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ARIHANT PRAKASHAN
KALINDI, T.P. NAGAR, MEERUT-250 002

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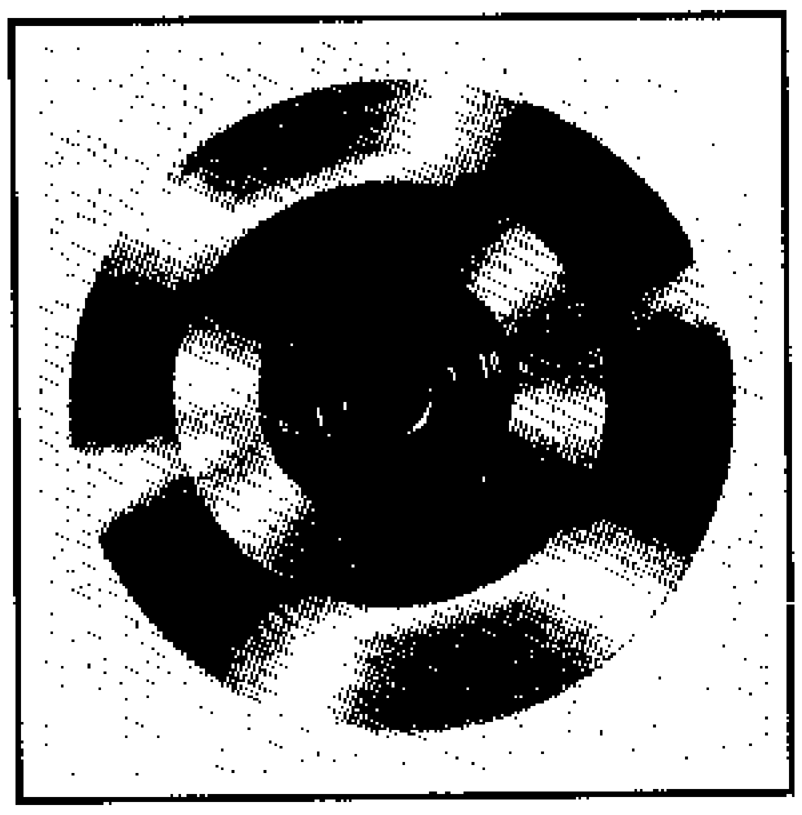
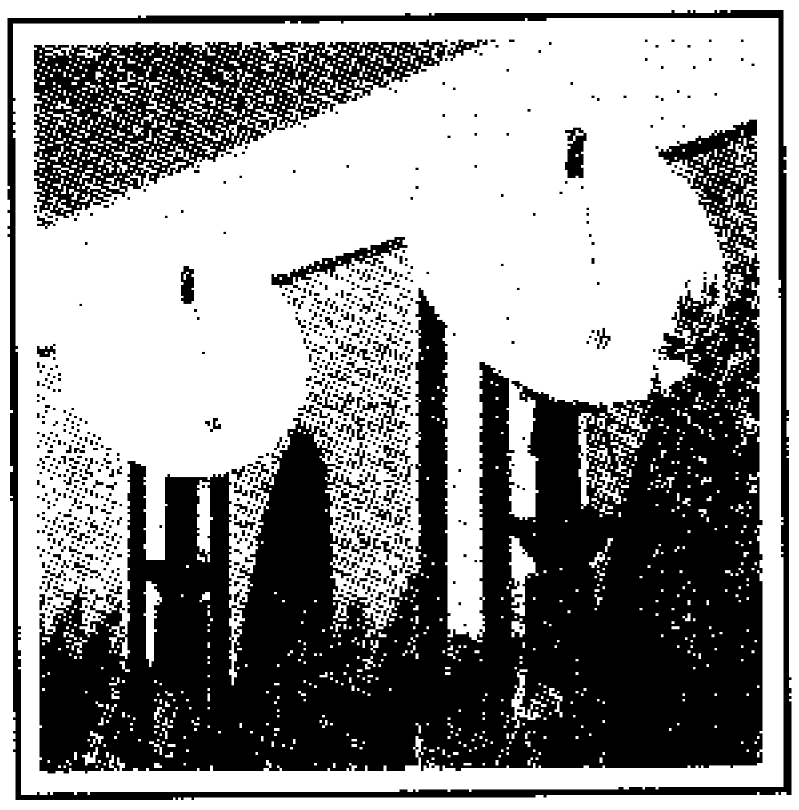
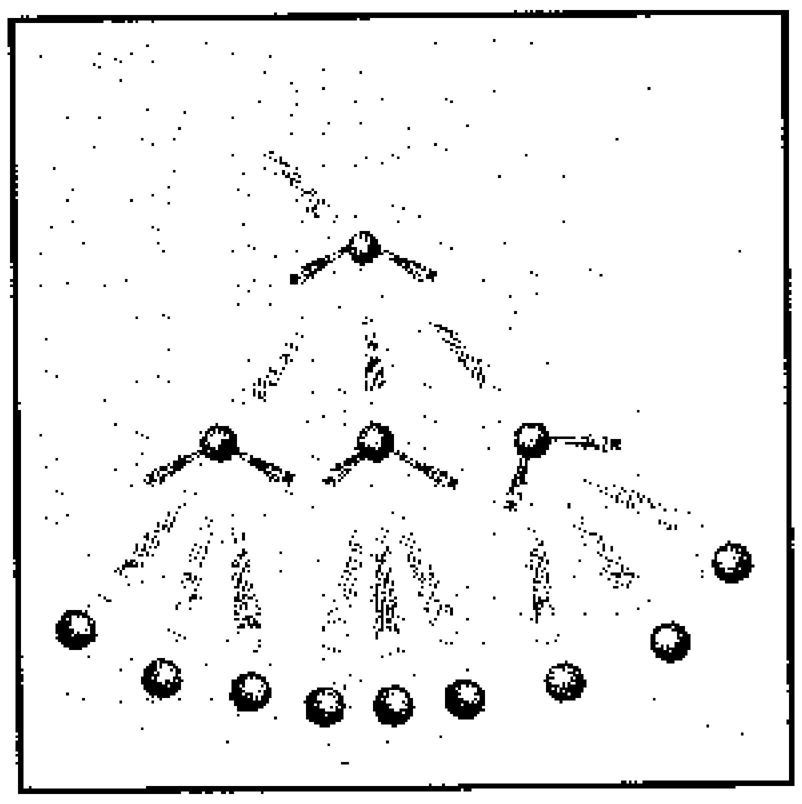






This book is dedicated to
my honourable grandfather
(Late) Sh. Pitamber Pandey;
a Kumaoni poet; resident of village
Dhaura (Almora) Uttaranchal

CHAPTER 22

Geometric Optics

Chapter Contents

22.1 INTRODUCTION

The branch of physics called **optics** deals with the behaviour of light and other electromagnetic waves. Light is the principal means by which we gain knowledge of the world. Consequently the nature of light has been the source of one of the longest debates in the history of science.

Electromagnetic radiation with wavelengths in the range of about 4000 Å to 7000 Å, to which eye is sensitive is called light.

Our investigation of light will revolve around two questions of fundamental importance (1) What is the nature of light and (2) How does it behave under various circumstances? The answers to these two questions can be found in Maxwell's field equations (which is out of JEE syllabus). These equations predict the existence of electromagnetic waves that travel at the speed of light. They also describe how these waves behave. Interestingly, not all light phenomena can be explained by Maxwell's theory. Experiments performed at the beginning of this century showed that light also has corpuscular, or particle like properties.

In the present and next chapter we investigate the behaviour of a beam of light when it encounters simple optical devices like mirrors, lenses and apertures. Under many circumstances, the wavelength of light is negligible compared with the dimensions of the device as in the case of ordinary mirrors and lenses. A light beam can then be treated as a ray whose propagation is governed by simple geometric rules. The part of optics that deals with such phenomena is known as **geometric optics**. However, if the wavelength is not negligible compared with the dimensions of the device (for example a very narrow slit), the ray approximation becomes invalid and we have to examine the behaviour of light in terms of its wave properties. This study is known as **physical optics**.

22.2 THE NATURE OF LIGHT

The question whether light is a wave or a particle has a very interesting and long history. Early theories considered light to be a stream of particles which emanated from a source and caused the sensation of vision upon entering the eye. The most influential proponent of this particle theory of light was **Newton**. Using it, he was able to explain the laws of reflection and refraction. The chief proponents of the wave theory of light propagation were **Christian Hygens** and **Robert Hooke**. Hygen's using his wave theory was also able to explain reflection and refraction. Newton saw the virtues of the wave theory of light particularly as it explained the colours formed by thin films, which Newton studied extensively. However, he rejected the wave theory because of the observed straight line propagation of light. Because of Newton's great reputation and authority, this reluctant rejection of the wave theory of light, based on lack of evidence of diffraction was strictly adhered to by Newton's followers. Newton's particle theory of light was accepted for more than a century.

In 1801 **Thomas Young** revived the wave theory of light. He was one of the first to introduce the idea of interference as a wave phenomenon in both light and sound. His observation of interference with light was a clear demonstration of the wave nature of light. Young's work went unnoticed by the scientific community for more than a decade.

Fresnel performed extensive experiments on interference and diffraction and put the wave theory on a mathematical basis. He showed, that the rectilinear propagation of light is a result of very short wavelength of visible light. In 1850 **Jean Foucault** measured the speed of light in water and showed that it is less than that in air, thus ruling out Newton's particle theory according to whom the speed of light is more in water.

But the drama was not yet over. The climax came when the wave theory of light failed to explain the photoelectric effect invented by **Albert Einstein** in 1905. He himself explained it on the basis of particle nature of light. An amicable understanding was ultimately reached in accepting that light has **dual nature.** It can behave as particles as well as waves depending on its interaction with the surrounding. Later it was found that even the well established particles such as electrons also have a dual character and can show interference and diffraction under suitable conditions.

Electromagnetic Waves

In Chapter–16 (Waves and Thermodynamics) we saw that a wave travelling along x-axis with a speed v satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{...(i)}$$

Maxwell was able to show that time dependent electric and magnetic fields also satisfy the wave equation. The changing electric and magnetic fields form the basis of electromagnetic waves. In free space, far from the source of the fields, the fields satisfy **Maxwell' wave equations**:

$$\frac{\partial^2 E}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} \quad \text{...(ii)}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2} \quad \text{...(iii)}$$

On comparing these with the standard wave equation, we see that the electromagnetic wave speed is

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \quad \text{...(iv)}$$

When the values $\mu_0 = 4\pi\times10^{-7}$ H/m and $\varepsilon_0 = 8.85\times10^{-12}$ F/m are inserted, we find

$$c = 3.00\times10^8 \text{ m/s}$$

This is speed of light in vacuum.

The simplest plane wave solutions to Eqs. (ii) and (iii) are

$$E = E_0 \sin(\omega t - kx) \quad \text{...(v)}$$

$$B = B_0 \sin(\omega t - kx) \quad \text{...(vi)}$$

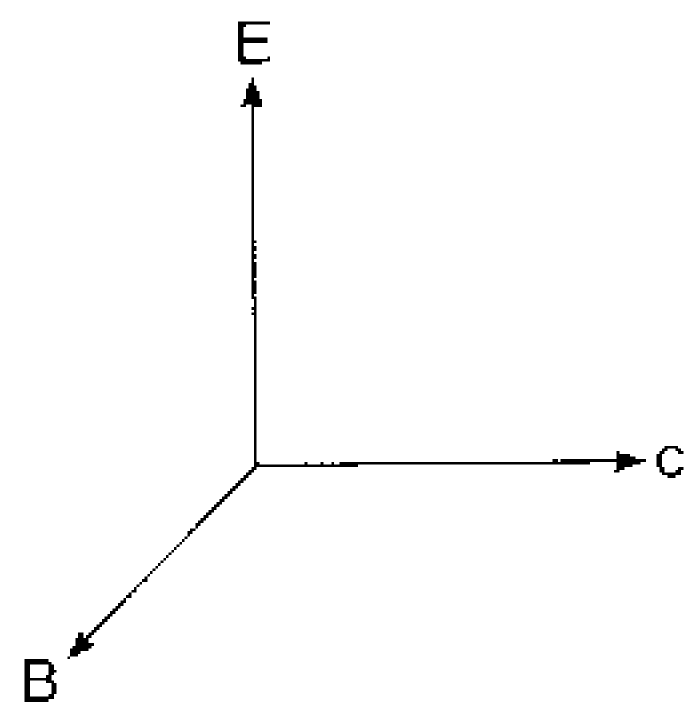


Fig. 22.1

From these equations we see that at any point E and B are in phase. The electric and magnetic fields in a plane electromagnetic wave are perpendicular to each other and also perpendicular to the direction of propagation of light as shown in figure. They are transverse electromagnetic waves. The magnitudes of the fields are related by

$$c = \frac{E}{B} \quad \text{or} \quad E = cB \quad \text{...(vii)}$$

According to the thinking of the 19th century, the constants μ_0 and ε_0 referred to properties of the **ether**, the medium through which the electromagnetic waves were assumed to propagate. This is not our present thinking. The ether does not exists and electromagnetic waves do not require any medium in which to propagate. However, when they travel through a substance, the fields do interact with charges in the medium. The strength of the interaction is related to the permittivity ε and the permeability μ of the substance. As a result the speed of light in medium is reduced to $\frac{1}{\sqrt{\varepsilon\mu}}$. Hence,

$$v = \frac{1}{\sqrt{\varepsilon\mu}} \quad \text{...(viii)}$$

The ratio of c and v ($< c$) is known as the refractive index of the substance. This is a pure ratio which has a value greater than or equal to one. Thus,

$$\text{Refractive index} = \frac{c}{v} \quad \text{...(ix)}$$

EXAMPLE 22.1 *The magnetic field of an electromagnetic wave in a substance is given by*

$$B = (2 \times 10^{-6}\ T)\ cos[\pi(0.04\,x + 10^7 t)]$$

Find the refractive index of the substance.

SOLUTION Comparing the given equation with the standard wave equation

$$B = B_0 \cos(\omega t + kx)$$

We have, $\omega = \pi \times 10^7$ rad/s and $k = \pi \times (0.04)\ \text{m}^{-1}$

$\therefore$ Speed of electromagnetic wave in this medium is

$$v = \frac{\omega}{k} = 2.5 \times 10^8\ \text{m/s}$$

Now, refractive index of substance $= \frac{c}{v} = \frac{3.0 \times 10^8}{2.5 \times 10^8}$

$$= 1.2$$ **Ans.**

INTRODUCTORY EXERCISE 22.1

1. Show that the unit of $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is m/s.
2. The magnetic field in a plane electromagnetic wave is given by
$$B_y = (2 \times 10^{-7}\ T) \sin[500x + 1.5 \times 10^{11} t]$$
   (a) What is the wavelength and frequency of the wave?
   (b) Write an expression for the electric field vector.

22.3 FEW GENERAL POINTS OF GEOMETRIC OPTICS

Here are few general points which I consider are important before studying the geometric optics. Students who have never studied the optics before are advised to read this article once more after finishing the present chapter.

1. Normally the object in kept on the left hand side of the optical instrument (mirror, lens etc.), *i.e.*, the ray of light travels from left to right. Sometimes it may happen that the light is travelling in opposite direction. See the figure.

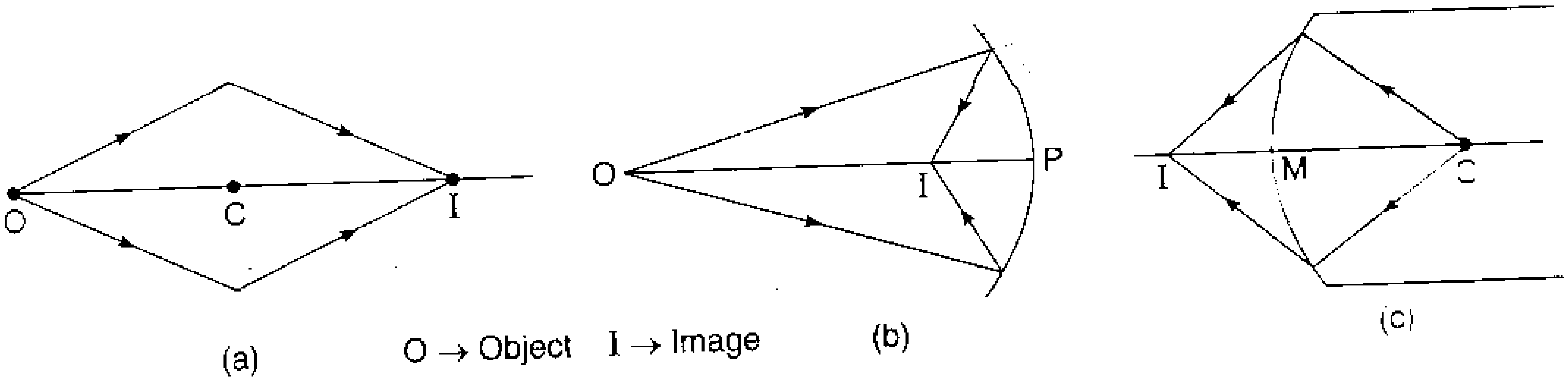


Fig. 22.2

In figures (a) and (b) light is travelling from left to right and in figure (c) it is travelling from right to left.

2. Whenever a silvered surface comes on the path of a ray of light it returns from there, otherwise it keeps on moving forwards.
3. **Sign convention :** The distances measured along the incident light are taken as positive while the distances against incident light are taken as negative. For example, in figures (a) and (b) the incident light travels from left to right. So the distances measured in this direction are positive. While in figure (c) the incident light travels from right to left. So in this case this direction will be positive. Distances are measured from pole of the mirror [point P in figure (b)], optical centre of the lens [point C in figure (a)] and the centre of the refracting surface [point M in figure (c)].

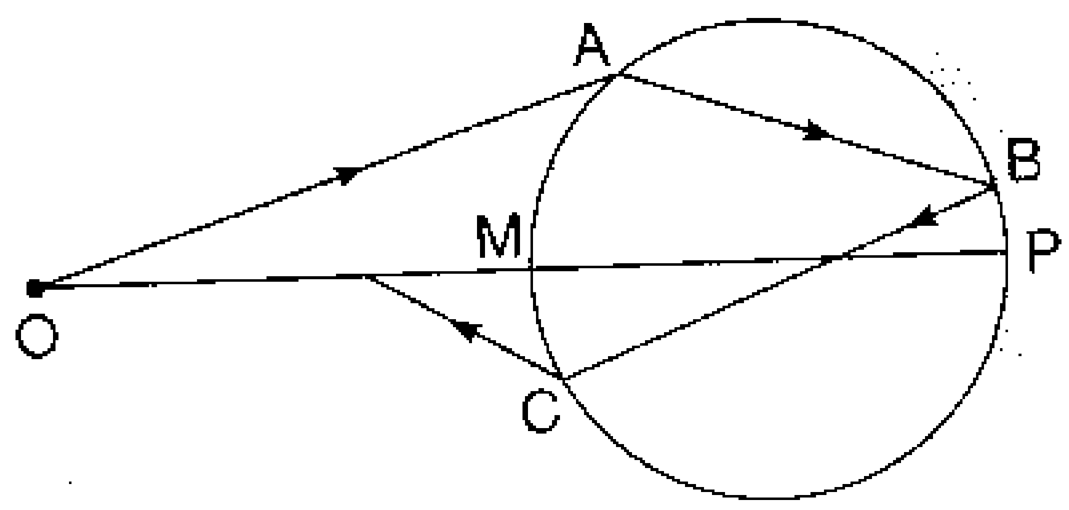


Fig. 22.3

It may happen in some problem that sign convention does not remain same for the whole problem.
For example, in the figure 22.3 shown, the ray of light emanating from O first undergoes refraction at A, then reflection at B and then finally refraction at C. For refraction and reflection at A and B the incident light is travelling from left to right, so distances measured along this direction are positive. For final refraction at C the incident light travels from right to left, so now the sign convention will change or right to left is positive.
4. Object distance (from P, C or M along the optic axis) is shown by u and image distance by v.
5. Image at infinity means rays after refraction or reflection have become parallel to the optic axis. If a screen is placed directly in between these parallel rays no image will be formed on the screen. But if a converging lens (convex) is placed on the path of the parallel rays and a screen is placed at the focus of the lens, image will be formed on the screen. Sometimes our eye plays the role of this converging lens and the retina is the screen.

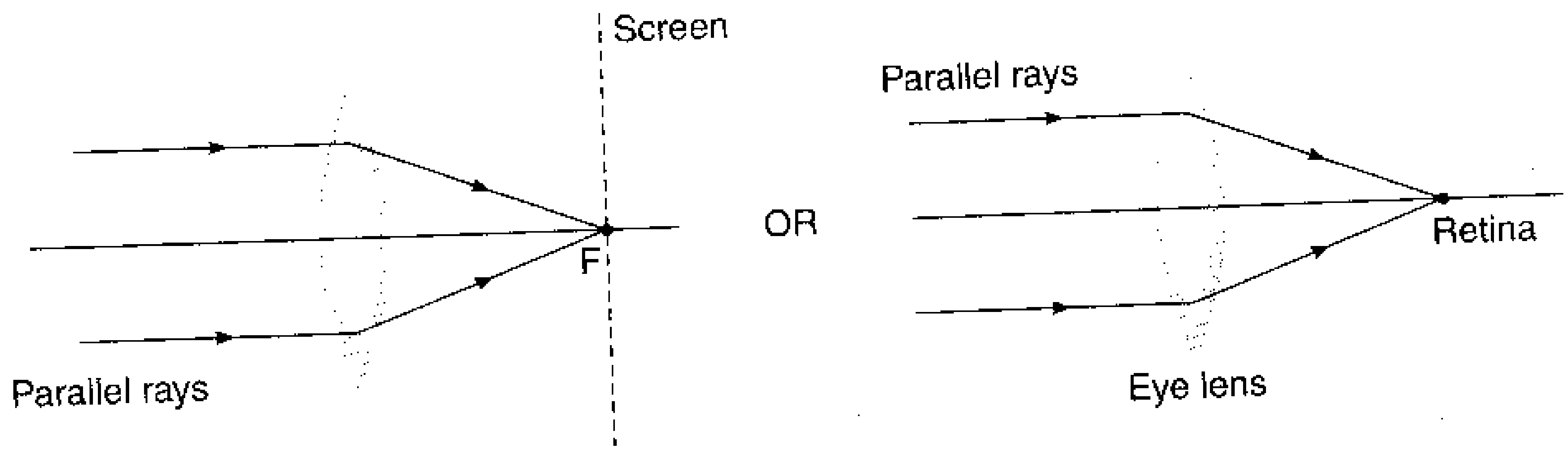


Fig. 22.4

6. **Real object, virtual object, real image, virtual image :** In figure (a), object is real while image is virtual. In figure (b), object is virtual while its image is real.

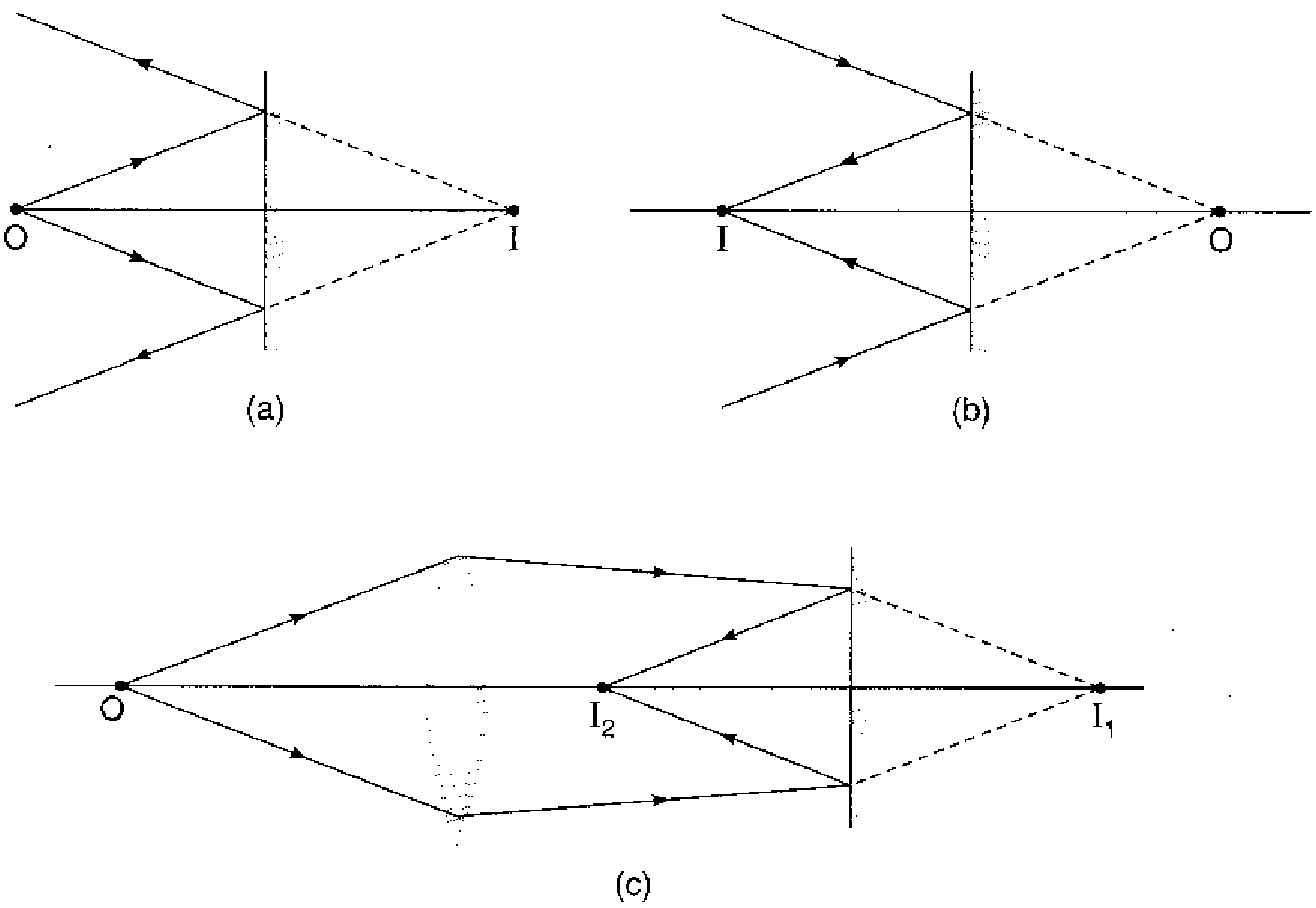


Fig. 22.5

In figure (c), the object O is real. Its image formed by the lens (*i.e.*, I_1) is real. But it acts as a virtual object for mirror which forms its real image I_2.

Note : The virtual images cannot be taken on screen. But they can be seen by our eye. Because our eye lens forms their real image on our retina. Thus, if we put a screen at I in figure (a) no image will be formed on it. At the same time if we put the screen at I in figure (b), image will be formed.

22.4 REFLECTION OF LIGHT

When waves of any type strike the interface between two optical materials, new waves are generated which move away from the barrier. Experimentally it is found that the rays corresponding to the incident and reflected waves make equal angles with the normal to the interface and that the reflected ray lies in the plane of incidence formed by the incident ray and the normal. Thus, the two laws of reflection can be summarised as under:

(1) $\angle i = \angle r$

(2) Incident ray, reflected ray and normal lie on the same plane.

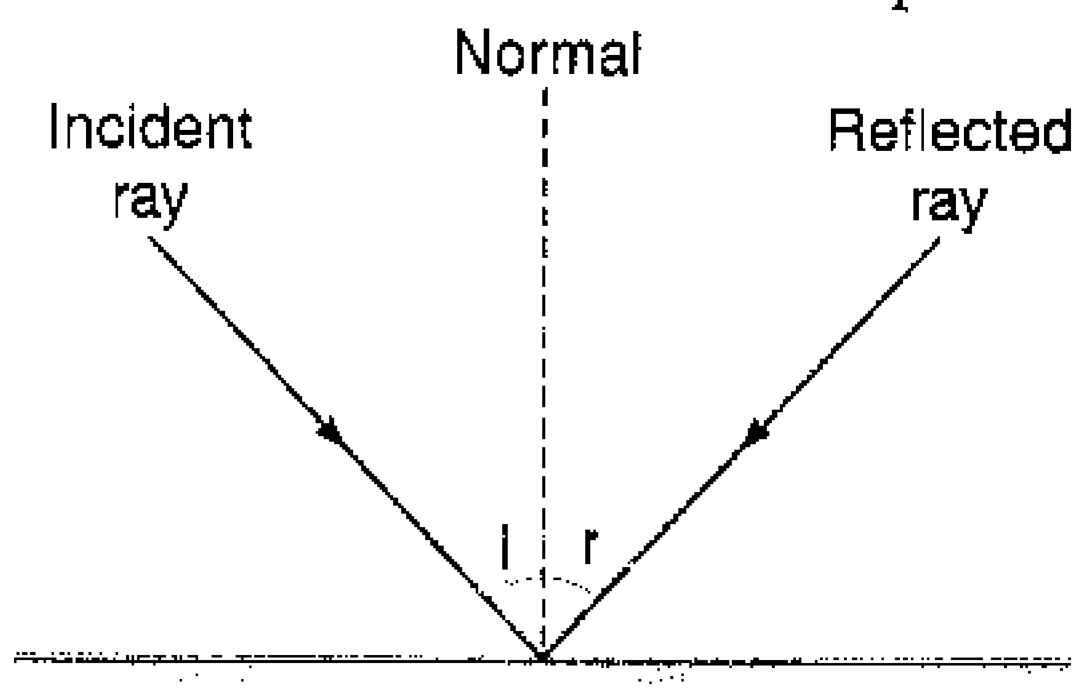


Fig. 22.6

Note : The above two laws of reflection can be applied to the reflecting surfaces which are not even horizontal. The following figures illustrates the point.

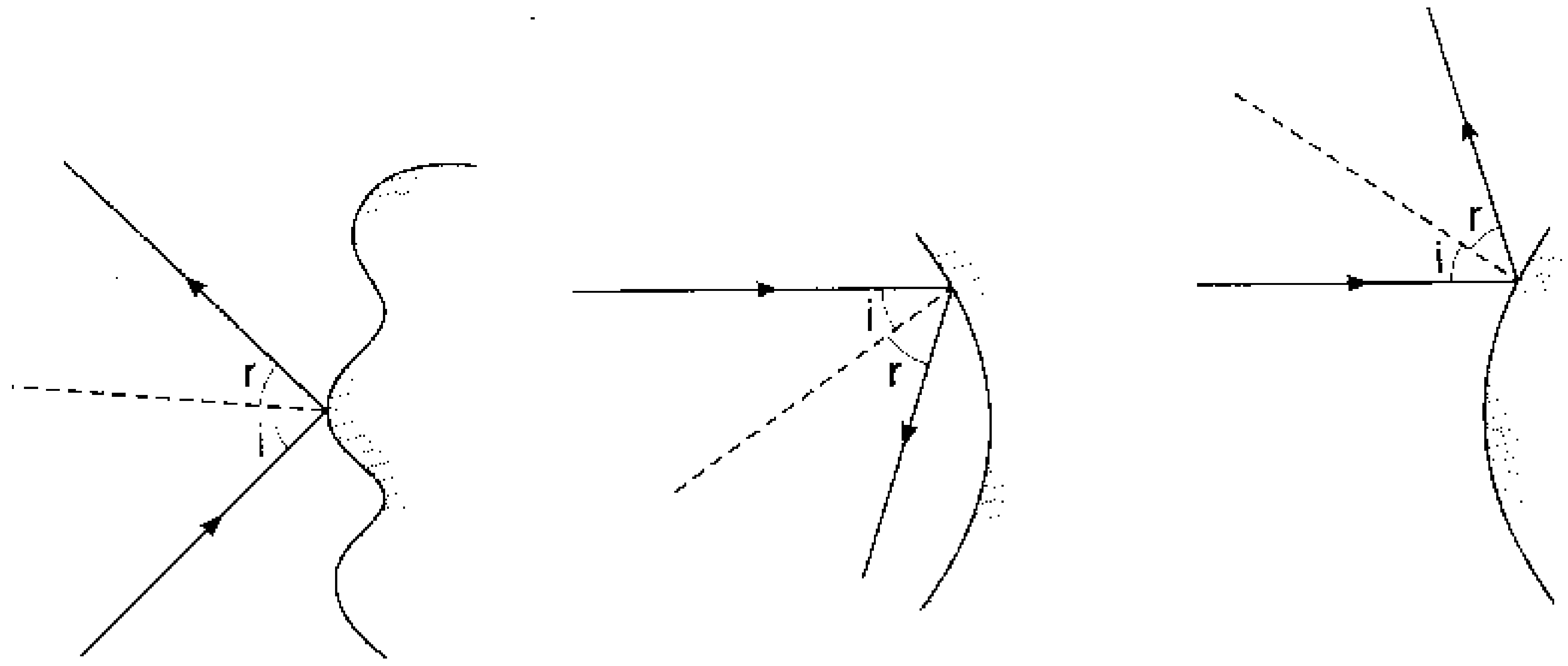


Fig. 22.7

Reflection from a plane surface (or plane mirror)

Almost everybody is familiar with the image formed by a plane mirror. If the object is real, the image formed by a plane mirror is virtual, erect, of same size and at the same distance from the mirror. The ray diagram of the image of a point object and of an extended object is as shown below.

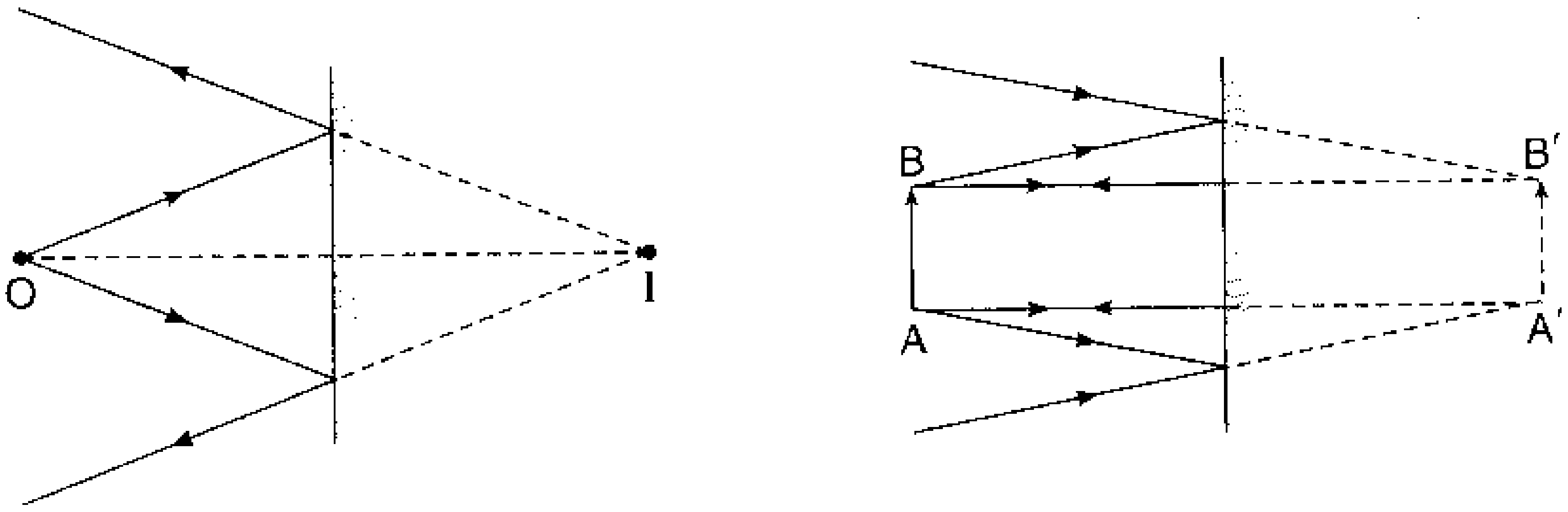


Fig. 22.8

IIT-JEE GALAXY 22.1

1. To find the location of image of an object from an inclined plane mirror, you have to see the perpendicular distance of the object from the mirror.

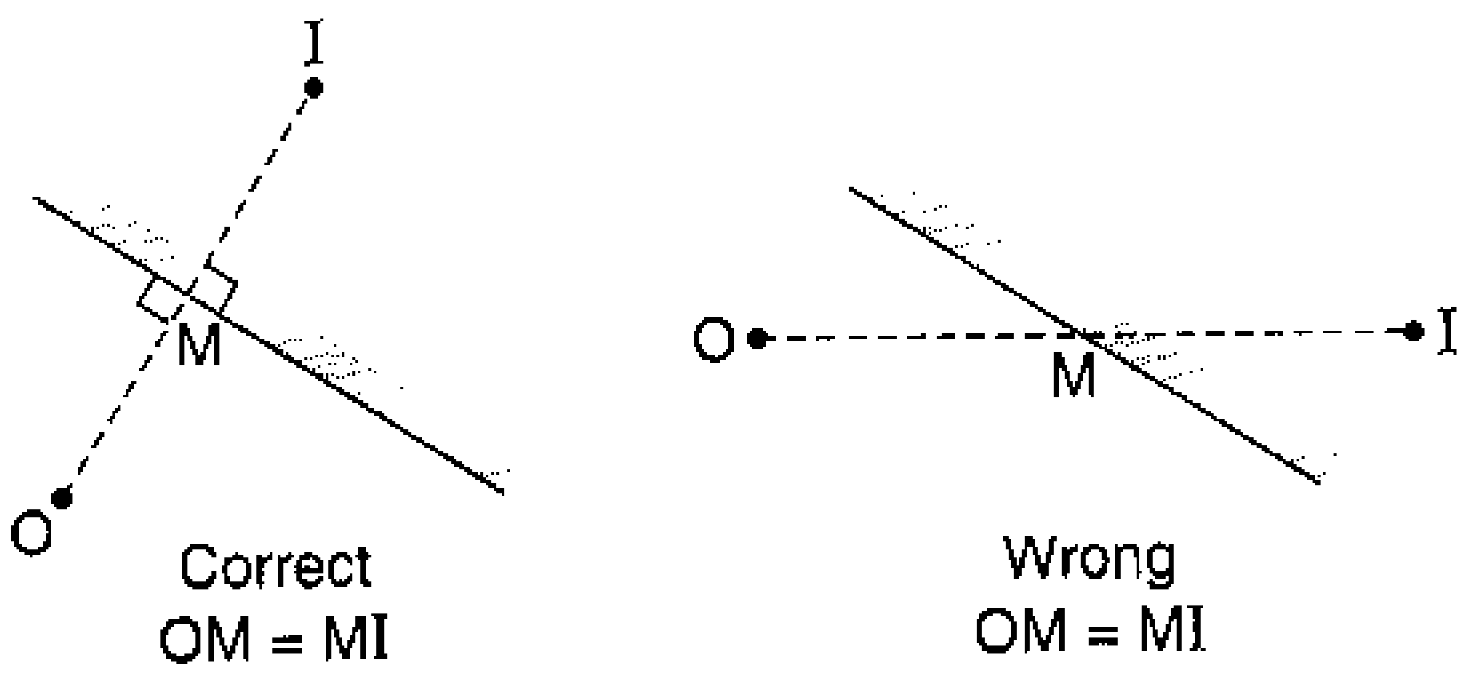


Fig. 22.9

2. Suppose an object O is placed in front of two plane mirrors M_1 and M_2. Size of M_2 is more than that of M_1. In this case the intensity of the images formed by M_2 (*i.e.*, I_2) will be more than that formed by M_1 (*i.e.* I_1). This is because I_2 is formed from more number of reflected

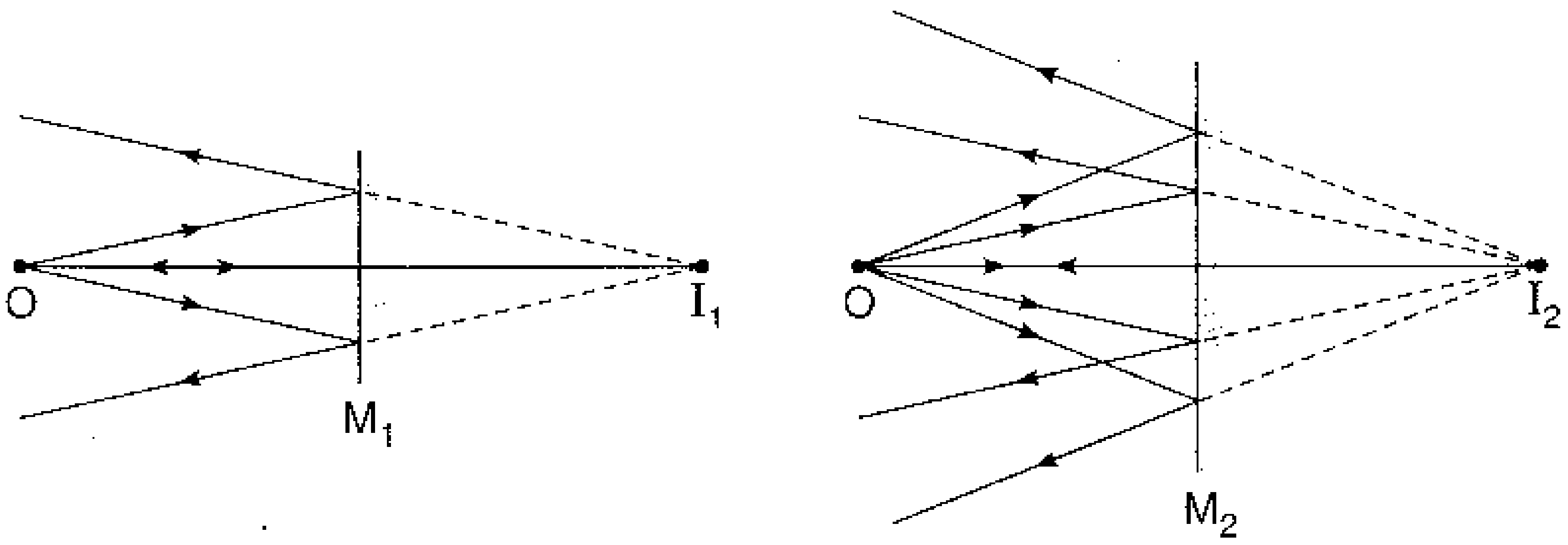


Fig. 22.10

rays. Or we can say it is formed from more light. The same is the case with an image formed by a lens of large aperture.

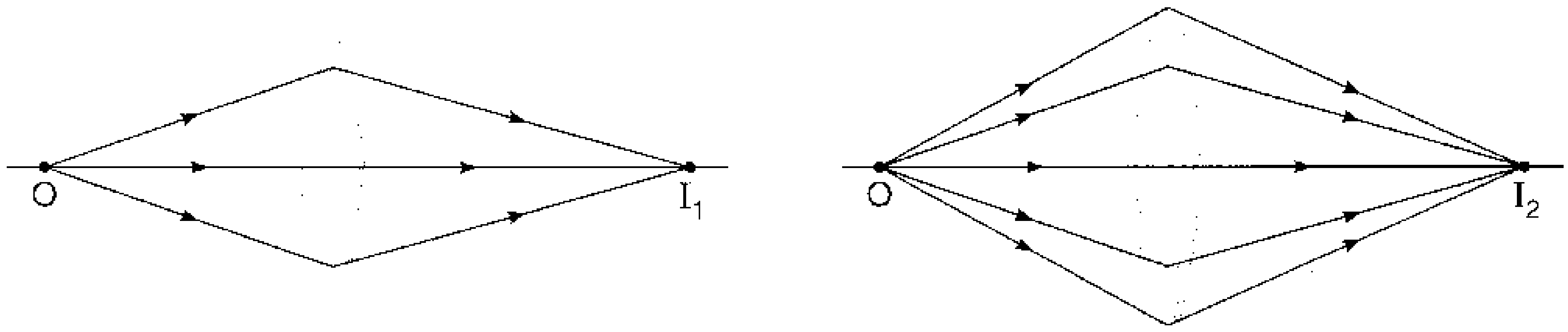


Fig. 22.11

In the figure shown, I_2 will be more intense. This is the reason why we choose a lens of large aperture in telescopes to see the distant objects. You might have heard that the strength of an observatory is measured from the aperture of its lens. Now, you can think if this is the case then why don't we take a lens of an aperture as large as we wish. Actually some technical problems arise in casting a lens of large aperture. So we have some limitations in doing so.

3. **Field of view of an object for a given mirror :** Suppose a point object O is placed in front of a small mirror as shown in Fig. 22.12 (a), then a question arises in mind whether this mirror will form the image of this object or not or suppose an elephant is standing in front of a small mirror, will the mirror form the image of the elephant or not. The answer is yes, it will form. A mirror whatever may be the size of it forms the images of all objects lying in front of it. But every object has its own field of view for the given mirror. The field of view is the region between the extreme reflected rays and depends on the location of the object

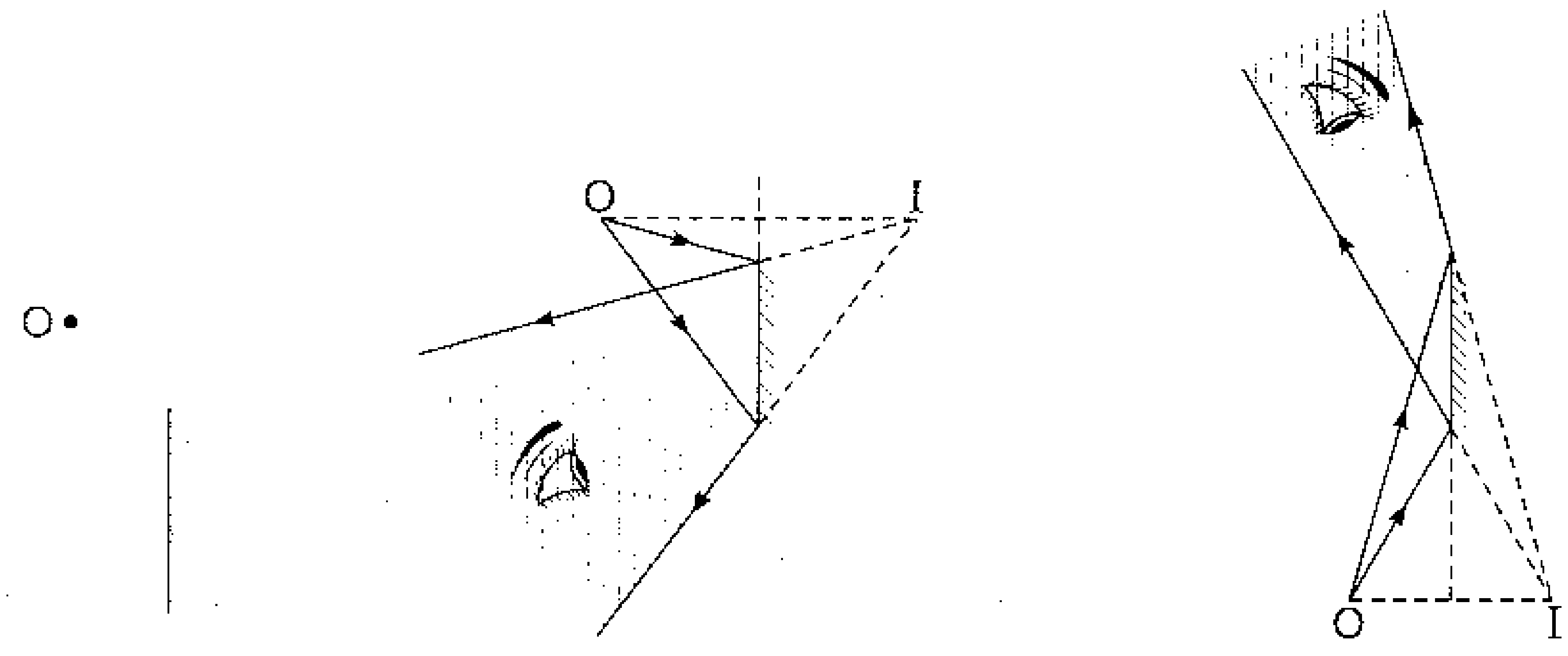


Fig. 22.12

in front of the mirror. If our eye lies in the field of view then only we can see the image of the object otherwise not. The field of view of an object placed at different locations in front of a plane mirror are shown in Fig. 22.12 (b) and (c).

4. Suppose a mirror is rotated by an angle θ (say anticlockwise), keeping the incident ray fixed then the reflected ray rotates by 2θ along the same sense, *i.e.*, anticlockwise.

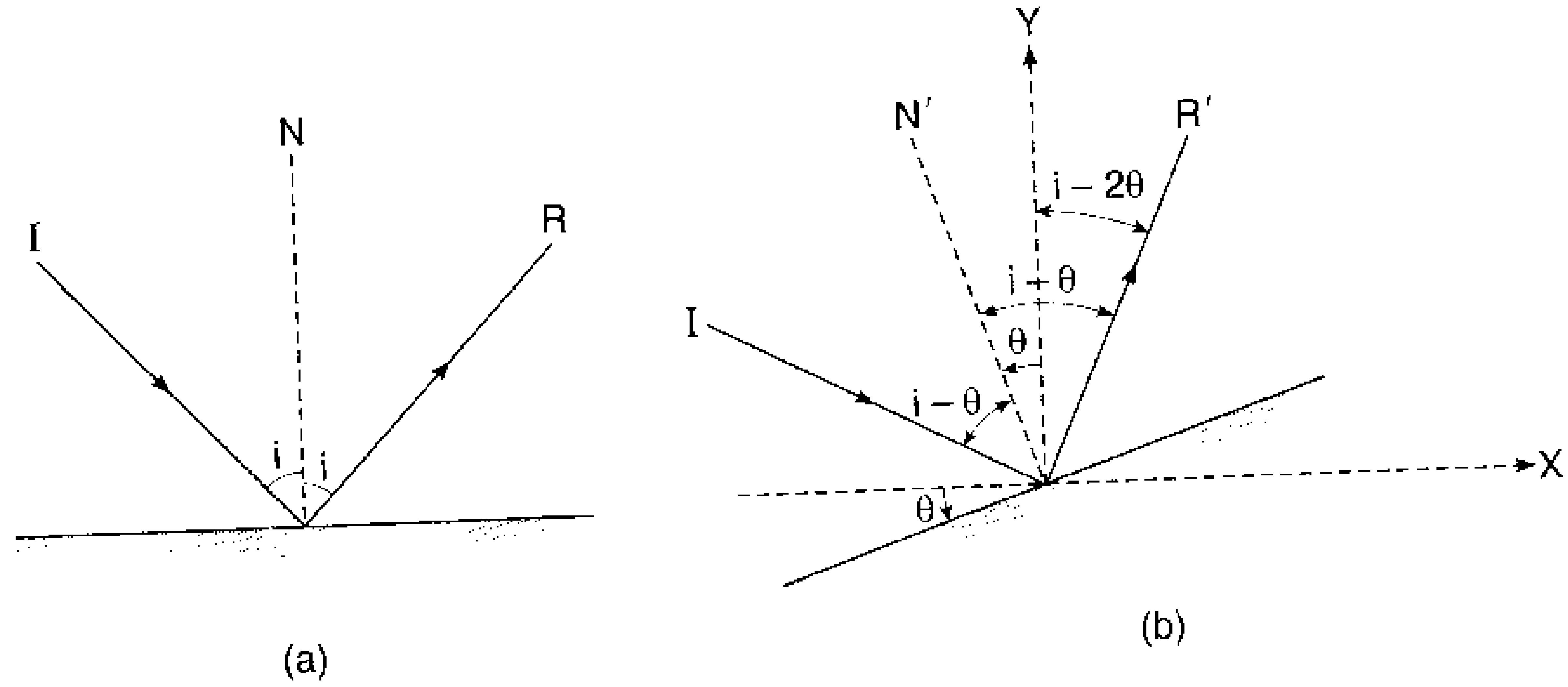


Fig. 22.13

In figure (a), *I* is the incident ray, *N* the normal and *R* the reflected ray.

In figure (b), *I* remains as it is. *N* and *R* shift to *N'* and *R'*.

From the two figures we can see that earlier the reflected ray makes an angle *i* with *y*-axis while after rotating the mirror it makes the angle $i - 2\theta$. Thus, we may conclude that the reflected ray has been rotated by angle 2θ.

5. The minimum length of a plane mirror to see one's full height in it is $\frac{H}{2}$, where

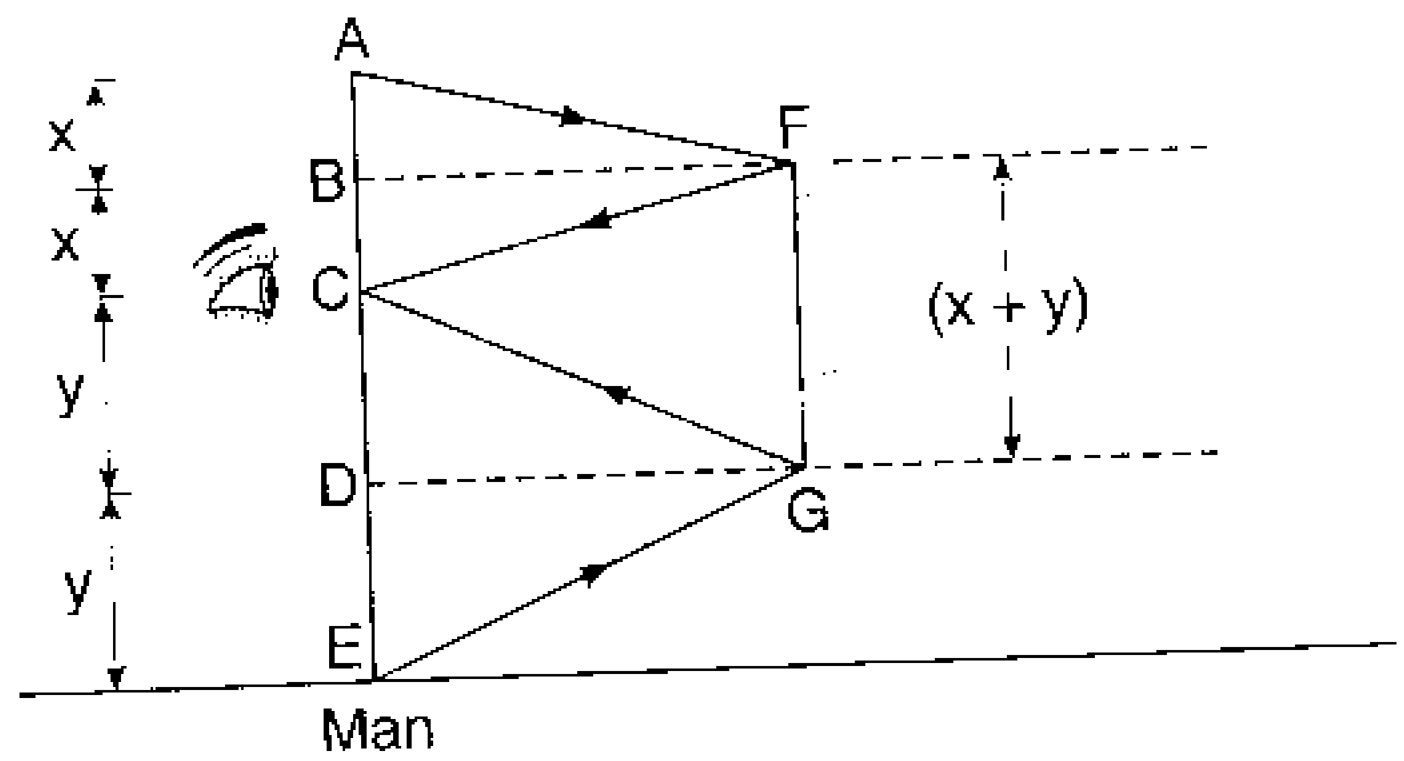


Fig. 22.14

H is the height of man. But the mirror should be placed in a fixed position which is shown in Fig. 22.14.

A ray starting from head (*A*) after reflecting from upper end of the mirror (*F*) reaches the eye at *C*. Similarly the ray starting from the foot (*E*) after reflecting from the lower end (*G*) also reaches the eye at *C*. In two similar triangles *ABF* and *BFC*, $AB = BC = x$ (say), Similarly in triangles *CDG* and *DGE*,

$$CD = DE = y \text{ (say)}$$

Now, we can see that height of the man is $2(x + y)$ and that of mirror is $(x + y)$, *i.e.*, height of the mirror is half the height of the man.

Note : The mirror can be placed anywhere between the centre line *BF* (of *AC*) and *DG* (of *CE*).

6. A man is standing exactly at midway between a wall and a mirror and he want to see the full height of the wall (behind him) in a plane mirror (in front of him). The minimum length of mirror in this case should be $\frac{H}{3}$, where *H* is the height of wall. The ray diagram in this case is drawn in Fig. 22.15.

In triangles HBI and IBC, $HI = IC = x$ (say). Now, in triangles HBI and ABF,

$$\frac{AF}{HI} = \frac{FB}{BI}$$

or $$\frac{AF}{x} = \frac{2d}{d}$$

or $$AF = 2x$$

Similarly we can prove that $DG = 2y$ if, $CK = KJ = y$

Now, we can see that height of the wall is $3(x + y)$ while that of the mirror is $(x + y)$.

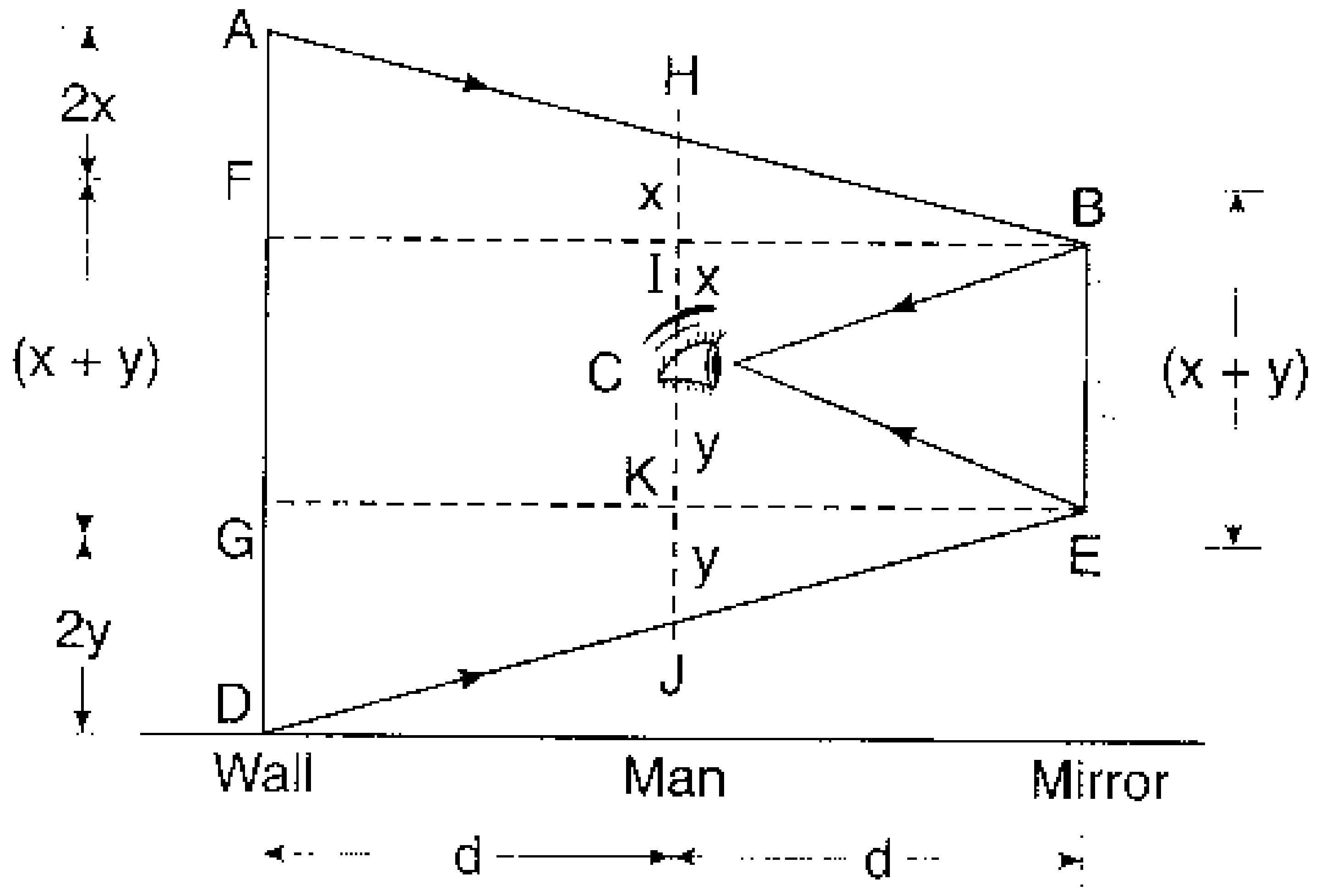


Fig. 22.15

EXAMPLE 22.2 *A point source of light S, placed at a distance L in front of the centre of a mirror of width d, hangs vertically on a wall. A man walks in front of the mirror along a line parallel to the mirror at a distance 2L from it as shown. The greatest distance over which he can see the image of the light source in the mirror is*

(a) $d/2$ (b) d
(c) $2d$ (d) $3d$

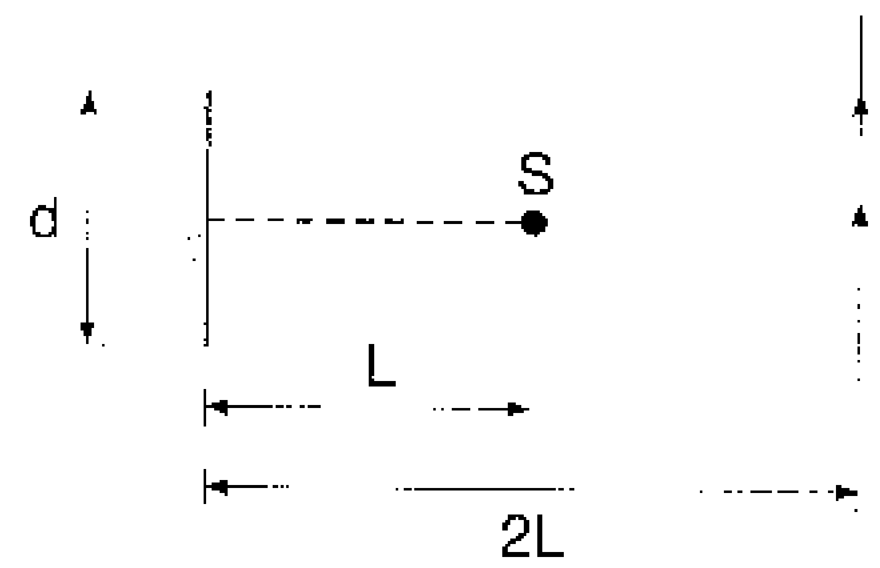


Fig. 22.16

SOLUTION (d) The ray diagram will be as shown in Fig. 22.17.

$$HI = AB = d$$

$$DS = CD = \frac{d}{2}$$

Since, $$AH = 2AD$$

$\therefore$ $$GH = 2\,CD = 2\frac{d}{2} = d$$

Similarly $$IJ = d$$

$\therefore$ $$GJ = GH + HI + IJ$$
$$= d + d + d = 3d$$

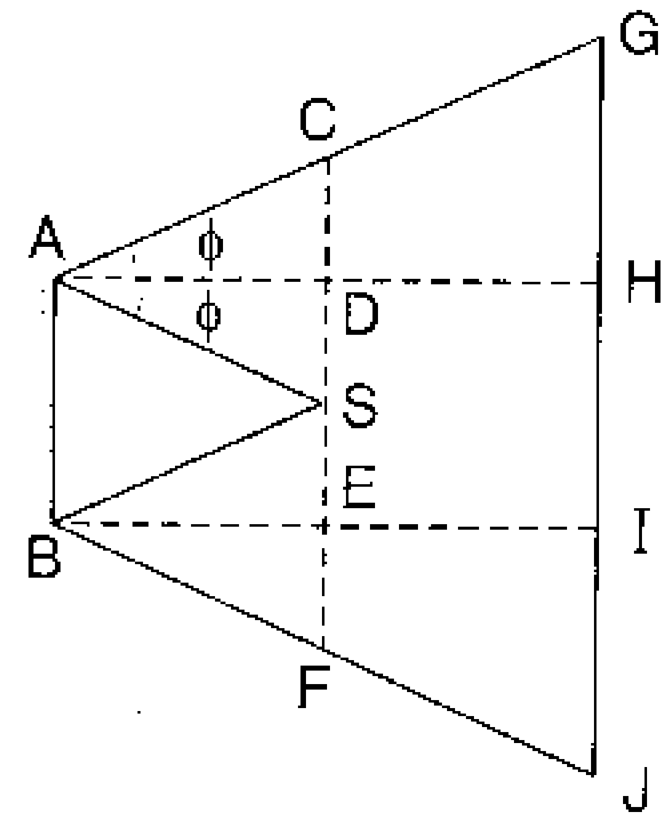


Fig. 22.17

EXAMPLE 22.3 *Two plane mirrors M_1 and M_2 are inclined at angle θ as shown. A ray of light 1, which is parallel to M_1 strikes M_2 and after two reflections, the ray 2 becomes parallel to M_2. Find the angle θ.*

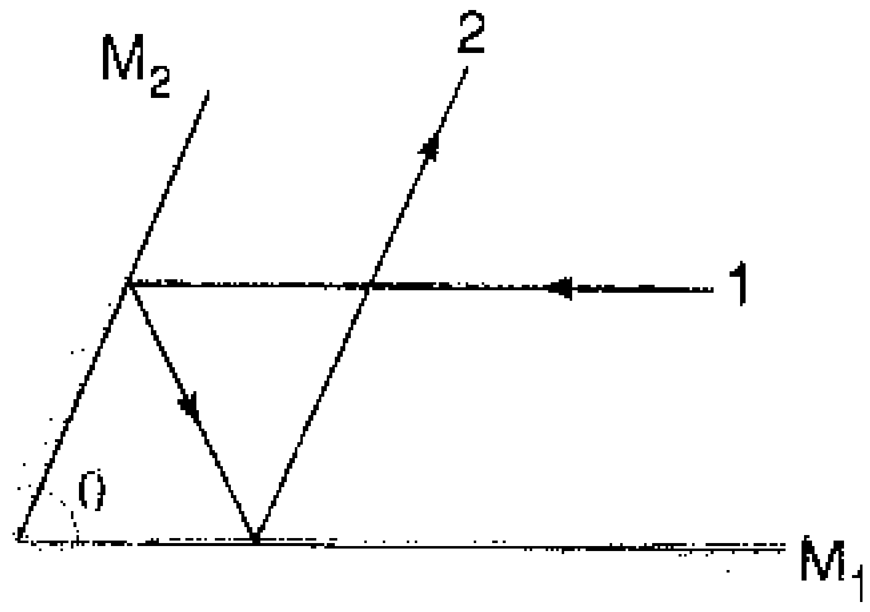


Fig. 22.18

SOLUTION Different angles are as shown in Fig. 22.19. In triangle ABC,

$$\theta + \theta + \theta = 180°$$

$$\therefore \qquad \theta = 60° \qquad \textbf{Ans.}$$

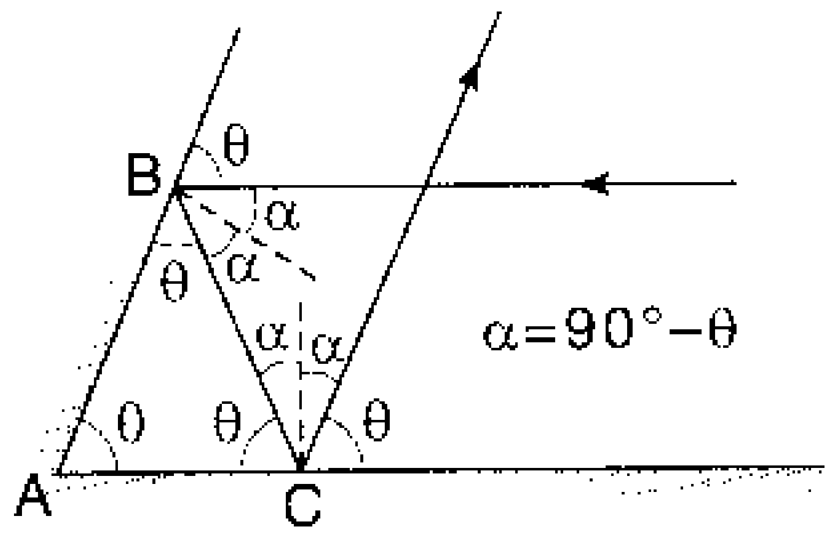


Fig. 22.19

INTRODUCTORY EXERCISE 22.2

1. Prove that for any value of angle i, rays 1 and 2 are parallel.

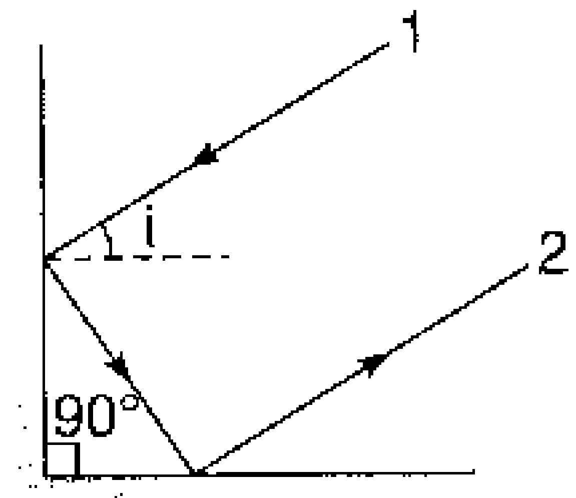


Fig. 22.20

2. A man approaches a vertical plane mirror at speed of 2 m/s. At what rate does he approach his image?
3. A pole of height 4 m is kept in front of a vertical plane mirror of length 2 m. The lower end of the mirror is at a height of 6 m from the ground. The horizontal distance between the mirror and the pole is 2 m. Upto what minimum and maximum heights a man can see the image of top of the pole at a distance of 4 m standing on the same horizontal line which is passing through the pole and the horizontal point below the mirror?

Reflection from a spherical surface

We shall mainly consider the spherical mirrors, *i.e.*, those which are part of a spherical surface.

(a) Terms and Definitions

There are two types of spherical mirrors, concave and convex.

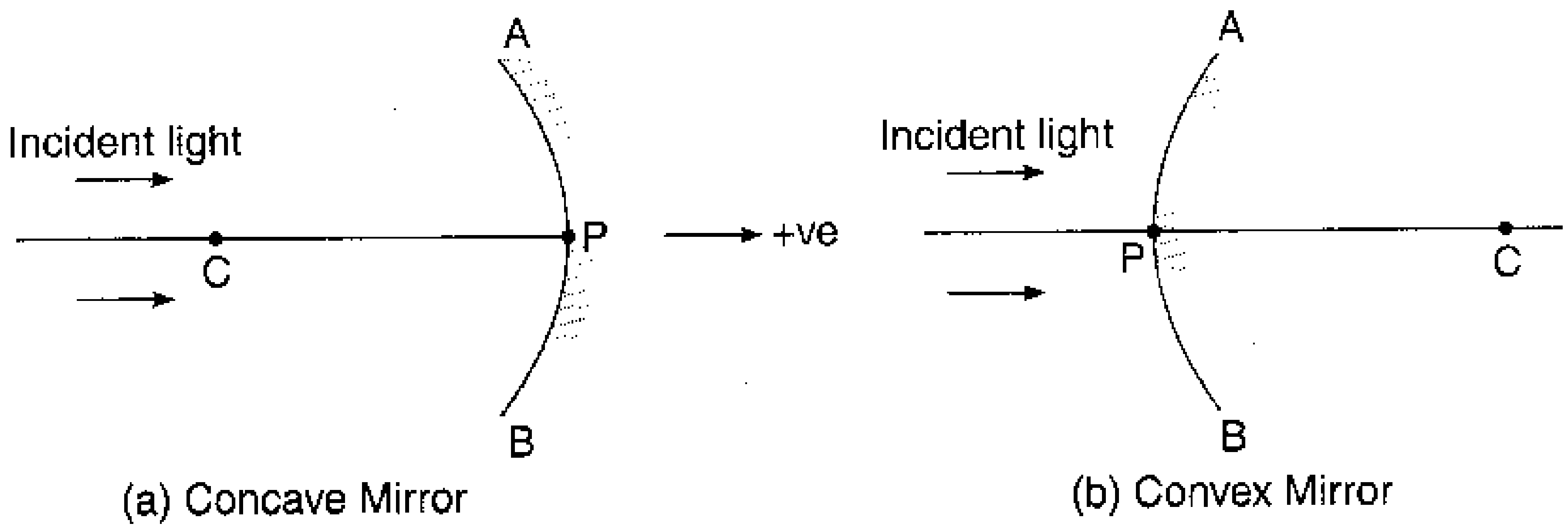


Fig. 22.21

Centre C of the sphere of which the mirror is a part is called the **centre of curvature** of the mirror and P the centre of the mirror surface, is called the **pole**. The line CP produced is the **principal axis** and AB is the **aperture** of the mirror. The distance CP is called the **radius of curvature (R)**. All distances are measured from point P. We can see from the two figures that R is positive for convex mirror and negative for concave mirror.

Principal focus

Observation shows that a narrow beam of rays, parallel and near to the principal axis, is reflected from a concave mirror so that all rays converge to a point F on the principal axis. F is called the **principal**

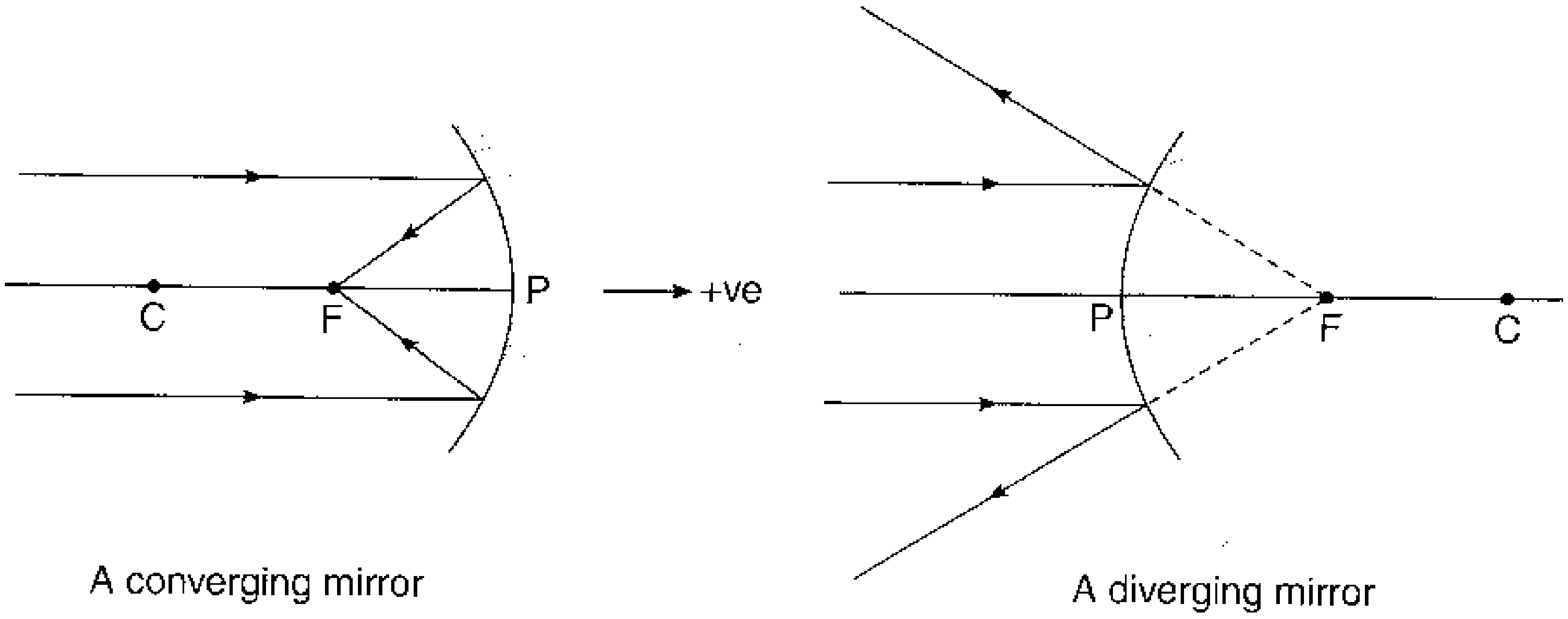


Fig. 22.22

focus of the mirror and it is a **real** focus, since, light actually passes through it. Concave mirrors are also known as converging mirrors because of their action on a parallel beam of light. They are used in car head-lights, search-lights and telescopes. A narrow beam of rays, parallel and near to the principal axis, falling on a convex mirror is reflected to form a divergent beam which appears to come from a point F behind the mirror. A convex mirror thus has a virtual principal focus. It is also called a diverging mirror. The distance FP is called the **focal length (f)** of the mirror. Further, we can see that f is negative for a concave mirror and positive for convex mirror. Later we will see that $f = R/2$.

Paraxial rays : Rays which are close to the principal axis and make small angles with it, *i.e.*, they are nearly parallel to the axis, are called paraxial rays. Our treatment of spherical mirrors will be restricted to such rays which means we shall consider only mirrors of small aperture. In diagrams, however, they will be made larger for clarity.

(b) Relation between f and R

A ray AM parallel to the principal axis of a concave mirror of small aperture is reflected through the principal focus F. If C is the centre of curvature, CM is the normal to the mirror at M because the radius of a spherical surface is perpendicular to the surface. From first law of reflection,

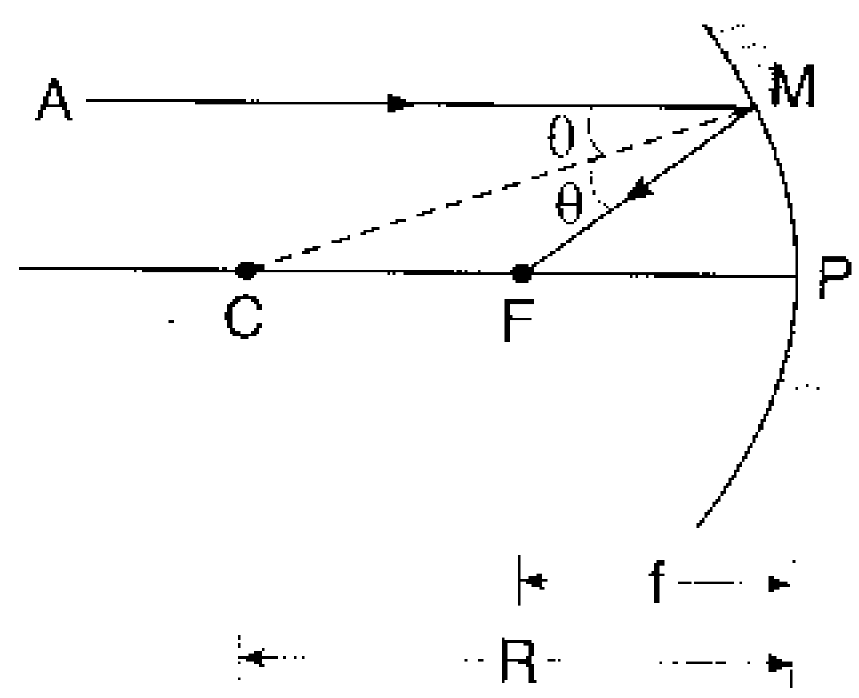


Fig. 22.23

$$\angle i = \angle r$$

or $$\angle AMC = \angle CMF = \theta \text{ (say)}$$

But $$\angle AMC = \angle MCF \text{ (alternate angles)}$$

$\therefore$ $$\angle CMF = \angle MCF$$

Therefore, $\Delta\ FCM$ is thus isosceles with $FC = FM$. The rays are paraxial and so M is very close to P. Therefore,

$$FM \approx FP$$

$\therefore$ $$FC = FP$$

or $$FP = \frac{1}{2} CP$$

or $$f = \frac{R}{2}$$

EXERCISE : Prove the above relation for convex mirror.

(c) Images formed by Spherical Mirrors

In general position of image and its nature (*i.e.*, whether it is real or virtual, erect or inverted, magnified or diminished) depend on the distance of object from the mirror.

Information about the image can be obtained either by drawing a **ray diagram** or by calculation using **formulae**.

Ray diagrams : We shall consider the small objects and mirrors of small aperture so that all rays are paraxial. To construct the image of a point object two of the following four rays are drawn passing through the object. To construct the image of an extended object the image of two end points is only drawn. The image of a point object lying on principles axis is formed on the principal axis itself. The four rays are as under:

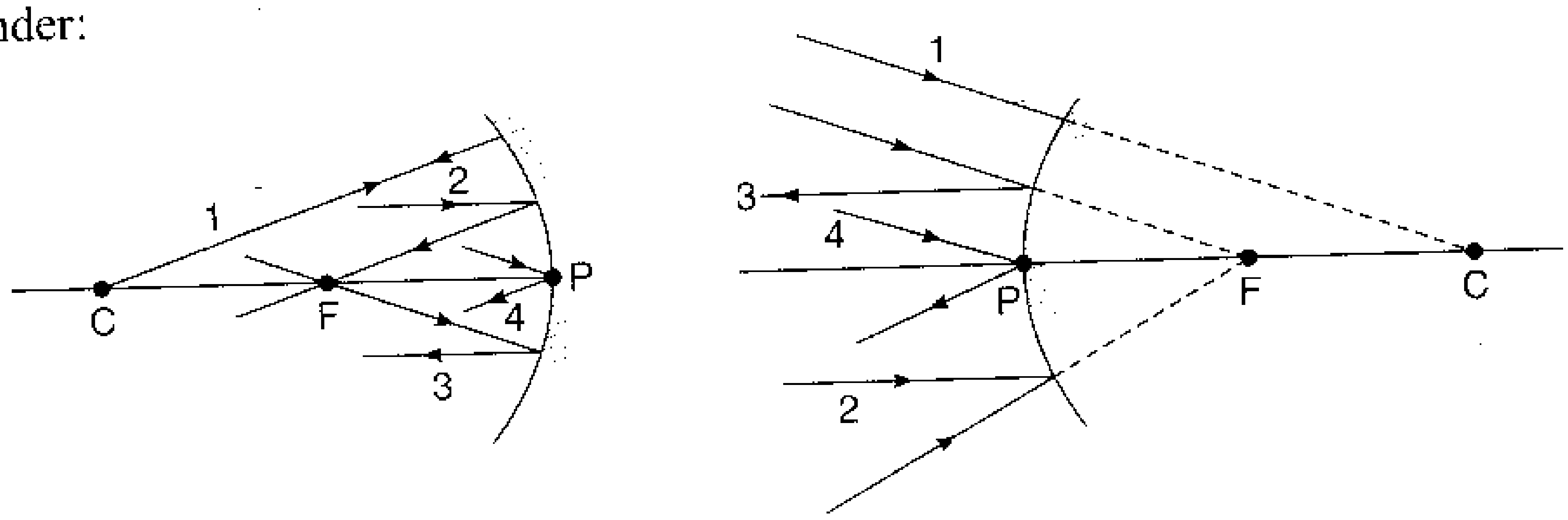


Fig. 22.24

Ray 1. A ray through the centre of curvature which strikes the mirror normally and is reflected back along the same path.

Ray 2. A ray parallel to principal axis after reflection either actually passes through the principal focus F or appears to diverge from it.

Ray 3. A ray passing through the principal focus F or a ray which appears to converge at F is reflected parallel to the principal axis.

Ray 4. A ray striking at pole P is reflected symmetrically back in the opposite side.

Note : 1. Image formed by convex mirror is always virtual, erect and diminished no matter where the object is.

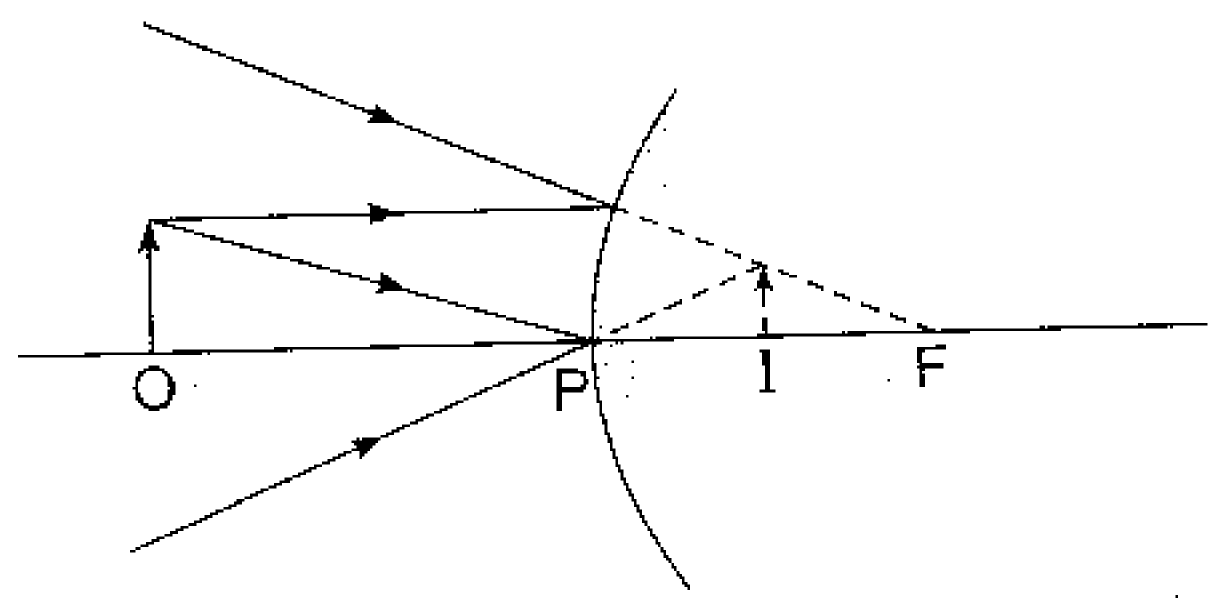


Fig. 22.25

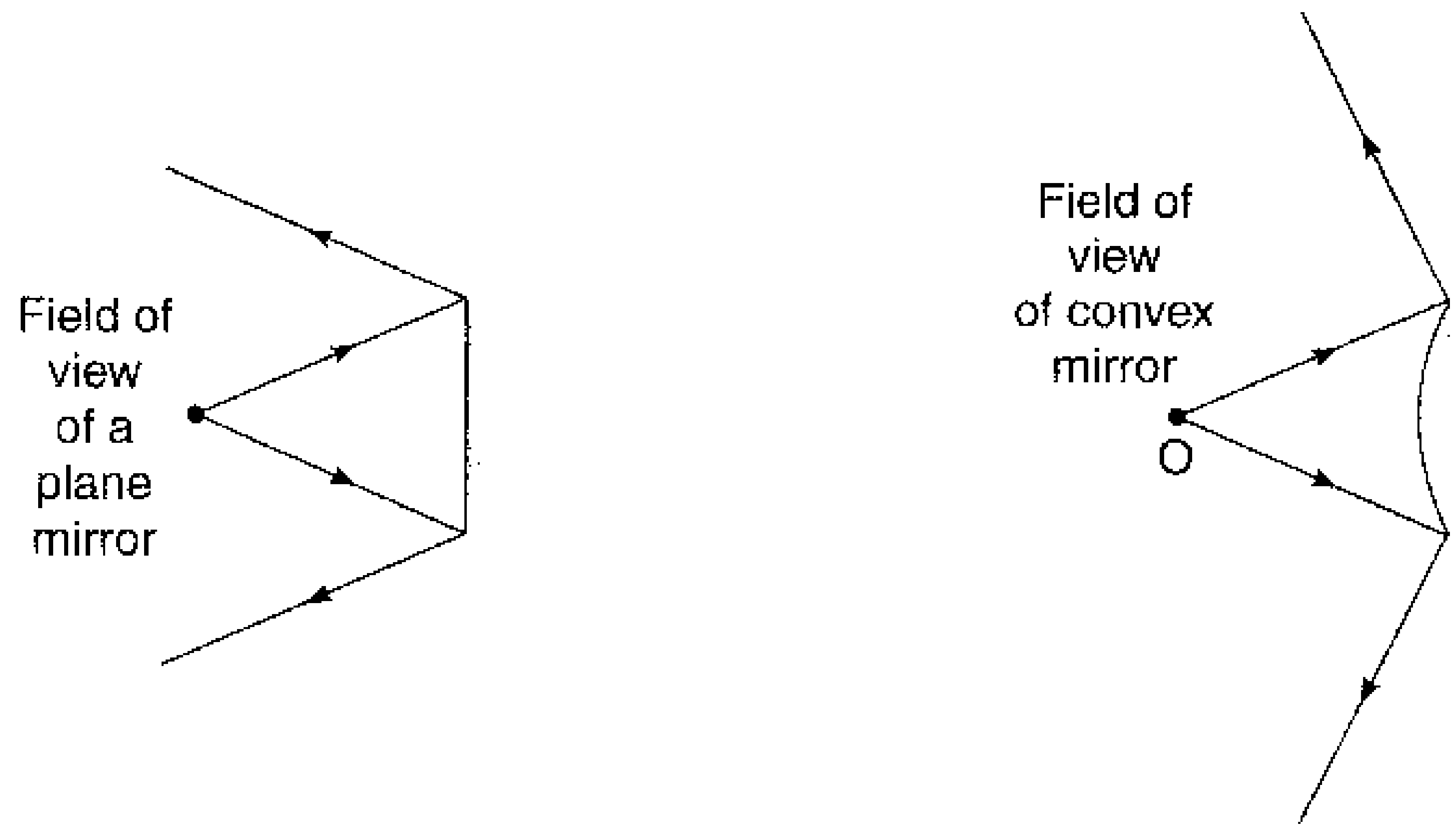


Fig. 22.26

Figure 22.26 shows that convex mirror gives a wider **field of view** than a plane mirror, convex mirrors are therefore, used as rear view mirrors in cars or scooters. Although they make the estimation of distances more difficult but still they are preferred because there is only a small movement of the image for a large movement of the object.

2. In case of a concave mirror the image is erect and virtual when the object is placed between F and P. In all other positions of object the image is real.

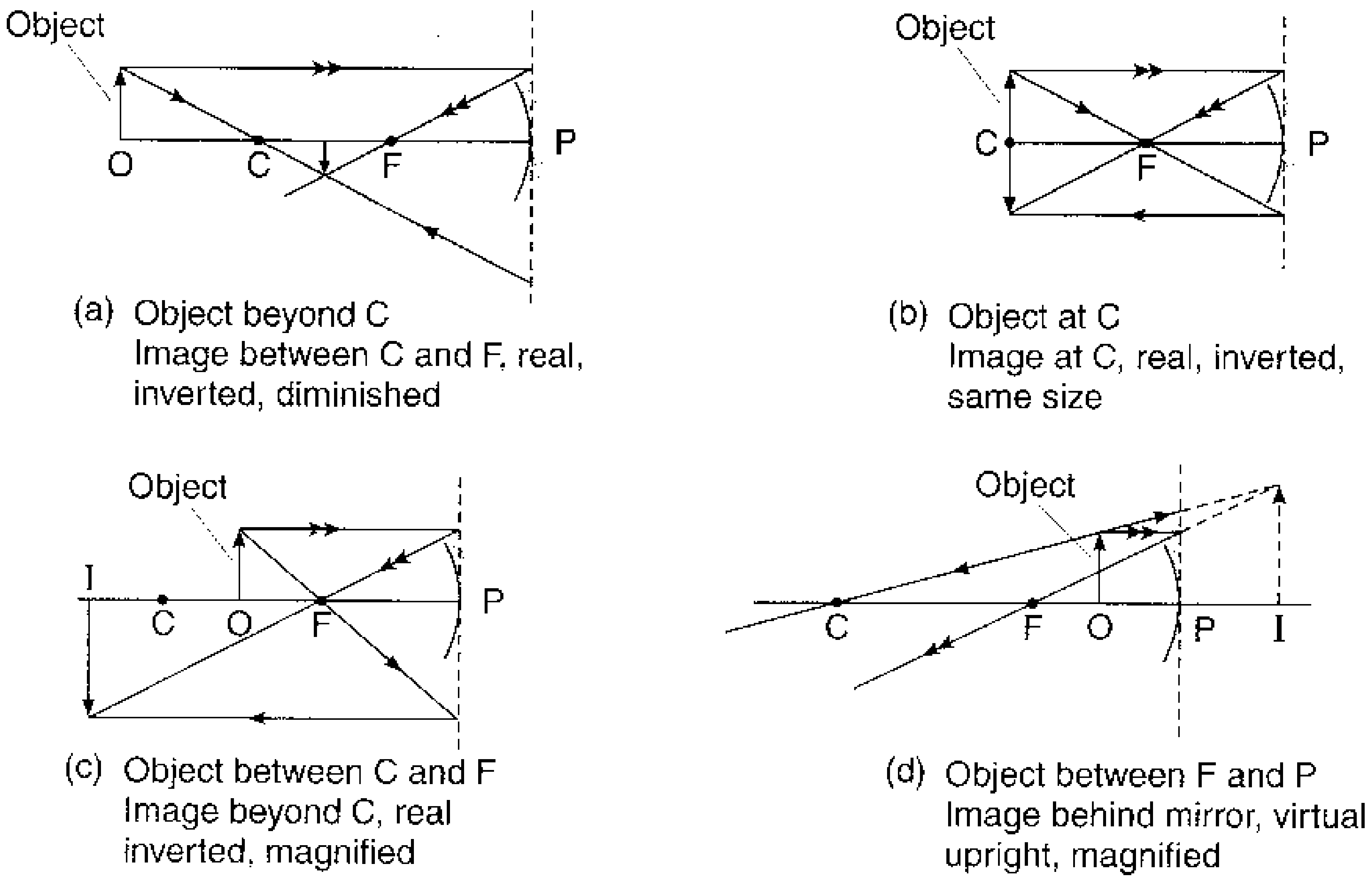


Fig. 22.27

EXAMPLE 22.4 *An image I is formed of a point object O by a mirror whose principal axis is AB as shown in figure.*

(a) State whether it is a convex mirror or a concave mirror.

(b) Draw a ray diagram to locate the mirror and its focus. Write down the steps of construction of the ray diagram. Consider the possible two cases:

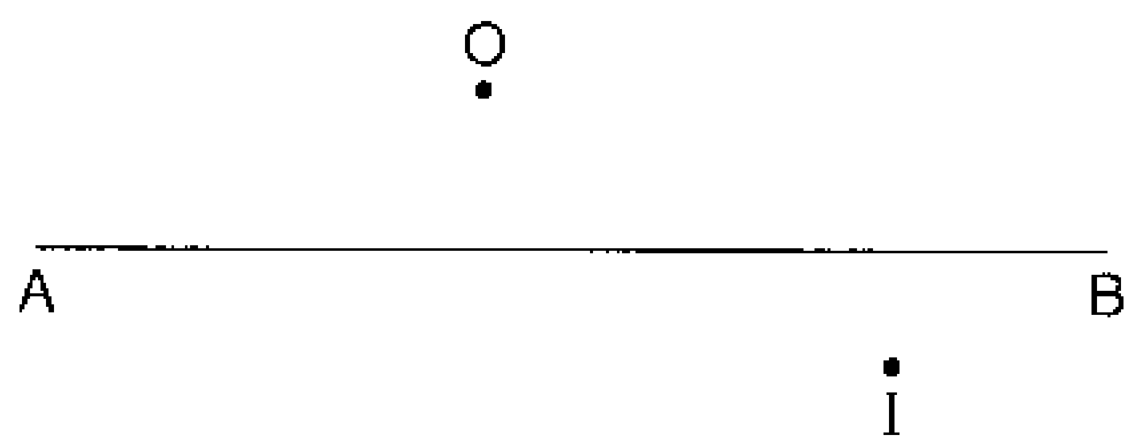


Fig. 22.28

(1) *When distance of I from AB is more than the distance of O from AB and*
(2) *When distance of O from AB is more than the distance of I from AB*

SOLUTION

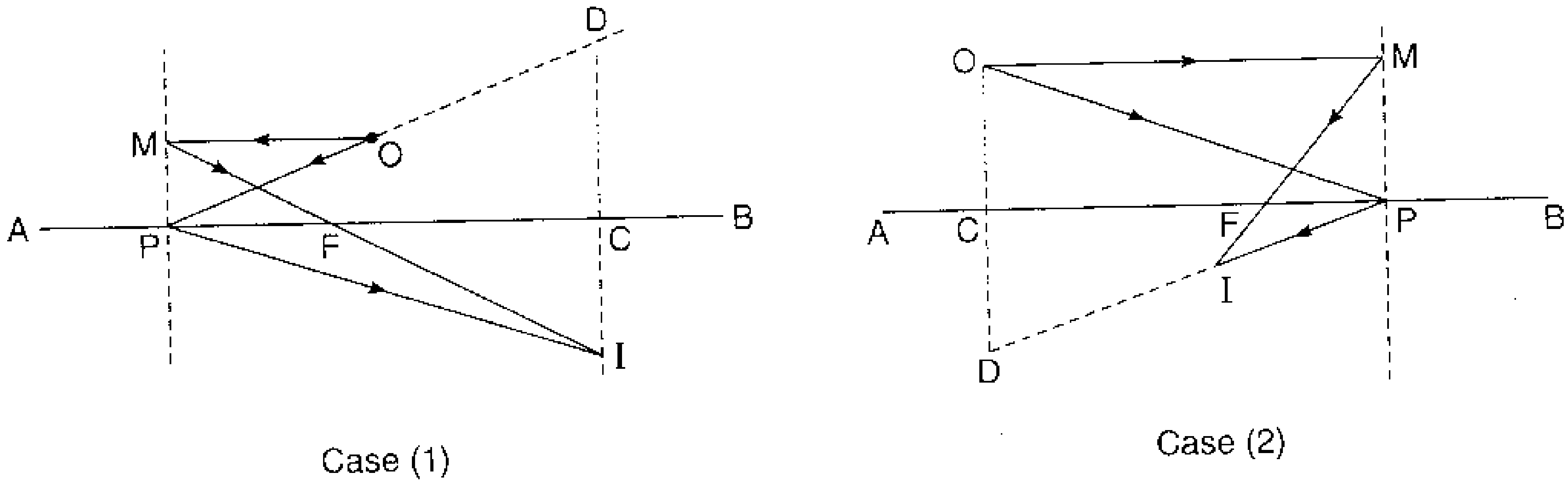


Fig. 22.29

(a) As the image is on the opposite side of the principal axis, the mirror is concave. Because convex mirror always forms an erect image.

(b) Two different cases are shown in figure. Steps are as under:

(i) From I or O drop a perpendicular on principal axis, such that $CI = CD$ or $OC = CD$.

(ii) Draw a line joining D and O or D and I so that it meets the principal axis at P. The point P will be the pole of the mirror as a ray reflected from the pole is always symmetrical about principal axis.

(iii) From O draw a line parallel to principal axis towards the mirror so that it meets the mirror at M. Join M to I, so that it intersects the principal axis at F. F is the focus of the mirror as any ray parallel to principal axis after reflection from the mirror intersects the principal axis at the focus.

The Mirror Formula

In figures 22.30 (a) and (b) a ray OM from a point object O on the principal axis is reflected at M so that the angle θ, made by the incident and reflected rays with the normal CM are equal. A ray OP strikes the mirror normally and is reflected back along PO. The intersection I of the reflected rays MI and PO in figure (a) gives a **real** point image of O and in figure (b) gives a **virtual** point image of O. Let α, β and γ be the angles as shown. As the rays are paraxial, these angles are small, we can take $\alpha \approx \tan\alpha = \frac{MP}{OP}$, $\beta = \frac{MP}{CP}$ and $\gamma = \frac{MP}{IP}$.

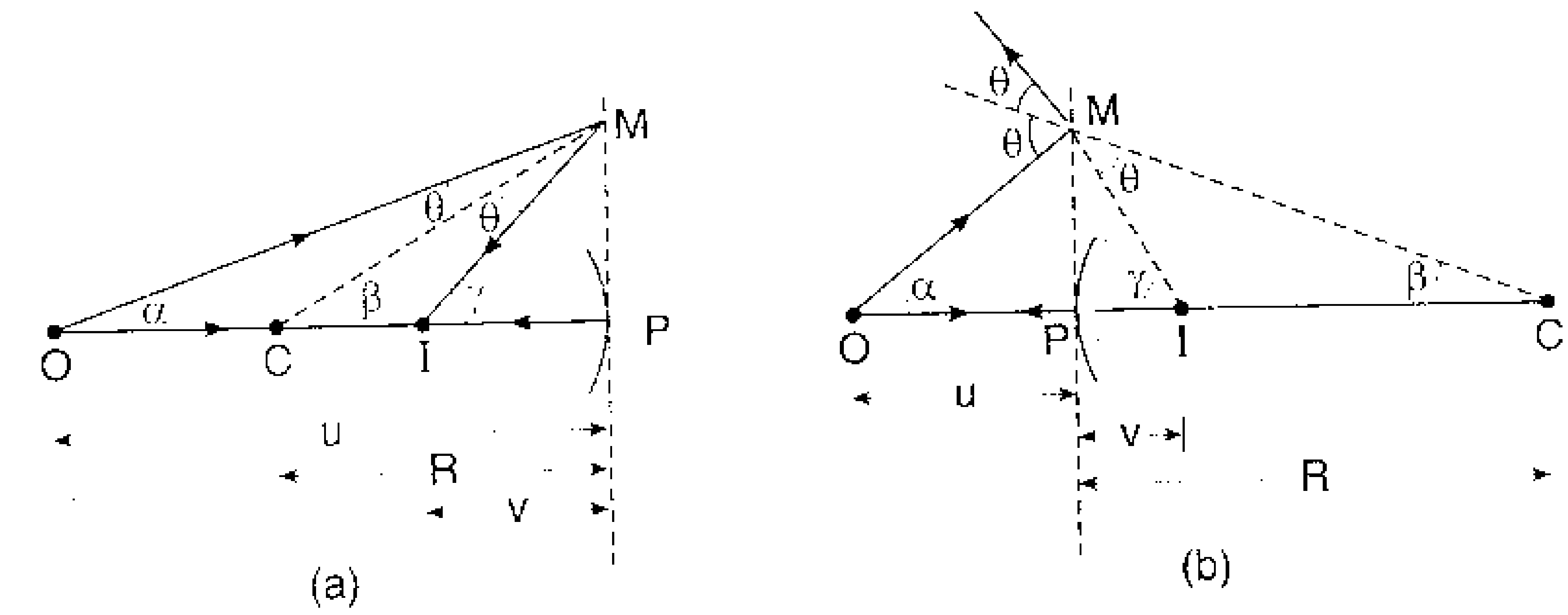


Fig. 22.30

Now, let us take the two figures simultaneously

Concave	Convex
In triangle CMO, $\beta = \alpha + \theta$ (the exterior angle) or $\theta = \beta - \alpha$ ...(i)	In triangle CMO, $\theta = \alpha + \beta$ ...(iv) (the exterior angle)
In ΔCMI, $\gamma = \beta + \theta$ $\therefore$ $\theta = \gamma - \beta$ ...(ii)	In ΔCMI $\gamma = \theta + \beta$ or $\theta = \gamma - \beta$ ...(v)
From Eqs. (i) and (ii) $2\beta = \gamma + \alpha$ ...(iii)	From Eqs. (iv) and (v) $2\beta = \gamma - \alpha$ ...(vi)
Substituting the values of α, β and γ, we get $\frac{2}{CP} = \frac{1}{IP} + \frac{1}{OP}$ ...(A)	Substituting the values of α, β and γ, we get $\frac{2}{CP} = \frac{1}{IP} - \frac{1}{OP}$ ...(B)
If we now substitute the values with sign, *i.e.*, $CP = -R$, $IP = -v$ and $OP = -u$	If we now substitute the values with sign, *i.e.*, $CP = +R$, $IP = +v$ and $OP = -u$, we get
we get, $\frac{2}{R} = \frac{1}{v} + \frac{1}{u}$	$\frac{2}{R} = \frac{1}{v} + \frac{1}{u}$
or $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ as $\left(f = \frac{R}{2}\right)$	or $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ as $\left(f = \frac{R}{2}\right)$

Note : Out of u, v and f values of two will be known to us and we will be asked to find the third. In such type of problems two cases are possible. Case 1 is when signs of all three will be known to us from the given information. Under this condition substitute all three with sign, then answer (*i.e.*, the third quantity) will come without sign. Only numerical value of the unknown comes. Case 2 is when the sign of third unknown quantity is not known to us. Under such situation substitute only the known quantities with sign. Sign with numerical value will automatically come in the answer.

Magnification

The lateral, transverse or linear magnification m is defined as,

$$m = \frac{\text{height of image}}{\text{height of object}} = \frac{I'I}{O'O} = \frac{IP}{OP} \quad ...(i)$$

(From similar triangles)

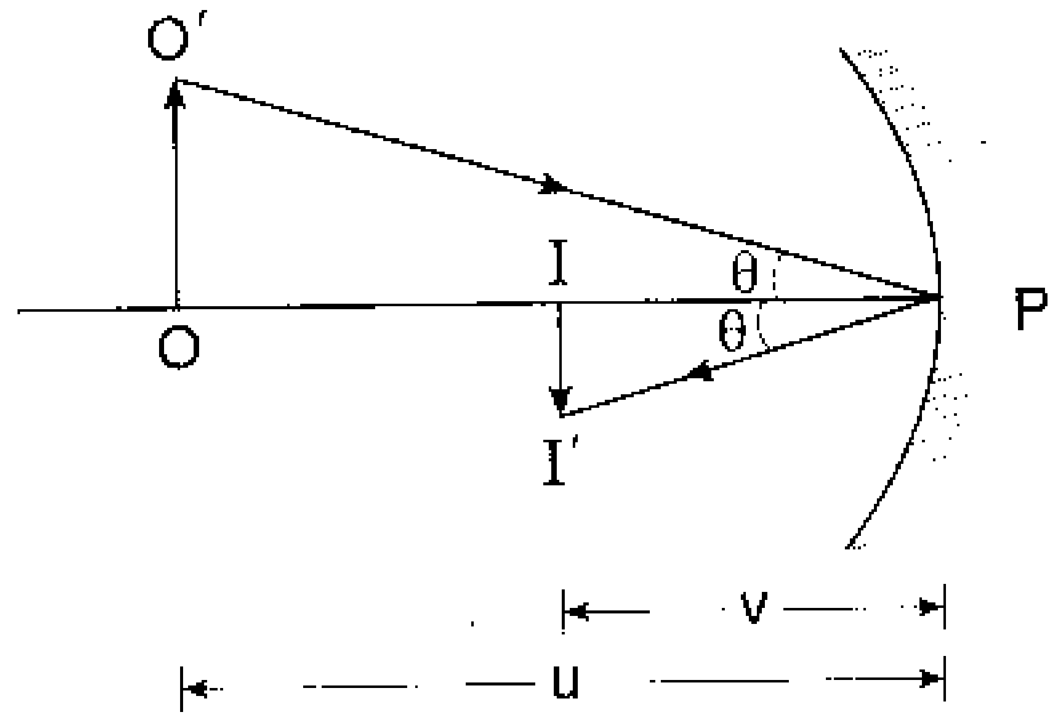


Fig. 22.31

Here, $IP = -v$ and $OP = -u$, further object is erect and image is inverted so we can take $I'I$ as negative and $O'O$ as positive and Eq. (i) will then become

$$\frac{I'I}{O'O} = -\frac{v}{u} \quad \text{or} \quad m = -\frac{v}{u}$$

Note : We have derived $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ and $m = -\frac{v}{u}$ for special cases of the position of object but the same result can be derived for other cases.

EXAMPLE 22.5 *Find the distance of object from a concave mirror of focal length 10 cm so that image size is four times the size of the object.*

SOLUTION Concave mirror can form real as well as virtual image. Here nature of image is not given in the question. So we will consider two possible cases.

Case 1 (when image is real) : Real image is formed on the same side of the object, *i.e.*, u, v and f all are negative. So let,

$$u = -x$$

then $\quad v = -4x \quad$ as $\quad \left|\frac{v}{u}\right| = |m| = 4 \quad$ and $\quad f = -10\text{ cm}$

Substituting in, $$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

we have $$\frac{1}{-4x} - \frac{1}{x} = \frac{1}{-10}$$

or $$\frac{5}{4x} = \frac{1}{10}$$

$\therefore$ $$x = 12.5\text{ cm}$$ **Ans.**

Note : $|x| > |f|$ and we know that in case of a concave mirror, image is real when object lies beyond F.

Case 2 (When image is virtual) : In case of a mirror image is virtual when it is formed behind the mirror, *i.e.*, u and f are negative while v is positive. So let

$$u = -y$$

then $\quad v = +4y \quad$ and $\quad f = -10\text{cm}$

Substituting in $$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

We have $$\frac{1}{4y} - \frac{1}{y} = \frac{1}{-10}$$

or $$\frac{3}{4y} = \frac{1}{10}$$

or $$y = 7.5\text{ cm}$$ **Ans.**

Note : Here $|y| < |f|$, as we know that image is virtual when the object lies between F and P.

IIT-JEE GALAXY 22.2

1. For spherical mirrors

$$m = -\frac{v}{u}$$

Positive value of m means v and u are of opposite sign. So if u is negative then v is positive and *vice-versa*. Thus, if $m = +2$ for a real object, it means image is virtual, erect and two times greater in size. Similarly $m = -\frac{1}{2}$ means image is real, inverted and $\frac{1}{2}$ times in size (that of object).

2. **Method of finding coordinates of image of a point object if the coordinates of object are known :** Suppose coordinates of a point object (x_0, y_0) with respect to the coordinate axes shown in figure are known to us. The coordinates of image (x_i, y_i) can be obtained using

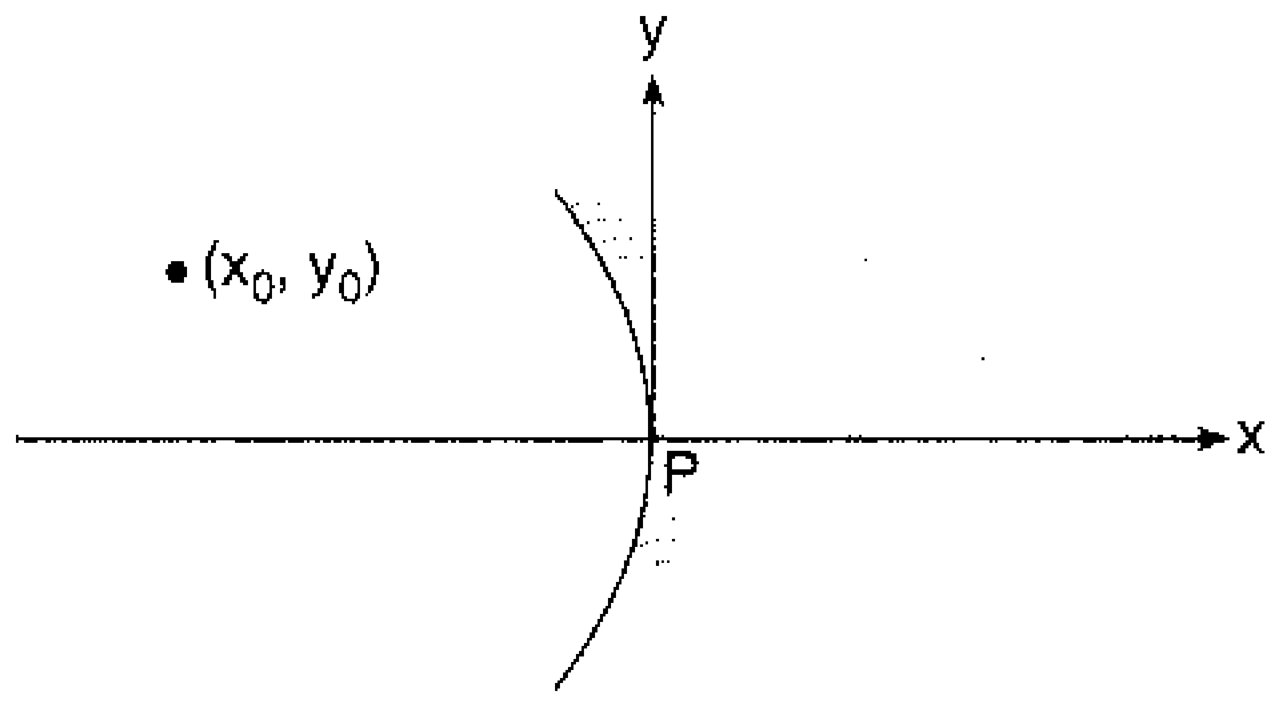


Fig. 22.32

$$\frac{1}{x_0} + \frac{1}{x_i} = \frac{1}{f} \qquad \left(\frac{1}{v} + \frac{1}{u} = \frac{1}{f}\right)$$

or $$x_i = \frac{fx_0}{x_0 - f} \qquad \text{...(i)}$$

Similarly $$m = \frac{y_i}{y_0} = -\frac{v}{u} = -\frac{x_i}{x_O}$$

or $$y_i = \frac{f\,y_0}{f - x_0} \qquad \text{...(iii)}$$

3. **For Concave Mirror**

S. No.	Position of object	Details of image
1.	At ∞	At F, real, inverted $\lvert m \rvert << 1$
2.	Between C and ∞	Between F and C, real, inverted, $\lvert m \rvert < 1$
3.	At C	At C, real, inverted, $\lvert m \rvert = 1$
4.	Between F and C	Between C and ∞, real, inverted, $\lvert m \rvert > 1$
5.	At F	At infinity, real, inverted, $\lvert m \rvert >> 1$
6.	Between F and P	Behind the mirror, virtual, erect $\lvert m \rvert > 1$

For Convex Mirror

S. No.	Position of object	Details of images
1.	At infinity	At F, virtual, erect, $\lvert m \rvert << 1$
2.	In front of mirror	Between P and F, virtual, erect, $\lvert m \rvert < 1$

4. If an object is placed with its length along the principal axis, then so called **longitudinal magnification** becomes,

$$m_L = \frac{I}{O} = -\left(\frac{v_2 - v_1}{u_2 - u_1}\right) = -\frac{dv}{du} \qquad \text{(for small objects)}$$

From, $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ we have

$$-v^{-2}dv - u^{-2}du = 0$$

or $$\frac{dv}{du} = -\left(\frac{v}{u}\right)^2$$

or $$m_L = -\frac{dv}{du} = \left(\frac{v}{u}\right)^2 = m^2$$

5. If we differentiate the mirror formula

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

with respect to time, we get

$$-v^{-2}.\frac{dv}{dt}-u^{-2}\frac{du}{dt}=0 \qquad \text{(as } f = \text{constant)}$$

or
$$\frac{dv}{dt}=-\left(\frac{v^2}{u^2}\right)\frac{du}{dt} \qquad \ldots\text{(iii)}$$

Here $\frac{du}{dt}$ is the rate by which u is changing. Or it is the object speed if mirror is stationary. Similarly, $\frac{dv}{dt}$ is the rate by which v (distance between image and mirror) is changing. Or it is image speed if mirror is stationary. So if at a known values of v and u, the object speed is given, we can find the image speed from the above formula.

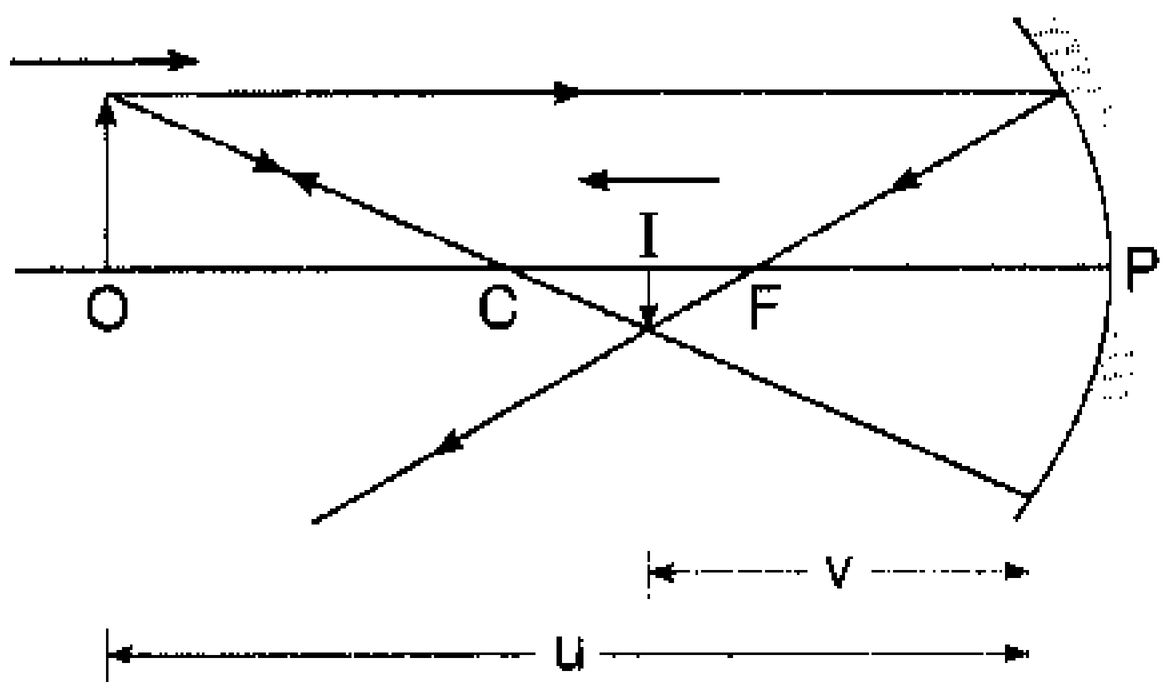


Fig. 22.33

Let us take an example for a concave mirror.

Suppose the object is moved from infinity towards focus.

As u is decreasing therefore,

$$\left(-\frac{du}{dt}\right)= \text{rate of decrease of } u \qquad \text{(speed of object)}$$

Therefore,
$$\left(\frac{dv}{dt}\right)= \text{rate of increase of } v \qquad \text{(speed of image)}$$

Further $v < u$ (when the object lies between ∞ and C)

$$\therefore \quad \left(\frac{dv}{dt}\right)<\left(-\frac{du}{dt}\right) \qquad \text{[from Eq. (iii)]}$$

Hence, as the object is moved towards mirror the image (which is real) will recede from the mirror with speed less than the speed of object.

When the object is at C, image is also at C, *i.e.*, $v = u$ or $\left(\frac{dv}{dt}\right)=\left(-\frac{du}{dt}\right)$. Hence, speed of image is equal to the speed of object. When the object lies between C and F. $v > u$, *i.e.*, image speed is more than the object speed when object comes inside F, image becomes virtual *i.e.*, u and f are negative while v is positive.

