\pi/2}^{\pi} f(\sin x)\,dx$

It is sufficient to prove that

$$\int_{\pi/2}^{\pi} f(\sin x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

In the left integral make the substitution

$$x = \pi - t \qquad \begin{array}{cc} x & t \\ \pi/2 & \pi/2 \\ \pi & 0 \end{array}$$

$$dx = -dt$$

$$\int_{\pi/2}^{\pi} f(\sin x)\,dx = -\int_{\pi/2}^{0} f[\sin(\pi - t)]\,dx$$

$$= \int_0^{\pi/2} f(\sin t)\,dt = \int_0^{\pi/2} f(\sin x)\,dx$$

6.4.14. *Prove the equality*

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

Solution. Let $$x = \pi - t \qquad \begin{array}{cc} x & t \\ 0 & \pi \\ \pi & 0 \end{array}$$

$$dx = -dt$$

Then, we get

$$\int_0^{\pi} x f(\sin x)\,dx = -\int_0^{\pi} (\pi - t) f[\sin(x - t)]\,dt$$

$$= \int_0^{\pi} \pi f(\sin t)\,dt - \int_0^{\pi} t f(\sin t)\,dt$$

$$= 2\int_0^{\pi} x f(\sin x)\,dx$$

$$= \pi \int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

6.4.15. *Using the equality*

$$\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} = \frac{1}{2} + \cos x + \cos 2x + \ldots.. + \cos nx$$

Prove that

$$\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)^x}{2\sin\frac{x}{2}}\,dx = \pi.$$

Solution.

$$\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx = \int_0^{\pi}\left[\frac{1}{2} + \cos x + \cos 2x + \ldots\ldots + \cos nx\right]dx$$

$$\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx = 2\left[\frac{1}{2}x + \sin x + \frac{\sin 2x}{2} + \ldots\ldots + \frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \pi$$

6.4.16. *Prove that if*

$$\phi(x) = \left(\frac{1}{2}\right)a_0 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + .. + a_n\cos nx + b_n\sin nx$$

then,

(a) $\int_0^{2\pi} \phi(x)\,dx = \pi a_0$

Solution.

$$\int_0^{2\pi} \phi(x)\,dx = \int_0^{2\pi}\left[\frac{1}{2}a_0 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \ldots\right]dx$$

$$= \left[\frac{1}{2}a_0 x + a_1\sin x - b_1\cos x + a_2\frac{\sin 2x}{2} + b_2\frac{\cos 2x}{2} + \ldots\right]_0^{2\pi}$$

$$= [\pi a_0]$$

(b) $\int_0^{2\pi} \phi(x) \cos kx \, dx = \pi a_k$

$$= \int_0^{2\pi} \left[\frac{1}{2} a_0 \cos kx + a_1 \cos x \cos kx + b_1 \sin x \cos kx + \ldots a_k \cos 2kx \cos kx + b_k \sin kx + \ldots \right] dx$$

$$= \left[\frac{1}{2} a_0 \frac{\sin kx}{k} + \frac{a_1}{2} \left[\frac{\sin(k+1)x}{k+1} + \frac{\sin(k-1)x}{k-1} \right]_0^{2\pi} + \ldots\ldots \frac{a_k}{2} \left[\frac{\sin 2k}{2} \right] + \frac{a_k}{2} x + \ldots\ldots \right]_0^{2\pi}$$

$= \pi \ a_k$

(c) $\int_0^{2\pi} \phi(x) \cdot \sin kx \, dx = \pi b_k \ (k = 1, 2, \ldots\ldots n)$

$$= \int_0^{2\pi} \left[\frac{1}{2} a_0 + a_1 \cos x + \ldots\ldots \right] \sin kx \, dx$$

$$= \int_0^{2\pi} \left[\frac{1}{2} a_0 \sin kx + a_1 \cos x \sin kx + \ldots\ldots + a_k \cos kx \sin kx + b_k \sin^2 kx + \ldots\ldots \right] dx$$

$$= \left[-\frac{1}{2} a_0 \frac{\cos kx}{k} + \ldots\ldots + \frac{b_k}{2} \left[x - \frac{\sin 2kx}{2k} \right] + \ldots\ldots \right]_0^{2\pi}$$

