CHAPTER 7

APPLICATIONS OF THE DEFINITE INTEGRAL

7.1 COMPUTING THE LIMITS OF SUMS WITH THE AID OF DEFINITE INTEGRALS

It is often necessary to compute the limit of a sum when the number of summands increases unlimitedly. In some cases, such limits can be found with the aid of the definite integral if it is possible to transform the given sum into an integral sum.

For instance, considering the points $\frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}$ as points of division of the interval [0, 1] into n equal parts of length $\Delta x = \frac{1}{n}$, for each continuous function $f(x)$, we have

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+...+f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,dx$$

EXAMPLES

7.1.1. *Compute* $\lim_{n\to\infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+......+\sin\frac{(n-1)\pi}{n}\right]$

Solution. The numbers in the brackets represent the values of the function $f(x) = \sin x$ at the points

$$x_1 = \frac{\pi}{n}, x_2 = \frac{2\pi}{n} + ...x_{n+1} = \frac{(n-1)\pi}{n}$$

Subdividing the interval $[0, \pi]$ into n equal parts of length $\Delta x = \frac{\pi}{n}$.

$\therefore$ If we add the summand $\sin\frac{\pi n}{n} = 0$ to our sum, the latter will be the integral sum for the function $f(x) = \sin x$ on the interval $[0, \pi]$.

By definition the limit of such an integral sum as $n \to \infty$ is definite integral of the function $f(x) = \sin x$ from 0 to π.

$$\lim_{n\to\infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+......+\sin\frac{n\pi}{n}\right] = \int_0^{\pi} \sin x\,dx = 2$$

7.1.2. *Compute the limit*

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^n}}+\ldots\ldots\frac{1}{\sqrt{4n^2-n^2}}\right).$$

Solution. $$\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{4-1/n^2}}+\frac{1}{\sqrt{4-2^2/n^2}}+\ldots\ldots\frac{1}{\sqrt{4-n^2/n^2}}\right)$$

The obtained sum is the integral sum for the function

$$f(x)=\frac{1}{\sqrt{(4-x^2)}}$$

on the interval [0, 1] subdivided into n equal parts

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots\ldots\frac{1}{\sqrt{4n^2-n^2}}\right]$$

$$=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\left[\sin^{-1}\frac{x}{2}\right]_0^1=\frac{\pi}{6}$$

7.1.3. *Compute* $$\lim_{n\to\infty}\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\ldots\ldots\sqrt{\frac{n}{n+3(n-1)}}\right].$$

Solution. $$\lim_{n\to\infty}\frac{3}{n}\left[1+\sqrt{\frac{1}{1+\frac{3}{n}}}+\sqrt{\frac{1}{1+\frac{6}{n}}}+\ldots\ldots\sqrt{\frac{1}{1+\frac{3(n-1)}{n}}}\right]$$

The obtained sum is the integral sum for the function $f(x)=\sqrt{\left[\frac{1}{1+x}\right]}$ on the interval [0, 3].

$$\therefore\quad \int_0^3\sqrt{\frac{1}{1+x}}\,dx=\int_0^3(1+x)^{-1/2}\,dx$$

$$=\left[2\sqrt{1+x}\right]_0^3$$

$$=2$$

7.1.4. *Using the definite integral, compute the following limits :*

(a) $$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots\frac{1}{n+n}\right).$$

Solution. $$\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n+1/n}+\frac{1}{n+2/n}+\ldots\ldots\frac{1}{n+n/n}\right)$$

The obtained sum is the integral sum for the function

$$f(x)=\frac{1}{1+x}$$

$$=\int_0^1\frac{dx}{1+x}$$

$$f(x)=\left[\ln(1+x)\right]_0^1$$

$$=\ln 2$$

(b) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots\ldots\sqrt{1+\frac{n}{n}}\right)$.

Solution.

$$f(x) = \sqrt{1+x}$$

$$= \int_0^1 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(x+1)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left(2^{3/2}-1\right)$$

(c) $\lim\limits_{n\to\infty} \frac{1+\sqrt[3]{2}+\sqrt[3]{3}+\ldots\ldots\sqrt[3]{n}}{\sqrt[3]{n^4}}$

Solution.

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^{1/3}}+\left(\frac{2}{n}\right)^{1/3}+\left(\frac{3}{n}\right)^{1/3}+\ldots\ldots\left(\frac{n}{n}\right)^{1/3}\right]$$

The obtained sum is the integral sum for the function

$$f(x) = \left(\frac{1}{x}\right)^{-1/3}$$

$$= x^{1/3} = \int_0^1 x^{+1/3}\,dx$$

$$= \left[\frac{3x^{4/3}}{4}\right]_0^1 = \frac{3}{4}$$

(d) $\lim\limits_{n\to\infty} \frac{\pi}{2n}\left(1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots\ldots\cos\frac{(n-1)\pi}{2n}\right)$

Solution. $\lim\limits_{n\to\infty} \frac{1}{n}\left[\frac{\pi}{2}\left\{1+\cos\frac{\pi}{2}\times\frac{1}{n}+\cos\frac{\pi}{2}\times\frac{2}{n}+\ldots\ldots\cos\frac{\pi}{2}\left(\frac{n-1}{n}\right)\right\}\right]$

The obtained sum is the integral sum for the function

$$f(x) = \cos x \qquad [0, \pi/2]$$

$$= \int_0^{\pi/2} \cos x\,dx$$

$$= [\sin x]_0^{\pi/2}$$

$$= 1$$

(e) $\lim\limits_{n\to\infty} n\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots\ldots\frac{1}{(2n)^2}\right]$

Solution.

