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CIRCULAR MOTION & W.P.E

THEORY AND EXERCISE BOOKLET

CONTENTS

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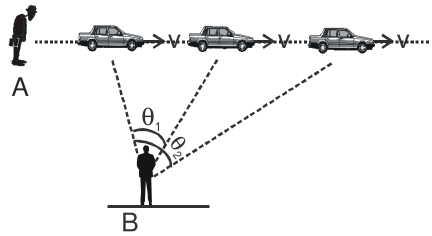
WORK POWER ENERGY

IIT - JEE Syllbabus :

Circular Motion (uniform and non-uniform), Work, Power, Kinetic Energy, Potential Energy, Conservation of Mechanical Energy.

1. CIRCULAR MOTION

When a particle moves in a plane such that its distance from a fixed (or moving) point remains constant then its motion is called as the circular motion with respect to that fixed (or moving) point. That fixed point is called centre and the distance between fixed point and particle is called radius.



The car is moving in a straight line with respect to the man A. But the man B continuously rotate his face to see the car. So with respect to man A $\frac{d\theta}{dt} = 0$

But with respect to man B $\frac{d\theta}{dt} \neq 0$

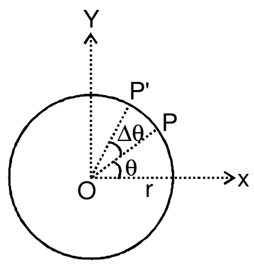
Therefore we conclude that with respect to A the motion of car is straight line but for man B it has some angular velocity

2. KINEMATICS OF CIRCULAR MOTION :

2.1 Variables of Motion :

(a) Angular Position :

The angle made by the position vector with given line (reference line) is called angular position Circular motion is a two dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and centre O. The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the angular position of the particle. As the particle moves on the circle its angular position θ change. Suppose the point rotates an angle $\Delta\theta$ in time Δt.



(b) Angular Displacement :

Definition :

Angle rotated by a position vector of the moving particle in a given time interval with some reference line is called its angular displacement.

Important point :

- It is dimensionless and has proper unit SI unit radian while other units are degree or revolution 2π rad = 360° = 1 rev
- Infinitely small angular displacement is a vector quantity but finite angular displacement is not because the addition of the small angular displacement is cummutative while for large is not.

  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ but $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$
- Direction of small angular displacement is decided by right hand thumb rule. When the fingers are directed along the motion of the point then thumb will represents the direction of angular displacement.
- Angular displacement can be different for different observers

(c) Angular Velocity ω

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} \; ;$$

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2 respectively.

(ii) Instantaneous Angular Velocity

The rate at which the position vector of a particle with respect to the centre rotates, is called as instantaneous angular velocity with respect to the centre.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Important points :

- It is an axial vector with dimensions $[T^{-1}]$ and SI unit rad/s.
- For a rigid body as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g., angular velocity of all points of earth about its own axis is $(2\pi/24)$ rad/hr.
- If a body makes 'n' rotations in 't' seconds then angular velocity in radian per second will be

  $$\omega_{av} = \frac{2\pi n}{t}$$

  If T is the period and 'f' the frequency of uniform circular motion

  $$\omega_{av} = \frac{2\pi \times 1}{T} = 2\pi f$$

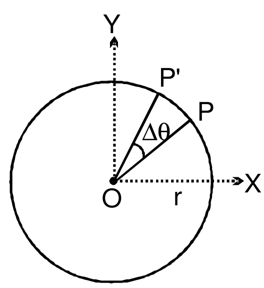
- If $\theta = a - bt + ct^2$ then $\omega = \frac{d\theta}{dt} = -b + 2ct$

Relation between speed and angular velocity :

$$\omega = \lim_{\Delta t \to \theta} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The rate of change of angular velocity is called the angular acceleration (α). Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$



The linear distance PP' travelled by the particle in time Δt is

$$\Delta s = r\Delta\theta \quad \text{or} \quad \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

$$= r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \quad \text{or} \quad \frac{\Delta s}{\Delta t} = r\frac{d\theta}{dt} \quad \text{or} \quad v = r\omega$$

Here, v is the linear speed of the particle

It is only valid for circular motion

$v = r\omega$ is a scalar quantity ($\vec{\omega} \neq \frac{\vec{v}}{r}$)

Ex.1 ***If θ depends on time t in following way***
$\theta = 2t^2 + 3$ then
(a) Find out ω average upto 3 sec. ***(b) ω at 3 sec***

Sol. $\omega_{avg} = \dfrac{\text{Total angular displacement}}{\text{total time}} = \dfrac{\theta_f - \theta_i}{t_2 - t_1}$

$\theta_f = 2\,(3)^2 + 3 = 21$ rad
$\theta_i = 2\,(0) + 3 = 3$ rad.

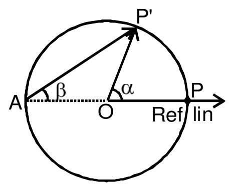
So, $\omega_{avg} = \dfrac{21-3}{3} = 6$ rad/sec

$\omega_{instantaneous} = \dfrac{d\theta}{dt} = 4t$

$\omega_{at\ t\ =\ 3\ sec} = 4 \times 3 = 12$ rad/sec

(d) Relative Angular Velocity

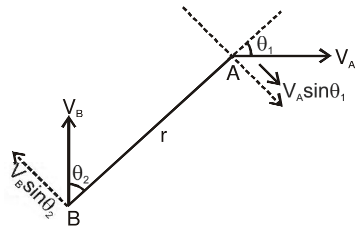
Angular velocity is defined with respect to the point from which the position vector of the moving particle is drawn Here angular velocity of the particle w.r.t. 'O' and 'A' will be different



$\omega_{PO} = \dfrac{d\alpha}{dt}$; $\omega_{PA} = \dfrac{d\beta}{dt}$

Definition :

Relative angular velocity of a particle 'A' with respect to the other moving particle 'B' is the angular velocity of the position vector of 'A' with respect to 'B'. That means it is the rate at which position vector of 'A' with respect to 'B' rotates at that instant



$\omega_{AB} = \dfrac{(V_{AB})_\perp}{r_{AB}}$ here $V_{AB\perp}$ = Relative velocity $\perp$ to position vector AB

$= \dfrac{\text{Relative velocity of A w.r.t. B perpendicular to line AB}}{\text{Seperation between A and B}}$

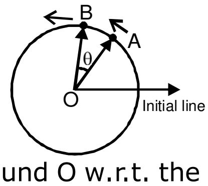
$(V_{AB})_\perp = V_A \sin\theta_1 + V_B \sin\theta_2$