Hence,
$$\frac{1}{v}-\frac{1}{u}=-\frac{1}{f}$$

or
$$\frac{1}{u}-\frac{1}{v}=\frac{1}{f}$$

or
$$-u^{-2}\left(\frac{du}{dt}\right)-v^{-2}\left(\frac{dv}{dt}\right)=0$$

$$\left(-\frac{dv}{dt}\right)=\left(\frac{v^2}{u^2}\right)\left(-\frac{du}{dt}\right)$$

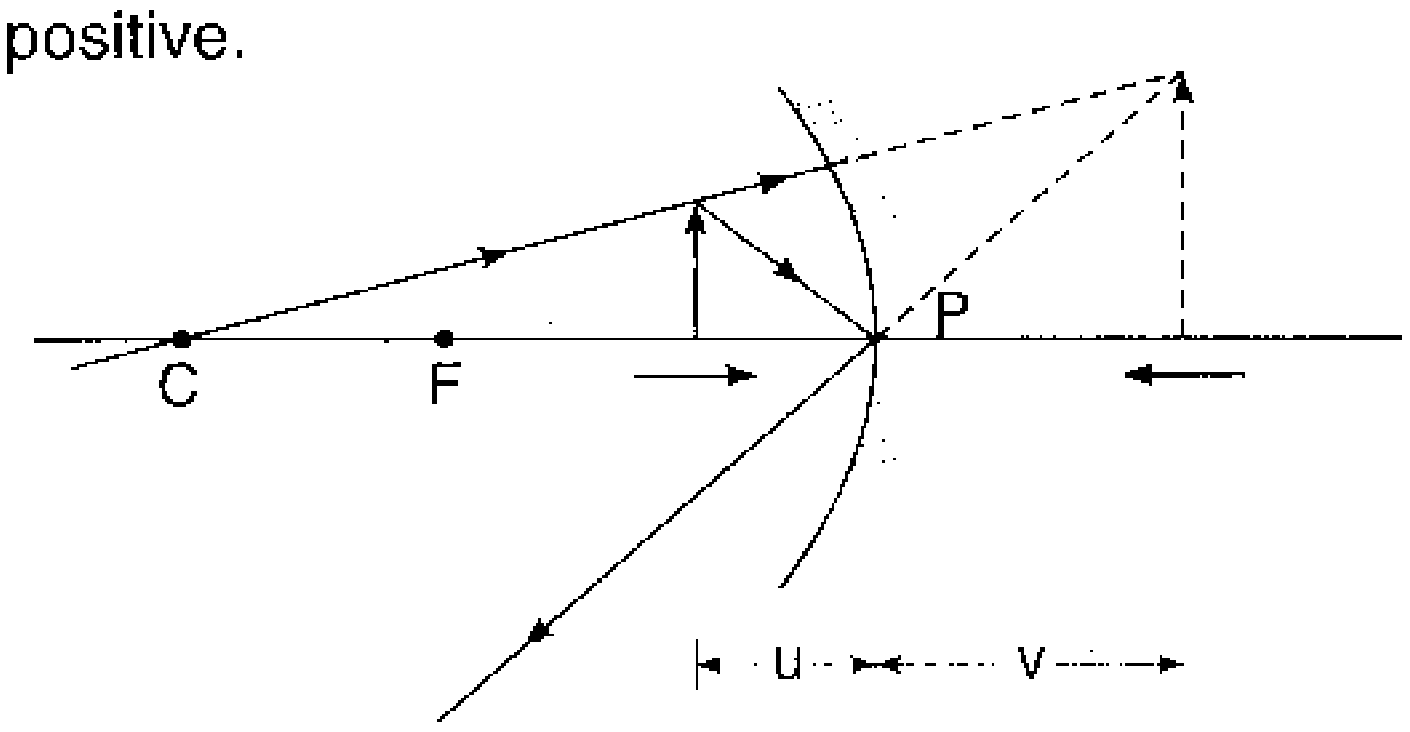


Fig. 22.34

Now as u is further decreased, v also decreases to keep $\frac{1}{f}$ constant. So, $-\frac{du}{dt}$ is rate with which object is approaching towards mirror and $\left(-\frac{dv}{dt}\right)$ is rate by which image is approaching towards mirror. Further in this case we know that image is always enlarged or $v > u$. Therefore, image speed is more than the object speed. Thus, we may conclude the above discussion as under:

When an object is moved from $-\infty$ to F, the image (real) moves from F to $-\infty$ and then when the object is further moved from F to P image (now virtual) moves from $+\infty$ to P.

Note : when the object is either at centre of curvature C or at pole P, the two speeds are equal. When the object is at pole it can be assumed as if the image is forming by a plane mirror due to the small aperture of the mirror.

Exercise : Do the above exercise with a convex mirror.

6. **Graph between $\frac{1}{v}$ versus $\frac{1}{u}$:** Let us first take the case of a concave mirror. Here, two cases are possible.

Case 1. When the image is real, *i.e.*, object lies between F and infinity. In such a situation u, v and f are negative. Hence, the mirror formula

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f} \quad \text{becomes}$$

$$-\frac{1}{v}-\frac{1}{u}=-\frac{1}{f}$$

or again
$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

or
$$\frac{1}{v}=-\frac{1}{u}+\frac{1}{f}$$

Comparing with $y=-x+c$, the desired graph will be a straight line with slope -1 and slope equal to $\frac{1}{f}$.

The corresponding graph is shown in figure 22.35.

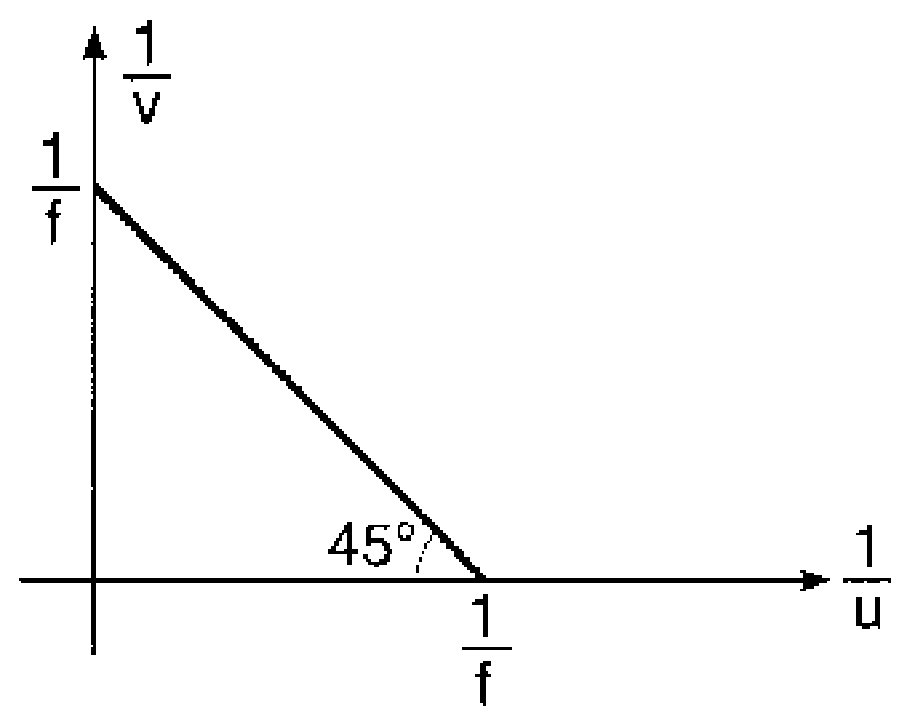


Fig. 22.35

Case 2. When the image is virtual, *i.e.*, object lies between F and P. Under such situation u and f are negative while v is positive. The mirror formula thus becomes

$$\frac{1}{v}=\frac{1}{u}-\frac{1}{f}$$

Comparing it with $y=x-c$ the desired graph is a straight line with slope $+1$ and intercept $-\frac{1}{f}$. The graph is thus shown in Fig. 22.36.

The two graphs can be drawn in one single graph as in Fig. 22.37.

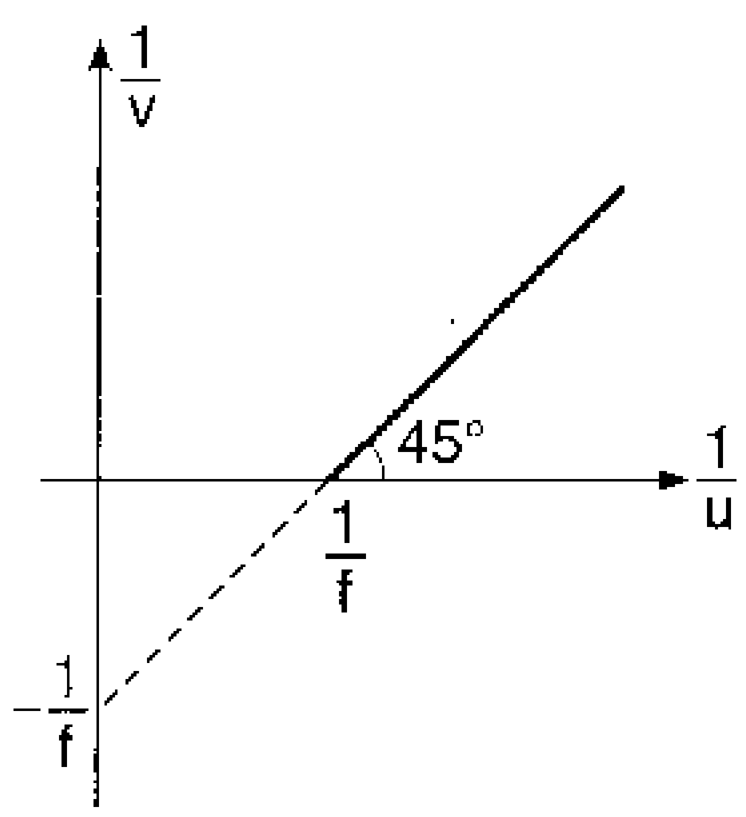


Fig. 22.36

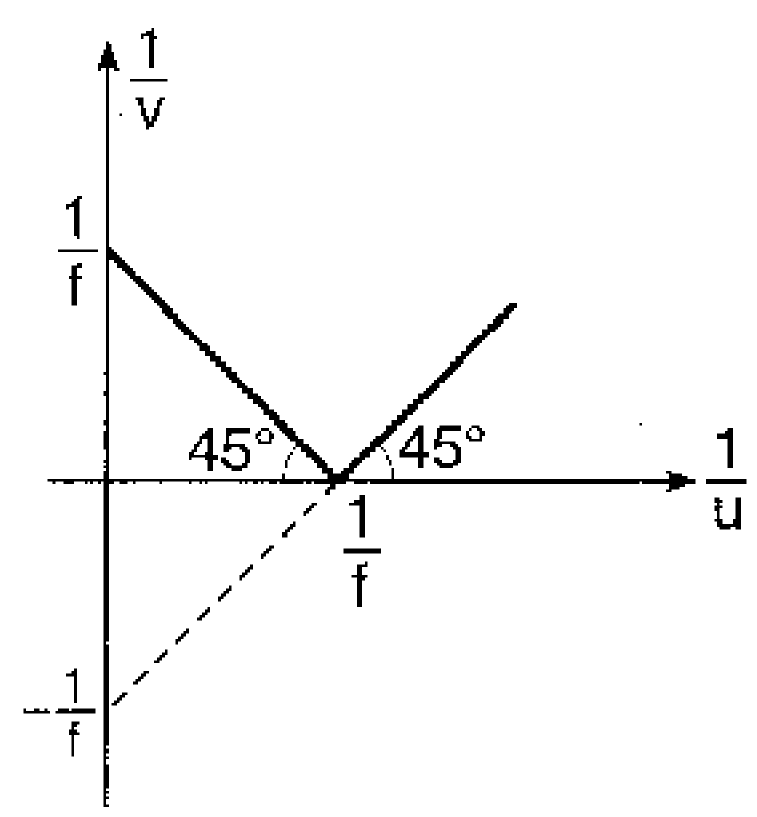


Fig. 22.37

Note : Here $\frac{1}{u}$ and $\frac{1}{v}$ are really the magnitudes of $\frac{1}{u}$ and $\frac{1}{v}$ (*i.e.*, without sign)

For a **convex mirror** image is always virtual, *i.e.*, u is negative while v and f are positive. Hence, the mirror formula becomes,

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{f}$$

Comparing with $y = x + c$, the desired graph is a straight line of slope + 1and intercept $\frac{1}{f}$. The graph is thus shown in Fig. 22.38.

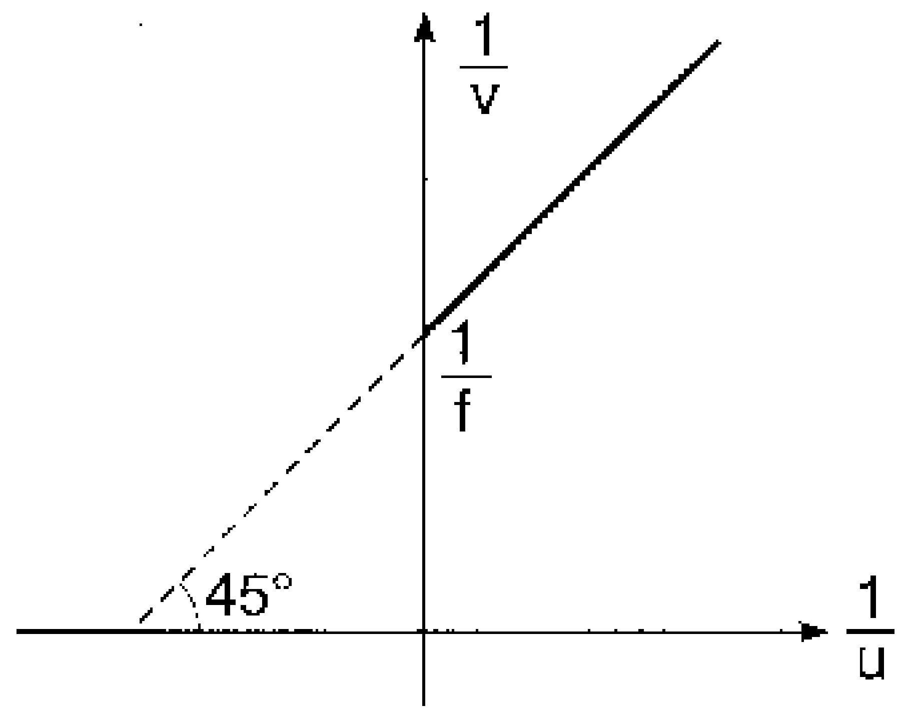


Fig. 22.38

EXAMPLE 22.6 *An object ABED is placed in front of a concave mirror beyond centre of curvature C as shown in figure. State the shape of the image.*

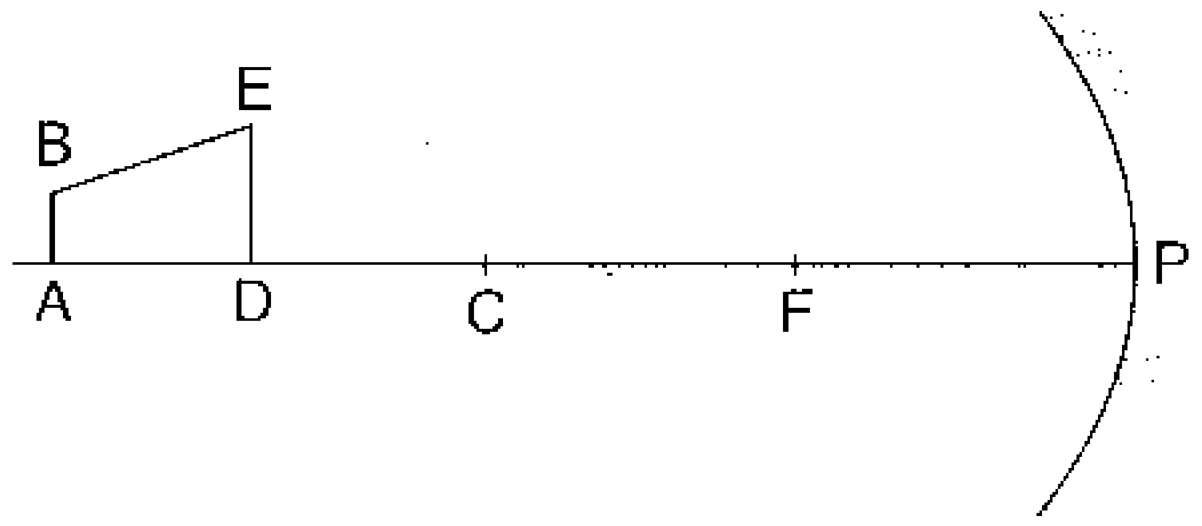


Fig. 22.39

SOLUTION Object is placed beyond C. Hence, the image will be real and it will lie between C and F. Further u, v and f all are negative, hence the mirror formula will become

$$-\frac{1}{v} - \frac{1}{u} = -\frac{1}{f}$$

or $$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{u-f}{uf}$$

or $$v = \frac{f}{1 - \frac{f}{u}}$$

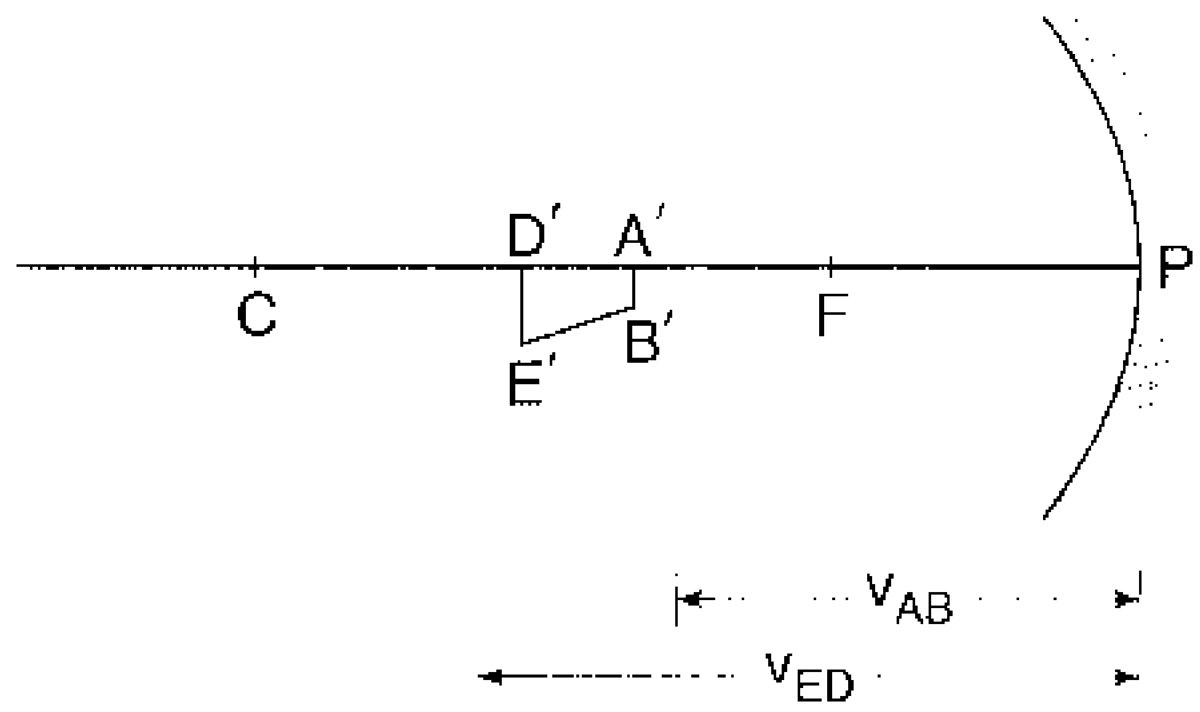


Fig. 22.40

Now $$u_{AB} > u_{ED}$$

∴ $$v_{AB} < v_{ED}$$

and $$|m_{AB}| < |m_{ED}| \qquad \left(\text{as } m = -\frac{v}{u}\right)$$

Therefore, shape of the image will be as shown in figure 22.40.

Also note that $v_{AB} < u_{AB}$ and $v_{ED} < u_{ED}$,

So $$|m_{AB}| < 1 \quad \text{and} \quad |m_{ED}| < 1$$

EXAMPLE 22.7 *A gun of mass m_1 fires a bullet of mass m_2 with a horizontal speed v_0. The gun is fitted with a concave mirror of focal length f facing towards a receding bullet. Find the speed of separations of the bullet and the image just after the gun was fired.*

SOLUTION Let v_1 be the speed of gun (or mirror) just after the firing of bullet. From conservation of linear momentum,

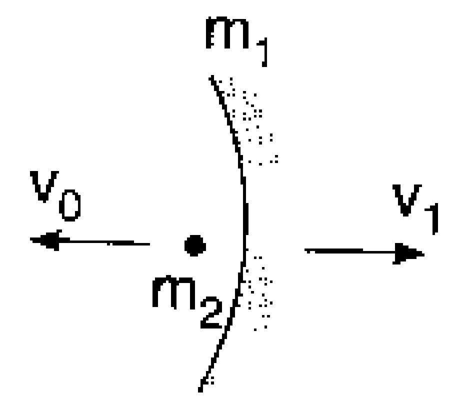


Fig. 22.41

$$m_2 v_0 = m_1 v_1$$

or
$$v_1 = \frac{m_2 v_0}{m_1} \qquad \text{...(i)}$$

Now, $\frac{du}{dt}$ = rate at which distance between mirror and bullet is increasing

$$= v_1 + v_0 \qquad \text{...(ii)}$$

From Eq. (iv) of III-JEE Galaxy 22.2 point number (5)

$$\therefore \qquad \frac{dv}{dt} = \left(\frac{v^2}{u^2}\right)\frac{du}{dt}$$

Here $\frac{v^2}{u^2} = m^2 = 1$ (as at the time of firing bullet is at pole).

$$\therefore \qquad \frac{dv}{dt} = \frac{du}{dt} = v_1 + v_0 \qquad \text{...(iii)}$$

Here $\frac{dv}{dt}$ is the rate at which distance between image (of bullet) and mirror is increasing. So if v_2 is the absolute velocity of image (towards right) then,

$$v_2 - v_1 = \frac{dv}{dt} = v_1 + v_0$$

or
$$v_2 = 2v_1 + v_0 \qquad \text{...(iv)}$$

Therefore, speed of separation of bullet and image will be,

$$v_r = v_2 + v_0 = 2v_1 + v_0 + v_0$$

or
$$v_r = 2(v_1 + v_0)$$

Substituting value of v_1 from equation (i) we have,

$$v_r = 2\left(1 + \frac{m_2}{m_1}\right)v_0$$

Ans.

INTRODUCTORY EXERCISE 22.3

1. Assume that a certain spherical mirror has a focal length of – 10.0 cm. Locate and describe the image for object distances of
   (a) 25.0 cm (b) 10.0 cm (c) 5.0 cm.
2. A ball is dropped from rest 3.0 m directly above the vertex of a concave mirror that has a radius of 1.0 m and lies in a horizontal plane.
   (a) Describe the motion of ball's image in the mirror.
   (b) At what time do the ball and its image coincide?
3. A spherical mirror is to be used to form on a screen 5.0 m from the object an image five times the size of the object.

(a) Describe the type of mirror required.
(b) Where should the mirror be positioned relative to the object?

4. Figure shows two rays P and Q being reflected by a mirror and going as P' and Q'. State which type of mirror is this?

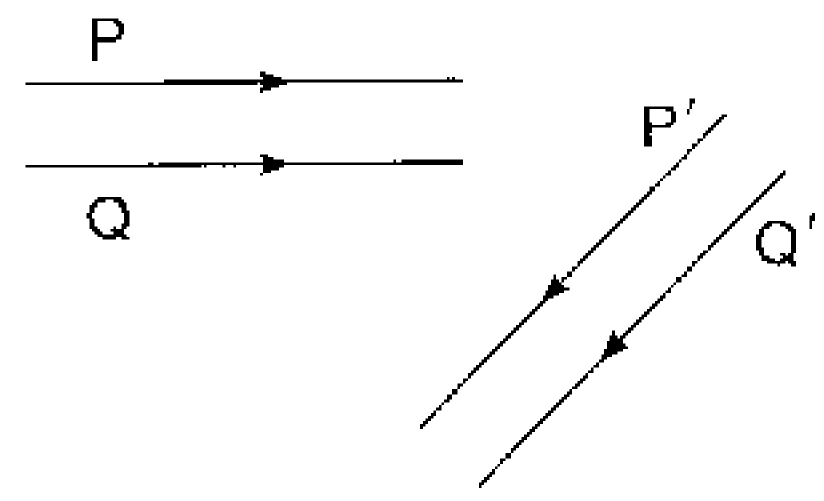


Fig. 22.42

5. The following table shows object distance, object size, and mirror focal length (all in centimeters) for various situations. Use ray tracing to determine the image size and location for each case.

Object Distance	Object Size	Focal Length
40	2.0	20
10	1.0	20
30	1.5	–30
20	2.0	–40

6. Complete the following table for spherical mirrors. All distances are in centimeters.

Mirror Type	Radius	Focal Length	Object Distance	Image Distance	Lateral Magnification	Real/Virtual Image
		– 25	40			
			20	– 30		
		+ 15		+ 30		
			20		– 2.0	
	+ 20			– 40		
		– 10			+ 2.0	

22.5 REFRACTION OF LIGHT

(a) Laws of Refraction

When light passes from one medium, say air, to another, say glass, part is reflected back into the first medium and the rest passes into the second medium. When it passes into the second medium, its direction of travel is changed. It either bends towards the normal or away from the normal. This phenomenon is known as **refraction**. There are two laws of refraction:

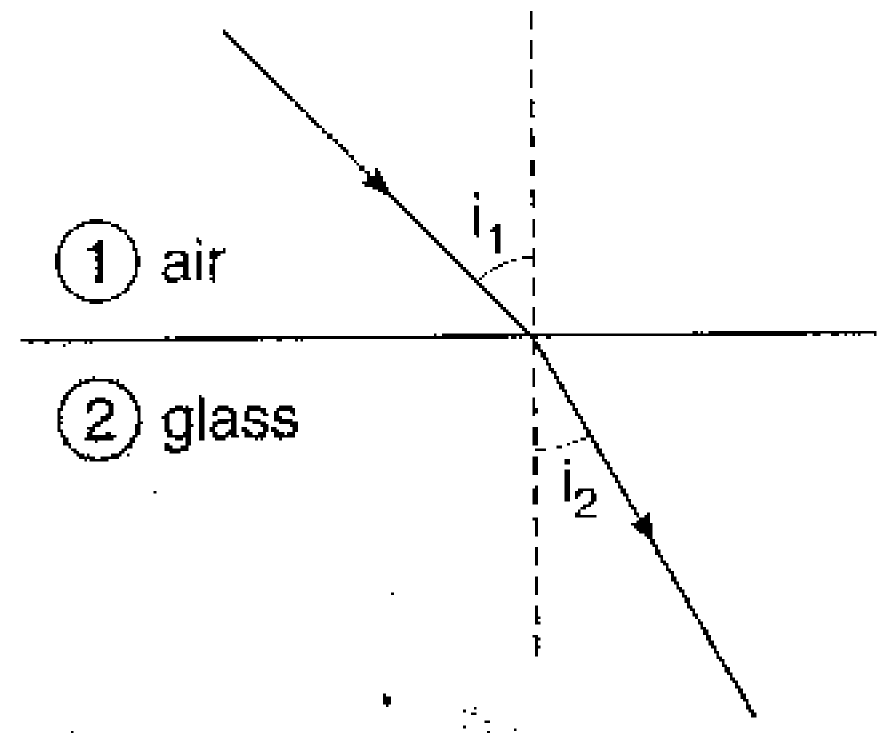


Fig. 22.43

1. For two particular media, the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant, *i.e.*,

$$\frac{\sin i_1}{\sin i_2} = \text{constant} \qquad \text{...(i)}$$

This is known as **Snell's law**.

2. The incident ray, the reflected ray and the refracted ray all lie in the same plane.

Notes : (i) The constant ratio $\frac{\sin i_1}{\sin i_2}$ is called the **refractive index** for light passing from the first to the second medium (or refractive index of 2 with respect to 1). It is denoted by ${}_1\mu_2$. Thus,

$${}_1\mu_2 = \frac{\sin i_1}{\sin i_2} \qquad \text{...(ii)}$$

(ii) If medium 1 is a vacuum (or, in practice air) we refer ${}_1\mu_2$ as the **absolute refractive index** of medium 2 and denote it by μ_2 or simply μ (if no other medium is there).

(iii) Now, we can write Snell's law as,

$$\mu \sin i = \text{constant} \qquad \text{...(iii)}$$

For two media,

$$\mu_1 \sin i_1 = \mu_2 \sin i_2$$

or

$$\frac{\mu_2}{\mu_1} = \frac{\sin i_1}{\sin i_2} = {}_1\mu_2 \qquad \text{...(iv)}$$

(iv) From Eq. (iii) we can see that $i_1 > i_2$ if $\mu_2 > \mu_1$, *i.e.*, if a ray of light passes from a rare to a denser medium it bends towards normal and *vice-versa*.

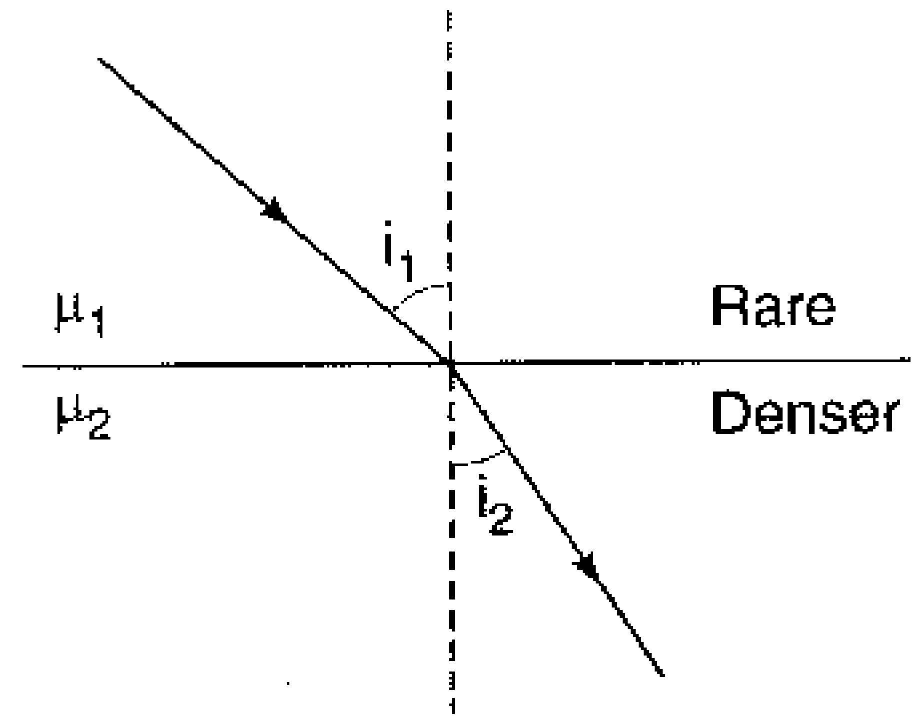


Fig. 22.44

(v) Eq. (iv) can be written as,

$${}_1\mu_2 = \frac{\sin i_1}{\sin i_2} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{\mu_2}{\mu_1} \qquad \text{...(v)}$$

Here v_1 is the speed of light in medium 1 and v_2 in medium 2. Similarly λ_1 and λ_2 are the corresponding wavelengths.

If $\mu_2 > \mu_1$ then $v_1 > v_2$ and $\lambda_1 > \lambda_2$, *i.e.*, in a rare medium speed and hence, wavelength of light is more.

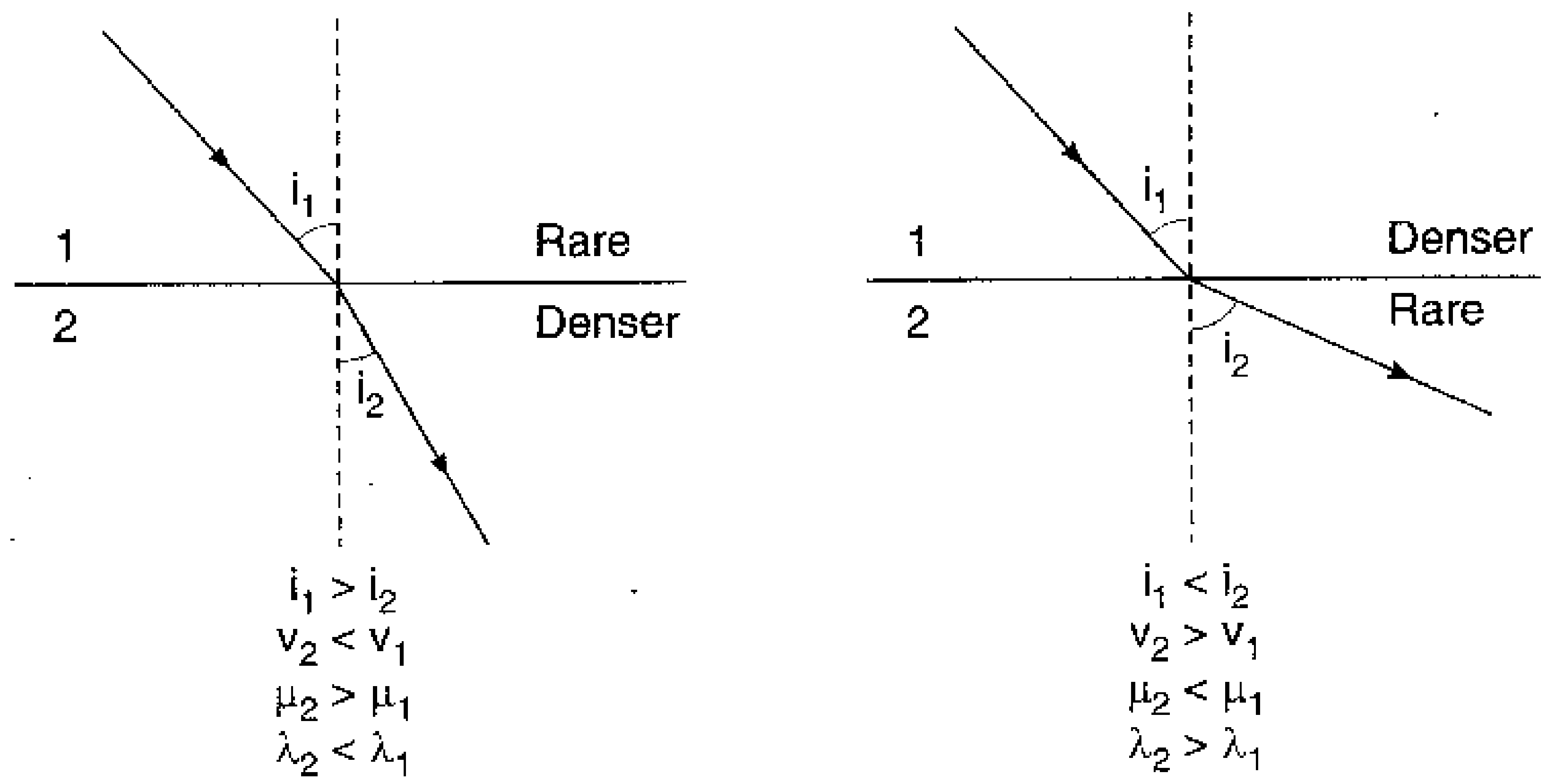


Fig. 22.45

(vi) In general speed of light in any medium is less than its speed in vacuum. It is convenient to define refractive index μ of a medium as,

$$\mu = \frac{\text{speed of light in vacuum}}{\text{speed of light in medium}} = \frac{c}{v}$$

(vii) As a ray of light moves from medium 1 to medium 2, its wavelength changes but its frequency remains constant.

$$\mu_2 > \mu_1,\ v_1 > v_2,\ \lambda_1 > \lambda_2$$

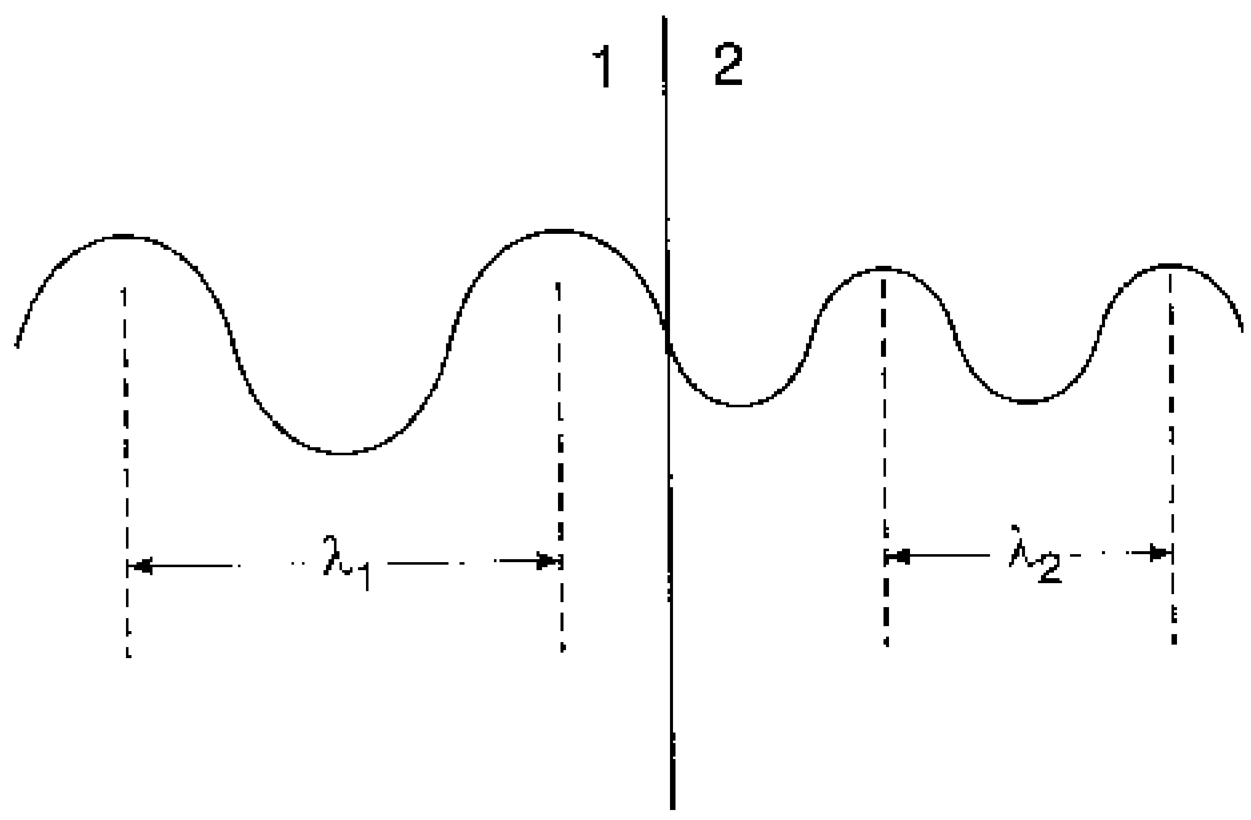


Fig. 22.46

(viii) ${}_1\mu_2 = \frac{\mu_2}{\mu_1}$ and ${}_2\mu_1 = \frac{\mu_1}{\mu_2}$

$\therefore$ $${}_1\mu_2 = \frac{1}{{}_2\mu_1}$$

(ix) ${}_1\mu_2 = \frac{\mu_2}{\mu_1}$, ${}_2\mu_3 = \frac{\mu_3}{\mu_2}$ and ${}_3\mu_1 = \frac{\mu_1}{\mu_3}$

$\therefore$ $${}_1\mu_2 \times {}_2\mu_3 \times {}_3\mu_1 = 1$$

(x) Experiments show that if the boundaries of the media are **parallel** the emergent ray *CD* although laterally displaced, is parallel to the incident ray *AB* if $\mu_1 = \mu_5$. We can also directly apply the Snell's law ($\mu \sin i$ = constant) in medium 1 and 5, *i.e.*,

$$\mu_1 \sin i_1 = \mu_5 \sin i_5$$

So, $i_1 = i_5$ if $\mu_1 = \mu_5$

If any of the boundary is not parallel we cannot use this law directly by jumping the intervening media.

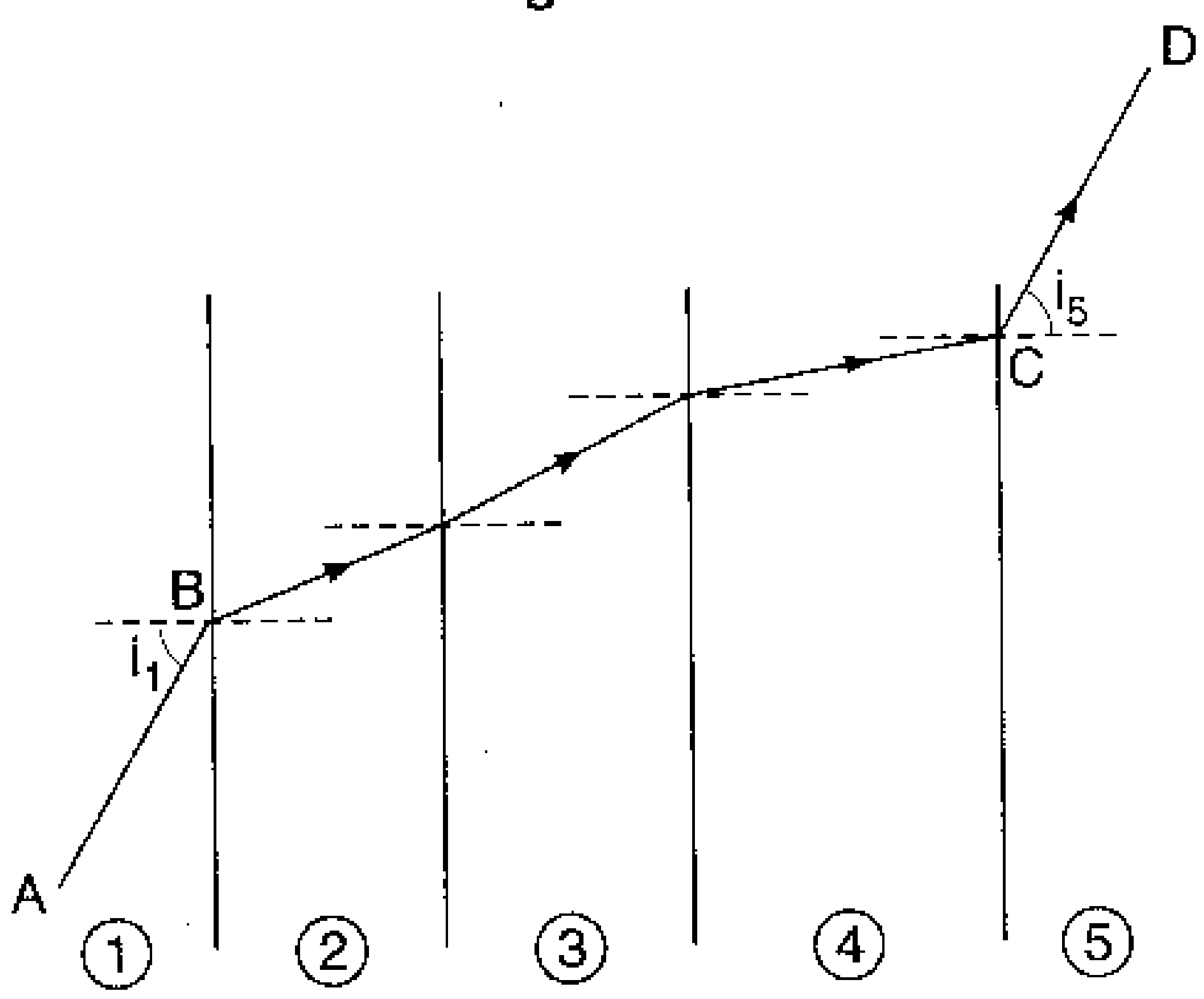


Fig. 22.47

EXAMPLE 22.8 *Refractive index of glass with respect to water is (9/8). Refractive index of glass with respect to air is (3/2). Find the refractive index of water with respect to air.*

SOLUTION Given, ${}_w\mu_g = 9/8$ and ${}_a\mu_g = 3/2$

As, $${}_a\mu_g \times {}_g\mu_w \times {}_w\mu_a = 1$$

$\therefore$ $$\frac{1}{{}_w\mu_a} = {}_a\mu_w = {}_a\mu_g \times {}_g\mu_w = \frac{{}_a\mu_g}{{}_w\mu_g}$$

$\therefore$ $${}_a\mu_w = \frac{3/2}{9/8} = \frac{4}{3}$$ **Ans.**

EXAMPLE 22.9 (a) *Find the speed of light of wavelength $\lambda = 780$ nm (in air) in a medium of refractive index $\mu = 1.55$.*

(b) *What is the wavelength of this light in the given medium?*

SOLUTION (a) $$v = \frac{c}{\mu} = \frac{3.0 \times 10^8}{1.55} = 1.94 \times 10^8 \text{ m/s}$$ **Ans.**

(b) $$\lambda_{\text{medium}} = \frac{\lambda_{\text{air}}}{\mu} = \frac{780}{1.55} = 503 \text{ nm}$$ **Ans.**

EXAMPLE 22.10 *A light beam passes from medium 1 to medium 2. Show that the emerging beam is parallel to the incident beam.*

SOLUTION Applying Snell's law at A and B,

$$\mu_1 \sin i_1 = \mu_2 \sin i_2$$

or $$\frac{\mu_1}{\mu_2} = \frac{\sin i_2}{\sin i_1} \qquad \ldots\text{(i)}$$

Similarly $$\mu_2 \sin i_2 = \mu_1 \sin i_3$$

$\therefore$ $$\frac{\mu_1}{\mu_2} = \frac{\sin i_2}{\sin i_3} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii) $$i_3 = i_1$$

i.e., the emergent ray is parallel to incident ray. **Proved**

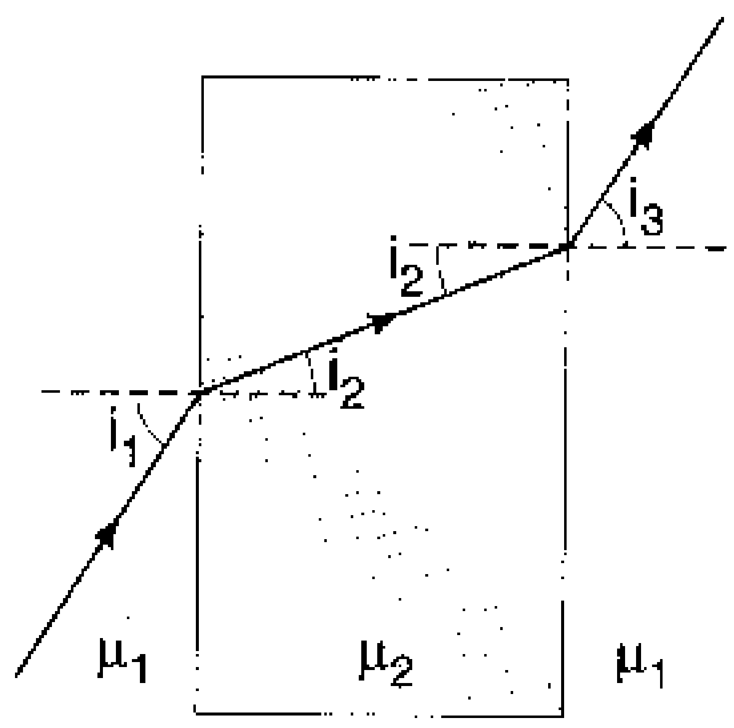


Fig. 22.48

(b) Single Refraction from a Plane Surface

Following four results can be drawn after refraction from a plane surface.

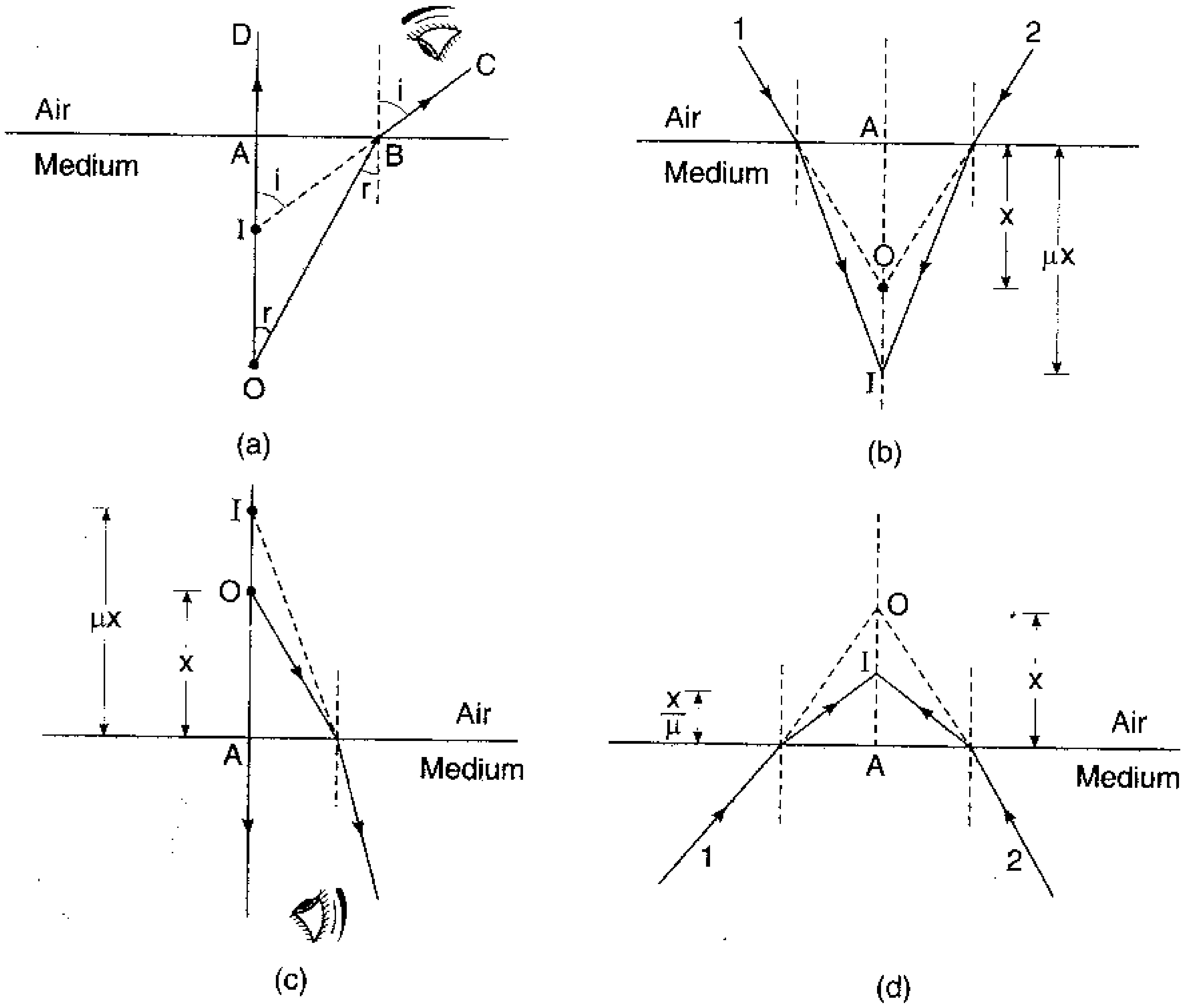


Fig. 22.49

In all the four figures, single refraction is taking place through a plane surface. Refractive index of medium (may be glass, water etc.) is μ. In figures (a) and (d) the ray of light is travelling from denser to rare medium and hence, it bends away from the normal. In figures (b) and (c) the ray of light is travelling from a rare to a denser medium and hence, it bends towards the normal. Now, let us take the four figures individually.

Refer figure (a) : Object O is placed at a distance x from A. Ray OA, which falls normally on the plane surface, passes undeviated as AB. Ray OB, which falls at angle r (with the normal) on the plane surface, bends away from the normal and passes as BC in air. Rays AD and BC meet at I after extending these two rays backwards. Let BC makes an angle i ($> r$) with normal.

In the figure $\angle AOB$ will be r and $\angle AIB$ is i. For normal incidence (*i.e.*, small angles of i and r)

$$\sin i \approx \tan i = \frac{AB}{AI} \qquad \ldots\text{(i)}$$

and

$$\sin r \approx \tan r = \frac{AB}{AO} \qquad \ldots\text{(ii)}$$

Dividing Eq. (i) by (ii), we have

$$\frac{\sin i}{\sin r} = \frac{AO}{AI}$$

or

$$\mu = \frac{AO}{AI} \qquad \left(\text{as } \frac{\sin i}{\sin r} = \mu\right)$$

$\therefore$

$$AI = \frac{AO}{\mu} = \frac{x}{\mu}$$

If point O is at a depth of d from a water surface, then the above result is also sometimes written as,

$$d_{\text{apparent}} = \frac{d_{\text{actual}}}{\mu}$$

or the apparent depth is μ times less than the actual depth.

Refer figure (b) : In the absence of the plane refracting surface the two rays 1 and 2 would had met at O. Proceeding in the similar manner we can prove that after refraction from the plane surface they will now meet a point I where,

$$AI = \mu x \qquad (\text{if } AO = x)$$

Refer figure (c) : In this case object is at O, a distance $AO = x$ from the plane surface. When seen from inside the medium it will appear at I, where

$$AI = \mu x$$

Refer figure (d) : The two rays 1 and 2 meeting at O will now meet at I after refraction from the plane surface. Where $AI = \frac{AO}{\mu} = \frac{x}{\mu}$.

Note : In all the four cases the change in the value of x is μ times whether it is increasing or decreasing. All the relations can be derived for small angles of incidence as done in part (a).

EXERCISE. Three immiscible liquids of refractive indices μ_1, μ_2 and μ_3 (with $\mu_3 > \mu_2 > \mu_1$) are filled in a vessel. Their depths are d_1, d_2 and d_3 respectively. Prove that the apparent depth (for normal incidence) when seen from top of the first liquid will be,

$$d_{app} = \frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} + \frac{d_3}{\mu_3}$$

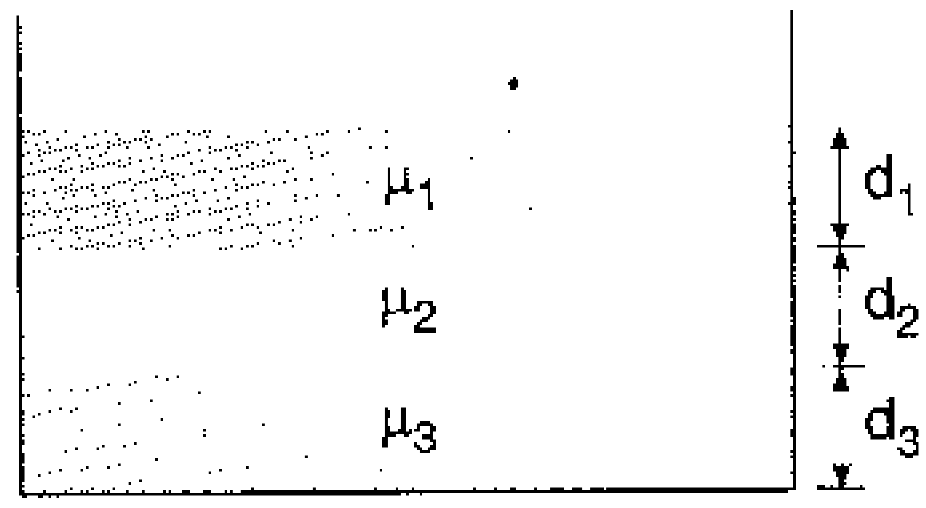


Fig. 22.50

(c) Shift due to a Glass Slab (Double Refraction from Plane Surfaces)

(i) Normal Shift : Here again two cases are possible.

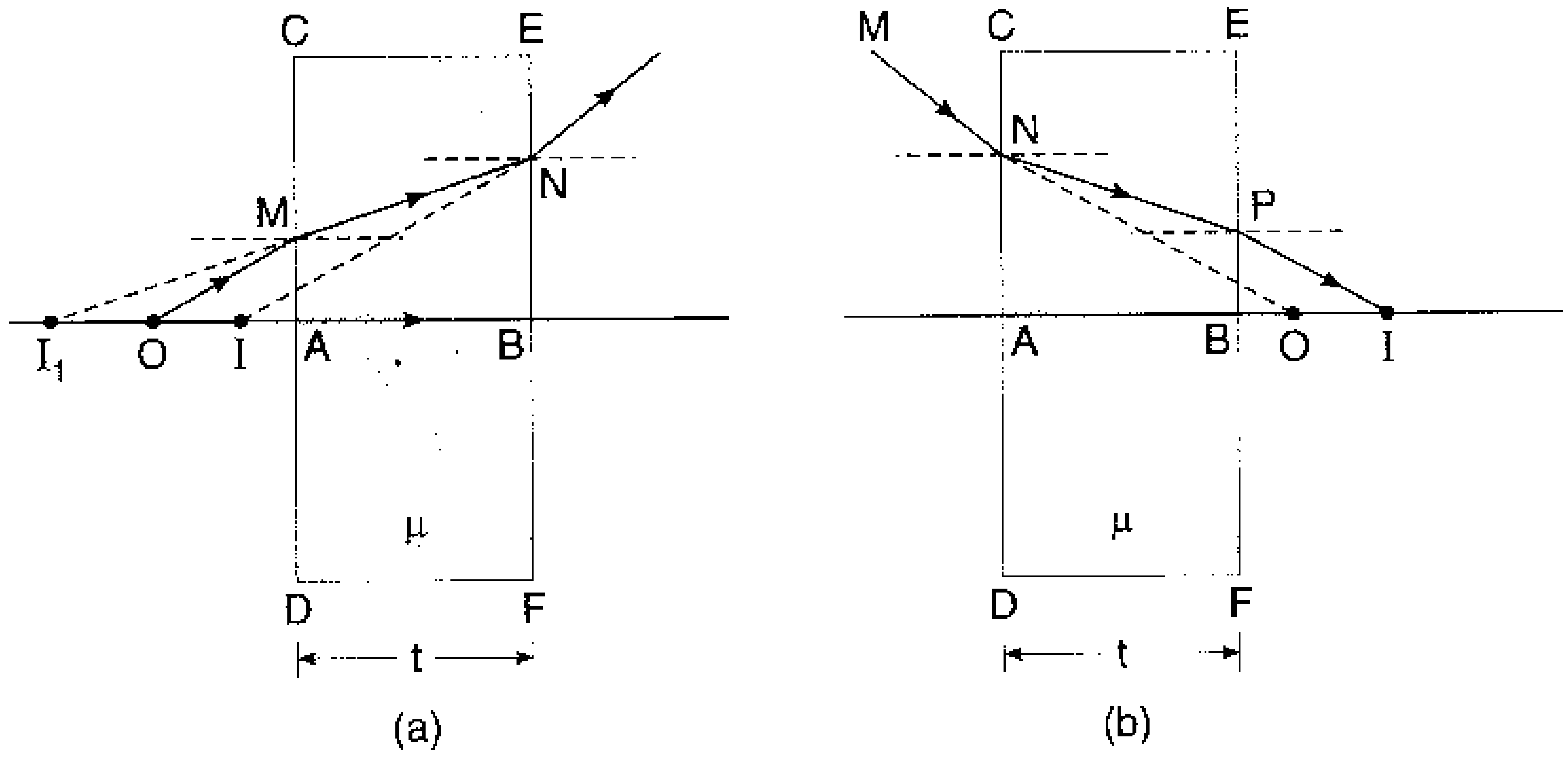


Fig. 22.51

Refer figure (a) : An object is placed at O. Plane surface CD forms its image (virtual) at I_1. This image acts as object for EF which finally forms the image (virtual) at I. Distance OI is called the normal shift and its value is,

$$OI = \left(1 - \frac{1}{\mu}\right)t$$

This can be proved as under:

Let $\quad OA = x$

then $\quad AI_1 = \mu x$ (Refraction from CD)

$$BI_1 = \mu x + t$$

$$BI = \frac{BI_1}{\mu} = x + \frac{t}{\mu} \quad \text{(Refraction from } EF\text{)}$$

$\therefore$

$$OI = (AB + OA) - BI$$

$$= (t + x) - \left(x + \frac{t}{\mu}\right)$$

$$= \left(1 - \frac{1}{\mu}\right)t$$

Hence Proved.

Refer figure (b) : The ray of light which would had met line AB at O will now meet this line at I after two times refraction from the slab. Here

$$OI = \left(1 - \frac{1}{\mu}\right)t$$

(ii) Lateral Shift : We have already discussed that ray MA is parallel to ray BN. But the emergent ray is displaced laterally by a distance d, which depends on μ, t and i and its value is given by the relation,

$$d = t\left(1 - \frac{\cos i}{\sqrt{\mu^2 - \sin^2 i}}\right)\sin i$$

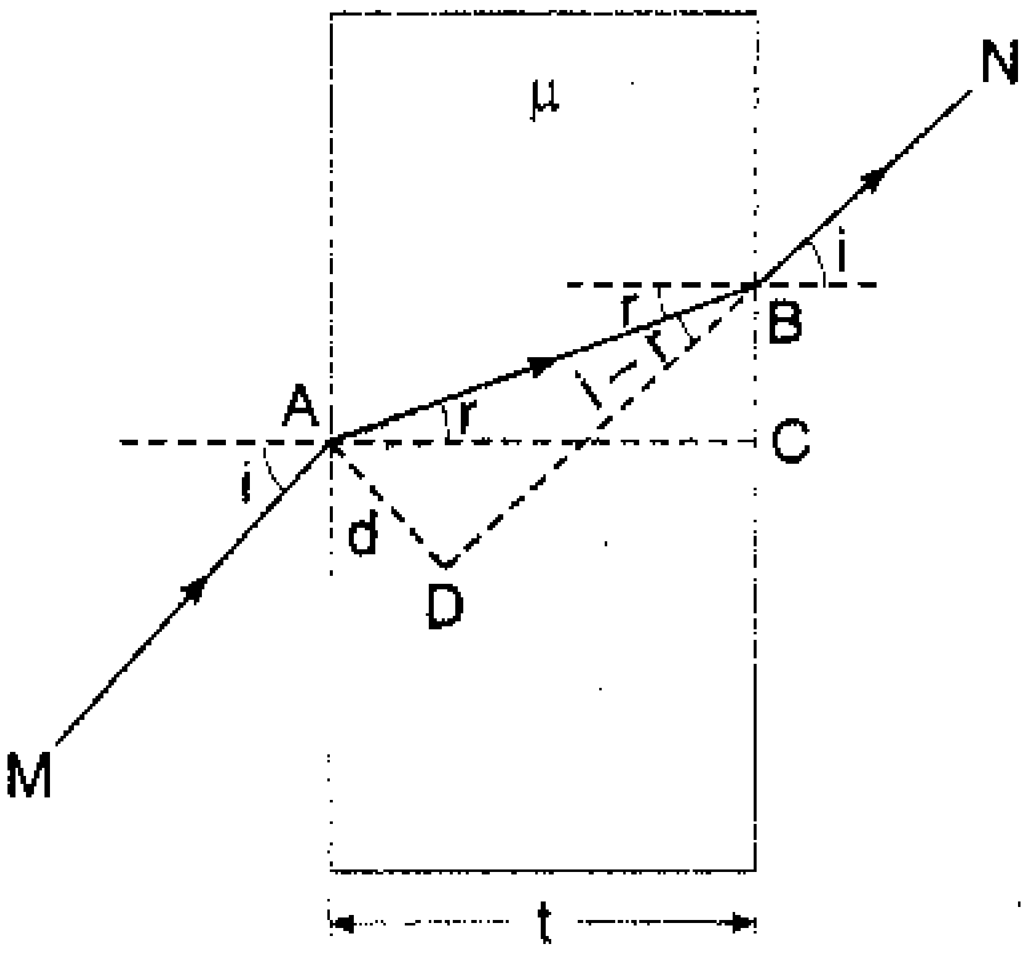


Fig. 22.52

Proof : $AB = \dfrac{AC}{\cos r} = \dfrac{t}{\cos r}$ (as $AC = t$)

Now, $d = AB \sin (i - r)$

$$= \frac{t}{\cos r}[\sin i \cos r - \cos i \sin r]$$

or $d = t\,[\sin i - \cos i \tan r]$ ...(i)

Further $\mu = \dfrac{\sin i}{\sin r}$ or $\sin r = \dfrac{\sin i}{\mu}$

$\therefore$ $\tan r = \dfrac{\sin i}{\sqrt{\mu^2 - \sin^2 i}}$

Substituting in Eq. (i), we get $d = \left[1 - \dfrac{\cos i}{\sqrt{\mu^2 - \sin^2 i}}\right] t \sin i$ **Hence Proved.**

EXERCISE. Show that for small angles of incidence, $d = ti\left(\dfrac{\mu - 1}{\mu}\right)$.

EXAMPLE 22.11 *A point object O is placed in front of a concave mirror of focal length 10 cm. A glass slab of refractive index $\mu = 3/2$ and thickness 6 cm is inserted between object and mirror. Find the position of final image when the distance x shown in figure is:*

(a) 5 *cm* *(b)* 20 *cm.*

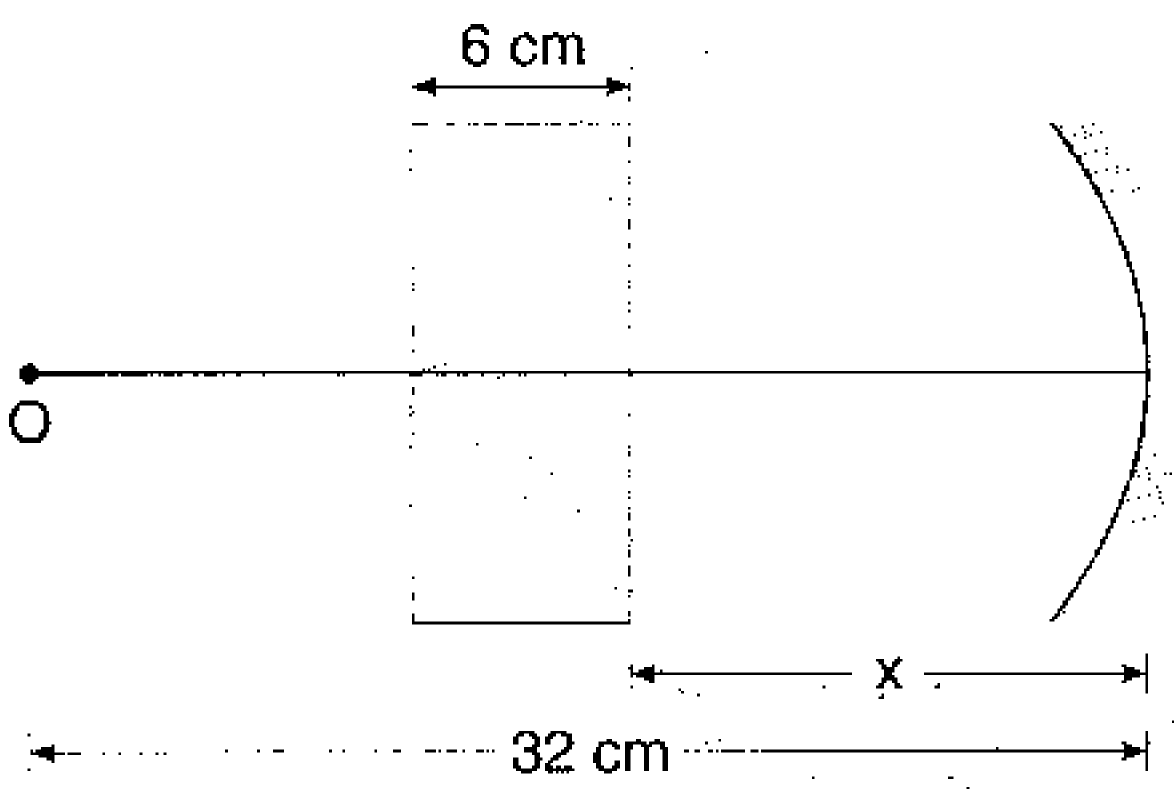


Fig. 22.53

SOLUTION As we have read in the above article the normal shift produced by a glass slab is,

$$\Delta x = \left(1 - \frac{1}{\mu}\right)t = \left(1 - \frac{2}{3}\right)(6) = 2\,\text{cm}$$

i.e., for the mirror the object is placed at a distance $(32 - \Delta x) = 30$ cm from it. Applying mirror formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

or
$$\frac{1}{v} - \frac{1}{30} = -\frac{1}{10}$$

or
$$v = -15\,\text{cm}$$

(a) When $x = 5$ cm : The light falls on the slab on its return journey as shown. But the slab will again shift it by a distance $\Delta x = 2$ cm. Hence, the final **real** image is formed at a distance $(15 + 2) = 17$ cm from the mirror.

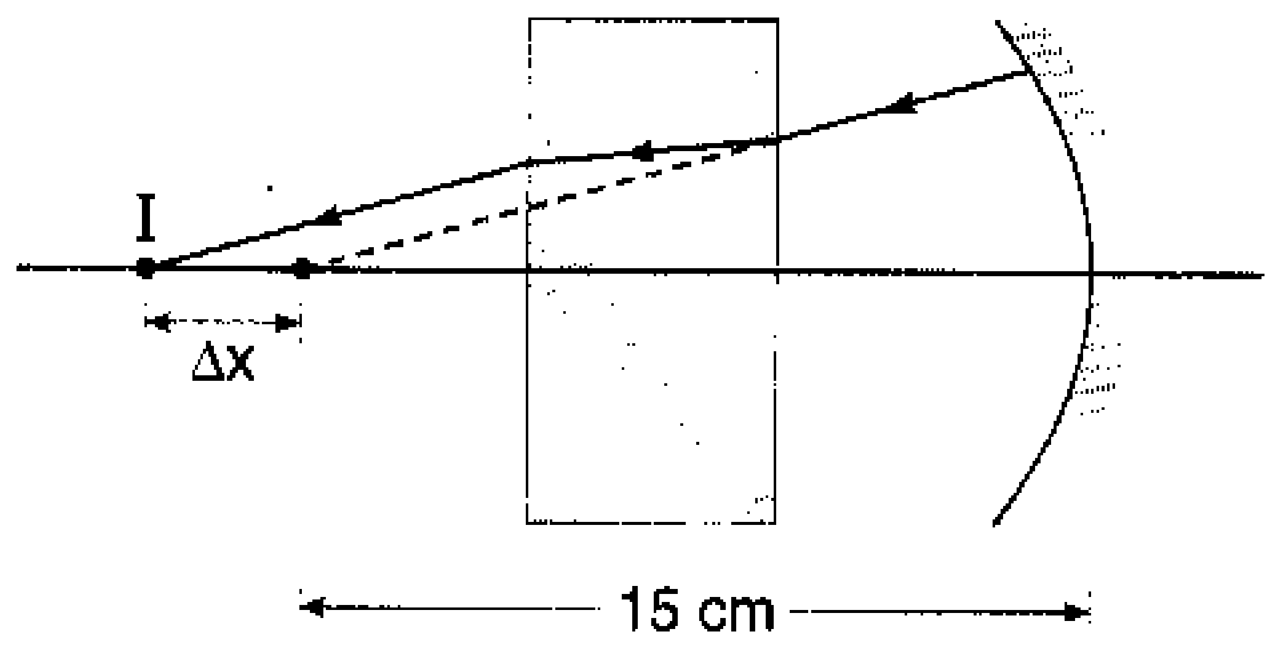


Fig. 22.54

(b) When $x = 20$ cm : This time also the final image is at a distance 17 cm from the mirror but it is **virtual** as shown.

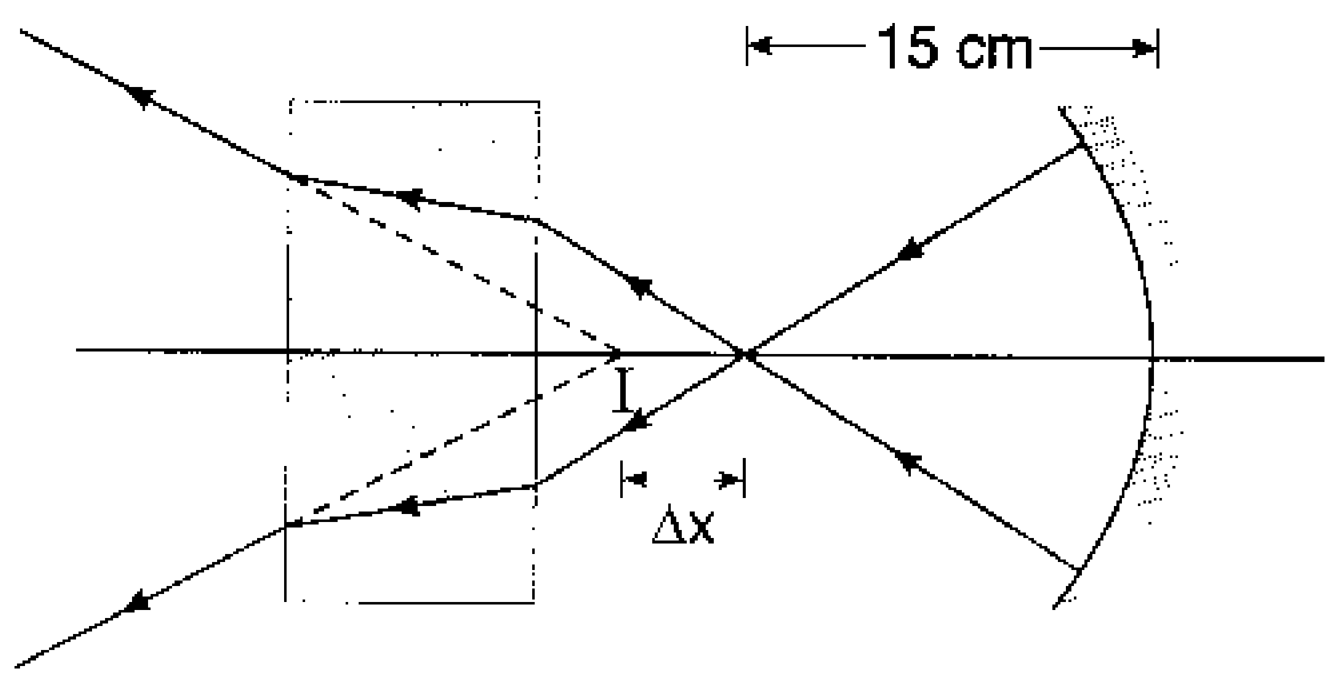


Fig. 22.55

INTRODUCTORY EXERCISE 22.4

1. There is a dust particle on a glass slab of thickness t and refractive index $\mu = 1.5$. When seen from one side of the slab, the dust particle appears at a distance 6 cm. From other side it appears at 4 cm. Find the thickness t of the glass slab.
2. Given that ${}_1\mu_2 = 4/3$, ${}_2\mu_3 = 3/2$. Find ${}_1\mu_3$.
3. What happens to the frequency, wavelengths and speed of light that crosses from a medium with index of refraction μ_1 to one with index of refraction μ_2?
4. A monochromatic light beam of frequency 6.0×10^{14} Hz crosses from air into a transparent material where its wavelength is measured to be 300 nm. What is the index of refraction of the material?

(d) Refraction from a Spherical Surface

Consider two transparent media having indices of refraction μ_1 and μ_2, where the boundary between the two media is a spherical surface of radius R. We assume that $\mu_1 < \mu_2$. Let us consider a single ray leaving point O and focussing at point I. Snell's law applied to this refracted ray gives,

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

Because θ_1 and θ_2 are assumed to be small, we can use the small angle approximation

$$\sin\theta \approx \theta$$

(angles in radians) and say that

$$\mu_1\theta_1 = \mu_2\theta_2 \quad \text{...(i)}$$

From the geometry shown in the figure,

$$\theta_1 = \alpha + \beta \quad \text{...(ii)}$$

and

$$\beta = \theta_2 + \gamma \quad \text{...(iii)}$$

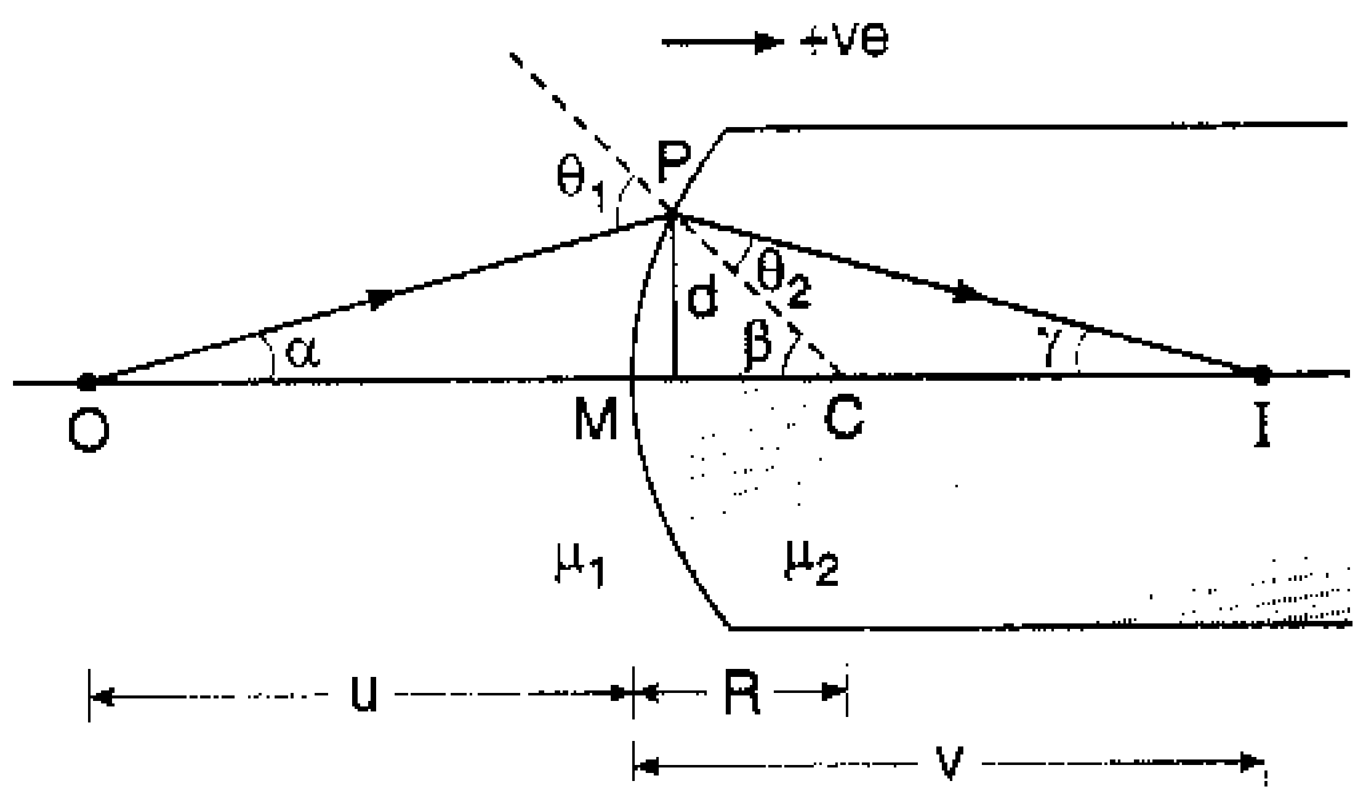


Fig. 22.56

Eqs. (i) and (iii) can be combined to express θ_2 in terms of α and β. Substituting the resulting expression into Eq. (ii) then yields

$$\beta = \frac{\mu_1}{\mu_2}(\alpha + \beta) + \gamma$$

so

$$\mu_1\alpha + \mu_2\gamma = (\mu_2 - \mu_1)\beta \quad \text{...(iv)}$$

Since, the arc PM (of length S) subtends an angle β at the centre of curvature,

$$\beta = \frac{S}{R}$$

Also in the paraxial approximation $\alpha = \frac{S}{u}$ and $\gamma = \frac{S}{v}$

Using these expressions in Eq. (iv) with proper signs, we are left with,

$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

or

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R} \quad \text{...(v)}$$

Although the formula (v) is derived for a particular situation, it is valid for all other situations of refraction at a single spherical surface.

Lateral Magnification : The lateral magnification may be obtained with the help of the adjacent figure, where two rays from the tip of an object of height h_0 meet at the corresponding point on an image of height h_i. One ray passes through the centre of curvature of the spherical surface so its direction is unchanged. The path of the second ray is obtained from Snell's law. With the paraxial approximation,

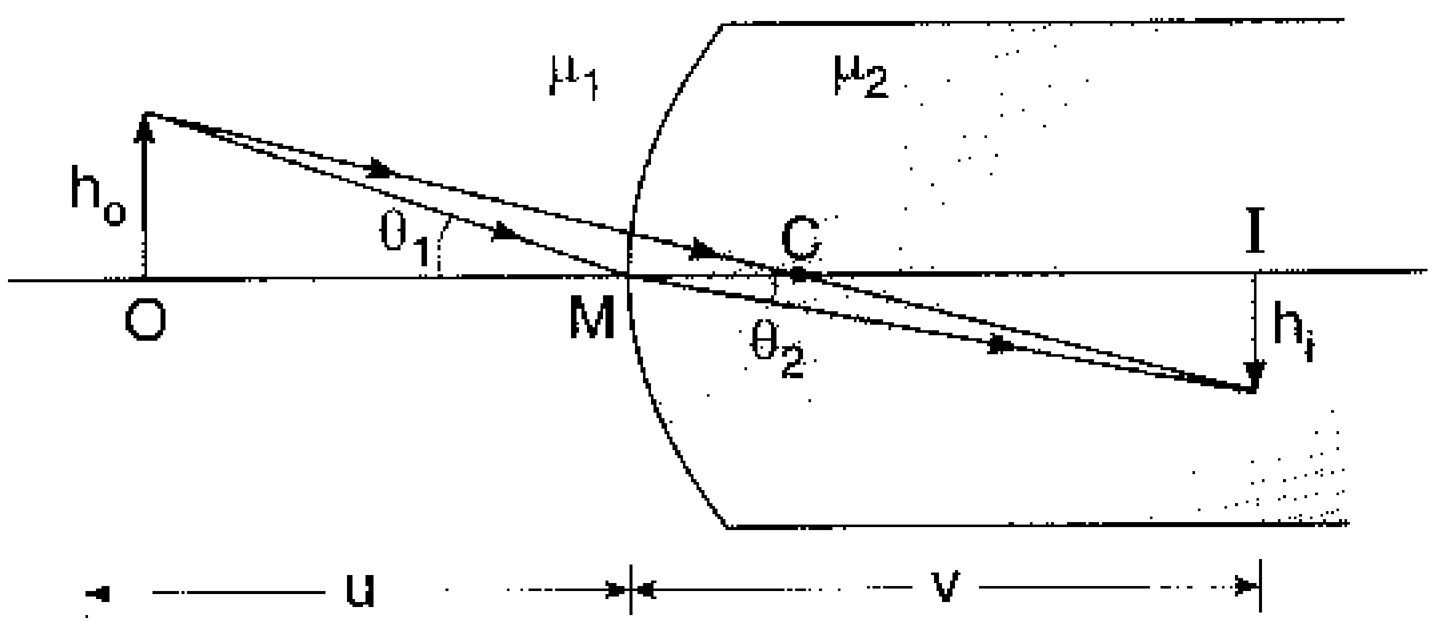


Fig. 22.57

$$\sin\theta_1 \approx \frac{h_o}{u} \quad \text{and} \quad \sin\theta_2 \approx \frac{h_i}{v}$$

Combining these equations with Snell's law then gives,

$$\mu_1\left(\frac{h_o}{u}\right) = \mu_2\left(\frac{h_i}{v}\right)$$

or
$$\frac{h_i}{h_o} = \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{v}{u}\right) \qquad \text{...(vi)}$$

The lateral magnification m is the ratio of the image height to the object height or $\frac{h_i}{h_o}$. We therefore, obtain

$$m = \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{v}{u}\right) \qquad \text{...(vii)}$$

Note : Here $v = +$ ve, $u = -$ ve, $h_i = -$ ve and $h_o = +$ ve (distances measured above the axis are taken positive). So if we put these sign conventions in Eq. (vi), we obtain the same result *viz*,. $m = \frac{\mu_1}{\mu_2}\frac{v}{u}$.

EXAMPLE 22.12 *A glass sphere of radius $R = 10$ cm is kept inside water. A point object O is placed at 20 cm from A as shown in figure. Find the position and nature of the image when seen from other side of the sphere. Also draw the ray diagram. Given $\mu_g = 3/2$ and $\mu_w = 4/3$.*

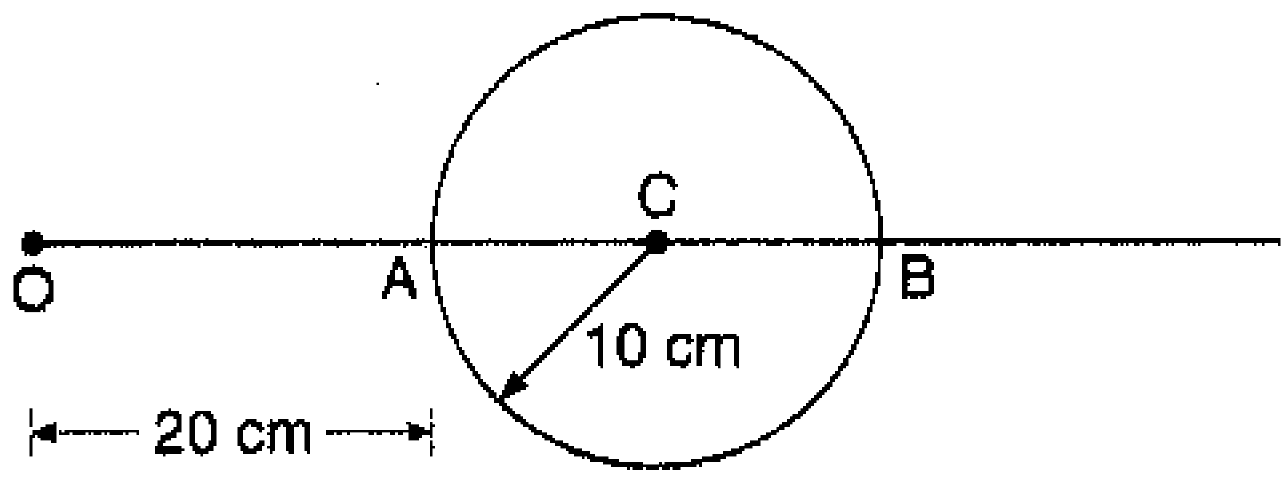


Fig. 22.58

SOLUTION A ray of light starting from O gets refracted twice. The ray of light is travelling in a direction from left to right. Hence, the distances measured in this direction are taken positive. Applying $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, twice with proper signs.

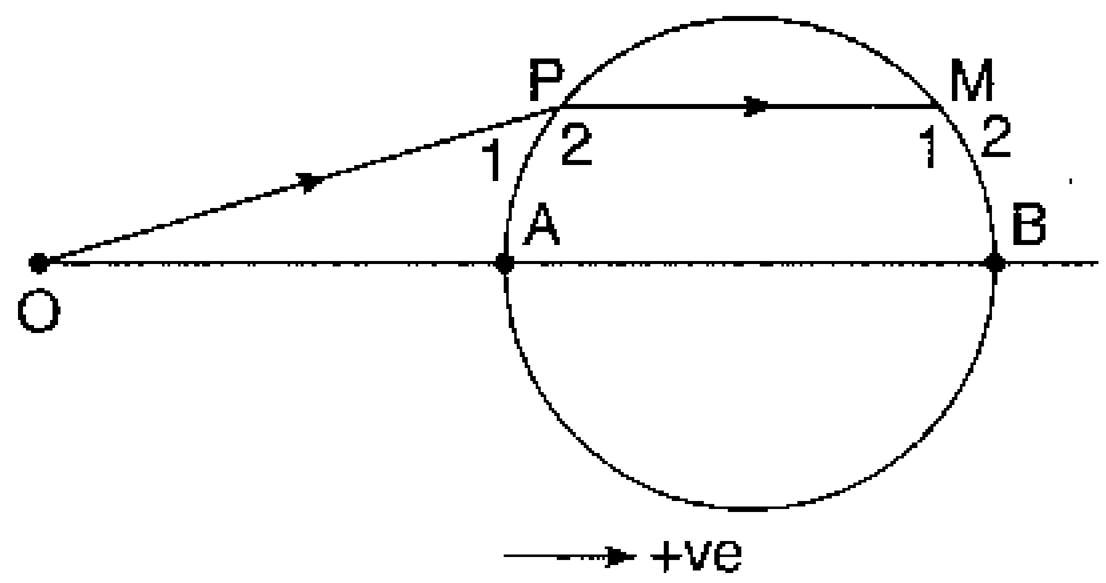


Fig. 22.59

We have,
$$\frac{3/2}{AI_1} - \frac{4/3}{-20} = \frac{3/2 - 4/3}{10}$$

or
$$AI_1 = -30 \text{ cm}$$

Now, the first image I_1, acts as an object for the second surface, where

$$BI_1 = u = -(30 + 20) = -50 \text{ cm}$$

$$\therefore \quad \frac{4/3}{BI_2} - \frac{3/2}{-50} = \frac{4/3 - 3/2}{-10}$$

$\therefore$ $BI_2 = -100$ cm

i.e., the final image I_2 is **virtual** and is formed at a distance 100 cm (towards left) from B. The ray diagram is as shown.

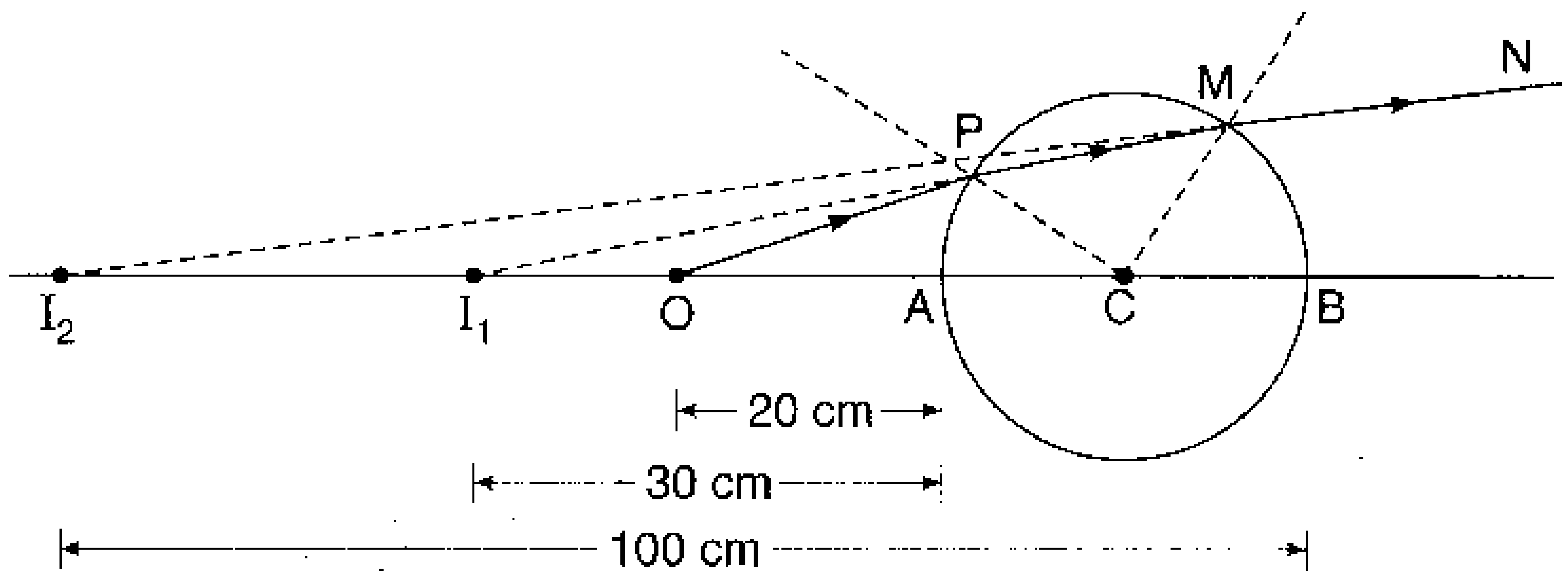


Fig. 22.60

Following points should be noted while drawing the ray diagram.

(i) At P the ray travels from rare to a denser medium. Hence, it will bend towards normal PC. At M, it travels from a denser to a rare medium, hence, moves away from the normal MC.

(ii) PM ray when extended backwards meets at I_1 and MN ray when extended meets at I_2.

Note : The refraction formula $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ can also be applied to plane refracting surfaces with $R = \infty$. Let us derive $d_{app} = \frac{d_{actual}}{\mu}$ using this.

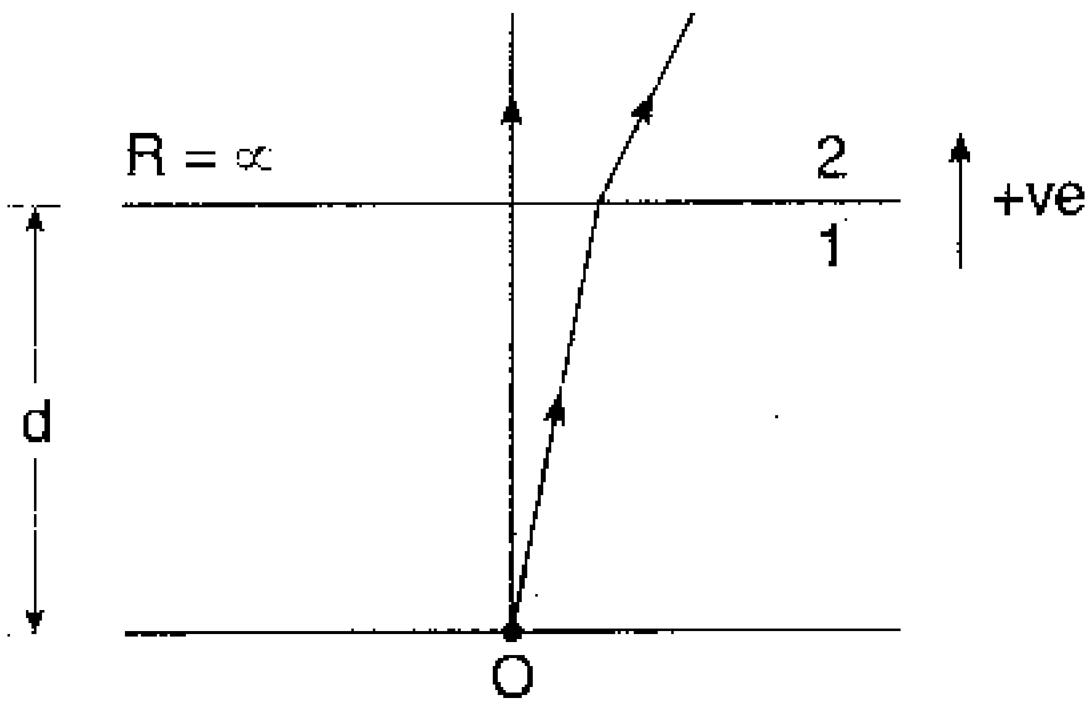


Fig. 22.61

Applying $$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

With proper sign and values, we have

$$\frac{1}{v} - \frac{\mu}{-d} = \frac{1-\mu}{\infty} = 0$$

or $$v = -\frac{d}{\mu}$$

i.e., image of object O is formed at a distance $\frac{d}{\mu}$ on same side.

or $$d_{app} = \frac{d_{actual}}{\mu}$$

INTRODUCTORY EXERCISE 22.5

1. A glass sphere ($\mu = 1.5$) with a radius of 15.0 cm has a tiny air bubble 5 cm above its centre. The sphere is viewed looking down along the extended radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?
2. One end of a long glass rod ($\mu = 1.5$) is formed into a convex surface of radius 6.0 cm. An object is positioned in air along the axis of the rod. Find the image positions corresponding to object distances of
   (a) 20.0 cm (b) 10.0 cm (c) 3.0 cm
   from the end of the rod.
3. A dust particle is inside a spherical plastic bowl of water of refractive index 1.33. If the dust particle is 10.0 cm from the wall of the 15.0 cm radius bowl, where does it appear to an observer outside the bowl.
4. A parallel beam of light enters a clear plastic bead 2.50 cm in diameter and index 1.440. At what point beyond the bead are these rays brought to a focus?
5. The left end of a long glass rod of index 1.6350 is ground and polished to a convex spherical surface of radius 2.50 cm. A small object is located in the air and on the axis 9.0 cm from the vertex. Find the lateral magnification.

22.6 THIN LENSES

A lens is one of the most familiar optical devices for a human being. A lens is an optical system with two refracting surfaces. The simplest lens has two spherical surfaces close enough together that we can neglect the distance between them (the thickness of the lens). We call this a **thin lens**.

Lenses are of two basic types **convex** which are thicker in the middle than at the edges and **concave** for which the reverse holds.

Figure shows examples of both types bounded by spherical or plane surfaces.

As there are two spherical surfaces, there are two centres of curvature C_1 and C_2 and correspondingly two radii of curvature R_1 and R_2.

The line joining C_1 and C_2 is called the **principal axis** of the lens. The centre P of the thin lens which lies on the principal axis, is called the **optical centre**.

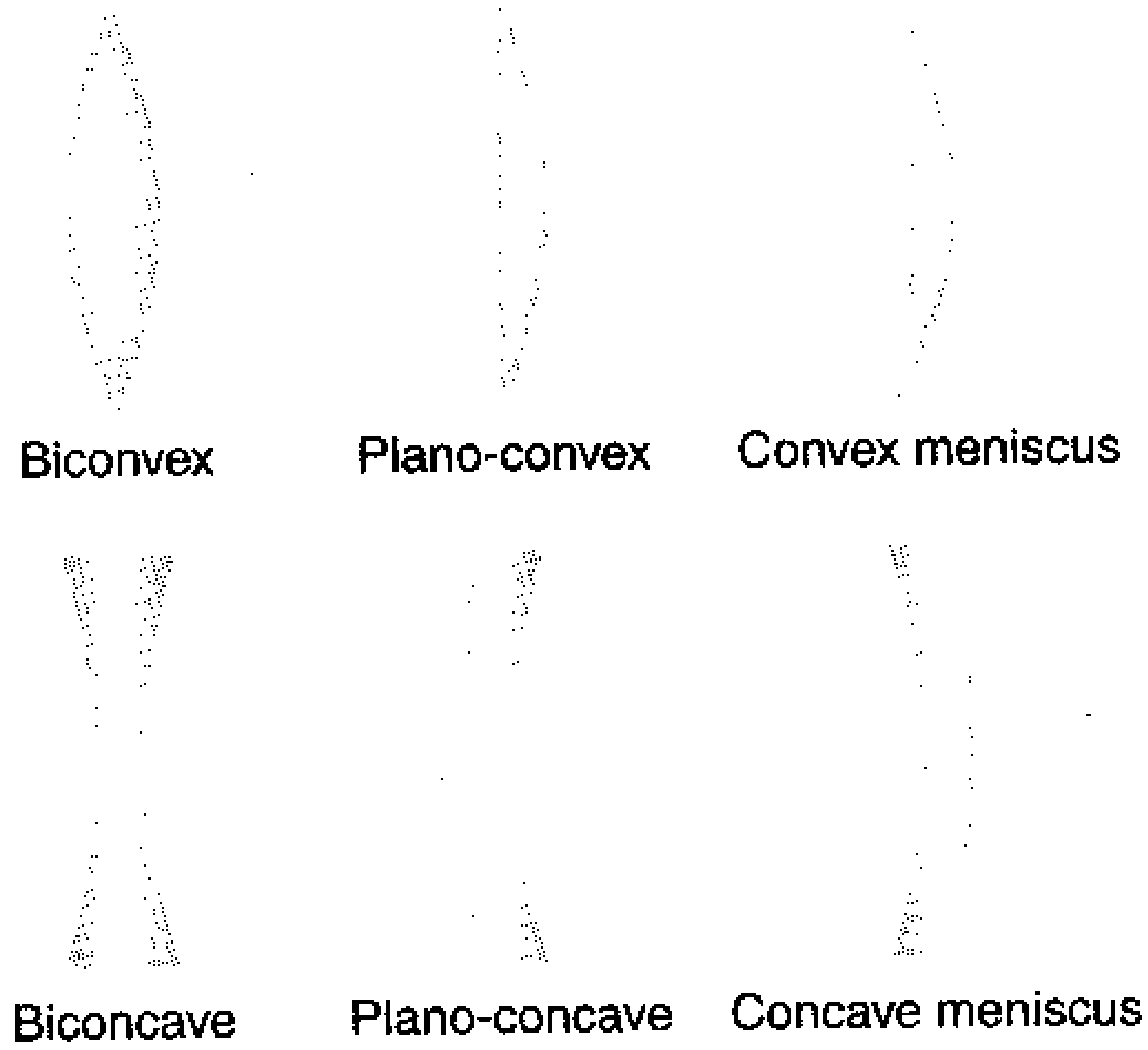


Fig. 22.62 Types of lens.

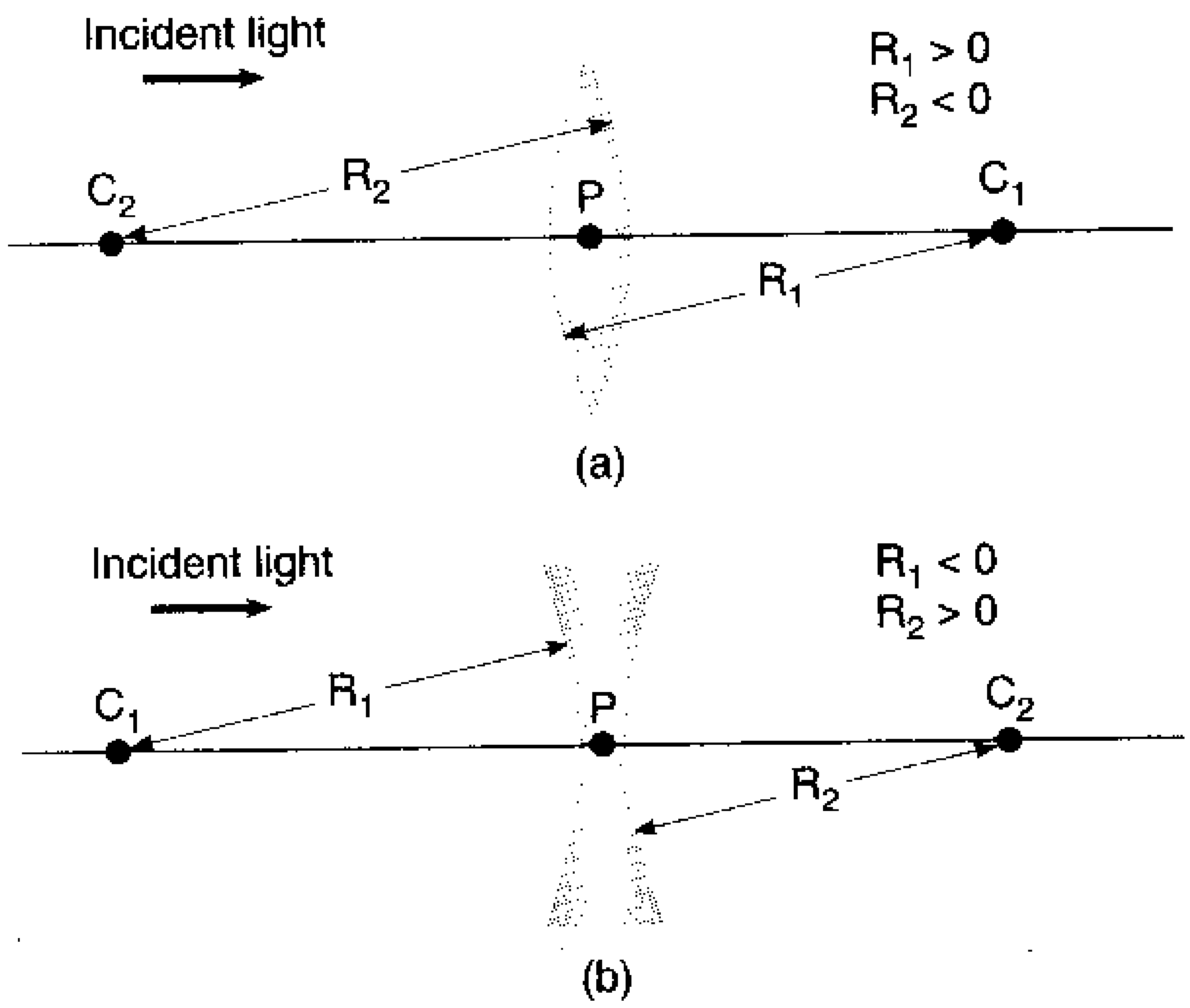
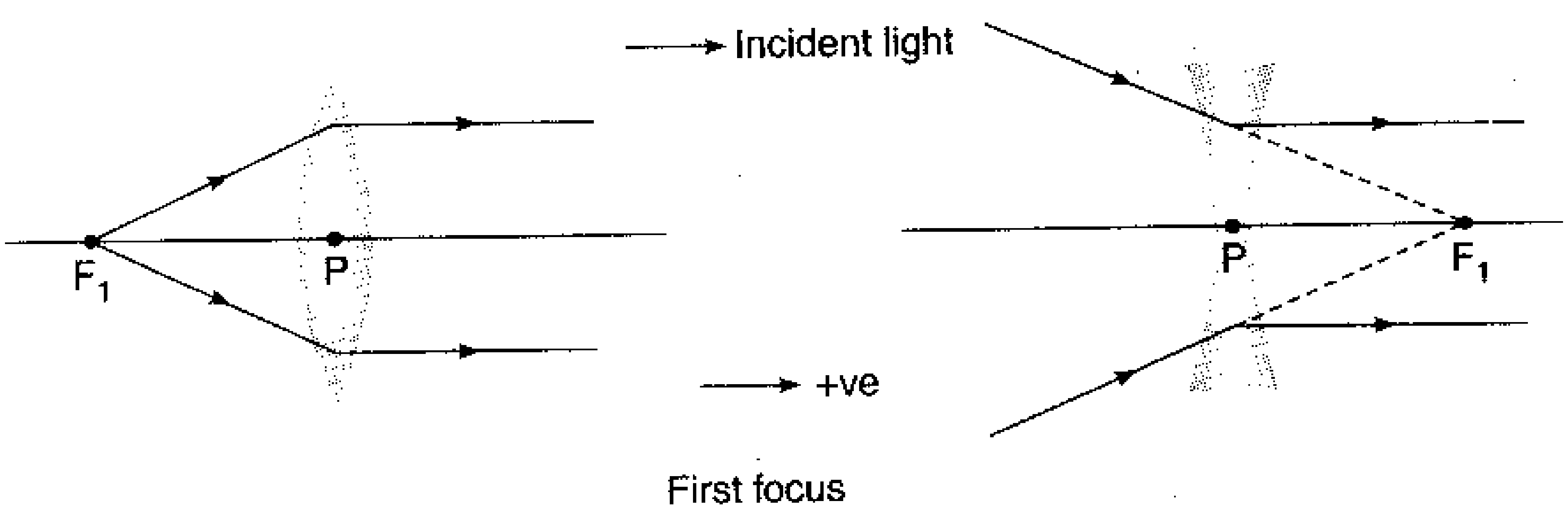
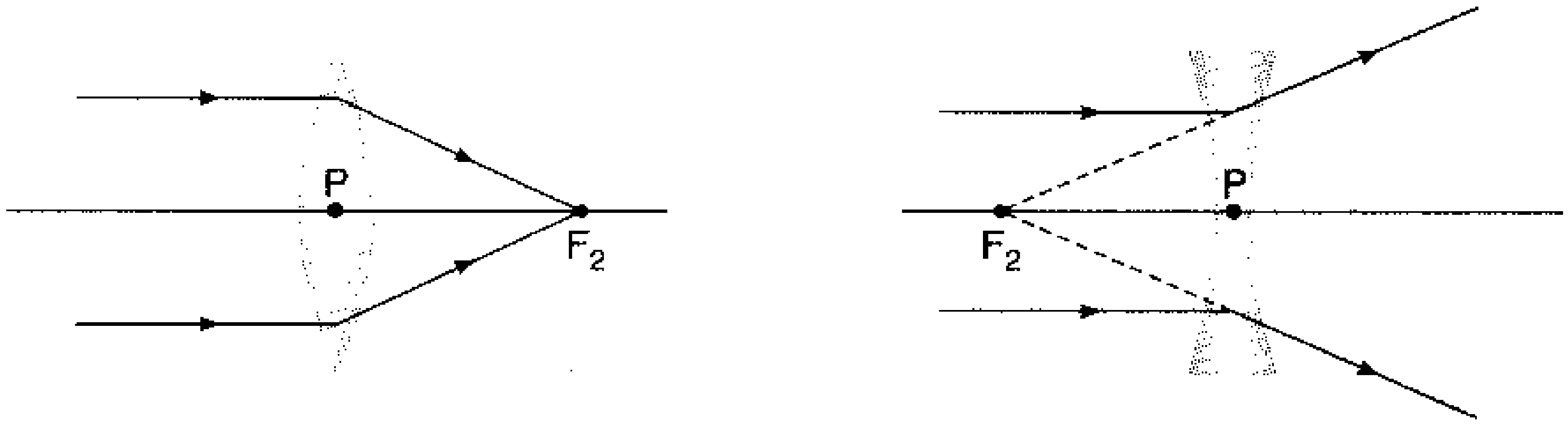


Fig. 22.63 (a) A converging thin lens and (b) a diverging thin-lens.

Focus :





Second focus or principal focus

Fig. 22.64

Unlike a mirror, a lens has two foci.

First focus (F_1) : It is defined as a point at which if an object (real in case of a convex lens and virtual for concave) is placed, the image of this object is formed at infinity. Or we can say, rays passing through F_1 become parallel to the principal axis after refraction from the lens. The distance PF_1 is the first focal length f_1.

Second focus or principal focus (F_2) : A narrow beam of light travelling parallel to the principal axis either converge (in case of a convex lens) or diverge (in case of a concave lens) at a point F_2 after refraction from the lens. This point F_2 is called the second or principal focus. If the rays converge at F_2, the lens is said a converging lens and if they diverge, they are called diverging lens. Distance PF_2 is the second focal length f_2.

Notes : (i) From the figure we can see that f_1 is negative for a convex lens and positive for a concave lens. But f_2 is positive for convex lens and negative for concave lens.

(ii) $|f_1| = |f_2|$ if the media on the two sides of a thin lens have same refractive index.

(iii) We are mainly concerned with the second focus f_2. Thus, wherever we write the focal length 'f', it means the second or principal focal length. Thus, $f = f_2$ and hence, f is positive for a convex lens and negative for a concave lens.

Lens maker's formula and lens formula

Consider an object O placed at a distance u from a convex lens as shown in figure. Let its image I after two refractions from spherical surfaces of radii R_1 (positive) and R_2 (negative) be formed at a distance v from the lens. Let v_1 be the distance of image formed by refraction from the refracting surface of radius R_1. This image acts as an object for the second surface. Using,

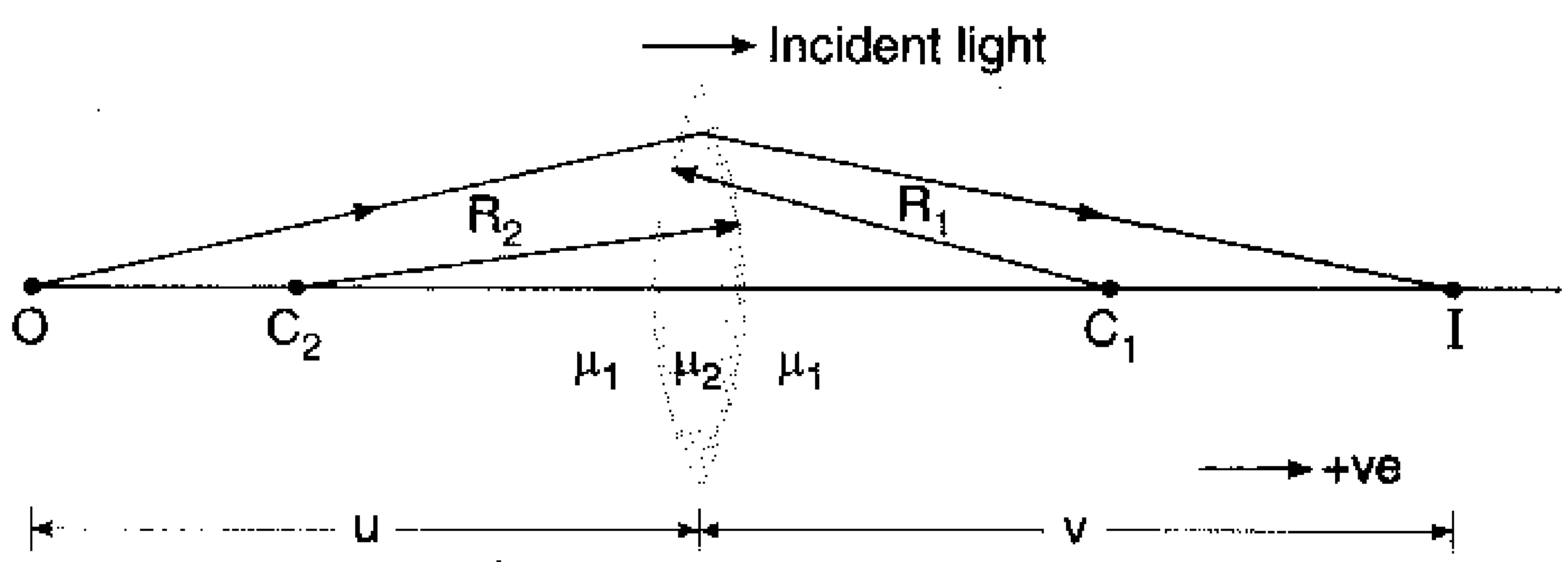


Fig. 22.65

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

twice, we have

or
$$\frac{\mu_2}{v_1} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R_1} \qquad \ldots(i)$$

and
$$\frac{\mu_1}{v} - \frac{\mu_2}{v_1} = \frac{\mu_1 - \mu_2}{-R_2} \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii) and then simplifying, we get
$$\frac{1}{v} - \frac{1}{u} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(iii)}$$

This expression relates the image distance v of the image formed by a thin lens to the object distance u and to the thin lens properties (index of refraction and radii of curvature). It is valid only for paraxial rays and only when the lens thickness is much less then R_1 and R_2. The **focal length** f of a thin lens is the image distance that corresponds to an object at infinity. So, putting $u = \infty$ and $v = f$ in the above equation, we have
$$\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(iv)}$$

If the refractive index of the material of the lens is μ and it is placed in air, $\mu_2 = \mu$ and $\mu_1 = 1$ so that Eq. (iv) becomes
$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{...(v)}$$

This is called the **lens maker's formula** because it can be used to determine the values of R_1 and R_2 that are needed for a given refractive index and a desired focal length f.

Combining Eqs. (iii) and (v), we get
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \text{...(vi)}$$

Which is known as the **lens formula**. Following conclusions can be drawn from Eqs. (iv), (v) and (vi):

1. For a converging lens, R_1 is positive and R_2 is negative. Therefore, $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ in Eq. (v) comes out a positive quantity and if the lens is placed in air, $(\mu - 1)$ is also a positive quantity. Hence, the focal length f of a converging lens comes out to be positive. For a diverging lens however, R_1 is negative and R_2 is positive and the focal length f becomes negative.

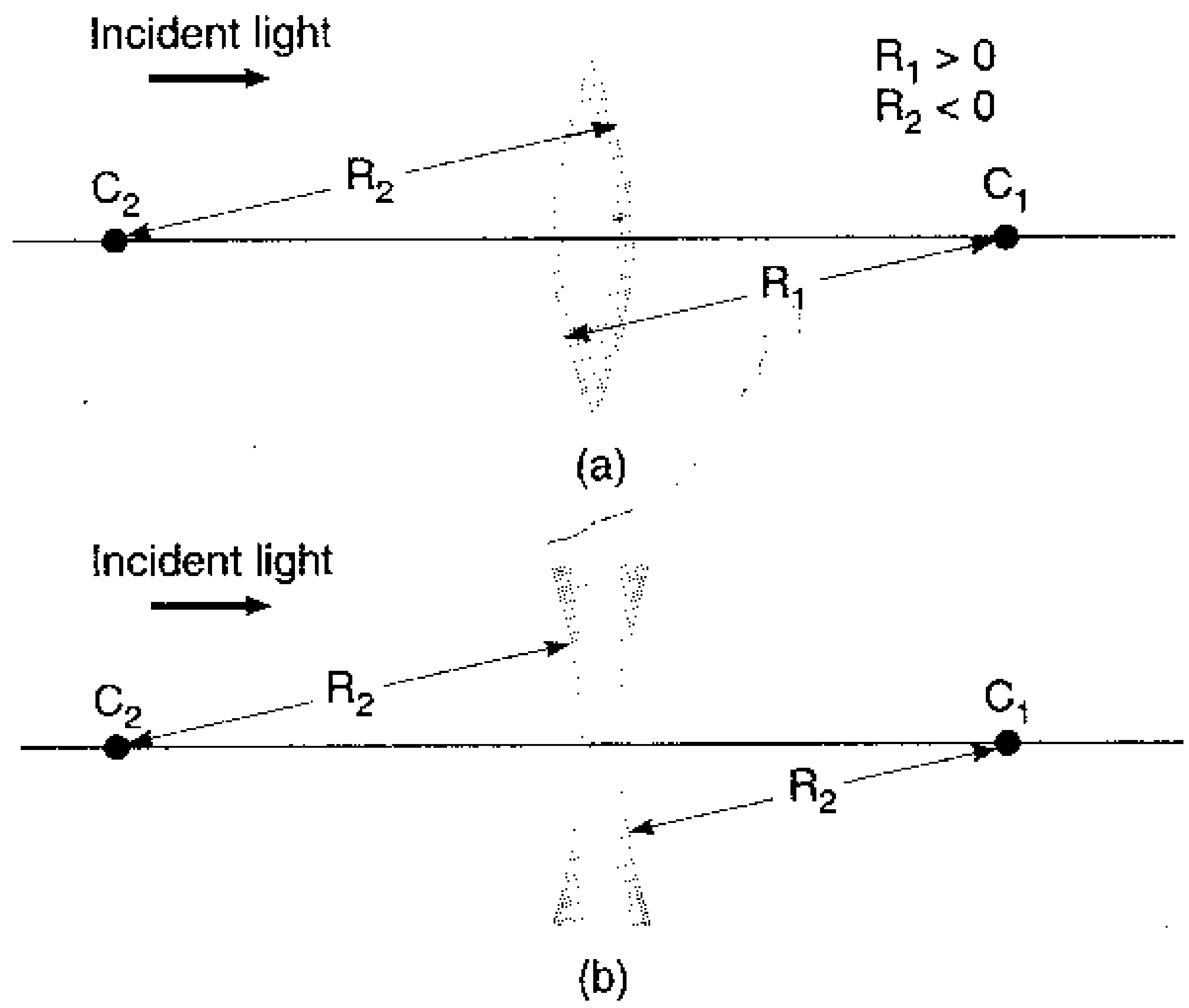


Fig. 22.66

2. Focal length of a mirror $\left(f_M = \frac{R}{2}\right)$ depends only upon the radius of curvature R while that of a lens [Eq. (iv)] depends on μ_1, μ_2, R_1 and R_2. Thus, if a lens and a mirror are immersed in some liquid, the focal length of lens would change while that of the mirror will remain unchanged.
3. Suppose $\mu_2 < \mu_1$ in Eq. (iv), *i.e.*, refractive index of the medium (in which lens is placed) is more than the refractive index of the material of the lens, then $\left(\frac{\mu_2}{\mu_1} - 1\right)$ becomes a negative quantity, *i.e.*, the lens changes its behaviour. A converging lens behaves as a diverging lens and *vice-versa*. An air bubble in water seems as a convex lens but behaves as a concave (diverging) lens.

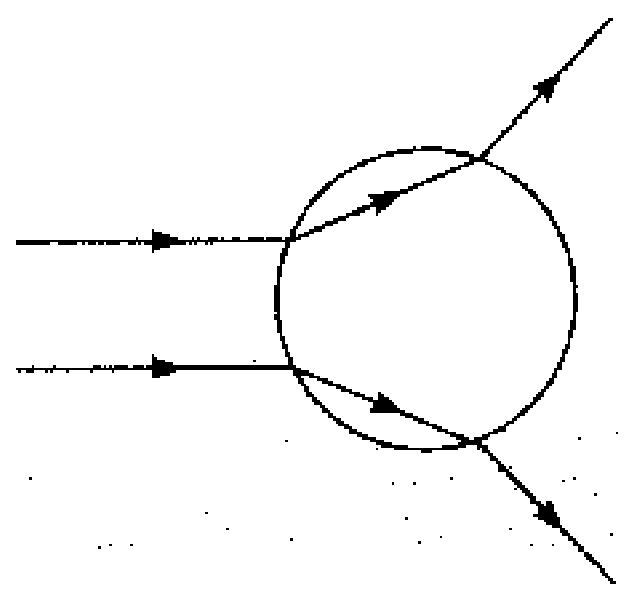


Fig. 22.67 Air bubble in water diverges the parallel beam of light incident

EXAMPLE 22.13 *Focal length of a convex lens in air is* 10 *cm. Find its focal length in water. Given that* $\mu_g = 3/2$ *and* $\mu_w = 4/3$.

SOLUTION

$$\frac{1}{f_{\text{air}}} = (\mu_g - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \ldots\text{(i)}$$

and

$$\frac{1}{f_{\text{water}}} = \left(\frac{\mu_g}{\mu_w} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \ldots\text{(ii)}$$

Dividing Eq. (i) by (ii), we get

$$\frac{f_{\text{water}}}{f_{\text{air}}} = \frac{(\mu_g - 1)}{(\mu_g/\mu_w - 1)}$$

Substituting the values, we have

$$f_{\text{water}} = \frac{(3/2 - 1)}{\left(\frac{3/2}{4/3} - 1\right)} f_{\text{air}}$$

$$= 4 f_{\text{air}} = 4 \times 10 = 40 \text{ cm}$$ **Ans.**

Note : Students can remember the result $f_{\text{water}} = 4\ f_{\text{air}}$, if $\mu_g = 3/2$ and $\mu_w = 4/3$.