$= \pi \ b_k$

6.5 INTEGRATION BY PARTS, REDUCTION FORMULAE

If u and v are functions of x and have continuous derivatives, then

$$\int_a^b u(x) v'(x) \, dx = [u(x) v(x)]_a^b - \int_a^b v(x) u'(x) \, dx$$

or

$$\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du$$

EXAMPLES

6.5.1. *Compute the integral* $\int_0^1 x e^x \, dx$

Solution.
$$= x \int_0^1 e^x dx - \int_0^1 \left(\frac{d}{dx}(x) \int e^x dx \right) dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x - e^x \right]_0^1 = 1$$

6.5.2. *Compute the integral* $\int_1^e \ln^3 x \, dx$

Solution.
$$I = \left[\ln^3 x \int 1 \cdot dx - \int \left(\frac{d}{dx} (\ln^3 x) \int 1 \cdot dx \right) \right]_1^e dx$$

$$= \left[x\cdot \ln^3 x - \int \frac{3\ln^2 x}{x}\cdot x\,dx\right]_1^e$$

$$I = \left[x\cdot \ln^3 x - 3\int \ln^2 x\,dx\right]_1^e$$

$$I = e - 3\left[\ln^2 x\int 1\cdot dx - \int\left(\frac{d}{dx}(\ln^2 x)\int 1\cdot dx\right)\right]_1^e dx$$

$$= e - 3e + 3\int_1^e \frac{2\ln x}{x}\cdot x\,dx$$

$$I = -2e + 6\int_1^e \ln x\,dx$$

$$= -2e + 6\left[x\ln x - x\right]_1^e$$

$$I = -2e + 6$$

6.5.3. ***Compute the integral*** $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx$.

Solution. $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

x	t
0	0
$\frac{\pi^2}{4}$	$\frac{\pi}{2}$

Hence, $\int_0^{\pi^2/4} \sin\sqrt{x}\,dx = 2\int_0^{\pi/2} t\sin t\,dt$

$$= 2[-t\cos t\,dt]_0^{\pi/2} + \left[\int_0^{\pi/2}\cos t\,dt\right]$$

$$= 2$$

6.5.4. ***Compute the integral***

$$I = \int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}$$

Solution.
$$I = \left[\sin^{-1} x\right]_0^1 \int_0^1 \frac{dx}{\sqrt{1+x}} - \int_0^1\left(\frac{d}{dx}(\sin^{-1} x)\cdot\int\frac{dx}{\sqrt{1+x}}\right)dx$$

$$= \left[2\sqrt{1+x}\cdot\sin^{-1} x + 4\sqrt{1-x}\right]_0^1$$

$$I = \sqrt{2}\,\pi - 4$$

6.5.5. ***Compute the integral*** $\int_0^{\pi/2} x^2 \sin x\,dx$.

Solution.
$$I = \left[x^2(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{d}{dx}(x^2)(-\cos x)\,dx$$

$$= 0 + \int_0^{\pi/2} 2x \cos x \, dx$$

$$= 2\left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{d(x)}{dx} \int \cos x \, dx$$

$$= 2\left[x \sin x \cos x\right]_0^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} - 1\right] = (\pi - 2)$$

6.5.6. ***Compute the integral*** $I_n = \int_0^a (a^2 - x^2)^n \, dx$ ***where n is a natural number.***

Solution.