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\left(1+\frac{1}{n}\right)^2}+\frac{1}{\left(1+\frac{2}{n}\right)^2}+\ldots\ldots\frac{1}{\left(1+\frac{n}{n}\right)^2}\right]$$

The obtained sum is the integral sum for the function

$$f(x) = \frac{1}{(1+x)^2}$$

$$= \int_0^1 \frac{dx}{(1+x)^2}$$

$$= \left[-\frac{1}{(1+x)}\right]_0^1$$

$$= -\frac{1}{2}+1 = \frac{1}{2}$$

7.1.5. *Compute the limit* $A = \lim\limits_{n\to\infty} \frac{(n!)^{1/n}}{n}$

Solution. Taking log both sides of above equation

$$\log A = \lim_{n\to\infty} \log\left(\frac{(n!)^{1/n}}{n}\right)$$

$$\log A = \lim_{n\to\infty} \frac{1}{n}\log(n!) - \log n$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \ldots\ldots \ln\left(\frac{n}{n}\right)\right]$$

The expression in brackets is the integral sum for the integral

$$\int_0^1 \ln x\, dx = [(x\ln x - x)]_0^1 = -1$$

$$\ln A = -1$$

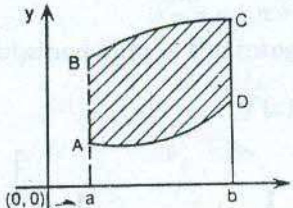
$$A = e^{-1} = \frac{1}{e}$$

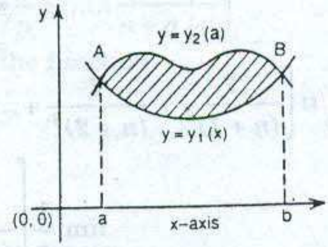
7.2 COMPUTING AREAS IN RECTANGULAR COORDINATES

If a plane figure is bounded by the straight lines $x = a$ and $x = b$ $(a < b)$ and the curves $y = y_1(x), y = y_2(x)$ provided $y_1(x) \le y_2(x)$ $(a \le x \le b)$, then its area is computed by the formula

$$S = \int_a^b [y_2(x) - y_1(x)]\, dx$$

In certain cases, the left boundary $x = a$ (or the right boundary $x = b$) can degenerate into a point of intersection of the curves $y = y_1(x)$ and $y = y_2(x)$. Then a and b are found as the abscissas of the points of intersection of the indicated curves.





EXAMPLES

7.2.1. ***Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.***

(i) $x = 0$

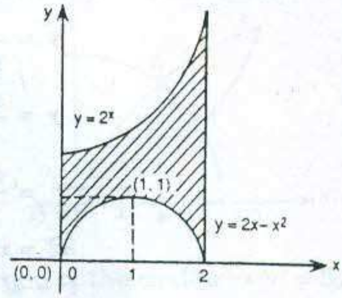
(ii) $x = 2$

(iii) $y = 2^x$

(iv) $y = 2x - x^2$

$$= -[(x-1)^2 - 1]$$

$(y-1) = -(x-2)^2$ [parabola (inverted) vertex (1, 1)]



$$S = \int_a^b \left[2^x - \left(2x - x^2\right)\right] dx$$

$$= \left[\frac{2^x}{\ln 2}\right]_0^2 - \left[\left(x^2 - \frac{x^3}{3}\right)\right]_0^2$$

$$= \frac{3}{\ln 2} - \frac{4}{3}$$

7.2.2. ***Compute the area of the figure bounded by the parabolas $x = -2y^2$ and $x = 1 - 3y^2$.***

Solution. $x = -2y^2 \qquad x = 1 - 3y^2$

$$y^2 = -\frac{x}{2}$$ [vertex (0, 0) symmetric about x-axis]

$$3y^2 = 1 - x$$

$$y^2 = \frac{1-x}{3}$$

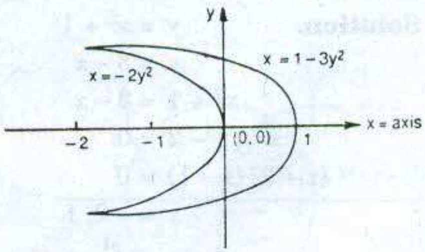
$$y^2 = -\left(\frac{x-1}{3}\right)$$

$$y^2 = -\frac{1}{3}(x-1)$$ [vertex (1, 0) symmetric about x-axis]

Solving system of equations

$$y^2 = -\frac{x}{2} \qquad \text{... (i)}$$

$$y^2 = -\left(\frac{x-1}{3}\right) \qquad \text{... (ii)}$$



$$\left(\frac{x-1}{3}\right) = \frac{x}{2}$$

$$2x - 2 = 3x$$

$$x = -2\,;\ y_1 = +1\,;\ y_2 = -1$$

Since, $1 - 3y^2 \geq -2y^2$ for $-1 \leq y \leq 1$

$$S = \int_{-1}^{1} \left[\left(1 - 3y^2\right) - \left(-2y^2\right)\right] dy$$

$$= 2\left(y - \frac{y^3}{3}\right)_0^1 = \frac{4}{3}$$

7.2.3. ***Find the area of the figure contained the parabola $x^2 = 4y$ and the width of Agnesi $y = 8/(x^2 + 4)$***

Solution. $x^2 = 4y$ [Parabola vertex (0, 0) symmetric about y-axis].

