COORDINATE GEOMETRY

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COORDINATE GEOMETRY

BY

HENRY BURCHARD FINE

AND

HENRY DALLAS THOMPSON

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PREFACE

In this book the several conics are treated early and in some detail, partly because of the value of a knowledge of their more important properties, partly because of the advantage, when presenting the analytic method to the student, of applying it in the first instance in the systematic study of a few interesting curves. In deference to usage, a chapter on the circle is introduced immediately after that on the straight line; but, if experience is to be trusted, it is better in a first course to proceed from the straight line directly to the parabola, so that, as early as possible, the student may get the impression which comes from seeing a method employed in the investigation of new material. The conics and the curves considered in Chapter XI afford illustrations of the study of locus problems by the method of coordinate geometry; and these illustrations are followed by a collection of exercises on loci in Chapter XII.

The part of the book devoted to solid geometry is more extended than is customary in elementary text-books; but it is desirable that the material here given should be easily accessible to students.

A pamphlet containing portions of the book has been in use at Princeton for three years. According to the experience thus gained, it should be possible for the better students to cover the text of the plane geometry, with the exception of Chapters III, VII, VIII, in a first-year course of three hours a week through half a year, and the remainder of the book in a second-year course of the same length.

Princeton, N.J.,
July 2, 1909.

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* This chapter may be omitted until after Chapter V.

COORDINATE GEOMETRY

COORDINATE GEOMETRY IN A PLANE

CHAPTER I

COORDINATES

1. Directed line segments. A line segment AB may be generated by the motion of a point from A to B or from B to A. In the first case the segment is called

$$AB,$$

in the second,

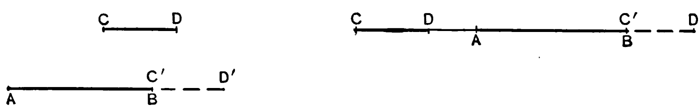
$$BA.$$

AB and BA have the same length, but are said to have opposite directions. To distinguish between them, it is customary to give them opposite algebraic signs and to write

$$AB = - BA.$$

Two segments are said to be *equal* when they have the same length and the same direction. The equal segments may be on the same line or on parallel lines.

2. Addition of line segments. Let AB and CD denote segments of the same line or of parallel lines. Shift the position



of CD, without changing its direction, so as to make C coincide with B, that is, into the position $C'D'$, as indicated in the figure. The resulting segment AD' is called the *sum* of AB and CD.

In particular, $AB + BA = 0$.

Again, if A, B, C denote any three points of the same line,

$$AB + BC = AC,$$

whether B lies between A and C or not.

According as AB and CD have the same, or opposite directions, the length of their sum, AD', will be the sum or the difference of the lengths of AB and CD.

3. Subtraction of line segments. If AB and CD denote segments of the same line or of parallel lines, $AB - CD$ is defined as $AB + (-CD)$. Hence [§ 1]

$$AB - CD = AB + DC,$$

and $$AB - (-CD) = AB - DC = AB + CD.$$

The associative and commutative laws [Alg.* §§ 33, 69, 177] hold good for addition and subtraction as just defined. Hence, so far as addition and subtraction are concerned, the general rules of reckoning are the same for line segments as for numbers represented by letters.

4. Numbers represented by line segments. The values of a single real variable, say x, may be represented as follows:

Choose some fixed line $x'x$, and on this line a fixed point O, as *origin*. Then, any value a of x being given, lay off the segment OA of length $|a|$† from O to the right when a is positive, from O to the left when a is negative. The value a of x may be represented by this segment OA, or by any segment equal to OA; for, the sign of a is indicated by the

* References in this form are to Fine's College Algebra, Ginn & Co., N.Y.

† $|a|$ is a symbol for the numerical value of a [Alg. § 63].

direction of OA, and its numerical value by the length of OA. It is customary to express this relation between a and OA by writing $a = OA$.

If the segments which represent two numbers, x_1 and x_2, are OA_1 and OA_2, respectively, the segment which represents their difference $x_2 - x_1$, is A_1A_2.

For $$x_2 - x_1 = OA_2 - OA_1 = A_1O + OA_2 = A_1A_2.$$

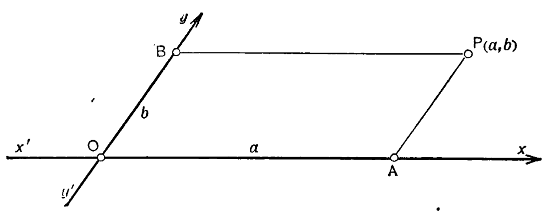
This is true whatever the signs of x_1 and x_2 may be, and therefore whatever the relative positions of the points A_1 and A_2.

5. Axes of coordinates. Let x and y denote two variables. As in the following figure, take two fixed lines $x'x$ and $y'y$ intersecting at O, and take O as origin on both lines.

(1) Any given value a of the single variable x will be represented by the segment OA of $x'Ox$, constructed as in the previous section.

(2) Similarly, any given value b of the single variable y will be represented by that segment OB of $y'Oy$ whose length is $|b|$, and which lies above or below O according as b is positive or negative.

(3) Any given *pair of values* of the two variables, as $x = a$, $y = b$, will be represented by the point P which is determined



as follows: On $x'Ox$ and $y'Oy$ construct the points A and B as in (1) and (2), and then through A and B take parallels to $y'Oy$ and $x'Ox$, respectively. The point P, in which these parallels meet, is the point required. It is called the *graph* of the value pair $(x = a, y = b)$.

The point P may also be found by first laying off the segment OA, and then the segment AP parallel and equal to OB.

If any point P be given, the value pair $(x = a,\ y = b)$, of which it is the graph, may be found by reversing the construction just described.

It is convenient to represent both the value pair $(x = a, y = b)$ and its graph P by the symbol (a, b), the value of x always being written first.

It is customary to call the number a, or one of the equal line segments OA or BP, the *abscissa* of P; and b, or one of the equal line segments OB or AP, the *ordinate* of P. The abscissa and ordinate together are called the *coordinates* of P.* Also, $x'Ox$ is called the *x-axis* or the *axis of abscissas*, and $y'Oy$, the *y-axis* or the *axis of ordinates.*

The axes, and the coordinates referred to them, are called *rectangular* or *oblique*, according as the angle xOy is a right angle or an oblique angle. When the axes are rectangular, the coordinates of a point may also be defined as its perpendicular distances from the axes.

6. Observe that this method of representing pairs of values of the two variables x, y by the points of a plane is such that: *When the axes of reference, $x'Ox$, $y'Oy$, and the unit for measuring lengths have once been chosen, to each pair of values of (x, y) there corresponds a single point P, and to each point P there corresponds a single pair of values of (x, y).*

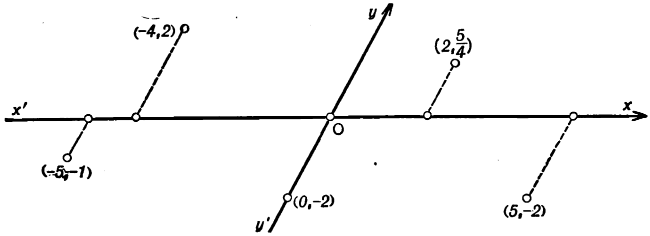
7. Exercises. Definition of coordinates.

1. In a figure, indicate the position of the following points: $(2, 5/4)$, $(0, -2)$, $(5, -2)$, $(-4, 2)$, $(-5, -1)$.

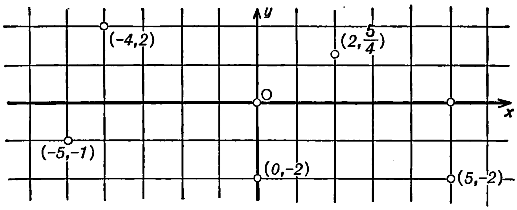
Construct the point $(2, 5/4)$ by taking the length 2 on the positive

* These are called *cartesian coordinates* after Descartes (1569–1650), who was the first to make a systematic use of them.

x-axis and then the length 5/4 parallel to the positive y-axis; and similarly for the other points. The following figure is thus obtained:



If rectangular axes are chosen, the points may be plotted also by measuring off lengths on squared paper as in the following figure:



2. In a figure, indicate the position of the following points: $(2, -1)$, $(0, 0)$, $(-3, -1)$, $(-2, 0)$, $(3, 2)$, $(\sqrt{5}, -\sqrt{2})$. Choose at random any two numbers, positive or negative, for the coordinates, and plot the point.

3. Plot on one figure $(-2, 4)$, $(-2, 3)$, $(-2, 1)$, $(-2, 0)$, $(-2, -2)$, $(-2, -3)$. If the abscissa of a point is -2 but its ordinate is not given, what is known about the position of the point?

4. What are the coordinates of the origin?

5. What are the coordinates of the point halfway between the origin and $(3, -8)$?

6. Prove that if A, B, C, D, E be any five points of the same straight line, then $AB + BC + CD + DE + EA = 0$.

7. If the axes are rectangular, prove that the points (a, b) and $(a, -b)$ are symmetric with respect to the x-axis; that (a, b) and $(-a, b)$ are symmetric with respect to the y-axis; and that $(a, -b)$ and $(-a, b)$ are symmetric with respect to the origin.

8. A line joining two given points is bisected at the origin. If one of the points is $(2, -3)$, what is the other?

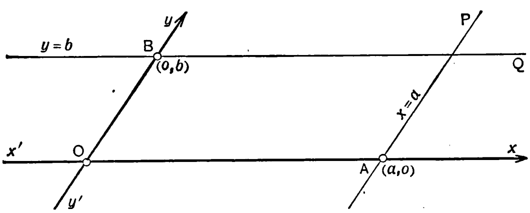
CHAPTER II

THE STRAIGHT LINE

8. Graphs of equations. An equation in the two variables x and y will ordinarily be satisfied by infinitely many pairs of real values of x and y. Every such pair is called a *real solution* of the equation. Suppose axes of coordinates to be taken as in § 5. Then each real solution of the equation will have its graph. The collection of all these graphs (which will usually form a curve) is called the *graph* or *locus of the given equation.*

9. *The graph of every equation of the first degree in x, y is a straight line.*

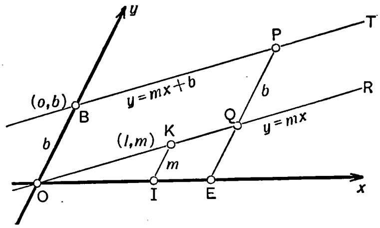
For consider the four particular forms of this equation: $x = a$, $y = b$, $y = mx$, $y = mx + b$.



First. The graph of $x = a$ is the line AP through the point $(a, 0)$ and parallel to the y-axis. For this line contains every point whose abscissa is a, and such points only.

Second. The graph of $y = b$ is the line BQ through the point $(0, b)$ and parallel to the x-axis. For this line contains every point whose ordinate is b, and such points only.

Third. The graph of $y = mx$ is the line OR through the origin and the point $(1, m)$. For this line contains every point, such as Q, whose ordinate EQ is m times its abscissa OE, and such points only.



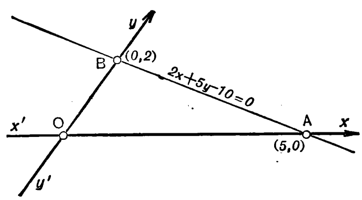
Fourth. The graph of $y = mx + b$ is BT, that parallel to the graph of $y = mx$ which passes through the point $(0, b)$. For BT contains every point, such as P, got by adding b to the ordinate EQ $(= mOE)$ of a point, Q, of OR; and it contains such points only.

Multiplying or dividing an equation by a constant (not 0) does not affect its solutions [Alg. § 338] and therefore does not affect its graph. And every equation of the first degree, $ax + by + c = 0$, may be reduced to one of the four forms just considered by dividing by the coefficient of x or y and transposing certain terms. Hence the graph of every equation of the first degree in x, y is a straight line.

10. Since a straight line is determined by any two of its points, the graph of an equation of the first degree may be obtained by finding any two of its solutions and plotting them.

Example 1. Find the graph of $2x + 5y - 10 = 0$.

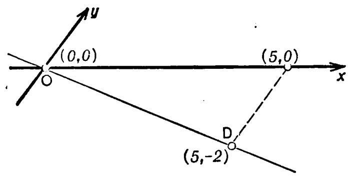
When $y = 0$, then $x = 5$; again, when $x = 0$, then $y = 2$. Hence two of the solutions are (5, 0) and (0, 2). Plot the corresponding points A (5, 0) and B (0, 2). The line AB is the graph required.



The two solutions A and B are numerically the simplest to find, but any two solutions of the equation give two points of the line, and thus determine it; for example, (5/2, 1) and (− 5, 4).

Example 2. Find the graph of $2x + 5y = 0$.

Two solutions of this equation must be found. One solution is seen to be (0, 0), and a second solution is found by inspection to be $(5, -2)$. Hence the line is that determined by the origin O (0, 0) and the point $D(5, -2)$. Plot these points; the graph is the line OD.



11. Exercises. Find and draw the graph of each of the following equations:

1. $x = 0$.	7. $y = -2$.	13. $3x + y = 0$.
2. $y = 0$.	8. $2x - 3 = 0$.	14. $2x - 3y = 0$.
3. $x = 2$.	9. $2x + 3 = 0$.	15. $x + y + 1 = 0$.
4. $y = 3$.	10. $y = x$.	16. $2x - y - 2 = 0$.
5. $x = -1$.	11. $2y - 3x = 0$.	17. $x + y - 1 = 0$.
6. $x = -4$.	12. $2y + 3x = 0$.	18. $3x - 2y + 6 = 0$.

12. *Two equations of the first degree*

$$ax + by + c = 0 \quad (1) \qquad \textit{and} \qquad a'x + b'y + c' = 0 \quad (2)$$

have the same graph when, and only when, their corresponding coefficients are proportional, that is, when

$$a/a' = b/b' = c/c'.$$

For the graph of (1) is the same as that of (2) when, and only when, the infinitely many solutions of (1) are the same as those of (2). But the solutions of (1) are the same as those of (2) when, and only when, (1) may be derived from (2) by multiplication by some constant, k, so that

$$ax + by + c \equiv k(a'x + b'y + c'),$$

or, $$a = ka', \quad b = kb', \quad c = kc',$$

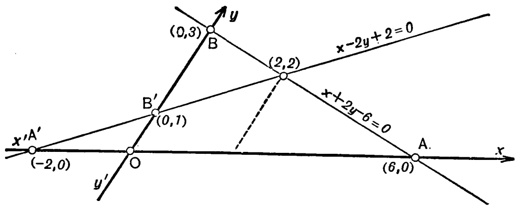
that is, $$a/a' = b/b' = c/c'.$$

Thus, the equations $4x + 2y + 10 = 0$, $6x + 3y + 15 = 0$ have the same graph, since $4/6 = 2/3 = 10/15$.

13. A pair of independent and consistent simultaneous equations,

$$ax + by + c = 0, \ (1) \qquad a'x + b'y + c' = 0, \ (2)$$

have one and but one solution in common. The graph of this solution is the point of intersection of the lines which are the graphs of the equations (1) and (2) themselves. For this point, and this point only, is the graph of a solution of both equations.



Example. The solution of the pair of equations $x + 2y - 6 = 0$ (1) and $x - 2y + 2 = 0$ (2) is (2, 2). The graphs of (1) and (2), found by the method of § 10, are the lines AB and $A'B'$ in the figure. And, as is indicated in the figure, these lines intersect at the point (2, 2).

It may be added that the equations (1) and (2) are both independent and consistent unless $a/a' = b/b'$. If $a/a' = b/b' = c/c'$, they are not independent [Alg. § 377, 1], and, as is proved in § 12, their graphs coincide throughout. If $a/a' = b/b' \neq c/c'$, they are not consistent [Alg. § 377, 2]; they have no finite solution in common, and their graphs are parallel lines.

14. The graph of the single equation

$$(ax + by + c)(a'x + b'y + c') = 0$$

consists of the graphs of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ jointly. For, since a product of integral factors vanishes when one of these factors vanishes and then only, the solutions of any integral equation of the form $C \cdot D = 0$ are the solutions of the equations $C = 0$ and $D = 0$ jointly. [See Alg. §§ 341, 346.]

Thus, the graph of $(x + 2y - 6)(x - 2y + 2) = 0$ is the *pair* of lines AB, $A'B'$ in the last figure.

15. Exercises. Graphs of one or more equations of the first degree.

1. Find and draw the graphs of the following equations:

(1) $2x - 3y + 4 = 0$.
(2) $4x - 6y + 8 = 0$.
(3) $3y - 2x - 4 = 0$.
(4) $x/3 - y/2 + 2/3 = 0$.

2. Find the graph of the *solution* of each of the following pairs of equations:

(1) $2x - 3y + 4 = 0$ and $x + y + 2 = 0$.
(2) $2x + 5y - 10 = 0$ and $2x - 5y = 0$.
(3) $2x + 5y - 10 = 0$ and $3x + 5y = 0$.

3. Draw on one figure the graphs of the equations $2x + 5y - 10 = 0$ and $2x + 5y = 0$. Is there a finite solution of these equations?

4. Find and draw the graphs of the following equations:

(1) $(2x - 3y + 4)(x + y + 2) = 0$.
(2) $(2x + 5y - 10)(2x - 5y) = 0$.
(3) $(2x + 5y - 10)(3x + 5y) = 0$.

What is the difference between this exercise and Ex. 2, where the graph of a *solution* of a pair of equations is sought?

5. Find and draw the graphs of the following equations:

(1) $x^2 + x - 12 = 0$.
(2) $2x^2 - 5xy - 12y^2 = 0$.

6. Prove that the graph of $ax^2 + 2hxy + by^2 = 0$ is a pair of straight lines (real or imaginary) through the origin.

7. What must be the values of a and b, if $ax + 8y + 4 = 0$ and $2x + ay + b = 0$ are to represent the same straight line?

16. Equations of straight lines. Given an equation of the first degree $ax + by + c = 0$, find two of its solutions and plot the points which are their graphs. As has already been seen, the straight line determined by these two points will be the graph of $ax + by + c = 0$.

Conversely, given any straight line, l, select two of its points and find their coordinates. Let these be (x', y') and (x'', y''). There is one, and but one, equation $ax + by + c = 0$ (1) of which x', y' and x'', y'' are solutions. For if (x', y') and (x'', y'') are to be solutions of (1), the equations $ax' + by' + c = 0$ (2) and $ax'' + by'' + c = 0$ (3) are true. Subtract (2) from (1), and (3) from (2); the results can be written $a(x - x') = -b(y - y')$ (4) and $a(x' - x'') = -b(y' - y'')$ (5). Divide (4) by (5); the result is $(x - x')/(x' - x'') = (y - y')/(y' - y'')$ (6). This is the equation of which l is the graph.

Example. Find the equation of which the line through the points (1, 4) and (– 1, – 2) is the graph.

The equation can be found by substituting in (6); but it can also be obtained as follows: Let $ax + by + c = 0$ (1) represent the required equation. Since (1, 4) and (– 1, – 2) are to be solutions of (1), the equations $a + 4b + c = 0$ (2) and $-a - 2b + c = 0$ (3) must be true. Solving the equations (2), (3) for a and b in terms of c gives $a = 3c$, $b = -c$. Substituting these values of a and b in (1) gives $3cx - cy + c = 0$, or dividing by c, $3x - y + 1 = 0$, which is the equation required.

Hence, *to every given straight line there corresponds an equation of the first degree in x and y of which the line is the graph: that is, an equation which is satisfied by the coordinates of each and every point on the line and by the coordinates of no other points:* or more briefly, *an equation which is true for every point on the line and false for every point off the line.* It is called *the equation of the line,* but this phrase is a mere abbreviation for the phrase: *the equation corresponding to the line.*

17. It follows from what has just been said that:

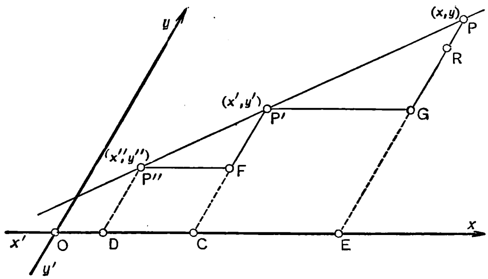
If a given equation of the first degree is true for two points of a certain line, it is the equation of that line.

18. If the equation of a line l is $ax + by + c = 0$, it is sometimes convenient to use l as a symbol for $ax + by + c$ and to write the equation of the line as $l = 0$.

19. The equation of a line may be obtained in various forms corresponding to the various pairs of conditions that may be given to determine the line. The more important of these forms will now be considered.

20. Line through two given points. Required the equation of the line determined by any two given points.

Let $P'(x', y')$ and $P''(x'', y'')$ be the two given points which determine the line $P'P''$. The equation of $P'P''$ has been derived by an *algebraic* method in the second paragraph of § 16. But it may also be derived *geometrically*, as follows:



Let P denote a *representative point* of $P'P''$, that is, a point which may lie anywhere on $P'P''$; and let x, y denote the coordinates of P. Let the line through P'' parallel to the x-axis meet the line through P' parallel to the y-axis at F, and let the line through P' parallel to the x-axis meet the line through P parallel to the y-axis at G; let D, C, E be the feet of the ordinates of P'', P', P, respectively.

From the similarity of the triangles $P'GP$ and $P''FP'$,

$$\frac{P'G}{GP} = \frac{P''F}{FP'}, \text{ and therefore } \frac{P'G}{P''F} = \frac{GP}{FP'}.$$

But $P'G = OE - OC = x - x'$, $\quad GP = EP - CP' = y - y'$,

$$P''F = OC - OD = x' - x'',\ FP' = CP' - DP'' = y' - y''.$$

Hence
$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''}, \tag{1}$$

which is the equation required. For besides x, y it involves only the known quantities x', y', x'', y''; it is true, as has just been proved, for every point P on $P'P''$; and it may be proved as follows to be false for every point not on $P'P''$: Take any such point R, and through R take ERP parallel to the y-axis, and meeting the line $P'P''$ at P. The left member of (1) will have the same value for R as for P, but the right member will have a different value for R than for P. Therefore, since (1) is true for P, it is false for R.

The derivation of the equation (1) of the line $P'P''$ fails when the line is parallel to either axis. But, if $P'P''$ be parallel to the x-axis, then $y' = y''$, and the equation of $P'P''$ is $y = y'$. Similarly, if $P'P''$ be parallel to the y-axis, its equation is $x = x'$.

By applying the theorem of § 17, it may also be proved by *inspection* that (1) is the equation of the line $P'P''$. For since (1) is an equation of the first degree in x, y, its graph is a straight line; and since (1) is satisfied by $x = x'$, $y = y'$, and by $x = x''$, $y = y''$, this straight line passes through the points $P'(x', y')$, $P''(x'', y'')$, and is therefore the line $P'P''$.

By the same method it can be proved that the equation of $P'P''$ may also be written in the determinant form:

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix} = 0. \tag{1'}$$

For (1′) is an equation of the first degree in (x, y), as may be seen by expanding the determinant, and it is satisfied by $x = x'$, $y = y'$, and by $x = x''$, $y = y''$, since a determinant vanishes when two of its rows are equal [Alg. § 903].

It is sometimes more convenient to write the equation (1) in the form

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \tag{1''}$$

The equation of the line through the origin (0, 0) and the point (x', y') is

$$y = \frac{y'}{x'}x. \tag{1'''}$$

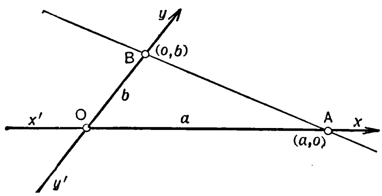
The equations (1), (1′), and (1″) are merely different *forms* of one and the same equation.

The equations (1), (1′), (1″), and (1‴) hold good whether the axes are rectangular or oblique.

21. Exercises. Lines through two points. In all exercises reduce each equation to its simplest form.

1. Obtain the equation of the line determined by (1, 3) and (2, 1).
2. By (– 1, 8) and (4, – 2).
3. By (1, – 1) and (– 5, – 2).
4. By (4, 0) and (0, 1).
5. By (0, 0) and (– 6, 1).
6. By (3, 2) and (3, 1).
7. By (1, – 1) and (– 1, – 1).
8. By any two points chosen at random.
9. Show by the theorem of § 17 that $(x-x')(y-y'')=(x-x'')(y-y')$ is the equation of the line determined by the points (x', y') and (x'', y'').

22. Intercept form of the equation. Let a straight line cut the x- and y-axes at A and B respectively. The segment OA is called the *intercept* on the x-axis, or the x-intercept, and may be represented in length and direction by the number a. Similarly, the segment OB is called the intercept on the y-axis, or the y-intercept, and may be represented by b. Evidently a line is determined when its intercepts a and b are given. Its equation may be found in terms of a and b as follows:



The coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively. Hence by § 20, (1) the equation of the line through A and B is

$$\frac{x-a}{a-0}=\frac{y-0}{0-b}, \text{ or } \frac{x}{a}-1=-\frac{y}{b}, \text{ or } \frac{x}{a}+\frac{y}{b}=1. \qquad (2)$$

This form of the equation also is true for both rectangular and oblique axes.

The general equation $Ax+By+C=0$ when reduced to the intercept form becomes

$$\frac{x}{-C/A}+\frac{y}{-C/B}=1. \qquad (3)$$

Example 1. If the intercepts of a line are – 3 and 2 respectively, the equation of the line is $x/(-3)+y/2=1$, or $2x-3y+6=0$.

Example 2. The equation of a certain line is $6x + 2y + 3 = 0$, which may be written in the form

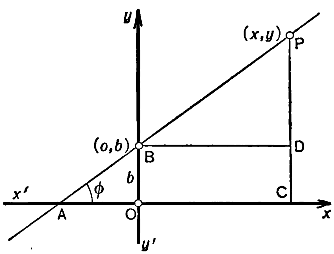
$$\frac{x}{-3/6} + \frac{y}{-3/2} = 1.$$

Its intercepts, therefore, are $-1/2$ and $-3/2$, respectively.

23. Equation in terms of slope and y-intercept.

The following forms of the equation of a straight line hold good for rectangular axes only.

Let ϕ denote the angle xAB which the line AB makes with the positive direction Ox on the x-axis, the angle being measured in the positive sense from Ox to AB, as indicated in the figure. The tangent of this angle ϕ is called the *slope* of the line AB, and is represented by m; so that $m = \tan \phi$. Evidently a line is determined when its slope m and its y-intercept b are given. Its equation in terms of m and b is found as follows:



Let $P(x, y)$ be a representative point of AB. Take PC, the perpendicular to Ox, and BD, the perpendicular to PC. Then $DP = BD \tan DBP$. But $DP = CP - CD = CP - OB = y - b$, $BD = OC = x$, and $\tan DBP = \tan xAP = \tan \phi = m$. Hence $y - b = mx$, or

$$y = mx + b, \tag{3}$$

which is the equation required. [Compare § 9.]

When the line passes through the origin, b is 0 and (3) becomes

$$y = mx. \tag{3'}$$

The equation $Ax + By + C = 0$ when reduced to the form $y = mx + b$ becomes $y = (-A/B)x + (-C/B)$. Hence the slope of the line represented by $Ax + By + C = 0$ is $-A/B$.

Example 1. The y-intercept of a line is $-3/2$, and it makes an angle of $60°$ with Ox; find its equation.

Since $\tan 60° = \sqrt{3}$, the equation is

$$y = \sqrt{3}\,x + (-3/2), \text{ or } 2\sqrt{3}\,x - 2y - 3 = 0.$$

Example 2. The equation of a line is $6x - 2y - 3 = 0$; find its slope and its y-intercept.

The equation may be reduced to the form $y = 3x + (-3/2)$; hence its y-intercept is $-3/2$, and its slope is 3.

24. Parallel lines. Evidently lines which have the same slope are parallel. Hence, the following theorems:

25. *The lines* $ax + by + c = 0$ (1) *and* $a'x + b'y + c' = 0$ (2) *are parallel, if* $a/a' = b/b'$.

For the slopes of (1) and (2) are $-a/b$ and $-a'/b'$, and if $a/a' = b/b'$, then $-a/b = -a'/b'$. [Compare § 13.]

26. *Every line parallel to the line* $ax + by + c = 0$ (1) *has an equation of the form* $ax + by + D = 0$ (2).

For if (x', y') be a point on a given line l parallel to (1), and D be given such a value that $ax' + by' + D \equiv 0$, that is, if $D \equiv -(ax' + by')$, then (2) will represent a line through (x', y') and parallel to (1), that is, the line l.

27. *The equation of the line which passes through the point* (x', y') *and which is parallel to the line* $ax + by + c = 0$ (1) *is* $a(x - x') + b(y - y') = 0$ (2).

For the line (2) has the same slope as the line (1), the coefficients of x and y being the same in (2) as in (1), and this line passes through the point (x', y') since (2) is satisfied by $x = x'$, $y = y'$.

Example. Find the equation of the line through the point (3, 4) and parallel to the line $2x - 3y + 5 = 0$.

The required equation has the form $2x - 3y + D = 0$; and since it is satisfied by the coordinates of the given point (3, 4), $2 \cdot 3 - 3 \cdot 4 + D = 0$, or $D = 6$. Hence the required equation is $2x - 3y + 6 = 0$. This is the method of § 26.

Or, using the method of § 27, it follows that the required equation is $2(x-3) - 3(y-4) = 0$, which reduces to $2x - 3y + 6 = 0$.

28. Equation of line in terms of slope and coordinates of a point. The equation of a line through the point (x', y') and having the slope m is

$$y - y' = m(x - x'). \tag{4}$$

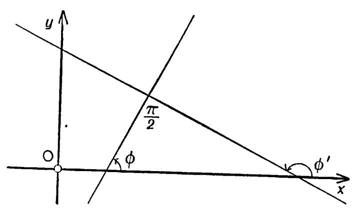
For, as in § 27, the equation (4) represents a line through the point (x', y') and parallel to the line $y = mx$ which has the slope m.

Example. A line makes an angle of 30° with the positive x-axis and passes through the point $(\sqrt{3}, 2)$; find its equation.

The slope m is $\tan 30° = 1/\sqrt{3}$, and therefore the required equation is $y - 2 = (1/\sqrt{3})(x - \sqrt{3})$, or $x - \sqrt{3}\,y + \sqrt{3} = 0$.

29. Perpendicular lines. *If two lines $y = mx + b$ and $y = m'x + b'$ are perpendicular, the slope of the one is the negative reciprocal of the slope of the other, that is, $m = -1/m'$; and conversely.*

For $m = \tan\phi$ and $m' = \tan\phi'$, where ϕ and ϕ' denote the angles made by the lines with the positive x-axis. Hence, if the lines are perpendicular, and if ϕ' denotes the larger of the two angles ϕ and ϕ', $\phi' = \phi + \pi/2$, and therefore $\phi = \phi' - \pi/2 = -(\pi/2 - \phi')$. Then it follows that $\tan\phi = -\tan(\pi/2 - \phi') = -\cot\phi' = -1/\tan'\phi$; or, $m = -1/m'$.



Conversely, if $m = -1/m'$, the lines are perpendicular. For $\tan\phi = -1/\tan\phi' = -\cot\phi' = -\tan(\pi/2 - \phi') = \tan(\phi' - \pi/2)$; therefore, $\phi = \phi' - \pi/2$, or $\phi' = \phi + \pi/2$; that is, the lines are perpendicular.

30. *The two lines* $ax + by + c = 0$ (1) *and* $a'x + b'y + c' = 0$ (2) *are perpendicular, if* $aa' + bb' = 0$.

For, if $aa' + bb' = 0$, then $-a/b = b'/a'$, or $-a/b$, the slope of (1), is the negative reciprocal of $-a'/b'$, the slope of (2).

31. *Every line perpendicular to the line* $ax + by + c = 0$ (1) *has an equation of the form* $bx - ay + D = 0$ (2).

For if (x', y') be a point on a given line l perpendicular to (1), and D be given such a value that $bx' - ay' + D \equiv 0$, or that $D \equiv -(bx' - ay')$, then (2) will represent a line through (x', y') and perpendicular to (1), that is, the line l.

32. *The equation of a line through the point* (x', y') *and perpendicular to the line* $ax + by + c = 0$ (1) *may be written* $b(x - x') - a(y - y') = 0$ (2).

For (2) represents a line through the point (x', y') and whose slope, namely, b/a, is the negative reciprocal of the slope of (1), namely, $-a/b$.

Example 1. The lines $3x + 2y = 0$ (1) and $2x - 3y = 0$ (2) are perpendicular, since the slope of (1) is $-3/2$ and that of (2) is $2/3$, and $-3/2 = \frac{-1}{2/3}$.

Example 2. Find the equation of the line through the point $N(2, 3)$ and perpendicular to the line $3x - 2y = 0$.

The required equation has the form $2x + 3y + D = 0$; and since it is to represent a line through $N(2, 3)$, $2 \cdot 2 + 3 \cdot 3 + D = 0$, or $D = -13$. Hence, the equation is $2x + 3y - 13 = 0$. This is the method of § 31.

Or, by § 32, the required equation is $2(x - 2) + 3(y - 3) = 0$ which reduces to $2x + 3y - 13 = 0$.

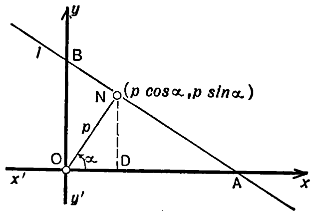
33. Exercises. Parallels and perpendiculars to a given line.

1. Find the equation of the line through $(-3, 1)$ and parallel to $2x + y - 1 = 0$.
2. Through $(0, 0)$ and parallel to $x - 2y + 3 = 0$.
3. Through $(-1, 1)$ and parallel to $x - 2y + 3 = 0$.
4. Through $(1, -1)$ and parallel to $x - 2y + 3 = 0$.

5. Find the equations of the lines perpendicular to $x + 2y + 1 = 0$, and through the points $(1, -1)$, $(-1, -1)$, $(1, 2)$, respectively.

6. Let A (1, 3), B (−2, −4), C (1, −2), be a triangle; find the equations of the perpendiculars from the vertices to the opposite sides.

34. Perpendicular form of the equation of the line. Let l be a given line, let N be the point in which the perpendicular to l through the origin O meets l, let α be the positive angle xON which ON makes with the positive x-axis, let p be the length of ON. Evidently the line l is determined when p and α are given. Its equation in terms of p and α is found as follows:



Take the perpendicular from N to the x-axis, meeting it in D; then $OD\ (= p \cos \alpha)$ is the x of the point N, and $DN\ (= p \sin \alpha)$ is the y of N; that is, N is the point $(p \cos \alpha, p \sin \alpha)$.

The equation of the line ON is $y = \tan \alpha \cdot x$ [§ 23, (3′)]; or, since $\tan \alpha = \sin \alpha / \cos \alpha$, its equation is $x \sin \alpha - y \cos \alpha = 0$.

Therefore, since l passes through $N(p \cos \alpha, p \sin \alpha)$ and is perpendicular to the line $x \sin \alpha - y \cos \alpha = 0$, its equation is [§ 32]

$$(x - p \cos \alpha) \cos \alpha + (y - p \sin \alpha) \sin \alpha = 0,$$

or

$$x \cos \alpha + y \sin \alpha - p (\cos^2 \alpha + \sin^2 \alpha) = 0,$$

or

$$x \cos \alpha + y \sin \alpha - p = 0. \qquad (5)$$

When l passes through the origin, the length of the perpendicular p is 0, the coordinates of N are (0, 0), and the equation of the line is

$$x \cos \alpha + y \sin \alpha = 0. \qquad (5')$$

When l does not pass through the origin, p is positive and α may have any value from 0 to 2π.

When l passes through the origin (so that p is 0), α is taken as the angle less than π which the perpendicular to l at O makes with the x-axis. Hence in the equation (5′), the coefficient of y, namely $\sin \alpha$, is always positive.

Any given equation, $Ax + By + C = 0$, may be reduced to the perpendicular form by the following method:

Since two equations of the first degree represent the same straight line when, and only when, they differ by a constant factor at most, [§ 12], the problem is to find three constants, λ, p, α, of which p is positive, such that

$$x \cos \alpha + y \sin \alpha - p \equiv \lambda (Ax + By + C).$$

But this will be a true identity, if the corresponding coefficients in its two members be equal, that is, if

$$\cos \alpha = \lambda A, \ (1) \qquad \sin \alpha = \lambda B, \ (2) \qquad -p = \lambda C. \ (3)$$

Since p is to be positive, (3) requires that λ shall have the opposite sign to that in C. Squaring (1) and (2), and adding, gives

$$\lambda^2(A^2 + B^2) = \cos^2 \alpha + \sin^2 \alpha = 1, \quad \therefore \lambda = 1/\pm\sqrt{A^2 + B^2}.$$

Substituting this value of λ in (1), (2), (3), gives

$$\cos \alpha = \frac{A}{\pm\sqrt{A^2 + B^2}}, \quad \sin \alpha = \frac{B}{\pm\sqrt{A^2 + B^2}}, \quad -p = \frac{C}{\pm\sqrt{A^2 + B^2}},$$

where the sign before the radical is opposite to that in C.

The method applies when C is 0. But in this case, to have the result in the form $x \cos \alpha + y \sin \alpha = 0$, where $\sin \alpha$ is positive, the radical $\sqrt{A^2 + B^2}$ must be given the same sign as that in B.

Hence the following rule:

To reduce any equation $Ax + By + C = 0$ to the perpendicular form, divide by $\pm\sqrt{A^2 + B^2}$, where the sign $\pm$ is opposite to that in C when $C \neq 0$, but the same as that in B when $C = 0$.

Example. Reduce $3x - 4y - 2 = 0$ to the perpendicular form.

In this case $\sqrt{A^2 + B^2}$ is 5, and since the absolute term of the original equation is negative, the divisor is positive. Hence the equation is

$$\frac{3x - 4y - 2}{5} = 0, \quad \text{or} \quad \tfrac{3}{5}x - \tfrac{4}{5}y - \tfrac{2}{5} = 0.$$

Here, $p = 2/5$, $\cos \alpha = 3/5$, and $\sin \alpha = -4/5$.

35. Recapitulation. It has been proved that the graph of every equation of the first degree $Ax + By + C = 0$ is a straight line; and conversely, it has been proved that to every straight line there corresponds a definite equation of the first degree, $Ax + By + C = 0$, which is true for every point on the line and false for every point off it, and which is therefore called the equation of the line. Various pairs of conditions may be given for determining the line; from such a pair of conditions the equation of the line may be obtained either geometrically (as in § 20) or algebraically (as in § 16), the latter method depending on the fact that two geometrical conditions which can be expressed by means of two homogeneous equations of the first degree in A, B, C give two of these letters in terms of the third, and therefore determine the equation $Ax + By + C = 0$. The forms in which the equation of a line has been derived are the following:

1. $\dfrac{x - x'}{x' - x''} = \dfrac{y - y'}{y' - y''}$ the *two-point* form.
2. $\dfrac{x}{a} + \dfrac{y}{b} = 1$ the *intercept* form.
3. $y = mx + b$ the *slope and y-intercept* form.
4. $y - y' = m(x - x')$ the *slope and one-point* form.
5. $x \cos \alpha + y \sin \alpha - p = 0$ the *perpendicular* form.

Obviously each of these five forms can be reduced to the form $Ax + By + C = 0$. Conversely, $Ax + By + C = 0$ can be reduced to each of these five forms.

36. Exercises. The equation of the straight line.

1. Draw the graphs of the following equations:

(1) $3x + 2 = 0$, (3) $8x + 3y = 5$, (5) $x/2 + y/5 = 1$,
(2) $2x + 3y = 0$, (4) $y = 2x + 3$, (6) $y - 3 = \sqrt{3}(x + 1)$,
(7) $(3/5)x - (4/5)y + 2 = 0$, (10) $x(x^2 - 1) = 0$,
(8) $(x + y - 5)(x - 2y) = 0$, (11) $xy = 0$,
(9) $x^2 - 4y^2 = 0$, (12) $x^2y - xy = 0$.

2. Given the following values of the constants, find in each case the equation of the line and draw the graph:

$$(1)\ m = 2,\ b = -5;\ (2)\ a = -3,\ b = 2;\ (3)\ p = 5,\ \alpha = 30^\circ.$$

3. Which of the following points are on the line $3x + 2y - 6 = 0$: (1, 1), (4, -3), (3, 0), (2, 0), (0, 2), (0, 3), (-2, 6), (1, $1\frac{1}{2}$)?

4. What is the equation of the x-axis?

5. Find the equations of the lines determined by the following pairs of points, and determine the intercepts on the axes: (2, -3), (-3, -2); (2, 4), (1, -1); (a, 0), (0, b); (2, -1), (-1, -1).

6. Do the following lines meet in the point (1, -1): $4x+5y+1=0$, $4x - 13y = 17$, $12x + 7y - 5 = 0$?

7. A straight line makes twice as great an intercept on the x-axis as on the y-axis and passes through the point (-2, 3); find its equation.

8. Find the equation of a straight line which passes through the intersection of the lines $x = a$ and $y + b = 0$, and through the origin.

9. Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (x_1, y_1).

10. Given the line $5x + 12y - 2 = 0$; find the slope, the intercepts on the axes, and the length of the perpendicular from the origin.

11. What are the equations of the diagonals of a rectangle whose vertices are (0, 0), (a, 0), (0, b), and (a, b)? Find also the point of intersection of the diagonals.

12. Find the equation of the line which passes through the point (2, -3) and makes an angle of 60° with the x-axis.

13. For each of the following lines find the slope, intercepts, perpendicular from the origin, and the angle which the perpendicular makes with the x-axis: $3x - 4y - 25 = 0$, $24x - 7y + 15 = 0$.

14. Prove that the following four points lie on the same straight line: (3, 2), (1, -2), (4, 4), (-2, -8).

37. Lines through the point of intersection of two given lines. *Let* $ax + by + c = 0$ (1) *and* $a'x + b'y + c' = 0$ (2) *be the equations of two given lines, and* λ *an arbitrary constant. Then*

$$(ax + by + c) + \lambda(a'x + b'y + c') = 0 \qquad (3)$$

will represent the system of lines through the point of intersection of (1) *and* (2).

For whatever the value of λ may be, (3) represents a straight line, since it is of the first degree in x, y; and this line will pass through the point of intersection of (1) and (2), since for this point both $ax+by+c$ and $a'x+b'y+c'$ are 0, and therefore (3) is satisfied.

And conversely, every given line, l, through the point of intersection of (1) and (2) is included among the lines represented by (3). For if (x', y') denote any second point of l, the constant, λ, can be given such a value that (3) will be satisfied by $x=x'$, $y=y'$; and when an equation of the first degree is true for two points of a line it is the equation of that line [§ 17].

Example 1. Find the equation of the line through the point of intersection of $2x-3y=0$ and $x+5y-4=0$, and the point (1, 2).

Since the required line passes through the point of intersection of $2x-3y=0$ and $x+5y-4=0$, it has an equation of the form

$$(2x-3y)+\lambda(x+5y-4)=0.$$

And since it passes through (1, 2), this equation must be satisfied by $x=1$, $y=2$.

Hence $(2-3\cdot 2)+\lambda(1+5\cdot 2-4)=0$, or $\lambda=4/7$.

Therefore the required equation is

$$(2x-3y)+(4/7)(x+5y-4)=0,$$

or

$$18x-y-16=0.$$

Example 2. Find the equation of the line through the point of intersection of $2x-3y=0$ and $x+5y-4=0$, and perpendicular to $4x-y+3=0$.

The required equation has the form

$$(2x-3y)+\lambda(x+5y-4)=0.$$

But its slope, namely $(2+\lambda)/(3-5\lambda)$, must be the negative reciprocal of the slope of $4x-y+3=0$, and this slope is 4.

Hence $(2+\lambda)/(3-5\lambda)=-1/4$, or $\lambda=11$.

Therefore the required equation is

$$(2x-3y)+11(x+5y-4)=0, \text{ or } 13x+52y-44=0.$$

Example 3. Prove that all lines which make intercepts on the x- and y-axes the sum of whose reciprocals is a constant k, pass through a fixed point.

The equation of every line of the system may be written

$$\frac{x}{a}+\frac{y}{b}-1=0 \quad (1), \quad \text{where } \frac{1}{a}+\frac{1}{b}=k. \qquad (2)$$

From (2) we have $1/b=k-1/a$. Substituting this in (1), gives

$$x/a+y(k-1/a)-1=0, \text{ or } (ky-1)+(1/a)(x-y)=0. \qquad (3)$$

But whatever the value of $1/a$ may be, (3) represents a line through the point of intersection of $ky-1=0$ and $x-y=0$; that is, through the point $(1/k, 1/k)$.

38. Exercises. Draw the graph in each case.

1. Find the equation of the line through the point of intersection of $x+y+1=0$ and $2x-3y-2=0$, and the point (3, 2).

2. Through the point of intersection of $2x-3y-2=0$ and $3x+2y-7=0$, and the point (3, 2).

3. Find the equation of the straight line which passes through the point (1, − 3) and is
   (*a*) parallel to the line $5x-2y+3=0$,
   (*b*) perpendicular to the line $3x+y=0$.

4. Find the equation of the line through the intersection of the lines $2x-2y+5=0$ and $4x+y-7=0$, and through the point (2, − 5).

5. Find the equation of the line perpendicular to $8y+5x-3=0$, which cuts the y-axis at a distance 8 from the origin.

6. Find the equation of the line through the intersection of the lines $5x+2y=8$ and $3y-4x=35$, which passes through the origin.

7. Find the equation of the line through the intersection of the lines $x-6y+4=0$ and $2x+2y-3=0$, and parallel to the y-axis.

8. Find the equations of the three lines through the point of intersection of $x-y+2=0$ and $4x+y-2=0$, and perpendicular, respectively, to the three lines: $x+y=0$, $x-4y+1=0$, $2x+5y-3=0$.

9. Find the equations of the two lines through the point of intersection of $x+7y-2=0$ and $2x-y+4=0$, and perpendicular, respectively, to these two lines.

39. Condition that three lines shall meet in a common point. The point of intersection of the lines represented by the equations $ax+by+c=0$ (1) and $a'x+b'y+c'=0$ (2) is the one point whose coordinates (x', y') satisfy both (1) and (2). It may therefore be found by regarding (1) and (2) as simultaneous, and solving for x and y [§ 13].

The lines represented by the equations $ax+by+c=0$ (1), $a'x+b'y+c'=0$ (2), and $a''x+b''y+c''=0$ (3) will pass through one common point when, and only when, the solution of two of the equations will satisfy the third, and, as is shown in algebra [Alg. § 922], this is true when, and only when, the coefficients of (1), (2), and (3) are connected by the relation:

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0. \qquad (4)$$

Example. Prove that the lines $2x-y+5=0$, $x+4y-1=0$, and $5x+2y+9=0$ meet in a common point.

In this case the determinant (4) is

$$\begin{vmatrix} 2 & -1 & 5 \\ 1 & 4 & -1 \\ 5 & 2 & 9 \end{vmatrix} = 72+5+10-100+9+4=0.$$

It may also be inferred that the lines (1), (2), (3) meet in one common point, if three constants k, l, m, not all 0, can be found such that

$$k(ax+by+c)+l(a'x+b'y+c')+m(a''x+b''y+c'')\equiv 0. \qquad (5)$$

This follows from § 37. It may also be proved thus:

Since, by hypothesis, (5) is an identity, its coefficients with respect to x and y must be 0, that is:

$$ka+la'+ma''=0,\ kb+lb'+mb''=0,\ kc+lc'+mc''=0. \qquad (6)$$

But since, by hypothesis, k, l, m are not all 0, and yet satisfy the three equations (6), the determinant of their coefficients in these equations must vanish [Alg. § 921], that is,

$$\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = 0. \tag{7}$$

But (7) is equivalent to (4) [Alg. § 899], and, as has already been proved, when (4) is satisfied, the lines (1), (2), (3) meet in a common point.

Example. Show that the perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.

Let the vertices be (x_1, y_1), (x_2, y_2), (x_3, y_3). The equation of the line joining (x_2, y_2) and (x_3, y_3) is

$$\frac{x - x_2}{x_2 - x_3} = \frac{y - y_2}{y_2 - y_3}, \quad \text{or} \quad y = \frac{y_2 - y_3}{x_2 - x_3}x + b.$$

The equation of the perpendicular from (x_1, y_1) to this line is [§ 32]

$$y - y_1 = -\frac{x_2 - x_3}{y_2 - y_3}(x - x_1), \quad \text{or}$$

$$x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0. \tag{1}$$

Hence, by symmetry, the equations of the other two perpendiculars are

$$x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0; \tag{2}$$

$$x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 - y_2) = 0. \tag{3}$$

Adding the left members of the equations (1), (2), (3), which is taking $k = l = m = 1$ in § 39 (5), gives $0 \cdot x + 0 \cdot y + 0$. Hence the lines represented by (1), (2), and (3) meet in one common point.

40. Exercises. Lines through a point.

1. Prove that the following lines meet in one common point:

$$x - 2y + 4 = 0, \ 2x + 3y - 3 = 0, \ 5x + 4y - 2 = 0.$$

2. Do the following lines meet in one common point:

$$3x + 2y + 8 = 0, \ x + 8y + 7 = 0, \ 7x - 32y - 3 = 0?$$

3. Do the following lines meet in one common point:

$$5x + 8y + 7 = 0, \ 4x + 3y + 5 = 0, \ 2x - 7y + 1 = 0?$$

4. What is the condition that the lines, $2y + 5x = a$, $3x - 7y = 8$, $10x + 13y = 21$ shall meet in a point?

5. For what values of a do the following lines meet in a point:

$$x + 2y + 3 = 0, \quad ax - y + 4 = 0, \quad 2x + 3y + a = 0\ ?$$

6. ABC is a triangle, right-angled at C. On AC and CB, and exterior to the triangle, are constructed the squares $ACDE$ and $CBFG$. Prove that AF, BE, and the perpendicular to AB through C meet in a common point. [Take CA and CB as axes.]

41. Problem. *To express the distance between two points P, P' in terms of their coordinates (x, y), (x', y'), the axes being rectangular.*

Let the line through P' parallel to the x-axis meet the line through P perpendicular to the x-axis in the point G; and let C and E be the feet of the ordinates of P' and P, respectively.

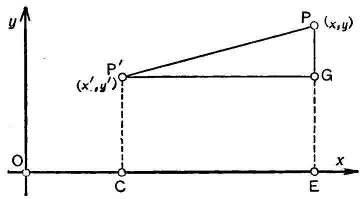
In the right-angled triangle $P'GP$, $P'P^2 = P'G^2 + GP^2$.

But

$$P'G = OE - OC = x - x',$$

and

$$GP = EP - CP' = y - y'.$$



Hence

$$P'P^2 = (x - x')^2 + (y - y')^2,$$

and therefore

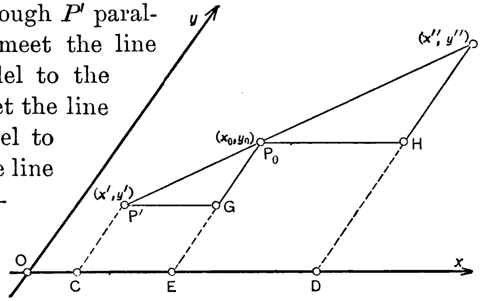
$$P'P = \sqrt{(x - x')^2 + (y - y')^2}.$$

42. The distance of a point $P'(x', y')$ from the origin O is $\sqrt{x'^2 + y'^2}$.

43. The equation $x^2 + y^2 = a^2$ is true for every point $P(x, y)$ on the circle whose center is at the origin and whose radius is a, and false for every point not on this circle; it is therefore called the *equation of this circle*. [Compare § 16, last paragraph.]

44. Problem. *To find the coordinates of the point which divides in a given ratio $k_1 : k_2$ the line segment joining two given points $P'(x', y')$ and $P''(x'', y'')$, the axes being rectangular or oblique.*

Let $P_0(x_0, y_0)$ be the point which divides $P'P''$ in the ratio $k_1 : k_2$, so that, $P'P_0 : P_0P'' = k_1 : k_2$.



Let the line through P' parallel to the x-axis meet the line through P_0 parallel to the y-axis in G, and let the line through P_0 parallel to the x-axis meet the line through P'' parallel to the y-axis in H; let C, E, D be the feet of the ordinates of P', P_0, P'', respectively. Then

$$k_1 : k_2 = P'P_0 : P_0P'' = P'G : P_0H = CE : ED.$$

Hence $k_1 \cdot ED = k_2 \cdot CE$; that is, $k_1 (x'' - x_0) = k_2 (x_0 - x')$.

Therefore, solving for x_0, $$x_0 = \frac{k_1x'' + k_2x'}{k_1 + k_2}. \quad (1)$$

In like manner, $$y_0 = \frac{k_1y'' + k_2y'}{k_1 + k_2}. \quad (2)$$

If P_0 be the mid-point of $P'P''$, then $k_1 = k_2$ and the formulas (1) and (2) reduce to

$$x_0 = \frac{x' + x''}{2}, \quad y_0 = \frac{y' + y''}{2}. \quad (3)$$

Notice that, if the point P_0 is interior to $P'P''$, so that $P'P_0$ and P_0P'' have the same direction, k_1 and k_2 have the same sign; but if the point P_0 be exterior to $P'P''$, so that $P'P_0$ and P_0P'' have opposite directions, k_1 and k_2 have opposite signs [§ 1].

In particular, if P_0 trisects $P'P''$, $k_1 = 2\,k_2$ or $2\,k_1 = k_2$; if P_0 is the point beyond P'' at which the line $P'P''$ is doubled, $k_1 = -2\,k_2$; and so on.

Example 1. Find the point where the segment from $P'(7, 4)$ to $P''(5, -6)$ is divided in the ratio 2 : 3.

The coordinates of the required point are

$$x_0 = \frac{2 \cdot 5 + 3 \cdot 7}{2 + 3} = \frac{31}{5}, \quad y_0 = \frac{2 \cdot (-6) + 3 \cdot 4}{2 + 3} = 0.$$

Example 2. The segment from $P'(5, 2)$ to $P''(6, 4)$ is produced through P' to a point P_0 such that $P_0P' = 2\,P'P''$; find the coordinates of P_0.

Here $k_1 : k_2 = -2 : 3$. Hence the coordinates of P_0 are

$$x_0 = \frac{(-2) \cdot 6 + 3 \cdot 5}{-2 + 3} = 3, \quad y_0 = \frac{(-2) \cdot 4 + 3 \cdot 2}{-2 + 3} = -2.$$

Example 3. Find the ratio in which the segment from (2, 5) to (5, − 1) is divided at the point (x_0, y_0), where it meets the line $2x - 3y - 5 = 0$.

The coordinates of the point of division can be expressed:

$$x_0 = \frac{5k_1 + 2k_2}{k_1 + k_2}, \quad y_0 = \frac{-k_1 + 5k_2}{k_1 + k_2}.$$

But the point (x_0, y_0) is on the line $2x - 3y - 5 = 0$. Hence

$$2\,\frac{5k_1 + 2k_2}{k_1 + k_2} - 3\,\frac{-k_1 + 5k_2}{k_1 + k_2} - 5 = 0,$$

or $$10k_1 + 4k_2 + 3k_1 - 15k_2 - 5k_1 - 5k_2 = 0,$$

or $$k_1 = 2k_2, \text{ that is, } k_1 : k_2 = 2 : 1.$$

The point (x_0, y_0) is (4, 1).

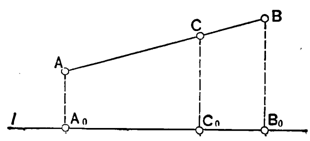
45. Projections. The foot A_0 of the perpendicular on the line l from the point A is called the *projection* on l of A, and the following notation is used to indicate this relation:

$$pr_l A = A_0, \tag{1}$$

which is read "the projection on l of A is A_0."

46. If A_0 be the projection on l of A, and B_0 that of B, then A_0B_0 is called the *projection* on l of AB, and the relation is indicated by

$$pr_l AB = A_0B_0. \tag{2}$$

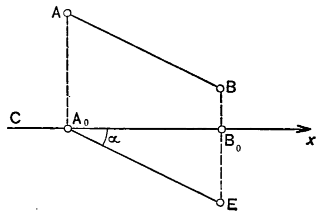


In this definition both AB and A_0B_0 are directed line segments.

47. Let AB and CD be two *directed* lines; by the angle α or (AB, CD) between these lines is meant the angle between the positive directions of these lines or parallel lines.

48. *Let Cx denote a line on which the positive direction is from C to x, and let AB be a segment of a line whose positive direction makes with Cx the angle α. Then the projection of AB on Cx is equal to AB* cos α.

Suppose that the positive direction on the line of which AB is a segment is from A to B; then AB is a positive segment.

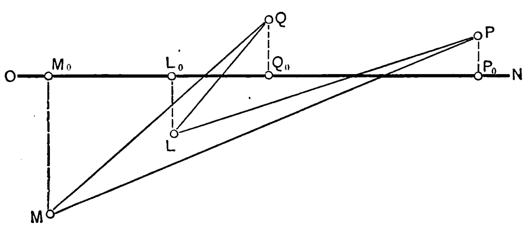


From A_0, the projection on Cx of A, take AE equal and parallel to AB. Then B_0, the projection on Cx of B, is also the projection on Cx of E, and α, or (AB, Cx), is the angle EA_0x.

Hence $$\cos\alpha = A_0B_0/A_0E = A_0B_0/AB,$$
and therefore $$A_0B_0 = AB\cos\alpha.$$

The theorem is therefore true for the positive segment AB. And it is true for the negative segment BA. For $AB = -BA$ and $A_0B_0 = -B_0A_0$, and therefore from the formula just obtained it follows that $B_0A_0 = BA\cos\alpha$.

49. The sum of the projections of the segments of any broken line MQ, QL, LP, on ON is equal to the projection on ON of



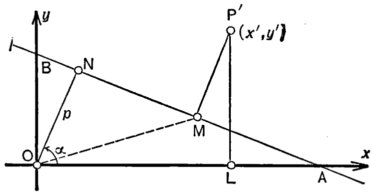
MP, the line segment from the initial point to the terminal point of the broken line. For if M_0, Q_0, L_0, P_0, be the projections on ON of M, Q, L, P, respectively, it is at once obvious [§ 2] that

$$M_0P_0 = M_0Q_0 + Q_0L_0 + L_0P_0,$$

or from the definition [§ 46, (2)],

$$pr_{ON}MP = pr_{ON}MQ + pr_{ON}QL + pr_{ON}LP. \qquad (4)$$

50. Perpendicular distance from a line to a point. From the origin O take ON perpendicular to the given line l, and meeting it at N; and let the positive direction on ON be fixed as that from O to N (from O upward, when l passes through O). Let $ON = p$ and $\angle xON = \alpha$ (where α is less than π when l passes through O).



Again, take $P'L$, the perpendicular to Ox, and $P'M$, the perpendicular to l. Then $OL = x'$, $LP' = y'$, and MP' is the perpendicular from l to P'.

Join OM. Then, by § 49,

$$pr_{ON}MP' = pr_{ON}MO + pr_{ON}OL + pr_{ON}LP'. \qquad (1)$$

But, by definition and by § 48,

$$pr_{ON}MP' = MP',$$

$$pr_{ON}MO = NO = -ON = -p,$$

$$pr_{ON}OL = OL\cos(ON, Ox) = x'\cos\alpha,$$

$$pr_{ON}LP' = LP'\cos(ON, Oy) = y'\cos\left(\frac{\pi}{2} - \alpha\right) = y'\sin\alpha.$$

When these values are substituted in (1), it becomes

$$MP' = x'\cos\alpha + y'\sin\alpha - p. \qquad (2)$$

The α and p in (2) are the same as the α and p in the perpendicular form of the equation of the line l, namely, $x\cos\alpha + y\sin\alpha - p = 0$ [§ 34]. Hence the perpendicular distance MP', or $x' \cos \alpha + y' \sin \alpha - p$, is the left member of that equation of the line with (x', y') substituted for (x, y).

If the equation of l is $ax + by + c = 0$, this equation can be reduced to the perpendicular form by dividing by $\pm\sqrt{a^2 + b^2}$ [§ 34, last paragraph]. Hence

The perpendicular distance of the point $P'(x', y')$ from the line $ax + by + c = 0$ is

$$\frac{ax' + by' + c}{\pm\sqrt{a^2 + b^2}}, \tag{3}$$

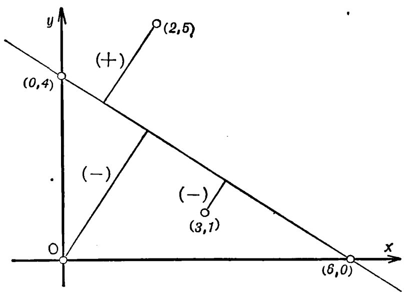
where the sign before the radical is opposite that in c when $c \neq 0$, but the same as that in b when $c = 0$.

Observe that MP', and therefore $x' \cos \alpha + y' \sin \alpha - p$, is positive or negative according as MP' has the same direction as ON or the opposite direction, that is, according as P' lies on the side of l remote from or toward the origin (or when l passes through O, according as P' lies above l or below it).

When P' is on l, MP', and therefore $x' \cos \alpha + y' \sin \alpha - p$, is 0. It has thus been demonstrated, independently of § 34, that the equation

$$x \cos \alpha + y \sin \alpha - p = 0$$

is true for all points on the line l determined by p and α, and false for all points not on this line, in other words, that it is the equation of l.



Example 1. Find the perpendicular distances of the points (3, 1) and (2, 5) from the line $2x + 3y - 12 = 0$. To which side of the line does each of the points lie?

The equation of the line when reduced to the perpendicular form is

$(2x + 3y - 12)/\sqrt{13} = 0$

and the perpendiculars from this line to (0, 0), (3, 1), and (2, 5) are

$$\frac{2 \cdot 0 + 3 \cdot 0 - 12}{\sqrt{13}}, \quad \frac{2 \cdot 3 + 3 \cdot 1 - 12}{\sqrt{13}}, \quad \frac{2 \cdot 2 + 3 \cdot 5 - 12}{\sqrt{13}},$$

or $-12/\sqrt{13}$, $-3/\sqrt{13}$, and $7/\sqrt{13}$, respectively. The perpendiculars from the line to (0, 0) and (3, 1) are of the same sign, that is, the two points are on the same side of the line. But (3, 1) and (2, 5) are on opposite sides of the line.

Example 2. Find the equations of the bisectors of the angles included between the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$.

Any point $P(x, y)$ on either bisector is equidistant from the given lines. Hence the required equations are

$$\frac{ax + by + c}{\pm\sqrt{a^2 + b^2}} = \frac{a'x + b'y + c'}{\pm\sqrt{a'^2 + b'^2}} \text{ and } \frac{ax + by + c}{\pm\sqrt{a^2 + b^2}} = -\frac{a'x + b'y + c'}{\pm\sqrt{a'^2 + b'^2}},$$

where the signs before the radicals are determined by the rule given above. The first of these equations represents the bisector of the angle which contains the origin, and the second equation, the other bisector.

51. The sign of the perpendicular distance from a line l to a point P', as expressed in § 50, corresponds to the conventions there made that the positive direction on lines perpendicular to l is that from the origin to l. But in the case of a line $x + a = 0$ parallel to the y-axis and at its left, this convention would be in conflict with the convention that the positive direction on all lines perpendicular to the y-axis, that is, parallel to the x-axis, is from left to right. The convention of § 50 is therefore not extended to such lines. For a similar reason it is not extended to lines $y + b = 0$ parallel to and below the x-axis.

Hence equations of the form $x + a = 0$ and $y + b = 0$, where a and b are positive, are to be left unchanged (and not reduced to the perpendicular forms $-x - a = 0$, $-y - b = 0$) when considering the perpendicular distances of points from the lines which they represent. The perpendicular distance of $P'(x', y')$ from the line $x + a = 0$ is $x' + a$, and according as $x' + a$ is positive or negative, $P'(x', y')$ lies to the right or left of the line $x + a = 0$. Similarly for equations of the form $y + b = 0$.

D

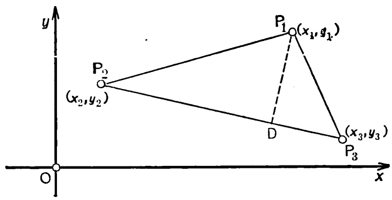
52. Problem. *To express the area of a triangle in terms of the coordinates of its vertices, the axes being rectangular.*

The area of the triangle $P_1P_2P_3$ is one half the product of a base P_2P_3 by its corresponding altitude DP_1.

The length of the base P_2P_3 is

$$P_2P_3 = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}, \qquad (1)$$

and DP_1 is the perpendicular from the line P_2P_3 to the point $P_1(x_1, y_1)$ and may be found as follows:



By § 20, the equation of P_2P_3 is

$$\frac{x - x_2}{x_2 - x_3} = \frac{y - y_2}{y_2 - y_3},$$

which when cleared of fractions and simplified becomes

$$x(y_2 - y_3) - y(x_2 - x_3) + (x_2y_3 - x_3y_2) = 0,$$

and by § 50, (3) the perpendicular distance of the point $P_1(x_1, y_1)$ from the line represented by this equation is

$$DP_1 = \frac{x_1(y_2 - y_3) - y_1(x_2 - x_3) + (x_2y_3 - x_3y_2)}{\pm\sqrt{(y_2 - y_3)^2 + (x_2 - x_3)^2}}. \qquad (2)$$

Therefore the area of the triangle is one half the numerical value of the product of the expressions (1) and (2); that is, except perhaps for sign,

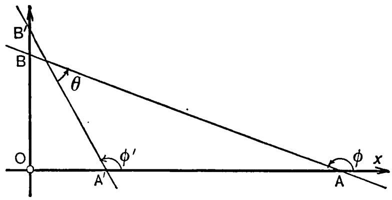
$$\Delta P_1P_2P_3 = \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \qquad (3)$$

53. The area of the triangle OP_1P_2, one of whose vertices is at the origin, is the numerical value of $\frac{1}{2}(x_1y_2 - x_2y_1)$.

54. Problem. *To find the angle made by a line AB with a line A'B', from the equations of AB and A'B', the axes being rectangular.*

Let AB and $A'B'$ be non-directed lines. Then the angle made by AB with $A'B'$ is defined as the positive angle θ through which $A'B'$ must be turned to bring it into coincidence or parallelism with AB.



If the equations of AB and $A'B'$ are $y = mx + b$ and $y = m'x + b'$, this angle θ can be found as follows:

In the case indicated in the figure here given, $\theta = \phi - \phi'$; and in every case, either $\theta = \phi - \phi'$ or $\theta = \pi + (\phi - \phi')$; and therefore $\tan\theta = \tan(\phi - \phi')$. Furthermore, $\tan\phi = m$, $\tan\phi' = m'$.

$$\text{Hence}\quad \tan\theta = \tan(\phi - \phi') = \frac{\tan\phi - \tan\phi'}{1 + \tan\phi\tan\phi'} = \frac{m - m'}{1 + mm'}.$$

Therefore the angle θ is given by the formula

$$\tan\theta = \frac{m - m'}{1 + mm'}. \qquad (1)$$

If the equations of AB and $A'B'$ are given in the form $ax + by + c = 0$, $a'x + b'y + c' = 0$, then $m = -a/b$, $m' = -a'/b'$, and therefore

$$\tan\theta = \frac{-a/b + a'/b'}{1 + aa'/bb'} = \frac{a'b - ab'}{aa' + bb'}. \qquad (2)$$

As was seen in §§ 25, 30, the lines are parallel when $a'b - ab' = 0$, and they are perpendicular when $aa' + bb' = 0$.

Example 1. Find the angle made by $2y - x = 0$ with $3y + x + 1 = 0$. Here $m = 1/2$, and $m' = -1/3$. Substituting these values in (1),

$$\tan\theta = \frac{1/2 + 1/3}{1 - 1/6} = 1, \text{ and therefore } \theta = \pi/4.$$

The same result can be obtained by substituting in (2),

$$a = -1,\ b = 2,\ a' = 1,\ b' = 3.$$

Example 2. Find the equation of the line through (2, 3) which makes an angle of 135° ($3\pi/4$) with $4x - 3y + 5 = 0$.

If m denote the slope of the required line, since $\tan(3\pi/4) = -1$, formula (1) gives $-1 = \dfrac{m - 4/3}{1 + 4m/3}$; or solving for m, $m = 1/7$.

Hence the required equation is $y - 3 = (x - 2)/7$, or $x - 7y + 19 = 0$.

55. Exercises. The straight line. Draw the graph in each exercise.

1. What is the distance between the points (2, − 3) and (3, 3)?

2. How far is the point ($a - b$, $a + b$) from the origin?

3. Find the areas of the triangles whose vertices are:

(*a*) (2, 3), (4, − 1), (− 5, 2);
(*b*) (3, 4), (0, 0), (4, − 3);
(*c*) (3, 2), (4, 4), (− 2, − 8).

4. Find the area of the quadrilateral whose vertices are: (2, 3), (4, − 1), (− 3, − 2), (0, 2).

5. Find the lengths of the sides of a triangle whose vertices are (1, 3), (− 2, −4), (1, − 2). Find also the lengths of the medians.

6. Prove that (4, 3) is the center of the circle circumscribing the triangle with the angular points (9, 3), (4, − 2), (8, 6).

7. Find the coordinates of the points of trisection of the line joining (3, − 2) and (− 2, − 1).

8. The line joining (2, 1) and (− 3, − 1) is produced through the latter point so as to be 4 times its original length; what are the coordinates of the extremity?

9. A vertex of a given square of side b is joined to the mid-point of one of the opposite sides; if this line is produced through the second point until the whole line is double its original length, how far is its extremity from each of the vertices? How far is it from the center of the square?

10. The line joining the points (1, $\frac{1}{2}$) and (2, $-\frac{1}{4}$) is divided in a certain ratio by the point ($\frac{1}{3}$, 1); find the ratio.

11. The line joining the points (2, 1) and (− 3, − 1) meets the line $3y - 9x = 11$. In what ratio does the point of intersection divide the line joining the original points?

12. Find the distances of the point (2, − 3) from the lines $2x + 3y - 5 = 0$ and $12x - 5y + 26 = 0$.

13. Prove that (2, 3) is the center of a circle which touches the three lines $4x + 3y - 7 = 0$, $5x + 12y - 20 = 0$, $3x + 4y - 8 = 0$.

14. Find the point of tangency of one of the lines in Ex. 13 with the circle.

15. Are $(2, -3)$ and $(-4, -2)$ on the same side of the line $5x - 8y + 2 = 0$?

16. Find the distance between the lines

$$5x + 4y - 3 = 0 \text{ and } 5x + 4y + 2 = 0.$$

17. Find the equation of the line parallel to $4x + 3y + 12 = 0$, and nearer the origin by a unit's distance.

18. How far from the origin is the line which passes through the point $(2, -3)$ and is parallel to the line $3y + x = 0$?

19. Find the angles between the following pairs of lines:

(*a*) $2x - 7y + 3 = 0$ and $5x + y + 1 = 0$,
(*b*) $2x - 3y + 2 = 0$ and $5x - y + 2 = 0$,
(*c*) $3x + y + 9 = 0$ and $3x + y - 10 = 0$,
(*d*) $7x - 2y + 1 = 0$ and $2x + 7y - 13 = 0$.

20. Find the equation of the line through the point $(2, -1)$ which makes an angle of $60°$ with the line $y = 2x$.

21. Prove that (2, 1), (0, 2), $(8/7, -2/7)$, $(6/7, 23/7)$, are the vertices of a parallelogram. Prove also that the diagonals bisect each other.

22. Prove that (2/5, 1/5), (0, 0), $(-1/5, 2/5)$, and (1/5, 3/5) are the vertices of a rectangle.

23. Find the equation of the line through the point $(a, 0)$ which makes an angle of $45°$ with the line $6x - 5y = 30$.

24. The sides, *AB*, *BC*, *CA*, of a triangle have the equations $x + 8y - 2 = 0$, $2x - 3y + 5 = 0$, $2x + 5y = 1$, respectively; verify [by using § 54] that the exterior angle at *A* is equal to the sum of interior angles at *B* and *C*.

25. What is the equation of the line joining the origin to the mid-point of the segment between the points $(2, -3)$ and $(4, -1)$?

26. Find the point of intersection of the lines joining the points (2, 1) and $(-3, -1)$, and $(-1, 2)$ and $(2, -2)$, respectively.

27. What angle do these lines (Ex. 26) make with each other?

28. Find the points which are equidistant from the points $(5, -2)$ and (6, 2), and at a distance of two units from the line $24x + 7y = 50$.

29. Find the center of the circle circumscribing the triangle whose vertices are $(1, 1)$, $(3, -1)$, and $(-1, -5)$.

30. Find the center of the circle circumscribing the triangle whose vertices are the points $(0, 0)$, $(4a, 0)$, $(2a, -2b)$.

31. Find the area of the triangle contained by the three straight lines

$$3x + 4y = 12, \quad 4x + 3y = 12, \quad x + y = 3.$$

32. Find the points on the y-axis, whose perpendicular distances from the line $3x + 4y = 6$ are 3 units each.

33. Find the equation of the line which makes an angle of 225° with the positive direction of the x-axis, and which is at a distance 5 from the origin.

34. What relation must hold good among the coefficients of the equation $ax + by + c = 0$ in order that

(1) it shall cut off an intercept -2 on the y-axis ?
(2) it shall cut off an intercept 3 on the x-axis ?
(3) it shall cut off equal intercepts on the axes ?
(4) it shall be perpendicular to $2x + 3y = 5$?
(5) it shall pass through the origin ?
(6) the perpendicular from the origin upon it may be 3 ?
(7) it shall pass through the point $(2, -3)$?
(8) the perpendicular distance of $(2, -1)$ from it may be 3 ?

35. Find the equation of the line through the point of intersection of $2x + 3y - 12 = 0$ and $3x - 4y - 1 = 0$, and

(*a*) through the origin,
(*b*) perpendicular to the line $4x - 5y = 0$,
(*c*) parallel to the x-axis,
(*d*) at the distance 3 from the point $(4, 5)$.

36. Prove that all lines represented by the equation $3x + \lambda y + 5 + 2\lambda = 0$, where λ is an arbitrary constant, pass through a common point, and find this point.

37. Does every equation of the first degree in x and y, which involves an arbitrary constant λ, denote a system of lines through a fixed point ?

38. Prove that all lines which make intercepts a and b on the x- and y-axes such that $1/a = 1/b + k$, where k is a constant, pass through a fixed point.

39. Find the equations of the bisectors of the angles made by the following pairs of lines, drawing a figure in each case:

(*a*) $3x - y = 0$ and $x - 2y = 0$,
(*b*) $2x + y = 0$ and $y - 3 = 0$,
(*c*) $6x + 8y - 41 = 0$ and $12x - 5y - 30 = 0$.

40. A line is taken through the origin perpendicular to $3x + y + 2 = 0$; find the equations of the lines bisecting the angles between the given line and this perpendicular line.

41. Find the center of the inscribed circle of the triangle whose sides are the lines:

(*a*) $y - 2x = 0,\ 2y - x = 0,\ x - 4 = 0,$
(*b*) $3x + 4y + 7 = 0,\ 4x + 3y - 21 = 0,\ 12x - 5y + 28 = 0.$

42. ABC is a triangle in which C is a right angle and $CA = a$ and $CB = b$. Squares are described on its three sides and exterior to the triangle. Find the coordinates of the angular points of these squares referred to CA and CB as x- and y-axes. Find also the equations of the diagonals of the square on AB and the coordinates of the point of intersection of these diagonals.

43. Prove [by using § 44 (3)] that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one half of it.

44. $ABCD$ is a parallelogram and E and F are the mid-points of AD and BC, respectively; prove that EB and DF trisect the diagonal AC. (Take AB and AD as axes.)

45. D is the mid-point of the side BC of the triangle ABC, and P is any point on AD. Through P the straight lines BPE and CPF are taken, meeting AC and AB at E and F, respectively. Prove that EF is parallel to BC. (Take BC and AD as axes.)

46. Through any point E on the diagonal AC of the parallelogram $ABCD$ the line FEG is taken parallel to AB and meeting AD at F and BC at G; and the line HEK is taken parallel to AD and meeting AB at H and DC at K. Prove that the lines AC, HG, and FK meet in a common point.

47. Prove that the perpendiculars from the vertices of a triangle to the opposite sides meet in a common point, taking one of the sides and the perpendicular to it as axes of reference.

48. Prove that in any triangle the perpendicular bisectors of the sides meet in a point.

49. Prove that in any triangle the medians meet in a point.

50. The points $O(0, 0)$, $A(a, 0)$, $B(0, b)$, and $C(h, k)$ are given (referred to rectangular or oblique axes). Let OA and BC produced meet at D, and let OB and AC produced meet at E. Prove that the mid-points of the line segments OC, AB, and DE lie in one and the same straight line.

51. The following lines and points are given: (*a*) $2x - y + 7 = 0$, (*b*) $x + 3y - 1 = 0$, (*c*) $2x - 3y + 2 = 0$, $D\ (1, -1)$, $E\ (0, 2)$, $F\ (-3, -2)$:

(1) Write the equation of the line *FD*. Of *DE*. Of *EF*.

(2) Find the distance *EF*. Find *FD*. Find *DE*.

(3) Write the equation of the line perpendicular to (*c*) and through *D*. Perpendicular to (*b*) and through *E*.

(4) Write the equation of the line parallel to (*c*) and through *D*. Parallel to (*b*) and through *E*.

(5) Write the equation of the line through the intersection of (*a*) and (*b*), and through *E*. Through the intersection of (*a*) and (*b*) and through *F*.

(6) How far are *D*, *E*, and *F* from (*b*)? Are they on the same side as the origin or on the opposite side?

(7) Find the tangent of the angle between (*a*) and (*b*). Between (*a*) and *FD*.

(8) Trisect the line segment *EF*. Bisect *FD*.

52. By aid of § 44 (3), prove that, if the angular points of a triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3), the point of intersection of the medians is $\{(x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3\}$.

53. Given the four points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$, prove that the lines joining the midpoints of each of the pairs of lines P_1P_2, P_3P_4; P_1P_3, P_2P_4; P_1P_4, P_2P_3, meet in the one common point $\{(x_1 + x_2 + x_3 + x_4)/4, (y_1 + y_2 + y_3 + y_4)/4\}$.