Images formed by thin lenses

Information as to the position and nature of the image in any case can be obtained either from a ray diagram or by calculation.

(a) Ray diagram : To construct the image of a small object perpendicular to the axis of a lens, two of the following three rays are drawn from the top of the object.

1. A ray parallel to the principal axis after refraction passes through the principal focus or appears to diverge from it.

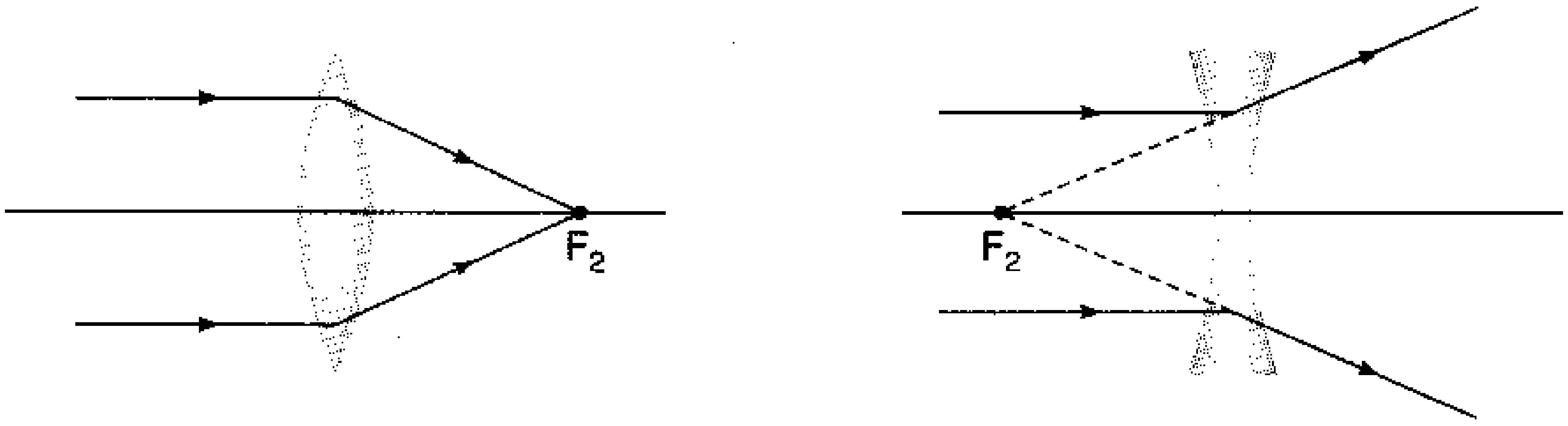


Fig. 22.68

2. A ray through the optical centre P passes undeviated because the middle of the lens acts like a thin parallel-sided slab.

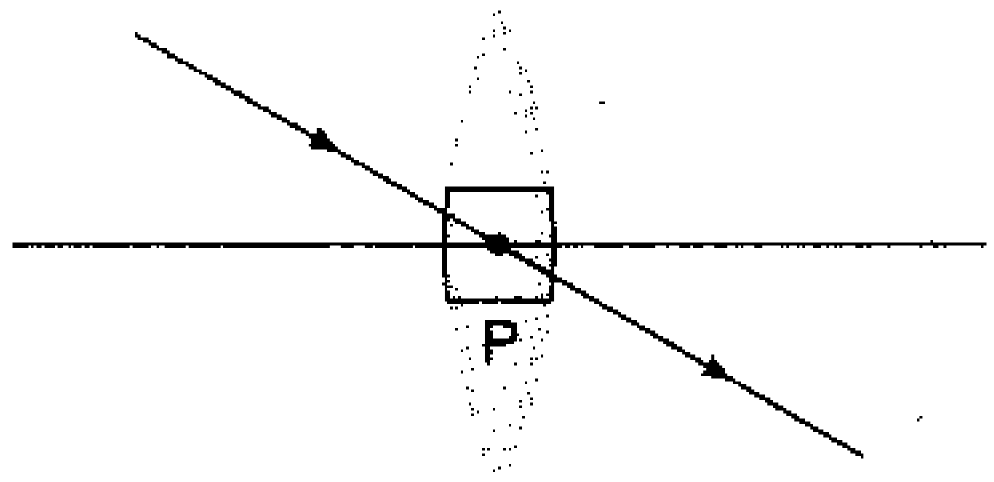


Fig. 22.69

3. A ray passing through the first focus F_1 become parallel to the principal axis after refraction.

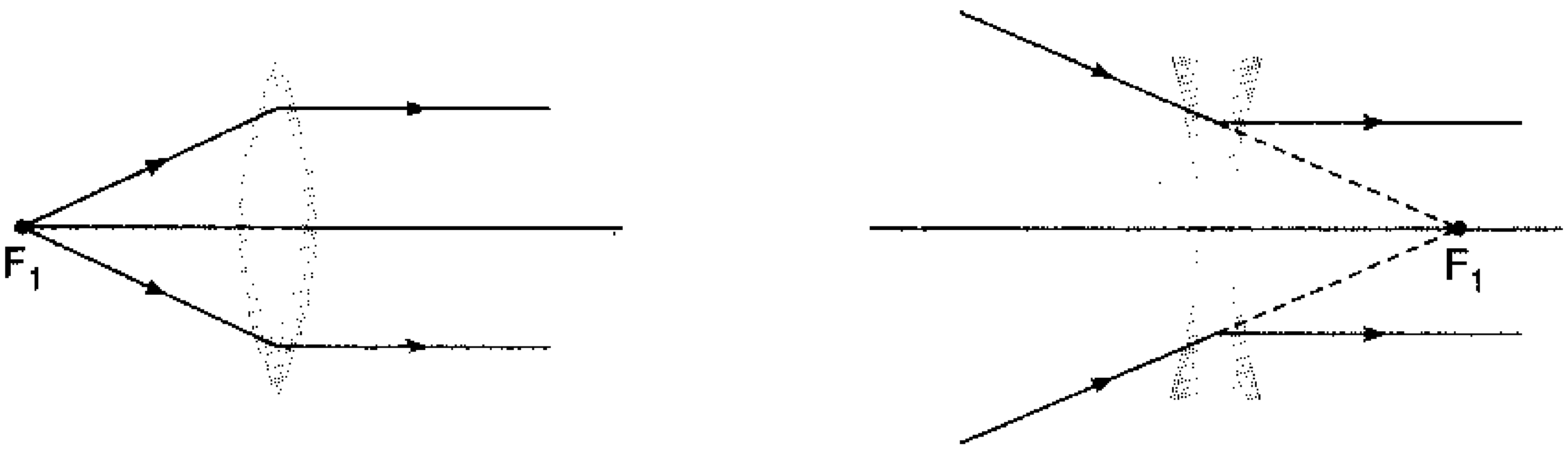
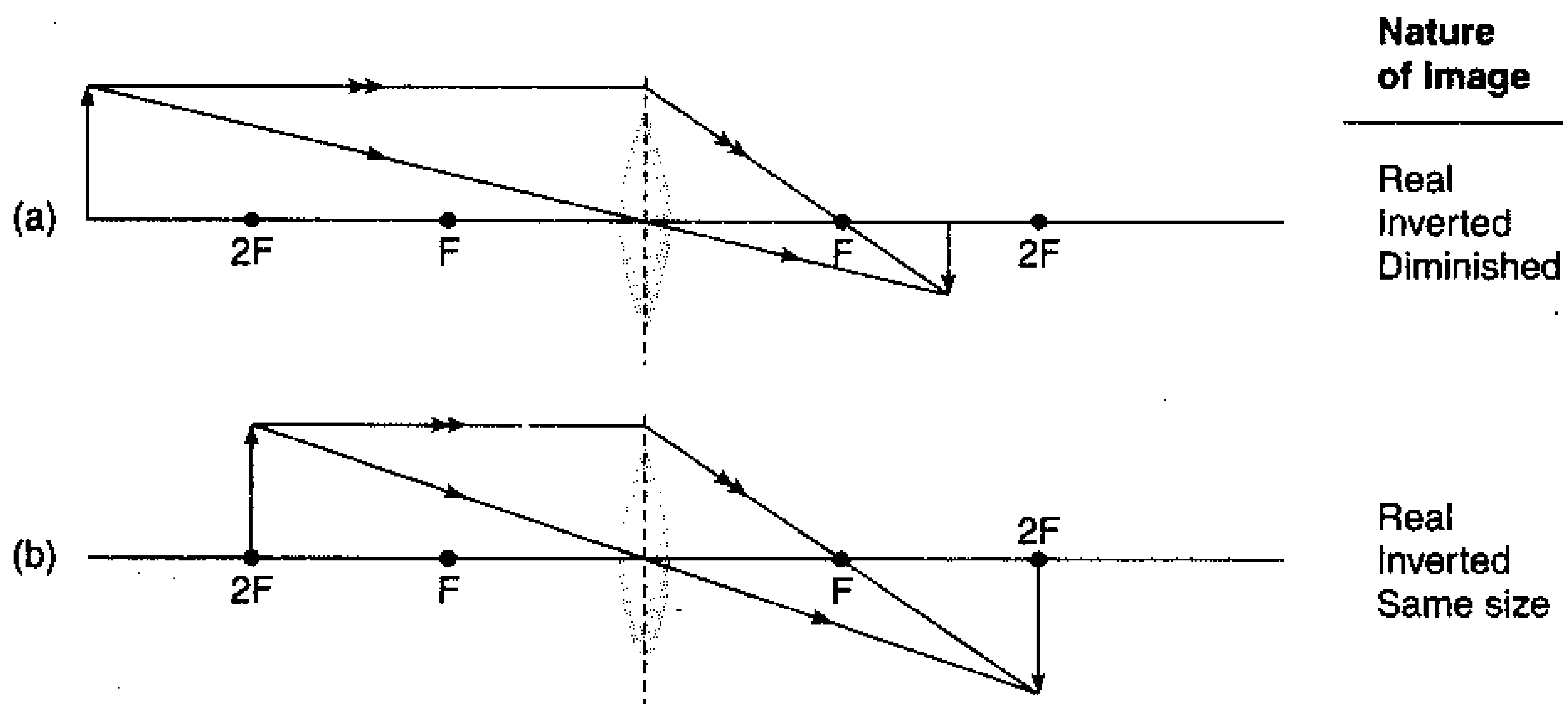


Fig. 22.70

Keeping the above three points in mind we can show that image formed by a concave lens is always virtual, erect and diminished (like a convex mirror) while the nature of image in case of a convex lens depends on the position of object. The ray diagrams for a convex and a concave lens are shown below. For a convex lens image is virtual when object lies between F and P. In all other cases it is real.



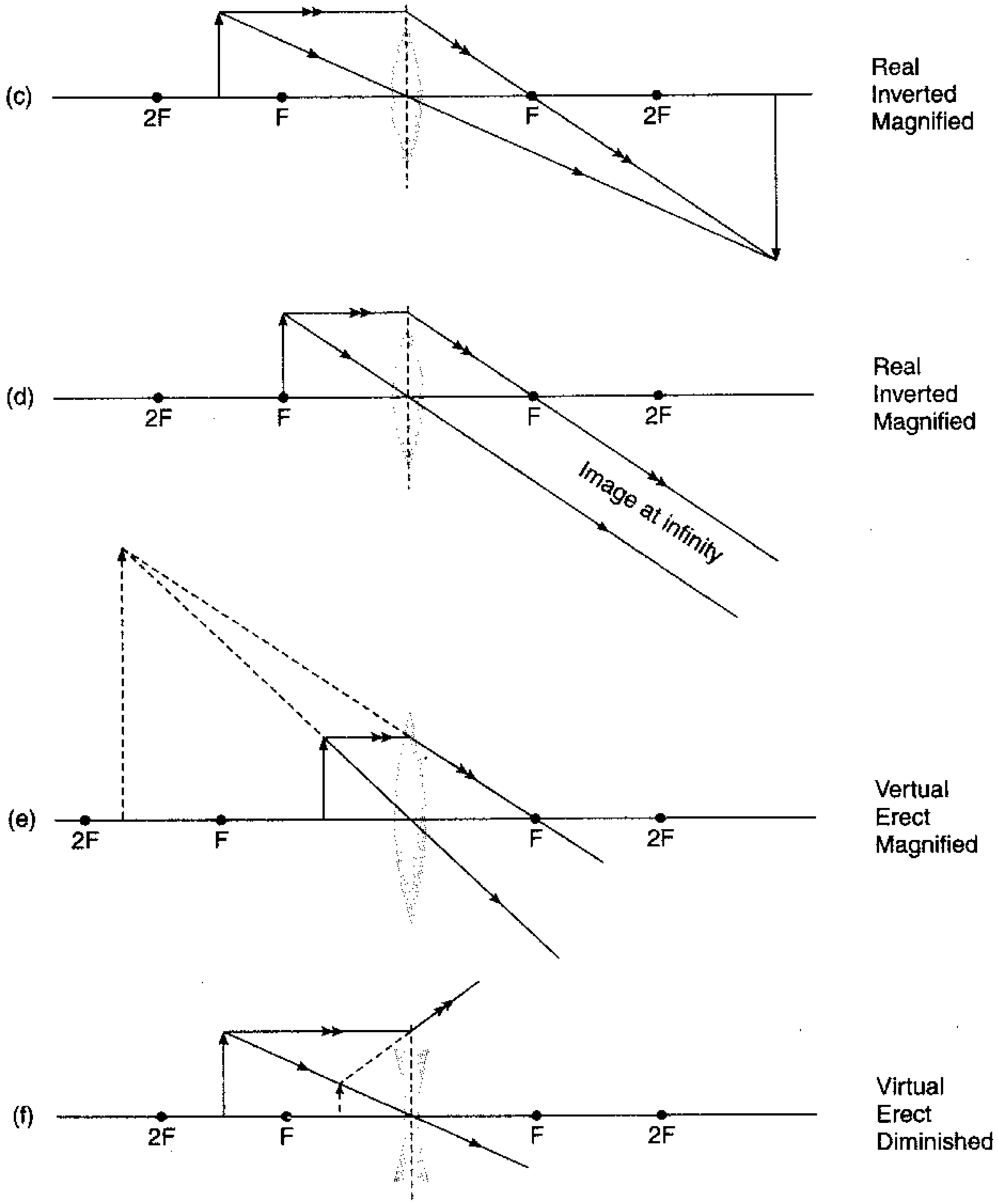


Fig. 22.71 Ray diagrams for a convex lens (a–e) and a concave lens (f).

EXAMPLE 22.14 *An image I is formed of point object O by a lens whose optic axis is AB as shown in figure.*

(a) State whether it is a convex lens or concave?

(b) Draw a ray diagram to locate the lens and its focus.

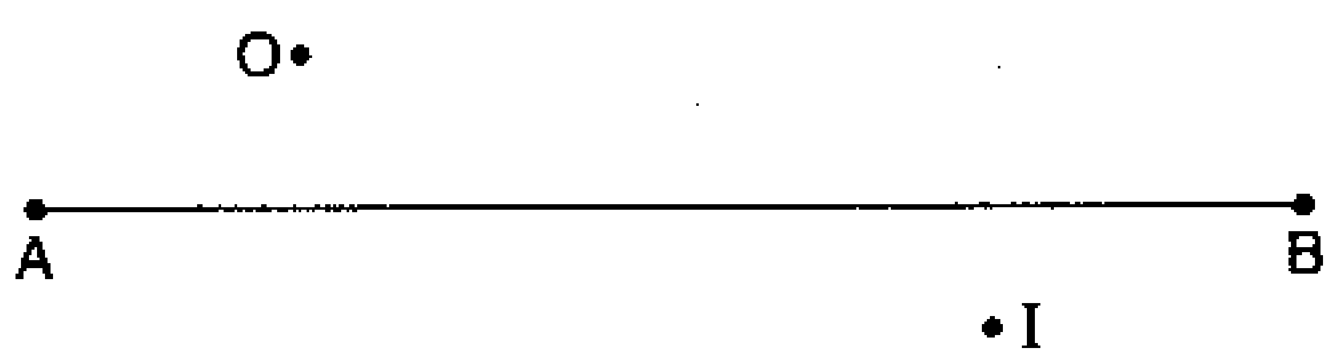


Fig. 22.72

SOLUTION (a) Concave lens always forms an erect image. The given image *I* is on the other side of the optic axis. Hence, the lens is **convex**.

(b) Join O with I. Line OI cuts the optic axis AB at pole (P) of the lens. The dotted line shows the position of lens.

From point O, draw a line parallel to AB. Let it cuts the dotted line at M. Join M with I. Line MI cuts the optic axis at focus (F) of the lens.

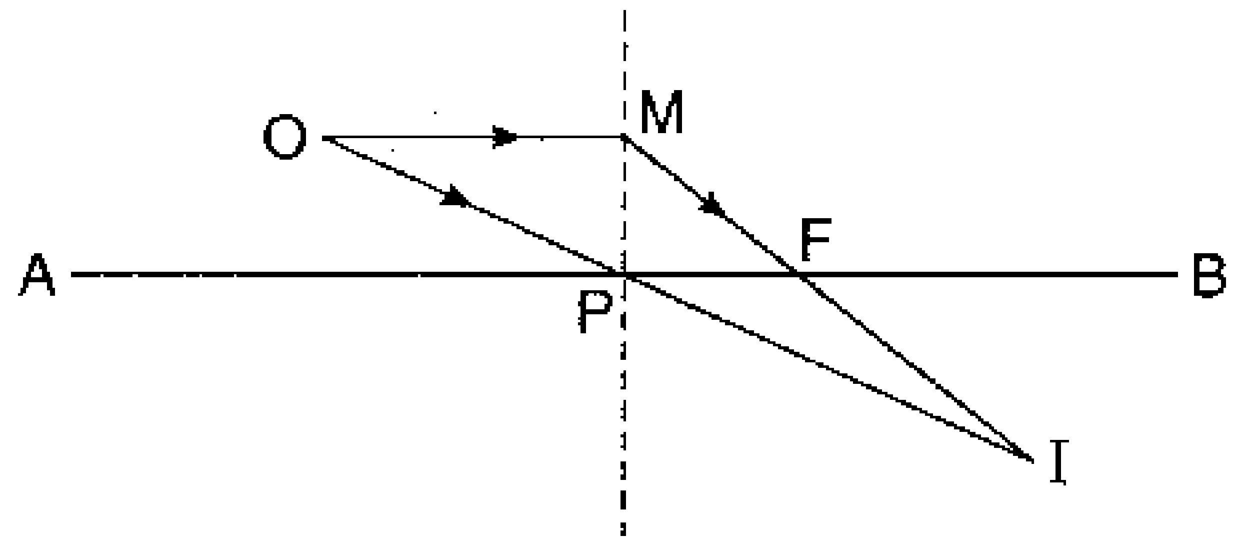


Fig. 22.73

(b) Lens formula : We have already discussed the lens formula,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

for finding the position of the lens. While using this formula appropriate sign of u, v and f must be included.

Magnification : The lateral, transverse or linear magnification m produced by a lens is defined by,

$$m=\frac{\text{height of image}}{\text{height of object}}=\frac{I}{O}$$

A real image II' of an object OO' formed by a convex lens is shown in figure.

$$\frac{\text{height of image}}{\text{height of object}}=\frac{II'}{OO'}=\frac{v}{u}$$

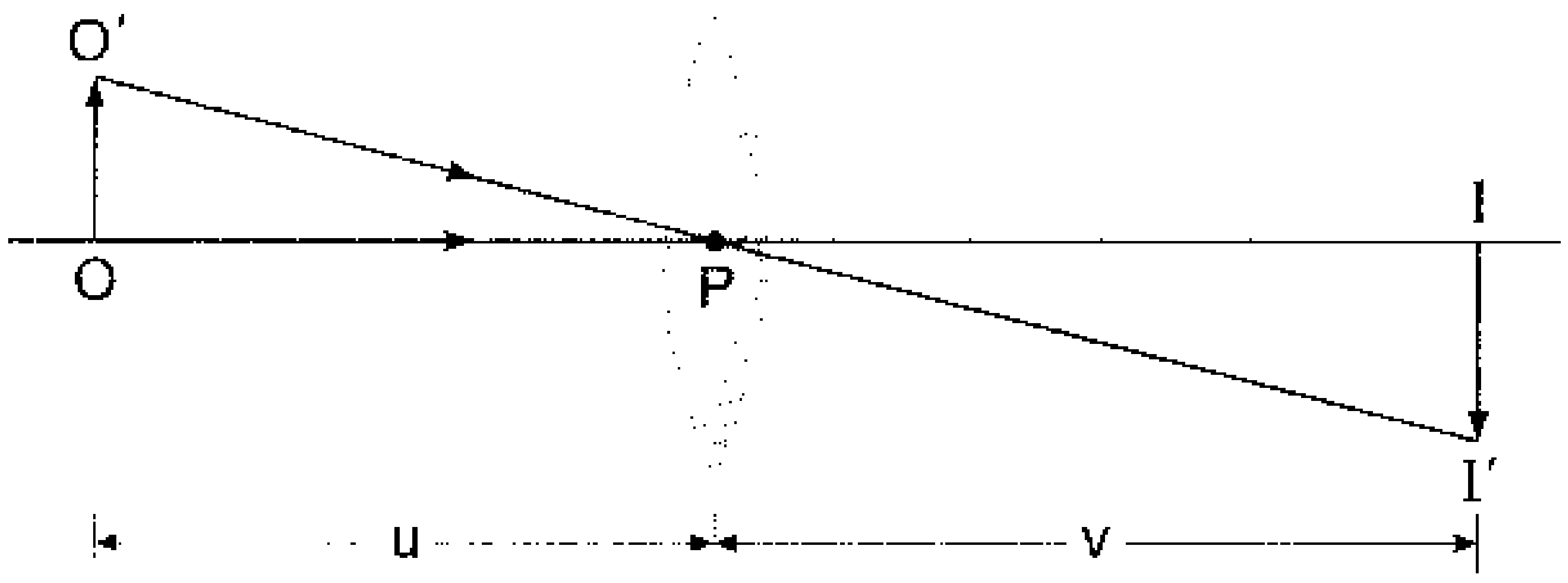


Fig. 22.74

Substituting v and u with proper sign,

$$\frac{II'}{OO'}=\frac{-I}{O}=\frac{v}{-u}$$

or $$\frac{I}{O}=m=\frac{v}{u}$$

Thus, $$m=\frac{v}{u}$$

Note : Suppose m is positive it means v and u are of same sign, *i.e.*, image and object are on the same side (left side), which implies that the image of a real object is virtual. Thus,
$m=+2$, means image is virtual, erect and two times magnified and $|v|=2|u|$.
Similarly $m=-\frac{1}{2}$ means image is real, inverted and diminished and $|v|=\frac{1}{2}|u|$.

EXAMPLE 22.15 *Find the distance of an object from a convex lens if image is two times magnified. Focal length of the lens is 10 cm.*

SOLUTION Convex lens forms both type of images real as well as virtual. Since, nature of the image is not mentioned in the question, we will have to consider both the cases.

When image is real : Means v is positive and u is negative with $|v|=2|u|$. Thus if

$$u=-x \quad \text{then} \quad v=2x \quad \text{and} \quad f=10\,\text{cm}$$

Substituting in $$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

We have $$\frac{1}{2x}+\frac{1}{x}=\frac{1}{10}$$

or $$\frac{3}{2x}=\frac{1}{10}$$

$\therefore$ $$x=15\text{ cm}$$ **Ans.**

$x=15$ cm, means object lies between F and $2F$.

When image is virtual : Means v and u both are negative. So let,

$$u=-y \quad \text{then} \quad v=-2y \quad \text{and} \quad f=10\text{ cm}$$

Substituting in, $$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

We have, $$\frac{1}{-2y}+\frac{1}{y}=\frac{1}{10}$$

or $$\frac{1}{2y}=\frac{1}{10}$$

$\therefore$ $$y=5\text{ cm}$$ **Ans.**

$y=5$ cm, means object lies between F and P.

Displacement method to determine the focal length of a convex lens

If the distance d between an object and screen is greater than 4 times the focal length of a convex lens, then there are two positions of the lens between the object and the screen at which a sharp image of the object is formed on the screen. This method is called **displacement method** and is used in laboratory to determine the focal length of convex lens.

To prove this, let us take an object placed at a distance u from a convex lens of focal length f. The distance of image from the lens $v=(d-u)$. From the lens formula,

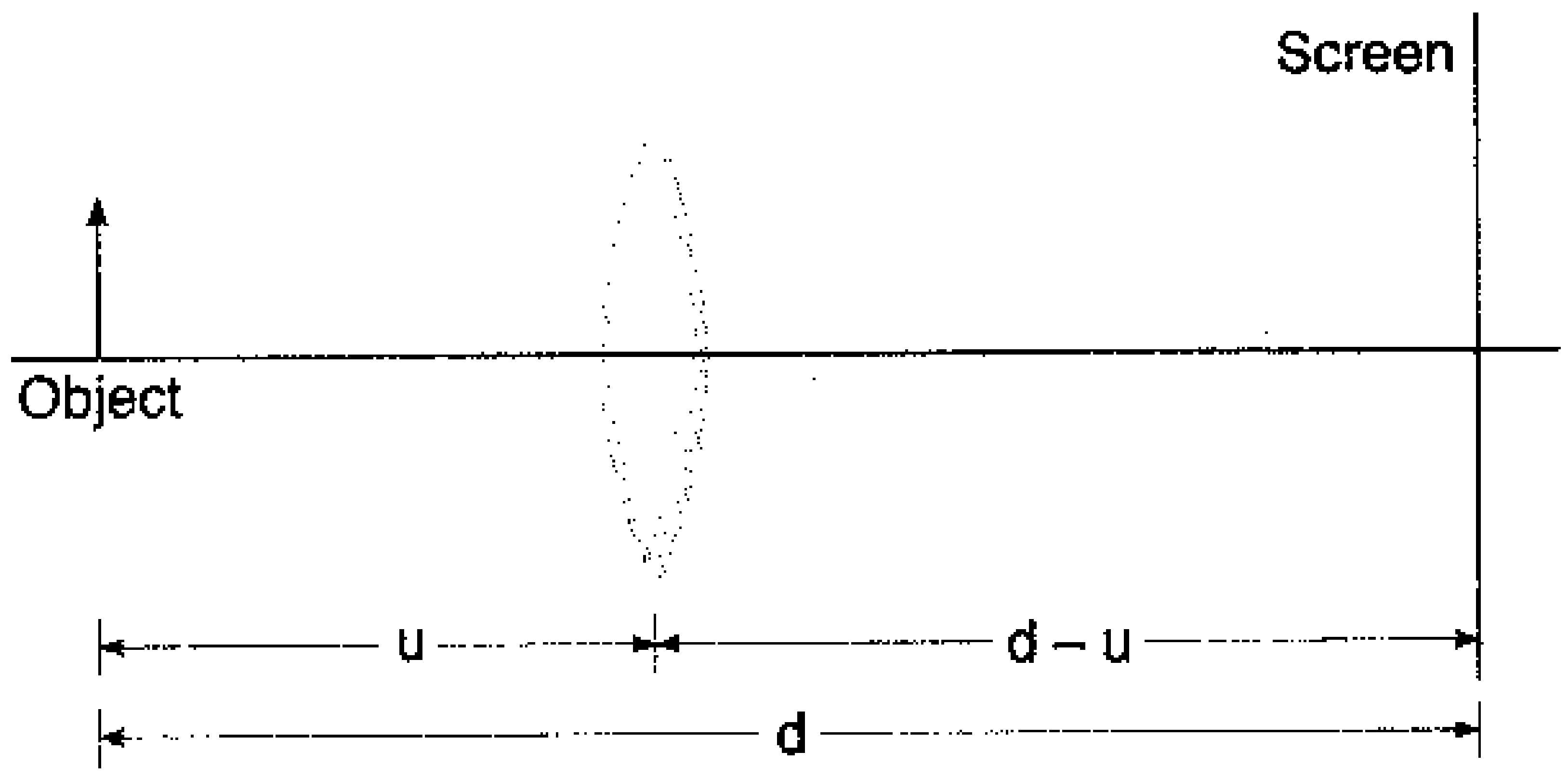


Fig. 22.75

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

We have, $$\frac{1}{d-u}-\frac{1}{-u}=\frac{1}{f}$$

or $$u^2-du+df=0$$

$\therefore$ $$u=\frac{d\pm\sqrt{d(d-4f)}}{2}$$

Now, there are following possibilities:

(i) If $d < 4f$, then u is imaginary.
So, physically no position of the lens is possible.

(ii) If $d = 4f$ then $u = \frac{d}{2} = 2f$. So only one position is possible. From here we can see that the **minimum distance between an object and its real image in case of a convex lens is $4f$.**

(iii) If $d > 4f$, there are two positions of lens at distances $\frac{d + \sqrt{d(d-4f}}{2}$ and $\frac{d - \sqrt{d(d-4f)}}{2}$ for which real image is formed on the screen.

(iv) Suppose I_1 is the image length in one position of the object and I_2 the image length in second position, then object length O is given by,

$$O = \sqrt{I_1 I_2}$$

This can be proved as under:

$$|u_1| = \frac{d + \sqrt{d(d-4f)}}{2} \qquad \therefore \ |v_1| = d - |u_1| = \frac{d - \sqrt{d(d-4f)}}{2}$$

$$|u_2| = \frac{d - \sqrt{d(d-4f)}}{2} \qquad \therefore \ |v_2| = d - |u_2| = \frac{d + \sqrt{d(d-4f)}}{2}$$

Now, $$|m_1 m_2| = \frac{I_1}{O} \times \frac{I_2}{O} = \frac{|v_1|}{|u_1|} \times \frac{|v_2|}{|u_2|}$$

Substituting the values, we get $$\frac{I_1 I_2}{O^2} = 1$$

or $$O = \sqrt{I_1 I_2}$$ **Hence Proved.**

Focal length of two or more thin lenses in contact

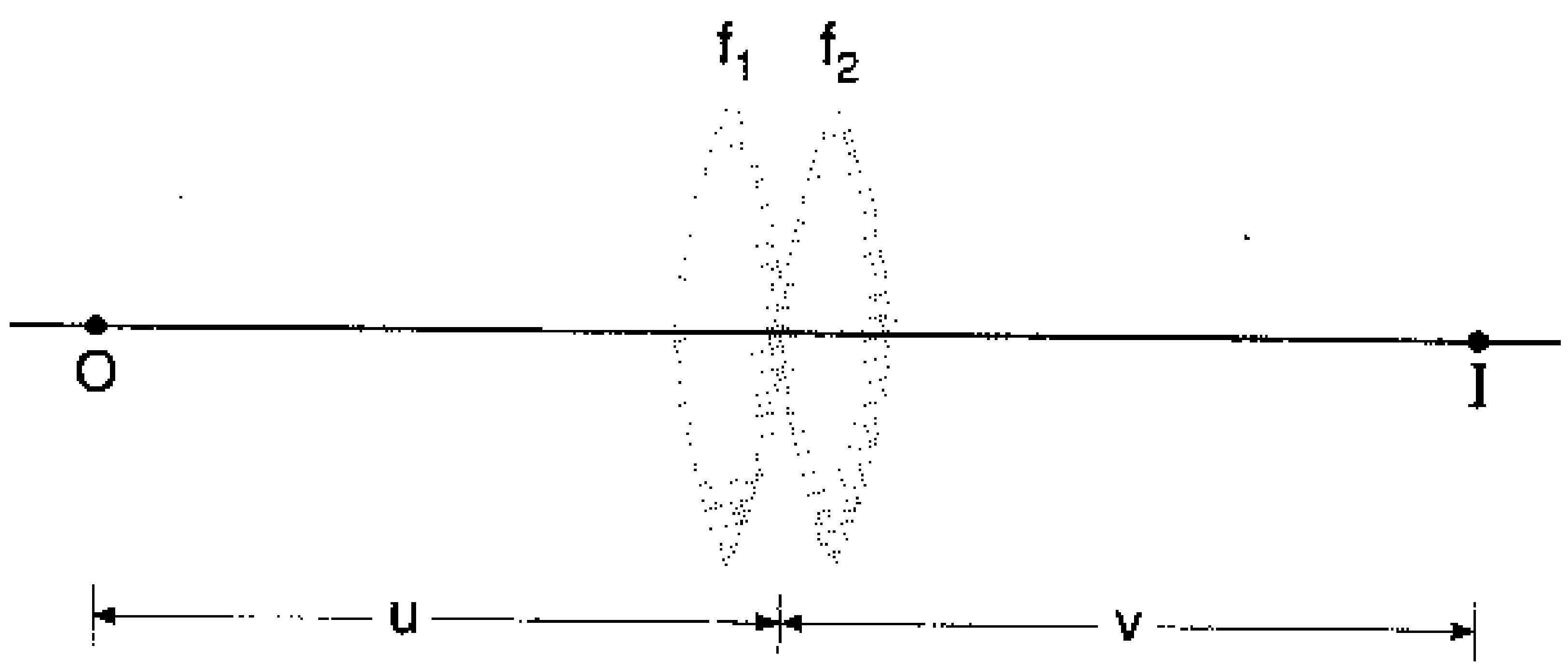


Fig. 22.76

Combinations of lenses in contact are used in many optical instruments to improve their performance.

Suppose two lenses of focal lengths f_1 and f_2 are kept in contact and a point object O is placed at a distance u from the combination. The first image (say I_1) after refraction from the first lens is formed at a distance v_1 (whatever may be the sign of v_1) from the combination. This image I_1 acts as an object for the second lens and let v be the distance of the final image from the combination. Applying the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

for the two lenses, we have $$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1} \qquad \text{...(i)}$$

and $$\frac{1}{v}-\frac{1}{v_1}=\frac{1}{f_2} \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we have $$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F}\text{ (say)}$$

Here F is the equivalent focal length of the combination. Thus,

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

Similarly for more than two lenses in contact, the equivalent focal length is given by the formula,

$$\frac{1}{F}=\sum_{i=1}^{n}\frac{1}{f_i}$$

Note : Here f_1, f_2 etc., are to be substituted with sign.

EXAMPLE 22.16 *A thin plano-convex lens of focal length f is split into two halves. One of the halves is shifted along the optical axis as shown in figure. The separation between object and image planes is 1.8 m. The magnification of the image, formed by one of the half lens is 2. Find the focal length of the lens and separation between the two halves. Draw the ray diagram for image formation.* **(JEE 1996)**

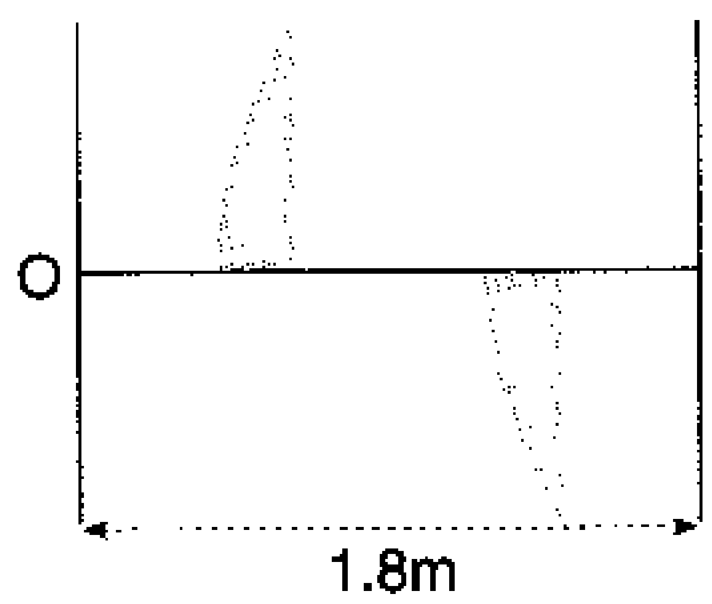


Fig. 22.77

SOLUTION For both the halves, position of object and image is same. Only difference is of magnification. Magnification for one of the halves is given as 2 (>1). This can be for the first one, because for this, $|v|>|u|$. Therefore, magnification, $|m|=|v/u|>1$. So, for the first half

$$|v/u|=2 \quad \text{or} \quad |v|=2|u|$$

Let $$u=-x, \quad \text{then } v=+2x$$

and $$|u|+|v|=1.8\text{ m}$$

i.e., $$3x=1.8\text{ m} \quad \text{or} \quad x=0.6\text{ m}$$

Hence, $$u=-0.6\text{ m} \quad \text{and} \quad v=+1.2\text{ m}$$

Using $$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}=\frac{1}{1.2}-\frac{1}{-0.6}=\frac{1}{0.4}$$

$$\therefore \quad f=0.4\text{ m}$$ **Ans.**

For the second half $$\frac{1}{f}=\frac{1}{1.2-d}-\frac{1}{-(0.6+d)}$$

or $$\frac{1}{0.4}=\frac{1}{1.2-d}+\frac{1}{(0.6+d)}$$

Solving this, we get $$d=0.6\text{ m}$$ **Ans.**

Magnification for the second half will be

$$m_2 = \frac{v}{u} = \frac{0.6}{-(1.2)} = -\frac{1}{2}$$

and magnification for the first half is $m_1 = \frac{v}{u} = \frac{1.2}{-(0.6)} = -2$

The ray diagram is as follows:

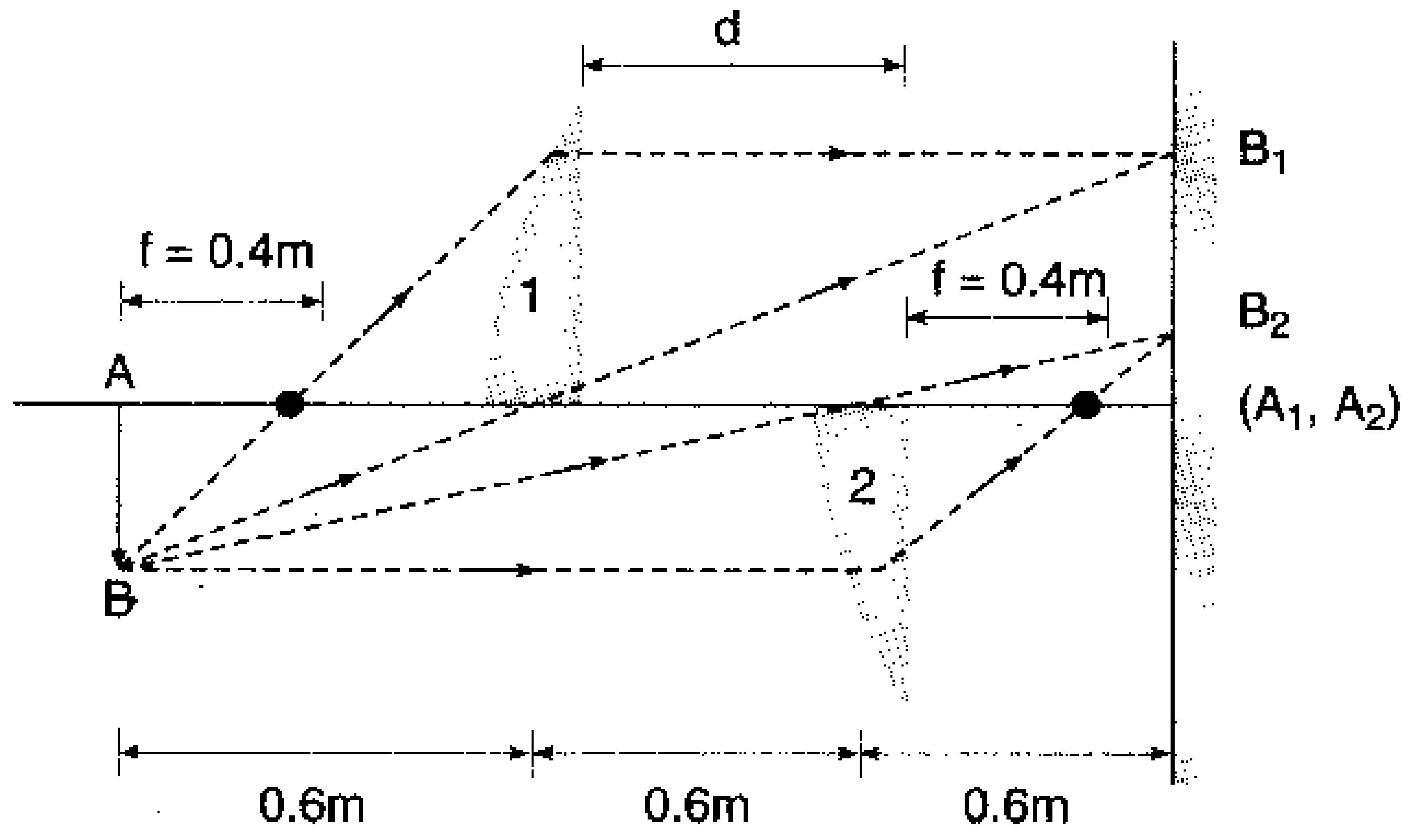


Fig. 22.78

EXAMPLE 22.17 *A converging lens of focal length 5.0 cm is placed in contact with a diverging lens of focal length 10.0 cm. Find the combined focal length of the system.*

SOLUTION Here, $f_1 = +5.0\,\text{cm}$, and $f_2 = -10.0\,\text{cm}$

Therefore, the combined focal length F is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{5.0} - \frac{1}{10.0} = +\frac{1}{10.0}$$

$\therefore$ $F = +10.0\,\text{cm}$ **Ans.**

i.e., the combination behaves as a converging lens of focal length 10.0 cm.

IIT-JEE GALAXY 22.3

1. **Power of an optical instrument :** By optical power of an instrument (whether it is a lens, mirror or a refractive surface) we mean the ability of the instrument to deviate the path of rays passing through it. If the instrument converges the rays parallel to the principal axis its power is said positive and if it diverges the rays it is said a negative power.

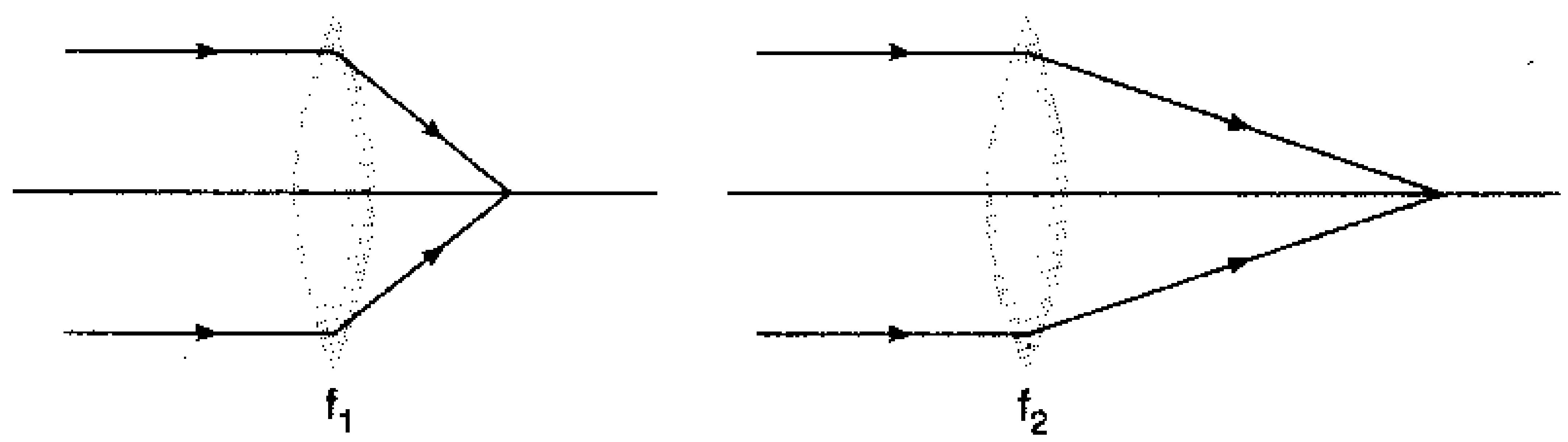


Fig. 22.79

The shorter the focal length of a lens (or a mirror) the more it converges or diverges light. As shown in the figure,

$$f_1 < f_2$$

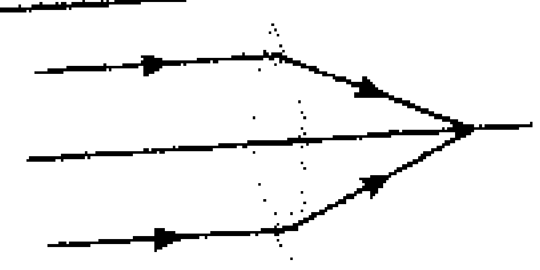
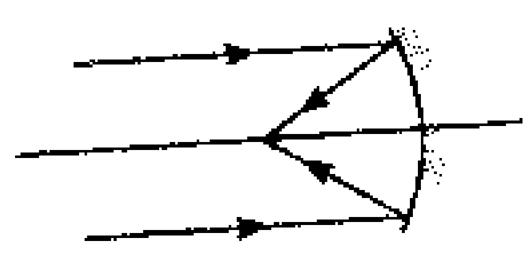
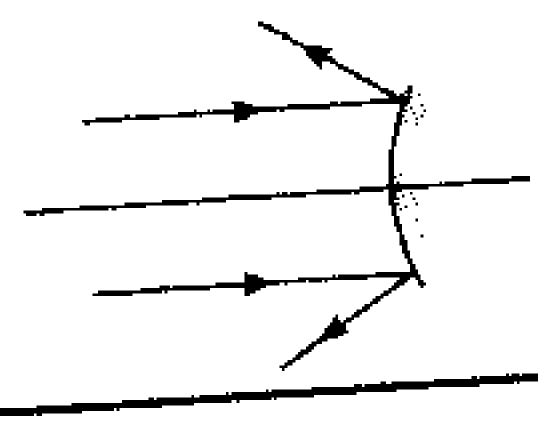
and hence the power $P_1 > P_2$, as bending of light in case 1 is more than that of case 2. For a lens,

$$P\text{ (in dioptre)} = \frac{1}{f\text{ (metre)}}$$

and for a mirror,

$$P\text{ (in dioptre)} = \frac{-1}{f\text{ (metre)}}$$

Following table gives the sign of P and f for different type of lens and mirror.

Nature of lens/mirror	**Focal length (f)**	**Power $P_L = \frac{1}{f}, P_M = -\frac{1}{f}$**	**Converging/ diverging**	**Ray diagram**
Convex lens	+ ve	+ ve	converging	
Concave mirror	– ve	+ ve	converging	
Concave lens	– ve	– ve	diverging	
Convex mirror	+ ve	– ve	diverging	

Thus, convex lens and concave mirror have positive power or they are converging in nature. Concave lens and convex mirror have negative power or they are diverging in nature.

2. **Under what conditions does an image coincides with object :** Real image coincide with object only when there is a mirror and the rays fall normally on the mirror. In case of a spherical mirror it is possible when object for the mirror lies at it centre of curvature, *i.e.*, at a distance $2f$ from the mirror. If

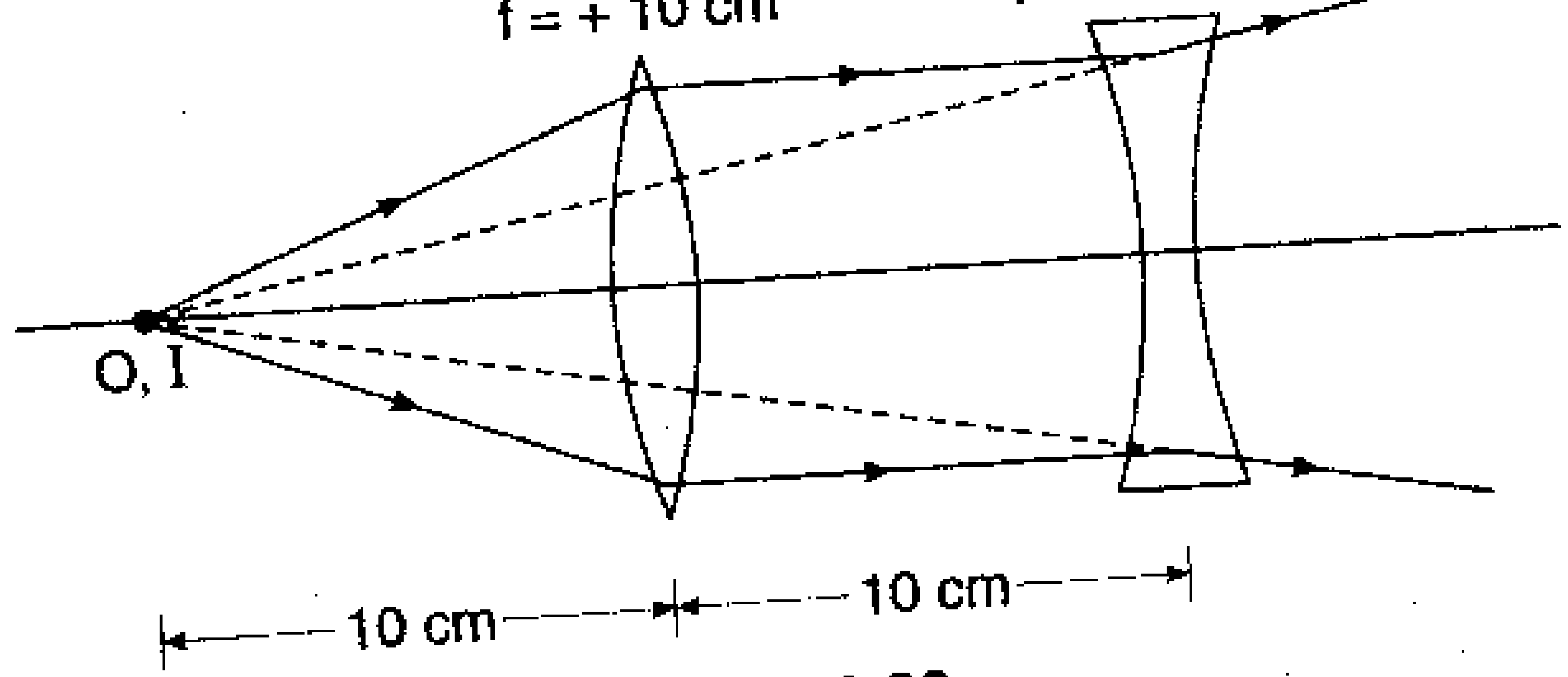


Fig. 22.80

the virtual image coincides with the object then there should be no mirror at all. Many possibilities may exist in this case. One of them is shown in figure.

3. To find the position of image when one face of a lens is silvered : Let us make a formula to find the position of image under such condition.

Ray of light is first refracted, then reflected and then again refracted. In first two steps light is travelling from left to right and in the last one direction of light is reversed. But we will take one sign convention, *i.e.*, left to right as positive and in the last step will take v, u and R as negative.

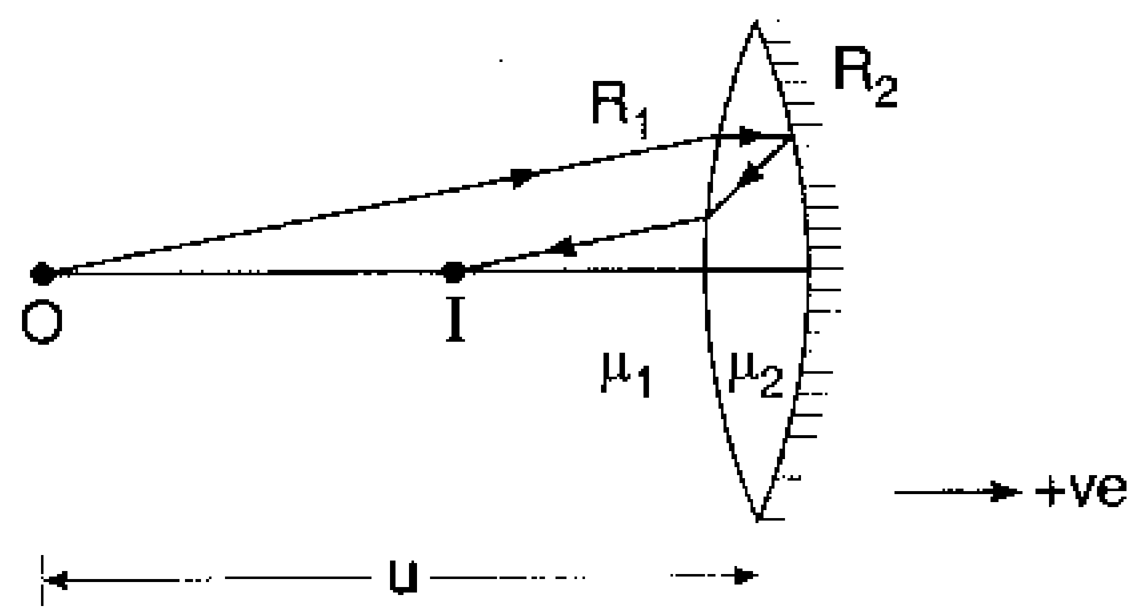


Fig. 22.81

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R_1} \quad \text{...(i)}$$

$$\frac{1}{v_2} + \frac{1}{v_1} = \frac{2}{R_2} \quad \text{...(ii)}$$

$$\frac{\mu_1}{-v} - \frac{\mu_2}{-v_2} = \frac{\mu_1 - \mu_2}{-R_1} \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\frac{1}{v} + \frac{1}{u} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1} \quad \text{...(iv)}$$

This is the desired formula for finding position of image for the given situation.

Note : The given system finally behaves as a mirror. Whose focal length can be found by comparing Eq. (iv) with mirror formula $1/v + 1/u = 1/f$.

$$\frac{1}{f} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1} \quad \text{...(v)}$$

EXAMPLE 22.18 *A spherical convex surface separates object and image space of refractive index* 1.0 *and* $\frac{4}{3}$. *If radius of curvature of the surface is* 10 *cm, find its power.*

SOLUTION Let us see where does the parallel rays converge (or diverge) on the principal axis. Let us call it the focus and the corresponding length the focal length f. Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with proper values and signs, we have

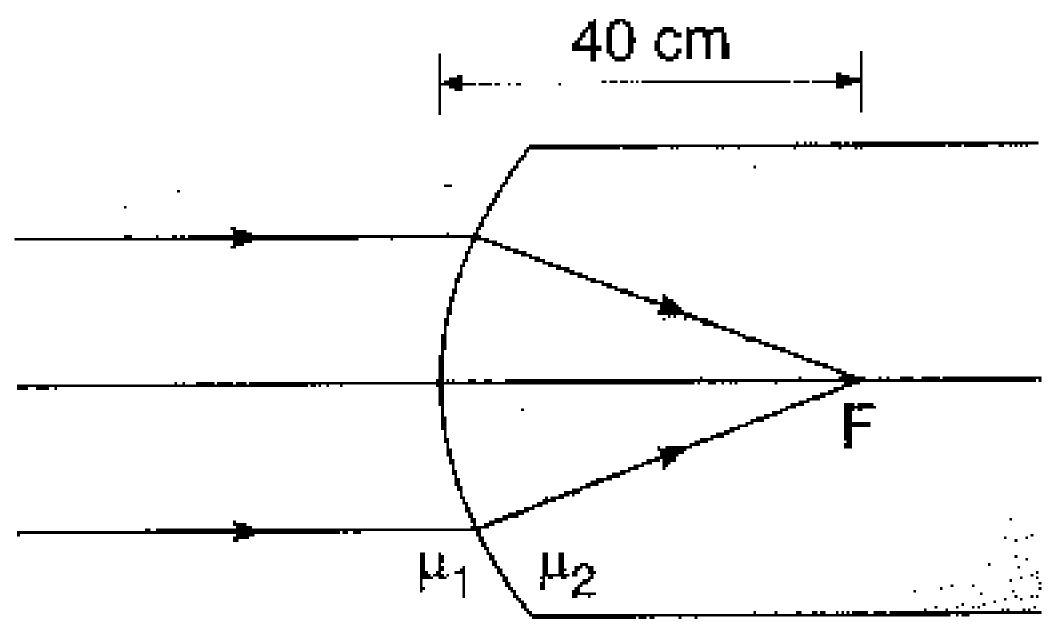


Fig. 22.82

$$\frac{4/3}{f} - \frac{1.0}{\infty} = \frac{4/3 - 1.0}{+10}$$

or $$f = 40 \text{ cm} = 0.4 \text{ m}$$

Since, the rays are converging, its power should be positive. Hence,

$$P \text{ (in dioptre)} = \frac{+1}{f \text{ (metre)}} = \frac{1}{0.4}$$

or $$P = 2.5 \text{ dioptre}$$ **Ans.**

EXAMPLE 22.19 *A biconvex thin lens is prepared from glass of refractive index 3/2. The two bounding surfaces have equal radii of 25 cm each. One of the surfaces is silvered from outside to make it reflecting. Where should an object be placed before this lens so that the image coincides with the object.*

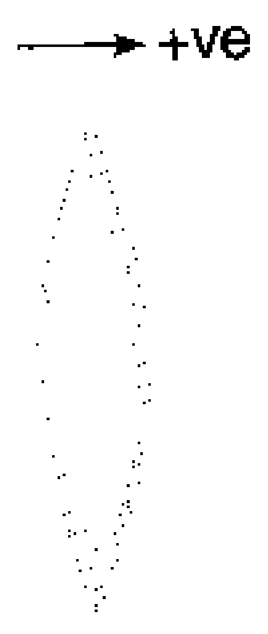


Fig. 22.83

SOLUTION Refer IIT-JEE Galaxy 22.3 point number (3).

Here $R_1 = +25\,\text{cm},\ R_2 = -25\,\text{cm}, \mu_1 = 1$ and $\mu_2 = 3/2$

Image coincides with object, hence, $u = v = -x$ (say)

Substituting in equation (iv), we have

$$\frac{1}{-x} - \frac{1}{x} = \frac{2(3/2)}{-25} - \frac{2(3/2-1)}{25}$$

or

$$\frac{2}{x} = \frac{3}{25} + \frac{1}{25} = \frac{4}{25}$$

$\therefore$

$$x = 12.5\ \text{cm}$$

Ans.

Hence, the object should be placed at a distance 12.5 cm in front of the silvered lens.

INTRODUCTORY EXERCISE 22.6

1. When an object is placed 60 cm in front of a diverging lens, a virtual image is formed 20 cm from the lens. The lens is made of a material of refractive index $\mu = 1.65$ and its two spherical surfaces have the same radius of curvature. What is the value of this radius?
2. A converging lens has a focal length of 30 cm. Rays from a 2.0 cm high filament that pass through the lens form a virtual image at a distance of 50 cm from the lens. Where is the filament located? What is the height of the image?
3. Show that the focal length of a thin lens is not changed when the lens is rotated so that the left and the right surfaces are interchanged.
4. As an object is moved from the surface of a thin converging lens to a focal point, over what range does the image distance vary?
5. A diverging lens is made of material with refractive index 1.3 and has identical concave surfaces of radius 20 cm. The lens is immersed in a transparent medium with refractive index 1.8.
   (a) What is now the focal length of the lens?
   (b) What is the minimum distance that an immersed object must be from the lens so that a real image is formed?
6. An object is located 20 cm to the left of a converging lens with $f = 10$ cm. A second identical lens is placed to the right of the first lens and then moved until the image it produces is identical in size and orientation to the object. What is the separation between the lenses?
7. Suppose an object has thickness du so that it extends from object distance u to $u + du$. Prove that the thickness dv of its image is given by $\left(-\frac{v^2}{u^2}\right) du$, so the longitudinal magnification $\frac{dv}{du} = -m^2$, where m is the lateral magnification.
8. Two thin similar convex glass pieces are joined together front to front, with its rear portion silvered such that a sharp image is formed 0.2 m for an object at infinity. When the air between the glass pieces is replaced by water $\left(\mu = \frac{4}{3}\right)$, find the position of image.

9. When a pin is moved along the principal axis of a small concave mirror, the image position coincides with the object at a point 0.5 m from the mirror. If the mirror is placed at a depth of 0.2 m in a transparent liquid, the same phenomenon occurs when the pin is placed 0.4 m from the mirror. Find the refractive index of the liquid.

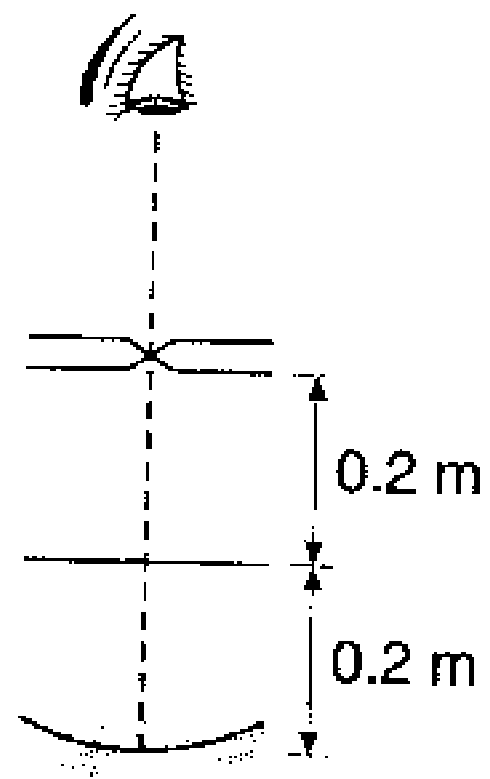


Fig. 22.84

10. When a lens is inserted between an object and a screen which are a fixed distance apart the size of the image is either 6 cm or $\frac{2}{3}$ cm. Find size of the object.

11. A lens of focal length 12 cm forms an upright image three times the size of a real object. Find the distance in cm between the object and images.

12. The distance between an object and its upright image is 20 cm. If the magnification is 0.5, what is the focal length of the lens that is being used to form the image?

13. A thin lens of focal length + 10.0 cm lies on a horizontal plane mirror. How far above the lens should an object be held if its image is to coincide with the object?

22.7 TOTAL INTERNAL REFLECTION (TIR)

Figure shows the reflection and refraction of a light ray at the interface between a denser and a rare medium, whose refractive indices are μ_D and μ_R. Angle of incidence in denser medium is i and angle of refraction is r.

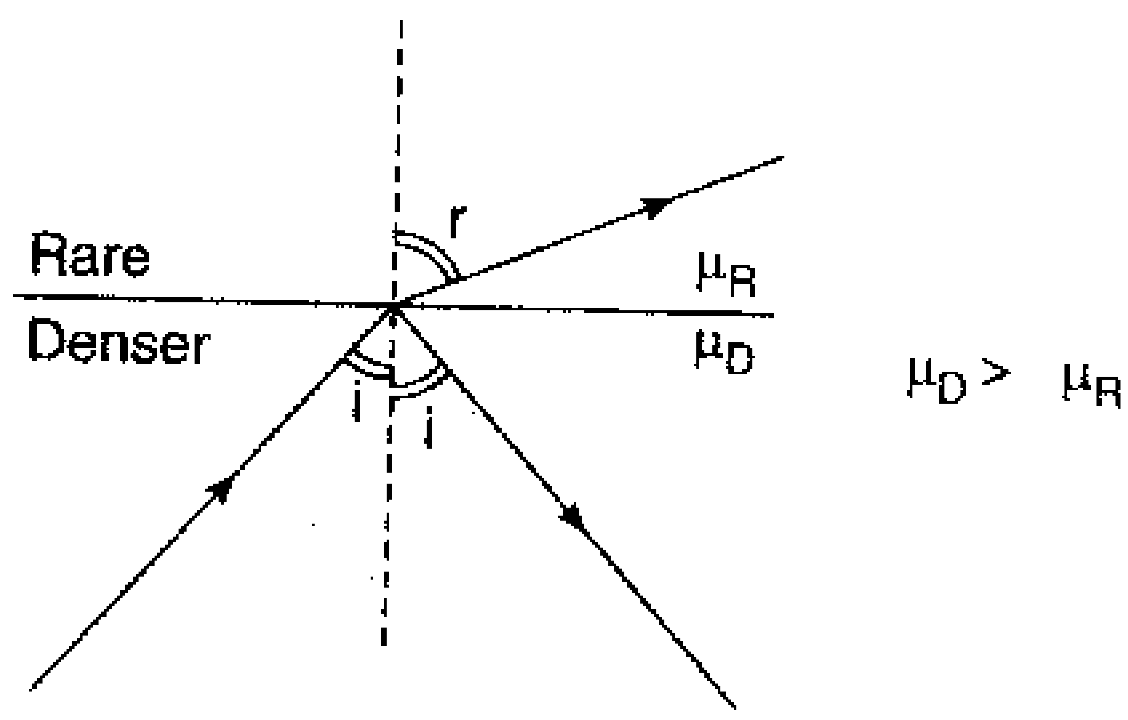


Fig. 22.85

From Snell's law ($\mu \sin i = \text{constant}$), we may write

$$\mu_D \sin i = \mu_R \sin r$$

or

$$\frac{\mu_D}{\mu_R} \sin i = \sin r$$

The right hand side of this equation is a sine function that has a range from 0 to 1. The left hand side must therefore, have the same range, *i.e.*,

$$1 \geq \frac{\mu_D}{\mu_R} \sin i \geq 0 \quad \text{or} \quad \frac{\mu_R}{\mu_D} \geq \sin i \geq 0$$

or

$$\theta_c \geq i \geq 0$$

Here

$$\theta_c = \sin^{-1}\left(\frac{\mu_R}{\mu_D}\right) \quad \ldots\text{(i)}$$

is called the **critical angle**. When the angle of incidence exceeds θ_c, no refracted beam is observed and the incident beam is completely reflected at the boundary. This phenomenon known as **total internal reflection (TIR)**, only occurs when the light travels from a denser medium to a rare medium. When the rare medium is air,

$$\mu_R = 1 \quad \text{and} \quad \mu_D = \mu$$

$$\therefore \qquad \theta_c = \sin^{-1}\left(\frac{1}{\mu}\right) \qquad \text{...(ii)}$$

TIR has following applications.

(i) Totally reflecting prisms : Refractive index of crown glass is 3/2. Hence,

$$\theta_c = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{2}{3}\right) \approx 42°$$

A ray *OA* incident normally on face *PQ* of a crown glass prism suffers TIR at face *PR* since, the angle of incidence in the optically denser medium is 45°. A bright ray *AB* emerges at right angles to face *QR*. The prism thus, reflects the ray through 90°. Light can be reflected through 180° and an erect image can be obtained of an inverted one if the prism is arranged as shown in figure (b).

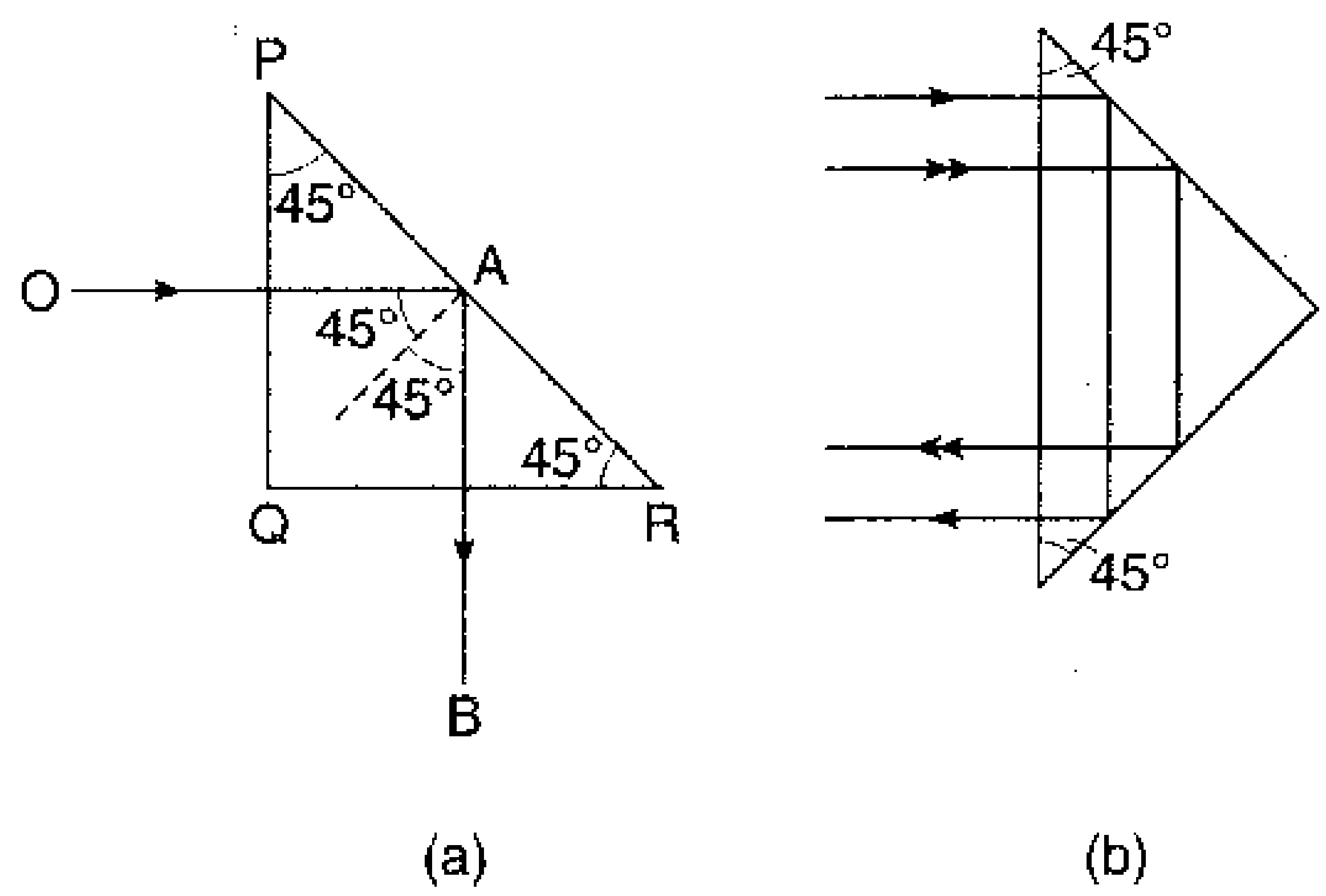


Fig. 22.86 Prism reflectors

(ii) Optical fibres : Light can be confined within a bent glass rod by TIR and so 'piped' along a twisted path as in figure. The beam is reflected from side to side practically without loss (except for that due to absorption in the glass) and emerges only at the end of the rod where it strikes the surface almost normally, *i.e.*, at an angle less than the critical angle. A single, very thin, solid glass fibre behaves in the same way and if several thousand are taped together a flexible light pipe is obtained that can be used, for example in medicine and engineering to illuminate an inaccessible spot. Optical fibres are now a days used to carry telephone, television and computer signals from one place to the other.

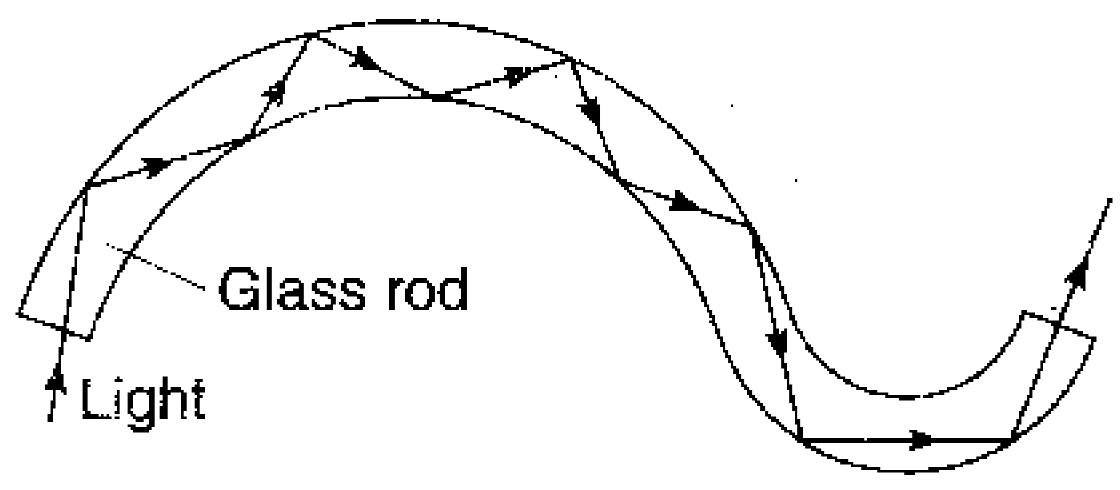


Fig. 22.87 Principle of an optical fibre.

Note : As we have seen $\qquad \theta_c = \sin^{-1}\left(\frac{\mu_R}{\mu_D}\right)$

Suppose we have two sets of media 1 and 2 and

$$\left(\frac{\mu_R}{\mu_D}\right)_1 < \left(\frac{\mu_R}{\mu_D}\right)_2$$

then $\qquad (\theta_c)_1 < (\theta_c)_2$

So a ray of light has more chances to have TIR in case 1.

EXAMPLE 22.20 *An isotropic point source is placed at a depth h below the water surface. A floating opaque disc is placed on the surface of water so that the bulb is not visible from the surface. What is the minimum radius of the disc. Take refractive index of water* = μ.

SOLUTION

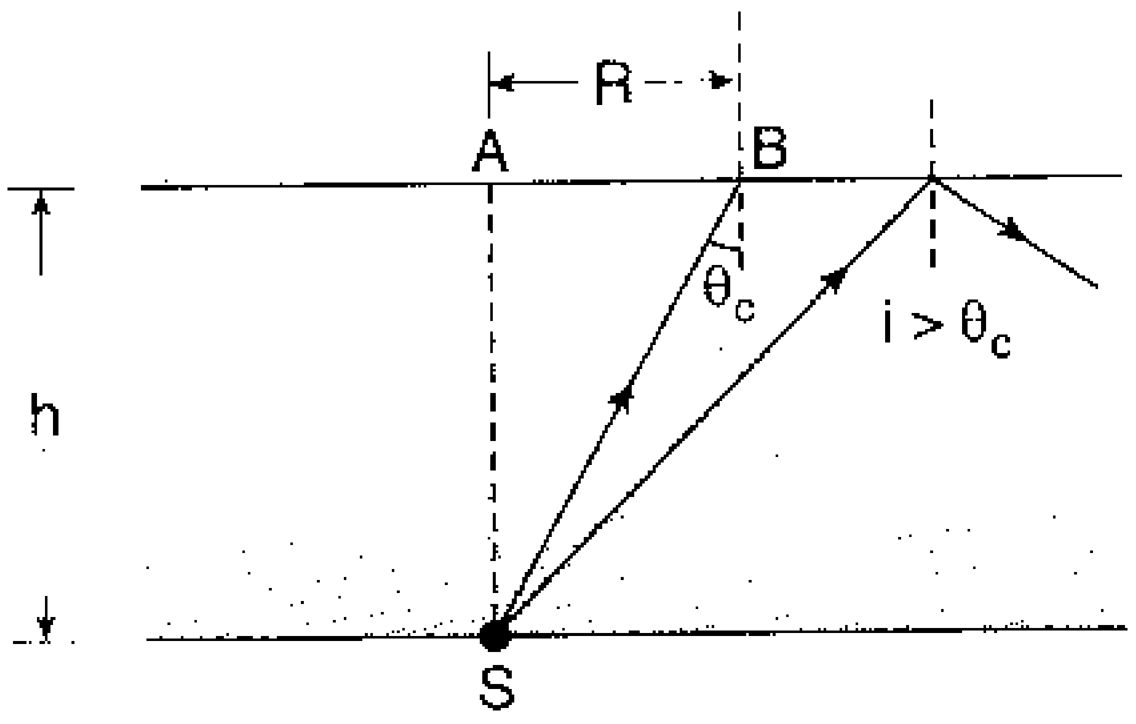


Fig. 22.88

As shown in figure light from the source will not emerge out of water if $i > \theta_c$.

Therefore, minimum radius R corresponds to $i = \theta_c$

In ΔSAB,

$$\frac{R}{h} = \tan\theta_c$$

$\therefore$ $$R = h\tan\theta_c$$

or $$R = \frac{h}{\sqrt{\mu^2 - 1}}$$ **Ans.**

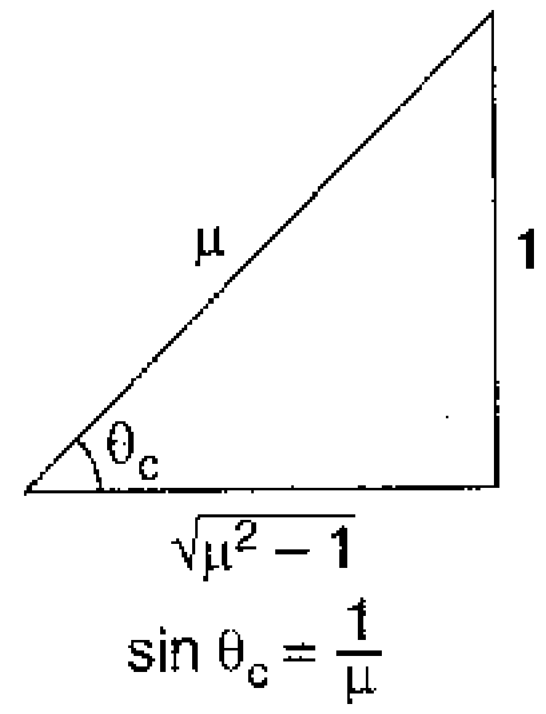


Fig. 22.89

EXAMPLE 22.21 *A point source of light is placed at a distance h below the surface of a large and deep lake. Show that the fraction f of light that escapes directly from water surface is independent of h and is given by,*

$$f = \frac{[1 - \sqrt{1 - 1/\mu^2}]}{2}$$

SOLUTION Due to TIR, light will be reflected back into the water if $i > \theta_c$. So only that portion of incident light will escape which passes through the cone of angle $\theta = 2\theta_c$.

So, the fraction of light escaping

$$f = \frac{\text{area } ACB}{\text{total area of sphere}}$$

$$= \frac{2\pi R^2(1 - \cos\theta_c)}{4\pi R^2} = \frac{1 - \cos\theta_c}{2}$$

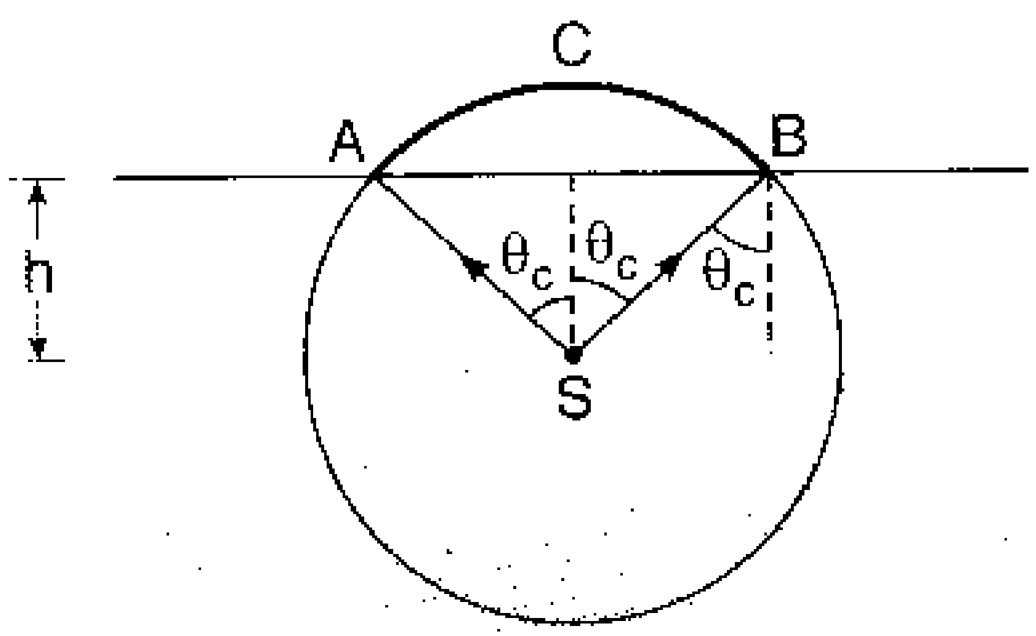


Fig. 22.90

Now, as f depends on θ_c and which depends only on μ, it is independent of h. **Proved.**

Further $$\cos\theta_c = \frac{\sqrt{\mu^2 - 1}}{\mu} = \sqrt{1 - 1/\mu^2}$$

$\therefore$ $$f = \frac{1 - \sqrt{1 - 1/\mu^2}}{2}$$ **Ans.**

Note : Area of $ACB = 2\pi R^2(1 - \cos\theta_c)$ can be obtained by integration.

22.8 REFRACTION THROUGH PRISM

A prism has two plane surfaces AB and AC inclined to each other as shown in figure. $\angle A$ is called the **angle of prism** or **refracting angle.**

The importance of the prism really depends on the fact that the angle of deviation suffered by light at the first refracting surface, say AB (in 2-dimensional figure) is not cancelled out by the deviation at the second surface AC (as it is in a parallel glass slab), but is added to it. This is why it can be used in a spectrometer, an instrument for analysing light into its component colours.

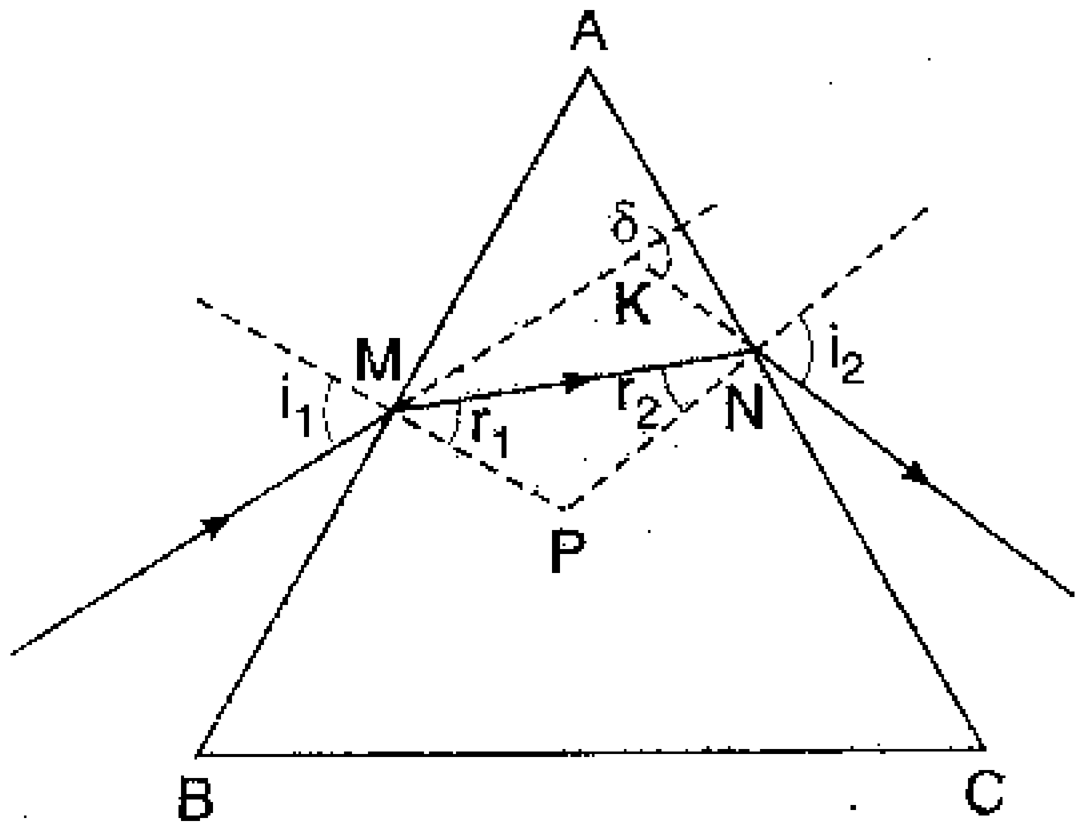


Fig. 22.91

General Formulae

(i) In quadrilateral $AMPN$, $\angle AMP + \angle ANP = 180°$

$\therefore$ $A + \angle MPN = 180°$ ...(i)

In triangle MNP, $r_1 + r_2 + \angle MPN = 180°$ ...(ii)

From Eqs. (i) and (ii), we have $r_1 + r_2 = A$ ...(iii)

(ii) Deviation : Deviation δ means angle between incident ray and emergent ray.

In reflection

$$\delta = 180 - 2i = 180 - 2r$$

in refraction

$$\delta = |i - r|$$

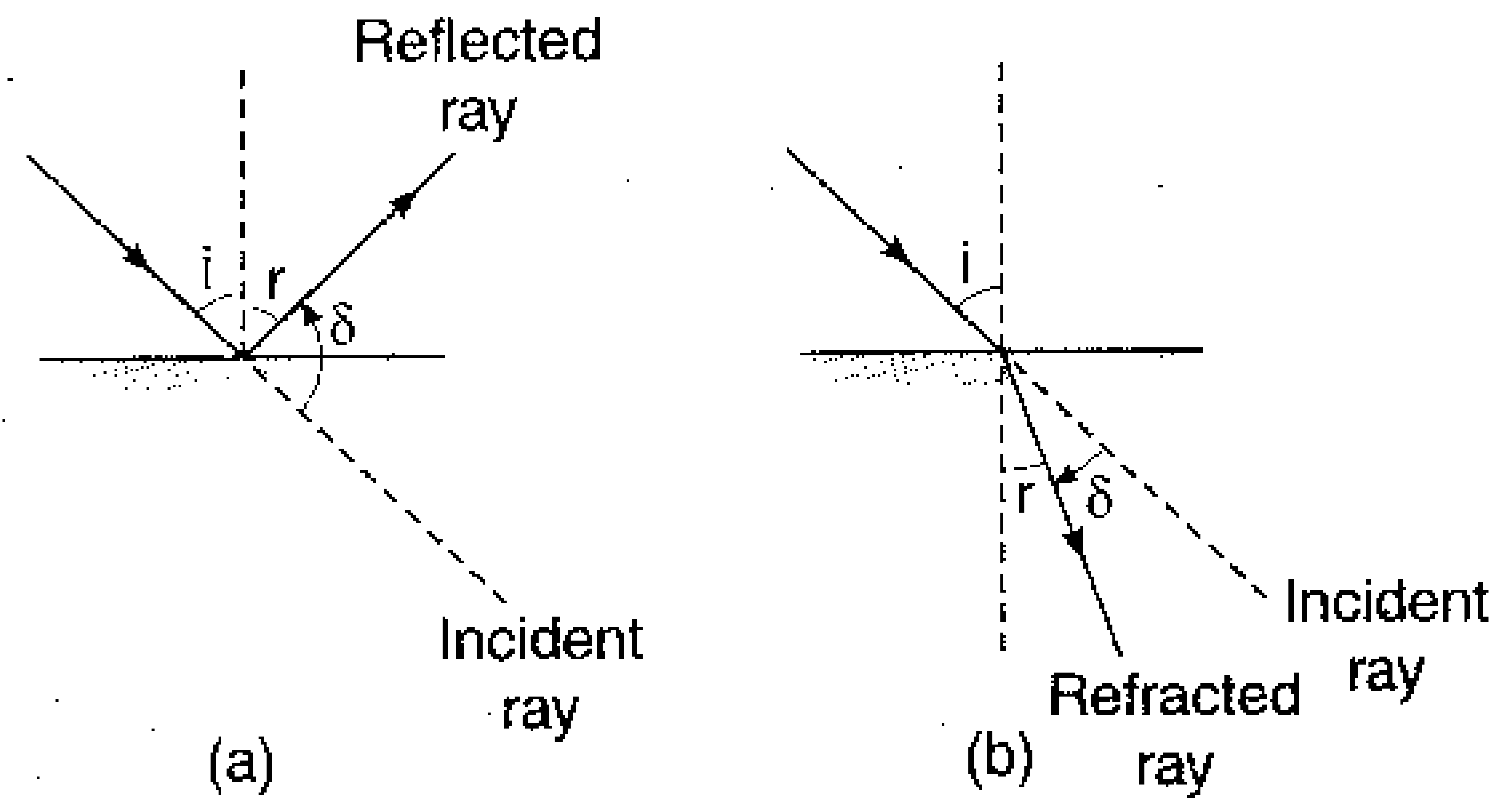


Fig. 22.92

In prism a ray of light gets refracted twice one at M and other at N. At M its deviation is $i_1 - r_1$ and at N it is $i_2 - r_2$. These two deviations are added. So the net deviation is,

$$\delta = (i_1 - r_1) + (i_2 - r_2) = (i_1 + i_2) - (r_1 + r_2) = (i_1 + i_2) - A$$

Thus, $$\delta = (i_1 + i_2) - A$$ ...(iv)

(iii) If A and i_1 are small : The expression for the deviation in this case is basically used for developing the lens theory. Consider a ray falling almost normally in air on a prism of small angle A (less than about 6° or 0.1 radian) so that angle i_1 is small. Now $\mu = \frac{\sin i_1}{\sin r_1}$, therefore, r_1 will also be small.

Hence, since sine of a small angle is nearly equal to the angle in radians, we have

$$i_1 = \mu r_1$$

Also, $A = r_1 + r_2$ and so if A and r_1 are small r_2 and i_2 will also be small. From $\mu = \frac{\sin i_2}{\sin r_2}$ we can say

$$i_2 = \mu r_2$$

Substituting these values in Eq. (iv), we have

$$\delta = (\mu r_1 + \mu r_2) - A = \mu(r_1 + r_2) - A = \mu A - A$$

or
$$\delta = (\mu - 1) A \qquad \text{...(v)}$$

This expression shows that for a given angle A all rays entering a small angle prism at small angles of incidence suffer the same deviation.

(iv) Minimum deviation : It is found that the angle of deviation δ varies with the angle of incidence i_1 of the ray incident on the first refracting face of the prism. The variation is shown in figure and for one angle of incidence it has a minimum value δ_{min}. At this value the ray passes symmetrically through the prism (a fact that can be proved theoretically as well as be shown experimentally), *i.e.*, the angle of emergence of the ray from the second face equals the angle of incidence of the ray on the first face.

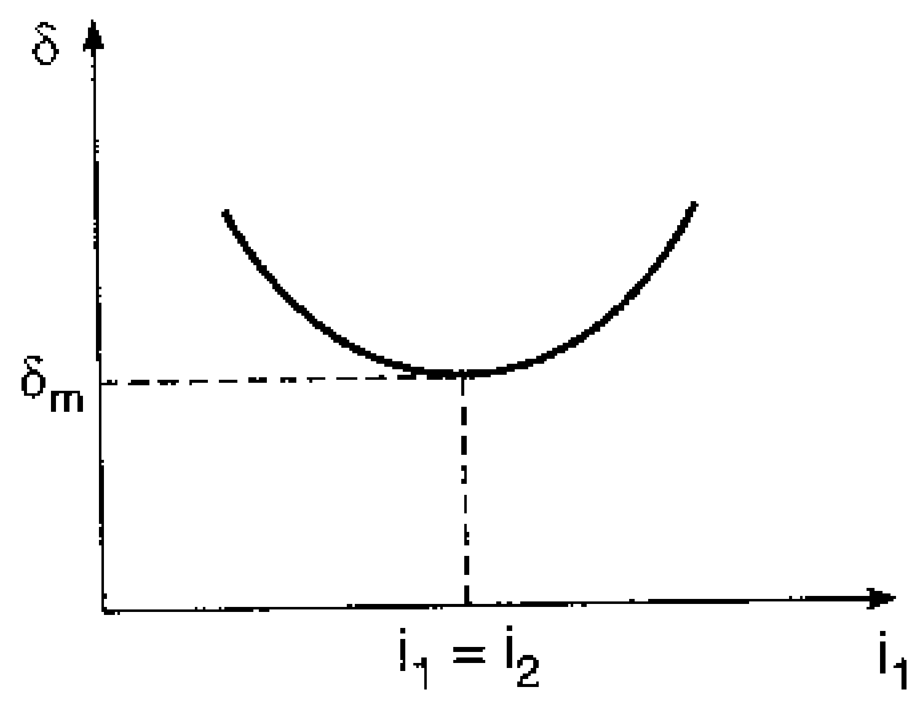


Fig. 22.93

$$i_2 = i_1 = i \qquad \text{...(vi)}$$

It therefore, follows that
$$r_1 = r_2 = r \qquad \text{...(vii)}$$

From Eqs. (iii) and (vii)
$$r = \frac{A}{2}$$

Further at
$$\delta = \delta_m = (i + i) - A$$

or
$$i = \frac{A + \delta_m}{2} \qquad \text{...(viii)}$$

$\therefore$
$$\mu = \frac{\sin i}{\sin r}$$

or
$$\mu = \frac{\sin\left(\dfrac{A + \delta_m}{2}\right)}{\sin \dfrac{A}{2}} \qquad \text{...(ix)}$$

(v) Condition of no emergence : In this section we want to find the condition such that a ray of light entering the face AB does not come out of the face AC for any value of angle i_1, *i.e.*, TIR takes place on AC

$$r_1 + r_2 = A$$

$\therefore$
$$r_2 = A - r_1$$

or
$$(r_2)_{min} = A - (r_1)_{max} \qquad \text{...(x)}$$

Now r_1 will be maximum when i_1 is maximum and maximum value of i_1 can be 90°.

Hence,
$$\mu = \frac{\sin(i_1)_{max}}{\sin(r_1)_{max}} = \frac{\sin 90°}{\sin (r_1)_{max}}$$

$\therefore$
$$\sin(r_1)_{max} = \frac{1}{\mu} = \sin \theta_c$$

$\therefore$
$$(r_1)_{max} = \theta_c$$

$\therefore$ From Eq. (x),
$$(r_2)_{min} = A - \theta_c \qquad \text{...(xi)}$$

Now, if minimum value of r_2 is greater than θ_c then obviously all values of r_2 will be greater than θ_c and TIR will take place under all conditions. Thus, the condition of no emergence is,

$$(r_2)_{\min} > \theta_c \qquad \text{or} \qquad A - \theta_c > \theta_c$$

or

$$A > 2\theta_c \qquad \ldots\text{(xii)}$$

(vi) Dispersion and deviation of light by a prism : White light is a superposition of waves with wavelengths extending throughout the visible spectrum. The speed of light in vacuum is the same for all wavelengths, but the speed in a material substance is different for different wavelengths. Therefore, the index of refraction of a material depends on wavelength. In most materials the value of refractive index μ decreases with increasing wavelength.

If a beam of white light, which contains all colours, is sent through the prism, it is separated into a spectrum of colours. The spreading of light into its colour components is called **dispersion**.

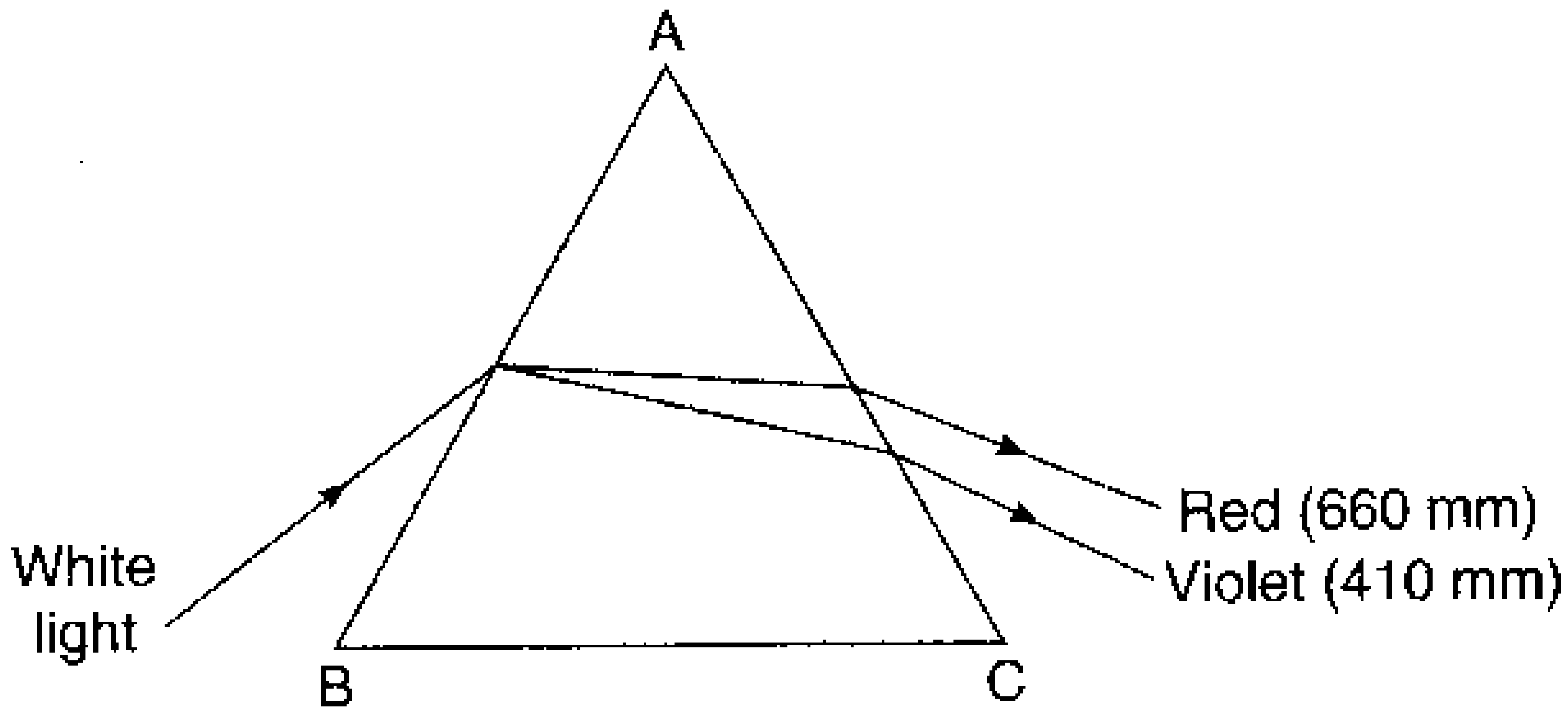


Fig. 22.94

Dispersive Power

When a beam of white light is passed through a prism of transparent material light of different wavelengths are deviated by different amounts. If δ_r, δ_y and δ_v are the deviations for red, yellow and violet components then average deviation is measured by δ_y as yellow light falls in between red and violet. $\delta_v - \delta_r$ is called **angular dispersion**. The **dispersive power** of a material is defined as the ratio of angular dispersion to the average deviation when a white beam of light is passed through it. It is denoted by ω. As we know

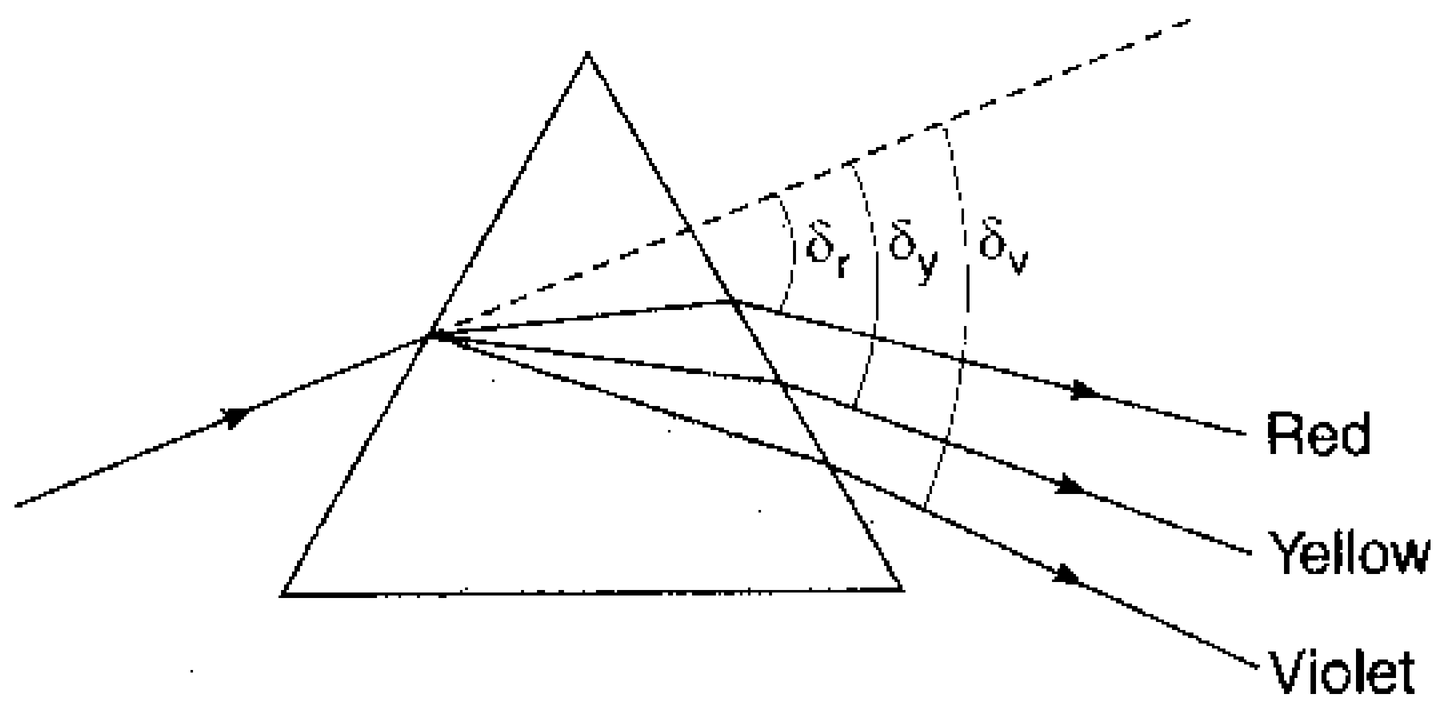


Fig. 22.95

$$\delta = (\mu - 1)\, A$$

This equation is valid when A and i are small. Suppose, a beam of white light is passed through such a prism, the deviation of red, yellow and violet light are

$$\delta_r = (\mu_r - 1)A, \qquad \delta_y = (\mu_y - 1)A$$

and

$$\delta_v = (\mu_v - 1)\, A$$

The angular dispersion is $\delta_v - \delta_r = (\mu_v - \mu_r)\, A$ and the average deviation is $\delta_y = (\mu_y - 1)\, A$. Thus, the dispersive power of the medium is,

$$\omega = \frac{\mu_v - \mu_r}{\mu_y - 1} \qquad \ldots\text{(i)}$$

Dispersion without average deviation and average deviation without dispersion

Figure shows two prisms of refracting angles A and A' and dispersive powers ω and ω' respectively. They are placed in contact in such a way that the two refracting angles are reversed with respect to each other. A ray of light passes through the combination as shown. The deviation produced by the two prisms are,

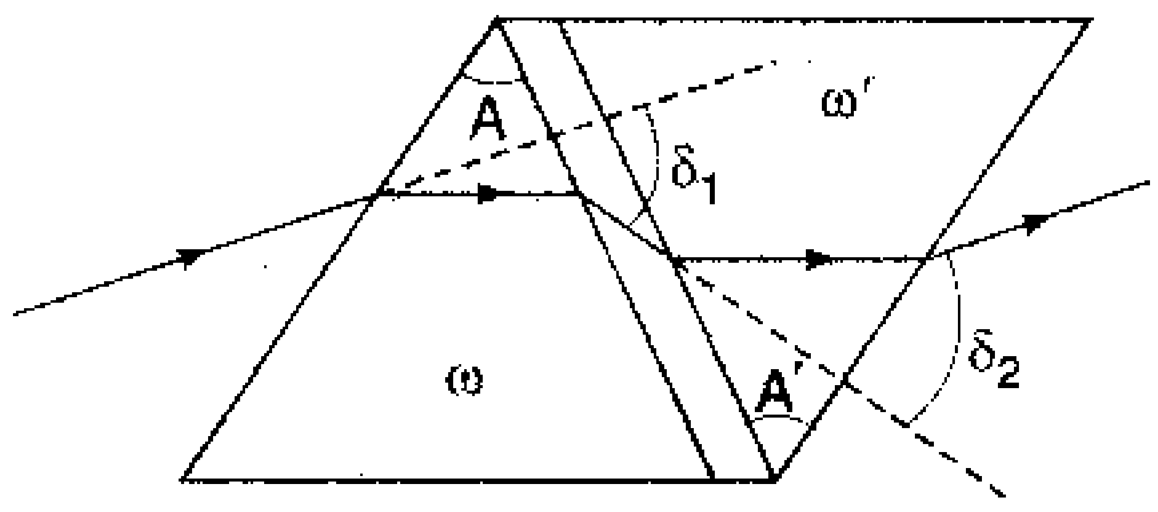


Fig. 22.96

$$\delta_1 = (\mu - 1) A$$

and $$\delta_2 = (\mu' - 1) A'$$

As the two deviations are opposite to each other, the net deviation is,

$$\delta = \delta_1 - \delta_2 = (\mu - 1)A - (\mu' - 1) A' \quad \text{...(ii)}$$

Using this equation, the average deviation produced by the combination if white light is passed is,

$$\delta_y = (\mu_y - 1)A - (\mu'_y - 1) A' \quad \text{...(iii)}$$

and the net angular dispersion is,

$$\delta_v - \delta_r = (\mu_v - \mu_r)A - (\mu'_v - \mu'_r)A'$$

But as $\mu_v - \mu_\gamma = \omega(\mu_y - 1)$ from Eq. (i), we have

$$\delta_v - \delta_r = (\mu_y - 1)\omega A - (\mu'_y - 1)\omega' A' \quad \text{...(iv)}$$

Dispersion without average deviation : From Eq. (iii),

$$\delta_\gamma = 0 \quad \text{if}$$

$$\frac{A}{A'} = \frac{\mu'_y - 1}{\mu_y - 1} \quad \text{...(v)}$$

This is the required condition of dispersion without average deviation. Using this in Eq. (iv) the net angular dispersion produced is:

$$\delta_v - \delta_r = (\mu_y - 1)A(\omega - \omega')$$

Average deviation without dispersion : From equation (iv),

$$\delta_v - \delta_r = 0 \text{ if}$$

$$\frac{A}{A'} = \frac{(\mu'_y - 1)\omega'}{(\mu_y - 1)\omega} = \frac{\mu'_v - \mu'_r}{\mu_v - \mu_r} \quad \text{...(vi)}$$

This is the required condition of average deviation without dispersion. Using the above condition in Eq. (iii) the net average deviation is,

$$\delta_\gamma = (\mu_y - 1)A\left(1 - \frac{\omega}{\omega'}\right)$$

IIT-JEE GALAXY 22.4

1. To find the total deviation of a ray of light, its sense of rotation (clockwise or anticlockwise) should be kept in mind in each reflection and refraction and they should be added and subtracted accordingly.

For example, in the figure shown

$$\delta_A = i_1 - r_1 \quad \text{(clockwise)}$$

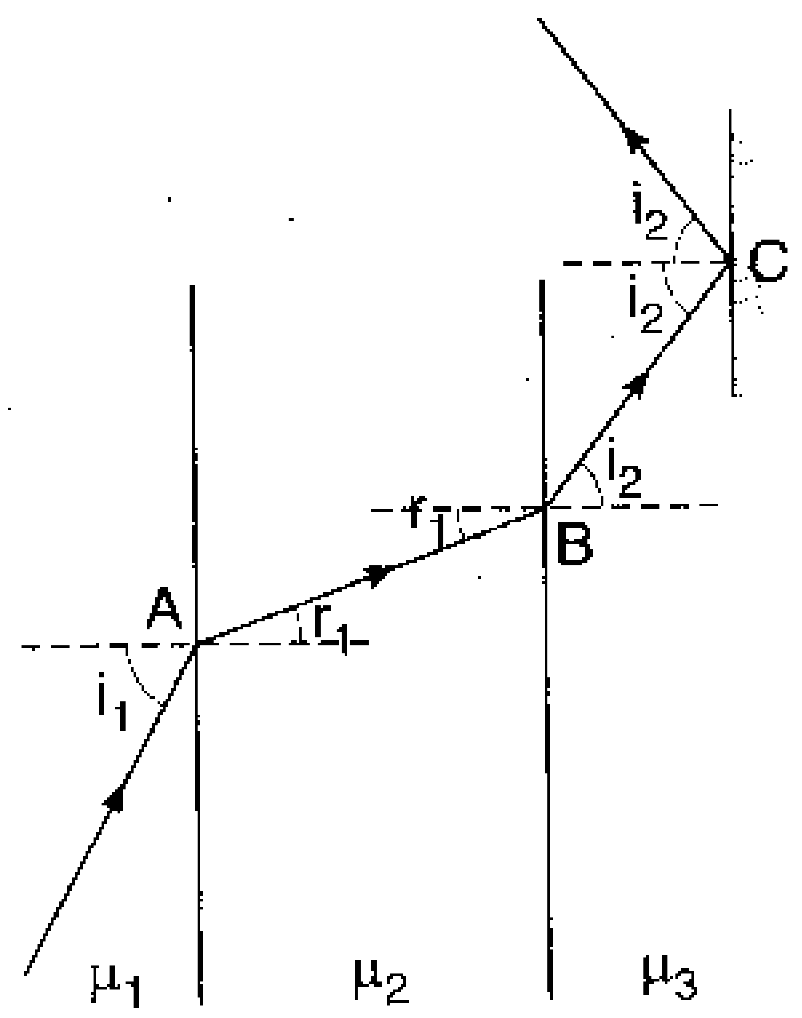
$$\delta_B = i_2 - r_1 \quad \text{(anticlockwise)}$$

$$\delta_C = 180° - 2i_2 \quad \text{(anticlockwise)}$$

$$\therefore \quad \delta_{Total} = \delta_A - \delta_B - \delta_C$$

$$= (i_1 - r_1) - (i_2 - r_1) - (180° - 2i_2)$$

(clockwise)



$\mu_2 > \mu_1 > \mu_3$

Fig. 22.97

2. Equation $r_1 + r_2 = A$ can be applied at any of the three vertices. For example in the figure shown, $r_1 + r_2 = B$.

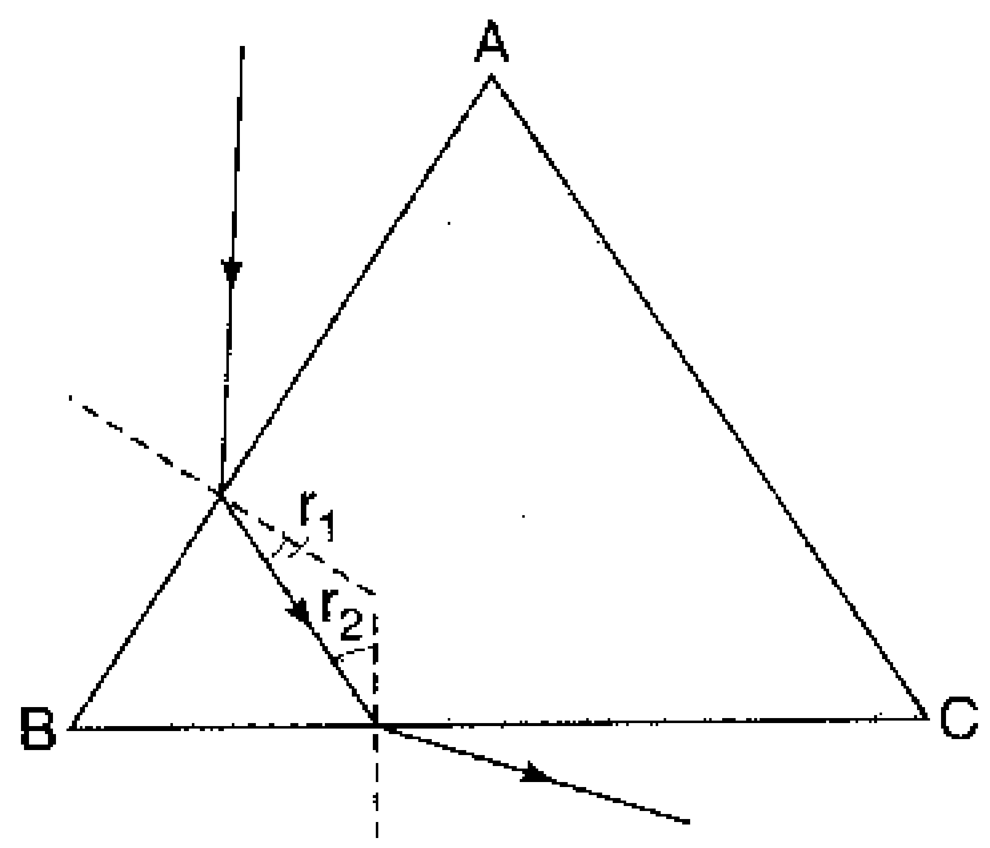


Fig. 22.98

3. Sometimes a part of a prism is given and we keep on thinking whether how should we proceed? To solve such problems first complete the prism then solve as the problems of prism are solved.

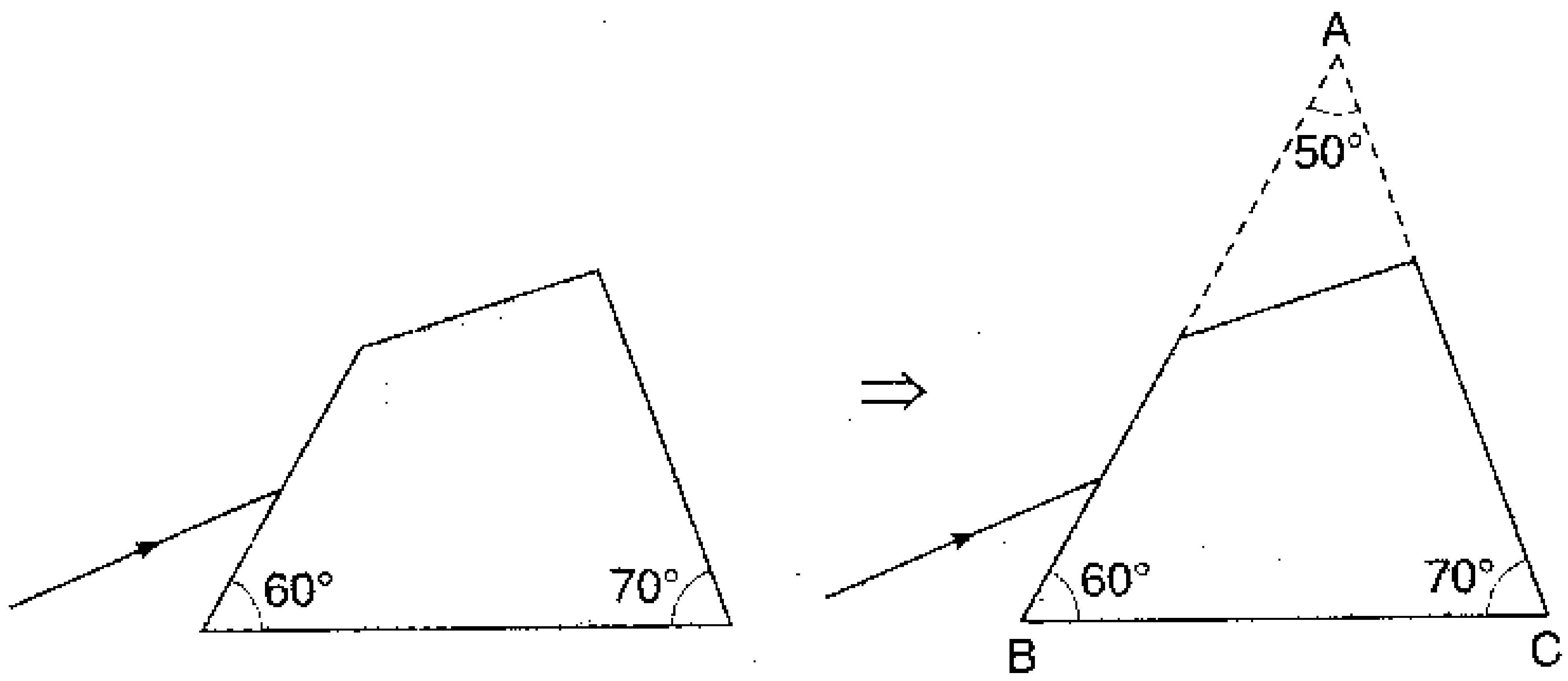


Fig. 22.99

EXAMPLE 22.22 *An isosceles glass prism has one of its faces coated with silver. A ray of light is incident normally on the other face (which is equal to the silvered face). The ray of light is reflected twice on the same sized faces and then emerges through the base of the prism perpendicularly. Find angles of prism.*

Note : Most of the problems of prims are easily solved by drawing proper ray diagram and then applying laws of geometry with the basic knowledge of prism formulae.

SOLUTION $r_1 = 0 \quad \therefore \quad r_2 = A = 180° - 2\theta$ ...(i)

$$\angle DFE = 180° - 90° - 2r_2$$
$$= 180° - 90° - 360° + 4\theta$$
$$= 4\theta - 270° \quad \text{...(ii)}$$
$$\therefore \quad r_3 = 90° - \angle DFE = 360° - 4\theta \quad \text{...(iii)}$$
$$\angle BFG = 90° - \theta = 90° - r_3$$
or
$$r_3 = \theta \quad \text{...(iv)}$$

From Eqs. (iii) and (iv)

$$5\theta = 360°$$
$$\therefore \quad \theta = 72°$$
and
$$180° - 2\theta = 36°$$

$\therefore$ Angles of prism are 72°, 72° and 36°. **Ans.**

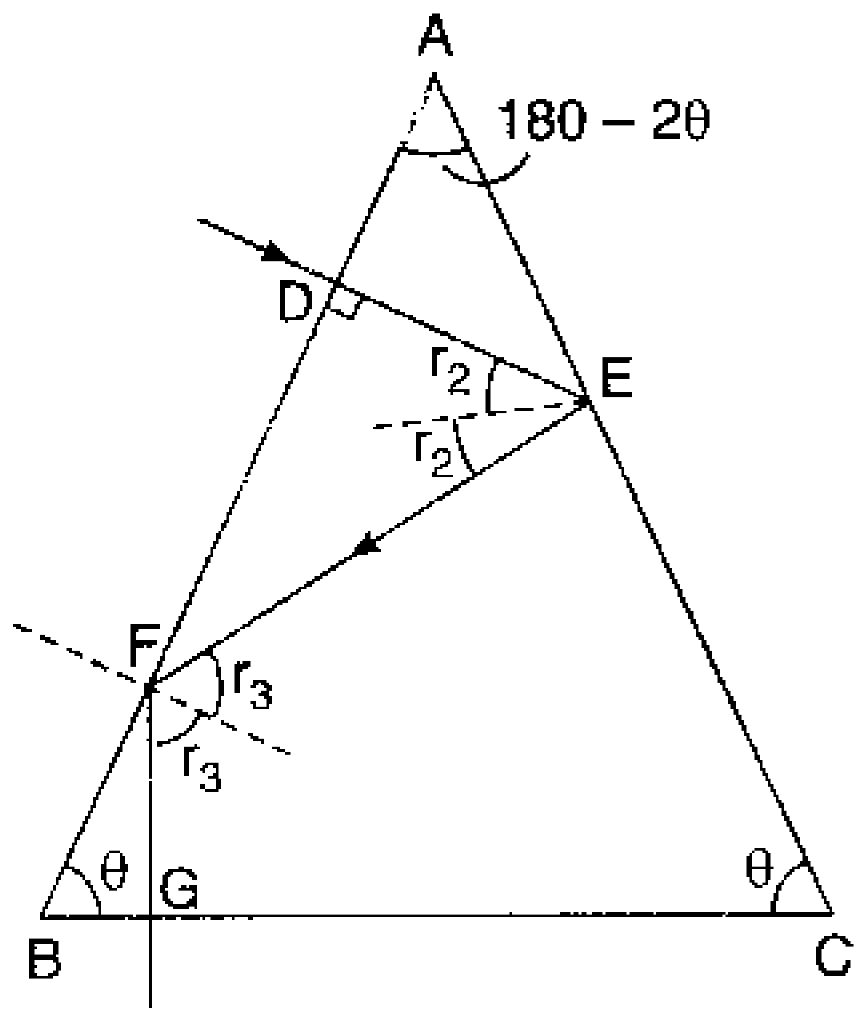


Fig. 22.100

INTRODUCTORY EXERCISE 22.7

1. The prism shown in figure has a refractive index of 1.60 and the angles *A* are 30°. Two light rays *P* and *Q* are parallel as they enter the prism. What is the angle between them after they emerge?