$$y = \frac{8}{x^2+4} \quad ...(i)$$

$$y = \frac{x^2}{4} \quad ...(ii)$$

$$\frac{8}{x^2+4} = \frac{x^2}{4}$$

$$32 = x^4 + 4x^2 \qquad \text{Let } x^2 = t$$

$$t^2 + 4t - 32 = 0$$

$$t^2 + 8t - 4t - 32 = 0$$

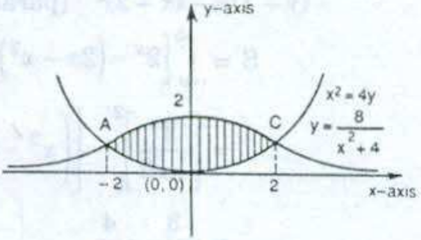
$$t(t+8) - 4(t+8) = 0$$

$$(t-4)(t+8) = 0$$

$$t = 4, -8$$

$$x^2 = 4 \qquad x^2 = -8$$

$$x = \pm 2 \qquad x = \pm i\sqrt{8} \text{ [imaginary]}$$



$$\frac{8}{x^2+4} \geq \frac{x^2}{4} \qquad [-2, 2]$$

$$S = \int_{-2}^{2}\left(\frac{8}{x^2+4} - \frac{x^2}{4}\right)dx$$

$$= 4\left[\tan^{-1}\frac{x}{2} - \frac{x^3}{12}\right]_{-2}^{2}$$

$$S = 2\pi - \frac{4}{3}$$

7.2.4. ***Find the area of the figure bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.***

Solution.
$$y = x^2 + 1$$

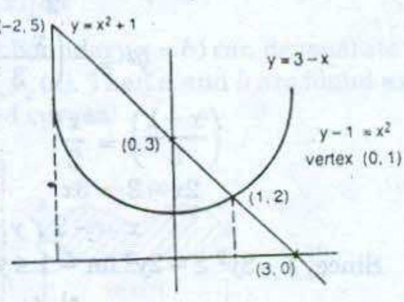
$$y = 3 - x$$

$$x^2 + 1 = 3 - x$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2, 1$$



$$S = \int_{-2}^{1}[(3-x) - (x^2+1)]\,dx$$

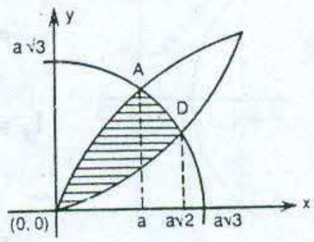
$$= \int_{-2}^{1}(2 - x - x^2)\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \frac{27}{6} \text{ sq. units.}$$

7.2.5. ***Compute the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3a^2$ and is bounded by the parabolas $x^2 = 2ay$, $y^2 = 2ax$ $(a > 0)$***

Solution.
$$x^2 + y^2 = 3a^2 \quad ...(i)$$
$$x^2 = 2ay \quad ...(ii)$$
$$y^2 = 2ax \quad ...(iii)$$



Solving *(i)* and *(ii)*, we get
$$2ay + y^2 = 3a^2$$
$$x^2 + 2ax - 3a^2 = 0$$

Hence, we get the only positive root $x_A = a$.

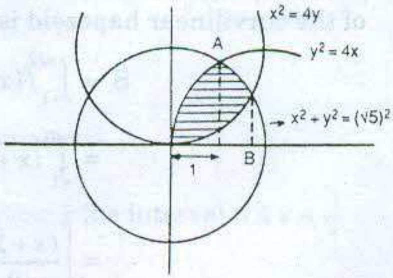
Analogously, we find the abscissa of the point D of intersection of the circle $x^2 + y^2 = 3a^2$ and the parabola $x^2 = 2ay$, $x_0 = a\sqrt{2}$
$$S = \int_0^{a\sqrt{2}} [y_2(x) - y_1(x)]\, dx$$

Hence,
$$y_1(x) = \frac{x^2}{2a}$$
$$y_2(x) = \begin{cases} \sqrt{2ax} & \text{for } 0 \le x \le a \\ 3a - x & \text{for } a < x \le a\sqrt{2} \end{cases}$$

By Additive property of integral
$$S = \int_0^a \left(\sqrt{2ax} - \frac{x^2}{2a}\right) dx + \int_0^{a\sqrt{2}} \left(\sqrt{3a^2 - x^2} - \frac{x^3}{2a}\right) dx$$
$$= \left[\sqrt{2a} \cdot \frac{2}{3} \cdot x^{3/2} - \frac{x^3}{6a}\right]_0^a + \left[\frac{x}{2} \cdot \sqrt{3a^2 - x^2} + \frac{3a^3}{2} \sin^{-1} \frac{x}{a\sqrt{3}} - \frac{x^3}{6a}\right]_a^{a\sqrt{2}}$$
$$= \left(\frac{\sqrt{2}}{3} + \frac{3}{2} \sin^{-1} \frac{1}{3}\right) a^2$$

7.2.6. ***Compute the area of the figure lying in the first quadrant and bounded by the curves $y^2 = 4x$, $x^2 = 4y$ and $x^2 + y^2 = 5$.***

Solution.