54. Prove that the equation $(ax + by + c)^2 - (a^2 + b^2)d^2 = 0$ represents two lines parallel to the line $ax + by + c = 0$ and at the distance d to either side of it.

55. Find the center and radius of each of the four circles which touch the three lines $4y - 3x = 0$, $5y - 12x = 0$, and $y - 6 = 0$.

56. Let m_1 and m_2 denote the slopes of the two lines through the origin represented by the equation $ax^2 + 2hxy + by^2 = 0$. Prove that $m_1 + m_2 = -2h/b$, $m_1m_2 = a/b$, and that, if θ denote the angle between the lines, $\tan\theta = 2\sqrt{h^2 - ab}/(a + b)$.

57. Prove that the equation $x^2 - \lambda xy - y^2 = 0$ represents, for every given value of λ, a pair of perpendicular lines through the origin.

CHAPTER III

THE CIRCLE *

56. Equation of the circle. The equation of the circle whose center is $C(x_0, y_0)$, and whose radius is r, is

$$(x - x_0)^2 + (y - y_0)^2 = r^2. \qquad (1)$$

For since the left member of (1) represents the square of the distance of the point $P(x, y)$ from the point $C(x_0, y_0)$, this equation is true for every point on the circle and false for every point not on the circle [§ 41].

When the center is at the origin, then $x_0 = 0$ and $y_0 = 0$, and (1) becomes

$$x^2 + y^2 = r^2. \qquad (2)$$

When the circle touches the y-axis at the origin and is at the right of this axis, then the coordinates of the center are $(r, 0)$, and (1) becomes $(x - r)^2 + y^2 = r^2$, which reduces to

$$x^2 + y^2 - 2\,rx = 0. \qquad (3)$$

Similarly, the equation of a circle which touches the x-axis at the origin and lies above this axis is

$$x^2 + y^2 - 2\,ry = 0. \qquad (4)$$

Thus, consider a circle whose radius is 3. If its center be the point $(-1, 2)$, its equation is

$$(x + 1)^2 + (y - 2)^2 = 9, \text{ or } x^2 + y^2 + 2\,x - 4\,y - 4 = 0.$$

If it touches the y-axis at the origin, its equation is

$$x^2 + y^2 - 6\,x = 0, \text{ or } x^2 + y^2 + 6\,x = 0,$$

according as it lies to the right or left of the y-axis.

* The chapter on the circle may be omitted until after the chapter on the ellipse. By proceeding first to the chapters on the parabola and ellipse, the student sooner realizes the power of the method of the coordinate geometry through seeing it employed in investigating *new* material.

57. Every equation of the second degree in x, y which lacks the xy term, and in which the coefficients of x^2 and y^2 are the same, can be reduced to the form

$$x^2 + y^2 + 2\,gx + 2\,fy + c = 0. \qquad (1')$$

Complete the square of the terms $x^2 + 2\,gx$ by adding g^2 to both members of (1′); similarly, complete the square of the terms $y^2 + 2\,fy$ by adding f^2 to both members; also transpose c. The result may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c. \qquad (1'')$$

If $(g^2 + f^2 - c)$ is positive, the equation (1″) is of the form (1) of § 56, and therefore represents a circle whose center is the point $(-g, -f)$ and whose radius is $\sqrt{g^2 + f^2 - c}$.

If $(g^2 + f^2 - c)$ is 0, the locus of (1′) is the single point $(-g, -f)$; it is sometimes called a *point-circle.*

If $(g^2 + f^2 - c)$ is negative, the locus is imaginary; it is sometimes called an *imaginary circle.*

Therefore, every equation which can be reduced to the form (1′), where $g^2 + f^2 - c$ is positive, represents a circle. The center of this circle is the point $(-g, -f)$, and its radius is $\sqrt{g^2 + f^2 - c}$.

Example 1. Show that $3\,x^2 + 3\,y^2 + 5\,x - 6\,y + 1 = 0$ represents a circle, and find its center and radius.

Dividing by the common coefficient of x^2 and y^2, and rearranging the terms,

$$\{x^2 + (5/3)x + \quad\} + \{y^2 - 2\,y + \quad\} = -1/3.$$

Completing the squares

$$(x^2 + \tfrac{5}{3}x + \tfrac{25}{36}) + (y^2 - 2\,y + 1) = -\tfrac{1}{3} + \tfrac{25}{36} + 1,$$

or

$$(x + \tfrac{5}{6})^2 + (y - 1)^2 = \tfrac{49}{36},$$

which represents a circle whose center is $(-5/6, 1)$ and whose radius is $7/6$.

Example 2. Find the locus of $4\,x^2 + 4\,y^2 - 4\,x + 8\,y + 7 = 0$.

The equation is equivalent to

$$(x^2 - x + 1/4) + (y^2 + 2\,y + 1) = 1/4 + 1 - 7/4,$$

or

$$(x - 1/2)^2 + (y + 1)^2 = -1/2.$$

There is no real pair of values of x, y which will satisfy this equation; the locus has no *real* point; the locus is *imaginary*.

Example 3. Find the locus of $4x^2 + 4y^2 - 4x + 8y + 5 = 0$.

This equation is equivalent to $(x^2 - x + 1/4) + (y^2 + 2y + 1) = 0$. The only real solution is $x = 1/2$, $y = -1$. The locus is the point $(1/2, -1)$.

58. Circles determined by three given conditions. Any three given points not in the same straight line determine a circle. Its equation may be found by substituting the coordinates of each of the given points in the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad (1')$$

solving the three equations thus obtained for g, f, and c, and substituting the resulting values in (1′).

Example. Find the equation of the circle which passes through the three points (0, 0), (1, 0), and (0, 1).

Substituting the coordinates (0, 0), (1, 0), (0, 1) in (1′) gives the three equations

$$c = 0, \quad 1 + 2g + c = 0, \quad 1 + 2f + c = 0.$$

Hence $$c = 0, \quad 2g = -1, \quad 2f = -1,$$

and the required equation is,

$$x^2 + y^2 - x - y = 0.$$

59. And, in general, since the equation (1′) involves three arbitrary constants, g, f, and c, if three conditions be given for determining a circle, and if these conditions can be expressed by three equations involving only g, f, c, and known quantities, and if these equations can be solved for g, f, c, then the results can be substituted in (1′), and the equation of the circle is known. If more than one set of real values be thus found for g, f, c, there is more than one circle satisfying the given conditions.

Example. Find the equation of the circle which passes through the two points $(-1, 0)$, $(2, 1)$, and whose center lies on the line $y - 2x + 3 = 0$.

Substituting the coordinates $(-1, 0)$, $(2, 1)$ in the general equation (1′) gives

$$1 - 2g + c = 0, \quad 5 + 4g + 2f + c = 0.$$

But since the center is to lie on $y - 2x + 3 = 0$, and the coordinates of the center are $(-g, -f)$,

$$-f + 2g + 3 = 0.$$

Solving these three equations in f, g, and c gives

$$g = -1,\ f = 1,\ \text{and } c = -3.$$

Hence the required equation is

$$x^2 + y^2 - 2x + 2y - 3 = 0.$$

60. Equation of the tangent to a circle. If the point $P'(x', y')$ is on the circle $x^2 + y^2 - r^2 = 0$, the slope of the line joining P' to the center $C(0, 0)$ is y'/x'; therefore, since the tangent to the circle at P' is perpendicular to this line, its equation is

$$y - y' = -\frac{x'}{y'}(x - x').$$

By clearing of fractions and replacing $x'^2 + y'^2$ by r^2, this equation can be reduced to the form

$$xx' + yy' - r^2 = 0. \tag{1}$$

And, in general, if the point $P'(x', y')$ is on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the slope of the line joining P' to the center $C(-g, -f)$ is $(y' + f)/(x' + g)$.

Hence the equation of the tangent at P' is

$$y - y' = -\frac{x' + g}{y' + f}(x - x'),$$

or clearing of fractions, expanding, and rearranging the terms,

$$xx' + yy' + gx + fy = x'^2 + y'^2 + gx' + fy'.$$

If $gx' + fy' + c$ be added to both members, the right member becomes $x'^2 + y'^2 + 2gx' + 2fy' + c$, which is 0, since P' is on the circle, and the equation itself therefore becomes

$$xx' + yy' + g(x + x') + f(y + y') + c = 0. \tag{2}$$

Hence the equation of the tangent can be obtained from the equation of the circle by replacing x^2 and y^2 by xx' and yy', and $2x$ and $2y$ by $x + x'$ and $y + y'$.

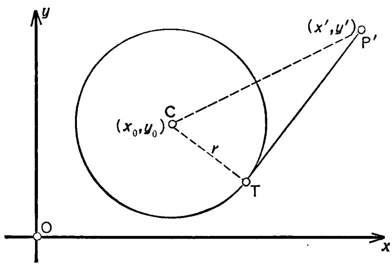
Thus, for example, the point $(-3, 2)$ is on the circle $x^2 + y^2 = 13$, since $(-3)^2 + 2^2 = 13$, and, by (1), the equation of the tangent at this point is $-3x + 2y - 13 = 0$, or $3x - 2y + 13 = 0$.

Again, the point $(2, -1)$ is on $x^2 + y^2 - 8x + 6y + 17 = 0$, and, by (2), the tangent at $(2, -1)$ is $2x - y - 4(x + 2) + 3(y - 1) + 17 = 0$, or simplifying, $x - y - 3 = 0$.

61. Length of the tangent from a point to a circle. Let $P'(x', y')$ be any point outside the circle,

$$(x - x_0)^2 + (y - y_0)^2 - r^2 = 0. \qquad (1)$$

From P' draw $P'T$, to touch the circle at T, and join the center C to T and to P'.



Then since CTP' is a right angle,

$$P'T^2 = P'C^2 - CT^2.$$

But [§ 41]

$$P'C^2 = (x' - x_0)^2 + (y' - y_0)^2,$$

and $\qquad CT^2 = r^2.$

Hence $$P'T^2 = (x' - x_0)^2 + (y' - y_0)^2 - r^2. \qquad (2)$$

Therefore the square of the length of the tangent $P'T$ from the point $P'(x', y')$ to a circle whose equation is given in the form (1), or in the equivalent form

$$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad (1')$$

is the result obtained by substituting the coordinates of P' in the left member of (1) or (1′).

Example 1. Find the square of the length of the tangent from the point (2, 1) to the circle

$$3x^2 + 3y^2 - 5x + 2y - 3 = 0.$$

Reducing the equation to the form (1′),

$$x^2 + y^2 - \tfrac{5}{3}x + \tfrac{2}{3}y - 1 = 0.$$

Hence the square of the length of the tangent from (2, 1) is

$$2^2 + 1^2 - \tfrac{5}{3} \cdot 2 + \tfrac{2}{3} \cdot 1 - 1, \text{ or } 4/3.$$

Example 2. Prove that if $P'(x', y')$ is within the circle whose equation is (1) or (1′), and $QP'R$ is any chord of the circle through P', the result of substituting (x', y') for (x, y) in the left member of (1) or (1′) is negative and equal numerically to the area of the rectangle $P'Q \cdot P'R$.

62. Systems of circles through two points. Radical axis. Radical center.

Let $$S \equiv x^2 + y^2 + 2\,gx + 2\,fy + c = 0, \quad (1)$$
and $$S' \equiv x^2 + y^2 + 2\,g'x + 2\,f'y + c' = 0, \quad (2)$$
represent two given circles, and λ *an arbitrary constant. Then*

$$S + \lambda S' = 0 \quad (3)$$

will represent the system of circles through the points of intersection of the given circles $S = 0$ *and* $S' = 0$.

For, when like terms in x and y are collected, $S + \lambda S' = 0$ becomes

$$(1 + \lambda)\,x^2 + (1 + \lambda)\,y^2 + 2\,(g + \lambda g')\,x + 2\,(f + \lambda f')\,y + (c + \lambda c') = 0 \quad (3')$$

which represents a circle for every value of λ (except -1), since the xy term is lacking and the coefficients of x^2 and y^2 are the same [§ 57].

Every such circle (3) or (3′) passes through the points of intersection of the circles (1) and (2), since for these points both S and S' are 0 and therefore the equation $S + \lambda S = 0$ is satisfied, whatever the value of λ may be.

Moreover, every given circle through the points of intersection of $S = 0$ and $S' = 0$ is included among the circles represented by $S + \lambda S' = 0$. For, if (x'', y'') denote any third point on such a circle, that is, any point distinct from the points of intersection of $S = 0$ and $S' = 0$, a value λ' can be found for λ such that $S + \lambda' S' = 0$ is satisfied by (x'', y''), and $S + \lambda' S' = 0$ will then represent the given circle, since it is satisfied by the coordinates of three points of this circle.

Example. Find the equation of the circle through the points of intersection of $x^2 + y^2 = 5$, $x^2 + y^2 - x = 0$, and the point (2, 3).

Substituting (2, 3) in $x^2 + y^2 - 5 + \lambda(x^2 + y^2 - x) = 0$,

gives $$8 + \lambda \cdot 11 = 0, \text{ or } \lambda = -8/11.$$

Hence the required equation is

$$11(x^2 + y^2 - 5) - 8(x^2 + y^2 - x) = 0, \text{ or } 3x^2 + 3y^2 + 8x - 55 = 0.$$

63. When $\lambda = -1$ the equation (3′) of § 62 becomes

$$2(g - g')x + 2(f - f')y + (c - c') = 0, \tag{4}$$

which represents a straight line, since it is of the first degree. This straight line passes through the points of intersection of $S = 0$ and $S' = 0$, since (4) is the expanded form of $S - S' = 0$. It is called the *radical axis* of the circles $S = 0$ and $S' = 0$.

The equation $S - S' = 0$ may be written $S = S'$. Hence, by § 61, the radical axis of two circles $S = 0$, $S' = 0$ may also be defined as the locus of points, the tangents from which to the circles $S = 0$ and $S' = 0$ are equal. This definition holds good even when $S = 0$ and $S' = 0$ do not intersect in real points.

64. The radical axes of the three circles $S = 0$, $S' = 0$, $S'' = 0$, taken in pairs, are $S - S' = 0$, $S' - S'' = 0$, $S'' - S = 0$. These three lines meet in a common point called the *radical center* of the three circles. This follows from § 39, since

$$(S - S') + (S' - S'') + (S'' - S) \equiv 0.$$

Example. Find the radical axes of the following three circles taken in pairs, also the radical center of these circles: $2x^2 + 2y^2 - 3 = 0$, (1) $x^2 + y^2 - 2x + 4y = 0$, (2) and $x^2 + y^2 + 3x + 5y - 1 = 0$. (3)

The radical axis of (1) and (2) is

$$(x^2 + y^2 - 3/2) - (x^2 + y^2 - 2x + 4y) = 0,$$

or $$2x - 4y - 3/2 = 0. \tag{4}$$

Similarly the radical axes of (1), (3), and of (2), (3), are

$$3x + 5y + 1/2 = 0, \tag{5}$$

and $$5x + y - 1 = 0, \text{ respectively.} \tag{6}$$

The radical center, since it is the point of intersection of any two of the radical axes, say, (4) and (5), is (1/4, − 1/4), and the coordinates of this point satisfy (6) as they should.

65. Orthogonal Circles. Two circles are said to be *orthogonal*, if they meet at right angles, that is, if their tangents at a point of intersection include a right angle.

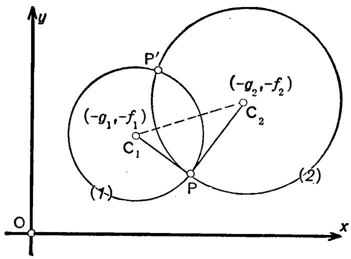
To find the condition that two circles may be orthogonal.

Let the equations of the circles be

$$x^2 + y^2 + 2\,g_1x + 2\,f_1y + c_1 = 0, \tag{1}$$

$$x^2 + y^2 + 2\,g_2x + 2\,f_2y + c_2 = 0, \tag{2}$$

and let C_1 and C_2 denote their centers, and P one of their points of intersection.



Join C_1C_2, C_1P, and C_2P. By hypothesis, the tangent to (1) at P is perpendicular to the tangent to (2) at P; but the radius C_2P is also perpendicular to the tangent to (2) at P; hence the tangent to (1) at P coincides with C_2P. Similarly, the tangent to (2) at P coincides with C_1P. Hence the angle C_1PC_2 is a right angle, and therefore

$$C_2C_1^2 = C_1P^2 + C_2P^2. \tag{3}$$

But since the coordinates of C_1 are $(-g_1, -f_1)$, and those of C_2 are $(-g_2, -f_2)$, it follows from § 41 that

$$C_2C_1^2 = (-g_1 + g_2)^2 + (-f_1 + f_2)^2$$

and since C_1P and C_2P are the radii of (1) and (2),

$$C_1P^2 = g_1^2 + f_1^2 - c_1 \text{ and } C_2P^2 = g_2^2 + f_2^2 - c_2.$$

Substituting these values in (3), gives

$$(-g_1 + g_2)^2 + (-f_1 + f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2,$$

or simplifying, $$2\,g_1g_2 + 2\,f_1f_2 = c_1 + c_2, \tag{4}$$

which is the condition required.

Example. Find the equation of the circle which is orthogonal to all three of the circles: $x^2 + y^2 - 4 = 0$, $x^2 + y^2 - x + 2y - 3 = 0$, and $x^2 + y^2 + 4x - 2y + 1 = 0$.

Let the required equation be $x^2 + y^2 + 2gx + 2fy + c = 0$. The circle represented by this equation will be orthogonal to the three given circles, if

$$-4 + c = 0, \quad -g + 2f = c - 3, \quad 4g - 2f = c + 1$$

Solving these equations, $g = 2$, $f = 3.\,2$, $c = 4$,
Hence the required equation is $x^2 + y^2 + 4x + 3y + 4 = 0$.

66. Exercises. The circle.

1. Find the equation of the circle:

(1) whose center is $(2, -3)$ and whose radius is 5.
(2) whose center is $(0, 1)$ and whose radius is $\sqrt{2}$.
(3) which touches the y-axis at the origin and is at its left and has the radius 3.
(4) which touches the x-axis at the point $(2, 0)$ and is above it and has the radius 4.

2. Find the center and radius of the circle represented by each of the following equations, drawing the graph in each case:

(1) $x^2 + y^2 - 5x + 4y = 0$.
(2) $x^2 + y^2 - 5y = 0$.
(3) $x^2 + y^2 + 4x - 8y + 11 = 0$.
(4) $2x^2 + 2y^2 - 5x + 6y = 0$.

3. The points of intersection of two circles or of a straight line and circle may be found by regarding their equations as simultaneous and solving for x, y. (Compare § 13.) By this method find the points where the line $x - y + 1 = 0$ cuts the circle $x^2 + y^2 - x - 3y = 0$, drawing a figure.

4. Find the points at which the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ cuts each of the axes.

5. Prove that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the x-axis in two coincident points, or touches it, if $c = g^2$, and that it touches the y-axis, if $c = f^2$.

6. Find the equation of the circle:

(1) through the three points $(0, 0)$, $(2, 1)$, $(0, 3)$.
(2) through the three points $(1, 1)$, $(2, -1)$, $(3, 2)$.

7. Find the equation of the circle whose center is on the line $x - 2y + 1 = 0$ and which passes through the two points $(0, 0)$ and $(3, 4)$.

8. Find the equation of the circle whose diameter is the line joining the two points $(-2, 1)$ and $(1, -3)$.

9. Find the equation of the circle which touches the x-axis and passes through the two points $(1, 1)$ and $(3, 1)$.

10. Find the equation of the circle whose center is $(3, 4)$ and which touches the x-axis.

11. Find the length of the tangent from the point $(1, 2)$ to the circle $3x^2 + 3y^2 - 2x + 5y + 2 = 0$; from the origin to the circle.

12. Of the circles through the points of intersection of the two given circles $x^2 + y^2 - 4x + 2y + 3 = 0$ and $x^2 + y^2 + 6x - 4y = 0$, find:

(1) that which passes through the point $(1, 2)$.
(2) that whose center lies on the x-axis.
(3) that which is orthogonal to $x^2 + y^2 - x + y = 0$.

13. Find the radical axes of the following circles taken by pairs; also their radical center: $x^2 + y^2 - 6x - 1 = 0$, $x^2 + y^2 - 2x + 6y = 0$, and $2x^2 + 2y^2 - 5 = 0$.

14. Prove that the radical axis of any two circles is perpendicular to the line joining their centers.

15. Find the equation of the circle through the points $(0, 0)$, $(1, 1)$ and orthogonal to $x^2 + y^2 - 4x + 2y - 3 = 0$.

16. Find the equation of the circle which passes through the point $(2, 0)$ and is orthogonal to the two circles: $x^2 + y^2 - 4x + 2y - 3 = 0$, and $x^2 + y^2 + 2x - 6y + 6 = 0$.

17. Find the equation of the circle orthogonal to the three circles: $x^2 + y^2 = 2$, $x^2 + y^2 - 4x + 2y - 3 = 0$, and $x^2 + y^2 + 2x - 6y + 6 = 0$.

18. Prove that the center of any circle of the system $S + \lambda S' = 0$ [§ 62] is on the line joining the centers of the circles $S = 0$ and $S' = 0$ and divides it in the ratio $\lambda : 1$.

19. Find the equation of the circle circumscribing the triangle whose sides are $x = 0$, $y = 2x$, $y + 2x = 8$.

20. Find the equation of the circle inscribed to the triangle whose sides are $3x + 4y + 8 = 0$, $4x - 3y + 12 = 0$, $4x + 3y - 36 = 0$.

21. Find the equation of the tangent to $x^2 + y^2 = 1$ at $(3/5, 4/5)$; to $x^2 + y^2 + 3x + 5y + 2 = 0$ at $(1, -3)$; to $2x^2 + 2y^2 - 4x + 5y = 0$ at $(0, 0)$.

CHAPTER IV

THE PARABOLA

67. Loci. If but one condition is given as to the situation of a point in the plane, so that it is free to occupy infinitely many different positions, the collection of all these positions is called the *locus* of the point.

To find such a locus by the methods of coordinate geometry, fix the attention upon the point P in some representative position, use (x, y) to denote its coordinates referred to conveniently chosen axes, and then express the given condition in terms of x and y. The resulting equation in x, y is called the *equation of the locus*. The graph of this equation will be the locus itself.

68. Thus, the equation of the locus of a point (x, y) at the constant distance r from the fixed point (x_0, y_0) is

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

which is therefore the equation of a circle whose center is the point (x_0, y_0) and whose radius is r [§ 41].

69. Conics. Let a fixed point F and a fixed line l be given, and suppose a point P to move in the plane of F and l in such a manner that its distance from F is in a constant ratio to its distance from l. This moving point will trace out a curve called a *conic*. The fixed point F is called the *focus* of this conic, the fixed line l is called the *directrix*, and the constant ratio is called the *eccentricity*.

It is customary to represent the eccentricity by e. This constant e is positive, and may be equal to 1, less than 1, or greater than 1.

If $e = 1$, the conic is called a *parabola*.

If $e < 1$, the conic is called an *ellipse*.

If $e > 1$, the conic is called an *hyperbola*.

70. The equation of the parabola. By definition, the parabola is the locus of a point equidistant from the focus and the directrix.

Let the point F be the focus, and the line SR the directrix. Through F, take FD perpendicular to SR at D. The point V where FD is bisected, being equidistant from F and SR, is a point of the parabola. It is called the *vertex* of the parabola.

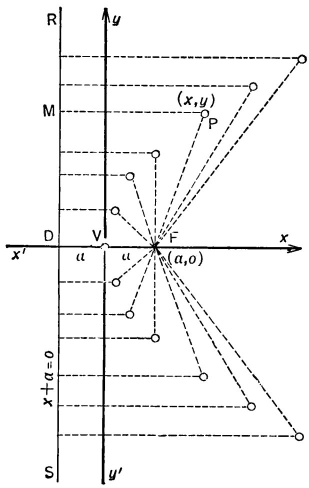
Take the line VF as the x-axis and the parallel to SR through V as the y-axis, thus making V the origin. The equation of the parabola referred to these axes is to be found.

Represent the length (and direction) of $DV(=VF)$ by a. Then the coordinates of F are $(a, 0)$ and the equation of SR is $x + a = 0$.

Let $P(x, y)$ denote any representative point of the parabola; join PF, and take PM perpendicular to SR.

By hypothesis,

$$FP^2 = MP^2. \qquad (1)$$



Since FP is the distance of the point (x, y) from the point $(a, 0)$, $FP^2 = (x-a)^2 + y^2$ [§ 41].

And since MP is the perpendicular distance of the point (x, y) from the line SR, whose equation is $x + a = 0$, it follows from § 51, that $MP^2 = (x + a)^2$.

Substituting these expressions for FP^2 and MP^2 in terms of the coordinates of P in (1), gives

$$(x - a)^2 + y^2 = (x + a)^2$$

or, transposing $(x - a)^2$ and simplifying,

$$y^2 = 4\,ax, \qquad (2)$$

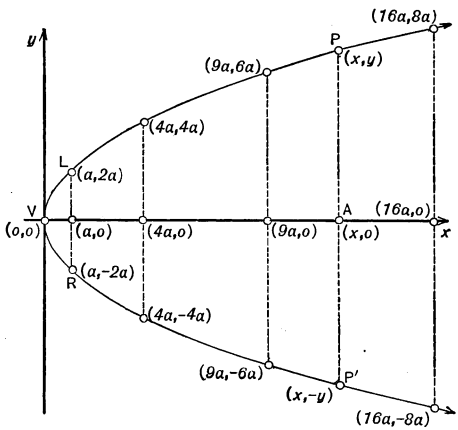
which is the equation required. For it has been proved to be true for every point P on the parabola; and it is false for every point off the parabola, since, if P is off the parabola, FP^2 is not equal to MP^2, therefore $(x-a)^2+y^2$ is not equal to $(x+a)^2$, and therefore finally y^2 is not equal to $4\,ax$.

In the equation (2), a represents the distance and direction from the vertex to the focus. When, as in the figure, the focus lies to the right of the directrix, a is positive; but when the focus lies to the left of the directrix, a is negative.

71. The shape of the parabola. The shape of the parabola and its position relative to the axes may readily be inferred from its equation.

From $y^2=4\,ax$, it follows that $y=\pm 2\sqrt{ax}$.

Hence, if a be *positive*, y is imaginary when x is negative, has the value 0 (to be counted twice) when x is 0, and has two real values equal numerically but of opposite sign when x is positive. Therefore, the curve lies wholly to the right of the y-axis, which it touches at the origin, and it is symmetric with respect to the x-axis, that is, to any point A on the positive x-axis there correspond two points P and P' on the parabola, vertically above and below A and at equal distances from it. Furthermore since $2\sqrt{ax}$ increases indefinitely with x, the curve extends indefinitely to the right of the y-axis and indefinitely above and below the x-axis. It may be said to consist of a single "infinite branch."



If a be *negative*, the curve extends in a similar manner from the y-axis indefinitely to the left.

A figure representing the parabola may be obtained as above by plotting a number of the solutions of $y^2 = 4ax$ and drawing a "smooth" curve through the points thus found.

72. The line through the focus perpendicular to the directrix, which has been taken as the x-axis, is called the *axis* of the parabola. As has just been proved, it bisects all chords of the parabola which are parallel to the directrix.

73. The chord LR through the focus and parallel to the directrix is called the *latus rectum.* Its length is $4a$, the coefficient of x in the equation $y^2 = 4ax$; for the abscissa of the focus is a, and when $x = a$, then $y = \pm 2a$.

74. Exercises. The equation of the parabola.

1. Find the coordinates of the focus and the equation of the directrix for each of the parabolas:

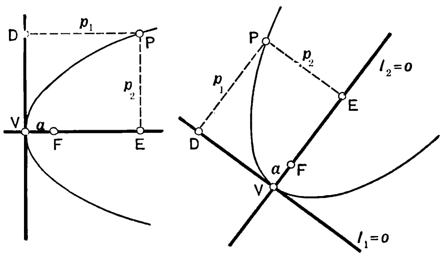
(1) $y^2 = 4x$, (2) $y^2 = 8x$, (3) $y^2 = -8x$, (4) $y^2 = 5x$.

2. Which of the following points are on the parabola $y^2 = 8x$: $(0, 1)$, $(0, -2)$, $(0, 0)$, $(2, 4)$, $(-2, 4)$, $(1/2, -2)$, $(3, -5)$?

3. The parabola $y^2 = -8x$ is given. Find the ordinates of the points on the curve whose abscissa is -2; find the abscissa of the point on the curve whose ordinate is -2; find the points where the curve is met by the lines $x + 2 = 0$ and $y + 2 = 0$.

4. Find the points of intersection of $y^2 = 5x$ and $y = 2x$. Of $y^2 = 6x$ and $3x - 4y + 6 = 0$.

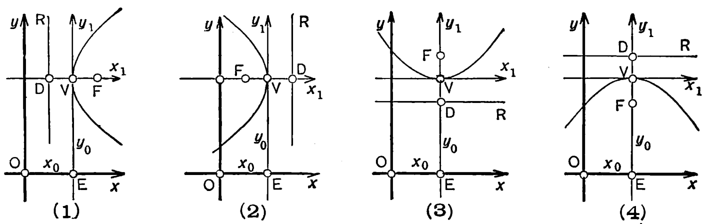
75. A more general form of the equation of the parabola. In the equation $y^2 = 4ax$, x denotes the distance p_1 of any point P of the parabola from the tangent at the vertex, and y the distance p_2 of P from the axis of the parabola, and the equation is equivalent to the statement that in any parabola these distances p_1 and p_2 are connected with a, the distance from the vertex to the focus, by the relation $p_2^2 = 4\,ap_1$. This property of the parabola is independent of the position of the curve in the plane. Hence the locus of a point P, whose distances p_1 and p_2 from *any* two perpendicular lines $l_1 = 0$ and $l_2 = 0$ are connected by the relation $p_2^2 = 4\,ap_1$ is a parabola equal to the parabola $y^2 = 4\,ax$ and placed with respect to the lines $l_1 = 0$, $l_2 = 0$ as the parabola $y^2 = 4\,ax$ is placed with respect to the lines $x = 0$, $y = 0$.



Thus the distances of a point $P(x, y)$ from the perpendicular lines $x - x_0 = 0$, $y - y_0 = 0$ are $x - x_0$, $y - y_0$ [§ 51]. Hence

$$(y - y_0)^2 = 4\,a(x - x_0) \tag{1}$$

represents a parabola having $y - y_0 = 0$ for axis, $x - x_0 = 0$ for tangent at the vertex, (x_0, y_0) for vertex, and a for distance from the vertex to the focus, and which lies to the right or left of $x - x_0 = 0$, according as a is positive or negative. See figures (1) and (2).



Similarly

$$(x - x_0)^2 = 4\,a(y - y_0) \tag{2}$$

represents a parabola having $x - x_0 = 0$ for axis and $y - y_0 = 0$ for tangent at vertex, and which lies above or below $y - y_0 = 0$ according as a is positive or negative. See figures (3), (4).

Every equation of the form $y^2 + 2\,gx + 2\,fy + c = 0$, *where* $g \neq 0$, *represents a parabola whose axis is parallel to the x-axis; and the equation* $x^2 + 2\,gx + 2\,fy + c = 0$, *where* $f \neq 0$, *represents a parabola whose axis is parallel to the y-axis.*

For, as will be seen from the following examples, the equations $y^2 + 2\,gx + 2\,fy + c = 0$, $x^2 + 2\,gx + 2\,fy + c = 0$ can be reduced to the forms $(y - y_0)^2 = 4\,a(x - x_0)$, $(x - x_0)^2 = 4\,a(y - y_0)$, respectively.

Example 1. Find the axis, vertex, focus, and directrix of the parabola $y^2 + 4\,x + 4\,y = 0$.

The equation may be written $$y^2 + 4\,y = -4\,x.$$

Completing the square $$y^2 + 4\,y + 4 = -4\,x + 4,$$

or $$(y + 2)^2 = -4(x - 1).$$

Hence $y + 2 = 0$ is the axis, $x - 1 = 0$ is the tangent at the vertex, and the point $(1, -2)$ is the vertex. Since $4\,a = -4$, we have $a = -1$. Hence the abscissa of the focus is $1 + (-1)$ or 0, its ordinate being -2, and the equation of the directrix is $x - 1 + (-1) = 0$ or $x - 2 = 0$. Since a is negative the parabola lies to the left of $x - 1 = 0$.

Example 2. Write $3\,x^2 - 4\,x - 6\,y + 8 = 0$ in the form (2).

The equation may be written $3(x^2 - \frac{4}{3}x \quad) = 6\,y - 8.$

Completing the square, $3(x^2 - \frac{4}{3}x + \frac{4}{9}) = 6\,y - 8 + 3 \cdot \frac{4}{9},$

or $$3(x - \tfrac{2}{3})^2 = 6(y - \tfrac{20}{18}),$$

or $$(x - \tfrac{2}{3})^2 = 4(\tfrac{1}{2})(y - \tfrac{10}{9}).$$

Hence $a = \frac{1}{2}$, and the parabola extends upward. The axis is $x - \frac{2}{3} = 0$, the tangent at the vertex is $y - \frac{10}{9} = 0$, the vertex is $(\frac{2}{3}, \frac{10}{9})$; the focus is $(\frac{2}{3}, \frac{29}{18})$, and the directrix is $y - \frac{11}{18} = 0$.

Example 3. Find the equation of the parabola whose axis is parallel to the y-axis and which passes through the points (0, 1), (1, 0), and (2, 0). The required equation is of the form

$$x^2 + 2\,gx + 2\,fy + c = 0. \qquad (1)$$

Since it has the solution (0, 1), $2\,f + c = 0$ (2). Similarly, since it has the solutions (1, 0) and (2, 0),

$$1 + 2\,g + c = 0 \ (3) \quad \text{and} \quad 4 + 4\,g + c = 0 \ (4).$$

Solving (2), (3), (4) for g, f, c, gives $2\,g = -3$, $2\,f = -2$, $c = 2$. Substituting these values in (1) gives $x^2 - 3\,x - 2\,y + 2 = 0$, the equation required.

Example 4. The two perpendicular lines $7x - 6y + 84 = 0$ (1) and $6x + 7y - 42 = 0$ (2) being given, find the equation of the parabola of latus rectum 4 which has (1) for axis and (2) for tangent at vertex, and which lies to the origin side of (2).

Let $P(x, y)$ denote any point on the parabola, and p_1, p_2 the distances of P from the tangent (2) and the axis (1), respectively; then [§ 50],

$$p_1 = \frac{6x + 7y - 42}{\sqrt{85}}, \qquad p_2 = \frac{7x - 6y + 84}{-\sqrt{85}}.$$

Therefore since p_1 is negative on the origin side of (2), the required equation $p_2^2 = 4\,ap_1$ is

$$\frac{(7x - 6y + 84)^2}{85} = -4\frac{6x + 7y - 42}{\sqrt{85}}.$$

Example 5. Find the equation of the parabola whose focus is the origin, and directrix the line $2x + y - 10 = 0$.

The distances of any point $P(x, y)$ of this parabola from the focus and directrix are $\sqrt{x^2 + y^2}$ and $(2x + y - 10)/\sqrt{5}$; these distances, and therefore their squares, are equal; hence the required equation is

$$x^2 + y^2 = \frac{(2x + y - 10)^2}{5}, \text{ or } x^2 - 4xy + 4y^2 + 40x + 20y - 100 = 0.$$

The student may verify the following facts regarding this parabola: The axis is the line $x - 2y = 0$; the point where the axis cuts the directrix is (4, 2); the vertex is the point (2, 1); the distance from focus to directrix is $10/\sqrt{5}$; the latus rectum is $20/\sqrt{5}$; the points where the curve cuts the x-axis, found by setting $y = 0$ in its equation and solving for x, are $(-20 \pm \sqrt{500}, 0)$; the points where it cuts the y-axis are $\{0, (-5 \pm \sqrt{125})/2\}$.

From these data the graph of the parabola is readily found.

76. Exercises. More general equation of the parabola.

Find the coordinates of the focus and vertex, and the equation of the directrix of each of the following parabolas, drawing the graph in each case:

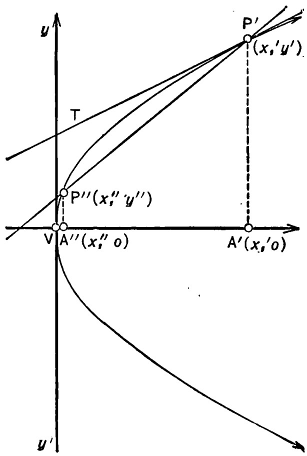
1. $x^2 - 2x - 4y + 5 = 0$.
2. $x^2 + 2x + 4y - 3 = 0$.
3. $y^2 - 3x - 2y - 5 = 0$.
4. $3y^2 - 4x + 12y = 0$.
5. Find the equation of the parabola whose latus rectum is 8 and which has the line $x + 2 = 0$ for axis and the line $y - 3 = 0$ for tangent at the vertex, and which lies below the tangent.
6. Find the equation of the parabola whose axis is parallel to the x-axis and which passes through the points (2, 0), (0, 1), and (3, 2). Also find the vertex of this parabola.

7. The perpendicular lines $y - x = 0$ (1) and $y + x = 0$ (2) are the axis and the tangent at the vertex of a parabola whose latus rectum is $4\sqrt{2}$, and which lies above the tangent (2); find the equation of the parabola.

8. The latus rectum of a parabola is 10; its axis is $3x - 4y + 12 = 0$; the tangent at its vertex is $4x + 3y - 10 = 0$, and it lies to the origin side of this line; find its equation.

9. Find the equation of the parabola whose focus is the point $(-1, 2)$ and whose directrix is the line $2x - 3y - 6 = 0$.

77. Definition of tangent. A line $P'P''$ which meets a conic in two distinct points P' and P'' is called a *secant.* If the point P'' be made to move along the curve into coincidence with P', the line $P'P''$ will turn about P' toward a definite limiting position $P'T$ in which it is called the *tangent* to the curve at P'. Hence the tangent at P' is said to meet the curve in *two coincident points* at P'.



78. Equation of tangent in terms of slope. The coordinates of the points $P'(x', y')$, $P''(x'', y'')$ where the line $y = mx + c$ (1) meets the parabola $y^2 = 4ax$ (2) may be found by solving (1) and (2) for x, y. [Compare § 13.] The abscissas of these points are therefore the roots x', x'' of the equation

$$(mx + c)^2 = 4ax,$$

or

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0. \quad (3)$$

Hence the line (1) will meet the parabola (2) in *coincident* points (or touch it), if the roots x', x'' of (3) be equal. But the roots of (3) will be equal, if the left member of (3) is a perfect square, that is [Alg. § 635], if $(mc - 2a)^2 = m^2c^2$, or

$-4\,amc + 4\,a^2 = 0$, or $mc = a$, or $c = a/m$. Hence, whatever the value of m may be, the line

$$y = mx + \frac{a}{m} \tag{4}$$

will touch the parabola $y^2 = 4\,ax$.

When $c = a/m$, the equation (3) gives $x = a/m^2$; and this set in the equation (4) gives $y = 2\,a/m$. Hence the point of contact of the tangent (4) with the parabola is $(a/m^2, 2\,a/m)$.

Example 1. Find the equation of the tangent to the parabola $y^2 = -6\,x$ which is parallel to the line $3\,x - y = 0$; also the point of contact.

Here $m = 3$ and $a = -6/4 = -3/2$. Hence the required equation is $y = 3\,x - 1/2$, or $6\,x - 2\,y - 1 = 0$. The point of contact is $(-1/6, -1)$.

Example 2. Find the equations of the tangents from the point (2, 3) to the parabola $y^2 = 4\,x$.

The equation of every tangent to this parabola is of the form

$$y = mx + 1/m, \text{ or } m^2x - my + 1 = 0. \tag{1}$$

But since the required tangents pass through the point (2, 3), m must have such a value that this equation is satisfied when $x = 2$, $y = 3$; that is, m must satisfy the equation $2\,m^2 - 3\,m + 1 = 0$, which gives $m = 1$ or $\frac{1}{2}$. Therefore the required tangents are

$$y = x + 1 \quad \text{and} \quad y = x/2 + 2.$$

Example 3. Find the tangent to the parabola $x^2 + 4\,x - 3\,y = 0$ which is parallel to the line $y = 2\,x$.

As the equation of the parabola is not given in the form $y^2 = 4\,ax$, the tangent cannot be found by substitution in (4). It may be found as follows: Every line parallel to the line $y = 2\,x$ has an equation of the form $y = 2\,x + \lambda$. The abscissas of the points where the line $y = 2\,x + \lambda$ meets the parabola $x^2 + 4\,x - 3\,y = 0$ are the roots of the equation

$$x^2 + 4\,x - 3(2\,x + \lambda) = 0, \quad \text{or} \quad x^2 - 2\,x - 3\,\lambda = 0.$$

These roots are equal when $1 + 3\,\lambda = 0$, or $\lambda = -1/3$. Hence, the required tangent is $y = 2\,x - 1/3$.

Example 4. Prove that the slope equation of the tangent to the parabola $(y - y_0)^2 = 4\,a(x - x_0)$ is $y - y_0 = m(x - x_0) + a/m$.

This follows by replacing x and y by $(x - x_0)$ and $(y - y_0)$ in the algebraic reckoning of § 78.

Example 5. Find the tangent to the parabola $y^2 + 6\,y - 7\,x = 0$ which is perpendicular to the line $x - 2\,y = 0$.

79 A.* Equation of tangent in terms of coordinates of point of contact. *First method of derivation.* The equation of the line through any two points (x', y') and (x'', y'') is

$$\frac{y - y'}{y' - y''} = \frac{x - x'}{x' - x''}. \tag{1}$$

But if the points (x', y') and (x'', y'') be on the parabola $y^2 = 4\,ax$, then $y'^2 \equiv 4\,ax'$ (2) and $y''^2 \equiv 4\,ax''$. (3)

Subtracting (3) from (2) gives $y'^2 - y''^2 = 4\,a(x' - x'')$. (4)

Multiplying (1) by (4), $(y - y')(y' + y'') = 4\,a(x - x')$. (5)

Hence, when the points (x', y') and (x'', y'') are on the parabola, the equation (1) of the line joining them can be reduced to the form (5); in other words, (5) is the equation of the secant through the two points (x', y') and (x'', y'') on the parabola.

If the point (x'', y'') be made to move along the curve into coincidence with the point (x', y'), the secant through (x', y') and (x'', y'') becomes, at the limit, the tangent at (x', y'), and the equation (5) becomes $2\,y'(y - y') = 4\,a(x - x')$, or $yy' - y'^2 = 2\,ax - 2\,ax'$, or, since $y'^2 \equiv 4\,ax'$,

$$yy' = 2\,a(x + x'). \tag{6}$$

Hence (6) is the equation of the tangent at (x', y').

79 B. Equation of tangent in terms of coordinates of point of contact. *Second method of derivation.* Let (x', y') and (x'', y'') denote two points on the parabola $y^2 - 4\,ax = 0$ (1), so that $y'^2 - 4\,ax' \equiv 0$ (2), and $y''^2 - 4\,ax'' \equiv 0$ (3), and consider the equation

$$(y - y')(y - y'') = y^2 - 4\,ax. \tag{4}$$

This is an equation of the first degree, for on being simplified it becomes

$$y(y' + y'') - 4\,ax - y'y'' = 0. \tag{5}$$

Moreover, it is satisfied by $x = x'$, $y = y'$ and $x = x''$, $y = y''$. For if (x', y') be substituted for (x, y) in (4), the left member becomes $(y' - y')(y' - y'')$, which is 0 identically, and the right member becomes $y'^2 - 4\,ax'$, which is 0 because (x', y') is on the

* Only one of the Sections 79 A, 79 B, 79 C need be taken.

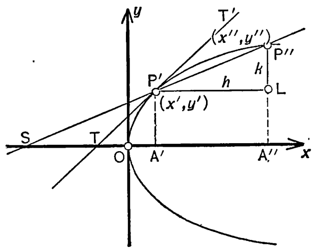
parabola. And in the same way it can be shown that (4) is satisfied by $x = x''$, $y = y''$. Hence (4), or its equivalent (5), is the equation of the secant through (x', y') and (x'', y'') [§ 17].

If the point (x'', y'') be made to move along the curve into coincidence with (x', y'), the secant will become the tangent at (x', y') and the equation (5) will become $2\,yy' - 4\,ax - y'^2 = 0$, or, since $y'^2 \equiv 4\,ax'$,

$$yy' = 2\,a(x + x'). \tag{6}$$

Hence (6) is the equation of the tangent at (x', y').

79 C. Equation of tangent in terms of coordinates of point of contact. *Third method of derivation.* Referring to the definition of the tangent [§ 77], represent the point P' by (x', y') and the point $P''(x'', y'')$ by $(x' + h, y' + k)$, that is, set $x'' = x' + h$ and $y'' = y' + k$.



When P'' moves along the curve into coincidence with P', and the secant $P'P''$ becomes the tangent at P', both h and k approach 0 as limit.

The slope of the secant $P'P''$ is k/h. Hence the slope of the tangent at P' is the limiting value of k/h; and this may be found as follows:

Since the points P' and P'' are on the parabola, $y^2 = 4\,ax$ (1), $y'^2 \equiv 4\,ax'$ (2) and $(y' + k)^2 \equiv 4\,a(x' + h)$ (3).

Expanding (3) and subtracting (2) from the result,

$$2\,y'k + k^2 = 4\,ah. \tag{4}$$

Hence $\dfrac{k}{h} = \dfrac{4\,a}{2\,y' + k}$, and therefore $\lim \dfrac{k}{h} = \dfrac{2\,a}{y'}$.

The tangent is the line through $P'(x', y')$, which has the slope $2\,a/y'$; hence its equation is $y - y' = (2\,a/y')(x - x')$, (5) which reduces to $yy' - y'^2 = 2\,ax - 2\,ax'$, or since $y'^2 \equiv 4\,ax'$, to

$$yy' = 2\,a(x + x'). \tag{6}$$

Hence (6) is the equation of the tangent at (x', y').

80. It may be added that, if $P'(x', y')$ and $P''(x'+h, y'+k)$ denote two points on *any* given curve, the equation of the tangent to the curve at P' is

$$y - y' = (\lim k/h)(x - x').$$

It is customary to represent $(\lim k/h)$ by the symbol $\frac{dy'}{dx'}$, the equation of the tangent being then written

$$y - y' = \frac{dy'}{dx'}(x - x').$$

81. The equation $yy' = 2a(x + x')$ may be obtained from the equation $y^2 = 4ax$ by replacing y^2 by yy' and $2x$ by $x + x'$. This is a special case of the rule which will be proved in § 171 for finding the equation of the tangent at the point (x', y') to the curve represented by any equation of the second degree in x, y: namely, replace x^2 and y^2 by xx' and yy'; $2xy$ by $x'y + y'x$; $2x$ and $2y$ by $x + x'$ and $y + y'$.

Example 1. Show that the point $(2, -3)$ is on the parabola

$$y^2 - 4x + 5y + 14 = 0,$$

and find the equation of the tangent at this point.

The substitution $x = 2$, $y = -3$ in the equation gives $9 - 8 - 15 + 14$, or 0; hence the point $(2, -3)$ is on the parabola.

The equation of the tangent to this parabola at (x', y') is

$$yy' - 2(x + x') + \tfrac{5}{2}(y + y') + 14 = 0,$$

and setting $(x', y') = (2, -3)$ in this equation, and simplifying, gives $4x + y - 5 = 0$, which is therefore the equation of the required tangent.

Setting $(x', y') = (2, -3)$ in $4x + y - 5 = 0$ gives $8 - 3 - 5$, or 0; that is, the line $4x + y - 5 = 0$ passes through the point $(2, -3)$, as it should. This is a partial check on the accuracy of the reckoning.

Example 2. Find the equation of the tangent to the graph of the equation

$$3x^2 - 2xy + y^2 - 6x + 5y - 4 = 0$$

at the point $(-2, -4)$.

Since one of the coefficients $2h$, $2g$, $2f$ is odd, to avoid fractions, multiply the equation throughout by 2 before applying the rule for finding the equation of the tangent. The equation thus becomes

$$6x^2 - 4xy + 2y^2 - 12x + 10y - 8 = 0.$$

Hence, by the rule, the equation of the tangent at the point (x', y') is

$$6\,xx' - 2\,(x'y + y'x) + 2\,yy' - 6\,(x + x') + 5\,(y + y') - 8 = 0.$$

Setting $x' = -2$ and $y' = -4$ in this equation,

$$6\,x(-2) - 2\{(-2)\,y + (-4)\,x\} + 2\,y(-4) - 6\,(x - 2) + 5\,(y - 4) - 8 = 0,$$

or simplifying, $$10\,x - y + 16 = 0,$$

which is the equation required.

82. **Exercises.** Tangent to the parabola.

Write the equation of the tangent to:

1. $y^2 = 2\,x$ at $(2, -2)$.
2. $y^2 = x$ at $(1, 1)$.
3. $y^2 = 8\,x$ at $(2, 4)$.
4. $y^2 = -8\,x$ at $(-2, -4)$.
5. $y^2 = 7\,x$ at $(0, 0)$.
6. $x^2 = 3\,y$ at $(2, 4/3)$.
7. $3\,x^2 = 25\,y$ at $(5, 3)$.
8. $y^2 + 6\,x - 8\,y = 0$ at $(0, 0)$.
9. Write the equation of the tangent to $y^2 = 8\,x$ which has the slope 2.
10. Find the tangent to $y^2 = 5\,x$ which is perpendicular to $3\,x + 2\,y = 0$. Also find the coordinates of the point of contact.
11. Find the tangent to $y^2 + 6\,x + 8\,y = 0$ which has the slope 3.
12. Find the tangent to $3\,x^2 - 3\,xy - y^2 + 15\,x + 10\,y - 18 = 0$ at the point $(-1, 3)$.
13. Find the tangent to $3\,x^2 - 4\,xy - 4\,y^2 + 5\,x + 6\,y + 8 = 0$ at the point $(2, -3)$.

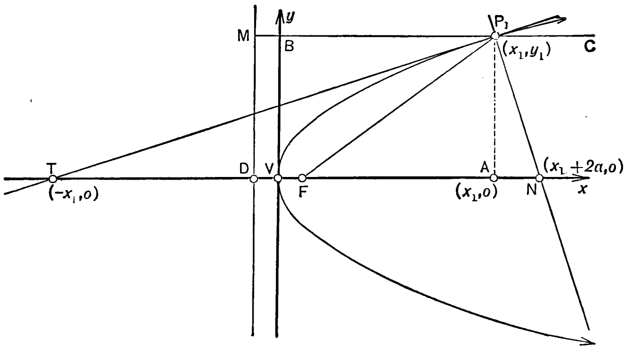
83. The normal to a curve at any point (x', y') on the curve is the line through (x', y') perpendicular to the tangent at (x', y').

Example. The tangent to the parabola $y^2 = 8\,x$ at the point $(2, 4)$ is $y \cdot 4 = 4\,(x + 2)$, or $x - y + 2 = 0$. The normal, by definition, is the perpendicular to this line through the point of contact, namely [§ 32], the line $(y - 4) + (x - 2) = 0$, or $x + y - 6 = 0$.

From § 79, (6), the slope of the tangent at (x', y') is $2\,a/y'$. Hence that of the normal is $-y'/2\,a$. Therefore the equation of the normal is $y - y' = (-y'/2\,a)(x - x')$, or

$$2\,a\,(y - y') + y'(x - x') = 0.$$

84. Geometric properties of the parabola. Let $P_1(x_1, y_1)$ be any point on the parabola $y^2 = 4\,ax$, and A the foot of its ordinate.



Let the tangent at P_1 meet the x-axis at T, and let the normal at P_1 meet the x-axis at N. The segment TA of the x-axis is called the *subtangent*, and the segment AN, the *subnormal*, corresponding to P_1.

The lengths of the subtangent and subnormal may be found as follows:

To find the abscissa of the point T, set $y = 0$ in the equation $yy_1 = 2\,a(x + x_1)$, which gives $x + x_1 = 0$ or $x = -x_1$. But $x = VT$ and $x_1 = VA$; hence $TA = 2\,x_1$, or *the subtangent is bisected at the vertex.*

Similarly, to find the abscissa of N, set $y = 0$ in the equation $2\,a(y - y_1) + y_1(x - x_1) = 0$, which gives $x = x_1 + 2\,a$. But $x = VN$ and $x_1 = VA$; hence $AN = 2\,a$, that is, *the subnormal is constant and equal to half the latus rectum.*

85. *The tangent at any point P_1 of a parabola bisects the angle contained by the line joining P_1 to the focus and the line through P_1 perpendicular to the directrix.*

For, referring to the preceding figure, join $P_1(x_1, y_1)$ to the focus $F(a, 0)$ and take P_1M perpendicular to the directrix at M and meeting the y-axis at B.

It has just been proved that $TV = x_1$. Hence $TF = x_1 + a$. But FP_1 is also equal to $x_1 + a$; for from the definition of the parabola, $FP_1 = MP_1 = MB + BP_1 = a + x_1$. Hence $TF = FP_1$, and therefore $\measuredangle FTP_1 = \measuredangle FP_1T$.

But since MP_1 is parallel to TF, $\measuredangle FTP_1 = \measuredangle TP_1M$. Therefore $\measuredangle FP_1T = \measuredangle TP_1M$, that is, the angle FP_1M is bisected by the tangent P_1T, as was to be demonstrated.

It also follows that if C be a point on MP_1 produced through P_1, *the normal* P_1N *bisects* the angle FP_1C.

It readily follows from § 84 that the line joining F and M meets P_1T at right angles at the point where P_1T meets VB. Hence, *the foot of the perpendicular from the focus to a tangent lies on the tangent at the vertex.*

Again, if Q be the point where P_1T meets the directrix DM, it is easily seen that the triangles QFP_1 and QMP_1 are equal, and therefore that $\measuredangle QFP_1$ is a right angle. Hence, *the portion of a tangent between the point of tangency and the directrix subtends a right angle at the focus.*

86. Diameters. The locus of the mid-points of a system of parallel chords of a conic is called a *diameter* of the conic.

Every diameter of a parabola is a straight line parallel to the axis.

For if $x = ny$ (1) denotes any *given* line through the origin, the equation of every line parallel to (1) will be of the form $x = ny + \lambda$ (2), [§ 26].

The ordinates, y_1 and y_2, of the points P_1 and P_2, where the line (2) cuts the parabola $y^2 = 4ax$, are the roots of the equation

$$y^2 = 4a(ny + \lambda), \quad \text{or} \quad y^2 - 4any - 4a\lambda = 0. \tag{3}$$

Hence $y_1 + y_2$ is the coefficient of y in (3) with its sign changed, that is, $y_1 + y_2 = 4an$ [Alg. § 636].

But if η denote the ordinate of the mid-point of the chord P_1P_2, then $\eta = (y_1 + y_2)/2$ [§ 44].

Hence $\eta = 2an$, or $y = 2an$ is the equation of the required locus; that is, the locus of the midpoint of P_1P_2 is a line parallel to the axis and at the distance $2an$ from it.

Example 1. Find the diameter of the parabola $y^2 = 12\,x$ which bisects all chords parallel to the line $3\,x - 2\,y + 1 = 0$.

Here $a = 3$ and $n = 2/3$. Hence the diameter is $y = 4$.

Example 2. Find the diameter of the parabola $x^2 - 4\,y + x = 0$ which bisects all chords parallel to the line $2\,x + 3\,y = 0$.

Every line parallel to $2\,x + 3\,y = 0$, or $y = -2\,x/3$, has an equation of the form $y = -2\,x/3 + \lambda$. The abscissas of the points where the line $y = -2\,x/3 + \lambda$ cuts the parabola $x^2 - 4\,y + x = 0$ are the roots of the equation $x^2 - 4(-2\,x/3 + \lambda) + x = 0$, or $x^2 + 11\,x/3 - 4\,\lambda = 0$. One-half the sum of the roots of this equation is $-11/6$. Hence the required diameter is $x = -11/6$.

87. Exercises. The parabola.

1. Which of the following points are on the parabola $y^2 = 6\,x$:

$$(0, 0),\ (-6, 6),\ (2/3, 2),\ (-2/3, 2),\ (3, -4)\ ?$$

2. Prove of a point (x', y'), not on the parabola $y^2 - 4\,ax = 0$, that it is inside or outside the parabola according as $y'^2 - 4\,ax'$ is negative or positive. To which side of the parabola $y^2 = 5\,x$ does each of the points $(3, 4)$, $(1, 1)$, $(-2, 3)$ lie?

3. Find the focus and directrix of each of the following:

(1) $y^2 = 12\,x$, (2) $x^2 = 8\,y$, (3) $y^2 = -10\,x$, (4) $x^2 = -y$.

4. Find the ordinates of the points of $y^2 = 12\,x$ whose abscissa is 2.

5. The axis of a parabola coincides with the x-axis, its vertex is at the origin, and its focus is the point $(3, 0)$. Find its equation.

6. Find the parabola $y^2 = 4\,ax$ which passes through the point $(-3, 4)$.

7. Prove that if the parabola $y^2 = 4\,ax$ passes through the point (h, k), the equation is $y^2 = k^2x/h$.

8. Given the lines $y - 5 = 0$ (1), $x + 3 = 0$ (2), find the parabola of latus rectum 8 which has these lines for axis and tangent at vertex,

(*a*) when (1) is axis and the parabola is at the right of (2),

(*b*) when (2) is axis and the parabola is below (1).

9. Given the lines $x - 2\,y + 3 = 0$ (1) and $2\,x + y - 8 = 0$ (2), find the parabola of latus rectum 6 which has (1) for axis, and (2) for tangent at vertex, and which lies to the side of (2) remote from the origin.

10. The line $12\,x - 5\,y + 6 = 0$ touches a parabola of latus rectum 4 at its vertex $(-3, -6)$, and the parabola lies to the origin side of this line; find the equation of the parabola.

11. Prove that each of the following equations represents a parabola whose focus is at the origin;

(1) $y^2 = 4a(x + a)$, (3) $y^2 = -4a(x - a)$,
(2) $x^2 = 4a(y + a)$, (4) $x^2 = -4a(y - a)$.

12. Find the coordinates of the focus and vertex, and the equation of the directrix of each of the following parabolas, drawing the graph in each case:

(1) $x^2 + 4x + 6y - 2 = 0$.
(2) $4y^2 + 4y - 32x - 63 = 0$.
(3) $2y^2 + 4y + 10x - 3 = 0$.
(4) $4y^2 - 4y - 36x - 11 - 0$.
(5) $x^2 + x + 2y + 7 = 0$.
(6) $3x^2 + 6x - 3y + 4 = 0$.
(7) $2y^2 + 2y + 8x + 17 = 0$.
(8) $4x^2 + 4x + 8y + 9 = 0$.

13. Find the equation of the parabola whose axis is parallel to the x-axis and which passes through the points $(0, 0)$, $(2, -1)$, $(2, 2)$.

14. Find the equation of the parabola whose axis is parallel to the y-axis and which passes through the points $(1, 1)$, $(2, -1)$, $(-1, 3)$.

15. Find the equation of the parabola whose axis is parallel to the y-axis, whose vertex is at the point $(2, -1)$, and which passes through the point $(-2, 0)$.

16. Find the equation of the parabola whose axis is the line $3x + 4y = 0$, and which passes through the points $(0, -1)$ and $(2, 0)$.

17. Find the equation of a parabola whose focus is the point $(1, -2)$ and whose directrix is the line $4x - 3y + 10 = 0$.

18. Find the equation of the tangent to

(1) $y^2 = x$ at $(1, -1)$;
(2) $y^2 = -4x$ at $(-2, 2\sqrt{2})$;
(3) $y^2 = 8x$ at $(2, 4)$;
(4) $y^2 = 5x$ at $(\frac{1}{20}, -\frac{1}{2})$;
(5) $3x^2 = 4y$ at $(-2, 3)$;
(6) $2x^2 - 3y - 6 = 0$ at $(-3, 4)$.

19. Find the equations of the tangent and normal to $y^2 = -8x$ at the point whose ordinate is -4.

20. Tangents are drawn to the parabola $y^2 = 20x$ at the two points whose abscissa is 5. Find the angle included by these tangents.

21. Find the points of intersection of (1) $y = x + 2$ and $y^2 = 9x$; (2) $2y + 3x + 2 = 0$ and $y^2 = 6x$; (3) $3y + 8x + 5 = 0$ and $y^2 = 18x$.

22. Prove that the line $3y - 9x = 2$ touches the parabola $y^2 = 8x$; also that the line $2y + x - 10 = 0$ touches the parabola $y^2 = -10x$.

23. Find the equation of the tangent to $y^2 = -6x$ which is parallel to $3y + 4x = 0$; also the equation of the tangent perpendicular to $3x - y + 2 = 0$.

24. Find the equation of the tangent to $y^2 = 8x$ which is perpendicular to the line joining the vertex to the upper end of the latus rectum.

25. What value must be given λ, if the line $2y + 3x + \lambda = 0$ is to touch the parabola $y^2 = -6x$?

26. Find the equation of the tangent to $y^2 = 4ax$ which makes an angle of 45° with the x-axis; find also the coordinates of the point of tangency.

27. Find the equation of the tangent

(1) to $3y^2 - 2y + x = 0$, perpendicular to $3x + y = 0$;
(2) to $3y^2 + 6y - 3x + 4 = 0$, parallel to $2y + 3x = 5$.

28. Find the equations of the tangent and normal to the parabola $y^2 + 6x - 8y - 2 = 0$ at the point (3, 4).

29. Find the equations of the tangents

(1) to $y^2 = 8x$ from the point $(-6, 4)$;
(2) to $y^2 = 12x$ from the point $(1, -4)$.

30. Prove that the tangent to the parabola $(x - x_0)^2 = 4a(y - y_0)$ is $(x - x_0) = n(y - y_0) + a/n$. [$n = \tan yBA$, § 23.]

31. What is the equation of the diameter of $y^2 = 8x$ which bisects all chords parallel to $3y + 2x = 0$?

32. What is the equation of the diameter of the parabola $y^2 + 3y + 2x = 0$ which bisects all chords parallel to the line $2y - x = 0$?

33. Prove that the area of a triangle inscribed in the parabola $y^2 = 4ax$ is $(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)/8a$, where y_1, y_2, y_3 denote the ordinates of the vertices.

34. Prove that $y = mx - am(2 + m^2)$ is the slope equation of the normal to the parabola $y^2 = 4ax$. [Use figure of § 84.]

35. Two equal parabolas have the same focus and axis but extend in opposite directions; prove that they cut one another at right angles.

36. Prove that the tangents at the extremities of the latus rectum of a parabola meet at right angles in the point of intersection of the axis and directrix.

37. Two equal parabolas have the same vertex and their axes are perpendicular; prove that they have a common tangent and that it touches each parabola at an extremity of its latus rectum.

38. Prove that any tangent to a parabola meets the directrix and the latus rectum produced in points which are equidistant from the focus.

39. From any point on the latus rectum of a parabola perpendiculars are drawn to the tangents at its extremities; prove that the line joining the feet of these perpendiculars touches the parabola.

40. If P, Q, R are three points on a parabola, whose ordinates are in geometrical progression, prove that the tangents at P and R meet on the ordinate of Q produced.

41. If r denote the distance of a point P on the parabola $y^2 = 4ax$ from the focus, and p the perpendicular distance of the tangent at P from the focus, prove that $p^2 = ar$.

42. The vertex V of a parabola is joined to any point P on the curve, and PQ is taken at right angles to VP and meeting the axis at Q; prove that the projection of PQ on the axis is equal to the latus rectum.

43. Two perpendicular lines VP and VQ pass through the vertex of a parabola and meet the curve again at P and Q; prove that PQ cuts the axis in a fixed point.

44. Prove that the tangents $y = m_1x + a/m_1$ and $y = m_2x + a/m_2$ meet in the point $\{a/m_1m_2,\ a(m_1 + m_2)/m_1m_2\}$.

45. Prove that the slope of the line joining the vertex of a parabola to the point of intersection of any two tangents is the sum of the slopes of the tangents.

46. Prove that the portion of any tangent to a parabola which is intercepted between two fixed tangents subtends at the focus an angle which is equal to the angle between the two fixed tangents.

47. Prove that the tangents at the extremities of any focal chord of a parabola meet at right angles on the directrix.

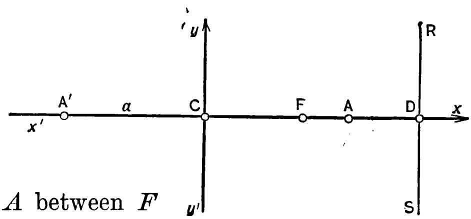
48. Two parabolas, $y^2 = 4ax$ and $x^2 = ay$, intersect at the origin; find the cube of the ordinate, MP, of the other point of intersection. (Since the cube of MP is twice the cube of half the latus rectum of $y^2 = 4ax$, this exercise gives a solution of the problem of the duplication of the cube, one of the famous problems of antiquity.)

CHAPTER V

THE ELLIPSE

88. The equation of the ellipse. By definition [§ 69], the ellipse is the locus of a point whose distance from a fixed point, the focus, divided by its distance from a fixed line, the directrix, is a constant e, less than 1.

Let F be the focus and SR the directrix.



Through F take FD perpendicular to SR at D.

There is a point A between F and D such that $FA / AD = e$.

Again, since $e < 1$, there is a point A' on FD produced through F, such that $A'F/A'D = e$. The points A and A' are on the ellipse. They are called its *vertices*.

Let C be the mid-point of $A'A$. Take the line CD as x-axis, and the line through C parallel to SR as y-axis; it is required to find the equation of the ellipse when referred to these axes.

Represent the length of $A'C$ $(= CA)$ by a, so that $A'A = 2a$.

To obtain the coordinates of F and the equation of SR, it is only necessary to express the lengths of CF and CD in terms of a and e. This may be done as follows:

Since $FA / AD = e$, $A'F/A'D = e$, and $A'C = CA = a$, it follows that

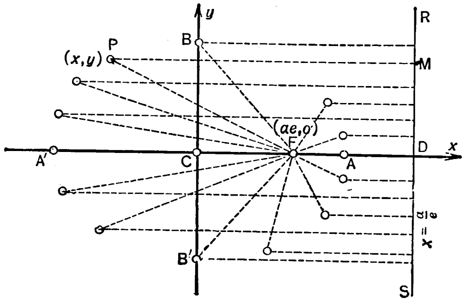
$$e = \frac{FA}{AD} = \frac{a - CF}{CD - a}, \quad \therefore CD\ e - ae = a - CF. \tag{1}$$

$$e = \frac{A'F}{A'D} = \frac{a + CF}{a + CD}, \quad \therefore CD \cdot e + ae = a + CF. \tag{2}$$

Adding (1) and (2) gives $2\,CD \cdot e = 2\,a, \therefore CD = a/e$.

Subtracting (1) from (2) gives $2\,ae = 2\,CF, \therefore CF = ae$.

Therefore, the coordinates of F will be $(ae,\ 0)$, and the equation of SR will be $x = a/e$.



The equation of the ellipse may now be derived as follows:

Let $P\ (x,\ y)$ denote any representative point of the ellipse. Join PF, and take PM perpendicular to SR.

Since P is on the ellipse, $FP/MP = e$, and therefore,

$$FP^2 = e^2\,MP^2. \tag{3}$$

But since FP is the distance of the point $(x,\ y)$ from the point $(ae,\ 0)$, $FP^2 = (x - ae)^2 + y^2$ [§ 41].

And since MP is the perpendicular distance of the point $(x,\ y)$ from the line $x - a/e = 0$, and this equation is in the perpendicular form, $MP^2 = (x - a/e)^2$ [§ 51].

The substitution of these expressions for FP^2 and MP^2 in (3) gives

$$(x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

or

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{4}$$

Hence (4) is the equation of the ellipse when referred to the axes above indicated. For it has been proved that (4) is true for every point P on the ellipse; and it is false for every point off the ellipse, since if the point P is off the ellipse, FP^2 is not equal to $e^2\,MP^2$, therefore $(x - ae)^2 + y^2$ is not equal to

$e^2(x-a/e)^2$, and therefore finally $x^2/a^2+y^2/a^2(1-e^2)$ is not equal to 1.

Since $e<1$, the quantity $a^2(1-e^2)$ is positive and less than a^2. It is represented by b^2. The substitution of b^2 for $a^2(1-e^2)$ in (4) gives

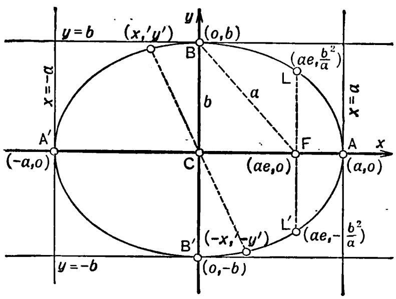
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad (5) \qquad \text{where} \quad b^2=a^2(1-e^2). \quad (6)$$

This is the form in which the equation of the ellipse is usually written.

89. The shape of the ellipse. The shape of the ellipse and its position with respect to the axes may be readily inferred from the equation (5).

Solving (5) for y, $\quad y=\pm\frac{b}{a}\sqrt{a^2-x^2}$.

Hence y has two real values, equal numerically but of opposite signs, when $x^2<a^2$, the value 0 counted twice when $x^2=a^2$, and imaginary values when $x^2>a^2$. Therefore the curve lies wholly between the lines $x=-a$ and $x=a$, which it touches at the points $(-a, 0)$ and $(a, 0)$, and it is symmetric with respect to the x-axis.



As x increases from $-a$ to a, the positive value of y first increases from 0 to b and then decreases from b to 0; and the negative value of y first decreases from 0 to $-b$ and then increases from $-b$ to 0.

Solving (5) for x, $\quad x=\pm\frac{a}{b}\sqrt{b^2-y^2}$.

Hence x has two real values, equal numerically but of opposite signs, when $y<b^2$, the value 0 counted twice when $y^2=b^2$,

and imaginary values when $y^2 > b^2$. Therefore the curve lies wholly between the lines $y = -b$ and $y = b$, which it touches at the points $(0, -b)$ and $(0, b)$, and it is symmetric with respect to the y-axis.

Thus the ellipse is a closed curve inclosed by the lines $x = a$, $x = -a$, $y = b$, $y = -b$ and cut into four equal parts by the x- and y-axes.

90. If the point (x', y') be on the ellipse, so also is the point $(-x', -y')$ on the ellipse; for if $x'^2/a^2 + y'^2/b^2 \equiv 1$, so also is $(-x')^2/a^2 + (-y')^2/b^2 \equiv 1$. But the two points (x', y') and $(-x', -y')$ are on the same straight line through the origin C and are equidistant from C; hence C is the mid-point of every chord of the ellipse which passes through it. It is therefore called the *center* of the ellipse.

91. The chord $A'A$ through the center and focus is called the *major axis* of the ellipse; its length is $2a$. $A'C = CA = a$ is called the *semimajor axis*.

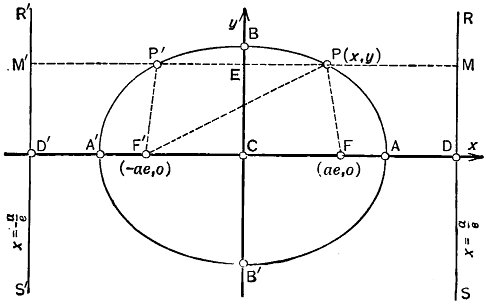
92. The chord $B'B$ through the center of the ellipse, and perpendicular to the major axis is called the *minor axis* of the ellipse; its length is $2b$. $B'C = CB = b$ is called the *semiminor axis*.

93. The chord $L'L$ through the focus and perpendicular to the major axis is called the *latus rectum*. Its length is $2b^2/a$; for when $x = ae$, the first equation of § 89 and the equation (6) of § 88, give

$$y = \pm \frac{b}{a}\sqrt{a^2 - a^2e^2} = \pm \frac{b^2}{a}.$$

The relation connecting a, b, and e is represented geometrically in the right triangle CFB; for since $CF = ae$ and $CB = b$, the hypotenuse $FB = \sqrt{b^2 + a^2e^2} = \sqrt{a^2 - a^2e^2 + a^2e^2} = a$. Hence, the distance from the focus to the extremity of the minor axis is equal to the semimajor axis. And $a^2e^2 = a^2 - b^2$.

94. The second focus and directrix. On the x-axis and to the left of C, lay off CF' equal to CF, and CD' equal to CD, and through D' take $S'R'$ parallel to SR. Then F' will be a second focus of the ellipse and $S'R'$ will be the corresponding directrix; as may be proved in the following manner:



Let P denote any point of the ellipse and through P take PM parallel to the x-axis and meeting SR in M. This line will meet the ellipse at a second point P' and the line $S'R'$ at a point M'. Join PF and $P'F'$. Then, from the symmetry of the ellipse with respect to the y-axis, it immediately follows that $PF = P'F'$ and $PM = M'P'$.

Hence from $FP/MP = e$, it follows that $F'P'/M'P' = e$, that is, the ellipse is also the locus of a point P' whose distance from the point F', divided by its distance from the line $S'R'$, is the constant e. Hence F' is a focus of the ellipse and $S'R'$ is the corresponding directrix. The coordinates of F' are $(-ae, 0)$, and the equation of $S'R'$ is $x + a/e = 0$.

Example. The equation $9x^2 + 16y^2 = 144$ can be written in the form $x^2/16 + y^2/9 = 1$, or $x^2/4^2 + y^2/3^2 = 1$. Hence the semimajor axis is 4, the semiminor axis is 3, $e^2 = 1 - b^2/a^2 = 1 - 9/16 = 7/16$, $e = \sqrt{7}/4$; and hence $ae = \sqrt{7}$, and $a/e = 16/\sqrt{7}$. The foci are $(\sqrt{7}, 0)$, $(-\sqrt{7}, 0)$ and the directrices are $x = 16/\sqrt{7}$, $x = -16/\sqrt{7}$. The vertices are $(4, 0)$, $(-4, 0)$.

95. Exercises. The equation of the ellipse.

1. Find the coordinates of the foci and vertices, and the equations of the directrices of the following ellipses:

(1) $2x^2 + 3y^2 = 6$, (2) $9x^2 + 25y^2 = 225$, (3) $9x^2 + 25y^2 = 1$.

2. Which of the following points are on the ellipse $2x^2 + 3y^2 = 6$: $(1, 1)$, $(0, \sqrt{2})$, $(0, -\sqrt{2})$, $(\pm\sqrt{3}, 0)$, $(0, 3/2)$, $(\sqrt{3}/\sqrt{2}, 1)$, $(1, 2/\sqrt{3})$?

3. The ellipse $2x^2 + 3y^2 = 6$ is given. Find the ordinates of the points on the curve whose abscissas are ± 1. Find the abscissas of the points on the curve whose ordinates are ± 1. Find the coordinates of the points where the ellipse is met by the lines $x + 1 = 0$ and $y + 1 = 0$.

96. *The distances of any point $P(x, y)$ of the ellipse from the foci $F(ae, 0)$ and $F'(-ae, 0)$, are $a - ex$ and $a + ex$, respectively, and the sum of these focal distances is the constant $2a$.*

For, referring to the preceding figure [§ 94], join $F'P$, and let $P'P$ cut the y-axis at E. Then from the definition of the ellipse,

$$PF = ePM = e(EM - EP) = e\left(\frac{a}{e} - x\right) = a - ex,$$

$$PF' = eM'P = e(M'E + EP) = e\left(\frac{a}{e} + x\right) = a + ex,$$

and adding gives $$PF + PF' = 2a.$$

97. *Hence, an ellipse may also be defined as the locus of a point the sum of whose distances from two fixed points is constant.*

If the ends of a piece of string of length $2a$ be fastened at two points in a sheet of paper, and the string be then drawn taut by a pencil, the point of the pencil can be made to trace the ellipse whose foci are the two points and whose major axis is $2a$.

98. A more general form of the equation of the ellipse. From § 88 it follows that the graph of every equation of the form $x^2/a^2 + y^2/b^2 = 1$, referred to rectangular axes, is an ellipse whose axes coincide with the axes of reference.

When $a > b$, the major axis coincides with the x-axis, the eccentricity is given by the relation $e^2 = 1 - b^2/a^2$, the foci are the points $(-ae, 0)$, $(ae, 0)$, and the directrices are the lines $x + a/e = 0$, $x - a/e = 0$.

When $b > a$, the major axis coincides with the y-axis, the eccentricity is given by the relation $e^2 = 1 - a^2/b^2$, the foci are the points $(0, -be)$, $(0, be)$, and the directrices are the lines $y + b/e = 0$, $y - b/e = 0$.

When $a = b$, the equation becomes $x^2/a^2 + y^2/a^2 = 1$, or $x^2 + y^2 = a^2$, which, as has already been seen [§ 43], represents a circle whose center is at the origin and whose radius is a. Since $e^2 = 1 - a^2/a^2 = 0$, a circle may be regarded as the limiting case of an ellipse whose eccentricity has become 0, whose foci have moved into coincidence with the center, and whose directrices have moved out to an infinite distance in the plane.