   [$\sin^{-1}(0.8) = 53°$]

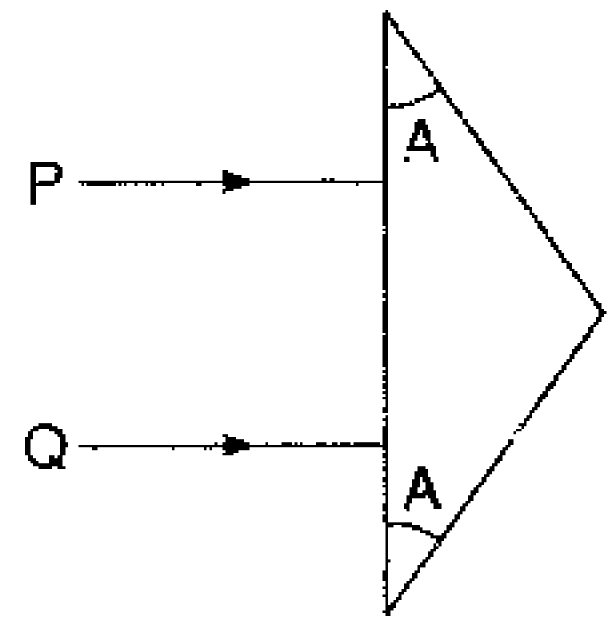


Fig. 22.101

2. Light is incident normally on the short face of a 30° – 60° – 90° prism. A liquid is poured on the hypotenuse of the prism. If the refractive index of the prism is $\sqrt{3}$, find the maximum refractive index of the liquid so that light is totally reflected.

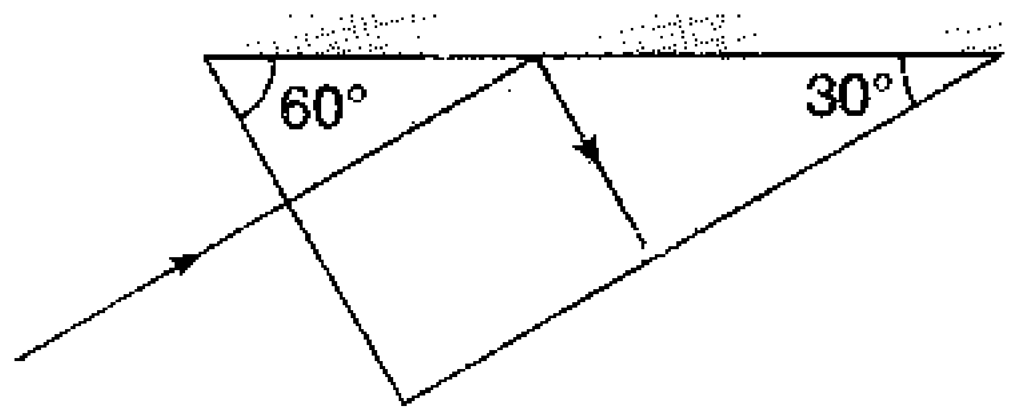


Fig. 22.102

3. A glass vessel in the shape of a triangular prism is filled with water, and light is incident normally on the face *XY*. If the refractive indices for water and glass are 4/3 and 3/2 respectively, total internal reflection will occur at the glass-air surface *XZ* only for sin θ greater than

   A 1/2 **B** 2/3 **C** 3/4
   D 8/9 **E** 16/27.

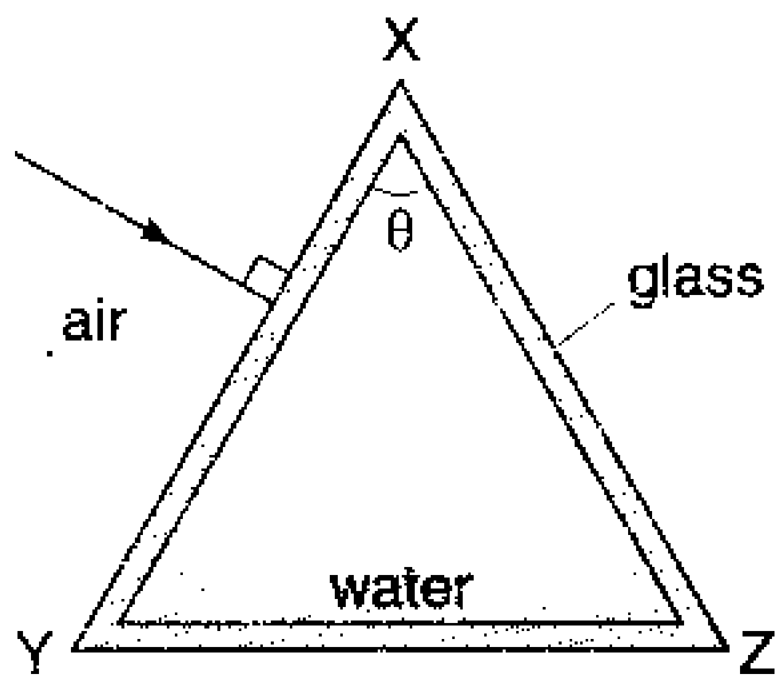


Fig. 22.103

4. A parallel beam of light is incident on a prism shown in figure. Through what angle should the mirror be rotated so that light returns back to its original path? Refractive index of prism is 1.5.

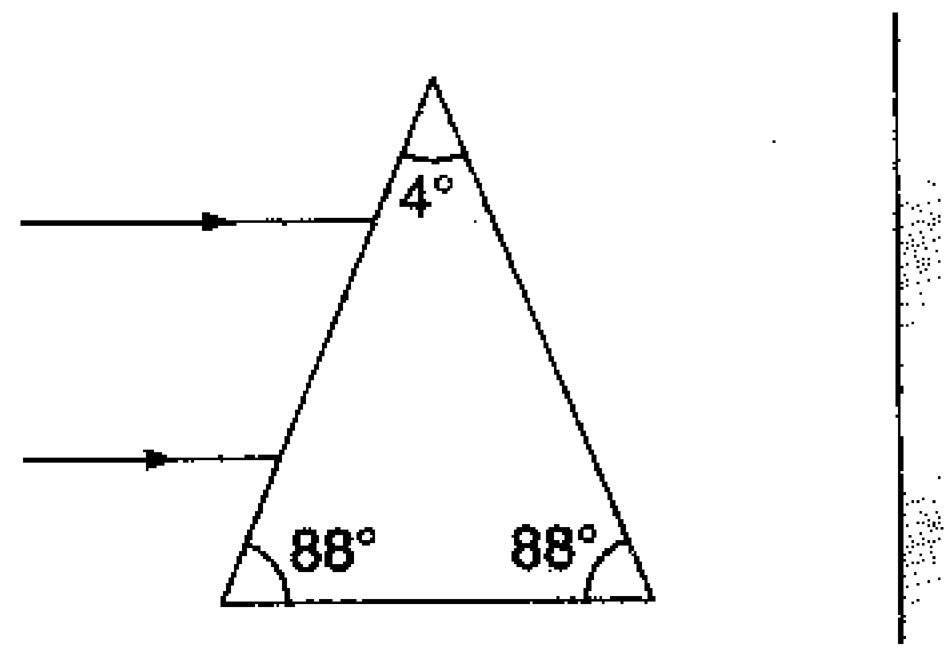


Fig. 22.104

5. A light ray going through a prism with the angle of prism 60°, is found to deviate by 30°. What is the range of the refractive index of the prism?
6. A ray of light falls normally on a refracting face of a prism. Find the angle of prism if the ray just fails to emerge from the prism ($\mu = 3/2$).
7. A ray of light is incident at an angle of 60° on one face of a prism which has an angle of 30°. The ray emerging out of the prism makes an angle of 30° with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges and calculate the refractive index of the material of prism.
8. A ray of light passing through a prism having refractive index $\sqrt{2}$ suffers minimum deviation. It is found that the angle of incidence is double the angle of refraction within the prism. What is the angle of prism?
9. A ray of light undergoes deviation of 30° when incident on an equilateral prism of refractive index $\sqrt{2}$. What is the angle subtended by the ray inside the prism with the base of the prism?

10. Light is incident at an angle *i* on one planar end of a transparent cylindrical rod of refractive index μ. Find the least value of μ so that the light entering the rod does not emerge from the curved surface of the rod irrespective of the value of *i*.

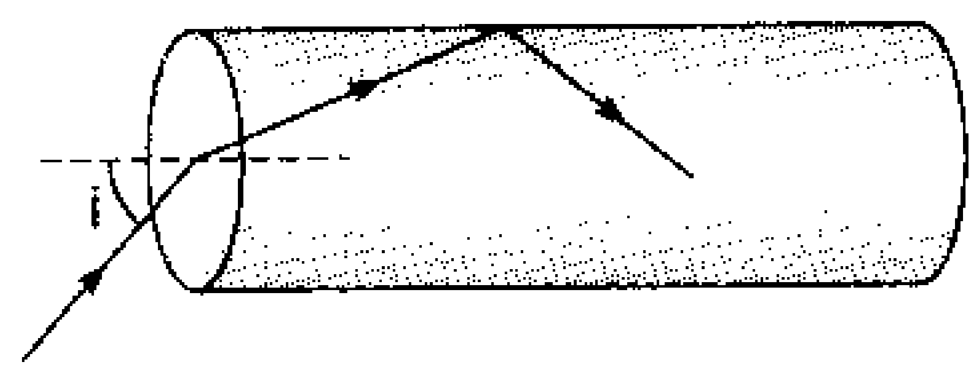


Fig. 22.105

11. The refractive index of the material of a prism of refracting angle 45° is 1.6 for a certain monochromatic ray. What will be the minimum angle of incidence of this ray on the prism so that no TIR takes place as the ray comes out of the prism.

MISCELLANEOUS EXAMPLES

EXAMPLE 1 *A light ray enters the atmosphere of a planet and descends vertically* 20.0 *km to the surface. The index of refraction where the light enters the atmosphere is* 1.0 *and it increases linearly to the surface where it has a value* 1.005. *How long does it take the ray to traverse this path.*

SOLUTION $x_0 = 20 \times 10^3$ m, $c =$ speed of light in vacuum $= 3 \times 10^8$ m/s.

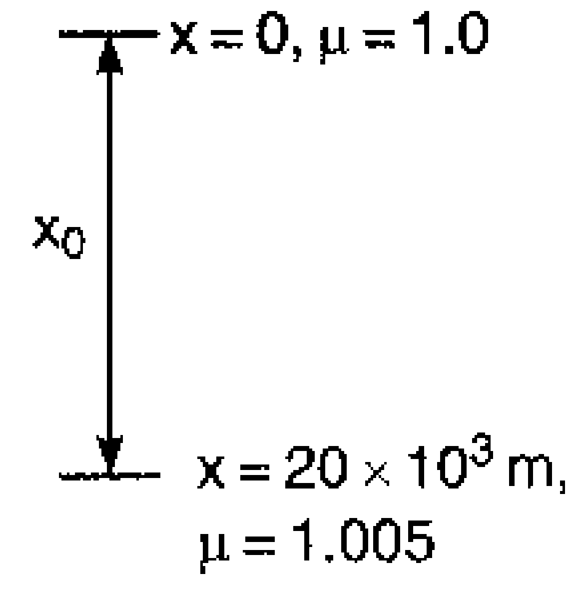


Fig. 22.106

$$\mu(x) = 1.0 + \left(\frac{0.005}{x_0}\right)x$$

$$\therefore \quad v(x) = \frac{c}{\mu(x)} = \frac{3.0 \times 10^8}{1 + \dfrac{0.005}{20 \times 10^3}x} = \frac{3.0 \times 10^8}{1 + 2.5 \times 10^{-7}x}$$

or
$$\frac{dx}{dt} = \frac{3.0 \times 10^8}{1 + 2.5 \times 10^{-7}x}$$

$$\therefore \quad dt = \frac{1}{3.0 \times 10^8}[(1 + 2.5 \times 10^{-7}x)dx]$$

or
$$\int_0^t dt = \frac{1}{3.0 \times 10^8}\int_0^{20 \times 10^3}(1 + 2.5 \times 10^{-7}x)dx$$

$$\therefore \quad t = \frac{1}{3.0 \times 10^8}\left[20 \times 10^3 + \frac{(20 \times 10^3)^2(2.5 \times 10^{-7})}{2}\right]$$

$$= 6.68 \times 10^{-5}\text{ s}$$ **Ans.**

EXAMPLE 2 *The index of refraction for violet light in silica flint glass is* 1.66 *and that for red light is* 1.62. *What is the angular dispersion of visible light passing through a prism of apex angle* 60° *if the angle of incidence is* 50°?

SOLUTION **For violet :**

$$\mu = \frac{\sin i_1}{\sin r_1}$$

$$\therefore \quad 1.66 = \frac{\sin 50°}{\sin r_1}$$

$$\therefore \quad r_1 = 27.5°$$

$$\therefore \quad r_2 = A - r_1 = 32.5°$$

Now,
$$\mu = \frac{\sin i_2}{\sin r_2}$$

$$\therefore \quad 1.66 = \frac{\sin i_2}{\sin 32.5°}$$

$$\therefore \quad i_2 = 63.1°$$

$$\therefore \qquad \delta_V = (i_1 + i_2) - A = 53.1°$$

For red :

$$1.62 = \frac{\sin 50°}{\sin r_1}$$

$$\therefore \qquad r_1 = 28.2°$$

$$r_2 = A - r_1 = 31.8°$$

Further,

$$1.62 = \frac{\sin i_2}{\sin 31.8°}$$

$$\therefore \qquad i_2 = 58.6°$$

$$\therefore \qquad \delta_R = (i_1 + i_2) - A = 48.6°$$

Angular dispersion

$$= \delta_V - \delta_R$$

$$= 4.5°$$

Ans.

EXAMPLE 3 *Determine the maximum angle* θ *for which the light ray incident on the end of pipe shown in figure are subject to TIR along the walls of the pipe. Assume that the pipe has an index of refraction of* 1.36 *and the outside medium is air.*

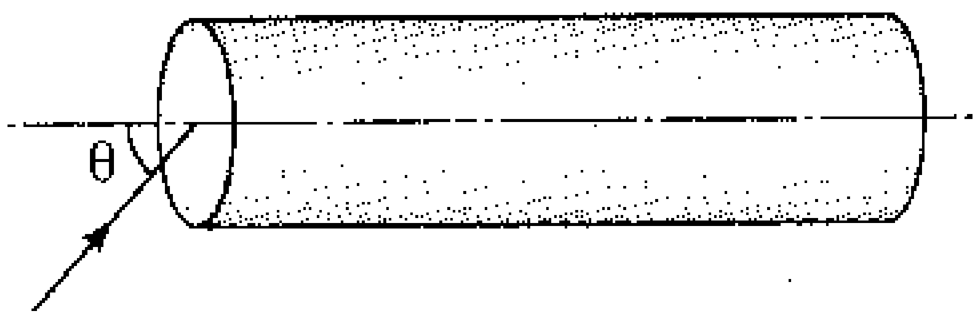


Fig. 22.107

SOLUTION Critical angle,

$$\theta_C = \sin^{-1}\left(\frac{1}{\mu}\right) = \sin^{-1}\left(\frac{1}{1.36}\right) = 47.3°$$

For TIR to take place at B,

$$\beta > \theta_C \quad \text{or} \quad \beta > 47.3°$$

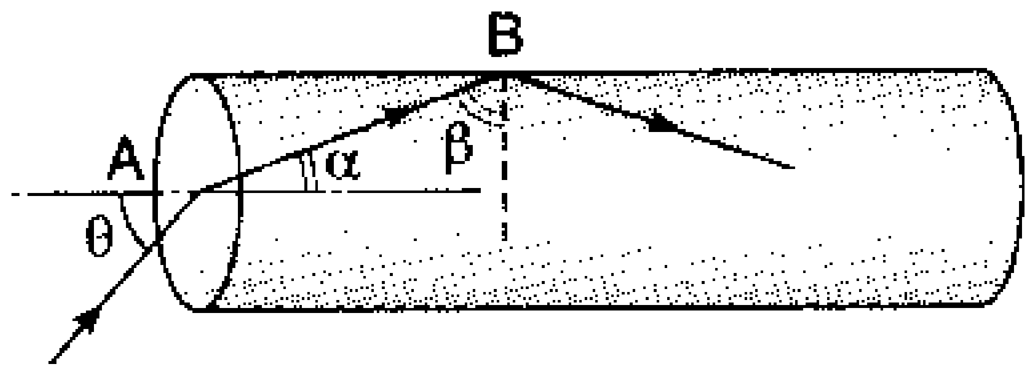


Fig. 22.108

For this

$$\alpha < 90° - 47.3°$$

or

$$\alpha < 42.7°$$

Using Snell's law at A,

$$\theta < \sin^{-1}(\mu \sin 42.7°)$$

or

$$\theta < \sin^{-1}(1.36 \sin 42.7°)$$

or

$$\theta < 67.3°$$

Thus, maximum value of θ for TIR to take place at B is 67.3°. **Ans.**

EXAMPLE 4 *An object is 5.0 m to the left of a flat screen. A converging lens for which the focal length is* $f = 0.8$ *m is placed between object and screen.*

(*a*) *Show that two lens positions exist that form images on the screen and determine how far these positions are from the object?*

(*b*) *How do the two images differ from each other?*

SOLUTION (a) Using the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have

$$\frac{1}{\quad} - \frac{1}{\quad} = \frac{1}{\quad}$$

or $$\frac{1}{5-u}+\frac{1}{u}=1.25$$

$\therefore$ $$u+5-u=1.25\,u\,(5-u)$$

or $$1.25\,u^2-6.25\,u+5=0$$

$\therefore$ $$u=\frac{6.25\pm\sqrt{39.0625-25}}{2.5}$$

or $$u=4\text{ m}\quad\text{and}\quad 1\text{ m}$$ **Ans.**

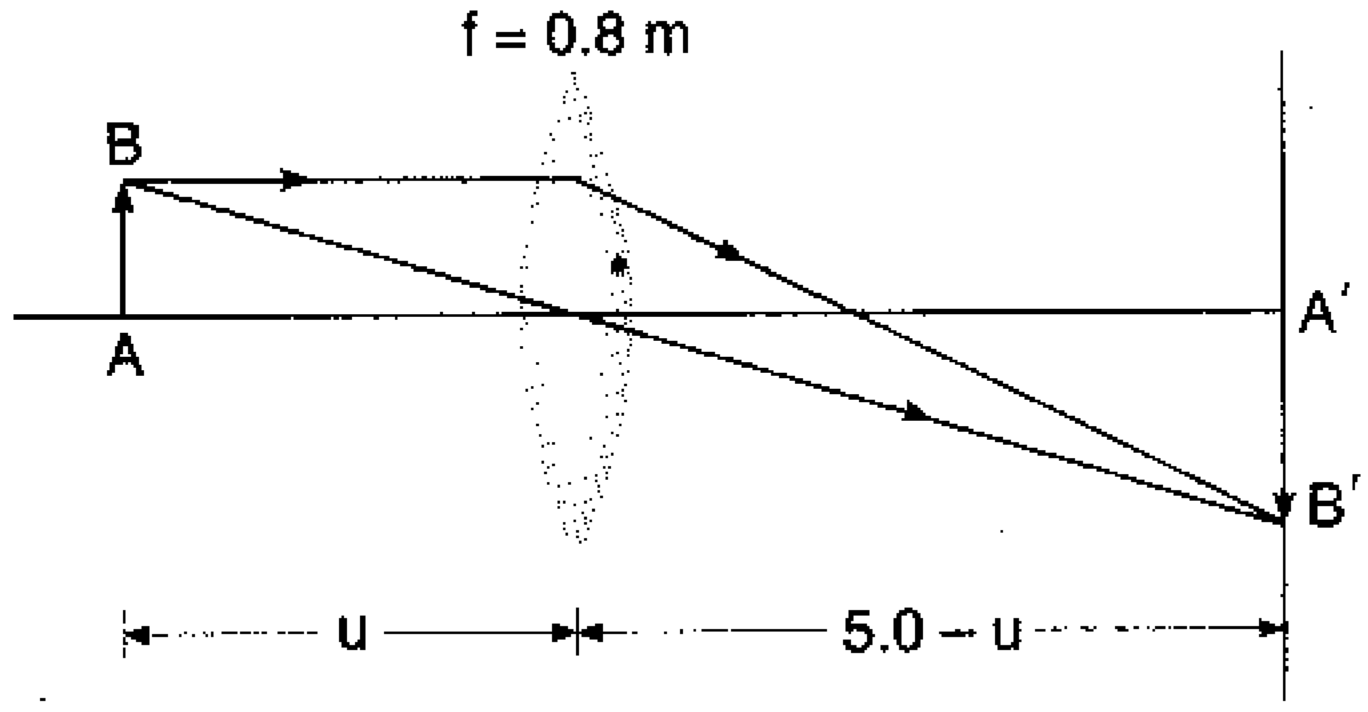


Fig. 22.109

Both the values are real, which means there exist two positions of lens that form images of object on the screen.

(b) $$m=\frac{v}{u}$$

$\therefore$ $$m_1=\frac{(5.0-4.0)}{(-4.0)}=-0.25$$

and $$m_2=\frac{(5.0-1.0)}{(-1.0)}=-4.00$$

Hence, both the images are real and inverted, the first has magnification – 0.25 and the second –4.00. **Ans.**

EXAMPLE 5 *The object is midway between the lens and the mirror. The mirror's radius of curvature is* 20.0 *cm and the lens has a focal length of* – 16.7 *cm. Considering only the rays that leaves the object and travels first toward the mirror, locate the final image formed by this system. Is this image real or virtual. Is it upright or inverted? What is the overall magnification?*

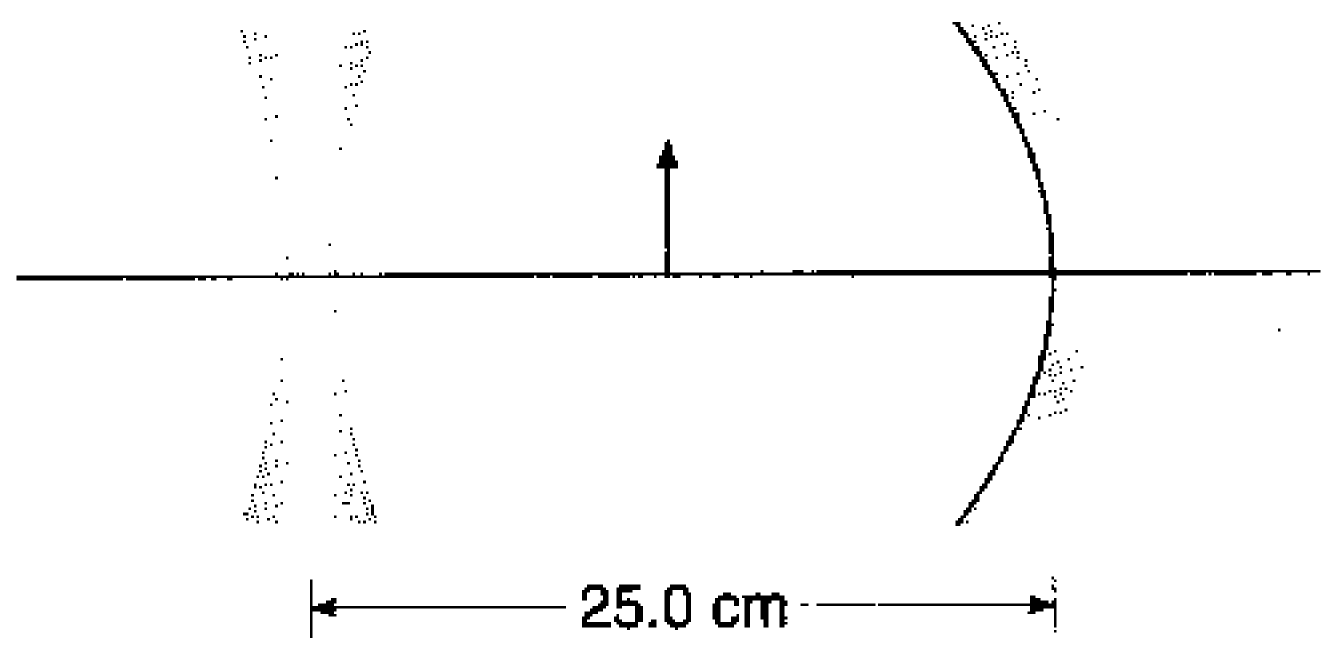


Fig. 22.110

SOLUTION **Image formed by mirror :** Using mirror formula

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}=\frac{2}{R}\qquad\left(\text{as } f=\frac{R}{2}\right)$$

We have, $$\frac{1}{v_1}+\frac{1}{-12.5}=\frac{2}{-20}$$

$\therefore$ $$v_1=-50\text{ cm}$$

$$m_1=-\frac{v}{u}=-\frac{(-50)}{(-12.5)}=-4$$

i.e., image formed by the mirror is at a distance of 50 cm from the mirror to the left of it. It is inverted and four times larger.

Image formed by lens : Image formed by mirror acts as an object for lens. It is at a distance of 25.0 cm to the left of lens using the lens formula,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

We have, $$\frac{1}{v_2} - \frac{1}{25} = \frac{1}{-16.7}$$

$\therefore$ $$v_2 = -50.3 \text{ cm}$$

and $$m_2 = \frac{v}{u} = \frac{-50.3}{25} = -2.012$$

overall magnification is $$m = m_1 \times m_2 = 8.048$$

Thus, the final image is at a distance 25.3 cm to the right of the mirror, virtual, upright enlarged and 8.048 times. Positions of the two images are shown in figure.

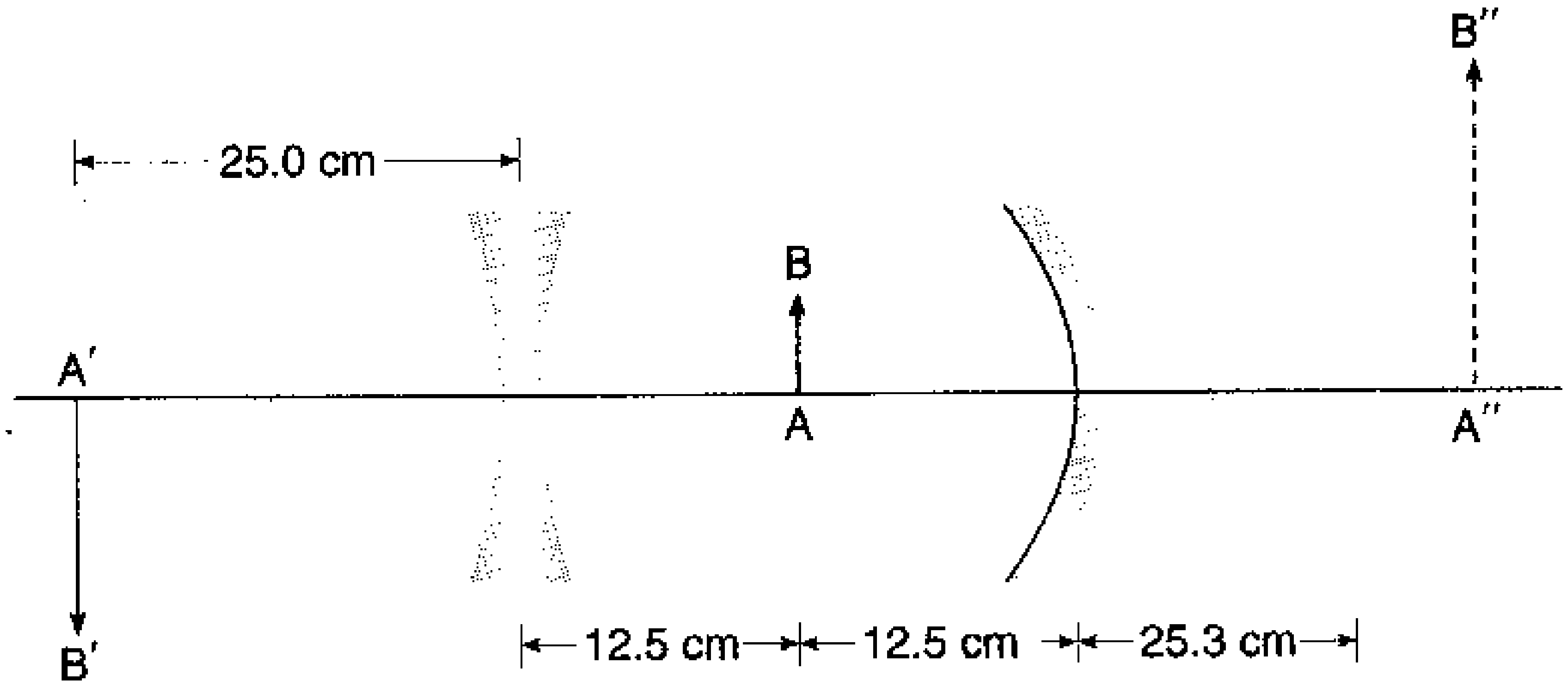


Fig. 22.111

EXAMPLE 6 *An object is placed* 12 *cm to the left of a diverging lens of focal length* – 6.0 *cm. A converging lens with a focal length of* 12.0 *cm is placed at a distance d to the right of the diverging lens. Find the distance d that corresponds to a final image at infinity.*

SOLUTION

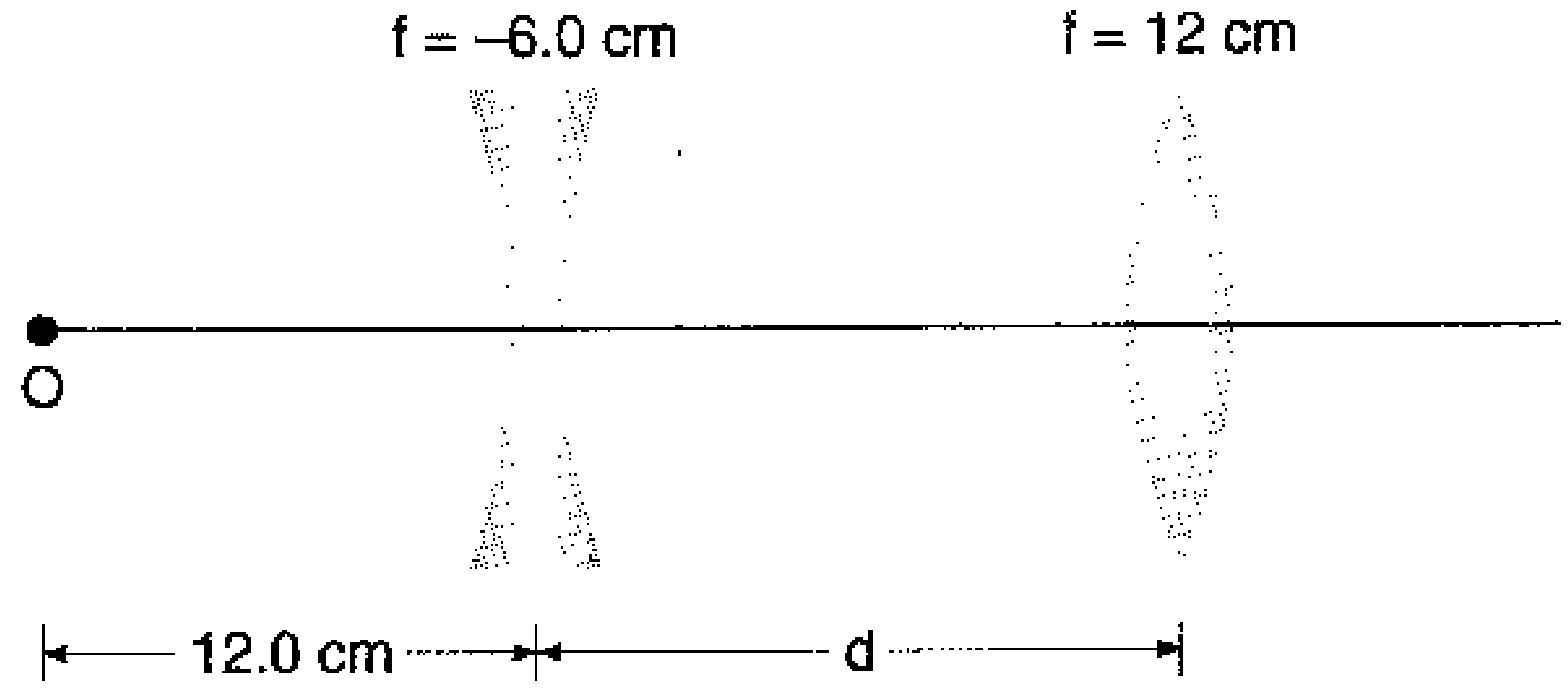


Fig. 22.112

Applying lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ twice we have,

$$\frac{1}{v_1} - \frac{1}{-12} = \frac{1}{-6} \quad \text{...(i)}$$

$$\frac{1}{\infty} - \frac{1}{v_1 - d} = \frac{1}{12} \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have

$$v_1 = -4\,\text{cm}$$

and $$d = 8\,\text{cm}$$ **Ans.**

EXAMPLE 7 *A solid glass sphere with radius R and an index of refraction 1.5 is silvered over one hemisphere. A small object is located on the axis of the sphere at a distance 2R to the left of the vertex of the unsilvered hemisphere. Find the position of final image after all refractions and reflections have taken place.*

SOLUTION The ray of light first gets refracted then reflected and then again refracted. For first refraction and then reflection the ray of light travels from left to right while for the last refraction it travels from right to left. Hence, the sign convention will change accordingly.

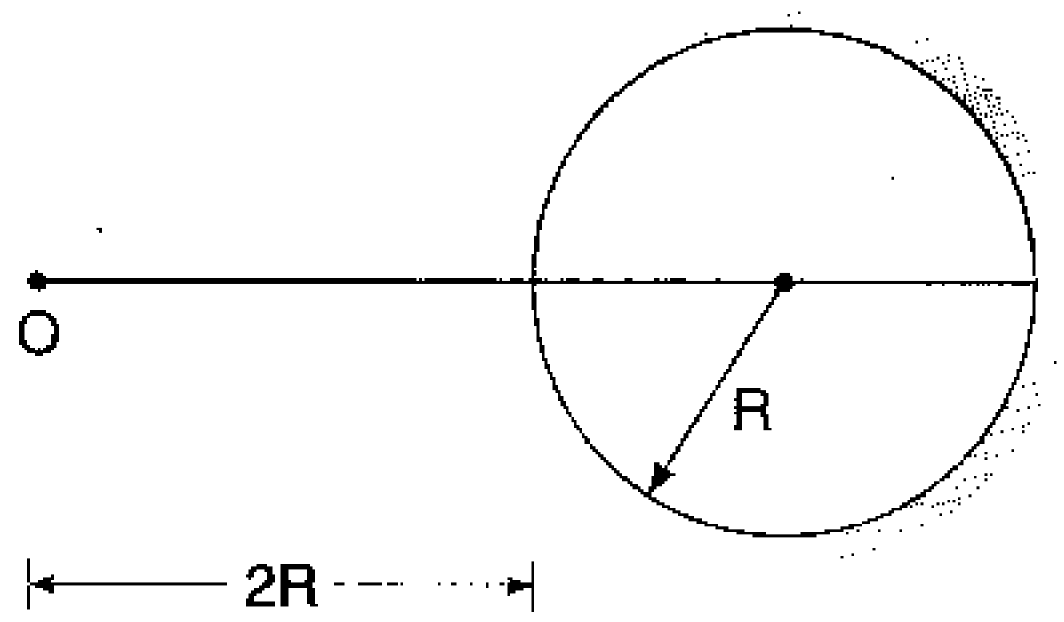


Fig. 22.113

First refraction : Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with proper sign conventions, we have

$$\frac{1.5}{v_1} - \frac{1.0}{-2R} = \frac{1.5 - 1.0}{+R}$$

$$\therefore \quad v_1 = \infty$$

Second reflection : Using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$ with proper sign conventions, we have

$$\frac{1}{v_2} + \frac{1}{\infty} = -\frac{2}{R} \qquad \therefore \quad v_2 = -\frac{R}{2}$$

Third refraction : Again using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with reversed sign convention, we have

$$\frac{1.0}{v_3} - \frac{1.5}{-1.5R} = \frac{1.0 - 1.5}{-R}$$

or $$v_3 = -2R$$

i.e., final image is formed on the vertex of the silvered face

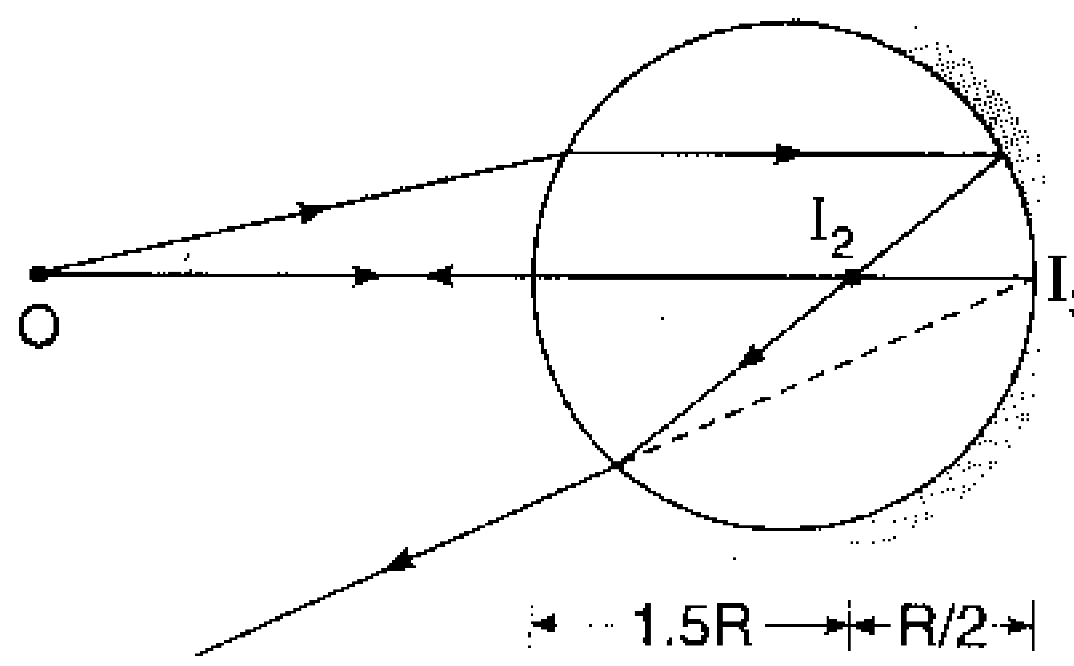


Fig. 22.114

EXAMPLE 8 *A material having an index of refraction μ is surrounded by vacuum and is in the shape of a quarter circle of radius R. A light ray parallel to the base of the material is incident from the left at a distance of L above the base and emerges out of the material at an angle θ. Determine an expression for θ.*

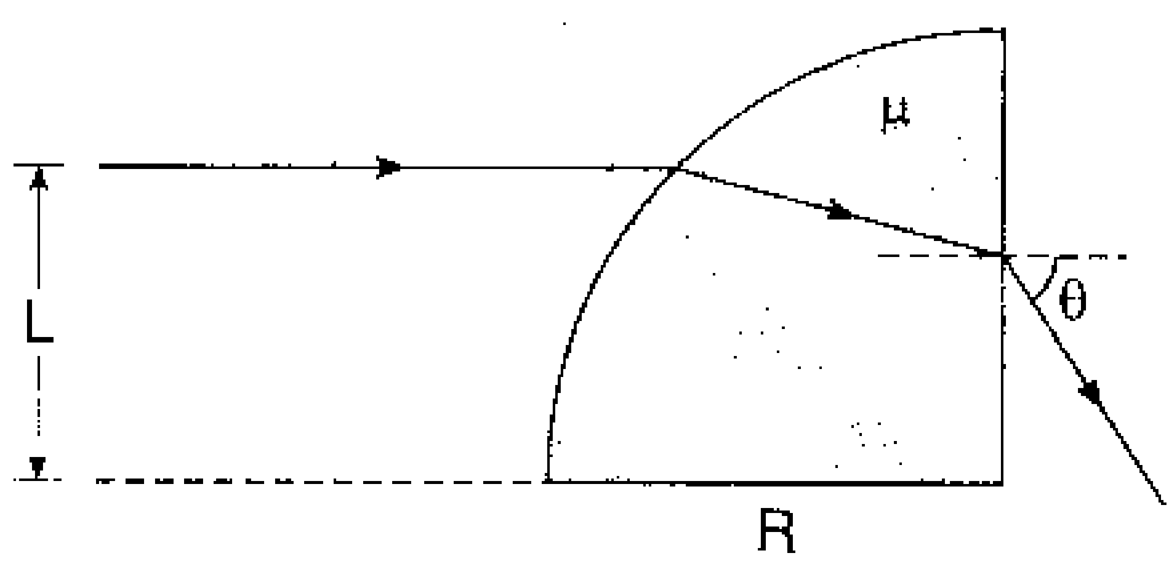


Fig. 22.115

SOLUTION $\sin i = \dfrac{L}{R}$

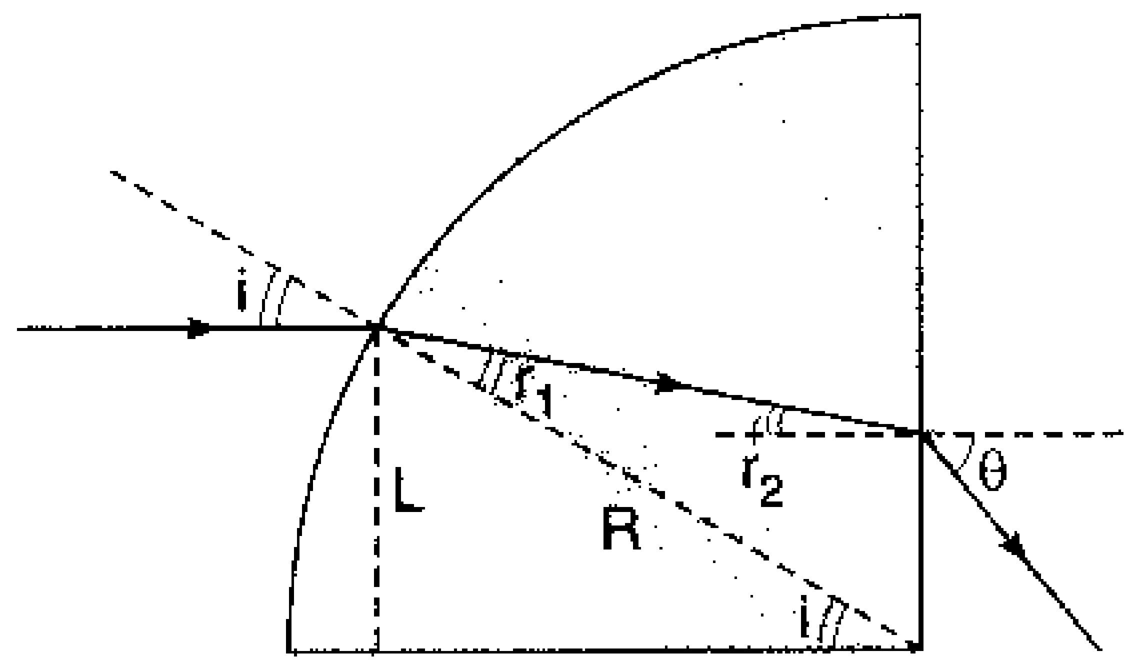


Fig. 22.116

$\therefore$ $$i = \sin^{-1}\left(\frac{L}{R}\right)$$

$$\mu = \frac{\sin i}{\sin r_1}$$

$\therefore$ $$r_1 = \sin^{-1}\left\{\frac{\sin i}{\mu}\right\} = \sin^{-1}\left(\frac{L}{\mu R}\right)$$

Deviation of the ray, $$\delta = i - r_1 = \sin^{-1}\left(\frac{L}{R}\right) - \sin^{-1}\left(\frac{L}{\mu R}\right)$$

This is also the angle r_2.

$\therefore$ $$r_2 = \sin^{-1}\left(\frac{L}{R}\right) - \sin^{-1}\left(\frac{L}{\mu R}\right)$$

Using, $$\sin^{-1}(C) - \sin^{-1}(D) = \sin^{-1}\left[C\sqrt{1-D^2} - D\sqrt{1-C^2}\right]$$

We have $$r_2 = \sin^{-1}\left[\frac{L}{R}\sqrt{1 - \frac{L^2}{\mu^2 R^2}} - \frac{L}{\mu R}\sqrt{1 - \frac{L^2}{R^2}}\right]$$

Now, $$\mu = \frac{\sin\theta}{\sin r_2}$$

$\therefore$ $$\theta = \sin^{-1}\left[\mu \sin r_2\right]$$

or $$\theta = \sin^{-1}\left[\frac{\mu L}{R}\sqrt{1 - \frac{L^2}{\mu^2 R^2}} - \frac{L}{R}\sqrt{1 - \frac{L^2}{R^2}}\right]$$

or $$\theta = \sin^{-1}\left[\frac{L}{R^2}\sqrt{\mu^2 R^2 - L^2} - \frac{L}{R^2}\sqrt{R^2 - L^2}\right]$$

or $$\theta = \sin^{-1}\left[\frac{L}{R^2}\left(\sqrt{\mu^2 R^2 - L^2} - \sqrt{R^2 - L^2}\right)\right]$$ **Ans.**

EXAMPLE 9 *A ray of light is incident on a plane mirror along a vector* $\hat{i} + \hat{j} - \hat{k}$. *The normal on incidence point is along* $\hat{i} + \hat{j}$. *Find a unit vector along the reflected ray.*

SOLUTION Reflection of a ray of light is just like an elastic collision of a ball with a horizontal ground. Component of incident ray along the inside normal gets reversed while the component perpendicular to it remains unchanged. Thus the component of incident ray vector $\vec{A} = \hat{i} + \hat{j} - \hat{k}$ parallel to normal, *i.e.*, $\hat{i} + \hat{j}$ gets reversed while perpendicular to it, *i.e.*, $-\hat{k}$ remains unchanged. Thus, the reflected ray can be written as,

$$\vec{R} = -\hat{i} - \hat{j} - \hat{k}$$

$\therefore$ A unit vector along the reflected ray will be,

$$\hat{r} = \frac{\vec{R}}{R} = \frac{-\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$$

or $$\hat{r} = -\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$ **Ans.**

Note : In this problem normal is inside the mirror surface. Think why?

EXAMPLE 10 *A converging lens forms a five fold magnified image of an object. The screen is moved towards the object by a distance $d = 0.5$ m, and the lens is shifted so that the image has the same size as the object. Find the lens power and the initial distance between the object and the screen.*

SOLUTION In the first case image is five times magnified. Hence, $|v| = 5|u|$. In the second case image and object are of equal size. Hence, $|v| = |u|$. From the two figures,

$$6x = 2y + d$$

or $$6x - 2y = 0.5 \quad \text{...(i)}$$

Using the lens formula for both the cases,

$$\frac{1}{5x} - \frac{1}{-x} = \frac{1}{f}$$

or $$\frac{6}{5x} = \frac{1}{f} \quad \text{...(ii)}$$

$$\frac{1}{y} - \frac{1}{-y} = \frac{1}{f}$$

or $$\frac{2}{y} = \frac{1}{f} \quad \text{...(iii)}$$

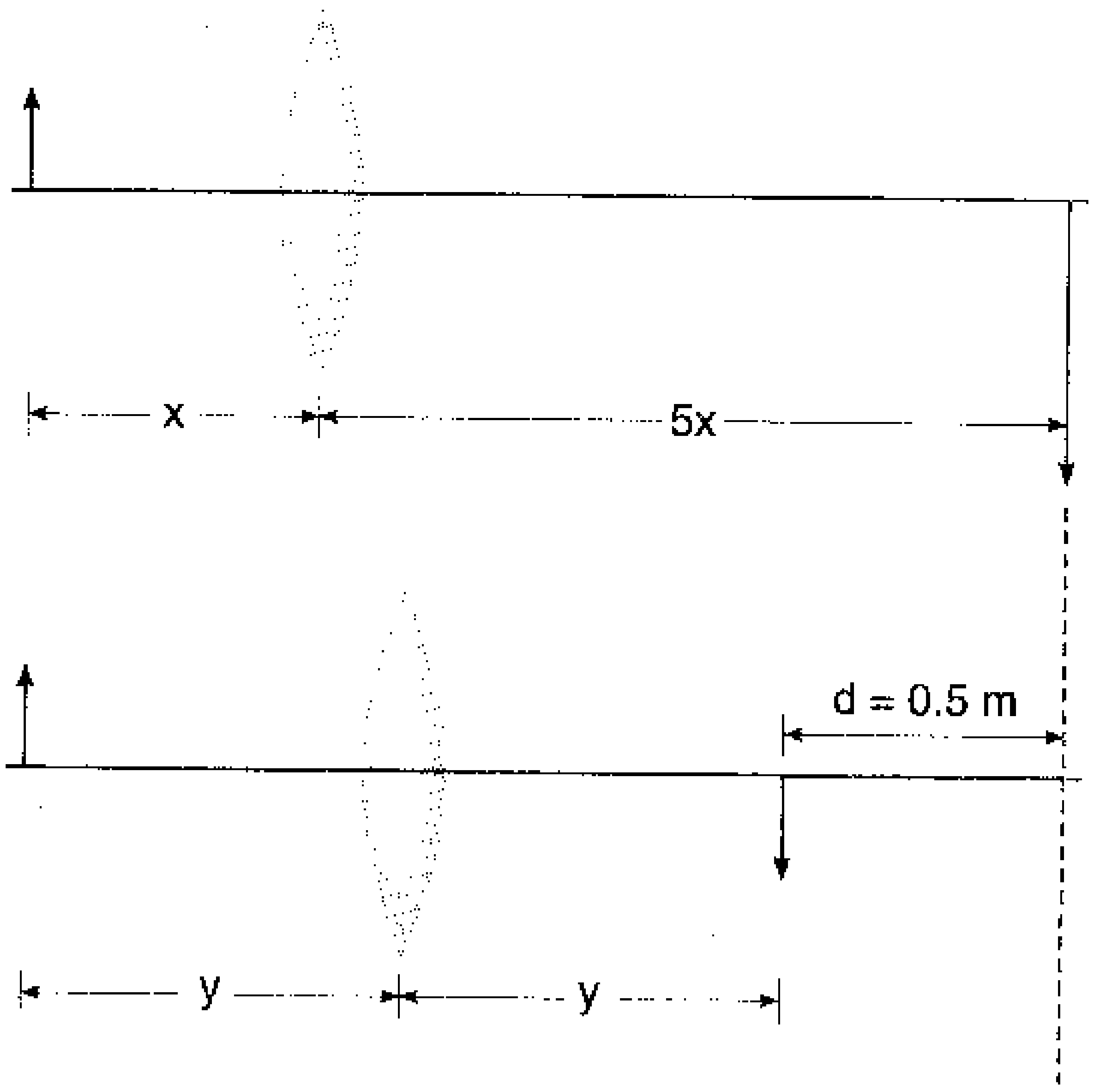


Fig. 22.117

Solving these three equations, we get

$$x = 0.1875 \text{ m} \quad \text{and} \quad f = 0.15625 \text{ m}$$

Therefore, initial distance between the object and the screen $= 6x = 1.125$ m **Ans.**

Power of the lens, $P = \frac{1}{f} = \frac{1}{0.15625}\,D = 6.4\,D$ **Ans.**

$$\vec{R} = -\hat{i} - \hat{j} - \hat{k}$$

∴ A unit vector along the reflected ray will be,

$$\hat{r} = \frac{\vec{R}}{R} = \frac{-\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$$

or $$\hat{r} = -\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$ **Ans.**

Note : In this problem normal is inside the mirror surface. Think why?

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$$6x = 2y + d$$

or $$6x - 2y = 0.5 \quad \text{...(i)}$$

Using the lens formula for both the cases,

$$\frac{1}{5x} - \frac{1}{-x} = \frac{1}{f}$$

or $$\frac{6}{5x} = \frac{1}{f} \quad \text{...(ii)}$$

$$\frac{1}{y} - \frac{1}{-y} = \frac{1}{f}$$

or $$\frac{2}{y} = \frac{1}{f} \quad \text{...(iii)}$$

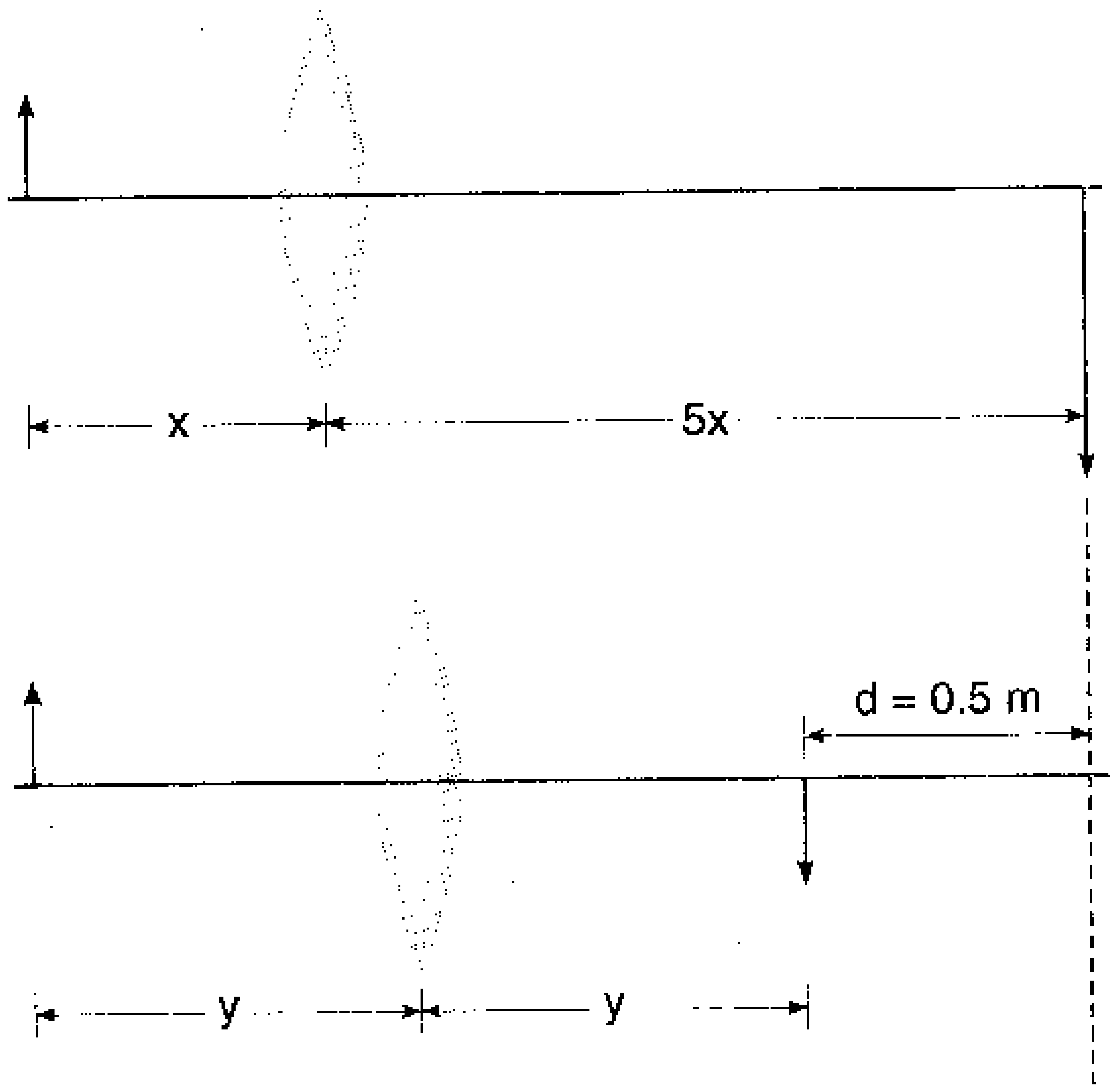


Fig. 22.117

Solving these three equations, we get

$$x = 0.1875 \text{ m} \quad \text{and} \quad f = 0.15625 \text{ m}$$

Therefore, initial distance between the object and the screen $= 6x = 1.125$ m **Ans.**

Power of the lens, $P = \frac{1}{f} = \frac{1}{0.15625} D = 6.4 D$ **Ans.**

EXAMPLE 11 *Two thin lenses $f_1 = 10$ cm and $f_2 = 20$ cm are separated by a distance $d = 5$ cm. Their optical centres are displaced a distance $\Delta = 0.5$ cm. A linear object of size 3 cm placed at 30 cm from the optical centre of left lens. Find the nature, position and size of final image.*

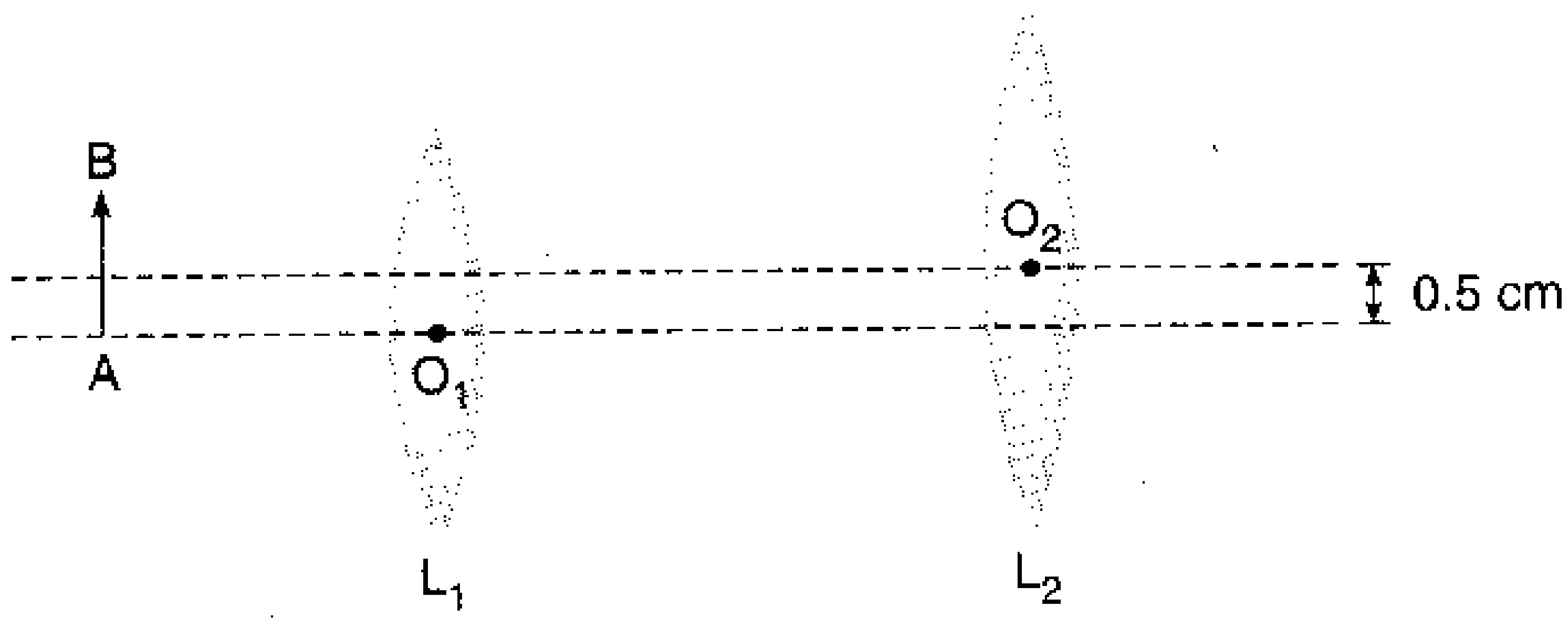


Fig. 22.118

SOLUTION **Refraction from the first lens L_1 :** Using the lens formula

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

we have, $$\frac{1}{v_1}-\frac{1}{-30}=\frac{1}{10}$$

or $$v_1=15\,\text{cm}$$

and $$m_1=\frac{v}{u}=\frac{15}{-30}=-\frac{1}{2}$$

Refraction from the second lens L_2 : Again using the lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we have

$$\frac{1}{v_2}-\frac{1}{(15-5)}=\frac{1}{20}$$

or $$v_2=\frac{20}{3}\,\text{cm}\approx 6.67\,\text{cm}$$

and $$m_2=\frac{v}{u}=\frac{20/3}{10}=\frac{2}{3}$$

Final image : Overall magnification $m=m_1m_2=-\frac{1}{3}$, *i.e.*, height of the image is $3\times\frac{1}{3}=1\,\text{cm}$. Since, the overall magnification is negative, final image is inverted. Further y coordinate of a point on the image will be,

$$y_I=my_0-m_2\Delta \qquad \text{...(i)}$$

w.r.t the principal axis of L_1.

$\therefore$ y coordinate of image of $A=\left(-\frac{1}{3}\right)(0)-\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=-\frac{1}{3}$ cm and

y coordinate of image of $B=\left(-\frac{1}{3}\right)(3)-\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=-\frac{4}{3}$ cm.

Thus final image is as shown below.

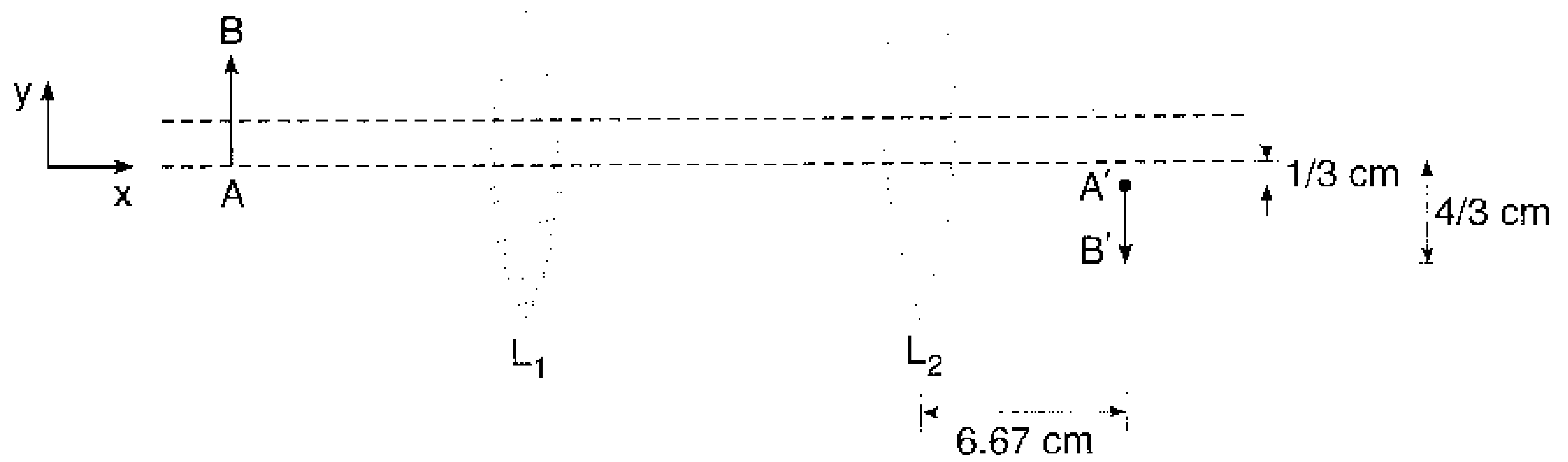


Fig. 22.119

EXERCISE : Derive Eq. (i).

EXAMPLE 12 *A ray of light is falling on face AB of a tetrahedral of refractive index μ at angle of incidence i. The ray after getting internally reflected on face BC emerges from AD perpendicularly to the incident beam. Find the range of μ and i.*

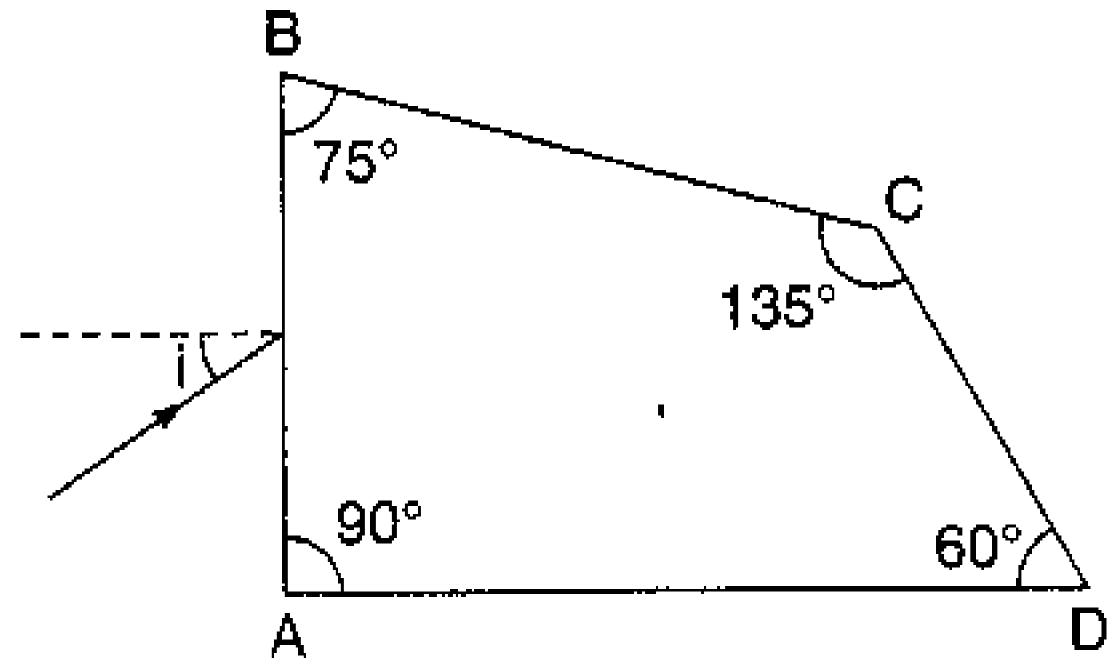


Fig. 22.120

SOLUTION

$$r_1 + r_2 = \angle B = 75° \quad \ldots\text{(i)}$$

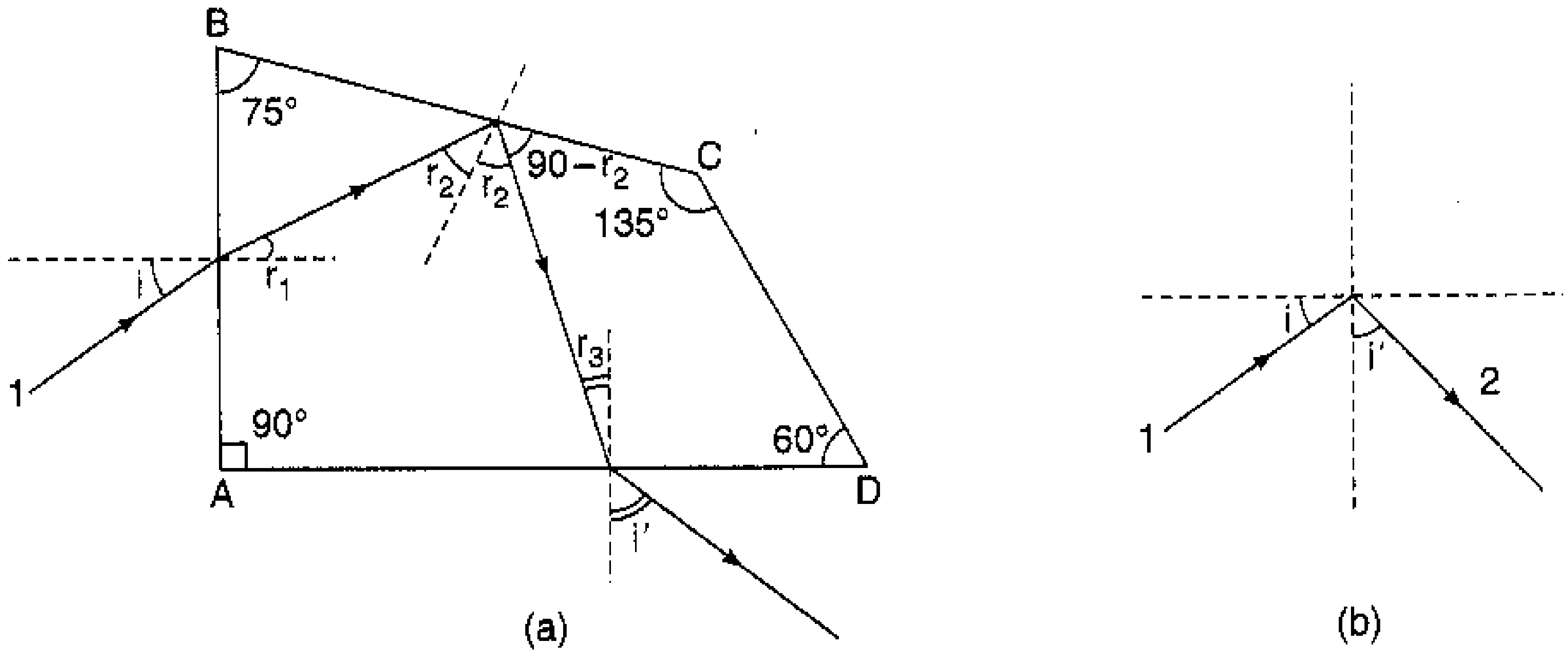


Fig. 22.121

From figure (b) : $i' = i$ as 1 and 2 are perpendicular.

and $$r_3 = 360° - 60° - 135° - (90° - r_2) - 90° = r_2 - 15° \quad \ldots\text{(ii)}$$

Further, $$\mu = \frac{\sin i}{\sin r_1} = \frac{\sin i'}{\sin r_3}$$

or $$r_3 = r_1 \quad (\text{because } i = i') \quad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$r_2 = 45° \quad \text{and} \quad r_1 = 30°$$

Now, for TIR (total internal reflection) to take place on face BC; we should have

$$r_2 > \theta_c \quad \text{or} \quad \sin r_2 > \sin\theta_c$$

$\therefore$ $$\sin 45° > \frac{1}{\mu} \quad \text{or} \quad \frac{1}{\sqrt{2}} > \frac{1}{\mu}$$

or $$\mu > \sqrt{2}$$ **Ans.**

Moreover, $$\mu = \frac{\sin i}{\sin r_1} = \frac{\sin i}{\sin 30°} = 2\sin i$$

As $$\mu > \sqrt{2}$$

$$\therefore \quad 2\sin i > \sqrt{2} \quad \text{or} \quad \sin i > \frac{1}{\sqrt{2}}$$

or $$i > 45°$$ **Ans.**

EXAMPLE 13 *In figure shown L is half part of an equiconvex glass lens* ($\mu = 1.5$) *whose surfaces have radius of curvature* $R = 40$ *cm and its right surface is silvered. Normal to its principal axis there is a plane mirror M placed on the right of the lens. Distance between two is b. An object O is placed to the left of the lens such that there is no parallax between final images formed by the lens and mirror. If transverse length of final image formed by lens is twice that of image formed by the mirror. Calculate distance a between lens and object and distance b.*

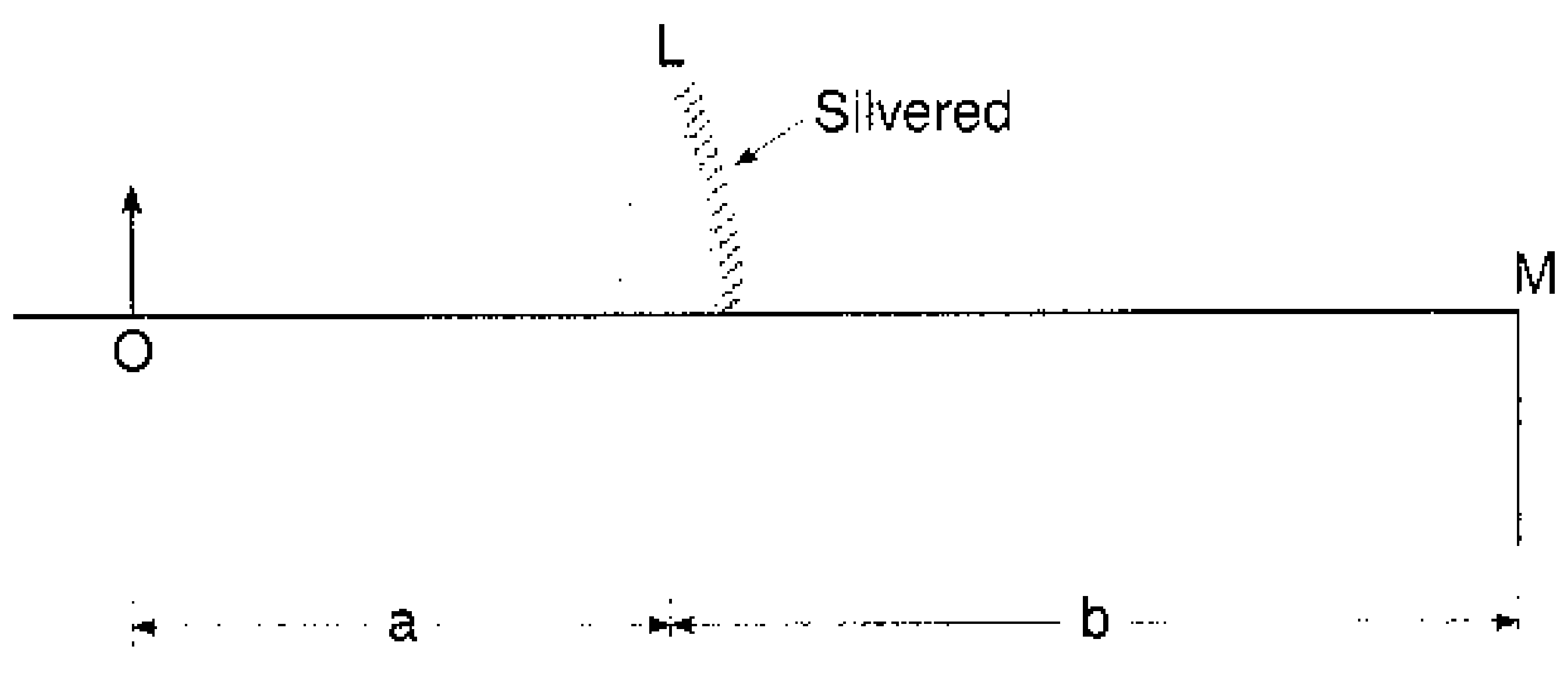


Fig. 22.122

SOLUTION Distance of image from plane mirror = '$a + b$'. As there is no parallax between the two images, distance of image from the silvered lens will be '$a + 2b$'. Further transverse length of final image formed by silvered lens is of twice length that of image formed by mirror. Hence,

$$|v| = 2|u|$$

or $$a + 2b = 2a$$

or $$a = 2b \quad \text{...(i)}$$

The image formed by silvered lens is given by the equation,

$$\frac{1}{v} + \frac{1}{u} = \frac{2(\mu_2/\mu_1)}{R_2} - \frac{2(\mu_2/\mu_1 - 1)}{R_1}$$ (IIT-JEE Galaxy 22.3, point 3)

Substituting the values, we have

$$\frac{1}{a+2b} - \frac{1}{a} = \frac{2 \times 1.5}{-40} - \frac{2(1.5-1)}{40}$$

Substituting $2b = a$, $$\frac{1}{2a} - \frac{1}{a} = -\frac{3}{40} - \frac{1}{40} = -\frac{1}{10}$$

or $$-\frac{1}{2a} = -\frac{1}{10}$$

$$\therefore \quad a = 5 \text{ cm}$$

and $$b = \frac{a}{2} = 2.5 \text{ cm}$$ **Ans.**

EXAMPLE 14 *Surfaces of a thin equiconvex glass lens have radius of curvature R. Paraxial rays are incident on it. If the final image is formed after n internal reflections, calculate distance of this image from pole of the lens. Refractive index of glass is* μ.

SOLUTION The rays will first get refracted, then n-times reflected and finally again refracted. So, using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ for first refraction, we have

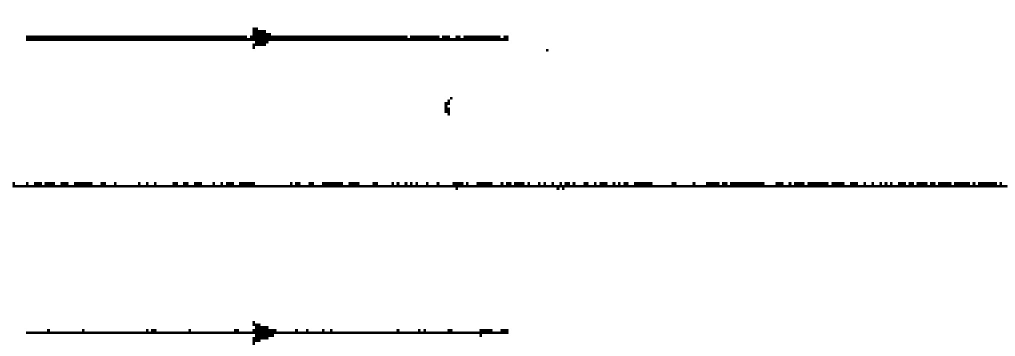


Fig. 22.123

$$\frac{\mu}{v_i} - \frac{1}{\infty} = \frac{\mu - 1}{R}$$

$$\therefore \qquad v_i = \left(\frac{\mu}{\mu - 1}\right) R$$

For first reflection, let us use $\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$

$$\therefore \qquad \frac{1}{v_1} + \left(\frac{\mu - 1}{\mu R}\right) = \frac{-2}{R} \qquad \text{or} \qquad \frac{1}{v_1} = -\left(\frac{3\mu - 1}{\mu R}\right)$$

For second reflection $\qquad \frac{1}{v_2} + \frac{3\mu - 1}{\mu R} = \frac{-2}{R} \qquad$ or $\qquad \frac{1}{v_2} = -\left(\frac{5\mu - 1}{\mu R}\right)$

Similarly after n^{th} reflections, $\qquad \frac{1}{v_n} = -\left[\frac{(2n+1)\mu - 1}{\mu R}\right]$

Finally using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, we have

$$\frac{1}{v_f} - \left\{\frac{(2n+1)\mu - 1}{R}\right\} = \frac{1 - \mu}{-R}$$

or

$$v_f = \frac{R}{2(\mu n + \mu - 1)} \qquad \textbf{Ans.}$$

ASSIGNMENT

LEVEL-I

1. An object is 30.0 cm from a spherical mirror, along the central axis. The absolute value of lateral magnification is $\frac{1}{2}$. The image produced is inverted. What is the focal length of the mirror?

2. A concave mirror has a radius of curvature of 24 cm. How far is an object from the mirror if an image is formed that is

(a) virtual and 3.0 times the size of the object.

(b) real and 3.0 times the size of the object and

(c) real and 1/3 the size of the object?

3. Two plane mirrors are placed parallel to each other and 40 cm apart. An object is placed 10 cm from one mirror. What is the distance from the object to the image for each of the five images that are closest to the object?

4. Prove that for spherical mirrors the product of the distances of the object and the image to the principal focus is always equal to the square of the principal focal length.

5. A light beam of wavelength 600 nm in air passes through film 1 ($n_1 = 1.2$) of thickness 1.0 μm, then through film 2 (air) of thickness 1.5 μm, and finally through film 3 ($n_3 = 1.8$) of thickness 1.0 μm.

(a) Which film does the light cross in the least time, and what is that least time?

(b) What are the total number of wavelengths (at any instant) across all three films together?

6. A ray of light falls on a glass plate of refractive index $n = 1.5$.

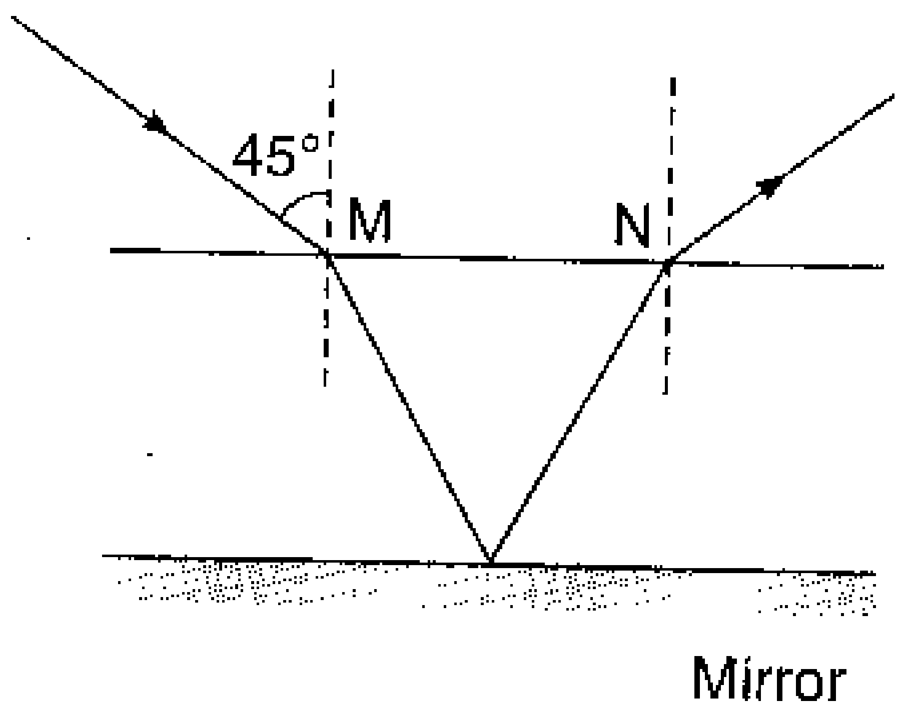
What is the angle of incidence of the ray if the angle between the reflected and refracted rays is 90°?

7. A point source of light is arranged at a height h above the surface of water. Where will the image of this source in the flat mirror-like bottom of a vessel be if the depth of the vessel full of water is d? Refractive index of water is $n = \frac{4}{3}$. Consider only two steps.

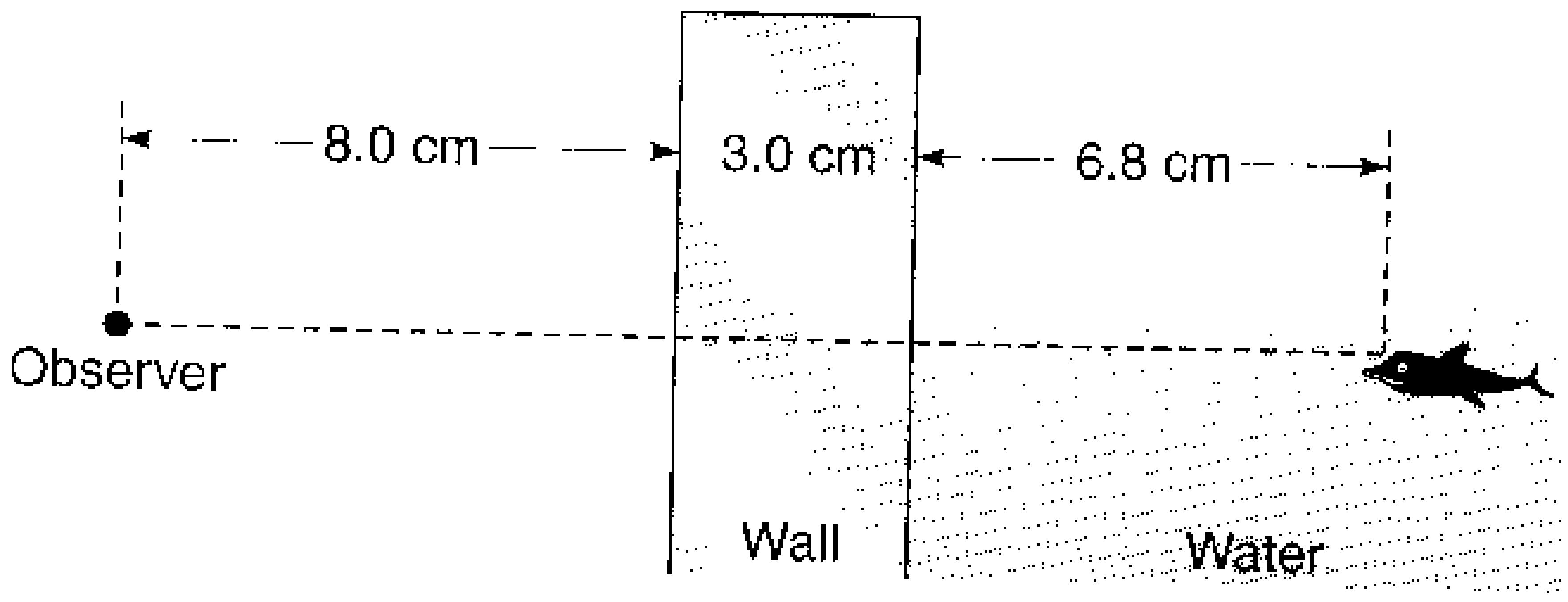
8. A pile 4 m high driven into the bottom of a lake is 1 m above the water.

Determine the length of the shadow of the pile on the bottom of the lake if the sunrays make an angle of 45° with the water surface. The refractive index of water is 4/3.

9. A plate with plane parallel faces having refractive index 1.8 rests on a plane mirror. A light ray is incident on the upper face of the plate at 60°. How far from the entry point will the ray emerge after reflection by the mirror of the plate is 6 cm thick?



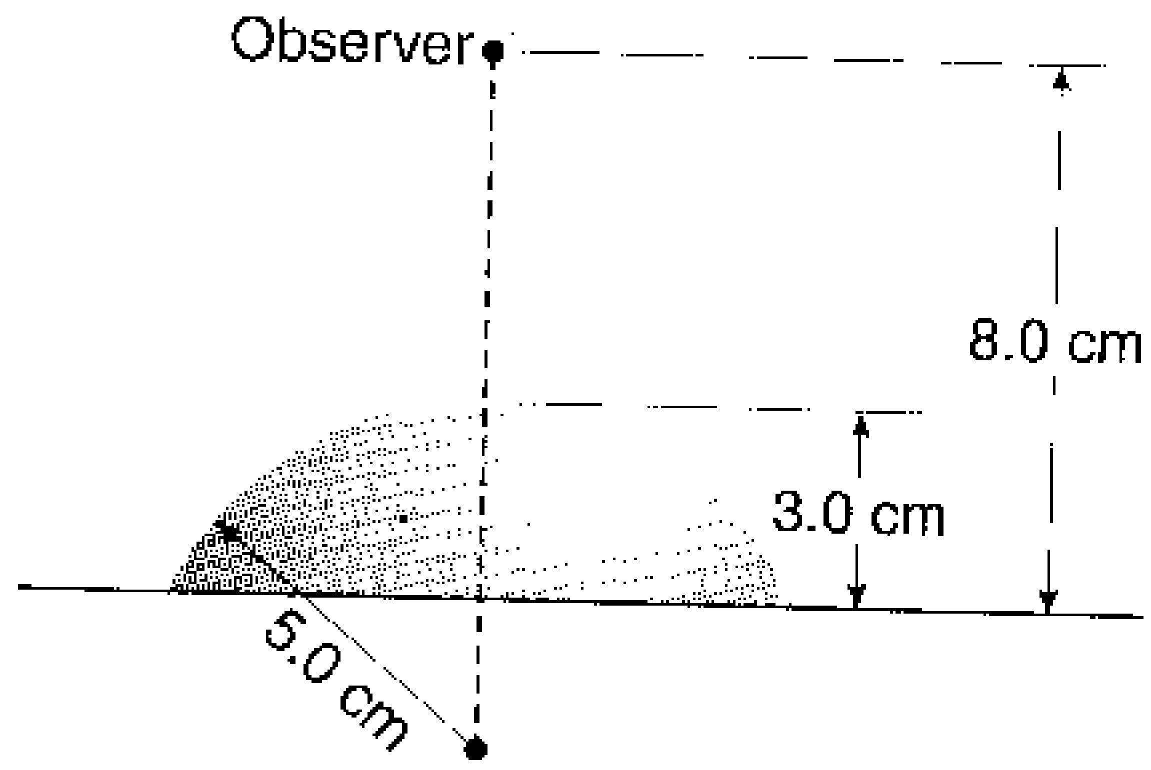
10. In figure, a fish watcher watches a fish through a 3.0 cm thick glass wall of a fish tank. The watcher is in level with the fish; the index of refraction of the glass is 8/5 and that of the water is 4/3.



(a) To the fish, how far away does the watcher appear to be?

(b) To the watcher, how far away does the fish appear to be?

11. A glass sphere has a radius of 5.0 cm and a refractive index of 1.6. A paperweight is constructed by slicing through the sphere on a plane that is 2.0 cm from the centre of the sphere and perpendicular to a radius of the sphere that passes through the center of the circle formed by the intersection of the plane and the sphere. The paperweight is placed on a table and viewed from directly above by an observer who is 8.0 cm from the tabletop, as shown in figure. When viewed through the paperweight, how far away does the tabletop appear to the observer?



12. One face of a rectangular glass plate 6 cm thick is silvered. An object held 8 cm infront of the unsilvered face forms an image 6 cm behind the silvered face. Find the refractive index of glass. Consider all the three steps.

13. A lens with a focal length of 16 cm produces a sharp image of an object in two positions which are 60 cm apart. Find the distance from the object to the screen.

14. The distance between an object and a divergent lens is m times greater than the focal length of the lens. How many times will the image be smaller than the object?

15. The height of a candle flame is 5 cm. A lens produces an image of this flame 15 cm high on a screen. Without touching the lens, the candle is moved over a distance of $l = 1.5$ cm away from the lens, and a sharp image of the flame 10 cm high is obtained again after shifting the screen. Determine the main focal length of the lens.

16. Compare the longitudinal and the lateral magnifications of a thin lens. Consider the case of small longitudinal dimensions of the object.

17. A thin converging lens of focal length f is moved between a candle and a screen. The distance between the candle and the screen is d ($> 4f$). Show that for two different positions of the lens, two different images can be obtained on the screen. If the ratio of dimensions of the image is β, find the value of $(\beta + 1/\beta)$.

18. Two glasses with refractive indices of 1.5 and 1.7 are used to make two identical double convex lenses.

(a) Find the ratio between their focal lengths.

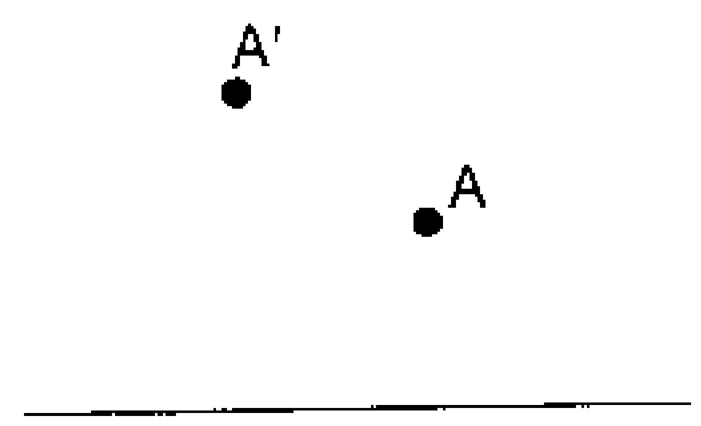
(b) How will each of these lenses act on a ray parallel to its optical axis if the lenses are submerged into a transparent liquid with a refractive index of 1.6?

19. A converging beam of rays is incident on a diverging lens. Having passed through the lens the rays intersect at a point 15 cm from the lens. If the lens is removed, the point where the rays meet, move 5 cm closer to the mounting that holds the lens.

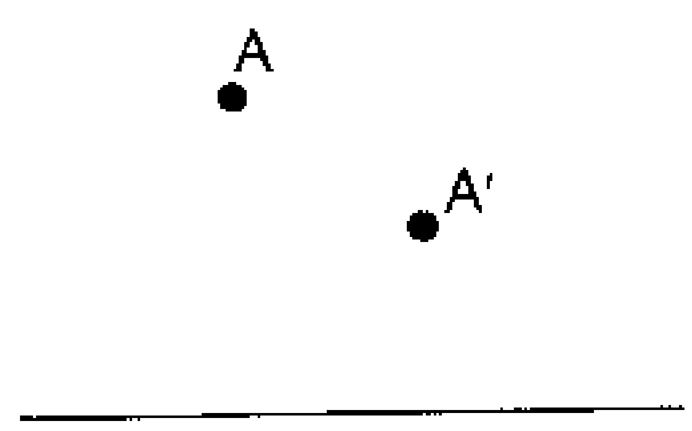
Find the focal length of the lens.

20. The distance between two point sources of light is 24 cm. Find out where would you place a converging lens of focal length 9 cm, so that the images of both the sources are formed at the same point.

21. Figure shows the optical axis of a lens, the point source of light A and its virtual image A'. Trace the rays to find the position of the lens and of its principal focus. What type of lens is it?



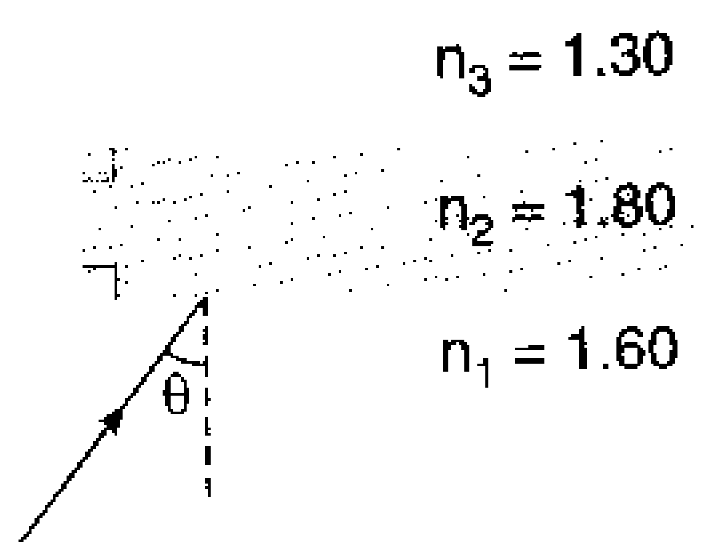
22. Solve the problem similar to the previous one if A and A' are interchanged as shown.



23. In figure, light refracts from material 1 into a thin layer of material 2, crosses that layer, and then is incident at the critical angle on the interface between materials 2 and 3.

(a) What is angle θ?

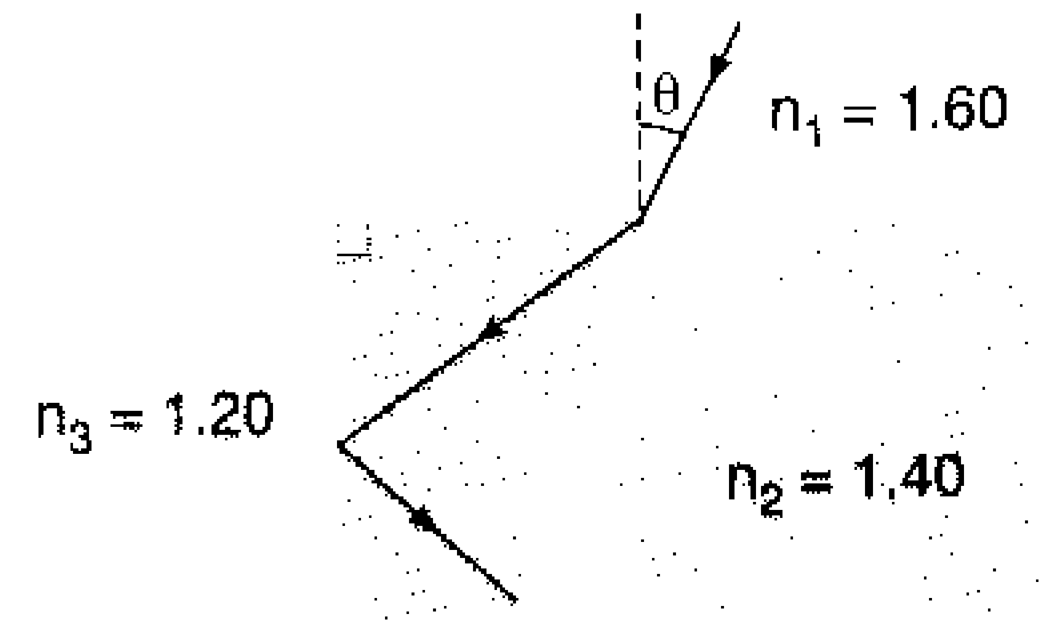
(b) If θ is decreased, is there refraction of light into material 3?



24. In figure, light refracts into material 2, crosses that material, and is then incident at the critical angle on the interface between materials 2 and 3.

(a) What is angle θ?

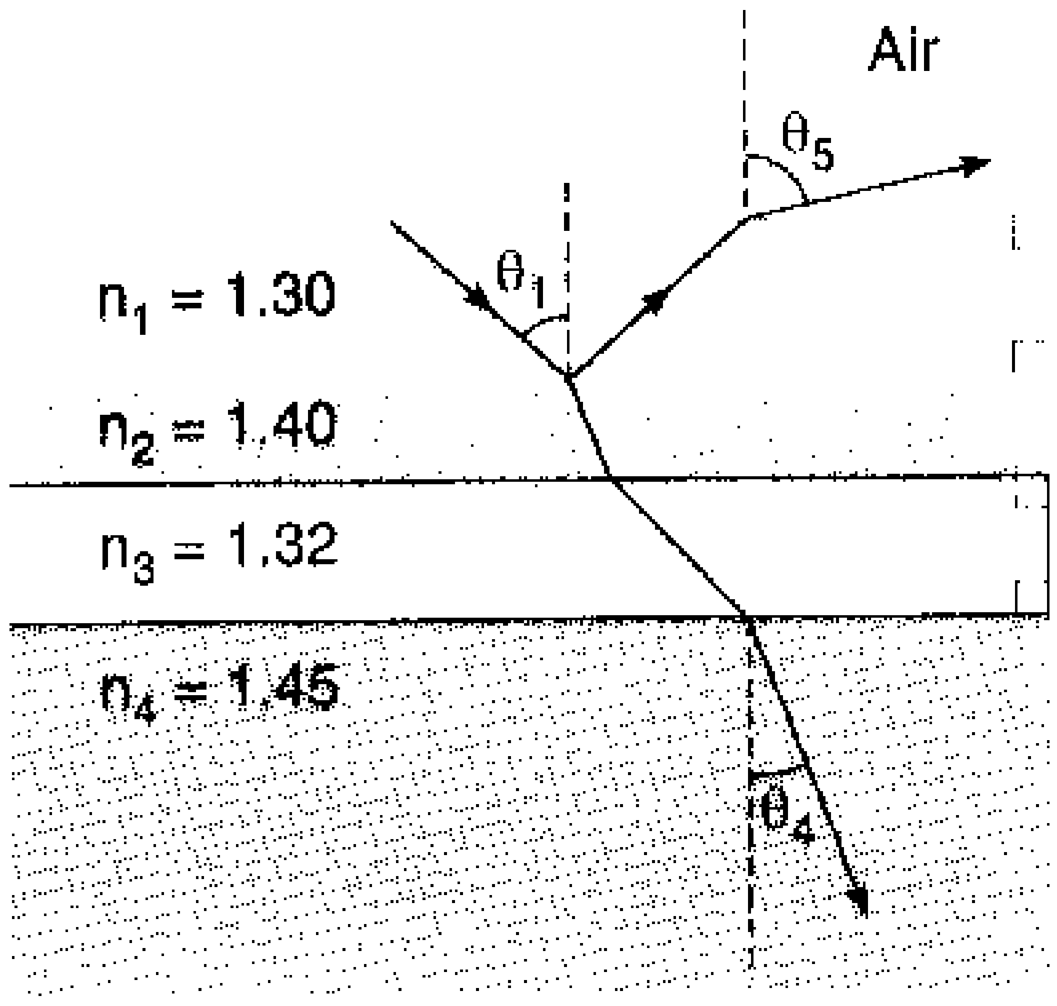
(b) If θ is increased, is there refraction of light into material 3?



25. In figure , light undergoes three refractions as it heads downward and a reflection and then a refraction to reach the air. The initial angle $\theta_1 = 30°$. What are the values of

(a) θ_5 and

(b) θ_4?



26. The angle of minimum deviation for a glass prism with $n = \sqrt{3}$ equals the refracting angle of the prism. What is the angle of the prism?

27. A ray of light is incident at an angle of 60° on the face of a prism having refracting angle 30°. The ray emerging out of the prism makes an angle 30° with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges.

28. The refracting angle of a glass prism is 30°. A ray is incident onto one of the faces perpendicular to it. Find the angle δ between the incident ray and the ray that leaves the prism. The refractive index of glass is $n = 1.5$.

29. The perpendicular faces of a right isosceles prism are coated with silver.

Prove that the rays incident at an arbitrary angle on the hypotenuse face will emerge from the prism parallel to the initial direction.

30. The refractive index of the material of a prism is 1.6 for a certain monochromatic ray. What should the maximum angle of incidence of this ray onto the prism be so that no total internal reflection occurs when the ray leaves the prism? The angle of refraction of the prism is 45°.

31. A ray of white light falls onto the side surface of an isosceles prism at such an angle that the red ray leaves the prism perpendicular to the second face. Find the deflection of the red and violet rays from the initial direction if the refraction angle of the prism is 45°. The refractive indices of the prism material for red and violet rays are 1.37 and 1.42, respectively.

32. Two thin converging lenses are placed on a common axis so that the centre of one of them coincides with the focus of the other. An object is placed at a distance twice the focal length from the left-hand lens. Where will its image be? What is the lateral magnification? The focal of each lens is f.

33. A concave spherical mirror with a radius of curvature of 0.2 m is filled with water. What is the focal power of this system? Refractive index of water is 4/3.

34. A convexo-convex lens has a focal length of $f_1 = 10$ cm. One of the lens surfaces having a radius of curvature of $R = 10$ cm is coated with silver. Construct the image of the object produced by the given optical system and determine the position of the image if the object is at a distance of $a = 15$ cm from the lens. Refractive index of lens = 1.5.

35. A source of light is located at double focal length from a convergent lens. The focal length of the lens is $f = 30$ cm. At what distance from the lens should a flat mirror be placed so that the rays reflected from the mirror are parallel after passing through the lens for the second time?

36. A parallel beam of rays is incident on a convergent lens with a focal length of 40 cm. Where should a divergent lens with a focal length of 15 cm be placed for the beam of rays to remain parallel after passing through the two lenses?

37. At what distance from a convexo-convex lens with a focal length of $f = 1$ metre should a concave spherical mirror with a radius of curvature of $R = 1$ metre be placed for a beam incident on the lens parallel to the major optical axis of the system to leave the lens, remaining parallel to the optical axis, after being reflected from the mirror? Find the image of the object produced by the given optical system.

38. An optical system consists of two convergent lenses with focal lengths $f_1 = 20$ cm and $f_2 = 10$ cm. The distance between the lenses is $d = 30$ cm. An object is placed at a distance of $a_1 = 30$ cm from the first lens. At what distance from the second lens will the image be obtained?

39. When observed from the Earth the angular diameter of the solar disk is $\phi = 32'$.

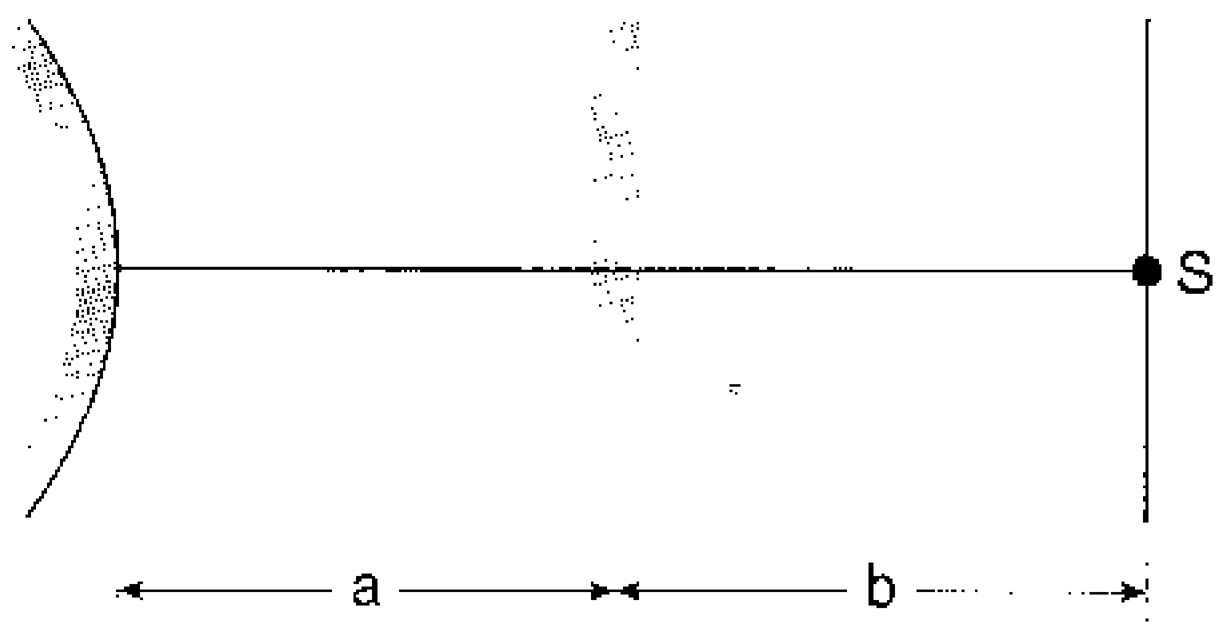
Determine the diameter of the image of the Sun formed by a convergent lens with a focal length $f = 0.25$ m.

40. Determine the position of the image produced by an optical system consisting of a concave mirror with a focal length of 10 cm and a convergent lens with a focal length of 20 cm. The distance from the mirror to the lens is 30 cm and from the lens to the object 40 is cm. Consider only two steps. Plot the image.

41. A convex lens of focal length 15 cm is placed infront of a convex mirror. Both are coaxial and the lens is 5 cm from the apex of the mirror. When an object is placed on the axis at a distance of 20 cm from the lens, it is found that image coincides with the object is 40 cm. Consider only two steps. Calculate the radius of curvature of mirror.

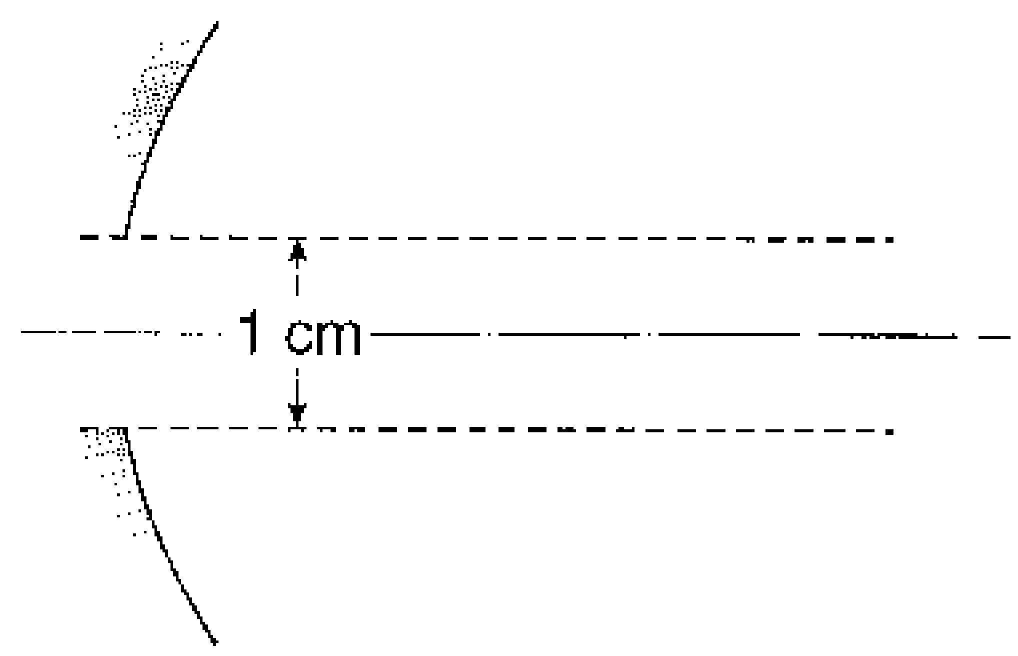
42. A lens with a focal length of $f = 30$ cm produces on a screen a sharp image of an object that is at a distance of $a = 40$ cm from the lens. A plane parallel plate with thickness of $d = 9$ cm is placed between the lens and the object perpendicular to the optical axis of the lens. Through what distance should the screen be shifted for the image of the object to remain distinct? The refractive index of the glass of the plate is $\mu = 1.8$.

43. A thin flat glass plate is placed in front of a convex mirror. At what distance b from the plate should a point source of light S be placed so that its image produced by the rays reflected from the front surface of the plate coincides with the image formed by the rays reflected from the mirror? The focal length of the mirror is $f = 20$ cm and the distance from the plate to the mirror $a = 5$ cm. How can the coincidence of the images be established by direct observation?



44. A concave mirror forms the real image of a point source lying on the optical axis at a distance of 50 cm from the mirror. The focal length of the mirror is 25 cm. The mirror is cut in two and its halves are drawn a distance of 1 cm apart in a direction perpendicular to the optical axis.

How will the images formed by the halves of the mirror be arranged?



45. Find the minimum and maximum angle of deviation for a light ray passing through a prism with refracting angle $A = 90°$. [Take $n = 1.3$]

LEVEL-II

1. A ray of light is incident upon one face of a prism (angle of prism $< \pi/2$) in a direction perpendicular to the other face. Prove that the ray will fail to emerge from the other face if $\cot A < \cot \theta_c - 1$, where θ_c is critical angle for the material of prism.

2. A ray of light falling on a glass sphere of $\mu = \sqrt{3}$ such that the directions of the incident ray and emergent ray when produced meet the surface at the same point on the surface. Draw the ray diagram and find the value of angle of incidence.

3. One face of a prism with a refractive angle of 30° is coated with silver. A ray incident on another face at an angle of 45° is refracted and reflected from the silver coated face and retraces its path.

What is the refractive index of the prism?

4. One side of radius of curvature $R_2 = 120$ cm of a convex lens of material of refractive index $\mu = 1.5$ and focal length $f_1 = 40$ cm is silvered. It is placed on a horizontal surface with silvered surface in contact with it. Another convex lens of focal length $f_2 = 20$ cm is fixed coaxial $d = 10$ cm above the first lens. A luminous point object O on the axis gives rise to an image coincident with it. Find its height above the upper lens.

5. A small object is placed on the principal axis of a concave spherical mirror of radius 20 cm at a distance of 30 cm. By how much will the position and size of the image alter, when a parallel-sided slab of glass of thickness 6 cm and refractive index 1.5 is introduced between the centre of curvature and the object? The parallel sides are perpendicular to the principal axis.

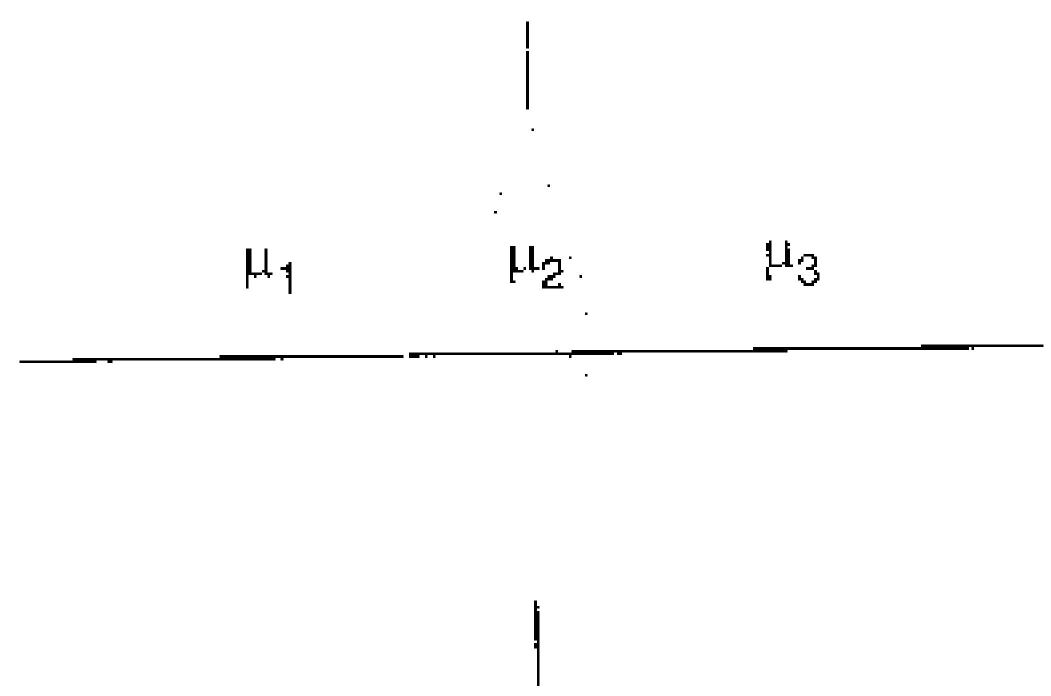
6. A convex lens of focal length f_1 is placed infront of a luminous point object. The separation between the object and the lens is $3f_1$. A glass slab of thickness t is placed between object and the lens A real image of the object is formed at the shortest possible distance from the object.

(a) Find the refractive index of the slab.

(b) If a concave lens of very large focal length f_2 is placed in contact with the convex lens, find the shifting of the image.

7. Light rays from a source are incident on a glass prism of index of refraction μ and angle of prism α. At near normal incidence, calculate the angle of deviation of the emerging rays.

8. A thin glass lens of refractive index $\mu_2 = 1.5$ behaves as a interface between two media of refractive indices $\mu_1 = 1.4$ and $\mu_3 = 1.6$ respectively. Determine the focal length of the lens for the shown arrangement of radius of curvature of both the surfaces is 20 cm.



9. A ray of light is incident at an angle of 60° on one face of a prism which has an angle of refraction 30°. The ray emerging out of the prism makes an angle of 30° with the incident ray. Show that the emergent ray is perpendicular to the face through which it emerges and calculate the refractive index of the material of the prism.

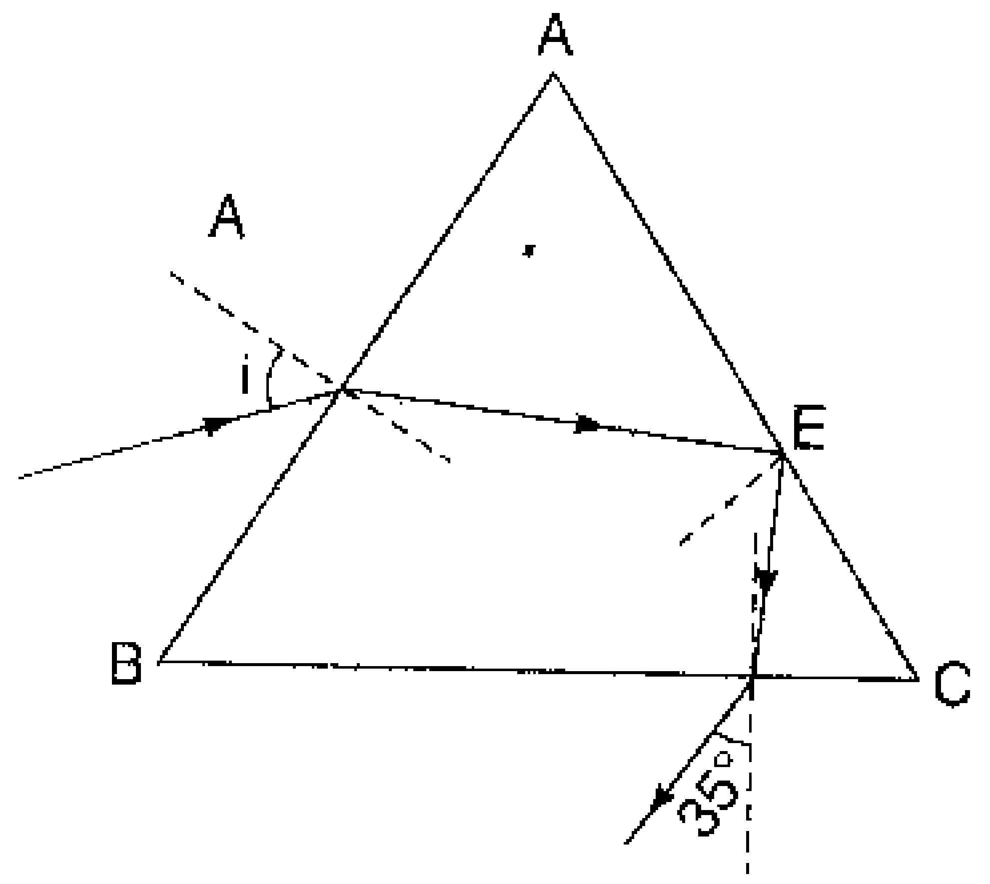
10. A point source of light S is placed at the bottom of a vessel containing a liquid of refractive index 5/3. A person is viewing the source from above the surface. There is an opaque disc of radius 1 cm floating on the surface. The center of the disc lies vertically above the source S. The liquid from the vessel is gradually drained out through a tap. What is the maximum height of the liquid for which the source cannot at all seen from above.

11. A rectangular glass block is placed on top of a sheet of paper on which there is a small cross. When the paper is soaked in alcohol and a sodium lamp is placed opposite to one vertical of the block the cross can be seen through the opposite vertical face up to a point where the angle of emergence of the light is 30°. If the refractive index of the glass is 1.5, find the refractive index of alcohol. Why can't the black cross be seen through the face when the paper is dry.

12. The path of a ray undergoing refraction in an equilateral prism is shown in figure. The ray suffers refraction at the face AB and the refracted ray is incident on the face AC at the critical angle and hence, totally reflected. After refraction at the face BC the emergent ray makes an angle of $35°$ with normal at BC at the point of emergence.

Find : (a) refractive index of the prism

(b) the corresponding angle of incidence i.



13. Three convergent thin lenses of focal lengths $4a$, a and $4a$ respectively are placed in order along the axis so that the distance between consecutive lenses is $4a$. Prove that this combination simply inverts every small object on the axis without change of magnitude or position.

14. A ray of light falls onto a plane-parallel glass plate 1 cm thick at an angle of 60°. The refractive index of the glass is $\sqrt{3}$. Some of the light is reflected and the rest, being refracted, passes into the glass is reflected from the bottom of the plate, refracted a second time and emerges back into the air parallel to the first reflected ray. Determine the distance l between the rays.

15. A glass hemisphere of radius 10 cm and $\mu = 1.5$ is silvered over its curved surface. There is an air bubble in the glass 5 cms from the plane surface along the axis. Find the position of the images of this bubble seen by observer looking along the axis into the flat surface of the hemisphere.

16. A prism of flint glass ($\mu_g = 3/2$) with an angle of refraction 30° is placed inside water ($\mu_w = 4/3$).

(a) At what angle should a ray of light fall on the face of the prism so that inside the prism the ray is perpendicular to the bisector of the angle of the prism.

(b) Through what angle will the ray turn after passing through both faces of the prism?

17. A vertical beam of light of cross-sectional radius r is incident symmetrically on the curved surface of a glass hemisphere ($\mu = 3/2$) of radius $2r$ placed with its base on a horizontal table. Find the radius of the luminous spot formed on the table.