$$\left.\begin{aligned} y^2 &= 4x \\ x^2 + y^2 &= 5 \end{aligned}\right] x = 1, -5$$
$$\left.\begin{aligned} x^2 &= 4y \\ x^2 + y^2 &= 5 \end{aligned}\right] x = 2, -2$$
$$I_1 = \int_0^1 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$
$$= \left[2x^{3/2} \cdot \frac{2}{3} - \frac{x^3}{12}\right]_0^1$$
$$= \frac{4}{3} - \frac{1}{12}$$

$$= \frac{15}{12}$$

$$= \frac{5}{4}$$

$$I_2 = \int_1^2 \left(\sqrt{5 - x^2} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{1}{2} x \cdot \sqrt{5 - x^2} + \frac{1}{2} \times 5 \cdot \sin^{-1} \frac{x}{\sqrt{5}} - \frac{x^3}{12} \right]_1^2$$

$$= 1 + \frac{5}{2} \sin^{-1} \frac{2}{\sqrt{5}} - \frac{8}{12} - 1 - \frac{5}{2} \sin^{-1} \frac{1}{\sqrt{5}} + \frac{1}{12}$$

$$I_1 + I_2 = I$$

$$= \frac{4}{3} + \frac{5}{2} \sin^{-1} \left[\frac{1}{\sqrt{5}} \times \sqrt{1 - \left(\frac{2}{\sqrt{5}} \right)^2} - \frac{2}{\sqrt{5}} \cdot \sqrt{1 - \left(\frac{1}{\sqrt{5}} \right)^2} \right]$$

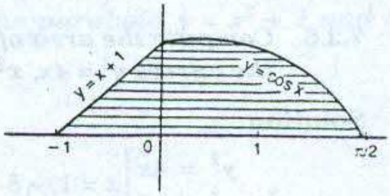
$$= \frac{4}{3} + \frac{5}{2} \sin^{-1} \left[\frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} \times \frac{2}{\sqrt{5}} \right] - \frac{8}{12}$$

$$= \frac{4}{3} + \frac{5}{2} \sin^{-1} \frac{3}{5} - \frac{2}{3}$$

$$= \frac{2}{3} + \frac{5}{2} \sin^{-1} \left(\frac{3}{5} \right)$$

7.2.7. ***Compute the area of the figure bounded by the lines $y = x + 1$, $y = \cos x$ and x-axis.***

Solution. $y = f(x) = \begin{cases} x + 1 & \text{if } -1 \le x \le 0 \\ \cos x & \text{if } 0 \le x \le \pi/2 \end{cases}$



$f(x)$ is continuous on the interval $[-1, \pi/2]$. The area of the curvilinear hapezoid is equal to

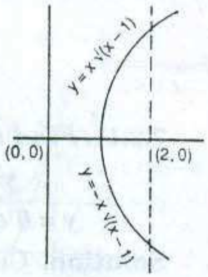
$$S = \int_{-1}^{\pi/2} f(x)\, dx$$

$$= \int_{1}^{0} (x + 1)\, dx + \int_0^{\pi/2} \cos x\, dx$$

$$= \left[\frac{(x+1)^2}{2} \right]_{-1}^{0} + [\sin x]_0^{\pi/2}$$

$$= \frac{3}{2}$$

7.2.8. ***Find the area of the segment of the curve $y^2 = x^3 - x^2$ if the line $x = 2$ is the chord determining the segment.***



Solution. $y^2 = x^2(x-1)$

$x^2(x-1) \ge 0$

So either $x \ge 0$ or $x \ge 1$.

Domain of the above function $[1, \infty]$

$$S = \int_1^2 [x\sqrt{x-1} - (-x\sqrt{x-1})]\,dx$$

$$= 2\int_1^2 x\sqrt{x-1}\,dx$$

$x - 1 = t^2$

x	t
1	0
2	1

$dx = 2t\,dt$

$$S = 4\int_1^2 (t^2+1)t^2\,dt$$

$$= 4\left[\frac{t^5}{5} + \frac{t^3}{3}\right]_0^1$$

$$S = \frac{32}{15}$$

7.2.9. ***Determine the area of the figure bounded by two branches of the curve $(y - x)^2 = x^3$ and the straight line $x = 1$.***

Solution.

It is implicit function of x $\qquad x \ge 0$

$y = x + x\sqrt{x}$ $\qquad x \ge 0 \quad x + x\sqrt{x} \ge x - x\sqrt{x}$

$\therefore$

$$S = \int_0^1 (x + x\sqrt{x} - x + x\sqrt{x})\,dx$$

$$= 2\int_0^1 x\sqrt{x}\,dx$$

$$= \frac{4}{5}\left[x^{5/2}\right]_0^1$$

$$= \frac{4}{5}$$

7.2.10. ***Compute the area enclosed by the loop of the curve***

$$y^2 = x(x-1)^2$$

Solution. The domain of definition of the implicit function y the interval $0 \le x < \infty$

$$y = y_1(x) = \sqrt{x}\,|x - 1|$$

$$= \begin{cases} \sqrt{x}(1-x), & 0 \le x \le 1 \\ \sqrt{x}(x-1), & x > 1 \end{cases}$$

The loop is formed by the curves $y = \sqrt{x}(1-x)$ and $y = -\sqrt{x}(1-x)$, $0 \le x \le 1$.

The area enclosed being $\qquad S = 2\int_0^1 \sqrt{x}(1-x)\,dx$

$$= 2\left[x^{3/2}\cdot\frac{2}{3} - x^{5/2}\cdot\frac{2}{5}\right]_0^1$$

$$= \frac{8}{15}$$

7.2.11. ***Find the area enclosed by the loop of the curve***

$$y^2 = (x-1)(x-2)^2$$

y = 0 at x = 1, 2

Solution. Curve is symmetric about x-axis forming a loop between $x = 1$ and $x = 2$.

$$S = 2\int_1^2 (x-2)\sqrt{x-1}\,dx$$

Let $x - 1 = t^2$ x t

$dx = 2t\,dt$ 1   0

$x - 2 = (t^2 - 1)$ 2   1

$$S = 2\int_0^1 (t^2-1)\cdot t\cdot 2t\,dt$$

$$= 4\int_0^1 (t^2-1)\cdot t^2\,dt$$

$$= 4\left[\frac{t^5}{5} - \frac{t^3}{3}\right]_0^1$$

$$= 4\left[\frac{1}{5} - \frac{1}{3}\right]$$

$$= -\frac{8}{3}$$

But area is always positive so $S = \frac{8}{3}$ units.