Observe that in every case the eccentricity of an ellipse is given by the relation

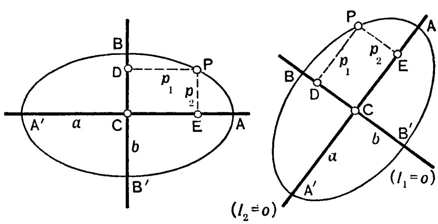
$$e^2 = 1 - \frac{(\text{semiminor axis})^2}{(\text{semimajor axis})^2}, \tag{1}$$

and the foci and directrices are found from

$CF = e$ (semimajor axis), (2) $CD =$ (semimajor axis)$/e$. (3)

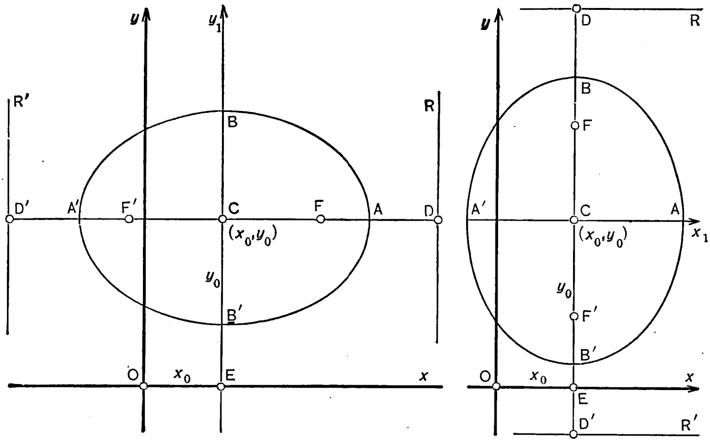
Thus, $x^2/16 + y^2/25 = 1$, in which $a = 4$, $b = 5$, represents an ellipse whose major axis ($= 2b$) coincides with the y-axis and whose eccentricity is $e = \sqrt{1 - 16/25} = 3/5$. Hence $be = 3$ and $b/e = 25/3$, and therefore the foci are $(0, -3)$, $(0, 3)$, and the directrices $y + 25/3 = 0$, $y - 25/3 = 0$.

99. In the equation $x^2/a^2 + y^2/b^2 = 1$, x and y denote the distances p_1 and p_2 of any point P on the ellipse from the axes of the ellipse, and the equation is equivalent to the statement that in any ellipse these distances, p_1 and p_2, are connected with a and b, the lengths of the semiaxes, by the relation $p_1^2/a^2 + p_2^2/b^2 = 1$. This property of the ellipse is independent of the position of the curve in the plane. Hence the locus of a point P, whose distances p_1 and p_2 from *any* two perpendicular lines $l_1 = 0$ and $l_2 = 0$ are connected by the relation $p_1^2/a^2 + p_2^2/b^2 = 1$, will be an ellipse which is equal to the



ellipse $x^2/a^2 + y^2/b^2 = 1$, and is placed with respect to the lines $l_1 = 0$, $l_2 = 0$ as the latter ellipse is placed with respect to the lines $x = 0$, $y = 0$ (*i.e.* the y- and x-axis, respectively).

Thus, in particular, $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$ represents an ellipse whose axes coincide with the lines $x - x_0 = 0$ and $y - y_0 = 0$



and whose center is therefore the point (x_0, y_0). The lengths of the semiaxes are a, b. If $a > b$, the major axis coincides with $y - y_0 = 0$; if $b > a$, it coincides with $x - x_0 = 0$. See the figures.

Every equation of the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, *and in which* a, b, *and* $(g^2/a + f^2/b - c)$ *are of the same sign, represents an ellipse whose axes are parallel to the axes of coordinates.*

For this equation can be reduced to the form just considered.

Example 1. Find the graph of the equation

$$2x^2 + 3y^2 + 12x + 6y + 15 = 0. \qquad (1)$$

The equation may be written $2(x^2 + 6x \quad) + 3(y^2 + 2y \quad) = -15$, or, completing the squares,

$$2(x^2 + 6y + 9) + 3(y^2 + 2y + 1) = -15 + 18 + 3,$$

that is,

$$2(x + 3)^2 + 3(y + 1)^2 = 6,$$

or,

$$(x + 3)^2/3 + (y + 1)^2/2 = 1. \qquad (2)$$

Here $a = \sqrt{3}$, $b = \sqrt{2}$, and since $a > b$, the major and minor axes coincide with the lines $y + 1 = 0$ and $x + 3 = 0$, respectively, the center being the point $(-3, -1)$. From § 98 (1), $e = \sqrt{1 - 2/3} = \sqrt{1/3}$, and therefore $ae = 1$, $a/e = 3$. Hence the foci are the points $(-3 + 1, -1)$, $(-3-1, -1)$; and the directrices are the lines $x+3-3=0$, $x+3+3=0$.

Applying this method to $ax^2 + by^2 + 2gx + 2fy + c = 0$, where a and b are of the same sign,

$$a(x + g/a)^2 + b(y + f/b)^2 = g^2/a + f^2/b - c.$$

Hence, if $(g^2/a + f^2/b - c)$ be of the same sign as a and b, the graph is an ellipse whose axes coincide with the lines $x + g/a = 0$, $y + f/b = 0$. But if $(g^2/a + f^2/b - c)$ be of the opposite sign, the equation has no real solution and therefore no graph; and if $(g^2/a + f^2/b - c)$ be 0, the graph is the single point $(-g/a, -f/b)$.

Example 2. Find the equation of the ellipse whose axes are parallel to the axes of coordinates and which passes through the points $(-1, 0)$, $(0, -1)$, $(3, 0)$, $(2, 2)$.

The required equation has the form $ax^2 + by^2 + 2gx + 2fy + c = 0$ (1). Since it has the solution $(-1, 0)$, $a - 2g + c = 0$ (2). Similarly, since it has the solutions $(0, -1)$, $(3, 0)$, and $(2, 2)$, $b - 2f + c = 0$ (3), $9a + 6g + c = 0$ (4), and $4a + 4b + 4g + 4f + c = 0$ (5). Solving (2), (3), (4), (5) for a, b, $2f$, $2g$ in terms of c, substituting the results in (1), and simplifying, we obtain $2x^2 + 3y^2 - 4x - 3y - 6 = 0$, the equation required.

Example 3. Find the equation of the ellipse whose major and minor axes coincide with the lines $x - 2y + 4 = 0$ (1) and $2x + y - 2 = 0$ (2), respectively, the semiaxes being 4 and 3.

Here $p_1 = (2x + y - 2)/\sqrt{5}$, $p_2 = (x - 2y + 4)/(-\sqrt{5})$, $a = 4$, $b = 3$. Hence the required equation is

$$\frac{(2x + y - 2)^2}{5 \cdot 16} + \frac{(x - 2y + 4)^2}{5 \cdot 9} = 1.$$

Example 4. Prove that the equation of the ellipse whose eccentricity is 2/3 and which has $(2, 0)$ and $x + y = 0$ for focus and corresponding directrix is

$$(x - 2)^2 + y^2 = \frac{4}{9}\left(\frac{x + y}{\sqrt{2}}\right)^2.$$

Example 5. Find the equation of the ellipse whose center is $(a, 0)$ and whose semiaxes are a and b.

Example 6. Find the graph of $3x^2 + 4y^2 - 12x - 8y - 8 = 0$.

Example 7. Find the equation of the ellipse whose axes are parallel to the axes of coordinates and which passes through the points (0, 0), (2, 0), (0, 3), (1, 4).

Example 8 Find the equation of the ellipse whose eccentricity is e, and for which the origin and the line $x - d = 0$ are a focus and the corresponding directrix.

100. The Circle. When $a = b$, by dividing by a the equation considered in the preceding section can be reduced to the form

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0,$$

and therefore to the form

$$(x + g')^2 + (y + f')^2 = g'^2 + f'^2 - c',$$

which, if $(g'^2 + f'^2 - c') > 0$, represents a circle whose center is the point $(-g', -f')$ and whose radius is $\sqrt{g'^2 + f'^2 - c'}$ [§ 68]. If $(g'^2 + f'^2 - c') < 0$, there is no graph.

Example 1. Prove that $3x^2 + 3y^2 + 5x - 6y - 9 = 0$ represents a circle, and find its center and radius.

Dividing by the common coefficient of x^2 and y^2, and rearranging the terms

$$\{x^2 + (5/3)x + \quad\} + \{y^2 - 2y + \quad\} = 3.$$

completing the squares

$$\{x^2 + (5/3)x + 25/36\} + \{y^2 - 2y + 1\} = 3 + 25/36 + 1,$$

or

$$\{x + 5/6\}^2 + \{y - 1\}^2 = 169/36,$$

which represents a circle whose center is $(-5/6, 1)$ and whose radius is 13/6.

Example 2. Find the equation of the circle which passes through the points (0, 0), (– 1, 0), and (0, 1).

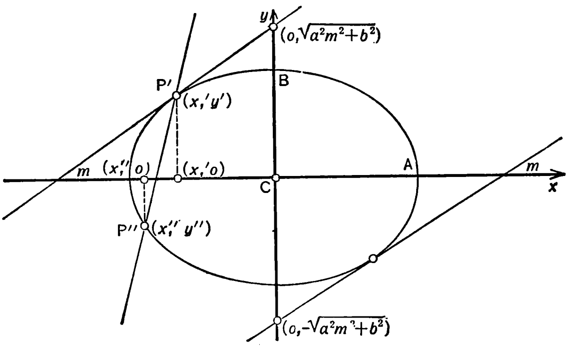
The required equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$. (1). Since it has the solution (0, 0), $c = 0$ (2). Similarly since it has the solutions (– 1, 0) and (0, 1), $1 - 2g + c = 0$ (3), and $1 + 2f + c = 0$ (4). Hence $c = 0$, $2g = 1$, and $2f = -1$, and the required equation is $x^2 + y^2 + x - y = 0$.

101. Equation of tangent in terms of slope. The abscissas of the points where the line $y = mx + c$ (1) cuts the ellipse $x^2/a^2 + y^2/b^2 = 1$ (2) are the roots of the equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

or $$(a^2m^2 + b^2)\,x^2 + 2\,a^2mc\,x + a^2(c^2 - b^2) = 0. \qquad (3)$$

Hence the line (1) will meet the ellipse (2) in *coincident* points, or touch it, if the roots of (3) are equal. But the roots of (3)



will be equal, if the left member of (3) is a perfect square, that is, if [compare § 78] $$(a^2mc)^2 = (a^2m^2 + b^2)\cdot a^2(c^2 - b^2)$$

or, $$a^4m^2c^2 = a^4m^2c^2 + a^2b^2c^2 - a^4m^2b^2 - a^2b^4,$$

or $$0 = a^2b^2(c^2 - a^2m^2 - b^2),$$

or, since neither a nor b is zero, $c^2 = a^2m^2 + b^2$,

or, finally, $$c = \pm\sqrt{a^2m^2 + b^2}.$$

Hence for any given value of m there are two tangents on the opposite sides of the ellipse and equidistant from its center, namely, $y = mx + \sqrt{a^2m^2 + b^2}$, and $y = mx - \sqrt{a^2m^2 + b^2}$.

Example 1. Find the equations of both of the tangents to the ellipse $3\,x^2 + 4\,y^2 - 12 = 0$ which are perpendicular to the line $y + 2\,x = 0$.

The equation of the ellipse may be written $x^2/4 + y^2/3 = 1$. Hence the tangents are $y = x/2 \pm \sqrt{4\,(1/4) + 3}$, or $y = x/2 \pm 2$.

Example 2. Find the equations of both of the tangents to the ellipse $3x^2 + y^2 - 2x = 0$ which are parallel to the line $y = x$.

Every line parallel to $y = x$ has an equation of the form $y = x + \lambda$. The abscissas of the points of intersection of the line $y = x + \lambda$ with the given ellipse are the roots of the equation $3x^2 + (x + \lambda)^2 - 2x = 0$, or $4x^2 + 2(\lambda - 1)x + \lambda^2 = 0$. These roots are equal, if $(\lambda - 1)^2 = 4\lambda^2$, that is, if $3\lambda^2 + 2\lambda - 1 = 0$, or solving, if $\lambda = -1$ or $1/3$. Hence the required equations are $y = x - 1$ and $y = x + 1/3$.

102 A.* Equation of tangent in terms of coordinates of point of contact. *First method of derivation.* The equation of the line through any two points (x', y') and (x'', y'') is

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''}. \tag{1}$$

But if the two points be on the ellipse $x^2/a^2 + y^2/b^2 = 1$,

then $x'^2/a^2 + y'^2/b^2 \equiv 1$ (2) and $x''^2/a^2 + y''^2/b^2 \equiv 1$. (3)

Subtracting (3) from (2), transposing, and factoring,

$$(x' - x'')(x' + x'')/a^2 \equiv -(y' - y'')(y' + y'')/b^2. \tag{4}$$

Multiplying (1) by (4), and transposing,

$$(x - x')(x' + x'')/a^2 + (y - y')(y' + y'')/b^2 = 0, \tag{5}$$

that is, when the points (x', y') and (x'', y'') are on the ellipse, the equation (1) of the line joining them can be reduced to the form (5); in other words, (5) is the equation of the secant through the two points (x', y') and (x'', y'') on the ellipse.

If (x'', y'') be made to move along the ellipse into coincidence with (x', y'), the secant becomes, at the limit, the tangent at (x', y'), and the equation (5) becomes, after dividing by 2,

$$\frac{(x - x')x'}{a^2} + \frac{(y - y')y'}{b^2} = 0; \text{ that is, } \frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

Hence,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \tag{6}$$

is the equation of the tangent at (x', y').

* Only one of § 102 A, § 102 B, § 102 C need be taken.

102 B. Equation of tangent in terms of coordinates of point of contact. *Second method of derivation.* Let (x', y') and (x'', y'') be two points on the ellipse $x^2/a^2+y^2/b^2-1=0$ (1), so that both $x'^2/a^2+y'^2/b^2-1\equiv 0$ (2), and $x''^2/a^2+y''^2/b^2-1\equiv 0$ (3), and consider the equation

$$\frac{(x-x')(x-x'')}{a^2}+\frac{(y-y')(y-y'')}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1. \tag{4}$$

This is an equation of the first degree; for, on being simplified, it reduces to

$$\frac{x'+x''}{a^2}x+\frac{y'+y''}{b^2}y=\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+1. \tag{5}$$

Moreover, it is satisfied by $x=x'$, $y=y'$ and by $x=x''$, $y=y''$. For if (x', y') be substituted for (x, y) in (4), the left member becomes $(x'-x')(x'-x'')/a^2+(y'-y')(y'-y'')/b^2$, which is identically 0, and the right member becomes $x'^2/a^2+y'^2/b^2-1$, which is 0 because (x', y') is on the ellipse. And it can be shown in the same manner that (4) is satisfied by $x=x''$, $y=y''$.

Therefore (4), or its equivalent (5), is the equation of the secant to the ellipse through the points (x', y') and (x'', y''). For if an equation of the first degree be true for two points of a given line, it is the equation of that line [§ 17]

When the point (x'', y'') moves along the ellipse into coincidence with the point (x', y'), the secant becomes the tangent at (x', y'), and (5) becomes

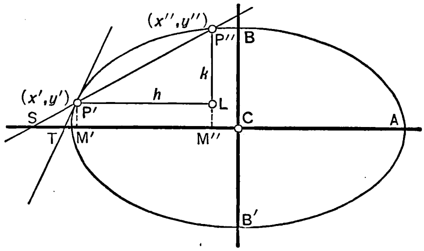
$$2\frac{xx'}{a^2}+2\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+1.$$

But since $x'^2/a^2+y'^2/b^2\equiv 1$, this equation may be reduced to the form

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1, \tag{6}$$

which is therefore the equation of the tangent to the ellipse $x^2/a^2+y^2/b^2=1$ at the point (x', y').

102 C. Equation of tangent in terms of coordinates of point of contact. *Third method of derivation.* Let P' and P'' be two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and represent the coordinates of P' by (x', y'), and those of P'' by $(x' + h, y' + k)$.



If P'' be made to move along the curve into coincidence with P', the secant $P'P''$ will become the tangent at P', and both h and k will approach 0 as limit.

The slope of the secant $P'P''$ is k/h. Hence the slope of the tangent at P' is the limiting value of k/h, which is represented by lim k/h; and this may be found as follows:

Since P' and P'' are on the ellipse,

$$x'^2/a^2 + y'^2/b^2 \equiv 1, \quad (1) \qquad (x' + h)^2/a^2 + (y' + k)^2/b^2 \equiv 1. \quad (2)$$

Expanding (2) and subtracting (1) from the result,

$$\frac{2\,x'h + h^2}{a^2} + \frac{2\,y'k + k^2}{b^2} = 0, \text{ or } h\frac{2\,x' + h}{a^2} + k\frac{2\,y' + k}{b^2} = 0. \quad (3)$$

Hence $\frac{k}{h} = -\frac{b^2}{a^2}\frac{2\,x' + h}{2\,y' + k}$, and therefore $\lim\frac{k}{h} = -\frac{b^2}{a^2}\frac{x'}{y'}$.

Therefore, since the slope of the tangent at $P'(x', y')$ is $-b^2x'/a^2y'$, the equation of the tangent is

$$y - y' = -\frac{b^2x'}{a^2y'}(x - x'), \quad (4)$$

which reduces to

$$\frac{yy'}{b^2} - \frac{y'^2}{b^2} = -\left(\frac{xx'}{a^2} - \frac{x'^2}{a^2}\right), \quad (5)$$

or, transposing and using $x'^2/a^2 + y'^2/b^2 \equiv 1$, to

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1. \quad (6)$$

Hence (6) is the equation of the tangent at (x', y').

103. When $b^2 = a^2$, the ellipse is the circle $x^2 + y^2 = a^2$, (1) and the equations for the tangent in § 101 and § 102 become

$$y = mx \pm a\sqrt{1+m^2}, \quad (2) \qquad xx' + yy' = a^2. \quad (3)$$

104. Exercises. Tangent to the ellipse.

Write the equation of the tangent to:

1. $9x^2 + 16y^2 = 144$ at $(3, \sqrt{63}/4)$ and at $(2, \sqrt{27}/2)$.
2. $2x^2 + 3y^2 = 6$ at $(1, 2/\sqrt{3})$, at $(\sqrt{3/2}, 1)$, and at $(\sqrt{3/2}, -1)$.
3. $9x^2 + 16y^2 = 144$ at the point whose ordinate is 2.
4. $9x^2 + 16y^2 = 144$ with the slope -1.
5. $3x^2 + 4y^2 - 2x + 6y - 25 = 0$ at the point $(3, -2)$.
6. $4x^2 + 3y^2 - 4y = 0$ with the slope 2.
7. Prove that $y - y_0 = m(x - x_0) \pm \sqrt{a^2m^2 + b^2}$ is the slope equation of the tangents to the ellipse $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$.

105. The Normal. Since the normal to the ellipse at the point (x', y') is perpendicular to the tangent, and § 102 (6) is the equation of the tangent, the equation of the normal is

$$(x - x')\frac{y'}{b^2} - (y - y')\frac{x'}{a^2} = 0.$$

Example. The tangent to the ellipse $9x^2 + 16y^2 = 145$ at the point (3, 2) is $9x\,3 + 16y\,2 = 145$, or $27x + 32y = 145$. The normal, by definition, is the perpendicular to this line through the point (3, 2), namely, $27(y - 2) = 32(x - 3)$, or $32x - 27y - 42 = 0$.

106. Geometrical properties of the ellipse. *The tangent at any point of an ellipse makes equal angles with the lines joining the point to the foci of the ellipse.*

Let $P'(x', y')$ be any point on the ellipse and let $T'P'T$ be the tangent at P'. Join $P'F$ and $P'F'$. The angles $FP'T$ and $F'P'T'$ are to be proved equal.

Draw FE and $F'E'$ perpendicular to $T'T$ at E and E', respectively.

The equation of $T'T$ may be written.

$$b^2x'x + a^2y'y - a^2b^2 = 0.$$

Therefore EF and $E'F'$, the perpendicular distances of $F(ae, 0)$ and $F'(-ae, 0)$ from $T'T$, are

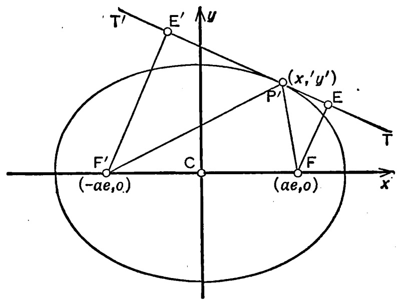
$$EF = \frac{b^2x' \cdot ae - a^2b^2}{\sqrt{b^4x'^2 + a^4y'^2}}, \quad E'F' = \frac{-b^2x' \cdot ae - a^2b^2}{\sqrt{b^4x'^2 + a^4y'^2}}.$$

Hence

$$EF : E'F' = \frac{b^2a(x'e - a)}{\sqrt{b^4x'^2 + a^4y'^2}} : \frac{-b^2a(x'e + a)}{\sqrt{b^4x'^2 + a^4y'^2}} = a - ex' : a + ex'.$$

But $a - ex' = FP'$ and $a + ex' = F'P'$ [§ 96].

Hence $FE : F'E' = FP' : F'P'$, or $FE : FP' = F'E' : F'P'$; and therefore the right-angled triangles $FP'E$ and $F'P'E'$ are similar, and $\measuredangle FP'E = \measuredangle F'P'E'$, which was to be proved.



107. *The perpendiculars from the foci to any tangent of an ellipse meet the tangent in points E and E' which lie on the circle* $x^2 + y^2 = a^2$.

The equation of the tangent $E'E$ is $\quad y - mx = \sqrt{a^2m^2 + b^2}$. (1)

The equation of FE is $\quad my + x = ae$. (2)

Regarding (1) and (2) as simultaneous (which is to make (x, y) the coordinates of E), square both equations, and add; the result is

$(m^2 + 1)(x^2 + y^2) = a^2m^2 + b^2 + a^2e^2 = a^2m^2 + a^2$, [Since $a^2e^2 = a^2 - b^2$.]

hence $$x^2 + y^2 = a^2.$$

The product of the perpendicular distances of the foci from any tangent is equal to the square of the semiminor axis, that is, $EF \cdot E'F' = b^2$.

For the equation of the tangent is $y - mx - \sqrt{a^2m^2 + b^2} = 0$.

Hence $$EF = \frac{-mae - \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}} \text{ and } E'F' = \frac{mae - \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}},$$

and $$EF \cdot E'F' = \frac{a^2m^2 + b^2 - m^2a^2e^2}{1 + m^2} = b^2.$$

108. Diameters. The locus of the mid-points of a system of parallel chords of an ellipse is called a *diameter* of the ellipse. [Compare § 86.] It is to be proved that every such diameter is a straight line through the center of the ellipse.

Example. Find the locus of the mid-points of the system of parallel chords of the ellipse $x^2 + 2y^2 - 2 = 0$ (1), whose slope is 4/3.

Every chord of the system has an equation of the form $y = 4x/3 + \lambda$ (2), where λ is an arbitrary constant. The abscissas of the points of intersection of (1) and (2), that is, the abscissas of the end-points of the chord (2), are given by

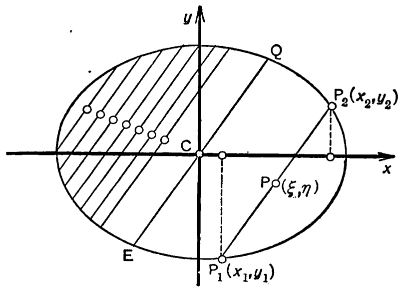
$$x^2 + 2(4x/3 + \lambda)^2 - 2 = 0, \text{ or } 41x^2 + 48\lambda x + 18\lambda^2 - 18 = 0. \quad (3)$$

Hence, if $P(\xi, \eta)$ denote the mid-point of the chord, its abscissa ξ is one-half the sum of the roots of (3). But the sum of the roots of (3) is $-48\lambda/41$. Hence $\xi = -24\lambda/41$. (4)

Again, $P(\xi, \eta)$ is on the line (2); hence $\eta = 4\xi/3 + \lambda = 9\lambda/41$. (5)

Eliminate λ by dividing (5) by (4); the result is $\eta/\xi = -3/8$. Hence, whatever the value of λ may be, the mid-point of the chord (2) lies on the line $y = -3x/8$, which passes through the center of the ellipse.

Consider the system of chords parallel to the line QE whose equation is $y = mx$. (1)



Let P_1P_2 represent any such chord, and let

$$P_1(x_1, y_1),\ P_2(x_2, y_2)$$

denote the points where it cuts the ellipse, and P the mid-point of P_1P_2. Represent the coordinates of P by (ξ, η), in order to distinguish them from the coordinates of the points of the curve. The equation of the locus of P may be found as follows:

Since P_1P_2 is parallel to QE ($y = mx$), but is otherwise undetermined, its equation is

$$y = mx + \lambda, \quad (2)$$

where λ is an arbitrary constant [§ 26].

Eliminating y between this equation and that of the ellipse, $x^2/a^2 + y^2/b^2 = 1$, gives

$$\frac{x^2}{a^2} + \frac{(mx+\lambda)^2}{b^2} = 1, \quad \text{or} \quad x^2 + \frac{2\,ma^2\lambda}{m^2a^2+b^2}x + \frac{(\lambda^2-b^2)a^2}{m^2a^2+b^2} = 0. \qquad (3)$$

The roots of this equation are x_1 and x_2, the abscissas of P_1 and P_2; hence $x_1 + x_2$ is the coefficient of x with its sign changed. And, since $P(\xi, \eta)$ is the mid-point of P_1P_2, $\xi = (x_1 + x_2)/2$; therefore

$$\xi = -\frac{ma^2}{m^2a^2+b^2}\lambda. \qquad (4)$$

Furthermore, $P(\xi, \eta)$ is on the line $y = mx + \lambda$; hence [§ 16] $\eta = m\xi + \lambda$, and therefore from (4),

$$\eta = -\frac{m^2a^2}{m^2a^2+b^2}\lambda + \lambda, \text{ or, simplifying, } \eta = \frac{b^2}{m^2a^2+b^2}\cdot\lambda. \qquad (5)$$

The elimination of the arbitrary constant λ, by dividing (5) by (4), gives

$$\eta = -\frac{b^2}{a^2m}\xi, \quad \text{or} \quad y = -\frac{b^2}{a^2}\frac{1}{m}x \qquad (6)$$

as the equation of the locus of P. The locus is therefore a straight line through the center, as was to be demonstrated.

Example 1. Find the diameter of the ellipse $5\,x^2 + 6\,y^2 - 10 = 0$ which bisects all chords parallel to the line $3\,x + 2\,y - 5 = 0$.

Here $a^2 = 2$, $b^2 = 5/3$, $m = -3/2$. Hence the required equation is
$$y = -(5/3)\cdot(1/2)\cdot(-2/3)\cdot x, \text{ or } y = (5/9)\,x.$$

Example 2. Find the diameter of the ellipse $3\,x^2 + 2\,y^2 + 6\,x - 5 = 0$ which bisects all chords which have the slope 2.

The equation of any chord having the slope 2 is $y = 2\,x + \lambda$. The abscissas of the points of intersection of the line $y = 2\,x + \lambda$ with the ellipse are the roots of the equation.

$$3\,x^2 + 2(2\,x+\lambda)^2 + 6\,x - 5 = 0, \text{ or } 11\,x^2 + (8\,\lambda + 6)\,x + 2\,\lambda^2 - 5 = 0.$$

Hence the abscissa of the mid-point $P(\xi, \eta)$ of the chord is $\xi = -(4\,\lambda + 3)/11$. Its ordinate η is given by $\eta = 2\,\xi + \lambda$ and is $\eta = (3\,\lambda - 6)/11$. The elimination of λ between these equations for ξ and η gives $33\,\xi + 44\,\eta + 33 = 0$. Hence the equation of the diameter is $3\,x + 4\,y + 3 = 0$.

109. Conjugate diameters. It has just been proved that all chords parallel to the line $y = mx$ are bisected by the line $y = m'x$, where $m' = -b^2/a^2m$ [§ 108, (6)], or

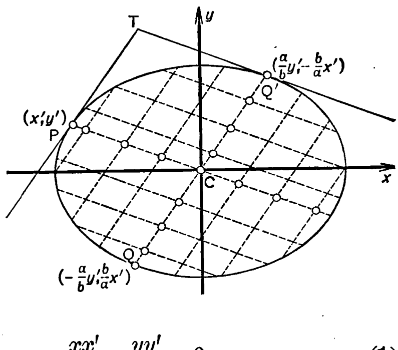
$$mm' = -\frac{b^2}{a^2}. \tag{1}$$

But (1) is symmetric with respect to m and m', and therefore also proves that all chords parallel to the line $y = m'x$ are bisected by the line $y = mx$. Or, to put the proof in another way, the argument in § 108 proves that the locus of the mid-points of chords parallel to $y = m'x$ is $y = -b^2x/a^2m'$, that is, $y = mx$, since $-b^2/a^2m' = m$.

Therefore, every line through the center of an ellipse is a diameter; and if the slopes m and m' of two diameters, $y = mx$ and $y = m'x$, are connected by the relation (1), each bisects all chords parallel to the other. Two such diameters are said to be *conjugate*.

110. *Let $P(x', y')$ denote any point of an ellipse, CP the diameter through P, and CQ the diameter conjugate to CP. The equation of CQ is*

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$



For the slope of CP is y'/x'; hence the slope of CQ is $-x'b^2/y'a^2$ [§ 109]; and therefore the equation of CQ is

$$y = -\frac{x'b^2}{y'a^2}x, \quad \text{or} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0. \tag{1}$$

From this result it follows that CQ is parallel to the tangent at P. See equation (6), § 102.

111. *To find the coordinates of the points Q and Q', where the diameter conjugate to CP cuts the ellipse, in terms of the coordinates of $P\,(x', y')$.*

The elimination of y/b between $x^2/a^2 + y^2/b^2 = 1$ and $xx'/a^2 + yy'/b^2 = 0$ [equation of QQ'] gives

$$\frac{y'^2}{b^2}\left(1 - \frac{x^2}{a^2}\right) = \frac{x'^2}{a^2}\frac{x^2}{a^2}, \quad \text{or} \quad \frac{y'^2}{b^2} = \frac{x^2}{a^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right),$$

or, since (x', y') is on the ellipse, and therefore

$$x'^2/a^2 + y'^2/b^2 \equiv 1,$$

$$\frac{x^2}{a^2} = \frac{y'^2}{b^2}, \text{ or finally } \frac{x}{a} = \pm\frac{y'}{b};$$

and this value of x/a set in the equation of QQ' gives

$$\frac{y}{b} = \mp\frac{x'}{a}.$$

Therefore the coordinates of Q and Q' are given by $(\pm\, y'a/b, \mp\, x'b/a)$; where, with the lettering of the figure of § 110, Q is $(-\,y'a/b,\, x'b/a)$ and Q' is $(y'a/b, -\,x'b/a)$.

Example. Find the diameter of the ellipse $4x^2 + 5y^2 - 21 = 0$ conjugate to that through the point $(2, 1)$.

Since the equation of the tangent at the point $(2, 1)$ is $8x + 5y - 21 = 0$, the equation of the required diameter is $8x + 5y = 0$. Its points of intersection with the ellipse are $(\sqrt{5}/2, -4/\sqrt{5})$ and $(-\sqrt{5}/2, 4/\sqrt{5})$.

112. *The area of the parallelogram bounded by CP, CQ', and the tangents at P and Q' is constant and equal to ab.*

Let the tangents at P and Q' meet at T. The parallelogram $CPTQ'$ is twice the triangle CPQ'. Therefore the substitution of the coordinates of $P\,(x', y')$ and $Q'\,(y'a/b, -\,x'b/a)$ in the formula of § 53 gives

$$CPTQ' = \frac{a}{b}y'^2 + \frac{b}{a}x'^2 = \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)ab = ab.$$

113. *The sum of the squares of CP and CQ is constant and equal to $a^2 + b^2$.*

For $$CP^2 = x'^2 + y'^2 \text{ and } CQ^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2.$$

Hence, $$CP^2 + CQ^2 = x'^2 + \frac{b^2}{a^2}x'^2 + y'^2 + \frac{a^2}{b^2}y'^2.$$

$$= \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)(a^2 + b^2) = a^2 + b^2.$$

114. Orthogonal projection. The foot of the perpendicular from any point P in space upon a plane γ is called the *orthogonal projection* or, more briefly, the *projection* of P on γ.

The projections of the points of a given curve C in space form a curve C' in γ called the projection of the given curve C.

Similarly, the projections of the points of a given surface A form a portion A' of γ called the projection of the given surface A, or $pr_\gamma A = A'$.

Evidently, if two given lines or curves intersect (or touch) at a point P, their projections intersect (or touch) at the projection of P.

It is also readily proved that a straight line projects into a straight line, and parallel lines into parallel lines.

115. *If AB denote a given line segment, $A'B'$ its projection on γ, and θ the angle made by the line AB with γ, then $A'B' = AB \cos\theta$.*

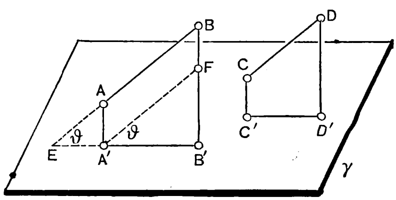
For, by definition, the angle made by the line AB with γ is the angle made by the line AB with its projection, the line $A'B'$, that is, the angle $A'EA$ in the figure. Hence $\theta = \measuredangle A'EA$.

Draw $A'F'$ parallel to AB, meeting $B'B$ at F. Then $A'F = AB$, and $\measuredangle B'A'F = \measuredangle A'EA = \theta$.

Therefore, since the triangle $A'B'F$ is right-angled,

$A'B' = A'F \cos\theta = AB \cos\theta$,

or $pr_\gamma AB = AB \cos\theta$, as was to be demonstrated.

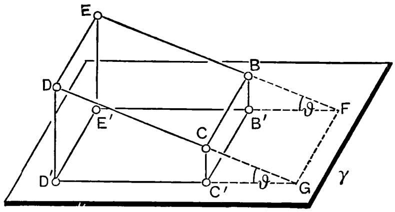


116. *The ratio of two parallel line segments AB and CD is the same as that of their projections $A'B'$ and $C'D'$.*

For since AB and CD are parallel, they make the same angle θ with the plane γ, and therefore $A'B' = AB \cos\theta$ and $C'D' = CD \cos\theta$. Hence, $AB : CD = A'B' : C'D'$.

117. *The projection of any plane area A is equal to the product of A by the cosine of the angle θ made by the plane of A with γ.*

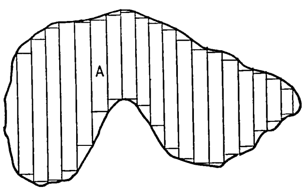
The theorem is obviously true of a rectangle, as $BCDE$ in the figure, one of whose sides CD is perpendicular to FG, the intersection of the two planes. For, in this case, θ is the angle $C'GC$ made by CD with its projection $C'D'$, so that $C'D' = CD \cos\theta$, and, since $B'C' = BC$, it follows that



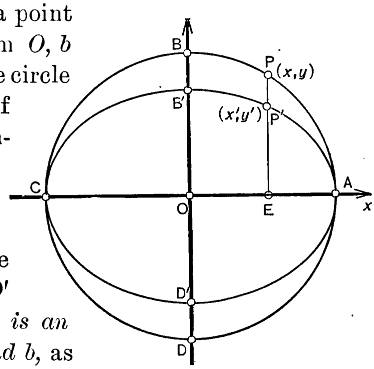
$$B'C' \cdot C'D' = BC \cdot CD \cdot \cos\theta,$$

which was to be proved.

To extend the theorem to any plane area A, divide A into strips of equal breadth by lines drawn perpendicular to the intersection of the plane of A with γ, and then inscribe rectangles in these strips in the manner indicated in the figure. If S denote the sum of the rectangles, the sum of their projections is $S \cos\theta$. But if the width of the strips be indefinitely decreased, S will approach A as limit, and $S \cos\theta$ will approach the projection of A as limit. Hence the area of the projection of A is $A \cos\theta$, or $pr_\gamma A = A \cos\theta$, as was to be demonstrated.



118. The ellipse the orthogonal projection of the circle. Let $ABCD$ be a circle of radius a, AC and BD a pair of perpendicular diameters, and B' a point on OB at the distance b from O, b being less than a. Suppose the circle to be turned out of the plane of the paper, about AC as axis, until B comes to lie vertically over B', and that the circle is then projected from its new position on to the plane of the paper. Let the curve $AB'CD'$ represent the projection. *It is an ellipse whose semiaxes are a and b*, as may be proved in the following manner.



Let P'' denote any point of the circle in its second position, and $P''E$ its ordinate; and let P' denote the projection of P'', and $P'E$ that of $P''E$. And let B'' denote the second position of B, so that B' is the projection of B''.

Let (x'', y'') denote the coordinates of P'' referred to the lines OA and OB'' as axes, and (x', y') the coordinates of P' referred to the lines OA and OB'.

Since $P''(x'', y'')$ is on the circle,

$$x''^2 + y''^2 = a^2, \text{ or } \frac{x''^2}{a^2} + \frac{y''^2}{a^2} = 1. \tag{1}$$

But $OE = x'' = x'$, and therefore, $\dfrac{x''}{a} = \dfrac{x'}{a}$; (2)

and since EP'' and OB'' are parallel,

$$\frac{EP''}{OB''} = \frac{EP'}{OB'} \text{ [§ 116], that is, } \frac{y''}{a} = \frac{y'}{b}; \tag{3}$$

and the substitution of these values, (2) and (3), for $\dfrac{x''}{a}$ and $\dfrac{y''}{a}$ in (1) gives

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \tag{4}$$

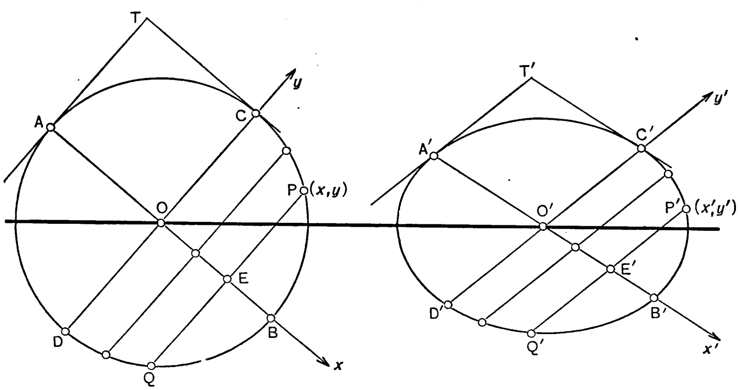
that is, the curve $AB'CD'$ is an ellipse whose semiaxes are a and b, as was to be proved.

Hence the following theorems, §§ 119, 120, 121.

119. *The area of an ellipse whose semiaxes are a and b is* $ab\pi$. For, if θ denote the angle through which the circle just considered is turned in bringing B vertically over B', then by § 115, $\cos\theta = b/a$. But the area of the ellipse is the area of the circle times $\cos\theta$ [§ 117], that is, $a^2\pi\cdot\cos\theta$, or $a^2\pi\cdot b/a$, or $ab\pi$, as was to be proved.

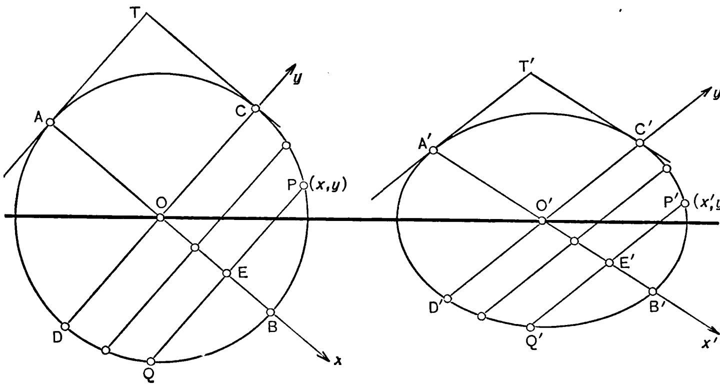
120. *When a circle is projected into an ellipse, every pair of perpendicular diameters of the circle is projected into a pair of conjugate diameters of the ellipse.*

For let the ellipse at the right in the figure represent the projection, by the method of § 118, of the circle at the left;



where, after projecting the circle, the figures are separated on the axis of rotation. And let AB, CD be any perpendicular diameters of the circle. Their projections $A'B'$, $C'D'$ are conjugate diameters of the ellipse. For any chord PEQ of the circle which is parallel to CD is bisected by AB. Hence its projection $P'E'Q'$, a chord of the ellipse which is parallel to

$C'D'$, is bisected by $A'B'$ [§ 116]. Similarly, every chord of the ellipse which is parallel to $A'B'$ is bisected by $C'D'$. And conjugate diameters are such that each bisects all chords parallel to the other [§ 109].



Since the tangents at the extremities of CD are parallel to AB, the tangents at the extremities of C'D' are parallel to A'B'.

Again, since the area of the square AOCT is constant for all perpendicular diameters of the circle, the area of the parallelogram A'O'C'T' is constant for all conjugate diameters of the ellipse.

121. *The equation of an ellipse referred to a pair of conjugate diameters, as oblique axes, is*

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1,$$

where a', b' denote the lengths of the semiconjugate diameters.

For, referring to the preceding figure, let (x, y) denote the coordinates of the point P of the circle referred to the lines OB and OC as axes, and (x', y') the coordinates of the point P' of the ellipse referred to the lines $O'B'$ and $O'C'$ as oblique axes, And let the lengths of OB, $O'B'$, $O'C'$ be a, a', b', respectively.

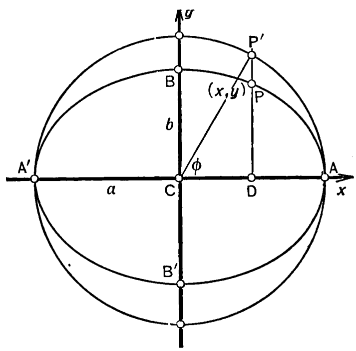
Then $OE = x$, $EP = y$, $O'E' = x'$, $E'P' = y'$.

But, by § 116, $$\frac{OE}{OB} = \frac{O'E'}{O'B'}, \text{ or } \frac{x}{a} = \frac{x'}{a'},$$

and $$\frac{EP}{OC} = \frac{E'P'}{O'C'}, \text{ or } \frac{y}{a} = \frac{y'}{b'},$$

therefore, since $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, it follows that $\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1$.

122. The eccentric angle. On the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ as diameter, describe a circle, which may be called the *auxiliary circle.* Let $P(x, y)$ be any point of the ellipse, and produce its ordinate DP to meet the circle at P'. Join P' to the center C. The angle ACP', or ϕ, is called the *eccentric angle of P.*



Since $CP' = a$,

$x = CD = CP' \cos\phi = a \cos\phi$.

Since $P(x, y)$ is on the ellipse,

$$y = DP = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \frac{b}{a}\sqrt{a^2 - a^2\cos^2\phi} = b \sin\phi.$$

Therefore the coordinates of any point P on the ellipse may be expressed in terms of the eccentric angle of the point by the formulas

$$x = a\cos\phi, \quad y = b\sin\phi.$$

123. *The equation of the tangent to an ellipse at a point $P'(x', y')$, whose eccentric angle is ϕ', is*

$$\frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1.$$

This equation is obtained by substituting $x' = a\cos\phi'$, $y' = b\sin\phi'$ in the equation $xx'/a^2 + yy'/b^2 = 1$, and simplifying the result.

124. Exercises. The ellipse.

1. Prove that, according as $x'^2/a^2 + y'^2/b^2 - 1$ is negative, 0, or positive, (x', y') is within, on, or without the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$.

2. Determine for each of the points $(1, 3/2)$, $(1, 3)$, $(-\sqrt{5}, -1)$, whether it lies within, on, or without the ellipse $4x^2 + 5y^2 = 25$.

3. Find the vertices, foci, and directrices of each of the following ellipses, and in each case draw the graph:

(1) $3x^2 + 4y^2 = 12$, (3) $9x^2 + 27y^2 = 2$,
(2) $5x^2 + 9y^2 = 45$, (4) $63x^2 + 144y^2 = 28$.

4. The distances from the center of an ellipse to a focus and a directrix are 3 and 12, respectively; find the eccentricity and the semiaxes of the ellipse.

5. Given two fixed points 6 units apart, find the locus of the point the sum of whose distances from these points is 12.

6. Find the equations of the lines joining a focus of the ellipse $5x^2 + 9y^2 = 45$ to the extremities of the latus rectum through the other focus. Find the angle between these lines.

7. Find the axes, center, vertices, foci, and directrices of each of the following ellipses, drawing the graph in each case:

(1) $3x^2 + 4y^2 + 12x - 16y + 16 = 0$,
(2) $4x^2 + 9y^2 - 8x + 18y + 12 = 0$,
(3) $4x^2 + y^2 + 4x - 6y + 9 = 0$.

8. Find the equation of the circle which passes through the three points $(1, 1)$, $(3, 1)$, $(-1, 2)$. Find the center and radius of this circle.

9. Find the equation of the circle which passes through the points $(1, 1)$ and $(1, 3)$ and whose center lies on the line $2y - x = 0$.

10. An ellipse whose axes are parallel to the axes of coordinates passes through the points $(0, 0)$, $(1, 2)$, $(1, -1)$, $(2, 1)$. Find its equation.

11. An ellipse whose axes are parallel to the axes of coordinates passes through the points $(-1, 0)$, $(3, 0)$, $(4, 1)$, $(1, 3)$. Find its equation.

12. Find the equation of the ellipse whose center is $(-2, 4)$ and whose major and minor axes are parallel to the x- and y-axis, respectively, the lengths of the semiaxes being 4 and 3.

13. Find the equation of the ellipse whose major and minor axes coincide with the lines $x + y = 0$ and $x - y = 0$, respectively, the lengths of the semiaxes being $\sqrt{5}$ and 2.

14. Find the equation of the ellipse one of whose foci is the origin and the corresponding directrix the line $x + y = 2$, the eccentricity being $1/2$.

15. Find the points where $3x^2 + 4y^2 = 19$ is cut by each of the lines (1) $3x + 2y + 1 = 0$; (2) $5x + y = 7$.

16. For what value of λ will $3y + 2x = \lambda$ touch $2x^2 + y^2 = 5$?

17. For what value of λ will $2y = 3x + \lambda$ touch $x^2 + 4y^2 = 1$?

18. Find the equations of the tangents to $5x^2 + 9y^2 = 45$ which are parallel to $3x + 4y - 5 = 0$. Find also the equations of the tangents which are perpendicular to this line.

19. Find the equations of the tangent and normal to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$; at the point $(-1, 2)$.

20. Find the equations of the tangent and normal to $5x^2 + 7y^2 = 73$ at the point $(3, -2)$.

21. Find the equations of the tangents and normals to $3x^2 + 4y^2 = 12$ at the extremities of one of the latera recta.

22. Find the angle between the lines which touch $3x^2 + 4y^2 = 16$ at the points $(2, 1)$ and $(0, 2)$.

23. At what angle does $2y^2 = x$ cut $3x^2 + 4y^2 = 16$?

24. Find the equations of the tangents common to the parabola $y^2 = 5x$ and the circle $9x^2 + 9y^2 = 16$.

25. Find the equations of the tangents to $x^2 + y^2 = 10$ which pass through the point $(-4, 2)$.

26. Find the equations of the tangents to

(1) $4x^2 + 9y^2 = 36$ from the point $(3, -3)$
(2) $3x^2 + 22y^2 = 66$ from the point $(3, 2)$.

27. Find the equations of the tangents to $4x^2 + 4y^2 - 4x - 3 = 0$ which make an angle of $45°$ with the x-axis.

28. Two circles pass through the points $(4, 1)$ and $(1, 5)$ and touch the y-axis; find their equations and also the angle at which they cut each other.

29. Prove that any tangent to an ellipse meets the tangents at the vertices in points the product of whose ordinates is equal to the square of the semiminor axis.

30. What is the equation of the line which bisects all chords of the ellipse $4x^2 + 9y^2 = 36$ which are parallel to the line $x + y = 7$?

31. What is the equation of the diameter of the ellipse $5x^2 + 3y^2 = 30$ which is conjugate to $3x - 2y = 0$?

32. Find the equation of the chord of the ellipse $4x^2 + 8y^2 = 1$ which passes through the point $(1/8, -1/4)$ and is parallel to the diameter conjugate to $2x + y = 0$.

33. Show that if FP and $F'P$ be the focal distances of a point P of an ellipse whose center is C, and CQ be the semidiameter conjugate to CP, then $FP \cdot F'P$ will equal CQ^2.

34. The tangent at a point P of an ellipse meets the tangent at A, one of the vertices, in the point Q; show that the line joining Q to the center is parallel to the line joining P to the other vertex.

35. Find the extremities of the diameter of $x^2 + 2y^2 = 4$ which is conjugate to that through the point (2, 1).

36. Prove that every ellipse has one pair of equal conjugate diameters and that they coincide with the diagonals of the rectangle whose sides touch the ellipse at the extremities of its axes.

37. Prove that the lines joining any point P of an ellipse to the extremities E and G of any diameter are parallel to a pair of conjugate diameters. (PE and PG are called *supplemental chords.*)

38. Find the equations of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the extremities of its latus rectum.

39. If the ordinate DP of any point P on an ellipse be produced to meet at Q the tangent at the extremity of the latus rectum through the focus F, prove that $DQ = FP$.

40. Prove that the points of tangency of parallel tangents to an ellipse are the extremities of a diameter.

41. Prove that the sides of any parallelogram inscribed in an ellipse are parallel to a pair of conjugate diameters; and that the diagonals of any parallelogram circumscribed to an ellipse are a pair of conjugate diameters.

42. Prove that the subtangent and subnormal to an ellipse are $(a^2 - x'^2)/x'$ and $x'(1 - e^2)$, where x' is the abscissa of the point of tangency. [Compare § 84.]

43. Prove that any tangent to an ellipse and the corresponding normal meet either axis in points T and N such that $CT \cdot CN = CF^2$.

44. Prove that the straight lines drawn from a focus of an ellipse to the end points of any diameter make equal angles with the tangents at these points.

45. Prove that the sum of the squares of the distances of the points $(0, ae)$ and $(0, -ae)$ from any tangent is equal to $2a^2$.

46. Prove that the perpendicular from the focus F upon the tangent at P will meet the line joining P to the center on the directrix corresponding to F.

47. Perpendiculars are taken through any point P on an ellipse to the lines joining P to the vertices; prove that they intercept a segment of the axis which is equal to the latus rectum.

48. Prove that the sum of the squares of the reciprocals of two diameters of an ellipse which are at right angles is constant.

49. Prove that the eccentric angles of two points which are extremities of a pair of conjugate diameters differ by $\pi/2$.

50. Prove that the equation of the normal to $x^2/a^2 + y^2/b^2 = 1$ at a point whose eccentric angle is ϕ is $ax/\cos\phi - by/\sin\phi = a^2 - b^2$.

51. Prove that the area of the triangle bounded by the x- and y-axes and the tangent to $x^2/a^2 + y^2/b^2 = 1$ at the point whose eccentric angle is ϕ is $ab/\sin 2\phi$.

52. Prove that the tangents at the points whose eccentric angles are ϕ and $\phi + \pi/2$ (which are extremities of conjugate diameters) meet at the point whose coordinates are $x = a(\cos\phi - \sin\phi)$, $y = b(\cos\phi + \sin\phi)$.

53. Prove that the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the extremities of a pair of conjugate diameters meet on the ellipse $x^2/a^2 + y^2/b^2 = 2$.

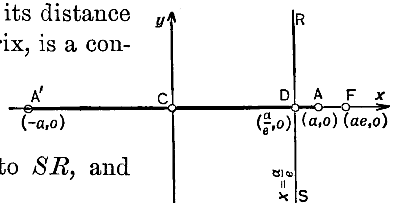
54. Assuming that the greatest triangle which can be inscribed in a given circle is equilateral, prove that the area of the greatest triangle which can be inscribed in an ellipse is $3\,ab\sqrt{3}/4$, where a, b are the semi-axes. Show also that the median lines of this triangle intersect at the center of the ellipse.

55. Through a given point A outside an ellipse whose center is C a line is drawn which meets the ellipse in the two points P, Q. Show that the area of the triangle CPQ is greatest when P and Q are the extremities of a pair of conjugate diameters.

CHAPTER VI

THE HYPERBOLA

125. The equation of the hyperbola. By definition [§ 69], the hyperbola is the locus of a point whose distance from a fixed point, the focus, divided by its distance from a fixed line, the directrix, is a constant, e, greater than 1.



Let F be the focus and SR the directrix.

Take FD perpendicular to SR, and meeting it at D.

There is a point A between F and D such that $FA/AD = e$.

Again, since $e > 1$, there is a point A' on FD, produced through D, such that $FA'/DA' = e$. The points A and A' are on the hyperbola; they are called its *vertices*.

Let C be the mid-point of $A'A$, and let a represent the length of $A'C\,(= CA)$. Take the line CF as x-axis, and the line through C parallel to SR as y-axis. The equation of the hyperbola is to be obtained, referred to these axes. The process is identical with that followed in the case of the ellipse [§ 88].

To obtain the coordinates of F and the equation of SR, it is only necessary to express the lengths of CF and CD in terms of a and e. This may be done as follows:

Since $FA/AD = e$, $FA'/DA' = e$, and $A'C = CA = a$, it follows that

$$e = \frac{AF}{DA} = \frac{CF - a}{a - CD}. \quad \therefore ae - CD \cdot e = CF - a. \qquad (1)$$

$$e = \frac{A'F}{A'D} = \frac{a + CF}{a + CD}. \quad \therefore ae + CD \cdot e = CF + a. \qquad (2)$$

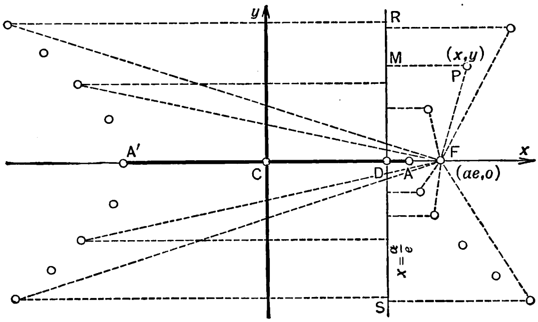
Adding (1) and (2), gives $2\,ae = 2\,CF$. $\therefore CF = ae$.

Subtracting (1) from (2), gives $2\,CD \cdot e = 2\,a$. $\therefore CD = a/e$.

Therefore the coordinates of F are $(ae, 0)$, and the equation of SR is $x - a/e = 0$. The equation of the hyperbola may now be derived as follows:

Let $P(x, y)$ denote any representative point of the hyperbola. Join FP and take MP perpendicular to SR.

Since P is on the hyperbola, $FP/MP = e$, and therefore



$$FP^2 = e^2 MP^2. \tag{3}$$

But [§ 41] $FP^2 = (x - ae)^2 + y^2$, and [§ 51] $MP^2 = (x - a/e)^2$. The substitution of these expressions for FP^2 and MP^2 in (3) gives

$$(x - ae)^2 + y^2 = e^2 (x - a/e)^2,$$

or

$$(1 - e^2) x^2 + y^2 = a^2 (1 - e^2),$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{4}$$

Hence (4) is the equation of the hyperbola, when referred to the axes above indicated; for it is true when (3) is true, that is, when P is on the hyperbola, and false when (3) is false, that is, when P is off the hyperbola.

Since $e > 1$, the quantity $a^2(1 - e^2)$ is negative; represent it by $-b^2$; then (4) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (5) \qquad \text{where } b^2 = a^2(e^2 - 1). \quad (6)$$

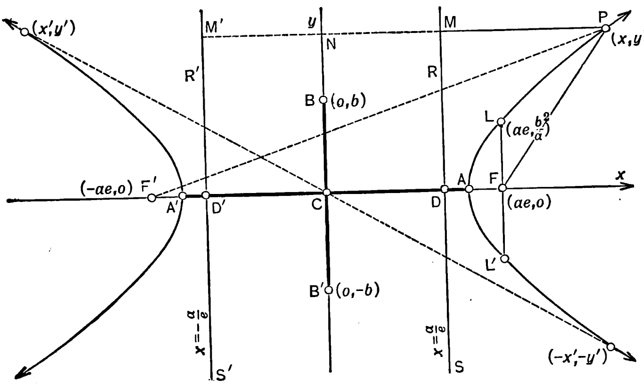
This is the form in which the equation of the hyperbola is usually written.

126. The shape of the hyperbola. The shape of the hyperbola and its position relative to the axes may readily be inferred from its equation [(5), § 125]

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solving this equation for y, $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$.

Hence, y has imaginary values when $x^2 < a^2$, the value 0, counted twice, when $x^2 = a^2$, and two real values, equal numerically but



of opposite signs, when $x^2 > a^2$, these values increasing indefinitely (numerically) with x. Therefore (as indicated in the figure) the curve consists of two infinite branches, one extending indefinitely to the right from the line $x = a$, which it touches at the point $(a, 0)$, the other extending indefinitely to the left from the line $x = -a$, which it touches at the point $(-a, 0)$. And it is symmetric with respect to the x-axis.

Solving the equation of the hyperbola for x,

$$x = \pm \frac{a}{b}\sqrt{y^2 + b^2}.$$

Hence, for every value of y, x has two real values equal numerically but of opposite sign, these values increasing numerically with y. Therefore the curve is symmetric with respect to the y-axis.

127. If (x', y') be a point on the hyperbola, the same is true of $(-x', -y')$; for if $x'^2/a^2 - y'^2/b^2 \equiv 1$, so also is $(-x')^2/a^2 - (-y')^2/b^2 \equiv 1$. But the points (x', y') and $(-x', -y')$ are on the same straight line through the origin C and are equidistant from C; hence the origin C is the mid-point of every chord of the hyperbola which passes through it; it is therefore called the *center* of the hyperbola.

128. The chord $A'A$ through the center and focus is called the *transverse axis* of the hyperbola; its length is $2a$. One half of $A'A = A'C = CA = a$ is called the *semitransverse axis.*

129. The line through C perpendicular to $A'A$ does not meet the hyperbola in real points; but that portion of it which lies between the points $B(0, b)$ and $B'(0, -b)$ is called the *conjugate axis;* its length is $2b$. $B'C = CB = b$ is called the *semi-conjugate axis.*

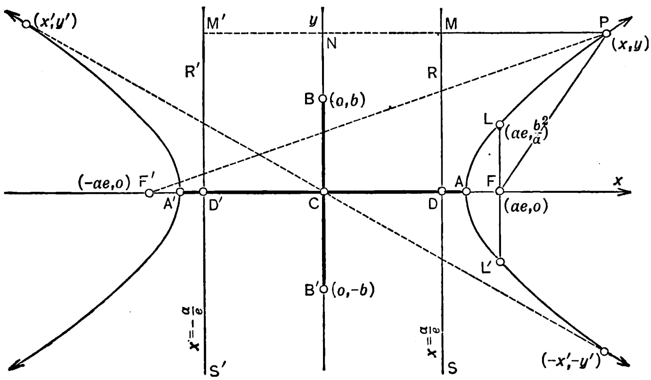
130. The chord $L'L$ through F perpendicular to $A'A$ is called the *latus rectum.* Its length is $2b^2/a$; for when $x = ae$, the equation of the hyperbola gives $y = \pm \frac{b}{a}\sqrt{a^2e^2 - a^2} = \pm \frac{b^2}{a}$.

131. The second focus and directrix. Referring to the preceding figure, on the x-axis and to the left of C lay off CF' equal to CF, and CD' equal to CD, and through D' take $S'R'$ parallel to SR. It will then follow from the symmetry of the curve with respect to the y-axis that F' is a second focus of the hyperbola, and $S'R'$ the corresponding directrix. (Compare § 94.)

The coordinates of F' are $(-ae, 0)$ and the equation of $S'R'$ is $x + a/e = 0$.

132. *The distances of any point $P(x, y)$ of the hyperbola from the foci $F(ae, 0)$ and $F'(-ae, 0)$ are $ex - a$ and $ex + a$, respectively, and the numerical difference of these distances is the constant $2a$.*

For, referring to the figure, join FP, and let MP produced



meet $S'R'$ in M', and the y-axis in N; then from the definition of the hyperbola

$$FP = eMP = e(NP - NM) = e\left(x - \frac{a}{e}\right) = ex - a,$$

$$F'P = eM'P = e(NP + M'N) = e\left(x + \frac{a}{e}\right) = ex + a,$$

and subtracting gives $$|F'P - FP| = 2a.$$

133. *Hence, an hyperbola may also be defined as the locus of a point the difference of whose distances from two fixed points is constant.*

134. Conjugate hyperbolas. The reasoning of § 98 applies in this connection also. Therefore, since the equation of the hyperbola having the segment $A'A(=2a)$ of the x-axis for transverse axis, and the segment $B'B(=2b)$ of the y-axis for conjugate axis, is $x^2/a^2 - y^2/b^2 = 1$, the equation of the hyperbola having $B'B$ for transverse axis and $A'A$ for conjugate axis, is $y^2/b^2 - x^2/a^2 = 1$. The two hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ (1) \text{ and } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \ (2)$$

are called *conjugate hyperbolas*. The transverse axis of each is the conjugate axis of the other.

The eccentricity of (1) is given by the relation $e_1^2 = 1 + b^2/a^2$, the foci are the points $(-ae_1, 0)$, $(ae_1, 0)$ and the directrices are the lines $x + a/e_1 = 0$, $x - a/e_1 = 0$.

The eccentricity of (2) is given by the relation $e_2^2 = 1 + a^2/b^2$, the foci are the points $(0, -be_2)$, $(0, be_2)$, and the directrices are the lines $y + b/e_2 = 0$, $y - b/e_2 = 0$.

In every case the eccentricity of an hyperbola is given by the relation

$$e^2 = 1 + \frac{(\text{semiconjugate axis})^2}{(\text{semitransverse axis})^2}. \qquad (3)$$

And the foci and directrices are found from

$$CF = e \text{ (semitransverse axis)}, \qquad (4)$$

$$CD = (\text{semitransverse axis})/e. \qquad (5)$$

Since $\quad e_1^2 = 1 + b^2/a^2 = (a^2 + b^2)/a^2$; or $a^2e_1^2 = a^2 + b^2$,

and $\quad e_2^2 = 1 + a^2/b^2 = (a^2 + b^2)/b^2$; or $b^2e_2^2 = a^2 + b^2$,

it follows that $ae_1 = be_2$, and therefore, since the foci of (1) are $(ae_1, 0)$, $(-ae_1, 0)$, and the foci of (2) are $(0, be_2)$, $(0, -be_2)$, that *the four foci lie on a circle* ($x^2 + y^2 = a^2 + b^2$), *whose center is the center of the two hyperbolas.*

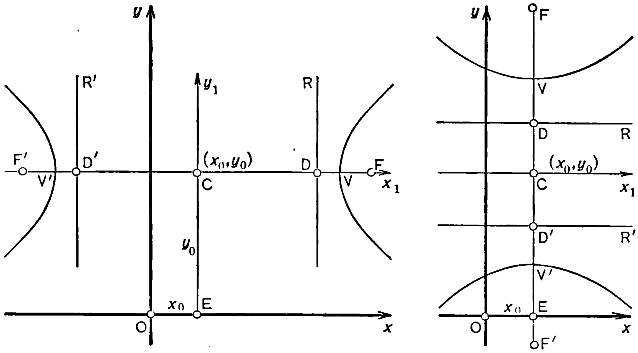
The circle $x^2 + y^2 = a^2 + b^2$ through the four foci cuts each hyperbola in points which lie on a directrix of the other hyperbola.

For the elimination of x^2 between the two equations $x^2 + y^2 = a^2 + b^2$ and $x^2/a^2 - y^2/b^2 = 1$ gives $y^2 = b^4/(a^2 + b^2)$, and therefore (since $b^2/(a^2 + b^2) = 1/e_2^2$), $y = \pm b/e_2$; and $y = b/e_2$, $y = -b/e_2$ are the directrices of the hyperbola $y^2/b^2 - x^2/a^2 = 1$.

The line joining the focus of an hyperbola to a focus of its conjugate passes through the point of intersection of the directrices of the two hyperbolas.

For it can be proved that the points $(ae_1, 0)$, $(a/e_1, b/e_2)$, $(0, be_2)$ are on a straight line.

135. A more general form of the equation of the hyperbola. It also follows, as in § 99, that $(x-x_0)^2/a^2-(y-y_0)^2/b^2=1$ represents an hyperbola whose transverse and conjugate axes coincide with the lines $y-y_0=0$ and $x-x_0=0$, their lengths being $2a$ and $2b$; and that $(y-y_0)^2/b^2-(x-x_0)^2/a^2=1$ represents the conjugate hyperbola. See the following figure.



Every equation of the form $ax^2+by^2+2gx+2fy+c=0$, in which a and b have opposite signs and $(g^2/a+f^2/b-c)\neq 0$, represents an hyperbola whose axes are parallel to the axes of coordinates.

For this equation can be reduced to one of the forms just considered.

Example 1. Find the graph of the equation,

$$x^2-3y^2-2x+18y-35=0.$$

The equation may be written $\quad (x^2-2x)-3(y^2-6y)=35 \qquad (1)$

or, completing squares, $(x^2-2x+1)-3(y^2-6y+9)=35+1-27,$

that is, $$(x-1)^2-3(y-3)^2=9,$$

or finally $$\frac{(x-1)^2}{9}-\frac{(y-3)^2}{3}=1, \qquad (2)$$

which represents an hyperbola whose transverse and conjugate axes coincide with the lines $y-3=0$ and $x-1=0$, respectively (the hyperbola at the left in the figure), its center being (1, 3).

Since $a=3$, $b=\sqrt{3}$, and therefore $e=\sqrt{1+1/3}=2/\sqrt{3}$, $ae=2\sqrt{3}$, $a/e=3\sqrt{3}/2$. Hence the foci F' and F are the points $(1-2\sqrt{3},\ 3)$,

$(1+2\sqrt{3},\ 3)$, and the two directrices $D'R'$ and DR are the lines $x-1+3\sqrt{3}/2=0$, $x-1-3\sqrt{3}/2=0$.

Applying this method to $ax^2+by^2+2\,gx+2\,fy+c=0$,

$$a(x+g/a)^2+b(y+f/b)^2=g^2/a+f^2/b-c.$$

Hence, if a and b have opposite signs, and $(g^2/a+f^2/b-c)\neq 0$, the graph is an hyperbola whose axes coincide with the lines $y+f/b=0$, $x+g/a=0$. But if $(g^2/a+f^2/b-c)=0$, the graph is a pair of straight lines. (Compare § 99.)

Example 2. Find the graph of $3\,x^2-18\,y^2+12\,x+24\,y+10=0$.

Example 3. Find the equation of the hyperbola whose center is $(2,\ -3)$, and which passes through the points $(3,\ -1)$ and $(-1,\ 0)$.

Example 4. Find the equation of the hyperbola whose axes are parallel to the axes of coordinates, and which passes through the points $(0,\ 1)$, $(0,\ -1)$, $(1,\ 0)$, $(3,\ 1)$.

Example 5. Find the equation of the hyperbola having the lines $y-3=0$, and $x+2=0$ for transverse and conjugate axes, respectively, the semiaxes being $\sqrt{3}$ and $\sqrt{5}$.

Example 6. Find the equations of the two hyperbolas which have the lines $3\,x+2\,y=0$ and $2\,x-3\,y=0$ for axes, 4 and 3 being the semiaxes.

Example 7. Find the equation of the hyperbola whose eccentricity is 2, and of which $(1,\ -1)$ and $3\,x+2\,y-2=0$ are a focus and the corresponding directrix.

136. The tangent and normal. The equations [§ 101, § 102] $y=mx\pm\sqrt{a^2m^2+b^2}$ and $xx'/a^2+yy'/b^2=1$ of the tangent to the ellipse were derived from the equation $x^2/a^2+y^2/b^2=1$ by purely algebraic considerations. The same considerations applied to the equation $x^2/a^2-y^2/b^2=1$ will lead to results differing only in the sign of b^2 from those obtained in the case of the ellipse. Hence, for the hyperbola $x^2/a^2-y^2/b^2=1$:

1. The equation of the tangent whose slope is m is

$$y=mx\pm\sqrt{a^2m^2-b^2}. \tag{1}$$

2. The equation of the tangent at the point $(x',\ y')$ is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1. \tag{2}$$

It should be observed that (1) represents a real line only when $|m| \geqq b/a$.

From (2) it follows that the equation of the *normal* at the point (x', y') is $a^2y'(x - x') + b^2x'(y - y') = 0.$ (3)

Example 1. Find the equations of the tangents to the hyperbola $5x^2 - 4y^2 - 10 = 0$ which are perpendicular to the line $x + 3y = 0$.

Example 2. Find the equations of the tangent and normal to the hyperbola $3x^2 - 4y^2 - 8 = 0$ at the point $(2, -1)$; to the hyperbola $y^2 - 4x^2 + 3y + 26 = 0$ at the point $(3, 2)$.

137. *The tangent at any point of an hyperbola bisects the angle included by the lines joining the point to the foci.*

This theorem may be proved by the method used in proving the corresponding theorem for the ellipse, § 106.

138. The asymptotes. The hyperbola has two tangents whose points of contact with the curve are at an infinite distance from its center. They are called its *asymptotes*. See figure in § 140. Their equations may be obtained as follows:

The line $y = mx + c$ (1) cuts the hyperbola $x^2/a^2 - y^2/b^2 = 1$ (2) in two points whose abscissas are the roots of the equation

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

or

$$(a^2m^2 - b^2)x^2 + 2ma^2cx + a^2(c^2 + b^2) = 0. \qquad (3)$$

The line (1) will therefore meet the hyperbola (2) in two infinitely distant coincident points, in other words, will be an asymptote of the hyperbola, if the two roots of (3) be infinite. But [Alg. § 638] the two roots of (3) are infinite if the coefficients of x^2 and of x are 0, that is, if

$$a^2m^2 - b^2 = 0 \quad \text{and} \quad 2ma^2c = 0,$$

or, if

$$m = \pm b/a \quad \text{and} \quad c = 0.$$

Hence the lines

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x \qquad (4)$$

are the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

The lines (4) are also the asymptotes of the conjugate hyperbola $y^2/b^2 - x^2/a^2 = 1$.

When $a = b$, the asymptotes (4) are the perpendicular lines $y = x$ and $y = -x$. Hence the hyperbola $x^2 - y^2 = a^2$ is often called the *rectangular hyperbola*. It is also called the *equilateral hyperbola*.

By the method employed in this section the asymptotes of an hyperbola may be found from its equation referred to any axes whatsoever.

Example 1. Find the asymptotes of the graph of $4x^2 - y^2 - 3x - y = 0$. The abscissas of the points where the line $y = mx + c$ cuts the graph are the roots of the equation

$$(m^2 - 4)x^2 + (2mc + m + 3)x + (c^2 + c) = 0.$$

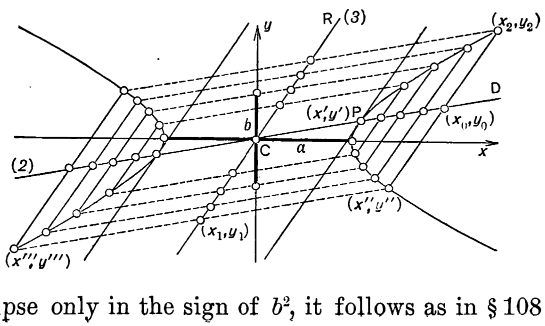
Both roots are infinite, if $m^2 - 4 = 0$ and $2mc + m + 3 = 0$, that is, if $m = 2$, $c = -5/4$, or if $m = -2$, $c = 1/4$.

Hence the asymptotes are $y = 2x - 5/4$ and $y = -2x + 1/4$.

Example 2. Find the asymptotes of the hyperbola $3x^2 - 4y^2 + 6 = 0$.

Example 3. The graph of $2x^2 - xy - 6y^2 + 2x - 11y = 0$ is an hyperbola; find the asymptotes.

139. Conjugate diameters. Since the equation $x^2/a^2 - y^2/b^2 = 1$ of the hyperbola differs from the equation $x^2/a^2 + y^2/b^2 = 1$ of



the ellipse only in the sign of b^2, it follows as in § 108 that if the slopes of two lines $y = mx$ and $y = m'x$ through the center C are connected by the relation

$$mm' = \frac{b^2}{a^2}, \tag{1}$$

each of these lines will bisect all chords of the hyperbola which are parallel to the other. Two such lines are called *conjugate diameters* of the hyperbola.

The portions of two conjugate diameters which are on the same side of the transverse axis are also on the same side of the conjugate axis. For since b^2/a^2 is positive, it follows from (1) that m and m' are of the same sign.

Of two conjugate diameters $y = mx$, $y = m'x$, only one meets the hyperbola in real points. For it follows from (1) that if $|m| < b/a$, then $|m'| > b/a$; but if $|m'| > b/a$, the line $y = m'x$ will not meet the hyperbola in real points.

140. Properties of conjugate diameters. 1) Let $P(x', y')$ denote any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Then the equation of CP, the diameter through P, and that of CQ, the diameter conjugate to CP, are

$$y = \frac{y'}{x'}x \quad (2) \qquad \text{and} \qquad \frac{xx'}{a^2} - \frac{yy'}{b^2} = 0, \quad (3)$$

respectively. For both (2) and (3) represent lines through the center C; (2) is satisfied by $x = x'$, $y = y'$; and the slopes of (2) and (3) are connected by the relation $mm' = b^2/a^2$.

Observe that (3) represents a line parallel to the tangent to (1) at (x', y'), namely $xx'/a^2 - yy'/b^2 = 1$. Hence the tangent at any point P of an hyperbola is parallel to the system of chords bisected by the diameter through P.

2) Since the diameter (2) meets the hyperbola (1) in real

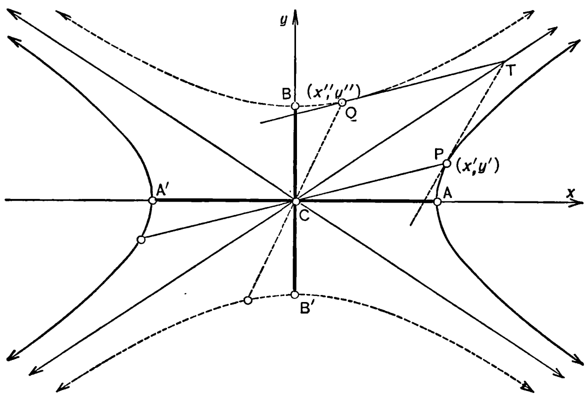
points, its conjugate (3) does not. But (3) does meet the conjugate hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{4}$$

in real points, namely the points whose abscissas x'' and ordinates y'' are given by the equations

$$\frac{x''}{a} = \pm \frac{y'}{b} \quad \text{and} \quad \frac{y''}{b} = \pm \frac{x'}{a}, \tag{5}$$

as may be shown by solving (3) and (4) for x, y, and taking



account of the fact that (x', y') lies on (1) so that $x'^2/a^2 - y'^2/b^2 \equiv 1$. [Compare § 111.]

3) Let $Q(x'', y'')$ denote one of the points where (3) meets (4), and let PT be the tangent to (1) at $P(x', y')$, and QT the tangent to (4) at $Q(x'', y'')$. The area of the parallelogram $CPTQ$ is constant and equal to ab.

For since the parallelogram $CPTQ$ is twice the triangle CPQ, it follows from § 53 and the equations (5) that

$$CPTQ = x'y'' - x''y' = x'^2\frac{b}{a} - y'^2\frac{a}{b} = \left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)ab = ab.$$

4) It can also be proved that $CP^2 - CQ^2$ is constant and equal to $a^2 - b^2$. For

$$CP^2 - CQ^2 = x'^2 + y'^2 - x''^2 - y''^2$$
$$= x'^2 + y'^2 - \frac{a^2}{b^2}y'^2 - \frac{b^2}{a^2}x'^2$$
$$= \left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)(a^2 - b^2) = a^2 - b^2.$$

5) If $CP = a'$ and $CQ = b'$, the equation of the hyperbola referred to the conjugate diameters CP and CQ, as oblique axes, is

$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1.$$

For, consider the cylindrical surface perpendicular to the plane of the hyperbola on the figure of page 111; and take that plane containing CT which cuts the cylindrical surface so that the lines, whose projections are CP and CQ, are at right angles. Then, exactly as in § 121, since the equation of the curve in this cutting plane is $x^2/a_1^2 - y^2/b_1^2 = 1$, the equation referred to the oblique axes CP and CQ is $x'^2/a'^2 - y'^2/b'^2 = 1$, as was to be proved.

Example 1. Find the diameter of the hyperbola $3x^2 - 2y^2 = 4$ which bisects all chords parallel to the line $2x - y + 3 = 0$.

Example 2. Referring to the figure of page 111, prove that a line drawn to join P and Q will be bisected by the asymptote $y = (b/a)x$, and that this asymptote passes through the point T where the tangents at P and Q meet, as is indicated in the figure.

Example 3. Find the equation of that diameter of the hyperbola $3x^2 - 4y^2 = 8$ which passes through the point $(2, -1)$; also the equation of the conjugate diameter; also the points where this conjugate diameter cuts the conjugate hyperbola $4y^2 - 3x^2 = 8$.

Example 4. Find the sine of the angle included by the pair of conjugate diameters obtained in the preceding example.

Example 5. Find the equation of that diameter of the hyperbola $x^2 - 2y^2 + 6x - 3y = 0$ which bisects all chords whose slope is -2.

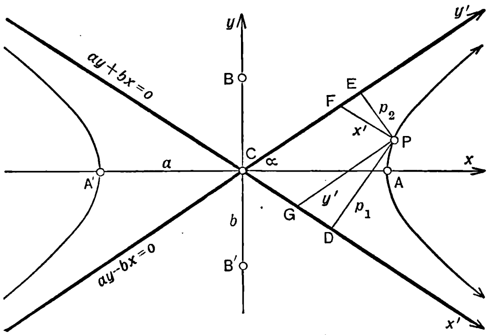
141. The equation of the hyperbola referred to its asymptotes. The equation of an hyperbola referred to its axes may be written

$$(ay + bx)(ay - bx) + a^2b^2 = 0. \qquad (1)$$

The asymptotes are the lines

$$ay + bx = 0 \quad (2) \quad \text{and} \quad ay - bx = 0. \quad (3)$$

Call the lines (2) and (3) Cx' and Cy', respectively; it is required to obtain the equation of the hyperbola when referred to Cx', Cy' as the axes of coordinates.



Let P denote any point on the hyperbola. Take PD and PE perpendicular to Cx' and Cy', respectively, and represent their lengths by p_1 and p_2.

Again, take PF and PG parallel to Cx' and Cy', respectively; if x', y' denote the coordinates of P referred to Cx', Cy', then $FP = x'$ and $GP = y'$.