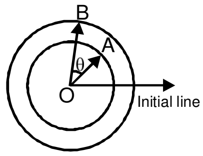
$r_{AB} = r$

$\omega_{AB} = \dfrac{V_A \sin\theta_1 + V_B \sin\theta_2}{r}$

Important points :

- If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively, the rate of change of angle between $\vec{OA}$ and $\vec{OB}$ is



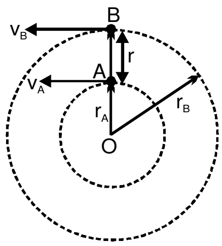


$$\frac{d\theta}{dt} = \omega_B - \omega_A$$

So the time taken by one to complete one revolution around O w.r.t. the other

$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2}$$

- If two particles are moving on two different concentric circles with different velocities then angular velocity of B relative to A as observed by A will depend on their positions and velocities. consider the case when A and B are closest to each other moving in same direction as shown in figure. In this situation



$$v_{rel} = |\vec{v}_B - \vec{v}_A| = v_B - v_A$$

$$r_{rel} = |\vec{r}_B - \vec{r}_A| = r_B - r_A$$

so, $$\omega_{BA} = \frac{(v_{rel})_\perp}{r_{rel}} = \frac{v_B - v_A}{r_B - r_A}$$

$(v_{rel})_\perp$ = Relative velocity perpendicular to position vector

Ex.2 Two particles move on a circular path (one just inside and the other just outside) with angular velocities ω and 5 ω starting from the same point. Then, which is incorrect.

(a) they cross each other at regular intervals of time $\frac{2\pi}{4\omega}$ when their angular velocities are oppositely directed

(b) they cross each other at points on the path subtending an angle of 60° at the centre if their angular velocities are oppositely directed

(c) they cross at intervals of time $\frac{\pi}{3\omega}$ if their angular velocities are oppositely directed

(d) they cross each other at points on the path subtending 90° at the centre if their angular velocities are in the same sense

Sol. If the angular velocities are oppositely directed, they meet at intervals of

$$\text{time } t = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{6\omega} = \frac{\pi}{3\omega}$$

Angle subtended at the centre by the crossing points

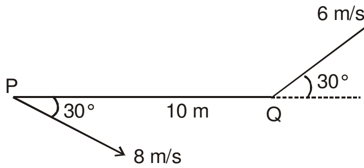
$$\theta = \omega t = \frac{\pi}{3} = 60°$$

When their angular velocities are in the same direction,

$$t' = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{4\omega} = \frac{\pi}{2\omega} \text{ and } \theta' = \frac{\pi}{2\omega} \times \omega = \frac{\pi}{2}$$

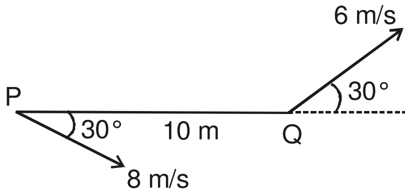
Ans. (a)

Ex.3 Two moving particles P and Q are 10 m apart at a certain instant. The velocity of P is 8 m/s making 30° with the line joining P and Q and that of Q is 6 m/s making 30° with PQ in the figure. Then the angular velocity of Q with respect to P in rad/s at that instant is



(A) 0 ***(B) 0.1*** ***(C) 0.4*** ***(D) 0.7***

Sol.



Angular velocity of Q relative to P = $\dfrac{\text{Projection of } V_{QP} \text{ perpendicular to the line PQ}}{\text{Separation between P and Q}}$

$\dfrac{V_Q \sin\theta_2 - V_P \sin\theta_1}{PQ} = \dfrac{6\sin 30° - (-8\sin 30°)}{10} = 0.7$ rad/s

$\therefore$ (D)

(e) Angular Acceleration α :

(i) Average Angular Acceleration :

Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

(ii) Instantaneous Angular Acceleration :

It is the limit of average angular acceleration as Δt approaches zero, i.e.,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta}$$

Important points :

- It is also an axial vector with dimension $[T^{-2}]$ and unit rad/s^2
- If $\alpha = 0$, circular motion is said to be uniform.
- As $\omega = \dfrac{d\theta}{dt}$, $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$,

  i.e., second derivative of angular displacement w.r.t time gives angular acceleration.
- α is a axial vector and direction of α is along ω if ω increases and opposite to ω if ω decreases

(f) Radial and tangential acceleration

Acceleration of a particle moving in a circle has two components one is along $\hat{e}_t$ (along tangent) and the other along $-\hat{e}_r$ (or towards centre). Of these the first one is the called the tangential acceleration. (a_t) and the other is called radial or centripetal acceleration (a_r). Thus.

$a_t = \frac{dv}{dt}$ = rate of change of speed

and $a_r = \omega^2 r = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$

Here, a_t is the component which is responsible for changing the magnitude of speed of the particle in circular motion. a_r is the component which is responsible for changing the direction of particle in circular motion.

the two component are mutually perpendicular. Therefore, net acceleration of the particle will be :

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(r\omega^2)^2 + \left(\frac{dv}{dt}\right)^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Following three points are important regarding the above discussion :

1. In uniform circular motion, speed (v) of the particle is constant, i.e., $\frac{dv}{dt} = 0$. Thus, $a_t = 0$ and $a = a_r = r\omega^2$

2. In accelerated circular motion, $\frac{dv}{dt}$ = positive, i.e., a_t is along $\hat{e}_t$ or tangential acceleration of particle is parallel to velocity $\vec{v}$ because $\vec{v} = r\omega\hat{e}_t$ and $\vec{a}_r = \frac{dv}{dt}\hat{e}_t$

3. In decelerated circular motion, $\frac{dv}{dt}$ = negative and hence, tangential acceleration is anti-parallel to velocity $\vec{v}$.

(g) Relation between angular acceleration and tangential acceleration

we know that

$v = r\omega$

Here, v is the linear speed of the particle

Differentiating again with respect to time, we have

$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$ or $a_t = r\alpha$

Here, $a_t = \frac{dv}{dt}$ is the rate of change of speed (not the rate of change of velocity).

Ex.4 A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes from 5.0 m/s to 6.0 m/s in 2.0s, find the angular acceleration.

Sol. The tangential acceleration is given by

$a_t = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$

($\because$ Here speed increases uniformly $a_t = \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$)

$= \frac{6.0 - 5.0}{2.0}$ m/s² = 0.5 m/s²

The angular acceleration is $\alpha = a_t/r$

$= \frac{0.5\,m/s^2}{20\,cm}$ = 2.5 rad/s²

Ex-5 A particle moves in a circle of radius 20 cm. Its linear speed at any time is given by v = 2t where v is in m/s and t is in seconds. Find the radial and tangential acceleration at t = 3 seconds and hence calculate the total acceleration at this time.