7.2.12. ***Find the area of the figure bounded by the parabola $y = -x^2 - 2x + 3$ the line tangent to it at the point M (2, –5) and y-axis.***

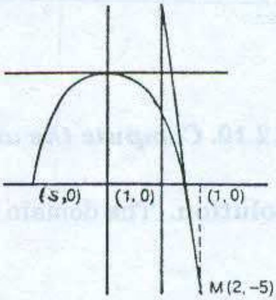
Solution. $y = -[(x+1)^2 - 2]$

$$y - 2 = -(x+1)^2$$

Equation of a tangent M (2, – 5)

$$y + 5 = -6(x-2)$$

or $$y = 7 - 6x$$



$$S = \int_0^2 [7 - 6x - (-x^2 - 2x + 3)]\,dx$$

$$= \int_0^2 (x^2 - 4x + 4)\,dx$$

$$= \frac{8}{3}$$

7.2.13. Find the area bounded by the parabola $y = x^2 - 2x + 2$ the line tangent to it at the point M (3, 5) and the axis of the ordinates.

Solution. $y = (x-1)^2 + 1$

$$y - 1 = (x-1)^2$$

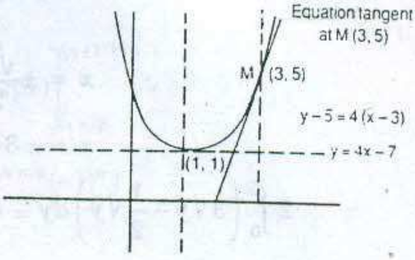
$$S = \int_0^3 [(x-1)^2 + 1 - (4x-7)]\,dx$$

$$= \left[\frac{(x-1)^3}{3} - \frac{4x^2}{2} + 8x\right]_0^3$$

$$= \frac{2^3}{3} - 2.9 + 8.3 + \frac{1}{3} - 0$$

$$= 3 - 18 + 124$$

$$= 9$$

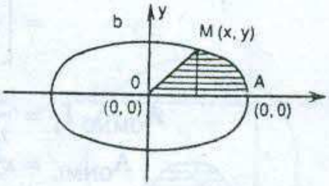


7.2.14. We take on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b$$

a point M (x, y) lying in the first quadrant.

Solution. From figure, $S_{OMAO} = S_{\Delta OMB} + S_{MABM}$



$$S_{\Delta OMB} = \frac{xy}{2}$$

$$= \frac{b}{2a} x \cdot \sqrt{a^2 - x^2}$$

$$S_{MABM} = \int_x^a y\,dx$$

$$= \int_x^a \frac{b}{2a}\sqrt{a^2 - t^2}\,dt$$

$$= \frac{b}{2a}\left(t.\sqrt{a^2 - t^2} + a^2 \sin^{-1}\frac{t}{a}\right)_x^a$$

$$= \frac{b}{2a}\left[-x \cdot \sqrt{a^2 - x^2} + a^2\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right)\right]$$

Since $\quad \frac{\pi}{2} - \sin^{-1}\frac{x}{a} = \cos^{-1}\frac{x}{a}$

$$S_{MABM} = \frac{b}{2a}\left[-x\sqrt{a^2 - x^2} + a^2 \cos^{-1}\frac{x}{a}\right]$$

$$S_{OMAO} = S_{\Delta OMB} + S_{MABM}$$

$$= \frac{ab}{2}\cos^{-1}\frac{x}{a}$$

At $x = 0$, the sector becomes quarter of the ellipse

$$\frac{1}{4} S_{ellipse} = \frac{ab}{2}\cos^{-1} 0$$

$$= \frac{ab}{4} \cdot \pi$$

So, $S = \Delta ab$ at $a = b$ it becomes circle

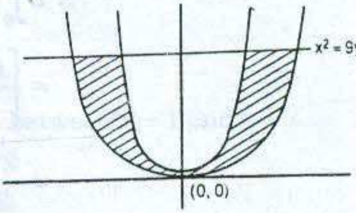
$$S_{circle} = \pi a^2.$$

7.2.15. ***Find the area bounded by the parabolas $y = 4x^2$, $y = x^2/9$ and the straight line $y = 2$.***

Solution. Both curves are symmetric about y-axis. It is advisable to integrate about y-axis.

$$x = \pm\frac{\sqrt{y}}{2}$$

$$x = \pm 3\sqrt{5}$$



$$2\int_0^2 \left(3\sqrt{y} - \frac{1}{2}\sqrt{y}\right) dy = 5\int_0^2 \sqrt{y}\, dy$$

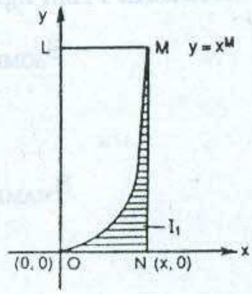
$$= \frac{20\sqrt{2}}{3}$$

7.2.16. ***From an arbitrary point M (x, y) of the curve $y = x^m$ (m > 0) perpendiculars MN and ML (x > 0) are dropped on to the co-ordinate axes. What part of the area of the rectangle ONML does the area ONMO constitute ?***

Solution.