18. The greatest thickness of a planoconvex lens when viewed normally through the plane surface appears to be 0.03 m and when viewed normally through the curved surface it appears to be 0.036 m. If the actual thickness is 0.045 m, find

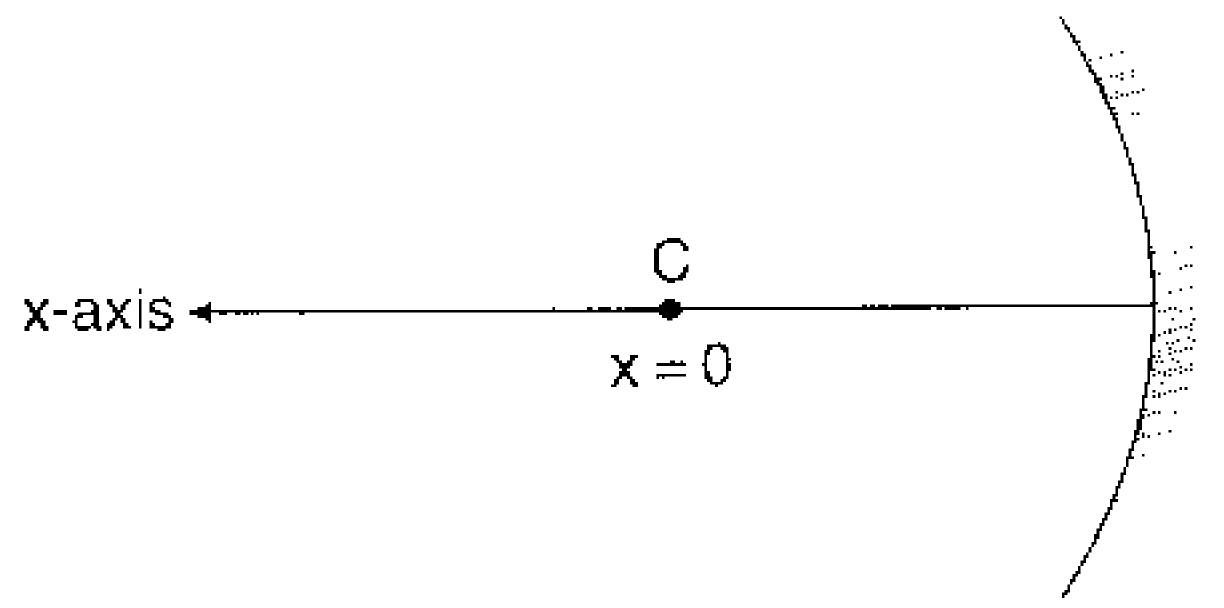
(a) the refractive index of the material of the lens

(b) the radius of curvature of lens

(c) what must be the focal length if its plane surface is silvered

(d) what must be the focal length when the curved surface is silvered

19. A ball swings back and forth infront of a concave mirror. The motion of the ball is described approximately by the equation $x = f \cos \omega t$, where f is the focal length of the mirror and x is measured along the axis of mirror. The origin is taken at the centre of curvature of the mirror.

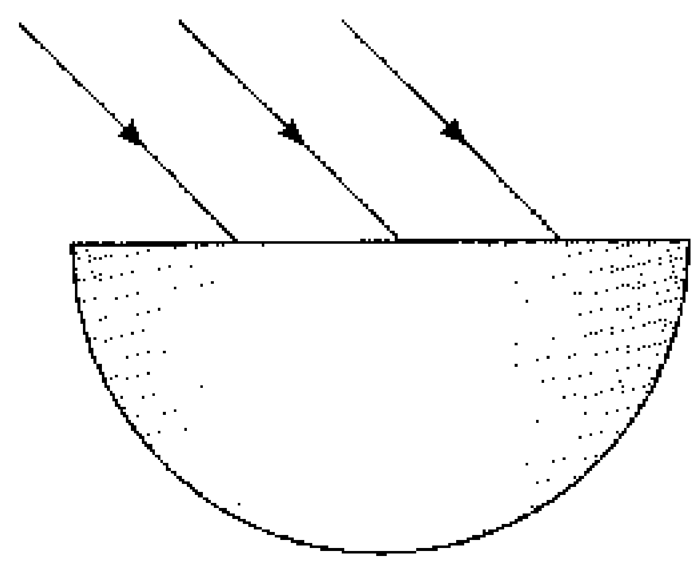


(a) Derive an expression for the distance from the mirror to the image of the swinging ball.

(b) At what point does the ball appear to coincide with its image.

(c) What will be the lateral magnification of the image of the ball at time $t = \frac{T}{2}$, where T is time period of oscillation?

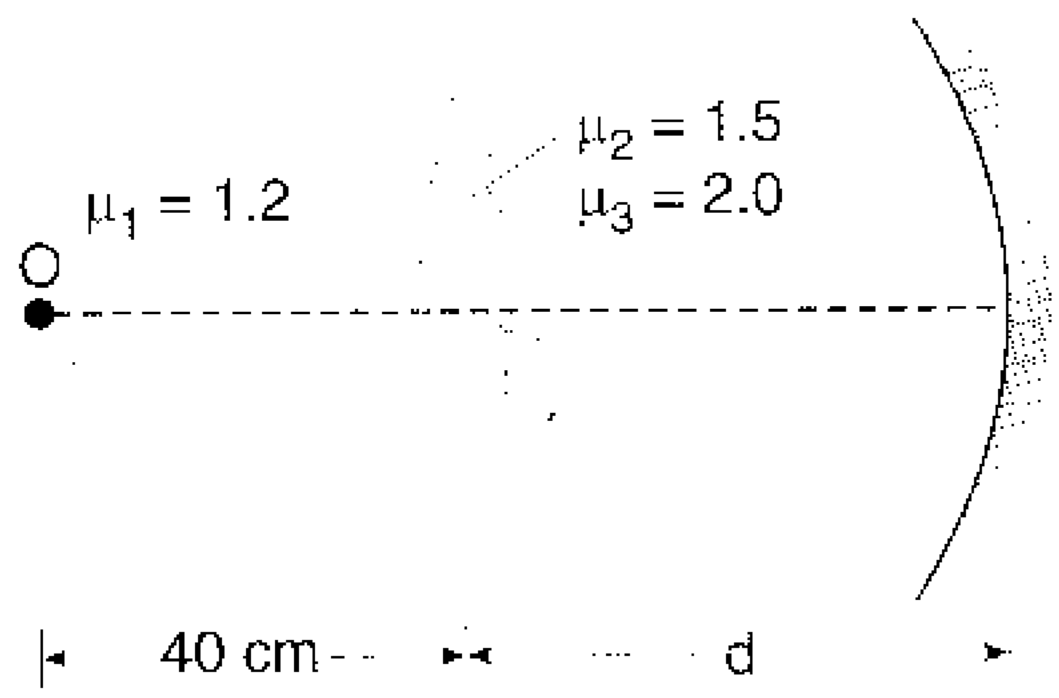
20. Rays of light fall on the plane surface of a half cylinder at an angle 45° in the plane perpendicular to the axis (see figure). Refractive index of glass is $\sqrt{2}$. Discuss the condition that the rays do not suffer total internal reflection.



21. Convex and concave mirrors have the same radii of curvature R. The distance between the mirrors is $2R$. At what point on the common optical axis of the mirrors should a point source of light A be placed for the rays to converge at the point A after being reflected first on the convex and then on the concave mirror?

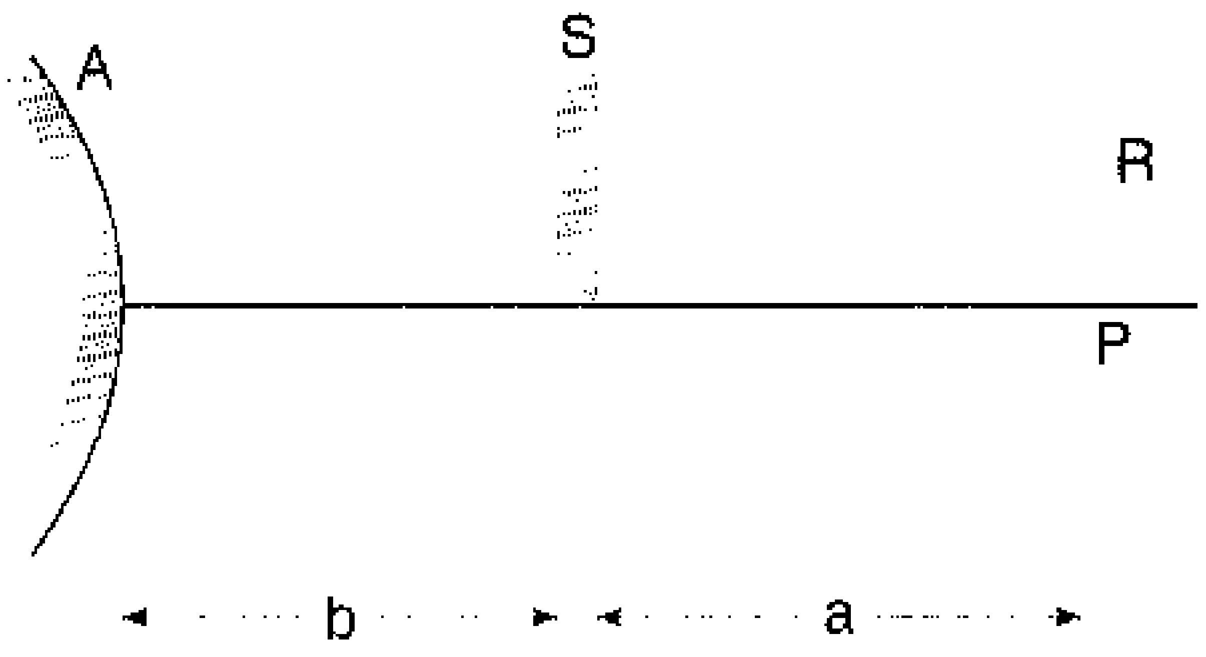
22. A point source of light S is placed on the major optical axis of concave mirror at a distance of 60 cm. At what distance from the concave mirror should a flat mirror be placed for the rays to converge again at the point S having been reflected from the concave mirror and then from the flat one? Will the position of the point where the rays meet change if they are first reflected from the flat mirror? The radius of the concave mirror is 80 cm.

23. The figure shows an arrangement of an equiconvex lens and a concave mirror. A point object O is placed on the principal axis at a distance 40 cm from the lens such that the final image is also formed at the position of the object. If the radius of curvature of the concave mirror is 80 cm, find the distance d. Also draw the ray diagram. The focal length of the lens in air is 20 cm.



24. A screen S is placed a distance $b = 5$ cm from a circular convex mirror as shown in figure. An object RP of height $h = 3$ cm is arranged a distance $a = 5$ cm from the screen.

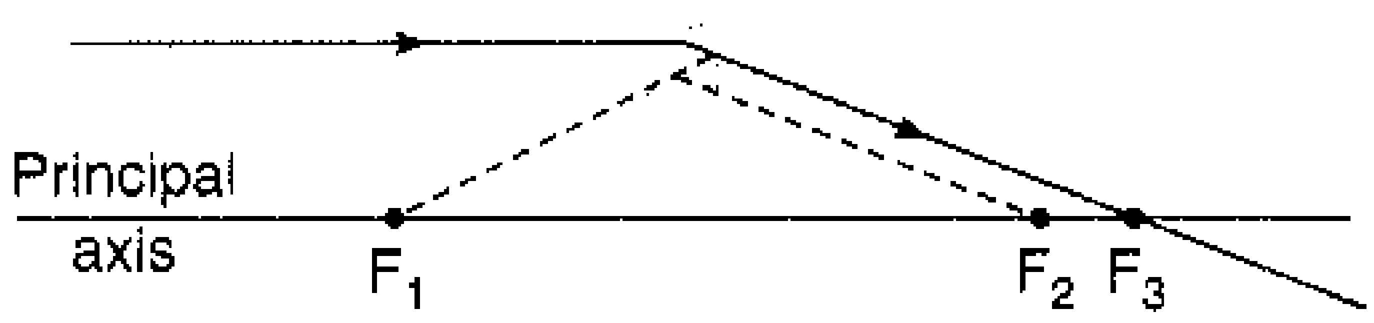
Where must an observer position himself to see the image of the entire object? What are the maximum dimensions of the object (with the given arrangement of the object, the mirror and the screen) for the mirror to reproduce an image of the entire object? The diameter of the mirror is $d = 10$ cm.



25. At what distance from one's face should a pocket convex mirror 5 cm in cross section be held to see all of the face if the focal length of the mirror is 7.5 cm and the length of the face is 20 cm?

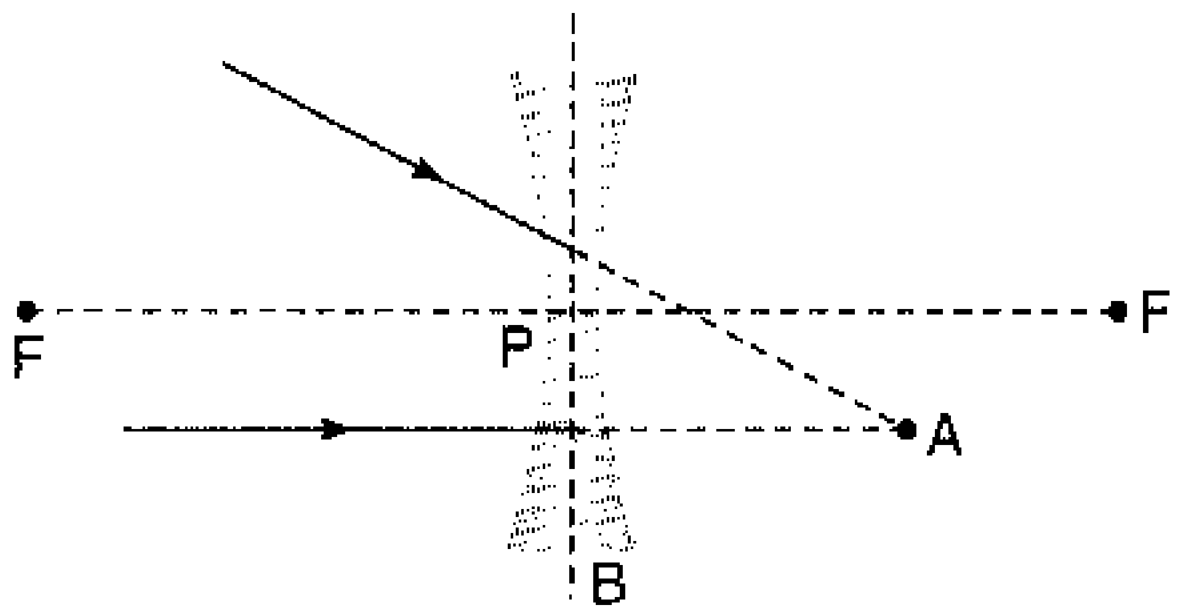
26. An intense beam parallel to the principal axis is incident on a convex lens. Multiple extra images $F_1, F_2, \ldots$ are formed due to feeble internal reflections, called flare spots as shown in the figure. The radius of curvature of the lens are 30 cm and 60 cm and the refractive index is 1.5.

Find the position of the first flare spot.



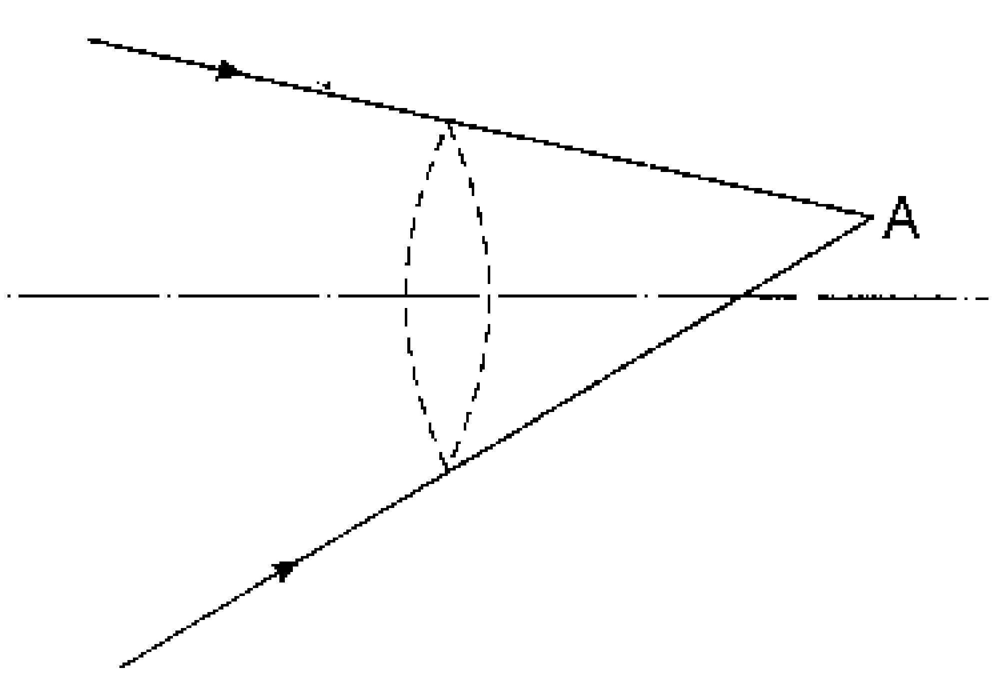
27. The rays of a converging beam meet at a point A. A diverging lens is placed in their path in the plane B (Fig).

Plot the position of the point where the rays meet after passing through the lens. The position of the principal foci PF is known.



28. A converging beam of rays passes through a round aperture in a screen as shown in figure. The apex of the beam A is at a distance of 15 cm from the screen.

How will the distance from the focus of the rays to the screen change if a convergent lens is inserted in the aperture with a focal length of 30 cm? Plot the path of the rays after the lens is fitted.



29. A convex lens is held 45 cm above the bottom of an empty tank. The image of a point at the bottom of a tank is formed 36 cm above the lens. Now a liquid is poured into the tank to a depth of 40 cm. It is found that the distance of the image of the same point on the bottom of the tank is 48 cm above the lens. Find the refractive index of the liquid.

30. In an isosceles prism of angle 45°, it is found that when the angle of incidence is same as the prism angle the emergent ray grazes the emergent surface. Find the refractive index of the material of the prism. For what angle of incidence the angle of deviation will be minimum?

31. An object is at a distance of $d = 2.5$ cm from the surface of a glass sphere with a radius $R = 10$ cm. Find the position of the image produced by the sphere. The refractive index of the glass is $\mu = 1.5$.

32. A concave mirror has the form of a hemisphere with a radius $R = 55$ cm. A thin layer of an unknown transparent liquid is poured into this mirror, and it was found that the given optical system produces, with the source in a certain position, two real images, one of which (formed by direct reflection) coincides with the source and the other is at a distance of $l = 30$ cm from it. Find the refractive index μ of the liquid.

33. A glass wedge with a small angle of refraction α is placed at a certain distance from a convergent lens with a focal length f, one surface of the wedge being perpendicular to the optical axis of the lens. A point source of light is on the other side of the lens in its focus. The rays reflected from the wedge produce after refraction in the lens, two images of the source displaced with respect to each other by d. Find the refractive index of the wedge glass.

34. A convexo-convex lens made of glass with a refractive index of $\mu = 1.6$ has a focal length of $f = 10$ cm in air. What will the focal length of this lens be if it is placed into a transparent medium with a refractive index of $\mu_1 = 1.5$? Also find the focal length of this lens in a medium with a refractive index of $\mu_2 = 1.7$.

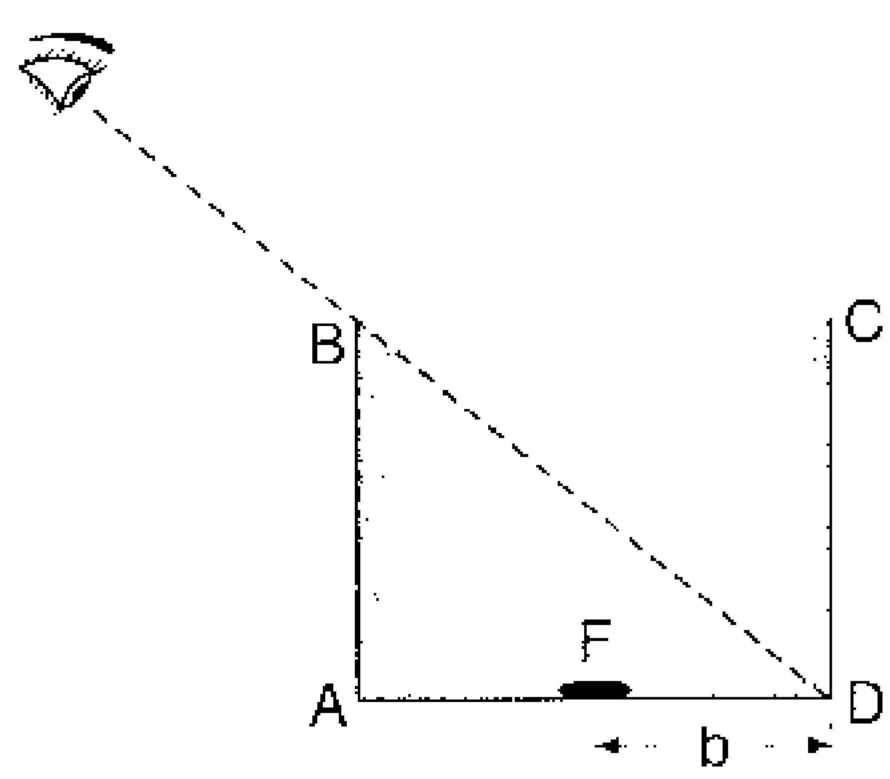
35. A beam of light enters a glass prism at an angle α and emerges into the air at an angle β. Having passed through the prism, the beam is deflected from the original direction by an angle γ. Find the angle of refraction of the prism and the refractive index of the material which it is made of.

36. A ray incident on the face of a prism is refracted and escapes through an adjacent face. What is the maximum permissible angle of refraction of the prism, if it is made of glass with a refractive index of $\mu = 1.5$?

37. A cubical vessel with non-transparent walls is so located that the eye of an observer does not see its bottom but sees all of the wall CD.

To what height should water be poured into the vessel for the observer to see an object F arranged at a distance of $b = 10$ cm from corner D? The face of the vessel is $a = 40$ cm.

Refractive index of water is $\frac{4}{3}$.

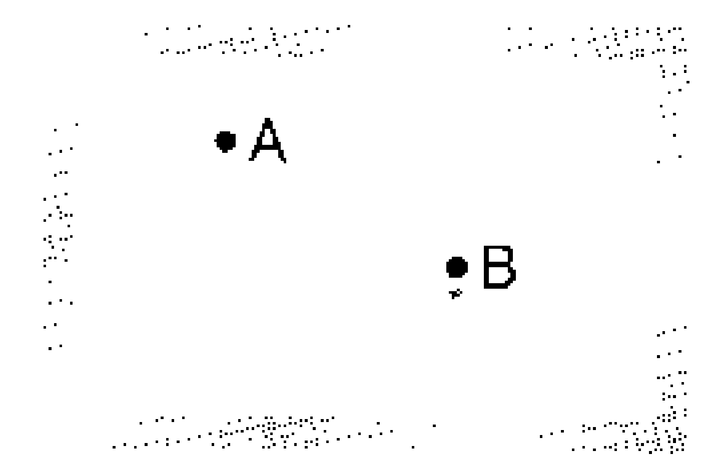


38. A lens with a focal length of $f = 30$ cm produces on a screen a sharp image of an object that is at a distance of $a = 40$ cm from the lens. A plane-parallel plate with a thickness of $d = 9$ cm is placed between the lens and the object perpendicular to the optical axis of the lens. Through what distance should the screen be shifted for the image of the object to remain distinct? The refractive index of the glass of the plate is $\mu = 1.8$.

39. A parallel beam of light is incident on a system consisting of three thin lenses with a common optical axis. The focal lengths of the lenses are equal to $f_1 = +10$ cm, $f_2 = -20$ cm and $f_3 = +9$ cm, respectively. The distance between the first and the second lenses is 15 cm and between the second and the third 5 cm. Find the position of the point at which the beam converges when it leaves the system of lenses.

40. In what direction should a beam of light be sent from point A (Fig.) contained in a mirror box for it to fall onto point B after being reflected once from all four walls?

Point A and B are in one plane perpendicular to the walls of the box (*i.e.*, in the plane of the drawing).



41. One end of a long glass rod ($\mu = 1.5$) is formed into the shape of a convex surface of radius 6.0 cm. An object is located in air along the axis of the rod, at a distance of 10 cm from the end of the rod. (a) How far apart are the object and the image formed by the glass rod? (b) For what range of distances from the end of the rod must the object be located in order to produce a virtual image?

42. A converging bundle of light rays in the shape of a cone with the vertex angle of 40° falls on a circular diaphragm of 20 cm diameter. A lens with a focal power of 5 diopters is fixed in the diaphragm. What will the new cone angle be?

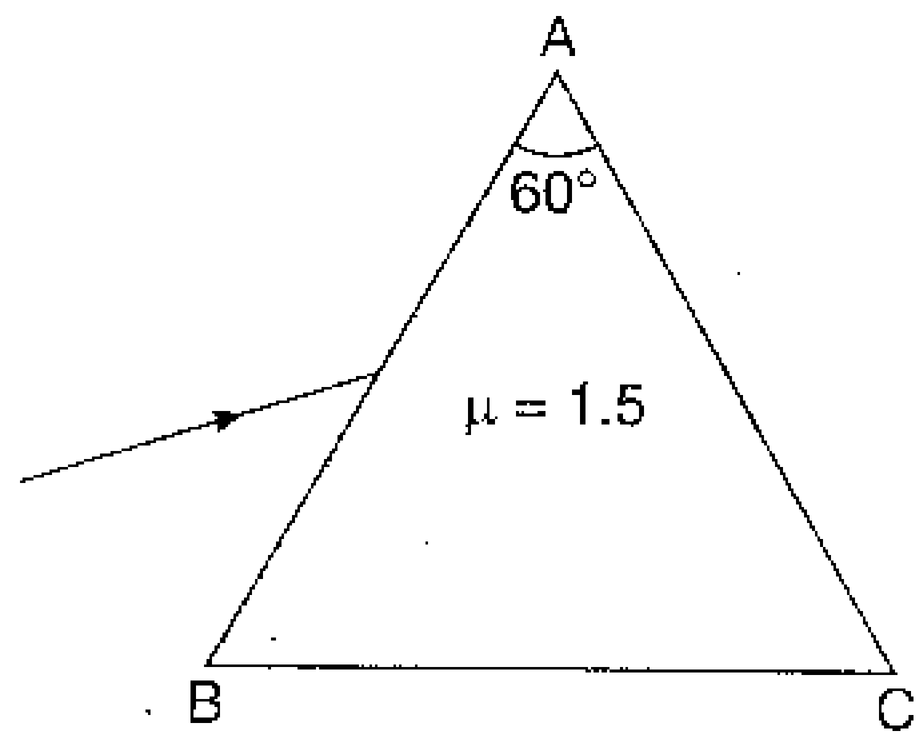
43. Two thin lenses with focal lengths of $f_1 = 7$ cm and $f_2 = 6$ cm are placed at a distance $d = 3$ cm apart. What is the distance of the focus of the system from the second lens? The system is a centred one.

44. Two equi-convex lenses of focal lengths 30 cm and 70 cm, made of material of refractive index = 1.5, are held in contact coaxially by a rubber band round their edges. A liquid of refractive index 1.3 is introduced in the space between the lenses filling it completely. Find the position of the image of a luminous point object placed on the axis of the combination lens at a distance of 90 cm from it.

45. In a 60° prism of $\mu = 1.5$, the condition for minimum deviation is fulfilled. If face AC is polished

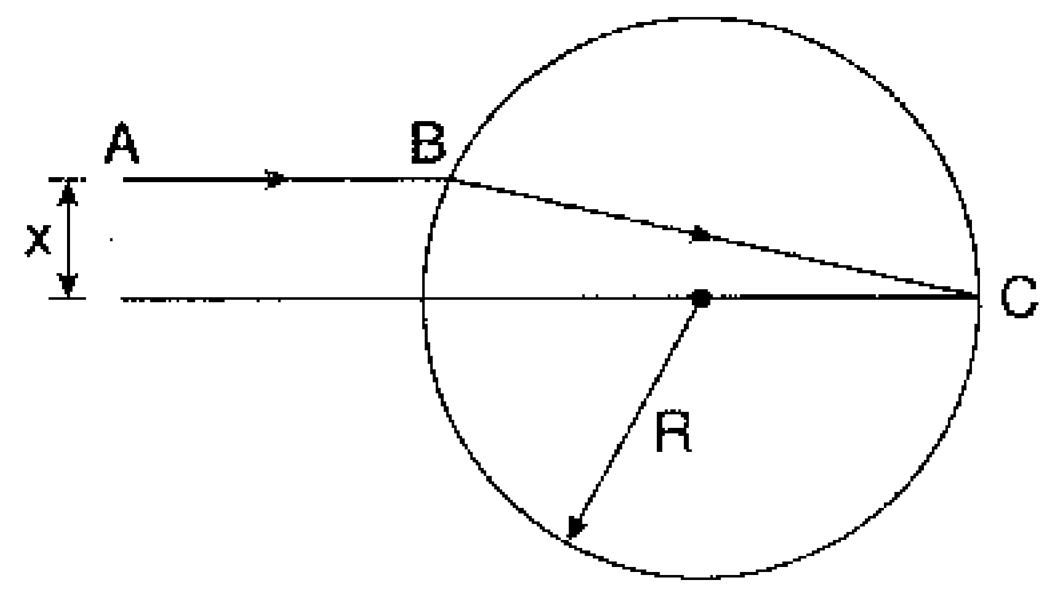
(a) Find the net deviation.

(b) If the system is placed in water what will be the net deviation? Refractive index of water $= \frac{4}{3}$.



46. A fish is rising up vertically inside a pond with velocity 4 cm/s, and notices a bird, which is diving vertically downward and its velocity appears to be 16 cm/s (to the fish). What is the real velocity of the diving bird, if refractive index of water is 4/3.

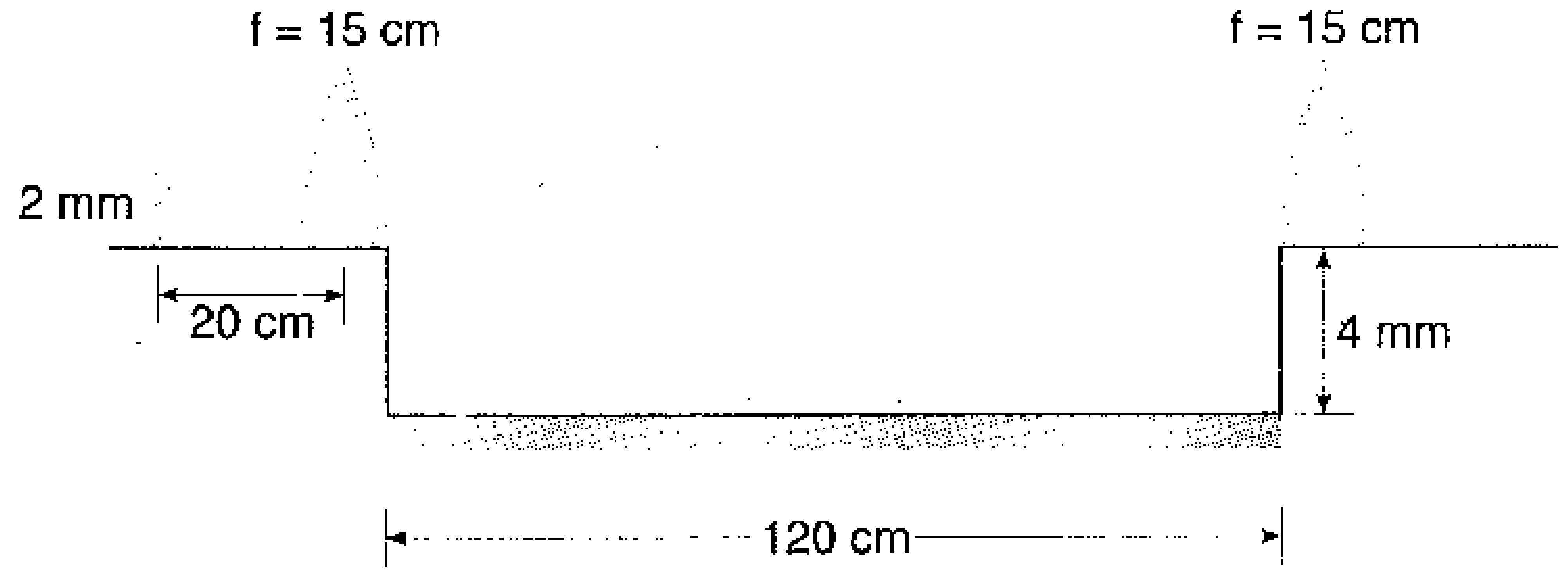
47. A spherical ball of of transparent material has index of refraction μ. A narrow beam of light AB is aimed as shown. What must the index of refraction be in order that the light is focussed at the point C on the opposite end of the diameter from where the light entered? Given that $x << R$.



48. A ray incident on a droplet of water at an angle of incidence i undergoes two reflections (not total) and emerges. If the deviation suffered by the ray within the drop is minimum and the refractive index of the droplet be μ, then show that $\cos i = \sqrt{\frac{\mu^2 - 1}{8}}$.

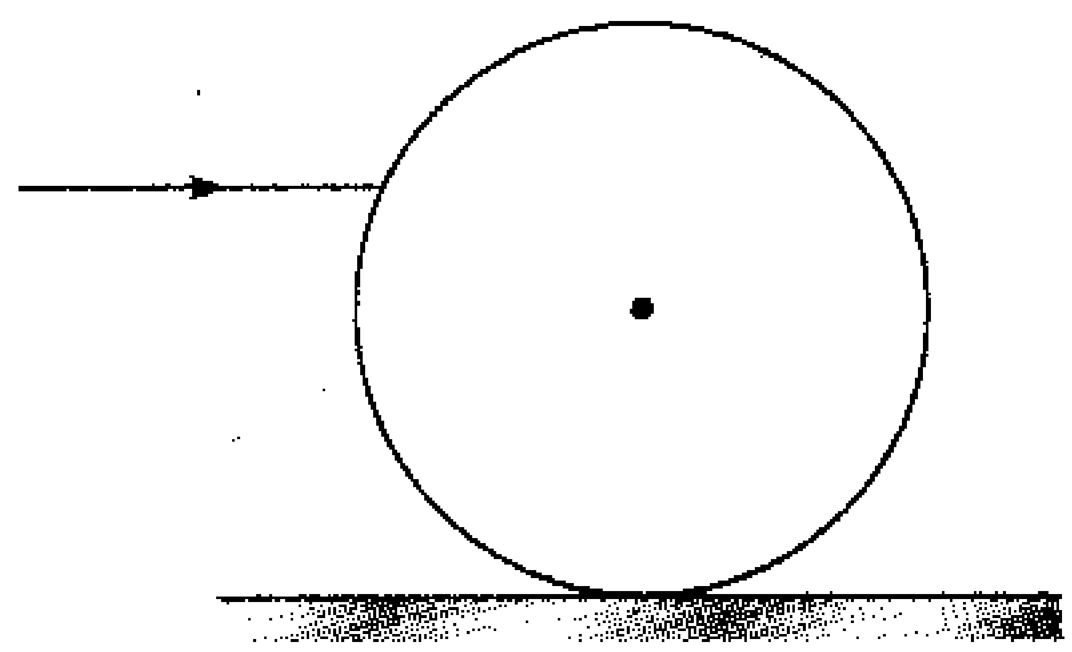
LEVEL-III

1. A convex lens of focal length 15 cm is split into two halves and the two halves are placed at a separation of 120 cm. Between the two halves of convex lens a plane mirror is placed horizontally and at a distance of 4 mm below the principal axis of the lens halves. An object of length 2 mm is placed at a distance of 20 cm from one half lens as shown in figure.

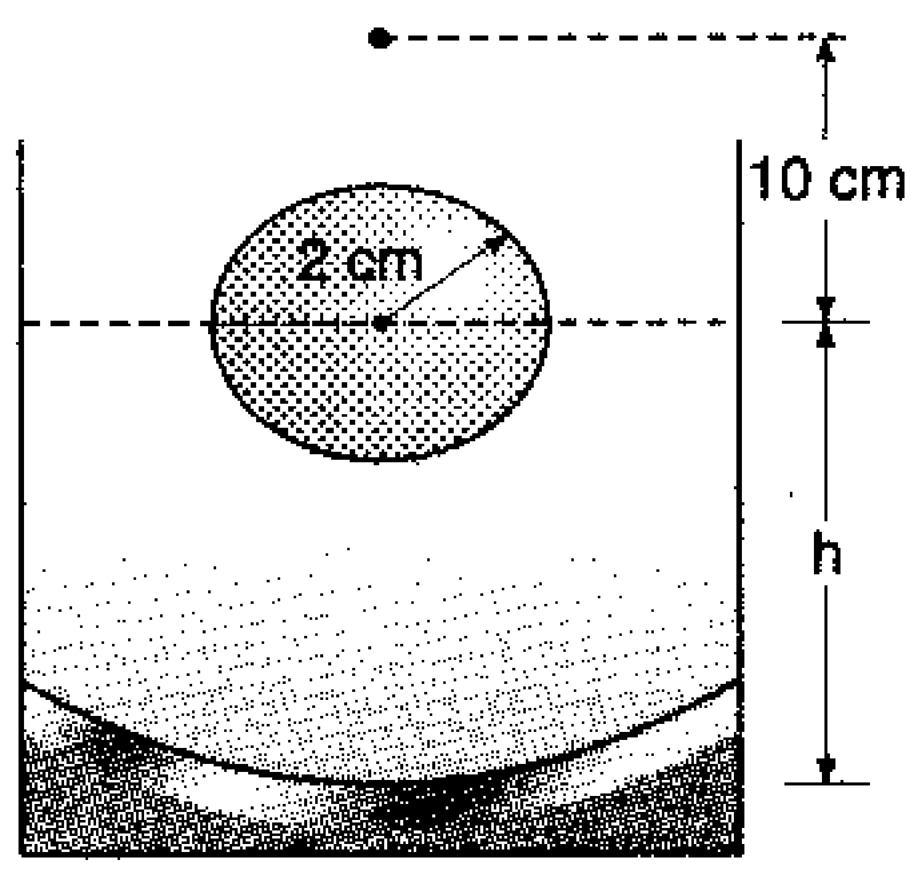


(a) Find the position and size of the final image.
(b) Trace the path of rays forming the image.

2. A cylindrical glass rod of radius 0.1 m and refractive index $\sqrt{3}$ lies on a horizontal plane mirror. A horizontal ray of light moving perpendicular to the axis of the rod is incident on it. At what height from the mirror should the ray be incident so that it leaves the rod at a height of 0.1 m above the plane mirror? At what distance a second similar rod, parallel to the first, be placed on the mirror, such that the emergent ray from the second rod is in line with the incident ray on the first rod?

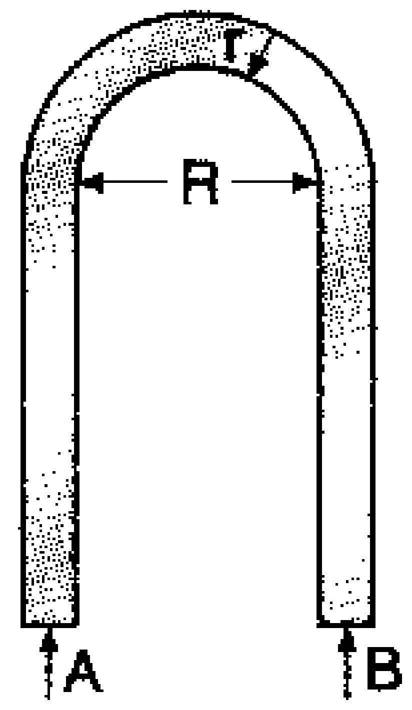


3. A transparent solid sphere of radius 2 cm and density ρ floats in a transparent liquid of density 2ρ kept in a beaker. The bottom of the beaker is spherical in shape with radius of curvature 8 cm and is silvered to make it concave mirror as shown in the figure. When an object is placed at a distance of 10 cm directly above the centre of the sphere its final image coincides with it. Find h (as shown in the figure), the height of the liquid surface in the beaker from the apex of the bottom. Consider the paraxial rays only. The refractive index of the sphere is 3/2 and that of the liquid is 4/3.



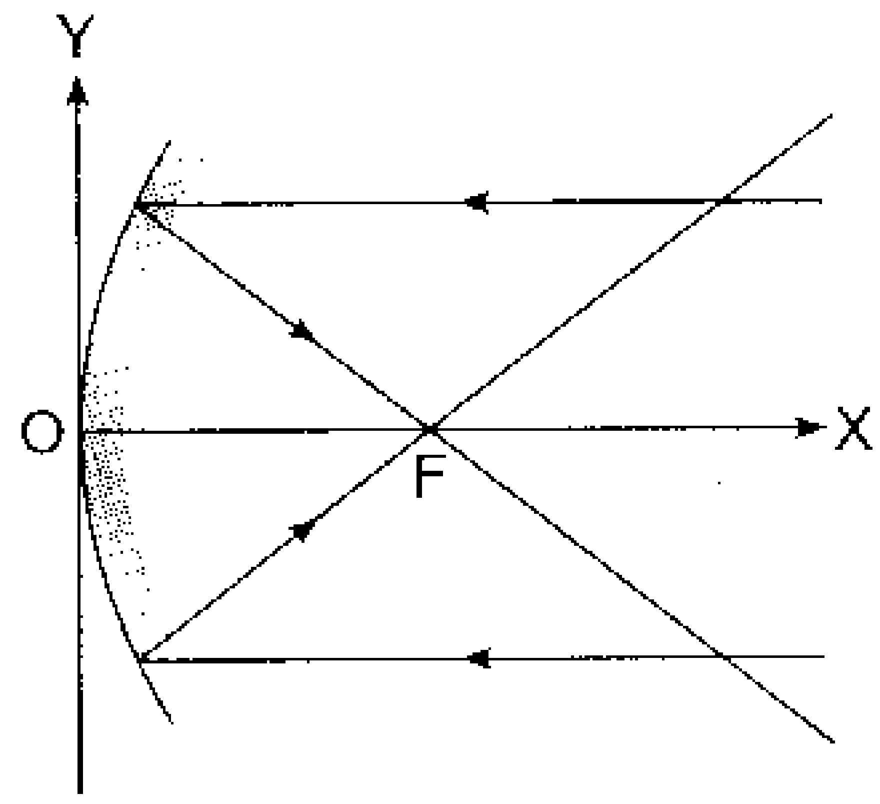
4. A convex lens of focal length f_1 is placed infront of a luminous point P so that the distance of the point P from lens is greater than focal length and the image formed is at the shortest possible distance. If now a concave lens of very large focal length f_2 be placed in contact with first, find the shift in the position of the image.

5. Find the maximum value of r/R, so that the beam of light incident normally at the face A of a U shaped glass tube emerges through B as shown in the figure. The refractive index of glass is $\mu = 1.5$.

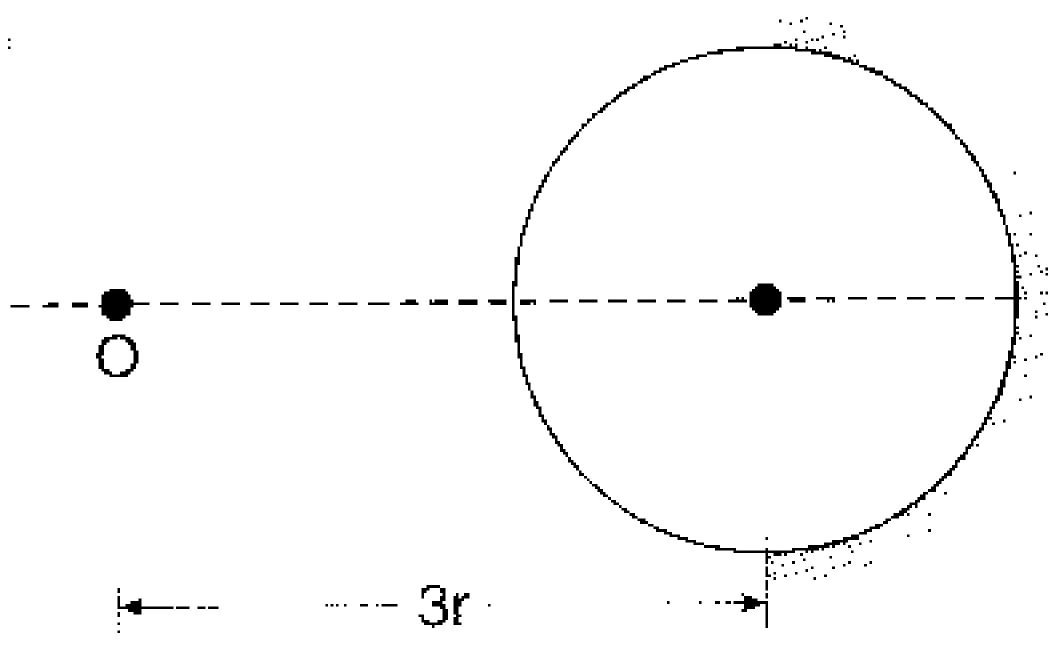


6. A hollow sphere of glass of refractive index μ has a small mark on its interior surface which is observed from a point outside the sphere on the side opposite the center. The inner cavity is concentric with external surface and the thickness of the glass is everywhere equal to the radius of the inner surface. Prove that the mark will appear nearer than it really is, by a distance $\frac{(\mu - 1)R}{(3\mu - 1)}$, where R is the radius of the inner surface.

7. Show that a parallel bundle of light rays parallel to the x-axis and incident on a parabolic reflecting surface given by $x = 2\,by^2$, will pass through a single point called focus of the reflecting surface. Also, find the focal length.



8. A hemispherical portion of the surface of a solid glass sphere of refractive index 1.5 and of radius r is silvered to make the inner side reflecting. An object is placed on the axis of the sphere at a distance $3r$ from the centre of the sphere. The light from the object is refracted at the unsilvered part, then reflected from the silvered part and again refracted at the unsilvered part. Locate the final image formed.

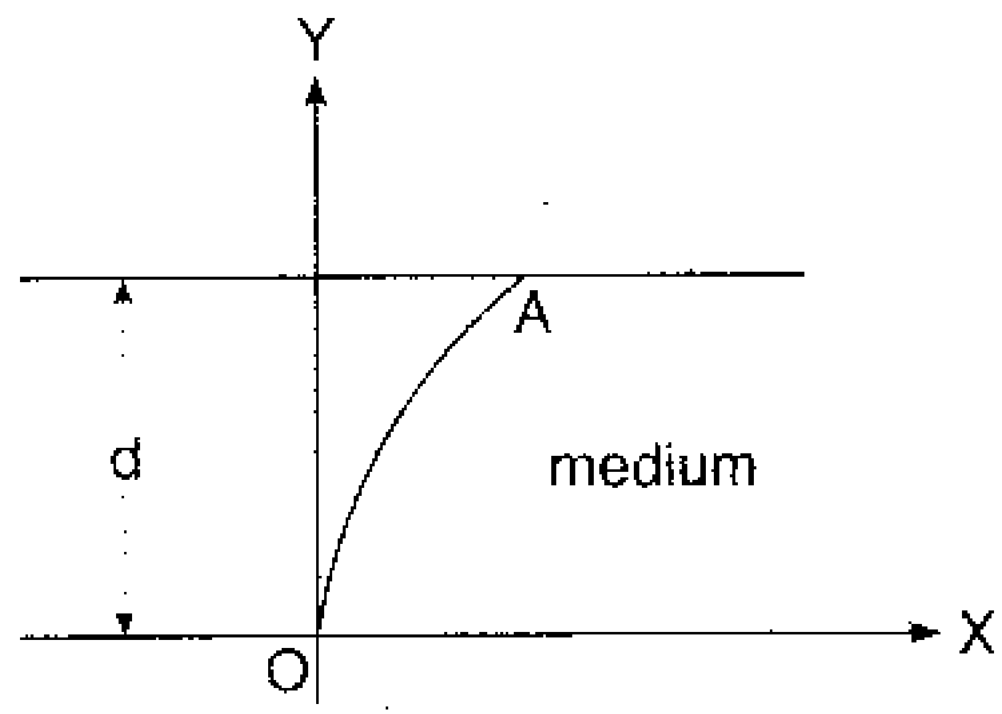


9. A long rectangular slab of transparent medium of thickness d is placed on a table with length parallel to the x-axis and width parallel to the y-axis. A ray of light is travelling along y-axis at origin. The refractive index μ of the medium varies as $\mu = \dfrac{\mu_0}{1-(x/r)}$, where μ_0 and $r\,(>1)$ are constants. The refractive index of air is 1.

(a) Determine the x-coordinate of the point A, where the ray intersects the upper surface of the slab-air boundary.

(b) Write down the refractive index of the medium at A.

(c) Indicate the subsequent path of the ray in air.



10. (a) If r be the radius of curvature of each face of thin converging lens whose one face is silvered and μ is the refractive index of lens material, show that the lens is equivalent to a concave mirror of focal length $\dfrac{r}{4\mu-2}$.

(b) Consider a planoconcave lens with one of the radii of curvature r made up of a transparent material whose refractive index varies with intensity (I) of incident light as $\mu = \mu_0 + aI$, where $a > 0$ and $3/2 > \mu_0 > 0$. Calculate the intensity when the focal length is equal to two times the radius of curvature r.

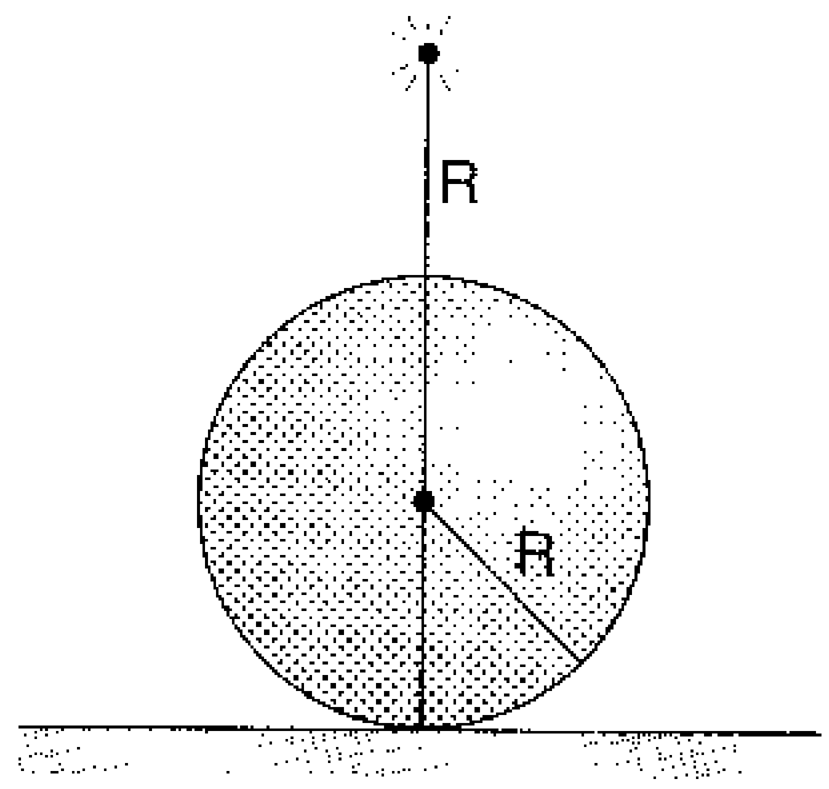
11. A ray of light is refracted through a sphere whose material has a refractive index μ in such a way that it passes through the extremities of two radii which make an angle β with each other. Prove that if α is the deviation of the ray caused by its passage through the sphere,

$$\cos\left(\frac{\beta-\alpha}{2}\right) = \mu\cos\left(\frac{\beta}{2}\right)$$

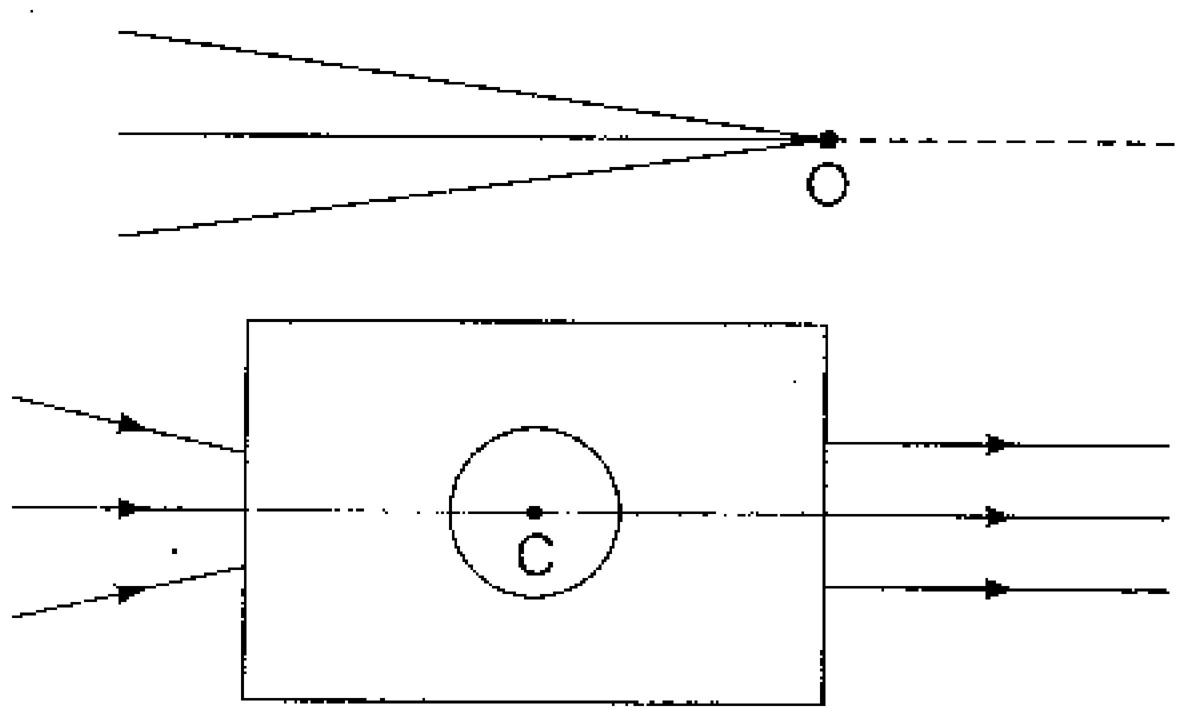
12. An opaque sphere of radius R lies on a horizontal plane. On the perpendicular through the point of contact there is a point source of light a distance R above the sphere.

(a) Show that the area of the shadow on the plane is $3\pi R^2$.

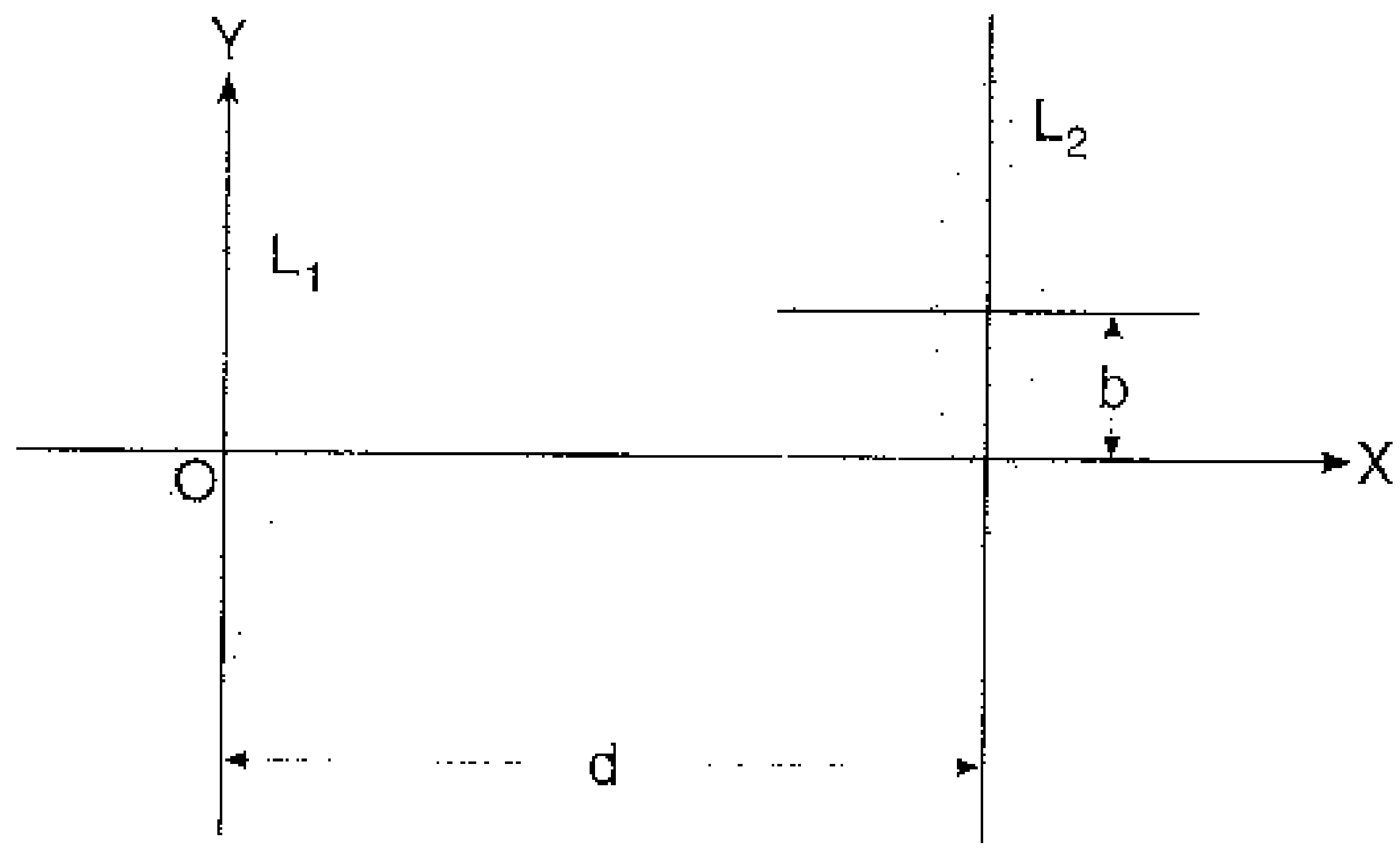
(b) A transparent liquid of refractive index $\sqrt{3}$ is filled above the plane such that the sphere is just covered with the liquid. Show that the area of shadow now becomes $2\pi R^2$.



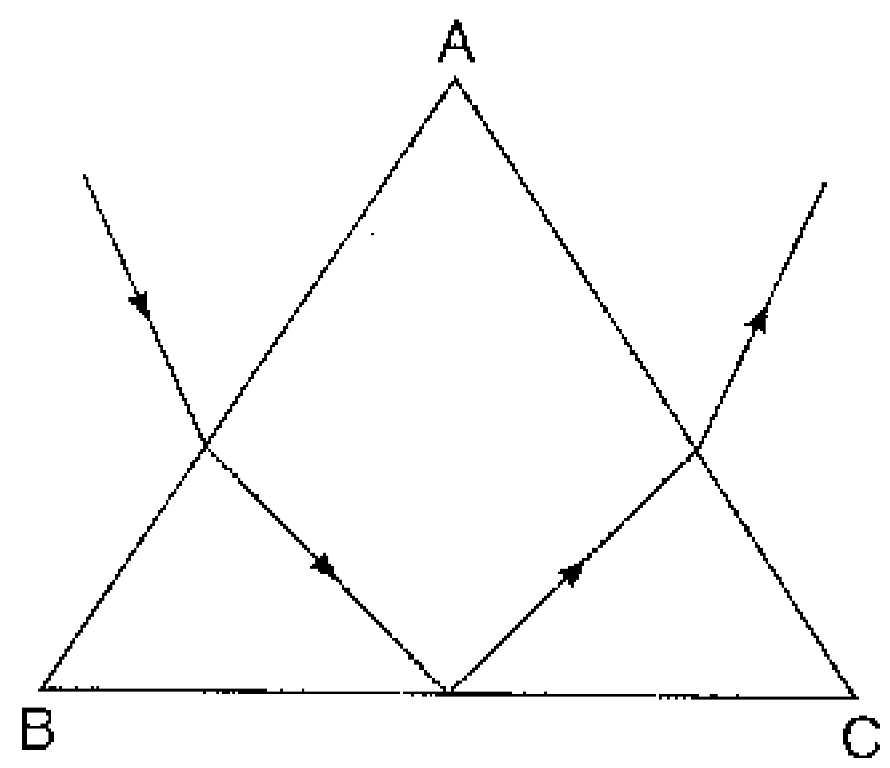
13. A beam of light rays converges to a point O on X-axis as shown. A cube of glass of side-length 40 cm containing a concentric spherical air-cavity of radius of 10 cm is to be placed in the path of the converging beam so that the beam emerging from the cube is parallel to X-axis. At what point C on X-axis should the centre of the cube be placed to achieve this? Give the coordinates of C taking O as origin. Refractive index of glass is 1.5.



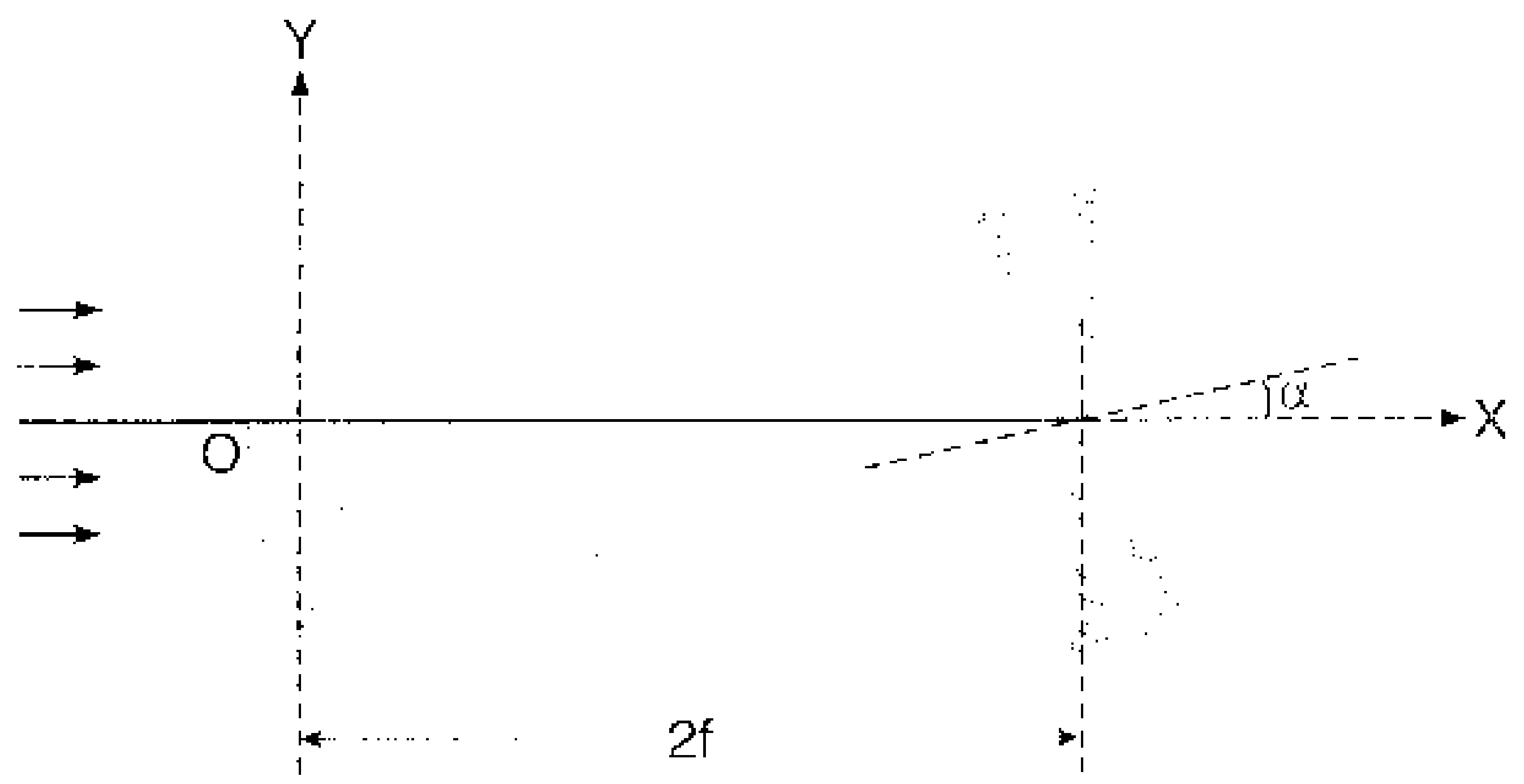
14. Two thin convex lenses of focal lengths f_1 and f_2 are separated by a horizontal distance 'd' (where $d < f_1$, $d < f_2$), and their principal axes are separated by a vertical distance b as shown in the figure. Taking the centre of the first lens (O) as the origin of co-ordinate system and considering a parallel beam of light coming from the left, find the x and y-coordinates of the focal point of this lens system.



15. The path of a ray of light passing through an equilateral glass prism ABC is shown in the figure. The ray of light is incident on face BC at the critical angle for just total internal reflection. The total angle of deviation after the refraction at face AC is 108°. Calculate the refractive index of the glass.



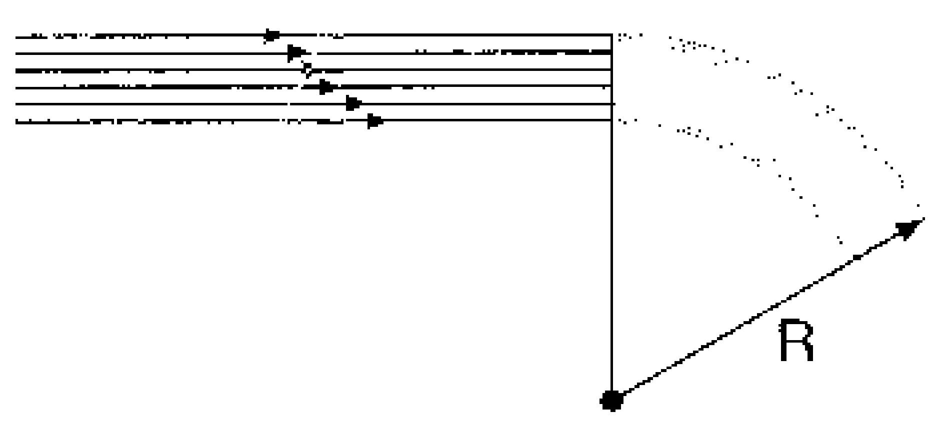
16. In the given figure there are two thin lenses of same focal length f arranged with their principal axes inclined at an angle α. The separation between the optical centers of the lenses is $2f$. A point object lies on the principal axis of the convex lens at a large distance to the left of convex lens.



(a) Find the co-ordinates of the final image formed by the system of lenses taking O as the origin of co-ordinate axes, and

(b) Draw the ray diagram.

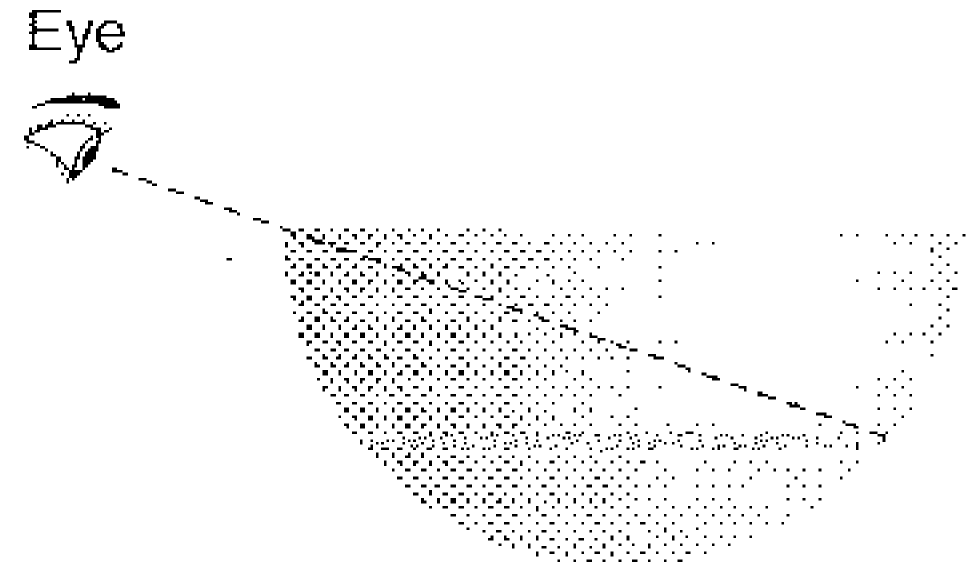
17. A portion of straight glass rod of diameter 4 cm and refractive index 1.5 is bent into an arc of circle of radius R. A parallel beam of light is incident on it as shown in the figure. Find the smallest value of R which permits all the light to pass around the arc.



18. When the plane surface of a plano-convex lens is silvered it is found that the image of the object pin is formed at the position of the object pin placed at a distance of x_1 from the silvered lens. When the same lens is silvered on the curved surface the image of the object pin is formed at the position of the object pin placed at a distance of x_2 from the silvered lens. Find the focal length of lens, the radius of curvature of the curved surface and the index of refraction of the medium of lens in terms of x_1 and x_2.

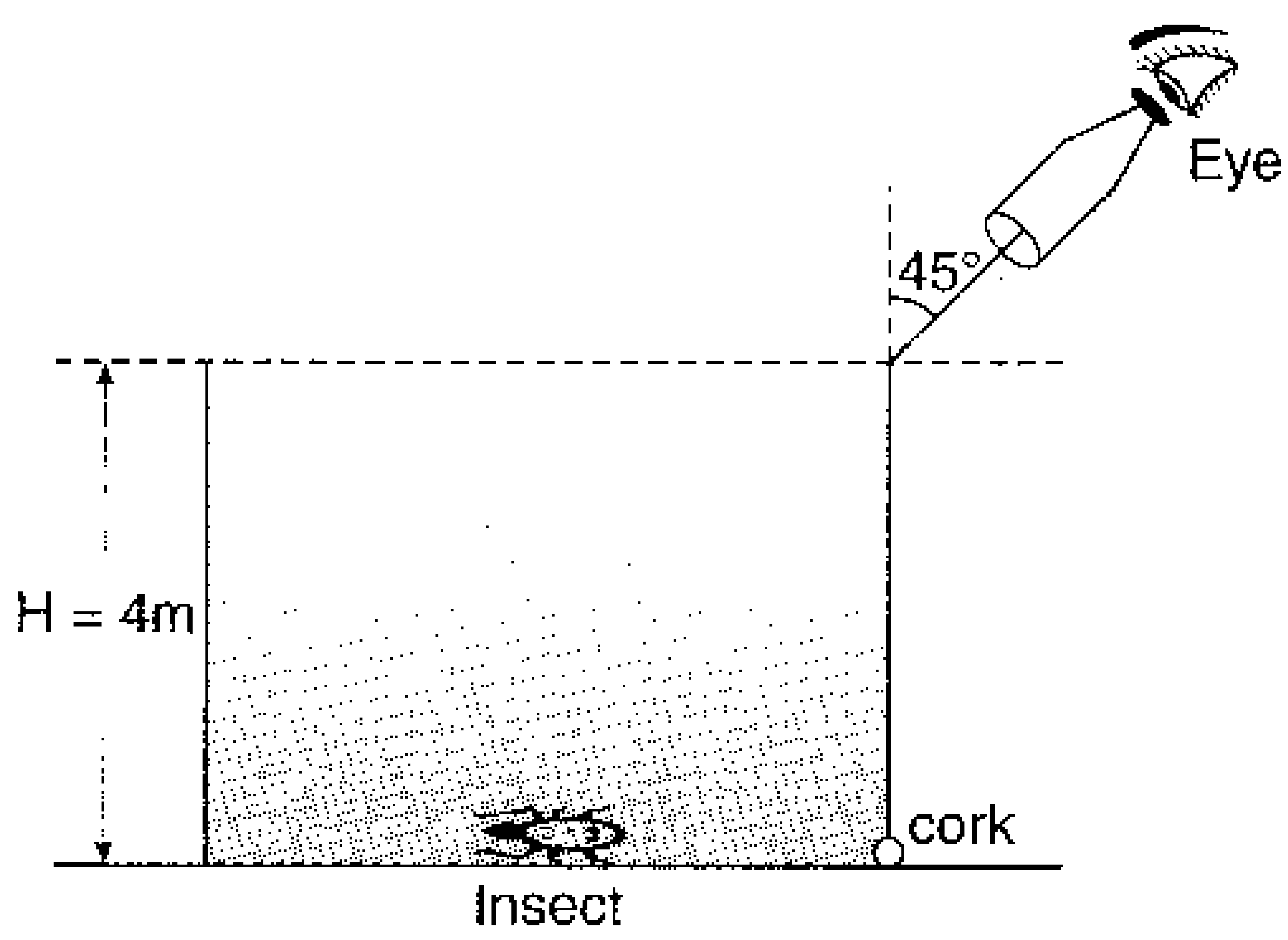
19. A circular disc of diameter d lies horizontally inside a metallic hemispherical bowl of radius a. The disc is just visible to an eye looking over the edge. The bowl is now filled with a liquid of refractive index μ. Now, the whole of the disc is just visible to the eye in the same position. Show that

$$d = 2a\,\frac{(\mu^2 - 1)}{(\mu^2 + 1)}$$



20. A man of height 2.0 m is standing on level road where because of temperature variation the refractive index of air is varying as $\mu = \sqrt{1 + ay}$, where y is height from road. If $a = 2.0 \times 10^{-6}\ \text{m}^{-1}$. Then find distant point that he can see on the road.

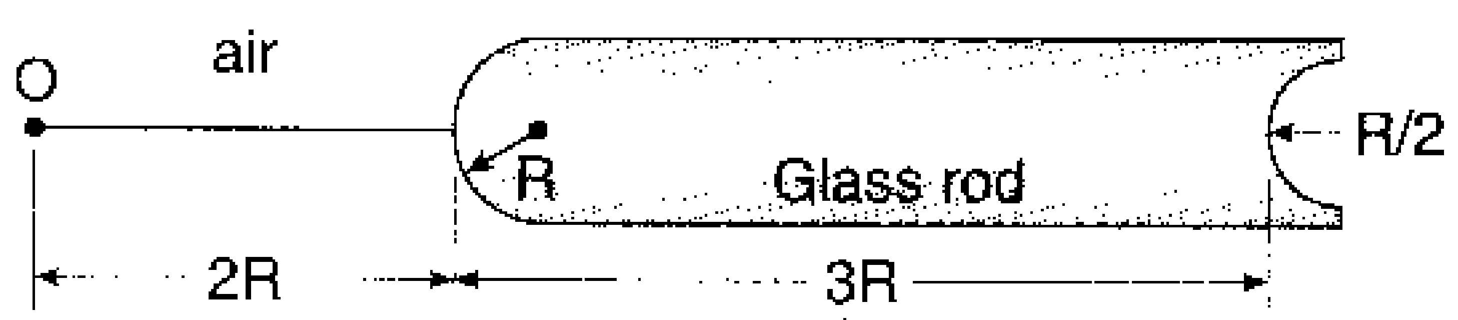
21. A fixed cylindrical tank of height $H = 4$ m and radius $R = 3$ m is filled up with a liquid. An observer observes through a telescope fitted at the top of the wall of the tank and inclined at $\theta = 45°$ with the vertical. When the tank is completely filled with liquid, he notices an insect, which is at the center of the bottom of the tank. At $t = 0$, he opens a cork of radius $r = 3$ cm at the bottom of tank. The insect moves in such a way that it is visible for a certain time. Determine



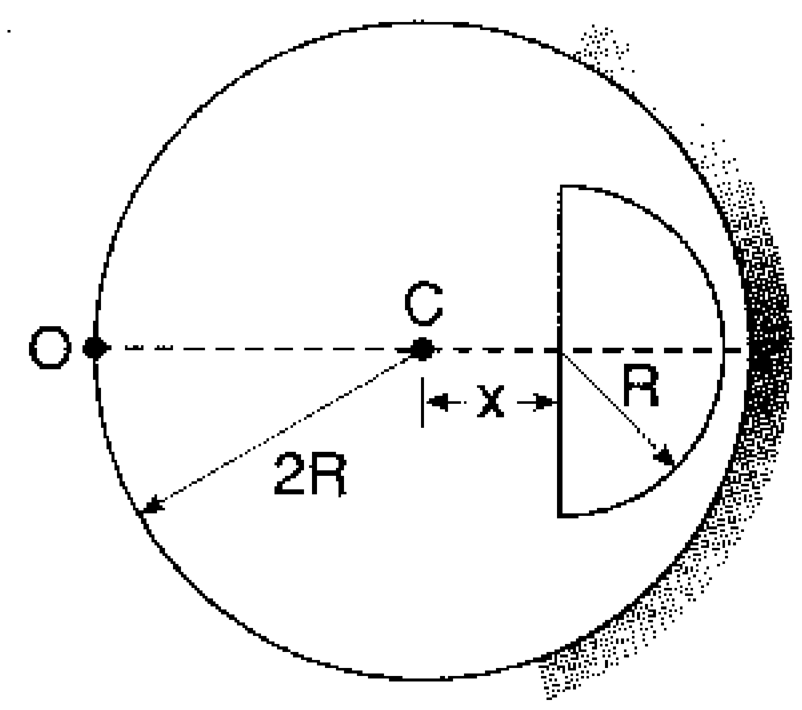
(a) the refractive index of the liquid
(b) the velocity of insect as a function of time.

22. A parallel beam of light falls normally on the first face of a prism of small angle. At the second face it is partly transmitted and partly reflected, the reflected beam striking at the first face again and emerging from it in a direction making an angle of $6° 30'$ with the reversed direction of the incident beam. The refracted beam is found to have undergone a deviation of $1° 15'$ from the original direction. Calculate the refractive index of the glass and the angle of the prism.

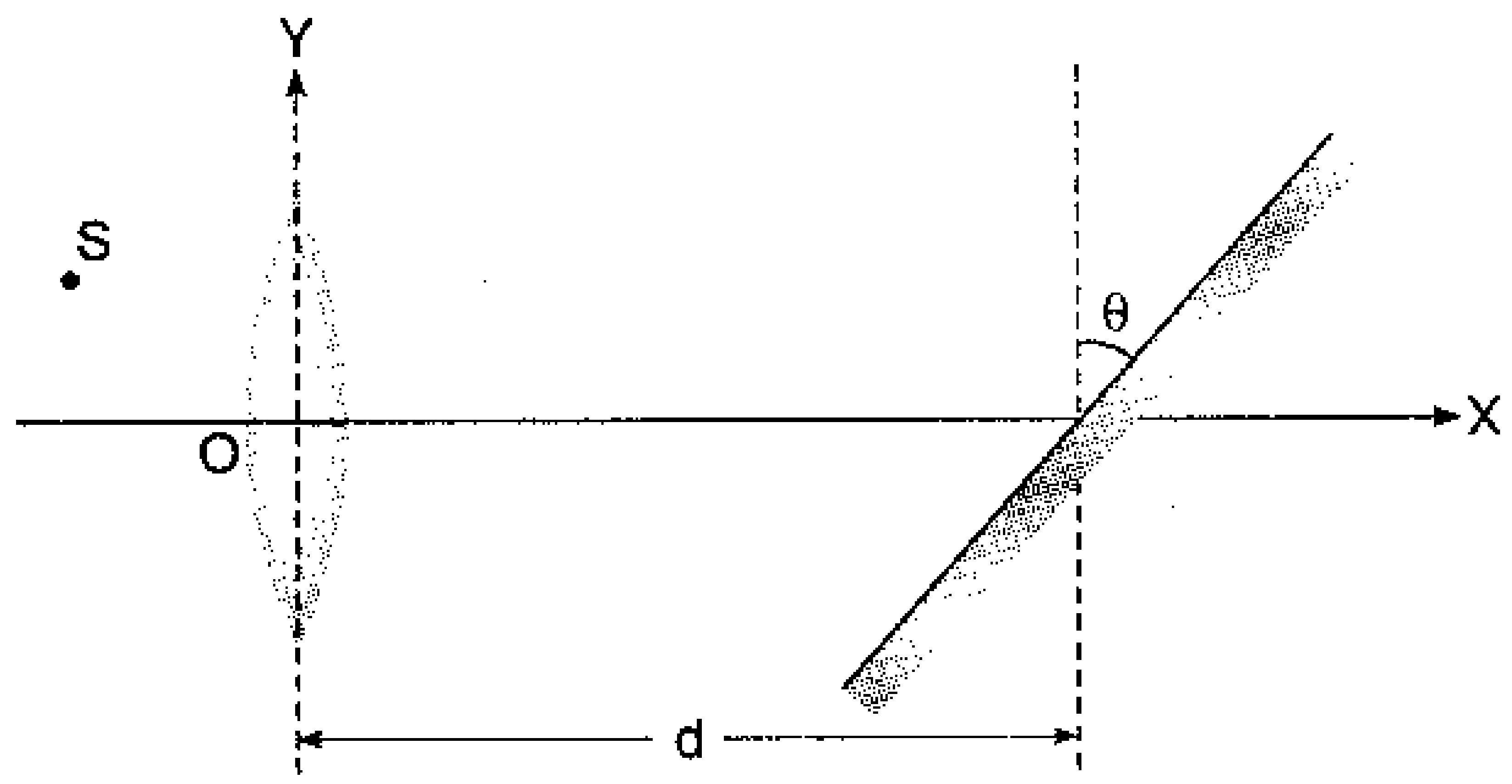
23. A glass rod has ends as shown in figure. The retractive index of glass is μ. The object O is at a distance $2R$ from the surface of larger radius of curvature. The distance between apexes of ends is $3R$. Find the distance of image formed of the point object from right hand vertex. What is the condition to be satisfied if the image is to be real?



24. Consider the figure shown. A hemispherical cavity of radius R is carved out from a sphere ($\mu = 1.5$) of radius $2R$ such that principal axis of cavity coincides with that of sphere. One side of sphere is silvered as shown. Find the value of x for which the image of an object at O is formed at O itself.



25. A thin converging lens of focal length $f = 1.5$ m is placed along y-axis such that its optical centre coincides with the origin. A small light source S is placed at (–2.0 m, 0.1 m). Where should a plane mirror inclined at an angle θ, $\tan\theta = 0.3$ be placed such that y co-ordinate of final image is 0.3 m, *i.e.*, find d. Also find x-co-ordinate of final image.



ANSWERS

Introductory Exercise 22.1

2. (a) 1.26 cm, 2.39×10^{10} Hz (b) $E_z = (60 \text{ V/m}) \sin (500x + 1.5 \times 10^{11} t)$

Introductory Exercise 22.2

2. 4 m/s **3.** 10 m, 16 m

Introductory Exercise 22.3

1. (a) – 16.7 cm, real (b) ∞ (c) + 10.0 cm, virtual

2. (a) A real image moves from – 0.6 m to – ∞, then a virtual image moves from + ∞ to 0.
(b) 0.639 sec and 0.782 sec.

3. (a) A concave mirror with radius of curvature 2.08 m
(b) 1.25 m from the object.

4. Plane mirror

Introductory Exercise 22.4

1. 15 cm **2.** 2

3. The frequency does not change, while the wavelength and speed change by the factor μ_1/μ_2.

4. 1.67

Introductory Exercise 22.5

1. 8.57 cm **2.** (a) 45.0 cm (b) –90.0 cm (c) –6.0 cm **3.** Inside the bowl at – 9.01 cm

4. 0.795 cm **5.** – 0.777

Introductory Exercise 22.6

1. 39 cm **2.** 18.8 cm from the lens, 5.3 cm **4.** From the surface to infinity on object side

5. (a) 36 cm (b) 36 cm **6.** 40 cm **8.** 10 cm **9.** 1.5 **10.** 2 cm **11.** 16 cm **12.** – 40.0 cm

13. 10.0 cm.

Introductory Exercise 22.7

1. 46° **2.** 1.5 **3.** C **4.** 2° **5.** $\mu \le \sqrt{2}$ **6.** $\sin^{-1}(2/3) = 42°$ **7.** $\mu = \sqrt{3}$ **8.** 90° **9.** 0°.

10. $\sqrt{2}$ **11.** 10°17′.

Assignment

Level-I

1. $f = -10$ cm **2.** (a) 8 cm (b) 16 cm (c) 48 cm **3.** 20 cm, 60 cm, 80 cm, 100 cm, 140 cm.

5. (a) 2^{nd} (air) film, $t_{min} = 2 \times 10^{-15}$ second (b) 7.5 **6.** $\tan^{-1}(1.5) \approx 56.3°$ **7.** $\left(\frac{4}{3}h + d\right)$ from the mirror

8. 2.88 m **9.** 6.6 cm **10.** (a) 13.3 cm (b) 14.975 cm **11.** 7.42 cm **12.** 1.5 **13.** 100 cm

14. The image will be $(m + 1)$ times smaller than the object. **15.** $f = 9$ cm **16.** $m_2 \approx m_1^2$

17. $\beta + \frac{1}{\beta} = \left(\frac{d}{f} - 2\right)^2 - 2$ **18.** (a) 1.4 (b) The first lens will be a diverging and the second a converging one.

19. $f = -30$ cm **20.** The lens should be kept at a distance of 6 cm from either of the object.

21. See the hint. Lens is convex. **22.** See the hint. Lens is concave

23. (a) $\theta = \sin^{-1}\left(\frac{13}{16}\right) \approx 54.34°$ (b) yes **24.** (a) 26.8° (b) yes **25.** (a) 40.54° (b) 26.6°

26. 60° **28.** 19° **30.** 10.1° **31.** $\delta_R = 30.6°$, $\delta_V = 33.4°$

32. Image is at a distance of $f/2$ from right-hand lens $m = -\frac{1}{2}$. **33.** $f = -7.5$ cm (concave mirror)

34. At a distance of 2.14 cm from the system. **35.** 45 cm **36.** 25 cm **37.** 2 m or 1 m **38.** 7.5 cm

39. 2.35 mm **40.** 5 cm from mirror towards the lens **41.** 55 cm **42.** 60 cm away from the lens

43. $b = 15$ cm **44.** At a distance of 50 cm from the mirror and 2 cm from each other

45. 22°, 56°

Level-II

2. 60° **3.** $\sqrt{2}$ **4.** 10 cm **5.** $\Delta v = 0.55$ cm, $\frac{m_2}{m_1} = 1.1$

6. (a) $\mu = \frac{t}{t - f_1}$ (b) No shifting of image will take place

7. $\sin^{-1}(\mu \sin \alpha) - \alpha$ **8.** $f = \infty$ **9.** $\mu = \sqrt{3}$ **10.** $\frac{4}{3}$ cm **11.** $\mu = \sqrt{2}$, TIR will take place

12. (a) 1.58 (b) 35° **14.** $\frac{1}{\sqrt{3}}$ cm

15. First image at a distance of 3.33 cm from flat surface and the second at infinity

16. (a) 34.2° (b) 8.4° **17.** 0.7r **18.** (a) 1.5 (b) 9 cm (c) –9 cm (d) –3 cm

19. (a) distance $= \left(\frac{2 + \cos \omega t}{1 + \cos \omega t}\right) f$ (b) At $x = 0$ (c) $m = \infty$

20. Maximum distance of the incident rays from the centre should be $\sqrt{\frac{2}{3}}R$, where R is the radius of hemisphere.

21. At a distance $\left(\frac{\sqrt{3} + 1}{2}\right) R$ from convex mirror **22.** 90 cm, Yes **23.** 30 cm

24. For position of observer see the hint. Maximum dimension of object $h_1 = 5$ cm **25.** 45 cm

26. The first spot is at 12 cm on the left side from the optical centre **27.** See the hint
28. It will move 5 cm closer to the screen **29.** 1.37 **30.** $\mu = \sqrt{5}, i = 58.8°$
31. Final image is formed at 65 cm from first face on the same side of the object **32.** 1.6 **33.** $\mu = \dfrac{d}{2\alpha f}$
34. 90 cm, –102 cm **35.** $A = \alpha + \beta - \gamma, \mu = \sin\beta\sqrt{\left\{\dfrac{\sin\alpha}{\sin\beta\sin(\alpha+\beta-\gamma)} + \dfrac{1}{\tan(\alpha+\beta-\gamma)}\right\}^2 + 1}$
36. $A_{max} = 83.62°$ **37.** 26.65 cm **38.** 60 cm away from the lens
39. Rays will become parallel to the optic axis **40.** See the hint **41.** (a) 80 cm (b) $|u| < 12$ cm **42.** 81°40′
43. 2.4 cm **44.** 45 cm from the lens combination **45.** (a) 157.2° (b) 128.4° **46.** 9 cm/s **47.** $\mu = 2$

Level-III

1. (a) Final image is at a distance of 20 cm behind the second half lens and at a distance of 2/3 mm above the principal axis. The size of image is 2 mm and is inverted as compared to the given object, (b) See the hint

2. 0.186 m, 0.315 m **3.** $h = 15$ cm **4.** Shift $= \dfrac{4f_1^2}{f_2}$ **5.** $\dfrac{1}{2}$ **8.** At pole of the mirror

9. (a) $x_A = r\left[1 - \sqrt{\mu_0^2 - \left(\dfrac{d}{r} + \sqrt{\mu_0^2 - 1}\right)^2}\right]$, (b) $\mu_A = \dfrac{\mu_0}{\sqrt{\mu_0^2 - \left(\dfrac{d}{r} + \sqrt{\mu_0^2 - 1}\right)^2}}$,

(c) Ray will become parallel to y-axis

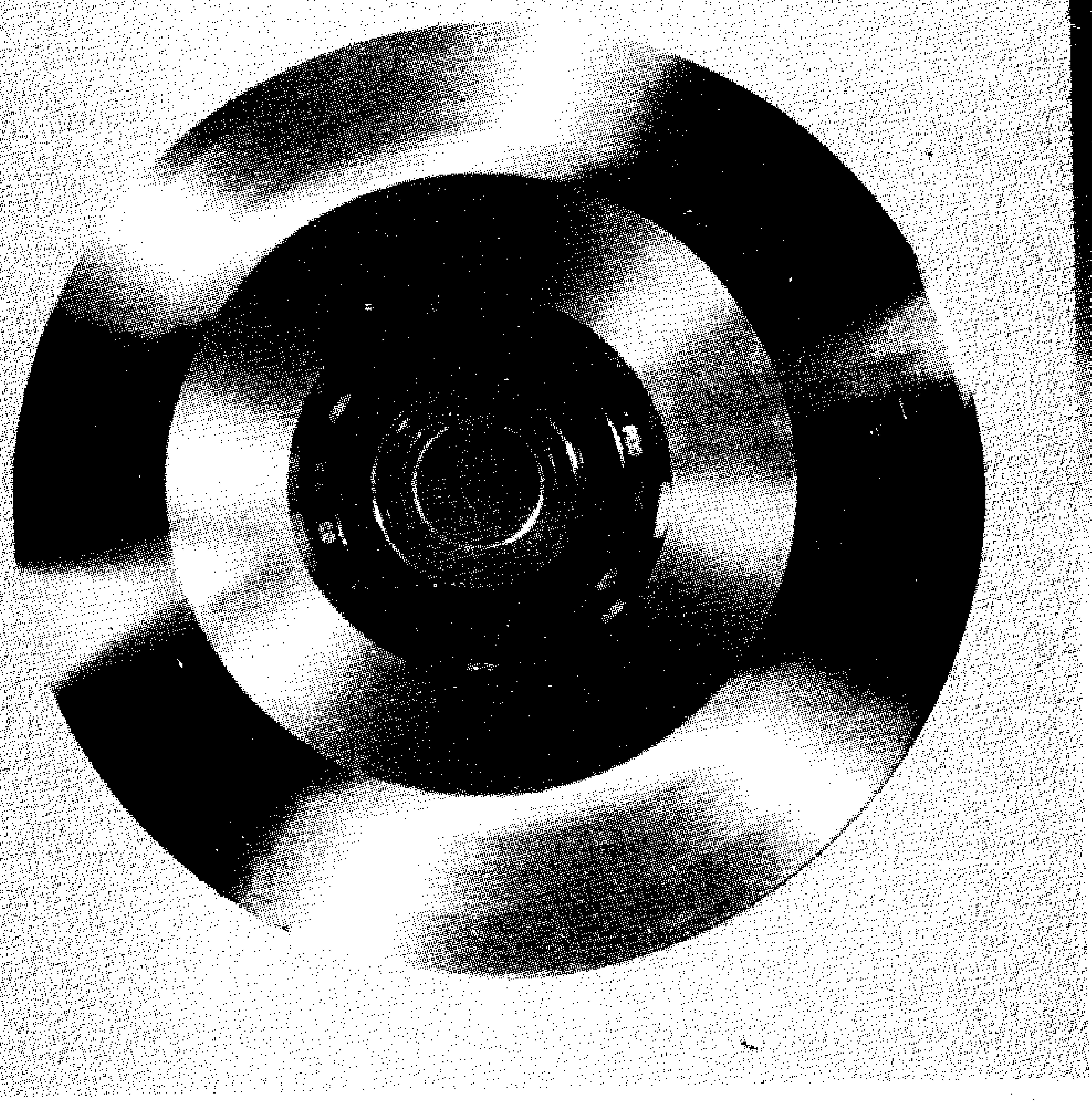
10. (b) $I = \dfrac{3 - 2\mu_0}{2a}$ **13.** (0, 0) **14.** $\left[\dfrac{f_1 f_2 + d(f_1 - d)}{f_1 + f_2 - d}, \dfrac{(f_1 - d)b}{f_1 + f_2 - d}\right]$ **15.** 1.447

16. $\left[f\left(\dfrac{2\cos\alpha + 1}{\cos\alpha + 1}\right), 0\right]$ **17.** $R \geq 12$ cm **18.** $f = x_1,\ R = \dfrac{x_1 x_2}{x_1 - x_2}, \mu = \dfrac{x_1}{x_1 - x_2}$ **20.** 2 km

21. (a) $\mu = \dfrac{5}{3\sqrt{2}}$, (b) $v = 1.1\,(2 - 2.21 \times 10^{-4}) \times 10^{-4}$ m/s **22.** $A = 2°,\ \mu = 1.62$

23. Distance $= \dfrac{R(9 - 4\mu)}{(10\mu - 9)(\mu - 2)}$, for final image to be real μ should lie between 2 and 2.25

24. $x = 0.75R$ **25.** $d = 5.0$ m, x co-ordinate of final image = 4.0 m



CHAPTER

23

Interference of Light Waves

Chapter Contents

Note. Readers are advised to go through articles 17.2 and 17.3 of the author's previous book "Waves and Thermodynamics" before studying this chapter just for better understanding of the current topic.

23.1 INTRODUCTION

In geometric optics the propagation of light is described by rays that travel in straight lines and obey the laws of reflection and refraction. However, the rules of geometric optics are only valid when the wavelength of light is much smaller than the dimensions of the apertures or barriers involved. When this is not the case, phenomena arising from the wave nature of light are observed. Such phenomena fall under the general heading of **physical optics**. We have already discussed how superposed mechanical waves can interfere constructively or destructively, depending on their phase difference. Since, light is an electromagnetic wave, it must also exhibit interference effects. In this chapter we'll look at interference phenomena in light. The colours seen in oil films and soap bubbles are a result of interference between light reflected from the front and back surfaces of a thin film of oil or soap solution. The first successful optical interference was carried out by **Thomas Young** in 1801.

23.2 ENERGY DISTRIBUTION IN INTERFERENCE

As we discussed in chapter 17, the term interference refers to any situation in which two or more waves overlap in space. When this occurs, the total wave at any point at any instant of time is governed by the principle of superposition. This principle states that:

> "When two or more waves overlap, the resultant displacement at any instant may be found by adding the instantaneous displacements that would be produced at the point by the individual waves if each were present alone."

If a single source of light is present in the medium, then the energy distribution is uniform. However, if there are two adjacent exactly similar sources then the distribution no longer remains uniform. At some places energy is maximum while at other places the energy is minimum, of course the total energy of the system remains the same.

This modification in energy distribution due to presence of two or more exactly similar sources is specific case of superposition of wave and known as interference.

If crest of a wave superposes over the crest of the other wave, then the resultant displacement becomes the sum of two amplitudes. It is double if two amplitudes are exactly equal. Naturally the intensity at such places becomes four times. ($I \propto A^2$). The same is true if a wave trough coincides with the trough due to other wave. However if a crest of a wave superposes over the trough of the other wave then the resultant displacement is the difference in the amplitudes of the two waves and the intensity is minimum at such places. The minimum energy is equal to zero if the amplitude of two waves are exactly equal. Thus the places of maxima have four times of the normal intensity while the places of minima have zero intensity. This condition of distribution of energy persists due to two waves having same amplitudes. However, in cases when the amplitudes due to two waves are not exactly equal, the interference occurs but intensity of minima is not zero.

23.3 CONDITIONS FOR INTERFERENCE

In order to obtain a sustained (permanent or stable) and observable interference pattern, the following conditions must be fulfilled.

(i) Sources must be coherent : In order to produce a stable interference pattern the individual waves must maintain a **constant phase** relationship with one another, *i.e.*, the two interfering sources must emit waves having a constant phase difference between them. If the phase difference between two sources does not remain constant then the places of maxima and minima shift. In case of mechanical waves it is possible to keep a constant phase relationship between two different sources. But in case of light two different light sources can't be coherent This is because of the way light is emitted. In ordinary light sources, atoms gain excess energy by thermal agitation or by impact with accelerated electrons. An atom that is 'excited' in such a way begins to radiate energy and continues until it has lost all the energy it can, typically in a time of the order of 10^{-8} sec. The many atoms in a source ordinarily radiate in an unsynchronized and random phase relationship, and the light that is emitted from two such sources has no definite phase relationship. Hence, to obtain a stable interference in light a single source is split into two coherent sources. Any random phase change in the source affects these secondary sources equally and does not change their relative phase. Laser light is much more coherent than ordinary light.

(ii) Same frequency or wavelength : Phase relationship between two waves can be kept constant only when their frequencies are same. Thus we can say, that two coherent sources must have the same frequency.

(iii) Equality of amplitudes : The amplitudes of two interfering waves should be equal or approximately equal. Otherwise the difference between the intensities of maximas and minimas will be too small and the contrast will be poor. Maximum contrast is, however obtained when $A_1 = A_2$, because then minimum intensity will be zero.

IIT-JEE GALAXY 23.1

1. Consider two coherent sources S_1 and S_2. Suppose two waves emanating from these two sources superpose at point P. The phase difference between them at P is ϕ (which is constant). If the amplitude due to two individual sources at P is A_1 and A_2, then resultant amplitude at P, will be,

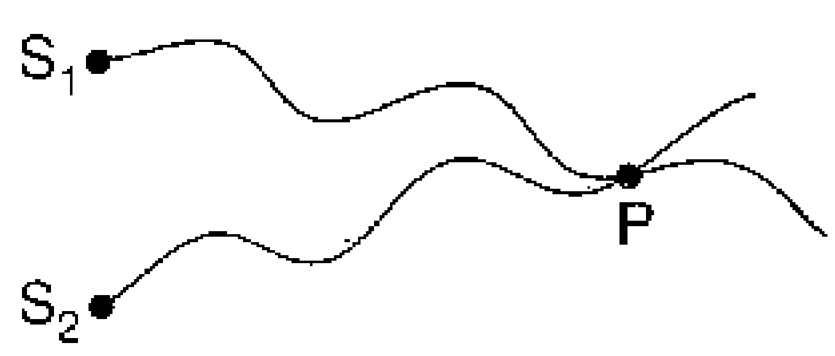


Fig. 23.1

$$A = \sqrt{A_1^2 + A_2^2 + 2\,A_1A_2\cos\phi} \qquad \text{...(i)}$$

Similarly the resultant intensity at P is given by,

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi \qquad \text{...(ii)}$$

Here I_1 and I_2 are the intensities due to independent sources. If the sources are incoherent then resultant intensity at P is given by,

$$I = I_1 + I_2 \qquad \text{...(iii)}$$

2. As we have read in article 23.3 a single source is split in two coherent sources to obtain sustained interference in light: This can be done in following different ways.

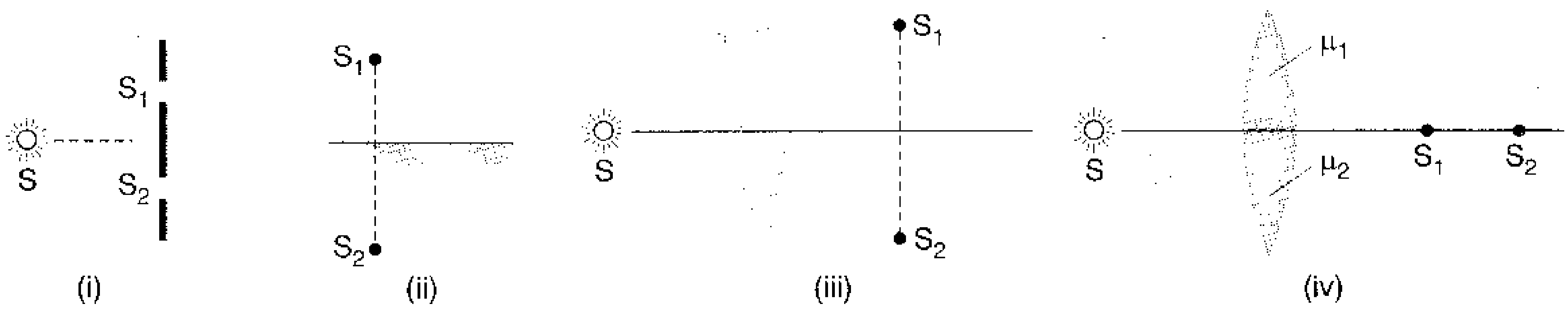


Fig. 23.2

23.4 YOUNG'S DOUBLE SLIT EXPERIMENT

One of the first to demonstrate the interference of light was Thomas Young in 1801.

Principle : Monochromatic light (single wavelength) from a narrow vertical slit S falls on two other narrow slits S_1 and S_2 which are very close together and parallel to S. S_1 and S_2 act as two coherent sources (both being derived from S). If S, S_1 and S_2 all are very narrow, diffraction (bending of light at openings whose width is of the order of wavelength of light) causes the emerging beams to spread into the region beyond the slits. Superposition occurs in the shaded area, where the diffracted beams overlap. Alternate bright and dark equally spaced vertical bands (interference fringes) can be observed on a screen placed at some distance from the slits. If either of S_1 or S_2 is covered, the fringes disappear.

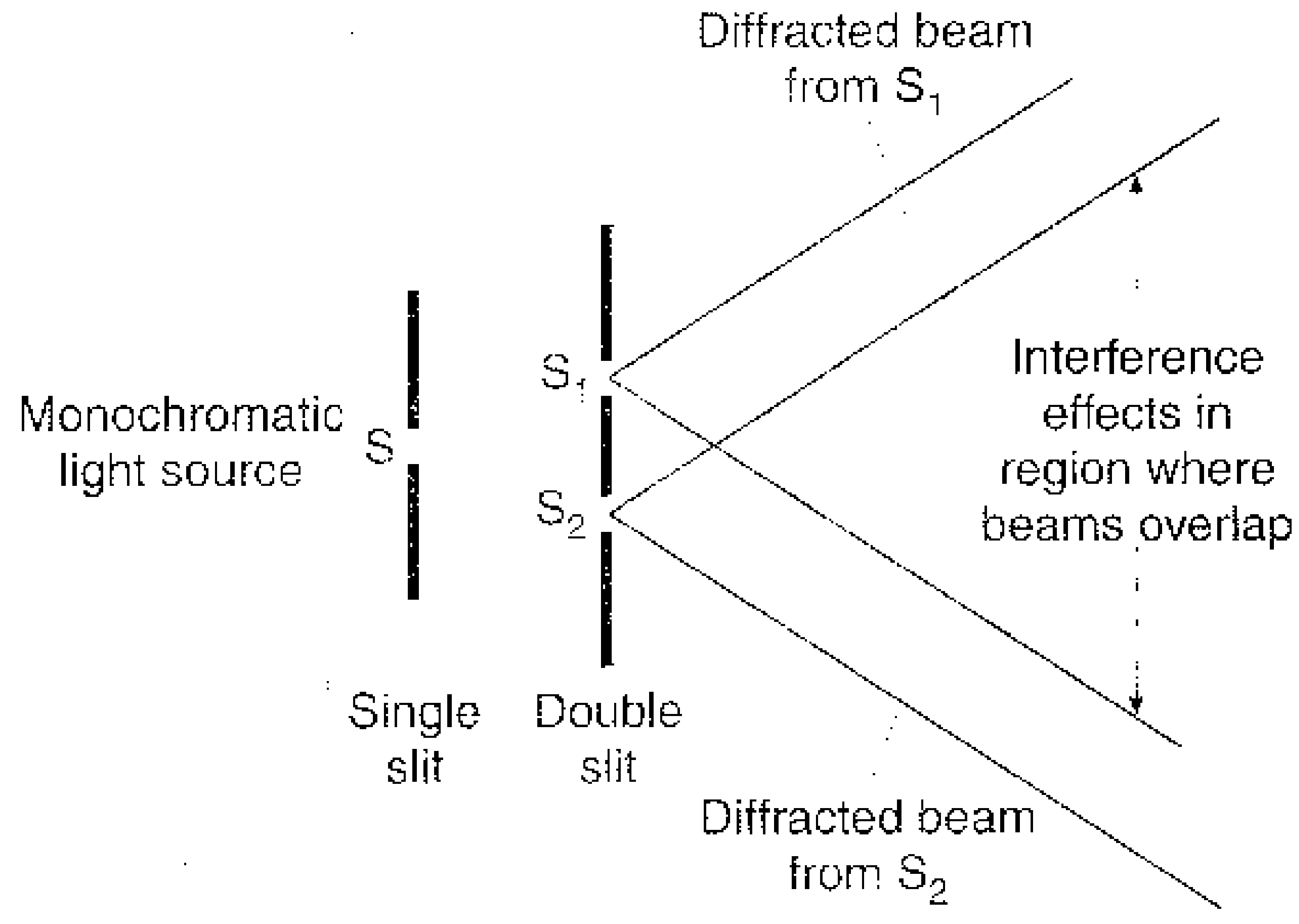


Fig. 23.3

Theory : An expression for the separation of two bright (or dark) fringes (called fringe width w) can be obtained as under.

Figure shows the light waves from S_1 and S_2 meeting at an arbitrary point P on the screen. Since $D >> d$, the two light rays are approximately parallel with a path difference,

$$\Delta x = S_2P - S_1P$$
$$\approx d \sin\theta \qquad \ldots\text{(i)}$$

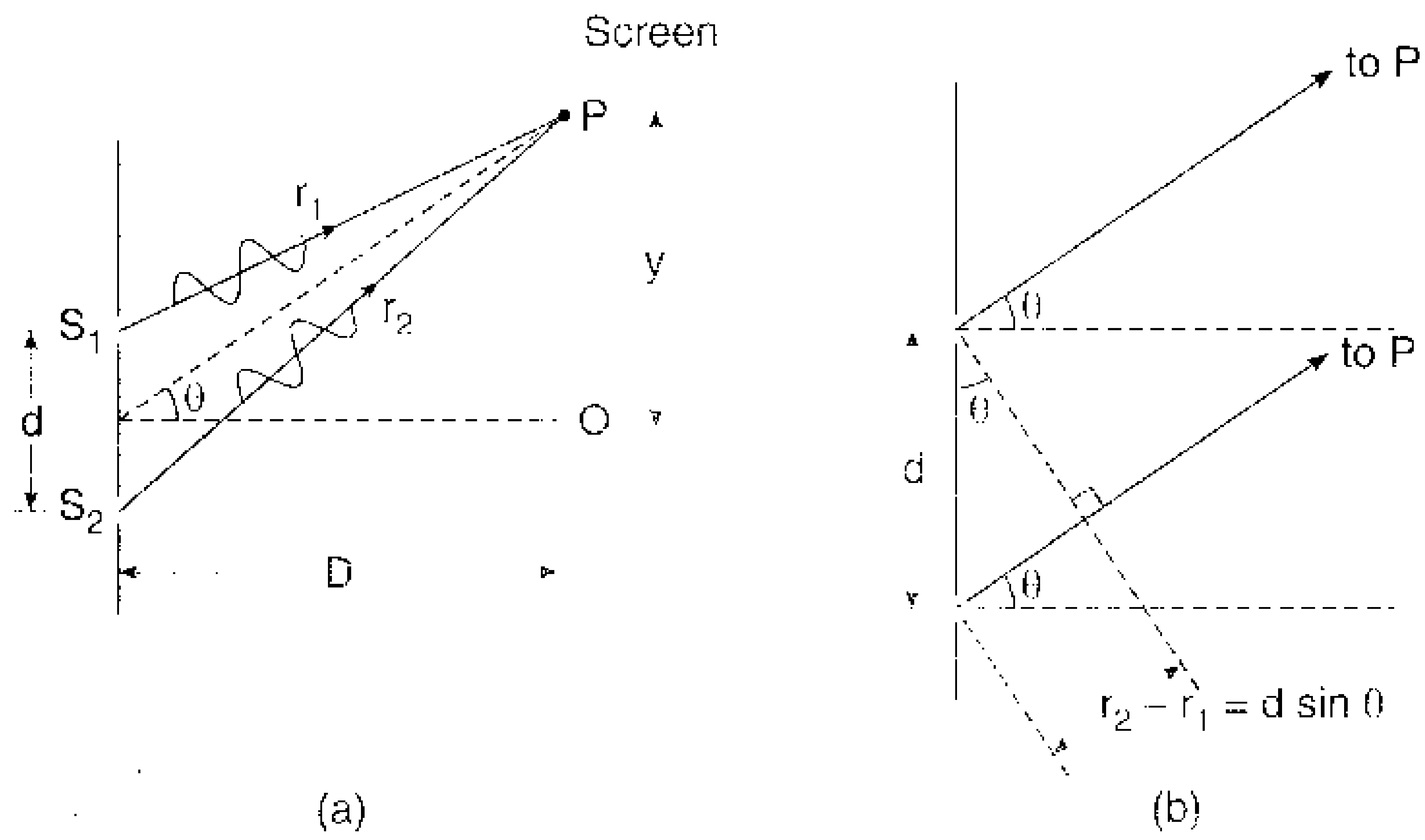


Fig. 23.4 (a) To reach P, the light waves from S_1 and S_2 must travel different distances. (b) The path difference between the two rays is $d \sin\theta$.

For **maximum** intensity at P,

$$\Delta x = n\lambda \quad (n = 0, \pm 1, \pm 2, \ldots)$$

or
$$d \sin\theta = n\lambda \quad (n = 0, \pm 1, \pm 2, \ldots) \qquad \ldots\text{(ii)}$$

The bright fringe corresponding to the integer n is called the n^{th} order (or just n^{th}) bright fringe. The bright fringe for $n = 0$, is known as the central fringe and its centre (point O) is called the central (or zero order) maximum. Higher order bright fringes are situated symmetrically about the central fringe.

For **minimum** intensity at P,

The first minima ($n = \pm 1$) are adjacent to the central maximum on either side. It is useful to obtain expressions for the positions of the bright and dark fringes measured vertically from O. In addition to our assumption that $D >> d$, we assume that $d >> \lambda$. These can be valid assumptions because in practice D is often of the order of 1 m, d a fraction of a milimetre and λ a fraction of a micrometre for visible light. Under these conditions θ is small, thus, we can use the approximation,

$$\sin\theta \approx \tan\theta = \frac{y}{D}$$

Substituting in Eqs. (ii) and (iii), we get

$$y_{\text{bright}} = \frac{n\lambda D}{d} \qquad (n = 0, \pm 1, \pm 2, \ldots) \qquad \ldots\text{(iv)}$$

and

$$y_{\text{dark}} = \frac{(2n-1)\lambda D}{2d} \qquad (n = \pm 1, \pm 2, \ldots) \qquad \ldots\text{(v)}$$

Fringe Width (w) : Distance between two adjacent bright (or dark) fringes is called the fringe width. It is denoted by w. Thus,

$$w = \frac{n\lambda D}{d} - \frac{(n-1)\lambda D}{d} = \frac{\lambda D}{d}$$

or

$$w = \frac{\lambda D}{d} \qquad \ldots\text{(vi)}$$

Alternatively we can show that distance between two successive dark fringes is also $\frac{\lambda D}{d}$.

Further Points

1. In YDSE (Young's double slit experiment) we are basically interested in finding the resultant intensity at a general point P at a distance y from O on the screen or at an angle θ from C as shown. Rather we can say we will find intensity I as a function of y or θ.

2. Since $d << D$, we can assume that intensity at P due to independent sources S_1 and S_2 are almost equal.

or

$$I_1 \approx I_2 = I_0 \qquad \text{(say)} \qquad \ldots\text{(vii)}$$

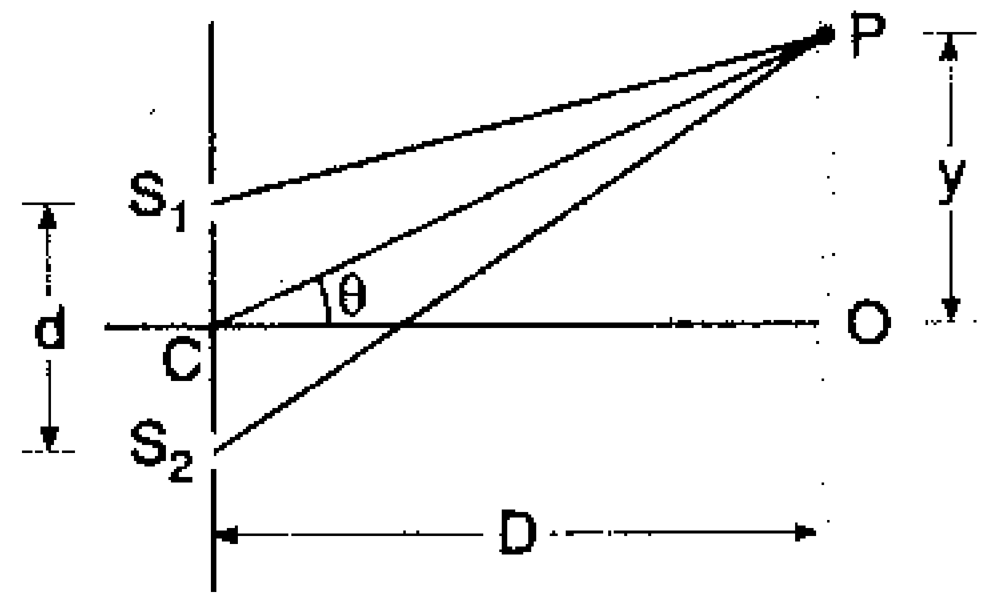


Fig. 23.5

3. Path difference between S_2P and S_1P as a function of y or θ is given by

$$\Delta x = d\sin\theta \approx \frac{yd}{D} \qquad \left(\text{as } \sin\theta \approx \tan\theta = \frac{y}{D}\right) \qquad \ldots\text{(viii)}$$

Note : $\Delta x = \frac{yd}{D}$ is applicable only when θ is small. However $\Delta x = d\sin\theta$ can be applied to larger values of θ also provided $d << D$.

4. Once path difference as a function of y or θ is known, corresponding phase difference (we will denote it by ϕ not by $\Delta\phi$) can be obtained by the relation,

$$\phi = \frac{2\pi}{\lambda} \cdot \Delta x \qquad \ldots\text{(ix)}$$

5. For two coherent sources the resultant intensity is given by,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\phi \qquad \text{(See III-JEE Galaxy 23.1)}$$

Putting, $I_1 = I_2 = I_0$

We have $I = I_0 + I_0 + 2\sqrt{I_0 \times I_0} \cos\phi$

Simplifying the above expression, we get

$$I = 4 I_0 \cos^2 \frac{\phi}{2} \qquad \ldots\text{(x)}$$

6. From equation (viii), we can see that intensity is **maximum** at points where,

$$\cos\frac{\phi}{2} = \pm 1 \quad \text{or} \quad \frac{\phi}{2} = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

or $\phi = 2n\pi$ or $\frac{2\pi}{\lambda}\Delta x = 2n\pi$

or $\Delta x = n\lambda$ or $d\sin\theta = n\lambda$

or $\frac{yd}{D} = n\lambda$ or $y = \frac{n\lambda D}{d}$

and this condition, we have already discussed earlier. Further, the maximum intensity is

$$I_{max} = 4I_0 \qquad \ldots\text{(xi)}$$

with this Eq. (viii), can also be written as,

$$I = I_{max} \cos^2 \frac{\phi}{2} \qquad \ldots\text{(xii)}$$

Minimum intensity on the screen is found at points where,

$$\cos\frac{\phi}{2} = 0 \quad \text{or} \quad \frac{\phi}{2} = \left(n - \frac{1}{2}\right)\pi \qquad (n = \pm 1, \pm 2, \pm 3, \ldots)$$

or $\phi = (2n-1)\pi$ or $\frac{2\pi}{\lambda}\Delta x = (2n-1)\pi$

or $\Delta x = (2n-1)\frac{\lambda}{2}$ or $d\sin\theta = (2n-1)\frac{\lambda}{2}$

or $\frac{yd}{D} = (2n-1)\frac{\lambda}{2}$ or $y = \frac{(2n-1)\lambda D}{2d}$

The minimum intensity is zero $I_{min} = 0$ ...(xiii)

A plot of light intensity *versus* $d\sin\theta$ is given in figure.

The above points will remain continued but first let us take some simple examples in support of the theory discussed above.

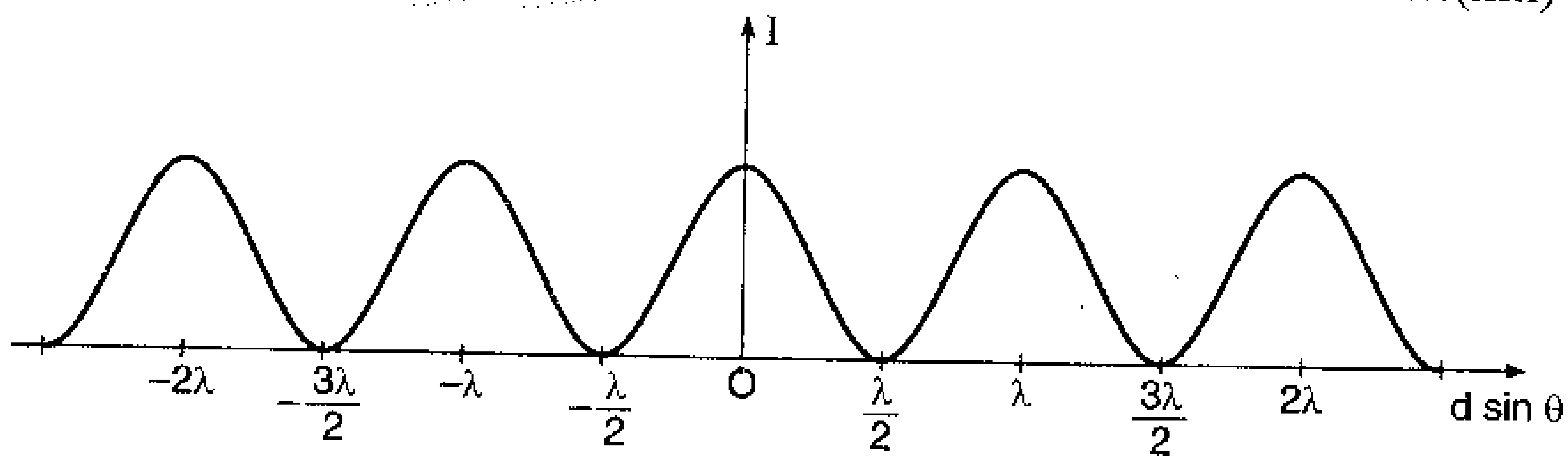


Fig. 23.6

EXAMPLE 23.1 *In YDSE, the two slits are separated by 0.1 mm and they are 0.5 m from the screen. The wavelength of light used is 5000 Å. Find the distance between 7^{th} maxima and 11^{th} minima on the screen.*

SOLUTION Given, $d = 0.1\text{ mm} = 10^{-4}$ m, $D = 0.5$ m and $\lambda = 5000\text{ Å} = 5.0\times10^{-7}$ m

$$\Delta y = (y_{11})_{\text{dark}} - (y_7)_{\text{bright}} = \frac{(2\times11-1)\lambda D}{2d} - \frac{7\lambda D}{d}$$

or

$$\Delta y = \frac{7\lambda D}{2d} = \frac{7\times5.0\times10^{-7}\times0.5}{2\times10^{-4}}$$

$$= 8.75\times10^{-3}\text{ m}$$

$$= 8.75\text{ mm}$$ **Ans.**

EXAMPLE 23.2 *Maximum intensity in YDSE is I_0. Find the intensity at a point on the screen where*

(*a*) *the phase difference between the two interfering beams is $\frac{\pi}{3}$.*

(*b*) *the path difference between them is $\frac{\lambda}{4}$.*

SOLUTION (a) From equation (xii), we have

$$I = I_{\max}\cos^2\left(\frac{\phi}{2}\right)$$

Here $I_{\max}$ is I_0 (*i.e.*, intensity due to independent sources is $I_0/4$). Therefore, at

$$\phi = \frac{\pi}{3} \quad\text{or}\quad \frac{\phi}{2} = \frac{\pi}{6}$$

$$I = I_0\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}I_0$$ **Ans.**

(b) Phase difference corresponding to the given path difference $\Delta x = \frac{\lambda}{4}$ is,

$$\phi = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) \qquad \left(\phi = \frac{2\pi}{\lambda}\Delta x\right)$$

$$= \frac{\pi}{2}$$

or

$$\frac{\phi}{2} = \frac{\pi}{4}$$

∴

$$I = I_0\cos^2\left(\frac{\pi}{4}\right) = \frac{I_0}{2}$$ **Ans.**

7. Fringe width (w) is the distance between two successive maximas or minimas. It is given by,

$$w = \frac{\lambda D}{d}$$

or $$w \propto \lambda$$

Two conclusions can be drawn from this relation

(i) If YDSE apparatus is immersed in a liquid of refractive index μ, then wavelength of light and hence, fringe width decreases μ times.