Finally let 2α denote the angle $x'Cy'$. Then by § 50, since P lies above Cx' and below Cy',

$$\frac{ay + bx}{\sqrt{a^2 + b^2}} = p_1 = y' \sin 2\alpha, \quad \frac{ay - bx}{\sqrt{a^2 + b^2}} = -p_2 = -x' \sin 2\alpha. \qquad (4)$$

Hence $\quad (ay + bx)(ay - bx) \equiv -x'y'(a^2 + b^2)\sin^2 2\alpha. \qquad (5)$

But $\tan\alpha = b/a$; hence, $\cos^2\alpha = a^2/(a^2+b^2)$, $\sin^2\alpha = b^2/(a^2+b^2)$, and therefore

$$\sin^2 2\alpha = 4\sin^2\alpha\cos^2\alpha = 4a^2b^2/(a^2 + b^2)^2.$$

Therefore, substituting in (5),

$$(ay + bx)(ay - bx) \equiv - 4\, a^2b^2x'y'/(a^2 + b^2). \tag{6}$$

Substituting this expression for $(ay + bx)(ay - bx)$ in (1) and simplifying

$$4\, x'y' = a^2 + b^2, \tag{7}$$

the equation required.

Observe that it also follows from the equations (1) and (4) that $p_1p_2 = a^2b^2/(a^2 + b^2)$, that is, that the product of the perpendicular distances of any point P of an hyperbola from its asymptotes is constant. This property of the hyperbola is independent of the position of the curve in the plane. Hence, *if* $l_1 = 0$, $l_2 = 0$ *denote any two intersecting lines, the locus of a point* $P(x, y)$, *the product of whose distances* p_1, p_2 *from* $l_1 = 0$, $l_2 = 0$ *is a constant, is an hyperbola having* $l_1 = 0$, $l_2 = 0$ *for asymptotes.*

Example 1. Find the equation of the hyperbola whose asymptotes are $x + y - 3 = 0$ and $y - 2x = 0$, and which passes through the point (1, 3).

Since the distances of any point $P(x, y)$ from the lines $x + y - 3 = 0$ and $y - 2x = 0$ are proportional to $x + y - 3$ and $y - 2x$, respectively [§ 50], the required equation has the form $(x + y - 3)(y - 2x) = c$. But since the hyperbola passes through the point (1, 3), this equation must be satisfied by $x = 1$, $y = 3$. Hence $c = 1$ and the required equation is $(x + y - 3)(y - 2x) = 1$.

Example 2. Find the equation of the hyperbola whose asymptotes are the lines $x + 2y - 3 = 0$ and $3x - y + 5 = 0$, and which passes through the point (3, − 2).

Example 3. Find the equation of the hyperbola whose asymptotes are the x-axis and a line parallel to the y-axis, and which passes through the points (0, 4) and (− 1, 2).

142. Exercises. The Hyperbola.

1. Find the vertices, foci, directrices, and asymptotes of each of the following hyperbolas, drawing the graph in each case:

(1) $5x^2 - 4y^2 = 20$, (3) $4y^2 - 5x^2 = 20$,

(2) $9x^2 - 16y^2 = 12$, (4) $9y^2 - 7x^2 = 7$.

2. Find the axes, center, foci, and directrices of each of the following, drawing the graph in each case :

(1) $3x^2 - y^2 + 4y - 7 = 0$,
(2) $5x^2 - 4y^2 + 10x + 4y - 16 = 0$,
(3) $9x^2 - 16y^2 - 18x - 64y + 19 = 0$,
(4) $2x^2 - 3y^2 - 5x - 7y + 20 = 0$.

3. Prove that $4x^2 - 4y^2 + 4x + 24y - 35 = 0$ represents a pair of straight lines, and find the equations of these lines.

4. Find the equation of the hyperbola whose transverse and conjugate axes coincide with the lines $x - 4 = 0$ and $y + 5 = 0$, respectively, their lengths being 6 and 8. Find the equation of the conjugate hyperbola also.

5. Find the equation of the hyperbola one of whose foci is the point (0, 1), and the corresponding directrix the line $x + y = 6$, the eccentricity being 2.

6. Find the equation of the hyperbola whose transverse and conjugate axes coincide with the lines $x - 2y = 0$ and $2x + y = 0$, respectively, their lengths being 2 and 6.

7. Find the equations of the tangents to $2x^2 - 3y^2 = 1$ which are

(1) parallel to the line $y = 4x$,
(2) perpendicular to the line $5y + x = 0$.

8. Find the equations of the tangents to $2x^2 - 3y^2 = 1$ which make an angle of 45° with the x-axis. Show that the slope of no real tangent to this hyperbola can be less than $\sqrt{2/3}$.

9. Prove that the equation of the tangent to $y^2/b^2 - x^2/a^2 = 1$ is $y = mx \pm \sqrt{-a^2m^2 + b^2}$.

10. For what value of λ will the line $2y = 3x + \lambda$ touch the hyperbola $x^2 - 3y^2 = 1$?

11. Find the equations of the tangent and normal to $9x^2 - 8y^2 = 1$ at the point $(1, -1)$; also at the point $(5/3, \sqrt{3})$.

12. Find the equations of the tangents to $x^2/16 - y^2/9, = 1$ at the extremities of the latera recta.

13. Find the equations of the lines which touch both the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $x^2 - y^2 = 16$.

14. An hyperbola whose axes are parallel to the axes of coordinates passes through the points (1, 0), (0, 2), (– 1, 2), (3, – 1). Find its equation.

15. An hyperbola whose axes are parallel to the axes of coordinates passes through the points (0, 0), (1, 1), (− 1, − 2), (2, − 2). Find its equation.

16. Find the equations of the asymptotes of each of the following, drawing the graph in each case:

(1) $xy = 4$,
(2) $(x - y)(x + y - 2) = 1$,
(3) $5y^2 - 4x^2 + 20y + 4x + 4 = 0$,
(4) $2x^2 + xy - 3y^2 + 3x + 7y + 1 = 0$,
(5) $2x^2 + 3xy - 2y^2 + 3x + 6y + 8 = 0$.

17. The asymptotes of an hyperbola are the lines $2x - y = 0$ and $2x + y = 0$; if the curve passes through the point (3, − [illegible]), what is its equation?

18. An hyperbola has the lines $3x + 2y - 1 = 0$ and $2x - 3y - 5 = 0$ for asymptotes and passes through the point (2, 1). Find its equation.

19. Find the equation of the hyperbola whose axes coincide with the coordinate axes and which passes through the two points (2, 3) and (− 1, 4).

20. An hyperbola whose asymptotes are parallel to the axes of coordinates passes through the points (2, 5), (3, 2), (− 2, 3). Find its equation.

21. If e and e' denote the eccentricities of two conjugate hyperbolas, prove that $1/e^2 + 1/e'^2 = 1$.

22. Prove that the perpendicular distance from a focus to an asymptote of $x^2/a^2 - y^2/b^2 = 1$ is b.

23. Prove that for all values of ϕ the point $(a \sec \phi, b \tan \phi)$ is on the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

24. Prove that an ellipse and an hyperbola which have the same foci intersect at right angles.

25. What diameter of $4x^2 - 5y^2 - 20$ is conjugate to $5x + 2y = 0$?

26. Find the points where the diameter of $x^2/2 - y^2 = 1$ which is conjugate to that through the point (− 2, 1) meets the conjugate hyperbola $y^2 - x^2/2 = 1$.

27. If a straight line cut an hyperbola at the points P and P' and its asymptotes at the points R and R', prove that the mid-point of PP' will also be the mid-point of RR'.

28. Prove that the distance of any point of a rectangular hyperbola $x^2 - y^2 = a^2$ from the center is a mean proportional to its distances from the foci.

29. Assuming that the equation of the tangent to the hyperbola $4xy = a^2 + b^2$ at the point (x_1, y_1) is $2(xy_1 + x_1y) = a^2 + b^2$, prove that the portion of the tangent intercepted by the asymptotes is bisected at the point of tangency, and that the area of the triangle bounded by the asymptotes and the tangent is constant for all positions of the tangent.

30. Prove that the diameter of $4xy = a^2 + b^2$ conjugate to $y - mx = 0$ is $y + mx = 0$.

31. The tangent to an hyperbola at P meets one of the asymptotes in the point T, and TQ is taken parallel to the other asymptote and meeting the curve in the point Q. Prove that if PQ meets the asymptotes in the points R and S, the line RS will be trisected at P and Q.

32. Through any two points P and Q of an hyperbola, lines are drawn parallel to both asymptotes, forming the parallelogram $PRQS$. Prove that the diagonal RS passes through the center.

33. The tangent to an hyperbola at the point P meets the conjugate hyperbola in the points R and S. Prove that P is the mid-point of RS.

34. Prove that if an hyperbola has a pair of equi-conjugate diameters, it is a rectangular hyperbola.

35. Prove that the eccentricity of an hyperbola whose asymptotes include the angle 2α is $\sec \alpha$.

36. Prove that the portion of an asymptote of an hyperbola which is intercepted between the directrices is equal to the transverse axis.

37. Prove that the tangents at the vertices of an hyperbola meet the asymptotes on the circle of which the line joining the foci is a diameter.

38. If the ordinate of any point P on an hyperbola be produced to meet the nearer asymptote at Q, and QR be then taken perpendicular to the asymptote to meet the transverse axis at R, prove that the line PR will be the normal at P.

39. Prove that the bisectors of the angles between the lines joining any point on a rectangular hyperbola to the vertices are parallel to the asymptotes.

40. The asymptotes of an hyperbola are the lines $y=0$ and $3y-4x=0$ and it passes through the point (2, 2). Find its equation, the equations and lengths of its axes, its vertices, eccentricity, foci, and directrices.

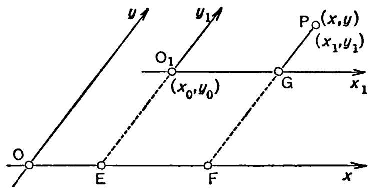
CHAPTER VII

TRANSFORMATION OF COORDINATES

143. Transformation of coordinates. From the equation of a curve referred to a given pair of lines as axes may be derived its equation referred to any second pair of lines as axes. The process is called the *transformation of coordinates.*

144. *To change the origin without changing the direction of the axes.*

Let Ox, Oy be the original axes, and O_1x_1, O_1y_1, parallel to Ox, Oy, respectively, the new axes. And let x_0, y_0 denote the coordinates of the new origin O_1 referred to the axes Ox, Oy.



Take any representative point P, and take PGF parallel to Oy and O_1y_1 and meeting Ox and O_1x_1 at F and G, respectively. Then, if x, y denote the coordinates of P referred to the axes Ox, Oy, and x_1, y_1 its coordinates referred to the axes O_1x_1, O_1y_1, the following relations hold good:

$$x = OF = OE + O_1G = x_0 + x_1,$$
$$y = FP = EO_1 + GP = y_0 + y_1.$$

Hence the equation of any curve referred to the axes Ox, Oy may be transformed into its equation referred to the axes O_1x_1, O_1y_1 by the substitution

$$x = x_1 + x_0, \quad y = y_1 + y_0. \tag{1}$$

The solution of these equations for x_1 and y_1 gives

$$x_1 = x - x_0, \quad y_1 = y - y_0. \tag{2}$$

Observe, as in the two following examples, that this transformation will leave unchanged the coefficients of the terms of highest degree in any equation to which it is applied.

Example 1. What does $y^2 - 4y + x = 0$ become when the origin is transferred to the point (4, 2), the directions of the axes remaining unchanged?

The transformed equation, obtained by setting $x = x_1 + 4$, $y = y_1 + 2$, is $(y_1 + 2)^2 - 4(y_1 + 2) + (x_1 + 4) = 0$, or simplifying, $y_1^2 + x_1 = 0$.

Example 2. By change of origin, transform $x^2 - 2xy + 2x - 6y = 0$ into an equation which lacks the terms of the first degree.

Let the coordinates of the new origin be (h, k), and in the given equation substitute $x = x_1 + h$, $y = y_1 + k$. It becomes

$$(x_1 + h)^2 - 2(x_1 + h)(y_1 + k) + 2(x_1 + h) - 6(y_1 + k) = 0,$$

or, expanding and collecting terms,

$$x_1^2 - 2x_1y_1 + 2(h - k + 1)x_1 - 2(h + 3)y_1 + (h^2 - 2hk + 2h - 6k) = 0.$$

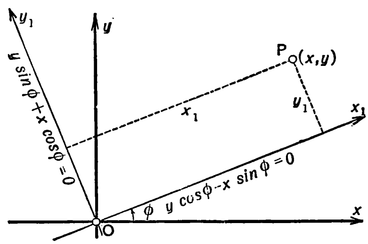
This equation will lack the x_1 and y_1 terms, if $h - k + 1 = 0$ and $h + 3 = 0$; that is, if $h = -3$ and $k = -2$. And when $h = -3$, $k = -2$, the transformed equation becomes

$$x_1^2 - 2x_1y_1 + 3 = 0.$$

145. *To change the directions of the axes without changing the origin, both pairs of axes being rectangular.*

Let Ox, Oy be the original axes, and Ox_1, Oy_1 the new axes, and let ϕ denote the angle $xOx_1 (= yOy_1)$.

Take any representative point P, whose coordinates referred to the axes Ox, Oy are x, y, and let x_1, y_1, denote its coordinates referred to the new axes Ox_1, Oy_1. Then y_1 and x_1 are the perpendicular distances of P from Ox_1 and Oy_1, respectively.



Since $\tan\phi = \sin\phi / \cos\phi$, the slope equations of the lines Ox_1 and Oy_1, referred to the axes Ox, Oy, may be reduced to

$$y\cos\phi - x\sin\phi = 0 \text{ and } y\sin\phi + x\cos\phi = 0.$$

Therefore, since the sum of the squares of the coefficients of x and y in each of these equations is 1, the perpendicular distances of $P(x, y)$ from Oy_1 and Ox_1, namely x_1 and y_1, are [§ 50],

$$x_1 = y \sin\phi + x\cos\phi, \quad y_1 = y\cos\phi - x\sin\phi. \tag{1}$$

Solving these equations for x, y in terms of x_1, y_1,

$$x = x_1\cos\phi - y_1\sin\phi, \quad y = x_1\sin\phi + y_1\cos\phi. \tag{2}$$

Hence the equation of a curve referred to the axes Ox, Oy may be transformed into its equation referred to the axes Ox_1, Oy_1 by the substitution (2).

Observe that this transformation will leave the constant term in the equation unchanged.

Example 1. Transform $x^2 + 4xy + y^2 = 2$ (referred to rectangular axes) to axes bisecting the angles between the given axes.

Here, the angle ϕ made by the axis Ox_1 with Ox is 45° or $\pi/4$; and $\sin 45° = \cos 45° = 1/\sqrt{2}$. Hence $x = (x_1 - y_1)/\sqrt{2}$, $y = (x_1 + y_1)/\sqrt{2}$, and this substitution transforms the given equation into one which when simplified is $3x_1^2 - y_1^2 = 2$.

Example 2. Prove that by the substitution $x = x_1\cos\phi - y_1\sin\phi$, $y = x_1\sin\phi + y_1\cos\phi$, any equation of the second degree referred to rectangular axes, namely:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be transformed into one which lacks the xy term.

The terms of the second degree, $ax^2 + 2hxy + by^2$, which alone need be considered, are thus transformed into

$$a(x_1\cos\phi - y_1\sin\phi)^2 + b(x_1\sin\phi + y_1\cos\phi)^2 \\ + 2h(x_1\cos\phi - y_1\sin\phi)(x_1\sin\phi + y_1\cos\phi).$$

The sum of the x_1y_1 terms in this expression when expanded is

$$x_1y_1[(b-a)2\sin\phi\cos\phi + 2h(\cos^2\phi - \sin^2\phi)];$$

that is, the coefficient of x_1y_1 is

$$(b-a)\sin 2\phi + 2h\cos 2\phi,$$

and this will be zero, if ϕ be given such a value that

$$(b-a)\sin 2\phi = -2h\cos 2\phi,$$

or

$$\tan 2\phi = \frac{2h}{a-b}.$$

Example 3. Transform the equation $5x^2 - 4xy + 2y^2 = 6$ (1) into one which lacks the product term.

Here $\tan 2\phi = \frac{2h}{a-b} = \frac{-4}{5-2} = -\frac{4}{3}$; and $\tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi}$.

Hence
$$\frac{2\tan\phi}{1-\tan^2\phi} = -\frac{4}{3}$$

or
$$2\tan^2\phi - 3\tan\phi - 2 = 0. \tag{2}$$

Solving,
$$\tan\phi = 2 \text{ or } -1/2.$$

Selecting the positive value, 2, of $\tan\phi$, so that Ox_1 may make a positive acute angle with Ox, we find $\cos\phi = 1/\sqrt{5}$ and $\sin\phi = 2/\sqrt{5}$.

Hence the required substitution is
$$x = (x_1 - 2y_1)/\sqrt{5}, \quad y = (2x_1 + y_1)/\sqrt{5}. \tag{3}$$

By this substitution the given equation (1) is transformed into
$$\frac{5(x_1 - 2y_1)^2}{5} - \frac{4(x_1 - 2y_1)(2x_1 + y_1)}{5} + \frac{2(2x_1 + y_1)^2}{5} = 6,$$

which, when simplified, becomes
$$x_1^2 + 6y_1^2 = 6, \tag{4}$$

an equation which lacks the product term, as it should.

146. The transformations (1), § 144, and (2), § 145, are the only ones often required in practice. But it is not difficult to show in general (by projection on the two perpendiculars to Oy and Ox, respectively) that the formulas for the transformation from any axes Ox, Oy, rectangular or oblique, through O, to any other axes Ox_1, Oy_1 through O are

$$\left.\begin{aligned} x\sin(xy) &= x_1\sin(x_1y) + y_1\sin(y_1y) \\ y\sin(yx) &= x_1\sin(x_1x) + y_1\sin(y_1x) \end{aligned}\right\}, \tag{3}$$

where (xy) denotes the angle made by Oy with Ox, and so on.

147. As the substitutions (1) of § 144, (2) of § 145, and (3) of § 146 are all of the first degree in both x, y and x_1, y_1, and any transformation of coordinates may be effected by these substitutions singly or combined, the degree of an equation is not raised by the transformation of coordinates. And it cannot be lowered; for if it could, the transformation back to the original axes would give an equation of lower degree than the original equation. But, as the examples given above have shown, it is often possible to simplify the form of an equation by this process of transformation.

148. Exercises. Transformation of coordinates.

1. Transform $2x + y - 1 = 0$ to axes parallel to the given axes through the point $(2, -3)$.

2. What do the equations $y - x - 1 = 0$, $y - 2x - 1 = 0$ become when, without changing the directions of the axes, the origin is transferred to the point of intersection of the lines which the equations represent?

3. Transform each of the following equations to axes parallel to the given axes, through the origin indicated:

(a) $x^2 + y^2 - 4x + 2y = 0$, $O_1(2, -1)$.
(b) $3x^2 - 2y^2 + 6x + 12y - 16 = 0$, $O_1(-1, 3)$.
(c) $4x^2 + 4x - y - 4 = 0$, $O_1(-1/2, -5)$.

4. By change of origin transform each of the following equations into one which lacks the terms of the first degree:

(a) $3x^2 + 2y^2 - 12x + 8y + 19 = 0$.
(b) $x^2 - 3y^2 - 10x + 12y + 12 = 0$.

5. Prove that the method employed in Ex. 4 fails in the case of an equation $(x + \lambda y)^2 + 2gx + 2fy + c = 0$ whose terms of the second degree form a perfect square.

6. What does the equation $x^2 + 2\sqrt{3}xy - y^2 = 4$ become when the rectangular axes are turned through an angle of $30°$?

7. Prove that every equation of the form $x^2 + 2hxy + y^2 + c = 0$ may be rid of the xy terms by turning the rectangular axes through the angle $\pi/4$.

8. Transform each of the following into an equation which lacks the product term:

(a) $3x^2 - 3xy - y^2 = 5$. (b) $3x^2 + 12xy - 2y^2 - 14x = 0$.

9. Find the substitution for transforming to the rectangular axes whose equations referred to the given axes are

$$Ox_1 : 4y - 3x = 0, \quad Oy_1 : 3y + 4x = 0.$$

CHAPTER VIII

THE GENERAL EQUATION OF THE SECOND DEGREE. SECTIONS OF A CONE. SYSTEMS OF CONICS

149. Graphs of equations of the second degree. The formulas (3) of § 146 will change any equation of the second degree in oblique coordinates to one in rectangular coordinates, and then the turning of the axes through an angle ϕ, given by the equation $\tan 2\phi = 2h/(a-b)$, will make the product term disappear [example 2 of § 145]; hence

Every equation of the second degree in oblique or rectangular coordinates can be reduced to the form

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

referred to rectangular axes.

But it has already been proved that every equation of this form represents a parabola [§ 75], an ellipse [§ 99], or an hyperbola [§ 135], except when it is a product of factors of the first degree or when it has no real solution. Hence, except in these latter cases:

The graph of every equation of the second degree in x, y, referred to oblique or rectangular axes, is a conic.

150. Condition that the equation represent a pair of straight lines. The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

will represent a pair of straight lines when, and only when, its left member is the product of two factors of the first degree. The condition for this may be found as follows:

If $b \neq 0$, solve (1) for y in terms of x; the result may be written

$$by = -(hx + f) \pm \sqrt{R}, \tag{2}$$

where $R = (h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)$,

and this will represent two straight lines (one for each sign before the radical) when, and only when, R is a perfect square. But the condition that R be a perfect square is [Alg. § 635]

$$(hf - bg)^2 - (h^2 - ab)(f^2 - bc) = 0.$$

This condition when expanded (and the factor b, which is not 0, is omitted) reduces to

$$abc - af^2 - bg^2 - ch^2 + 2fgh = 0, \tag{3}$$

which may be written in the determinant form:

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0. \tag{3'}$$

If $b = 0$, but $a \neq 0$, the same conclusion may be reached by solving (1) for x in terms of y.

If both a and b are 0, the equation (1) will have the form $2hxy + 2gx + 2fy + c = 0$ where $h \neq 0$. If the left member of this equation be the product of two factors of the first degree, these must be of the form $x + \lambda$ and $y + \mu$, and such that

$$\begin{aligned} 2hxy + 2gx + 2fy + c &\equiv 2h(x + \lambda)(y + \mu) \\ &\equiv 2hxy + 2h\mu x + 2h\lambda y + 2h\lambda\mu. \end{aligned}$$

But this identity will be satisfied when, and only when,

$$g = h\mu, \quad f = h\lambda, \quad c = 2h\lambda\mu,$$

and values of λ and μ satisfying these three equations exist when, and only when,

$$2gf = hc,$$

a relation which is equivalent to (3), when $a = 0$, $b = 0$, and $h \neq 0$.

Hence, in every case (3), or (3′), is the condition that the equation (1) shall represent a pair of straight lines. The determinant D is called the *discriminant* of the equation (1).

Example 1. Show that $3x^2 + 2xy - y^2 + 14x + 2y + 15 = 0$ represents a pair of straight lines, and find the equations of these lines.

Substituting in (3) gives

$$D = 3 \cdot (-1) \cdot 15 - 3 \cdot 1^2 + 1 \cdot 7^2 - 15 \cdot 1^2 + 2 \cdot 1 \cdot 7 \cdot 1 = 0.$$

Hence the equation represents a pair of lines. To find the equations of these lines, solve the equation for y. It may be written

$$y^2 - 2(x+1)y - (3x^2 + 14x + 15) = 0.$$

Hence $\quad y = x + 1 \pm \sqrt{(x+1)^2 + 3x^2 + 14x + 15},$

or $\quad y = x + 1 \pm \sqrt{4x^2 + 16x + 16},$

that is, $\quad y = 3x + 5, \text{ or } y = -x - 3.$

Therefore, the lines are $y - 3x - 5 = 0$ and $y + x + 3 = 0$.

Example 2. Show that $9x^2 - 6xy + y^2 + 6x - 2y - 15 = 0$ represents a pair of straight lines, and find the equations of these lines.

Substituting in (3′) gives

$$D = \begin{vmatrix} 9 & -3 & 3 \\ -3 & 1 & -1 \\ 3 & -1 & -15 \end{vmatrix} = 0,$$

since the first row (or column) is the second multiplied by -3.

Hence the equation represents a pair of straight lines. The equations of these lines may be found as in Ex. 1, or by factoring the left member of the given equation by inspection. They are

$$3x - y + 5 = 0 \text{ and } 3x - y - 3 = 0,$$

which represent *parallel* lines.

Observe that whenever, as in this case, $D = 0$, and the terms of the second degree (here $9x^2 - 6xy + y^2$) form a perfect square, the equation represents a pair of parallel lines.

151. It may be added that when $D \neq 0$ and the equation § 150, (1) therefore represents a conic, (2) supplies a convenient means of finding points on this conic and so constructing it. For (2) gives two real values for y for each value of x for which R is positive. Corresponding to these values of y there are two points on the conic which may be found by drawing the line

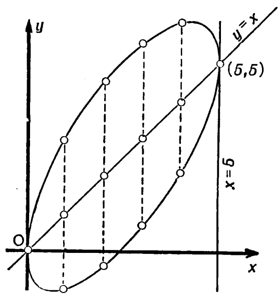
$$by = -(hx + f),$$

and then increasing and diminishing its ordinate for the value of x in question by the value of $\sqrt{R}/b$. From the following examples it will be seen that, according as the coefficient of x^2 in R, namely $h^2 - ab$, is $<, >$, or $= 0$, the graph is an ellipse, an hyperbola, or a parabola. The axes may be rectangular or oblique. [Compare also Alg. § 668.]

Example 1. Find the graph of $y^2 - 2xy + 2x^2 - 5x = 0$. (1)

Solving for y, $y = x \pm \sqrt{5x - x^2}$. (2)

The values of y given by (2) are real when $5x - x^2$, or $x(5 - x)$, is positive (or 0), that is, when x lies between 0 and 5. Hence, the graph of (1) lies between the lines $x = 0$ and $x = 5$.



When $x = 0$ and when $x = 5$, the values of y are equal, both being 0 when $x = 0$ and both being 5 when $x = 5$. Hence the graph touches the line $x = 0$ at (0, 0) and the line $x = 5$ at (5, 5). The line $y = x$ joins these points of tangency.

For each value of x between 0 and 5 the equation (2) gives two real values of y, obtained by increasing and diminishing the value of x by that of $\sqrt{5x - x^2}$. The corresponding points on the graph may be obtained by drawing the line $y = x$ and then increasing and diminishing its ordinate for the value of x in question by the value of $\sqrt{5x - x^2}$.

Thus, when	$x = 0$,	1,	2,	3,	4,	5.
we have on line $y = x$,	$y = 0$,	1,	2,	3,	4,	4.
and on graph of (1),	$y = 0$,	1 ± 2,	$2 \pm \sqrt{6}$,	$3 \pm \sqrt{6}$,	4 ± 2,	5.

The graph is therefore the ellipse represented in the figure. As the line $y = x$ bisects all chords parallel to $x = 0$, it is a diameter of the ellipse [§ 108].

Example 2. The equation $y^2 - 2xy + 5x = 0$ when solved for y gives $y = x \pm \sqrt{x^2 - 5x}$; prove that the graph is an hyperbola.

Example 3. The equation $y^2 - 2xy + x^2 - 5x = 0$ when solved for y gives $y = x \pm \sqrt{5x}$; prove that the graph is a parabola.

152. Central Conics. Determination of the center. It remains to show how to transform the equation of a conic given in the general form,

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0 \qquad (1)$$

into its equation referred to its axes, if it be an ellipse or hyperbola, or to its axis and the tangent at the vertex, if it be a parabola.

By the substitution $x = x_1 + x_0$, $y = y_1 + y_0$, the equation (1) will be transformed to axes parallel to the original axes and passing through the point (x_0, y_0). The result may be written

$$ax_1^2 + 2\,hx_1y_1 + by_1^2 + 2\,x_1(ax_0 + hy_0 + g) + 2\,y_1(hx_0 + by_0 + f)$$
$$+ ax_0^2 + 2\,hx_0y_0 + by_0^2 + 2\,gx_0 + 2\,fy_0 + c = 0. \qquad (2)$$

The coefficients of both x_1 and y_1 in (2) will be zero, if x_0, y_0 can be so chosen that

$$ax_0 + hy_0 + g = 0 \quad (3) \quad \text{and} \quad hx_0 + by_0 + f = 0; \quad (4)$$

and finite values of x_0, y_0 satisfying (3) and (4) always exist except when $ab - h^2 = 0$; for solving (3), (4),

$$x_0 = \frac{hf - bg}{ab - h^2}, \qquad y_0 = \frac{hg - af}{ab - h^2}. \qquad (5)$$

Hence, if $ab - h^2 \neq 0$, the equation (1) can be transformed into an equation of the form

$$ax_1^2 + 2\,hx_1y_1 + by_1^2 + c' = 0 \qquad (6)$$

where

$$\begin{aligned} c' &= ax_0^2 + 2\,hx_0y_0 + by_0^2 + 2\,gx_0 + 2\,fy_0 + c \\ &= x_0(ax_0 + hy_0 + g) + y_0(hx_0 + by_0 + f) + (gx_0 + fy_0 + c) \\ &= gx_0 + fy_0 + c \text{ [by (3) and (4)]} \qquad (7) \\ &= g\frac{hf - bg}{ab - h^2} + f\frac{hg - af}{ab - h^2} + c \text{ [by (5)]} \\ &= \frac{abc - af^2 - bg^2 - ch^2 + 2\,fgh}{ab - h^2}, \end{aligned}$$

that is,

$$c' = \frac{D}{ab - h^2}. \qquad (8)$$

The new origin $O_1(x_0, y_0)$ is the *center* of the conic represented by (1) or (6). For if (6) be satisfied by $x_1 = x_1'$, $y_1 = y_1'$, it is also satisfied by $x_1 = -x_1'$, $y_1 = -y_1'$; that is, all chords of the conic (6) which pass through O_1 are bisected at O_1. The conic is therefore either an ellipse, or an hyperbola, unless $c' = 0$, when (6) or (1) will represent a pair of straight lines intersecting at $O_1(x_0, y_0)$.

Example. Transform $5x^2 - 4xy + 2y^2 - 16x + 4y + 8 = 0$ to the center of the conic represented by the equation as origin.

Here the equations [(3), (4)] for finding the center are

$$5x_0 - 2y_0 - 8 = 0, \qquad -2x_0 + 2y_0 + 2 = 0,$$

which, when solved, give $x_0 = 2$, $y_0 = 1$.

Substituting these values for x_0, y_0 in (7), gives

$$c' = (-8) \cdot 2 + 2 \cdot 1 + 8 = -6.$$

Hence the required transformed equation is

$$5x^2 - 4xy + 2y^2 - 6 = 0.$$

153. If $ab - h^2 = 0$, the equations (3), (4) of § 152, have no finite solution, and therefore the conic represented by (1) has no center in the finite region of the plane. It is a parabola [see § 158], or, if $D = 0$, a pair of parallel lines [§ 150, Example 2].

154. Central conics. Determination of the axes. It will next be shown how to transform the equation of a conic referred to any rectangular axes through its center [§ 152, (6)], into its equation referred to the axes of the conic as axes of coordinates. First consider the following example.

Example. Find the equation of the conic $5x^2 - 4xy + 2y^2 = 6$ (1) when referred to its axes as coordinate axes.

Call the axes of the conic Ox_1, Oy_1. Since they are perpendicular, their equations have the form:

$$Ox_1\colon\ y - \lambda x = 0, \quad (2) \qquad Oy_1\colon\ \lambda y + x = 0. \quad (3)$$

The coordinates (x_1, y_1) of any point $P(x, y)$ referred to Ox_1, Oy_1 as axes, are the perpendicular distances of P from Oy_1, Ox_1, respectively;

hence $$x_1 = \frac{\lambda y + x}{(\lambda^2 + 1)^{\frac{1}{2}}}, \qquad y_1 = \frac{y - \lambda x}{(\lambda^2 + 1)^{\frac{1}{2}}}. \quad (4)$$

It follows [§ 99, § 135] that the equation of the conic (1) when referred to its axes Ox_1, Oy_1 as axes of coordinates is of the form

$$Ax_1^2 + By_1^2 - 6 = 0. \tag{5}$$

Hence it is required to find values of λ, A, B, such that

$$5x^2 - 4xy + 2y^2 \equiv A\frac{(\lambda y + x)^2}{\lambda^2 + 1} + B\frac{(y - \lambda x)^2}{\lambda^2 + 1}. \tag{6}$$

When cleared of fractions and expanded, (6) becomes

$$5(\lambda^2 + 1)x^2 - 4(\lambda^2 + 1)xy + 2(\lambda^2 + 1)y^2$$
$$\equiv (A + B\lambda^2)x^2 + 2(A - B)\lambda xy + (A\lambda^2 + B)y^2. \tag{7}$$

The identity (7) will be satisfied if the corresponding coefficients are equal, that is, if

$$A + B\lambda^2 = 5(\lambda^2 + 1), \tag{8}$$
$$A\lambda^2 + B = 2(\lambda^2 + 1), \tag{9}$$
$$(A - B)\lambda = -2(\lambda^2 + 1). \tag{10}$$

To solve these equations for λ, A, B, first subtract (9) from (8). The result is

$$(A - B)(1 - \lambda^2) = 3(\lambda^2 + 1). \tag{11}$$

Then divide (11) by (10). The result when simplified is

$$2\lambda^2 - 3\lambda - 2 = 0. \tag{12}$$

Solving (12), $\lambda = 2$, or $-1/2$.

Substituting $\lambda = 2$ in (8) and (9) and solving for A and B,

$$A = 1, \quad B = 6.$$

Since $\lambda = 2$, $A = 1$, $B = 6$, the equations (2), (3), of the axes are

$$Ox_1: \; y - 2x = 0, \qquad Oy_1: \; 2y + x = 0,$$

and the equation (5) of the conic is

$$x_1^2 + 6y_1^2 = 6, \quad \text{or} \quad \frac{x_1^2}{6} + \frac{y_1^2}{1} = 1.$$

155. In general, if the equation of the conic referred to rectangular axes be given in the form

$$ax^2 + 2hxy + by^2 + c' = 0, \tag{1}$$

its equation referred to its axes may be found as follows:

Call the axes of the conic Ox_1, Oy_1; since they are perpendicular, their equations have the form:

$$Ox_1: \; y - \lambda x = 0, \; (2) \qquad Oy_1: \; \lambda y + x = 0. \tag{3}$$

The coordinates (x_1, y_1) of any point $P(x, y)$, referred to Ox_1, Oy_1 as axes, are the perpendicular distances of P from Oy_1 and Ox_1 respectively. Hence

$$x_1 = \frac{\lambda y + x}{(\lambda^2+1)^{\frac{1}{2}}}, \qquad y_1 = \frac{y - \lambda x}{(\lambda^2+1)^{\frac{1}{2}}}. \tag{4}$$

The equation (1), when referred to Ox_1, Oy_1 as axes, is of the form

$$Ax_1^2 + By_1^2 + c' = 0. \tag{5}$$

Hence it is required to find values of λ, A, B, such that

$$ax^2 + 2\,hxy + by^2 \equiv A\,\frac{(\lambda y + x)^2}{\lambda^2+1} + B\,\frac{(y - \lambda x)^2}{\lambda^2+1}. \tag{6}$$

When cleared of fractions and expanded, (6) becomes

$$\begin{aligned} a(\lambda^2+1)x^2 + 2\,h(\lambda^2+1)xy + b(\lambda^2+1)y^2 \\ \equiv (A + B\lambda^2)x^2 + 2(A - B)\lambda xy + (A\lambda^2 + B)y^2. \end{aligned} \tag{7}$$

This identity will be true if its corresponding coefficients are equal, that is, if λ, A, B satisfy the following equations:

$$A + B\lambda^2 = a(\lambda^2+1), \tag{8}$$

$$A\lambda^2 + B = b(\lambda^2+1), \tag{9}$$

$$(A - B)\lambda = h(\lambda^2+1). \tag{10}$$

To solve these equations for λ, A, B, subtract (9) from (8). The result is

$$(A - B)(1 - \lambda^2) = (a - b)(\lambda^2+1). \tag{11}$$

Then divide (11) by (10). The result, when simplified, is

$$h\lambda^2 + (a - b)\lambda - h = 0. \tag{12}$$

The discriminant of (12), namely $(a-b)^2 + 4\,h^2$, is positive; hence the roots of (12) are real [Alg. § 635]. Moreover, since the coefficients of λ^2 and the absolute term are equal numerically but have opposite signs, the product of the roots is -1; that is, one of the roots is the negative reciprocal of the other [Alg. § 636].

Take the positive root of (12) as the value of λ, substitute this value of λ in any two of the equations (8), (9), (10), and then solve these two equations for A and B. Finally, substitute the values λ, A, B thus obtained in (2), (3), and (5). The results will be the equations of the axes Ox_1, Oy_1, and the equation of the conic referred to Ox_1, Oy_1 as coordinate axes.*

156. Returning to the equations (8), (9), (10), of § 155 add (8) and (9) and simplify; the result is

$$A + B = a + b. \tag{13}$$

Again multiply (8) by (9), and from the result subtract the square of (10); the final result, when simplified, is

$$AB = ab - h^2. \tag{14}$$

If in the transformed equation (5), A and B have the same sign, (5) represents an ellipse (real or imaginary, according as the sign of c' is opposite to or the same as the sign of A and B). On the other hand, if A and B have opposite signs, (5) represents an hyperbola. Therefore, since AB is positive or negative according as A and B have the same or opposite signs, it follows from (14) that

If the graph of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *is a proper conic, that is, not a pair of straight lines or imaginary, this conic is an ellipse when* $ab - h^2 > 0$, *and an hyperbola when* $ab - h^2 < 0$.

157. The values of A and B may be found without carrying out the reckoning of § 155, namely, by forming and solving the equations (13) and (14) of § 156. Two solutions will be thus obtained, but since λ is positive, it follows from § 155, (10) that the one to be selected is that for which $A - B$ has the same sign as h. Hence the rule:

* It should be observed that the equations (4) are the same as the substitution of § 145, (1), expressed in terms of $\lambda = \tan\phi$ instead of $\sin\phi$ and $\cos\phi$, and that the value of $\lambda = \tan\phi$ obtained by solving (12) is the same as that found in § 145, Ex. 2, to meet the requirement that the transformed equation shall lack the x_1y_1 term. [§ 154, Ex. and § 145, Ex. 3.]

To transform $ax^2+2hxy+by^2+c'=0$ into the equation referred to the axes of the conic, form the equations

$$h\lambda^2+(a-b)\lambda-h=0, \quad (1) \qquad A+B=a+b, \quad (2)$$
$$AB=ab-h^2; \quad (3)$$

find the positive root of (1) *and call this* λ; *solve* (2) *and* (3) *for* A *and* B, *selecting the solution for which* $A-B$ *has the same sign as* h. *The equations of the axes and of the conic are then*

$$Ox_1:\ y-\lambda x=0, \quad Oy_1:\ \lambda y+x=0, \quad Ax_1^2+By_1^2+c'=0.$$

Example. Analyze the equation

$$19x^2+4xy+16y^2-212x+104y-356=0.$$

Here

$$ab-h^2=19\cdot 16-2^2=300, \qquad D=\begin{vmatrix}19 & 2 & -106\\ 2 & 16 & 52\\ -106 & 52 & -356\end{vmatrix}=-360000.$$

Hence the equation represents an ellipse.

The center is found by solving the equations

$$19x_0+2y_0-106=0,$$
$$2x_0+16y_0+52=0,$$

and is $C(6,-4)$.

And $c'=-360000/300=-1200$.

Hence the equation, referred to axes through C and parallel to the given axes, is

$$19x^2+4xy+16y^2-1200=0.$$

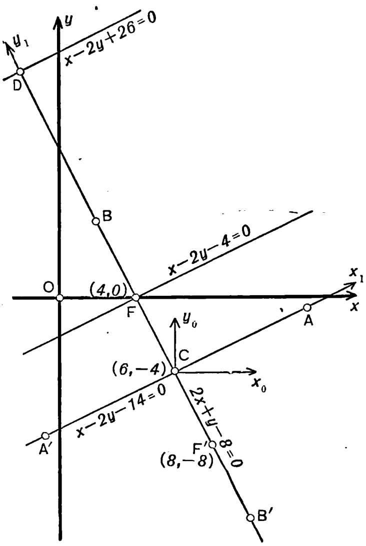
The equation giving the direction of the axes of the conic is

$$2\lambda^2+3\lambda-2=0$$

the positive root of which is 1/2.

Hence the equations of the axes (referred to C as origin) are

$$Cx_1:\ x-2y=0,$$
$$Cy_1:\ 2x+y=0.$$



The equations $A + B = a + b$, $AB = ab - h^2$ are here

$$A + B = 35 \text{ and } AB = 300,$$

and $A - B$ is positive since h is positive.

These equations therefore give $A = 20$ and $B = 15$.

Hence the equation of the conic referred to its axes Cx_1, Cy_1 is

$$20\,x^2 + 15\,y^2 - 1200 = 0,$$

or

$$x^2/60 + y^2/80 = 1.$$

The lengths of its semiaxes are $2\sqrt{15}$ and $4\sqrt{5}$.

Hence [§ 98] $e^2 = 1 - 60/80 = 1/4$, $e = 1/2$; and the distances from center to focus and directrix are $CF = 4\sqrt{5}/2 = 2\sqrt{5}$; and $CD = 8\sqrt{5}$.

The equations of Cx_1 and Cy_1, referred to axes through C and parallel to the given axes, were found to be $x - 2\,y = 0$ and $2\,x + y = 0$, respectively. Hence, their equations referred to the given axes have the form $x - 2\,y + k = 0$ and $2\,x + y + l = 0$. But these lines pass through C, whose coordinates referred to these same axes are $(6, -4)$; hence, $6 + 8 + k = 0$, or $k = -14$, and $12 - 4 + l = 0$, or $l = -8$. Therefore, referred to the given axes, the equations of Cx_1 and Cy_1 are

$$Cx_1: \; x - 2\,y - 14 = 0, \qquad Cy_1: \; 2\,x + y - 8 = 0.$$

The directrix through D, in the figure, is parallel to Cx_1 and at the distance CD $(= 8\sqrt{5})$ in the negative direction from it; hence the equation of this directrix is [§ 50, § 94, last line].

$$(x - 2\,y - 14)/\sqrt{5} + 8\sqrt{5} = 0, \text{ or } x - 2\,y + 26 = 0.$$

The corresponding focus F is the point where the line parallel to Cx_1 and at the distance CF $(= 2\sqrt{5})$ in the negative direction from it, namely

$$(x - 2\,y - 14)/\sqrt{5} + 2\sqrt{5} = 0, \text{ or } x - 2\,y - 4 = 0,$$

cuts the line Cy_1, or $2\,x + y - 8 = 0$, and is $F\,(4, 0)$.

To verify the correctness of the reckoning, find the equation of the ellipse which has the directrix $x - 2\,y + 26 = 0$, the focus $(4, 0)$, and the eccentricity $e = 1/2$.

As in § 88, the equation is $PF^2 = e^2 PM^2$, or

$$(x - 4)^2 + y^2 = \frac{1}{4}\left(\frac{x - 2\,y + 26}{-\sqrt{5}}\right)^2,$$

or $$20\,x^2 - 160\,x + 320 + 20\,y^2 = x^2 - 4\,xy + 4\,y^2 + 52\,x - 104\,y + 676,$$

or $$19\,x^2 + 4\,xy + 16\,y^2 - 212\,x + 104\,y - 356 = 0,$$

the equation given to be analyzed.

158. The parabola. When the terms of the second degree in the equation

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$$

are connected by the relation $ab - h^2 = 0$, that is, when they form a perfect square, the preceding transformations fail, since the curve represented by the equation has no center [§ 153]. It is a parabola (or a pair of parallel lines). The equation of this parabola, referred to its axis and the tangent at its vertex, may be found as in the following example.

Example. Prove that $9\,x^2 + 24\,xy + 16\,y^2 - 52\,x + 14\,y - 6 = 0$ represents a parabola, and find the equation of this parabola referred to the axis and the tangent at the vertex as coordinate axes.

As the terms of the second degree form a perfect square, the equation may be written

$$(3\,x + 4\,y)^2 = 52\,x - 14\,y + 6. \tag{1}$$

The lines represented by $3\,x + 4\,y = 0$ and $52\,x - 14\,y + 6 = 0$ are not perpendicular, but $3\,x + 4\,y$ and $52\,x - 14\,y + 6$ may be replaced by expressions which represent perpendicular lines when set equal to zero, by the following procedure.

Add λ to the expression $3\,x + 4\,y$ in the left member of (1), and add to the right member the terms thus added to the left. This gives

$$(3\,x + 4\,y + \lambda)^2 = (6\,\lambda + 52)\,x + (8\,\lambda - 14)\,y + \lambda^2 + 6. \tag{2}$$

The lines represented by the expressions on the left and right, when set equal to 0, will be perpendicular, if [§ 30]

$$3\,(6\,\lambda + 52) + 4\,(8\,\lambda - 14) = 0, \quad \text{or} \quad \lambda = -2 \tag{3}$$

and when this value of λ is substituted in (2), this equation becomes

$$(3\,x + 4\,y - 2)^2 = 10\,(4\,x - 3\,y + 1). \tag{4}$$

If y_1 and x_1 denote the perpendicular distances of any point $P(x, y)$ from the perpendicular lines $3\,x + 4\,y - 2 = 0$ and $4\,x - 3\,y + 1 = 0$, then [§ 50]

$$y_1 = \frac{3\,x + 4\,y - 2}{5}, \tag{5}$$

and

$$x_1 = \frac{4\,x - 3\,y + 1}{-5}. \tag{6}$$

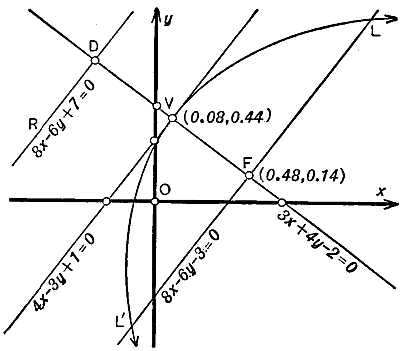
Substituting these expressions in (4) gives

$$y_1^2 = -2x_1 \tag{7}$$

which is the equation of a parabola whose axis is the line $y_1 = 0$, or $3x + 4y - 2 = 0$, and the tangent at the vertex the line $x_1 = 0$, or $4x - 3y + 1 = 0$; the equation (7), expressed in terms of the original coordinates, being

$$\left(\frac{3x + 4y - 2}{5}\right)^2 = 4\left(-\frac{1}{2}\right)\left(\frac{4x - 3y + 1}{-5}\right). \tag{8}$$

Since the positive direction on perpendiculars to the line $4x-3y+1=0$ is *from* the origin *to* the line [§ 50, lines 4 and 5], therefore the positive direction for x_1 in (6) and (7) is from V to D in the figure. And since in (7) a negative sign appears in the right member, the parabola is on that side of the tangent at the vertex which is opposite to VD; that is, the parabola lies on the origin side of this tangent. The position of the parabola also can be fixed as follows: Every real pair of values of (x, y) which satisfy (4) will make its left member, and therefore its right member, positive; and $4x - 3y + 1$ is positive only for points which lie on the origin side of the line $4x - 3y + 1 = 0$.



The vertex V is the point of intersection of the lines $3x + 4y - 2 = 0$ and $4x - 3y + 1 = 0$, or (0.08, 0.44).

Since, in (8), $a = -1/2$, the distance from the vertex to the focus and to the directrix is 1/2. And the equations of the directrix and of the line through the focus parallel to the directrix are

$$\frac{4x - 3y + 1}{-5} - \frac{1}{2} = 0 \quad \text{and} \quad \frac{4x - 3y + 1}{-5} + \frac{1}{2} = 0,$$

or $$8x - 6y + 7 = 0 \quad \text{and} \quad 8x - 6y - 3 = 0.$$

The focus is the point of intersection of $8x - 6y - 3 = 0$ and $3x + 4y - 2 = 0$, or (0.48, 0.14).

159. And, in general, if $ab - h^2 = 0$, so that the terms of the second degree in the equation

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0 \qquad (1)$$

form a perfect square, the equation may be written

$$(\alpha x + \beta y)^2 = -\,(2\,gx + 2\,fy + c), \qquad (2)$$

where $\alpha = \sqrt{a}$ and $\beta = \sqrt{b}$.

If the lines $\alpha x + \beta y = 0$ and $2\,gx + 2\,fy + c = 0$ are not perpendicular, replace $\alpha x + \beta y$ in (2) by $\alpha x + \beta y + \lambda$, where λ denotes a constant, at the same time adding to the right member of (2) the terms thus added to the left. The equation thus becomes

$$(\alpha x + \beta y + \lambda)^2 = 2\,(\lambda\alpha - g)\,x + 2\,(\lambda\beta - f)\,y + \lambda^2 - c. \qquad (3)$$

The lines
$$\alpha x + \beta y + \lambda = 0, \qquad (4)$$

and
$$2\,(\lambda\alpha - g)\,x + 2\,(\lambda\beta - f)\,y + \lambda^2 - c = 0, \qquad (5)$$

will be perpendicular, if [§ 30]

$$\alpha(\lambda\alpha - g) + \beta\,(\lambda\beta - f) = 0,$$

that is, if
$$\lambda = \frac{\alpha g + \beta f}{\alpha^2 + \beta^2}. \qquad (6)$$

Assign this value to λ in (3), (4), (5), and then take the perpendicular lines (4) and (5) as new axes of reference O_1x_1 and O_1y_1. The coordinates (x_1, y_1) of any point $P(x, y)$ referred to O_1x_1 and O_1y_1 as axes are the perpendicular distances of P from O_1y_1 and O_1x_1, respectively; hence

$$\left.\begin{aligned} y_1 &= \frac{\alpha x + \beta y + \lambda}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}, \\ x_1 &= \frac{2\,(\lambda\alpha - g)\,x + 2\,(\lambda\beta - f)\,y + \lambda^2 - c}{2\{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2\}^{\frac{1}{2}}}. \end{aligned}\right\} \qquad (7)$$

By the substitution (7) the equation (3) becomes

$$(\alpha^2 + \beta^2)\,y_1^2 = 2\,\{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2\}^{\frac{1}{2}}\,x_1,$$

or, replacing λ by its value (6), and α, β by $\sqrt{a}$, $\sqrt{b}$, and simplifying,

$$y_1^2 = \frac{2(f\sqrt{a} - g\sqrt{b})}{(a+b)^{\frac{3}{2}}} x_1 \qquad (8)$$

which represents a parabola having $y_1 = 0$ for its axis and $x_1 = 0$ for the tangent at the vertex.

160. Recapitulation. The preceding discussion has proved that any given equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be analyzed as follows:

Calculate the values of

$$ab - h^2 \qquad \text{and} \qquad D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The character of the graph of the given equation is indicated by the values of $ab - h^2$ and D. There are the following four cases:

1. *If* $D = 0$ *and* $ab - h^2 = 0$, *two parallel lines.*
2. *If* $D = 0$ *and* $ab - h^2 \neq 0$, *two intersecting lines.*
3. *If* $D \neq 0$ *and* $ab - h^2 = 0$, *a parabola.*
4. *If* $D \neq 0$ *and* $ab - h^2 \neq 0$, *a central conic; namely, an ellipse (real or imaginary) when* $ab - h^2 > 0$, *an hyperbola when* $ab - h^2 < 0$.

To find the graph in cases 1 and 2, proceed as in § 150, Examples 2 and 1.

To find the graph in case 3, proceed as in § 158, Example.

To find the graph in case 4, proceed as in § 157, Example; namely, the following steps are to be taken:

Find the center $C(x_0, y_0)$ by solving the equations

$$ax_0 + hy_0 + g = 0,$$
$$hx_0 + by_0 + f = 0,$$

where the coefficients are the same as in the first two rows of D.

Also compute $c' = D/(ab - h^2)$.

Referred to axes through $C\ (x_0, y_0)$ and parallel to the original axes, the equation of the conic is

$$ax^2 + 2\,hxy + by^2 + c' = 0.$$

Referred to these same axes, the *equations* of the axes of the conic are

$$Cx_1:\ y - \lambda x = 0, \qquad Cy_1:\ \lambda y + x = 0,$$

where λ is the *positive* root of

$$h\lambda^2 + (a - b)\,\lambda - h = 0.$$

Referred to the axes Cx_1, Cy_1, the equation of the conic is $Ax^2 + By^2 + c' = 0$, where A and B are obtained from

$$A + B = a + b \text{ and } AB = ab - h^2,$$

and the condition that $A - B$ has the same sign as h.

The equation of the conic referred to its axes may then be written

$$\frac{x^2}{-c'/A} + \frac{y^2}{-c'/B} = 1.$$

From this equation the eccentricity and the distances from the center to the foci and directrices can be found [§ 98, § 134].

161. Exercises. Draw a figure for each exercise.

1. What does each of the following equations represent?

(1). $3\,x^2 - 2\,xy + y^2 - 6 = 0.$
(2) $9\,x^2 - 20\,xy + 11\,y^2 - 50 = 0.$
(3) $xy + x - 3\,y + 7 = 0.$
(4) $3\,x^2 - 2\,xy + y^2 + 6 = 0.$
(5) $9\,x^2 - 30\,xy + 25\,y^2 - 10\,x = 0.$
(6) $x^2 - xy + 5\,x - 2\,y + 6 = 0.$

2. Prove that $2\,x^2 - xy - 6\,y^2 + 13\,x + 9\,y + 15 = 0$ represents a pair of straight lines, and find the equation of each of these lines.

3. For what value of λ does $x^2 + 2\,xy + 2\,y^2 + x + \lambda = 0$ represent a pair of straight lines? Are these lines real or imaginary?

4. Transform each of the following equations to axes through the center of the conic which it represents:

(1) $x^2 - xy + y^2 + 3\,x = 0.$
(2) $2\,x^2 + xy + y^2 - 5\,x - 10\,y + 18 = 0.$
(3) $3\,x^2 - 5\,xy + y^2 + 16\,x - 9\,y + 11 = 0.$

5. What are the equations of the following conics referred to their axes?

(1) $x^2 + xy + y^2 - 1 = 0.$
(2) $x^2 + 3xy - 3y^2 - 4 = 0.$
(3) $2x^2 - 12xy - 3y^2 + 14 = 0.$
(4) $43x^2 + 30xy + 59y^2 - 68 = 0.$

6. Transform each of the following equations first to the center, and then to the axes, of the conic which it represents.

(1) $x^2 + 6xy + y^2 - 4x - 12y + 10 = 0.$
(2) $3x^2 + 12xy - 2y^2 - 14x = 0.$
(3) $3x^2 - 3xy - y^2 + 15x + 10y - 24 = 0.$
(4) $7x^2 + 4xy + 4y^2 + 10x + 4y - 25 = 0.$
(5) $2x^2 - 4xy - y^2 - 20x + 8y - 40 = 0.$
(6) $3x^2 + 2xy + 3y^2 - 16x + 16y + 52 = 0.$
(7) $2x^2 - 4xy + 5y^2 - 38x + 64y + 167 = 0.$

7. Prove that each of the following equations represents a parabola, and transform it to the axis of the parabola and the tangent at the vertex as coordinate axes:

(1) $x^2 - 2xy + y^2 - 10x - 6y + 25 = 0.$
(2) $x^2 - 4xy + 4y^2 - 4x - 2y + 8 = 0.$
(3) $y^2 - 2xy + x^2 + 2x = 0.$

8. Prove that the centers of all conics represented by

$$ax^2 + 2hxy + by^2 + 2g\lambda x + 2f\lambda y + c = 0,$$

where a, b, c, h, g, f, are given, but λ is arbitrary, lie on a straight line which passes through the origin.

9. Prove that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines if $ab - h^2 = 0$ and $g^2/f^2 = a/b$.

10. Analyze the equation $3x^2 - 4xy - 4y^2 + 10x + 6y - 4 = 0$ (namely, determine the center, focus, and directrix of the graph, and draw the graph).

11. Analyze the equation

$$3x^2 - 4xy - 4y^2 + 5x + 6y - 2 = 0.$$

12. Analyze the equation

$$2x^2 - 8xy + 8y^2 - 72x - 56y - 7 = 0.$$

13. Analyze the equation

$$8x^2 - 12xy + 3y^2 - 9 = 0.$$

14. Analyze the equation

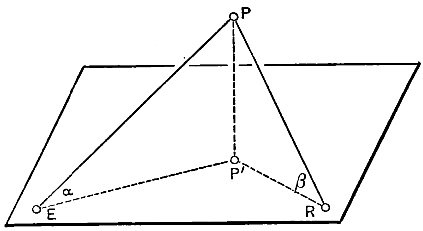
$$2x^2 - 4xy + 5y^2 + 4x - 16y + 8 = 0.$$

15. Analyze the equation

$$y^2 - 2xy + x^2 - 5x = 0.$$

162. Conics obtained as plane sections of a cone. It will be proved that every conic is a plane section of a right circular cone, a fact to which these curves owe their name. In the proof the following theorem is used:

The lengths of any two lines from a point to a plane are inversely proportional to the sines of the angles which the lines make with the plane.



For, let the two lines PE and PR from P meet the plane in the points E and R, and let P' denote the projection of P on the plane. Then,

$$EP \sin P'EP = P'P = RP \sin P'RP,$$

and therefore,
$$\frac{EP}{RP} = \frac{\sin P'RP}{\sin P'EP}.$$

163. *Every plane section of a right circular cone is a conic.*

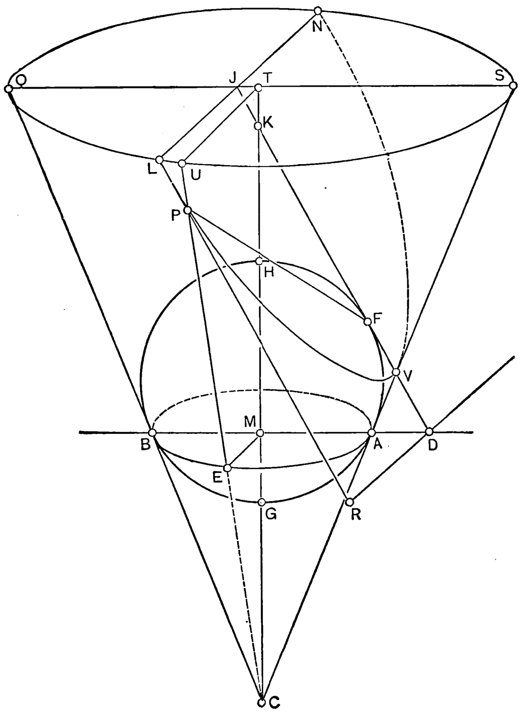
Let C be the vertex of any right circular cone $C\text{–}QUS$; let any tangent sphere touch the cone along the circle BEA; and let any plane, tangent to the sphere at F, cut the plane of the circle BEA in the line DR and the cone in the curve $LPVN$. It is to be proved that $LPVN$ is a conic whose focus is the point F and whose directrix is the line DR.

Through P, any representative point on the section, take the element of the cone $UPEC$, tangent to the sphere at E. Then PE equals PF, since two tangents to a sphere from the same point are equal. And PE makes with the plane of BEA a fixed angle ($= DAV$) no matter where P is taken on the section $LPVN$, for this is a property of all elements of the cone.

Take PR perpendicular to DR. PR is parallel to FD, the perpendicular from F to DR, and therefore no matter where the point P is taken in the section $LPVN$, the perpendicular PR makes a fixed angle $(=ADV)$ with the plane BEA.

Therefore [§ 162],

$$\frac{PF}{PR}=\frac{PE}{PR}=\frac{\sin VDA}{\sin VAD}=\text{const.}$$



that is, the section $LPVN$ is a conic having F for its focus and DR for its directrix [§ 69].

If the cutting plane is inclined to the plane of the base of the cone at the same angle as an element of the cone, the section is a parabola; if the inclination is less, the section is an ellipse; if greater, it is an hyperbola.

164. Systems of conics. *If U and V denote two expressions of the second degree in x, y, and λ a constant, then $U + \lambda V = 0$ will represent a conic which passes through the points of intersection of the conics represented by $U = 0$ and $V = 0$.*

For, $U + \lambda V = 0$ represents a conic, since it is of the second degree in x, y, and this conic will pass through the points of intersection of the conics $U = 0$ and $V = 0$, since for these points both U and V are 0, and therefore $U + \lambda V = 0$ is satisfied. [Compare § 37, § 62.]

The conics may be pairs of straight lines, and if the terms of the second degree in U and V are proportional, λ may have such a value that $U + \lambda V = 0$ will represent one straight line. [Compare § 63.]

Example. Prove that there are two parabolas which pass through the points of intersection of the circle $x^2 + y^2 - x - 9 = 0$ and the hyperbola $xy = 1$, and find their equations.

The equation $x^2 + y^2 - x - 9 + \lambda(xy - 1) = 0$ represents a conic through the points of intersection of the conics $x^2 + y^2 - x - 9 = 0$ and $xy - 1 = 0$, whatever the value of λ may be. And this conic will be a parabola if the terms of the second degree in the equation, namely $x^2 + \lambda xy + y^2$, form a perfect square [§ 158], that is, if $\lambda = 2$ or -2. Hence there are two parabolas through the points of intersection of the given conics, and their equations are

$$x^2 + 2xy + y^2 - x - 11 = 0 \text{ and } x^2 - 2xy + y^2 - x - 7 = 0.$$

The equation $U + \lambda l^2 = 0$ (1), in which U is an expression of the second degree, l one of the first degree, and λ a constant, represents a conic which touches the conic $U = 0$ where it is met by the line $l = 0$. For if ϵ denotes a variable quantity (not involving x or y) whose limit is 0, the original equation $U + \lambda l^2 = 0$ is the limiting form of the equation $U + \lambda l(l + \epsilon) = 0$ (2); the conic (2) passes through the four points in which the conic $U = 0$ is met by the two lines $l = 0$ and $l + \epsilon = 0$; and these four points coincide in pairs when ϵ becomes 0 and the conic (2) becomes the conic (1).

Thus the conic $y^2-4x=0$ is met by the line $y-x=0$ in the points (0, 0) and (4, 4); and the conic $y^2-4x+2(y-x)^2=0$, or $3y^2-4xy+2x^2-4x=0$, touches the conic $y^2-4x=0$ at these two points. In fact it may be proved by the method of § 81 that the tangent to both conics at (0, 0) is $x=0$, and that the tangent to both at (4, 4) is $x-2y+4=0$.

165. Conic through five points. As the general equation of the second degree

$$ax^2+2hxy+bx^2+2gx+2fy+c=0$$

has six terms, and the solutions of the equation are not changed by dividing throughout by one of the coefficients, it contains five independent constants. From this it follows (compare § 16) that if any five points in the plane be given, no three of which lie in the same straight line, there is one and but one conic which passes through these five points. The simplest method of finding its equation is that illustrated in the following example:

Example. Find the equation of the conic which passes through the five points $A(1, 0)$, $B(2, 1)$, $C(1, 2)$, $D(0, 1)$, $E(0, 0)$.

Select any four of the points, as A, B, C, D, and find the equations of two of the three pairs of straight lines which pass through these four points, as the pair AB, CD and the pair AC, BD.

The equation of the line AB is $x-y-1=0$.

The equation of the line CD is $x-y+1=0$.

Hence, the equation of the pair AB, CD is

$$(x-y-1)(x-y+1)=0. \qquad (1)$$

Similarly, the equation of the pair AC, BD is found to be

$$(x-1)(y-1)=0. \qquad (2)$$

Hence $$(x-y-1)(x-y+1)+\lambda(x-1)(y-1)=0 \qquad (3)$$

is the equation of a conic through A, B, C, D, whatever the value of λ may be. And the conic will pass through the fifth point $E(0, 0)$ if (3) be satisfied by $x=0$, $y=0$, that is, if $\lambda=1$.

Hence

$$(x-y-1)(x-y+1)+(x-1)(y-1)=0, \text{ or } x^2-xy+y^2-x-y=0,$$

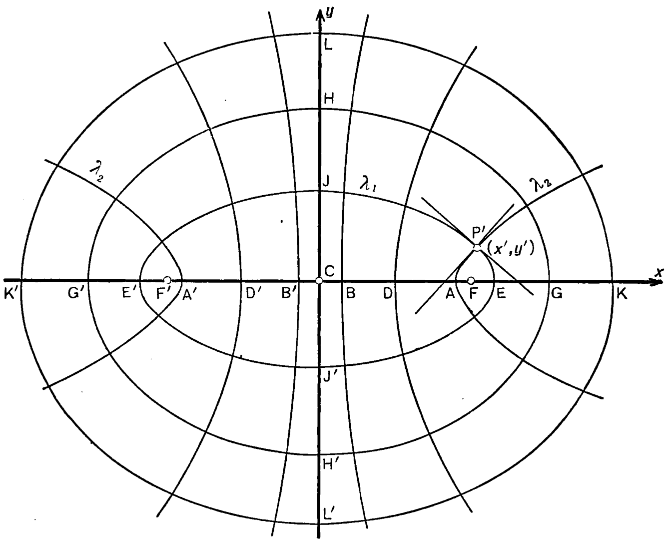
is the equation required.

166. Confocal conics. In the conic $x^2/a^2 + y^2/b^2 = 1$ (1), where $a > b$, the foci are on the x-axis and at the distance $ae = a\sqrt{1 - b^2/a^2} = \sqrt{a^2 - b^2}$ to the right and left of the origin. Let λ denote an arbitrary constant; then the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad (2)$$

will represent the system of conics which have the same foci as the conic (1); for in any conic (2) the distance from the center to a focus is $\{(a^2+\lambda)-(b^2+\lambda)\}^{\frac{1}{2}}$, that is, $\sqrt{a^2-b^2}$.

167. For all positive values of λ, and all negative values between 0 and $-b^2$, both $a^2+\lambda$ and $b^2+\lambda$ are positive, and (2) therefore represents ellipses; for all values of λ between $-b^2$ and $-a^2$, $a^2+\lambda$ is positive, and $b^2+\lambda$ is negative, and (2)



therefore represents hyperbolas; for all values of λ between $-a^2$ and $-\infty$, both $a^2+\lambda$ and $b^2+\lambda$ are negative, and the locus of (2) is therefore imaginary.

168. *Through every point* (x', y') *there pass two conics of the system* (2), *the one being an ellipse, the other an hyperbola.*

For substitute (x', y') for (x, y) in (2), and clear of fractions; the result is $(\lambda + a^2)(\lambda + b^2) - x'^2(\lambda + b^2) - y'^2(\lambda + a^2) = 0.$ (3)

This is a quadratic equation in λ with real roots, one lying between $+\infty$ and $-b^2$, the other between $-b^2$ and $-a^2$; for when $\lambda = \infty$, the left member of (3) is positive; when $\lambda = -b^2$, the left member of (3) becomes $-y'^2(-b^2 + a^2)$ which is negative; and when $\lambda = -a^2$, the left member of (3) becomes $-x'^2(-a^2 + b^2)$, which is positive [Alg. § 833].

These roots may be found by solving (3), or

$$\lambda^2 + (a^2 + b^2 - x'^2 - y'^2)\lambda + (a^2b^2 - b^2x'^2 - a^2y'^2) = 0, \qquad (3')$$

for λ; and if λ_1 denote the root between ∞ and $-b^2$, and λ_2 that between $-b^2$ and $-a^2$, the equations

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1 \quad (4) \qquad \text{and} \quad \frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1 \qquad (5)$$

will represent an ellipse and an hyperbola passing through the point (x', y').

169. *The two conics of the system* (2) *through any point* (x', y') *cut each other at right angles.*

For since the conics pass through the point (x', y') and are represented by the equations (4) and (5),

$$\frac{x'^2}{a^2 + \lambda_1} + \frac{y'^2}{b^2 + \lambda_1} \equiv 1 \qquad \text{and} \quad \frac{x'^2}{a^2 + \lambda_2} + \frac{y'^2}{b^2 + \lambda_2} \equiv 1,$$

and therefore (subtracting and simplifying),

$$\frac{x'^2}{(a^2 + \lambda_1)(a^2 + \lambda_2)} + \frac{y'^2}{(b^2 + \lambda_1)(b^2 + \lambda_2)} = 0. \qquad (6)$$

But (6) is the condition that the tangents at (x', y') to (4) and (5) meet at right angles; for the equations of these tangents

are $\quad \dfrac{xx'}{a^2 + \lambda_1} + \dfrac{yy'}{b^2 + \lambda_1} = 1, \quad (7) \qquad \dfrac{xx'}{a^2 + \lambda_2} + \dfrac{yy'}{b^2 + \lambda_2} = 1, \qquad (8)$

and the left member of (6) is the sum of the products of the coefficients of x and y in the left members of (7) and (8) [§ 30].

170. Exercises. Systems of conics and confocals.

1. Find the equations of the conics which pass through the following sets of points:

(1) $(0, 0),\ (1, 0),\ (2, 1),\ (1, 3),\ (-1, -4)$.

(2) $(1, 1),\ (3, 2),\ (0, 4),\ (-4, 0),\ (-2, -2)$.

2. Find the equation of the conic which passes through the points of intersection of the conics $4x^2 - y^2 + 3 = 0$ and $x^2 - 3xy + y^2 - 6x = 0$ and the point $(3, -2)$.

3. Find the equations of the conics which touch the x-axis, and which pass through the points of intersection of $x^2 + 2xy + 3y^2 + 18x + 5 = 0$ and $x^2 + xy - y^2 - 6x + y - 1 = 0$.

4. Find the equations of the two parabolas which pass through the points where $x^2 - 3xy + 4y^2 - x - 2 = 0$ cuts the x- and y-axes (that is $xy = 0$).

5. Find the equation of the conic which passes through the point $(1, 3)$ and touches the circle $x^2 + y^2 - 4 = 0$ in both points where it is cut by the line $y - 2x = 0$. (The equation is of the form $x^2+y^2-4+\lambda(y-2x)^2=0$.)

6. Find the equation of the conic which passes through the point $(1, -2)$ and touches the x- and y-axes where they are met by the line $x + 2y - 4 = 0$.

7. Find the equation of the conic which passes through the point $(5, 6)$ and touches the x- and y-axes at the points $(4, 0)$ and $(0, -2)$.

8. Prove that the centers of all conics of the system

$$\lambda xy + (lx + my - 1)(l'x + m'y - 1) = 0$$

lie on a conic, and find its equation.

9. Prove that it follows from Ex. 8 that the centers of all conics through four given points (no three of which are on the same straight line) lie on a conic.

10. Find the two conics of the confocal system $x^2/(3+\lambda) + y^2/(2+\lambda) = 1$ which pass through the point $(2, 1)$.

11. Prove that the equation of the hyperbola confocal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and meeting the ellipse at the point whose eccentric angle is ϕ is $x^2/\cos^2\phi - y^2/\sin^2\phi = a^2 - b^2$.

CHAPTER IX

TANGENTS AND POLARS OF THE CONIC

171. Equation of tangent to any conic. The equation of the tangent to a conic whose equation is given in the general form

$$f(x, y) \equiv ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0 \qquad (1)$$

may be found by the method used in § 79 B, § 102 B.

Let (x', y') and (x'', y'') denote two points on the conic, so that $f(x', y') \equiv 0$ and $f(x'', y'') \equiv 0$. To find the equation of the secant through (x', y') and (x'', y''), proceed as follows:

The terms of the second degree in $f(x, y)$ are the same as the terms of the second degree in the expression

$$a(x - x')(x - x'') + 2\,h(x - x')(y - y'') + b(y - y')(y - y'') \qquad (2)$$

which, like $f(x, y)$, vanishes when $x = x'$, $y = y'$, and when $x = x''$, $y = y''$.

Hence the equation formed by setting $f(x, y)$ equal to the expression (2), namely the equation,

$$a(x-x')(x-x'') + 2\,h(x-x')(y-y'') + b(y-y')(y-y'') = f(x, y),$$

will, when simplified, be of the first degree, and it will be satisfied when $x = x'$, $y = y'$, and when $x = x''$, $y = y''$. It will therefore be the equation of the secant through (x', y'), (x'', y'').

When the point (x'', y'') is moved along the curve into coincidence with (x', y'), the secant becomes the tangent at (x', y') and the equation just described becomes the equation of this tangent. Hence the equation of the tangent at (x', y') is

$$a(x - x')^2 + 2\,h(x - x')(y - y') + b(y - y')^2 = f(x, y),$$

or, $$2\,axx' + 2\,h(xy' + yx') + 2\,byy' + 2\,gx + 2\,fy + c = ax'^2 + 2\,hx'y' + by'^2.$$

If $2gx' + 2fy' + c$ be added to both members of this equation, the right member will vanish, since $f(x', y') \equiv 0$, and the equation, after dividing by 2, will become

$$axx' + h(xy' + yx') + byy' + g(x + x') + f(y + y') + c = 0. \quad (3)$$

Hence, to obtain the equation of the tangent at the point (x', y') from the equation of the curve, it is only necessary to replace x^2 and y^2 by xx' and yy', $2xy$ by $xy' + x'y$, and $2x$ and $2y$ by $x + x'$ and $y + y'$. (This is true for oblique axes also.)

Thus, the equation of the tangent at (x', y') to the curve

$$2x^2 - 5xy + y^2 + 4x - 3y + 7 = 0$$

is $$2xx' - \tfrac{5}{2}(xy' + x'y) + yy' + 2(x + x') - \tfrac{3}{2}(y + y') + 7 = 0.$$

172. Poles and polars. The equation (3) of § 171 represents a tangent to the conic (1) only when the point (x', y') is on the conic. But, whether the point lies on the conic or not, the equation represents a definite straight line. This line is called the *polar* of the point (x', y') with respect to the conic (1), and (x', y') is called the *pole* of the line.

From the symmetry of the equation (3) with respect to x, y on the one hand, and x', y' on the other hand, and the fact that (3) represents the tangent at (x', y') when (x', y') is on the curve, it is not difficult to infer the geometric relation between any point (x', y') not on the curve and its polar (3). Only the case in which the curve is the circle $x^2 + y^2 = r^2$ will be considered here, but the reasoning will be general and will apply to any conic. (The axes may be rectangular or oblique.)

173. *If the polar of the point $P_1(x_1, y_1)$ passes through the point $P_2(x_2, y_2)$, then will the polar of P_2 pass through P_1.*

For, the equations of the polars of P_1 and P_2 are

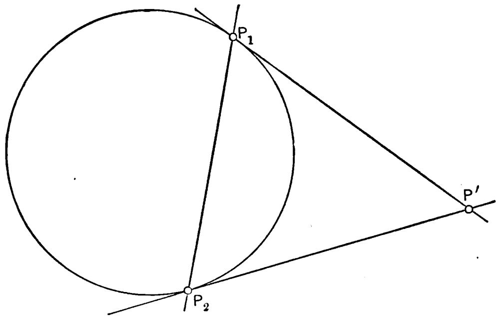
$$xx_1 + yy_1 = r^2 \quad (1) \qquad \text{and} \qquad xx_2 + yy_2 = r^2. \quad (2)$$

But since $P_2(x_2, y_2)$ lies on (1), $\quad x_2x_1 + y_2y_1 \equiv r^2$,
which may also be written $\quad x_1x_2 + y_1y_2 \equiv r^2$,
and therefore states that $P_1(x_1, y_1)$ lies on (2).

174. *If the polars of two points P_1 and P_2 meet at P, then P is the pole of the line P_1P_2.*

For, since P lies on the polars of both P_1 and P_2, its polar must pass through both P_1 and P_2 and must therefore be the line P_1P_2.

175. *To find the pole of a line which cuts the circle in two real points.*

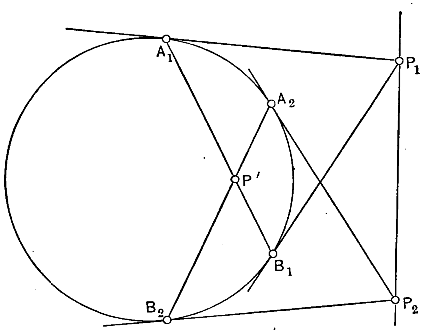


Let the given line meet the circle in the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

By the preceding theorem [§ 174], the pole of P_1P_2 is the point of intersection of the polars of P_1 and P_2.

But since P_1 and P_2 are on the circle, their polars are the tangents at P_1 and P_2. Hence the point P', where these tangents meet, is the pole of the given line P_1P_2.

176. *To find the pole of a line which lies wholly without the circle.*



Take any two points P_1 and P_2 on the given line and from these points draw P_1A_1 and P_1B_1, P_2A_2 and P_2B_2, to touch the circle at A_1 and B_1, A_2 and B_2.

Join A_1B_1 and A_2B_2. Then [§ 175] A_1B_1 is the polar of P_1, and A_2B_2 is the polar of P_2. Hence [§ 174] the point P' where these lines meet is the pole of the given line P_1P_2.

177. Therefore, the following theorems have been proved:

If the point $P'(x', y')$ lies without the circle $x^2 + y^2 = r^2$, its polar $xx' + yy' = r^2$ is the line joining the points of contact of the tangents from P' to the circle.

If the point $P'(x', y')$ lies within the circle $x^2 + y^2 = r^2$, its polar $xx' + yy' = r^2$ is the locus of the point of intersection of the tangents at the extremities of every chord of the circle which passes through P'.

And, since the reasoning is general, these theorems hold good for any conic. [§ 172.]

Example 1. Find the polar of the point (2, 3) with respect to the conic $2x^2 + y^2 - 4x + 3 = 0$.

Substituting $x' = 2$, $y' = 3$ in the equation of the polar to this conic, namely $2xx' + yy' - 2(x + x') + 3 = 0$, gives $4x + 3y - 2(x + 2) + 3 = 0$, or $2x + 3y - 1 = 0$, the polar required.

Example 2. Find the pole of the line $3x - y + 4 = 0$ with respect to the conic $2xy + 3y^2 - 8x = 0$.

The equation of the polar of the point (x', y') with respect to this conic is

$$xy' + yx' + 3yy' - 4(x + x') = 0, \text{ or } (y' - 4)x + (x' + 3y')y - 4x' = 0.$$

If this equation is to represent the same line as $3x - y + 4 = 0$, the corresponding coefficients in the two equations must be proportional [§ 12], that is,

$$\frac{y' - 4}{3} = \frac{x' + 3y'}{-1} = \frac{-4x'}{4},$$

whence $x' = 4/3$ and $y' = 0$, the pole required.