Sol. The linear speed at 3 seconds is

$v = 2 \times 3 = 6$ m/s

The radial acceleration at 3 seconds

$$= \frac{v^2}{r} = \frac{6 \times 6}{0.2} = 180 \text{ m/s}^2$$

The tangential acceleration is given by

$\frac{dv}{dt} = 2$, because v = 2t.

$\therefore$ tangential acceleration is 2 m/s^2.

Net Acceleration = $\sqrt{{a_r}^2 + {a_t}^2} = \sqrt{(180)^2 + (2)^2}$ = 180.01 m/s^2

T-1 Is it possible for a car to move in a circular path in such a way that it has a tangential acceleration but no centripetal acceleration ?

Ex.6 A particle moves in a circle of radius 2.0 cm at a speed given by v = 4t, where v is in cm/s and t in seconds.

(a) Find the tangential acceleration at t = 1 s.

(b) Find total accleration at t = 1 s.

Sol. (a) Tangential acceleration

$$a_t = \frac{dv}{dt} \quad \text{or} \quad a_t = \frac{d}{dt}(4t) = 4 \text{ cm/s}^2$$

$$a_c = \frac{v^2}{R} = \frac{(4)^2}{2} = 8 \Rightarrow \quad a = \sqrt{a_t^2 + a_c^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5}\, m/s^2$$

Ex.7 A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above level ground. The string breaks, and the stone files off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the cetripetal acceleration of the stone while in circular motion ?

Sol. $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2}{9.8}}$ = 0.64 s

$v = \frac{10}{t}$ = 15.63 m/s

$a = \frac{v_B^2}{R}$ = 0.45 m/s^2

Ex.8 Find the magnitude of the acceleration of a particle moving in a circle of radius 10 cm with uniform speed completing the circle in 4 s.

Sol. The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$$v = 2\pi r/t = \frac{2\pi \times 10\text{cm}}{4\text{s}} = 5\pi \text{ cm/s}.$$

The acceleration is $a_r = \frac{v^2}{r} = \frac{(5\pi\text{cm/s})^2}{10\text{cm}} = 2.5\pi^2$ cm/s^2

Ex.9 A particle moves in a circle of radius 20 cm. Its linear speed is given by v = 2t where t is in second and v in meter/second. Find the radial and tangential acceleration at t = 3s.

Sol. The linear speed at t = 3s is

v = 2t = 6 m/s

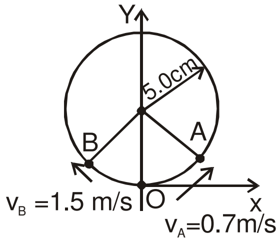
The radial acceleration at t = 3s is

$$a_r = v^2/r = \frac{36m^2/s^2}{0.20\,m} = 180 \text{ m/s}^2$$

The tangential acceleration is

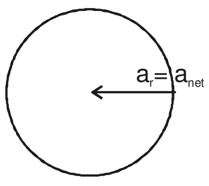
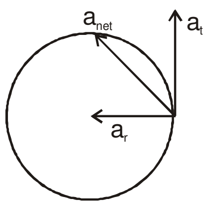
$$a_t = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2 \text{ m/s}^2$$

Ex.10 Two particles A and B start at the origin O and travel in opposite directions along the circular path at constant speeds v_A = 0.7 m/s and v_B = 1.5 m/s, respectively. **Determine the time when they collide and the magnitude of the acceleration of B just before this happens.**



Sol. $1.5\,t + 0.7\,t = 2\pi R = 10\,\pi \quad \therefore\ t = \frac{10\pi}{2.2} = 14.3$ s

$$a = \frac{v_B^2}{R} = 0.45 \text{ m/s}^2$$

Uniform circular motion	Non-uniform circular motion
(1) Speed of the particle is constant i.e., ω is constant	speed of the particle is not constant i.e. ω is not constant
(ii) $a_t = \frac{d\lvert\vec{v}\rvert}{dt} = 0$	$a_t = \frac{d\lvert\vec{v}\rvert}{dt} \neq 0$
$a_r = \frac{v^2}{r} \neq 0$	$a_r \neq 0$
$\therefore\ a_{net} = a_r$	$\vec{a}_{net} = \vec{a}_r + \vec{a}_t$
	

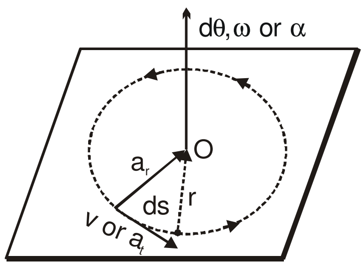
(h) Relations among Angular Variables

These relations are also referred as equations of rotational motion and are -

$\omega = \omega_0 + \alpha t$ ...(1)

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ ...(2)

$\omega^2 = \omega_0^2 + 2\alpha\theta$ ...(3)



These are valid only if angular acceleration is constant and are analogous to equations of translatory motion, i.e.,

$v = u + at$; $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$

3. DYNAMICS OF CIRCULAR MOTION :

In circular motion or motion along any curved path Newton's law is applied in two perpendicular directions one along the tangent and other perpendicular to it. i.e., towards centre. The compnent of net force towards the centre is called **centripetal force.** The component of net force along the tangent is called **tangential force.**

tangential force (F_t) = $Ma_t = M\dfrac{dv}{dt} = M\,\alpha\,r$; where α is the angular acceleration

centripetal force (F_c) = $m\,\omega^2 r = \dfrac{mv^2}{r}$

Ex.11 A small block of mass 100 g moves with uniform speed in a horizontal circular groove, with vertical side walls, of radius 25 cm. If the block takes 2.0s to complete one round, find the normal contact force by the slide wall of the groove.

Sol. The speed of the block is

$$v = \frac{2\pi \times (25\,\text{cm})}{2.0\,\text{s}} = 0.785 \text{ m/s}$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0.785\text{m/s})^2}{0.25} = 2.46 \text{ m/s}^2$$

towards the center. The only force in this direction is the normal contact force due to the side walls. Thus from Newton's second law, this force is

$$N = ma = (0.100 \text{ kg})\,(2.46 \text{ m/s}^2) = 0.246 \text{ N}$$

3.1 Centripetal Force :

Concepts : This is necessary resultant force towards the centre called the centripetal force.

$$F = \frac{mv^2}{r} = m\omega^2 r$$

(i) A body moving with constant speed in a circle is not in equilibrium.

(ii) It should be remembered that in the absence of the centripetal force the body will move in a straight line with constant speed.