$$I_1 = \int_0^{x_1} x^m\, dx$$

$$= \left[\frac{x^{m+1}}{m+1}\right]_0^{x_1}$$



$$A_{OMNO}\ I_1 = \frac{x_1^{m+1}}{m+1}$$

$$A_{ONML} = xy = x\, x^m = x^{m+1}$$

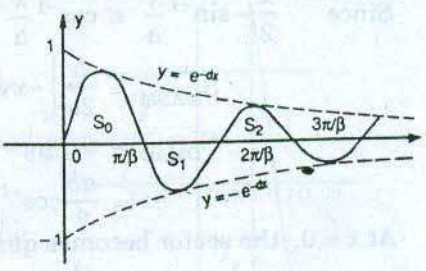
$$A_{OMNO} : A_{ONML} = \frac{x^{m+1}}{m+1} : x^{m+1}$$

$$= \frac{1}{m+1}$$

7.2.17. ***Prove that the areas S_0, S_1, S_2, S_3.... bounded by the x-axis and half-waves of the curve $y = e^{-\alpha x} \sin \beta x$, $x \geq 0$ form a geometric progression with common ratio $q = e^{-\alpha\pi}$.***

Solution. The curve intersects the positive semi-axis OX at the points, where $\sin bx = 0$.

Hence, $x_n = \frac{n\pi}{\beta}$, $n = 0, 1, 2$ ....



The function $y = e^{-\alpha x} \sin \beta x$ is positive in the intervals (x_{2k}, x_{2k+1}) and negative in (x_{2k+1}, x_{2k+2}), the sign of the function in the interval (x_n, x_{n+1}) coincides with that of the number $(-1)^n$.

$$\therefore \quad S_n = \int_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} |y|\, dx$$

$$= (-1)^n \int_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} e^{-\alpha x} \cdot \sin \beta x\, dx$$

But the indefinite integral is equal to

$$\int e^{-\alpha x}\cdot \sin\beta x\, dx = -\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha \sin \beta x + \beta\cos\beta x) + c$$

Consequently, $$S_n = (-1)^{n+1}\left[\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x+\beta\cos\beta x)\right]_{n\pi/\beta}^{(n+1)\pi/\beta} + c$$

$$= \frac{(-1)^{n+1}}{\alpha^2+\beta^2}[e^{-\alpha(n+1)\pi/\beta}\beta(-1)^{n+1} - e^{\alpha n\pi/\beta}(-1)^n]$$

$$= \frac{\beta}{\alpha^2+\beta^2}e^{-\alpha n\pi/\beta}(1+e^{\alpha\pi/\beta})$$

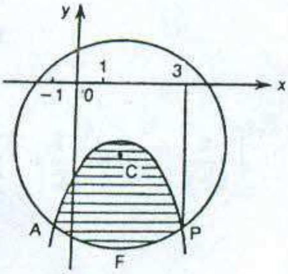
Hence, $$q = \frac{S_{n+1}}{S_n}$$

$$= \frac{e^{-\alpha(n+1)\pi/\beta}}{e^{-\alpha n\pi/\beta}}$$

$$= e^{-\alpha\pi/\beta}$$

7.2.18. ***Find the areas enclosed between the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3}$.***

Solution.



We know that equation of circle may be written as

$(x-1)^2 + (y+2)^2 = 16$ with centre $(1, -2)$ and radius $= 4$

Similarly, the equation of parabola may also be written as

$$y = -(x-1)^2 - \sqrt{3} + 2$$

The axis of parabola coincides with the straight line $x = 1$ and its vertex lies at the point $i\ (1, 2, -2\sqrt{3})$.

$$\text{Area of ABDFA} = \int_{x_A}^{x_D}(y_{par} - y_{circle})\,dx$$

Where x_A and x_D are determined from the system of equations

$$\begin{cases}(x-1)^2+(y+2)^2 = 16\\ y+2 = -(x-1)^2 - 2\sqrt{3}+4\end{cases}$$

Hence $x_A = -1, x_D = 3$

$$S_{ABDFA} = \int_{-1}^{3}[(-x^2+2x+1-2\sqrt{3}) + (2+\sqrt{16-(x-1)^2})]\,dx$$

$$= \left[-\frac{x^3}{3} + x^2 + (3-2\sqrt{3})x + \frac{(x-1)}{2}\sqrt{16-(x-1)^2} + \frac{16}{2}\sin^{-1}\frac{(x-1)}{4}\right]_1^3$$

$$= \frac{32}{3} + 8\sqrt{3} + 2\sqrt{12} + 16\sin^{-1}\frac{1}{2}$$

$$= \frac{32}{3} - 4\sqrt{3} + \frac{8}{3}\pi$$

7.2.19. ***Compute the area bounded by the curves $y = (x-4)^2$; $y = 16 - x^2$ and the x-axis.***