Example 3. Suppose a conic given, and let O denote any point not on this conic. Through O take any two lines meeting the conic at A_1, A_2 and B_1, B_2, respectively. Let A_1B_1 and A_2B_2 meet at P, and let A_1B_2 and A_2B_1 meet at Q. Prove that the line PQ is the polar of the point O.

Take O as origin, OA_1A_2 as x-axis, OB_1B_2 as y-axis, and let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1) be the equation of the conic referred to these axes. Then OA_1, OA_2 are the intercepts which the conic makes on the x-axis; represent them by a_1, a_2; they are the roots of the equation $ax^2 + 2gx + c = 0$ (2) got by setting $y = 0$ in (1); hence

$a_1 + a_2 = -2\,g/a$, $a_1 a_2 = c/a$, and therefore $1/a_1 + 1/a_2 = -2\,g/c$ (3). Similarly, it can be proved that if $OB_1 = b_1$ and $OB_2 = b_2$, then $1/b_1 + 1/b_2 = -2\,f/c$ (4).

The equation of the polar of $O(0, 0)$ with respect to (1) is $gx+fy+c=0$, or, by (3), (4), $x(1/a_1 + 1/a_2) + y(1/b_1 + 1/b_2) = 2$ (6). But the equations of A_1B_1, A_2B_2 are $x/a_1 + y/b_1 = 1$ (7), $x/a_2 + y/b_2 = 1$ (8), and (6) is the sum of (7) and (8); hence [§ 39] the point P where A_1B_1, A_2B_2 meet is on (6). Similarly, it can be proved that the point Q where A_1B_2, A_2B_1 meet is on (6). Hence PQ is the line (6), that is, the polar of O.

This construction affords a solution of the problem of drawing the tangent to a conic from a point O without it.

178. Exercises. Tangents and polars to a conic.

1. Find the following polars:

(1) of (2, 3) with respect to $3\,x^2 + 2\,xy - y + 5 = 0$.
(2) of (0, 0) with respect to $x^2 + 3\,y^2 - 2\,x + 4\,y - 6 = 0$.
(3) of (x', y') with respect to $(x - \alpha)^2 + (y - \beta)^2 = r^2$.

2. Find the following poles:

(1) of $2\,x - 3\,y + 3 = 0$ with respect to $4\,x^2 - y^2 + 2\,xy - 3 = 0$.
(2) of $x - 2\,y = 7$ with respect to $xy = 10$.
(3) of $x + 2\,y + 3 = 0$ with respect to $x^2 + y^2 - 2\,y = 0$.

3. Prove that the polar of a focus of a conic is the corresponding directrix.

4. Find the point of intersection of the tangents to the conic $x^2 - 3\,y^2 + 4\,x - 2 = 0$ at the points where it is cut by the line $x - 2\,y + 5 = 0$.

5. If the polar of (x', y') with respect to the circle $x^2 + y^2 = a^2$ touches the circle $x^2 + y^2 - 2\,ax = 0$, prove that $y'^2 + 2\,ax' = a^2$.

6. The polar of any point on the circle $x^2 + y^2 - 2\,ax = 3\,a^2$ with respect to the circle $x^2 + y^2 + 2\,ax = 3\,a^2$ will touch the parabola $y^2 + 4\,ax = 0$.

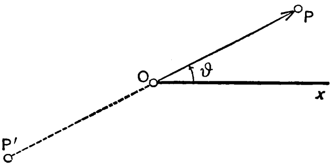
7. Prove that if the polars of P with respect to the circle $x^2 + y^2 = a^2$ and the hyperbola $2\,xy = b^2$ meet at right angles, P lies on one of the axes of reference.

8. The mid-points of a certain system of chords of a parabola lie on a fixed line perpendicular to the axis. Prove that the poles of all these chords lie on another parabola.

CHAPTER X

POLAR COORDINATES

179. Polar coordinates. The position of a point in a plane can be defined in other ways than by reference to a pair of lines as axes. The following method is often useful.



Let O be a given point, called the *pole* or *origin*, and Ox a given directed line from O, called the *polar axis.* The *polar coordinates* of any point P, referred to O and Ox, are r, the length of OP, and θ, the measure of the angle xOP; and r is called the *radius vector* of P, and θ its *vectorial angle.*

To construct a point whose polar coordinates r, θ are given, draw from O a half line making the angle θ with Ox, and then on this half line itself or produced through O, according as r is positive or negative, lay off OP or OP' of length $|r|$.

Observe that the polar coordinates of the point $P(r, \theta)$ may also be written $(r, -2\pi+\theta)$, $(-r, \pi+\theta)$, $(-r, -\pi+\theta)$.

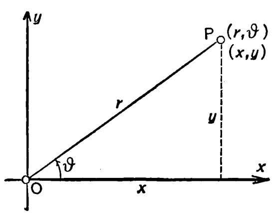
180. If the polar axis Ox be taken as the x-axis of a rectangular system, and Oy as the corresponding y-axis, the relations connecting the coordinates of any point P, referred to the two systems, are

$$\left.\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}\right\} \qquad (1)$$

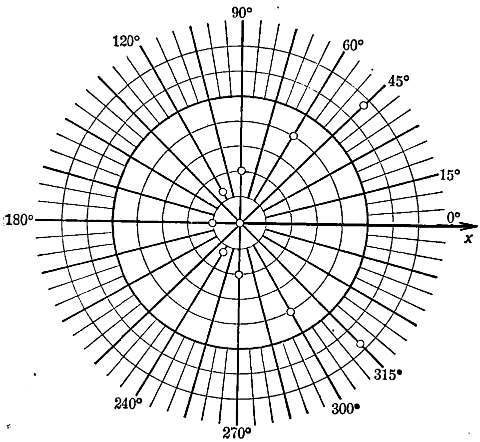
$$\left.\begin{aligned} r^2 &= x^2+y^2, \\ \tan\theta &= y/x. \end{aligned}\right\} \qquad (2)$$

$$\sin\theta = y/\sqrt{x^2+y^2}. \qquad (3)$$

$$\cos\theta = x/\sqrt{x^2+y^2}. \qquad (4)$$



181. Graphs in polar coordinates. The *graphs of points* given in polar coordinates are obtained by taking the length r on the terminal line of the angle θ, these lengths being measured on the terminal line itself or on this line produced through the origin according as r is positive or negative. It is often convenient to use paper prepared for the purpose, as in the figure, where the graphs are indicated of (6.828, 45°), (4, 60°), (2, 90°), (4/3, 120°), (1, 180°), (4/3, 240°), (2, 270°), (4, 300°), (6.828, 315°). Observe that the point (2, 90°) is the same as (−2, 270°); and so on.



182. *The graph of an equation* in r and θ is the collection of the graphs of all the solutions of the equation.

For example, the coordinates of the nine points in the previous section are solutions of the equation $r = 2/(1 - \cos\theta)$; and by giving other values to θ and obtaining the corresponding values of r, as many other points on the graph of the equation may be found as are desired. For this purpose it is more convenient to take the equation in the form $r = 1/\sin^2(\theta/2)$. [$\cos\theta = 1 - 2\sin^2(\theta/2)$.] Thus, if $\theta \doteq 75°$, $r = 2.698$; if $\theta = 285°$, $r = 2.698$; if $\theta = 105°$ or $255°$, $r = 1.589$; if $\theta = 135°$ or $225°$, $r = 1.174$; and so on.

The equation of this graph may be obtained in rectangular coordinates by the substitution [§ 180]: $r = \sqrt{x^2 + y^2}$, $\cos\theta = x/\sqrt{x^2 + y^2}$, which changes $r = 2/(1 - \cos\theta)$ into

$\sqrt{x^2+y^2}(1-x/\sqrt{x^2+y^2})=2$, or $\sqrt{x^2+y^2}-x=2$; or, transposing and squaring, $x^2+y^2=x^2+4x+4$, or $y^2=4(x+1)$; which represents a parabola with the focus at the origin. [§ 87, 11.]

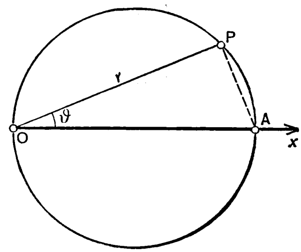
183. Exercises. Graphs in polar coordinates.

1. Indicate the graphs of the following points: $(8, -15°)$, $(4\sqrt{2}, 0°)$, $(8/\sqrt{3}, 15°)$, $(4, 45°)$, $(8/\sqrt{3}, 75°)$, $(4\sqrt{2}, 90°)$, $(8, 105°)$.

2. Obtain several points of the graph of $r = 4/\cos(\theta - 45°)$.

3. By the substitution $x = r\cos\theta$, $y = r\sin\theta$, change the equation of the straight line $x\cos\alpha + y\sin\alpha - p = 0$ to the form $r = p/\cos(\theta - \alpha)$.

184. Graphs of $r = a\cos\theta$, $r = a\cos 2\theta$, $r = a\cos 3\theta$. These graphs can be plotted by finding conveniently chosen solutions of the equations.



185. Graph of $r = a\cos\theta$.

If $\theta=0$, $r=a$, and therefore $(a, 0)$ is on the locus. Call this point A. Then the equation is equivalent to $OP/OA=\cos\theta$; whence, if P and A be joined, the angle OPA is a right angle, and P is on a circle with the diameter a coinciding with the initial line.

186. Graph of $r = a\cos 2\theta$.

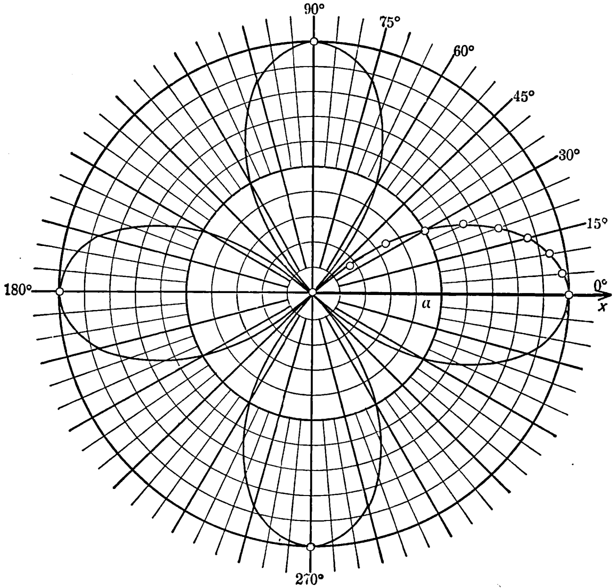
If $r=0$, $\cos 2\theta = 0$, whence $2\theta = 90°, 270°, 450°, 630°, \cdots$, that is, $\theta = 45°, 135°, 225°, 315°, \cdots$.

If $r = a$, $\cos 2\theta = 1$, whence $2\theta = 0, 360°, 720° \cdots$, that is, $\theta = 0, 180°, 360°, \cdots$.

If $r = -a$, $\cos 2\theta \doteq -1$, whence $2\theta = 180°, 540°, \cdots$, that is, $\theta = 90°, 270°, \cdots$.

Arranging these solutions in order of increasing values of θ, the following points are on the locus: $(a, 0)$, $(0, 45°)$, $(-a, 90°)$, $(0, 135°)$, $(a, 180°)$, $(0, 225°)$, $(-a, 270°)$, $(0, 315°)$, $(a, 360°)$, $\cdots$.

From a table of natural cosines [Table E], by taking $\theta = 0$, $5°, 10°, 15°, \cdots$, and computing r to two places of decimals, the following also are found to be points of the locus: $(a, 0)$, $(0.98\,a, 5°)$, $(0.94\,a, 10°)$, $(0.87\,a, 15°)$, $(0.77\,a, 20°)$, $(0.64\,a, 25°)$, $(0.5\,a, 30°)$, $(0.34\,a, 35°)$, $(0, 17\,a, 40°)$, $(0, 45°)$, and the r

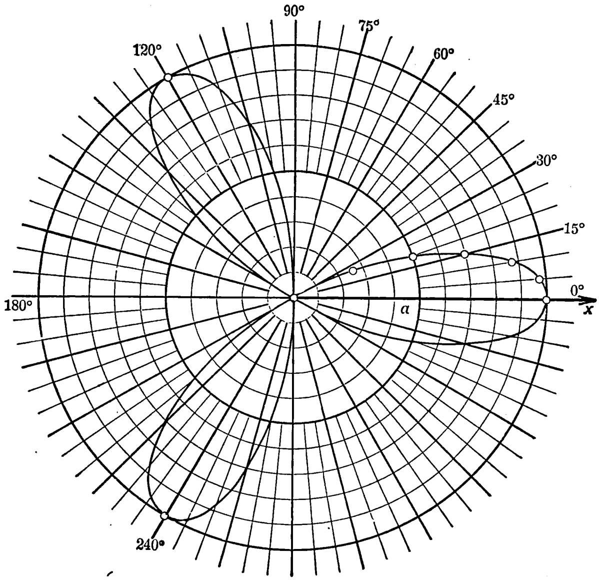


diminishes continuously through this set of solutions; which gives the upper part of the right portion of the figure.

Moreover, since $\cos 2(-\theta) = \cos 2\theta$, the curve is symmetric as to the initial line. Again, since $\cos 2\theta = -\cos 2(270° \pm \theta) = \cos 2(180° \pm \theta) = -\cos 2(90° \pm \theta)$, any arc of the curve is repeated when rotated about the pole through one, two, or three right angles. Hence, the curve is composed of *four lobes*, as indicated in the figure. When θ increases from 0° to 45°, the upper part of the right lobe is generated to the left; when θ increases from 45° to 90°, the left half of the lower lobe is generated downward, r being negative and therefore to be produced through the origin; when θ increases from 90° to 135°, the right half of the lower lobe is generated upward, and so on.

187. Graph of $r = a \cos 3\theta$.

If $\theta = 0°, 120°, 240°, \cdots, r = a$; if $\theta = 30°, 90°, 150°, \cdots, r = 0$; if $\theta = 60°, 180°, 300°, \cdots, r = -a$.



Arranging these solutions in order of increasing values of θ, the following points are on the locus: $(a, 0)$, $(0, 30°)$, $(-a, 60°)$ or $(a, 240°)$, $(0, 90°)$, $(a, 120°)$, $(0, 150°)$, $(-a, 180°)$ or $(a, 360°)$, $\cdots$.

From a table of natural cosines [Table E], by taking $\theta = 0$, $5°, 10°, 15°, \cdots$, the following also (to two places of decimals) are found to be points of the locus: $(a, 0)$, $(0.97\,a, 5°)$, $(0.87\,a, 10°)$, $(0.71\,a, 15°)$, $(0.5\,a, 20°)$, $(0.26\,a, 25°)$, $(0, 30°)$, $(-0.26\,a, 35°)$, $(-0.5\,a, 40°)$, $(-0.71\,a, 45°)$, $(-0.87\,a, 50°)$, $(-0.97\,a, 55°)$, $(-a, 60°)$, $(-0.97\,a, 65°)$, $\cdots$.

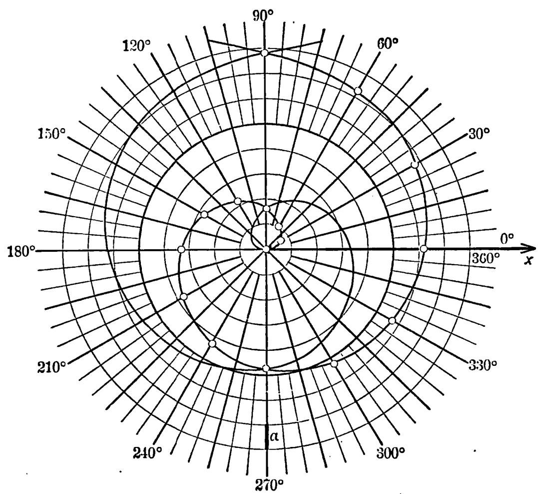
Since $\cos 3(-\theta) = \cos 3\theta$, the curve is symmetric as to the initial line. Also, $\cos 3\theta = \cos 3(\theta + 120°) = \cos 3(\theta + 240°)$, hence any arc of the curve is repeated when rotated as to the pole through the angle 120° or 240°. Therefore, the curve is composed of three lobes, as indicated in the figure.

188. The spirals. The graphs of the equations $r = a\theta$, $r = e^{a\theta}$, $r = a/\theta$, $r^2 = a^2/\theta$ belong to a class of curves called *spirals.* They can be plotted by obtaining conveniently chosen solutions of each equation.

189. The graph of $r = a\theta$ is called the *spiral of Archimedes.* The angle θ must be expressed in circular measure. From a table of arc lengths [Table E], to two places of decimals,

RADIANS

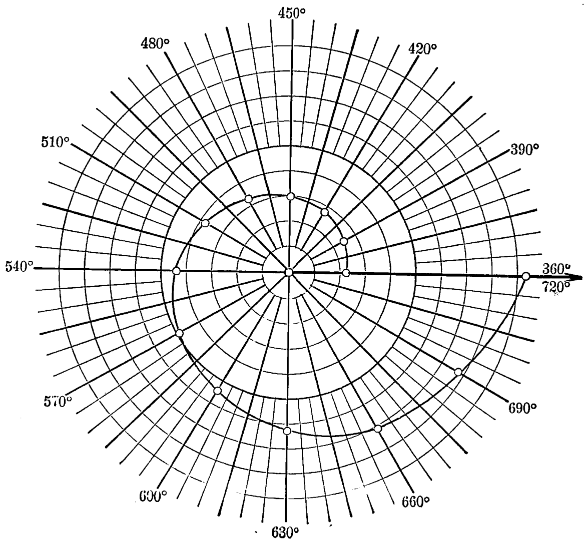
$0° = 0$
$30° \doteq 0.52$
$60° = 1.05$
$90° = 1.57$
$120° = 2.09$
$150° = 2.62$
$180° = 3.14$
$210° = 3.67$
$240° = 4.19$
$270° = 4.71$
$300° = 5.24$
$330° = 5.76$
$360° = 6.28$
$390° = 6.81$
$420° = 7.33$
$450° = 7.85$



These values give points on the graph of the right-hand spiral, as indicated in the figure. The corresponding negative values give the points on the left-hand spiral; for when θ is negative, r also is negative, and must therefore be produced through the origin [§ 179].

190. The graph of $r = e^{a\theta}$ or $\log_e r = a\theta$ is called the *logarithmic* or *equiangular spiral.*

Let θ be given in circular measure. Set $a = bc$, and let b be so taken that $e^b = 10$; then the equation is $r = 10^{c\theta}$ or $\log_{10} r = c\theta$, which is adapted to easy calculation with an ordinary table of



logarithms and arc lengths. For convenience take $c = 0.1$, then, from the tables D and E, since $r =$ antilogarithm of $c\theta$,

$c\theta$	r	$c\theta$	r	$c\theta$	r
360° = .628,	4.2	510° = .890,	7.8	660° = 1.152,	14.2
390° = .681,	4.8	540° = .942,	8.8	690° = 1.204,	16.0
420° = .733,	5.4	570° = .995,	9.9	720° = 1.257,	18.1
450° = .785,	6.1	600° = 1.047,	11.1	735° = 1.283,	19.2
480° = .838,	6.9	630° = 1.100,	12.6	. . . . .	. . .

These values give points on the arc of the spiral from a radius vector, $\theta = 360°$, around to the *same* radius vector, $\theta = 720°$.

When θ varies from $360°$ to 0, r decreases from 4.2 to 1 and the point describes an arc which starts at the initial point (4.2, 360°) of the arc in the figure and ends at the point (1, 0).

When θ runs through the negative values from 0 to $-\infty$, it follows from the equation $r = e^{a\theta}$ that r decreases from 1 to 0, the radius vector in the meantime turning clockwise an infinite number of times about the origin, which the point is said to approach asymptotically.

191. The graph of the equation $r = a/\theta$ is called the *hyperbolic spiral.*

192. The graph of the equation $r^2 = a^2/\theta$ is called the *lituus.*

193. Exercises. Graphs in polar coordinates.

1. What is the graph of $r = \text{const}$?

2. What is the graph of $\theta = \text{const}$?

3. What is the graph of $\theta = 0$?

4. By the substitution $x = r \cdot \cos\theta$, $y = r \cdot \sin\theta$, change the formula of § 41 for the distance between two points, $P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, to the formula $P_1P_2^2 = r_2^2 + r_1^2 - 2\, r_1 r_2 \cos(\theta_2 - \theta_1)$.

5. By the substitution $x = r \cdot \cos\theta$, $y = r \cdot \sin\theta$, change the equation of the central conic to the form $r^2 = a^2b^2/(b^2\cos^2\theta \pm a^2\sin^2\theta)$.

6. Find the graph of $r = a\sin\theta$.

7. Find the graph of $r = a\sin 2\,\theta$.

8. Find the graph of $r = a\sin 3\,\theta$.

9. Find the graph of $r/a = \sin^3(\theta/3)$.

10. Find the graph of $r/a = \sec 2\,\theta$.

11. Find the graph of $r/a = \cos\theta - \sin\theta$.

12. Find the graph of $r/a = \sec 2\,\theta + \tan 2\,\theta$.

13. Find the graph of $r\theta = a$.

14. Find the graph of $r^2\theta = a^2$.

15. Find the graph of $r^2 = a^2\sin\theta$.

194. In the preceding pages of this chapter the graphs of certain equations in polar coordinates have been plotted. The reciprocal problem of obtaining the *equation in polar coordinates* of the locus of a point satisfying a given condition will now be illustrated. [Compare § 67.]

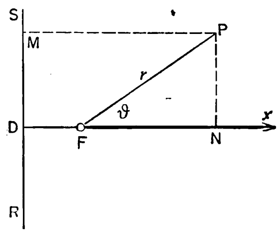
195. Polar equation of a conic. The polar equation of a conic referred to the focus as pole, and the perpendicular from the directrix through the pole as polar axis is

$$r = \frac{p}{1 - e\cos\theta},$$

where p is half the latus rectum, and e is the eccentricity.

Let F be the focus and SR the directrix, and P a representative point of the locus. Let M be the foot of the perpendicular from P to the directrix, N the foot of the perpendicular from P to the polar axis, and D the point of intersection of the directrix and the polar axis. Then, by definition [§ 69], $FP = e \cdot MP$. Or

$$\begin{aligned} r &= e \cdot MP \\ &= e \cdot DN \\ &= e \cdot DF + e \cdot FN \\ &= e \cdot DF + e \cdot r\cos\theta. \end{aligned}$$



Let $e \cdot DF$ be represented by p, then

$$r = p + er\cos\theta,$$

or $$r(1 - e\cos\theta) = p,$$

or finally, $$r = \frac{p}{1 - e\cos\theta}.$$

When $\theta = 90°$, $r = p$; hence, p is half the latus rectum [§§ 73, 93, 130]. [Compare the equation of § 182, where $p = 2$, and $e = 1$.]

CHAPTER XI

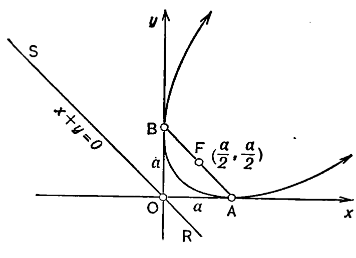
EQUATIONS AND GRAPHS OF CERTAIN CURVES

196. The parabola. The equation of the parabola referred to the tangents at the extremities of its latus rectum as axes of coordinates is

$$x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}},$$

where a denotes the distance from the origin to each point of tangency.

For the tangents at the extremities of the latus rectum (AB) meet at right angles at the point of intersection of the axis and directrix of the parabola. Hence, if these tangents be taken as axes of reference, and $a = OA = OB$, the equation of the directrix is $x + y = 0$, and the coordinates of the focus are $(a/2, a/2)$.



But, by the definition of the parabola, the equation of the parabola whose directrix is $x + y = 0$, and whose focus is $(a/2, a/2)$, is

$$\frac{(x+y)^2}{2} = \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2,$$

and this equation can be reduced to the form

$$(x+y)^2 - 2\,a(x+y) + a^2 = 4\,xy,$$

or $\quad x + y - a = \mp 2\,x^{\frac{1}{2}}y^{\frac{1}{2}}, \quad$ or $\quad x \pm 2\,x^{\frac{1}{2}}y^{\frac{1}{2}} + y = a,$

or finally, $\quad x^{\frac{1}{2}} \pm y^{\frac{1}{2}} = \pm a^{\frac{1}{2}},$

where the four combinations of the $\pm$ signs are related to the graph as follows:

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is true for the arc between A and B,

$x^{\frac{1}{2}} - y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is true for the arc beyond A,

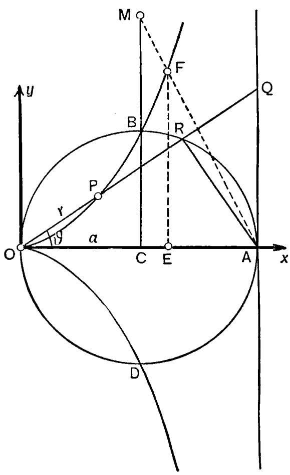
$x^{\frac{1}{2}} - y^{\frac{1}{2}} = -a^{\frac{1}{2}}$ is true for the arc beyond B,

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = -a^{\frac{1}{2}}$ has no real points.

197. The cissoid. A circle of radius a passes through the origin O and has its diameter OCA on the polar axis. Through O, any chord OR is taken and produced to meet, at Q, the tangent to the circle at A. On the line OR the point P is then taken such that $PQ = OR$. The locus of P is a curve called the *cissoid*.

Its equation may be found as follows:

Let AOP be θ and OP be r.



Then

$$OA/OQ = \cos\theta, \text{ or } OQ = 2\,a/\cos\theta,$$

and

$$OR/OA = \cos\theta, \text{ or } OR = 2\,a\cos\theta.$$

Therefore,

$$\begin{aligned} r &= OP \\ &= OQ - PQ \\ &= OQ - OR \\ &= 2\,a/\cos\theta - 2\,a\cos\theta \\ &= 2\,a(1-\cos^2\theta)/\cos\theta, \end{aligned}$$

or

$$r = 2\,a\frac{\sin^2\theta}{\cos\theta}, \qquad (1)$$

which is the equation required.

By the substitution,

$$r\cos\theta = x, \quad \sin^2\theta = y^2/(x^2+y^2),$$

the equation becomes in rectangular coordinates:

$$x = 2\,ay^2/(x^2 + y^2), \quad \text{or} \quad x^3 + xy^2 = 2\,ay^2, \quad \text{or} \quad y^2(2\,a - x) = x^3,$$

or, finally,
$$y^2 = \frac{x^3}{2\,a - x}. \tag{2}$$

From the definition, or from equation (1) or (2), it follows that the curve has the form indicated in the figure. It is symmetric with respect to the x-axis (since (2) involves no odd powers of y); it lies between the lines $x = 0$ and $x = 2\,a$ (since y^2 would be negative for $x < 0$ or $> 2\,a$); and the line $x = 2\,a$ is an asymptote (y^2 being ∞ when $x = 2\,a$). At the origin it has a peculiar sharp point called a *cusp*. It was called the cissoid from the fancied resemblance to an ivy leaf of the figure bounded by the semicircle BAD and the portion of the cissoid $DOPB$. (The Greek word κίσσος = ivy.)

198. Note. The cissoid was used to solve the problem of the duplication of the cube, that is, of finding the edge of a cube whose volume is twice that of a given cube, one of the famous problems of antiquity.

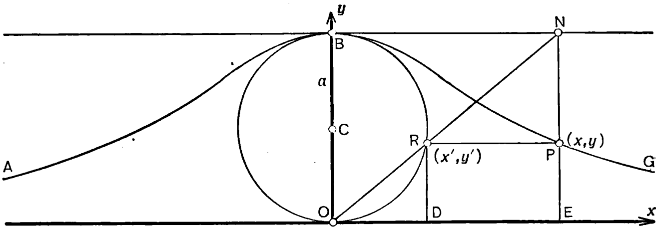
Let M be taken on CB so that $CM = 2\,CB$, let the line AM cut the cissoid in F, and let OE and EF be the coordinates of F.

Then, from the similarity of the triangles EFA and CMA, $EF/EA = CM/CA$; therefore, since $CM = 2\,CB = 2\,CA$, it follows that $EF = 2\,EA$.

From the equation of the curve $y^2(2\,a - x) = x^3$, it follows that $EF^2 \cdot EA = OE^3$, or since $EA = \frac{1}{2}\,EF$, it follows that $\frac{1}{2}\,EF^3 = OE^3$; or, finally, that $EF^3 = 2\,OE^3$. Hence, if OE be the edge of the given cube, EF is that of a cube of twice the volume.

In the same way, if CM be taken as the nth multiple of CB, the construction gives the solution of the problem of finding the nth multiple of a given cube.

199. The witch of Agnesi. A circle of radius a touches the x-axis at the origin and cuts the y-axis at O and B. Through O, any chord OR is taken and produced to meet, at N, the tangent to the circle at B. Through R a line is taken parallel to the x-axis, and through N a line is taken parallel to the y-axis. The locus of the point P, where these lines meet, is called the *witch*. Its equation may be found as follows:



Let the coordinates of P be (x, y), and let those of R be (x', y'). The point $R(x', y')$ is on the given circle and therefore [§ 56],

$$x'^2 + y'^2 - 2\,ay' = 0. \tag{1}$$

Let NP meet Ox at E, and take RD perpendicular to Ox. Then $DR = EP$, and therefore,

$$y' = y. \tag{2}$$

Again, $OD/DR = RP/PN$, or $x'/y' = (x - x')/(2\,a - y')$,

therefore,
$$x' = \frac{xy'}{2\,a} = \frac{xy}{2\,a} \tag{3}$$

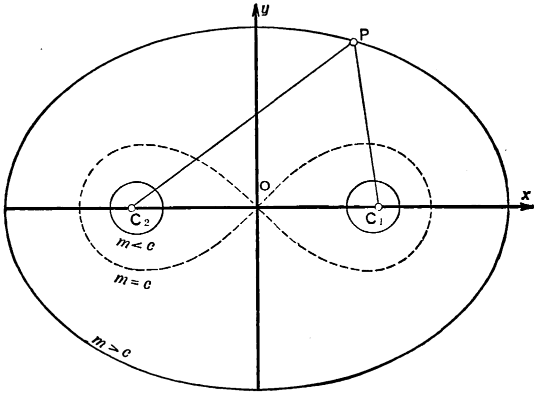
Substitute (2) and (3) in (1) and simplify; discarding the solution $y = 0$, the result is

$$y(x^2 + 4\,a^2) = 8\,a^3, \quad \text{or} \quad y = \frac{8\,a^3}{x^2 + 4\,a^2}, \tag{4}$$

which is the equation required.

The curve has the form indicated in the figure. It is symmetric with respect to the y-axis (since (4) involves no odd powers of x); it lies between the lines $y = 0$ and $y = 2\,a$; and the line $y = 0$ is an asymptote.

200. Cassini's oval. The locus of a point P, the product of whose distances from two fixed points is a constant, is called *Cassini's oval.*



To obtain its equation, let $C_1(c, 0)$ and $C_2(-c, 0)$ be the two fixed points, and let $PC_1^2 \cdot PC_2^2 = m^4$. The formula of § 41 gives, after reduction, for the locus of P the equation:

$$(x^2 + y^2 + c^2)^2 - 4c^2x^2 = m^4. \qquad (1)$$

And this equation by the transformation,

$$x = r\cos\theta, \ y = r\sin\theta,$$

becomes in polar coordinates:

$$r^2 = c^2\cos 2\theta + \sqrt{m^4 - c^4\sin^2 2\theta}. \qquad (2)$$

201. The lemniscate. When $m = c$, Cassini's oval is called the *lemniscate.* Its equation is

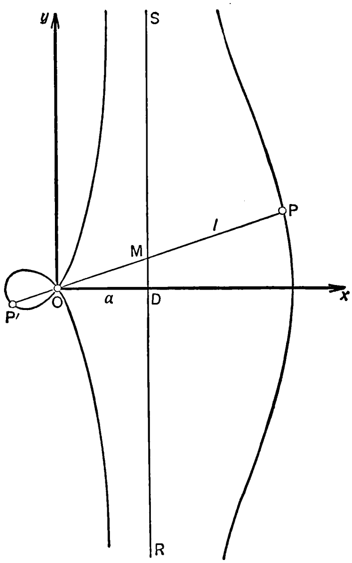
$$(x^2 + y^2)^2 + 2c^2(y^2 - x^2) = 0, \qquad (1)$$

or

$$r^2 = 2c^2\cos 2\theta. \qquad (2)$$

202. The conchoid. Let SR be a fixed line perpendicular to the polar axis and meeting it in D; let a radius vector, OM, meet SR in M; let a fixed length $MP = -MP' = l$ be added to and subtracted from OM; the locus of P and P' is called the *conchoid.*

Let OP be r, xOP be θ, and OD be a; then $OM = a/\cos\theta$, and $r = OP = OM + MP$, or $r = a/\cos\theta + l$.



In the same way, the locus of P' is $r = OP' = OM + MP'$, or $r = a/\cos\theta - l$.

Therefore the equation of the conchoid is

$$r = a/\cos\theta \pm l. \quad (1)$$

By the substitution $r = \sqrt{x^2 + y^2}$, $\cos\theta = x/\sqrt{x^2 + y^2}$, the equation in rectangular coordinates is obtained, namely, $\sqrt{x^2+y^2} = a\sqrt{x^2+y^2}/x \pm l$; or, simplifying,

$$(x^2 + y^2)(x - a)^2 = l^2x^2. \quad (2)$$

The curve is symmetric with respect to the polar axis; and SR is an asymptote to both branches. When $l > a$, the curve has the form indicated in the figure.

203. The limaçon. Let ODM be a circle, with the diameter OD coinciding with the polar axis; let a radius vector, OM, meet the circle in M, and let a fixed length $MP = -MP' = l$ be added to and subtracted from OM; the locus of P and P' is called the *limaçon.*

Let OP be r, xOP be θ, and OD be $2a$; then $OM = 2a\cos\theta$, and $r = OP = OM + MP = 2a\cos\theta + l$.

In the same way, for P'

$$r = OP' = OM + MP' = 2\,a\cos\theta - l.$$

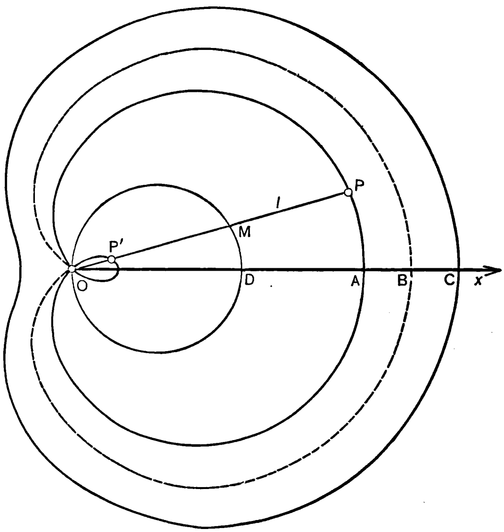
Therefore the equation of the limaçon is

$$r = 2\,a\cos\theta \pm l. \tag{1}$$

By the substitution $r = \sqrt{x^2+y^2}$, $\cos\theta = x/\sqrt{x^2+y^2}$ the equation in rectangular coordinates is obtained, namely, $\sqrt{x^2+y^2} = 2\,ax/\sqrt{x^2+y^2} \pm l$; or, simplifying,

$$(x^2+y^2-2\,ax)^2 = l^2(x^2+y^2). \tag{2}$$

The locus is a closed curve symmetric with respect to the polar axis. When $l < 2\,a$, the curve has an internal loop, and has the form indicated by the solid line through A in the figure. When $l > 2\,a$ the curve has the form indicated by the outside solid line through C in the figure.



204. The cardioid. The limaçon, for which $l = 2\,a$, is called the *cardioid*. Its equations are

$$r = 2\,a(\cos\theta \pm 1), \tag{1}$$

and

$$(x^2+y^2-2\,ax)^2 = 4\,a^2(x^2+y^2). \tag{2}$$

The cardioid has the form indicated by the dotted line in the figure.

205. Parametric equations of a curve. Sometimes the most convenient method of representing a curve analytically is by a pair of equations of the form

$$x = \phi(t), \quad (1) \qquad y = \psi(t), \quad (2)$$

where t denotes a variable called a *parameter*.

By assigning a series of values to t, reckoning out the corresponding values of (x, y), and plotting their graphs, it is possible to obtain any number of points on the curve, and therefore a figure which will represent the curve with any degree of accuracy that may be required.

By eliminating t between (1) and (2), an equation of the form $f(x, y) = 0$ is obtained. This will be the equation of the curve in rectangular (or oblique) coordinates.

Thus, $x = t^2$, $y = 2t$ are parametric equations of the parabola $y^2 = 4x$; for $y^2 = 4x$ follows from $x = t^2$, $y = 2t$ by eliminating t.

Assigning values to t and computing the corresponding values of (x, y) as given by $x = t^2$, $y = 2t$:

$t = \cdots$	$-3,$	$-2,$	$-1,$	$-\frac{1}{2},$	$0,$	$\frac{1}{2},$	$1,$	$2,$	$3, \cdots$
$x = \cdots$	$9,$	$4,$	$1,$	$\frac{1}{4},$	$0,$	$\frac{1}{4},$	$1,$	$4,$	$9, \cdots$
$y = \cdots$	$-6,$	$-4,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$4,$	$6, \cdots.$

Plot the points $\cdots (1, -2), (1/4, -1), (0, 0), (1/4, 1), (1, 2), \cdots$ thus determined, and as many more as may be desired, and through them draw a smooth curve. This curve will represent the parabola $y^2 = 4x$ (see the figure, § 71). Observe that as t varies from $-\infty$ to ∞ the corresponding point P will trace out the entire curve, coming in from ∞ on the lower half, and going out to ∞ on the upper half.

The parametric equations

$$x = x_0 + at, \quad y = y_0 + bt$$

represent the *straight line* which passes through the point (x_0, y_0) and has the slope b/a; for the equation obtained by eliminating t is

$$(x - x_0)b = (y - y_0)a.$$

Similarly, by eliminating the parameter t (or ϕ) in each case, it may be proved that

(1) The parametric equations

$$x = at^2, \ y = 2\,at$$

represent the parabola $y^2 = 4\,ax$.

(2) The parametric equations

$$x = a \cos \phi, \ \ y = b \sin \phi$$

represent the ellipse $x^2/a^2 + y^2/b^2 = 1$. [Compare § 122.]

(3) The parametric equations

$$x = a \sec \phi, \ \ y = b \tan \phi$$

represent the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

Example 1. Find the points where the line through the point (3, 2), and having the slope 2, cuts the circle $x^2 + y^2 - 5 = 0$.

The parametric equations of the line are $x = 3 + t$, $y = 2 + 2\,t$. Hence, for the required points, $(3 + t)^2 + (2 + 2\,t)^2 - 5 = 0$, or $5\,t^2 + 14\,t + 8 = 0$, whence $t = -2$, or $-4/5$. Therefore the points are $(1, -2)$ and $(11/5, 2/5)$.

Example 2. Prove that the equations of the tangent and normal to the parabola $y^2 = 4\,ax$ at the point $(at^2, 2\,at)$ are $x - ty + at^2 = 0$ and $tx + y - 2\,at - at^3 = 0$.

The equation of the tangent at (x', y') is $yy' = 2\,a(x + x')$. Substituting $x' = at^2$, $y' = 2\,at$ in this equation, and simplifying, gives $x - ty + at^2 = 0$.

The equation of the normal at (x', y') is $y'(x - x') + 2\,a\,(y - y') = 0$. Substituting $x' = at^2$, $y' = 2\,at$ in this equation, and, simplifying, gives $tx + y - 2\,at - at^3 = 0$.

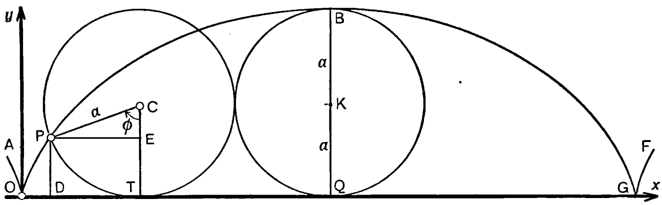
Example 3. Prove that the tangents to the parabola $y^2 = 4\,ax$ at the points $(at_1{}^2, 2\,at_1)$ and $(at_2{}^2, 2\,at_2)$ meet at the point $\{at_1t_2, a(t_1 + t_2)\}$.

Example 4. Prove that the area of the triangle whose angular points are $(at_1{}^2, 2\,at_1)$, $(at_2{}^2, 2\,at_2)$, and $(at_3{}^2, 2\,at_3)$ is $a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$. Also that this area is double that of the triangle whose sides are the tangents to the parabola at the three given points.

206. The cycloid. This is the curve traced by a point on the circumference of a circle, when the circle is made to roll (without sliding) on a straight line. Parametric equations of the cycloid may be found as follows:

Take the line on which the circle rolls as x-axis, and one of the positions in which the tracing point P is on this line as the origin O. Let C be the center of the circle, and a its radius.

Then, taking the circle in any representative position, as in the figure, join C to P and to the point of tangency T. Also



take PD and PE perpendicular to Ox and CT, respectively, and represent the circular measure of the angle TCP by ϕ.

The position of the circle is such that, were it rolled back to the left, the tracing point P would come into coincidence with O; hence $OT = \text{arc } TP = a\phi$.

Therefore, if (x, y) denote the coordinates of P,

$$x = OD = OT - PE = a\phi - a\sin\phi,$$
$$y = DP = TC - EC = a - a\cos\phi.$$

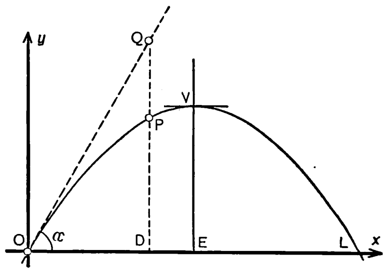
Hence the parametric equations of the cycloid, referred to the axes above indicated, are

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi).$$

As ϕ varies from 0 to 2π, P traces out the arch indicated in the figure. The entire curve consists of this arch and repetitions of it to the right and left corresponding to the values of ϕ already considered increased or diminished by the multiples of 2π.

207. Path of a projectile. It is required to find the curve traced by a projectile whose initial velocity is given, on the assumption that the resistance of the air is to be disregarded.

Take the initial position of the projectile as the origin O, and the horizontal and vertical lines through O as the x- and y-axes. Let v denote the magnitude of the initial velocity, and α the angle which its direction makes with the x-axis.



Take the projectile in any representative position P, and let t denote the time which has elapsed since it left the initial position O.

Through O draw a line making the angle α with Ox, and let this line be met by the perpendicular to Ox through P at Q.

If the projectile were not acted upon by gravity, it would move along the line OQ and in the time t would describe the distance vt; hence $OQ = vt$.

But since the projectile is acted upon by gravity, its distance from Ox at the end of the time t is not DQ, but DQ diminished by a distance (represented by PQ in the figure) which in Mechanics is shown to be $gt^2/2$, where g is a constant.

Hence, if the coordinates of P are (x, y),

$$x = OD = OQ \cos \alpha = vt \cos \alpha,$$

$$y = DP = DQ - PQ = vt \sin \alpha - \tfrac{1}{2} gt^2.$$

Therefore the parametric equations of the path of P are

$$x = vt \cos \alpha, \quad y = vt \sin \alpha - \tfrac{1}{2} gt^2.$$

These equations represent a parabola; for eliminating t,

$$y = x \tan \alpha - \frac{g}{2 v^2 \cos^2 \alpha} x^2,$$

which may be reduced to the form

$$\left(x - \frac{v^2 \sin \alpha \cos \alpha}{g}\right)^2 = -\frac{2\,v^2 \cos^2 \alpha}{g}\left(y - \frac{v^2 \sin^2 \alpha}{2\,g}\right).$$

This equation represents a parabola whose axis is parallel to Oy, whose vertex V (the highest point which P reaches) is $(v^2 \sin 2\,\alpha/2\,g,\ v^2 \sin^2 \alpha/2\,g)$; and whose latus rectum is $2\,v^2 \cos^2 \alpha/g$.

From the equation of the path, in the figure, $EV = v^2 \sin^2 \alpha/2\,g$, and the distance from V to the directrix is $v^2 \cos^2 \alpha/2\,g$, therefore the distance from E to the directrix is $v^2/2\,g$, which does not contain α. Therefore all the paths with a common velocity v going out from O with different angles α have a common directrix.

The distance from O to L in the figure is called the *range*. And $OL = \sin 2\alpha\, v^2/g$. The greatest value which $\sin 2\,\alpha$ can have is 1, and then $2\,\alpha = 90°$, and $\alpha = 45°$. Therefore the *maximum range* is v^2/g, which is obtained when $\alpha = 45°$. In this case E is the focus.

Example. Prove that in putting the shot, if one of two men of similar figure and of equal strength and knack is three inches taller than the other, the taller man should win by about two inches.

208. The graphs of the trigonometric functions, $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$.

From a table of arc lengths and natural trigonometric functions, the following values can be found [see Table E]:

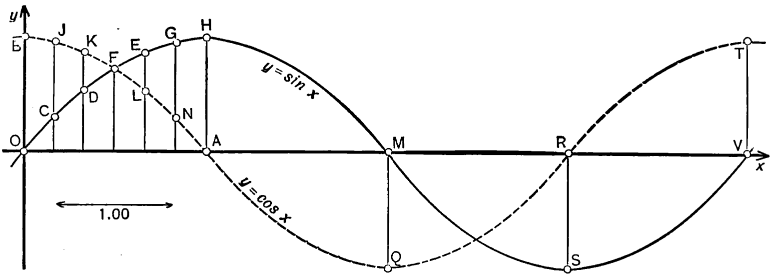
angle	0°	15°	30°	45°	60°	75°	90°
arc	0.0	0.262	0.52	0.79	1.05	1.31	1.57
sin	0.0	0.259	0.50	0.71	0.87	0.97	1.00
cos	1.0	0.966	0.87	0.71	0.50	0.26	0.00
tan	0.0	0.268	0.58	1.00	1.73	3.73	∞

209. Hence the following points are on the graph of $y = \sin x$; $O(0, 0)$, $C(0.262, 0.259)$, $D(0.52, 0.5)$, $F(0.79, 0.71)$, $E(1.05, 0.87)$, $G(1.31, 0.97)$, $H(1.57, 1)$; and, for intermediate points, y increases with x.

The curve from $x = 0$ to $x = \pi/2 = 1.57$ is the part from O to H in the figure. Since $\sin x = \sin(\pi - x)$, the curve from

$x=\pi/2$ to $x=\pi$ will be symmetric with respect to AH to the preceding part, that is, it will be of the form HM in the figure.

Since $\sin x = -\sin(x-\pi)$, as x increases from π to 2π, y will run through the negative values equal numerically to the positive values through which it ran as x increased from 0 to



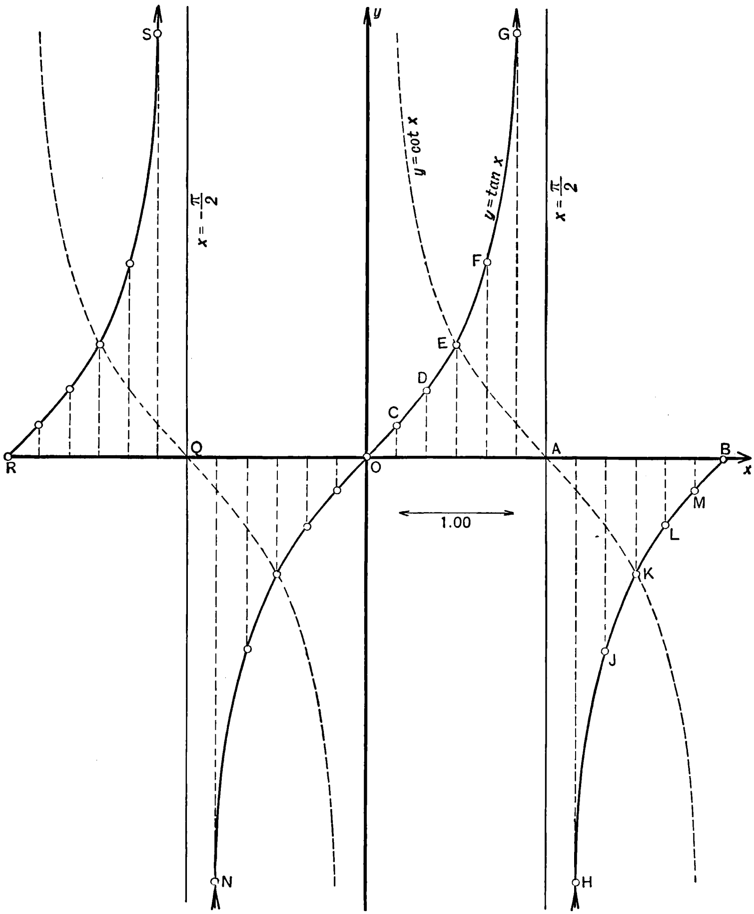
π; hence the curve from $x=\pi$ to $x=2\pi$ will be of the form MSV in the figure.

Since $\sin x=\sin(x\pm 2m\pi)$, the complete graph consists of the part already described and repetitions of it to the right and left.

210. Since $\cos x=\sin(x+\pi/2)$, the *graph of* $y=\cos x$ is obtained by shifting the graph of $y=\sin x$ a distance $\pi/2$ to the left. The graph will be the dotted curve $BFAQRT$ with repetitions to the right and left.

But the *graph of* $y=\cos x$ can be found independently. From the table of arc lengths and natural cosines the following points are on the curve: $B(0, 1)$, $J(0.26, 0.97)$, $K(0.52, 0.87)$, $F(0.79, 0.71)$, $L(1.05, 0.5)$, $N(1.31, 0.26)$, $A(1.57, 0)$; and for intermediate points y decreases as x increases. Since $\cos x = -\cos(\pi - x)$, the y will be negative for all points on the curve from $x=\pi/2$ to $x=\pi$, and the y will numerically increase through the same values it ran through between A and B; this part of the graph is represented by the dotted line from A to Q in the figure. Since $\cos x = \cos(2\pi - x)$, the curve from $x=\pi$ to $x=2\pi$ will be symmetric to the part BAQ; namely, it will be of the form of the dotted line QRT in the figure. The graph of $y=\cos x$ will be represented by the dotted line $BFAQRT$ and repetitions to the right and left.

211. *The graph of* $y = \tan x$. From the table of arc lengths and natural tangents [Table E] the following points are found



to be on the curve: $O(0, 0)$, $C(0.262, 0.268)$, $D(0.52, 0.58)$, $E(0.79, 1)$, $F(1.05, 1.73)$, $G(1.31, 3.73)$, $(\pi/2 = 1.57, \infty)$; and since $\tan x = -\tan(\pi - x)$, the following points also are on the curve: $H(1.83, -3.73)$, $J(2.09, -1.73)$, $K(2.36, -1)$, $L(2.62, -0.58)$, $M(2.88, -0.268)$, $B(3.14, 0)$. The graph is of the form *OCDEFG*, and so out to infinity on the line $x = \pi/2$, and then from infinity on this line through H, J, K, L, M, to B.

Since $\tan x = \tan(x - \pi)$ and $\tan x = \tan(x + \pi)$, the curve will consist of repetitions to the left and right.

212. *The graph of* $y = \cot x$. Since $\cot x = \tan(\pi/2 - x)$, the graph of $y = \cot x$ is symmetric to the graph of $y = \tan x$ with respect to the line $x = \pi/4$. It has the form indicated by the dotted line in the figure.

213. Representation of a function. If the variable y depends on the variable x in such a manner that to each value of x there corresponds a definite value or set of values of y, y is called a *function* of x. Thus, if $y = x^2$ or $y = \sin x$, y is a "one-valued" function of x, that is, to each value of x there corresponds a single value of y; if $y^2 = x$, y is a "two-valued" function of x; and similarly, whenever x and y are connected by an equation, y is a function (one or many valued) of x.

There is no better method of exhibiting the "functional relation" between y and x defined by a given equation than by the graph of this equation. If the equation can be reduced to the form $y = f(x)$, where $f(x)$ denotes a definite expression in x, the graph of the equation can be obtained by computing the values of y corresponding to a set of assigned values of x, plotting the graphs of the solutions of the equation thus found, and then drawing a smooth curve through the points so constructed. Most of the graphs in this book were obtained by this method. The graph of the equation not only exhibits the manner in which y varies with x, but enables one by mere measurement to obtain approximately correct values of y for values of x intermediate to those used in constructing the graph. The method presents fewest difficulties when, in the equation $y = f(x)$, $f(x)$ is a rational integral function of x.

It frequently happens in the application of mathematics to physical problems that while it is known of two variables, y and x, that the first is a function of the second, and while it is possible by experiment and measurement to find the values of y corresponding to certain assigned values of x, the equation connecting x and y is not known. It is then sometimes found useful to obtain the simplest equation of the form $y = f(x)$, where $f(x)$ is rational and integral, which has for solutions the known pairs of values of (x, y). This equation will represent in a simple manner what is known as to the relation between the two variables; and it may be used to compute approximately the values of y corresponding to values of x intermediate to the given values, or such values of y may be obtained by measurement from its graph.

Thus, if it is known that when $x = 1, 2, 3, 4$, then $y = 1, 1, -5, -23$, set

$$y = b_0 + b_1x + b_2x^2 + b_3x^3.$$

Since this equation is to have the four solutions $(1, 1)$, $(2, 1)$, $(3, -5)$, $(4, -23)$, the coefficients b_0, b_1, b_2, b_3 must satisfy the four equations:

$$\begin{aligned} 1 &= b_0 + b_1 + b_2 + b_3, \\ 1 &= b_0 + 2\,b_1 + 4\,b_2 + 8\,b_3, \\ -5 &= b_0 + 3\,b_1 + 9\,b_2 + 27\,b_3, \\ -23 &= b_0 + 4\,b_1 + 16\,b_2 + 64\,b_3. \end{aligned}$$

Solving these equations gives $b_0 = 1$, $b_1 = -2$, $b_2 = 3$, $b_3 = -1$. Hence the required equation is

$$y = 1 - 2\,x + 3\,x^2 - x^3.$$

By this method, if the number of known pairs of values of (x, y) is n, an equation can be found of the form $y = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ which has these known pairs for solutions.

Observe that the method gives a solution of the problem of finding a curve which will pass through n given points.

CHAPTER XII

PROBLEMS ON LOCI

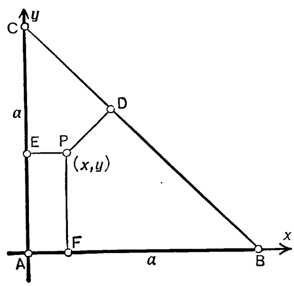
214. Illustrative problems. The method employed in coordinate geometry for finding the locus of a point satisfying a given condition [§ 67] has been illustrated in deriving and interpreting the equations of the several conics [§§ 70, 88, 125], and their diameters [§§ 86, 108], and also in deriving the equations of certain other curves [Chapter XI]. The following pages contain some further illustrations of this important method.

215. *Example* 1. Find the locus of a point P the square of whose distance from the base of a given right-angled isosceles triangle is equal to the product of its distances from the other two sides.

Let ABC be the triangle, right-angled at A, and having the equal sides AB and AC of length a.

Take the lines AB and AC as x- and y-axis, respectively, and let (x, y) denote the coordinates of P referred to these axes.

If PD, PE, PF denote the perpendiculars from P to BC, AC, AB, respectively, by hypothesis



$$DP^2 = EP \cdot FP.$$

But $EP = x$; $FP = y$; and, since the equation of BC is $x + y - a = 0$,

$$DP = (x + y - a)/\sqrt{2}.$$

Therefore, $$\frac{(x+y-a)^2}{2} = xy, \text{ or } x^2 + y^2 - 2\,ax - 2\,ay + a^2 = 0$$

is the equation of the locus of P. Hence the locus is a circle whose center is the point (a, a) and whose radius is a.

216. *Example* 2. Find the locus of a point P the square of whose distance from the origin equals the product of its distances from the axes.

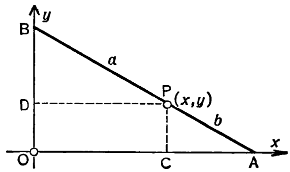
If the coordinates of P are (x, y), the equation of the locus is

$$x^2 + y^2 = xy;$$

which, solved for y, is $y = x(1 \pm \sqrt{-3})/2.$

Therefore the equation has no real solution except (0, 0). Hence the locus of P consists of a single point, the origin.

217. *Example* 3. The extremities A and B of a line segment AB of given length move along the x- and y-axis, respectively, the axes being rectangular. Find the locus of the point P where AB is divided into two parts PA and PB whose lengths are b and a.



Take AB in any representative position, as in the figure, and let x, y denote the coordinates of P; that is, let $DP = x$ and $CP = y$. The problem is to obtain the equation connecting x, y, a, and b.

By considering the figure it is obvious that $DP : BP = CA : PA$. Also $DP = x$, $BP = a$, $CA = \sqrt{b^2 - y^2}$, $PA = b$. The substitution of these values in the proportion gives $x/a = \sqrt{b^2 - y^2}/b$, or, squaring,

$$x^2/a^2 = (b^2 - y^2)/b^2 = 1 - y^2/b^2,$$

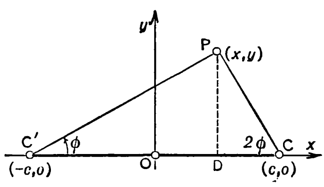
or, finally,

$$x^2\ a^2 + y^2/b^2 = 1.$$

Hence the locus of P is an ellipse with the semi-axes a and b, and with its axes on the coordinate axes.

218. *Example* 4. The base of a triangle is given in length and position, and one of the angles at the base is double the other; find the locus of the vertex.

Let $PC'C$ be the triangle with the vertex P in any representative position in which the angle at C is double that at C'.



Take the mid-point of the base $C'C$ as the origin O, the line $C'C$ as x-axis, and the line through O perpendicular to $C'C$ as y-axis. Then, if $2c$ denote the length of the base, the coordinates of C' and C are $(-c, 0)$ and $(c, 0)$. Let (x, y) denote the coordinates of P.

Since the angle at C is double that at C', if the angle at C' be represented by ϕ, that at C will be represented by 2ϕ. These angles cannot themselves be immediately expressed in terms of (x, y) but their tangents can be so expressed. For, take PD perpendicular to Ox; then

$$\tan\phi = \frac{DP}{C'D} = \frac{y}{c+x}, \text{ and } \tan 2\phi = \frac{DP}{DC} = \frac{y}{c-x}.$$

Between the functions $\tan 2\phi$ and $\tan\phi$ there exists the relation $\tan 2\phi = 2\tan\phi/(1-\tan^2\phi)$. In this equation substitute the expressions just obtained for $\tan\phi$ and $\tan 2\phi$. The result is

$$\frac{y}{c-x} = \frac{2y/(c+x)}{1-y^2/(c+x)^2},$$

which is equivalent to $y = 0$ and $3x^2 - y^2 + 2cx = c^2$, the equation required. It represents an hyperbola, whose center is the point $(-c/3, 0)$ and whose transverse axis coincides with the x-axis; the vertices are the points $(-c, 0)$ and $(c/3, 0)$. Since the interior base angles only are considered, only the branch of this hyperbola through $(c/3, 0)$ is to be taken.

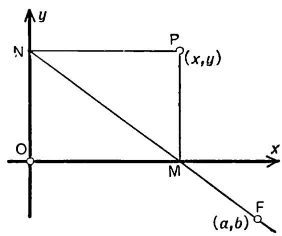
219. *Note.* Any given angle $\alpha(<\pi)$ may be trisected by aid of this hyperbola. For on $C'C$ describe a segment of a circle containing the angle $\pi - \alpha$. Then, if P denote one of the points of intersection of this circle with the hyperbola, $\measuredangle C'PC = \pi - \alpha$. $\therefore \measuredangle PC'C + \measuredangle PCC' = \alpha$. $\therefore \measuredangle PC'C = \alpha/3$.

The trisection of an angle was one of the famous problems of antiquity.

220. *Example* 5. From a point P perpendiculars PM and PN are taken to the rectangular axes Ox and Oy; the line MN passes through the fixed point (a, b); find the locus of P.

Represent the coordinates of P by (x, y); then $OM = x$ and $ON = y$. Therefore, if (ξ, η) denote the coordinates of any point on the line MN, its equation (in the intercept form) is

$$\frac{\xi}{x} + \frac{\eta}{y} = 1.$$



But, by hypothesis, (a, b) is a solution of this equation. Hence

$$\frac{a}{x} + \frac{b}{y} = 1, \text{ or } xy - bx - ay = 0$$

is the equation required. It represents an hyperbola passing through the origin and having asymptotes parallel to Ox and Oy.

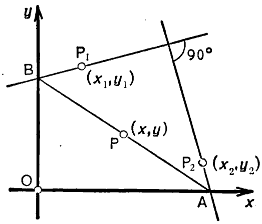
221. *Example* 6. The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are fixed. Through P_1 a line passes which meets the y-axis at B, and through P_2 a line passes which meets the x-axis at A. If these lines are perpendicular, find the locus of P, the mid-point of AB.

The first line passes through the point $P_1(x_1, y_1)$ and has a variable slope. Call this slope λ. The equation of the line is then

$$y - y_1 = \lambda(x - x_1). \qquad (1)$$

Since the second line passes through the point $P_2(x_2, y_2)$ and is perpendicular to the line (1), its equation is

$$y - y_2 = -\frac{1}{\lambda}(x - x_2). \qquad (2)$$



The coordinates of the point B where the line (1) meets the y-axis are $(0, y_1 - \lambda x_1)$.

The coordinates of the point A where the line (2) meets the x-axis are $(x_2 + \lambda y_2, 0)$.

Hence the coordinates of $P(x, y)$, the mid-point of AB, are

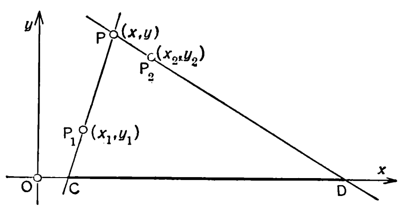
$$x = \frac{x_2 + \lambda y_2}{2}, \qquad y = \frac{y_1 - \lambda x_1}{2}. \qquad (3)$$

The elimination of the "parameter" λ between these two equations gives

$$2xx_1 + 2yy_2 - (x_1x_2 + y_1y_2) = 0. \qquad (4)$$

The locus of P is the straight line represented by this equation.

222. *Example* 7. The point P is the intersection of two lines PP_1 and PP_2 which pass through the fixed points P_1 and P_2 and intercept the constant length l on a given line; find the locus of this point P.



Take the given line as x-axis, any point O on this line as origin, and the line through O perpendicular to the given line as y-axis. Represent the coordinates of P, P_1, P_2 referred to these axes by (x, y), (x_1, y_1), and (x_2, y_2).

If C and D denote the points where PP_1 and PP_2 meet Ox, then $CD = l$. But $CD = OD - OC$. Hence the equation of the locus of $P(x, y)$ will be obtained if expressions for OC and OD in terms of x, y can be found.

Represent the variable lengths OC and OD by c and d, respectively; then the coordinates of C are $(c, 0)$ and those of D are $(d, 0)$ and the equations of the two lines CP_1 and DP_2 are

$$\frac{x - x_1}{x_1 - c} = \frac{y - y_1}{y_1}, \quad (1) \qquad \frac{x - x_2}{x_2 - d} = \frac{y - y_2}{y_2}. \quad (2)$$

Regard these equations as simultaneous; then, in both equations, (x, y) will denote the coordinates of P.

Solving (1) for c in terms of x, y, x_1, y_1, and (2) for d in terms of x, y, x_2, y_2, gives

$$c = \frac{x_1 y - y_1 x}{y - y_1} \quad \text{and} \quad d = \frac{x_2 y - y_2 x}{y - y_2}.$$

Therefore, since $d - c = l$,

$$\frac{x_2 y - y_2 x}{y - y_2} - \frac{x_1 y - y_1 x}{y - y_1} = l,$$

or $\quad (y - y_1)(x_2 y - y_2 x) - (y - y_2)(x_1 y - y_1 x) = l(y - y_1)(y - y_2),$

which is the equation required. It is of the second degree in (x, y) and therefore represents a conic.

223. *Example* 8. To find the locus of a point P the tangents from which to the parabola $y^2 = 4ax$ include an angle of 45°.

Let (x, y) denote the coordinates of P, and m_1, m_2 the slopes of the tangents from $P(x, y)$ to the parabola.

To solve the problem three equations connecting x, y, m_1, m_2 must be found, and m_1 and m_2 eliminated.

One of these equations may be obtained directly; for since the tangents include an angle of 45°, and $\tan 45° = 1$, by § 54,

$$\frac{m_1 - m_2}{1 + m_1 m_2} = 1. \qquad (1)$$

To find the other two equations, proceed as follows: The equation of the tangent to $y^2 = 4ax$ in terms of its slope is $y = mx + a/m$, which, when written as an equation for determining m in terms of the coordinates (x, y) of any point on the tangent, becomes

$$xm^2 - ym + a = 0. \qquad (2)$$

Hence, if in (2), x, y denote the coordinates of the point P whose locus is sought, the roots of (2) will be m_1 and m_2, the slopes of the tangents through P. Therefore

$$m_1 + m_2 = \frac{y}{x}, \quad (3) \qquad \text{and} \qquad m_1 m_2 = \frac{a}{x}. \quad (4)$$

The equation of the locus may therefore be found by eliminating m_1

and m_2 from the equations (1), (3), and (4). From (3) and (4) it follows that $m_1 - m_2 = \sqrt{y^2 - 4ax}/x$. Substituting this result and (4) in (1) and simplifying, gives

$$x^2 - y^2 + 6ax + a^2 = 0, \tag{5}$$

which is the equation required. It represents a rectangular hyperbola.

224. *Example* 9. Find the locus of a point P the tangents from which to the ellipse $x^2/a^2 + y^2/b^2 = 1$ include a right angle.

As in the preceding example, let (x, y) denote the coordinates of P, and m_1, m_2 the slopes of the tangents from P to the ellipse.

The equation of the tangent to the ellipse in terms of its slope is $y = mx \pm \sqrt{a^2m^2 + b^2}$, which when written as an equation for determining m becomes

$$(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0. \tag{1}$$

Hence, if in (1) x, y denote the coordinates of P, the roots of (1) will be m_1 and m_2, the slopes of the tangents from P; therefore

$$m_1m_2 = (y^2 - b^2)/(x^2 - a^2).$$

But since the tangents are perpendicular, $m_1m_2 = -1$. Therefore

$$\frac{y^2 - b^2}{x^2 - a^2} = -1, \quad \text{or} \quad x^2 + y^2 = a^2 + b^2$$

is the equation of the locus of P. It represents a circle whose center is at the center of the ellipse and the square of whose radius is the sum of the squares of the semimajor and semiminor axes of the ellipse. It is called the *director circle* of the ellipse.

225. Loci problems are so varied in character that it is possible to give general directions only for dealing with them. Take the point P whose locus is sought in some representative position, and construct a figure containing the several points and lines mentioned in the statement of the problem. Then choose axes of reference related as simply as possible to these points and lines, and represent the coordinates of P with respect to these axes by (x, y).

If the given condition involves, directly or indirectly, the distance of P from certain *fixed* points and lines only, the equation of the locus can be found at once by expressing these distances in terms of the coordinates of $P(x, y)$ and known quantities, and substituting the expressions thus found in the statement of the given condition. Thus, the equation of the locus of a point which is twice as far from the origin as it is from the line $y-1=0$ is $\sqrt{x^2+y^2}=2(y-1)$.

On the other hand, the given condition may connect P in some way with certain *movable* points or lines [§ 221]. The coordinates of these points or certain of the quantities which appear in the coefficients of the equations of these lines are then themselves variables; they are called *parameters*. In this case the statement of the problem leads to a system of equations involving these parameters and the coordinates of $P(x, y)$, there being one more equation than there are parameters, if the problem be a locus problem in the proper sense of the word. The elimination of the parameters from this system of equations gives the equation in x, y, and known quantities, which represents the required locus. Care must be taken that *all* the given conditions are expressed in the equations, or in the method of combining them [§§ 221, 222, 223].

It is sometimes easier to obtain the equation of a locus in polar coordinates [§§ 197, 202, 203].

226. What in this book has been called *Coordinate Geometry* is so called because a point is determined by its coordinates and *vice versa*. The subject is sometimes called by other names; namely, *Analytic Geometry*, *Algebraic Geometry*, *Cartesian Geometry*, or *Conic Sections*. It is called algebraic or analytic geometry, because it represents geometric relations by equations; Cartesian geometry, because the method was used by Descartes (Latin, Cartesius) in his *Géométrie* published in 1636; conic sections, because these curves are studied by this method.

227. Exercises. The locus of a point.

1. Find the locus of a point which is twice as far from the x-axis as from the y-axis.

2. Find the locus of a point the square of whose distance from the origin is equal to the sum of its distances from the axes.

3. Find the locus of a point the sum of the squares of whose distances from the sides of a given square is constant.

4. In the rectangle *AMPN*, the perimeter, the position of A, and the directions of the sides are given; find the locus of P.

5. A point moves in such a manner that its distance from $(a, 0)$ is equal to its distance from a straight line through $(-a, 0)$ and parallel to the y-axis; find its locus.

6. Find the locus of a point whose distance from the point $(1, -1)$ is one half its distance from the line $x + 2y = 0$.

7. Find the locus of a point the sum of whose distances from the points $(-2, 0)$ and $(2, 0)$ is 6.

8. Given the base AB of the triangle ABC, find the locus of its vertex C

(1) when $CA^2 - CB^2$ is given.
(2) when $CA^2 + CB^2$ is given.
(3) when CA/CB is given.
(4) when the vertical angle C is given.
(5) when the difference of the base angles A and B is given.

9. Given the base AB and the vertical angle C of the triangle ABC, find the locus of the point of intersection of the perpendiculars from A and B to the opposite sides.

10. The hypotenuse of a right-angled triangle is given in position and length. Find the locus of the center of the inscribed circle of the triangle.

11. Find the locus of a point P which is twice as far from the line $2x + 3y + 1 = 0$ as it is from the line $x - 2y - 6 = 0$.

12. The sum of the squares of the distances of a point P from two given intersecting lines is constant; prove that its locus is an ellipse.

13. Find the locus of a point the sum of the squares of whose distances from the angular points of a given square is constant.

14. The sum of the distances of the point P from the sides of the triangle $y = 0$, $3y - 4x = 0$, $12x + 5y - 60 = 0$ is constant; find the equation of its locus.

15. Two given parallel lines l_1 and l_2 are perpendicular to a third given line l_3. If the perpendicular distances of the point P from these lines are p_1, p_2, and p_3, respectively, and $p_1p_2 = cp_3^2$, where c is constant, prove that the locus of P is an ellipse when c is negative, and an hyperbola when c is positive.

16. Find the locus of the center of a circle which passes through a given point and touches a given line.

17. Prove that the locus of the center of a circle which touches two given circles, of which one is not within the other, is a pair of hyperbolas of which the centers of the given circles are the foci. What is the locus when one of the given circles lies within the other?

18. Find the locus of the center of a circle which touches a given line and a given circle.

19. Prove that the locus of the foot of the perpendicular to a tangent of the ellipse $x^2/a^2 + y^2/b^2 = 1$ from either focus is the circle $x^2 + y^2 = a^2$, or the circle $x^2 + y^2 = b^2$, according as a is greater or less than b.

20. The base of a triangle is fixed in length and position and its vertex moves along a given line; find the locus of the point of intersection of the perpendiculars from the angular points of the triangle to the opposite sides.

21. From a point P perpendiculars PM and PN are drawn to the rectangular axes Ox and Oy. Find the locus of P

(1) when the length of MN is constant.

(2) when MN is parallel to the given line $y = mx$.

22. Two fixed points $A(a, 0)$ and $B(0, b)$ are taken on the x- and y-axes. Two variable points $A'(a', 0)$ and $B'(0, b')$ are also taken on these axes. Find the locus of the point of intersection of AB' and $A'B$

(1) when $a' + b' = a + b$.

(2) when $1/a' - 1/b' = 1/a - 1/b$.

23. A line OP is drawn joining the origin O to any point P on the line $2x - 3y = 6$; find the locus of the mid-point of OP; also that of the point Q where OP is divided in the ratio $k : l$.

24. Through a given point (x', y') two lines are drawn which meet the axes of reference in the points A, B, and A', B', respectively; find the locus of the point of intersection of the lines AB' and $A'B$.

25. A variable line makes with two fixed lines a triangle of constant area; find the locus of the mid-point of that portion of the variable line which lies between the two fixed lines.

26. Through the point (2, 0) a line is drawn which meets the lines $y = x$ and $y = 3x$ in the points R and S; find the locus of the mid-point of RS.

27. Find the locus of the point of intersection of the diagonals of a rectangle inscribed in a given triangle.

28. From a point P tangents are drawn to the parabola $y^2 = 4\,ax$, which make the angles θ_1 and θ_2 with the axis. Find the locus of P.

(1) when $\tan\theta_1 \tan\theta_2$ is constant.

(2) when $\cot\theta_1 + \cot\theta_2$ is constant.

(3) when $\theta_1 + \theta_2$ is constant.

(4) when $\cos\theta_1 \cos\theta_2$ is constant.

29. Two tangents to the parabola $y^2 = 4\,ax$ meet at an angle of $60°$; find the locus of their point of intersection.

30. Find the locus of the point of intersection of two tangents to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ which are mutually perpendicular.

31. The chord of a parabola passes through a fixed point; prove that the locus of its mid-point is a parabola.

32. Prove that the locus of a point whose polars with respect to the parabola $y^2 = 4\,ax$ and the circle $x^2 + y^2 = a^2$ meet at right angles is the parabola $y^2 = 2\,ax$.

33. Prove that the locus of poles of tangents to the parabola $y^2 = 4\,ax$ with respect to the parabola $y^2 = 4\,bx$ is the parabola $y^2 = (4\,b^2/a)x$.

34. From a point P a perpendicular is drawn to the polar of P with respect to the parabola $y^2 = 4\,ax$. This perpendicular meets the polar of P at M and the axis of the parabola at N. Prove that if $PM \cdot PN$ is constant, the locus of P is a parabola.

35. Prove that the locus of the extremities of the minor axes of all ellipses which have a given point and line for focus and directrix is a parabola.

36. From the focus F of an ellipse a line is drawn perpendicular to any diameter, and from the focus F' a line is drawn perpendicular to the conjugate diameter. If these two lines meet at P, prove that the locus of P is another ellipse.

37. The points A, B, C, and D are given on a straight line; find the locus of the point P at which AB and CD subtend equal angles.

38. A fixed point A is joined to any point P on a given straight line and on AP the point Q is taken such that $AP \cdot AQ$ is constant; prove that the locus of Q is a circle.

39. If P be any point on an ellipse whose center is C and on CP the point Q be taken such that $CQ = k\ CP$, where k is constant, the locus of Q is an ellipse. (These two ellipses are called *similar ellipses.*)

40. Find the equation of the locus of the centers of the systems of conics which pass through the points of intersection of the lines $x + y - 1 = 0$ and $x + y - 3 = 0$ with the lines $x = 0$ and $y = 0$.

41. Prove that the locus of the centers of the system of conics represented by the equation $ax^2 + 2hxy + \lambda y^2 + 2fy = 0$, where λ is an arbitrary constant, is a straight line.

42. Prove that the locus of the centers of the system of conics $x^2 + 2\lambda xy - y^2 + 2fy = 0$ is a circle.

43. Find the locus of the point of intersection of two tangents to an ellipse, (1) when the sum of their slopes is constant, (2) when the product of their slopes is constant.

44. Find the locus of the center of the inscribed circle of the triangle whose angular points are any point on an ellipse and the foci.

45. The normal to an ellipse at any point P meets the major axis at N; prove that the locus of the mid-point of PN is an ellipse.

46. Prove that the locus of the mid-point of a chord joining the extremities of a pair of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is the ellipse $x^2/a^2 + y^2/b^2 = 1/2$.

47. Prove that the locus of the mid-point of the chord of contact of tangents to the circle $x^2 + y^2 = a^2$ from a point on the line $x = c$ is the circle $c(x^2 + y^2) = a^2x$.

48. Given the rectangle $ABCD$. On AD and DC the points P and Q are taken such that $AP : DQ = AD : DC$; prove that the locus of the point of intersection of BP and AQ is an ellipse whose axes are equal to AD and DC.

49. The perpendicular from the center of an ellipse to the tangent at P meets the line through P and the focus F in the point Q; prove that the locus of Q is a circle.

50. Find the locus of the point of intersection of normals to an ellipse at the extremities of a pair of conjugate diameters.

51. Prove that the locus of the poles of chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ which touch the circle $x^2 + y^2 = a^2e^2$ is the ellipse $x^2/a^4 + y^2/b^4 = 1/(a^2 + b^2)$.

52. Prove that the locus of the poles of a given line with respect to a system of confocal conics is a line perpendicular to the given line.

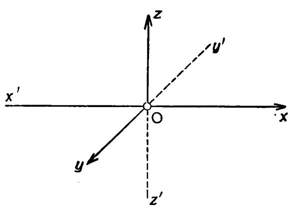
COORDINATE GEOMETRY IN SPACE

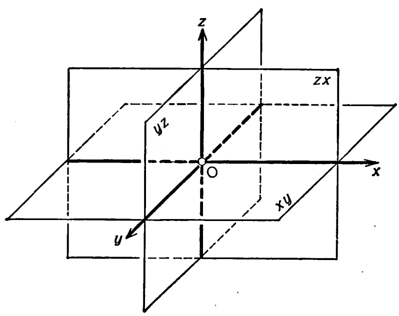
CHAPTER XIII

COORDINATES AND DIRECTION COSINES

228. Axes of coordinates. The methods of coordinate geometry may be extended to space as follows:

Through any point O in space, chosen as *origin*, take three mutually perpendicular lines as *axes of coordinates*, one horizontal, one vertical, and the third perpendicular to the plane of these two lines, and call the horizontal line the x-axis, the vertical line the z-axis, and the third line the y-axis.