(iii) It is not a new kind of force which acts on bodies. In fact, any force which is directed towards the centre may provide the necessary centripetal force.

Ex.12 A small block of mass 100 g moves with uniform speed in a horizontal circular groove, with vertical side walls, of radius 25 cm. If the block takes 2.0s to complete one round, find the normal contact force by the slide wall of the groove.

Sol. The speed of the block is

$$v = \frac{2\pi \times (25\,\text{cm})}{2.0\,\text{s}} = 0.785 \text{ m/s}$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0.785\text{m/s})^2}{0.25} = 2.5 \text{ m/s}^2$$

towards the center. The only force in this direction is the normal contact force due to the slide walls. Thus from Newton's second law, this force is

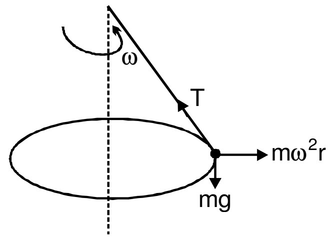
$$N = ma = (0.100 \text{ kg})\,(2.5 \text{ m/s}^2) = 0.25 \text{ N}$$

3.2 Centrifugal Force :

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B, as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.)

Its magnitude is equal to that of the centripetal force = $\frac{mv^2}{r}$. Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame)

FBD of ball w.r.t non inertial frame rotating with the ball.

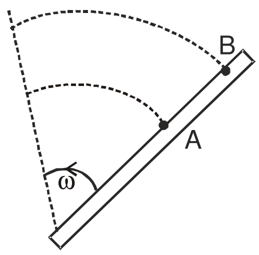


Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ act radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

T-2 A particle of mass m rotates in a circle of radius r with a uniform angular speed ω. It is viewed from a frame rotating about same axis with a uniform angular speed ω_0. The centrifugal force on the particle is

(A) $m\omega^2 r$ (B) $m\omega_0^2 r$ (C) $m\left(\frac{\omega+\omega_0}{2}\right)^2 r$ (D) $m\omega_0\omega r$

: A rod move with ω angular velocity then we conclude following for point A & B in a rod.

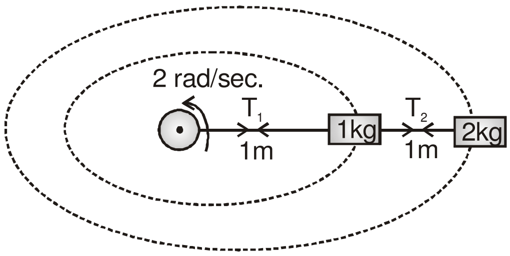


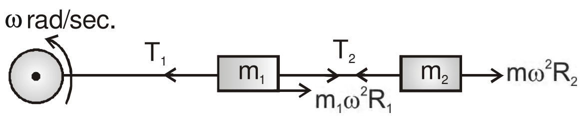
$\alpha_A = \alpha_B$ $s_B > s_A$

$\theta_A = \theta_B$ $v_B > v_A$

$\omega_A = \omega_B$ $a_{tB} > a_{tA}$

Ex.13 Find out the tension T_1, T_2 is the string as shown in figure





We know that $\omega_{m_1} = \omega_{m_2}$

$\Rightarrow$ $T_1 = m_1\omega^2 R_1 + T_2$

$T_2 = m\,\omega^2\,R_2$

$T_2 = 2 \times 4 \times 2 = 16$ N

So $T_1 = (1)\,(2)^2\,(1) + 16\text{ N} = 4 + 16\text{ N}$

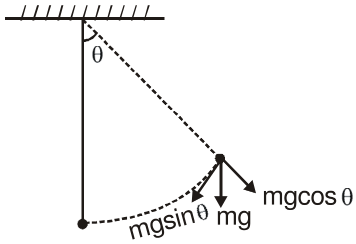
$T_1 = 20$ N

4. SIMPLE PENDULUM

Ex.14 A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle θ with the vertical. Find the tension in the string at this instant.

Sol. The force acting on the bob are (figure)

(a) the tension T (b) the weight mg.



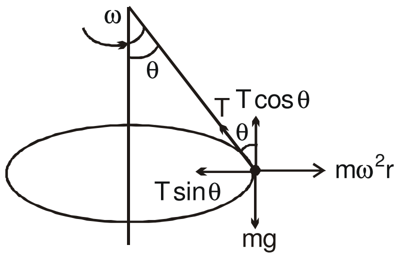
As the bob moves in a vertical circle with centre at O, the radial acceleration is v^2/L towards O. Taking the components along this radius and applying Newton's second law, we get

$$T - mg\cos\theta = mv^2/L \quad \text{or,} \quad T = m(g\cos\theta + v^2/L)$$

$$|\vec{F}_{net}| = \sqrt{(mg\sin\theta)^2 + \left(\frac{mv^2}{L}\right)^2} \qquad = m\sqrt{g^2\sin^2\theta + \frac{v^4}{L^2}}$$

5. CIRCULAR MOTION IN HORIZONTAL PLANE

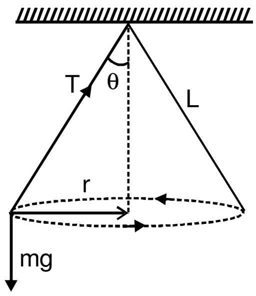
A ball of mass m attached to a light and inextensible string rotates in a horizontal circle of radius r with an angular speed ω about the vertical. If we draw the force diagram of the ball. We can easily see that the component of tension force along the centre gives the centripetal force and component of tension along vertical balances the gravitation force. Such a system is called a conical pendulum.



FBD of ball w.r.t ground

Ex. 15 A particle of mass m is suspended from a ceiling through a string of length L. The particle moves in a horizontal circle of radius r. Find (a) the speed of the particle and (b) the tension in the string.

Sol. The situation is shown in figure.



The angle θ made by the string with the vertical is given by

$\sin\theta = r/L$ ... (i)

The forces on the particle are

(a) the tension T along the string and

(b) the weight mg vertically downward.

The particle is moving in a circle with a constant speed v. Thus, the radial acceleration towards the centre has magnitude v^2/r. Resolving the forces along the radial direction and applying. Newton's second law,

$T\sin\theta = m(v^2/r)$ ...(ii)

As there is no acceleration in vertical directions, we have from Newton's law,

$T\cos\theta = mg$ ...(iii)

Dividing (ii) by (iii),

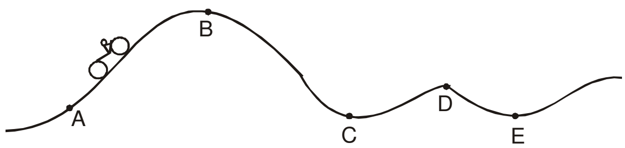
$\tan\theta = \dfrac{v^2}{rg}$ or $v = \sqrt{rg\tan\theta}$

And from (iii), $T = \dfrac{mg}{\cos\theta}$

Using (i), $v = \dfrac{r\sqrt{g}}{(L^2 - r^2)^{1/4}}$ and $T = \dfrac{mgL}{(L^2 - r^2)^{1/2}}$

6. MOTION OF A MOTORCYCLIST ON A CURVED PATH.

A cylist having mass m move with constant speed v on a curved path as shown in figure.