Solution. The equation of curve are $y = (x-4)^2$ and $y = 16 - x^2$

$$= -(x^2 - 16)$$

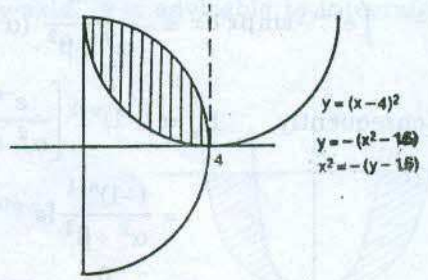
$$x^2 = -(y - 16)$$

$$I = \int_0^4 [(16 - x^2) - (x-4)^4]\, dx$$

$$I = \left[16x - \frac{x^3}{3} - \frac{(x-4)^3}{3}\right]_0^4$$

$$= 16 \times 4 - \frac{64}{3} - 0 - \frac{64}{3}$$

$$= \frac{64}{3} \text{ sq. units}$$



7.2.20. *Compute the area enclosed between the parabolas.*

Solution. $x = y^2 \quad x = \frac{3}{4}y^2 + 1$

$x = y^2$ – vertex (0, 0) symmetric about x-axis

$\frac{4}{3}(x-1)' = y^2$ – vertex (1, 0) symmetric about x-axis

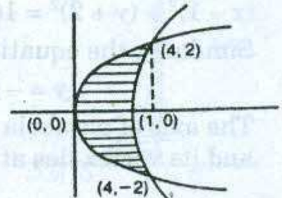
$$y^2 = x$$

$$y^2 = \frac{4}{3}(x-1)$$

$$x = \frac{4}{3}x - \frac{4}{3}$$

$$\frac{4}{3} = \frac{x}{3}$$

$$x = 4$$



$$I = 2\left[\int_0^4 \sqrt{x} - \int_0^4 \sqrt{\frac{4}{3}(x-1)}\right] dx$$

$$= 2\left[x^{3/2} \cdot \frac{2}{3} - \int_0^4 \frac{2}{\sqrt{3}} \times \frac{2}{3}(x-1)^{3/2}\right]_0^4$$

$$I = 2 \times \frac{16}{3} - 2\left[\frac{4}{3\sqrt{3}} \cdot 3\sqrt{3}\right]$$

$$= \frac{32}{3} - 8$$

$$I = \frac{8}{3} \text{ sq units.}$$

7.2.21. *Compute the area of the portions cut off by the hyperbola $x^2 - 3y^2 = 1$ from the ellipse $x^2 + 4y^2 = 8$.*

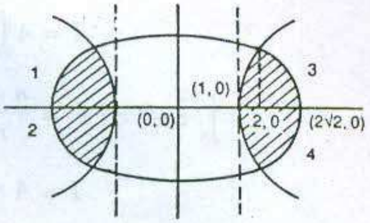
Solution. The equation of hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{1/3} = 1$$

And the equation of ellipse

$$\frac{x^2}{8}+\frac{y^2}{2} = 1$$

$$\frac{x^2}{1^2}+\frac{y^2}{(1\sqrt{3})^2} = 1$$



As curves are symmetric areas 1, 2, 3, 4 are equal.

$$\left.\begin{aligned}\frac{1}{3}(x^2-1) &= y^2\\ \frac{8-x^2}{4} &= y^2\end{aligned}\right]$$

$$\frac{1}{3}x^2-\frac{1}{3} = 2-\frac{x^2}{4}$$

$$\frac{7}{12}x^2 = \frac{7}{3}$$

$$x^2 = 4$$

$$x = \pm 2$$

$$I_1 = \int_1^2 \frac{1}{\sqrt{3}}\sqrt{x^2-1}\,dx+\frac{1}{2}\int_2^{2\sqrt{2}}\sqrt{8-x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\ln[x+\sqrt{x^2-1}]\right]_1^2+\frac{1}{2}\left[\frac{1}{2}x\sqrt{8-x^2}+\frac{1}{2}\times 8\sin^{-1}\frac{x}{2\sqrt{2}}\right]_2^{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{3}}\left[\sqrt{3}+\frac{1}{2}\ln[2+\sqrt{3}]\right]+\frac{1}{2}\left[4\cdot\frac{\pi}{2}-2-4\times\frac{\pi}{4}\right]$$

$$I_1 = 1+\frac{1}{2\sqrt{3}}\ln[2+\sqrt{3}]+\pi-1-\frac{\pi}{2}$$

$$I_1 = \frac{\pi}{2}+\frac{1}{2\sqrt{3}}\ln[2+\sqrt{3}]$$

$$I = 4I_1$$

$$= 2\pi+\frac{2}{\sqrt{3}}\ln[2+\sqrt{3}]$$ **sq. units.**

7.2.22. ***Compute the area enclosed by the curve $y^2 = (1-x^2)^3$.***

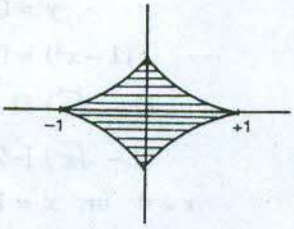
Solution. Curve is symmetric about x-axis, y-axis

$x = -1, \quad +1 \to y = 0 \qquad x = 0 \qquad y = \pm 1$

$$I_1 = \int_0^1 (1-x^2)^{3/2}\,dx$$

$$I = 4I_1$$

$$I = 4\int_0^1 (1-x^2)^{3/2}\,dx$$



$x = \sin\theta$

$dx = \cos\theta\,d\theta$

x	θ
1	$\pi/2$
0	0

$$I = 4\int_0^{\pi/2} \cos^3\theta . \cos\theta \, d\theta = 4\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{(n-3)}{n-2} \ldots\ldots \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2}, \qquad n \text{ is even}$$

$$I = 4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3}{4}\pi = 0.75\,\pi$$