(ii) If white light is used in place of a monochromatic light then coloured fringes are obtained on the screen with red fringes of larger size than that of violet, because $\lambda_{red} > \lambda_{violet}$.

But **note** that centre is still white because path difference there is zero for all colours. Hence, all the wavelengths interfere constructively. At other places light will interfere destructively for those wavelengths for whom path difference is $\lambda/2, 3\lambda/2, \ldots$ etc. and they will interfere constructively for the wavelengths for whom path difference is $\lambda, 2\lambda, \ldots$ etc.

8. Path difference produced by a slab : Consider two light rays 1 and 2 moving in air parallel to each other. If a slab of refractive index μ and thickness t is inserted between the path of one of the rays then a path difference

$$\Delta x = (\mu - 1)\, t \qquad \ldots\text{(xiv)}$$

is produced among them. This can be shown as under,

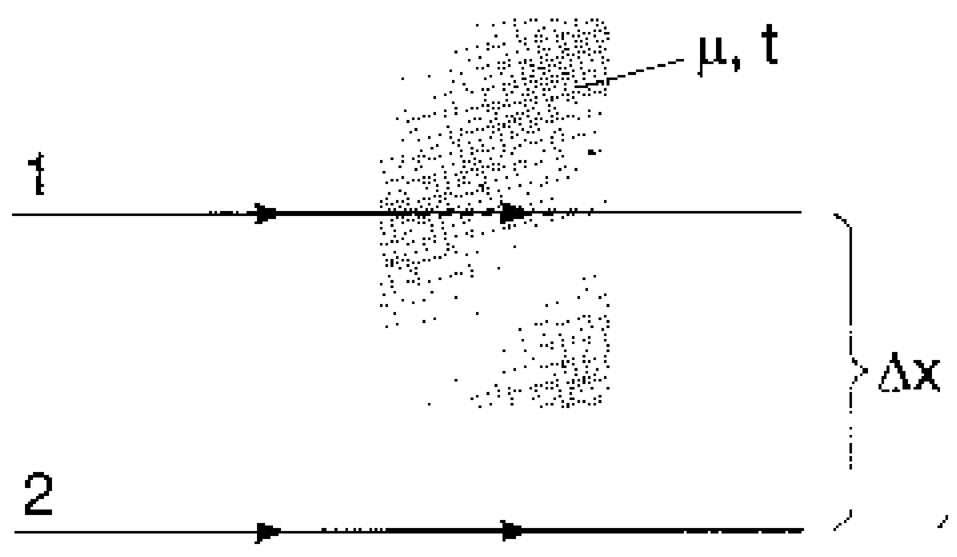


Fig. 23.7

$$\text{Speed of light in air} = c$$

$$\text{Speed of light in medium} = \frac{c}{\mu}$$

time taken by ray 1 to cross the slab, $$t_1 = \frac{t}{c/\mu} = \frac{\mu t}{c}$$

and time taken by ray 2 to cross the same thickness t in air will be,

$$t_2 = \frac{t}{c} \quad \text{as} \quad t_1 > t_2$$

difference in time $$\Delta t = t_1 - t_2 = (\mu - 1)\frac{t}{c}$$

During this time ray 2 will travel an extra distance, $\Delta x = (\Delta t)\, c = (\mu - 1)\, t$, which is same as equation (xiv).

EXERCISE : A slab of thickness t and refractive index μ_2 is kept in a medium of refractive index μ_1 ($< \mu_2$). Prove that, if two rays parallel to each other passes through such a system, with one ray passing through the slab then the path difference produced between them due to the slab will be

$$\Delta x = \left(\frac{\mu_2}{\mu_1} - 1\right) t$$

Optical path length : Now we can show that a thickness t in a medium of refractive index μ is equivalent to a length μt in vacuum (or air). This is called optical path length. Thus,

$$\text{Optical path length} = \mu t$$

9. Shifting of fringes : Suppose a glass slab of thickness t and refractive index μ is inserted onto the path of the ray emanating from source S_1 then the whole fringe pattern shifts upwards by a distance $\frac{(\mu - 1)\, tD}{d}$. This can be shown as under,

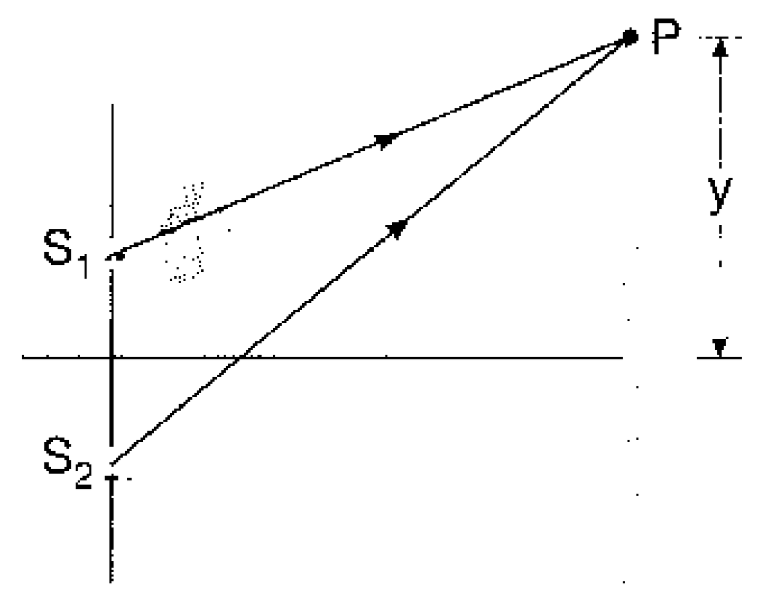


Fig. 23.8

Geometric path difference between S_2P and S_1P is,

$$\Delta x_1 = S_2P - S_1P = \frac{yd}{D}$$

Path difference produced by the glass slab,

$$\Delta x_2 = (\mu - 1)\, t$$

Note : Due to the glass slab path of ray 1 gets increased by Δx_2.

Therefore, net path difference between the two rays is,

$$\Delta x = \Delta x_1 - \Delta x_2$$

or $$\Delta x = \frac{yd}{D} - (\mu - 1)\, t$$

For n^{th} maxima on upper side, $\Delta x = n\lambda$

or $$\frac{yd}{D} - (\mu - 1)\, t = n\lambda$$

$\therefore$ $$y = \frac{n\lambda D}{d} + \frac{(\mu - 1)\, tD}{d}$$

Earlier it was $\frac{n\lambda D}{d}$

$\therefore$ $$\text{Shift} = \frac{(\mu - 1)t\, D}{d} \qquad \text{...(xv)}$$

Following three points are important with regard to Eq. (xv).

(a) Shift is independent of n, (the order of the fringe), *i.e.*,

shift of zero order maximum = shift of 7^{th} order maximum

or shift of 5^{th} order maximum = shift of 9^{th} order minimum and so on.

(b) Shift is independent of λ, *i.e.*, if white light is used then,

shift of red colour fringe = shift of violet colour fringe.

(c) $$\text{Number of fringes shifted} = \frac{\text{shift}}{\text{fringe width}}$$

$$= \frac{(\mu - 1)\, t\, D/d}{\lambda D/d} = \frac{(\mu - 1)t}{\lambda}$$

These numbers are inversely proportional to λ. This is because shift is same for all colours but fringe width of the colour having smaller value of λ is small, so more number of fringes will shift of this colour.

EXAMPLE 23.3 *Bichromatic light is used in YDSE having wavelengths* $\lambda_1 = 400$ *nm and* $\lambda_2 = 700$ *nm. Find minimum order of* λ_1 *which overlaps with* λ_2.

SOLUTION Let n_1 bright fringe of λ_1 overlaps with n_2 bright fringe of λ_2. Then,

$$\frac{n_1\lambda_1 D}{d} = \frac{n_2\lambda_2 D}{d}$$

or
$$\frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1} = \frac{700}{400} = \frac{7}{4}$$

The ratio $\frac{n_1}{n_2} = \frac{7}{4}$ implies that 7^{th} bright fringe of λ_1 will overlap with 4^{th} bright fringe of λ_2.

Similarly 14^{th} of λ_1 will overlap with 8^{th} of λ_2 and so on.

So the minimum order of λ_1 which overlaps with λ_2 is 7.

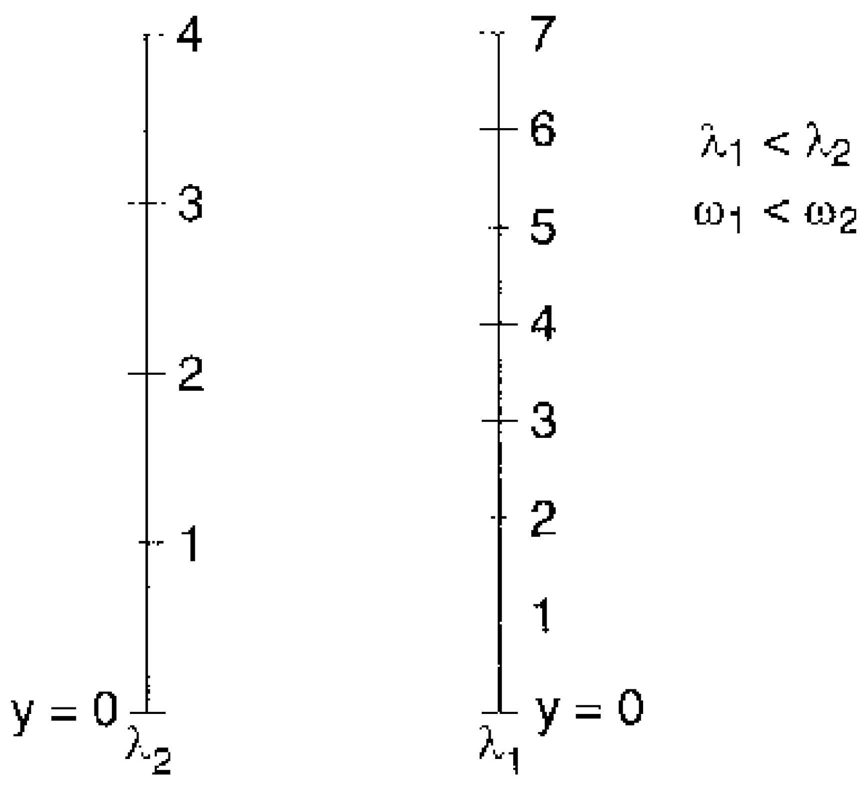


Fig. 23.9

EXAMPLE 23.4 *In YDSE find the thickness of a glass slab* ($\mu = 1.5$) *which should be placed before the upper slit* S_1 *so that the central maximum now lies at a point where* 5^{th} *bright fringe was lying earlier (before inserting the slab). Wavelength of light used is* 5000 *Å*.

SOLUTION According to the question,

$$\text{Shift} = 5 \text{ (fringe width)}$$

$$\therefore \quad \frac{(\mu - 1)\, tD}{d} = \frac{5\lambda D}{d}$$

$$\therefore \quad t = \frac{5\lambda}{\mu - 1} = \frac{25000}{1.5 - 1}$$

$$= 50{,}000 \text{ Å}$$

Ans.

IIT-JEE GALAXY 23.2

1. In the problems of *YDSE* our first task is to find the path difference. Let us take a typical case. In the figure shown, path of ray 1 is more than path of ray 2 by a distance,

$$\Delta x_1 = d \sin\alpha \quad \text{and} \quad \Delta x_2 = (\mu_1 - 1)\, t_1$$

and path of ray 2 is greater than path of ray 1 by a distance,

$$\Delta x_3 = d \sin\beta \quad \text{and} \quad \Delta x_4 = (\mu_2 - 1)\, t_2$$

Therefore, net path difference is,

$$\Delta x = (\Delta x_1 + \Delta x_2) \sim (\Delta x_3 + \Delta x_4)$$

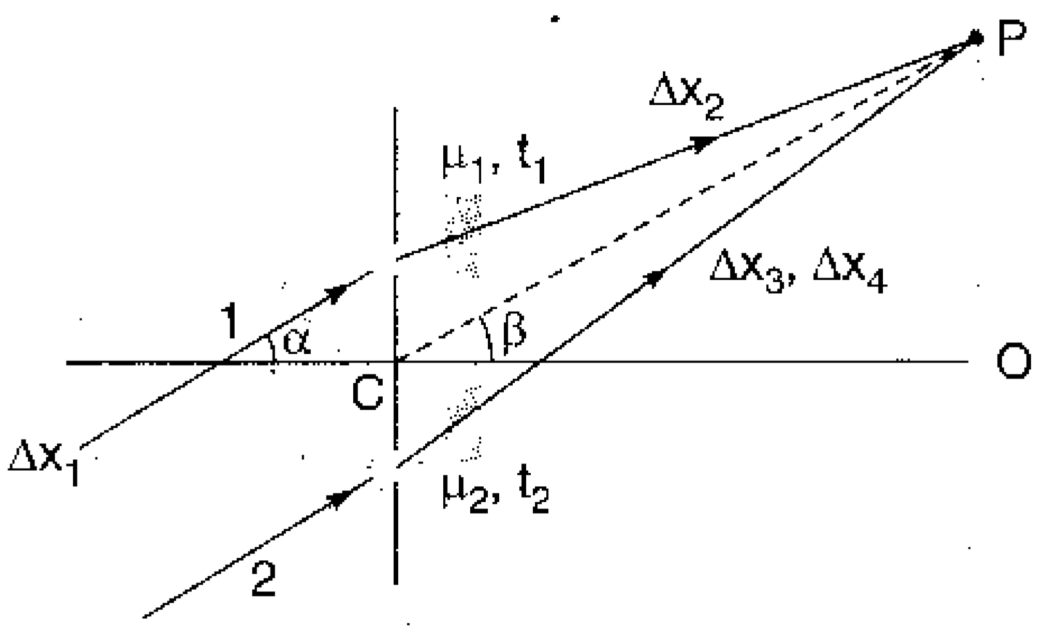


Fig. 23.10

2. Once the path difference is known, put

$$\Delta x = n\lambda \qquad \text{(for maximum intensity)}$$

and

$$\Delta x = (2n-1)\frac{\lambda}{2} \qquad \text{(for minimum intensity)}$$

3. If the slits are vertical [as in figure (a)] path difference is,

$$\Delta x = d \sin\theta$$

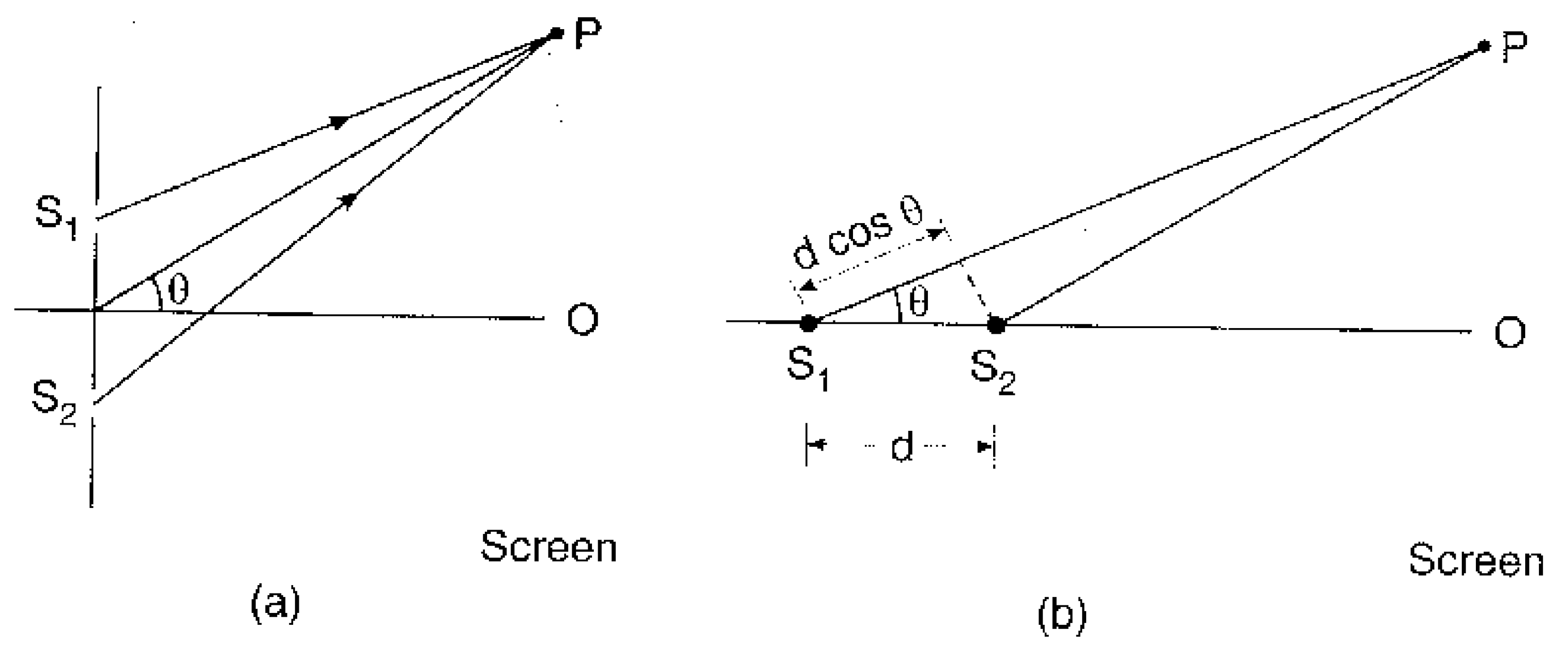


Fig. 23.11

This path difference increases as θ increases. Hence, order of fringe ($d \sin\theta = n\lambda$ or $n = \frac{d\sin\theta}{\lambda}$) increases as we move away from point O on the screen.

Opposite is the case when the slits are horizontal [as in figure (b)]. Here path difference is

$$\Delta x = d \cos\theta$$

This path difference decreases as θ increases. Hence, order of fringe $\left(n = \frac{d\cos\theta}{\lambda}\right)$ decreases as we move away from point O.

See the figure below.

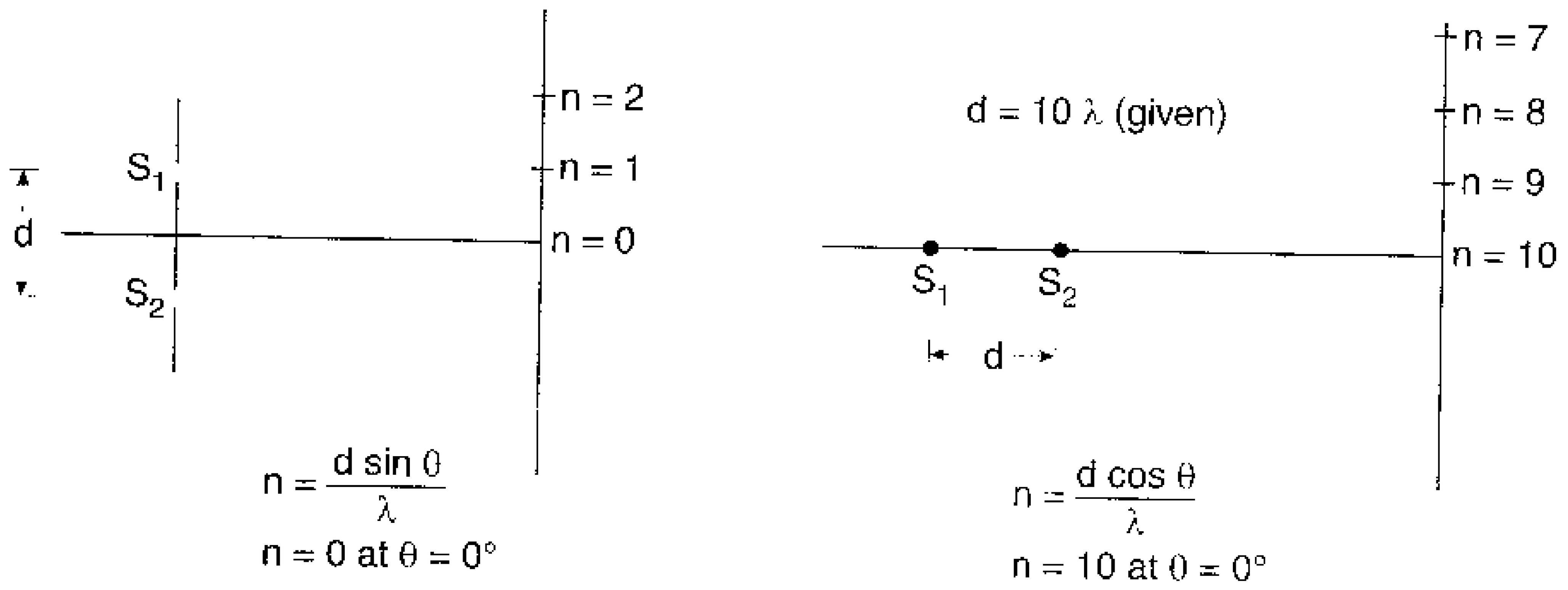


Fig. 23.12

4. Sometimes maximum number of maximas or minimas are asked in the question which can be obtained on the screen. For this we use the fact that value of $\sin\theta$ (or $\cos\theta$) can't be greater than 1. For example in the first case when the slits are vertical.

$$\sin\theta = \frac{n\lambda}{d} \qquad \text{(for maximum intensity)}$$

$$\sin\theta \not> 1$$

$$\therefore \quad \frac{n\lambda}{d} \not> 1 \quad \text{or} \quad n \not> \frac{d}{\lambda}$$

Suppose in some question $\frac{d}{\lambda}$ comes out say 4.6, then total number of maximas on the screen will be 9. Corresponding to $n = 0, \pm 1, \pm 2, \pm 3$ and ± 4.

5. **Lloyd's Mirror :** A plane glass plate (acting as a mirror) is illuminated at almost grazing incidence by a light from a slit S_1. A virtual image S_2 of S_1 is formed closed to S_1 by reflection and these two act as coherent sources. The expression giving the fringe width is the same as for the double slit, but the fringe system differs in one important respect. In Lloyd's mirror, if the point P, for example, is such that the path difference $S_2P - S_1P$ is a whole number of wavelengths, the fringe at P is dark not bright. This is due to 180° phase change which occurs when light is reflected from a denser medium. This is equivalent to adding an extra half wavelength to the path of the reflected wave. At grazing incidence a fringe is formed at O, where the geometrical path difference between the direct and reflected waves is zero and it follows that it will be dark rather than bright.

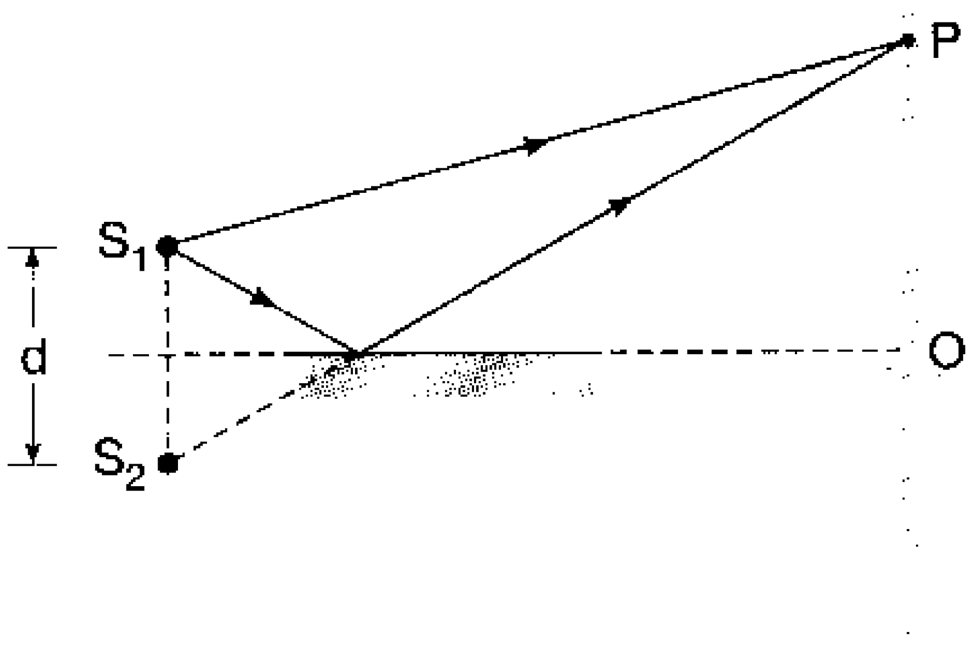


Fig. 23.13

Thus, whenever there exists a phase difference of π between the two interfering beams of light, conditions of maximas and minimas are interchanged, *i.e.*,

$$\Delta x = n\lambda \qquad \text{(for minimum intensity)}$$

and

$$\Delta x = (2n - 1)\lambda/2 \qquad \text{(for maximum intensity)}$$

6. **Multiple Slit Interference Patterns :** The argument used to determine the interference maxima for two slits can be extended to N slits. As figure shows, the path difference between adjacent sources to a point on the screen at angle θ is $d \sin\theta$. If this path difference is $n\lambda$, the waves from all N sources are in phase and interfere constructively. Therefore,

$$d \sin\theta = n\lambda \qquad \ldots\text{(i)}$$

also gives the locations of the interference maxima for N slits.

Determining the intensity at an arbitrary angle θ requires the addition of N harmonic waves, all with the same frequency and amplitude but with different phases. Let us find the resultant intensity due to $N = 6$, sources which are separated with each other by a distance

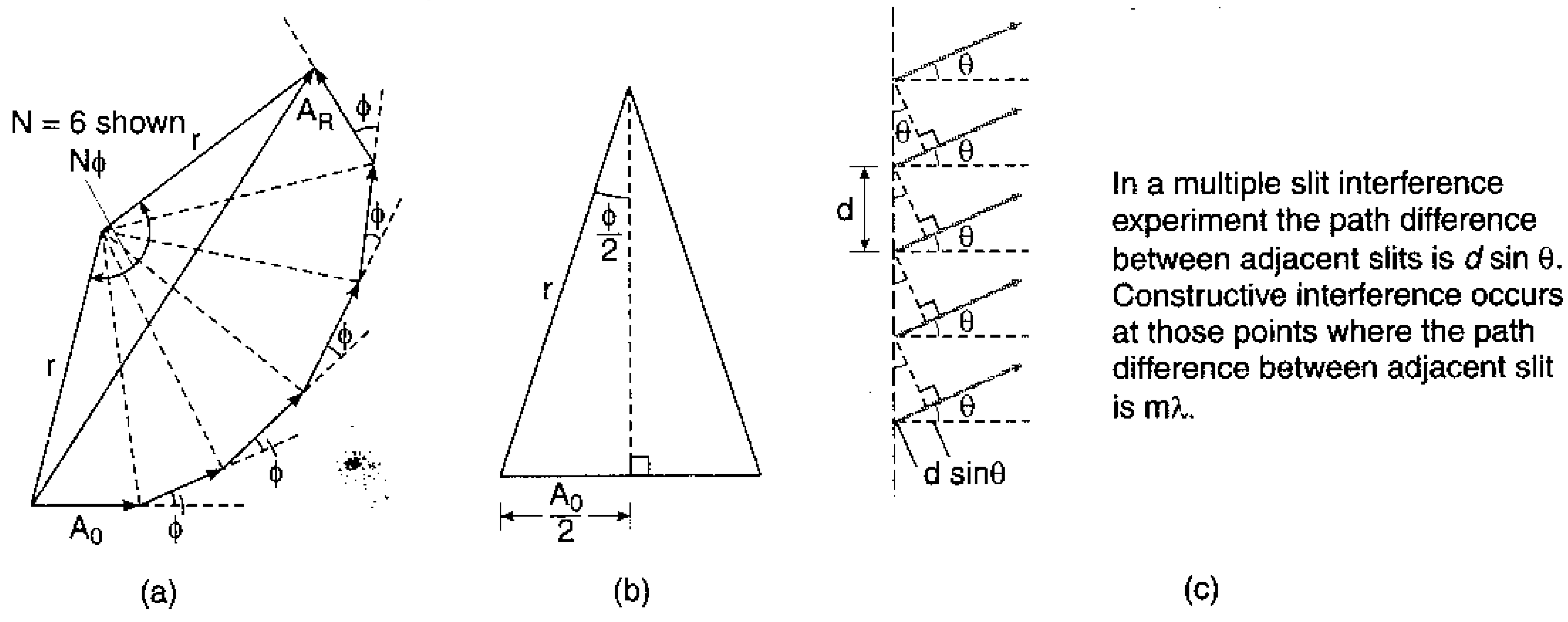


Fig. 23.14

d. At angle θ path difference between two successive slits is $\Delta x = d \sin\theta$. Corresponding phase difference ϕ can be obtained by the relation

$$\phi = \frac{2\pi}{\lambda}\Delta x = 2\pi\left(\frac{d\sin\theta}{\lambda}\right)$$

Let the amplitude and intensity due to individual source at the point under consideration be A_0 and I_0 respectively.

Let the resultant amplitude be A_R and intensity I_R. From figure (a)

$$\frac{A_R}{2} = r\sin\left(\frac{N\phi}{2}\right) \qquad \text{...(ii)}$$

and figure (b)

$$\frac{A_0}{2} = r\sin\left(\frac{\phi}{2}\right) \qquad \text{...(iii)}$$

Dividing Eq. (ii) by (iii), we get $\dfrac{A_R}{A_0} = \dfrac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)}$ or $A_R = A_0\left(\dfrac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}}\right)$

As $I \propto A^2$, we can write

$$I_R = I_0\left(\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}}\right)^2 \qquad \text{...(iv)}$$

For $N = 2$, Eq. (iv) reduces to, $I_R = 4I_0\cos^2\frac{\phi}{2}$

which we have already studied in article 23.4.

What we have read above was actually the **primary maxima.** There are **secondary maximas** as well, but as far as JEE is concerned this much theory is enough.

In fact, the number of secondary maxima is always $N-2$, where N is the number of slits.

Let us take an example for $N = 3$. Primary maximas occurs when $\phi = 0, \pm 2\pi, \pm 3\pi \ldots$. Resultant intensity at primary maxima will be $9I_0$. One secondary maxima will be obtained in this case. The intensity at secondary maxima in this case will be I_0, at points where $\phi = \pm\pi, \pm 3\pi, \ldots$ etc. For these points, the wave from one slit exactly cancels that from another slit. This means that only light from the third slit

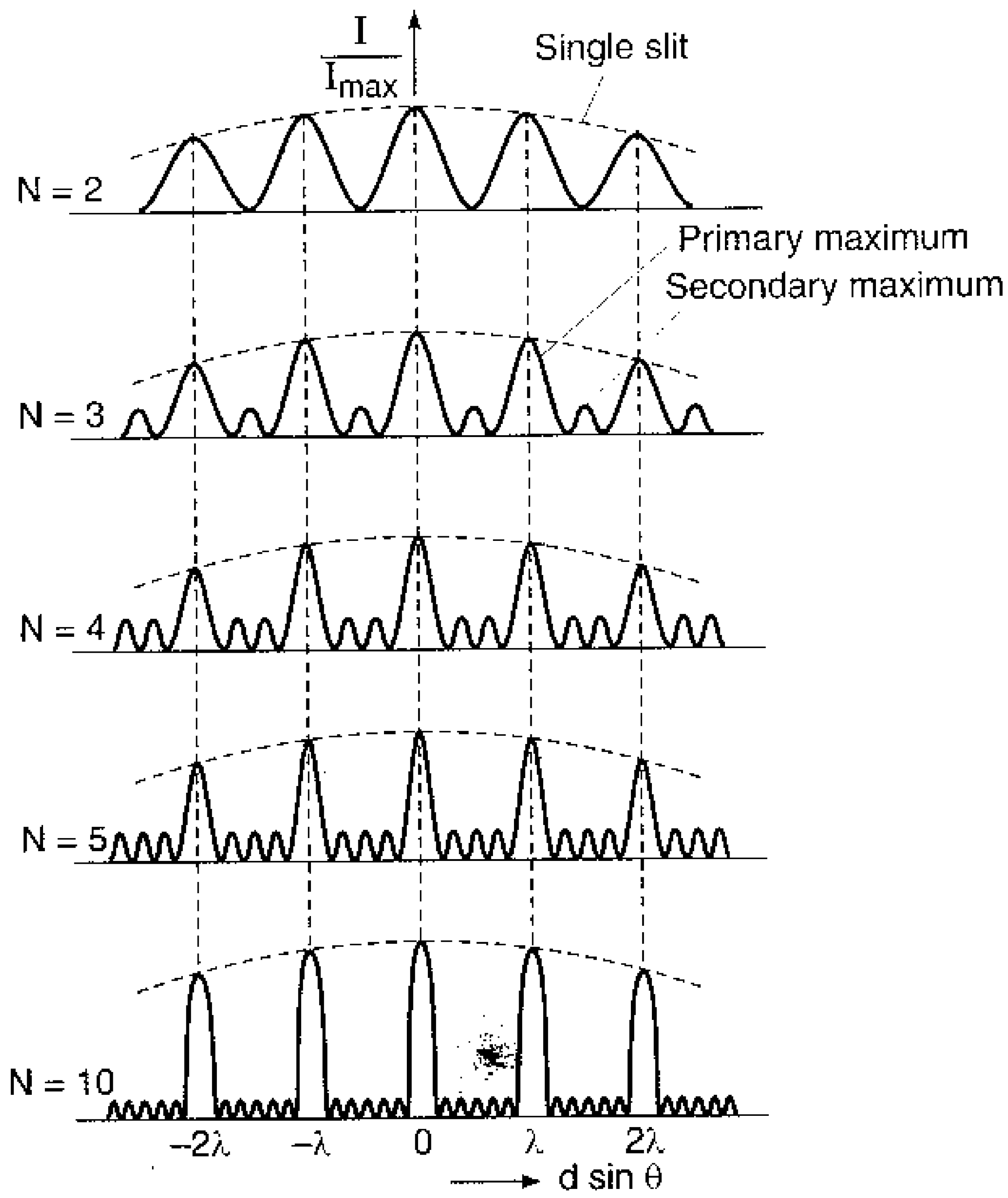


Fig. 23.15

d. At angle θ path difference between two successive slits is $\Delta x = d \sin\theta$. Corresponding phase difference ϕ can be obtained by the relation

$$\phi = \frac{2\pi}{\lambda}\Delta x = 2\pi\left(\frac{d\sin\theta}{\lambda}\right)$$

Let the amplitude and intensity due to individual source at the point under consideration be A_0 and I_0 respectively.

Let the resultant amplitude be A_R and intensity I_R. From figure (a)

$$\frac{A_R}{2} = r\sin\left(\frac{N\phi}{2}\right) \quad \text{...(ii)}$$

and figure (b)

$$\frac{A_0}{2} = r\sin\left(\frac{\phi}{2}\right) \quad \text{...(iii)}$$

Dividing Eq. (ii) by (iii), we get

$$\frac{A_R}{A_0} = \frac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)} \quad \text{or} \quad A_R = A_0\left(\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}}\right)$$

As $I \propto A^2$, we can write

$$I_R = I_0\left(\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}}\right)^2 \quad \text{...(iv)}$$

For $N = 2$, Eq. (iv) reduces to, $I_R = 4I_0\cos^2\frac{\phi}{2}$

which we have already studied in article 23.4.

What we have read above was actually the **primary maxima**. There are **secondary maximas** as well, but as far as JEE is concerned this much theory is enough.

In fact, the number of secondary maxima is always $N - 2$, where N is the number of slits.

Let us take an example for $N = 3$. Primary maximas occurs when $\phi = 0, \pm 2\pi, \pm 3\pi...$. Resultant intensity at primary maxima will be $9I_0$. One secondary maxima will be obtained in this case. The intensity at secondary maxima in this case will be I_0, at points where $\phi = \pm\pi, \pm 3\pi,...$ etc. For these points, the wave from one slit exactly cancels that from another slit. This means that only light from the third slit

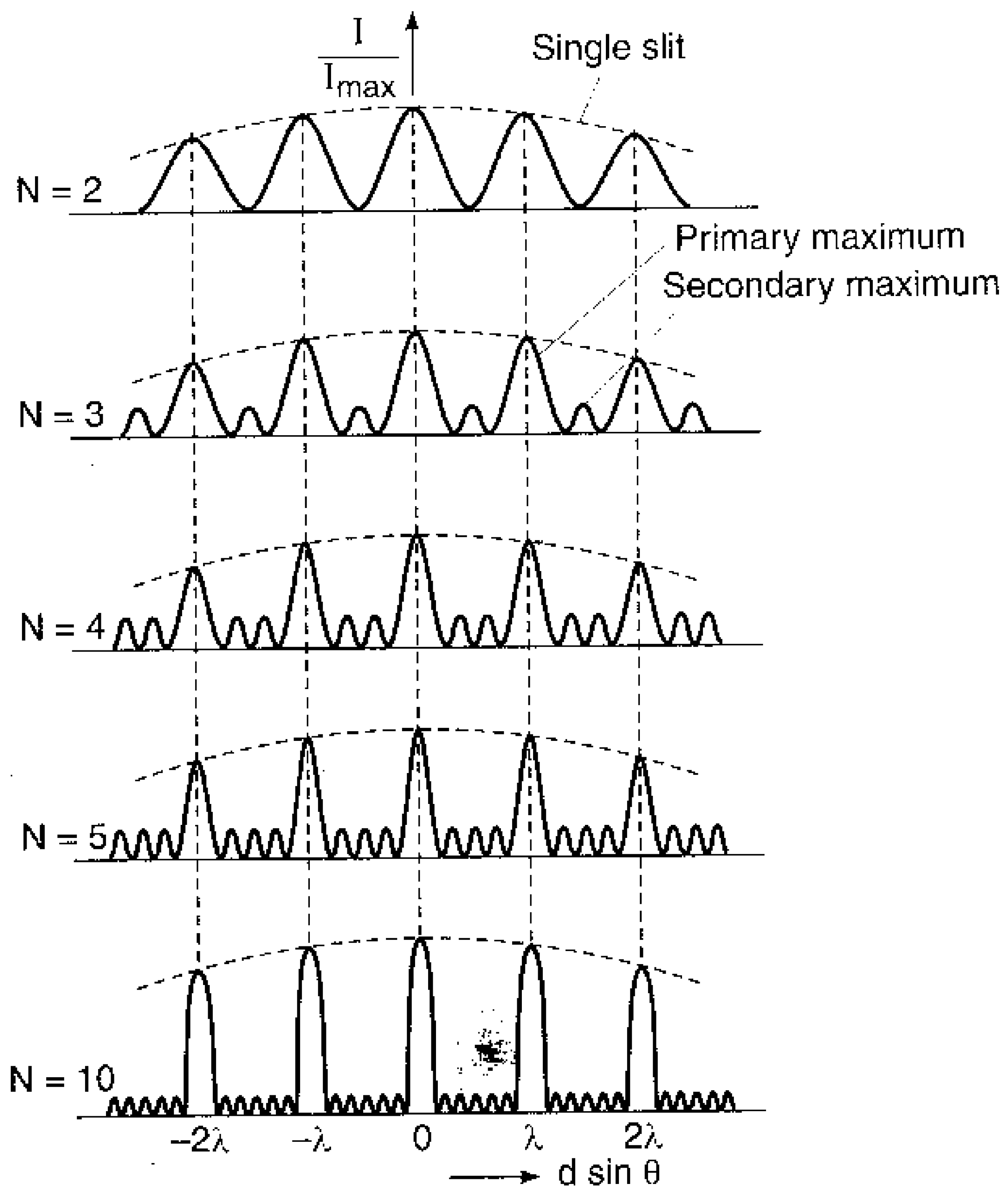


Fig. 23.15

contributes to the resultant, which consequently has a value I_0.

Thus maximum intensity (primary) for N slits is,

$$I_{max} = N^2 I_0 \qquad (\phi = 0, \pm 2\pi, \pm 4\pi, \ldots)$$

$$\Rightarrow \qquad A_R = NA_0 \quad \text{or} \quad I_R = N^2 I_0$$

The minimum intensity for N slits is still zero.

$$I_{min} = 0$$

7. The number of fringes obtained depends on the amount of diffraction occuring at the slits and this in turn depends on their width. The narrower the slits, the greater will be the number of fringes due to the increased diffraction but the fainter they will be, since less light gets through. In practice, to give easily seen fringes, the slits have to be many wavelengths wide.

8. Resultant intensity due to two coherent sources is given by the equation,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2 \qquad \text{where} \quad \cos\phi = +1$$

and

$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \text{where} \quad \cos\phi = -1$$

$$\therefore \qquad \frac{I_{max}}{I_{min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 \qquad \ldots(v)$$

If both the slits are of equal width

$$I_1 \approx I_2 = I_0 \text{ (say)}$$

and in that case, $\quad I_{max} = 4 I_0$ and $I_{min} = 0$

If the slits are of unequal width, then

$$I_1 \neq I_2 \qquad I_{min} \neq 0$$

According to some authors, amplitude of the light coming from a slit is proportional to the slit width, *i.e.*, intensity is proportional to square of the slit width.

9. **Interference in thin films :** Interference effects are commonly observed in thin films, such as thin layers of oil on water or the thin surface of a soap bubble.

The varied colors observed when white light is incident on such films result from the interference of waves reflected from the two surfaces of the film.

Consider a film of uniform thickness t and index of refraction μ, as shown in figure. Let us assume that the light rays travelling in air are nearly normal to the two surfaces of the film. To determine whether the reflected rays interfere constructively or destructively, we first note the following facts.

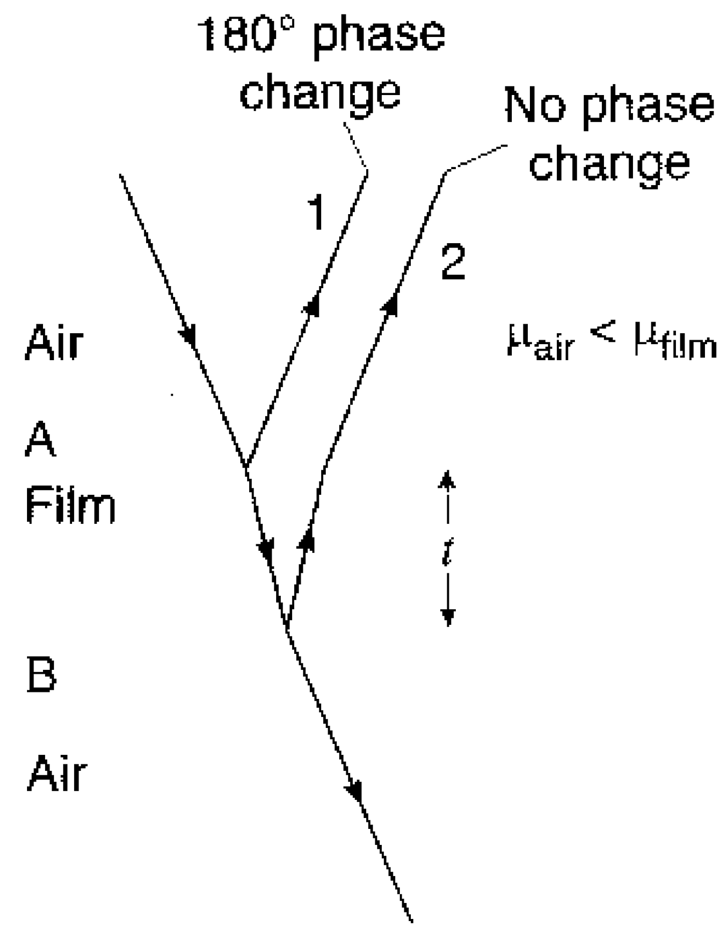


Fig. 23.16 Interference in light reflected from a thin film is due to a combination of rays reflected from the upper and lower surfaces of the film.

(i) The wavelength of light in a medium whose refractive index is μ is,

$$\lambda_\mu = \frac{\lambda}{\mu}$$

where λ is the wavelength of light in vacuum (or air)

(ii) If a wave is reflected from a denser medium it undergoes a phase change of 180°. Let us apply these rules to the film shown in figure. The path difference between the two rays 1 and 2 is $2t$ while the phase difference between them is 180°. Hence, condition of **constructive** interference will be,

$$2t = (2n-1)\frac{\lambda_\mu}{2} \qquad n = 1, 2, 3, \ldots$$

or $$2\mu t = \left(n - \frac{1}{2}\right)\lambda \qquad \text{as} \quad \lambda_\mu = \frac{\lambda}{\mu}$$

Similarly, condition of **destructive** interference will be,

$$2\mu t = n\lambda \qquad n = 0, 1, 2, \ldots$$

10. All the fringes in YDSE are predicted to have the same maximum intensity. However this is not what is observed. The brightness of the fringes decreases with increasing order and some of the fringes are missing. This is because of the finite width of the slits. Diffraction is always present in interference experiments. Thus, two effects (interference and diffraction) are occuring simultaneously. Further discussion is beyond the JEE syllabus.

EXAMPLE 23.5 *Calculate the minimum thickness of a soap bubble film* ($\mu = 1.33$) *that results in constructive interference in the reflected light if the film is illuminated with light whose wavelength in free space is* $\lambda = 600$ *nm.*

SOLUTION For constructive interference in case of soap film,

$$2\mu t = \left(n - \frac{1}{2}\right)\lambda \qquad n = 1, 2, 3, \ldots$$

For minimum thickness t, $n = 1$

or $$2\mu t = \frac{\lambda}{2}$$

or $$t = \frac{\lambda}{4\mu} = \frac{600}{4 \times 1.33}$$

$$= 112.78 \text{ nm}$$ **Ans.**

EXAMPLE 23.6 *In solar cells a silicon solar cell* ($\mu = 3.5$) *is coated with a thin film of silicon monoxide* SiO ($\mu = 1.45$) *to minimize reflective losses from the surface. Determine the minimum thickness of* SiO *that produces the least reflection at a wavelength of* 550 *nm, near the centre of the visible spectrum.*

SOLUTION The reflected light is a minimum when rays 1 and 2 (shown in figure) meet the condition of destructive interference.

Note : Both rays undergo a 180° phase change upon reflection. The net change in phase due to reflection is therefore zero, and the condition for a reflection minimum requires a path difference of $\lambda_\mu/2$. Hence,

$$2t = \frac{\lambda}{2\mu}$$

or $$t = \frac{\lambda}{4\mu} = \frac{550}{4\,(1.45)} = 94.8 \text{ nm}$$ **Ans.**

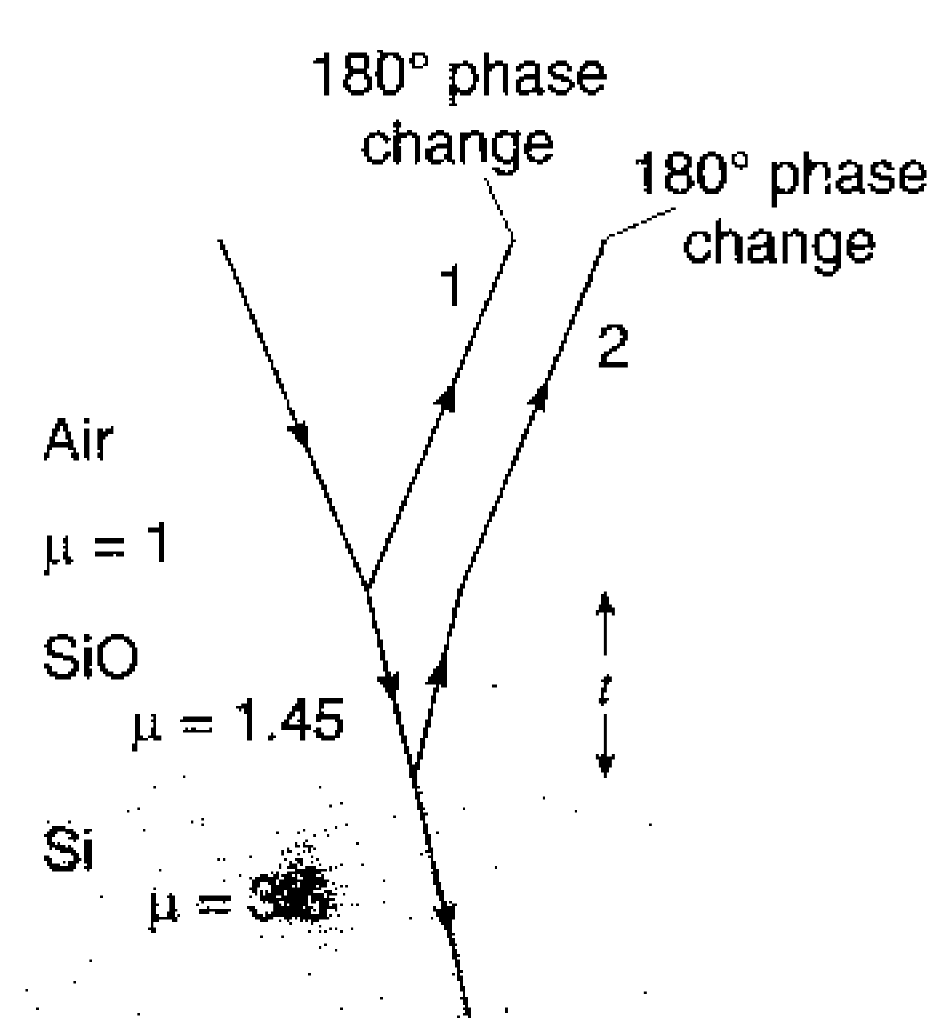


Fig. 23.17

EXAMPLE 23.7 *Two slits are separated by 0.32 mm. A beam of 500 nm light strikes the slits producing an interference pattern. Determine the number of maxima observed in the angular range* $-30° < \theta < 30°$.

SOLUTION Fringe width $w = \frac{\lambda D}{d}$ and $y = \frac{D}{\sqrt{3}}$

Therefore, number of fringe widths in a distance y is

$$n = \frac{y}{w} = \frac{d}{\sqrt{3}\lambda}$$

$$= \frac{0.32 \times 10^{-3}}{(\sqrt{3})(500 \times 10^{-9})} = 369.5$$

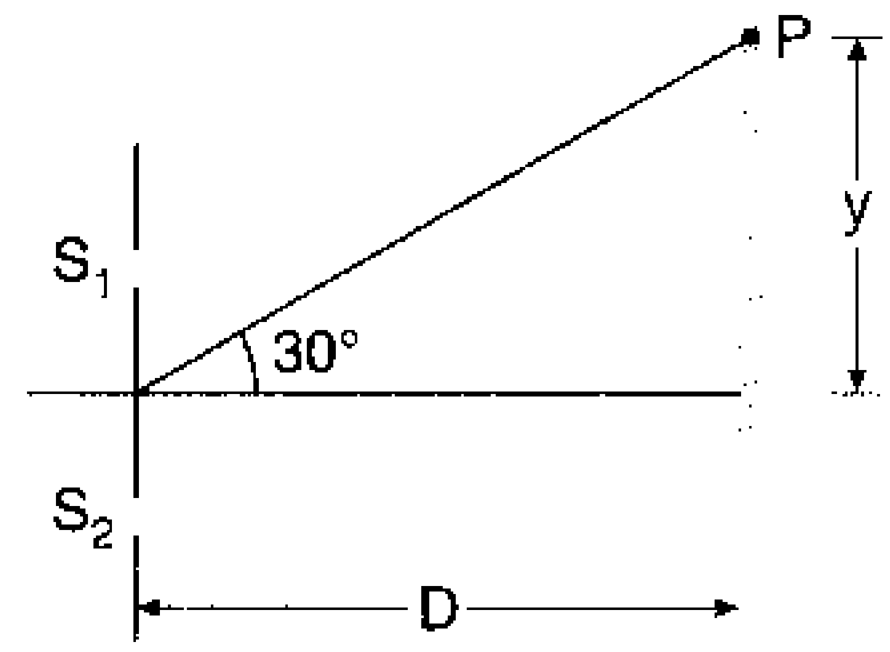


Fig. 23.18

Therefore, total number of maximas obtained in the angular range $-30° < \theta < 30°$ (including the central one) is

$$N = 2 \times 369 + 1 = 739$$ **Ans.**

EXAMPLE 23.8 *Distance between the slits shown in figure is* $d = 20\lambda$ *where* λ *is the wavelength of light used. Find the angle* θ *where*

(*a*) *central maxima (where path difference is zero) is obtained*

(*b*) *third order maxima is obtained.*

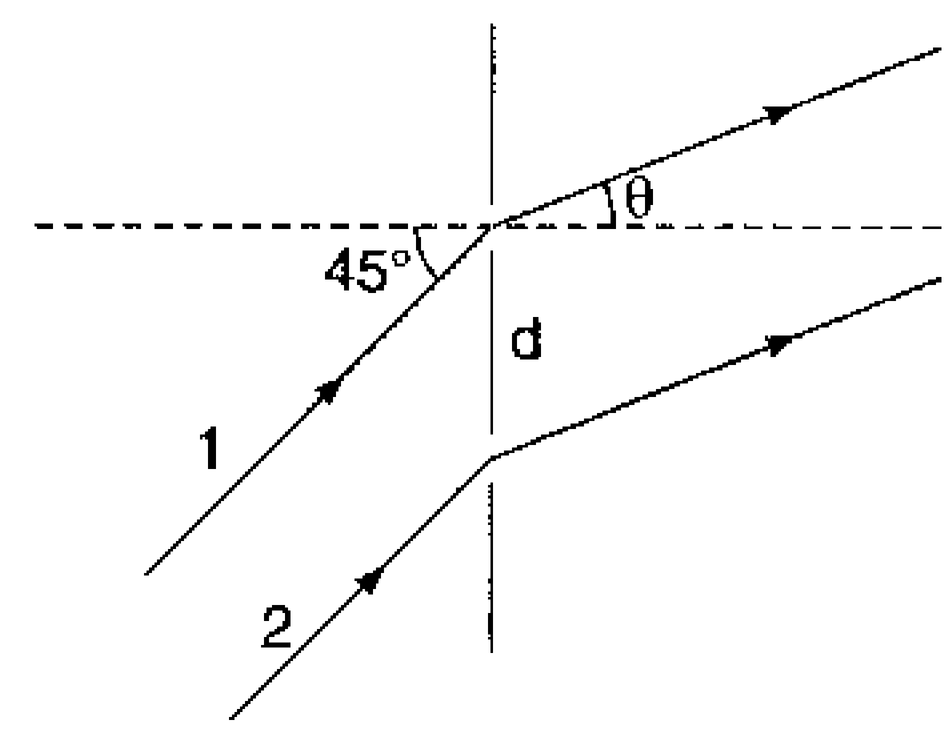


Fig. 23.19

SOLUTION Ray 1 has a longer path than that of ray 2 by a distance $d \sin 45°$, before reaching the slits. Afterwards ray 2 has a path longer than ray 1 by a distance $d \sin \theta$. The net path difference is therefore, $d \sin\theta - d \sin 45°$.

(a) Central maximum is obtained where, net path difference is zero,

or $$d \sin\theta - d \sin 45° = 0$$

or $$\theta = 45°$$ **Ans.**

(b) Third order maxima is obtained where net path difference is 3λ, or

$$d \sin\theta - d \sin 45° = 3\lambda$$

$\therefore$ $$\sin\theta = \sin 45° + \frac{3\lambda}{d}$$

Putting $$d = 20\lambda$$

we have $$\sin\theta = \sin 45° + \frac{3\lambda}{20\lambda}$$

or $$\theta \approx 59°$$ **Ans.**

INTRODUCTORY EXERCISE 23.1

1. Explain why two flashlights held close together do not produce an interference pattern on a distant screen.
2. Why it is so much easier to perform interference experiments with a laser than with an ordinary light source.
3. In *YDSE*, $D = 1.2$ m and $d = 0.25$ cm. The slits are illuminated with coherent 600 nm–light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75% of the maximum.
4. A possible means for making an airplane invisible to radar is to coat the plane with an antireflective polymer. If radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $\mu = 1.5$. How thick is the oil film?
5. Slit 1 of a double slit is wider than slit 2, so that the light from slit 1 has an amplitude three times that of the light from slit 2. Show that Eq. (xii) in article 23.4 is replaced by the equation,
$$I = \left(\frac{4\,I_{max}}{9}\right)\left(1 + 3\cos^2\frac{\phi}{2}\right)$$
6. Determine what happens to the double slit interference pattern if one of the slits is covered with a thin, transparent film whose thickness is $\frac{\lambda}{2(\mu - 1)}$, where λ is the wavelength of the incident light and μ is the index of refraction of the film.
7. Two slits 4.0×10^{-6} m apart are illuminated by light of wavelength 600 nm. What is the highest order fringe in the interference pattern.
8. Consider an interference experiment using eight equally spaced slits. Determine the smallest phase difference in the waves from adjacent slits such that the resultant wave has zero amplitude.

MISCELLANEOUS EXAMPLES

EXAMPLE 1 *A Young double slit apparatus is immersed in a liquid of refractive index μ_1. The slit plane touches the liquid surface. A parallel beam of monochromatic light of wavelength λ (in air) is incident normally on the slits.*

(a) *Find the fringe width*

(b) *If one of the slits (say S_2) is covered by a transparent slab of refractive index μ_2 and thickness t as shown, find the new position of central maxima.*

(c) *Now the other slit S_1 is also covered by a slab of same thickness and refractive index μ_3 as shown in figure due to which the central maxima recovers its position find the value of μ_3.*

(d) *Find the ratio of intensities at O in the three conditions (a), (b) and (c).*

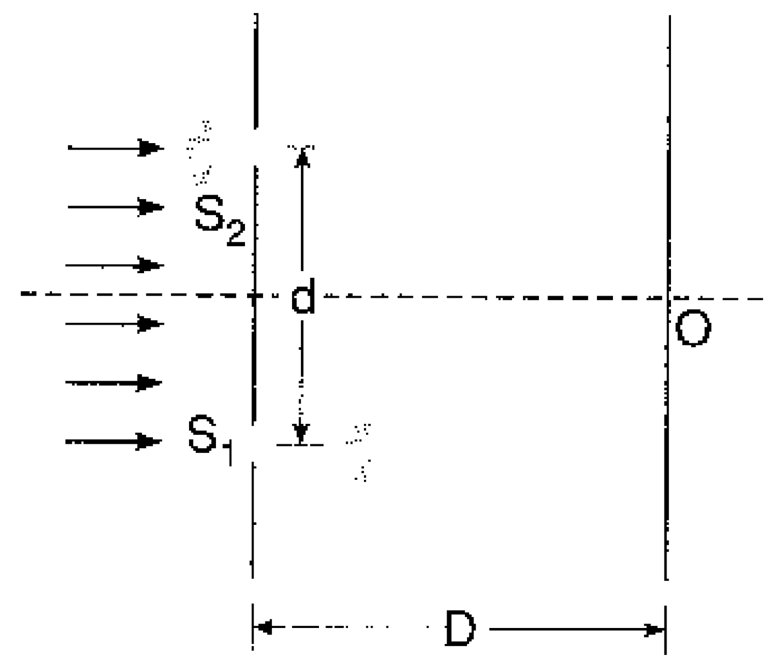


Fig. 23.20

SOLUTION (a) Fringe width

$$w = \frac{\lambda_{\mu_1} D}{d} = \frac{\lambda D}{\mu_1 d}$$ **Ans.**

(b) Position of central maximum is shifted upwards by a distance

$$= \frac{(\mu_2 - 1)\, tD}{d}$$ **Ans.**

(c)
$$\frac{(\mu_2 - 1)\, tD}{d} = \frac{\left(\frac{\mu_3}{\mu_1} - 1\right) tD}{d}$$

$\therefore$
$$\frac{\mu_3}{\mu_1} = \mu_2$$

or
$$\mu_3 = \mu_1 \mu_2$$ **Ans.**

(d)
$$I = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$$

where
$$\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x \quad \text{or} \quad \frac{\phi}{2} = \left(\frac{\pi}{\lambda}\right)\Delta x$$

$$I \propto \cos^2\left(\frac{\phi}{2}\right)$$

In the first and third case, $\Delta x = 0$ while in second case, $\Delta x = (\mu_2 - 1)\, t$. Therefore, the desired ratio is,

$$I_1 : I_2 : I_3 = 1 : \cos^2\left\{\frac{\pi(\mu_2 - 1)\, t}{\lambda}\right\} : 1$$ **Ans.**

EXAMPLE 2 *Young's double slit experiment is carried out using microwaves of wavelength $\lambda = 3$ cm. Distance between the slits is $d = 5$ cm and the distance between the plane of slits and the screen is $D = 100$ cm.*

(*a*) *Find the number of maximas and*
(*b*) *Their positions on the screen.*

SOLUTION (a) The maximum path difference that can be produced = distance between the sources or 5 cm.

Thus, in this case we can have only three maximas, one central maxima and two on its either side for a path difference of λ or 3 cm.

(b) For maximum intensity at P,

$$S_2P - S_1P = \lambda$$

or

$$\sqrt{(y + d/2)^2 + D^2} - \sqrt{(y - d/2)^2 + D^2} = \lambda^2$$

Substituting $d = 5$ cm, $D = 100$ cm

and $\lambda = 3$ cm and solving the above equation, we get

$$y = \pm 75 \text{ cm}$$

Thus, the three maximas will be at

$$y = 0 \quad \text{and} \quad y = \pm 75 \text{ cm}.$$

Ans.

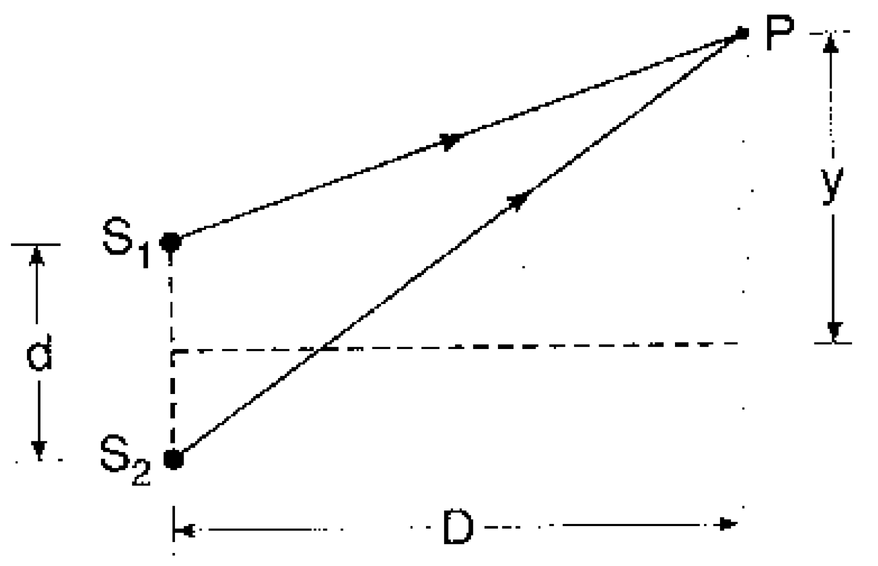


Fig. 23.21

EXAMPLE 3 *Interference fringes are produced by a double slit arrangement and a piece of plane parallel glass of refractive index 1.5 is interposed in one of the interfering beam. If the fringes are displaced through 30 fringe widths for light of wavelength 6×10^{-5} cm, find the thickness of the plate.*

SOLUTION Path difference due to the glass slab, $\Delta x = (\mu - 1)\,t$

Thirty fringes are displaced due to the slab. Hence,

$$\Delta x = 30\,\lambda$$

$$\therefore \quad (\mu - 1)\,t = 30\,\lambda$$

$$\therefore \quad t = \frac{30\lambda}{\mu - 1} = \frac{30 \times 6 \times 10^{-5}}{1.5 - 1}$$

$$= 3.6 \times 10^{-3} \text{ cm}$$

Ans.

EXAMPLE 4 *The arrangement for a mirror experiment is shown in the figure. 'S' is a point source of frequency 6×10^{14} Hz. D and C represent the two ends of a mirror placed horizontally, and LOM represents the screen.*

Determine the position of the region where the fringes will be visible and calculate the number of fringes.

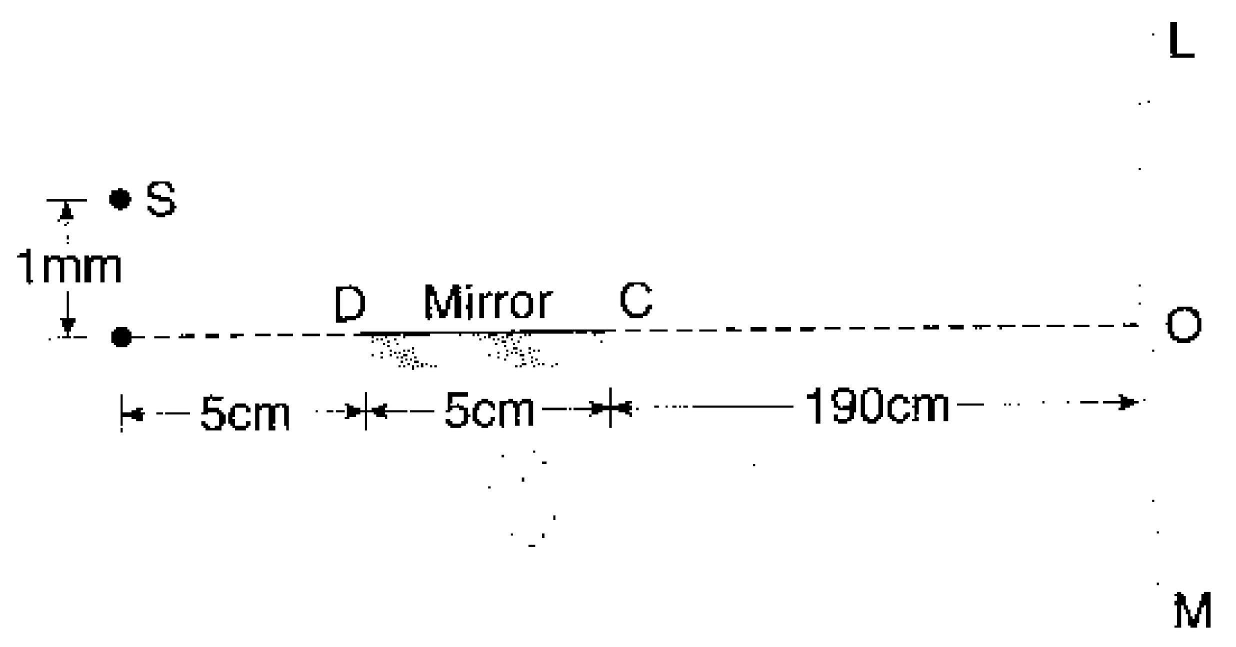


Fig. 23.22

SOLUTION Fringes will be observed in the region between P_1 and P_2 because the reflected rays lie only in this region.

From similar triangles BDS' and $S'P_2A$,

$$\frac{AP_2}{BS'} = \frac{AS'}{BD}$$

$$\therefore \quad AP_2 = \frac{(AS')(BS')}{BD}$$

$$= \frac{(190+5+5)(0.1)}{5} = 4\,\text{cm}$$

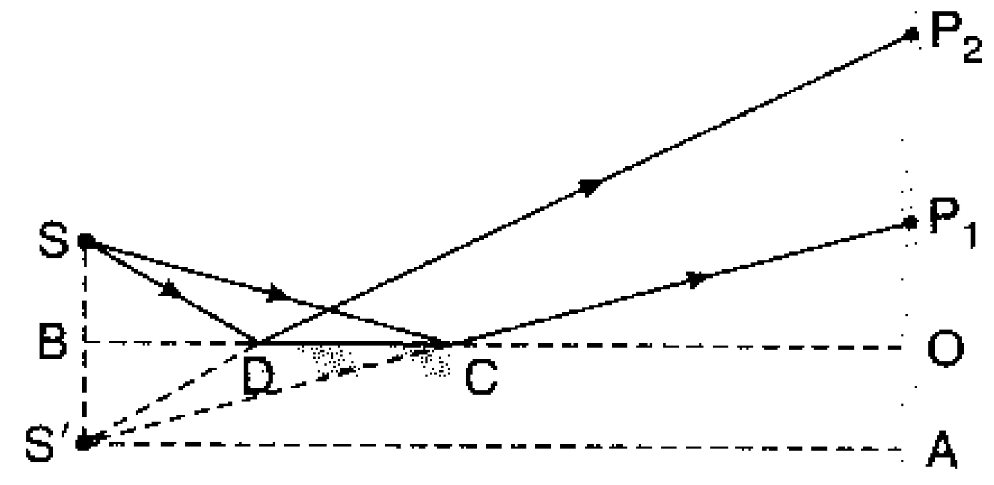


Fig. 23.23

Similarly in triangles BCS' and $S'P_1A$,

$$\frac{AP_1}{BS'} = \frac{AS'}{BC}$$

$$\therefore \quad AP_1 = \frac{(AS')(BS')}{BC} = \frac{(190+5+5)(0.1)}{10} = 2\,\text{cm}$$

$$\therefore \quad P_1P_2 = AP_2 - AP_1 = 2\,\text{cm}$$ **Ans.**

Wavelength of the light $\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^{14}} = 5\times10^{-7}$ m

Fringe width $w = \frac{\lambda D}{d}$

Here $D = S'A = (190+5+5) = 200\,\text{cm} = 2.0\,\text{m}$, $d = SS' = 2\,\text{mm} = 2\times10^{-3}$ m

$$\therefore \quad w = \frac{(5\times10^{-7})(2.0)}{2\times10^{-3}} = 5\times10^{-4}\,\text{m}$$

$$= 0.05\,\text{cm}$$

$$\therefore \quad \text{Number of fringes} = \frac{P_1P_2}{w} = 40$$ **Ans.**

EXAMPLE 5 *In a Young experiment the light source is at distance $l_1 = 20\,\mu m$ and $l_2 = 40\,\mu m$ from the slits. The light of wavelength $\lambda = 500$ nm is incident on slits separated at a distance $10\,\mu m$. A screen is placed at a distance $D = 2$ m away from the slits as shown in figure. Find*

(a) *the values of θ relative to the central line where maxima appear on the screen?*

(b) *how many maxima will appear on the screen?*

(c) *What should be minimum thickness of a slab of refractive index 1.5 be placed on the path of one of the ray so that minima occurs at C?*

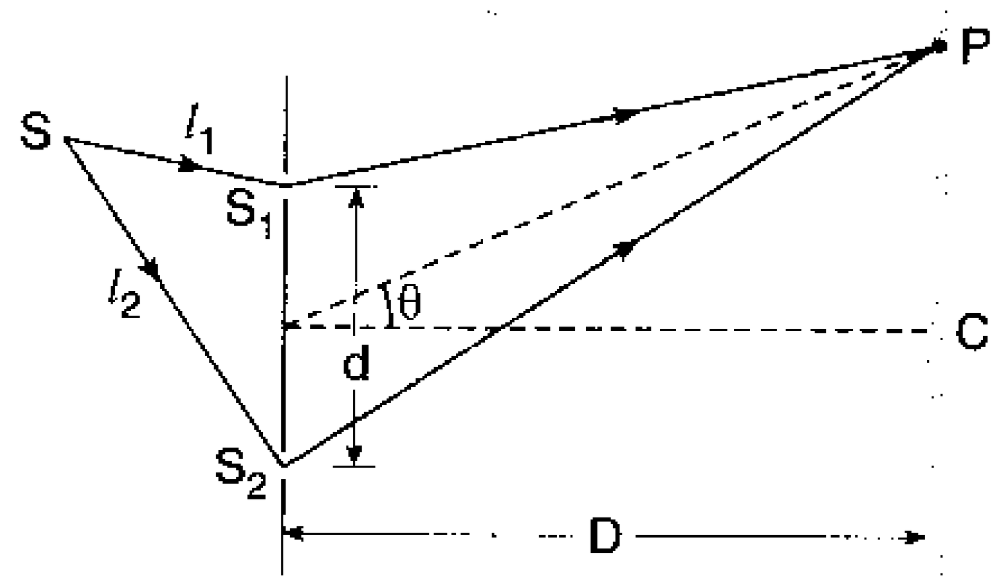


Fig. 23.24

SOLUTION (a) The optical path difference between the beams arriving at P,

$$\Delta x = (l_2 - l_1) + d\sin\theta$$

The condition for maximum intensity is,

$$\Delta x = n\lambda \qquad n = 0, \pm 1, \pm 2, \ldots$$

Thus,

$$\sin\theta = \frac{1}{d}[\Delta x - (l_2 - l_1)] = \frac{1}{d}[n\lambda - (l_2 - l_1)]$$

$$= \frac{1}{10\times 10^{-6}}[n\times 500\times 10^{-9} - 20\times 10^{-6}]$$

$$= 2\left[\frac{n}{40} - 1\right]$$

Hence,

$$\theta = \sin^{-1}\left[2\left(\frac{n}{40} - 1\right)\right]$$ **Ans.**

(b) $$|\sin\theta| \le 1$$

$\therefore$ $$-1 \le 2\left[\frac{n}{40} - 1\right] \le 1 \quad \text{or} \quad -20 \le (n - 40) \le 20$$

or $$20 \le n \le 60$$

Hence, number of maxima $= 60 - 20 = 40$ **Ans.**

(c) At C, phase difference,

$$\phi = \left(\frac{2\pi}{\lambda}\right)(l_2 - l_1)$$

$$= \left(\frac{2\pi}{500\times 10^{-9}}\right)(20\times 10^{-6})$$

$$= 80\,\pi$$

Hence, maximum intensity will appear at C. For minimum intensity at C,

$$(\mu - 1)\,t = \frac{\lambda}{2}$$

or $$t = \frac{\lambda}{2(\mu - 1)} = \frac{500\times 10^{-9}}{2\times 0.5} = 500 \text{ nm}$$ **Ans.**

EXAMPLE 6 *Light of wavelength $\lambda = 500$ nm falls on two narrow slits placed a distance $d = 50\times 10^{-4}$ cm apart, at an angle $\phi = 30°$ relative to the slits shown in figure. On the lower slit a transparent slab of thickness 0.1 mm and refractive index 3/2 is placed. The interference pattern is observed on a screen at a distance $D = 2$m from the slits. Then calculate*

(a) position of the central maxima?

(b) the order of minima closest to centre C of screen?

(c) How many fringes will pass over C, if we remove the transparent slab from the lower slit?

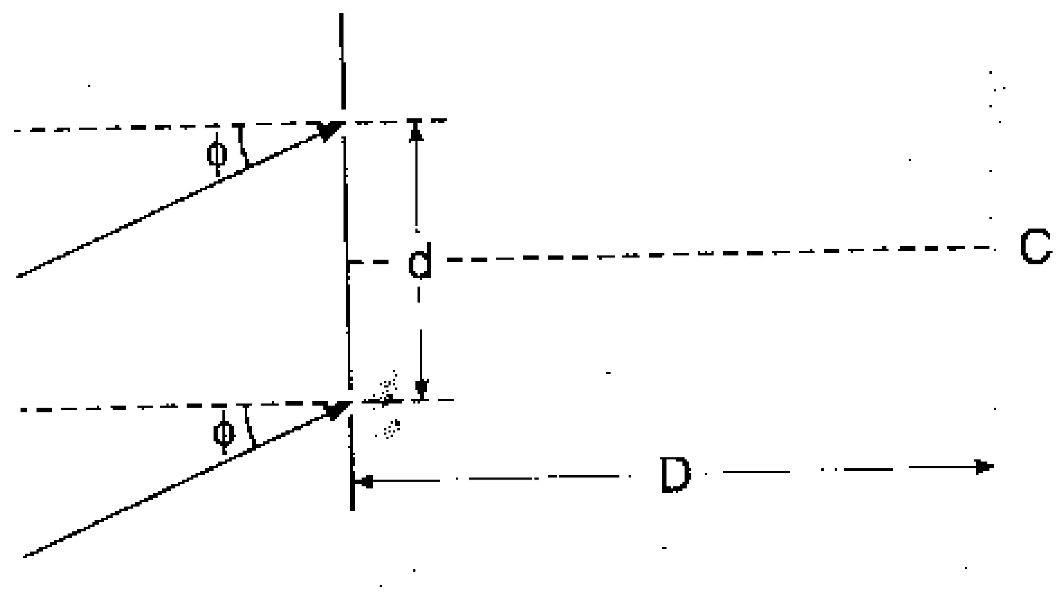


Fig. 23.25

SOLUTION (a) Path difference,

$$\Delta x = d\sin\phi + d\sin\theta - (\mu - 1)\,t$$

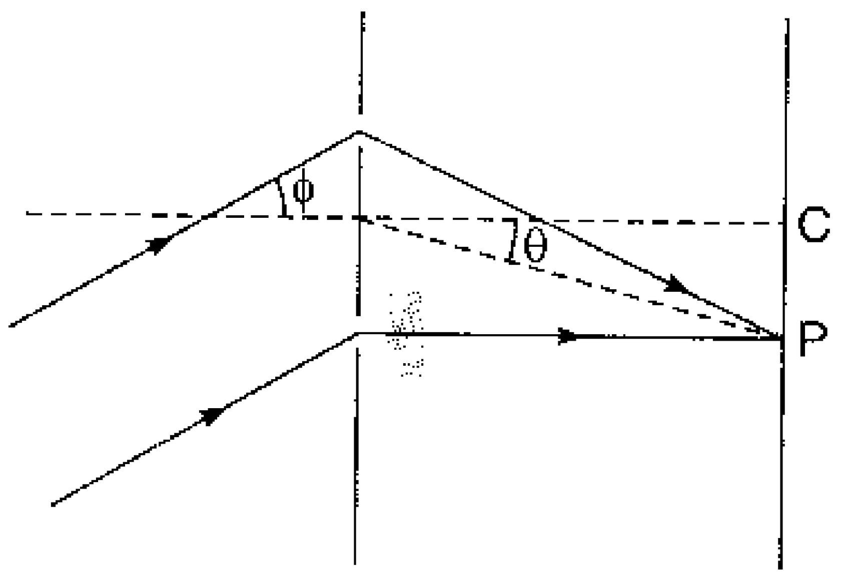


Fig. 23.26

For central maxima, $\Delta x = 0$

$$\therefore \quad \sin\theta = \frac{(\mu - 1)\,t}{d} - \sin\phi$$

$$= \frac{(3/2 - 1)\,(0.1)}{50 \times 10^{-3}} - \sin 30^\circ = \frac{1}{2}$$

$$\therefore \quad \theta = 30^\circ$$

Ans.

(b) At C, $\theta = 0^\circ$, therefore,

$$\Delta x = d\sin\phi - (\mu - 1)\,t$$

$$= (50 \times 10^{-3})\left(\frac{1}{2}\right) - (3/2 - 1)\,(0.1)$$

$$= 0.025 - 0.05 = -0.025 \text{ mm}$$

Substituting $\Delta x = n\lambda$, we get

$$n = \frac{\Delta x}{\lambda} = \frac{-0.025}{500 \times 10^{-6}} = -50$$

Hence, at C, there will be maxima. Therefore, closest to C order of minima are –49.

(c) Number of fringes shifted upwards $= \dfrac{(\mu - 1)\,t}{\lambda} = \dfrac{(3/2 - 1)\,(0.1)}{500 \times 10^{-6}} = 100$ **Ans.**

EXAMPLE 7 *In a Young's double slit experiment a parallel beam containing wavelengths $\lambda_1 = 4000$ Å and $\lambda_2 = 5600$ Å incident at an angle $\phi = 30^\circ$ on a diaphragm having narrow slits at a separation $d = 2$ mm. The screen is placed at a distance $D = 40$ cm from slits. A mica slab of thickness $t = 5$ mm is placed in front of one of the slits and whole the apparatus is submerged in water. If the central bright fringe is observed at C, calculate*

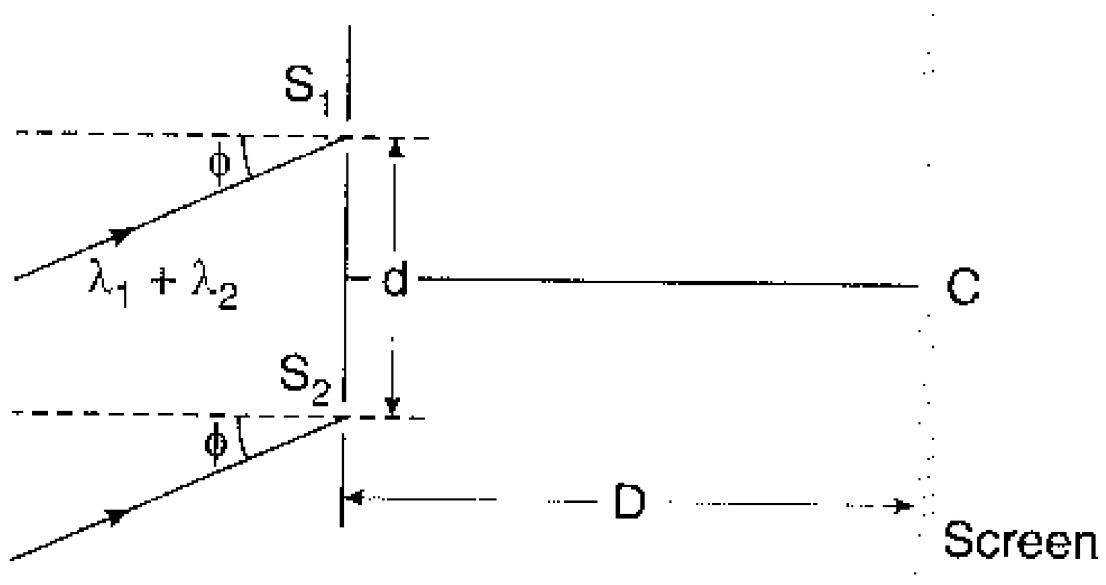


Fig. 23.27

(*a*) *the refractive index of the slab.*

(*b*) *the distance of the first black line from C. Both wavelengths are in air. Take $\mu_w = 4/3$.*

SOLUTION (a) The mica slab should be placed in front of S_2 to observe bright fringe at C. In that case net path difference at C is,

$$\Delta x = d\sin\phi - (\mu_r - 1)\,t \qquad \left(\mu_r = \frac{\mu_{slab}}{\mu_w}\right)$$

For central bright at C, $\Delta x = 0$

$$\therefore \quad d\sin\phi = (\mu_r - 1)\,t$$

or
$$(\mu_r - 1) = \frac{d \sin\phi}{t} = \frac{(2\times10^{-3})\sin 30^\circ}{5\times10^{-3}} = 0.2$$

$$\therefore \qquad \mu_r = 1.2 \qquad \text{or} \qquad \frac{\mu_{slab}}{\mu_w} = 1.2$$

$$\therefore \qquad \mu_{slab} = 1.2\times\mu_w = 1.2\times\frac{4}{3} = 1.6$$
Ans.

(b) A black line is formed at the position where both the wavelengths interfere destructively. Distance of n^{th} dark fringe from C,
$$y = \frac{(2n-1)\lambda D}{2d}$$

For black line,
$$\frac{(2n_1 - 1)\lambda'_1 D}{2d} = \frac{(2n_2 - 1)\lambda'_2 D}{2d} \qquad \ldots(i)$$

Here λ'_1 and λ'_2 are wavelengths in water.
$$\frac{\lambda'_1}{\lambda'_2} = \frac{\lambda_1/\mu_w}{\lambda_2/\mu_w} = \frac{\lambda_1}{\lambda_2} = \frac{4000}{5600}$$

Substituting these values in Eq. (i), we get
$$\frac{2n_1 - 1}{2n_2 - 1} = \frac{7}{5}$$

For minimum value $n_1 = 4$ and $n_2 = 3$
Hence, distance of first black line,
$$y = \frac{(2\times4-1)(4000\times10^{-10})\,40\times10^{-2}\times3}{2\times2\times10^{-3}\times4}$$
$$= 2.1\times10^{-4}\text{ m}$$
$$= 0.21\text{ mm}$$
Ans.

EXAMPLE 8 *The interference pattern of a Young's double slit experiment is observed in two ways by placing the screen as shown in figure (a) and (b). The distance between two consecutive right most minima on the screen of figure. (a) using light of wavelength $\lambda_1 = 4000$ Å is observed to be 600 times the fringe width in the screen of figure (b) using the wavelength $\lambda_2 = 6000$ Å. If D (as shown in figure) is 1 m then find the separation between the coherent sources S_1 and S_2. Given that $d > 3\lambda_1/2$.*

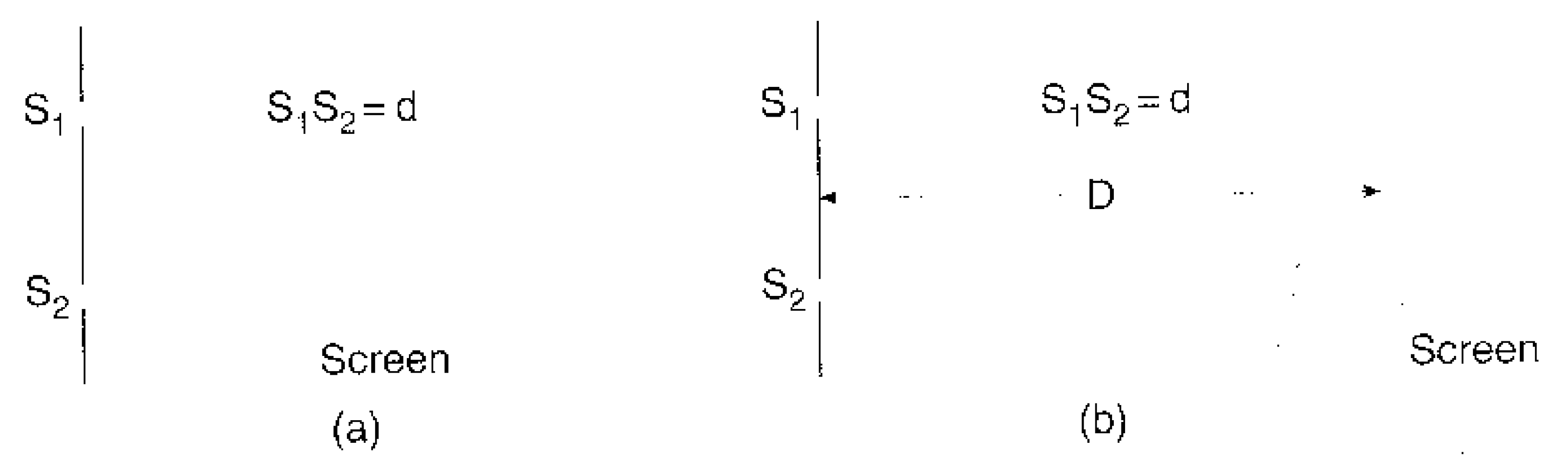


Fig. 23.28

SOLUTION P and Q' would have been the positions of first and second minima (last two), had the screen be perpendicular to S_2P.

Since, the angular positions of minima do not depend on the position of the screen. Therefore, second minima is formed at Q on the screen.

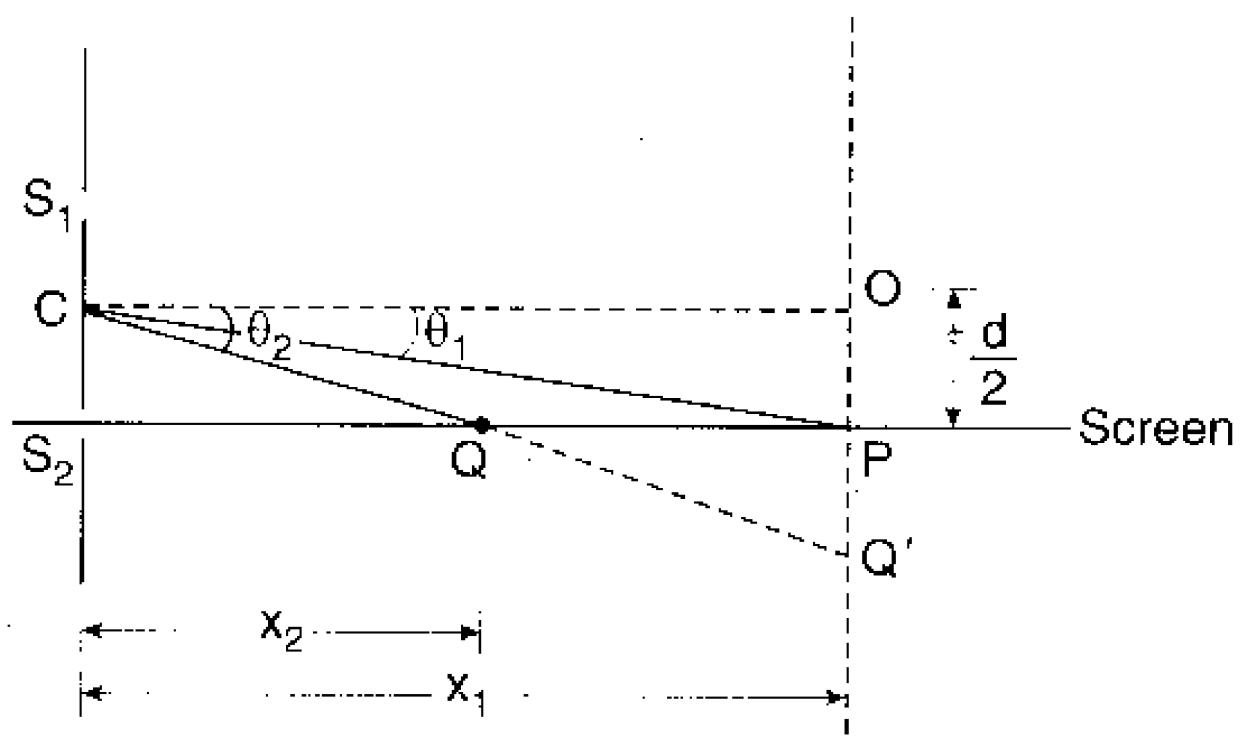


Fig. 23.29

For right most minima at P,

$$d\sin\theta_1 = \lambda_1/2 \quad \text{...(i)}$$

For small angles

$$\sin\theta_1 \approx \tan\theta_1 = \frac{d/2}{x_1}$$

Substituting in Eq. (i), we get
$$x_1 = \frac{d^2}{\lambda_1} \quad \text{...(ii)}$$

For next minima at Q,
$$d\sin\theta_2 = \frac{3}{2}\lambda_1 \quad \text{...(iii)}$$

For small angles
$$\sin\theta_2 \approx \tan\theta_2 = \frac{d/2}{x_2}$$

Substituting in Eq. (iii), we have
$$x_2 = \frac{d^2}{3\lambda_1} \quad \text{...(iv)}$$

$\therefore$
$$PQ = x_1 - x_2 = \frac{2d^2}{3\lambda_1} \quad \text{...(v)}$$

In the second case, fringe width
$$w = \frac{\lambda_2 D}{d} \quad \text{...(vi)}$$

Given that
$$PQ = 600\,w$$

$\therefore$
$$\frac{2d^2}{3\lambda_1} = 600\frac{\lambda_2 D}{d}$$

$\therefore$
$$d^3 = 900\lambda_1\lambda_2 D$$
$$= 900 \times 4000 \times 6000 \times 10^{-20} \times 1$$
$$= 216 \times 10^{-12}$$

$\therefore$
$$d = 6 \times 10^{-4}\text{ m} = 0.6\text{ mm}$$ **Ans.**

EXAMPLE 9 *In Young's double slit experiment set-up with light of wavelength $\lambda = 6000$ Å, distance between the two slits is 2 mm and distance between the plane of slits and the screen is 2 m. The slits are of equal intensity. When a sheet of glass of refractive index 1.5 (which permits only a fraction η of the incident light to pass through) and thickness 8000 Å is placed in front of the lower slit, it is observed that the intensity at a point P, 0.15 mm above the central maxima does not change. Find the value of η.*

SOLUTION Without inserting the slab, path difference at P,

$$\Delta x = \frac{yd}{D} = \frac{0.15\times10^{-3}\times2\times10^{-3}}{2}$$

$$=1.5\times10^{-7}\text{ m}$$

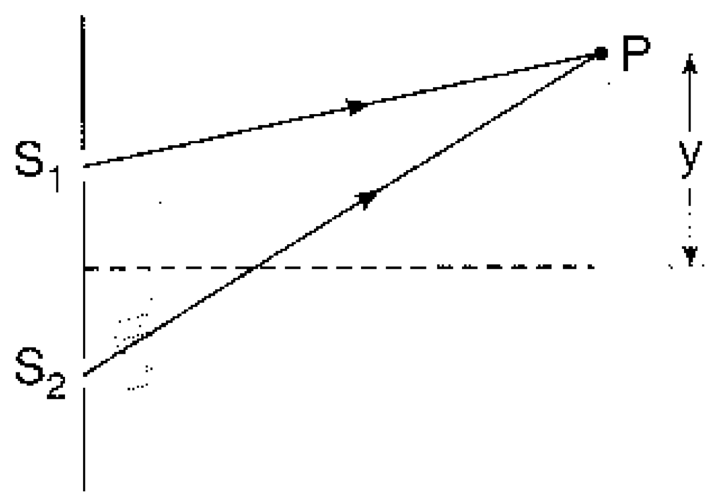


Fig. 23.30

Corresponding phase difference at P,

$$\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x)$$

$$=\left(\frac{2\pi}{6000\times10^{-10}}\right)(1.5\times10^{-7}) = \frac{\pi}{2}$$

$\therefore$ $$\frac{\phi}{2} = \frac{\pi}{4}$$

$\therefore$ Intensity at P, $$I = 4I_0\cos^2\frac{\phi}{2} = 2I_0$$

Phase difference after placing the glass sheet,

$$\phi' = \phi + \frac{2\pi}{\lambda}(\mu-1)t$$

$$=\frac{\pi}{2} + \frac{2\pi}{6000\times10^{-10}}(1.5-1)\ (8000\times10^{-10})$$

$$=\frac{11\pi}{6}$$

The intensity at P is now, $$I' = I_0 + \eta I_0 + 2\sqrt{\eta I_0^2}\cos\frac{11\pi}{6} = 2I_0$$ (given)

Solving this equation, we get, $\eta = 0.21$ **Ans.**

√**EXAMPLE 10** *In a Young's double slit experiment set-up source S of wavelength* 5000 Å *illuminates two slits* S_1 *and* S_2, *which act as two coherent sources. The source S oscillates about its shown position according to the equation* $y = 0.5\sin\pi t$, *where y is in millimetres and t in seconds. Find*

(*a*) *position of the central maxima as a function of time.*

(*b*) *minimum value of t for which the intensity at point P on the screen exactly in front of the upper slit becomes maximum.*

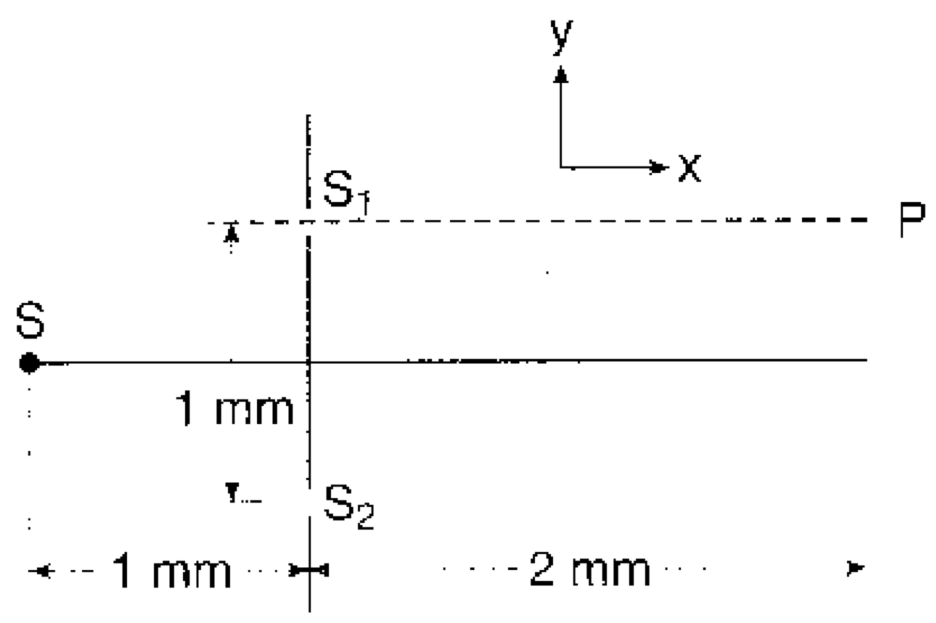


Fig. 23.31

SOLUTION (a) Net path difference at Q,

$$\Delta x = \frac{yd}{D} + \frac{y'd}{D'}$$

For central maximum, $\Delta x = 0$

or $y' = -\frac{D'}{D} y$

$= -\left(\frac{2}{1}\right)(0.5 \sin \pi t)$

$= -(\sin \pi t)$ mm **Ans.**

(b) $y' = \frac{d}{2}$, at point P exactly in front of S_1

$\therefore \quad \Delta x = \left(\frac{yd}{D}\right) + \frac{(d^2/2)}{D'}$

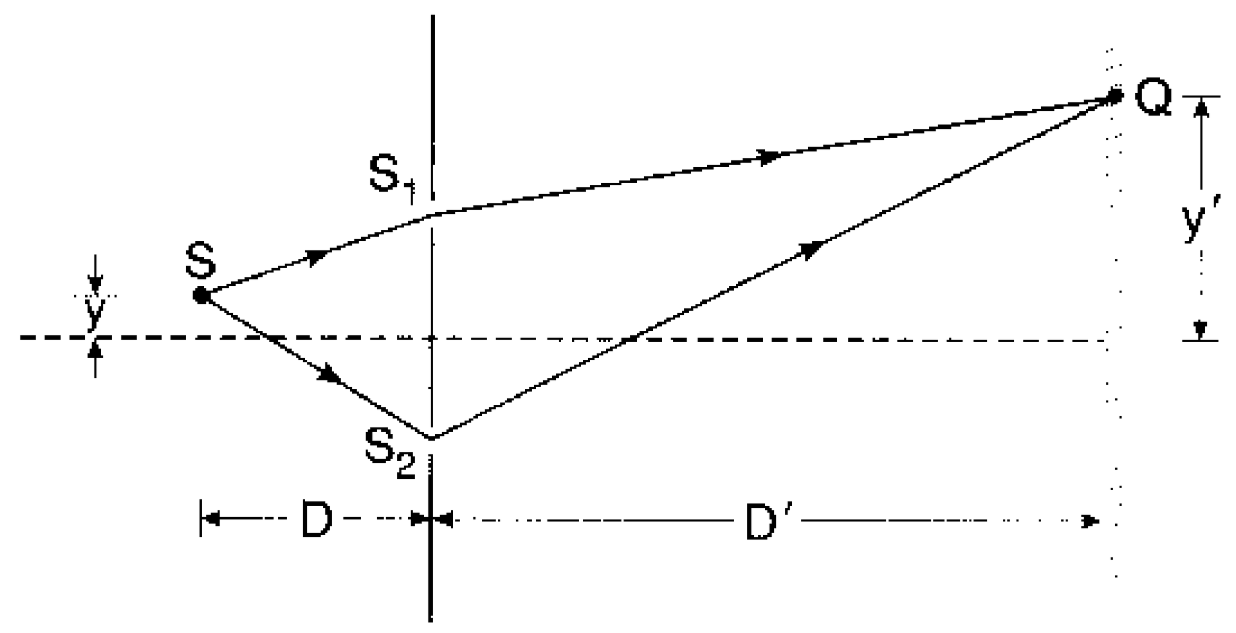


Fig. 23.32

For maximum intensity,

$$\Delta x = n\lambda$$

Putting the values, we get $0.5 \sin \pi t + 0.25 = 0.5\, n$

or $\sin \pi t = \frac{0.5\, n - 0.25}{0.5}$

For minimum value of t, $n = 1$

$\therefore \quad \sin \pi t = 0.5$

or $\pi t = \frac{\pi}{6}$

$\therefore \quad t = \frac{1}{6} = 0.167\,\text{s}$ **Ans.**

EXAMPLE 11 *Two coherent narrow slits emitting light of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The light is collected on a screen S which is placed at a distance $D (>> \lambda)$ from the slit S_1 as shown in figure. Find the finite distance x such that the intensity at P is equal to intensity at O.*

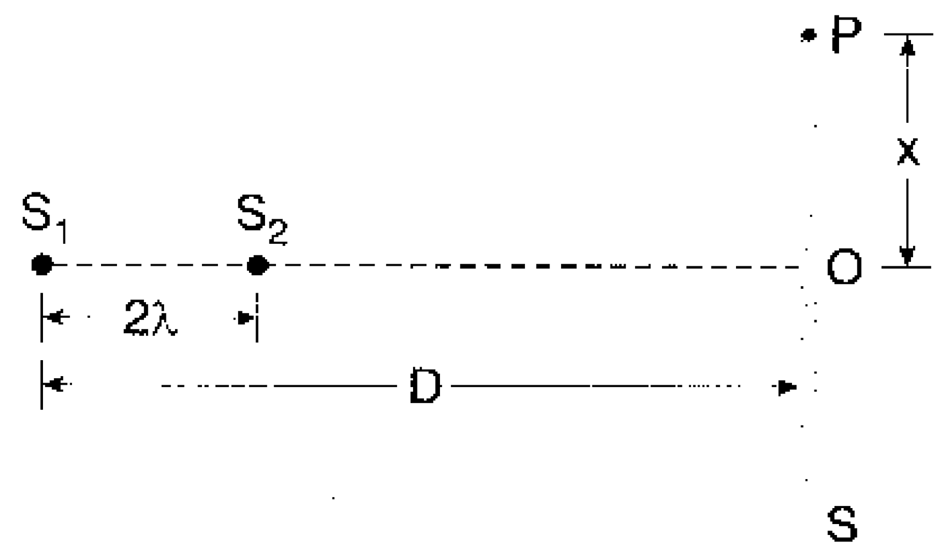


Fig. 23.33

SOLUTION Path difference at O, $S_1O - S_2O = 2\lambda$

i.e., maximum intensity is obtained at O. Next maxima will be obtained at point P where,

$$S_1P - S_2P = \lambda$$

or $d \cos\theta = \lambda$

or $(2\lambda)\cos\theta = \lambda$

or $\cos\theta = \frac{1}{2}$

$\therefore \quad \theta = 60°$

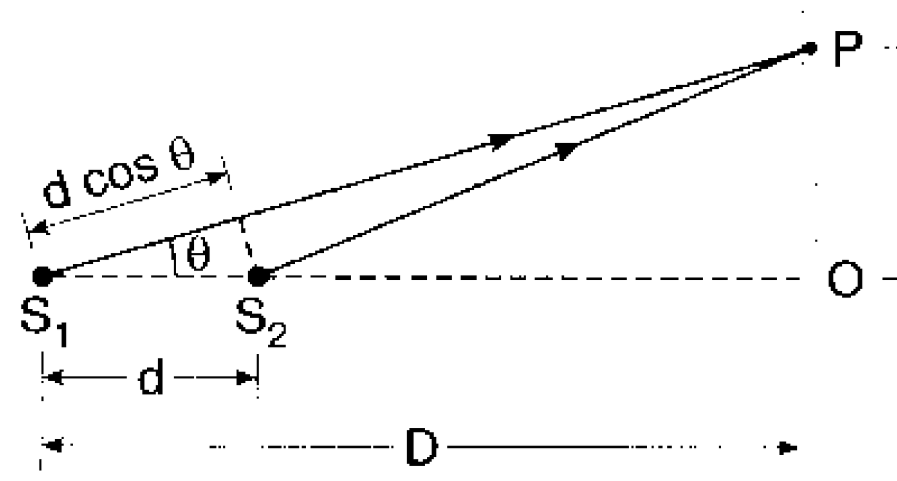


Fig. 23.34

Now in ΔS_1PO, $\frac{PO}{S_1O} = \tan\theta$ or $\frac{x}{D} = \tan 60° = \sqrt{3}$

$\therefore$ $x = \sqrt{3}D$ **Ans.**

Note : At point O, path difference is 2λ, *i.e.*, we obtain second order maxima. At point P, where path difference is λ (*i.e.*, $x = \sqrt{3}D$) we get first order maxima. The next, *i.e.*, zero order maxima will be obtained where path difference, *i.e.*, $d\cos\theta = 0$ or $\theta = 90°$. At $\theta = 90°$, $x = \infty$. So, our answer, *i.e.*, finite distance of x should be $x = \sqrt{3}D$, corresponding to first order maxima.

EXAMPLE 12 *An interference is observed due to two coherent sources S_1 placed at origin and S_2 placed at* $(0,\ 3\lambda,\ 0)$*. Here λ is the wavelength of the sources. A detector D is moved along the positive x-axis. Find x-coordinates on the x-axis (excluding $x = 0$ and $x = \infty$) where maximum intensity is observed.*

SOLUTION At $x = 0$, path difference is 3λ. Hence, third order maxima will be obtained. At $x = \infty$, path difference is zero. Hence, zero order maxima is obtained. In between first and second order maximas will be obtained.

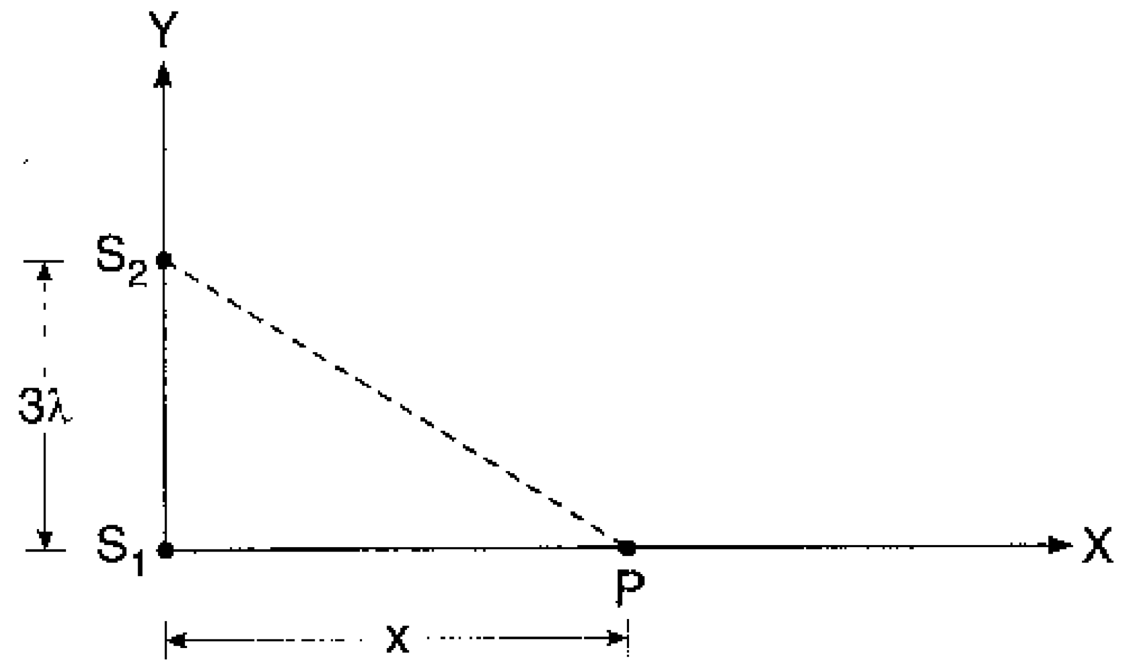


Fig. 23.35

First order maxima:

$$S_2P - S_1P = \lambda$$

or $$\sqrt{x^2 + 9\lambda^2} - x = \lambda$$

or $$\sqrt{x^2 + 9\lambda^2} = x + \lambda$$

Squaring both sides, we get $$x^2 + 9\lambda^2 = x^2 + \lambda^2 + 2x\lambda$$

Solving this, we get $$x = 4\lambda$$

Second order maxima: $$S_2P - S_1P = 2\lambda$$

or $$\sqrt{x^2 + 9\lambda^2} - x = 2\lambda$$

or $$\sqrt{x^2 + 9\lambda^2} = (x + 2\lambda)$$

Squaring both sides, we get

$$x^2 + 9\lambda^2 = x^2 + 4\lambda^2 + 4x\lambda$$

Solving, we get

$$x = \frac{5}{4}\lambda = 1.25\lambda$$

Hence, the desired x coordinates are,

$$x = 1.25\lambda \quad \text{and} \quad x = 4\lambda$$ **Ans.**

Note : (i) As we move on positive x-axis (from origin) order of maxima decreases from $n = 3$ to $n = 0$

(ii) Here we can not take the path difference $d\cos\theta$ or $d\sin\theta$. Think why?

EXAMPLE 13 *A light wave of wavelength 500 nm falls upon three slits a distance 0.5 mm each from one another. A screen is placed at a distance 2 m from slits. Find*

(*a*) *the distances from P where intensity reduces to zero*

(*b*) *the distances from P where next bright fringe are observed*

(*c*) *the ratio of intensities of bright fringes observed on the screen.*

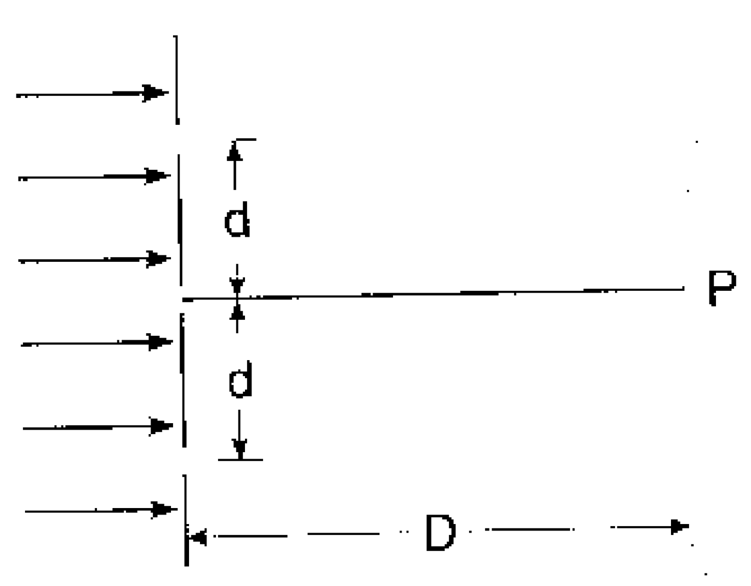


Fig. 23.36

SOLUTION (a) In case of three slits, intensity becomes zero, when phase difference between any two waves is,

$$\phi = \frac{2\pi}{3} + 2n\pi \qquad n = 0, 1, 2, \ldots$$

The corresponding path difference,

$$\Delta x = \left(\frac{\lambda}{2\pi}\right)\phi$$

or

$$d\sin\theta = \left(\frac{\lambda}{2\pi}\right)\left(\frac{2\pi}{3} + 2\pi n\right) = n\lambda + \frac{\lambda}{3}$$

or

$$\sin\theta = \frac{\lambda}{d}\left(n + \frac{1}{3}\right)$$

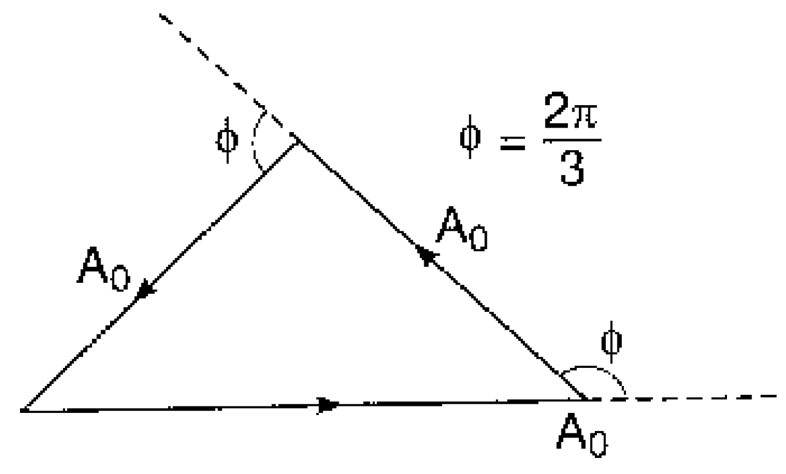


Fig. 23.37

For small angles,

$$\sin\theta \approx \tan\theta = \frac{y}{D}$$

∴

$$\frac{y}{D} = \frac{\lambda}{d}\left(n + \frac{1}{3}\right)$$

or

$$y = \frac{\lambda D}{d}\left(n + \frac{1}{3}\right)$$

Substituting the values, we have

$$y = \frac{500 \times 10^{-9} \times 2}{0.5 \times 10^{-3}}\left(n + \frac{1}{3}\right) = 2 \times 10^{-3}\left(n + \frac{1}{3}\right) \text{m}$$

$$= 2\left(n + \frac{1}{3}\right) \text{mm}$$

Ans.

Here, $n = 0, 1, 2, \ldots$

For example, $y = \frac{2}{3}$ mm for $n = 0$, $y = \frac{8}{3}$ mm for $n = 1$ etc.

(b) Bright fringes are obtained on the screen where

(i) $\phi = 2n\pi$, $n = 1, 2, 3, \ldots$

or

$$\Delta x = \left(\frac{\lambda}{2\pi}\right)(\phi) = n\lambda \quad \text{or} \quad d\sin\theta = n\lambda$$

$\therefore \quad \sin\theta = \dfrac{n\lambda}{d}$

For small angles, $\quad \sin\theta \approx \tan\theta = \dfrac{y}{D}$

$\therefore \quad \dfrac{y}{D} = \dfrac{n\lambda}{d}$

or $\quad y = \dfrac{n\lambda D}{d} = \dfrac{n(500\times10^{-9})(2)}{(0.5\times10^{-3})}$

$= 2n\times10^{-3}$ m

$= (2n)$ mm $\quad$ where $n = 1, 2, 3, \ldots$ etc.

These are called **primary** maximas.

(ii) $\phi = (2n+1)\pi, \quad n = 1, 2, \ldots$

Proceeding in the similar manner, we have

$$y = 2\left(n + \frac{1}{2}\right) \text{mm} \qquad \text{where } n = 1, 2, 3, \ldots$$

These are called **secondary** maximas.

Note that $y = 0$ is also a secondary maxima because at P, $\phi = \pi$.

(c) At principal; maximas ($\phi = 2\pi, 4\pi, \ldots$, etc.)

Fig. 23.38

Resultant amplitude $= 3A_0$

$\therefore \quad I = 9I_0 \quad (\text{as } I \propto A^2)$

While at secondary maximas, ($\phi = \pi, 3\pi, 5\pi \ldots$)

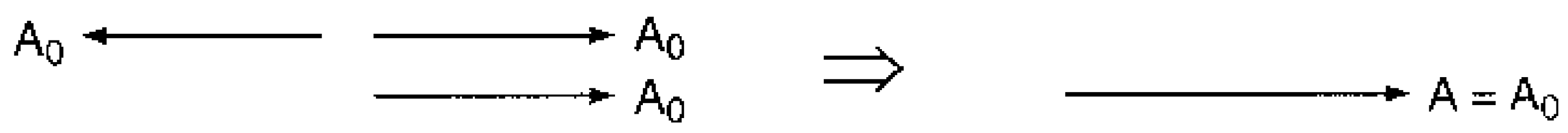


Fig. 23.39

Resultant amplitude, $A = A_0 \quad \therefore \quad I = I_0$

$\therefore$ The desired ratio is therefore, 9 : 1. **Ans.**

7. S_1 and S_2 are two point sources of radiation that are radiating waves in phase with each other of wavelength 400 nm. The sources are located on x-axis at $x = 6.5$ μm and $x = -6.0$ μm, respectively. (a) Determine the phase difference (in radians) at the origin between the radiation from S_1 and the radiation from S_2. (b) Suppose a slab of transparent material with thickness 1.5 μm and index of refraction $\mu = 1.5$ is placed between $x = 0$ and $x = 1.5$ μm. What then is the phase difference (in radians) at the origin between the radiation from S_1 and the radiation from S_2?

8. A parallel beam of white light falls on a thin film whose refractive index is equal to 4/3. The angle of incidence $i = 53°$. What must be the minimum film thickness if the reflected light is to be coloured yellow (λ of yellow $= 0.6$ μm) most intensively? ($\tan 53° = 4/3$)

9. A thick glass slab ($\mu = 1.5$) is to be viewed in reflected white light. It is proposed to coat the slab with a thin layer of a material having refractive index 1.3 so that the wavelength 6000 Å is suppressed. Find the minimum thickness of the coating required.

10. In a Young's double slit experiment using monochromatic light the fringe pattern shifts by a certain distance on the screen when a mica sheet of refractive index 1.6 and thickness 1.964 microns is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the slits and the screen is doubled. It is found that the distance between successive maxima (or minima) now is the same as the observed fringe shift upon the introduction of the mica sheet. Calculate the wavelength of the monochromatic light used in the experiment.

11. In a double slit pattern ($\lambda = 6000$ Å), the first order and tenth order maxima fall at 12.50 mm and 14.75 mm from a particular reference point. If λ is changed to 5500 Å, find the position of zero order and tenth order fringes, other arrangements remaining the same.

12. A narrow stream of 100 eV electrons is fired at two parallel slits very close to each other. The distance between the slits is 10 Å. The electron waves after passing through the slits interfere on a screen, 3 metres away from slits and form interference fringes. Find the width of the fringe.

13. Two light rays, initially in phase and having wavelength 6.00×10^{-7} m, go through different plastic layers of the same thickness, 7.00×10^{-6} m. The indices of refraction are 1.65 for one layer and 1.49 for the other.

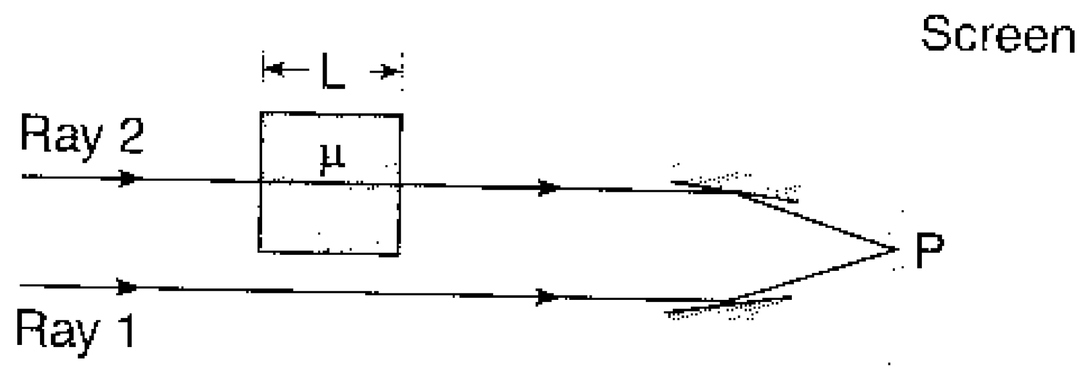
(a) What is the equivalent phase difference between the rays when they emerge?

(b) If those two rays then reach a common point, does the interference result in complete darkness, maximum brightness, intermediate illumination but closer to complete darkness, or intermediate illumination but closer to maximum brightness?

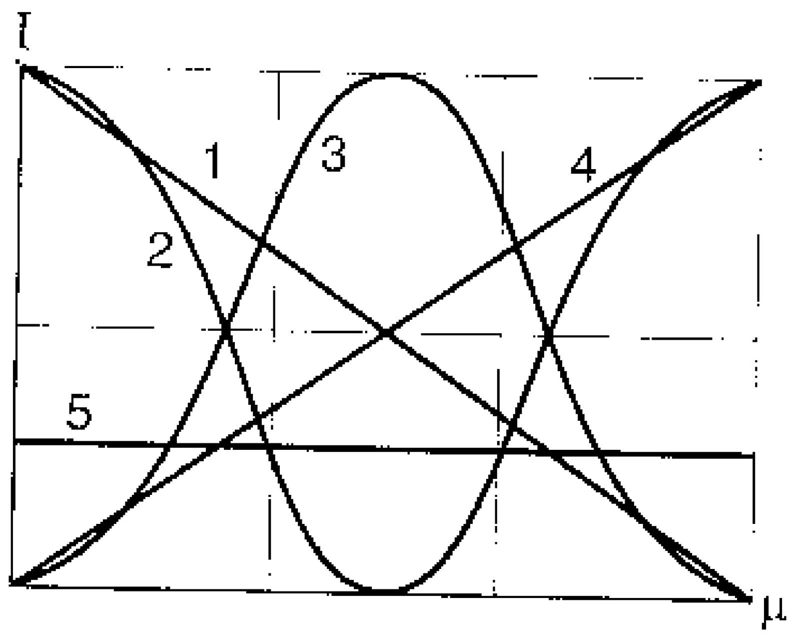
(c) If the two rays are, instead, initially exactly out of phase, what are the answers to (a) and (b)?