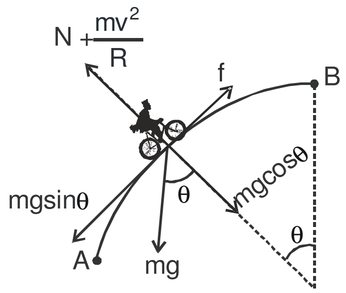
We divide the motion of cyclist in four parts :
(1) from A to B (2) from B to C
(3) from C to D (4) from D to E
(1 and 3 are same type of motion)

(A) Motion of cyclist from A to B

$$N + \frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow \quad N = mg\cos\theta - \frac{mv^2}{R} \qquad \text{...(1)}$$

$$f = mg\sin\theta \qquad \text{..(2)}$$



(1) As cyclist move upward
$\because$ θ decreases & $\cos\theta$ increases
$\therefore$ N increases
and
$\because$ θ decreases $\sin\theta$ decreases
$\therefore$ friction force required to balance mg sin θ (As cyclist is moving with constant speed) also decreases

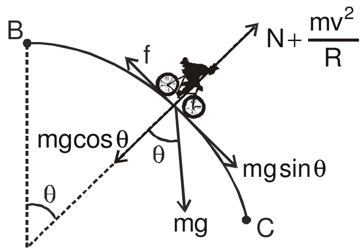
(B) Motion of cyclist from B to C

$$N + \frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow N = mg\cos\theta - \frac{mv^2}{R} \qquad \text{...(1)}$$

$f = mg\sin\theta$ ...(2)

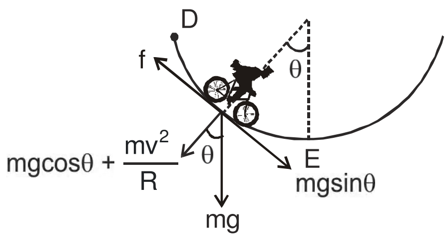
Therefore from B to C Normal force decrease but friction force increase becuse θ increases.



(C) Motion of cyclist from D to E

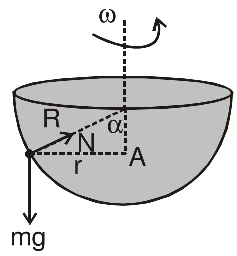
$$N = \frac{mv^2}{R} + mg\cos\theta \qquad f = mg\sin\theta$$

from D to E θ decreases
therefore mg cos θ increase
So N increase but f decreases



Ex.16 A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.

Sol. Let ω be the angular speed of rotation of the bowl.
Two force are acting on the ball.
1. normal reaction N 2. weight mg
The ball is rotating in a circle of radius r (= R sin α) with centre at A at an angular speed ω. Thus,

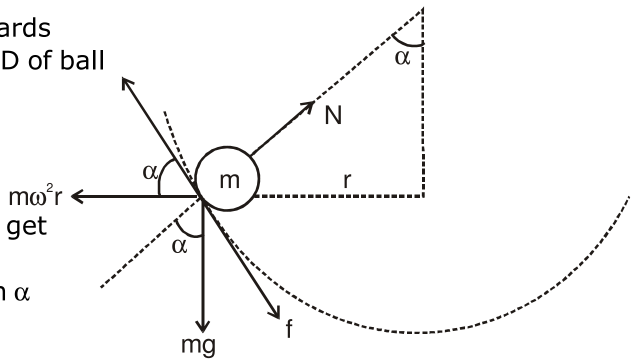


$N \sin\alpha = mr\omega^2 = mR\omega^2 \sin\alpha$
$N = mR\omega^2$ ...(i)
and $N\cos\alpha = mg$ ...(ii)
Dividing Eqs. (i) by (ii),

we get $\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$ $\therefore$ $\omega = \sqrt{\frac{g}{R\cos\alpha}}$

Ex.17 If friction is present between the surface of ball and bowl then find out the range of ω for which ball does not slip (μ is the friction coefficient)
Friction develop a range of ω for which the particle will be at rest.

Sol.
(a) When $\omega > \omega_0$
In this situation ball has a tendency to slip upwards so the friction force will act downwards. So F.B.D of ball

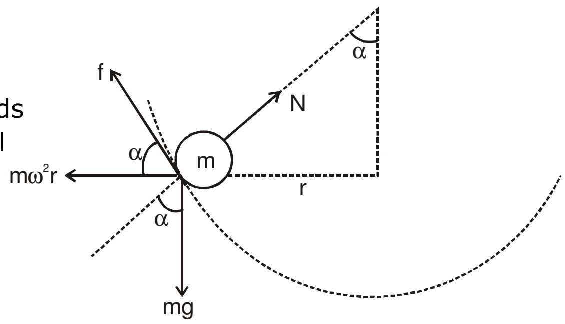


$N = m\omega^2 r \sin\alpha + mg\cos\alpha$. ...
$f + mg\sin\alpha = m\omega^2 r\cos\alpha$ ...(2)
$\therefore f_{max} = \mu N = \mu(m\omega^2 r \sin\alpha + mg\cos\alpha)$
$r = R\sin\alpha$
Substituting the values of f_{max} & r in eq. (2) we get

$\Rightarrow \mu\,(m\omega^2 r\sin\alpha + mg\cos\alpha) \geq m\omega^2 r\cos\alpha - mg\sin\alpha$
$\therefore \mu(m\omega^2 R\sin 2\alpha + mg\cos\alpha) \geq m\omega^2 R\sin\alpha\cos\alpha - mg\sin\alpha$

$$\omega \leq \sqrt{\frac{\mu g\cos\alpha + g\sin\alpha}{R\sin\alpha(\cos\alpha - \mu\sin\alpha)}}$$

(b) when $\omega < \omega_0$
In this situation ball has a tendency to slip downwards so the friction force will act upwards. So F.B.D of ball



$\Rightarrow$ $f + m\omega^2 r\cos\alpha = mg\sin\alpha$
$f = m\,(g\sin\alpha - \omega^2 r\cos\alpha)$ ...(1)
$N = mg\cos\alpha + m\omega^2 r\sin\alpha$ ...(2)
$f_{max} = \mu N = \mu(mg\cos\alpha + m\omega^2 r\sin\alpha)$
for equillibrium

$\Rightarrow \mu(g \cos \alpha + \omega^2 r \sin\alpha) \geq (g\sin\alpha - \omega^2 r \cos \alpha)$

$\Rightarrow$ Substituting $r = R\sin\alpha$ then

$$\omega \geq \sqrt{\frac{g(\sin\alpha - \mu\cos\alpha)}{R\sin\alpha(\mu\sin\alpha + \cos\alpha)}}$$

7. CIRCULAR TURNING ON ROADS :

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

1. By Friction only
2. By Banking of Roads only
3. By Friction and Banking of Roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both. Now let us write equations of motion in each of the three cases separately and see what are the constant in each case.