$$I_n = \int_0^a (a^2 - x^2)^{n-1} (a^2 - x^2) \, dx$$

$$= a^2 I_{n-1} - \underbrace{\int_0^a x (a^2 - x^2)^{n-1} \, dx}_{I_1}$$

Integrating I_1 by parts, we get

$$I_n = a^2 I_{n-1} + \left[\frac{1}{2n} x (a^2 - x^2)^n\right]_0^a - \frac{1}{2n} \int_0^a (a^2 - x^2)^n \, dx$$

$$= a^2 I_{n-1} - \frac{1}{2n} I_n$$

$$I_n = a^2 \cdot \frac{2n}{2n+1} I_{n-1}$$

This formula is valid at any real n other than 0 and $-\frac{1}{2}$. In particular, at natural n, taking into account that

$$I_0 = \int_0^a dx = a$$

We get
$$I_n = \frac{a^{2n+1}\, 2n (2n-2)(2n-4) \ldots\ldots 6 \cdot 4 \cdot 2}{(2n+1)(2n-1)(2n-3) \ldots\ldots 5 \cdot 3}$$

$$= a^{2n+1} \frac{(2n)!}{(2n+1)!}$$

Using the result of the preceding problem, we get the formula

$$1 - \frac{C_n^1}{3} + \frac{C_n^2}{5} - \ldots\ldots + \frac{(-1)^n C_n^2}{2n+1} = \frac{(2n)!}{(2n+1)!}$$

where C_n^k are binomial coefficients.

6.5.7. ***Consider the integral***

$$I_n = \int_0^1 (1 - x^2)^n \, dx = \frac{(2n)!}{(2n+1)!}$$

Solution. Expanding the integrand by the formula of the Newton binomial and integrating within the limits from 0 to 1, we get

$$I_n = \int_0^1 (1-x^2)^n dx$$

$$= \int_0^1 \left(1 - C_n^1 x^2 + C_n^2 x^4 + \ldots\ldots (-1)^n C_n^n x^{2n}\right) dx$$

$$= \left[x - \frac{C_n^1 x^3}{3} + \frac{C_n^2 x^5}{5} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1}\right]_0^1$$

$$I_n = 1 - \frac{C_n^1}{3} + \frac{C_n^2}{5} - \frac{C_n^3}{7} + \ldots\ldots + \frac{(-1)^n}{2n+1}$$

6.5.8. ***Compute the integral***

$$H_m = \int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \cos^m x\, dx, \text{ where } m \text{ a natural number.}$$

Solution. Let $\sin x = t,$
$\cos x\, dx = dt$

x	t
0	0
$\frac{\pi}{2}$	1

The integral reduces the second integral to the integral

$$H_m = \int_0^{\pi/2} \left(1-\sin^2 x\right)^{\frac{m-1}{2}} \cos x\, dx$$

$$= \int_0^1 \left(1-t^2\right)^{\frac{m-1}{2}} dt$$

considered $a = 1$ and $n = \frac{m-1}{2}$. Therefore, the reduction formula

$$H_m = \frac{m-1}{m} H_{m-2} \quad (m \neq 0, m \neq 1)$$

is valid here as

$$H_m = I_{\frac{m-1}{2}} = \frac{2 \cdot \frac{m-1}{2}}{2 \cdot \frac{m-1}{2} + 1} I_{\frac{m-1}{2} - 1}$$

$$= \frac{m-1}{m} I_{\frac{m-3}{2}} = \frac{m-1}{m} H_{m-2}$$

If m is an odd number, the obtained reduction formula becomes into H_m

$$H_1 = \int_0^{\pi/2} \cos x\, dx = 1$$

therefore, $$H_m = \frac{(m-1)!!}{m!!}$$

If m is an even number then the reduction formula transforms H_m into

$$H_0 = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

therefore,
$$H_m = \frac{(m-1)!!}{m!!}\cdot\frac{\pi}{2}.$$

6.5.9. ***Compute the integral***

$$I = \int_0^{\pi} x \sin^m x\, dx, \quad \textit{where m a natural number}$$

Solution. We know from earlier examples that

$$I = \int_0^{\pi} x \sin^m x\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \sin^m x\, dx$$

$$= \pi \int_0^{\pi/2} \sin^m x\, dx,$$

Taking consideration from previous example, we get

$$I = \int_0^{\pi} x \sin^m x\, dx = \begin{cases} \dfrac{\pi^2}{2}\cdot\dfrac{(m-1)!!}{m!!}, & \text{if } m \text{ is even,} \\ \pi\dfrac{(m-1)!!}{m!!}, & \text{if } m \text{ is odd.} \end{cases}$$