The x- and z-axes determine an upright or vertical plane, called the zx-plane, which may be supposed parallel to the plane of the paper in front of the observer. The x- and y-axes determine a horizontal plane, called the xy-plane, and the y- and z-axes a vertical plane, called the yz-plane, both of which are perpendicular to the zx-plane. This is indicated in the accompanying figure, which corresponds to the case in which the eye of the observer is in front of the zx-plane, above the xy-plane, and to the right of the yz-plane.

The xy-, yz-, and zx-planes are called the *coordinate planes.*

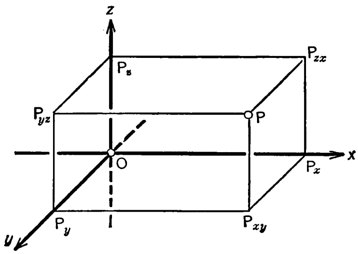
As the positive direction along the x-axis, take that from left to right; along the z-axis, that from below upwards; along the y-axis, that towards the observer, the eye being placed as just indicated.

The system of axes just described is a *rectangular* or *orthogonal* system.

229. Any three lines through O which are not in the same plane may be taken as axes, these axes being called *oblique* when they are not rectangular. Oblique systems will be employed very rarely in this book, and when they are used, ordinarily it is explicitly so stated.

230. The drawing of a plane figure which will correctly represent a solid figure as it appears to the eye is in most cases a difficult process. It is therefore convenient, as in the figures herewith, to use what is called cabinet-makers' or parallel projection. In applying this method, figures in the vertical zx-plane, and in planes parallel to it, are represented as they are, not as they would be seen by the eye. Some convenient direction in the plane of the paper is then chosen to represent the direction of the y-axis and of lines parallel to it, the one often taken, when the axes are orthogonal, being that which makes an angle of 135° with each of the lines representing the other two axes. Line segments along the x- and z-axes, or parallel to them, are then to be drawn correctly to scale. But, as the y-axis is perpendicular (or oblique) to the zx-plane, line segments along this axis or parallel to it will be foreshortened; it is convenient sometimes to represent them on a scale one half that used for segments along the x- and z-axes. When cross-section paper is used, the lines representing the x- and z-axis may be taken on the ruling, and that representing the y-axis in the direction of a diagonal of the squares, and it is then convenient to take $1/\sqrt{2}$ as the unit of the scale for the y-axis, so that the diagonal of a square will be two units. Of course this is merely a conventional method of representation, but it is easily applied, and it gives figures which are accurate enough for most purposes.

231. Orthogonal projections of a point. If through a point P in space a plane be taken perpendicular to a given line, the point in which this plane meets the line is called the *projection of P upon the line.*



Let P_x, P_y, P_z, denote the projections of the point P upon the x-, y-, and z-axes, respectively.

232. Again, defining the *projection of a point upon a plane* as the foot of the perpendicular from the point to the plane [§ 114], let P_{xy}, P_{yz}, P_{zx} denote the projections of the point P upon the xy-, yz-, zx-planes, respectively.

These points are exhibited in the accompanying figure, constructed by taking through P planes parallel to the coordinate planes. These planes, together with the coordinate planes, bound a parallelepiped whose corners are O, P, and the points P_x, P_y, P_z, P_{yz}, P_{zx}, P_{xy}.

233. Coordinates of a point. The distances of a point P from the yz-, zx-, and xy-planes are denoted by x, y, and z, respectively, and are called the *coordinates* of P. These distances are represented as follows by the line segments which form the edges of the parallelepiped just constructed:

The x of P is $OP_x = P_yP_{xy} = P_{yz}P = P_zP_{zx}$.
The y of P is $OP_y = P_zP_{yz} = P_{zx}P = P_xP_{xy}$.
The z of P is $OP_z = P_xP_{zx} = P_{xy}P = P_yP_{yz}$.

In reasoning about a point P it is usually convenient to take for its coordinates (x, y, z) either the set OP_x, OP_y, OP_z, or the set $P_{yz}P$, $P_{zx}P$, $P_{xy}P$, but in drawing figures it is often more convenient to use the set OP_x, P_xP_{xy}, $P_{xy}P$. (See the figures in the following exercises.)

The x-coordinate of P is positive or negative according as P is to the right of the yz-plane or to its left; the y-coordinate is positive or negative according as P is in front of the zx-plane or behind it; the z-coordinate is positive or negative according as P is above the xy-plane or below it [§ 228].

If the coordinates of a point be given, the point itself can be obtained by reversing the construction just explained.

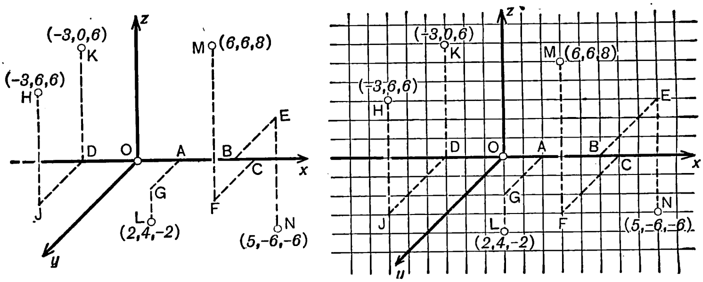
Thus the method of coordinates enables one to establish such a relation between sets of real values of the variables (x, y, z) *and the points of space, that to each set of values of* (x, y, z) *there will correspond one point in space, and to each point in space, one set of values of* x, y, z [*compare* § 6].

234. It may be added that the coordinates of a point with respect to a system of *oblique axes* [§ 229] are its distances from each coordinate plane measured in the direction of the coordinate axis not in that plane.

235. Exercises. Definition of coordinates.

(The following graphs may be obtained by using squared or cross-section paper or not, as in the first exercise.)

1. Plot to scale the following points: $O(0, 0, 0)$, $M(6, 6, 8)$, $N(5, -6, -6)$, $L(2, 4, -2)$, $A(2, 0, 0)$, $K(-3, 0, 6)$, $H(-3, 6, 6)$.



2. Using cross-section paper, plot to scale the following points, putting as many as may be convenient on one figure: (1, 1, 1), (2, 0, 3),

$(-4, -1, -4)$, $(-3, -4, 1)$, $(4, 4, -1)$, $(-7, 2, 3)$, $(-1, 5, -5)$, $(-4, 2, 8)$, $(3, -4, -1)$, $(2, 1, -3)$, $(-1, 0, 0)$, $(4, -2, 2)$, $(0, 0, 2)$, $(0, -1, 0)$, $(-3, 0, 0)$, $(0, 0, 0)$.

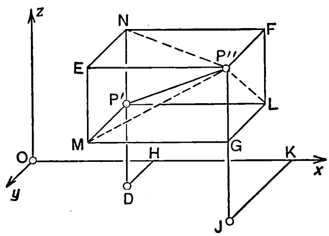
3. Plot to scale on one figure the following nine points: $(3, 2, 3)$, $(3, 2, 0)$, $(3, 0, 0)$, $(3, 0, 3)$, $(3, -2, 3)$, $(3, -2, 0)$, $(3, -2, -2)$, $(3, 0, -2)$, $(3, 2, -2)$.

4. What are the relative positions of the following eight points: (a, b, c), $(a, b, -c)$, $(a, -b, c)$, $(-a, b, c)$, $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$, $(-a, -b, -c)$?

Which of these points are symmetric with regard to the origin? Which are symmetric with regard to the coordinate planes?

236. Problem. *To express the distance between two points* P', P'', *in terms of their coordinates* (x', y', z'), (x'', y'', z'').

Through P' and P'' take planes parallel to the coordinate planes. These six planes bound a parallelepiped of which $P'P''$ is a diagonal. Let the lines through P' parallel to the axes meet the planes through P'' parallel to the coordinate planes in L, M, N, respectively; let the lines through P'' parallel to the axes meet the planes through P' parallel to the coordinate planes in E, F, G, respectively; and let the planes of the parallelepiped through P' and P'' parallel to the yz-plane meet the x-axis in H and K, and the xy-plane in the lines DH and JK, as in the figure. Then, OH is x', and OK is x''; and $P'L = HK = OK - OH = x'' - x'$. And similarly $LG = y'' - y'$, and $GP'' = z'' - z'$.



Since $P'LP''$ is a right angle, $P'P''^2 = P'L^2 + LP''^2$. Again, since LGP'' is a right angle, $LP''^2 = LG^2 + GP''^2$. Hence $P'P''^2 = P'L^2 + LG^2 + GP''^2$. In this equation substitute the expressions just obtained for $P'L$, LG, GP'' in terms of the coordinates; the result is (compare § 41):

$$P'P''^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2. \qquad (1)$$

o

237. In particular, the distance of a point $P(x, y, z)$ from the origin is given by the formula

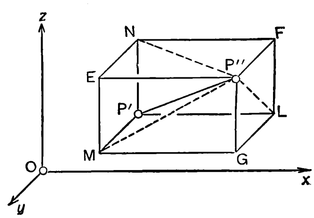
$$OP^2 = x^2 + y^2 + z^2. \qquad (2)$$

238. Angle between two lines in space. Two lines in space will ordinarily not intersect. If parallels to a pair of non-intersecting and non-parallel lines be taken through any point, the angle made by these intersecting lines is called the *angle made by the non-intersecting lines.*

In particular, this angle may be constructed by taking a parallel to one of the non-intersecting lines through any point on the other.

239. Direction cosines of a line. The cosines of the angles which either direction along a line makes with the positive directions of the coordinate axes are called its *direction cosines.* It is customary to represent the angles by α, β, γ, and their cosines by λ, μ, ν.

Observe that if the direction cosines of the line $P'P''$ in the direction from P' to P'' be λ, μ, ν, its direction cosines in the direction from P'' to P' are $-\lambda$, $-\mu$, $-\nu$; for the angles made by the directions $P'P''$ and $P''P'$ with each axis are supplementary. Hence, if it is unnecessary to distinguish between the two directions along the line, either λ, μ, ν or $-\lambda$, $-\mu$, $-\nu$ may be taken for the direction cosines, as may be the more convenient.



Let the line $P'P''$ be any line in space, and take a parallelepiped $P'P''EFGLMN$, of which P', P'' are opposite vertices, as in § 236. Let the length of $P'P''$ be represented by r; then, by definition [§ 238], the angles which $P'P''$ makes with the axes are $\alpha = LP'P''$, $\beta = MP'P''$, $\gamma = NP'P''$;

and since the angles $P'LP''$, $P'MP''$, $P'NP''$ are right angles, the direction cosines of the line are:

$$\lambda = \cos \alpha = \frac{P'L}{P'P''} = \frac{x'' - x'}{r}, \tag{1}$$

$$\mu = \cos \beta = \frac{P'M}{P'P''} = \frac{y'' - y'}{r}, \tag{2}$$

$$\nu = \cos \gamma = \frac{P'N}{P'P''} = \frac{z'' - z'}{r}. \tag{3}$$

When one of the points is the origin, from equations (1), (2), (3) it follows that the direction cosines of the line from the origin O to the point $P'(x', y', z')$ are

$$\lambda = x'/r, \quad \mu = y'/r, \quad \nu = z'/r. \tag{4}$$

Example. Find the direction cosines of the line from the point (3, 2, 1) to the point (1, – 2, 2).

Here,

$$x'' - x' = 1 - 3 = -2, \quad y'' - y' = -2 - 2 = -4, \quad z'' - z' = 2 - 1 = 1.$$

Hence $\quad r^2 = (-2)^2 + (-4)^2 + 1^2 = 21$, or $r = \sqrt{21}$. [§ 236.]

Therefore $\quad \lambda = -2/\sqrt{21}, \ \mu = -4/\sqrt{21}, \ \nu = 1/\sqrt{21}$;

the direction cosines of the line are

$$(-2/\sqrt{21}, \ -4/\sqrt{21}, \ 1/\sqrt{21}), \text{ or } (2/\sqrt{21}, \ 4/\sqrt{21}, \ -1/\sqrt{21}).$$

240. *The sum of the squares of the direction cosines of any line is unity.*

As in the preceding section, let $P'P''$ be any line in space, of length r; its direction cosines are

$$\lambda = (x'' - x')/r, \quad \mu = (y'' - y')/r, \quad \nu = (z'' - z')/r.$$

Squaring and adding these three equations gives

$$\lambda^2 + \mu^2 + \nu^2 = \{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2\}/r^2,$$

or, since the numerator is equal to the denominator [§ 236],

$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

241. If there be given three numbers a, b, c, which are known to be proportional to the direction cosines λ, μ, ν of a certain line, λ, μ, ν themselves can be found as follows:

The hypothesis that a, b, c are proportional to λ, μ, ν, can be expressed:

$$ka = \lambda, \quad kb = \mu, \quad kc = \nu;$$

where k denotes a constant, as yet unknown.

Squaring and adding these equations,

$$k^2(a^2 + b^2 + c^2) = \lambda^2 + \mu^2 + \nu^2 = 1.$$

Hence $$k = 1/\sqrt{a^2 + b^2 + c^2},$$ and therefore

$$\lambda = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \mu = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad \nu = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Only the plus sign is taken before the radical for the reason explained in § 239.

Thus, if $\lambda : \mu : \nu = 2 : 1 : -3$, then $\lambda = 2/\sqrt{14}$, $\mu = 1/\sqrt{14}$, $\nu = -3/\sqrt{14}$.

242. Exercises. Direction cosines, and distance between two points.

1. Find the distances apart of the following pairs of points, and the direction cosines of the lines which they determine:

$(5, 2, 2)$ and $(8, 5, 4)$; $(1, 1, 1)$ and $(2, 0, 3)$;
$(-4, -1, -4)$ and $(-3, -4, 1)$; $(-1, 5, -5)$ and $(-1, 2, -5)$;
$(0, 0, 0)$ and $(2, 2, -2)$; $(0, 0, 0)$ and $(0, 0, 1)$.

2. Find the equation of the locus of points equidistant from the two points $(8, 5, 4)$ and $(5, 2, 2)$.

3. Show that the equation $x^2 + y^2 + z^2 = 9$ represents a sphere whose center is the origin and whose radius is 3.

4. Show that the point $(\frac{13}{5}, 1, \frac{9}{5})$ is the center of a sphere which passes through the four points $(2, 3, 4)$, $(4, 3, 0)$, $(0, 2, 3)$ and $(2, 0, -1)$.

5. If the direction cosines of a line are in the ratio $3 : 2 : -5$, what are their values?

6. Find the direction cosines of a line which is equally inclined to the three coordinate axes.

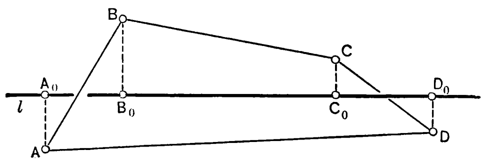
7. Find the length and direction cosines of the line joining the origin to the point $(-2, 3, -5)$.

243. Projections of line segments. Let AB denote a line segment and l any other line in space. If the projections of A and B upon l are A_0 and B_0 respectively [§ 231], the segment A_0B_0 is called the *projection of* AB *upon* l. This is expressed: $A_0B_0 = pr_l AB$, which is read: A_0B_0 is the projection of AB on the line l [compare § 46].

244. Thus, if O be the origin and P a point whose coordinates are x, y, z, the projections of OP upon the x-, y-, and z-axes are OP_x, OP_y, and OP_z, that is, x, y, and z, respectively. See the figure in § 231.

245. Again, if P' and P'' are two points whose coördinates are x', y', z', and x'', y'', z'', respectively, the projection of $P'P''$ upon the x-axis is $HK = OK - OH = x'' - x'$. See the figure in § 236. Similarly, the projections of $P'P''$ on the y- and z-axes are $y'' - y'$ and $z'' - z'$, respectively.

246. The projection of a broken line, made up of line segments, upon any line l is defined as the algebraic sum [§ 2] of the projections upon l of the segments of which the broken line consists. This sum is readily seen to be the same as the projection upon l of the segment from the initial extremity of the broken line to its final extremity. For example, if the projections of the points A, B, C, D upon l are A_0 B_0, C_0, D_0, then the projection upon l of the broken line $ABCD$ is $A_0B_0 + B_0C_0 + C_0D_0$. But this sum is A_0D_0, the projection of AD upon l. Hence



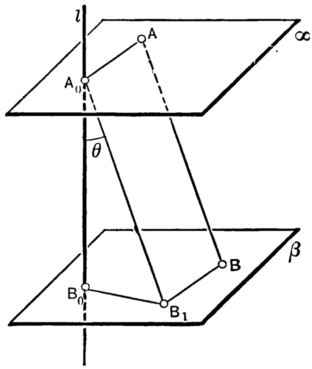
$$A_0D_0 = A_0B_0 + B_0C_0 + C_0D_0;$$

that is, $$pr_l AD = pr_l AB + pr_l BC + pr_l CD.$$

It is to be understood, of course, that the points A, B, C, D are not restricted to one and the same plane.

247. The projection upon l of any *closed* line made up of line segments is zero.

248. *The projection of a line segment upon another line is equal to the line segment multiplied by the cosine of the angle which it makes with the other line.*



For, let l be any line, and AB any line segment in space. Let α and β be the planes perpendicular to l through A and B, respectively, and A_0 and B_0 the points where these planes meet l. Then, by definition, A_0 is the projection of A and B_0 of B, and A_0B_0 is the projection of AB upon l.

Through A_0 take a line parallel to AB, and let B_1 denote the point where it meets β, and θ the angle which it makes with l. Join B_0B_1. Then $A_0B_1 = AB$ [parallels between parallel planes], and since $A_0B_0B_1$ is a right angle,

$$A_0B_0 = A_0B_1 \cos\theta = AB \cos\theta,$$

which may be written $pr_l AB = AB\cos\theta$, as was to be demonstrated.

249. The projections of segments of the same line, or of parallel lines, upon a given line are proportional to the segments themselves.

250. The projections of a line segment upon the three coordinate axes are proportional to the three direction cosines of the line segment. For, in the figure of § 236, $P'L$ is equal to the projection of $P'P''$ upon the x-axis, and

$$P'L = P'P'' \cos LP'P'' = \lambda \cdot P'P'';$$

and so on.

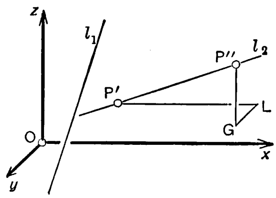
Example. Find the projections on the axes, and the direction cosines of the line segment joining the point $A(3, 2, 1)$ to the point $B(1, -2, 2)$.

The projections of the line segment AB on the x-, y-, and z-axes are [§ 245],
$$1 - 3 = -2, \quad -2 - 2 = -4, \quad \text{and} \quad 2 - 1 = 1.$$
Hence
$$\lambda : \mu : \nu = -2 : -4 : 1.$$
Therefore [§ 241], $\lambda = -2/\sqrt{21}$, $\mu = -4/\sqrt{21}$, $\nu = 1/\sqrt{21}$;
or, $\lambda = 2/\sqrt{21}$, $\mu = 4/\sqrt{21}$, $\nu = -1/\sqrt{21}$ [§ 239].

Observe that this is the example of § 239 expressed in the language of projections.

251. Problem. *To find the angle between two lines in terms of their direction cosines.*

Let l_1 and l_2 be two lines in space whose direction cosines are λ_1, μ_1, ν_1, and λ_2, μ_2, ν_2, respectively, and let θ denote the angle between the lines. It is required to express θ in terms of λ_1, μ_1, ν_1; λ_2, μ_2, ν_2.



On l_2 take any two points P', P'', and let $P'LGP''$ be the broken line from P' to P'', made up of segments $P'L$, LG, GP'' parallel to Ox, Oy, Oz, respectively [compare the figure in § 236].

By § 246, the projections of $P'P''$ and $P'LGP''$ upon l_1 are equal, that is,
$$pr_{l_1} P'P'' = pr_{l_1} P'L + pr_{l_1} LG + pr_{l_1} GP''.$$
Therefore, since l_1 and l_2 make the same angles with $P'L$, LG, GP'' as with Ox, Oy, Oz, respectively, and the cosines of these angles are λ_1, μ_1, ν_1 for l_1, and λ_2, μ_2, ν_2 for l_2, by § 248,
$$P'P'' \cos\theta = P'L \cdot \lambda_1 + LG \cdot \mu_1 + GP'' \cdot \nu_1;$$
or, since $P'L = P'P''\lambda_2$, $LG = P'P''\mu_2$, $GP'' = P'P''\nu_2$ [§ 250],
$$P'P'' \cos\theta = P'P''\lambda_2 \cdot \lambda_1 + P'P''\mu_2 \cdot \mu_1 + P'P''\nu_2 \cdot \nu_1;$$
or, dividing both members by $P'P''$,
$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2.$$
Thus, the angle between the two lines whose direction cosines are $(2/\sqrt{14},\ 1/\sqrt{14},\ -3/\sqrt{14})$ and $(1/\sqrt{6},\ 2/\sqrt{6},\ -1/\sqrt{6})$ is given by
$$\cos\theta = \{(2)(1) + (1)(2) + (-3)(-1)\}/\sqrt{14}\,\sqrt{6} = \tfrac{1}{2}\sqrt{7/3}.$$

252. Any two lines whose direction cosines are λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 are perpendicular, if

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0.$$

Thus, the two lines are perpendicular whose direction cosines are proportional to (2, − 1, 3) and (− 1, 1, 1), since

$$\cos\theta = \{(2)(-1) + (-1)(1) + (3)(1)\}/\sqrt{42} = 0.$$

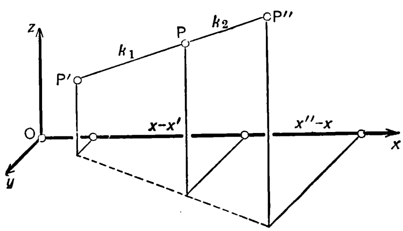
Since the numerator here is zero, it is unnecessary to calculate the denominator.

253. The sine of the angle between two lines may be expressed as follows in terms of the direction cosines of the lines:

$\sin^2\theta = 1 - \cos^2\theta$, or using § 240 and § 251,

$$\begin{aligned}\sin^2\theta &= (\lambda_1^2 + \mu_1^2 + \nu_1^2)(\lambda_2^2 + \mu_2^2 + \nu_2^2) - (\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2)^2\\ &= \lambda_1^2\lambda_2^2 + \lambda_1^2\mu_2^2 + \lambda_1^2\nu_2^2 + \mu_1^2\lambda_2^2 + \mu_1^2\mu_2^2 + \mu_1^2\nu_2^2\\ &\qquad + \nu_1^2\lambda_2^2 + \nu_1^2\mu_2^2 + \nu_1^2\nu_2^2 - \lambda_1^2\lambda_2^2 - \mu_1^2\mu_2^2 - \nu_1^2\nu_2^2\\ &\qquad\qquad - 2\,\mu_1\mu_2\nu_1\nu_2 - 2\,\nu_1\nu_2\lambda_1\lambda_2 - 2\,\lambda_1\lambda_2\mu_1\mu_2\\ &= (\mu_1^2\nu_2^2 - 2\,\mu_1\mu_2\nu_1\nu_2 + \nu_1^2\mu_2^2) + (\nu_1^2\lambda_2^2 - 2\,\nu_1\nu_2\lambda_1\lambda_2 + \lambda_1^2\nu_2^2)\\ &\qquad + (\lambda_1^2\mu_2^2 - 2\,\lambda_1\lambda_2\mu_1\mu_2 + \mu_1^2\lambda_2^2)\\ &= (\mu_1\nu_2 - \nu_1\mu_2)^2 + (\nu_1\lambda_2 - \lambda_1\nu_2)^2 + (\lambda_1\mu_2 - \mu_1\lambda_2)^2\\ &= \begin{vmatrix}\mu_1 & \nu_1\\ \mu_2 & \nu_2\end{vmatrix}^2 + \begin{vmatrix}\nu_1 & \lambda_1\\ \nu_2 & \lambda_2\end{vmatrix}^2 + \begin{vmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{vmatrix}^2.\end{aligned}$$

254. Problem. *To find the coordinates of the point where the line segment joining two given points is divided in a given ratio.*



Let $P'(x', y', z')$ and $P''(x'', y'', z'')$ be the given points, and let $P(x, y, z)$ denote the point where $P'P''$ is divided in the given ratio $k_1 : k_2$.

The projections of the segments $P'P$ and PP'' upon the x-axis are proportional to the segments $P'P$ and PP'' themselves [§ 249], and these projections are $x - x'$ and $x'' - x$, respectively [§ 245]. Hence

$$k_1 : k_2 = P'P : PP'' = x - x' : x'' - x.$$

Therefore

$$k_1(x'' - x) = k_2(x - x')$$

or solving for x,

$$x = \frac{k_2 x' + k_1 x''}{k_1 + k_2}.$$

Similarly

$$y = \frac{k_2 y' + k_1 y''}{k_1 + k_2}, \quad z = \frac{k_2 z' + k_1 z''}{k_1 + k_2}.$$

255. In particular, if P be the mid-point of $P'P''$,

$$x = \frac{x' + x''}{2}, \quad y = \frac{y' + y''}{2}, \quad z = \frac{z' + z''}{2}.$$

256. Observe that k_1 is measured *from* P' *to* P, and that k_2 is measured *from* P *to* P'' so that when k_1 and k_2 have the same sign, P lies between P' and P'', but when k_1 and k_2 have opposite signs, P lies on $P'P''$ produced through P' or P''.

Example 1. The coordinates of the point where the line segment joining the points (5, 2, 2) and (8, 5, 4) is divided in the ratio (1 : 2) are found as follows (the line is divided internally, and is trisected):

$$\frac{(2)(5) + (1)(8)}{1 + 2} = 6, \quad \frac{(2)(2) + (1)(5)}{1 + 2} = 3, \quad \frac{(2)(2) + (1)(4)}{1 + 2} = \frac{8}{3}.$$

The other point of trisection is that at which the segment is divided in the ratio 2 : 1 and is

$$\frac{(1)(5) + (2)(8)}{2 + 1} = 7, \quad \frac{(1)(2) + (2)(5)}{2 + 1} = 4, \quad \frac{(1)(2) + (2)(4)}{2 + 1} = \frac{10}{3}.$$

Example 2. The point where the same line segment is divided in the ratio (4 : − 1) is

$$\frac{(-1)(5) + (4)(8)}{4 - 1} = 9, \quad \frac{(-1)(2) + (4)(5)}{4 - 1} = 6, \quad \frac{(-1)(2) + (4)(4)}{4 - 1} = \frac{14}{3}.$$

The line is divided externally beyond the point (8, 5, 4); one third of the segment is added to the line at the extremity (8, 5, 4).

257. Exercises. (When convenient draw a figure.)

1. What are the direction cosines of the positive half of the z-axis? of the negative half? of the bisector of the angle between the positive half of the x-axis and the negative half of the y-axis? of the bisector of the angle between this line and the positive half of the z-axis?

2. Find the length and direction cosines of the line segment from the origin to the point (3, 4, 12); also the projections of this line segment upon each of the coordinate axes and upon each of the coordinate planes.

3. Obtain the corresponding results for the line segment from the point (5, − 2, 7) to the point (2, 2, − 5).

4. Find the cosine of the angle between two lines whose direction cosines have the ratios 1 : 2 : 3 and 2 : 3 : 4, respectively.

5. Show that the lines whose direction cosines have the ratios 7 : − 2 : 4 and 2 : 1 : − 3 are perpendicular.

6. Find the direction cosines of the line which is perpendicular to the two lines whose direction cosines have the ratios 2 : − 1 : 1 and 1 : 2 : 3.

7. A line is drawn from the origin to each of the points (1, 2, 3) and (1, 3, 2). Find the direction cosines of the bisector of the angle between these two lines.

8. Find the projection of the line segment from the point (1, 4, 1) to the point (2, 2, − 1) upon the line whose direction cosines have the ratios 2 : 3 : 6.

9. Find the projection of the line segment from the point (3, 4, 5) to the point (2, − 1, 3) upon the line determined by the two points (1, 5, 4) and (3, 7, 2).

10. Find the coordinates of the points where the line segment from the point (1, − 1, 1) to the point (2, − 3, 2) is divided in each of the following ratios:

(1) 1 : 2, (2) 2 : 1, (3) 4 : − 1, (4) − 1 : 4.

11. In what ratio is the line segment joining the points (2, 3, 1) and (1, 5, − 2) cut by the xy-plane? What are the coordinates of the point of intersection?

12. Find the coordinates of the mid-point of the line segment joining the points (5, − 2, 7) and (2, 2, − 5); also the coordinates of the points where this line segment is trisected.

13. If the line segment from (2, 3, − 4) to (1, 4, 6) be produced until its length is doubled, what will the coordinates of its extremity be? What will the coordinates of the extremity be if the length be trebled?

14. Prove that the set of coordinates of the mass-center of a triangle whose angular points are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\tfrac{1}{3}(x_1 + x_2 + x_3),\ \tfrac{1}{3}(y_1 + y_2 + y_3),\ \tfrac{1}{3}(z_1 + z_2 + z_3).$$

15. Prove that the three lines joining the mid-points of opposite edges of a tetrahedron meet in a common point, whose coordinates are $\frac{1}{4}(x_1 + x_2 + x_3 + x_4)$, $\frac{1}{4}(y_1 + y_2 + y_3 + y_4)$, $\frac{1}{4}(z_1 + z_2 + z_3 + z_4)$, when the vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4). Prove also that this point lies on the line joining any angular point to the mass-center of the opposite face, and divides that line in the ratio 3 : 1; also that it is the mass-center of the tetrahedron.

16. Prove that the angle θ between two lines whose direction cosines are $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ is given by the equation:

$$4 \sin^2(\theta/2) = (\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2 + (\nu_1 - \nu_2)^2.$$

(HINT. Use $\cos\theta = 1 - 2\sin^2(\theta/2)$ and $\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$.)

17. In the figure of § 239 prove that $2\,P'P''^2 = P'E^2 + P'F^2 + P'G^2$.

18. A point P lies below the xy-plane, OP is 2, xOP is 45°, yOP is 60°; find the angle zOP, and obtain the coordinates of P.

19. Find the direction cosines of the bisector of the angle between the lines joining the points $A(4, 4, 7)$ and $B(3, 4, 12)$ to the origin O.

(HINT. The bisector passes through the point at which AB is divided in the ratio $OA : OB$.)

CHAPTER XIV

PLANES AND STRAIGHT LINES IN SPACE

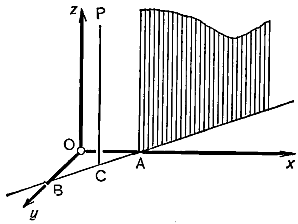
258. Loci of equations in x, y, z. An equation in any or all of the variables x, y, z will ordinarily be satisfied by infinitely many sets of real values of x, y, z. Every such set of values is called a *real solution* of the equation. Axes of reference and a unit of measurement having been chosen, to each real solution $x = a$, $y = b$, $z = c$, of the equation there will correspond a point of which (a, b, c) are the coordinates. The collection of all such points is called the *locus* of the equation, or the locus represented by the equation. Conversely, the equation is called the equation of this locus, or collection of points.

259. If an equation be multiplied throughout by a constant, its solutions and therefore its locus will not be affected.

260. The locus of the equation $x = a$ is the plane parallel to the yz-plane and at the distance $|a|$ from it, to the right or left according as a is positive or negative. For the equation $x=a$ is satisfied by the value a of x taken with any values whatsoever of y and z, and the collection of all points whose x-coordinate is a and whose y- and z-coordinates are any numbers whatsoever forms the plane just described.

Similarly, the locus of $y=b$ is a plane parallel to the zx-plane, and that of $z=c$ is a plane parallel to the xy-plane.

261. Cylindrical surfaces. The locus of $lx+my+n=0$ is a plane parallel to the z-axis and passing through the straight line which $lx+my+n=0$ represents in the xy-plane. For, if through any point C on this line a parallel to the z-axis be taken, the x- and y-coordinates of every point P on this parallel will be the same as those of the point C and will therefore satisfy the equation $lx+my+n=0$.



262. And, in general, the locus of any equation $f(x, y) = 0$, from which z is absent, is the surface which would be generated by a parallel to the z-axis, if made to move along the curve represented by $f(x, y) = 0$ in the xy-plane. Such a surface is called a *cylindrical surface*, and the line its *generating line*.

Equations of the forms $f(y, z) = 0$ and $f(z, x) = 0$ represent surfaces similarly related to the x- and y-axes, respectively. That is:

Every equation in but two of the three coordinates x, y, z represents a cylindrical surface parallel to one of the axes.

263. Equation of a surface. As an example of an equation containing all three variables, take $z = x^2 + y^2$. This equation is satisfied, if any pair of values a, b be assigned to x, y and the value $a^2 + b^2$ to z; hence, to every point $C(a, b, 0)$ in the xy-plane there corresponds a point $P(a, b, a^2+b^2)$ of the locus of $z = x^2 + y^2$, this point P being vertically above C and at the distance $a^2 + b^2$ from it. The collection of these points forms a surface. This particular surface $z = x^2 + y^2$ passes through the origin and lies above the xy-plane.

264. And, in general, the locus of an algebraic equation $f(x, y, z) = 0$, which contains all three variables, is a surface which is met by parallels to each axis in a finite number of points.

265. Simultaneous equations. The locus of a pair of simultaneous equations $f(x, y, z) = 0$, $\phi(x, y, z) = 0$ consists of the points which are common to the loci of the individual equations. If the loci of $f\lambda(x, y, z) = 0$ and $\phi(x, y, z) = 0$ are intersecting surfaces, the locus of the *pair* will be the curve or curves in which the surfaces intersect.

Thus, the locus of the pair of equations $x = a$, $y = b$ is the straight line in which the planes represented by $x = a$ and $y = b$ intersect. It is the parallel to the z-axis through the point $(a, b, 0)$.

266. If from a pair of equations $f(x, y, z) = 0$, $\phi(x, y, z) = 0$ one of the variables, as z, be eliminated, an equation of the form $F(x, y) = 0$ is obtained. This equation represents a cylindrical surface whose generating line is parallel to the z-axis and which passes through the curve of intersection of the surfaces represented by $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$. This cylindrical surface meets the xy-plane in the curve which the equation $F(x, y) = 0$ represents in that plane, and which is therefore the *projection* of the curve of intersection of the surfaces represented by $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$ [§ 114].

Thus, if between $x^2 + y^2 + z^2 = 4$ (1) and $z = x$ (2), z be eliminated, the equation $2x^2 + y^2 = 4$ (3) is obtained.

Equation (1) represents a sphere whose center is at the origin and whose radius is 2 [§ 237]. Equation (2) represents a plane through the y-axis. This sphere and plane intersect in a circle. Equation (3) represents in space the cylindrical surface which passes through this circle and whose generating line is parallel to the z-axis; and, with $z = 0$, it represents the ellipse in the xy-plane which is the projection of the circle upon that plane.

267. A set of three simultaneous algebraic equations $f(x, y, z) = 0$, $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$ will ordinarily have a finite number of solutions. Such of these solutions as are real have corresponding to them real points of intersection of the surfaces represented by the three equations.

268. Equations of the first degree. *The locus of every equation of the first degree in x, y, z is a plane.*

The general equation of the first degree in x, y, z is of the form

$$Ax + By + Cz + D = 0. \qquad (1)$$

Let (x', y', z') and (x'', y'', z'') denote any two solutions of this equation, so that

$$Ax' + By' + Cz' + D \equiv 0, \qquad (2)$$

$$Ax'' + By'' + Cz'' + D \equiv 0. \qquad (3)$$

Take any two constants k_1, k_2; multiply (2) by $k_2/(k_1 + k_2)$ and (3) by $k_1/(k_1 + k_2)$, and add. The result is

$$A\frac{k_1x'' + k_2x'}{k_1 + k_2} + B\frac{k_1y'' + k_2y'}{k_1 + k_2} + C\frac{k_1z'' + k_2z'}{k_1 + k_2} + D \equiv 0. \qquad (4)$$

It has thus been proved that, if (x', y', z') and (x'', y'', z'') be any two solutions of (1), so also is (x_0, y_0, z_0), where

$$x_0 = \frac{k_1x'' + k_2x'}{k_1 + k_2}, \quad y_0 = \frac{k_1y'' + k_2y'}{k_1 + k_2}, \quad z_0 = \frac{k_1z'' + k_2z'}{k_1 + k_2}.$$

But since k_1, k_2, denote *any* constants whatsoever, it follows from this [§ 254] that the locus of (1) has the property that, if any two of its points be joined by a straight line, all points of that line will be points of the locus. Therefore, since the locus of (1) is known to be a surface [§ 264], and the plane is the only surface having the property just mentioned, the locus is a plane, as was to be demonstrated.

269. In particular, the locus of $Ax + By + Cz = 0$ is a plane through the origin; that of $Ax + By + D = 0$ is a plane parallel to the z-axis; that of $Ax + By = 0$ is a plane containing the z-axis; that of $Ax + D = 0$ is a plane parallel to the yz-plane, and so on [compare § 260 and § 261].

270. To find the plane corresponding to any given equation of the first degree, proceed as follows:

If the equation lacks one or two of the variable terms, as is the case with $Ax + By + D = 0$, and $Ax + D = 0$, the methods explained in § 260 and § 261 are followed. If the equation is complete, the points where it meets the three coordinate axes are found. If it lacks only the constant term, the lines in which it cuts two of the coordinate planes are found; or any three of its points, not in a straight line, are found.

The straight line in which the plane corresponding to $Ax + By + Cz + D = 0$ cuts the xy-plane is found by setting $z = 0$, which gives $Ax + By + D = 0$, $z = 0$. The required line is that represented by $Ax + By + D = 0$ in the xy-plane. Similarly for the other coordinate planes.

The point in which the plane corresponding to

$$Ax + By + Cz + D = 0$$

cuts the x-axis is found by setting $y = 0$, $z = 0$, which gives $Ax + D = 0$, $y = 0$, $z = 0$. Hence the point is $(-D/A, 0, 0)$. Similarly for the other coordinate axes.

Example 1. Find three points in the plane corresponding to the equation $3x - y + 4z - 12 = 0$, and thus determine the plane.

When $y = 0$ and $z = 0$, then $x = 4$; when $z = 0$ and $x = 0$, then $y = -12$; when $x = 0$ and $y = 0$, then $z = 3$.

Hence the required plane is that determined by the three points $(4, 0, 0)$, $(0, -12, 0)$, $(0, 0, 3)$.

Example 2. Find two lines in the plane corresponding to the equation $3x - y + 4z = 0$, and thus determine the plane.

Setting first $x = 0$ and then $y = 0$ in this equation, we obtain $-y + 4z = 0$ and $3x + 4z = 0$. The line represented by $-y + 4z = 0$ in the plane $x = 0$, and that represented by $3x + 4z = 0$ in the plane $y = 0$, determine the required plane.

The equation $3x - y + 4z = 0$ is also satisfied for the points $(0,0,0)$, $(-1, 1, 1)$, and $(2, 2, -1)$; hence these three points also determine the required plane.

271. The locus of the equation

$$(Ax + By + Cz + D)(A'x + B'y + C'z + D') = 0$$

is the pair of planes corresponding to the two equations

$$Ax + By + Cz + D = 0 \quad \text{and} \quad A'x + B'y + C'z + D' = 0.$$

Thus, the locus of $(x - a)(x - b) = 0$ is the pair of planes corresponding to $x - a = 0$ and $x - b = 0$, both of which are parallel to the yz-plane.

272. Equations of planes. It has been proved that the locus of every equation of the first degree in x, y, z is a plane [§ 268]. Conversely, to every given plane there corresponds an equation of the first degree of which the plane is the locus; that is, *an equation which is true for every point on the plane and false for every point off the plane.* It is called *the equation of the plane* [compare § 16].

For, on the given plane take any three points which are not in the same straight line, and find their coordinates. There is one equation of the first degree, and but one, of which these three sets of coordinates are solutions, and this equation will be satisfied by the coordinates of every other point on the given plane.

Thus, suppose that the points (2, 0, 0), (0, 1, 1), (1, 1, −1) are to lie on the given plane, and let

$$Ax + By + Cz + D = 0 \tag{1}$$

represent the required equation.

Since (2, 0, 0), (0, 1, 1), and (1, 1, −1) are to be solutions of (1),

$$2A + D = 0, \quad (2) \quad B + C + D = 0, \quad (3) \quad A + B - C + D = 0. \quad (4)$$

Solving (2), (3), (4) for A, B, and C in terms of D, gives $A = -D/2$, $B = -3D/4$, $C = -D/4$. Substituting these values of A, B, and C in (1), and simplifying, gives $2x + 3y + z - 4 = 0$, the equation required.

273. It follows from what has just been said that, if a given equation of the first degree be true for three points of a certain plane, and these three points are not in a straight line, the given equation is the equation of that plane.

274. Plane through three given points. A plane is determined by any three given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), which are not in the same straight line. Its equation may be found as in the example above [§ 272], or as follows:

By § 268, the required equation is of the form

$$Ax + By + Cz + D = 0. \tag{1}$$

And since the plane is to pass through the given points, the equation must be satisfied by their coordinates. Hence

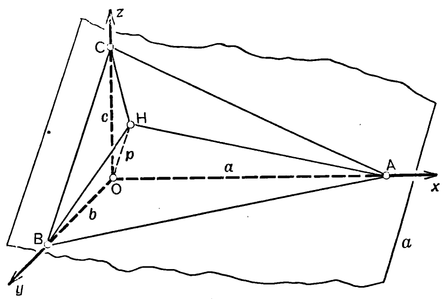
$$Ax_1 + By_1 + Cz_1 + D \equiv 0, \tag{2}$$
$$Ax_2 + By_2 + Cz_2 + D \equiv 0, \tag{3}$$
$$Ax_3 + By_3 + Cz_3 + D \equiv 0. \tag{4}$$

Eliminating A, B, C, D from these four equations,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

which is the required equation; as, indeed, is also obvious by inspection. [Compare § 20, Eq. (1′).]

275. Intercept form of the equation. Let a plane α meet the x-, y-, and z-axes in the points A, B, and C, respectively. Then OA, OB, and OC are called the x-, y-, and z-intercepts of the plane α, and are represented by a, b, and c, respectively. Evidently the plane is determined when its intercepts a, b, c are given. Its equation in terms of the intercepts may be obtained as follows:



The plane has an equation of the form [§ 272],

$$Ax + By + Cz + D = 0. \tag{1}$$

And since its intercepts are a, b, and c, it passes through the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. Hence (1) has the solutions $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, that is:

$$Aa + D = 0, \text{ or } A = -D/a, \tag{2}$$
$$Bb + D = 0, \text{ or } B = -D/b, \tag{3}$$
$$Cc + D = 0, \text{ or } C = -D/c. \tag{4}$$

Substituting these values for A, B, C in (1), dividing the resulting equation throughout by $-D$, and transposing, gives

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{5}$$

the equation required.

276 A.* Perpendicular form of the equation of the plane. *First Proof.* Let H be the foot of the perpendicular from the origin to the plane α, and let p and λ, μ, ν denote the length and direction cosines, respectively, of this perpendicular. Evidently the plane α is determined when λ, μ, ν, and p are given, and its equation in terms of λ, μ, ν, and p may be obtained as follows:

The equation in the intercept form [§ 275 (5)] may be written:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0, \tag{1}$$

and this equation multiplied throughout by p becomes:

$$\frac{p}{a}x + \frac{p}{b}y + \frac{p}{c}z - p = 0. \tag{2}$$

Since AOH is a right triangle, $\cos AOH = OH/OA$; or from the definition of the symbols, $\lambda = p/a$; and, similarly, $\mu = p/b$, $\nu = p/c$. Setting these values in the equation (2), gives:

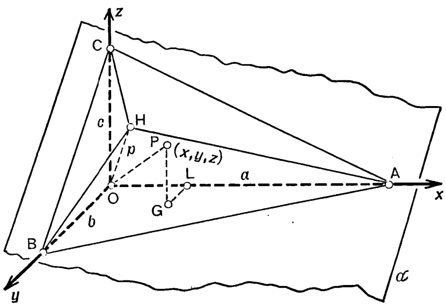
$$\lambda x + \mu y + \nu z - p = 0, \tag{3}$$

which is the equation required.

This proof fails when the plane passes through the origin or is parallel to one of the axes.

* Only one of § 276 A, § 276 B, § 276 C need be taken.

276 B. Perpendicular form of the equation of the plane. *Second Proof.* Let H be the foot of the perpendicular from the origin to the plane α, and let p and λ, μ, ν denote the length and direction cosines of this perpendicular. Evidently the plane α is determined when λ, μ, ν, and p are given; and its equation in terms of λ, μ, ν, and p may be obtained as follows:



Take any representative point $P(x, y, z)$ in the plane α and connect O with P, first by the line segment OP, and second by the broken line $OLGP$ made up of the line segments OL, LG, and GP, which represent the x-, y-, and z-coordinates of P, respectively.

Then [§ 246] the projection of OP upon OH is equal to the sum of the projections of OL, LG, and GP upon OH, or

$$pr_{OH}OP = pr_{OH}OL + pr_{OH}LG + pr_{OH}GP. \tag{1}$$

But since OH is perpendicular to the plane α, the projection of OP upon OH is OH itself, or p; and the projections of OL, LG, and GP upon OH are λx, μy, and νz, respectively [§ 248]. Hence equation (1) gives:

$$p = \lambda x + \mu y + \nu z, \tag{2}$$

or

$$\lambda x + \mu y + \nu z - p = 0, \tag{3}$$

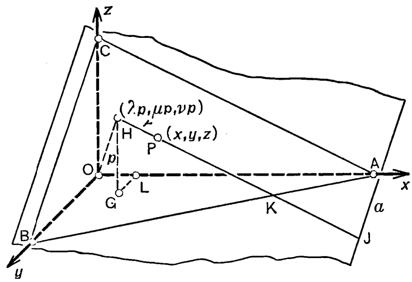
which is the equation required.

When α passes through the origin, p is zero, and (3) becomes

$$\lambda x + \mu y + \nu z = 0, \tag{3'}$$

where λ, μ, ν are the direction cosines of the perpendicular to α.

276 C. Perpendicular form of the equation of the plane. *Third Proof.* Take any line segment from the origin O to a point H. Let p be the length, and λ, μ, ν the direction cosines of OH. Then [§ 239 (4)], the coordinates of H are $(\lambda p,\ \mu p,\ \nu p)$. Let $P(x, y, z)$ be any point in space such that when P is joined to H, the angle OHP is a right angle. The direction cosines



$(\lambda_1, \mu_1, \nu_1)$ of HP are proportional to $(x-\lambda p,\ y-\mu p,\ z-\nu p)$ [§ 250]; and the direction cosines of OH are (λ, μ, ν). Since HP and OH are perpendicular, $\lambda\lambda_1+\mu\mu_1+\nu\nu_1=0$; and therefore

$$\lambda(x-\lambda p)+\mu(y-\mu p)+\nu(z-\nu p)=0, \tag{1}$$

or

$$\lambda x+\mu y+\nu z-(\lambda^2+\mu^2+\nu^2)p=0, \tag{2}$$

or

$$\lambda x+\mu y+\nu z-p=0. \tag{3}$$

Hence (3) is the equation of the locus of all points P, such that the lines joining them to H are perpendicular to OH. But this is the plane α, determined by p, λ, μ, ν.

The plane can be regarded as generated by the rotation of the line HP (produced indefinitely) about the perpendicular OH. The plane is sometimes defined as the surface generated in this manner.

277. *From the equation of a plane given in the general form $Ax+By+Cz+D=0$, its equation in the perpendicular form is derived by dividing by $\pm\sqrt{A^2+B^2+C^2}$, where the sign before the radical is opposite to that in D.*

Two equations which represent the same plane can only differ by a constant factor [§ 259]; hence, if the equation of the given plane in the perpendicular form be $\lambda x + \mu y + \nu z - p = 0$, then,

$$k(Ax + By + Cz + D) \equiv \lambda x + \mu y + \nu z - p,$$

where k denotes some constant.

This identity will be satisfied [Alg. § 285], if the following four equations are true:

$$k \cdot A = \lambda \ (1), \quad k \cdot B = \mu \ (2), \quad k \cdot C = \nu \ (3), \quad k \cdot D = -p \ (4).$$

The equation (4) requires that k shall have the algebraic sign opposite to that in D. Squaring (1), (2), and (3), and adding the results gives [§ 240],

$$k^2(A^2 + B^2 + C^2) = \lambda^2 + \mu^2 + \nu^2 = 1;$$

whence,
$$k = \frac{1}{\pm\sqrt{A^2 + B^2 + C^2}},$$

where the $\pm$ sign is opposite to the sign in D. The substitution of this value of k in (1), (2), (3), and (4), gives the expressions for λ, μ, ν, and p, in terms of the coefficients A, B, C, D.

Example. Reduce the equation $2x - 3y + 6z - 12 = 0$ to the perpendicular form.

Here, $\pm\sqrt{A^2 + B^2 + C^2} = \pm\sqrt{4 + 9 + 36} = \pm 7$, and since $D\ (= -12)$ is negative, the $\pm$ sign is to be taken as positive, and $k = 1/7$. Hence the required equation is:

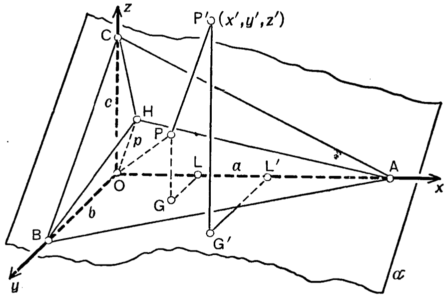
$$\frac{2x - 3y + 6z - 12}{7} = 0, \text{ or } \frac{2}{7}x - \frac{3}{7}y + \frac{6}{7}z - \frac{12}{7} = 0,$$

and the values of λ, μ, ν, and p are 2/7, −3/7, 6/7, and 12/7, respectively.

278. From § 277 (1), (2), (3) the important conclusion follows:

In the equation of any plane $Ax + By + Cz + D = 0$, the coefficients A, B, C are proportional to the direction cosines λ, μ, ν of a line perpendicular to the plane.

279. Perpendicular distance from a plane to a point. Let $P'(x', y', z')$ be a given point, and α a given plane; and let P be the foot of the perpendicular through P' to the plane. It is required to find an expression for the length of the perpendicular PP'.



Join the point P and the origin O. Let x', y', z', the coordinates of P', be OL', $L'G'$, $G'P'$, respectively.

Let H be the foot of the perpendicular from the origin on the plane, and let the length and direction cosines of OH be p, λ, μ, ν, respectively. Then [§ 246] the projection on OH of the broken line PO, OL', $L'G'$, $G'P'$ is the same as the projection on OH of the line PP'; that is:

$$pr_{OH}PP' = pr_{OH}PO + pr_{OH}OL' + pr_{OH}L'G' + pr_{OH}G'P'. \quad (1)$$

But since PP' and OH are parallel, $pr_{OH}PP'$ is PP' itself; and, since OH is perpendicular to α, $pr_{OH}PO$ is HO or $-p$; also, $pr_{OH}OL'$ is $\cos L'OH \cdot OL'$ [§ 248] or $\lambda x'$; similarly, $pr_{OH}L'G'$ is $\mu y'$, and $pr_{OH}G'P'$ is $\nu z'$. Setting these values in (1) and transposing, gives

$$PP' = \lambda x' + \mu y' + \nu z' - p. \quad (2)$$

If P' be on the plane, then $PP' = 0$, and $P'(x', y', z')$ coincides with $P(x, y, z)$, and the equation (2) reduces to

$$\lambda x + \mu y + \nu z - p = 0, \quad (3)$$

the equation of the plane α [§ 276 (3)]. Therefore, comparing the equations (2) and (3), it follows that the perpendicular distance from the plane represented by $\lambda x + \mu y + \nu z - p = 0$ to the point $P'(x', y', z')$ is obtained by merely substituting x', y', z' for x, y, z in the left member of this equation.

If the equation of the plane be $Ax + By + Cz + D = 0$, it can be reduced to the perpendicular form by dividing by $\pm\sqrt{A^2 + B^2 + C^2}$ [§ 277]; and therefore:

The perpendicular distance of the point $P'(x', y', z')$ from the plane $Ax + By + Cz + D = 0$ is given by

$$PP' = \frac{Ax' + By' + Cz' + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \tag{4}$$

where the sign before the radical is opposite to that in D.

280. It is obvious from § 279 (2) and the figure that the number represented by $(\lambda x' + \mu y' + \nu z' - p)$ is negative or positive according as $P'(x', y', z')$ lies on the same side of the plane as the origin O (0, 0, 0) or on the opposite side; and conversely. And in the same way, from § 279 (4), it follows that, when D is negative (and therefore the sign before the radical is $+$), the point $P'(x', y', z')$ will lie on the origin side of the plane $Ax + By + Cz + D = 0$ or on the opposite side, according as $(Ax' + By' + Cz' + D)$ is negative or positive.

Example. Find the perpendicular distances of the points $E(1, 3, 4)$, $J(1, 2, -1)$, $K(-3, 3, 2)$ from the plane $2x - 3y + 6z + 3 = 0$.

Here, $\pm\sqrt{A^2+B^2+C^2} = \pm\sqrt{4 + 9 + 36} = -7$ (since $D = +3$). Hence, the perpendicular distance to E is $(2 \cdot 1 - 3 \cdot 3 + 6 \cdot 4 + 3)/(-7) = -20/7$; the perpendicular to J is $\{2 \cdot 1 - 3 \cdot 2 + 6 \cdot (-1) + 3\}/(-7) = +1$; and finally that to K is $\{2 \cdot (-3) - 3 \cdot 3 + 6 \cdot 2 + 3\}/(-7) = 0$; that is, the point K is on the plane. As the sign of the perpendicular to E is negative, the point E lies on the same side of the plane as the origin; as the sign of the perpendicular to J is positive, J lies on the side of the plane remote from the origin; and therefore E and J are on opposite sides of the plane.

281. Parallel planes. Since two planes perpendicular to the same line are parallel, it follows from § 278 that

The planes represented by the equations $Ax + By + Cz + D = 0$, $A'x + B'y + C'z + D' = 0$ are parallel, if $A : B : C = A' : B' : C'$.

Hence, in particular, the following two theorems, § 282 and § 283.

282. Every plane parallel to the plane $Ax+By+Cz+D=0$ may be represented by an equation of the form

$$Ax+By+Cz+k=0,$$

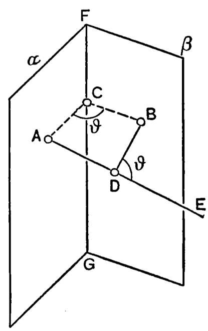
where k is an arbitrary constant.

283. The equation of the plane through the point (x', y', z') and parallel to the plane $Ax+By+Cz+D=0$ (1) is

$$A(x-x')+B(y-y')+C(z-z')=0; \qquad (2)$$

for (2) is satisfied by $x=x'$, $y=y'$, $z=z'$, and the coefficients of x, y, z in the two equations are identical.

284. The angle between two planes. Let C be any point in the line of intersection FG of two planes α and β, and let the plane through C perpendicular to FG cut the planes α and β in CA and CB, respectively; the angle ACB or θ is called the plane angle of the two planes.



The perpendiculars to the two planes α and β at A and B, respectively, will meet, say at D. Then in the plane quadrilateral $ACBD$, since the angles at A and B are right angles, the angle at C equals the exterior angle at D. Therefore the angle between the two planes is the same as the angle between any two perpendiculars to the planes.

Hence, when the equations of the planes are given in the perpendicular form,

$$\lambda x+\mu y+\nu z-p=0, \qquad \lambda' x+\mu' y+\nu' z-p'=0,$$

since λ, μ, ν and λ', μ', ν' are the direction cosines of the perpendiculars to the planes, the angle θ between the two planes is given by the formula [§ 251],

$$\cos\theta=\lambda\lambda'+\mu\mu'+\nu\nu';$$

and it follows [§ 277] that for the planes

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0,$$

the corresponding formula is

$$\cos\theta = \frac{AA' + BB' + CC'}{\pm\sqrt{A^2 + B^2 + C^2} \cdot \pm\sqrt{A'^2 + B'^2 + C'^2}}.$$

285. Perpendicular planes. The planes $Ax + By + Cz + D = 0$ and $A'x + B'y + C'z + D' = 0$ are *perpendicular* ($\cos\theta = 0$), if

$$AA' + BB' + CC' = 0.$$

286. Direction cosines of the line of intersection of two planes. The direction cosines λ, μ, ν of the line of intersection of two given planes

$$Ax + By + Cz + D = 0 \quad \text{and} \quad A'x + B'y + C'z + D' = 0$$

may be found as follows:

Since a line perpendicular to a plane is perpendicular to every line in that plane, the perpendiculars to the two given planes are perpendicular to their line of intersection. Therefore, since the direction cosines of the perpendiculars to the given planes are proportional to A, B, C and A', B', C', respectively [§ 278], it follows [§ 252] that:

$$A\lambda + B\mu + C\nu = 0,$$
$$A'\lambda + B'\mu + C'\nu = 0.$$

Solving these equations for $\lambda : \mu : \nu$ [Alg., § 921] gives:

$$\lambda : \mu : \nu = \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} : \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} : \begin{vmatrix} A & B \\ A' & B' \end{vmatrix},$$

and the ratios of λ, μ, ν being thus known, λ, μ, ν themselves can be found by dividing by the square root of the sum of the squares [§ 241].

Example. Find the direction cosines of the line of intersection of the planes

$$2x - 3y + z + 2 = 0, \quad x + 4y - 2z + 3 = 0.$$

Here

$$\lambda : \mu : \nu = \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix} : \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} : \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 2 : 5 : 11;$$

and the square root of the sum of the squares of these numbers, 2, 5, 11, is $\sqrt{150}$; hence

$$\lambda = 2/\sqrt{150}, \quad \mu = 5/\sqrt{150}, \quad \nu = 11/\sqrt{150}.$$

287. Planes through the line of intersection of two given planes. *If E and E_0 denote two expressions of the first degree in x, y, z, and k is an arbitrary constant, then $E + kE_0 = 0$ will represent the system of planes through the line of intersection of the planes represented by $E = 0$ and $E_0 = 0$.*

For, whatever the value of k may be, $E + kE_0 = 0$ represents a plane, since it is of the first degree in x, y, z; and this plane will pass through the line of intersection of the planes $E = 0$ and $E_0 = 0$, since for points of this line both E and E_0 are 0, and therefore $E + kE_0 = 0$ is satisfied.

Conversely, every plane, α, through the line of intersection of the planes $E = 0$ and $E_0 = 0$, is included among the planes represented by $E + kE_0 = 0$. For, if (x', y', z') denote any point of α not on the line of intersection of $E = 0$ and $E_0 = 0$, such a value can be given to k that $E + kE_0 = 0$ will be true for this point, and since $E + kE_0 = 0$ will then be true for three points of α which are not in the same straight line, it will be the equation of α [§ 273].

Example 1. Find the equation of the plane through the line of intersection of $x + 2y + 3z - 4 = 0$ and $2x + y - 4z + 5 = 0$, and through the point (2, 1, 4).

The required plane has an equation of the form

$$x + 2y + 3z - 4 + k(2x + y - 4z + 5) = 0,$$

and this equation has the solution (2, 1, 4). Hence

$$2 + 2 + 3 \cdot 4 - 4 + k(2 \cdot 2 + 1 - 4 \cdot 4 + 5) = 0, \quad \text{or} \quad k = 2.$$

Therefore the required equation is

$$x + 2y + 3z - 4 + 2(2x + y - 4z + 5) = 0, \text{ or } 5x + 4y - 5z + 6 = 0.$$

Example 2. Find the equation of the plane through the line of intersection of $2x + y + 3z = 0$ and $x - 2y = 0$, and perpendicular to the plane $3x + y - 2z = 0$.

The required plane has an equation of the form

$$2x + y + 3z + k(x - 2y) = 0, \quad \text{or} \quad (2 + k)x + (1 - 2k)y + 3z = 0,$$

and, since it is perpendicular to the plane $3x + y - 2z = 0$, by § 285,

$$3(2 + k) + (1 - 2k) - 2 \cdot 3 = 0, \quad \text{or} \quad k = -1.$$

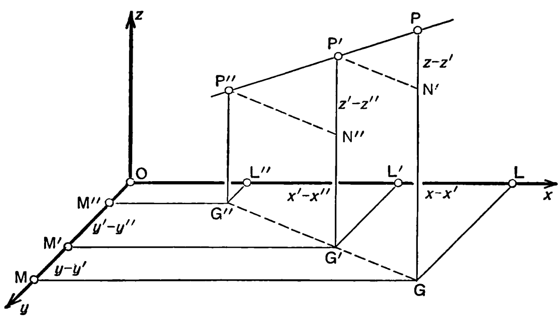
Therefore the required equation is

$$2x + y + 3z - (x - 2y) = 0, \quad \text{or} \quad x + 3y + 3z = 0.$$

288. Equations of a line. It has been seen already [§ 265] that the locus of a pair of simultaneous equations of the first degree in x, y, z is a straight line, namely the line of intersection of the planes represented by the individual equations.

Conversely, to represent any given straight line in space, a *pair* of simultaneous equations is required, but these may be the equations of *any* two planes through the given line. Any such pair of equations are called *the equations of the line*, since both of them are true for every point on the line and at least one of them is false for every point off the line.

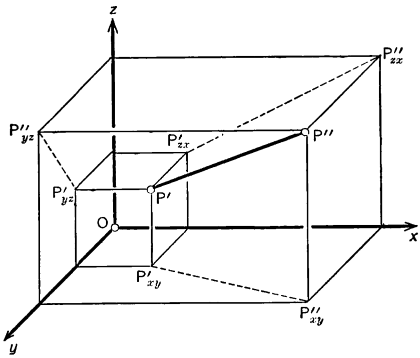
289. Equations of the line through two points. Let $P'(x', y', z')$ and $P''(x'', y'', z'')$ be two given points, and $P(x, y, z)$ any



representative point on the line determined by P' and P''. The ratio of the line segments $P'P$ and $P''P'$ is equal to that of their projections on each of the three coordinate axes [§ 249]. Hence the latter three ratios are equal to one another, that is [§ 245],

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''}=\frac{z-z'}{z'-z''}. \qquad (1)$$

These are the equations of the line $P'P''$. For besides x, y, z they involve only the known quantities, x', y', z', x'', y'', z''; they are true for every point on $P'P''$; and, as may readily be shown, they are false for every point off $P'P''$.



Observe that the individual equations

$$\frac{y-y'}{y'-y''}=\frac{z-z'}{z'-z''} \qquad (2),$$

$$\frac{z-z'}{z'-z''}=\frac{x-x'}{x'-x''} \qquad (3),$$

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''} \qquad (4),$$

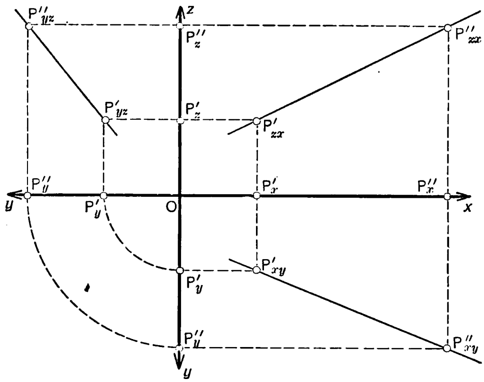
represent the planes through the given line and perpendicular to the yz-, zx-, and xy-planes, respectively, that is, the planes $P'P''P_{yz}''P_{yz}'$, $P'P''P_{zx}''P_{zx}'$, $P'P''P_{xy}''P_{xy}'$, respectively; these are called the projecting planes of the line. The equation (2) combined with $x=0$ represents the projection of the line on the yz-plane; and similarly for the other two equations.

The equations (3) and (4) can be solved for y or z in terms of x, and they then take the form:

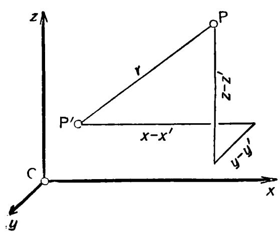
$$z=mx+b \ (3'), \qquad y=nx+c \ (4'),$$

where m, b, n, and c depend only on the coordinates (x', y', z'), (x'', y'', z'') of the points determining the line. These equations (3′) and (4′) involve four, and only four, constants, and therefore prove that four conditions are necessary and sufficient to determine a straight line in space.

290. The *Descriptive Geometry*, invented by Monge in 1794, represents a figure by its orthogonal projections in two perpendicular planes, drawn correctly to scale. Instead of attempting to depict a figure in space, it is the practice to indicate it by its *plan*, namely, its projection on the horizontal (xy) plane, and its *elevations*, the projections on the upright (zx or yz) planes. It is customary to consider the xy-plane turned down about the x-axis until it lies with the zx-plane in the plane of the drawing, and the yz-plane turned back to the left about the z-axis until it also lies with the zx-plane. The situation of the line $P'P''$ is thus indicated by the figure here given, where the line $P_{zx}'P_{zx}''$ is the line given by $y = 0$ and the equation (3) of § 289, and the line $P_{xy}'P_{xy}''$ is the line given by $z = 0$ and the equation (4) of § 289.



291. Line through a given point and having a given direction. Symmetric equations. If a point $P'(x', y', z')$ on a line, and the direction cosines λ, μ, ν of the line be given, its equations may be found as follows:



Let $P(x, y, z)$ denote any representative point on the line. The projections of the line segment $P'P$ on the x-, y-, and z-axes are equal to $x - x'$, $y - y'$, and $z - z'$, respectively [§ 245]. But since $P'P$ makes with the axes angles whose cosines are λ, μ, ν, respectively, it follows [§ 248] that these projections are equal to $\lambda \cdot P'P$, $\mu \cdot P'P$, $\nu \cdot P'P$, respectively.

Hence, if $P'P$ be represented by r,

$$x - x' = \lambda r, \quad y - y' = \mu r, \quad z - z' = \nu r. \tag{1}$$

Equating the values of r given by these three equations,

$$\frac{x - x'}{\lambda} = \frac{y - y'}{\mu} = \frac{z - z'}{\nu} (= r), \tag{2}$$

which are the equations required. They are often called the *symmetric equations* of a line. It should be noted that the denominators in the equations of § 289 are only *proportional* to the direction cosines of the line.

292. Observe that it also follows from the equations (1) of the preceding section that the coordinates of any point P on the line may be expressed in terms of r, the distance of P from the fixed point P', by the formulas

$$x = x' + \lambda r, \quad y = y' + \mu r, \quad z = z' + \nu r. \tag{1}$$

293. From any given equations of a line, its symmetric equations may be derived as in the following example.

Example. Find the symmetric equations of the line of intersection of the planes

$$2x + 3y - z + 4 = 0 \quad \text{and} \quad 2x - 3y - 5z - 8 = 0.$$

Combining the given equations, first so as to eliminate y, and second so as to eliminate x, the following equivalent pair is obtained,

$$2x - 3z - 2 = 0 \quad \text{and} \quad 3y + 2z + 6 = 0.$$

Equating the values of z given by these equations,

$$\frac{2x - 2}{3} = \frac{3y + 6}{-2} = z, \quad \text{or} \quad \frac{2(x-1)}{3} = \frac{3(y+2)}{-2} = z,$$

and therefore

$$\frac{x-1}{3/2} = \frac{y+2}{-2/3} = \frac{z}{1}, \quad \text{or} \quad \frac{x-1}{9} = \frac{y-(-2)}{-4} = \frac{z-0}{6}.$$

Hence the line passes through the point $(1, -2, 0)$. Its direction cosines are proportional to 9, -4, 6, and are therefore [§ 241] equal to $9/\sqrt{133}$, $-4/\sqrt{133}$, $6/\sqrt{133}$.

Therefore the required equations in the symmetric form [§ 291, (2)] are

$$\frac{x-1}{9/\sqrt{133}}=\frac{y-(-2)}{-4/\sqrt{133}}=\frac{z-0}{6/\sqrt{133}}.$$

The reduction to the symmetric form may also be made by finding any solution (x', y', z') of the given equations, and λ, μ, ν by the method of § 286.

Thus, setting $z=0$ in the given equations and then solving for x, y, the solution $(x'=1, y'=-2, z'=0)$ is obtained. Again, by § 286, $\lambda:\mu:\nu:=9:-4:6$ and therefore $\lambda=9/\sqrt{133}$, $\mu=-4/\sqrt{133}$, $\nu=6/\sqrt{133}$. Substituting these values of x', y', z', λ, μ, ν in the equations, § 291 (2), the same result is obtained as before.

294. Intersections of lines and planes. A system of three simultaneous equations of the first degree

$$A_1x+B_1y+C_1z+D_1=0$$
$$A_2x+B_2y+C_2z+D_2=0$$
$$A_3x+B_3y+C_3z+D_3=0$$

will ordinarily have one, and but one, solution and this solution will be finite. The point corresponding to this one solution is the one point of intersection of the planes represented by the three equations, or of the line represented by any two of them with the plane represented by the third.

295. But the following two exceptional cases may present themselves [Alg. § 394]:

1. The three equations *may not be independent*, that is, their left members E_1, E_2, E_3 may be connected by an identical relation of the form

$$k_1E_1+k_2E_2+k_3E_3\equiv 0, \qquad (1)$$

where k_1, k_2, k_3 denote constants (one of which may be 0).

In this case every solution of two of the equations is a solution of the third, the geometrical meaning of which is that the planes represented by the three equations meet in a common line.

The same thing may be seen by writing the equation (1) in the form $E_1 + (k_2/k_1)E_2 \equiv (-k_3/k_1)E_3$, which states that $E_1 + (k_2/k_1)E_2 = 0$ [§ 259] is the plane $E_3 = 0$; or, what is the same, that the plane $E_3 = 0$ is a plane through the intersection of $E_1 = 0$ and $E_2 = 0$.

2. The three equations *may not be consistent*, that is, their left members E_1, E_2, E_3 may be connected by an identical relation of the form

$$k_1 E_1 + k_2 E_2 + k_3 E_3 + l \equiv 0, \quad \text{where} \quad l \neq 0. \tag{2}$$

In this case the equations have no common finite solution, since it would follow from $E_1 = E_2 = E_3 = 0$ that $l = 0$, which is contrary to hypothesis. The geometrical meaning of this is, that the lines in which the planes represented by the equations intersect, two and two, are parallel, or that two of the planes themselves are parallel.

The same thing may be seen by writing (2) in the form $E_1 + (k_2/k_1)E_2 \equiv (-k_3/k_1)(E_3 + l/k_3)$, which states that the plane $E_1 + (k_2/k_1)E_2 = 0$ [§ 259] is the plane $E_3 + l/k_3 = 0$ [§ 282]. That is, (if the planes $E_1 = 0$ and $E_2 = 0$ intersect) the plane $E_1+(k_2/k_1)\ E_2 = 0$ is parallel to $E_3 = 0$, or the intersection of $E_1 = 0$ and $E_2 = 0$ is parallel to $E_3 = 0$.

One may readily find whether the given equations are consistent and independent or not by solving them. Thus, if they be combined so as to eliminate x and y, an equation in z of the form $az = b$ is obtained. If $a \neq 0$, the given equations are independent and consistent; if $a = 0$ and $b \neq 0$, they are not consistent; if $a = 0$ and $b = 0$, they are not independent.

Example 1. Find the intersection of the line

$$x - 2y + 4z + 4 = 0, \quad (1) \qquad x + y + z - 8 = 0, \quad (2)$$

with the plane

$$x - y + 2z + 1 = 0. \tag{3}$$

The solution of the system of equations (1), (2), (3) is $x = 2$, $y = 5$,

$z=1$. Hence the line (1), (2) *meets* the plane (3) in the point (2, 5, 1).

Example 2. Find the intersection of the line

$$x-2y+4z+4=0, \quad (1) \qquad x+y+z-8=0, \quad (2)$$

with the plane $$x+2z-4=0. \qquad (4)$$

From (1) and (4), $x=4-2z$, $y=4+z$.

The substitution of these values, $x=4-2z$, $y=4+z$ in (2) gives

$$4-2z+4+z+z-8=0.$$

But this is an identity (it may be written $0 \cdot z+0=0$) and is therefore satisfied by every value of z.

Hence the line (1), (2) *lies in* the plane (4).

The left members of (1), (2), (4) must therefore be connected by an identity of the form $k_1E_1+k_2E_2+k_3E_3 \equiv 0$. And, in fact,

$$(x-2y+4z+4)+2(x+y+z-8)-3(x+2z-4) \equiv 0.$$

Example 3. Find the intersection of the line

$$x-2y+4z+4=0, \quad (1) \qquad x+y+z-8=0, \quad (2)$$

with the plane $$x+2z=0. \qquad (5)$$

From (1) and (5), $x=-2z$, $y=z+2$.

The substitution of these values, $x=-2z$, $y=z+2$, in (2), gives

$$-2z+z+2+z-8=0 \quad \text{or} \quad 0 \cdot z-6=0.$$

But this equation has no finite root; its root is ∞ [Alg. § 522].

Hence the line (1), (2) is *parallel to* the plane (5).

The left members of (1), (2), (5) are connected by an identity of the form $k_1E_1+k_2E_2+k_3E_3+l \equiv 0$, namely

$$(x-2y+4z+4)+2(x+y+z-8)-3(x+2z)+12 \equiv 0.$$

296. The necessary and sufficient condition that the four planes represented by the equations

$$A_1x+B_1y+C_1z+D_1=0, \quad A_2x+B_2y+C_2z+D_2=0,$$
$$A_3x+B_3y+C_3z+D_3=0, \quad A_4x+B_4y+C_4z+D_4=0,$$

shall meet in a common point is

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}=0.$$

For, the condition that the four planes shall meet in a common point is that their equations shall have a common solution [Alg. § 922].

This, of course, is also the condition that the line represented by any two of the equations shall meet the line represented by the remaining two.

297. Exercises. Planes and straight lines.

1. Find the equation of the plane whose x-, y-, and z-intercepts are 3, 2, and -1, respectively.

2. Find the equation of the plane through the three points (0, 0, 0), (0, 1, 2), and (1, -1, 3). What are the x-, y-, and z-intercepts of this plane?

3. Find the equation of the plane through the three points (1, 1, 1), (1, -1, 1), and (-7, -3, -5). Show that the plane is parallel to the y-axis, and find its x- and z-intercepts.

4. Prove that the four points (1, 2, 3), (2, 4, 1), (-1, 0, 1), (0, 0, 5) lie in one plane, and find the equation of this plane.

5. For what value of z' will the four points (1, 2, -1), (3, -1, 2), (2, -2, 3), (1, -1, z') lie in one plane, and what is the equation of this plane?

6. Find the equations of the lines in which the three coordinate planes are cut by the plane $3x - 2y + 7z + 5 = 0$.

7. Find the equation of the plane the direction cosines of whose normal are proportional to 3, -1, 2, and whose distance from the origin is 5.

8. Find the distance from the origin, and the x-, y-, z-intercepts of the plane $8x - 4y + z - 72 = 0$.

9. Find the distance of the point (3, -1, 2) from the plane $2x - y + 2z - 15 = 0$.

10. Find the distance of the point (2, 1, -3) from the plane $x + 2y + 3z - 5\sqrt{14} = 0$.

11. What is the equation of the plane through the point (5, -2, 7) and parallel to the plane $2x + 3y - 2z + 4 = 0$?

12. Find the equation of the plane through the point (4, -2, 5) and parallel to the plane $2x - y + 2z + 7 = 0$. What is the distance between these two parallel planes?

13. What is the character of the locus of each of the following equations: $f(y)=0$, $ay+bz+c=0$, $x^2+y^2=a^2$, $y^2+z^2=b^2$, $z^2+x^2=c^2$, $f(z, x)=0$, $x^2+y^2+z^2=a^2$?

14. How do the points $(1, 2, -1)$, $(3, -1, 2)$, $(2, -2, 3)$, and $(1, -1, 2)$ lie with respect to the plane $x+2y-z+3=0$?

15. Find the equation of the plane through the origin and through the intersection of the planes $2x+3y-2z+4=0$ and $x-2y+4z-3=0$.

16. A plane passes through the line of intersection of the planes $3x+4y-2z+5=0$ and $x-2y+z+7=0$, and its z-intercept is -3; find its equation.

17. A plane passes through the line of intersection of the planes $2x-3y-z-3=0$ and $x+2y+4z+1=0$, and its x- and y-intercepts are equal; find its equation.

18. Find the equation of the plane determined by the origin and the line $(x-2)/1=(y+2)/2=(z-1)/-2$.

19. Prove that the line $x+3y-z+1=0$, $2x-y+2z-3=0$ lies in the plane $7x+7y+z-3=0$.

20. Find the equation of the plane through the two points $(2, 0, 1)$, $(-1, 1, 2)$, and perpendicular to the plane $3x+y-z=0$.

21. The two planes $2x+3y-6z+3=0$ and $8x-y+4z-5=0$ are given. Find the cosine of the angle between them. Does the origin lie in the acute angle or in the obtuse angle between these planes?

22. Find the sine of the angle made by the line $x/2=y/3=z/-1$ with the plane $2x+y-3z=0$.

23. Find the cosine of the angle between the two lines

$$\frac{x-2}{2}=\frac{y+2}{2}=\frac{z-1}{1} \quad \text{and} \quad \frac{x+1}{1}=\frac{y-1}{-1}=\frac{z+3}{1}.$$

24. Prove that the following two lines are perpendicular:

$$\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-1}{-1} \quad \text{and} \quad \frac{x+2}{1}=\frac{y-4}{2}=\frac{z+3}{8}.$$

25. Do the following points lie on a line: $(2, 4, 6)$, $(4, 6, 2)$, $(1, 3, 8)$?

26. For what value of k will the following three points lie on one line: $(k, -3, 10)$, $(2, -2, 3)$, $(6, -1, -4)$?

27. Is there a value of k for which the following three points lie on one line: $(k, 1, -2)$, $(2, -2, k)$, $(-2, -1, 3)$?

28. Find the equations of the lines through each of the following pairs of points: (1, 1, 1) and (2, 0, 3); (– 1, 5, – 5) and (– 1, 2, – 5); (8, 5, 4) and (5, 2, 2).