7.1 By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards center

Thus, $f = \frac{mv^2}{r}$

Further, limiting value of f is μN

or $f_L = \mu N = \mu mg$ ($N = mg$)

Therefore, for a safe turn without sliding $\frac{mv^2}{r} \leq f_L$

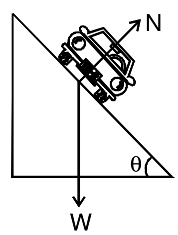
or $\frac{mv^2}{r} \leq \mu mg$ or $\mu \geq \frac{v^2}{rg}$ or $v \leq \sqrt{\mu rg}$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

7.2 By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved to avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

Applying Newton's second law along the radius and the first law in the vertical direction.



$$N\sin\theta = \frac{mv^2}{r} \quad \text{or} \quad N\cos\theta = mg$$

from these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg\tan\theta}$$

7.3 By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these force, the first force, i.e., weight (mg) is fixed both in magnitude and direction.

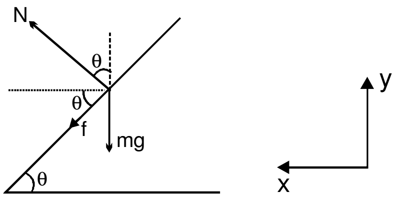


Figure (ii)

The direction of second force i.e., normal reaction N is also fixed (perpendicular or road) while the direction of the third i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_L = \mu N$). So the magnitude of normal reaction N and directions plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the center. Of these m and r are also constant. Therefore, magnitude of N and directions plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that :

(i) Friction f will be outwards if the vehicle is at rest v = 0. Because in that case the component weight mg sinθ is balanced by f.

(ii) Friction f will be inwards if

$$v > \sqrt{rg\tan\theta}$$

(iii) Friction f will be outwards if

$$v < \sqrt{rg\tan\theta}$$ and

(iv) Friction f will be zero if

$$v = \sqrt{rg\tan\theta}$$

(v) For maximum safe speed (figure (ii)

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{...(i)}$$

$$N\cos\theta - f\sin\theta = mg \qquad \text{...(ii)}$$

As maximum value of friction

$$f = \mu N$$

$$\therefore \quad \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg} \qquad \therefore v_{max} = \sqrt{\frac{rg(\mu + \tan\theta)}{(1 - \mu\tan\theta)}}$$

Similarly ; $v_{min} = \sqrt{\frac{rg(\mu - \tan\theta)}{(1 + \mu\tan\theta)}}$

:
- The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.
- The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical to negotiate a safe turn.

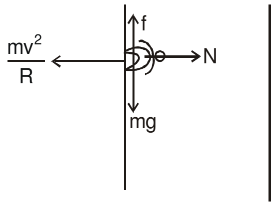
8. DEATH WELL :

A motor cyclist is driving in a horizontal circle on the inner surface of vertical cyclinder of radius R. Friction coefficient between tyres of motorcyclist and surface of cylinder is μ. Find out the minimum velocity for which the motorcyclist can do this. v is the speed of motor cyclist and m is his mass.

$$N = \frac{mv^2}{R}$$

$$f = mg$$

$$f_{max} = \frac{\mu mv^2}{R}$$



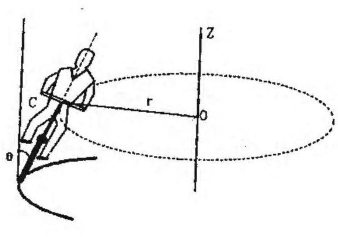
Cyclist does not drop down when

$$f_{max} \geq mg \;\Rightarrow\; \frac{\mu mv^2}{R} \geq mg$$

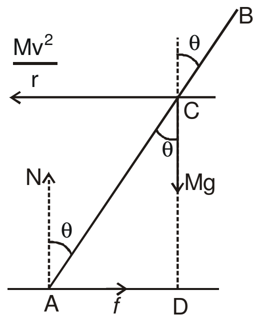
$$v \geq \sqrt{\frac{gR}{\mu}}$$

9. MOTION OF A CYCLIST ON A CIRCULAR PATH :

Suppose a cyclist is going at a speed v on a circular horizontal road of radius r which is not banked. Consider the cycle and the rider together as the system. The centre of mass C (figure shown) of the system is going in a circle with the centre at O and radius r.



(a)



(b)

Let us choose O as the origin, OC as the X-axis and vertically upward as the Z-axis. This frame is rotating at an angular speed $\omega = \frac{v}{r}$ about the Z-axis. In this frame the system is at rest. Since we are working from a rotating frame of reference, we will have to apply a centrifugal force on each particle. The net centrifugal force on the system will be M $\omega^2 r = Mv^2/r$, where M is the total mass of the system. This force will act through the centre of mass. Since the system is at rest in this frame, no other pseudo force is needed.

Figure in shows the forces. The cycle is bent at an angle θ with the vertical. The forces are

(i) weight Mg,

(ii) normal force N

(iii) friction *f* and

(iv) centrifugal force $\frac{Mv^2}{r}$

In the frame considered, the system is at rest. Thus, the total external force and the total external torque must be zero. Let us consider the torques of all the forces about the point A. The torques of N and f about A are zero because these forces pass through A. The torque of Mg about A is Mg(AD) in the clockwise direction and that of $\frac{Mv^2}{r}$ is $\frac{Mv^2}{r}(CD)$ in the anticlockwise direction. For rotational equilibrium,

$$Mg(AD) = \frac{Mv^2}{r}(CD)$$

or, $$\frac{AD}{CD} = \frac{v^2}{rg}$$

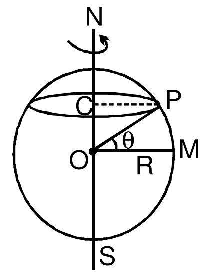
or, $$\tan\theta = \frac{v^2}{rg} \qquad ...(10.9)$$

Thus, the cyclist bends at an angle $\tan^{-1}\left[\frac{v^2}{rg}\right]$ with the vertical.