14. In figure (a), the waves along rays 1 and 2 are initially in phase. Ray 2 goes through a material with length L and index of refraction μ. The rays are then reflected by mirrors to a common point P on a screen

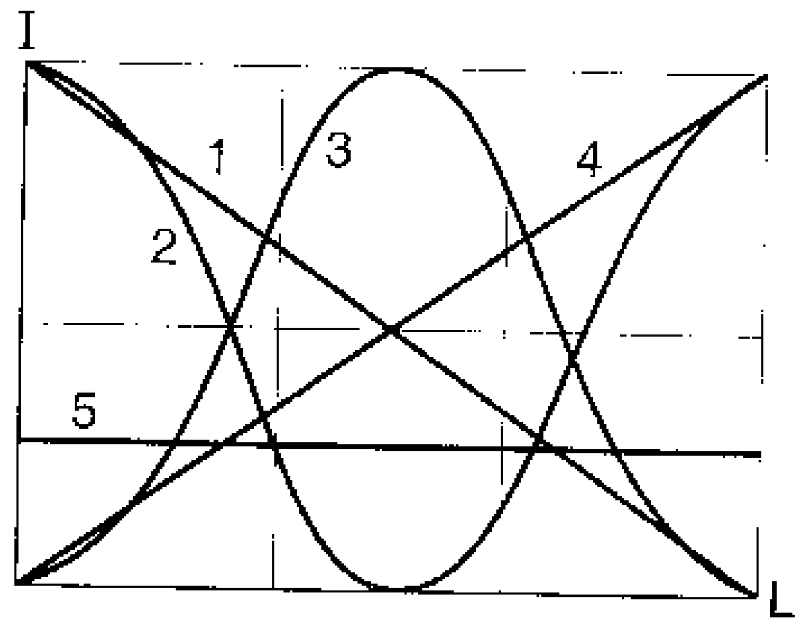
(a) Suppose that we can vary μ from an initial value matching that of air to larger values. Which of the curves in figure (b) would then best give the intensity I of the light at point P versus μ?



(a)

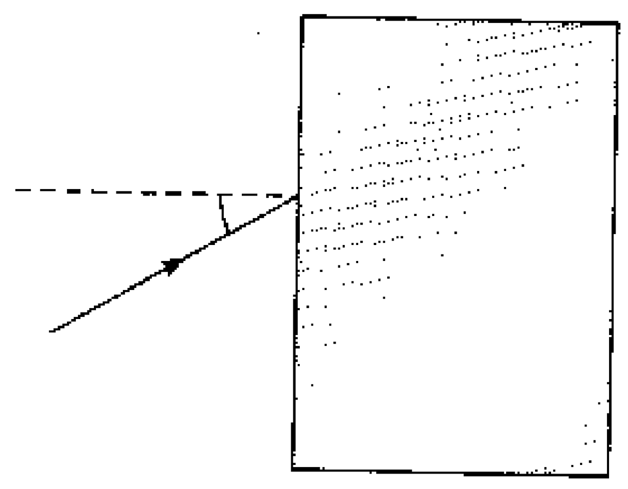


(b)



(c)

(b) Suppose, instead, that we can vary L from an initial value of zero to larger values. Then which of the curves in figure (c) would best give the intensity I of the light at point P versus L?



15. A ray of light is incident on the left vertical face of the glass slab. If the incident light has an intensity I and on each reflection the intensity decreases by 90% and on each refraction the intensity decreases by 10%, find the ratio of the intensities of maximum to minimum in the reflected pattern.

LEVEL-II

1. Two point sources are $d = n\lambda$ apart. A screen is held at right angles to the line joining the two sources at a distance D from the nearest source. Calculate the distance of the point on the screen, where the first bright fringe is observed. (excluding the centre one)Assume $D >> d$.

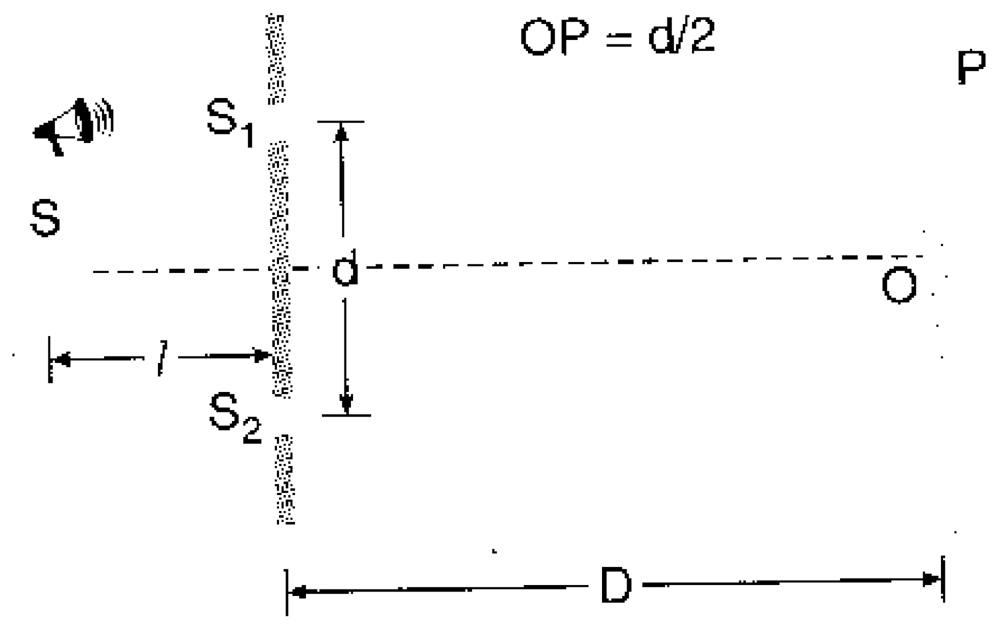
2. A convergent lens with a focal length of $f = 10$ cm is cut into two halves that are then moved apart to a distance of $d = 0.5$ mm (a double lens). Find the fringe width on screen at a distance of 60 cm behind the lens if a point source of monochromatic light ($\lambda = 5{,}000$ Å) is placed in front of the lens at a distance of $a = 15$ cm from it.

3. Two coherent radio point sources that are separated by 2.0 m are radiating in phase with a wavelength of 0.25 m. If a detector moves in a large circle around their midpoint, at how many points will the detector show a maximum signal?

4. A source S of wavelength λ is kept directly behind the slit S_1 in a double-slit apparatus. Find the phase difference at a point O which is equidistant from S_1 and S_2. What will be the phase difference at P if a liquid of refraction index μ is filled: ($D >> d$)

(a) between the screen and the slits?

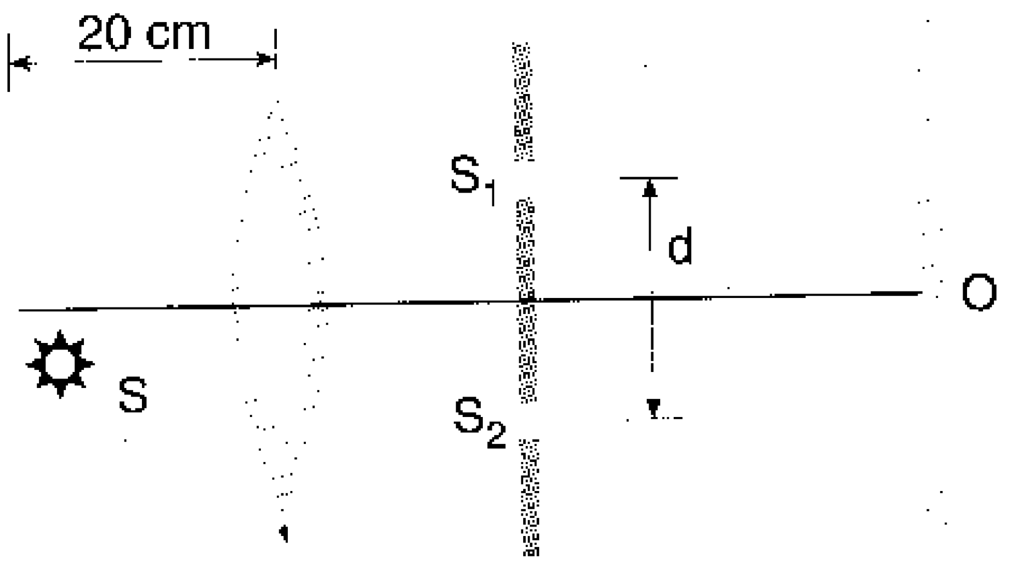
(b) between the slit and the source S?



5. A central portion with a width of $d = 0.5$ mm is cut out of a convergent lens having a focal length of $f = 10$ cm, as shown in figure. Both halves are tightly fitted against each other. The lens receives monochromatic light ($\lambda = 5000$ Å) from a point source at a distance of 5 cm from it. At what distance should a screen be fixed on the opposite side of the lens to observe three interference bands on it?

What is the maximum possible number of interference bands that can be observed in this installation?

6. A point source is placed at a distance $d/2$ below the principal axis of an equiconvex lens of refractive index $\frac{3}{2}$ and radius 20 cm. The emergent light from lens fall on the slits S_1 and S_2 placed symmetrically with the principal axis. The resulting interference pattern is observed on the screen kept at a distance $D = 1$ m from the slit plane. Find ($d = 1$ mm)



(a) the position of central maxima and its width

(b) the intensity at point O.

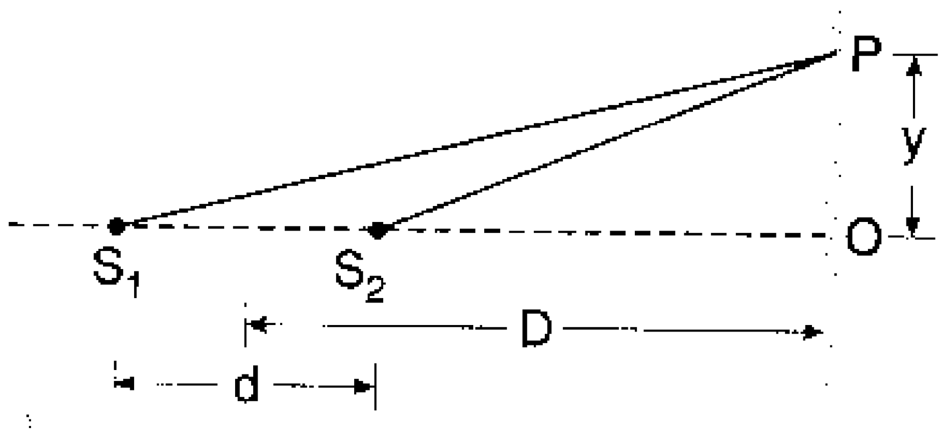
Given, $\lambda = 5000$ Å

7. A large opaque sheet placed parallel to the yz plane at $x = 0.03$ m. The region $x \geq 0$ is filled with a transparent liquid of refractive index 3/2. A wide monochromatic beam of light of wavelength 900 nm falls on the yz-plane at $x = 0$ making an angle of 30° with the x-axis. The sheet has two slits parallel to z-axis at $y = \pm 0.9$ mm. The intensity of the wave is measured on a screen placed at $x = 1.03$ m parallel to the sheet.

(a) Find the intensity at a point 'P' on the screen where $y = z = 0$.

(b) The lower slit is covered by a transparent strip of refractive index 1.4 and thickness 4.2 mm. Now find the intensity at point 'P'.

8. In the figure shown, a screen is placed normal to the line joining the two point coherent sources S_1 and S_2. The interference pattern consists of concentric circles.

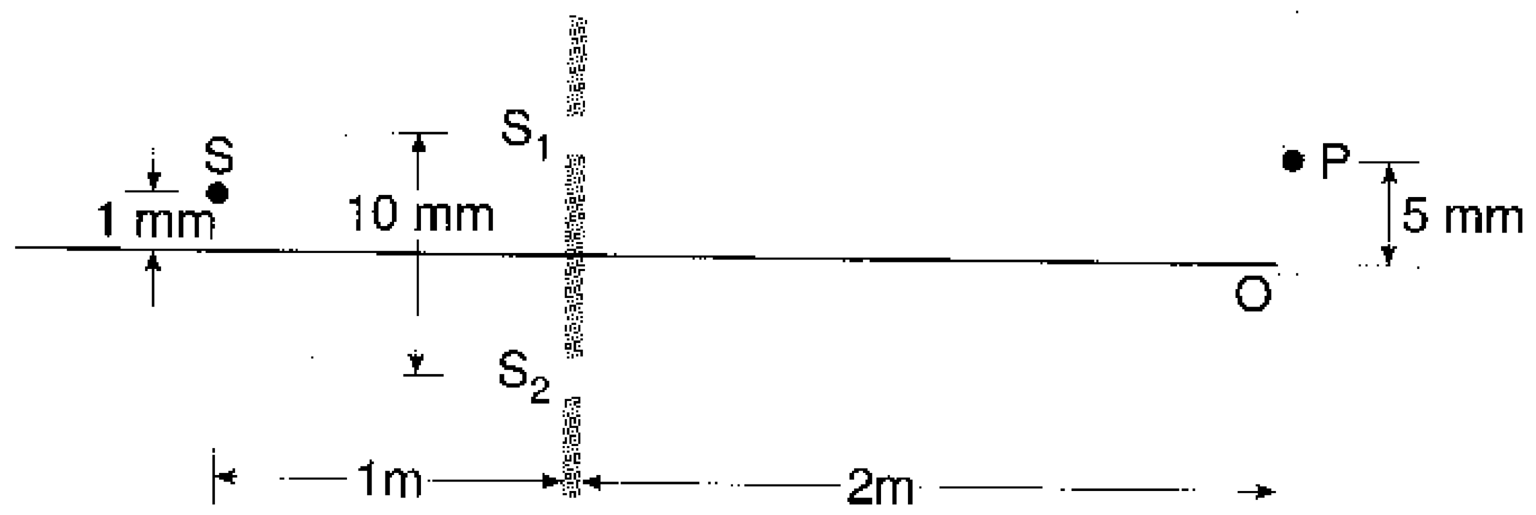


(a) Find the radius of the nth bright ring.

(b) If $d = 0.5$ mm, $\lambda = 5000$ Å and $D = 100$ cm, find the radius of the closest second bright ring.

(c) Also, find the value of n for this ring.

9. In the Young's Double Slit experiment a point source of $\lambda = 5000$ Å is placed slightly off the central axis as shown in the figure.

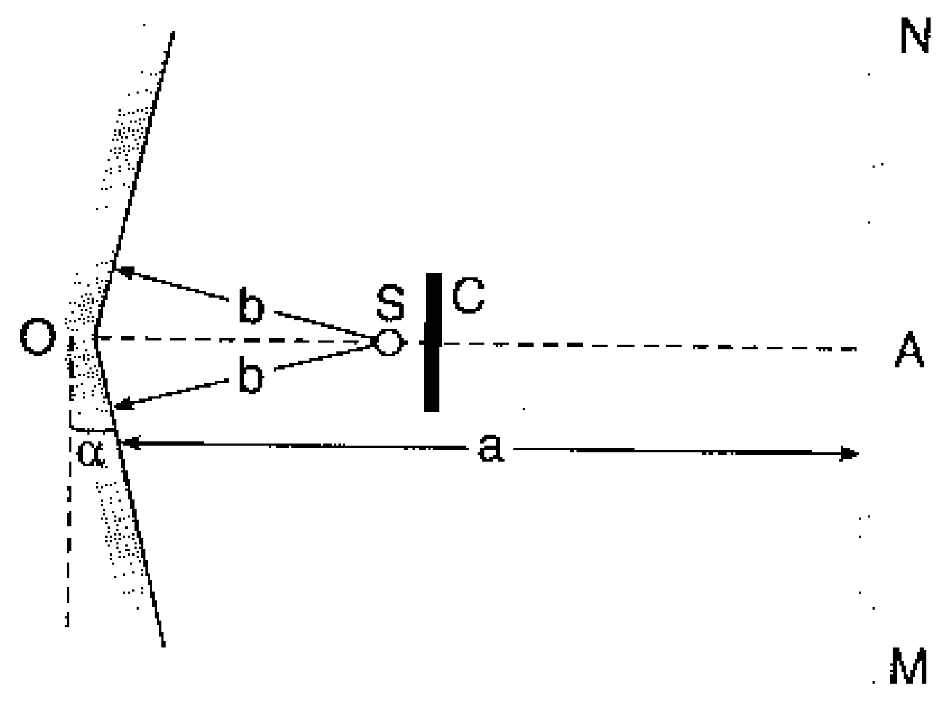


(a) Find the nature and order of the interference at the point P.

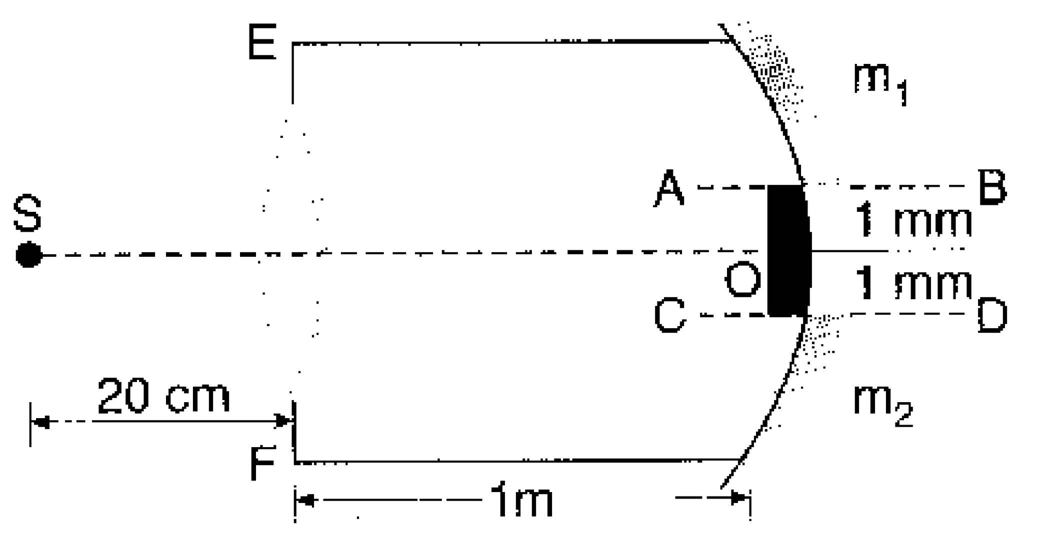
(b) Find the nature and order of the interference at O.

(c) Where should we place a film of refractive index $\mu = 1.5$ and what should be its thickness so that a maxima of zero order is placed at O.

10. Two flat mirrors form an angle close to 180°. A source of light S is placed at equal distances b from the mirrors. Find the interval between adjacent interference bands on screen MN at a distance $OA = a$ from the point of intersection of the mirror. The wavelength of the light wave is known and equal to λ. Shield C does not allow the light to pass directly from the source to the screen.

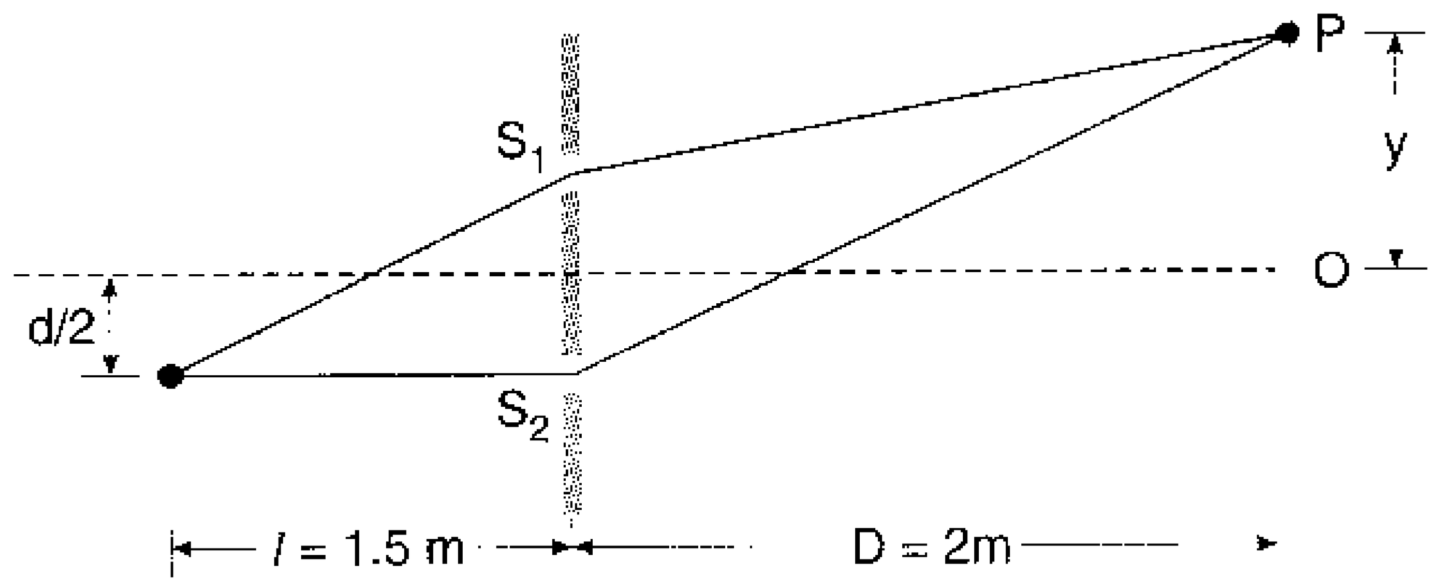


11. An equiconvex lens of focal length 10 cm (in air) and refractive index 3/2 is put at a small opening on a tube of length 1 m fully filled with liquid of refractive index 4/3. A concave mirror of radius of curvature 20 cm is cut into two halves m_1 and m_2 and placed at the end of the tube. m_1 and m_2 are placed such that their principal axes AB and CD respectively are separated by 1 mm each from the principle axes of the lens. A slit S placed in air illuminates the lens with light of frequency 7.5×10^{14} Hz. The light reflected from m_1 and m_2 forms interference pattern on the left end EF of the tube. O is an opaque substance to cover the hole left by m_1 and m_2. Find:



(a) The position of the image formed by lens water combination.
(b) The distance between the images formed by m_1 and m_2.
(c) Width of the fringes on EF.

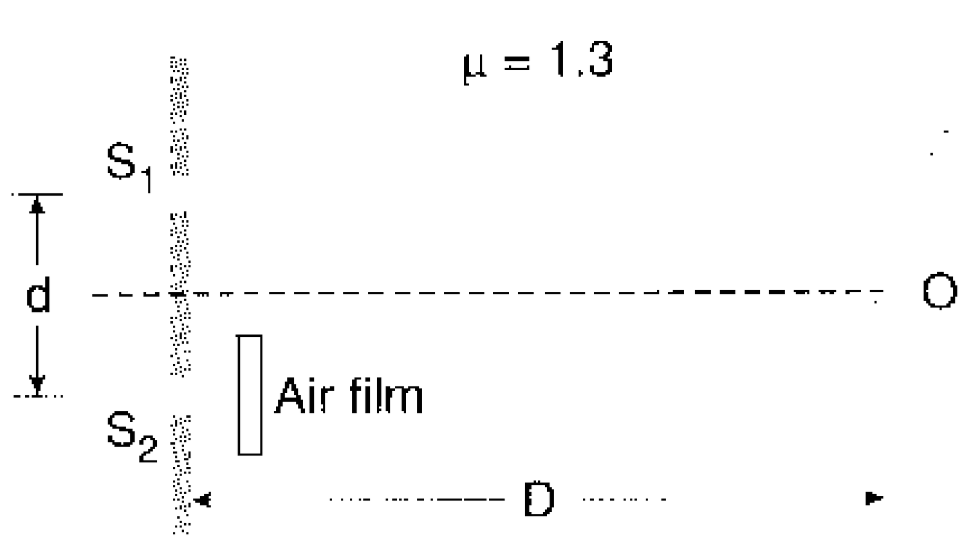
12. In the $YDSE$ the monochromatic source of wavelength λ is placed at a distance $\frac{d}{2}$ from the central axis (as shown in the figure), where d is the separation between the two slits S_1 and S_2.



(a) Find the position of the central maxima
(b) Find the order of interference formed at O.
(c) Find the minimum thickness of the film of refractive index $\mu = 1.5$ to be placed infront of S_2 so that intensity at O becomes $\frac{3}{4}$th of the maximum intensity.

Take $\lambda = 6000$ Å; $d = 6$ mm.

13. An $YDSE$ is carried out in a liquid of refractive index $\mu = 1.3$ and a thin film of air is formed in front of the lower slit as shown in the figure. If a maxima of third order is formed at the origin O, find the thickness of the air film. Find the positions of the *fourth* maxima. The wavelength of light in air is $\lambda_0 = 0.78\ \mu m$ and $D/d = 1000$.



ANSWERS

Introductory Exercise 23.1

1. Because they are incoherent **2.** Because they are highly coherent **3.** 48.0 μm **4.** 0.500 cm
6. Bright and dark fringes interchange positions **7.** 6 **8.** 45°

Assignment

Level-I

1. 0.546 mm **2.** 0.8 cm, 6.67×10^{-7} m **3.** 4.5 mm **4.** (a) $I_P = 3.0 \times 10^{-4} I_0$ (b) 5.49

5. 3 mm **6.** 1.2 μm **7.** (a) $\frac{5\pi}{2}$, (b) 6.25π **8.** 0.14 μm **9.** 1154 Å **10.** 589 nm

11. 12.25 mm, 14.55 mm **12.** 36.6 cm

13. (a) 11.72 radian, (b) Intermediate illumination closer to maximum brightness, (c) 14.86 radian or 8.56 radian, intermediate illumination but closer to darkness

14. (a) Curve 2, (b) Curve 2 **15.** 361

Level-II

1. $\sqrt{\frac{2D(D+n\lambda)}{n}}$ **2.** 0.1 mm **3.** 32

4. $\frac{2\pi}{\lambda}(\sqrt{d^2+l^2}-l)$ (a) $\frac{2\pi}{\lambda}\left[(\sqrt{d^2+l^2}-l)+\frac{\mu d^2}{2D}\right]$ (b) $\frac{2\pi}{\lambda}\left[\mu(\sqrt{d^2+l^2}-l)+\frac{d^2}{2D}\right]$

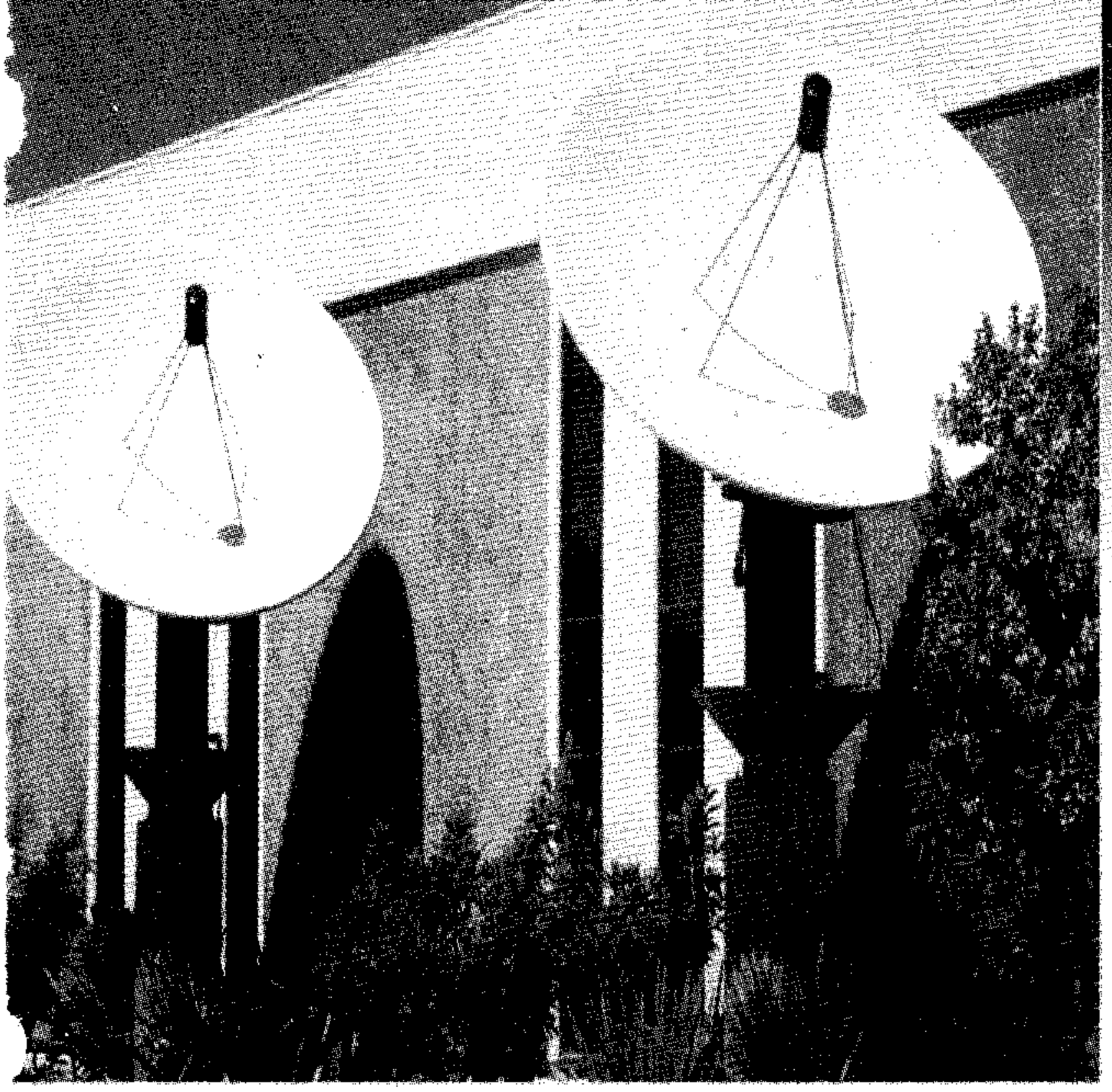
5. 15 cm **6.** (a) 2.5 mm above 0, (b) $I = I_{max}$ **7.** (a) $I = I_{max}$, (b) Again $I = I_{max}$

8. (a) $D\sqrt{2\left(1-\frac{n\lambda}{d}\right)}$, (b) 6.32 cm, (c) 998

9. (a) 70th order maxima (b) 20th order maxima (c) $t = 20$ μm, infront of S_1.

10. $\frac{\lambda(a+b)}{2b\alpha}$ **11.** (a) 80 cm from lens, (b) 4 mm (c) 60 μm **12.** (a) 4 mm above 0 (b) 20 (c) 2000 Å

13. (a) 7.8 μm, (b) 4.2 mm, –0.6 mm



CHAPTER 24

Modern Physics-I

Chapter Contents

24.1 DUAL NATURE OF ELECTROMAGNETIC WAVES

Classical physics treats particles and waves as separate components. The mechanics of particles and the optics of waves are traditionally independent disciplines, each with its own chain of principles based on their results. We regard electrons as particles because they possess charge and mass and behave according to the laws of particle mechanics in such familiar devices as television picture tubes. We shall see, however, that it is just as correct to interpret a moving electron as a wave manifestation as it is to interpret it as a particle manifestation. We regard electromagnetic waves as waves because under suitable circumstances they exhibit diffraction, interference and polarization. Similarly we shall see that under other circumstances they behave as a streams of particles. Rather we can say they have the **dual nature.**

The wave nature of light (a part of electromagnetic waves) was first demonstrated by Thomas Young, who observed the interference pattern of two coherent sources. The particle nature of light was first proposed by Albert Einstein in 1905 in his explanation of the photoelectric effect. A particle of light called a **photon** has energy E that is related to the frequency f and wavelength λ of light wave by the Einstein equation,

$$E = hf = \frac{hc}{\lambda} \qquad \text{...(i)}$$

where c is the speed of light (in vacuum) and h is Planck's constant.

$$h = 6.626 \times 10^{-34} \text{ J-s}$$
$$= 4.136 \times 10^{-15} \text{ eV-s}$$

Since, energies are often given in electron volt $(1 \text{ eV} = 1.6 \times 10^{-19} \text{ J})$ and wavelengths are in Å, it is convenient to the combination hc in eV-Å. We have,

$$hc = 12375 \text{ eV-Å}$$

Hence, Eq. (i), in simpler form can be written as,

$$E \text{ (in eV)} = \frac{12375}{\lambda \text{ (in Å)}} \qquad \text{...(ii)}$$

The propagation of light is governed by its wave properties, whereas the exchange of energy between light with matter is governed by its particle properties. The wave-particle duality is a general property of nature. For example, electrons (and other so called particles) also propagate as waves and exchange energy as particles.

24.2 ELECTROMAGNETIC SPECTRUM

The basic source of electromagnetic wave is an accelerated charge. This produces the changing electric and magnetic fields which constitute an electromagnetic wave. An electromagnetic wave may have its wavelength varying from zero to infinity. Not all of them are known till date. Today we are familiar with electromagnetic waves having wavelengths as small as 30 fm $(1 \text{ fm} = 10^{-15} \text{ m})$ to as large as 30 km. The boundaries separating different regions of spectrum are not sharply defined, with the exception of the visible part of the spectrum. The visible part of the electromagnetic spectrum covers from 4000 Å to 7000 Å. An approximate range of wavelengths is associated with each colour : violet (4000 Å- 4500 Å), blue (4500 Å–5200 Å), green (5200 Å–5600 Å), yellow (5600 Å–6000 Å), orange (6000 Å–6250 Å) and red (6250 Å–7000 Å).

Figure shows the spectrum of electromagnetic waves. The classification are based roughly on how the waves are produced and/or detected.

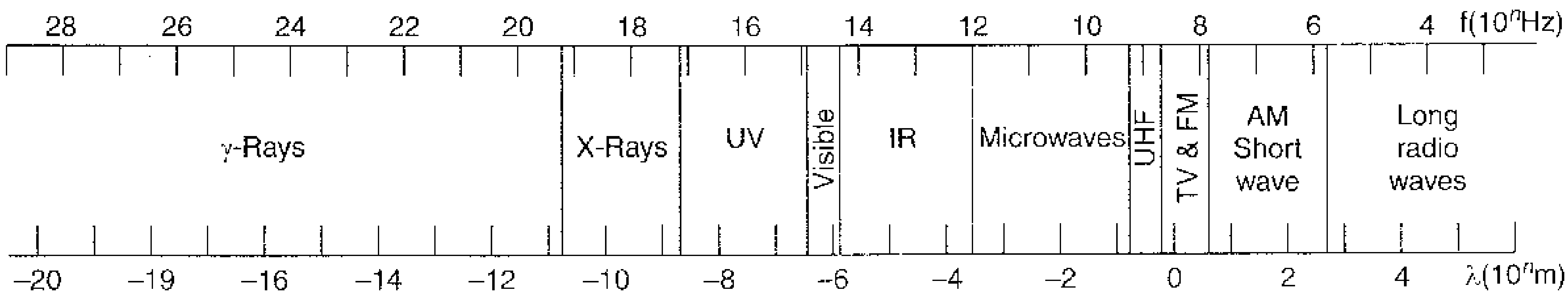


Fig. 24.1 The electromagnetic spectrum.

γ-Rays : These were identified by P. Villiard in 1900. These are usually produced within the nucleus of an atom and extremely energetic by atomic standards. They cover the range from 0.1 Å down, or, equivalently, from 10^{20} Hz up.

***X*-Rays :** *X*-rays, discovered in 1895 by W. Roentgen extend from 100 Å to 0.1 Å. These are produced by the rapid deceleration of electrons that bombard a heavy metal target. These are also produced by electronic transitions between the energy levels in an atom. *X*-rays are used to study the atomic structure of crystals or molecules such as DNA. Besides their diagnostic and therapeutic use in medicine they have become an important tool in studying the universe.

Ultraviolet radiation : Ultraviolet (UV) rays were first discovered by J.W. Ritter in 1801. The ultraviolet region extends from 4000 Å to 100 Å. It plays a role in the production of vitamin D in our skins. But prolonged doses of UV radiation can induce cancers in humans. Glass absorbs UV radiation and hence, can provide some protection against the sun's rays. If the ozone in our atmosphere did not absorb the UV below 3000 Å, there would be a large number of cell mutations, especially cancerous ones, in humans. For this reason, the depletion of the ozone in our atmosphere by chlorofluorocarbons (CFCs) is now a matter of international concern.

Visible light : A lot of discussion has already been done on visible light in previous two chapters. As electrons undergo transitions between energy levels in an atom, light is produced at well defined wavelengths. Light covering a continuous range of wavelengths is produced by the random acceleration of electrons in hot bodies. Our sense of vision and the process of photosynthesis in plants have evolved within the range of those wavelengths of sunlight that our atmosphere does not absorb.

Infrared radiation : The infrared region (IR) starts at 7000 Å and extends to about 1 mm. It was discovered in 1800 by M. Herchel. It is associated with the vibration and rotation of molecules and is perceived by us as heat. IR is used in the early detection of tumors.

Microwaves : Microwaves cover wavelengths from 1 mm to about 15 cm. Microwaves upto about 30 GHz (1 cm) may be generated by the oscillations of electrons in a device called klystron. Microwave ovens are used in kitchens. Modern intercity communications such as phone conversations and TV programs are often carried via a cross country network of microwave antennas.

Radio and TV signals : Radiowaves are generated when charges are accelerating through conducting wires. Their wavelengths lie in the range 10^{14} m to 10 cm. They are generated by *LC* oscillators and are used in radio and television communication systems.

24.3 MOMENTUM AND RADIATION PRESSURE

An electromagnetic wave transports linear momentum. We state, without proof that the linear momentum carried by an electromagnetic wave is related to the energy it transports according to,

$$p=\frac{E}{c} \quad \ldots(i)$$

If the wave is incident in the direction perpendicular to a surface and is completely absorbed then Eq. (i), tells us the linear momentum imparted to the surface. If surface is perfectly reflecting, the momentum change of the wave is doubled. Consequently the momentum imparted to the surface is also doubled.

According to Newton's second law the force exerted by an electromagnetic wave on a surface may be related by the equation,

$$F=\frac{\Delta p}{\Delta t}$$

From Eq. (i),

$$\frac{\Delta p}{\Delta t}=\frac{1}{c}\left(\frac{\Delta E}{\Delta t}\right)$$

$\therefore$

$$F=\frac{1}{c}\left(\frac{\Delta E}{\Delta t}\right) \quad \ldots(ii)$$

Intensity (I) of a wave is the energy transported per unit area per unit time.

or

$$I=\left(\frac{1}{S}\right)\frac{\Delta E}{\Delta t}$$

$\therefore$

$$\frac{\Delta E}{\Delta t}=I\cdot S$$

Substituting in Eq. (ii),

$$F=\frac{IS}{c}$$

or

$$\frac{F}{S}=\text{pressure}=\frac{I}{c}$$

or

$$P_{\text{rad}}=\frac{I}{c}$$

$\frac{I}{c}$ is also equal to the energy density (energy per unit volume) u.

Hence,

$$P_{\text{rad}}=u \quad \ldots(iii)$$

The radiation pressure is thus equal to the energy density ($N/m^2 = J/m^3$). At a perfectly reflecting surface the pressure on the surface is doubled. Thus, we can write,

$$P_{\text{rad}}=\frac{I}{c}=u \quad \text{(wave totally absorbed)}$$

and

$$P_{\text{rad}}=\frac{2I}{c}=2u \quad \text{(wave totally reflected)}$$

EXAMPLE 24.1 *The intensity of direct sunlight before it passes through the earth's atmosphere is* 1.4 kW/m^2. *If it is completely absorbed find the corresponding radiation pressure.*

SOLUTION For completely absorbing surface,

$$P_{\text{rad}}=\frac{I}{c}=\frac{1.4\times10^3}{3.0\times10^8}=4.7\times10^{-6}\ \text{Nm}^{-2}$$ **Ans.**

24.4 de-BROGLIE WAVELENGTH OF MATTER WAVE

The wave-particle nature of electromagnetic waves discussed in article 24.1, led de-Broglie (pronounced de Broy) to suggest that matter might also exhibit this duality and have wave properties. His ideas can be expressed quantitatively by first considering electromagnetic radiation. A photon of frequency f and wavelength λ has energy.

$$E = hf = \frac{hc}{\lambda}$$

By Einstein's energy mass relation, $E = mc^2$ the equivalent mass m of the photon is given by,

$$m = \frac{E}{c^2} = \frac{hf}{c^2} = \frac{h}{\lambda c} \quad \text{...(i)}$$

or

$$\lambda = \frac{h}{mc} \quad \text{or} \quad \frac{h}{p} \quad \text{...(ii)}$$

Here p is the momentum of photon. By analogy de-Broglie suggested that a particle of mass m moving with speed v behaves in some ways like waves of wavelength λ given by,

$$\lambda = \frac{h}{mv} = \frac{h}{p} \quad \text{...(iii)}$$

where p is the momentum of the particle. Momentum is related to the kinetic energy by the equation,

$$p = \sqrt{2Km}$$

and a charge q when accelerated by a potential difference V gains a kinetic energy $K = qV$. Combining all these relations Eq. (iii), can be written as,

$$\lambda = \frac{h}{mv} = \frac{h}{p} = \frac{h}{\sqrt{2Km}} = \frac{h}{\sqrt{2qVm}} \quad \text{(de-Broglie wavelength)} \quad \text{...(iv)}$$

de-Broglie wavelength for an electron

If an electron (charge = e) is accelerated by a potential difference of V volts, it acquires a kinetic energy,

$$K = eV$$

Substituting the values of h, m and q in Eq. (iv), we get a simple formula for calculating de-Broglie wavelength of an electron. This is,

$$\lambda \text{ (in Å)} = \sqrt{\frac{150}{V \text{ (in volts)}}} \quad \text{...(v)}$$

EXAMPLE 24.2 *An electron is accelerated by a potential difference of 25 volt. Find the de-Broglie wavelength associated with it.*

SOLUTION For an electron, de-Broglie wavelength is given by,

$$\lambda = \sqrt{\frac{150}{V}} = \sqrt{\frac{150}{25}} = \sqrt{6}$$

$$\approx 2.5 \text{ Å}$$

Ans.

24.5 EARLY ATOMIC STRUCTURES

Every atom consists of a small nucleus of protons and neutrons with a number of electrons some distance away.

In the present article and in the next our chief concern will be the structure of the atom, since it is this structure that is responsible for nearly all the properties of matter. In nineteenth century many models were present by different scientists, but ultimately the first theory of the atom to meet with any success was put forward in 1913 by Neils Bohr. But before studying Bohr's model of atom let us have a look on other two models of the period one presented by J.J. Thomson in 1898 and the other by Ernest Rutherford in 1911.

J.J. Thomson suggested that atoms are just positively charged lumps of matter with electrons embedded in them like raisins in a fruit cake. Thomson's model called the 'plum pudding' model is illustrated in Fig. 24.2.

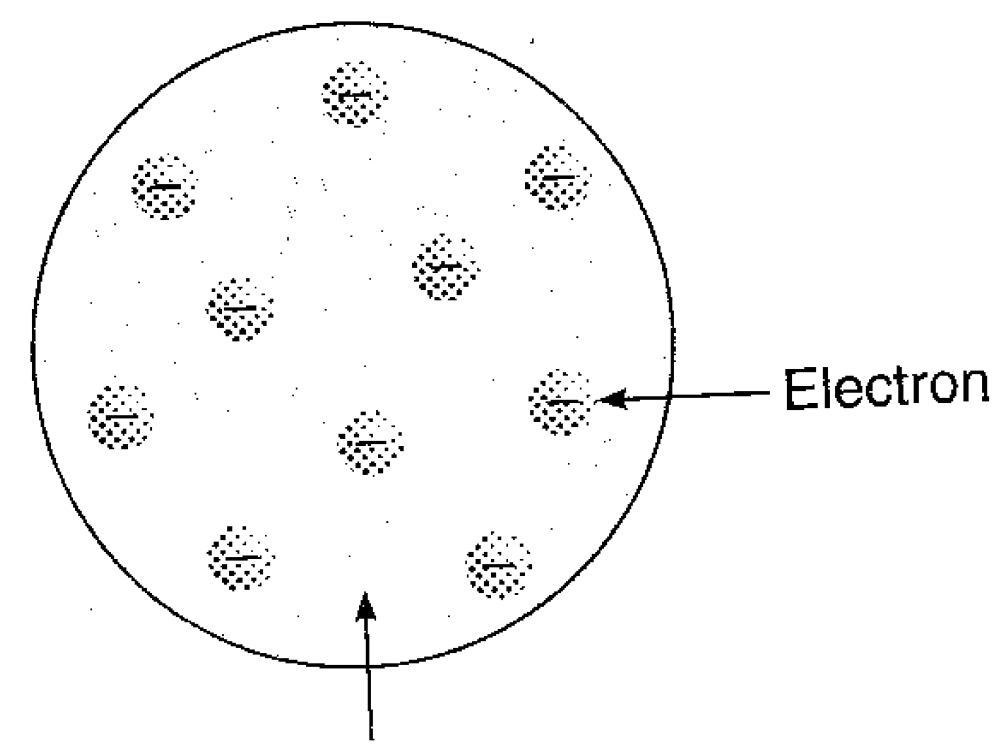


Fig. 24.2 The Thomson model of the atom. The Rutherford scattering experiment showed it to be incorrect.

Thomson had played an important role in discovering the electron, his idea was taken seriously. But the real atom turned out to be quite different.

Rutherford's nuclear atom

The nuclear atom is the basis of the modern theory of atomic structure and was proposed by Rutherford in 1911. He, with his two assistants Geiger and Marsden did an experiment in which they directed a narrow beam of α-particles onto gold foil about 1 μm thick and found that while most of the particles passed straight through, some were scattered appreciably and a very few-about 1 in 8000 suffered deflection of more than 90°.

To account for this very surprising result Rutherford suggested that : "All the positive charge and nearly all the mass were concentrated in a very small volume or nucleus at the centre of the atom. The electrons were supposed to move in circular orbits round the nucleus (like planets round the sun). The electrostatic attraction between the two opposite charges being the required centripetal force for such motion.

The large-angle scattering of α-particles would then be explained by the strong electrostatic repulsion from the nucleus.

Rutherford's model of the atom, although strongly supported by evidence for the nucleus, is inconsistent with classical physics. An electron moving in a circular orbit round a nucleus is accelerating and according to electromagnetic theory it should therefore, emit radiation continuously and thereby lose energy. If this happened the radius of the orbit would decrease and the electron would spiral into the nucleus in a fraction of second. But atoms do not collapse. In 1913 an effort was made by Neils Bohr to overcome this paradox.

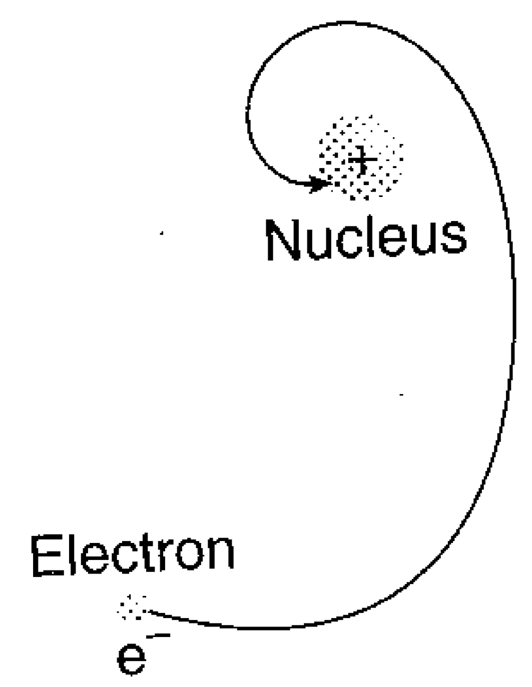


Fig. 24.3 An atomic electron should, classically, spiral rapidly into the nucleus as it radiates energy due to its acceleration.

24.6 THE BOHR HYDROGEN ATOM

After Neils Bohr obtained his doctorate in 1911, he worked under Rutherford for a while. In 1913, he presented a model of the hydrogen atom, which has one electron. He postulated that an electron moves only in certain circular orbits, called stationary orbits. In stationary orbits electron does not emit radiation, contrary to the predictions of classical electromagnetic theory. According to Bohr, there is a definite energy associated with each stable orbit and an atom radiaties energy only when it makes a transition from one of these orbits to another. The energy is radiated in the form of a photon with energy and frequency given by,

$$\Delta E = hf = E_i - E_f \qquad \text{...(i)}$$

Bohr found that the magnitude of the electron's angular momentum is quantized, and this magnitude for the electron must be integral multiple of $\frac{h}{2\pi}$. The magnitude of the angular momentum is $L = mvr$ for a particle with mass m moving with speed v in a circle of radius r. So, according to Bohr's postulate,

$$mvr = \frac{nh}{2\pi} \qquad (n = 1, 2, 3\ldots)$$

Each value of n corresponds to a permitted value of the orbit radius, which we will denote by r_n and the corresponding speed v_n. The value of n for each orbit is called **principal quantum number** for the orbit. Thus,

$$mv_n r_n = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

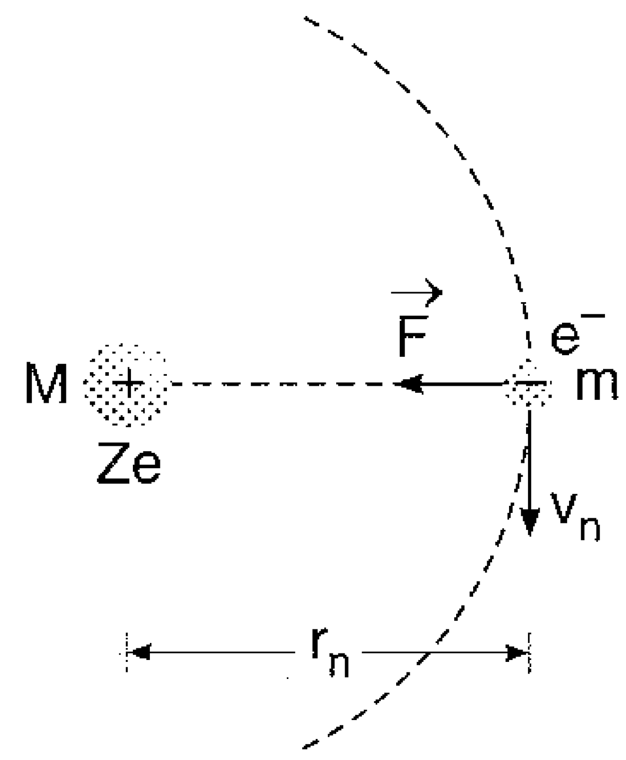


Fig. 24.4

According to Newton's second law a radially inward centripetal force of magnitude $F = \frac{mv^2}{r_n}$ is needed to the electron which is being provided by the electrical attraction between the positive proton and the negative electron.

Thus,
$$\frac{mv_n^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n^2} \qquad \text{...(iii)}$$

Solving Eqs. (ii) and (iii), we get

$$r_n = \frac{\varepsilon_0 n^2 h^2}{\pi m e^2} \quad \begin{pmatrix} n\text{th orbit radius} \\ \text{in Bohr model} \end{pmatrix} \qquad \text{...(iv)}$$

and
$$v_n = \frac{e^2}{2\varepsilon_0 nh} \quad \begin{pmatrix} n\text{th orbital speed} \\ \text{in Bohr model} \end{pmatrix} \qquad \text{...(v)}$$

The smallest orbit radius corresponds to $n = 1$. We'll denote this minimum radius, called the **Bohr radius** as a_0. Thus,

$$a_0 = \frac{\varepsilon_0 h^2}{\pi m e^2}$$

Substituting values of ε_0, h, π, m and e, we get

$$a_0 = 0.529 \times 10^{-10}\ \text{m} = 0.529\ \text{Å} \qquad \text{...(vi)}$$

Eq. (iv), in terms of a_0 can be written as,

$$r_n = n^2 a_0 \quad \text{or} \quad r_n \propto n^2 \qquad \text{...(vii)}$$

Similarly, substituting values of e, ε_0 and h with $n = 1$ in Eq. (v), we get

$$v_1 = 2.19 \times 10^6 \text{ m/s} \approx \frac{c}{137} \qquad \text{...(viii)}$$

This is the greatest possible speed of the electron in the hydrogen atom. Which is approximately equal to $c/137$ where c is the speed of light in vacuum.

Eq. (v), in terms of v_1 can be written as,

$$v_n = \frac{v_1}{n} \quad \text{or} \quad v_n \propto \frac{1}{n} \qquad \text{...(ix)}$$

Energy levels : Kinetic and potential energies K_n and U_n in n^{th} orbit are,

$$K_n = \frac{1}{2} m v_n^2 = \frac{me^4}{8\varepsilon_0^2 n^2 h^2}$$

and

$$U_n = -\frac{1}{4\pi\varepsilon_0} \frac{e^2}{r_n} = -\frac{me^4}{4\varepsilon_0^2 n^2 h^2}$$

The total energy E_n is the sum of the kinetic and potential energies.

$$E_n = K_n + U_n = -\frac{me^4}{8\varepsilon_0^2 n^2 h^2}$$

Substituting values of m, e, ε_0 and h with $n = 1$, we get the least energy of the atom in first orbit, which is -13.6 eV. Hence,

$$E_1 = -13.6 \text{ eV} \qquad \text{...(x)}$$

and

$$E_n = \frac{E_1}{n^2} = -\frac{13.6}{n^2} \text{ eV} \qquad \text{...(xi)}$$

Substituting $n = 2, 3, 4 \ldots$, etc., we get energies of atom in different orbits.

$$E_2 = -3.40 \text{ eV}, \; E_3 = -1.51 \text{ eV}, \ldots E_\infty = 0$$

Ionization energy of the hydrogen atom is the energy required to remove the electron completely. In ground state ($n = 1$) energy of atom is -13.6 eV and energy corresponding to $n = \infty$ is zero. Hence, energy required to remove the electron from ground state is 13.6 eV.

Emission spectrum of hydrogen atom

Under normal conditions the single electron in hydrogen atom stays in ground state ($n = 1$). It is excited to some higher energy state when it acquires some energy from external source. But it hardly stays there for more than 10^{-8} second.

A photon corresponding to a particular spectrum line is emitted when an atom makes a transition from a state in an excited level to a state in a lower excited level or the ground level.

Let n_i be the initial and n_f the final energy state, then depending on the final energy state following series are observed in the emission spectrum of hydrogen atom.

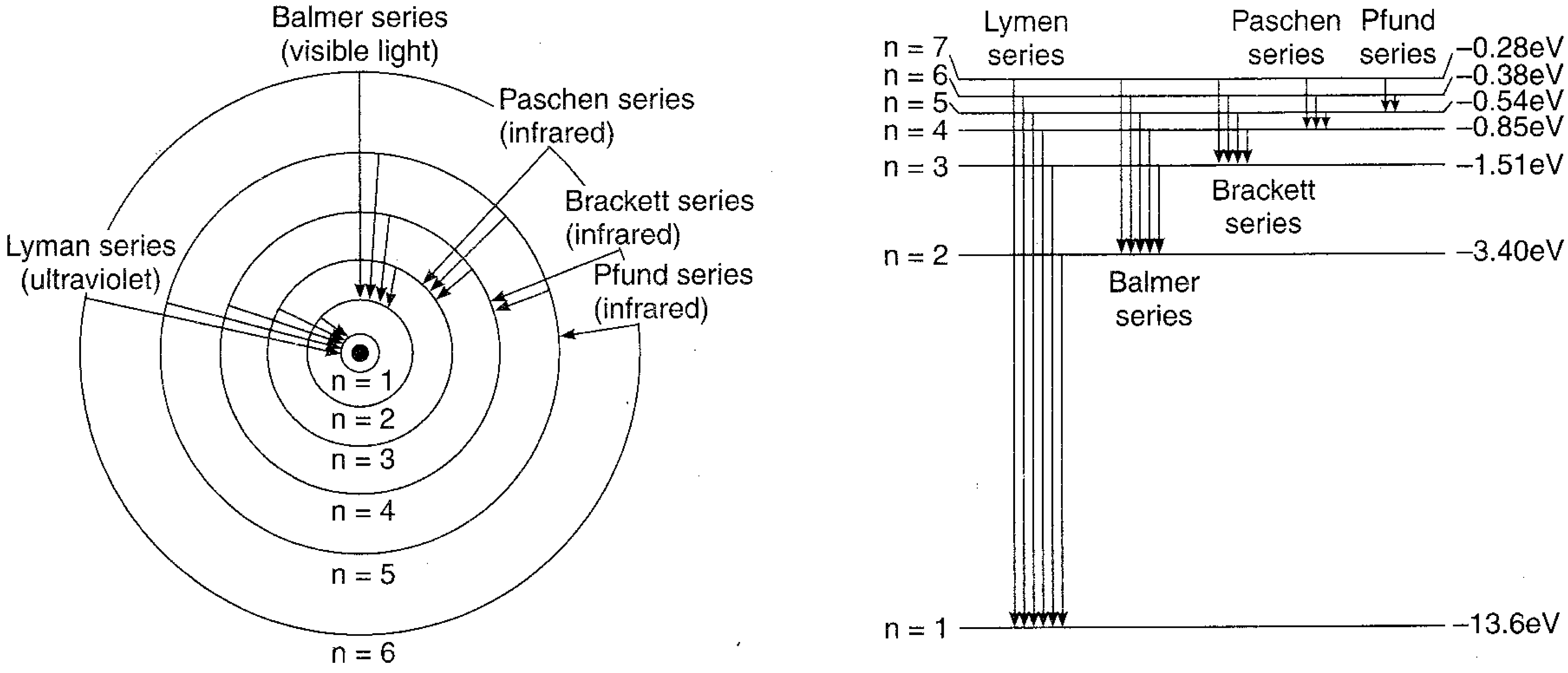


Fig. 24.5

For the Lymen series $n_f = 1$, for Balmer series $n_f = 2$ and so on. The relation of the various spectral series to the energy levels and to electron orbits is shown in figure.

Wavelength of Photon Emitted in De-excitation

According to Bohr when an atom makes a transition from one energy level to a lower level it emits a photon with energy equal to the energy difference between the initial and final levels. If E_i is the initial energy of the atom before such a transition, E_f is its final energy after the transition, and the photon's energy is $hf = \frac{hc}{\lambda}$, then conservation of energy gives,

$$hf = \frac{hc}{\lambda} = E_i - E_f \qquad \text{(energy of emitted photon)} \quad \ldots\text{(xii)}$$

By 1913, the spectrum of hydrogen had been studied intensively. The visible line with longest wavelength, or lowest frequency is in the red and is called H_α, the next line, in the blue-green is called H_β and so on.

In 1885, Johann Balmer, a swiss teacher found a formula that gives the wave lengths of these lines. This is now called the Balmer series. The Balmer's formula is,

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \quad \ldots\text{(xiii)}$$

Here, $n = 3, 4, 5 \ldots$, etc.

R = Rydberg constant $= 1.097 \times 10^7\ \text{m}^{-1}$

and λ is the wavelength of light/photon emitted during transition.

For $n = 3$, we obtain the wavelength of H_α line.

Similarly, for $n = 4$, we obtain the wavelength of H_β line. For $n = \infty$, the smallest wavelength (=3646 Å) of this series is obtained. Using the relation, $E = \frac{hc}{\lambda}$ we can find the photon energies corresponding to the wavelength of the Balmar series. Multiplying Eq. (xiii) by hc, we find

$$E = \frac{hc}{\lambda} = hcR\left(\frac{1}{2^2} - \frac{1}{n^2}\right) = \frac{Rhc}{2^2} - \frac{Rhc}{n^2}$$
$$= E_n - E_2$$

This formula suggests that,

$$E_n = -\frac{Rhc}{n^2}, n = 1, 2, 3 \ldots \quad \ldots\text{(xiv)}$$

Comparing this with Eq. (xi), of the same article, we have

$$Rhc = 13.60 \text{ eV} \quad \ldots\text{(xv)}$$

The wavelengths corresponding to other spectral series (Lymen, Paschen , etc.) can be represented by formulas similar to Balmer's formula.

Lymen Series : $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right), n = 2, 3, 4 \ldots$

Paschen Series : $\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right), n = 4, 5, 6 \ldots$

Brackett Series : $\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right), n = 5, 6, 7 \ldots$

Pfund Series : $\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right), n = 6, 7, 8 \ldots$

The Lymen series is in the ultraviolet, and the Paschen, Brackett and Pfund series are in the infrared region.

EXAMPLE 24.3 *Calculate (a)the wavelength and (b) the frequency of the H_β line of the Balmer series for hydrogen.*

SOLUTION (a) H_β line of Balmer series corresponds to the transition from $n = 4$ to $n = 2$ level. Using Eq. (xiii), the corresponding wavelength for H_β line is,

$$\frac{1}{\lambda} = (1.097 \times 10^7)\left(\frac{1}{2^2} - \frac{1}{4^2}\right)$$
$$= 0.2056 \times 10^7$$

$\therefore$ $\lambda = 4.9 \times 10^{-7}$ m **Ans.**

(b) $$f = \frac{c}{\lambda} = \frac{3.0 \times 10^8}{4.9 \times 10^{-7}}$$
$$= 6.12 \times 10^{14} \text{ Hz}$$ **Ans.**

EXAMPLE 24.4 *Find the largest and shortest wavelengths in the Lymen series for hydrogen. In what region of the electromagnetic spectrum does each series lie?*

SOLUTION The transition equation for Lymen series is given by,

$$\frac{1}{\lambda}=R\left(\frac{1}{1^2}-\frac{1}{n^2}\right),\quad n=2,3,\ldots$$

The largest wavelength is corresponding to $n=2$

$$\therefore \quad \frac{1}{\lambda_{max}}=1.097\times10^7\left(\frac{1}{1}-\frac{1}{4}\right)$$

$$=0.823\times10^7$$

$$\therefore \quad \lambda_{max}=1.2154\times10^{-7}\text{ m}$$

$$=1215\text{ Å}$$ **Ans.**

The shortest wavelength corresponds to $n=\infty$

$$\therefore \quad \frac{1}{\lambda_{min}}=1.097\times10^7\left(\frac{1}{1}-\frac{1}{\infty}\right)$$

or $$\lambda_{min}=0.911\times10^{-7}\text{ m}=911\text{ Å}$$ **Ans.**

Both of these wavelengths lie in ultraviolet (UV) region of electromagnetic spectrum.

24.7 HYDROGEN LIKE ATOMS

The Bohr model of hydrogen can be extended to hydrogen like atoms, *i.e.*, one electron atoms, such as singly ionized helium (He^+), doubly ionized lithium (Li^{+2}) and so on. In such atoms, the nuclear charge is $+ze$, where z is the atomic number, equal to the number of protons in the nucleus.

The effect in the previous analysis is to replace e^2 everywhere by ze^2. Thus, the equations for, r_n, v_n and E_n are altered as under:

$$r_n=\frac{\varepsilon_0 n^2h^2}{\pi mze^2}=\frac{n^2}{z}\cdot a_0 \quad\text{or}\quad r_n\propto\frac{n^2}{z} \qquad \ldots(\text{i})$$

where $a_0=0.529$ Å (radius of first orbit of H)

$$v_n=\frac{ze^2}{2\varepsilon_0 nh}=\frac{z}{n}v_1 \quad\text{or}\quad v_n\propto\frac{z}{n} \qquad \ldots(\text{ii})$$

where $v_1=2.19\times10^6$ m/s (speed of electron in first orbit of H)

$$E_n=-\frac{mz^2e^4}{8\varepsilon_0^2n^2h^2}=\frac{z^2}{n^2}E_1 \quad\text{or}\quad E_n\propto\frac{z^2}{n^2} \qquad \ldots(\text{iii})$$

where $E_1=-13.60$ eV (energy of atom in first orbit of H)

Fig. 24.6 compares the energy levels of H and He^+ which has $z=2$. H and He^+ have many spectrum lines that have almost the same wavelengths.

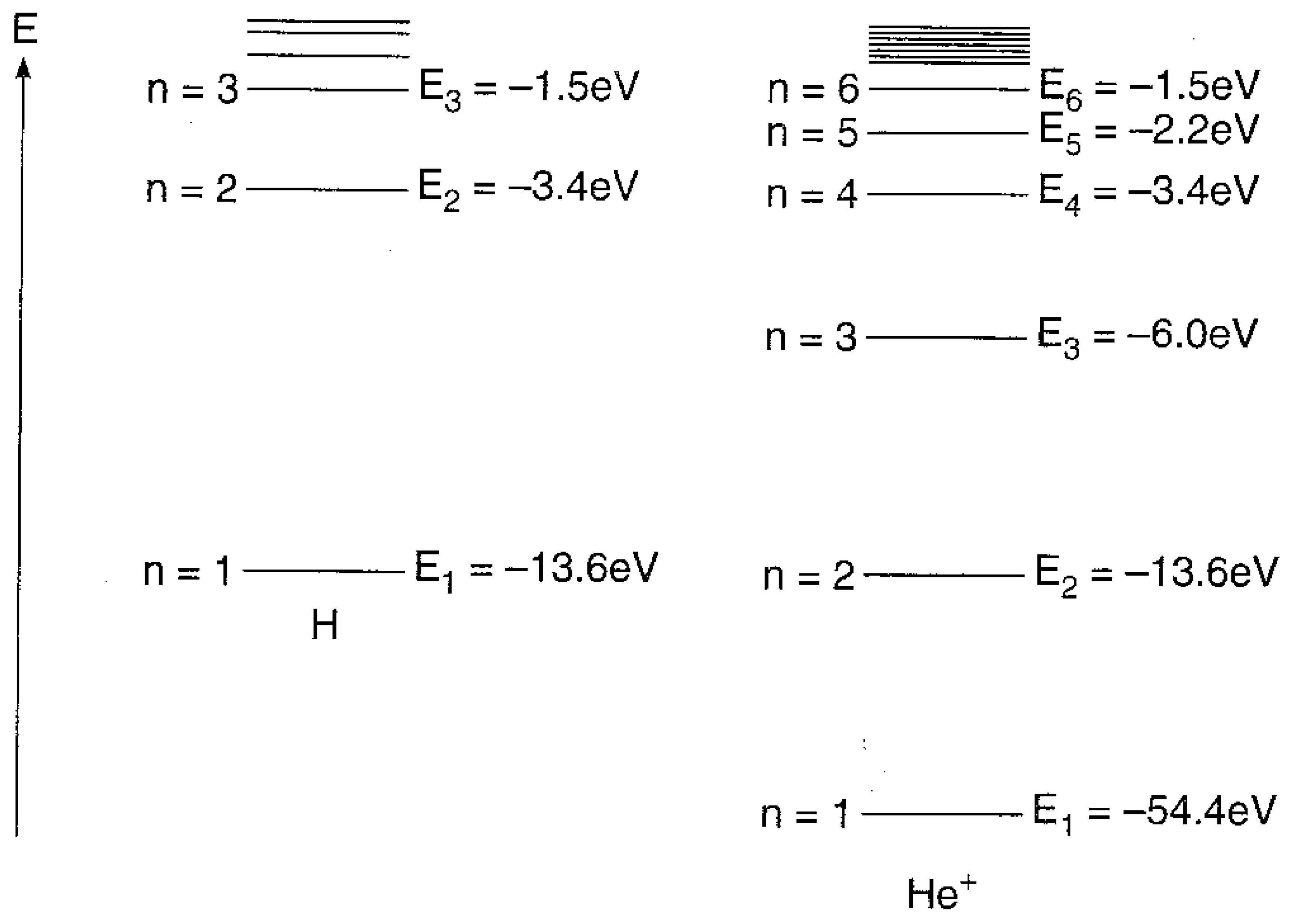


Fig. 24.6 Energy levels of H and He$^+$. Because of the additional factor Z^2 in the energy expression, the energy of the He$^+$ ion with a given n is almost exactly four times that of the H atom with the same n. There are small differences (of the order of 0.05%) because the reduced masses are slightly different.

EXAMPLE 24.5 *Using the known values for hydrogen atom, calculate*
(*a*) *radius of third orbit for* Li^{+2}
(*b*) *speed of electron in fourth orbit for* He^+.

SOLUTION (a) $z=3$ for Li^{+2}. Further we know that $r_n = \frac{n^2}{z} a_0$

substituting, $n=3$, $z=3$ and $a_0 = 0.529$ Å

we have r_3 for $Li^{+2} = \frac{(3)^2}{(3)}(0.529)$ Å $= 1.587$ Å **Ans.**

(b) $z=2$ for He^+. Also we know that

$$v_n = \frac{z}{n} v_1$$

Substituting $n=4$, $z=2$ and $v_1 = 2.19\times 10^6$ m/s

we get, v_4 for $He^+ = \left(\frac{2}{4}\right)(2.19\times 10^6)$m/s

$$= 1.095\times 10^6 \text{ m/s}$$ **Ans.**

24.8 REDUCED MASS

In our earlier discussion we have assumed that the nucleus (a proton in case of hydrogen atom) remains at rest. With this assumption the values of the Rydberg constant R and the ionization energy of hydrogen predicted by Bohr's analysis are within 0.1% of the measured values.

Rather the proton and electron both revolve in circular orbits about their common centre of mass. We can take the motion of the nucleus into account simply by replacing the mass of electron m by the reduced mass μ of the electron and the nucleus.

Here $$\mu = \frac{Mm}{M+m} \qquad ...(i)$$

where M = mass of nucleus. The reduced mass can also be written as,

$$\mu = \frac{m}{1+\frac{m}{M}}$$

Now, when $M >> m$, $\frac{m}{M} \to 0$ or $\mu \to m$

For ordinary hydrogen we let $M = 1836.2\,m$. Substituting in Eq. (i), we get $\mu = 0.99946\,m$ when this value is used instead of the electron mass m in the Bohr equations, the predicted values are well within 0.1% of the measured values.

The concept of reduced mass has other applications. A positron has the same rest mass as an electron but a charge $+e$.

A positronium atom consists of an electron and a positron, each with mass m, in orbit around their common centre of mass. This structure lasts only about 10^{-6} s before two particles annihiliate (combine) one another and disappear, but this is enough time to study the positronium spectrum. The reduced mass is $m/2$, so the energy levels and photon frequencies have exactly half the values for the simple Bohr model with infinite proton mass.

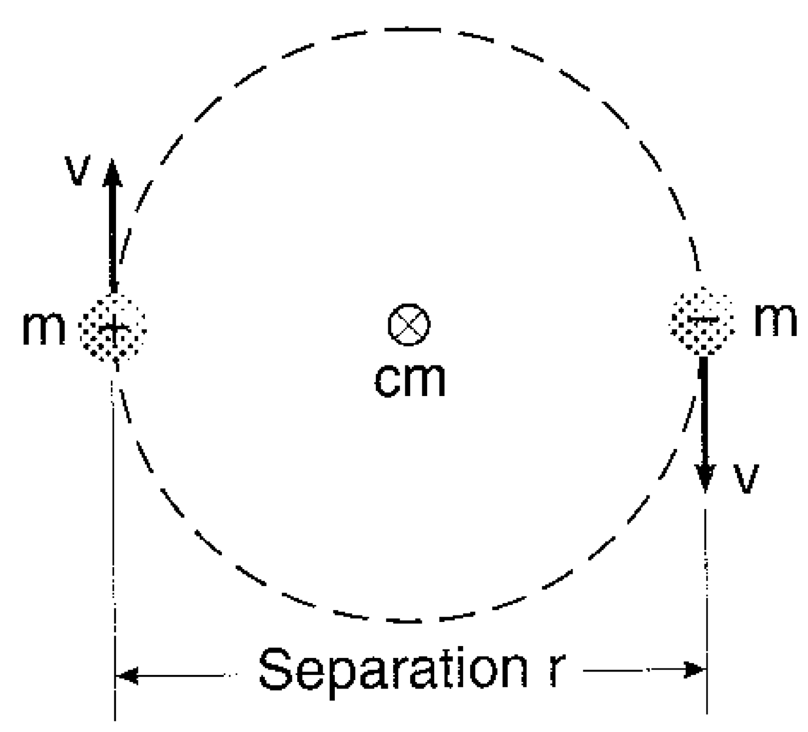


Fig. 24.7 Applying the Bohr model to positronium. The electron and the positron revolve about their common centre of mass, which is located midway between them because they have equal mass.

Now, let us prove why m is replaced by the reduced mass μ when motion of nucleus (proton) is also to be considered.

In Fig. 24.8, both the nucleus (mass = M, charge = e) and electron (mass = m, charge = e) revolve about their centre of mass (CM) with same angular velocity (ω) but different linear speeds. Let r_1 and r_2 be the distance of CM from proton and electron.

Let r be the distance between the proton and the electron. Then,

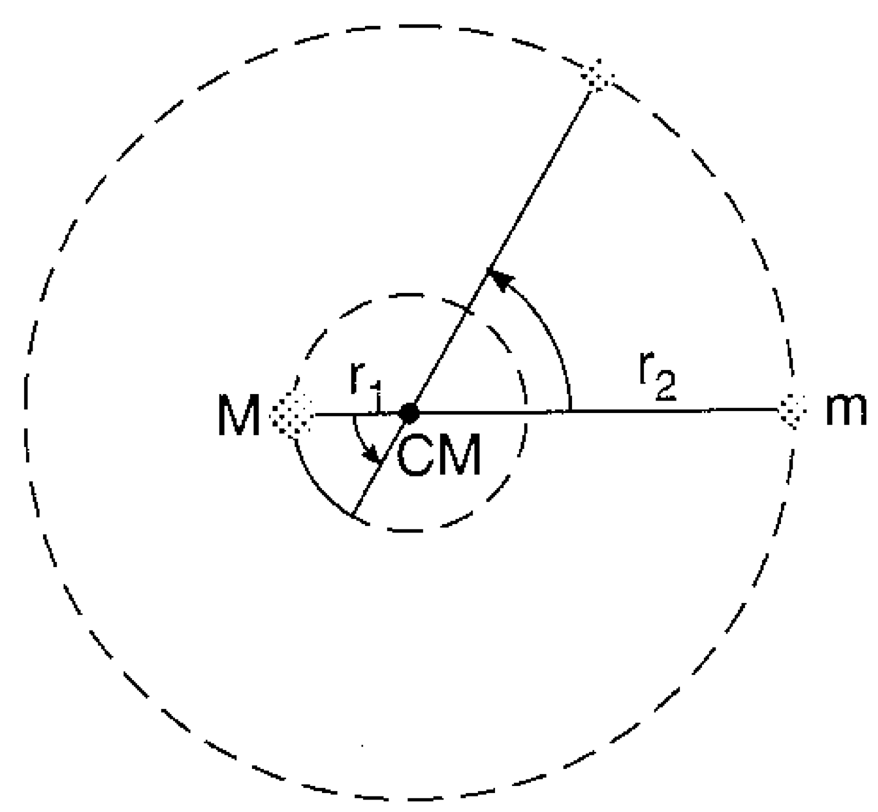


Fig. 24.8

$$Mr_1 = mr_2 \qquad ...(ii)$$

$$r_1 + r_2 = r \qquad ...(iii)$$

$$\therefore \qquad r_1 = \frac{mr}{M+m} \quad \text{and} \quad r_2 = \frac{Mr}{M+m} \qquad ...(iv)$$

Centripetal force to the electron is provided by the electrostatic force. So,

$$mr_2\omega^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

or $$m\left(\frac{Mr}{M+r}\right)\omega^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{r^2}$$

or
$$\left(\frac{Mm}{M+m}\right)r^3\,\omega^2 = \frac{e^2}{4\pi\varepsilon_0}$$

or
$$\mu r^3\omega^2 = \frac{e^2}{4\pi\varepsilon_0} \quad \text{...(v)}$$

where
$$\frac{Mm}{M+m} = \mu$$

Moment of inertia of atom about CM,

$$I = Mr_1^2 + mr_2^2 = \left(\frac{Mm}{M+M}\right)r^2 = \mu r^2 \quad \text{...(vi)}$$

According to Bohr's theory,
$$\frac{nh}{2\pi} = I\,\omega$$

or
$$\mu r^2\omega = \frac{nh}{2\pi} \quad \text{...(vii)}$$

Solving Eqs. (v) and (vii) for r, we get

$$r = \frac{\varepsilon_0 n^2 h^2}{\pi\mu e^2} \quad \text{...(ix)}$$

Comparing this equation with Eq. (iv) of article 24.6 we see that m has been replaced by μ:
Further electrical potential energy of the system,

$$U = \frac{-e^2}{4\pi\varepsilon_0 r}$$

and kinetic energy,
$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\mu r^2\omega^2$$

From equation (v),
$$\omega^2 = \frac{e^2}{4\pi\varepsilon_0\,\mu r^3}$$

$\therefore$
$$K = \frac{e^2}{8\pi\varepsilon_0 r}$$

$\therefore$ Total energy of the system,
$$E = K + U = -\frac{e^2}{8\pi\varepsilon_0 r}$$

Substituting value of r from Eq. (ix), we have

$$E = -\frac{\mu e^4}{8\varepsilon_0^2 n^2 h^2} \quad \text{...(x)}$$

The expression for E_n without considering the motion of proton is $E_n = -\dfrac{me^4}{8\varepsilon_0^2 n^2 h^2}$, *i.e.*, m is replaced by μ while considering the motion of proton.

IIT-JEE GALAXY 24.1

1. Total number of emission lines from some excited state n_1 to another energy state n_2 $(< n_1)$ is given by $\frac{(n_1 - n_2)(n_1 - n_2 + 1)}{2}$.

   For example total number of lines from $n_1 = n$ to $n_2 = 1$ are $\frac{n(n-1)}{2}$.

2. As the principal quantum number n is increased in hydrogen and hydrogen like atoms, some quantities are decreased and some are increased. The table given below shows which quantities are increased and which are decreased.

Table 24.1

Increased	Decreased
Radius	Speed
Potential energy	Kinetic energy
Total energy	Angular speed
Time period	
Angular momentum	

3. Whenever the force obeys inverse square law $\left(F \propto \frac{1}{r^2}\right)$ and potential energy is inversely proportional to r $\left(U \propto \frac{1}{r}\right)$, kinetic energy ($K$), potential energy ($U$) and total energy ($E$) have the following relationships.

$$K = \frac{|U|}{2} \quad \text{and} \quad E = -K = \frac{U}{2}$$

   If force is not proportional to $\frac{1}{r^2}$ or potential energy is not proportional to $\frac{1}{r}$, the above relations do not hold good. In JEE problems, this situation arises at two places, in an atom (between nucleus and electron) and in solar system (between sun and planet).

   See example number 24.7.

4. Total energy of a closed system is always negative and the modulus of this is the binding energy of the system. For instance, suppose a system has a total energy of –100 J. It means that this system will separate if 100 J of energy is supplied to this. Hence, binding energy of this system is 100 J. Thus, total energy of an open system is either zero or greater than zero.

5. Kinetic energy of a particle can't be negative, while the potential energy can be zero, positive or negative. It basically depends on the reference point where we have taken it zero. It is customary to take zero potential energy when the electron is at infinite distance from the nucleus. In some problem suppose we take zero potential energy in first orbit ($U_1 = 0$), then the modulus of actual potential energy in first orbit (when reference point was at infinity) is added in U and E in all energy states, while K remains unchanged. See example number 24.6.

6. Variation of r_n, v_n and E_n with mass of electron is as under,

$$r_n \propto \frac{1}{m}, \quad v_n = \text{independent of } m \quad \text{and} \quad E_n \propto m$$

Sometimes the electron is replaced by some another particle which has a charge $-e$ but mass different from the mass of electron. Here, two cases are possible.

Case 1. Let say mass of the replaced particle is x times the mass of the electron and nucleus is still very heavy compared to the replaced particle, *i.e.*, the motion of the nucleus is not to be considered. In this case r_n will become $\frac{1}{x}$ times, v_n will remain unchanged and E_n becomes x times.

Case 2. In this case motion of nucleus is also to be considered, *i.e.*, mass of the replaced particle is comparable to the mass of the nucleus. In this case the mass of the electron is replaced by the reduced mass of the nucleus and the replaced particle. Let say the reduced mass is y times the mass of the electron. Then, r_n will become $\frac{1}{y}$ times, v_n remains unchanged and E_n becomes y-times.

7. Reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ of m_1 and m_2 is less than both the masses.

EXAMPLE 24.6 *Find the kinetic energy, potential energy and total energy in first and second orbit of hydrogen atom if potential energy in first orbit is taken to be zero.*

SOLUTION $E_1 = -13.60\,\text{eV} \quad K_1 = -E_1 = 13.60\,\text{eV} \quad U_1 = 2E_1 = -27.20\,\text{eV}$

$$E_2 = \frac{E_1}{(2)^2} = -3.40\,\text{eV} \quad K_2 = 3.40\,\text{eV} \quad \text{and} \quad U_2 = -6.80\,\text{eV}$$

Now $U_1 = 0$, *i.e.*, potential energy has been increased by 27.20 eV. So, we will increase U and E in all energy states by 27.20 eV while kinetic energy will remain unchanged. Changed values in tabular form are as under.

Table 24.2

Orbit	K (eV)	U (eV)	E (eV)
First	13.60	0	13.60
Second	3.40	20.40	23.80

EXAMPLE 24.7 *A small particle of mass m moves in such a way that the potential energy $U = ar^2$ where a is a constant and r is the distance of the particle from the origin. Assuming Bohr's model of quantization of angular momentum and circular orbits, find the radius of n^{th} allowed orbit.*

SOLUTION The force at a distance r is,

$$F = -\frac{dU}{dr} = -2ar$$

Suppose r be the radius of n^{th} orbit. Then the necessary centripetal force is provided by the above force. Thus,

$$\frac{mv^2}{r} = 2ar \quad \text{...(i)}$$

Further, the quantization of angular momentum gives,

$$mvr = \frac{nh}{2\pi} \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii) for r, we get

$$r = \left(\frac{n^2h^2}{8\,am\,\pi^2}\right)^{1/4} \quad \textbf{Ans.}$$

EXAMPLE 24.8 *An imaginary particle has a charge equal to that of an electron and mass 100 times the mass of the electron. It moves in a circular orbit around a nucleus of charge + 4e. Take the mass of the nucleus to be infinite. Assuming that the Bohr's model is applicable to this system.*

(*a*) *Derive an expression for the radius of n^{th} Bohr orbit.*

(*b*) *Find the wavelength of the radiation emitted when the particle jumps from fourth orbit to the second orbit.*

SOLUTION (a) We have

$$\frac{m_p v^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\frac{ze^2}{r_n^2} \quad \text{...(i)}$$

The quantization of angular momentum gives,

$$m_p v r_n = \frac{nh}{2\pi} \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$r = \frac{n^2h^2\varepsilon_0}{z\pi m_p e^2}$$

Substituting $m_p = 100\text{ m}$

where m = mass of electron and $z = 4$

we get,

$$r_n = \frac{n^2h^2\varepsilon_0}{400\,\pi\,me^2} \quad \textbf{Ans.}$$

(b) As we know, $E_1^H = -13.60\text{ eV}$

and

$$E_n \propto \left(\frac{z^2}{n^2}\right)m$$

For the given particle, $E_4 = \frac{(-13.60)(4)^2}{(4)^2} \times 100 = -1360 \text{ eV}$

and $E_2 = \frac{(-13.60)(4)^2}{(2)^2} \times 100 = -5440 \text{ eV}$

$$\Delta E = E_4 - E_2 = 4080 \text{ eV}$$

$\therefore$ $\lambda \text{ (in Å)} = \frac{12375}{\Delta E \text{ (in eV)}} = \frac{12375}{4080}$

$$= \mathbf{3.0 \text{ Å}}$$ **Ans.**

EXAMPLE 24.9 *A moving hydrogen atom makes a head on collision with a stationary hydrogen atom. Before collision both atoms are in ground state and after collision they move together. What is the minimum value of the kinetic energy of the moving hydrogen atom, such that one of the atoms reaches one of the excitation state.*

SOLUTION Let K be the kinetic energy of the moving hydrogen atom and K', the kinetic energy of combined mass after collision.

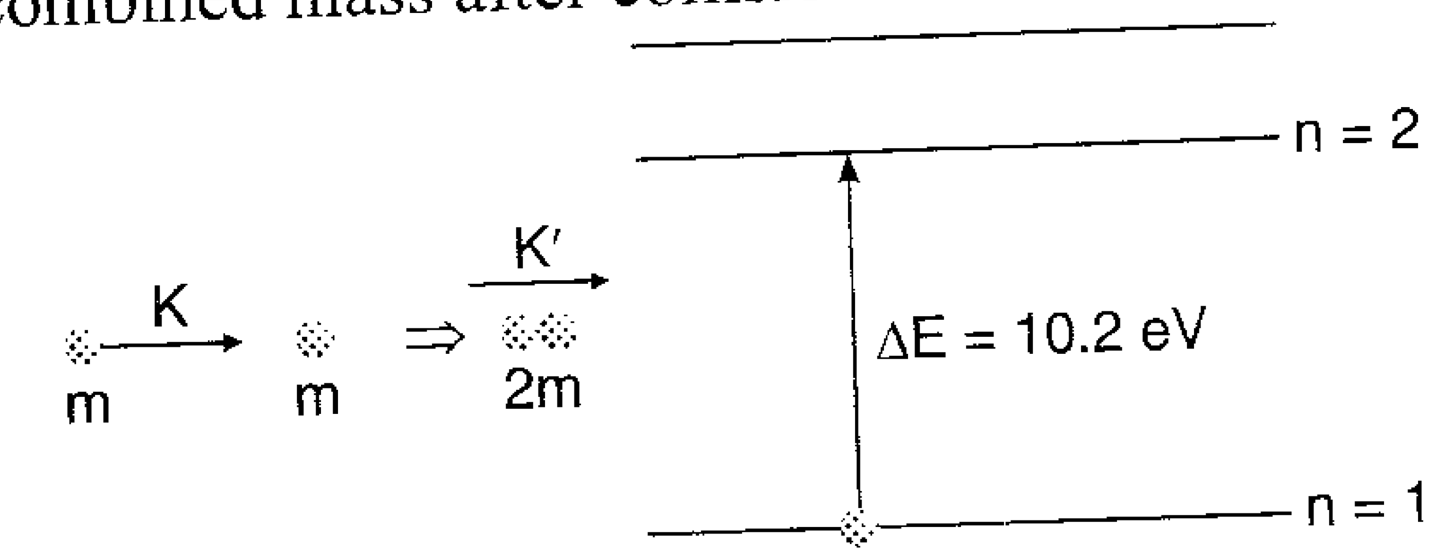


Fig. 24.9

From conservation of linear momentum,

$$p = p' \quad \text{or} \quad \sqrt{2Km} = \sqrt{2K'(2m)}$$

or $K = 2K'$ ...(i)

From conservation of energy, $K = K' + \Delta E$ ...(ii)

Solving Eqs. (i) and (ii), we get $\Delta E = \frac{K}{2}$

Now minimum value of ΔE for hydrogen atom is 10.2 eV.

or $\Delta E \geq 10.2 \text{ eV}$

$\therefore$ $\frac{K}{2} \geq 10.2$

$\therefore$ $K \geq 20.4 \text{ eV}$

Therefore, the minimum kinetic energy of moving hydrogen is 20.4 eV. **Ans.**

24.9 X-RAYS

Electromagnetic radiation with wavelengths from 0.1 Å to 100 Å falls into the category of X-rays. The boundaries of this category are not sharp. The shorter wavelength end overlaps gamma rays and the longer wavelength end overlaps ultraviolet rays. Photoelectric effect (will be discussed later) provide convincing evidence that photons of light can transfer energy to electrons. Is the inverse process also possible? That is, can part or all of the kinetic energy of a moving electron be converted into a photon? Yes, it is possible. In 1895 **Wilhelm Roentgen** found that a highly penetrating radiation of unknown nature is produced when fast moving electrons strike a target of high atomic number and high melting point. These radiations were given a name X-rays as their nature was unknown (in mathematics an unknown quantity is normally designated by X). Later it was discovered that these are high energy photons (or electromagnetic waves)

Production of X-Rays : Figure shows a diagram of a X-ray tube, called the coolidge tube. A cathode (a plate connected to negative terminal of a battery), heated by a filament through which an electric current is passed, supplies electrons by thermionic emission. The high potential difference V maintained between the cathode and a metallic target accelerate the electrons toward the latter. The face of the target is at an angle relative to the electron beam, and the X-rays that leave the target pass through the side of the tube. The tube is evacuated to permit the electrons to get to the target unimpeded.

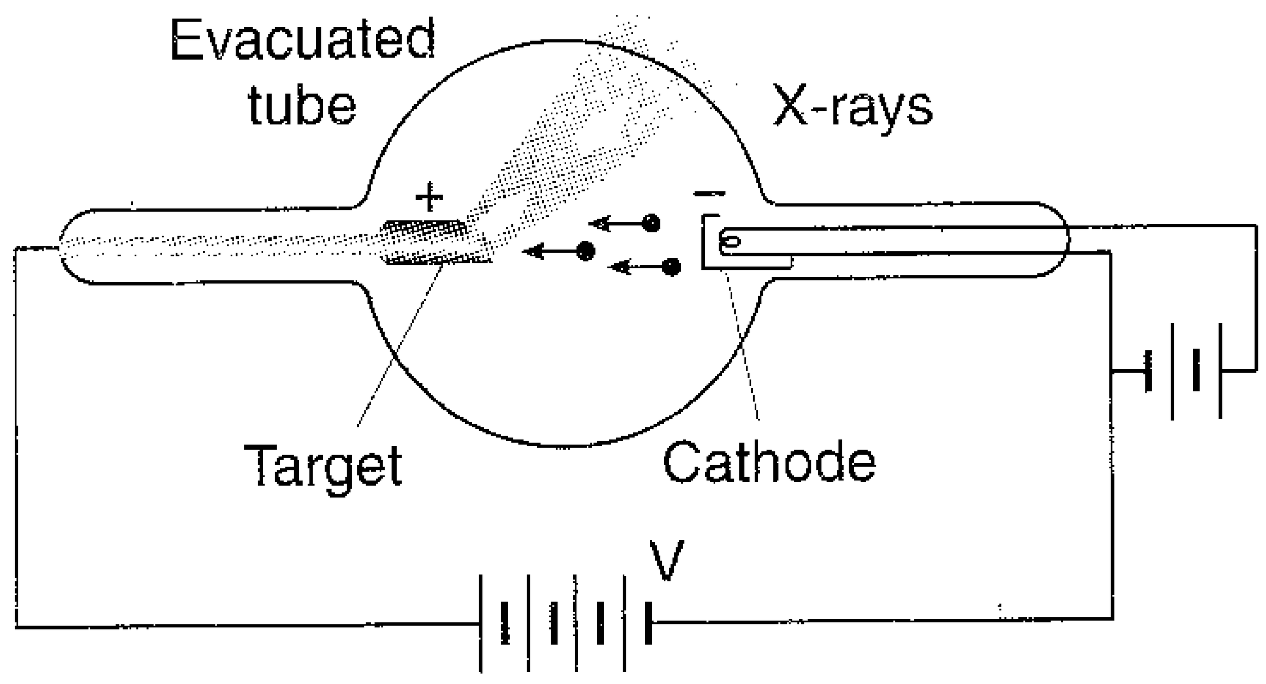


Fig. 24.10 An X-ray tube. The higher the accelerating voltage V, the faster the electrons and the shorter the wavelengths of the X-rays.

Continuous and characteristic X-rays : X-rays so produced by the coolidge tube are of two types, continuous and characteristic. While the former depends only on the accelerating voltage V the later depends on the target used.

Continuous X-rays : Electromagnetic theory predicts that an accelerated electric charge will radiate electromagnetic waves, and a rapidly moving electrons when suddenly brought to rest is certainly accelerated (of course negative). X-rays produced under these circumstances is given the German name **bremsstrahlung** (braking radiation). Energy loss due to bremsstrahlung is more important for electrons than for heavier particles because electrons are more violently accelerated when passing near nuclei in their paths. The continuous X-rays (or bremsstrahlung X-rays) produced at a given accelerating potential V vary in wavelength, but none has a wavelength shorter than a certain value λ_{min}. This minimum wavelength corresponds to the maximum energy of the X-rays which in turn is equal to the maximum kinetic energy qV or eV of the striking electrons. Thus,

$$\frac{hc}{\lambda_{min}} = eV$$

or

$$\lambda_{min} = \frac{hc}{eV}$$

After substituting values of h, c and e we obtain the following simple formula for λ_{min}.

$$\lambda_{min}\ (\text{in Å}) = \frac{12375}{V(\text{in volts})} \qquad \ldots\text{(i)}$$

Increasing V decreases λ_{min}. This wavelength is also known as the cutoff wavelength or the threshold wavelength.

Characteristic X-rays : The X-ray spectrum typically consists of a broad continuous band containing a series of sharp lines, as shown in Fig. 24.11.

As discussed above the continuous spectrum is the result of collisions between incoming electrons and atoms in the target. The kinetic energy lost by the electrons during the collisions emerges as the energy of the X-ray photons radiated from the target.

The sharp lines superimposed on the continuous spectrum are known as **characteristic X-rays** because they are characteristic of the target material. They were discovered in 1908, but their origin remained unexplained until the details of atomic structure, particularly the shell structure of the atom, were discovered.

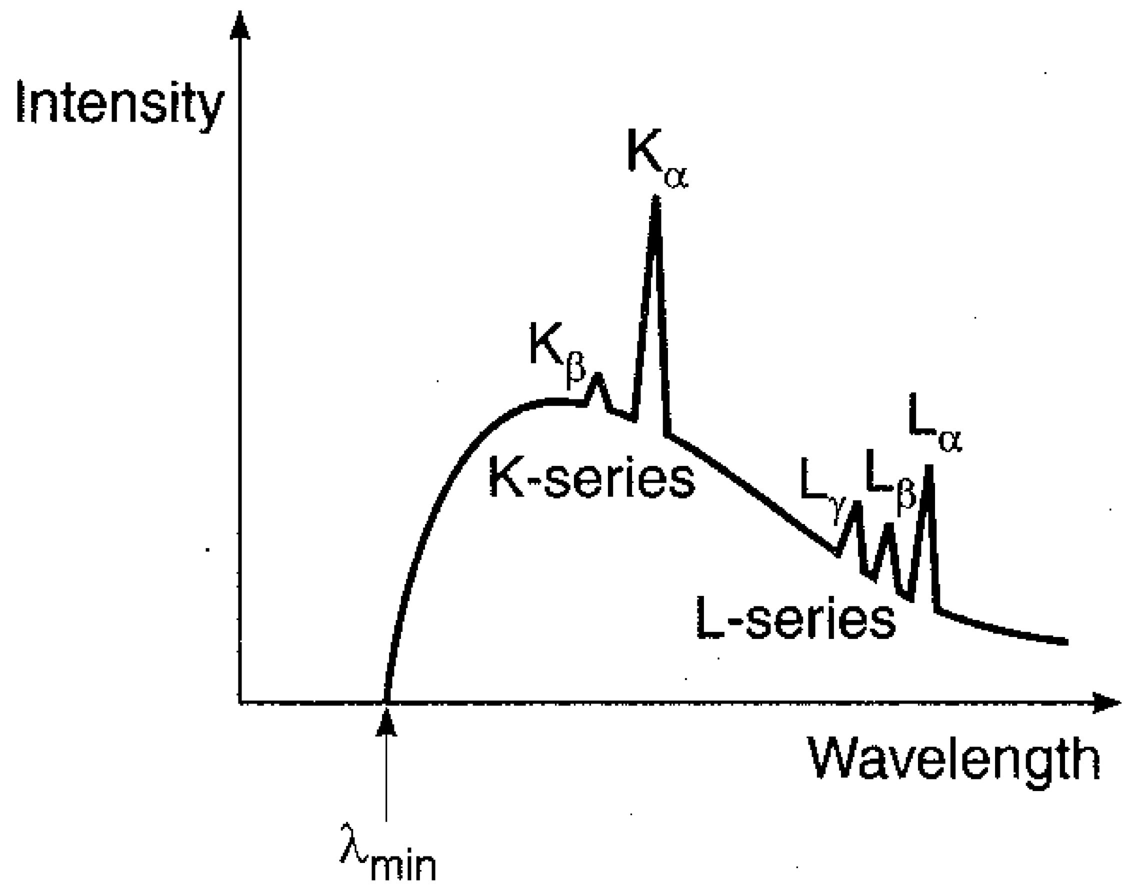


Fig. 24.11 X-ray spectrum

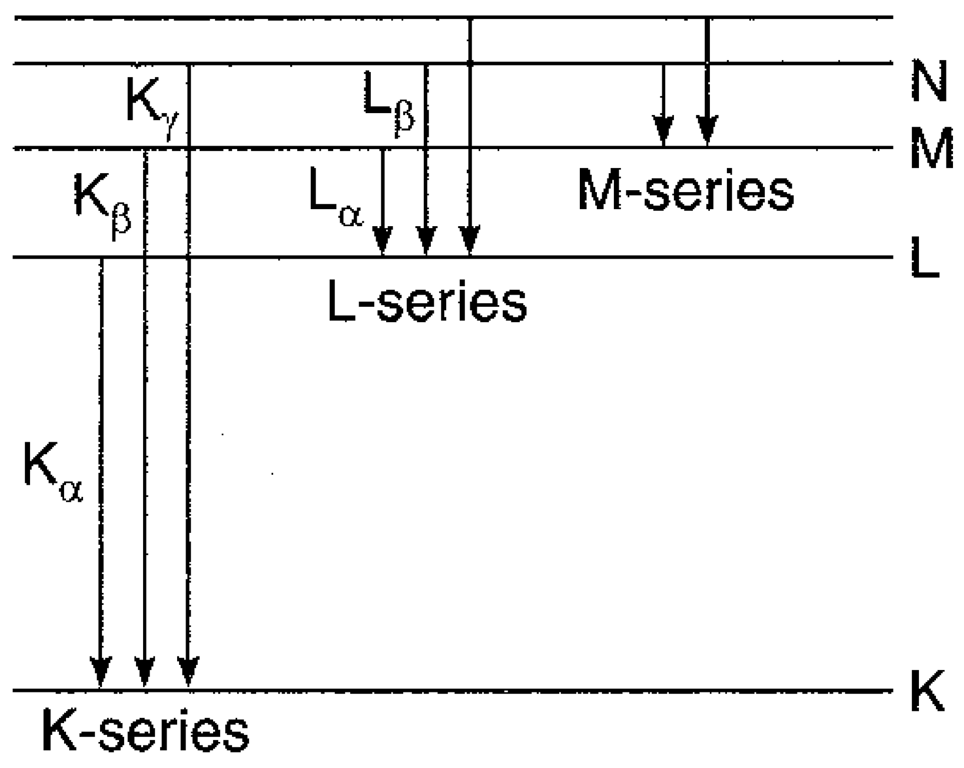


Fig. 24.12

Characteristic X-ray emission occurs when a bombarding electron that collides with a target atom has sufficient energy to remove an inner shell electron from the atom. The vacancy created in the shell is filled when an electron from a higher level drops down into it. This transition is accompanied by the emission of a photon whose energy equals the difference in energy between the two levels.

Let us assume that the incoming electron his dislodged an atomic electron from the innermost shell-the K shell. If the vacancy is filled by an electron droping from the next higher shell-the L shell-the photon emitted has an energy corresponding to the K_α characteristic X-ray line. If the vacancy is filled by an electron dropping from the M shell, the K_β line is produced. An L_α line is produced as an electron drops from the M shell to the L-shell, and an L_β line is produced by a transition from the N-shell to the L-shell.

Moseley's law for characteristic spectrum

Although multielectron atoms cannot be analyzed with the Bohr model, Henery G.J. Moseley in 1914 made an effort towards this. Moseley measured the frequencies of characteristic X-rays from a large number of elements and plotted the square root of the frequency $\sqrt{f}$ against the atomic number z of the element. He discovered that the plot is very close to a straight line. He plotted the square root of the frequency of the K_α line versus the atomic number z.

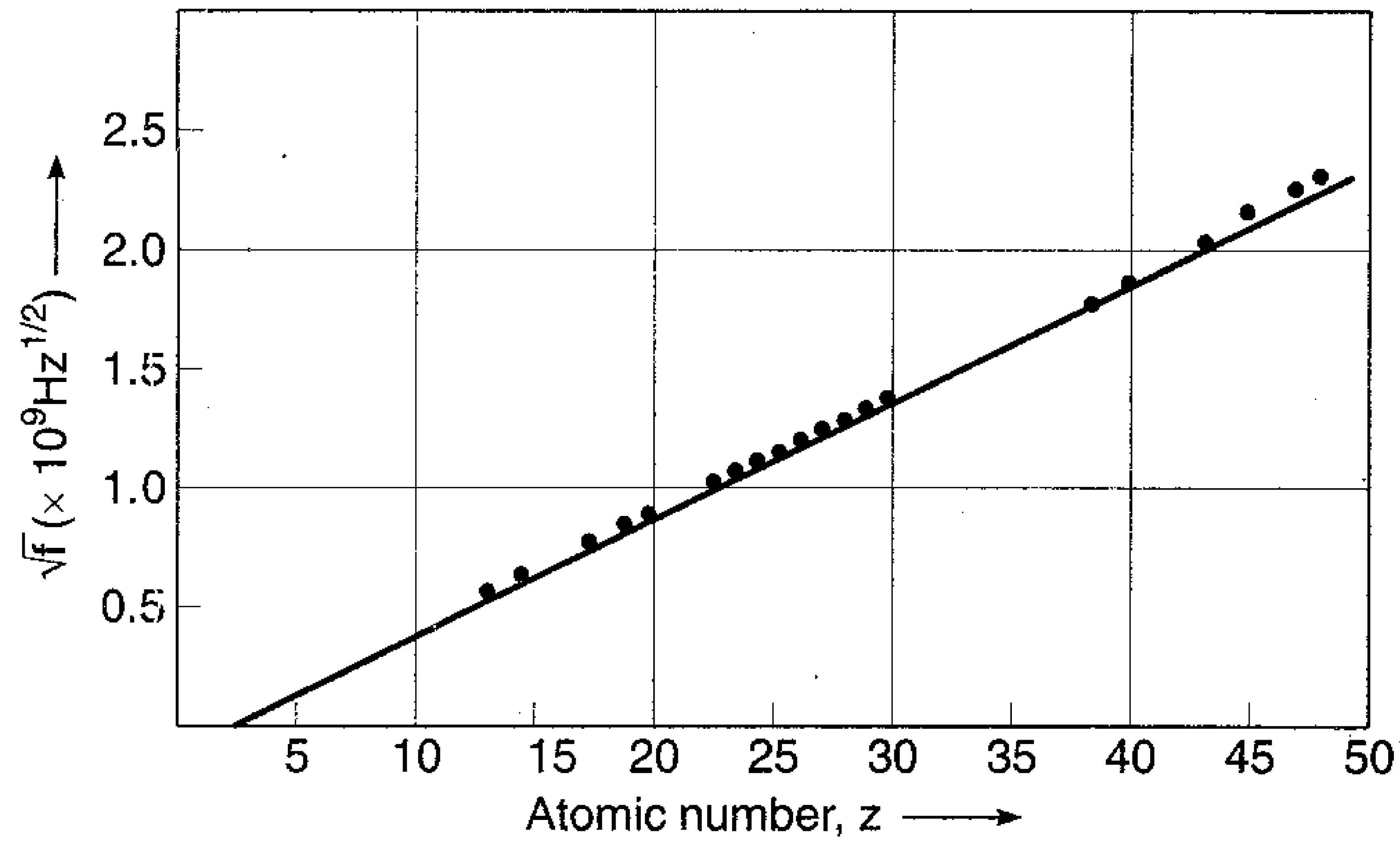


Fig. 24.13 A plot of the square root of the frequency of the K_α lines versus atomic number using Moseley's data.

As figure shows, Moseley's plot did not pass through the origin. Let us see why. It can be understood from Gauss's law. Consider an atom of atomic number Z in which one of the two electrons in the K-shell has been ejected. Imagine that we draw a Gaussian sphere just inside the most probable radius of the L-electrons. The effective charge inside the Gaussian surface is the positive nuclear charge and one negative charge due to the single K-electron. If we ignore the interactions between L-electrons, a single L electron behaves as if it experiences an electric field due to a charge $(Z-1)$ e enclosed by the Gaussian surface.

Thus, Moseley's law of the frequency of K_α line is,

$$\sqrt{f_{K_\alpha}} = a\,(Z-1) \qquad \ldots\text{(ii)}$$

Where a is a constant that can be related to Bohr's theory.

The above law in general can be stated as under,

$$\sqrt{f} = a\,(Z-b) \qquad \ldots\text{(iii)}$$

For K_α line,

$$\Delta E = hf = Rhc\,(Z-1)^2\left(\frac{1}{1^1}-\frac{1}{2^2}\right)$$

or

$$\sqrt{f} = \sqrt{\frac{3Rc}{4}}\,(Z-1)$$

or

$$a = \sqrt{\frac{3Rc}{4}} \quad \text{and} \quad b = 1$$

After substituting values of R and c, we get

$$a = 4.98 \times 10^7\ (\text{Hz})^{1/2}$$

Eq. (iii) can also be written as,

$$f = a^2\,(Z-b)^2 \qquad \ldots\text{(iv)}$$

For K_α line,

$$a^2 = \frac{3Rc}{4} = (2.48 \times 10^{15}\ \text{Hz}) \quad \text{and} \quad b = 1$$

Hence,

$$f_{K_\alpha} = (2.48 \times 10^{15}\ \text{Hz})\,(z-1)^2$$

Note : We have studied above Moseley's law only for K_α line for which $b = 1$ and $a = 4.98 \times 10^7\ (\text{Hz})^{1/2}$. For other spectral lines (K_β, K_γ, L_α etc.) the theory requires a good enough discussion and I personally feel it is not required at all as far as JEE problems are concerned. Even though for interested readers a brief idea has been given in IIT-JEE Galaxy 24.2.

EXAMPLE 24.10 *Find the cutoff wavelength for the continuous X-rays coming from an X-ray tube operating at* 40 *kV*.

SOLUTION Cutoff wavelength λ_{min} is given by,

$$\lambda_{\text{min}}\ (\text{in Å}) = \frac{12375}{V(\text{in volts})} = \frac{12375}{40 \times 10^3}$$

$$= 0.31\ \text{Å}$$ **Ans.**

EXAMPLE 24.11 *Use Moseley's law with $b = 1$ to find the frequency of the K_α X-rays of* La $(Z = 57)$ *if the frequency of the K_α X-rays of* Cu $(Z = 29)$ *is known to be* 1.88×10^{18} *Hz.*

SOLUTION Using the equation, $\sqrt{f} = a(Z - b)$ $(b = 1)$

$$\frac{f_{\text{La}}}{f_{\text{Cu}}} = \left(\frac{Z_{\text{La}} - 1}{Z_{\text{Cu}} - 1}\right)^2$$

or

$$f_{\text{La}} = f_{\text{Cu}}\left(\frac{Z_{\text{La}} - 1}{Z_{\text{Cu}} - 1}\right)^2$$

$$= 1.88 \times 10^{18}\left(\frac{57 - 1}{29 - 1}\right)^2$$

$$= 7.52 \times 10^{18} \text{ Hz}$$ **Ans.**

IIT-JEE GALAXY 24.2

1. The energy levels, in general, depend on principal quantum number (n) and orbital quantum number (l). *Let us take sodium ($z = 11$) as an example. According to Gauss's law, for any spherically symmetric charge distribution the electric field magnitude at a distance r* from the centre is $\frac{1}{4\pi\varepsilon_0}\frac{q_{encl}}{r^2}$, where q_{encl} is the total charge enclosed within a sphere with radius r. Mentally remove the outer (valence) electron from a sodium atom. What you have left is a spherically symmetric collection of 10 electrons (filling the K and L shells) and 11 protons. So,

$$q_{encl} = -10e + 11e = +e$$

If the eleventh is completely outside this collection of charges, it is attracted by an effective charge of $+e$, not $+11e$.

This effect is called **screening**, the 10 electrons screen 10 of the 11 protons leaving an effective net charge of $+e$. In general, an electron that spends all its time completely outside a positive charge $z_{\text{eff}}e$ has energy levels given by the hydrogen expression with e^2 replaced by $z_{\text{eff}}\, e^2$. *i.e.*,

$$E_n = -\frac{z_{\text{eff}}^2}{n^2}(13.6 \text{ eV}) \quad \text{(energy levels with screening)}$$

If the eleventh electron in the sodium atom is completely outside the remaining charge distribution, then $z_{\text{eff}} = 1$.

We can estimate the frequency of K_α X-ray photons using the concept of screening. A K_α X-ray photon is emitted when an electron in the L shell ($n = 2$) drops down to fill a hole in the K-shell ($n = 1$). As the electron drops down, it is attracted by the z protons in the nucleus screened by one remaining electron in the K-shell. Thus,

$$z_{\text{eff}} = (z - 1) \qquad n_i = 2 \quad \text{and} \quad n_f = 1$$

The energy before transition is,

$$E_i = -\frac{(z-1)^2}{2^2}(13.6 \text{ eV}) = -(z-1)^2(3.4 \text{ eV})$$

and energy after transition is,

$$E_f = -\frac{(z-1)^2}{1^2}(13.6 \text{ eV}) = -(z-1)^2(13.6 \text{ eV})$$

The energy of the K_α X-ray photon is

$$E_{K_\alpha} = E_i - E_f = (z-1)^2 (10.2\,\text{eV})$$

The frequency of K_α X-ray photon is therefore,

$$f_{K_\alpha} = \frac{E_{K_\alpha}}{h} = \frac{(z-1)^2 (10.2\,\text{eV})}{(4.136 \times 10^{-15}\,\text{eV-s})} = (2.47 \times 10^{15}\,\text{Hz})(z-1)^2$$

This relation agrees almost exactly with Moseley's experimental law.

2. The target (or anode) used in the Coolidge tube should be of high melting point. This is because less than 0.5% of the kinetic energy of the electrons is converted into X-rays. The rest of the kinetic energy becomes internal energy of the target which simultaneously has to be kept cool by circulating oil or water.
3. Atomic number of the target should be high. This is because X-rays are high energy photons and as we have seen above energy of the X-rays increases as z increases.
4. X-rays are basically electromagnetic waves. So they possess all the properties of electromagnetic waves.

EXAMPLE 24.12 *Determine the energy of the characteristic X-ray (K_β) emitted from a tungsten ($z = 74$) target when an electron drops from the M shell ($n = 3$) to a vacancy in the K-shell ($n = 1$).*

SOLUTION Energy associated with the electron in the K-shell is approximately

$$E_K = -(74-1)^2 (13.6\,\text{eV}) = -72474\,\text{eV}$$

An electron in the M-shell is subject to an effective nuclear charge that depends on the number of electrons in the $n = 1$ and $n = 2$ states because these electrons shield the M electrons from the nucleus. Because there are eight electrons in the $n = 2$ state and one remaining in the $n = 1$ state, roughly nine electrons shield M electrons from the nucleus, so

$$z_{\text{eff}} = z - 9$$

Hence, the energy associated with an electron in the M shell is,

$$E_M = \frac{-13.6\, z_{\text{eff}}^2}{3^2}\,\text{eV} = \frac{-13.6\,(z-9)^2}{3^2}\,\text{eV}$$

$$= -\frac{(13.6)(74-9)^2}{9}\,\text{eV} = -6384\,\text{eV}$$

Therefore, emitted X-ray has an energy equal to

$$E_M - E_K = \{-6384 - (-72474)\}\,\text{eV} = 66090\,\text{eV}$$ **Ans.**

INTRODUCTORY EXERCISE 24.1

1. The wavelength for $n = 3$ to $n = 2$ transition of the hydrogen atom is 656.3 nm. What are the wavelengths for this same transition in (a) positronium, which consists of an electron and a positron (b) singly ionized helium (Note: A positron is a positively charged electron).
2. Find the longest wavelength present in the Balmer series of hydrogen.

3. (a) Find the frequencies of revolution of electrons in $n = 1$ and $n = 2$ Bohr orbits.
   (b) What is the frequency of the photon emitted when an electron in an $n = 2$ orbit drops to an $n = 1$ orbit?
   (c) An electron typically spends about 10^{-8} s in an excited state before it drops to a lower state by emitting a photon. How many revolutions does an electron in an $n = 2$ Bohr orbit make in 1.00×10^{-8} s?
4. A **muon** is an unstable elementary particle whose mass is 207 m_e and whose charge is either ^{+}e or ^{-}e. A negative muon (μ^{-}) can be captured by a nucleus to form a muonic atom.
   (a) A proton captures a μ^{-}. Find the radius of the first Bohr orbit of this atom.
   (b) Find the ionization energy of the atom.
5. Find the de-Broglie wavelengths of
   (a) a 46 gm golf ball with a velocity of 30 m/s.
   (b) an electron with a velocity of 10^7 m/s.
6. (a) A gas of hydrogen atoms in their ground state is bombarded by electrons with kinetic energy 12.5 eV. What emitted wavelengths would you expect to see.
   (b) What if the electrons were replaced by photons of same energy.
7. For molybdenum the wavelength of the K_α line is 0.71 nm and of the K_β line it is 0.63 nm. Use this information to find wavelength of the L_α line.
8. The energy of the $n = 2$ state in molybdenum is $E_2 = -2870$ eV. Given that the wavelengths of the K_α and K_β lines are 0.71 nm and 0.63 nm, respectively determine the energies E_1 and E_3.
9. The energy levels of a certain atom are shown in figure. If a photon of frequency f is emitted when there is an electron transition from $5E$ to E, what frequencies of photons could be produced by other energy level transitions.

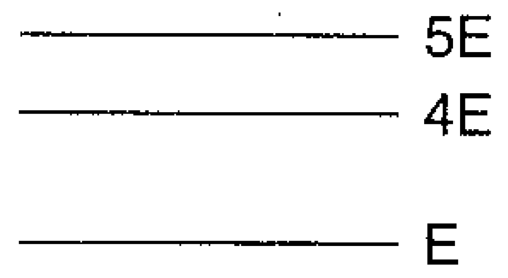


Fig. 24.14

10. An excited atom of mass m and initial speed v emits a photon in its direction of motion. If $v << c$, use the requirement that linear momentum and energy must both be conserved to show that the frequency of the photon is higher by $\frac{\Delta f}{f} \approx \frac{v}{c}$ than it would have been if the atom had been at rest.

24.10 EMISSION OF ELECTRONS

At room temperature the free electrons move randomly within the conductor, but they don't leave the surface of the conductor due to attraction of positive charges. Some external energy is required to emit electrons from a metal surface. Minimum energy is required to emit the electrons which are just on the surface of the conductor. This minimum energy is called the **work function** (denoted by W) of the conductor. Work function is the property of the metallic surface.

The energy required to liberate an electron from metal surface may arise from various sources such as heat, light, electric field etc. Depending on the nature of source of energy, the following methods are possible.

(i) Thermionic emission : The energy to the free electrons can be given by heating the metal. The electrons so emitted are known as **thermions.**

(ii) Field emission : When a conductor is put under strong electric field the free electrons on it experience an electric force in the opposite direction of field. Beyond a certain limit electrons start coming out of the metal surface. Emission of electrons from a metal surface by this method is called the field emissio

(iii) Secondary emission : Emission of electrons from a metal surface by the bombardment of high speed electrons or other particles is known as secondary emission.

(iv) Photoelectric emission : Emission of free electrons from a metal surface by falling light (or any other electromagnetic wave which has an energy greater than the work function of the metal) is called photoelectric emission. The electrons so emitted are called **photoelectrons**.

24.11 PHOTOELECTRIC EFFECT

When light of an appropriate frequency (or correspondingly of an appropriate wavelength) is incident on a metallic surface, electrons are liberated from the surface. This observation is known as **photoelectric effect**. Photoelectric effect was first observed in 1887 by Hertz. For photoemission to take place, energy of incident light photons should be greater than or equal to the work function of the metal.

or $$E \geq W \qquad \ldots(i)$$

$\therefore$ $$hf \geq W$$

or $$f \geq \frac{W}{h}$$

Here $\frac{W}{h}$ is the minimum frequency required for the emission of electrons. This is known as threshold frequency f_0.

Thus, $$f_0 = \frac{W}{h} \quad \text{(threshold frequency)} \qquad \ldots(ii)$$

Further Eq. (i) can be written as, $$\frac{hc}{\lambda} \geq W$$

or $$\lambda \leq \frac{hc}{W}$$

Here $\frac{hc}{W}$ is the largest wavelength beyond which photoemission does not take place. This is called the threshold wavelength λ_0.

Thus, $$\lambda_0 = \frac{hc}{W} \quad \text{(threshold wavelength)} \qquad \ldots(iii)$$

Hence, for the photoemission to take place either of the following conditions must be satisfied.

$$E \geq W \quad \text{or} \quad f \geq f_0 \quad \text{or} \quad \lambda \leq \lambda_0 \qquad \ldots(iv)$$

Stopping potential and maximum kinetic energy of photoelectrons

When the frequency f of the incident light is greater than the threshold frequency, some electrons are emitted from the metal with substantial initial speeds. Suppose E is the energy of light incident on a metal surface and W ($< E$) the work function of metal. As minimum energy is required to extract electrons from the surface, they will have the maximum kinetic energy which is $E - W$.

Thus, $$K_{max} = E - W \qquad \ldots(v)$$

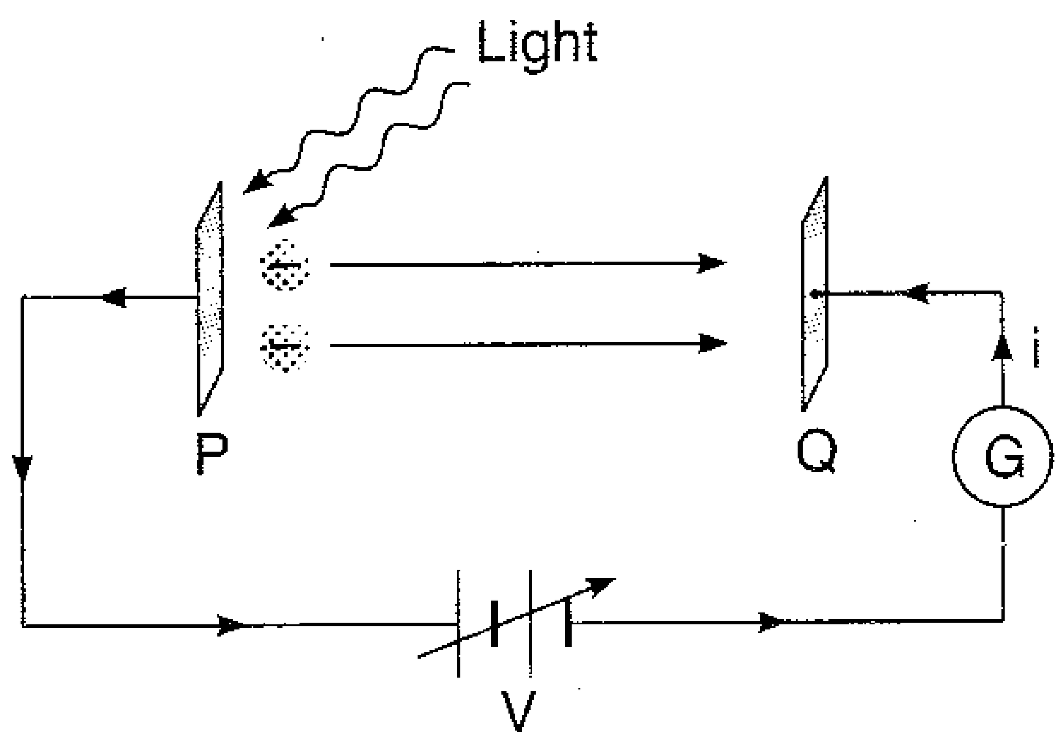


Fig. 24.15

This value K_{max} can experimentally be found by keeping the metal plate P (from which electrons are emitting) at higher potential relative to an another plate Q placed infront of P. Some electrons after emitting from plate P, reach the plate Q despite the fact that Q is at lower potential and it is repelling the electrons from reaching in itself. This is because the electrons emitted from plate P possess some kinetic energy and due to this energy they reach the plate Q and current i flows in the circuit in the direction shown in figure.