29. Find the equations of the line which passes through the point (2, – 1, 5), and whose direction cosines are proportional to 1, – 2, 2.

30. Find the cosine of the angle between the line joining the points (1, – 2, 4), (2, – 1, 3) and the line $x/3 = (y+1)/1 = (z-9)/-2$.

31. Find the cosine of the angle between the line joining the points (3, – 1, 0), (1, 2, 1) and the line joining the points (–2, 0, 1), (1, 2, 0).

32. Find the projection of the line segment from the point (2, – 5, 1) to the point (4, – 1, 5) upon the line $(x-1)/2 = (y+2)/-1 = (z+5)/2$; also upon the plane $2x - y + 2z = 0$.

33. Find the equation of the plane through the point (1, – 2, 1) and perpendicular to the line $x - 2 = (y+1)/-4 = z/8$.

34. Find the equation of the plane through the origin and perpendicular to the line $3x - y + 4z + 5 = 0$, $x + y - z = 0$.

35. Find the length of the perpendicular from the point (5, – 2, – 1) to the plane $8x - y + 4z + 27 = 0$; also the equations of the line of which this perpendicular is a segment.

36. Reduce the equations of the line $x - y + z - 5 = 0$, $2x - y - z - 4 = 0$ to the symmetric form.

37. Reduce to the symmetric form the equations of the line of intersection of the planes $1 - 2x + 3y - 5z = 0$ and $1 + x - y + 3z = 0$.

38. Find the value of k for which the following lines are perpendicular:

$$\frac{x-3}{2k} = \frac{y+1}{k+1} = \frac{z-3}{5} \text{ and } \frac{x-1}{3} = y + 5 = \frac{z+2}{k-2}.$$

39. Find the values of k for which the following planes are perpendicular: $kx - 5y + (k+6)z + 3 = 0$ and $(k-1)x + ky + z = 0$.

40. Find the equations of the line l through the point (2, 3, 4) and, (1) equally inclined to the axes of reference, (2) meeting the y-axis at right angles, (3) perpendicular to the zx-plane.

41. Find the equation of the plane through the point (1, 4, 3) and perpendicular to the line of intersection of the planes $3x + 4y + 7z + 4 = 0$ and $x - y + 2z + 2 = 0$. Also the equations of the line through the given point and parallel to the line of intersection of the given planes.

42. Find the equations of the projections of the line $3x - 2y + z - 4 = 0$, $x - 2y - 3z + 1 = 0$ upon each of the coordinate planes.

43. Find the equations of the planes which bisect the angles between the planes $2x - 3y + 4z = 0$, and $4x - 2y - 3z - 2 = 0$.

44. Find the direction cosines of the line of intersection of the planes given in the last exercise; also the cosine of the obtuse angle between the planes. Does the point (1, 1, 1) lie in this angle?

45. Find the point of intersection of the three planes $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$, and $2x - y + z - 2 = 0$.

46. Find the point (or points) where the plane $x + 2y - z + 3 = 0$ is met by the line $3x - y + 2z + 1 = 0$, $2x - 3y + 3z - 2 = 0$.

47. For what value of D do the following four planes meet in a common point: $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$, $2x - y + z - 2 = 0$, $x + y - z + D = 0$?

48. Prove that the line $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$ meets the line $2x - 2y + 3z - 2 = 0$, $x - y - z + 3 = 0$.

49. Find the point where the line $(x - 3)/2 = (y - 4)/3 = (z - 5)/6$ meets the plane $x + y + z = 0$. How far is this point from the point (3, 4, 5)?

50. Do the planes $2x + 5y + 3z = 0$, $7y - 5z + 4 = 0$, and $x - y + 4z - 2 = 0$ pass through the same straight line? Prove that the first two of these planes and the plane $x - y + 4z = 8$ intersect in parallel lines.

51. Find the perpendicular distance from the point (1, 4, − 4) to the line $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$. Also the distance from the point (− 2, 1, 3) to this line.

52. Find the equation of the plane through the points (1, − 1, 2), (3, 0, 1) and parallel to the line $x + y - z = 0$, $2x + y + z = 0$.

53. Find the equation of the locus of a point whose distance from the z-axis is twice its distance from the xy-plane.

54. Find the equation of the locus of a point whose distance from the origin is three times its distance from the plane $x - 2y + 2z = 0$.

55. Prove that the equation of the plane through the line $x/l = y/m = z/n$ and perpendicular to the plane of the two lines $x/m = y/n = z/l$ and $x/n = y/l = z/m$ is $(m - n)x + (n - l)y + (l - m)z = 0$.

56. Let $(x - \alpha)/\lambda = (y - \beta)/\mu = (z - \gamma)/\nu$ (1) be any line in space, and let the point (α, β, γ) be called C, and let A (x', y', z') be any point in space. Find the length and direction cosines of CA; then find the square of the sine of the angle between (1) and CA; and thus prove that the square of the distance from the point (x', y', z') to the line $(x-\alpha)/\lambda = (y-\beta)/\mu = (z-\gamma)/\nu$ is

$$\begin{vmatrix} y'-\beta & z'-\gamma \\ \mu & \nu \end{vmatrix}^2 + \begin{vmatrix} z'-\gamma & x'-\alpha \\ \nu & \lambda \end{vmatrix}^2 + \begin{vmatrix} x'-\alpha & y'-\beta \\ \lambda & \mu \end{vmatrix}^2.$$

57. Let $\dfrac{x-a_1}{\lambda_1} = \dfrac{y-b_1}{\mu_1} = \dfrac{z-c_1}{\nu_1} \; (l_1)$ and $\dfrac{x-a_2}{\lambda_2} = \dfrac{y-b_2}{\mu_2} = \dfrac{z-c_2}{\nu_2} \; (l_2)$

be two lines. Prove

(1) The equation of the plane through the point (α, β, γ) and parallel to both the lines (l_1) and (l_2) is

$$\begin{vmatrix} x-\alpha & \lambda_1 & \lambda_2 \\ y-\beta & \mu_1 & \mu_2 \\ z-\gamma & \nu_1 & \nu_2 \end{vmatrix} = 0.$$

(2) The condition that the lines (l_1) and (l_2) intersect is

$$\begin{vmatrix} a_1-a_2 & \lambda_1 & \lambda_2 \\ b_1-b_2 & \mu_1 & \mu_2 \\ c_1-c_2 & \nu_1 & \nu_2 \end{vmatrix} = 0.$$

(3) If θ denote the angle between the lines (l_1) and (l_2), the length of the shortest line from a point on the one to a point on the other is

$$\frac{1}{\sin\theta}\begin{vmatrix} a_1-a_2 & \lambda_1 & \lambda_2 \\ b_1-b_2 & \mu_1 & \mu_2 \\ c_1-c_2 & \nu_1 & \nu_2 \end{vmatrix}.$$

(4) The equations of the planes perpendicular to the plane of (1) and containing the lines (l_1) and (l_2), respectively, are

$$\begin{vmatrix} x-a_1 & \lambda_1 & (\mu_1\nu_2-\mu_2\nu_1) \\ y-b_1 & \mu_1 & (\nu_1\lambda_2-\nu_2\lambda_1) \\ z-c_1 & \nu_1 & (\lambda_1\mu_2-\lambda_2\mu_1) \end{vmatrix} = 0, \qquad \begin{vmatrix} x-a_2 & \lambda_2 & (\mu_1\nu_2-\mu_2\nu_1) \\ y-b_2 & \mu_2 & (\nu_1\lambda_2-\nu_2\lambda_1) \\ z-c_2 & \nu_2 & (\lambda_1\mu_2-\lambda_2\mu_1) \end{vmatrix} = 0.$$

And these are the equations of the line of the shortest distance between (l_1) and (l_2).

58. Prove that the two straight lines whose direction cosines are given by the equations $\mu\nu + \nu\lambda + \lambda\mu = 0$, $2\lambda + 2\mu - \nu = 0$ are at right angles. [If $\lambda_1, \mu_1, \nu_1, \lambda_2, \mu_2, \nu_2$ are the two solutions, the elimination of λ gives the quadratic $2\mu^2 - \mu\nu - \nu^2 = 0$, for whose roots $\mu_1\mu_2/\nu_1\nu_2 = -1/2$; and the elimination of μ in the same way gives $\lambda_1\lambda_2/\nu_1\nu_2 = -1/2$. Addition gives $\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0$.]

59. Prove that the two straight lines whose direction cosines are given by the equations $\lambda^2+\mu^2-\nu^2=0$, $\lambda+\mu+\nu=0$ make an angle of 60°. [Let λ_1, μ_1, ν_1, λ_2, μ_2, ν_2 be the two solutions; then the elimination of λ gives the quadratic $\mu^2+\mu\nu=0$, for whose roots $\mu_1=0$, and $\mu_2+\nu_2=0$. These values set in $\lambda+\mu+\nu=0$, give $\lambda_1+\nu_1=0$, and $\lambda_2=0$. The direction cosines are $(1/\sqrt{2}, 0, -1/\sqrt{2})$, $(0, 1/\sqrt{2}, -1/\sqrt{2})$.]

60. Prove that the two straight lines whose direction cosines are given by the equations $\lambda^2+\lambda\mu+\mu^2-\nu^2=0$ and $\lambda-\mu-\nu=0$ make an angle of 60°.

61. Prove that the equations of the two bisectors of the angles between the two lines $x/\lambda_1=y/\mu_1=z/\nu_1$ and $x/\lambda_2=y/\mu_2=z/\nu_2$ are $x/(\lambda_1\pm\lambda_2)=y/(\mu_1\pm\mu_2)=z/(\nu_1\pm\nu_2)$.

62. If the line $x/\lambda=y/\mu=z/\nu=r$, through the origin O, meets the plane $Ax+By+Cz+D=0$ in the point H, prove that the length of OH is $-D/(A\lambda+B\mu+C\nu)$. Prove that the line is parallel to the plane when $A\lambda+B\mu+C\nu=0$, even when the coordinates are oblique.

63. If a, b, c, be the intercepts of any plane, and p be the perpendicular from the origin, prove that $1/a^2+1/b^2+1/c^2=1/p^2$.

64. Interpret $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=(\lambda x+\mu y+\nu z-p)^2$, when $\lambda^2+\mu^2+\nu^2=1$. [The locus of a point P (x, y, z) equally distant from the point P_1 (x_1, y_1, z_1) and the plane $\lambda x+\mu y+\nu z-p=0$.]

65. Interpret $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=(Ax+By+Cz+D)^2$.

66. For the cases where $A^2+B^2+C^2$ is >1, $=1$, <1, interpret $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=\{A(x-x_1)+B(y-y_1)+C(z-z_1)\}^2$. [Locus is real cone; a straight line; the point (x_1, y_1, z_1).]

67. If A be any plane area, and A_x, A_y, A_z its projections on the three coordinate planes, prove that the projection of A on any plane β is $pr_\beta A=pr_\beta A_x+pr_\beta A_y+pr_\beta A_z$.

[Let θ be the angle between the original plane and β, and λ, μ, ν and λ', μ', ν' the direction cosines of the perpendiculars to the two planes, respectively; then by § 117, $A_x=\lambda A$, and so on, and hence by § 284, $\cos\theta\cdot A=(\lambda'\lambda+\mu'\mu+\nu'\nu)A=\lambda' A_x+\mu' A_y+\nu' A_z$.]

68. If A be any plane surface, and A_x, A_y, A_z its projections on the coordinate planes, prove that $A^2=A_x^2+A_y^2+A_z^2$ [§ 117].

69. If a plane cut the axes in A, B, C, and if a, b, c be the intercepts, prove that the area of ABC is $\frac{1}{2}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}$.

70. The area of a triangle with the angular points $P_2(x_2, y_2, z_2,)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ is [Ex. 68 and § 52]

$$\frac{1}{2}\left\{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}.$$

71. The perpendicular distance from the plane $P_2\ P_3\ P_4$ to the point $P_1(x_1, y_1, z_1)$ is [§ 274, § 277, § 279]

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div \left\{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}.$$

72. Since the volume of a tetrahedron is one third a base by the corresponding altitude, the volume of a tetrahedron with the angular points P_1, P_2, P_3, P_4 is [by the two preceding exercises]

$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

CHAPTER XV

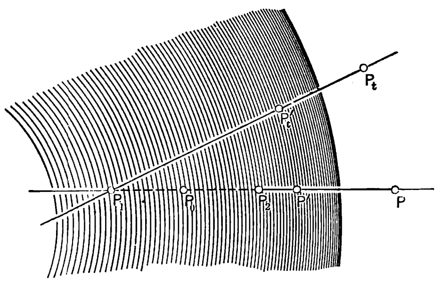
THE SHAPE OF THE CONICOIDS. CONFOCALS

298. The conicoid, and its points of intersection with a line. Let $F(x, y, z) = 0$ (1) denote an algebraic equation of the second degree, namely:

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy$$
$$+ 2\,lx + 2\,my + 2\,nz + d = 0. \qquad (1')$$

As has been seen already [§ 264], the locus of this equation is a surface; and, since the equation is of the second degree, the surface is called a *surface of the second degree*, or a *conicoid.*

299. To find the points where any line meets this surface, the equations of the line and the equation of the surface, taken as simultaneous equations, are solved for x, y, and z. The solution may be found as follows: Let the equations of the line be [§ 291, (2)]



$$\frac{x - x'}{\lambda} = \frac{y - y'}{\mu} = \frac{z - z'}{\nu} = r. \qquad (2)$$

These equations are equivalent to the following [§ 292]:

$$x = x' + \lambda r, \quad y = y' + \mu r, \quad z = z' + \nu r. \qquad (2')$$

For the points which are common to the line and the surface, these values of x, y, z [Eqs. (2')] must satisfy the equation $F(x, y, z) = 0$; that is, the following equation is true:

$$F(x' + \lambda r,\ y' + \mu r,\ z' + \nu r) = 0. \qquad (3)$$

The unknown quantity r occurs in this equation to the second power, since $F(x, y, z)$ is of the second degree; and the equation may therefore be written in the form:

$$Ar^2 + 2\,Br + C = 0. \qquad (3')$$

The two values of r obtained by solving this quadratic (3′) are the distances from the point (x', y', z') to the points P_1, P_2, where the line (2) cuts the surface (1′). The coordinates of P_1, P_2, are found by substituting the two values of r in (2′).

300. The quadratic (3′) may have two real roots; in this case the line (2) meets the surface (1′) in two real points.

Example. Find the points where the surface $x^2 + y^2 - z^2 + 7 = 0$ (1) is met by the line $(x - 1)/2 = (y - 2)/3 = (z + 1)/2$ (2).

Represent each of the equal fractions (2) by r'; then for any point (x, y, z) on the line (2), $x = 2\,r' + 1$, $y = 3\,r' + 2$, $z = 2\,r' - 1$. (2′)

The substitution of these values of x, y, and z in $x^2 + y^2 - z^2 + 7 = 0$ gives: $9\,r'^2 + 20\,r' + 11 = 0$. Hence, $r'_1 = -1$, $r'_2 = -11/9$. These values of r' set in the equations (2′) give the coordinates of the points of intersection, namely, $(-1, -1, -3)$ and $(-13/9, -5/3, -31/9)$. (Prove that these values satisfy both (1) and (2), and thus check the numerical work.)

Since the denominators in the equation of the line are not λ, μ, ν, but are merely proportional to λ, μ, ν, the r' here used has not the geometrical meaning of the r in the text.

301. The quadratic (3′) may have equal roots; in this case the line meets the conicoid in two coincident points (and is called a *tangent*).

Example. Show that the surface $x^2 - 2\,xy + 3\,z^2 - 5\,y + 10 = 0$ (1) is met by the line $(x - 3)/1 = (y + 2)/(-2) = (z - 3)/2 = r'$ (2) in one point only, and find this point.

The coordinates of any point on the line are $x = r' + 3$, $y = -2\,r' - 2$, $z = 2\,r' + 3$ (2′); and these values of (x, y, z) substituted in (1) give the quadratic $r'^2 + 4\,r' + 4 = 0$, which has equal roots, $r' = -2, -2$. This value of r' set in the equations (2′) gives the coordinates of the required point $(1, 2, -1)$.

302. The quadratic (3′) may have imaginary roots; in this case the points P_1 and P_2 have imaginary coordinates, and the line does not meet the conicoid in real points.

Example. Show that the surface $x^2 + y^2 - z^2 + 8 = 0$ (1) is met by the line $(x - 1)/2 = (y - 2)/3 = (z + 1)/2 = r'$ (2) in no real point.

From the equations (2): $x = 2r' + 1, y = 3r' + 2, z = 2r' - 1$ (2′); and the setting of these values in the equation (1) gives the quadratic $9r'^2 + 20r' + 12 = 0$, whose roots are $r' = (-10 \pm 2\sqrt{-2})/9$; and the corresponding values of x, y, z obtained from (2′) are imaginary. Since these values of x, y, z, though imaginary, algebraically satisfy the equation (1), the line may be said to meet the conicoid in two imaginary points.

303. The quadratic (3′) may be an *identity*, that is, the coefficient of r^2, the coefficient of r, and the absolute term may all three be zero; in this case *every* value of r will satisfy (3′), which means geometrically that every point on the line (2) is on the surface, or that the line lies wholly on the surface.

Example. Prove that the line $x - 1 = y - 2 = z + 1 = r'$ (2) lies entirely on the surface $z^2 - xy + 2x + y + 2z - 1 = 0$ (1).

The equations of the line (2) become: $x = r' + 1$, $y = r' + 2$, $z = r' - 1$ (2′), and substituting these expressions for x, y, z in the equation of the surface (1), and reducing, a quadratic in r is obtained, which has the form: $0 \cdot r'^2 + 0 \cdot r' + 0 = 0$. Every finite value of r' satisfies this equation, and therefore every point on the line is on the surface; that is, the line itself lies on the surface.

304. The section of a conicoid by a plane. Since any line of a plane cutting a conicoid, in general, meets the surface in two points only, *every plane section of a conicoid is a conic.*

Example. The section of $x^2/9 + y^2/4 + z^2/1 = 1$ (1) by the plane $z = k$ (2) is represented by the two equations, obtained by taking (1) and (2) as simultaneous equations,

$$x^2/9 + y^2/4 = (1 - k^2) \text{ and } z = k.$$

And these two equations give the section by the plane $z = k$ of the cylinder

$$x^2/9 + y^2/4 = (1 - k^2) \quad [\S\ 262].$$

Therefore, the section of the conicoid (1) by the plane $z = k$ (for $k^2 < 1$) is equal to the ellipse $x^2/9 + y^2/4 = (1 - k^2)$ in the plane $z = 0$.

305. Exercises.

Find the points of intersection of the following surfaces and lines:

1. $x^2 + 2y^2 - 2z^2 + 3yz + 2zx - 4xy + 3x - 3y + 2z - 4 = 0$, and $(x-1)/2 = (y-2)/3 = (z+1)/2$.

2. $3x^2 - 4y^2 + z^2 + 4xy - 2yz - 6xz - 6x - 10y + 2z - 223 = 0$, and $(x+1)/3 = (y+2)/2 = (z-2)/(-1)$.

3. $3x^2 - 4y^2 + z^2 - 2yz - 6xz + 4xy + 8x - 10y + 2z + 8 = 0$, and $(x+1)/3 = (y+2)/2 = (z-2)/(-1)$.

4. $2x^2 + 3y^2 - 2z^2 + 2xy - 4yz - 6xz + 3x - 5y - 2z - 21 = 0$, and $(x-1)/2 = (y+2)/2 = (z+1)$.

5. The surface of Ex. 4. and $(x-1)/2 = (y-1)/2 = (z+7)/(-13)$.

6. $x^2 - 4z^2 + 5y - x + 8z = 0$, and $(x-1)/(-4) = (y-8)/12 = (z-5)/3$.

7. $x^2 - 4z^2 + 5y - x + 8z = 0$, and $(x+3)/10 = y/(-2) = (z+1)/5$.

306. The shape of particular conicoids. Certain forms of the equation of the second degree will now be considered. In a later chapter [§ 364] it will be proved that every equation of the second degree can be reduced to one or the other of these forms.

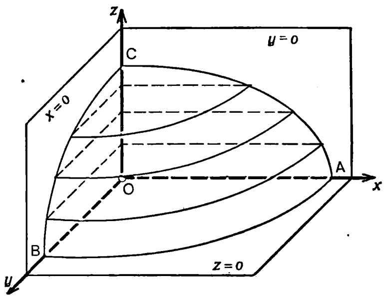
307. The ellipsoid. The conicoid which is the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an *ellipsoid.*

Since the equation involves only even powers of x, y, z, the surface is symmetric with respect to each of the coordinate planes $x = 0$, $y = 0$, and $z = 0$, which are called its *principal planes.* It is also symmetric with respect to the origin O, which is therefore called the *center* of the surface.

On account of the symmetry here noted, the shape of the whole ellipsoid can be inferred from the shape of that part for which all the coordinates are positive. See the figure herewith, and also Figure 1.



It is at once evident from the equation of the ellipsoid that no point of the surface can have an x-coordinate which is numerically greater than a, a y-coordinate greater than b, or a z-coordinate greater than c. Hence the surface lies wholly between the planes $x=a$, $x=-a$, $y=b$, $y=-b$, $z=c$, $z=-c$. Therefore, since all plane sections of conicoids are conics [§ 304], in the case of the ellipsoid these sections, being curves of limited extent, must be ellipses.

The sections by planes $z=k$, $|k|<c$, parallel to the coordinate plane $z=0$, are similar ellipses having equations of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}, \quad z=k,$$

and which diminish in size as k varies from 0 to c. And the like is true of sections by planes parallel to the other coordinate planes.

To find the points where the x-axis meets the ellipsoid, set $y=z=0$, and $x=\pm a$ is obtained. And similarly for the y- and x-axes. Hence the ellipsoid intercepts segments of the x-, y-, and z-axes whose lengths are $2a$, $2b$, and $2c$, respectively. These segments are called the *axes* of the ellipsoid. And a, b, and c are called the *semiaxes*. The ellipsoid passes through the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

In the general case a, b, and c are unequal. It is then convenient to suppose $a>b>c$.

308. If $a>b=c$, all sections parallel to the plane $x=0$ are circles. In this case the surface can be generated by revolving the ellipse which the equation $x^2/a^2+z^2/c^2=1$ represents in the xz-plane, about the x-axis, which is its major axis.

Similarly, if $a=b>c$, all sections parallel to the plane $z=0$ are circles. In this case the surface can be generated by revolving the ellipse $x^2/a^2+z^2/c^2=1$ about the z-axis, which is its minor axis.

The surface generated by revolving an ellipse about its major axis is called a *prolate spheroid;* that generated by revolving an ellipse about its minor axis is called an *oblate spheroid.* Both surfaces are called *ellipsoids of revolution.* Hence, if $a>c$, the equations

$$\frac{x^2}{a^2}+\frac{y^2}{c^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$$

represent a prolate spheroid and an oblate spheroid, respectively.

309. If $a=b=c$, the equation becomes $x^2+y^2+z^2=a^2$, and the surface is a *sphere* [§ 237].

310. Since a sum of squares of real numbers cannot equal a negative number, there is no real solution of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1,$$

which is said to represent an *imaginary ellipsoid.*

311. Hyperboloid of one sheet. This is the name given to the surface which is the locus of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

As in § 307, the following inferences can be drawn from the equation:

The surface is met by the x- and y-axes in real points, but not by the z-axis.

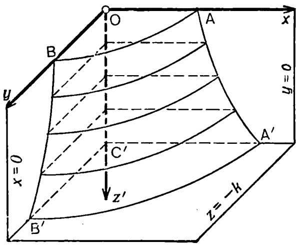
The sections by the planes $x=0$ and $y=0$, and planes parallel to these, are hyperbolas.

The sections by planes $z = k$, parallel to the plane $z = 0$, are similar ellipses having equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad z = k,$$

which increase in size indefinitely as k increases numerically.

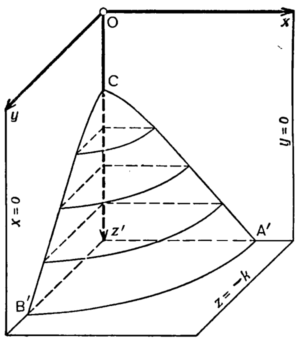
Since the equation involves only even powers of x, y, z, the surface is symmetric with respect to each of the coordinate planes; and the shape of the whole hyperboloid can be inferred from the shape of that part for every point of which x and y are positive and z is negative. See the figure herewith, and also Figure 2.



312. When $a = b$, the sections by the planes $z = k$ are circles. In this case the surface can be generated by revolving the hyperbola $x^2/a^2 - z^2/c^2 = 1$ about the z-axis, which is its conjugate axis.

313. Hyperboloid of two sheets. This name is given to the surface which is the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$



The z-axis meets the surface in real points, but the x- and y-axes do not meet it in real points.

The sections by the planes $x = 0$, $y = 0$, and planes parallel to these, are hyperbolas.

No point of the surface lies between the planes $z = -c$ and $z = c$. But every plane $z = k$, for which $|k| > c$, cuts the surface in an ellipse, the size of which increases with $|k|$. Hence the surface consists of two separate parts, one extending indefinitely above the plane $z = c$, the other indefinitely below the plane $z = -c$.

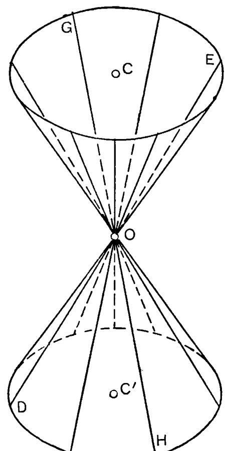
Since the equation involves only even powers of x, y, z, the surface is symmetric with respect to each of the coordinate planes, and the shape of the whole hyperboloid can be inferred from the shape of that eighth of the surface every point of which has a positive x and y and a negative z. See the figure herewith, and also Figure 3.

314. If $a = b$, the sections by planes $z = k$, where $|k| > c$, are circles. In this case the surface can be generated by revolving the hyperbola $x^2/a^2 - z^2/c^2 = -1$ about the z-axis, which is its transverse axis.

315. The Cone. Any surface generated by the motion in space of a line which passes through a fixed point is called a *cone*.

The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



is a cone for which the fixed point mentioned in the definition is the origin.

For, evidently, if (x', y', z') be any solution of the equation, so also is (kx', ky', kz') a solution, whatever the value of k may be. But every point of the line joining the point (x', y', z') to the origin has coordinates of the form (kx', ky', kz'). Hence the surface under consideration has the property that the line determined by the origin and any one of its points lies wholly on the surface. It is therefore a cone.

The sections by planes $z = \pm k$, parallel to $z = 0$, are similar ellipses of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}, \quad z = k.$$

316. There is no real solution, except (0, 0, 0), of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

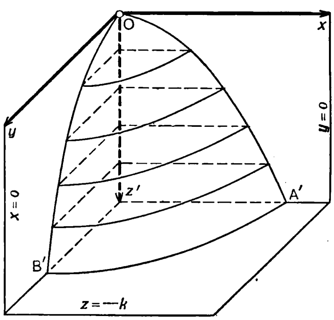
which is called an *imaginary cone.*

317. The conicoids of §§ 307–316 are called *central conicoids.*

318. The elliptic paraboloid. This name is given to the surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-2z.$$

The equation has no constant term, and no real solution in which z is positive. Hence the surface passes through the origin, and lies wholly below the plane $z=0$, which it touches.



The sections by the planes $x=0$ and $y=0$, and planes parallel to them, are parabolas.

The sections by planes $z=k$, parallel to and below the plane $z=0$, are ellipses which increase in size indefinitely with $|k|$.

Since the equation involves only even powers of x and y, the surface is symmetric with respect to each of the coordinate planes $x=0$ and $y=0$, and the shape of the whole elliptic paraboloid can be inferred from the quarter of the surface in front of the $y=0$ plane and to the right of the $x=0$ plane. See the figure herewith, and also Figure 4.

319. If $a=b$, the elliptic sections are circles. In this case, the surface can be generated by revolving the parabola $x^2/a^2=-2z$, or $x^2=-2a^2z$, about the z-axis, which is the axis of this parabola.

320. The hyperbolic paraboloid. This is the name of the surface which is the locus of the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z.$$

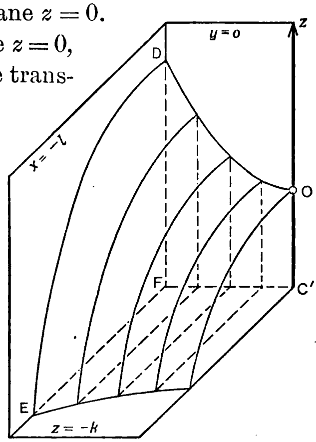
The plane $z=0$ cuts the surface in the pair of lines represented by

$$\frac{x}{a}+\frac{y}{b}=0,\ z=0,\ \text{and}\ \frac{x}{a}-\frac{y}{b}=0,\ z=0.$$

Every other line through the origin O and in the plane $z=0$ crosses each of these lines at O and therefore meets the surface in two coincident points, or touches it. Hence $z=0$ is the tangent plane to the surface at O [§§ 301, 328].

In that one of the angles between the planes $x/a+y/b=0$, $x/a-y/b=0$ which contains the x-axis, $x^2/a^2-y^2/b^2$, and therefore z, is positive. Hence that portion of the surface which is in this dihedral angle is above the plane $z=0$.

Similarly, the portion of the surface which is in that dihedral angle between the planes $x/a+y/b=0$, $x/a-y/b=0$ which contains the y-axis, is below the plane $z=0$.



Planes $z=k$, parallel to the plane $z=0$, cut the surface in hyperbolas whose transverse axes are parallel to the x-axis when k is positive, but parallel to the y-axis when k is negative, the asymptotes of each hyperbola being the lines of intersection of the plane $z=k$ with the pair of planes $x^2/a^2-y^2/b^2=0$.

Since the equation involves only even powers of x and y, the surface is symmetric with respect to each of the coordinate planes $x=0$ and $y=0$, and the shape of the whole hyperbolic paraboloid can be inferred from that quarter of the surface in front of the plane $y=0$ and to the left of the plane $x=0$. See the figure herewith, and also Figure 5.

Every plane parallel to (or coincident with) the plane $x=0$ cuts the surface in a parabola. For convenience in drawing the figure, take the sections by the planes $x=-l$; the equations of these sections will be

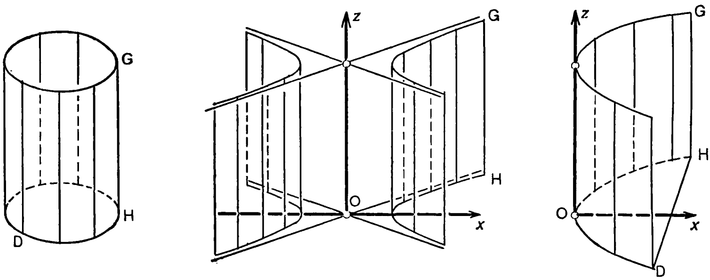
$$y^2=-2\,b^2\left(z-\frac{l^2}{2\,a^2}\right),\quad x=-l.$$

This is a parabola whose axis is parallel to the z-axis, whose vertex, the point $(-l,\,0,\,l^2/2\,a^2)$, is above the plane $z=0$, and which extends downward, its latus rectum $(=-2\,b^2)$ being negative.

Similarly, every plane $y=h$, parallel to the plane $y=0$, cuts the surface in a parabola whose axis is parallel to the z-axis whose vertex is below the plane $z=0$, and which extends upward.

The surface is saddle-shaped.

321. The cylinders and planes. As has been seen in § 262, an equation in but two of the coordinates, say x and y, represents



a cylindrical surface parallel to a coordinate axis. The loci of the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\quad y^2=4\,ax$$

are called the *elliptic, hyperbolic, parabolic cylinders,* respectively.

322. The equation $x^2/a^2+y^2/b^2=-1$ has no real solution; it is said to represent an *imaginary cylinder.*

323. The equation $x^2/a^2 - y^2/b^2 = 0$ represents the two planes $x/a + y/b = 0$, $x/a - y/b = 0$.

324. The equation $x^2/a^2 + y^2/b^2 = 0$ has no real solutions, except $(0, 0, z)$; it is true for points on the line $x = 0$, $y = 0$, and these points only; it is said to represent a pair of *imaginary planes.*

325. The locus of $y^2 = a$ is two *parallel planes*, real or imaginary according as a is positive or negative.

326. Finally, $y^2 = 0$ represents two *coincident planes.*

327. The following is a list of the forms of the equations just considered. In a later chapter [§ 364] it will be proved that this is a *complete* list of the various forms to which an equation of the second degree can be reduced. Opposite each equation is placed the name of the conicoid which is its locus.

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, ellipsoid [§ 307].

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$, imaginary ellipsoid [§ 310].

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, hyperboloid of one sheet [§ 311].

4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$, hyperboloid of two sheets [§ 313].

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, cone [§ 315].

6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$, imaginary cone [§ 316].

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -2z$, elliptic paraboloid [§ 318].

8. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$, hyperbolic paraboloid [§ 320].

9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, elliptic cylinder [§ 321].

10. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, hyperbolic cylinder [§ 321].

11. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$, imaginary cylinder [§ 322].

12. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$, pair of intersecting planes [§ 323].

13. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$, pair of imaginary planes [§ 324].

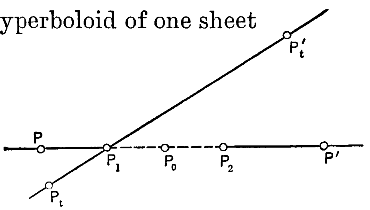
14. $y^2=4\,ax$, parabolic cylinder [§ 321].

15. $y^2=a$, two parallel planes, real or imaginary [§ 325].

16. $y^2=0$, two coincident planes [§ 326].

328. Tangent planes. The hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0 \qquad (1)$$



is met by the line

$$\frac{x-x'}{\lambda}=\frac{y-y'}{\mu}=\frac{z-z'}{\nu}=r, \qquad (2)$$

or $$x=\lambda r+x', \qquad y=\mu r+y', \qquad z=\nu r+z', \qquad (2')$$

in points, P_1, P_2, whose distances, r_1, r_2, from the point $P'(x', y', z')$ are the roots of the equation, obtained by substituting (2′) in (1),

$$\frac{(\lambda r+x')^2}{a^2}+\frac{(\mu r+y')^2}{b^2}-\frac{(\nu r+z')^2}{c^2}-1=0, \qquad (3)$$

or

$$r^2\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}-\frac{\nu^2}{c^2}\right)+2r\left(\frac{\lambda x'}{a^2}+\frac{\mu y'}{b^2}-\frac{\nu z'}{c^2}\right)+\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{z'^2}{c^2}-1\right)=0. \qquad (3')$$

If P' is on (1), so that

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{z'^2}{c^2}-1\equiv 0, \qquad (4)$$

one of the roots of (3′) is 0, which is as it should be since P' is then itself one of the points of intersection of (1) and (2), say the point P_1. The second root of (3′) will be 0 if the coefficient of r is also 0, that is, if λ, μ, ν have such values that

$$\frac{\lambda x'}{a^2}+\frac{\mu y'}{b^2}-\frac{\nu z'}{c^2}=0. \tag{5}$$

In this case the point P_2 will coincide with P' (that is, P_1) and the line (2) will meet the surface (1) in two coincident points at P_1, or be *tangent* to it at P_1 [§ 301]. The point P_1 is then called the *point of tangency* of the line with the surface.

Eliminate λ, μ, ν from (5) by aid of the equations (2). The result is, P' being P_1,

$$\frac{(x-x_1)x_1}{a^2}+\frac{(y-y_1)y_1}{b^2}-\frac{(z-z_1)z_1}{c^2}=0, \tag{6}$$

which represents a plane through (x_1, y_1, z_1) [§ 283]. From the manner in which (6) was derived, it follows that every line which touches the surface (1) at (x_1, y_1, z_1) lies in this plane; and, conversely, since (5) is a consequence of (2) and (6), every line through (x_1, y_1, z_1) which lies in the plane touches the surface. The plane (6) is therefore called the *tangent plane* to the surface at the point (x_1, y_1, z_1). As just said, it has the property that the line joining any point $P_t(x_t, y_t, z_t)$ in it to the point (x_1, y_1, z_1) is a tangent to the surface.

When the multiplications indicated in (6) are carried out, and $x_1^2/a^2+y_1^2/b^2-z_1^2/c^2$ is replaced by its equal, 1, the equation becomes

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}-1=0, \tag{7}$$

which is therefore the equation of the tangent plane to the hyperboloid (1) in its simplest form.

329. Observe that (7) can be obtained from the equation of the hyperboloid by replacing x^2, y^2, z^2 by xx_1, yy_1, zz_1, respectively.

In a similar manner [§ 330], it can be proved that the tangent plane to any conicoid at the point x_1, y_1, z_1 can be obtained by the rule: replace x^2, y^2, z^2 by xx_1, yy_1, zz_1; $2\,xy$, $2\,yz$, $2\,zx$ by x_1y+y_1x, y_1z+z_1y, z_1x+x_1z; $2\,x$, $2\,y$, $2\,z$ by $x+x_1$, $y+y_1$, $z+z_1$. (Compare § 81 and § 171.)

330. The equation of the tangent plane to any conicoid

$$F(x,\ y,\ z)\equiv ax^2+by^2+cz^2+2\,fyz+2\,gzx+2\,hxy \\ +2\,lx+2\,my+2\,nz+d=0 \qquad (1)$$

can be obtained by a method similar to that of § 328.

The conicoid is met by the line

$$\frac{x-x'}{\lambda}=\frac{y-y'}{\mu}=\frac{z-z'}{\nu}=r, \qquad (2)$$

or
$$x=\lambda r+x',\quad y=\mu r+y',\quad z=\nu r+z', \qquad (2')$$

in points P_1, P_2, whose distances r_1, r_2 from the point $P'(x',\ y',\ z')$ are the roots of the equation

$$F(\lambda r+x',\ \mu r+y',\ \nu r+z')=0, \qquad (3)$$

which, when expanded and arranged in descending powers of r, is

$$\{a\lambda^2+b\mu^2+c\nu^2+2\,f\mu\nu+2\,g\nu\lambda+2\,h\lambda\mu\}r^2 \\ +2\{(ax'+hy'+gz'+l)\lambda+(hx'+by'+fz'+m)\mu \\ +(gx'+fy'+cz'+n)\nu\}r \\ +\{ax'^2+by'^2+cz'^2+2\,fy'z'+2\,gz'x'+2\,hx'y' \\ +2\,lx'+2\,my'+2\,nz'+d\}=0. \qquad (3')$$

In the coefficient of r, represent the quantities by which λ, μ, ν are multiplied, by $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$, $\frac{\partial F}{\partial z'}$, or $\partial F/\partial x'$, $\partial F/\partial y'$, $\partial F/\partial z'$, respectively.* The equation can then be written

$$(a\lambda^2+b\mu^2+c\nu^2+2\,f\mu\nu+2\,g\nu\lambda+2\,h\lambda\mu)\,r^2 \\ +(\partial F/\partial x'\cdot\lambda+\partial F/\partial y'\cdot\mu+\partial F/\partial z'\cdot\nu)\,r+F(x',\ y',\ z')=0. \qquad (3'')$$

Hence if the point P' is on the conicoid (1), so that $F(x',\ y',\ z')\equiv 0$ (4), the line (2) will meet the surface in two coincident points at P', or be tangent to it, when λ, μ, ν have such values that

* $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ are called the partial derivatives of $F(x,y,z)$ with respect to x, y, z, respectively. $\partial F/\partial x$ is the sum of the terms obtained by multiplying each term of $F(x,\ y,\ z)$ which contains an x by the exponent of x in that term, and then diminishing the exponent by 1. $\partial F/\partial x'$ is then obtained from $\partial F/\partial x$ by priming all the variables. (This is the notation of the calculus.)

$$\frac{\partial F}{\partial x'}\lambda + \frac{\partial F}{\partial y'}\mu + \frac{\partial F}{\partial z'}\nu = 0. \tag{5}$$

The elimination of λ, μ, ν between (5) and (2) gives

$$\frac{\partial F}{\partial x'}(x - x') + \frac{\partial F}{\partial y'}(y - y') + \frac{\partial F}{\partial z'}(z - z') = 0. \tag{6}$$

Or, since P' is now the same as P_1,

$$\frac{\partial F}{\partial x_1}(x - x_1) + \frac{\partial F}{\partial y_1}(y - y_1) + \frac{\partial F}{\partial z_1}(z - z_1) = 0, \tag{7}$$

which represents a plane through the point $P_1(x_1, y_1, z_1)$ and containing all the tangent lines to the surface at that point. It is the tangent plane.

When $\partial F/\partial x_1$, $\partial F/\partial y_1$, $\partial F/\partial z_1$ are replaced by the expressions which they represent, and the indicated multiplications are carried out, and the result is simplified by aid of the relation $F(x_1, y_1, z_1) \equiv 0$, the equation (7) becomes

$$axx_1 + byy_1 + czz_1 + f(yz_1 + y_1z) + g(zx_1 + xz_1) + h(xy_1 + yx_1) \\ + l(x + x_1) + m(y + y_1) + n(z + z_1) + d = 0. \tag{8}$$

331. The normal. The line through a point P_1 on a surface and perpendicular to the tangent plane at P_1 is called the *normal* to the surface at P_1.

The equations of the normal to the surface $F(x, y, z) = 0$ at the point (x_1, y_1, z_1) are [§§ 278, 330]

$$\frac{x - x_1}{\partial F/\partial x_1} = \frac{y - y_1}{\partial F/\partial y_1} = \frac{z - z_1}{\partial F/\partial z_1}.$$

Example. Find the equation of the tangent plane and the equations of the normal to the hyperboloid $x^2 + 3y^2 - z^2 - 3 = 0$ at the point $(2, -1, 2)$. The values $(2, -1, 2)$ satisfy the equation, and the point is therefore on the surface.

The equation of the tangent plane at the point (x', y', z') is

$$xx' + 3yy' - zz' - 3 = 0.$$

Setting $(x', y', z') = (2, -1, 2)$ in this equation gives $2x - 3y - 2z - 3 = 0$.

The equations of the normal are $(x - 2)/2 = (y+1)/-3 = (z - 2)/-2$.

332. The polar plane. The equation (8) of § 330 represents a plane, whether the point $P_1(x_1, y_1, z_1)$ lies on the conicoid $F(x, y, z) = 0$ or not. This plane is called the *polar plane* of the point (x_1, y_1, z_1) with respect to the conicoid $F(x, y, z) = 0$. The point (x_1, y_1, z_1) is called the *pole* of the plane. By the reasoning employed in Chapter IX, it can be proved that the polar plane of a point P cuts the conicoid in a conic which is the locus of the points of tangency of all tangent lines from P to the conicoid. If the plane does not cut the surface in real points, the tangents from P are imaginary.

333. Exercises. Tangent planes and normals.

1. Find the equation of the tangent plane and the equations of the normal to $x^2 + 4xy + 2yz - 3x = 0$ at the point $A(-1, -2, 3)$; also at the point $B(4, 2, -9)$.

2. Find the equation of the tangent plane and the equations of the normal to $x^2 + 2y^2 - 2z^2 + 3yz + 2zx - 4xy + 3x - 3y + 2z - 4 = 0$ at the following points: $A(-2, -5/2, -4)$, $B(1, 1, 1)$, $C(1, 1, 5/2)$, $D(1, 0, 0)$, $E(-4, 0, 0)$, $F(1, 0, 2)$, $G(-8, 0, 2)$, $H(-1, -2, 0)$, $J(-1, 3/2, 0)$.

3. Find the equation of the polar plane of the point $(1, -1, 2)$ with respect to the conicoid $x^2 + 2z^2 - 6xy + 2yz - 4x = 0$.

4. Find the pole of the plane $2x - 3y + z + 4 = 0$ with respect to the conicoid $x^2 - 2y^2 + 4zx - 2xy + 8 = 0$.

5. Find the equation of the tangent cone from the origin to the sphere $(x-1)^2 + (y-2)^2 + (z+1)^2 = 3$.

334. Surfaces of revolution. A surface generated by revolving a plane curve about a straight line in its plane is called a *surface of revolution*.

Thus, the right circular cone, the sphere, the prolate and oblate spheroids, and the surfaces mentioned in §§ 312, 314, 319, are surfaces of revolution.

335. If the equation of a curve in the xy-plane is $f(x, y) = 0$, the equation of the surface got by revolving the curve about the x-axis is $f(x, \sqrt{y^2 + z^2}) = 0$. For, in the equation $f(x, y) = 0$, y denotes the distance from the x-axis of a point of the curve whose abscissa is x. And $\sqrt{y^2 + z^2}$ is the expression for this same distance for every position taken by the point as the curve turns about the x-axis.

Similarly the equation of the surface got by revolving the curve $f(x, y) = 0$ about the y-axis is $f(\sqrt{x^2 + z^2}, y) = 0$.

It is evident that, if the curve is symmetric with respect to the line about which it is revolved, the degree of the surface will be the same as that of the curve; but, if the curve is not symmetric with respect to this line, the degree of the surface will be twice that of the curve.

Examples. The equation of the right circular cone got by revolving the line $y - 2x - 1 = 0$ of the xy-plane about the x-axis is

$$\sqrt{y^2 + z^2} - 2x - 1 = 0, \text{ or } y^2 + z^2 - (2x + 1)^2 = 0.$$

The equation of the cone got by revolving this same line about the y-axis is

$$y - 2\sqrt{x^2 + z^2} - 1 = 0, \text{ or } 4x^2 + 4z^2 - (y - 1)^2 = 0.$$

Again, if the circle $x^2 + y^2 = a^2$ be revolved about either the x- or y-axis, the sphere $x^2 + y^2 + z^2 = a^2$ is obtained; if the parabola $y^2 = 4ax$ be revolved about the x-axis, the paraboloid $y^2 + z^2 = 4ax$ is obtained; and so on.

Finally, consider the surface generated by revolving the circle $x^2 + (y - b)^2 = a^2$, where $a < b$, about the x-axis. It is a ring-shaped surface called the *torus* or *anchor ring*. Its equation is

$$x^2 + (\sqrt{y^2 + z^2} - b)^2 = a^2, \text{ or } 4b^2(y^2 + z^2) = (a^2 - b^2 - x^2 - y^2 - z^2)^2.$$

336. Ruled surfaces. A surface of such a character that through every one of its points there is a straight line which lies entirely on the surface is called a *ruled surface,* and the straight line is called a *generating line.* [§ 303.]

337. Evidently cones and cylinders are ruled surfaces. The hyperboloid of one sheet and the hyperbolic paraboloid are also ruled surfaces, as will now be proved.

338. The equation of the hyperboloid of one sheet, namely $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, can be written

$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right). \qquad (1)$$

Let λ denote an arbitrary constant, and consider the following pair of simultaneous equations:

$$\frac{x}{a}+\frac{z}{c}=\lambda\cdot\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\cdot\left(1-\frac{y}{b}\right). \tag{2}$$

For any given value of λ this pair of equations represents a straight line; and this straight line must lie on the hyperboloid. This follows from the fact that any set of values of x, y, z which satisfies both equations (2) must satisfy (1), since, if the equations (2) be multiplied together, member by member, the equation (1) is obtained. For every real value of λ there is one such line, and these lines together completely cover the surface.

But the factors of the two members of (1) can be combined so as to form a second pair of equations involving an arbitrary constant, namely:

$$\frac{x}{a}+\frac{z}{c}=\mu\cdot\left(1-\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\mu}\cdot\left(1+\frac{y}{b}\right) \tag{3}$$

from which (1) may be derived by eliminating this constant, μ.

This pair of equations, like the pair (2), represents a system of straight lines which entirely cover the surface. See Figure 6.

Through every point P of the surface there will pass one line, and but one, of each of the systems of generating lines (2) and (3.) Moreover, the plane α determined by these two lines is the tangent plane to the surface at P. For, if Q denote any point of α not on either of the generating lines, the line QP, since it crosses both generating lines at P, meets the surface in two coincident points at P, or touches it [§ 328]. It is because this line QP cannot meet the surface in *more* than two points that we have the right to conclude that not more than two generating lines, one of each system, pass through P.

Example. Consider the hyperboloid

$$\frac{x^2}{4}+\frac{y^2}{9}-z^2=1. \tag{1}$$

Here the two systems of generating lines are

$$\frac{x}{2}+z=\lambda\cdot\left(1+\frac{y}{3}\right),\quad \frac{x}{2}-z=\frac{1}{\lambda}\cdot\left(1-\frac{y}{3}\right), \tag{2}$$

and

$$\frac{x}{2}+z=\mu\cdot\left(1-\frac{y}{3}\right),\quad \frac{x}{2}-z=\frac{1}{\mu}\cdot\left(1+\frac{y}{3}\right). \tag{3}$$

It will be found that the point (2, 6, 2) lies on (1).

To find the generating line of the system (2) which passes through this point, substitute $x=2$, $y=6$, $z=2$ in either of the equations (2), and solve for λ. The result is $\lambda=1$. Hence the equations of the line are

$$\frac{x}{2}+z=1+\frac{y}{3},\quad \frac{x}{2}-z=1-\frac{y}{3}. \tag{2'}$$

In the same manner, it is found that the equations of the generating line of the system (3) through the given point (2, 6, 2) are

$$\frac{x}{2}+z=-3\left(1-\frac{y}{3}\right),\quad \frac{x}{2}-z=-\frac{1}{3}\left(1+\frac{y}{3}\right). \tag{3'}$$

By § 328 the equation of the tangent plane at the point (2, 6, 2) is

$$3x+4y-12z-6=0. \tag{4}$$

Eliminating x and z between the three equations (2′) and (4), the equation $6+4y-4y-6=0$ is obtained, which is an identity. Hence [§ 303] the line (2′) lies in the plane (4). And in the same way it can be proved that the line (3′) lies in this plane.

339. The equation of the hyperbolic paraboloid, namely $x^2/a^2-y^2/b^2=2z$, can be written

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z.$$

Hence, it can be inferred, as above, that the pair of equations

$$\frac{x}{a}+\frac{y}{b}=2\lambda z,\quad \frac{x}{a}-\frac{y}{b}=\frac{1}{\lambda},$$

in which λ is an arbitrary constant, represents a system of generating lines which entirely covers the surface, and that the pair of equations

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{\mu},\quad \frac{x}{a}-\frac{y}{b}=2\mu z$$

represents a second system of such lines. See Figure 7.

And, as in the case of the hyperboloid of one sheet, it can be proved that one line of each system, and but one, passes through each point of the surface, and that the plane determined by these lines is the tangent plane at the point.

340. Confocal Conicoids. The system of surfaces represented by the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \tag{1}$$

in which λ is an arbitrary constant, is called a system of *confocal conicoids.* The principal sections of the system, that is, the sections by the planes $x=0$, $y=0$, and $z=0$, are confocal conics [§ 166].

Suppose $a>b>c$. Then for all positive values of λ, and for all negative values between 0 and $-c^2$, (1) represents ellipsoids; for all values of λ between $-c^2$ and $-b^2$, (1) represents hyperboloids of one sheet; for all values of λ between $-b^2$ and $-a^2$, (1) represents hyperboloids of two sheets; for all values of λ between $-a^2$ and $-\infty$, the locus of (1) is imaginary.

341. The two conics (corresponding to $\lambda=-c^2$ and $\lambda=-b^2$ in (1) of § 340),

$$z=0,\ \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1, \quad \text{and } y=0,\ \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1,$$

are called the *focal conics* of the system (1). The first is an ellipse, the second an hyperbola. See Figure 8.

342. *Through every point (x', y', z') there pass three conicoids of the system* (1), *namely, an ellipsoid, an hyperboloid of one sheet, and an hyperboloid of two sheets.*

For, substitute (x', y', z') for (x, y, z) in (1), and clear of fractions; the result is

$$(\lambda+a^2)(\lambda+b^2)(\lambda+c^2)-x'^2(\lambda+b^2)(\lambda+c^2)-y'^2(\lambda+c^2)(\lambda+a^2)$$
$$-z'^2(\lambda+a^2)(\lambda+b^2)=0. \tag{2}$$

When $\lambda = \infty$, the left member of (2) is positive;
when $\lambda = -c^2$, the left member is $-z'^2(-c^2+a^2)(-c^2+b^2)$, which is negative;
when $\lambda = -b^2$, the left member is $-y'^2(-b^2+c^2)(-b^2+a^2)$, which is positive;
when $\lambda = -a^2$, the left member is $-x'^2(-a^2+b^2)(-a^2+c^2)$, which is negative. Hence, the three values of λ which are the roots of (2) are real, and one of them lies between ∞ and $-c^2$, one between $-c^2$ and $-b^2$, and one between $-b^2$ and $-a^2$ [Alg. § 833]. Let these three roots be $\lambda_1, \lambda_2, \lambda_3$, respectively. The three equations obtained by substituting $\lambda_1, \lambda_3, \lambda_3$, successively, for λ in (1), namely

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}+\frac{z^2}{c^2+\lambda_1}=1, \qquad \infty > \lambda_1 > -c^2, \qquad (3)$$

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}+\frac{z^2}{c^2+\lambda_2}=1, \quad -c^2 > \lambda_2 > -b^2, \qquad (4)$$

$$\frac{x^2}{a^2+\lambda_3}+\frac{y^2}{b^2+\lambda_3}+\frac{z^2}{c^2+\lambda_3}=1, \quad -b^2 > \lambda_3 > -a^2, \qquad (5)$$

represent three surfaces of the system (1), all passing through the point (x', y', z'), the first, (3), being an ellipsoid, the second, (4), an hyperboloid of one sheet, and the third, (5), an hyperboloid of two sheets. See Figure 9.

343. *The three conicoids of the system* (1) *which pass through any given point* (x', y', z') *are orthogonal, that is, their tangent planes at* $(x', y,' z')$ *are perpendicular to one another.*

For, using the notation of the preceding section, since the conicoids (3) and (4) pass through the point (x', y', z'),

$$\frac{x'^2}{a^2+\lambda_1}+\frac{y'^2}{b^2+\lambda_1}+\frac{z'^2}{c^2+\lambda_1}\equiv 1, \quad \frac{x'^2}{a^2+\lambda_2}+\frac{y'^2}{b^2+\lambda_2}+\frac{z'^2}{c^2+\lambda_2}\equiv 1,$$

and therefore (subtracting and simplifying),

$$\frac{x'^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y'^2}{(b^2+\lambda_1)(b^2+\lambda_2)}+\frac{z'^2}{(c^2+\lambda_1)(c^2+\lambda_2)}\equiv 0. \qquad (6)$$

But the equations of the tangent planes to (3) and (4) at the point (x', y', z') are

$$\frac{xx'}{a^2+\lambda_1}+\frac{yy'}{b^2+\lambda_1}+\frac{zz'}{c^2+\lambda_1}=1, \quad \frac{xx'}{a^2+\lambda_2}+\frac{yy'}{b^2+\lambda_2}+\frac{zz'}{c^2+\lambda_2}=1,$$

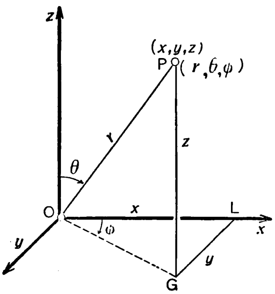
and (6) is the condition that these two planes be perpendicular to each other [§ 285]. And it can be proved in the same manner that the tangent planes to (3) and (5), and those to (4) and (5), at the point (x', y', z') are perpendicular to each other.

CHAPTER XVI

POLAR COORDINATES

344. Polar coordinates. The position of a point in space can be defined in other ways than by reference to an orthogonal system of axes such as has been used in the preceding pages. The following method is often employed:

As in the figure, let Ox, Oy, Oz represent the positive half axes of a rectangular system, P any point in space, and OG the projection of OP upon the xy-plane.



The position of P is defined by its distance r from O, the angle θ which OP makes with Oz, and the angle ϕ ($= xOG$) which the plane OzP makes with the plane Ozx. When P is defined in this way, the system of reference is the point O, the half-line Oz, and the plane Ozx; and r, θ, ϕ are called the *polar coordinates* of P referred to this system O, Oz, Ozx. As such a system of reference there may be taken *any* point O in space, any half-line Oz from O, and any plane Ozx containing Oz.

To construct a point $P(r, \theta, \phi)$ whose polar coordinates are given, take in the xy-plane a half-line OG making the angle ϕ with Ox, the angle being measured from Ox toward Oy when positive, in the contrary sense when negative; then, in the plane OzG thus determined, take the half-line OP making the angle θ with Oz, the angle being measured from Oz toward OG when positive, in the contrary sense when negative; and finally on this half-line OP itself or produced through O, according as r is positive or negative, lay off OP of length $|\,r\,|$.

345. The formulas connecting the rectangular coordinates of P (referred to Ox, Oy, Oz) and its polar coordinates (referred to O, Oz, Ozx) are easily found. Complete the figure by taking GL perpendicular to Ox. Then

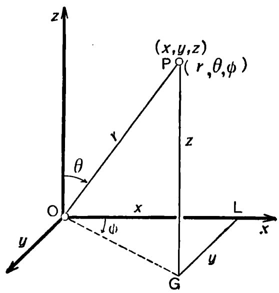
$$x = OL = OG \cos\phi = OP \sin\theta \cos\phi,$$
$$y = LG = OG \sin\phi = OP \sin\theta \sin\phi,$$
$$z = GP = OP \cos\theta.$$

Hence the required formulas are

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta.$$

Conversely, r, θ, ϕ are given by the formulas:

$$r^2 = x^2 + y^2 + z^2,$$
$$\tan^2\theta = (x^2 + y^2)/z^2,$$
$$\tan\phi = y/x.$$



346. Let the length of the line OG be represented by r'. The point P is sometimes considered as determined by (r', ϕ, z), which are then called the *cylindrical coordinates* of the point.

347. Exercises. Polar coordinates.

1. Find the rectangular coordinates of the points whose polar coordinates are: $(3, 30°, 60°)$, $(2, \pi/4, \pi)$, $(1, 45°, 45°)$.

2. Find the polar coordinates of the points whose rectangular coordinates are: $(2, 3, 4)$, $(3, 3, -2)$, $(-1, -2, 1)$.

3. What is represented by $r =$ const.?

4. What is represented by $\theta =$ const.?

5. What is represented by $\phi =$ const.?

6. What is represented by $\theta =$ const. and $\phi =$ const.?

7. What is represented by $\phi =$ const. and $r =$ const.?

8. What is represented by $r =$ const. and $\theta =$ const.?

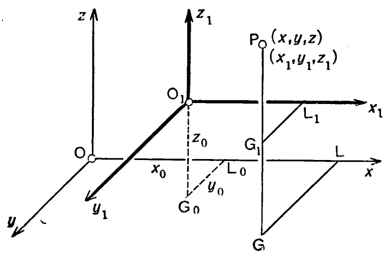
CHAPTER XVII

TRANSFORMATION OF COORDINATES

348. Transformation of coordinates. The formulas connecting the coordinates of a point referred to two different sets of rectilinear axes can be found. The process of changing from one set of axes to another is called the *transformation of coordinates.*

349. *Two parallel sets of axes, rectangular or oblique.*

Let Ox, Oy, Oz be a first set of axes, O_1 a point whose coordinates referred to these axes are x_0, y_0, z_0, and O_1x_1, O_1y_1, O_1z_1 a second set of axes parallel to Ox, Oy, Oz, respectively. Then, if P be any point in space, and x, y, z denote its coordinates referred to the system Ox, Oy, Oz, and x_1, y_1, z_1 its coordinates referred to the system O_1x_1, O_1y_1, O_1z_1, exactly as in § 144,



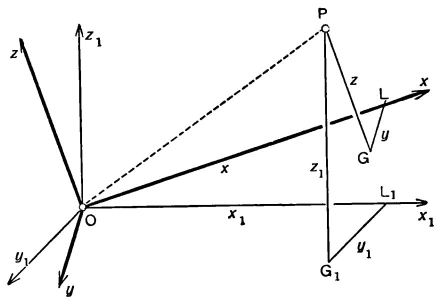
$$x = x_0 + x_1, \qquad x_1 = -x_0 + x,$$
$$y = y_0 + y_1, \quad \text{and} \quad y_1 = -y_0 + y,$$
$$z = z_0 + z_1, \qquad z_1 = -z_0 + z.$$

Since $(-x_0, -y_0, -z_0)$ are the coordinates of O referred to the system $O_1 - x_1y_1z_1$, these two sets of formulas are of precisely the same form when changing from the system $O - xyz$ to the system $O_1 - x_1y_1z$, and when changing from the system $O_1 - x_1y_1z_1$ to the system $O - xyz$.

350. *Two sets of rectangular axes with the same origin.*

Let Ox, Oy, Oz and Ox_1, Oy_1, Oz_1 be two systems of axes, both rectangular, and having the common origin O. Also, let P be any point in space, and let the coordinates of P in the first system be (x, y, z), in the second (x_1, y_1, z_1). The formulas connecting (x, y, z), and (x_1, y_1, z_1) can be obtained as follows:

As in the figure, connect P with O by the line segment, OP and by the two broken lines made up of the x, y, z of P, and its x_1, y_1, z_1, respectively. Then the projection of OP upon any line l will equal the projection of each of these broken lines upon this same line l [§ 246], and therefore the projection upon l of the broken line OL, LG, GP is equal to the projection upon l of the broken line OL_1, L_1G_1, G_1P; that is,



$$pr_l\, x + pr_l\, y + pr_l\, z = pr_l\, x_1 + pr_l\, y_1 + pr_l\, z_1.$$

Taking Ox, Oy, Oz, successively as l in this equation, which is to project upon Ox, Oy, Oz, successively, gives [§ 248]

$$x = x_1 \cos(xx_1) + y_1 \cos(xy_1) + z_1 \cos(xz_1),$$
$$y = x_1 \cos(yx_1) + y_1 \cos(yy_1) + z_1 \cos(yz_1),$$
$$z = x_1 \cos(zx_1) + y_1 \cos(zy_1) + z_1 \cos(zz_1),$$

where (xx_1) denotes the angle xOx_1, and so on.

And similarly, projecting upon Ox_1, Oy_1, Oz_1, successively, and inverting the members, gives

$$x_1 = x \cos(x_1x) + y \cos(x_1y) + z \cos(x_1z),$$
$$y_1 = x \cos(y_1x) + y \cos(y_1y) + z \cos(y_1z),$$
$$z_1 = x \cos(z_1x) + y \cos(z_1y) + z \cos(z_1z).$$

In the first equation of the first set, the coefficients are the direction cosines of Ox with respect to the axes Ox_1, Oy_1, Oz_1. Similarly in the remaining two equations of this set the coefficients are the direction cosines of Oy and Oz, respectively, with respect to Ox_1, Oy_1, Oz_1; and in the three equations of the second set, they are the direction cosines of Ox_1, Oy_1, and Oz_1, respectively, with respect to the axes Ox, Oy, Oz. The direction cosines of Ox, Oy, and Oz with respect to Ox_1, Oy_1 Oz_1 will be represented by $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and $(\lambda_3, \mu_3, \nu_3)$, respectively. Then the direction cosines of Ox_1, Oy_1, and Oz_1 with respect to Ox, Oy, Oz will be $(\lambda_1, \lambda_2, \lambda_3)$, (μ_1, μ_2, μ_3), and (ν_1, ν_2, ν_3), respectively, and the two sets of equations may be written

$$\begin{aligned} x &= \lambda_1 x_1 + \mu_1 y_1 + \nu_1 z_1, & x_1 &= \lambda_1 x + \lambda_2 y + \lambda_3 z, \\ y &= \lambda_2 x_1 + \mu_2 y_1 + \nu_2 z_1, & y_1 &= \mu_1 x + \mu_2 y + \mu_3 z, \\ z &= \lambda_3 x_1 + \mu_3 y_1 + \nu_3 z_1, & z_1 &= \nu_1 x + \nu_2 y + \nu_3 z. \end{aligned}$$

These equations and the meanings of the coefficients λ, μ, ν are exhibited in the accompanying scheme.

	x_1	y_1	z_1
x	λ_1	μ_1	ν_1
y	λ_2	μ_2	ν_2
z	λ_3	μ_3	ν_3

The nine cosines λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3, which appear as coefficients in these equations are connected by several (in number 22) important relations. Thus, since the coefficients in each set of equations are the direction cosines of mutually perpendicular lines [§ 240, § 252],

$$\begin{aligned} \lambda_1^2 + \mu_1^2 + \nu_1^2 &= 1, & \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 &= 0, \\ \lambda_2^2 + \mu_2^2 + \nu_2^2 &= 1, & \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 &= 0, \\ \lambda_3^2 + \mu_3^2 + \nu_3^2 &= 1, & \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 &= 0, \end{aligned} \qquad (1)$$

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= 1, & \mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3 &= 0, \\ \mu_1^2 + \mu_2^2 + \mu_3^2 &= 1, & \nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3 &= 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1, & \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 &= 0. \end{aligned} \qquad (2)$$

Again, taking the pair of equations

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0,$$
$$\lambda_1\lambda_3 + \mu_1\mu_3 + \nu_1\nu_3 = 0,$$

and solving for the ratios $\lambda_1 : \mu_1 : \nu_1$, gives

$$\lambda_1 : \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} = \mu_1 : \begin{vmatrix} \nu_2 & \lambda_2 \\ \nu_3 & \lambda_3 \end{vmatrix} = \nu_1 : \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}.$$

Calling each of these equal ratios $1 : k$,

$$\begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} = k\lambda_1, \quad \begin{vmatrix} \nu_2 & \lambda_2 \\ \nu_3 & \lambda_3 \end{vmatrix} = k\mu_1, \quad \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix} = k\nu_1. \tag{3}$$

Squaring each of these equations (3) and adding,

$$\begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix}^2 + \begin{vmatrix} \nu_2 & \lambda_2 \\ \nu_3 & \lambda_3 \end{vmatrix}^2 + \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}^2 = k^2(\lambda_1^2 + \mu_1^2 + \nu_1^2).$$

But [§ 253], the left number of this equation is equal to $\sin^2 yOz$, and therefore to unity, since the angle yOz is a right angle; and $\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1$. Hence

$$k^2 = 1, \text{ or } k = \pm 1. \tag{4}$$

Again, multiplying the equations (3) by λ_1, μ_1, ν_1, respectively, and adding,

$$\lambda_1 \cdot \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} + \mu_1 \cdot \begin{vmatrix} \nu_2 & \lambda_2 \\ \nu_3 & \lambda_3 \end{vmatrix} + \nu_1 \cdot \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix} = k(\lambda_1^2 + \mu_1^2 + \nu_1^2),$$

that is,

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = k = \pm 1. \tag{5}$$

Moreover, setting the value $k = \pm 1$ from (4) in the equations (3), gives

$$\lambda_1 = \pm \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix}, \quad \mu_1 = \pm \begin{vmatrix} \nu_2 & \lambda_2 \\ \nu_3 & \lambda_3 \end{vmatrix}, \quad \nu_1 = \pm \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}, \tag{6}$$

and the like can be shown true for the other elements λ_2, μ_2, ν_2, λ_3, μ_3, ν_3 of the above determinant (5), which is called the *determinant of the transformation.* Hence

The determinant of the transformation is equal to ± 1. *When the value of the determinant is* 1, *each element is equal to its minor; when the value of the determinant is* -1, *each element is equal to minus its minor.*

It only remains to find when the value of the determinant is 1, and when -1. If the two sets of axes are *congruent,* and are made to coincide, $\lambda_1 = \mu_2 = \nu_3 = 1$ and $\mu_1 = \nu_1 = \nu_2 = \lambda_2 = \lambda_3 = \mu_3 = 0$, and the value of the determinant is 1. But if the two sets of axes are *symmetric,* that is, are so situated that, when Ox is made to coincide with Ox_1, and Oy with Oy_1, Oz and Oz_1 have opposite directions, then, after this displacement, $\lambda_1 = \mu_2 = 1$, $\nu_3 = -1$, and $\mu_1 = \nu_1 = \nu_2 = \lambda_2 = \lambda_3 = \mu_3 = 0$, and the value of the determinant is -1. *Hence the value of the determinant is* 1 *or* -1 *according as the two sets of axes are congruent or symmetric.*

The nine quantities λ_1, μ_1, ν_1, λ_2, μ_2, ν_2, λ_3, μ_3, ν_3, which satisfy the 22 relations, in (1), (2), (5), and (6), are called the coefficients of an *orthogonal substitution.*

351. The equations above given for x, y, z in terms of x_1, y_1, z_1, namely

$$x = \lambda_1 x_1 + \mu_1 y_1 + \nu_1 z_1, \quad y = \lambda_2 x_1 + \mu_2 y_1 + \nu_2 z_1, \quad z = \lambda_3 x_1 + \mu_3 y_1 + \nu_3 z_1,$$

may also be used to transform from a rectangular system Ox, Oy, Oz, to an *oblique* system Ox_1, Oy_1, Oz_1, in which the direction cosines of Ox_1, with respect to Ox, Oy, Oz, are λ_1, λ_2, λ_3, those of Oy_1 are μ_1, μ_2, μ_3, and those of Oz_1 are ν_1, ν_2, ν_3.

By solving these equations for x_1, y_1, z_1, expressions for x_1, y_1, z_1 are obtained in terms of x, y, z which are of the first degree, but lack the simplicity of form they have when the system Ox_1, Oy_1, Oz_1 is rectangular.

352. Since all the equations of transformation in §§ 349, 350, 351, are of the first degree in both x, y, z and x_1, y_1, z_1, and any transformation of rectilinear coordinates may be effected by these equations singly or combined, the degree of an equation is not increased by a transformation of coordinates. And it cannot be decreased; for if it could, the transformation back to the original axes would give an equation of lower degree than the original equation.

353. Exercises. Transformation of coordinates.

1. Transform the equation $x^2 - 3yz + y^2 - 6x + z = 0$ to parallel axes through the point $(1, -1, 2)$.

2. Apply the transformation $x = x_0 + x_1$, $y = y_0 + y_1$, $z = z_0 + z_1$ to the equation $x^2 - 2y^2 + z^2 + 2x - 3y + z = 0$, and give such values to x_0, y_0, z_0 that the transformed equation shall lack all terms of the first degree.

3. Prove that the three planes $x + 2y + 2z = 0$, $2x + y - 2z = 0$, $2x - 2y + z = 0$ are perpendicular to one another, and, calling their lines of intersection Ox_1, Oy_1, Oz_1, find the equations of transformation from the system Ox, Oy, Oz to the system Ox_1, Oy_1, Oz_1; and *vice versa*.

4. Solve the same problem for the planes $x + y + z = 0$, $x - 2y + z = 0$, $x - z = 0$.

CHAPTER XVIII

GENERAL EQUATION OF THE SECOND DEGREE

354. Centers. As in §§ 299, 328, and 330, the distances from the point P_0 to the points P_1, P_2 where the conicoid

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz + 2\,lx + 2\,my + 2\,nz + d = 0 \qquad (1)$$

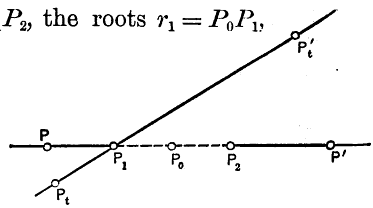
is met by the line

$$\frac{x - x_0}{\lambda} = \frac{y - y_0}{\mu} = \frac{z - z_0}{\nu} = r, \qquad (2)$$

are the roots of the equation in r,

$$\begin{aligned}\{a\lambda^2 + b\mu^2 + c\nu^2 + 2\,h\lambda\mu + 2\,g\lambda\nu + 2\,f\mu\nu\}r^2 \\ + 2\{(ax_0 + hy_0 + gz_0 + l)\lambda \\ + (hx_0 + by_0 + fz_0 + m)\mu \\ + (gx_0 + fy_0 + cz_0 + n)\nu\}r \\ + (ax_0^2 + by_0^2 + cz_0^2 + 2\,hx_0y_0 + 2\,gx_0z_0 + 2\,fy_0z_0 \\ + 2\,lx_0 + 2\,my_0 + 2\,nz_0 + d) = 0,\end{aligned} \qquad (3)$$

or, $$(a\lambda^2 + b\mu^2 + c\nu^2 + 2\,h\lambda\mu + 2\,g\lambda\nu + 2\,f\mu\nu)r^2 + (\partial F/\partial x_0 \cdot \lambda + \partial F/\partial y_0 \cdot \mu + \partial F/\partial z_0 \cdot \nu)r + F(x_0, y_0, z_0) = 0. \qquad (3')$$



If P_0 is the mid-point of P_1P_2, the roots $r_1 = P_0P_1$, $r_2 = P_0P_2$ are equal in length; but since P_0 is between P_1 and P_2, P_0P_1 and P_0P_2 are of opposite sign; and therefore $r_1 = -r_2$, or, $r_1 + r_2 = 0$. But, in any quadratic equation in r, in which the sum of the roots is zero, the coefficient of r is zero; therefore in (3′)

$$\partial F/\partial x_0 \cdot \lambda + \partial F/\partial y_0 \cdot \mu + \partial F/\partial z_0 \cdot \nu = 0. \qquad (4)$$

This equation (4) will be true for *all* values of λ, μ, ν, if x_0, y_0, z_0 have such values that $\partial F/\partial x_0$, $\partial F/\partial y_0$, $\partial F/\partial z_0$ are each zero; that is, if

$$\left.\begin{aligned} ax_0 + hy_0 + gz_0 + l &= 0, \\ hx_0 + by_0 + fz_0 + m &= 0, \\ gx_0 + fy_0 + cz_0 + n &= 0. \end{aligned}\right\} \tag{5}$$

355. If the equations (5) are *both independent and consistent* [§ 294], they have a single solution (x_0, y_0, z_0), and this is finite. Hence, in this case, there exists a point $P_0(x_0, y_0, z_0)$, which bisects every chord through it. This point is called the *center* of the conicoid.

It is convenient to use the following notation in the solution of (5): The determinant

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix} \tag{6}$$

is called the determinant of the coefficients of the general equation (1). The co-factor of any element in Δ will be represented by the capital letter corresponding to that element. With this notation the solution of (5),

$$\frac{x}{\begin{vmatrix} h & g & l \\ b & f & m \\ f & c & n \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a & g & l \\ h & f & m \\ g & c & n \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}}, \tag{7}$$

becomes:

$$x = L/D,\ y = M/D,\ z = N/D. \tag{8}$$

Example. Find the coordinates of the center of the conicoid

$$x^2 + y^2 - z^2 + 2\,zx + 4\,xy + 4\,yz - 2\,y + 4\,z - 4 = 0.$$

The equations (5) for the center of this conicoid are,

$$\begin{aligned} x + 2\,y + z &= 0, \\ 2\,x + y + 2\,z - 1 &= 0, \\ x + 2\,y - z + 2 &= 0. \end{aligned}$$

The center is the point $(-1/3, -1/3, 1)$.

356. The equations (5) may *not be consistent* [§ 295, (2)], and in this case the equations (5) have no finite solution in common. The geometric statement of this is that the center is at an infinite distance. Algebraically, in the equations (8), $D=0$, and at least one of L, M, N, is not zero.

Example. Find the coordinates of the center of the conicoid

$$x^2+4y^2-z^2+4xy+4yz+2xz+2x+6y-3z-4=0.$$

The equations (5) for the center of this conicoid are,

$$\begin{aligned} x+2y+z+1&=0,\\ 2x+4y+2z+3&=0,\\ x+2y-z-3/2&=0. \end{aligned}$$

Here $D=0$, $L=-4$, $M=2$, $N=0$; and the equations (8) give for the center $(\infty, \infty, 0/0)$; the center is at an infinite distance. (The surface is a paraboloid.)

357. The equations (5) may not be independent [§ 295, (1)], and in this case every solution of two of the equations (5) is a solution of the third; that is, the three planes represented by (5) pass through a line. (This line may be at infinity.) The geometric statement of this is, that there is a *line of centers*. The algebraic statement of the condition is that $D=0$, $L=0$, $M=0$, $N=0$.

Example 1. Find the coordinates of the center of the conicoid

$$x^2+4y^2-z^2+4xy+4yz+2xz+2x+4y-2z+d=0.$$

The equations (5) for the center of this conicoid are

$$\begin{aligned} x+2y+z+1&=0,\\ 2x+4y+2z+2&=0,\\ x+2y-z-1&=0. \end{aligned}$$

These equations are equivalent to $x=-2y$, $z=-1$; that is, any point on the line $\frac{x-0}{2/\sqrt{5}}=\frac{y-0}{-1/\sqrt{5}}=\frac{z+1}{0}$ is a center of the conicoid. (The surface is a cylinder.)

If $d=-1$, the point $(0, 0, -1)$ lies on the surface, and it will be seen that the original equation represents two planes, its left member being the product of the factors indicated below:

$$\{x + 2y + (1 + \sqrt{2})z + (1 + \sqrt{2})\}\{x + 2y + (1 - \sqrt{2})z + (1 - \sqrt{2})\} = 0.$$

Example 2. Find the coordinates of the center of the conicoid

$$x^2 + 4y^2 + z^2 + 4xy + 2xz + 4yz + 2x + 4y + 2z - 3 = 0.$$

The equations (5) for the center of this conicoid are

$$x + 2y + z + 1 = 0,$$
$$2x + 4y + 2z + 2 = 0,$$
$$x + 2y + z + 1 = 0.$$

That is, every point on the plane $x + 2y + z + 1 = 0$ is a center of the conicoid; and it will be seen that the original equation can be written

$$(x + 2y + z + 3)(x + 2y + z - 1) = 0.$$

Example 3. Find the coordinates of the center of the conicoid

$$x^2 + z^2 - 2zx + x + 4y = 0.$$

The second equation of (5) is $2 = 0$. Hence in this case the derivation of the equations (5) from (4) fails. But (4) will be true if $\mu = 0$, $\partial F/\partial x_0 = 0$, $\partial F/\partial z_0 = 0$. That is, the line P_1P_2 is parallel to the xz-plane, and the coordinates of the center P_0 satisfy the first and third equations of (5), namely, $x_0 - z_0 + 1/2 = 0$, $-x_0 + z_0 = 0$. The center is on a line perpendicular to the y-axis (since $\mu = 0$), and at infinity in the plane $x - z = 0$. There is a line of centers at infinity; in fact, by a method similar to that of § 158 the equation can be written $(x - z + k)^2 = (-1 + 2k)x - 4y - 2kz + k^2$. The planes $x - z + k = 0$ and $(-1 + 2k)x - 4y - 2kz + k^2 = 0$ will be perpendicular to each other, if $(-1 + 2k) - 4.0 + 2k = 0$, or $4k = 1$, or $k = 1/4$, and the given equation then becomes $(x - z + 1/4)^2 = -(1/2)(x + 8y + z - 1/8)$, or, finally,

$$\left(\frac{x - z + 1/4}{\sqrt{2}}\right)^2 = -4\left(\frac{\sqrt{66}}{16}\right)\left(\frac{x + 8y + z - 1/8}{\sqrt{66}}\right).$$

(The surface is a parabolic cylinder.)