T.3 A car driver going at a speed of v suddenly finds a wide wall at a distance r. Should he apply breaks or turn the car in a circle of radius r to avoid hitting the wall ?

10. EFFECT OF EARTHS ROTATION ON APPARENT WEIGHT :

The earth rotates about its axis at an angular speed of one revolution per 24 hours. The line joining the north and the south poles is the axis of rotation.



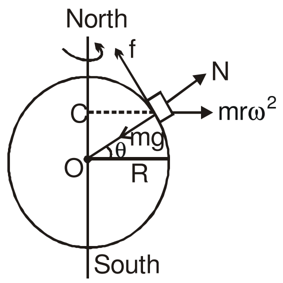
Every point on the earth moves in a circle. A point at equator moves in a circle of radius equal to the radius of the earth and the centre of the circle is same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place point on the earth (figure.)

Drop a perpendicular PC from P to the axis SN. The place P rotates in a circle with the centre at C. The radius of this circle is CP. The angle between the line OM and the radius OP through P is called the latitude of the place point. We have

CP = OP $\cos\theta$ or, $r = R\cos\theta$

where R is the radius of the earth.

If we calculate work from the frame of reference of the earth, we shall have to assume the existence of pseudo force. In Particular, a centrifugal force $m\omega^2 r$ has to be assumed on any particle of mass m placed at P. If we consider a block of mass m at point P then this block is at rest with respect to earth. If resolve the forces along and perpendicular to the line joining the centre of earth then



$$N + mr\omega^2\cos\theta = mg \Rightarrow N = mg - mr\omega^2\cos\theta$$
$$\Rightarrow N = mg - mR\omega^2\cos^2\theta$$

Ex.18 A body weighs 98N on a spring balance at the north pole. What will be its weight recorded on the same scale if it is shifted to the equator ? Use $g = GM/R^2 = 9.8\ m/s^2$ and the radius of the earth R = 6400 km.

Sol. At poles, the apparent weight is same as the true weight.

Thus, $98N = mg = m(9.8\ m/s^2)$

At the equator, the apparent weight is

$$mg' = mg - m\omega^2 R$$

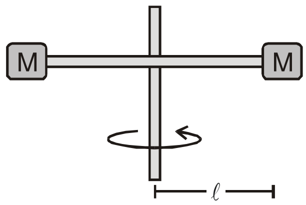
The radius of the earth is 6400 km and the angular speed is

$$\omega = \frac{2\pi \text{ rad}}{24 \times 60 \times 60 \text{s}}$$
$$= 7.27 \times 10^{-6} \text{ rad/s}$$

mg' = 98N – (10 kg) $(7.27 \times 10^{-5} \text{ s}^{-1})^2$ (6400 km) = 97.66 N

SOME SOLVED EXAMPLES

Ex.1 ***Two blocks each of mass M are connected to the ends of a light frame as shown in figure. The frame is rotated about the vertical line of symetry. The rod breaks if the tension in it exceeds T_0. Find the maximum frequency with which the frame may be rotated without breaking the rod.***

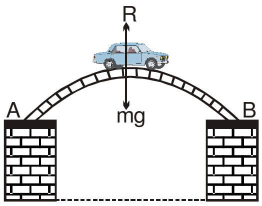


Sol. Consider one of the blocks. If the frequency of revolution is f, the angular velocity is $\omega = 2\pi f$. The acceleration towards the centre is $\omega^2 \ell = 4\pi^2 f^2 \ell$. The only horizontal force on the block is the tension of the rod. At the point of breaking, this force is T_0. So from Newton's law,

$$T_0 = M.4\pi^2 f^2 l \text{ or,} \qquad f = \frac{1}{2\pi}\left[\frac{T_0}{M\ell}\right]^{1/2}$$

Ex.2 ***Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.***

Sol. The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (figure);



The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.

$$\therefore \quad mg - R = \frac{mv^2}{r} \quad \text{or} \quad R = mg - \frac{mv^2}{r}$$

Clearly R < mg, i.e., the weight of the moving car is less than the weight of the stationary car.

Ex.3 ***A body weighing 0.4 kg is whirled in a vertical circle with a string making 2 revolutions per second. If the radius of the circle is 1.2m. Find the tension (a) at the top of the circle, (b) at the bottom of the circle. Give : $g = 10 \text{ m s}^{-2}$ and $\pi = 3.14$***

Sol. Mass, m = 0.4 kg ;

time period = $\frac{1}{2}$ second, radius, r = 1.2 m

Angular velocity, $\omega = \frac{2\pi}{1/2} = 4\pi \text{ rad s}^{-1} = 12.56 \text{ rad s}^{-1}$

(a) At the top of the circle, $T = \frac{mv^2}{r} - mg$

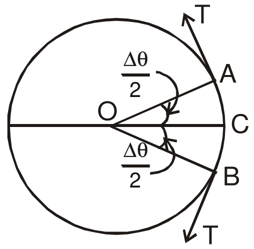
$= mr\omega^2 - mg = m(r\omega^2 - g)$

$= 0.4 (1.2 \times 12.56 \times 12.56 - 9.8)$ N = 71.2 N

(b) At the lowest point, $T = m(r\omega^2 + g) = 80$ N

Ex.4 A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of ring moves with velocity v. Find the tension in the ring.

Sol. Consider a small part ACB of the ring that subtends an angle $\Delta\theta$ at the centre as shown in figure.
Let the tension in the ring be T.
The forces on this elementary portion ACB are
(i) tension T by the part of the ring left to A
(ii) tension T by the part of the ring to B
(iii) weight (Δm) g
(iv) nomal force N by the table.