7.2.23. Compute the area enclosed by the loop of the curve $4(y^2 - x^2) + x^3 = 0$.

Solution. The given curve may be written as

$$4y^2 = 4x^2 - x^3$$
$$4y^2 = x^2 [4 - x]$$
$$y^2 = \frac{x^2}{4}(4 - x)$$

At $y = 0 \quad x = 0, 4$

$$I_1 = \frac{1}{2}\int_0^4 x . \sqrt{4 - x} \, dx$$

$$4 - x = t^2 \qquad x \quad t$$
$$-dx = 2t\,dt \qquad 4 \quad 0$$
$$\qquad 0 \quad 2$$

$$I_1 = \frac{1}{2}\int_0^2 \left(4 - t^2\right) 2t^2 \, dt = \left[\frac{4t^3}{3} - \frac{t^5}{5}\right]_0^2$$

$$= \frac{4}{3} \times 8 - \frac{32}{5} = \frac{64}{15}$$

$$I = 2I_1 = 2 \times \frac{64}{15}$$

$$I = \frac{128}{15}$$

7.2.24. Compute the area of the figure enclosed by the curve $\sqrt{x} + \sqrt{y} = 1$ and straight line $x + y = 1$.

Solution. $\sqrt{y} = 1 - \sqrt{x}$ $\qquad \begin{bmatrix} x > 0 \\ y > 0 \end{bmatrix} 1^{st}$ quadrant

$$y = (1 - \sqrt{x})^2$$
$$y = (1 - x)$$
$$(1 - x^2) = (1 - \sqrt{x})(1 + \sqrt{x})$$
$$(1 - \sqrt{x})[1 - \sqrt{x} - 1 - \sqrt{x}] = 0$$
$$(1 - \sqrt{x})[-2\sqrt{x}] = 0$$
$$x = 0 \quad \text{or} \quad x = 1$$

$$I = \int_0^1 \left[(1 - x) - \left(1 - \sqrt{x}\right)^2\right] dx$$

$$= \left[-\frac{(1-x)^2}{2}\right]_0^1 - \int_0^1 \left(1 + x - 2\sqrt{x}\right) dx$$

$$= \frac{1}{2} - \left[x + \frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2}\right]_0^1$$

$$= \frac{1}{2} - \left[1 + \frac{1}{2} - \frac{4}{3}\right] = \frac{1}{2} - 1 - \frac{1}{2} + \frac{4}{3}$$

$$= \frac{1}{3} \text{ sq. units.}$$

7.2.25. ***Compute the area enclosed by the curve $y^2 = x^2(1-x^2)$ curve symmetric both x as well as y–axis.***

Solution.

$$I_1 = \int_0^1 x\sqrt{1-x^2}\, dx$$

$$I = 4I_1$$

Let

$$1 - x^2 = t^2$$

$$-2x\, dx = 2t\, dt$$

$$x\, dx = -t\, dt$$

x	t
0	1
1	0

$$I_1 = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$$I = 4 \times \frac{1}{3} = \frac{4}{3} \text{ sq. units.}$$

7.2.26. ***Compute the area enclosed by the loop of the $x^3 + x^2 - y^2 = 0$.***

Solution.

$$y^2 = x^3 + x^2$$

$$y^2 = x^2(x+1)$$

$y = 0$ $x = 0, -1$ symmetric about x-axis

$$I_1 = \int_{-1}^0 x.\sqrt{x+1}\, dx$$

$x + 1 = t^2$

$$= \int_0^1 \left(t^2 - 1\right) t.\, 2t\, dt$$

$x = t^2 - 1$

$dx = 2t\, dt$

$$I_1 = 2\int_0^1 \left[t^4 - t^2\right] dt = 2\left[\frac{t^5}{5} - \frac{t^3}{3}\right]_0^1$$

$$= 2\left[-\frac{2}{15}\right] = -\frac{4}{15}$$

$$I = 2I_1 = -\frac{8}{15}$$

As area is always +ive,

$$I = \frac{8}{15} \text{ sq. units.}$$

7.2.27. *Compute the area bounded by the axis of ordinates and the curve $x = y^2(1-y)$.*

Solution. Given, the equation of curve is $x = y^2(1-y)$

$$x = 0 \quad y = 0, 1$$

$$I = \int_0^1 y^2(1-y)\,dy$$

$$= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \text{ sq. units.}$$

7.2.28. *Compute the area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$*

Solution. The equation of curve is $y = x^4 - 2x^3 + x^2 + 3$, the axis of abscissas and two ordinates corresponding to the points of minimum of the function $y(x)$

$$y = x^4 - 2x^3 + x^2 + 3$$

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

$$\frac{dy}{dx} = 0 \qquad 2x(x^2 - 3x + 1) = 0$$

$$2x[2x^2 - 2x - x + 1] = 0$$

$$2x(2x-1)(x-1) = 0$$

$$x = 0, \quad x = 1/2 \text{ and } x = 1$$

$$y = 3 \quad \text{at} \quad x = 0 \quad \text{[Minima]}$$

$$y = 3 \quad \text{at} \quad x = 1$$

$$y = 3 + 2 - 4 \quad \text{at} \quad x = 1/2 \quad \text{[Maxima]}$$

$$I = \int_0^1 \left(x^4 - 2x^3 + x^2 + 3\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + 3x\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{16}{5} - \frac{1}{6} = \frac{91}{30} \text{ sq. units.}$$

7.4 COMPUTING AREAS WITH PARAMETRIC REPRESENTED BOUNDARIES

If the boundary of a figure is represented by parametric equations

$$x = x(t)$$
$$y = y(t)$$

then the area of the figure is evaluated by one of the three formulae

$$S = -\int_\alpha^\beta y(t)\,x'(t)\,dt;$$

$$S = \int_\alpha^\beta x(t)\,y'(t)\,dt;$$

$$S = \frac{1}{2}\int_\alpha^\beta (xy' - yx')\,dt$$

where α and β are the values of the parameter t corresponding respectively to the beginning and the end of the traversal of the contour in the positive direction (the figure remains on the left).