358. Conicoid referred to center. When the conicoid has a finite center, the equation of the surface referred to the center as origin is obtained as follows:

Let the center be $C(x_0, y_0, z_0)$. The transformation of § 349, namely, $x = x_1 + x_0$, $y = y_1 + y_0$, $z = z_1 + z_0$, changes

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz + 2\,lx + 2\,my + 2\,nz + d = 0 \quad (1)$$

into

$$ax_1^2 + by_1^2 + cz_1^2 + 2\,hx_1y_1 + 2\,gx_1z_1 + 2\,fy_1z_1 + 2(ax_0 + hy_0 + gz_0 + l)\cdot x_1 + 2(hx_0 + by_0 + fz_0 + m)\cdot y_1 + 2(gx_0 + fy_0 + cz_0 + n)\cdot z_1 + F(x_0, y_0, z_0) = 0. \quad (2)$$

Let $$d' \equiv F(x_0, y_0, z_0). \quad (3)$$

Then, since (x_0, y_0, z_0) is the center, the coefficients of x_1, y_1, z_1 are zero [§ 354, (5)], namely,

$$ax_0 + hy_0 + gz_0 + l = 0, \quad (4)$$

$$hx_0 + by_0 + fz_0 + m = 0, \quad (5)$$

$$gx_0 + fy_0 + cz_0 + n = 0, \quad (6)$$

and the equation (2) of the surface *referred to the center* becomes, after dropping the subscripts,

$$ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz + d' = 0. \quad (7)$$

Multiplying (4) by x_0, (5) by y_0, (6) by z_0, and subtracting from $d' = F(x_0, y_0, z_0)$ gives

$$d' = lx_0 + my_0 + nz_0 + d, \quad (8)$$

which can be written

$$lx_0 + my_0 + nz_0 + (d - d') = 0. \quad (9)$$

The determinant of the equations (4), (5), (6), (9) is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & (d-d') \end{vmatrix} = 0. \quad (10)$$

Therefore, $$d' \cdot \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}, \quad (11)$$

or, using the symbols defined in § 355,

$$d' = \Delta/D. \tag{12}$$

When the coordinates of the center have been found, the value of d' can be obtained from (8); or it may be obtained from (12). The equation (7) of the surface referred to its center as origin is then known.

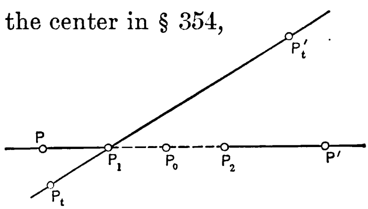
359. Exercises. Centers of conicoids.

Find the centers of the conicoids represented by the following equations; and, when there is one center at a finite distance from the original origin, transform the equation to the center as origin.

1. $2x^2 + y^2 - z^2 - 2zx - 4xy + 4yz + 2y - 4z - 4 = 0.$
2. $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy - x + y - z = 0.$
3. $y^2 + zx + 3xy + 2yz + 3x + 2y = 0.$
4. $5x^2 + 9y^2 + 9z^2 - 12xy - 6yz + 12x - 36z = 0.$
5. $z^2 - xz - yz - z = 0.$
6. $2x^2 + 4y^2 - z^2 - 8xy + 8x - 8y + 4 = 0.$
7. $xy + yz + xz - 9 = 0.$
8. $6x^2 + 28y^2 + 5z^2 - 8xy - 4xz - 12x + 8y + 4z = 0.$

360. Diametral and principal planes. By a *chord* of a conicoid is meant the line joining any two of its points.

361. *The locus of the mid-points of any system of parallel chords of a conicoid is a plane.*



In finding the equations for the center in § 354, the equation (4) of that section was considered true for *all* values of (λ, μ, ν); but if, on the other hand, the direction cosines (λ, μ, ν) in the equations of the line (2) of § 354 are considered *given*, then the equation § 354, (4), or

$$(ax_0 + hy_0 + gz_0 + l)\lambda + (hx_0 + by_0 + fz_0 + m)\mu$$
$$+ (gx_0 + fy_0 + cz_0 + n)\nu = 0, \qquad (4')$$

states that $P_0(x_0, y_0, z_0)$ will be the mid-point of a chord having the given direction cosines λ, μ, ν, if it lies anywhere on the plane [obtained by changing the order of the terms in (4′)]

$$(a\lambda + h\mu + g\nu)x + (h\lambda + b\mu + f\nu)y + (g\lambda + f\mu + c\nu)z$$
$$+ (l\lambda + m\mu + n\nu) = 0. \qquad (4'')$$

(Compare § 108, (6).) Hence, all chords of the system (2) of § 354 are bisected by the plane (4″), as was to be proved.

362. A plane which bisects a system of parallel chords of a conicoid is called a *diametral plane.* If such a plane be perpendicular to the chords which it bisects, it is called a *principal plane.*

363. *To determine the principal planes of a conicoid.*

The plane (4″) of § 361 will be perpendicular to the chords (2) of § 354 which it bisects, if λ, μ, ν have such values that

$$\frac{a\lambda + h\mu + g\nu}{\lambda} = \frac{h\lambda + b\mu + f\nu}{\mu} = \frac{g\lambda + f\mu + c\nu}{\nu}. \qquad (5)$$

If k denote the value of these equal fractions, the equations (5) are equivalent to the following:

$$\left.\begin{aligned} (a-k)\lambda + h\mu + g\nu &= 0 \\ h\lambda + (b-k)\mu + f\nu &= 0 \\ g\lambda + f\mu + (c-k)\nu &= 0 \end{aligned}\right\}. \qquad (6)$$

The elimination of λ, μ, ν gives

$$\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix} = 0, \qquad (7)$$

or, expanding and collecting terms,

$$k^3 - (a+b+c)k^2 + (bc + ca + ab - f^2 - g^2 - h^2)k$$
$$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0. \qquad (7')$$

For any value of k which satisfies (7′) the three equations (6) in λ, μ, ν are consistent. Hence if k_1 denote a real root of (7′) [equation (7′) has at least one real root since it is a cubic] and if k_1 be substituted for k in any two of the equations (6) and these equations be solved for $\lambda : \mu : \nu$, values of these ratios will be obtained for which (4″) will represent a principal plane.

As a matter of fact all three roots of (7′) are real.*

364. Classification of conicoids. It has just been seen that every conicoid has a principal plane. Take any point O_1 in this plane, and through O_1 take the line O_1x_1 perpendicular to the plane and any two lines O_1y_1, O_1z_1 in the plane which are at right angles to each other. And suppose the equation $F(x, y, z) = 0$ (1) transformed [Chapter XVII] to O_1x_1, O_1y_1, O_1z_1, as new axes of reference. The transformed equation will have the form

$$a'x_1^2 + b'y_1^2 + c'z_1^2 + 2f'y_1z_1 + 2m'y_1 + 2n'z_1 + d' = 0, \quad (2)$$

that is, it will lack all terms in which x_1 enters to the first

*Cauchy's proof of the reality of the three roots of the discriminating cubic, that is, the equation (7), is as follows:

Let the equation (7) be written in the form

$$K \equiv (k-a)\{(k-b)(k-c)-f^2\} - \{g^2(k-b) + h^2(k-c) + 2fgh\} = 0. \quad (7'')$$

Let $b > c$, or $b = c$, and first consider the expression $\{(k-b)(k-c) - f^2\}$.

When	$k = +\infty$	b	c	$-\infty$,
then	$\{(k-b)(k-c)-f^2\} = +\infty$	$-f^2$	$-f^2$	$+\infty$.

Therefore $\{(k-b)(k-c) - f^2\}$ is zero for a value of k between $+\infty$ and b, inclusive, and for a value of k between $-\infty$ and c, inclusive; let these values of k be, α and γ, respectively; then $+\infty > \alpha \geqq b \geqq c \geqq \gamma > -\infty$.

1. Suppose $\alpha \neq \gamma$. If in the original cubic (7″), k be set equal to $+\infty$, α, γ, $-\infty$, successively, then the left member of (7″) will become positive, negative, positive again, negative again, successively. This can be proved as follows:

(1) When $k = +\infty$, then $K = +\infty$.

(2) When $k = \alpha$, then $(\alpha - b)(\alpha - c) - f^2 = 0$, and the expression K becomes

$$-\{g^2(\alpha - b) + 2fgh + h^2(\alpha - c)\} = -\{\pm g\sqrt{\alpha - b} \pm h\sqrt{\alpha - c}\}^2$$
$$= -\text{ (a perfect square)} = \text{a negative number.}$$

(3) When $k = \beta$, then $(b - \gamma)(c - \gamma) - f^2 = 0$, and the expression K becomes

power. For, by hypothesis, the plane $x_1 = 0$ is a principal plane and therefore bisects all chords of the conicoid which are parallel to O_1x_1. Hence, if the point (x_1', y_1', z_1') be on the surface, the point $(-x_1', y_1', z_1')$ will also be on the surface. But this requires that the equation shall lack the terms just mentioned.

But the equation (2) can be rid of the y_1z_1 term, if present, by the method explained in § 145, Ex. 2. It is merely necessary to take in the plane $x_1 = 0$ two lines O_1y_2, O_1z_2 which make the angle $\frac{1}{2}\tan^{-1} 2f' / (b' - c')$ with O_1y_1 and O_1z_1, respectively, and then to transform the equation to O_1x_1, O_1y_2, O_1z_2 as axes of reference.

It has thus been proved that, by a transformation of coordinates, every equation of the second degree can be reduced to the form

$$Ax^2 + By^2 + Cz^2 + 2My + 2Nz + D = 0, \tag{3}$$

where $A \neq 0$, but any of the other coefficients may be 0.

$$-\{-g^2(b-\gamma) + 2fgh - h^2(c-\gamma)\} = +\{g^2(b-\gamma) - 2fgh + h^2(c-\gamma)\}$$
$$= +\{\pm g\sqrt{b-\gamma} \mp h\sqrt{c-\gamma}\}^2 = + \text{(a perfect square)} = \text{a positive number.}$$

(4) When $k = -\infty$, then $K = -\infty$.

Therefore the cubic (7″) has one real root between $+\infty$ and α, another real root between α and γ, and a third root between γ and $-\infty$. That is, the cubic has three real roots.

2. When $\alpha = \gamma$, then $\{(k-b)(k-c) - f^2\}$ is a perfect square, namely, $(k-\alpha)^2$. But the condition that $\{k^2 - (b+c)k + (bc - f^2)\}$ be a perfect square [Alg. § 635, 2] is $b^2 + 2bc + c^2 - 4(bc - f^2) = 0$, or $(b-c)^2 + 4f^2 = 0$, or finally, $b = c$ and $f = 0$. In this case the cubic (7″) becomes

$$(k-b)\{(k-a)(k-b) - g^2 - h^2\} = 0.$$

One root of this cubic is b, and the other two are obtained from the quadratic factor, that is, from

$$k^2 - (a+b)k + \{ab - (g^2 + h^2)\} = 0,$$

the solution of which gives

$$2k = a + b \pm \sqrt{a^2 + 2ab + b^2 - 4ab + 4(g^2 + h^2)} = a + b \pm \sqrt{(a-b)^2 + 4(g^2 + h^2)}.$$

These last two values of k are also real, and again the cubic has three real roots, as was to be proved.

1. Suppose both B and C to be different from 0.
The equation can be written in the form

$$Ax^2 + B\left(y + \frac{M}{B}\right)^2 + C\left(z + \frac{N}{C}\right)^2 = \frac{M^2}{B} + \frac{N^2}{C} - D.$$

Call the right member D', and transform [by § 349] to the origin $(0, -M/B, -N/C)$. Then the equation becomes

$$Ax^2 + By^2 + Cz^2 = D'.$$

If $D' \neq 0$, divide the equation by D' and so reduce it to one of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

where a^2 is written for $|D'/A|$, and so on.

If $D' = 0$, the equation will have one of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

2. Suppose C, one of the coefficients B, C, to be 0. The equation can be written in the form

$$Ax^2 + B\left(y + \frac{M}{B}\right)^2 + 2\,Nz + D - \frac{M^2}{B} = 0.$$

Hence when $N \neq 0$, by a change of origin,

$$Ax^2 + By^2 + 2\,Nz = 0,$$

which may be reduced to one of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 2\,z, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 2\,z.$$

Similarly when $N = 0$, by a change of origin,

$$Ax^2 + By^2 + D' = 0,$$

where D' may be 0, and this may be reduced to one of the forms:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1, \qquad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 0.$$

3. Finally, suppose both B and C to be 0. The equation is then

$$Ax^2 + 2My + 2Nz + D = 0.$$

By a transformation of coordinates in which the plane $2My + 2Nz + D = 0$ is taken as a new plane of reference $y = 0$, this equation can be reduced to the form

$$x^2 = 4ay,$$

or interchanging x and y, to

$$y^2 = 4ax.$$

But if both M and N are 0, it has the form

$$x^2 = a,$$

or interchanging x and y,

$$y^2 = a,$$

where, in particular, a may be 0.

These are the forms of the equation of the second degree given in the list of § 327.

365. The cone. When an equation of the second degree represents a cone with the center (or vertex) as origin, it follows, as in § 315, that if (x', y', z') be any solution of the equation, so also is (kx', ky', kz') a solution, whatever the value of k may be. Therefore, the most general equation of a cone of the second degree referred to the center as origin must be homogeneous, or of the form

$$ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz = 0.$$

That is, in the equation (7) of § 358, $d' = 0$; and therefore

[§ 358, (12)] $\Delta = 0$, since D is not ∞. Conversely, when $\Delta = 0$, and $D \neq 0$, then $d' = 0$, and the surface is a cone.

When there is a line of centers [§ 357], and the surface is a cylinder or a pair of planes, the equations giving the center are not independent, and $D = 0$, $L = 0$, $M = 0$, $N = 0$; whence $\Delta = 0$. Conversely, if $\Delta = 0$ and $D = 0$, then will $L = 0$, $M = 0$, $N = 0$, and the surface therefore will be a cylinder or a pair of planes; this can be proved as follows:

Since a determinant with two identical rows is zero, it follows that

$$aL + hM + gN + lD \equiv 0, \tag{1}$$

$$hL + bM + fN + mD \equiv 0, \tag{2}$$

$$gL + fM + cN + nD \equiv 0, \tag{3}$$

and, by definition,

$$lL + mM + nN + dD \equiv \Delta. \tag{4}$$

When $\Delta = 0$ and $D = 0$, these four equations become

$$aL + hM + gN = 0, \tag{1'}$$

$$hL + bM + fN = 0, \tag{2'}$$

$$gL + fM + cN = 0, \tag{3'}$$

$$lL + mM + nN = 0. \tag{4'}$$

In the equations (2′), (3′), (4′), either the elements L, M, N themselves are all zero, or the determinant of the coefficients is zero, that is, L is 0; and in the same way, from (1′), (3′), and (4′), it follows that $M = 0$, and from (1′), (2′), and (4′), that $N = 0$.

366. *Therefore, $\Delta = 0$ is the necessary and sufficient condition that the general equation of the second degree represents a cone; when also $D = 0$, this cone has its center or vertex at infinity and is therefore either a cylinder or a pair of planes.*

367. Invariants. There are four expressions made up of the coefficients of the general equation of the second degree, the values of which remain unchanged when the equation of the surface is changed by a transformation from one orthogonal system of axes to any other such system [§ 349, § 350]. On account of this property these expressions are called *invariants.* The four invariants of

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2lx + 2my + 2nz + d = 0$$

are

$$I \equiv a + b + c,$$

$$J \equiv bc + ca + ab - f^2 - g^2 - h^2,$$

$$D \equiv abc + 2fgh - af^2 - bg^2 - ch^2,$$

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix},$$

where Δ is called the *discriminant of* $F(x, y, z)$.

368. *Proof that I, J, and D are invariants.*

Any orthogonal transformation can be regarded as made up of one transformation to parallel axes [§ 349] and of one orthogonal transformation about the origin [§ 350].

The transformation to parallel axes [§ 349] affects only the absolute term and the coefficients of the terms of the first degree of $F(x, y, z)$, but does not change the coefficients of the terms of the second degree, and therefore leaves I, J, and D unchanged.

The orthogonal transformation of § 350 does not give a term of the second degree from a term not originally of the second degree, and changes

$$U_2 \equiv ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$$

into $$U'_2 = a'x'^2 + b'y'^2 + c'z'^2 + 2h'x'y' + 2g'x'z' + 2f'y'z'.$$

Build the function $\phi(x, y, z) \equiv U_2 - k(x^2 + y^2 + z^2)$. Then, since $(x^2 + y^2 + z^2)$ is the square of the distance from the origin to the representative point (x, y, z), it remains unchanged when the axes are twisted about the origin, and U_2 changes into U'_2, and $\phi(x, y, z)$ changes into $\phi'(x', y', z') \equiv U'_2 - k(x'^2 + y'^2 + z'^2)$. If now k have such a value that $\phi(x, y, z)$ splits up into the product of two linear factors [§ 150], and $\phi(x, y, z) = 0$, therefore, represents two planes, then $\phi'(x', y', z') = 0$ also represents two planes, and $\phi'(x', y', z')$ also splits up into the product of two factors. Therefore, $\phi(x, y, z)$ and $\phi'(x', y', z')$ will split up into the product of two linear factors for the same value of k.

The condition that $\phi(x, y, z)$ shall split up into two linear factors is [§ 150, (3)],

$$(a-k)(b-k)(c-k)-(a-k)f^2-(b-k)g^2-(c-k)h^2+2fgh=0,$$

or, $$k^3-(a+b+c)k^2+(bc+ca+ab-f^2-g^2-h^2)k-D=0,$$

or [§ 367], $$k^3 - Ik^2 + Jk - D = 0. \tag{1}$$

The condition that $\phi'(x', y', z')$ shall split up into two linear factors is, in the same way,

$$k^3 - I'k^2 + J'k - D' = 0, \tag{2}$$

where I', J', and D' are the same functions of the coefficients of U'_2 as I, J, and D are of the coefficients of U_2.

Since the roots of (1) and (2) are the same, their coefficients are in proportion, and since the coefficients of k^3 are equal, the other coefficients are equal in pairs, namely,

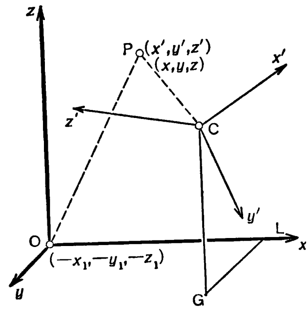
$$I = I', \quad J = J', \quad D = D',$$

which was to be proved.

369. The equation (1) of § 368 is called the *discriminating cubic* of $F(x, y, z)$.

370. *Proof that Δ is an invariant.*

Let (x, y, z) be the coordinates of a point P when referred to a coordinate system $O\text{–}xyz$, and let (x', y', z') be the coordinates of the same point when referred to another coordinate system $C\text{–}x'y'z'$. Let the coordinates of C in the $O\text{–}xyz$ system be OL, LG, GC; and let the coordinates of O in the $C\text{–}x'y'z'$ system be $(-x_1, -y_1, -z_1)$.



Let the equation of any conicoid referred to the $O\text{–}xyz$ system be

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2lx + 2my + 2nz + d = 0, \quad (1)$$

and let this equation become

$$F'(x', y', z') \equiv a'x'^2 + b'y'^2 + c'z'^2 + 2h'x'y' + 2g'x'z' + 2f'y'z' + 2l'x' + 2m'y' + 2n'z' + d' = 0, \quad (2)$$

when referred to the $C\text{–}x'y'z'$ system.

Build the equation

$$\phi(x, y, z) \equiv F(x, y, z) - k\{x^2 + y^2 + z^2 - 1\} = 0. \quad (3)$$

The sum $x^2 + y^2 + z^2$ is the square of the length of the line OP in the $O\text{–}xyz$ system [§ 237]; and, in the $C\text{–}x'y'z'$ system, the square of the length of this same line OP is $(x' + x_1)^2 + (y' + y_1)^2 + (z' + z_1)^2$ [§ 236]. Therefore the transformation which changes $F(x, y, z) = 0$, (1), into $F'(x', y', z') = 0$, (2), also changes

$$F(x, y, z) - k\{x^2 + y^2 + z^2 - 1\} = 0 \quad (3)$$

into

$$F'(x', y', z') - k\{(x' + x_1)^2 + (y' + y_1)^2 + (z' + z_1)^2 - 1\} = 0. \quad (4)$$

Let the discriminants of (1), (2), (3), and (4) be represented by Δ_1, Δ_2, Δ_3, and Δ_4, respectively; namely:

$$\Delta_1 \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}, \qquad \Delta_2 \equiv \begin{vmatrix} a' & h' & g' & l' \\ h' & b' & f' & m' \\ g' & f' & c' & n' \\ l' & m' & n' & d' \end{vmatrix},$$

$$\Delta_3 \equiv \begin{vmatrix} a-k & h & g & l \\ h & b-k & f & m \\ g & f & c-k & n \\ l & m & n & d+k \end{vmatrix},$$

$$\Delta_4 \equiv \begin{vmatrix} a'-k & h' & g' & l'-kx_1 \\ h' & b'-k & f' & m'-ky_1 \\ g' & f' & c'-k & n'-kz_1 \\ l'-kx_1 & m'-ky_1 & n'-kz_1 & d'-k(x_1^2+y_1^2+z_1^2-1) \end{vmatrix}.$$

Let, now, k have such a value that (3) represents a cone, then $\Delta_3 = 0$ [§ 366]; for this value of k, (4) also represents the same cone, and therefore $\Delta_4 = 0$. That is, the roots of $\Delta_3 = 0$ and $\Delta_4 = 0$, regarded as equations in k, are equal, and therefore the coefficients of k in $\Delta_3 = 0$ and $\Delta_4 = 0$ are in proportion. The coefficient of k^4 in $\Delta_3 = 0$ is -1, and the coefficient of k^4 in $\Delta_4 = 0$ is

$$\begin{vmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ x_1 & y_1 & z_1 & (x_1^2+y_1^2+z_1^2-1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1.$$

Hence all the corresponding coefficients are not only in proportion but equal. The absolute term of $\Delta_3 = 0$ is Δ_1, and the absolute term of $\Delta_4 = 0$ is Δ_2. Therefore $\Delta_1 = \Delta_2$; that is, from the definition, the discriminant is an invariant, as was to be proved.

371. Classification of conicoids. In § 364, (3) it has been proved that by a transformation of coordinates the terms of the second degree in the general equation can be reduced to $Ax^2 + By^2 + Cz^2$; that is, with the notation of § 368, the transformation of § 364 changes U_2 into $U'_2 \equiv Ax'^2 + By'^2 + Cz'^2$; and the equation (2) of § 368 becomes

$$k^3 - (A+B+C)k^2 + (BC+CA+AB)k - ABC = 0,$$

the roots of which are $k_1 = A$, $k_2 = B$, $k_3 = C$. But the roots of equations (1) and (2) of § 368 are the same; therefore the roots of the discriminating cubic of $F(x, y, z) = 0$ are A, B, C, the coefficients of x^2, y^2, z^2 in § 364, Eq. (3); and [§ 364, 1., 2., and 3.] the equation $F(x, y, z) = 0$, represents a central conicoid, when the discriminating cubic has no zero root; a paraboloid, an elliptic or hyperbolic cylinder, or a pair of planes (real or imaginary), when this cubic has one zero root; a parabolic cylinder or a pair of parallel or coincident planes, when this cubic has two zero roots.

If two of the roots of the discriminating cubic are equal and different from zero, the surface is a surface of revolution.

372. Recapitulation. The following symbols, definitions, and equations are used in connection with the equation

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz$$
$$+ 2\,lx + 2\,my + 2\,nz + d = 0. \qquad (1)$$

The *discriminating cubic* is

$$K(k) \equiv k^3 - (a + b + c)k^2 + (bc + ca + ab - f^2 - g^2 - h^2)k$$
$$- (abc + 2\,fgh - af^2 - bg^2 - ch^2) = 0. \qquad (2)$$

The *invariants* are I, J, D, Δ.

$$I \equiv (a + b + c). \qquad (3)$$

$$J \equiv (bc + ca + ab - f^2 - g^2 - h^2). \qquad (4)$$

$$D \equiv (abc + 2\,fgh - af^2 - bg^2 - ch^2). \qquad (5)$$

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}. \qquad (6)$$

$$\Delta \equiv lL + mM + nN + dD. \qquad (6')$$

L, M, N, D, $\cdots$ are the cofactors of l, m, n, d, $\cdots$ in Δ.
Δ is the *discriminant* of the equation.
D is the *discriminant* of the terms of the *second* degree.

The equations giving the *center* are

$$\left.\begin{aligned}\tfrac{1}{2}\,\partial F_0/\partial x_0 &\equiv ax_0 + hy_0 + gz_0 + l = 0,\\ \tfrac{1}{2}\,\partial F_0/\partial y_0 &\equiv hx_0 + by_0 + fz_0 + m = 0,\\ \tfrac{1}{2}\,\partial F_0/\partial z_0 &\equiv gx_0 + fy_0 + cz_0 + n = 0,\end{aligned}\right\} \tag{7}$$

with

$$lx_0 + my_0 + nz_0 + d = d', \tag{8}$$

where d' is the absolute term of the equation of the conicoid when transformed to the center as origin.

Also,

$$d' = \Delta/D. \tag{9}$$

(*a*) and (*b*). When k_1, k_2, k_3, the roots of the discriminating cubic (2), are all different from zero, they give, with d', the equation of the conicoid referred to its axes, namely,

$$k_1x^2 + k_2y^2 + k_3z^2 + d' = 0; \tag{10}$$

and these roots, k_1, k_2, k_3, set for k in any two of the equations,

$$\left.\begin{aligned}(a-k)\lambda + h\mu + g\nu &= 0,\\ h\lambda + (b-k)\mu + f\nu &= 0,\\ g\lambda + f\mu + (c-k)\nu &= 0,\end{aligned}\right\} \tag{11}$$

give the direction cosines of the perpendiculars to the principal planes, that is, of the axes of the conicoid.

(*c*) When one of the roots of the discriminating cubic is zero, the other two roots, k_1 and k_2, set successively in any two of the equations (11) give the directions of the two diametral planes; and (when $\Delta \neq 0$) set in

$$\Delta = \begin{vmatrix} k_1 & 0 & 0 & 0\\ 0 & k_2 & 0 & 0\\ 0 & 0 & 0 & n'\\ 0 & 0 & n' & 0\end{vmatrix}, \text{ or } \Delta = -k_1k_2n'^2, \tag{12}$$

they give n' in the transformed equation of the paraboloid, which (1) then represents, namely,

$$k_1x^2 + k_2y^2 + 2n'z = 0. \tag{13}$$

(*d*) When $D = 0$ and $\Delta = 0$, and k_1 and k_2, two of the roots of the discriminating cubic, are different from zero, there is a line of centers. In this case, the values of k_1 and k_2, and the corresponding principal planes obtained from (11) determine the cylinder or pair of planes which (1) represents.

(*e*) When $D = 0$ and $\Delta = 0$, and two roots of the discriminating cubic are zero, a method similar to the method for the parabola in the plane will give the parabolic cylinder or parallel planes which (1) represents.

The sixteen forms of § 347 can be classified as follows:

	D	Δ	k_1	k_2	k_3	
(*a*)	not 0	not 0	not 0	not 0	not 0	Ellipsoid, real or imaginary, or hyperboloid of one or two sheets.
(*b*)	not 0	0	not 0	not 0	not 0	Cone, real or imaginary.
(*c*)	0	not 0	not 0	not 0	0	Elliptic or hyperbolic paraboloid.
(*d*)	0	0	not 0	not 0	0	Elliptic, hyperbolic, or imaginary cylinder, or intersecting planes, real or imaginary.
(*e*)	0	0	not 0	0	0	Parabolic cylinder, or parallel or coincident planes.

373. The analysis of the general equation of the second degree. In the study of a given equation of the second degree it is ordinarily better to proceed as follows:

Derive the equations § 372, (7),

$$\partial F_0/\partial x_0 = 0,\ \partial F_0/\partial y_0 = 0,\ \partial F_0/\partial z_0 = 0.$$

I. ($D \neq 0$, $\Delta \neq 0$). If these equations (7) give one center at a finite distance, find this center; obtain d' from (8) or (9); then solve the discriminating cubic (taking the roots, when not integral, to one place of decimals), and write the equation (10) in one of the forms 1–6, § 327; from (11) obtain the direction cosines of the corresponding principal axes. This is the case in which $D \neq 0$, $\Delta \neq 0$.

II. ($D = 0$, $\Delta \neq 0$). If in attempting to solve the equations (7), it is found that $D = 0$, calculate Δ by (6′) or (6), and k_1 and k_2 from (2), ($k_3 = 0$). When $\Delta \neq 0$, find n' from (12), and write the equation (13) in one of the forms 7 or 8, § 327.

III. ($D = 0$, $\Delta = 0$). If in attempting to solve the equations (7), it is found that $D = 0$, calculate Δ by (6′) or (6), and k_1 and k_2 from (2), ($k_3 = 0$). When $\Delta = 0$, find the line of centers; and the cylinder or intersecting planes represented by the equation, if two roots of (2) are different from zero; but, if a second root of (2) is zero, the terms of the second degree form a perfect square, and a method similar to that of § 158 can be followed to find the parabolic cylinder or pair of parallel planes represented by the equation in this case.

Example 1. Analyze the equation

$$x^2 - 2y^2 + 6z^2 + 12xz - 16x - 4y - 36z + 62 = 0.$$

The equations giving the center are

$$x + 6z - 8 = 0, \quad -2y - 2 = 0, \quad 6x + 6z - 18 = 0.$$

The center is $(2, -1, 1)$, and d' is $+30$, D is 60, $I = 5$, $J = -44$, and the discriminating cubic is

$$k^3 - 5k^2 - 44k - 60 = 0,$$

the roots of which are,

$$k_1 = 10, \quad k_2 = -2, \quad k_3 = -3;$$

hence, by transformation, the equation becomes

$$10x^2 - 2y^2 - 3z^2 + 30 = 0.$$

Corresponding to the root $k_1 = 10$, the equations (11) give the values $(2/\sqrt{13}, 0, 3/\sqrt{13})$ as the direction cosines of the new x-axis, and the roots $k_2 = -2$, $k_3 = -3$ give $(0, 1, 0)$, $(3/\sqrt{13}, 0, -2/\sqrt{13})$, respectively, as the direction cosines of the new y- and z-axes.

That is, the given equation represents the hyperboloid of one sheet

$$-x^2/(\sqrt{3})^2 + y^2/(\sqrt{15})^2 + z^2/(\sqrt{10})^2 = 1,$$

with the center at $(2, -1, 1)$, and the direction cosines of the x-, y-, and z-axes, $(2/\sqrt{13}, 0, 3/\sqrt{13})$, $(0, 1, 0)$, $(3/\sqrt{13}, 0, -2/\sqrt{13})$, respectively.

Observe that in this example the roots of the discriminating cubic are integral, which is not the case in the following example.

Example 2. Analyze the equation

$$2x^2 + 3y^2 - 2z^2 + 6xy - 2xz + 2yz - 4x - 8y + 10z + 3/5 = 0.$$

The equations giving the center are

$$2x + 3y - z - 2 = 0, \quad 3x + 3y + z - 4 = 0, \quad -x + y - 2z + 5 = 0.$$

The center is $(24/5, -3, -7/5)$, and d' is -4. $D = -5$, $I = 3$, $J = -15$; hence the discriminating cubic is $K(k) \equiv k^3 - 3k^2 - 15k + 5 = 0$, which has one negative and two positive roots. Since $K(-2)$ is positive and $K(-3)$ is negative, the negative root lies between -2 and -3, and further since $K(-2.9)$ is negative and $K(-2.8)$ is positive, the negative root to one place of decimals is $k_1 = -2.8$. $K(0.4)$ is negative, $K(0.3)$ is positive, therefore $k_2 = 0.3$. Since $D = k_1k_2k_3$, therefore $k_3 = -5/k_1k_2 = 6$, approximately; and since $I = (k_1 + k_2 + k_3)$, therefore $k_3 = 3 + 2.8 - 0.3 = 5.5$, approximately; and since $K(5.5)$ is negative and $K(5.6)$ is positive, $k_3 = 5.5$. The equation of the conicoid referred to its axes is therefore

$$(-2.8)x^2 + (0.3)y^2 + (5.5)z^2 - 4 = 0,$$

or $$\frac{x^2}{10/7} - \frac{y^2}{40/3} - \frac{z^2}{8/11} = -1, \text{ or, } \frac{x^2}{(1.2)^2} - \frac{y^2}{(3.7)^2} - \frac{z^2}{(0.9)^2} = -1.$$

The first and last of the equations (11) are here $(2 - k)\lambda + 3\mu - \nu = 0$ and $\lambda - \mu + (2 + k)\nu = 0$, and they give for the direction cosines of the three axes: when $k_1 = -2.8$, $(0.38, -0.32, 0.87)$; when $k_2 = 0.3$, $(0.66, -0.55, -0.52)$; when $k_3 = 5.5$, $(0.65, 0.76, 0.01)$.

That is, the equation represents *approximately* the hyperboloid of one sheet $x^2/(1.2)^2 - y^2/(3.7)^2 - z^2/(0.9)^2 = -1$, with the center at the point $(24/5, -3, -7/5)$, and with direction cosines of the x-, y-, and z-axes, $(0.38, -0.32, 0.87)$, $(0.66, -0.55, -0.52)$, $(0.65, 0.76, 0.01)$, respectively.

Example 3. Analyze the equation

$$2x^2 + 2y^2 - 4z^2 - 2yz - 2zx - 5xy - 2x - 2y + z = 0.$$

The equations giving the center indicate that the center is at infinity. Calculation gives $D = 0$, $\Delta = 9 \cdot 9 \cdot 9/16$, $I = 0$, $J = -81/4$. The discriminating cubic is $k^3 - (81/4)k = 0$, the roots of which are $k_1 = 9/2$, $k_2 = -9/2$, $k_3 = 0$. The equation (12) gives $n' = \pm 3/2$ and the given equation therefore represents the hyperbolic paraboloid

$$\frac{x^2}{(\sqrt{2/3})^2} - \frac{x^2}{(\sqrt{2/3})^2} = \pm z.$$

See also the examples of § 357 (centers).

374. Exercises. Conicoids.

Analyze the following equations:

1. $11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$.
2. $x^2 + 2y^2 + 3z^2 - 4xy - 4xz + 4 = 0$.
3. $4x^2 + y^2 + 4z^2 - 4xy - 4yz + 8zx + 2x - 4y + 3z + 1 = 0$.
4. $32x^2 + y^2 + 4z^2 - 8xy - 16xz + 96x - 20y - 8z + 103 = 0$.
5. $3z^2 - 6yz - 6zx - 7x - 5y + 6z + 3 = 0$.
6. $32x^2 + y^2 + z^2 - 16xy - 16xz + 6yz - 6x - 12y - 12z + 18 = 0$.
7. $x^2 - 2y^2 + 2z^2 + 3zx - xy - 2x + 7y - 5z - 3 = 0$.
8. $x^2 + y^2 + z^2 + 4xy - 2xz + 4yz - 1 = 0$.
9. $x^2 - 2xy - 2yz - 2xz - 4 = 0$.
10. $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$.
11. $2x^2 + 5y^2 + z^2 - 4xy - 2x - 4y - 8 = 0$.
12. $x^2 + 2y^2 - 3z^2 - 12xy + 8xz - 4yz + 1 = 0$.
13. $2x^2 + 2y^2 - 4z^2 - 5xy - 2zx - 2yz - 2x - 2y + z = 0$.
14. $5x^2 - y^2 + z^2 + 4xy + 6xz + 2x + 4y + 6z - 8 = 0$.
15. $2x^2 + 3y^2 + 5xy + 2xz + 3yz - 4y + 8z - 32 = 0$.
16. $x^2 + y^2 + z^2 + xy + yz + xz - 1 = 0$.
17. $3z^2 - x^2 - y^2 + 4xy - 9 = 0$.
18. $x^2 + y^2 + z^2 - 2xy + 2xz + 2yz - 1 = 0$.
19. $x^2 - (y - 2)^2/3 + (z + 1)^2/4 = 1$.
20. $(x - 1)^2 - (y - 2)^2 + (z - 3)^2 = 0$.
21. $x^2 + y^2 + z^2 + 2x - 4y - 6z = 0$.
22. $2y^2 + 3z^2 + x - 4y + 6z = 0$.
23. $x^2 - 4z^2 + 5y - x + 8z = 0$.
24. $z^2 + x + y + 2z + 1 = 0$.
25. $(x + 2y + 2z)^2 - (2x + y - 2z)^2 + (2x - 2y + z)^2 = 0$.

375. Exercises (including loci problems).

1. Prove that the line $x + y = 0,\ z = 0$ lies wholly on the surface $x^3 + y^3 + z^3 = 0$.

2. From the equations of the tangent planes to the cone and the hyperbolic paraboloid, prove that when a point P is made to move along a generating line of a cone, the tangent plane at P remains stationary; but that when a point P is made to move along a generating line of a hyperbolic paraboloid, the tangent plane at P revolves about the generating line.

3. Prove that, if $E_1 = 0,\ E_2 = 0,\ E_3 = 0$ represent three mutually perpendicular planes, and p_1, p_2, p_3 the perpendicular distances of the point $P(x, y, z)$ from these planes, the equation $F(p_1, p_2, p_3) = 0$ represents a surface related to the planes $E_1 = 0,\ E_2 = 0,\ E_3 = 0$ precisely as the surface represented by the equation $F(x, y, z) = 0$ is related to the coordinate planes $x = 0,\ y = 0,\ z = 0$.

4. Find the equations of the projections upon each of the coordinate planes of the curve of intersection of the plane $x - y + 2z - 4 = 0$ with the conicoid $x^2 - yz + 3x = 0$.

5. Prove that the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where $a > b > c$, is cut by the sphere $x^2 + y^2 + z^2 = b^2$ in two circular plane sections. [When the equations are simultaneous, $x^2(1/b^2 - 1/a^2) - z^2(1/c^2 - 1/b^2) = 0$, which has two *real* factors.]

6. Find the equation of the plane which is the locus of the mid-points of all chords of the conicoid $x^2 + 3y^2 = 2z$ whose direction cosines have the ratios 1 : 2 : 3.

7. The normal to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point P meets the plane $z = 0$ in the point Q; show that the locus of the mid-point of PQ is an ellipsoid.

8. Prove that the cone, cylinder, hyperboloid of one sheet, and hyperbolic paraboloid are the only ruled surfaces of the second degree.

9. A line of constant length has its extremities on two fixed straight lines; prove that the locus of the middle point is an ellipse. [Take for z-axis the common perpendicular to the given lines, and for xy-plane the plane midway between the given lines and parallel to them, and for the x- and y-axes the lines which bisect the angles made by the projections of the given lines on the xy-plane. Then the given lines are: $y = mx$, $z = c$; and $y = -mx$, $z = -c$. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the extremities of the line of constant length, $2l$; then $y_1 = mx_1$, $z_1 = c$, and

$y_2 = -mx_2$, $z_2 = -c$. For the representative middle point (x, y, z), then $x = (x_1 + x_2)/2 = (y_1 - y_2)/2m$, $y = (y_1 + y_2)/2 = m(x_1 - x_2)/2$, and $z = (z_1 + z_2)/2 = 0$. Moreover, $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 4l^2$. Therefore, $4y^2/m^2 + 4m^2x^2 + 4c^2 = 4l^2$. Hence the locus of the mid-point is given by: $z = 0$ and $m^2x^2 + y^2/m^2 = (l^2 - c^2)$, which represent an ellipse.]

10. A line, l, moves so as always to intersect three given straight lines, l_1, l_2, l_3, which are not all parallel to the same plane; find the equation of the surface generated by the straight line. [Let l be $(x - x')/\lambda =$ etc., and l_1 be $(x - a_1)/\lambda_1 =$ etc. Then l will meet l_1 [§ 297, 57, (2)], if

$$\{\nu_1(y' - b_1) - \mu_1(z' - c_1)\}\lambda + \{\lambda_1(z' - c_1) - \nu_1(x' - a_1)\}\mu + \{\mu_1(x' - a_1) - \lambda_1(y' - b_1)\}\nu = 0.$$

The conditions that l meet l_2 and l_3 are similar equations with the subscripts 2 and 3, respectively. Therefore, calling (x, y, z) the coordinates of the *representative point* on the generating line, which have been thus far (x', y', z'), the locus of the representative point is given by the vanishing of a determinant, the first row only of which will be written, namely,

$$|\nu_1(y - b_1) - \mu_1(z - c_1),\ \lambda_1(z - c_1) - \nu_1(x - a_1),\ \mu_1(x - a_1) - \lambda_1(y - b_1)| = 0.$$

The coefficient of xyz in the determinant is (writing first rows only),

$$|\nu_1,\ \lambda_1,\ \mu_1| + |-\mu_1,\ -\nu_1,\ -\lambda_1|$$

which is zero; and the coefficient of y^2z is $|\nu_1,\ \lambda_1,\ -\lambda_1|$ which is zero, and so for all the other terms of the third degree. Therefore the locus is a conicoid.]

11. Determine the locus of a point which moves so as always to be equally distant from two given straight lines. [Take axes as in Ex. 9.]

12. Through two straight lines given in space two planes are taken at right angles to one another; find the locus of their line of intersection. [Take axes as in Ex. 9.]

13. Find the surface generated by a straight line which is parallel to a fixed plane and which meets two given straight lines.

14. Any two finite straight lines are divided in the same ratio by a straight line; find the equation of the surface which this straight line generates.

15. If any chord of a conicoid through a point O meets its polar plane in R and the surface in P_1 and P_2, prove that $1/OP_1 + 1/OP_2 = 2/OR$, and therefore that the chord is cut harmonically. [Take O for origin, and the chord as $x/\lambda = y/\mu = z/\nu = r$, and find the equations giving OR, OP_1, and OP_2.]

16. The locus of the centers of all plane sections of a conicoid which pass through a fixed point is a conicoid. [The equation of the locus is $(x-x_0)\partial F/\partial x_0 + (y-y_0)\partial F/\partial y_0 + (z-z_0)\partial F/\partial z_0 = 0$, where now (x, y, z) is the *fixed point* and (x_0, y_0, z_0) is the *representative point* of the locus.]

17. The locus of the centers of parallel sections of a conicoid is a straight line. [The locus is $\dfrac{\partial F/\partial x_0}{\lambda} = \dfrac{\partial F/\partial y_0}{\mu} = \dfrac{\partial F/\partial z_0}{\nu}$ where (λ, μ, ν) are *constant* and the *representative point* of the locus is (x_0, y_0, z_0).]

18. Prove that the line $x/\lambda = y/\mu = z/\nu = r$ meets the central conicoid $Ax^2 + By^2 + Cz^2 = 1$, in points given by $1/r^2 = A\lambda^2 + B\mu^2 + C\nu^2$. Thence prove that the lines which pass through the origin and meet the conicoid in two coincident points at infinity (*asymptotic lines*, § 138) satisfy the relation $A\lambda^2 + B\mu^2 + C\nu^2 = 0$, and that any *representative* point on such a line satisfies the relation $Ax^2 + By^2 + Cz^2 = 0$, the locus of which is a cone. (This cone is called the *asymptotic cone.*)

19. The sum of the squares of the reciprocals of any three semidiameters of an ellipsoid which are mutually perpendicular is a constant. [If r_1 is the semidiameter with the direction cosines $(\lambda_1, \mu_1, \nu_1)$, then $1/r_1^2 = \lambda_1^2/a^2 + \mu_1^2/b^2 + \nu_1^2/c^2$, and similarly for the others. The sum is a constant.]

20. If three fixed points on a straight line are each on one of three mutually perpendicular planes, prove that the locus of any fourth fixed point on the line is an ellipsoid. [Let A, B, C be the first three fixed points each on one of the mutually perpendicular planes, which will be taken as the coordinate planes, and let the fourth fixed point be $P(x, y, z)$ when the line has any *representative* position; let AP, BP, CP be a, b, c, respectively; and let the direction cosines of the line be (λ, μ, ν). Then $\lambda = x/a$, $\mu = y/b$, $\nu = z/c$, and the locus is an ellipsoid.]

21. Find the equation of the cone whose vertex is at the center of an ellipsoid and which passes through all the points of intersection of the ellipsoid and a given plane. [Let the plane be $Ax + By + Cz = 1$ and the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; then the equation of the required cone is $x^2/a^2 + y^2/b^2 + z^2/c^2 - (Ax + By + Cz)^2 = 0$.]

22. Find the equation of the cone whose vertex is at the center of an ellipsoid and which passes through all the points of intersection of the

ellipsoid and a fixed concentric sphere. [Let the equation of the sphere be $x^2+y^2+z^2=1/R^2$, and that of the ellipsoid, $A^2x^2+B^2y^2+C^2z^2=1$, then the equation of the required cone is

$$(A^2-R^2)x^2+(B^2-R^2)y^2+(C^2-R^2)z^2=0.]$$

23. Let l, m, n be half the coefficients of the first degree terms of any equation of the second degree. Prove that, if the rectangular axes be twisted in any manner about the origin, $(l^2+m^2+n^2)$ will be an invariant.

24. Prove that, if three chords of a conicoid have the same middle point, they all lie in a plane, or intersect in the center of the conicoid.

25. Prove that nine points, in general, determine a conicoid, and that a single infinity of conicoids pass through eight given points. Prove that all conicoids through eight given points have a common curve of intersection. [Consider $S_1=0$, $S_2=0$, $S_1-\lambda S_2=0$.] Prove that any three conicoids have eight common points (real or imaginary).

26. From the equation $x^2+y^2+z^2+2\,lx+2\,my+2\,nz+d=0$ of a sphere, prove that four points, in general, determine a sphere.

27. Let S be a symbol for $x^2+y^2+z^2+2\,lx+2\,my+2\,nz+d$, and S_1 be a symbol for the same expression when the coefficients have the subscript 1, and so on. Prove that $S_1-S_2=0$ represents a plane. This plane is called the *radical plane* of the spheres $S_1=0$, $S_2=0$.

Prove that the radical plane is the locus of the points the tangents from which to the two spheres are equal.

The *point*, $S_1=S_2=S_3=S_4$ is called the *radical center* of the spheres $S_1=0$, $S_2=0$, $S_3=0$, $S_4=0$.

Prove that the spheres $S_1=0$, $S_2=0$ are *orthogonal* at all the points of intersection if $2\,l_1l_2+2\,m_1m_2+2\,n_1n_2-d_1-d_2=0$. [Compare §§ 62–65.]

28. Prove that the plane $\lambda x+\mu y+\nu z-p=0$ (1) will be tangent to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2-1=0$, if $p^2=a^2\lambda^2+b^2\mu^2+c^2\nu^2$. [If (x', y', z') be the point of tangency, the plane (1) and the tangent plane $xx'/a^2+yy'/b^2+zz'/c^2-1=0$ will be the same, when

$$\frac{x'/a^2}{\lambda}=\frac{y'/b^2}{\mu}=\frac{z'/c^2}{\nu}=\frac{1}{p},$$

or when, $x'/a=a\lambda/p$, $y'/b=b\mu/p$, $z'/c=c\nu/p$, or, squaring and adding, when $a^2\lambda^2/p^2+b^2\mu^2/p^2+c^2\nu^2/p^2=1$.]

29. The locus of the point of intersection of three mutually perpendicular tangent planes of an ellipsoid is a sphere, called the *director sphere*. [From Ex. 28, the equation

$$\lambda_1x+\mu_1y+\nu_1z=\sqrt{a^2\lambda_1{}^2+b^2\mu_1{}^2+c^2\nu_1{}^2};$$

represents one tangent plane; similar equations with the subscripts 2 and 3 represent the other two. Squaring and adding the three equations gives $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$.]

30. From any point in space, six normals can be dropped to an ellipsoid. [The equation of the normal is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} = \frac{z - z_1}{z_1/c^2} = k,$$

and therefore

$$x_1/a = ax/(a^2 + k), \quad y_1/b = by/(b^2 + k), \quad z_1/c = cz/(c^2 + k). \qquad (1)$$

Let (x, y, z) be a *fixed* point, then k is given by the equation

$$a^2x^2/(a^2 + k)^2 + b^2y^2/(b^2 + k)^2 + c^2z^2/(c^2 + k)^2 - 1 = 0,$$

which is of the sixth degree in k. Each of the six roots of this equation, set in (1) gives the foot of a normal.]

31. Let P_1, P_2, P_3 be three points on an ellipsoid. Prove that, if P_1 is on the diametral plane of the system of chords parallel to OP_2, then will P_2 be on the diametral plane of OP_1. Let OP_3 be the line of intersection of the diametral planes of OP_1 and OP_2; prove that the diametral plane of OP_3 is OP_1P_2, so that the plane through any two of the three lines OP_1, OP_2, OP_3 is diametral to the third. These three planes are called *conjugate planes*, and the three lines OP_1, OP_2, OP_3 are called *conjugate* semidiameters. [The condition that the point (x_2, y_2, z_2) is on the diametral plane of OP_1 is $x_1x_2/a^2 + y_1y_2/b^2 + z_1z_2/c^2 = 0$.]

32. If P_1, P_2, P_3 are extremities of three conjugate diameters, prove that $(x_1/a, y_1/b, z_1/c)$, $(x_2/a, y_2/b, z_2/c)$, $(x_3/a, y_3/b, z_3/c)$ are the direction cosines of three straight lines perpendicular in pairs, and that therefore $x_1^2 + x_2^2 + x_3^2 = a^2$, etc. Prove that the sum of the squares of three conjugate semidiameters of an ellipsoid is constant, and equal to $a^2 + b^2 + c^2$. Prove also that the volume of the parallelepiped which has three conjugate semidiameters of an ellipsoid for conterminous edges is constant and equal to abc.

33. Prove that the equation of the ellipsoid referred to three conjugate diameters as oblique axes is $x^2/a'^2 + y^2/b'^2 + z^2/c'^2 = 1$, where a', b', c' are the lengths of the semiconjugate diameters. [The equation will be of the form $ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gxz + 2\,fyz = 1$. (See § 351.) From the definition of the conjugate diameters, if (x', y', z') is on the surface, so also will $(-x', y', z')$, $(x, -y', z')$, $(x', y', -z')$ be on the surface, and therefore h, g, and f are all zero. And in the resulting equation

$ax^2 + by^2 + cz^2 = 1$, if $y = 0$, $z = 0$, then $a = 1/x^2$, which in this case is $a = 1/a'^2$, etc.]

34. If a parallelepiped be inscribed in an ellipsoid, its edges will be parallel to a set of conjugate diameters.

35. If two conicoids have one plane section in common, their other points of intersection lie on another plane. [Let the common plane section be $z = 0$, and $ax^2 + by^2 + 2hxy + 2lx + 2my + d = 0$; then the most general conicoid which passes through this conic is

$$(ax^2 + by^2 + 2hxy + 2lx + 2my + d) + z(\lambda x + \mu y + \nu z - p) = 0.]$$

36. Prove that four cones, real or imaginary, will pass through the curve of intersection of two conicoids. [Compare Ex. 25.]

37. All conicoids which pass through seven given points pass through another fixed point. [Consider $S_1=0$, $S_2=0$, $S_3=0$, $S_1+\lambda S_2+\mu S_3= 0$.]

TABLE A

Certain Algebraic Symbols, Definitions, and Theorems

1. The symbol $|a|$ means the "absolute" or numerical value of a. Thus, $|3| = 3$, and $|-3| = 3$.

2. The symbol $\neq$ means "not equal to." Thus, $a \neq 0$ means that a is not equal to 0.

3. The symbol a/b has the same meaning as $\frac{a}{b}$. The slant line is called the *solidus*.

4. The *absolute term* of an equation is the term which does not involve the unknown letter or letters. Thus, in the equation $x^2 + 3x - 2 = 0$, the absolute term is -2.

5. The *identity* or *identical equation*, $A \equiv B$, means that the expression A can be transformed into the expression B by the rules of reckoning. Thus, $(x+y)^2 \equiv x^2 + 2xy + y^2$; similarly $2^2 - 3 \cdot 2 + 2 \equiv 0$.

An identity in one or more letters does not impose any restriction on the values of these letters; it is true for all values of these letters.

On the contrary, an *equation of condition*, as $x - 2 = 0$, or $x + y = 0$, is the statement of a condition which a certain letter or certain letters are to satisfy, and it restricts the letter or letters to values which satisfy this condition. Thus, $x - 2 = 0$ restricts x to the value 2; and $x + y = 0$ restricts x and y to pairs of values which are equal numerically but of opposite signs.

It is customary to call both identical equations and equations of condition "equations" simply, and to use the symbol $=$ in both, instead of $\equiv$ in the one and $=$ in the other.

6. *Quadratic equations.* The roots, x_1, x_2, of a quadratic equation in the form

$$ax^2 + bx + c = 0 \quad \text{are} \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots and coefficients are connected by the relations

$$x_1 + x_2 = -b/a, \qquad x_1x_2 = c/a.$$

The roots are equal, when $b^2 - 4ac = 0$; real and distinct when $b^2 - 4ac > 0$; imaginary, when $b^2 - 4ac < 0$.

One root is 0, if c is 0; both roots are 0, if both c and b are 0.

One root is ∞, if a is 0; both roots are ∞, if both a and b are 0.

If the equation has the form $ax^2 + 2b_1x + c = 0$, the roots are $(-b_1 \pm \sqrt{b_1^2 - ac})/a$; and the roots are equal when $b_1^2 - ac = 0$.

7. It is customary to represent an expression involving the single unknown or variable letter x by the symbol $f(x)$. The value which the expression takes for $x = a$ is then represented by $f(a)$. Thus, if $f(x) = x^2 + 3x - 2$, then $f(0) = -2$, and $f(1) = 1 + 3 - 2 = 2$, and so on.

Similarly, an expression involving the two variables x, y may be represented by the symbol $f(x, y)$, and the value which it takes when $x = a$, $y = b$, by $f(a, b)$. Thus, for example, if $f(x, y) = x^2 - 2xy + y^2$, then $f(1, 2) = 1^2 - 2 \cdot 1 \cdot 2 + 2^2 = 1$, and so on. In like manner, $f(x, y, z)$ is used to represent an expression involving the three variables x, y, z.

8. If $f(x) = 0$ denote a rational, integral equation, and the numbers $f(a)$ and $f(b)$ have opposite signs, the equation has at least one root between a and b. Thus, in the case of the equation $f(x) = x^3 - 4x^2 + 2 = 0$, $f(-1) = -3$, $f(0) = 2$, $f(1) = -1$, $f(2) = -6$, $f(3) = -7$, $f(4) = 2$; hence the three roots of the equation lie between -1 and 0, 0 and 1, 3 and 4, respectively.

TABLE B

CERTAIN TRIGONOMETRIC DEFINITIONS AND FORMULAS

1. The angle subtended at the center of a circle by an arc equal to the radius is called a *radian*.

The measure of any angle in terms of the radian is called the *circular measure* of the angle.

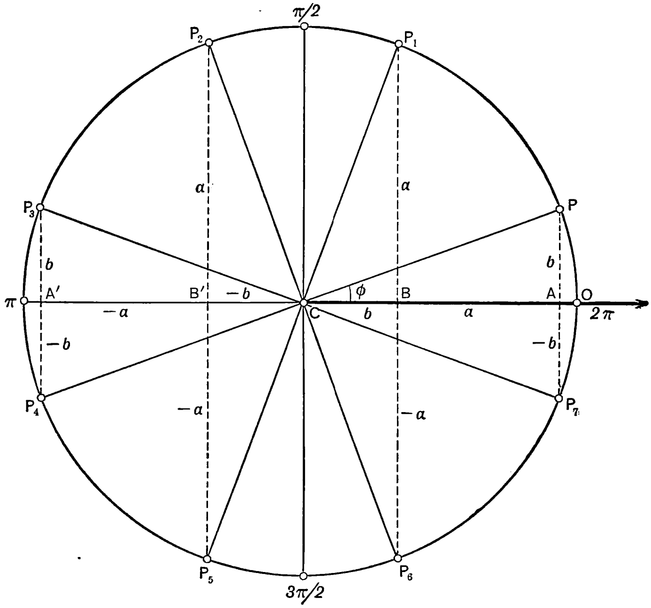
Since angles at the center of a circle are proportional to the arcs which they subtend, the ratio of any angle to the radian, that is, its circular measure, is equal to the ratio of the arc which it subtends at the center of any circle to the radius of that circle. Hence the circular measure of an angle is equal to the length of the arc which it subtends at the center of a circle of unit radius.

The length of the circumference of a circle of unit radius is 2π. Hence the circular measure of an angle equal to four right angles, or 360°, is 2π; that of an angle of 180°, 90°, 45°, 1°, is π, $\pi/2$, $\pi/4$, $\pi/180$, respectively.

2. A line CP turned about C from the initial position CO is said to generate the angle OCP having the *initial line* CO and the *terminal line* CP. If the rotation is counter-clockwise, the angle is said to be positive; if in the contrary sense, negative. See the figure on the next page.

3. Let P be any point on the terminal line of the angle OCP, and take PA perpendicular to the initial line CO. Then, for all positions of the terminal line, A is called the *projection of* P, and CA the *projection of* CP on CO, and AP is called the *projecting line*. The projection CA is positive or negative according as A lies to the right or left of C. The projecting line AP is positive or negative according as P lies above or below CO. The terminal line CP is always considered posi-

tive. The six ratios which can be formed with *CP*, *CA*, and



AP are called the *trigonometric functions* of the angle *OCP*. They are defined as follows:

$$\sin OCP = \frac{AP}{CP} = \frac{\text{projecting line}}{\text{terminal line}},$$

$$\cos OCP = \frac{CA}{CP} = \frac{\text{projection}}{\text{terminal line}},$$

$$\tan OCP = \frac{AP}{CA} = \frac{\text{projecting line}}{\text{projection}},$$

$$\cot OCP = \frac{CA}{AP} = \frac{\text{projection}}{\text{projecting line}},$$

$$\sec OCP = \frac{CP}{CA} = \frac{\text{terminal line}}{\text{projection}},$$

$$\operatorname{cosec} OCP = \frac{CP}{AP} = \frac{\text{terminal line}}{\text{projecting line}}.$$

The algebraic signs of these functions depend upon the signs of CA and AP, the latter being determined by the rule already given.

Using the notation indicated in the figure, it will be seen that

$$AP/CP = \sin\phi = \cos\left(\frac{\pi}{2} - \phi\right) = -\cos\left(\frac{\pi}{2} + \phi\right)$$
$$= \sin(\pi - \phi) = -\sin(\pi + \phi) = -\cos\left(\frac{3\pi}{2} - \phi\right)$$
$$= \cos\left(\frac{3\pi}{2} + \phi\right) = -\sin(2\pi - \phi) = -\sin(-\phi).$$

$$CA/CP = \cos\phi = \sin\left(\frac{\pi}{2} - \phi\right) = \sin\left(\frac{\pi}{2} + \phi\right)$$
$$= -\cos(\pi - \phi) = -\cos(\pi + \phi) = -\sin\left(\frac{3\pi}{2} - \phi\right)$$
$$= -\sin\left(\frac{3\pi}{2} + \phi\right) = \cos(2\pi - \phi) = \cos(-\phi).$$

4. The following is a list of some of the more important formulas connecting the trigonometric functions of one or two angles:

$\sin^2 A + \cos^2 A = 1,\quad \tan^2 A + 1 = \sec^2 A,\quad \cot^2 A + 1 = \operatorname{cosec}^2 A.$

$\sin A = 1/\operatorname{cosec} A,\quad \cos A = 1/\sec A,\quad \tan A = 1/\cot A.$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A.$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$
$$\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B).$$
$$\sin 2A = 2 \sin A \cos A,\qquad \tan 2A = 2\tan A/(1 - \tan^2 A).$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$
$$2\sin^2 A = 1 - \cos 2A,\quad 2\cos^2 A = 1 + \cos 2A.$$

$\cos x = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2) = 2\cos^2(x/2) - 1.$

$2\sin^2(x/2) = 1 - \cos x,\quad 2\cos^2(x/2) = 1 + \cos x.$

$$\sin(x/2) = \sqrt{(1 - \cos x)/2},\ \cos(x/2) = \sqrt{(1 + \cos x)/2}.$$
$$\tan(x/2) = \sqrt{(1 - \cos x)/(1 + \cos x)}.$$

$\sin x \pm \sin y = 2\sin\frac{1}{2}(x \pm y)\cos\frac{1}{2}(x \mp y).$

$\cos x + \cos y = 2\cos\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y).$

$\cos x - \cos y = -2\sin\frac{1}{2}(x + y)\sin\frac{1}{2}(x - y).$

TABLE C

Derivatives and Partial Derivatives

1. Let $f(x)$ denote an expression which is rational and integral with respect to x. Multiply each term involving x by the exponent of x in the term and then diminish the exponent by 1. The algebraic sum of the results thus obtained is called the *derivative* of $f(x)$, and is represented by the symbol $f'(x)$ or the symbol

$$\frac{df(x)}{dx}.$$

Thus, if $f(x) = x^3 - 3x^2 + 2$, then $f'(x) = 3x^2 - 6x$.

2. Let $f(x, y)$ denote an expression which is rational and integral with respect to x and y. The expression obtained, as in 1, by multiplying each term involving x by the exponent of x in the term and then diminishing the exponent by 1, is called the *partial derivative of* $f(x, y)$ *with respect to* x and is represented by the symbol $\frac{\partial f(x, y)}{\partial x}$, or $\partial f(x, y)/\partial x$. The expression similarly related to y is called the *partial derivative of* $f(x, y)$ *with respect to* y, and is represented by $\frac{\partial f(x, y)}{\partial y}$, or $\partial f(x, y)/\partial y$.

Thus, if $f(x, y) = x^2y - 2xy^2 + 3x - 2$, then

$$\frac{\partial f(x, y)}{\partial x} = 2xy - 2y^2 + 3, \qquad \frac{\partial f(x, y)}{\partial y} = x^2 - 4xy.$$

3. The partial derivatives of $f(x, y, z)$ with respect to x, y, and z have meanings similar to those explained in 2, and are represented by $\frac{\partial f(x, y, z)}{\partial x}$, or $\partial f(x, y, z)/\partial x$, and so on.

TABLE D

Four-place Logarithms of Numbers from 1.0 to 9.9

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956

TABLE E

Length of Arcs in Radians, and Natural Trigonometric Functions for Intervals of 5°

Deg.	Arc	Sin	Tan	Cot	Cos		
0	0.000	0.000	0.000	∞	1.000	1.571	90
5	0.087	0.087	0.087	11.430	0.996	1.484	85
10	0.175	0.174	0.176	5.671	0.985	1.396	80
15	0.262	0.259	0.268	3.732	0.966	1.309	75
20	0.349	0.342	0.364	2.747	0.940	1.222	70
25	0.436	0.423	0.466	2.145	0.906	1.134	65
30	0.524	0.500	0.577	1.732	0.866	1.047	60
35	0.611	0.574	0.700	1.428	0.819	0.960	55
40	0.698	0.643	0.839	1.192	0.766	0.873	50
45	0.785	0.707	1.000	1.000	0.707	0.785	45
		Cos	Cot	Tan	Sin	Arc	Deg.

TABLE F

THE LETTERS OF THE GREEK ALPHABET, WITH THEIR NAMES

Α α α	alpha	Ι ι	iota	Ρ ρ	rho
Β β	beta	Κ κ	kappa	Σ σ ς	sigma
Γ γ	gamma	Λ λ	lambda	Τ τ	tau
Δ δ	delta	Μ μ	mu	Υ υ	upsilon
Ε ϵ	epsilon	Ν ν	nu	Φ φ ϕ	phi
Ζ ζ	zeta	Ξ ξ	xi	Χ χ	chi
Η η	eta	Ο ο	omicron	Ψ ψ	psi
Θ ϑ θ	theta	Π π	pi	Ω ω	omega

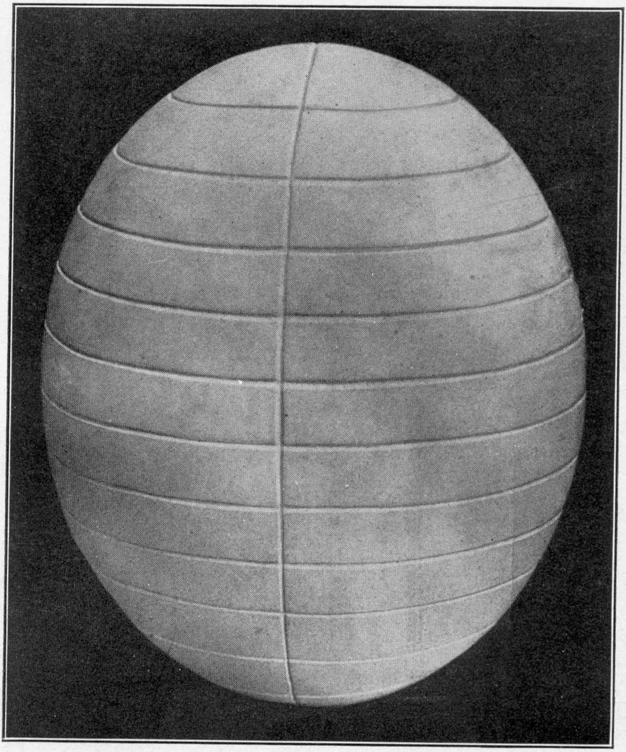


Figure 1

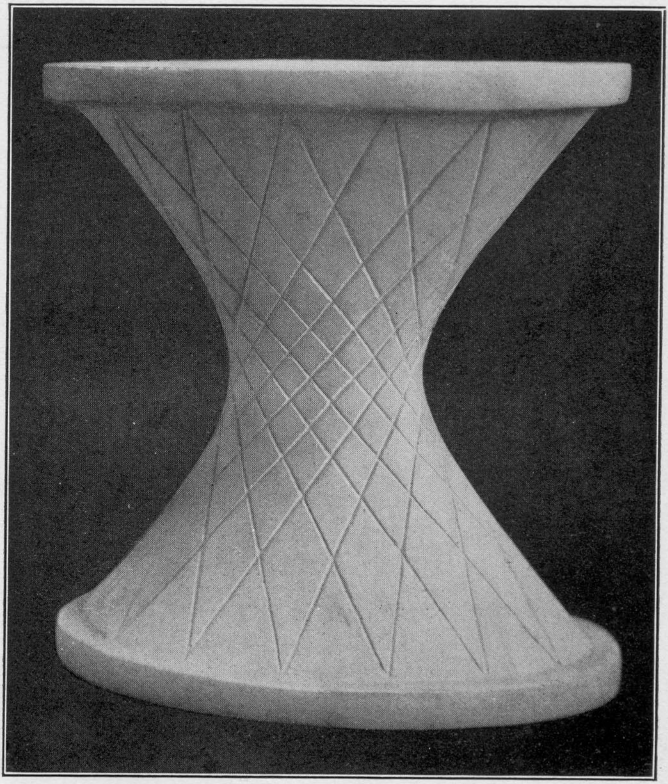


Figure 2

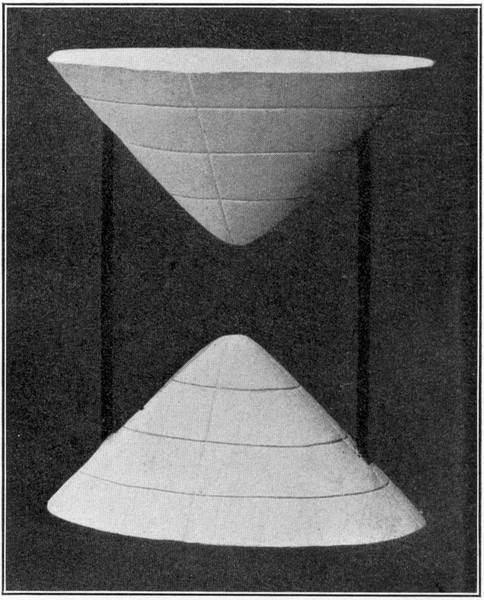


Figure 3

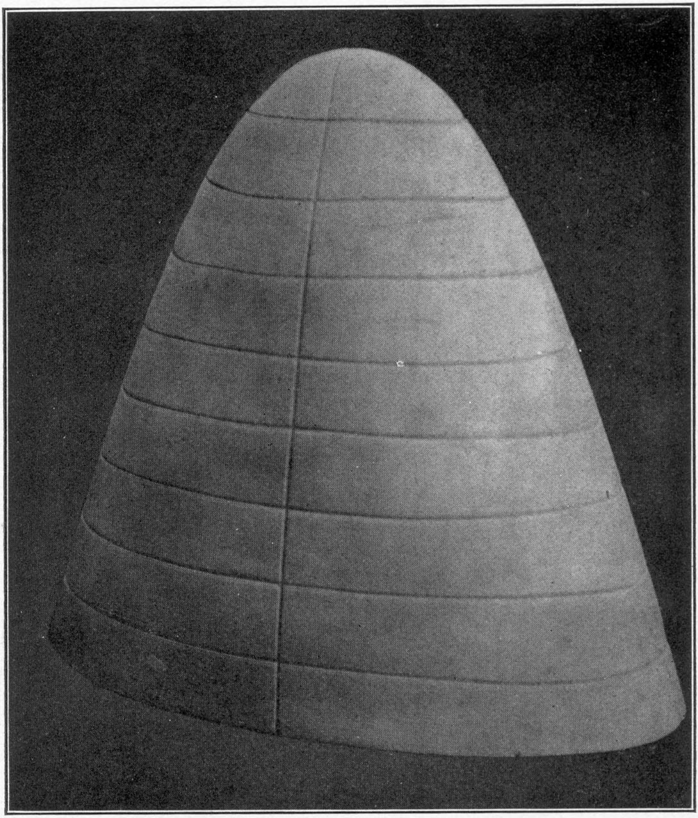


Figure 4

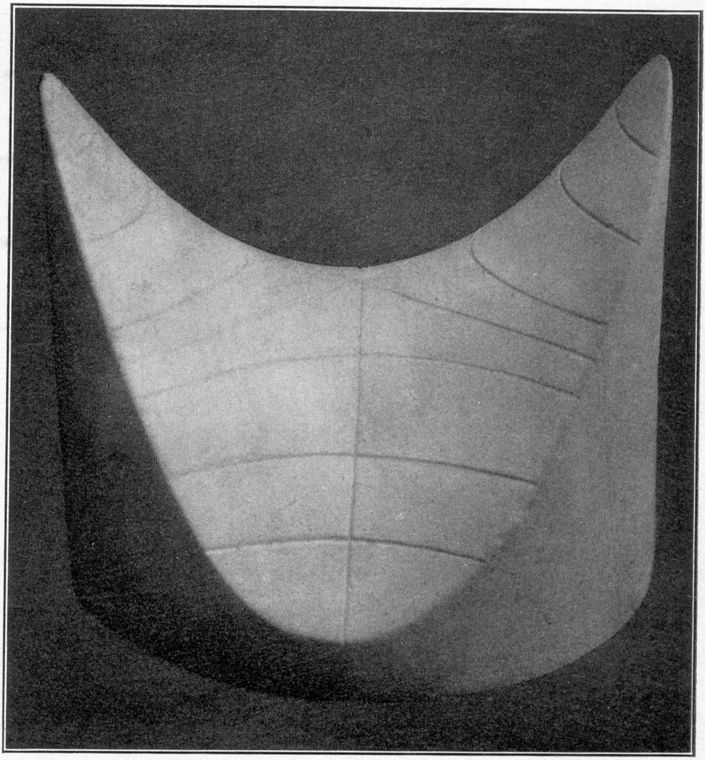


Figure 5

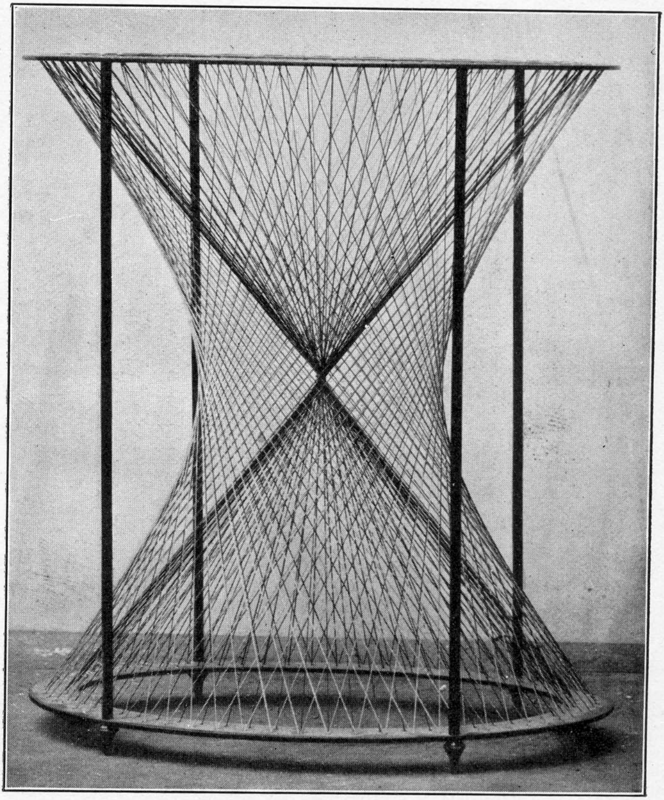


Figure 6

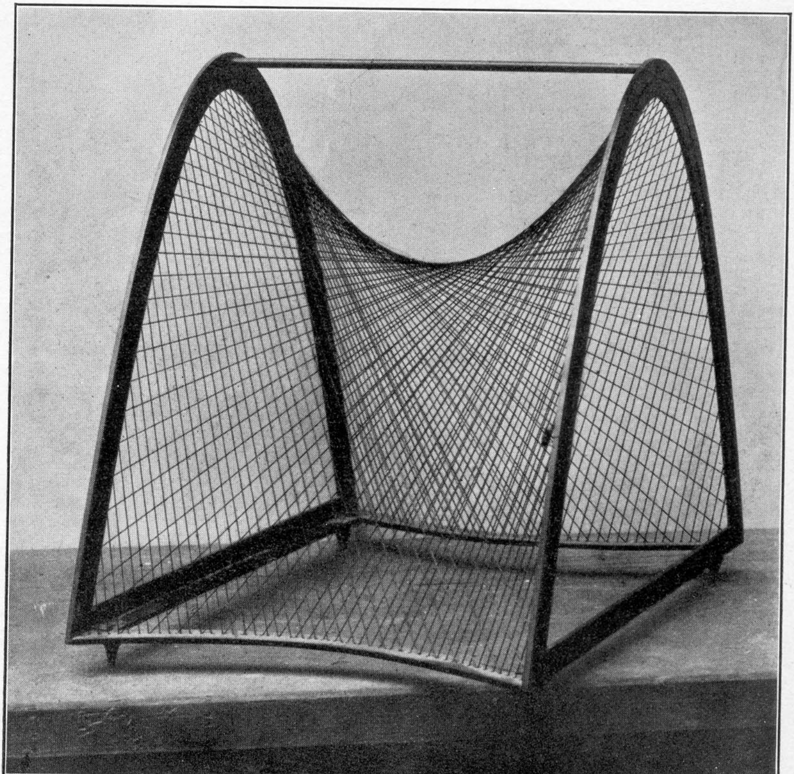


Figure 7

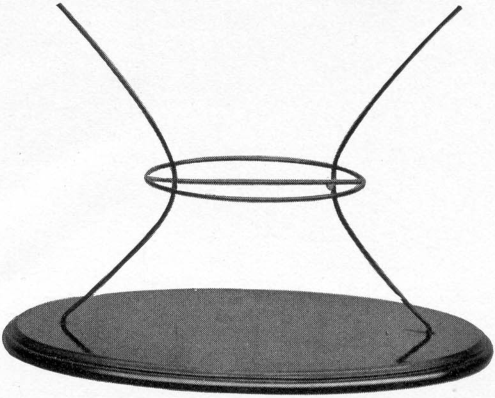


Figure 8

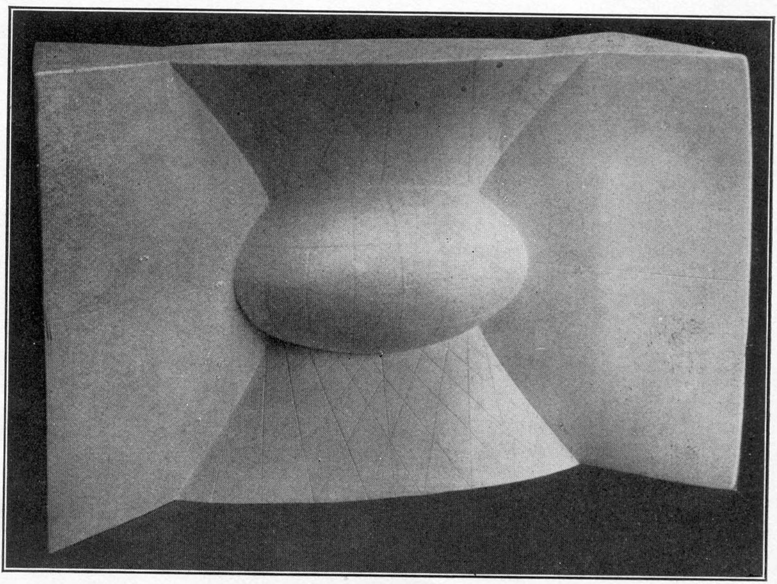


Figure 9