As the elementary portion ACB moves in a circle of radius R at constant speed v its acceleration towards centre is $\frac{(\Delta m)v^2}{R}$.
Resolving the force along the radius CO

$$T\cos\left(90° - \frac{\Delta\theta}{2}\right) + T\cos\left(90° - \frac{\Delta\theta}{2}\right) = \Delta m\frac{v^2}{R} \qquad ...(i)$$

$$2T\sin\frac{\Delta\theta}{2} = \Delta m\frac{v^2}{R} \qquad ...(ii)$$

Length of the part ACB = $R\Delta\theta$. The mass per unit length of the ring is $\frac{m}{2\pi R}$

$\therefore$ mass of this portion ACB, $\Delta m = \frac{R\Delta\theta m}{2\pi R} = \frac{m\Delta\theta}{2\pi}$

Putting this value of Δm in (ii),

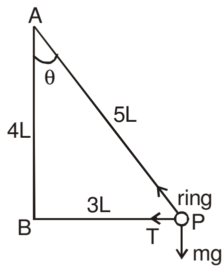
$$2T\sin\frac{\Delta\theta}{2} = \frac{m\Delta\theta v^2}{2\pi R}$$

$$\therefore\ T = \frac{mv^2}{2\pi R}\left(\frac{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)}\right)$$ Since $\Delta\theta$ is small So $\left(\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)$ is equal to 1,

$$T = \frac{mv^2}{2\pi R}$$

Ex.5 A small smooth ring of mass m is threaded on a light inextensible string of length 8L which has its ends fixed at points in the same vertical line at a distance 4L apart. The ring describes horizontal circles at constant speed with both parts of the string taut and with the lower portion of the string horizontal. Find the speed of the ring and the tension in the string. The ring is then tied at the midpoint of the string and made to perform horizontal circles at constant speed of $3\sqrt{gL}$. Find the tension in each part of the string.

Sol. When the string passes through the ring, the tension in the string is the same in both parts. Also from geometry
BP = 3L and AP = 5L



$$T\cos\theta = \frac{4}{5}T = mg \qquad ...(i)$$

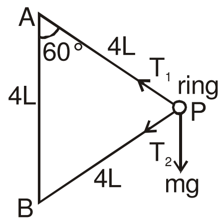
$$T + T\sin\theta = T\left(1 + \frac{3}{5}\right) = \frac{8}{5}T$$

$= \frac{mv^2}{BP} = \frac{mv^2}{3L}$ ...(ii)

$\frac{v^2}{3Lg} = 2$

$v = \sqrt{6Lg}$

From (i) $T = \frac{mg}{4/5} = \frac{5}{4}mg$



In the second case, ABP is an equilateral triangle.

$T_1\cos60° = mg + T_2 \cos60°$

$T_1 - T_2 = \frac{mg}{\cos 60°} = 2mg$ ...(iii)

$T_1\sin60° + T_2\sin60° = \frac{mv^2}{r} = \frac{9mgL}{4L\sin 60°}$

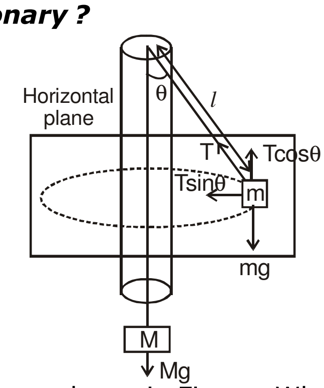
$T_1 + T_2 = \frac{9mg}{4\sin^2 60°} = 3\ mg$ ...(iv)

Solving equation (iii) and (iv)

$T_1 = \frac{5}{2}mg$; $T_2 = \frac{1}{2}mg$

Ex.6 ***A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in Figure. The mass m moves around in a circular path, which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is ℓ and θ is the angle this length makes with vertical. What should be the frequency of rotation of mass m so that M remains stationary ?***

Sol.



The forces acting on mass m and M are shown in Figure. When mass M is stationary

$T = Mg$ ..(i)

where T is tension in string.

For the smaller mass, the vertical component of tension $T\cos\theta$ balances mg and the horizontal component $T\sin\theta$ supplies the necessary centripetal force.

$T\cos\theta = mg$ ...(ii)

$T\sin\theta = mr\omega^2$ ...(iii)

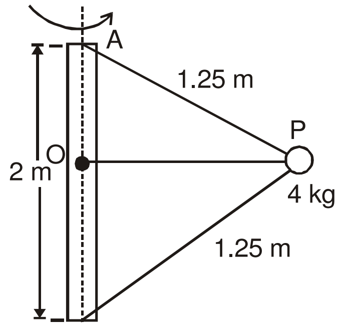
ω being the angular velocity and r is the radius of horizontal circular path.

Form (i) and (iii), $Mg\sin\theta = mr\omega^2$

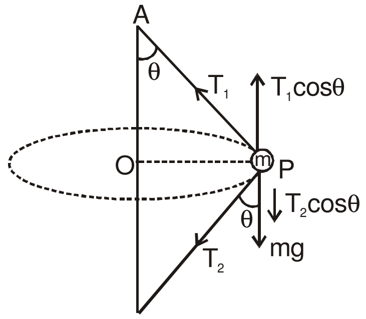
$\omega = \sqrt{\frac{Mg\sin\theta}{mr}} = \sqrt{\frac{Mg\sin\theta}{m\ell\sin\theta}} = \sqrt{\frac{Mg}{m\ell}}$

Frequency of rotation $= \frac{1}{T} = \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}$ $\quad\therefore$ Frequency $= \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$

Ex.7 ***The 4 kg block in the figure is attached to the vertical rod by means of two strings. When the system rotates about the axis of the rod, the two strings are extended as indicated in Figure. How many revolutions per minute must the system make in order that the tension in upper string is 60 N. What is tension in the lower string ?***



Sol. The forces acting on block P of mass 4 kg are shown in the Figure. If θ is the angle made by strings with vertical, T_1 and T_2 tensions in strings for equilibrium in the vertical direction



$T_1\cos\theta = T_2\cos\theta + mg$

$(T_1 - T_2)\cos\theta = mg$

$$\cos\theta = \frac{1}{1.25} = \frac{4}{5} \qquad \left[\because \cos\theta = \frac{OA}{AP} = \frac{1}{1.25}\right]$$

$$\therefore T_1 - T_2 = \frac{mg}{\cos\theta} = \frac{5mg}{4} = \frac{5}{4} \times 4 \times 9.8 = 49 \text{ N}$$

Given $T_1 = 60$ N

$T_2 = T_1 - 49 = 60 \text{ N} - 49 \text{ N} = 11 \text{ N}$

The net horizontal force ($T_1\sin\theta + T_2\sin\theta$) provides the necessary centripetal force $m\omega^2 r$.

$\therefore (T_1 + T_2)\sin\theta = m\omega^2 r$

$$\Rightarrow \omega^2 = \frac{(T_1 + T_2)\sin\theta}{mr}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(4/5)^2} = \frac{3}{5}$$

$$r = OP = \sqrt{1.25^2 - 1^2} = 0.75 \quad \therefore \omega^2 = \frac{(60+11)\frac{3}{5}}{4\times 0.75} = 0.75 \quad \Rightarrow \quad \omega = \sqrt{14.2} = 3.768 \text{ rad/s}$$

$$\text{Frequency of revolution} = \frac{\omega}{2\pi} = \frac{3.768}{2\times 3.14} = 0.6 \text{ rev/s} \quad \text{or } 36 \text{ rev/min}$$