As the potential V is increased, the force of repulsion to the electrons gets increased and less number of electrons reach the plate Q and current in the circuit gets decreased. At a certain value V_0 electrons having maximum kinetic energy (K_{max}) also get stopped and current in the circuit becomes zero. This is called the **stopping potential**.

As an electron moves from P to Q, the potential decreases by V_0 and negative work $-eV_0$ is done on the (negatively charged) electron, the most energetic electron leaves plate P with kinetic energy $K_{max} = \frac{1}{2} mv_{max}^2$ and has zero kinetic energy at Q. Using the work energy theorem, we have

$$W_{ext} = -eV_0 = \Delta K = 0 - K_{max}$$

or

$$K_{max} = \frac{1}{2} mv_{max}^2 = eV_0 \quad \text{...(vi)}$$

Photoelectric current

Figure shows an apparatus used to study the variation of photo current i with the intensity and frequency of light falling on metal plate P. Photoelectrons are emitted from plate P which are being attracted by the positive plate Q and a photoelectric current i flows in the circuit, which can be measured by the galvanometer G.

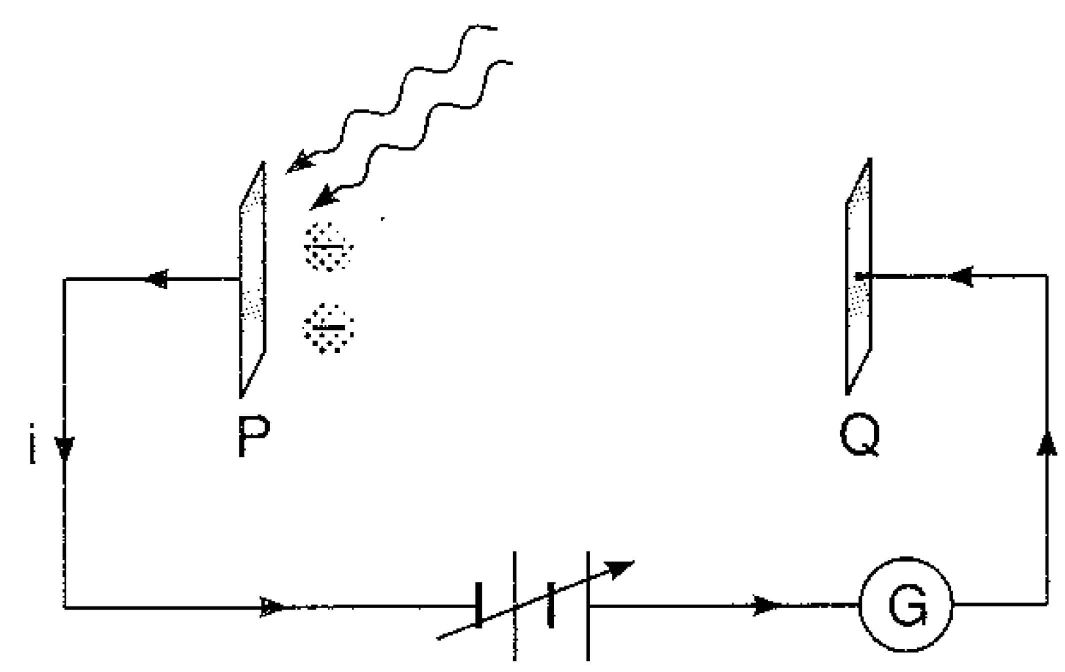


Fig. 24.16

Figure (a) shows graphs of photocurrent as a function of potential difference V_{QP} for light of constant frequency and two different intensities. When V_{QP} is sufficiently large and positive the current becomes constant, showing that all the emitted electrons are being collected by the anode plate Q. The stopping potential difference $-V_0$ needed to reduce the current to zero is shown. If the intensity of light is

(a) Photocurrent i as a function of the potential V_{QP} of the anode with respect to the cathode for a constant light frequency f, the stopping potential V_0 is independent of the light intensity I.

(b) Photocurrent i as a function of the potential V_{QP} of an anode with respect to a cathode for two different light frequencies f_1 and f_2 with the same intensity. The stopping potential V_0 (and therefore the maximum kinetic energy of the photoelectrons) increases linearly with frequency.

Fig. 24.17

increased, (or we can say the number of photons incident per unit area per unit time is increased) while its frequency is kept the same, the current becomes constant at a higher value, showing that more electrons are being emitted per unit time. But the stopping potential is found to be the same.

Figure (b) shows current as a function of potential difference for two different frequencies, with the same intensity in each case. We see that when the frequency of the incident monochromatic light is increased, the stopping potential V_0 gets increased, Of course, V_0 turns out to be a linear function of the frequency f.

Note : There major features of the photoelectric effect could not be explained by the wave theory of light which were later explained by Einstein's photon theory.

(i) Wave theory suggests that the kinetic energy of the photoelectrons should increase with the increase in intensity of light. However, Eq. (iv), $K_{max} = eV_0$ suggests that it is independent of the intensity of light.

(ii) According to wave theory, the photoelectric effect should occur for any frequency of the light, provided that the light is intense enough. However Eq. (iv) suggests that photo emission is possible only when frequency of incident light is either greater than or equal to the threshold frequency f_0.

(iii) If the energy to the photo electrons is obtained by soaking up from the incident wave, it is not likely that the effective target area for an electron in the metal is much more than a few atomic diameters. (see example 24.13) between the impinging of the light on the surface and the ejection of the photo electrons. During this interval the electron should be "soaking up" energy from the beam until it had accumulated enough energy to escape. However no detectable time lag has ever been measured.

EXAMPLE 24.13 *A metal plate is placed 5 metres from a monochromatic light source whose power output is 10^{-3} W. Consider that a given ejected photoelectron may collect its energy from a circular area of the plate as large as ten atomic diameters (10^{-9} m) in radius. The energy required to remove an electron through the metal surface is about 5.0 eV. Assuming light to be a wave, how long would it take for such a 'target' to soak up this much energy from such a light source.*

SOLUTION The target area is $S_1 = \pi(10^{-9})^2 = \pi \times 10^{-18}\ \text{m}^2$. The area of a 5 metre sphere centered on the light source is, $S_2 = 4\pi(5)^2 = 100\ \pi\ \text{m}^2$. Thus, if the light source radiates uniformly in all directions the rate P at which energy falls on the target is given by,

$$P = (10^{-3}\ \text{watt})\left(\frac{S_1}{S_2}\right) = (10^{-3})\left(\frac{\pi \times 10^{-18}}{100 \times \pi}\right) = 10^{-23}\ \text{J/s}$$

Assuming that all power is absorbed, the required time is,

$$t = \left(\frac{5\ \text{eV}}{10^{-23}\ \text{J/s}}\right)\left(\frac{1.6 \times 10^{-19}\ \text{J}}{1\text{eV}}\right) \approx 20\ \text{hrs}$$

Ans.

Einstein's photon theory

Einstein succeeded in explaining the photoelectric effect by making a remarkable assumption, that the energy in a light beam travels through space in concentrated bundles, called photons. The energy E of a single photon is given by

$$E = hf$$

Applying the photon concept to the photoelectric effect, Einstein wrote

$$hf = W + K_{max} \qquad \text{(already discussed)}$$

Consider how Einstein's photon hypothesis meets the three objections raised against the wave theory interpretation of the photoelectric effect.

As for objection 1 (the lack of dependence of K_{max} on the intensity of illumination), doubling the light intensity merely doubles the number of photons and thus doubles the photoelectric current, it does not change the energy of the individual photons

Objection 2 (the existence of a cutoff frequency) follows from equation $hf = W + K_{max}$. If K_{max} equals zero, We have $hf_0 = W$ which asserts that the photon has just enough energy to eject the photoelectrons and none extra to appear as kinetic energy. The quantity W is called the work function of the substance. If f is reduced below f_0, the individual photons, no matter how many of them there are (that is, no matter how intense the illumination), will not have enough energy to eject photo electrons.

Objection 3 (the absence of a time lag) follows from the photon theory because the required energy is supplied in a concentrated bundle. It is not spread uniformly over a large area, as in the wave theory. Although the photon hypothesis certainly fits the facts of photoelectricity, it seems to be in direct conflict with the wave theory of light. Out modern view of the nature of light is that it has a dual character, behaving like a wave under some circumstances and like a particle, or photon, under others.

Graph between K_{max} and f

Let us plot a graph between maximum kinetic energy K_{max} of photoelectrons and frequency f of incident light. The equation between K_{max} and f is,

$$K_{max} = hf - W$$

comparing it with $y = mx + c$, the graph between K_{max} and f is a straight line with positive slope and negative intercept.

From the graph we can note the following points.

(i) $K_{max} = 0$ at $f = f_0$

(ii) Slope of the straight line is h, a universal constant. *i.e.*, if graph is plotted for two different metals 1 and 2, slope of both the lines is same.

(iii) The negative intercept of the line is W, the work function, which is characteristic of a metal, *i.e.*, intercepts for two different metals will be different. Further,

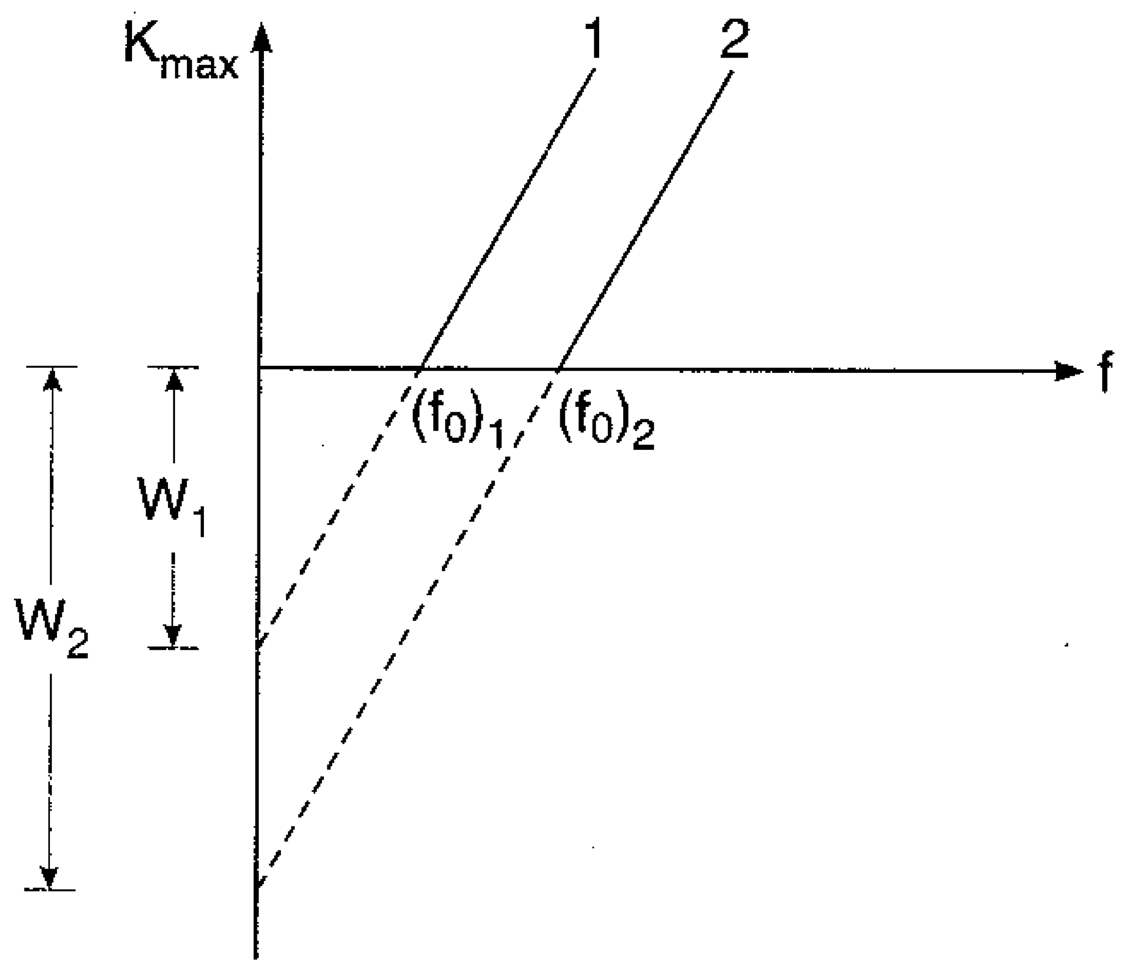


Fig. 24.18

$$W_2 > W_1 \quad \therefore \quad (f_0)_2 > (f_0)_1 \qquad \text{as } W = hf_0$$

Here $\qquad f_0 =$ threshold frequency

Graph between V_0 and f

Let us now plot a graph between the stopping potential V_0 and the incident frequency f. The equation between them is,

$$eV_0 = hf - W$$

or $$V_0 = \left(\frac{h}{e}\right) f - \left(\frac{W}{e}\right)$$

Again comparing with $y = mx + c$, the graph between V_0 and f is a straight line with positive slope $\frac{h}{e}$ (a universal constant) and negative intercept $\frac{W}{e}$ (which depends on the metal). The corresponding graph is shown in figure.

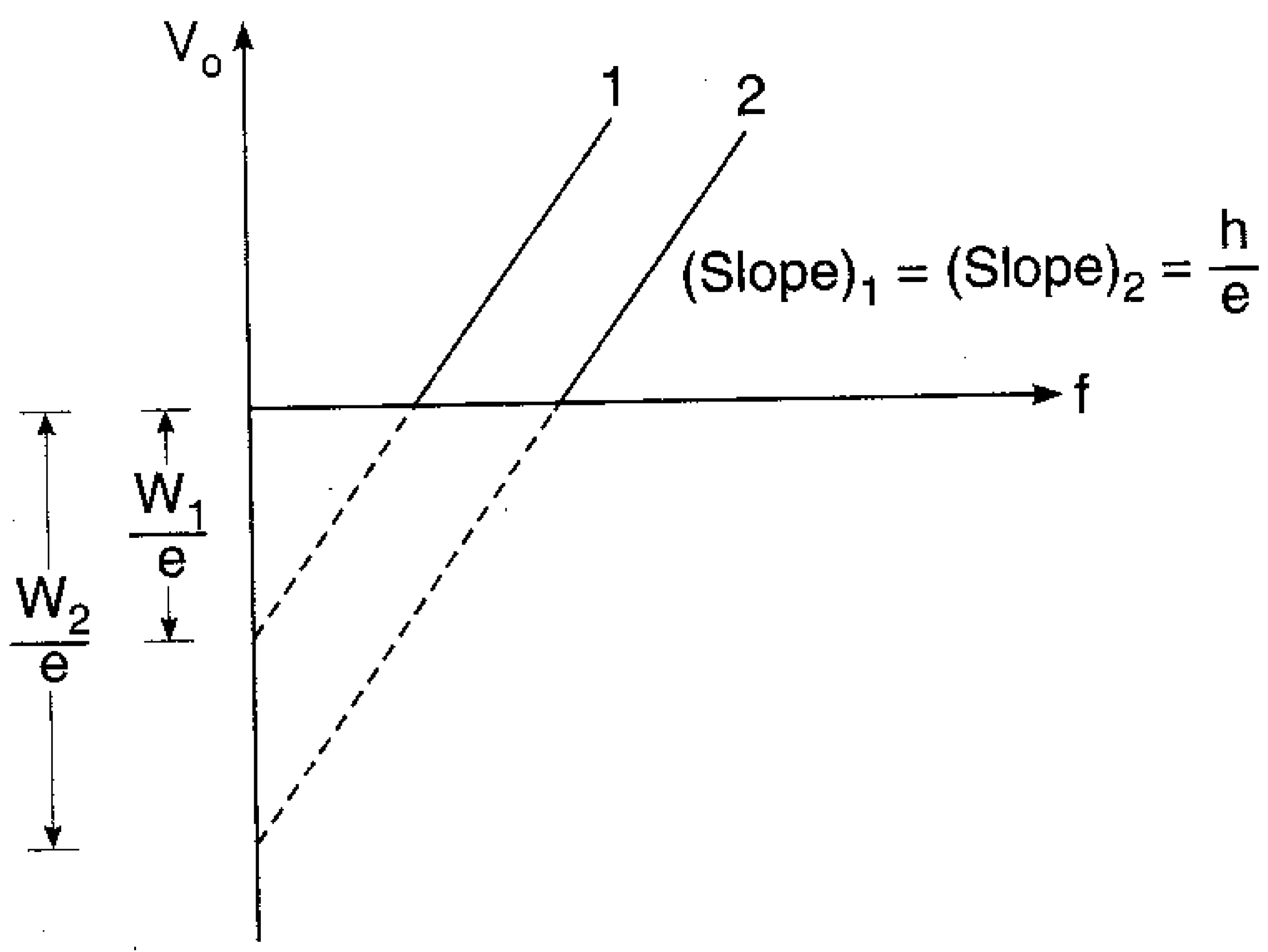


Fig. 24.19

EXAMPLE 24.14 *The photoelectric work function of potassium is 2.3 eV. If light having a wavelength of 2800 Å falls on potassium, find*
(a) the kinetic energy in electron volts of the most energetic electrons ejected.
(b) the stopping potential in volts

SOLUTION Given, $W = 2.3\text{ eV}$ $\lambda = 2800\text{ Å}$

$\therefore$ $$E\text{ (in eV)} = \frac{12375}{\lambda\text{ (in Å)}} = \frac{12375}{2800} = 4.4\text{ eV}$$

(a) $$K_{max} = E - W$$
$$= (4.4 - 2.3)\text{ eV}$$
$$= 2.1\text{ eV}$$ **Ans.**

(b) $$K_{max} = eV_0$$
$\therefore$ $$2.1\text{ eV} = eV_0$$
or $$V_0 = 2.1\text{ volt}$$ **Ans.**

INTRODUCTORY EXERCISE 24.2

1. A silver ball is suspended by a string in a vacuum chamber and ultraviolet light of wavelength 2000 Å is directed at it. What electrical potential will the ball acquire as a result? Work function of silver is 4.3 eV.
2. 1.5 mW of 400 nm light is directed at a photoelectric cell. If 0.1% of the incident photons produce photo electrons, find the current in the cell.
3. Is it correct to say that K_{max} is proportional to f. If not, what would a correct statement of the relationship between K_{max} and f.
4. Light of wavelength 2000 Å is incident on a metal surface of work function 3.0 eV. Find the minimum and maximum kinetic energy of the photoelectrons.
5. When a metal is illuminated with light of frequency f the maximum kinetic energy of the photoelectrons is 1.2 eV. When the frequency is increased by 50% the maximum kinetic energy increases to 3.6 eV. What is the threshold frequency for this metal.

MISCELLANEOUS EXAMPLES

EXAMPLE 1 *The nucleus of a deuterium has a mass of* 3.34×10^{-27} *kg as compared to* 1.67×10^{-27} *kg for the hydrogen. Calculate the wavelength difference between the first Balmer line emitted by hydrogen and the first Balmer line emitted by deuterium.* ($m_e = 9.109 \times 10^{-31}$ *kg)*

SOLUTION First Balmer line corresponds to transition from $n = 3$ to $n = 2$. In case of hydrogen,

$$\Delta E = E_3 - E_2$$

$$= \left\{ -\frac{13.6}{3^2} - \left(\frac{-13.6}{2^2} \right) \right\} \text{eV} = 1.89 \text{ eV}$$

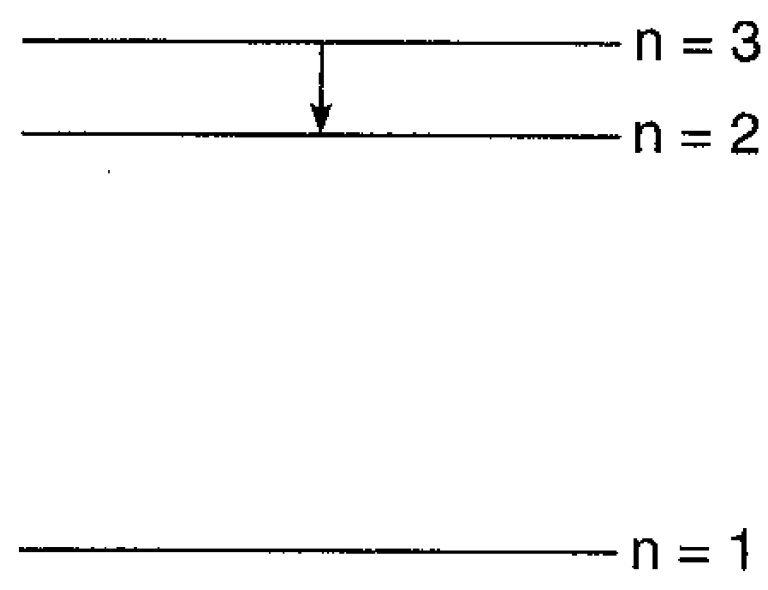


Fig. 24.20

$\therefore$ Wavelength
$$\lambda = \frac{12375}{\Delta E \text{ (in eV)}}$$

$$= \frac{12375}{1.89} = 6547.6 \text{ Å}$$

For ordinary hydrogen reduced mass of proton and electron is,

$$\mu_1 = \frac{(1.67 \times 10^{-27})(9.109 \times 10^{-31})}{(1.67 \times 10^{-27} + 9.109 \times 10^{-31})}$$

$$= 9.10408 \times 10^{-31} \text{ kg}$$

For deuterium atom reduced mass of nucleus and electron is,

$$\mu_2 = \frac{(3.34 \times 10^{-27})(9.109 \times 10^{-31})}{(3.34 \times 10^{-27} + 9.109 \times 10^{-31})}$$

$$= 9.10654 \times 10^{-31} \text{ kg}$$

All energies are proportional to μ, while wavelengths are inversely proportional to μ, wavelength of photon emitted in case of hydrogen

$$\lambda_1 = \frac{(6547.6)(9.109 \times 10^{-31})}{(9.10408 \times 10^{-31})}$$

$$= 6551 \text{ Å}$$

Similarly in case of deuterium, wavelength of photon emitted is,

$$\lambda_2 = \frac{(6547.6)(9.109 \times 10^{-31})}{(9.10654 \times 10^{-31})} = 6549 \text{Å}$$

$\therefore$
$$\Delta\lambda = \lambda_2 - \lambda_1 = 2 \text{ Å}$$
Ans.

EXAMPLE 2 *Stopping potential of* 24, 10, 110 *and* 115 *kV are measured for photoelectrons emitted from a certain element when it is radiated with monochromatic X-ray. If this element is used as a target in an X-ray tube, what will be the wavelength of* K_α *line?*

SOLUTION Stopping potentials are 24, 100, 110 and 115 kV.

i.e., if the electrons are emitted from conduction band, maximum kinetic energy of photoelectrons would be 115×10^3 eV. If they are emitted from next inner shell maximum kinetic energy of photoelectrons would be 110×10^3 eV and so on.

For photoelectrons of L shell it would be 100×10^3 3 eV and for K shell it is 24×10^3 eV. Therefore, difference between energy of L shell and K-shell is,

$$\Delta E = E_L - E_K = (100 - 24) \times 10^3 \text{ eV}$$
$$= 76 \times 10^3 \text{ eV}$$

$\therefore$ Wavelength of K_α line (transition of electron from L-shell to K-shell) is,

$$\lambda_{K_\alpha} (\text{in Å}) = \frac{12375}{\Delta E \text{ (in eV)}}$$
$$= \frac{12375}{76 \times 10^3}$$
$$= 0.163 \text{ Å}$$

Ans.

EXAMPLE 3 *A gas of hydrogen like ions is prepared in such a way that the ions are only in the ground state and the first excited state. A monochromatic light of wavelength 1216 Å is absorbed by the ions. The ions are lifted to higher excited states and emit radiations of six wavelengths, some higher, some lower or some greater than the incident wavelength.*

(*a*) *Find the principle quantum number of the final excited state.*
(*b*) *Identify the nuclear charge on the ions*
(*c*) *Calculate the value of the maximum and minimum wavelengths.*

SOLUTION (a) Energy corresponding to $\lambda = 1216$ Å is,

$$E = \frac{12375}{1216} \text{ eV} = 10.177 \text{ eV}$$

Now, total six wavelengths are obtained in the emission spectrum, hence from

$$\frac{n(n-1)}{2} = 6$$

We have $n = 4$, *i.e.*, after excitation the single electron jumps to 3rd excited state or $n = 4$. **Ans.**

(b) Now it may jump either from $n = 1$ or $n = 2$.

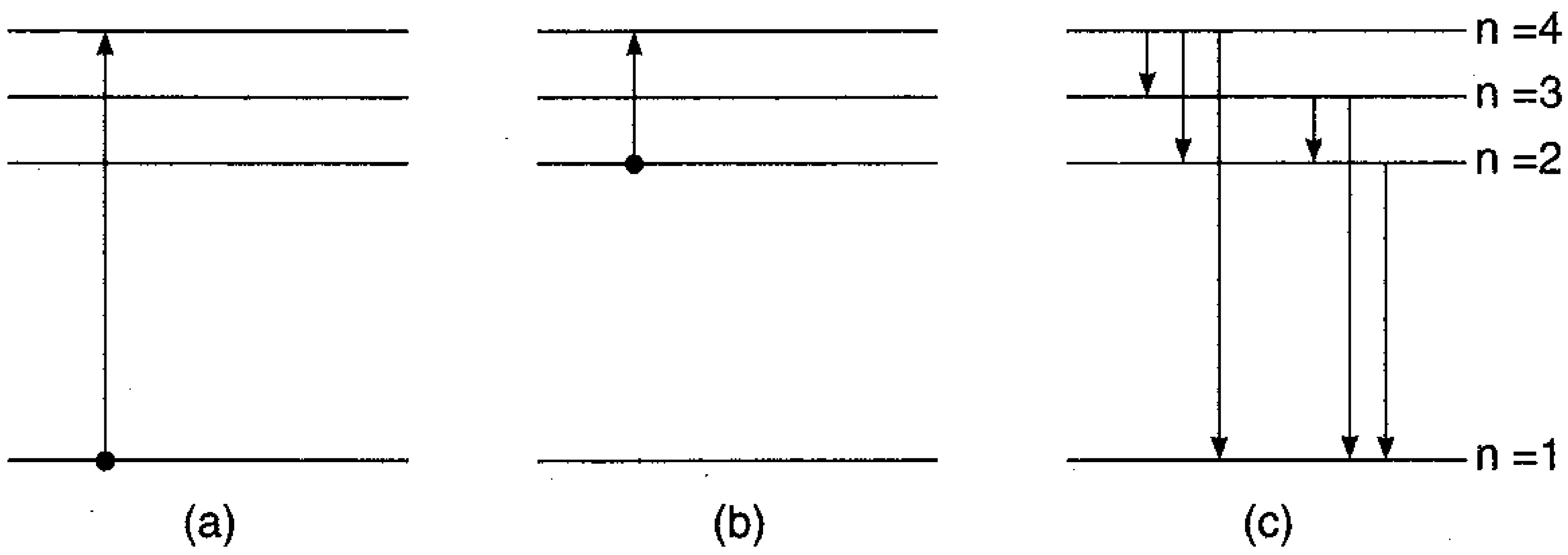


Fig. 24.21

If it jumps from $n = 1$, then in emission spectrum all the six photons have energy equal to or less than the energy of absorbed photon or the wavelength of emitted photon is either equal to or greater than the wavelength of absorbed photon. While in the question it is given that the emitted wavelengths are either less than or greater than or smaller than the wavelength of absorbed photon. Which is possible only in the second case, *i.e.*, when electron jumps from $n = 2$ to $n = 4$.

Hence, $$E_4 - E_2 = 10.177 \text{ eV}$$

or $$\frac{-13.6\, z^2}{4^2} - \left(\frac{-13.6\, z^2}{2^2}\right) = 10.177$$

Solving this, we get $$z \approx 2$$ **Ans.**

(c) Maximum wavelength corresponds to minimum energy, *i.e.*, a transition from $n = 4$ to $n = 3$.

Thus, $$\Delta E_{\text{min}} = E_4 - E_3$$

$$= \frac{(-13.6)\,(2)^2}{(4)^2} - \left[\frac{(-13.6)\,(2)^2}{(3)^2}\right]$$

$$= 2.64 \text{ eV}$$

$\therefore$ $$\lambda_{\text{max}} = \frac{12375}{2.64} = 4687 \text{ Å}$$ **Ans.**

Minimum wavelength corresponds to maximum energy, *i.e.*, a transition from $n = 4$ to $n = 1$.

Hence, $$\Delta E_{\text{max}} = E_4 - E_1$$

$$= \frac{(-13.6)\,(2)^2}{(4)^2} - \left[\frac{(-13.6)\,(2)^2}{(1)^2}\right]$$

$$= 51.0 \text{ eV}$$

$\therefore$ $$\lambda_{\text{min}} = \frac{12375}{51.0}$$

$$= 242 \text{ Å}$$ **Ans.**

EXAMPLE 4 *Light from a discharge tube containing hydrogen atoms falls on the surface of a piece of sodium. The kinetic energy of the fastest photo-electrons emitted from sodium is* 0.73 *eV. The work function for sodium is* 1.82 *eV. Find:*

(*a*) *the energy of the photons causing the photo-electric emission.*

(*b*) *the quantum number of the two levels involved in the emission of these photons.*

(*c*) *the change in the angular momentum of the electron in the hydrogen atom in the above transition.*

(*d*) *the recoil speed of the emitting atom assuming it to be at rest before the transition (Take mass of hydrogen* $= 1.6 \times 10^{-27}$ *kg)*

SOLUTION Given $K_{\text{max}} = 0.73$ eV and $W = 1.82$ eV.

(a) $$E = K_{\text{max}} + W = (0.73 + 1.82) \text{ eV} = 2.55 \text{ eV}$$ **Ans.**

(b) $$E_n = -\frac{13.6}{n^2} \text{ eV}$$ (for hydrogen atom)

$\therefore$ $E_1 = -13.6$ eV, $E_2 = -3.4$ eV, $E_3 = -1.51$ eV and $E_4 = -0.85$ eV

Clearly $$E_4 - E_2 = -0.85 - (-3.4) = 2.55\,\text{eV}$$

Hence, quantum levels involved are 4 and 2. **Ans.**

(c) $$\Delta L = L_4 - L_2 = 4\left(\frac{h}{2\pi}\right) - 2\left(\frac{h}{2\pi}\right)$$

$$= \frac{h}{\pi}$$ **Ans.**

(d) If p = linear momentum

then $$p = \frac{E}{c} = \frac{2.55 \times 1.6 \times 10^{-19}}{3.0 \times 10^{8}} = 1.36 \times 10^{-27}\ \text{kg–m/s}$$

If v = recoil speed of hydrogen atom of mass M then

from conservation of linear momentum $p = Mv$

$\therefore$ $$v = \frac{p}{M} = \frac{1.36 \times 10^{-27}}{1.6 \times 10^{-27}} = 0.85\ \text{m/s}$$ **Ans.**

EXAMPLE 5 *A hydrogen-like atom of atomic number z is in an excited state of quantum number* $2n$. *It can emit a maximum energy photon of* $204\,eV$. *If it makes a transition to quantum state n, a photon of energy* $40.8\ eV$ *is emitted. Find n, z and the ground state energy (in eV) for this atom. Also, calculate the minimum energy (in eV) that can be emitted by this atom during de-excitation. Ground state energy of hydrogen atom is* $13.6\ eV$.

SOLUTION Given $E_{2n} - E_1 = 204\,\text{eV}$

$\therefore$ $$(13.6)z^2\left(1 - \frac{1}{4n^2}\right) = 204 \qquad \text{...(i)}$$

$$E_{2n} - E_n = 40.8\,\text{eV}$$

$\therefore$ $$13.6\,z^2\left(\frac{1}{n^2} - \frac{1}{4n^2}\right) = 40.8 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have $n = 2$ and $z = 4$ **Ans.**

$$E_1 = (-13.6)\,z^2\ \text{eV}$$

$$= (-13.6)\,(4)^2\ \text{eV}$$

$$= -217.6\,\text{eV}$$ **Ans.**

During de-excitation, minimum energy emitted is,

$$E_{\min} = E_{2n} - E_{2n-1} = E_4 - E_3$$

$$= \frac{-217.6}{4^2} - \left(\frac{-217.6}{3^2}\right)$$

$$= 10.58\,\text{eV}$$ **Ans.**

EXAMPLE 6 *When a beam of* $10.6\ eV$ *photons of intensity* $2.0\ W/m^2$ *falls on a platinum surface of area* $1.0\times10^{-4}\ m^2$ *and work function* $5.6\ eV$, 0.53% *of the incident photons eject photo electrons. Find the number of photoelectrons emitted per second and their minimum and maximum energies (in eV). Take* $1\ eV = 1.6\times10^{-19}\ J$.

SOLUTION Number of photoelectrons emitted per second

$$=\frac{\text{(Intensity) (Area)}}{\text{(Energy of each photon)}}\times\frac{0.53}{100}$$

$$=\frac{(2.0)(1.0\times10^{-4})}{(10.6\times1.6\times10^{-19})}\times\frac{0.53}{100}$$

$$=6.25\times10^{11}$$ **Ans.**

Minimum kinetic energy of photoelectrons,

$$K_{\min}=0$$

and maximum kinetic energy is, $K_{\max}=E-W=(10.6-5.6)\ \text{eV}$

$$=5.0\ \text{eV}$$ **Ans.**

EXAMPLE 7 *A monochromatic light source of frequency illuminates a metallic surface and ejects photoelectrons. The photo electrons having maximum energy are just able to ionize the hydrogen atoms in ground state. When the whole experiment is repeated with an incident radiation of frequency* $\frac{5}{6}f$, *the photoelectrons so emitted are able to excite the hydrogen atom beam which then emits a radiation of wavelength* $1215\ Å$.

(*a*) *What is the frequency of radiation.*
(*b*) *Find the work function of the metal.*

SOLUTION (a) Using Einstein's equation of photoelectric effect,

$$K_{\max}=hf-W$$

Here $K_{\max}=13.6\ \text{eV}$

$\therefore$ $hf-W=13.6\ \text{eV}$ ...(i)

Further, $h\left(\frac{5}{6}f\right)-W=\frac{12375}{1215}=10.2\ \text{eV}$ ...(ii)

Solving Eqs. (i) and (ii), we have

$$\frac{hf}{6}=3.4\ \text{eV}$$

or $f=\frac{(6)(3.4)(1.6\times10^{-19})}{(6.63\times10^{-34})}=4.92\times10^{15}\ \text{Hz}$ **Ans.**

(b) $W=hf-13.6$ [from Eq. (i)]

$$=6(3.4)-13.6$$

$$=6.8\ \text{eV}$$ **Ans.**

EXAMPLE 8 *In Moseley's equation $\sqrt{f} = a(z-b)$, a and b are constant. Find their values with the help of the following data.*

Element	z	Wavelength of K_α X-rays
Mo	42	0.71 Å
Co	27	1.785 Å

SOLUTION $$\sqrt{f} = a(z-b)$$

or $$\sqrt{\frac{c}{\lambda_1}} = a(z_1 - b) \quad \text{...(i)}$$

and $$\sqrt{\frac{c}{\lambda_2}} = a(z_2 - b) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\sqrt{c}\left[\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right] = a(z_1 - z_2) \quad \text{...(iii)}$$

Solving above three equations with $c = 3.0 \times 10^8$ m/s, $\lambda_1 = 0.71 \times 10^{-10}$ m
$\lambda_2 = 1.785 \times 10^{-10}$ m, $z_1 = 42$ and $z_2 = 27$, we get

$$a = 5 \times 10^7 \text{ (Hz)}^{1/2} \quad \text{and} \quad b = 1.37$$ **Ans.**

EXAMPLE 9 *The energy levels of a hypothetical one electron atom are given by*

$$E_n = -\frac{18.0}{n^2} eV$$

where $n = 1, 2, 3, \ldots$

(*a*) *Compute the four lowest energy levels and construct the energy level diagram.*

(*b*) *What is the excitation potential of the stage $n = 2$?*

(*c*) *What wavelengths (Å) can be emitted when these atoms in the ground state are bombarded by electrons that have been accelerated through a potential difference of 16.2 V.*

(*d*) *If these atoms are in the ground state, can they absorb radiation having a wavelength of 2000 Å?*

(*e*) *What is the photoelectric threshold wavelength of this atom?*

SOLUTION (a) $$E_1 = \frac{-18.0}{(1)^2} = -18.0 \text{ eV}$$

$$E_2 = \frac{-18.0}{(2)^2} = -4.5 \text{ eV}$$

$$E_3 = \frac{-18.0}{(3)^2} = -2.0 \text{ eV}$$

and $$E_4 = \frac{-18.0}{(4)^2} = -1.125 \text{ eV}$$

The energy level diagram is shown below.

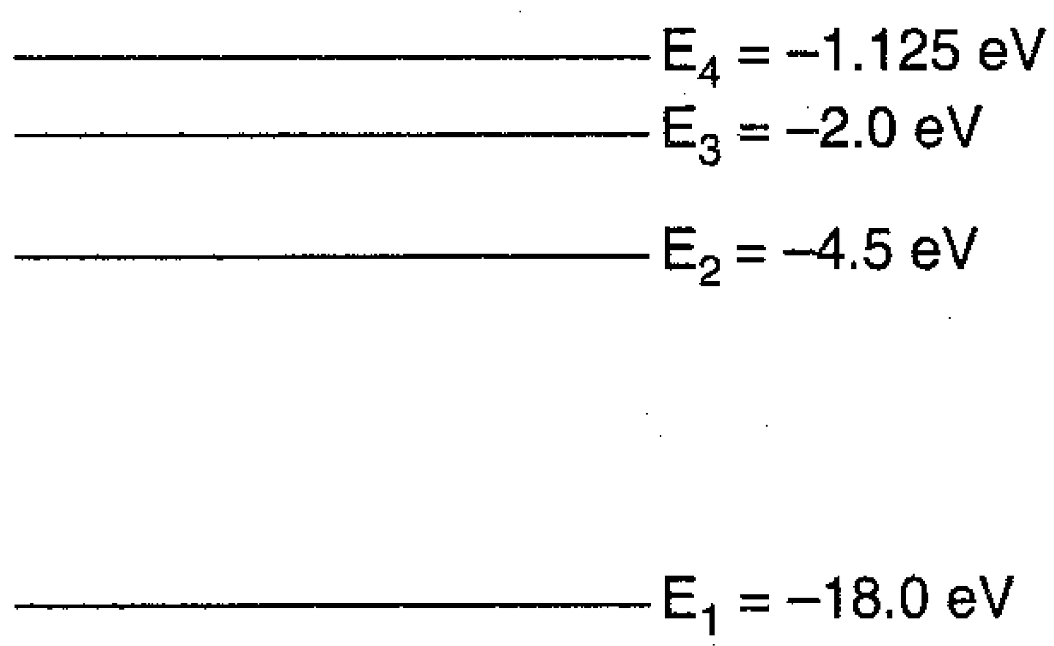


Fig. 24.22

(b) The excitation potential of stage $n = 2$ is, $18.0 - 4.5 = 13.5$ volt **Ans.**

(c) Energy of the electron accelerated by a potential difference of 16.2 V is 16.2 eV. With this energy the electron can excite the atom from $n = 1$ to $n = 3$ as

$$E_4 - E_1 = -1.125 - (-18.0) = 16.875\,\text{eV} > 16.2\,\text{eV}$$

and

$$E_3 - E_1 = -2.0 - (-18.0) = 16.0\,\text{eV} < 16.2\,\text{eV}$$

Now,

$$\lambda_{32} = \frac{12375}{E_3 - E_2} = \frac{12375}{-2.0 - (-4.5)}$$

$$= 4950 \text{ Å}$$ **Ans.**

$$\lambda_{31} = \frac{12375}{E_3 - E_1} = \frac{12375}{16} = 773 \text{ Å}$$ **Ans.**

and

$$\lambda_{21} = \frac{12375}{E_2 - E_1} = \frac{12375}{-4.5 - (-18.0)}$$

$$= 917 \text{ Å}$$ **Ans.**

(d) No, the energy corresponding to $\lambda = 2000$ Å is,

$$E = \frac{12375}{2000} = 6.1875\,\text{eV}$$ **Ans.**

The minimum excitation energy is 13.5 eV ($n = 1$ to $n = 2$).

(e) Threshold wavelength for photoemission to take place from such an atom is,

$$\lambda_{\text{min}} = \frac{12375}{18} = 687.5 \text{ Å}$$ **Ans.**

EXAMPLE 10 *If an X-ray tube operates at the voltage of* $10kV$, *find the ratio of the de-Broglie wavelength of the incident electrons to the shortest wavelength of X-rays produced. The specific charge of electron is* $1.8 \times 10^{11}\ C/kg$.

SOLUTION de Broglie wavelength when a charge q is accelerated by a potential difference of V volts is

$$\lambda_b = \frac{h}{\sqrt{2qVm}} \qquad \text{...(i)}$$

For cutoff wavelength of X-rays, we have $qV = \dfrac{hc}{\lambda_m}$

or
$$\lambda_m = \frac{hc}{qV} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii)
$$\frac{\lambda_b}{\lambda_m} = \frac{\sqrt{\frac{qV}{2m}}}{c}$$

For electron $\frac{q}{m} = 1.8 \times 10^{11}$ C/kg (given). Substituting the values the desired ratio is

$$\frac{\lambda_b}{\lambda_m} = \frac{\sqrt{\frac{1.8 \times 10^{11} \times 10 \times 10^3}{2}}}{3 \times 10^8}$$

$$= 0.1$$ **Ans.**

EXAMPLE 11 *In a photocell the plates P and Q have a separation of* 10 *cm, which are connected through a galvanometer without any cell. Bichromatic light of wavelengths* 4000 *Å and* 6000 *Å are incident on plate Q whose work function is* 2.39 *eV. If a uniform magnetic field B exists parallel to the plates, find the minimum value of B for which the galvanometer shows zero deflection.*

SOLUTION Energy of photons corresponding to light of wavelength $\lambda_1 = 4000$ Å is

$$E_1 = \frac{12375}{4000} = 3.1\,\text{eV}$$

and that corresponding to $\lambda_2 = 6000$ Å is,

$$E_2 = \frac{12375}{6000} = 2.06\,\text{eV}$$

As $E_2 < W$ and $E_1 > W$ (W = Work function)

Photoelectric emission is possible with λ_1 only.

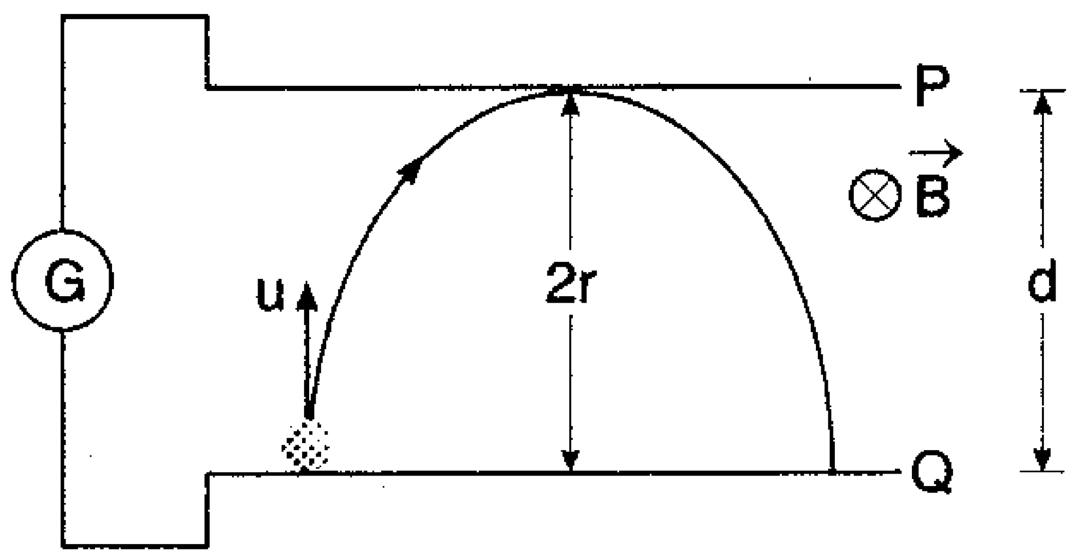


Fig. 24.23

Photoelectrons experience magnetic force and move along a circular path. The galvanometer will indicate zero deflection if the photoelectrons just complete semi-circular path before reaching the plate P.

Thus, $d = 2r = 10$ cm $\quad \therefore r = 5$ cm $= 0.05$ m

Further
$$r = \frac{mv}{Bq} = \frac{\sqrt{2Km}}{Bq}$$

$\therefore$ $$B_{min} = \frac{\sqrt{2Km}}{rq}$$

Here $$K = E_1 - W = (3.1 - 2.39) = 0.71\,\text{eV}$$

Substituting the values, we have

$$B_{min} = \frac{\sqrt{2 \times 0.71 \times 1.6 \times 10^{-19} \times 9.109 \times 10^{-31}}}{(0.05)\,(1.6 \times 10^{-19})}$$

$$= 5.68 \times 10^{-5}\ \text{Tesla}$$ **Ans.**

EXAMPLE 12 *A doubly ionized lithium atom is hydrogen like with atomic number 3. Find the wavelength of the radiation required to excite the electron in* Li^{++} *from the first to the third Bohr orbit. The ionization energy of the hydrogen atom is* 13.6 *eV*.

SOLUTION $$E_n = -\frac{z^2}{n^2}(13.6\,\text{eV})$$

By putting $z = 3$, we have $$E_n = -\frac{122.4}{n^2}\,\text{eV}$$

$$E_1 = -\frac{122.4}{(1)^2} = -122.4\,\text{eV}$$

and $$E_3 = -\frac{122.4}{(3)^2} = -13.6\,\text{eV}$$

$\therefore$ $$\Delta E = E_3 - E_1 = 108.8\,\text{eV}$$

The corresponding wavelength is

$$\lambda = \frac{12375}{\Delta E\ (\text{in eV})}\,\text{Å} = \frac{12375}{108.8}\,\text{Å}$$

$$= 113.74\ \text{Å}$$ **Ans.**

EXAMPLE 13 *The wavelength of the first line of Lyman series for hydrogen is identical to that of the second line of Balmer series for some hydrogen like ion x. Calculate energies of the first four levles, of x.*

SOLUTION Wavelength of the first line of Lymen series for hydrogen atom will be given by the equation

$$\frac{1}{\lambda_1} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3R}{4} \qquad \text{...(i)}$$

The wavelength of second Balmer line for hydrogen like ion X is

$$\frac{1}{\lambda_2} = Rz^2\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = \frac{3Rz^2}{16} \qquad \text{...(ii)}$$

Given that $$\lambda_1 = \lambda_2 \quad \text{or} \quad \frac{1}{\lambda_1} = \frac{1}{\lambda_2}$$

i.e., $$\frac{3R}{4} = \frac{3Rz^2}{16}$$

$$\therefore \qquad z = 2$$

i.e., x ion is He^+. The energies of first four levels of x are,

$$E_1 = -(13.6)\, z^2 = -54.4\,\text{eV}$$

$$E_2 = \frac{E_1}{(2)^2} = -13.6\,\text{eV}$$

$$E_3 = \frac{E_1}{(3)^2} = -6.04\,\text{eV}$$

and $$E_4 = \frac{E_1}{(4)^2} = -3.4\,\text{eV}$$ **Ans.**

EXAMPLE 14 *In a photoelectric effect setup, a point source of light of power 3.2×10^{-3} W emits monoenergetic photons of energy 5 eV. The source is located at a distance of 0.8 m from the centre of a stationary metallic sphere of work function 3 eV and of radius 8×10^{-3}. The efficiency of photoelectron emission is one for every 10^6 incident photons. Sphere is initially neutral, and that the photoelectrons are instantly swept away after emission.*

(a) Calculate the number of photoelectrons emitted per second.

(b) It is observed that the photoelectrons emission stops at a certain time t after the light source is switched on. Evaluate time 't'.

SOLUTION (a) Intensity of light at a distance 0.8 m from the source

$$I = \frac{(3.2 \times 10^{-3}\ \text{J/s})}{4\pi (0.8)^2\ \text{m}^2} \approx 4.0 \times 10^{-4}\ \frac{\text{watt}}{\text{m}^2}$$

$\therefore$ Energy incident on the metallic sphere in unit time

$$E_1 = \pi (8 \times 10^{-3})^2\, (4.0 \times 10^{-4}) = 8.04 \times 10^{-8}\ \text{watt}$$

Energy of one single photon

$$E_2 = 5.0 \times 1.6 \times 10^{-19}\ \text{J} = 8.0 \times 10^{-19}\ \text{J}$$

Therefore, total number of photons incident on the sphere per second

$$n_1 = \frac{E_1}{E_2} = \frac{8.04 \times 10^{-8}}{8.0 \times 10^{-19}} \approx 10^{11}$$

Since, the efficiency fo photoelectric emission is one for every 10^6. Hence, total number of photoelectrons per second,

$$n_2 = \frac{n_1}{10^6} = \frac{10^{11}}{10^6} = 10^5$$ **Ans.**

(b) Maximum kinetic energy of photoelectrons

$$K_{\text{max}} = E - W = 2\,\text{eV}$$

$\therefore$ Stopping potential $V_0 = 2$ volt.

ASSIGNMENT

LEVEL-I

Useful data: $h = 4.316 \times 10^{-15}$ eV-sec $= 6.6 \times 10^{-34}$ J-s

$R = 1.09 \times 10^{7}\ \text{m}^{-1}$.

Ionization energy of hydrogen $= -13.6$ eV.

$$E\ (\text{in eV}) = \frac{12375}{\lambda\ (\text{in Å})}$$

1. Find the energy, the mass and the momentum of a photon of ultraviolet radiation of 280 nm wavelength.

2. Find the ionization energy of a doubly ionized lithium atom.

3. What is the energy of a H-atom in the first excited state if the potential energy is taken to be zero in the ground state.

4. The binding energy of an electron in the ground state of He atom is equal to $E_0 = 24.6$ eV. Find the energy required to remove both electrons from the atom.

5. Find the de Broglie wavelength corresponding to the root-mean-square velocity of hydrogen molecules at room temperature (20° C).

6. From what material is the anode of an X-ray tube made, if the K_α-line wavelength of the characteristic spectrum is 0.76 Å?

7. What will be the maximum kinetic energy of the photoelectrons ejected from magnesium (for which the work function $W = 3.7$ eV) when irradiated by ultraviolet light of frequency $1.5 \times 10^{15}\ \text{sec}^{-1}$.

8. A voltage applied to an X-ray tube being increased $\eta = 1.5$ times, the short wave limit of an X-ray continuous spectrum shifts by $\Delta\lambda = 26$ pm. Find the initial voltage applied to the tube.

9. If the wavelength of the incident radiation is increased from 3000 Å to 3010 Å, find the corresponding change in the stopping potential V.

10. Hydrogen atom in its ground state is excited by means of monochromatic radiation of wavelength 1023 Å. How many different lines are possible in the resulting spectrum? Calculate the longest wavelength among them. You may assume the ionization energy of hydrogen atom as 13.6 eV.

11. Suppose a monochromatic X-ray beam of wavelength 0.01 Å is sent through a Young's double slit and the interference pattern is observed on a photographic plate placed 40 cm away from the slit. What should be the separation between the slits so that the successive maxima on the screen are separated by a distance of 0.1 μm?

12. A metallic surface is irradiated with monochromatic light of variable wavelength. Above a wavelength of 5000 Å, no photoelectrons are emitted from the surface. With an unknown wavelength, stopping potential of 3 V is necessary to eliminate the photocurrent. Find the unknown wavelength.

13. In an experiment on photo electric emission, following observations were made:

(i) Wavelength of the incident light $= 1.98 \times 10^{-7}$ m; (ii) Stopping potential $= 2.5$ volt.

Find : (a) Threshold frequency; (b) Work function and

(c) Energy of photo electrons with maximum speed.

14. Illuminating the surface of a certain metal alternately with light of wavelengths $\lambda_1 = 0.35\ \mu m$ and $\lambda_2 = 0.54\ \mu m$, it was found that the corresponding maximum velocities of photo electrons differ by a factor $\eta = 2$. Find the work function of that metal.

15. A beam of light consists of four wavelength 4000 Å, 4800 Å, 6000 Å and 7000 Å, each of intensity $1.5 \times 10^{-3}\ Wm^{-2}$. The beam falls normally on an area $10^{-4}\ m^2$ of a clean metallic surface of work function 1.9 eV. Assuming no loss of light energy calculate the number of photoelectrons liberated per second.

16. A source emits monochromatic light of frequency 5.5×10^{14} Hz at a rate of 0.1 W. Of the photons given out, 0.15% fall on the cathode of a photocell which gives a current of 6 μA in an external circuit.

(a) Find the energy of a photon

(b) Find the number of photons leaving the source per second

(c) Find the percentage of the photons falling on the cathode which produce photoelectrons.

17. Radiation of wavelength 5461 Å falls on a photo cathode and electrons with a maximum kinetic energy of 0.18 eV are emitted. When radiation of wavelength 1849 Å falls on the same surface a (negative) potential of 4.6 V has to be applied to the collector electrode to reduce the photoelectric current to zero. Find the value of h and cutoff wavelength.

18. A 40 W ultraviolet light source of wavelength 2480 Å illuminates a magnesium (Mg) surface placed 2 m away. Determine the number of photons emitted from the source per second and the number incident on unit area of the Mg surface per second. The photoelectric work function for Mg is 3.68 eV. Calculate the kinetic energy of the fastest electrons ejected from the surface. Determine the maximum wavelength for which the photoelectric effect can be observed with a Mg surface.

19. A light source, emitting three wavelengths 5000 Å, 6000 Å and 7000 Å, has a total power of 10^{-3} W and a beam diameter 2×10^{-3} m. The power density is distributed equally amongst the three wavelengths. The beam shines normally on a metallic surface of area on 10^{-4} m^2 and having a work function of 1.9 eV. Assuming that each photon liberates an electron, calculate the charge emitted per second from the metal surface.

20. A small plate of a metal is placed at a distance of 2 m from a monochromatic light source of wavelength 4.8×10^{-7} m and power 1.0 watt. The light falls normally on the plate. Find the number of photons striking the metal plate per square meter per second.

21. Light of wavelength 180 nm ejects photoelectrons from a plate of a metal whose work function is 2 eV. If a uniform magnetic field of 5×10^{-5} tesla is applied parallel to the plate, what would be the radius of the path followed by electrons ejected normally from the plate with maximum energy.

22. The electric current in an X-ray tube operating at 40 kV is 10 mA. Assume that on an average 1% of the total kinetic energy of the electrons hitting the target are converted into X-rays.

(a) What is the total power emitted as X-rays and

(b) How much heat is produced in the target every second?

23. Find the quantum number n corresponding to nth excited state of He^+ ion if on transition to the ground state the ion emits two photons in succession with wavelengths 108.5 nm and 30.4 nm. The ionization energy of the hydrogen atom is 13.6 eV.

24. A hydrogen-like atom (described by the Bohr model) is observed to emit six wavelengths, originating from all possible transitions between a group of levels. These levels have energies between –0.85 eV and –0.544 eV (including both these values).

(a) Find the atomic number of the atom.

(b) Calculate the smallest wavelength emitted in these transitions.

(Take ground state energy of hydrogen atom = –13.6 eV.)

25. The hydrogen atom in its ground state is excited by means of monochromatic radiation. Its resulting spectrum has six different lines. These radiations are incident on a metal plate. It is observed that only two of them are responsible for photoelectric effect. If the ratio of maximum kinetic energy of photoelectrons in the two cases is 5 then find the work function of the metal.

[Take ground state energy of H-atom = – 13.6 eV]

26. The energy levels of a hypothetical one electron atom are shown in the figure.

(a) Find the ionization potential of this atom.

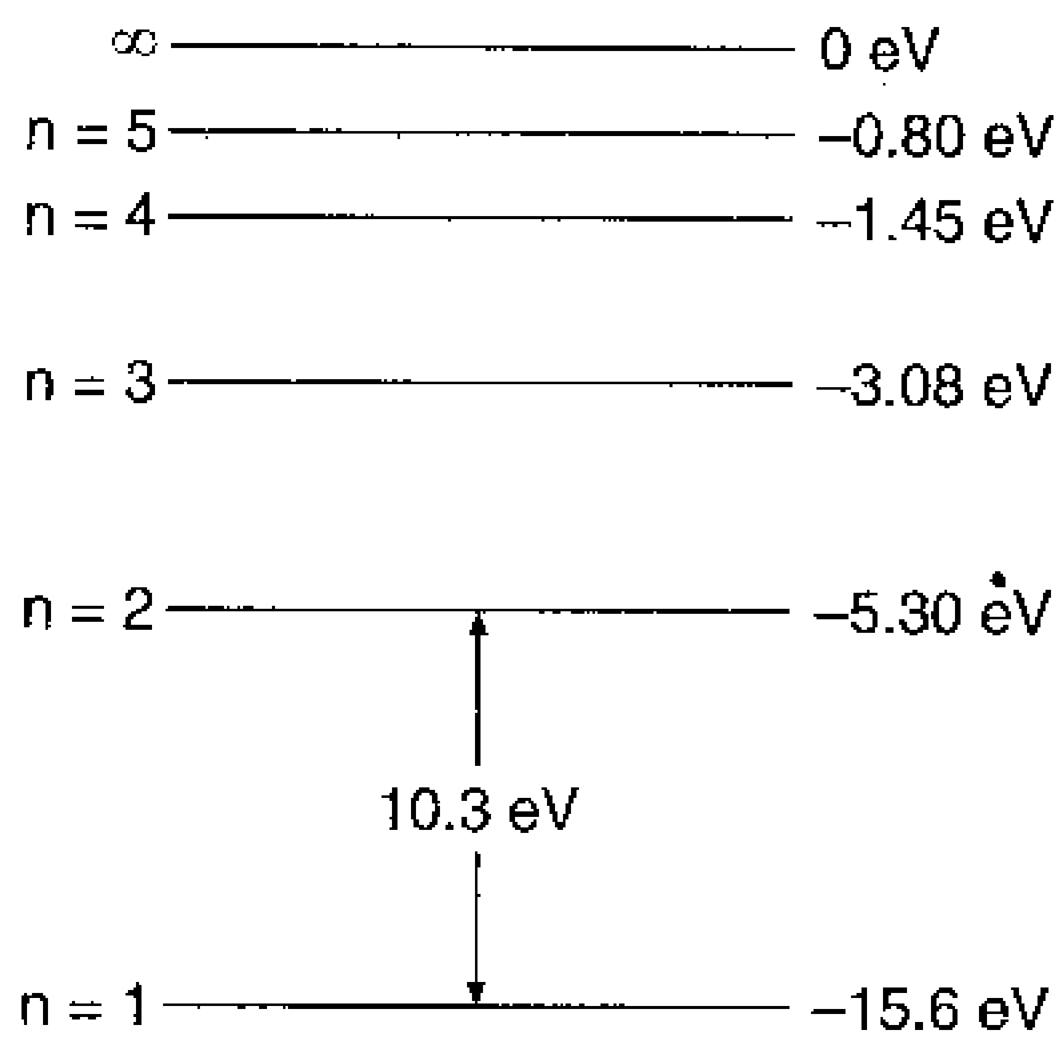
(b) Find the short wavelength limit of the series terminating at $n = 2$.

(c) Find the excitation potential for the state $n = 3$.

(d) Find wave number of the photon emitted for the transition $n = 3$ to $n = 1$.

(e) What is the minimum energy that an electron will have after interacting with this atom in the ground state if the initial kinetic energy of the electron is

(i) 6 eV (ii) 11 eV



27. A doubly ionised lithium atom is hydrogen-like with atomic number 3.

(a) Find the wavelength of the radiation require to excite the electron in Li^{++} from the first to the third Bohr orbit (Ionisation energy of the hydrogen atom equals 13.6 eV).

(b) How many spectral lines are observed in the emission spectrum of the above excited system.

28. Electrons in hydrogen like atoms ($Z = 3$) make transitions from the fifth to the fourth orbit and from the fourth to the third orbit. The resulting radiation are incident normally on a metal plate and eject photoelectrons. The stopping potential for the photoelectrons ejected by the shorter wavelength is 3.95 volts. Calculate the work function of the metal and the stopping potential for the photoelectrons ejected by the longer wavelength.

29. Emission spectrum of hydrogen atom has two lines of Balmer series with wavelength 4102 Å and 4861 Å. To what series does a spectral line belong if its wave number is equal to the difference of wave numbers of the above two lines? What is the wavelength of this line? $(R = 1.097 \times 10^7 \text{ m}^{-1})$

30. An electron of a stationary hydrogen atom passes from the fifth energy level to the fundamental state. What velocity did the atom acquire as the result of photon emission? What is the recoil energy? Express you answer in terms of Rydberg constant R mass of hydrogen atom M and universal constants.

31. Find an expression for the magnetic dipole moment and magnetic field induction at the center of a Bohr's hypothetical hydrogen atom in the n^{th} orbit of the electron in terms of universal constants.

32. A mercury arc lamp provides 0.2 watt of ultraviolet radiation at a wavelength of $\lambda = 2537$ Å.

Assume no other wavelength to be present. The cathode of photoelectric device consists of potassium and has an effective area of 4 cm^2. The arc lamp is at a distance of 1 m from the cathode. Given that work function for potassium is $W = 2.22$ eV.

(a) According to classical theory, what time of exposure to the radiation should be required for a potassium atom (radius 2 Å) to accumulate sufficient energy to eject a photoelectron?

(b) What is the energy of a single photon from the source?

(c) What is the flux of photons (number per second) at the cathode? To what saturation current does this flux correspond if the photo conversion efficiency is 10%. Photo conversion efficiency is the probability of a photon being successful in knocking out an electron.

(d) Find the cut-off potential.

LEVEL II

1. An electron and a proton are separated by a large distance and the electron approaches the proton with a kinetic energy of 2 eV. If the electron is captured by the proton to form a hydrogen atom in the ground state, what wavelength photon would be given off?

2. Assume a hypothetical hydrogen atom in which the potential energy between electron and proton at separation r is given by $U = [k \ln r - (k/2)]$ where k is a constant. For such a hypothetical hydrogen atom, calculate the radius of n^{th} Bohr's orbit and energy levels.

3. An electron is orbiting in a circular orbit of radius r under the influence of a constant magnetic field of strength B. Assuming that Bohr's postulate regarding the quantisation of angular momentum holds good for this electron, find

(a) the allowed values of the radius 'r' of the orbit.

(b) the kinetic energy of the electron in orbit

(c) the potential energy of interaction between the magnetic moment of the orbital current due to the electron moving in its orbit and the magnetic field B.

(d) the total energy of the allowed energy levels.

(e) the total magnetic flux due to the magnetic field B passing through the nth orbit.

(Assume that the charge on the electron is $-e$ and the mass of the electron is m).

4. A stream of α-particles is incident on a sample of hydrogen gas. What should be the minimum kinetic energy of α-particles to ionize the hydrogen atoms.

5. Two hydrogen like atoms A and B are of different masses and each atom contains equal number of protons and neutrons. The difference in the energies between the first Balmer lines emitted by A and B is 5.667 eV. When the atom A and B, moving with the same velocity, strike a heavy target they rebound back with the same velocity. In the process atoms B imparts twice momentum to the target than that A imparts. Identify the atoms A and B.

6. Electrons are emitted from an electron gun at almost zero velocity and are accelerated by an electric field E through a distance of 1 m. The electrons are now scattered by an atomic hydrogen sample in grounds state. What should be the minimum value of E so that red light of wavelength 6563 Å may be emitted in the hydrogen?

7. Certain gas of identical hydrogen like atoms has all its atoms in a particular upper energy level. The atoms make transition to a higher energy level when a monochromatic radiation, having wavelength 1654 Å, is incident upon it. Subsequently, the atoms emit radiation of only three different photon energies.

(a) Identify the atom

(b) Obtain the ionization energy for the gas atoms.

(c) If the atoms of the gas are to be excited to such a level which gives radiation of only six different photon energies, what should be energy of incident radiation.

8. In a photoelectric effect set-up, a point source of light of power 3.2×10^{-3} W emits mono energetic photons of energy 5.0 eV. The source is located at a distance of 0.8 m from the centre of a stationary metallic sphere of work function 3.0 eV and of radius 8.0×10^{-3} m. The efficiency of photoelectron emission is one for every 10^6 incident photons. Assume that the sphere is isolated and electrons are instantly swept away after emission.

(a) Calculate the number of photoelectrons emitted per second.

(b) Find the ratio of the wavelength of incident light to the de Broglie wavelength of the fastest photoelectrons emitted.

(c) It is observed that the photoelectron emission stops at a certain time t after the light source is switched on. Why?

(d) Evaluate the time t.

9. How many head-on, elastic collisions must a neutron have with deuterium nucleus to reduce its energy from 1 MeV to 0.025 eV.

10. Hydrogen gas in the atomic state is excited to an energy level such that the electrostatic potential energy of H-atom becomes -1.7 eV. Now a photoelectric plate having work function $W = 2.3$ eV is exposed to the emission spectra of this gas. Assuming all the transitions to be possible, find the minimum de-Broglie wavelength of the ejected photoelectrons.

11. A moving H-atom makes a head on perfectly inelastic collision with a stationary Li^{++} ion. Before collision H-atom and Li^{++} ion are both in their first excited states. What is the velocity of the moving H atom if after collision H is found in its ground state and Li^{++} ion in its second excited state. Take mass of hydrogen atom $= 1.66 \times 10^{-27}$ kg.

[mass of Li^{++} $= 7 \times$ mass of H atom]

12. Using Bohr's theory show that when n is very large the frequency of radiation emitted by hydrogen atom due to transition of electron from n to $(n-1)$ is equal to frequency of revolution of electron in its orbit.

13. A mixture of hydrogen atoms (in their ground state) and hydrogen like ions (in their first excited state) are being excited by electrons which have been accelerated by same potential difference V volts. After excitation when they come directly into ground state, the wavelengths of emitted light are found in the ratio 5 : 1. Then find:

(a) The minimum value of V for which both the atoms get excited after collision with electrons.

(b) Atomic number of other ion.

(c) The energy of emitted light.

14. When a surface is irradiated with light of $\lambda = 4950$ Å a photocurrent appears which vanishes if a retarding potential 0.6 V is applied. When a different source of light is used it is found that critical retarding potential is changed to 1.1 volt. Find the work function of emitting surface and wavelength of second source. If photoelectrons after emission from surface are subjected to a magnetic field of 10 tesla, what changes will be observed in the above two retarding potentials?

15. In an experiment on photoelectric effect light of wavelength 400 nm is incident on a caesium plate at the rate of 5 W. The potential of the collector plate is made sufficiently positive with respect to emitter so that the current reaches the saturation value. Assuming that on the average one out of every 10^6 photons is able to eject a photoelectron, find the photocurrent in the circuit.

16. A light beam of wavelength 400 nm is incident on a metal of work function 2.2 eV. A particular electron absorbs a photon and makes 2 collisions before coming out of the metal

(a) Assuming that 10% of extra energy is lost to the metal in each collision find the final kinetic energy of this electron as it comes out of the metal.

(b) Under the same assumptions find the maximum number of collisions the electron should suffer before it becomes unable to come out of the metal.

17. A stationary He^+ emitted a photon corresponding to the first line of Lyman series. This photon liberated a photo electron from a stationary hydrogen atom in the ground state. Find the velocity of the photoelectron.

18. Calculate the separation between the particles of a system in the ground state, the corresponding binding energy and wavelength of first line in Lyman series if such a system is positronium consisting of an electron and positron revolving round their common centre of mass.

19. A neutron with an energy of 4.6 MeV collides with protons and is retarded. Assuming that upon each collision neutron is deflected by 45° find the number of collisions which will reduce its energy to 0.23 eV.

20. A hot gas emits radiation of wavelengths 460 Å, 831 Å d 1035 Å, only. Assume that the atoms have only two excited states and the difference between consecutive energy levels decreases as energy is increased. Taking the energy of the highest energy state to be zero. Find the energies of the ground state and the first excited state.

21. A gas of hydrogen like atoms can absorb radiations of 68 eV. Consequently, the atoms emit radiations of only three different wavelengths. All the wavelengths are equal or smaller than that of the absorbed photon.

(a) Determine the initial state of the gas atoms.

(b) Identify the gas atoms.

(c) Find the minimum wavelength of the emitted radiations.

(d) Find the ionization energy and the respective wavelength for the gas atoms.

22. (a) If two times the λ_{min} of continuous *X*-ray spectra of target atom "*A*" at 34.3 kV is same as the wavelength of K_α line of target atom "*B*" at 40 kV, then determine the atomic number of the atom "*B*".

(b) Stopping potentials of 24, 100, 110 and 115 kV are measured for photoelectrons emitted from a certain element when it is radiated with monochromatic X-ray. If this element is used as a target in an X-ray tube, what will be the wavelength of K_α line?

23. A photon with an energy of 4.9 eV ejects photoelectrons from tungsten. When the ejected electron enters a constant magnetic field of strength $B = 2.5$ mT at an angle of 60° with the field direction, the maximum pitch of the helix described by the electron is found to be 2.7 mm. Find the work function of the metal in electron-volts. Given that specific charge of electron is 1.76×10^{11} C/kg.

24. According to the maxwell theory of electrodynamics an electron going in a circle should emit radiations of frequency equal to its frequency of revolution. What would be the wavelength of the radiation emitted by a hydrogen atom in ground state if this rule is followed?

25. For a certain hypothetical one-electron atom, the wavelength (in Å) for the spectral lines for transitions originating at $n = p$ and terminating at $n = 1$ are given by

$$\lambda = \frac{1500\, p^2}{p^2 - 1} \qquad \text{where } p = 2, 3, 4 \ldots$$

(a) Find the wavelength of the least energetic and the most energetic photons in this series.
(b) Construct an energy level diagram for this element showing the energies of the lowest three levels.
(c) What is the ionization potential of this element.

26. A photocell is operating in saturation mode with a photocurrent 4.8 mA when a monochromatic radiation of wavelength 3000 Å and power 1 mW is incident. When another monochromatic radiation of wavelength 1650 Å and power 5 mW is incident, it is observed that maximum velocity of photoelectron increases to two times. Assuming efficiency of photoelectron generation per incident photon to be same for both the cases, calculate,
(a) threshold wavelength for the cell
(b) saturation current in second case
(c) efficiency of photoelectron generation per incident photon.

ANSWERS

Introductory Exercise 24.1

1. (a) 1.31 μm, (b) 164 nm 2. 656 nm

3. (a) 6.58×10^{15} Hz, 0.823×10^{15} Hz; (b) 2.88×10^{15} Hz; (c) 8.23×10^{6} revolutions

4. (a) 2.85×10^{-13} m, (b) 2.53 keV 5. (a) 4.8×10^{-34} m, (b) 7.3×10^{-11} m

6. (a) 102 nm, 122 nm, 653 nm (b) No lines 7. 5.59 nm 8. $E_1 = -4260$ eV, $E_3 = -2650$ eV

9. $\frac{3f}{4}, \frac{f}{4}$

Introductory Exercise 24.2

1. 1.9 V 2. 0.48 μA 3. $K_{max} \propto (f - f_0)$ 4. Zero, 3.19 eV 5. 1.16×10^{15} Hz

Assignment

Level-I

1. 4.6 eV, 8.2×10^{-36} kg, 2.45×10^{-27} kg-m/s 2. 122.4 eV 3. 23.8 eV 4. 79 eV 5. 1.04 Å

6. $Z \approx 41$ 7. 2.8 eV 8. 15865 volt 9. 0.012 volt 10. 3, 1023 Å 11. 4 μm 12. 2260 Å

13. (a) 10^{15} Hz, (b) 3.75 eV, (c) 2 eV 14. 1.9 eV 15. 1.12×10^{12} 16. (a) 2.27 eV, (b) 2.75×10^{17}, (c) 9%

17. 6.6×10^{-34} J-s, 5990 Å 18. 5.0×10^{19}, 10^{18}, 1.32 eV, 3363 Å 19. 9.28×10^{-3} C/s

20. 4.82×10^{16} per m^2-sec 21. 15 cm 22. (a) 4 watt, (b) 396 J/s 23. $n = 5$

24. (a) $Z = 4$, (b) $\lambda_{min} = 40441$ 25. $W = 11.93$ eV

26. (a) 15.6 volt (b) 2335 Å (c) 12.52 V (d) 1.01×10^{7} m^{-1} (e) (i) 6 eV, (ii) 0.7 eV

27. (a) 113.74 Å, (b) 3 28. 2 eV, 0.754 volt 29. Second line of brackett series, 26206 Å

30. $\frac{24hR}{25M}, \frac{h^2R^2}{2.17M}$ 31. $\frac{neh}{4\pi m}, \frac{\mu_0 \pi m^2 e^7}{8\varepsilon_0 h^5 n^5}$

32. (a) 177.6 sec (b) 4.87 eV (c) 8.12×10^{12} photons/sec, 65 nA (d) 2.65 volt

Level-II

1. 793.3 Å 2. $r_n = \frac{nh}{2\pi\sqrt{mk}}$, $E_n = k \ln\left\{\frac{nh}{2\pi\sqrt{mk}}\right\}$

3. (a) $r_n = \sqrt{\frac{nh}{2\pi Be}}$ (b) $K = \frac{nhBe}{4\pi m}$ (c) $U = \frac{nheB}{4\pi m}$ (d) $E = \frac{nheB}{2\pi m}$ (e) $\frac{nh}{2e}$ 4. 68 eV

5. A is ${}_1H^2$ and B is ${}_2He^4$ 6. 12.1 volt/m 7. (a) Helium (b) 54.4 eV (c) 10.2 eV

8. (a) 10^5 sec^{-1} (b) 286 (d) 111 second 9. 8 10. 3.8 Å 11. 3.9×10^{4} m/s

13. (a) 10.2 volt (b) $Z = 2$ (c) 10.2 eV and 51 eV **14.** 1.9 eV, 4125 Å, No change is observed

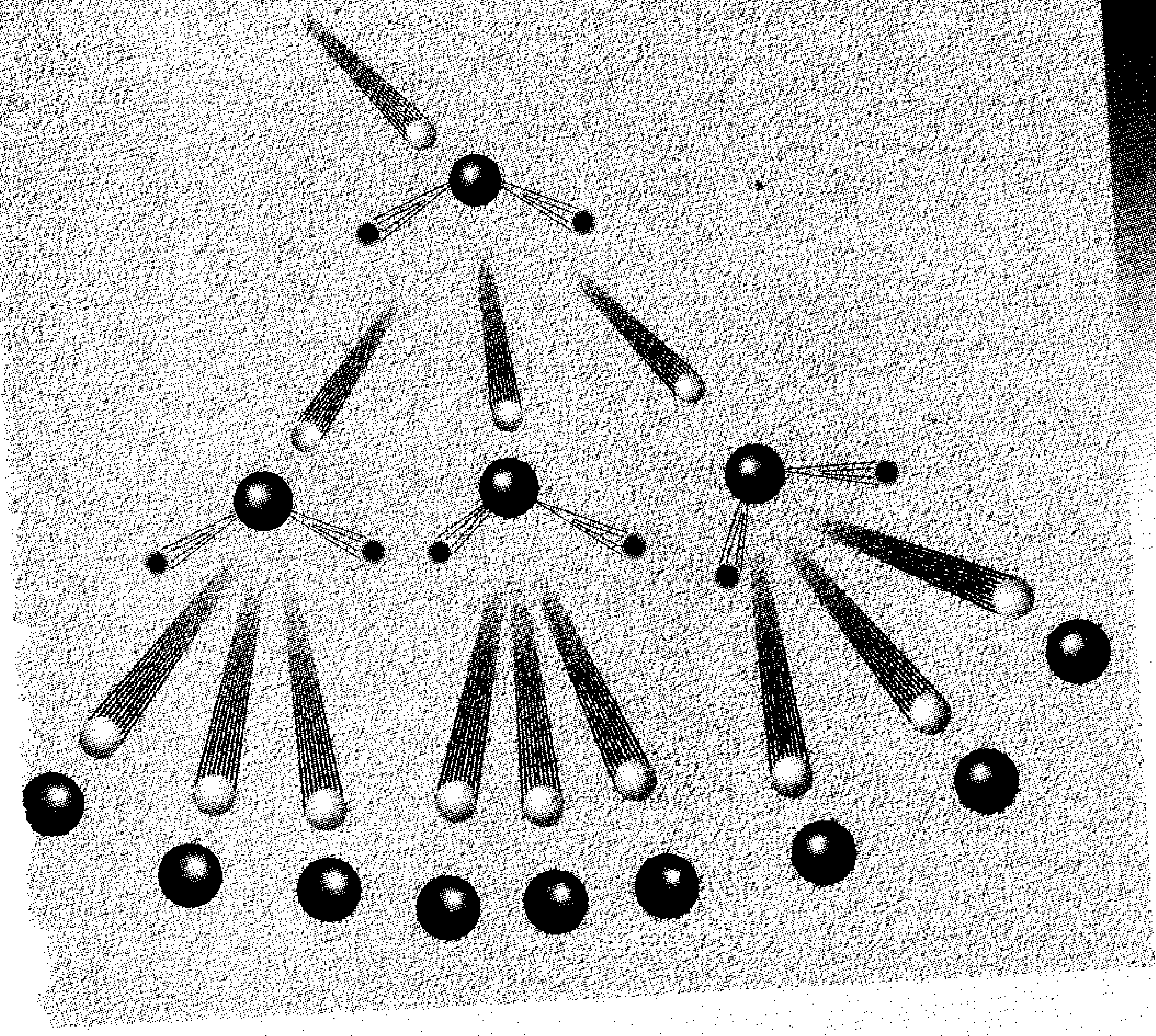
15. 1.6 μA **16.** (a) 0.31 eV (b) 4 **17.** 3.1×10^6 m/s **18.** 1.06 Å, 6.8 eV, 2426 Å **19.** $n = 24$

20. – 26.9 eV, –12.0 eV **21.** (a) $n_i = 2$ (b) $Z = 6$ (c) 28.43 Å (d) 489.6 eV, 25.3 Å

22. (a) $z = 42$ (b) 0.163 Å **23.** 4.5 eV **24.** 453 Å

25. (a) 2000 Å, 1500 Å (b) $E_1 = -8.25$ eV, $E_2 = -2.05$ eV and $E_3 = -0.95$ eV (c) 8.25 volt

26. (a) 4125 Å (b) 34 μA (c) 5.1%



CHAPTER 25

Modern Physics-II

Chapter Contents

25.1 NUCLEAR STABILITY AND RADIOACTIVITY

Among about 1500 known nuclides, less than 260 are stable. The others are unstable that decay to form other nuclides by emitting α and β-particles and γ-electromagnetic waves. This process is called radioactivity. It was discovered in 1896 by Henry Becquerel.

Whilst the chemical properties of an atom are governed entirely by the number of protons in the nucleus (*i.e.*, the proton number Z), the stability of an atom appears to depend on both the number of protons and the number of neutrons. For light nuclei, the greatest stability is achieved when the numbers of protons and neutrons are approximately equal ($N \approx Z$).

For heavier nuclei, instability caused by electrostatic repulsion between the protons is minimized when there are more neutrons than protons.

Figure shows a plot of N versus Z for the stable nuclei. For mass numbers upto about $A = 40$, we see that $N \approx Z$. 40Ca is the heaviest stable nucleus for which $N = Z$. For larger values of Z, the (short range) nuclear force is unable to hold the nucleus together against the (long-range) electrical repulsion of the protons unless the number of neutrons exceeds the number of protons. At Bi ($Z = 83$, $A = 209$), the neutron excess is $N - Z = 43$. There are no stable nuclides with $Z > 83$.

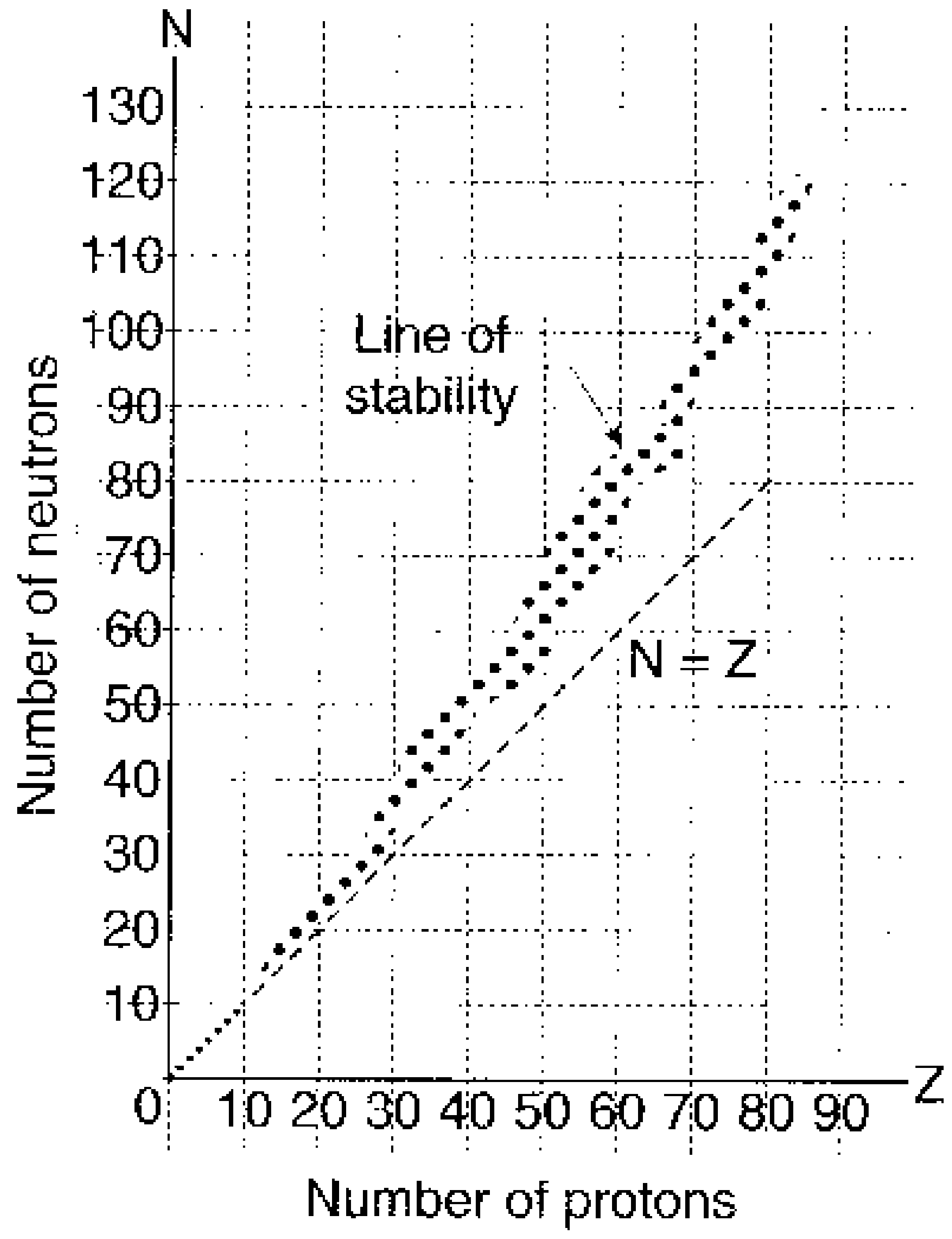


Fig. 25.1 The stable nuclides plotted on a graph of neutron number, N, versus proton number, Z. Note that for heavier nuclides, N is larger relative to Z. The stable nuclides group along a curve called the line of stability.

The nuclide $^{209}_{83}Bi$ is the heaviest stable nucleus.

Atoms are radioactive if their nuclei are unstable and spontaneously (and randomly) emit various particles, the α, β and/or γ radiations. When naturally occuring nuclei are unstable, we call the phenomena **natural radioactivity.** Other nuclei can be transformed into radioactive nuclei by various means, typically involving irradiation by neutrons, this is called **artificial radioactivity.**

A radioactive nucleus is called a **parent nucleus,** the nucleus resulting from its decay by particle emission is called **daughter nucleus**. Daughter nuclei also might be granddaughter nuclei, an so on. There are no son or grandson nuclei. For unstable nuclides and radioactivity following points can be made.

(i) Disintegrations tend to produce new nuclides near the stability line and continue until a stable nuclide is formed.

(ii) Radioactivity is a nuclear property, *i.e.*, α, β and γ emission take place from the nucleus.

(iii) Nuclear processes involve huge amount of energy so the particle emission rate is independent of temperature and pressure. The rate depends solely on the concentration of the number of atoms of the radioactive substance.

(iv) A radioactive substance is either an α-emitter or a β-emitter. γ-rays emit with both.

Alpha Decay

An alpha particle is a helium nucleus. Thus a nucleus emitting an alpha particle loses two protons and two neutrons. Therefore, the atomic number Z decreases by 2, the mass number A decreases by 4 and the neutron number N decreases by 2. The decay can be written as,

$$^A_Z X^A = {}^{A-4}_{Z-2}Y + {}^4_2\text{He}$$

where X is the parent nucleus and Y the daughter nucleus. As examples U^{238} and Ra^{226} are both alpha emitters and decay according to,

$$^{238}_{92}\text{U} \longrightarrow {}^{234}_{90}\text{Th} + {}^4_2\text{He}$$

$$^{226}_{88}\text{Ra} \longrightarrow {}^{222}_{86}\text{Rn} + {}^4_2\text{He}$$

As a general rule in any decay sum of mass numbers A and atomic numbers Z must be the same on both sides.

Note that a nuclide below the stability line in Fig. 25.1 disintegrates in such a way that its proton number decreases and its neutron to proton ratio increases. In heavy nuclides this can occur by alpha emission.

If the original nucleus has a mass number A that is 4 times an integer, the daughter nucleus and all those in the chain will also have mass numbers equal to 4 times an integer. (Because in α-decay A decreases by 4 and in β-decay it remains the same). Similarly, if the mass number of the original nucleus is $4n+1$, where n is an integer, all the nuclei in the decay chain will have mass numbers given by $4n+1$ with n decreasing by 1 in each α-decay. We can see therefore, that there are four possible α-decay chains, depending on whether A equals $4n$, $4n+1$, $4n+2$ or $4n+3$ where n is an integer.

Series $4n+1$ is now not found. Because its longest lived member (other than the stable end product Bi^{209}) is Np^{237} which has a half life of only 2×10^6 years. Because this is much less than the age of the earth this series has disappeared.

Figure shows the Uranium $(4n+2)$ series.

The series branches at Bi^{214} which decays either by α-decay to Tl^{210} or β-decay to Po^{214}. The branches meet at the lead isotope Pb^{210}. Table 25.1 lists the four radioactive series.

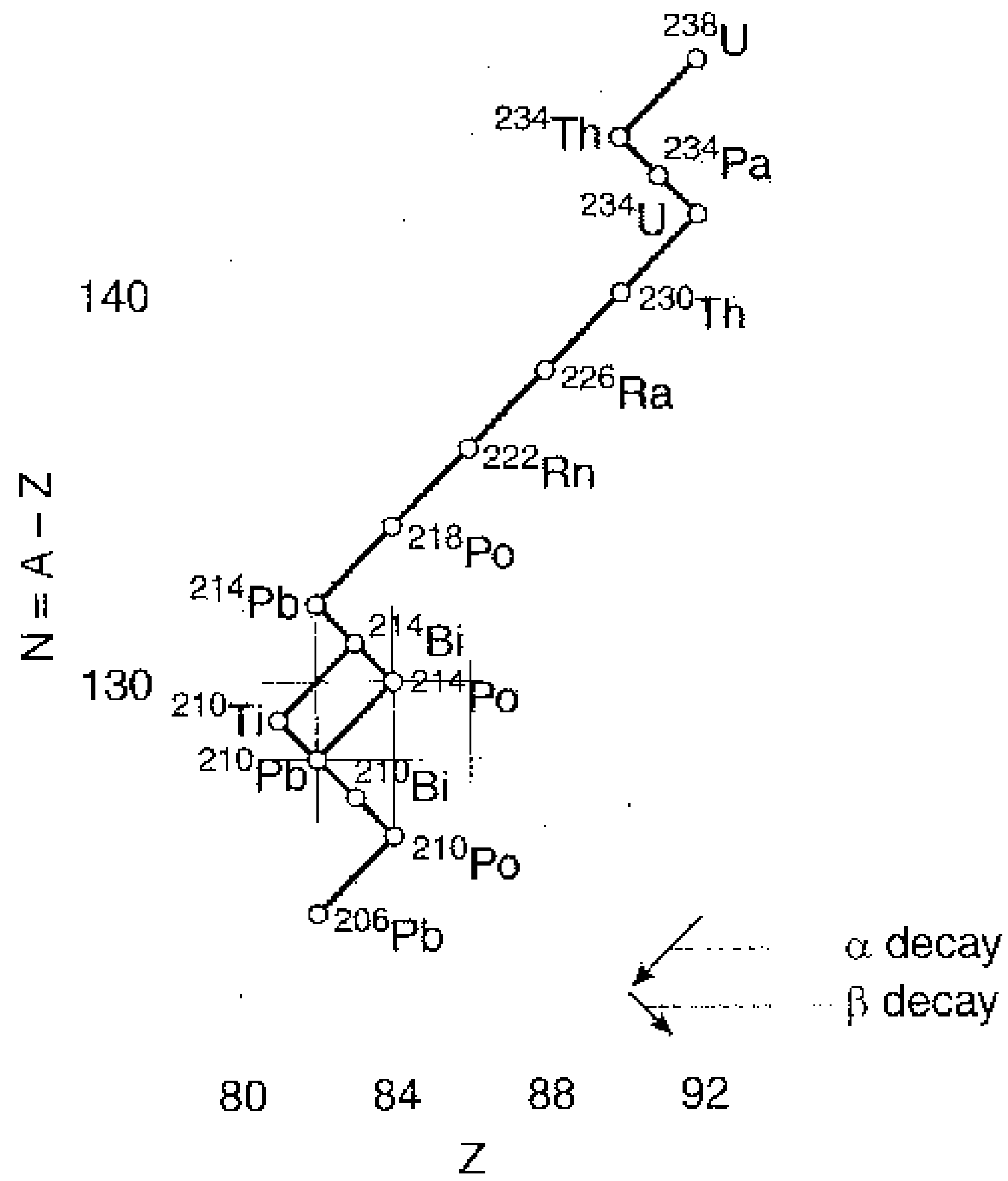


Fig. 25.2 The uranium decay series ($A = 4n + 2$). *The decay of* $^{214}_{83}$Bi may proceed either by alpha emission and then beta emission or in the reverse order.

Table 25.1 Four Radioactive Series.

Mass Numbers	Series	Parent	Half-Life, Years	Stable Product
$4n$	Thorium	$^{232}_{90}$Th	1.39×10^{10}	$^{208}_{82}$Pb
$4n+1$	Neptunium	$^{237}_{93}$Np	2.25×10^6	$^{209}_{83}$Bi
$4n+2$	Uranium	$^{238}_{92}$U	4.47×10^9	$^{206}_{82}$Pb
$4n+3$	Actinium	$^{235}_{92}$U	7.07×10^8	$^{207}_{82}$Pb

Beta Decay

Beta decay can involve the emission of either electrons or positrons. A positron is a form of antimatter. Which has a charge equal to $+e$ and mass equal to that of an electron. The electrons or positrons emitted in β-decay do not exist inside the nucleus. They are only created at the time of emission, just as photons are created when an atom makes a transition from a higher to a lower energy state.

In β^- decay a neutron in the nucleus is transformed into a proton, an electron and an antineutrino.

$$n \longrightarrow P + e^- + \bar{\nu}$$

To conserve energy and momentum in the process, the emission of an antineutrino ($\bar{\nu}$) (alongwith proton and electron) was first suggested by W. Pauli in 1930, but it was first observed experimentally in 1957.

Thus a parent nucleus with atomic number Z and mass number A decays by β^- emission into a daughter with atomic number $Z+1$ and the same mass number A.

$$^{A}_{Z}X \xrightarrow{\beta^-} {}^{A}_{Z+1}Y$$

β^- decay occurs in nuclei that have too many neutrons. An example of β^- decay is the decay of carbon 14 into nitrogen,

$$^{14}_{6}\text{C} \longrightarrow {}^{14}_{7}\text{N} + e^- + \bar{\nu}$$

In β^+ decay, a proton changes into a neutron with the emission of a positron (and a neutrino)

$$P \longrightarrow n + e^+ + \nu$$

Positron (e^+) emission from a nucleus decreases the atomic number Z by 1 while keeping the same mass number A.

$$^{A}_{Z}X \xrightarrow{\beta^+} {}^{A}_{Z-1}Y$$

B^+ decay occurs in nuclei that have two few neutrons. A typical β^+ decay is,

$$^{13}_{7}\text{N} \longrightarrow {}^{13}_{6}\text{C} + e^+ + \nu$$

Electron capture : Electron capture is competitive with positron emission since both processes lead to the same nuclear transformation. This occurs when a parent nucleus captures one of its own orbital electrons and emits a neutrino.

$$^{A}_{Z}X + e^{-1} \longrightarrow {}_{Z-1}^{A}Y + \nu$$

In most cases, it is a K-shell electron that is captured, and for this reason the process is referred to as ***K*-capture.** One example is the capture of an electron by $_4\text{Be}^7$

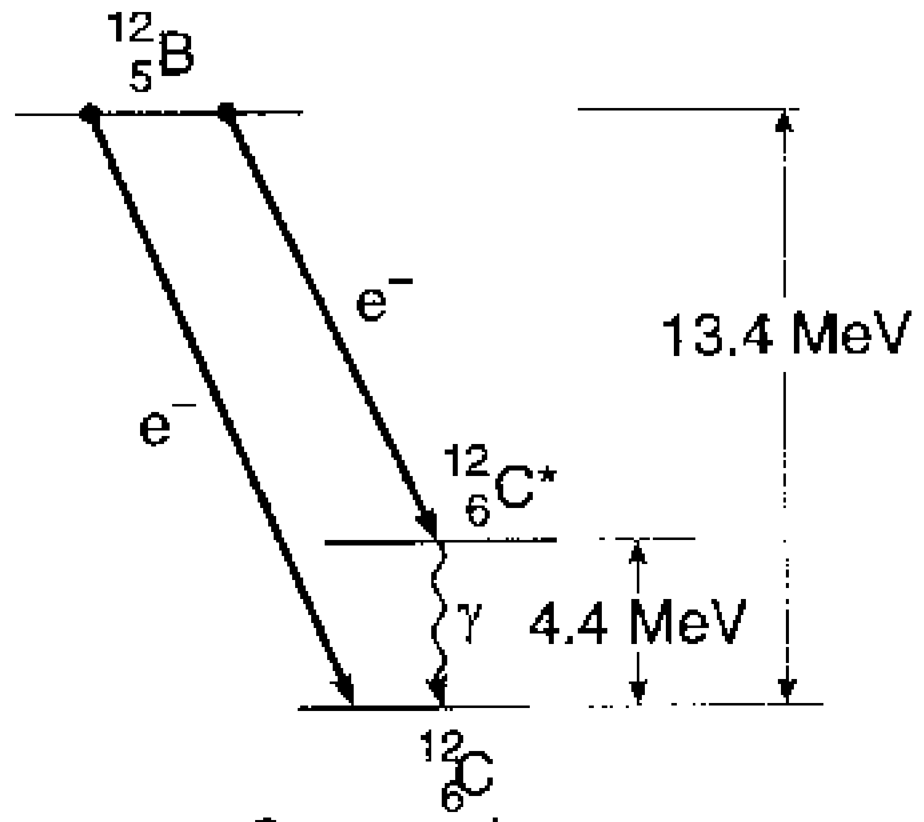
$$^{7}_{4}\text{Be} + e^- \longrightarrow {}^{7}_{3}\text{Li} + \nu$$

Gamma Decay

Very often a nucleus that undergoes radioactive decay (α or β decay) is left in an excited energy state (analogous to the excited states of the orbiting electrons, except that the energy levels associated with the nucleus have much larger energy differences than those involved with the atomic electrons). The typical half life of an excited nuclear state is 10^{-10} sec. The excited nucleus (X^*) then undergoes to a lower energy state, by emitting a high energy photon, called the γ-ray photon. The following sequence of events represents a typical situation in which γ-decay occurs.

$$ {}^{12}_{5}B \longrightarrow {}^{12}_{6}C^* + e^- + \bar{\nu} $$

$$ {}^{12}_{6}C^* \longrightarrow {}^{12}_{6}C + \gamma $$



Gamma decay

Fig. 25.3

Figure shows decay of B^{12} nucleus, which undergoes β-decay to either of two levels of C^{12}. It can either decay directly to the ground state of C^{12} by emitting a 13.4 MeV electron or undergo β-decay to an excited state of ${}^{12}_{6}C^*$ followed by γ-decay to the ground state. The latter process results in the emission of a 9.0 MeV electron and a 4.4 MeV photon. The various pathways by which a radioactive nucleus can undergo decay are summarized in Table 25.2.

Note : In both α and β decay, the Z value of a nucleus changes and the nucleus of one element becomes the nucleus of a different element. In γ-decay, the element does not change, the nucleus merely goes from an excited state to a less excited state.

Table 25.2. Various Decay Pathways

Alpha decay	${}^{A}_{Z}X^{A} \longrightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He$
Beta decay (β^-)	${}^{A}_{Z}X \longrightarrow {}_{Z+1}{}^{A}Y + e^- + \bar{\nu}$
Beta decay (β^+)	${}^{A}_{Z}X \longrightarrow {}_{Z-1}{}^{A}Y + e^+ + \nu$
Electron capture	${}^{A}_{Z}X + e^- \longrightarrow {}_{Z-1}{}^{A}Y + \nu$
Gamma decay	${}^{A}_{Z}X^* \longrightarrow {}^{A}_{Z}X + \gamma$

25.2 THE RADIOACTIVE DECAY LAW

Radioactive decay is a random process. Each decay is an independent event and one cannot tell when a particular nucleus will decay. When a particular nucleus decays, it is transformed into another nuclide, which may or may not be radioactive. When there is a very large number of nuclei in a sample, the rate of decay is proportional to the number of nuclei, N, that are present,

$$ \left(-\frac{dN}{dt}\right) \propto N \quad \text{or} \quad \left(-\frac{dN}{dt}\right) = \lambda N $$

where λ is called the **decay constant**. This equation may be expressed in the form $\frac{dN}{N} = -\lambda dt$ and integrated,

$$ \int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt \quad \text{or} \quad \ln\left(\frac{N}{N_0}\right) = -\lambda t $$

where N_0 is the initial number of parent nuclei at $t = 0$. The number that survive at time t is therefore,

$$N = N_0 e^{-\lambda t} \qquad \text{...(i)}$$

This function is plotted in Fig. 25.4.

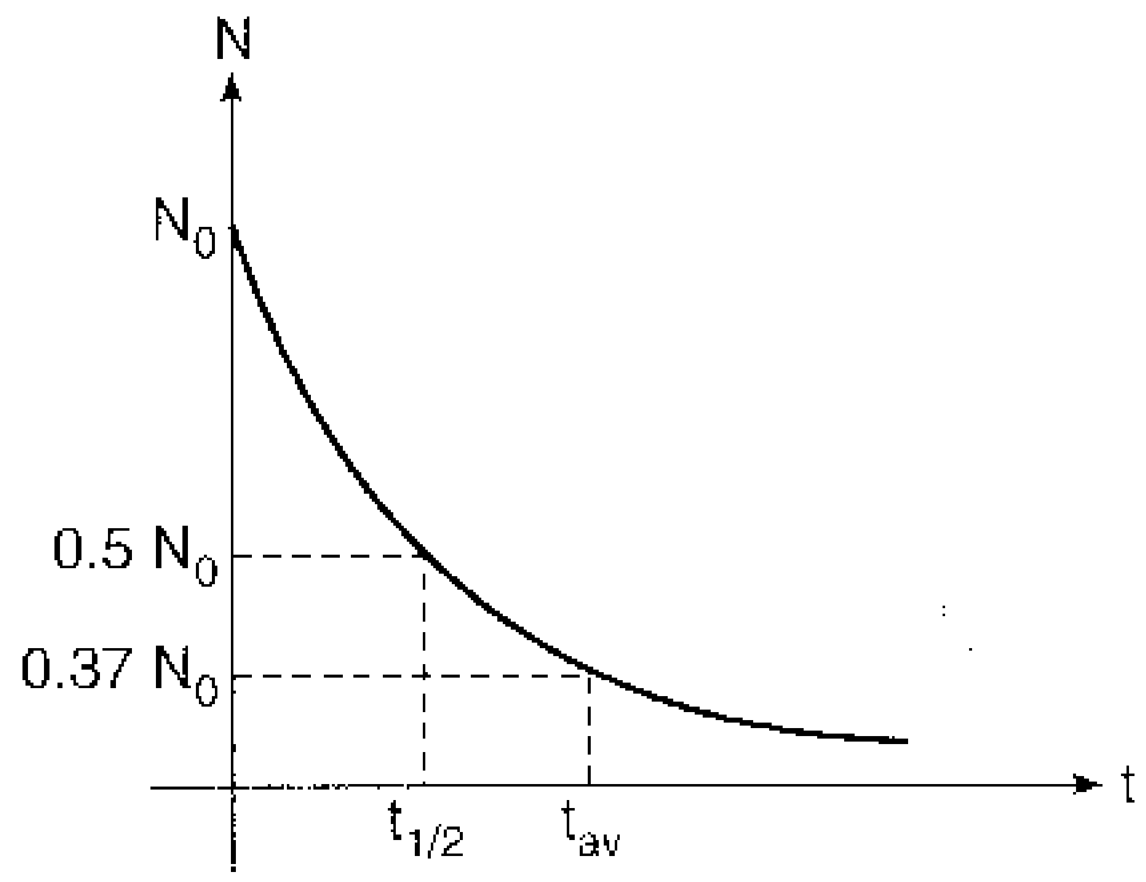


Fig. 25.4

Half Life : The time required for the number of parent nuclei to fall to 50% is called half life $t_{1/2}$ and may be related to λ as follows. Since,

$$0.5 N_0 = N_0 e^{-\lambda t_{1/2}}$$

We have

$$\lambda t_{1/2} = \ln(2) = 0.693$$

$\therefore$

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{0.693}{\lambda} \qquad \text{...(ii)}$$

Mean life : The average or mean life t_{av} is the reciprocal of the decay constant.

$$t_{av} = \frac{1}{\lambda} \qquad \text{...(iii)}$$

The mean life is analogous to the time constant in the exponential decrease in the charge on a capacitor in an RC circuit. After a time equal to the mean life time, the number of radioactive nuclei and the decay rate have each decreased to 37% of their original values.

Activity of a radioactive substance

The decay rate R of a radioactive substance is the number of decays per second. And as we have seen above

$$-\frac{dN}{dt} \propto N \quad \text{or} \quad -\frac{dN}{dt} = \lambda N$$

Thus,

$$R = -\frac{dN}{dt} \quad \text{or} \quad R \propto N$$

or

$$R = \lambda N \quad \text{or} \quad R = \lambda N_0 e^{-\lambda t}$$

or

$$R = R_0 e^{-\lambda t} \qquad \text{...(iv)}$$

where $R_0 = \lambda N_0$ is the activity of the radioactive substance at time $t = 0$. The activity versus time graph is shown in Fig. 25.5.

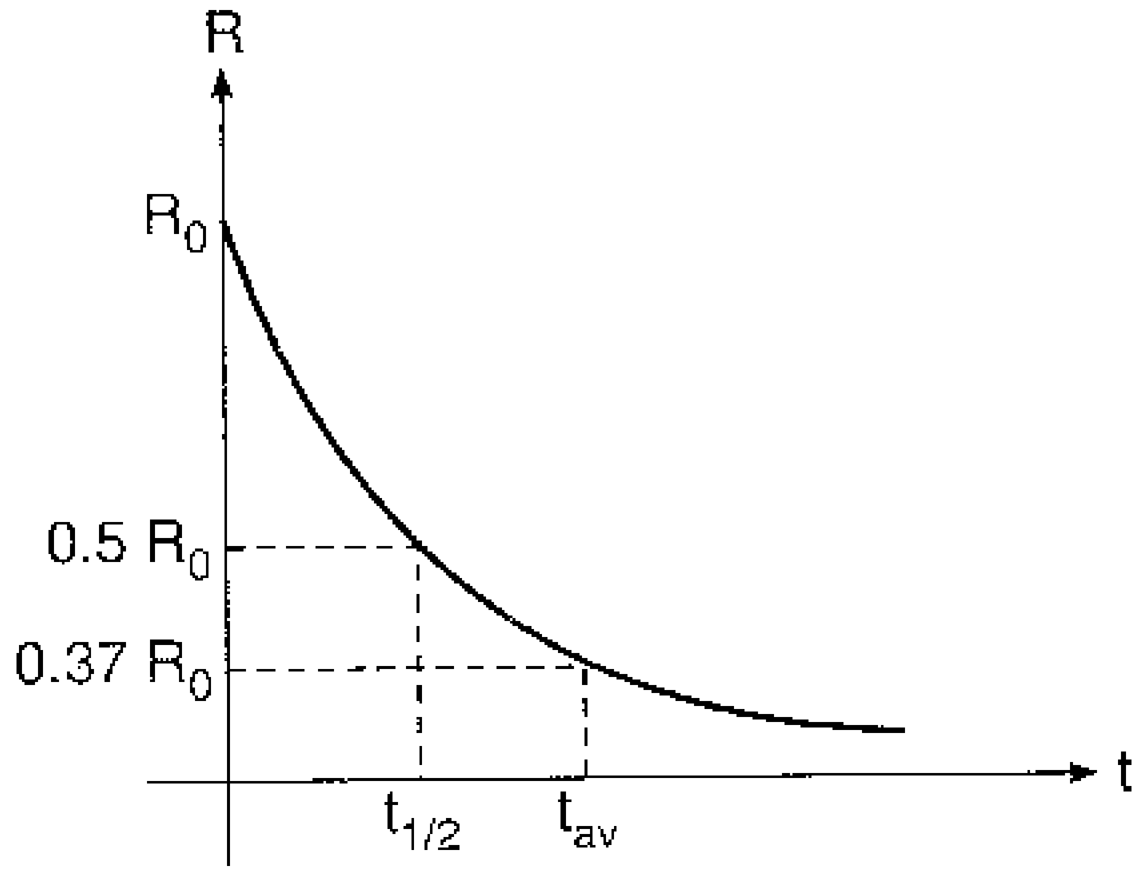


Fig. 25.5

Thus, the number of nuclei and hence the activity of the radioactive substance decrease exponentially with time.

Units of activity : The SI unit for the decay rate is the becqueral (Bq), but the curie (ci) and rutherford (rd) are often used in practice.

$$1\text{ Bq} = 1\text{ decays/s}, \quad 1\text{ ci} = 3.7\times 10^{10}\text{ Bq} \quad \text{and} \quad 1\text{ rd} = 10^{6}\text{ Bq}$$

IIT-JEE GALAXY 25.1

1. After n half lives,

   (a) number of nuclei left $= N_0\left(\dfrac{1}{2}\right)^n$

   (b) fraction of nuclei left $= \left(\dfrac{1}{2}\right)^n$ and

   (c) percentage of nuclei left $= 100\left(\dfrac{1}{2}\right)^2$

2. Number of nuclei decayed after time t,

   $$= N_0 - N$$
   $$= N_0 - N_0 e^{-\lambda t} = N_0\,(1 - e^{-\lambda t})$$

   The corresponding graph is as shown in figure.

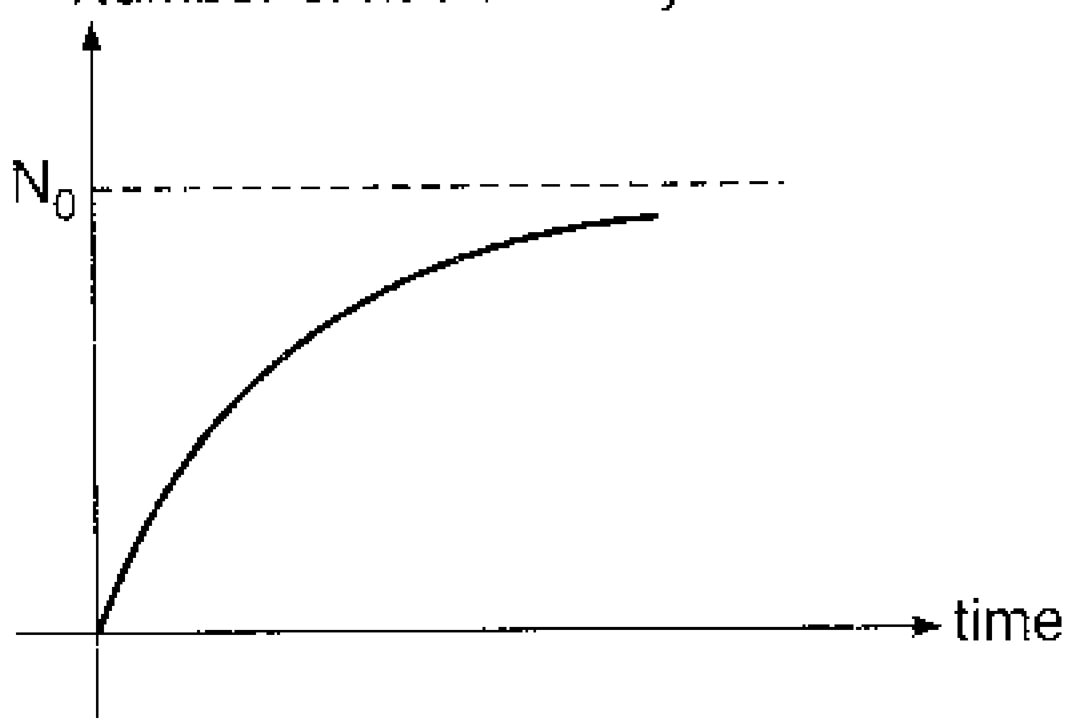


Fig. 25.6

3. Probability of a nucleus for survival of time t,

   $$P\,(\text{survival}) = \frac{N}{N_0} = \frac{N_0 e^{-\lambda t}}{N_0} = e^{-\lambda t}$$

   The corresponding graph is shown in figure.

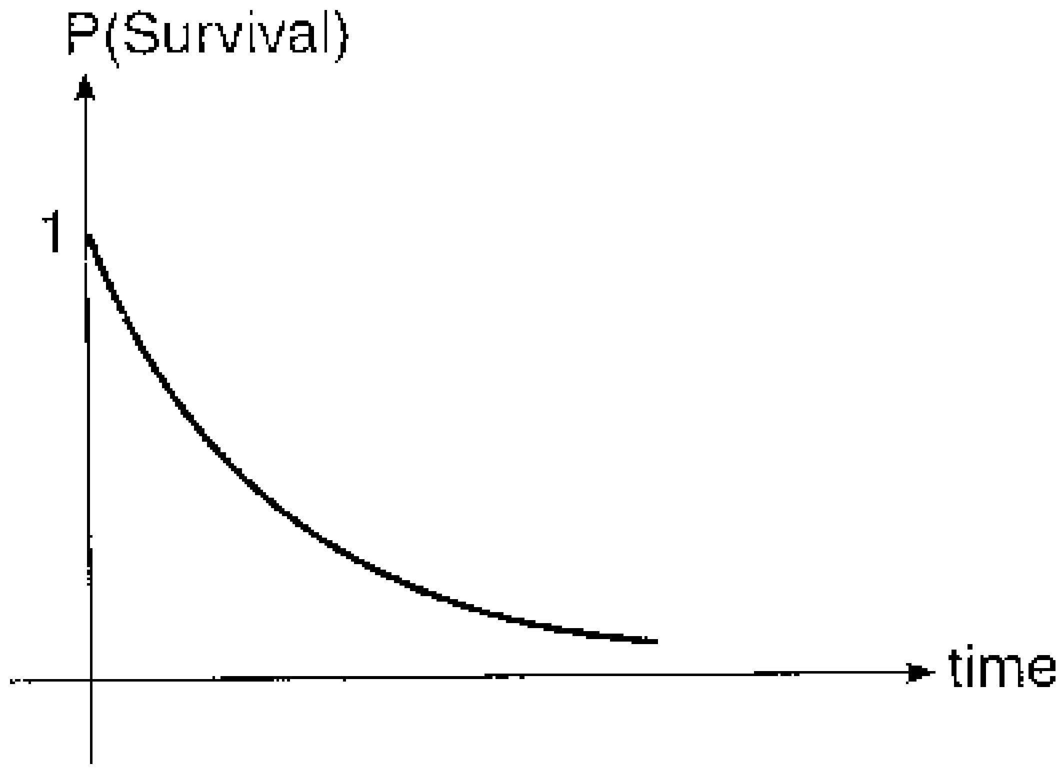


Fig. 25.7

4. Probability of a nucleus to disintegrate in time t is,

   P (disintegration) = 1 – P (survival) = $1 - e^{-\lambda t}$

   The corresponding graph is as shown.

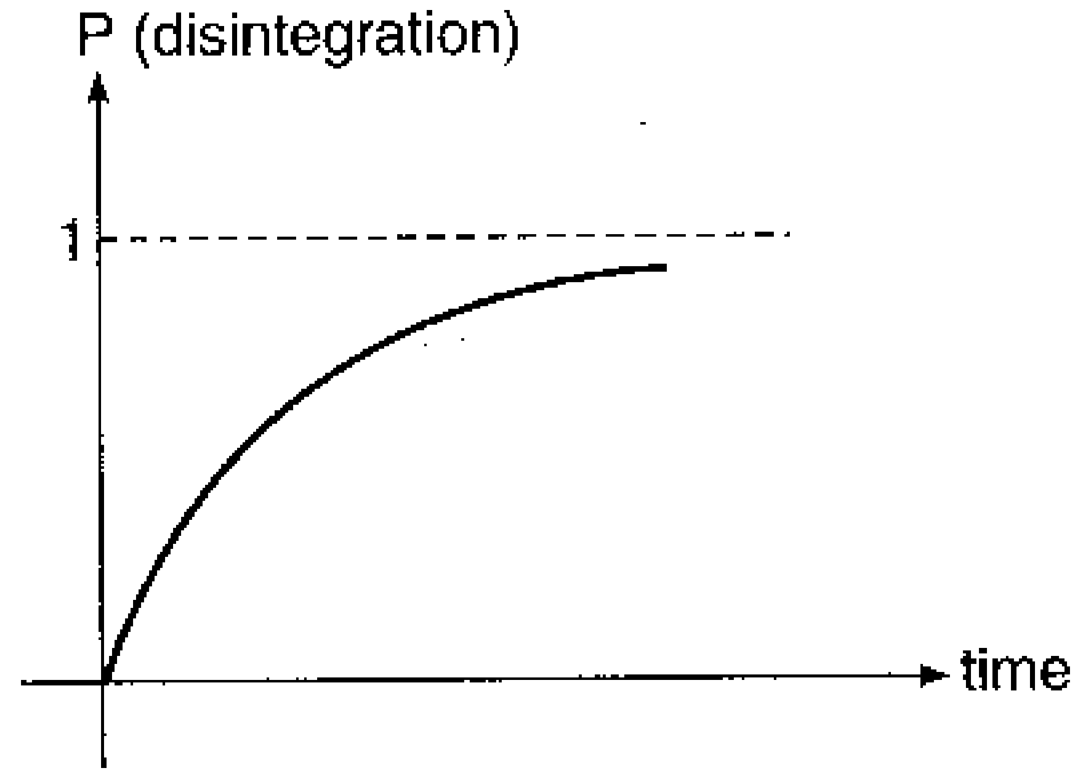


Fig. 25.8

5. Half life and mean life are related to each other by the relation,

$$t_{1/2} = 0.693\, t_{av} \quad \text{or} \quad t_{av} = 1.44 t_{1/2}$$

6. As we said in point number (2), number of nuclei decayed in time t are $N_0\,(1 - e^{-\lambda t})$. This expression involves power of e.

   So, to avoid it we can use,

$$\Delta N = \lambda N \Delta t$$

   where, ΔN are the number of nuclei decayed in time Δt, at the instant when total number of nuclei are N. But this can be applied only when $\Delta t << t_{1/2}$.

7. In same interval of time, equal percentage (or fraction) of nuclei are decayed (or left undecayed).

EXAMPLE 25.1 *At time $t = 0$, number of nuclei of a radioactive substance are 100. At $t = 1$ s these numbers become 90. Find the number of nuclei at $t = 2$ s.*

SOLUTION In 1 second 90% of the nuclei have remained undecayed, so in another 1 second 90% of 90 *i.e.*, 81 nuclei will remain undecayed.

EXAMPLE 25.2 *At time $t = 0$, activity of a radioactive substance is 1600 Bq, at $t = 8$ s activity remains 100 Bq. Find the activity at $t = 2$s.*

SOLUTION
$$R = R_0 \left(\frac{1}{2}\right)^n$$

Here, n is the number of half lives.

Given,
$$R = \frac{R_0}{16}$$

$\therefore$
$$\frac{R_0}{16} = R_0 \left(\frac{1}{2}\right)^n$$

or
$$n = 4$$

Four half lives are equivalent to 8 s. Hence, 2 s is equal to one half life. So in one half life activity will remain half of 1600 Bq *i.e.*, 800 Bq.

EXAMPLE 25.3 *Prove mathematically that mean life or average life of a radioactive substance is $t_{av} = 1/\lambda$.*

SOLUTION Let N be the number of atoms that exist at time t. Between t and $t + dt$ let dN atoms are decayed, then

$$\text{Mean or average life} = \frac{\int_{N_0}^{0} t\,dN}{\int_{N_0}^{0} dN}$$

Further, $-\frac{dN}{dt} = \lambda N$ or $dN = -\lambda N dt$

$$\therefore \quad \text{Mean average life} = \frac{-\int_0^{\infty} t\,\lambda N dt}{-N_0}$$

But $N = N_0 e^{-\lambda t}$. Hence

$$\text{Mean life} = \frac{\int_0^{\infty} t\lambda N_0 e^{-\lambda t}\,dt}{N_0}$$

This integration is done by parts. The result is,

$$t_{av} = \frac{1}{\lambda}$$ **Hence proved.**

EXAMPLE 25.4 *Uranium ores on the earth at the present time typically have a composition consisting of 99.3% of the isotope* $_{92}U^{238}$ *and* 0.7% *of the isotope* $_{92}U^{235}$. *The half lives of these isotopes are* 4.47×10^9 *y and* 7.04×10^8 *y respectively. If these isotopes were equally abundant when the earth was formed, estimate the age of the earth.*

SOLUTION Let N_0 be number of atoms of each isotope at the time of formation of the earth ($t = 0$) and N_1 and N_2 the number of atoms at present ($t = t$). Then

$$N_1 = N_0 e^{-\lambda_1 t} \quad \ldots\text{(i)}$$

and

$$N_2 = N_0 e^{-\lambda_2 t} \quad \ldots\text{(ii)}$$

$$\therefore \quad \frac{N_1}{N_2} = e^{(\lambda_2 - \lambda_1)t} \quad \ldots\text{(iii)}$$

Further it is given that

$$\frac{N_1}{N_2} = \frac{99.3}{0.7} \quad \ldots\text{(iv)}$$

Equating (iii) and (iv) and taking log both sides, we have

$$(\lambda_2 - \lambda_1)\,t = \ln\left(\frac{99.3}{0.7}\right)$$

$$\therefore \quad t = \left(\frac{1}{\lambda_2 - \lambda_1}\right)\ln\left(\frac{99.3}{0.7}\right)$$

Substituting the values, we have

$$t = \frac{1}{\dfrac{0.693}{7.04\times10^{8}} - \dfrac{0.693}{4.47\times10^{9}}} \ln\left(\frac{99.3}{0.7}\right)$$

or $$t = 5.97\times10^{9}\ \text{y}$$ **Ans.**

INTRODUCTORY EXERCISE 25.1

1. Activity of a radioactive substance decreases from 8000 Bq to 1000 Bq in 9 days. What is the half life and average life of the radioactive substance.
2. A radioactive substance has a half life of 64.8 hr. A sample containing this isotope has an initial activity ($t = 0$) of 40 μCi. Calculate the number of nuclei that decay in the time interval between $t_1 = 10.0$ hr and $t_2 = 12.0$ hr.
3. A freshly prepared sample of a certain radioactive isotope has an activity of 10 mCi. After 4.0 hr its activity is 8.00 mCi.
   (a) Find the decay constant and half life
   (b) How many atoms of the isotope were contained in the freshly prepared sample.
   (c) What is the sample's activity 30.0 hr after it is prepared.
4. A radioactive substance contains 10^{15} atoms and has an activity of 6.0×10^{11} Bq. What is its half life.
5. Two radioactive elements X and Y have half life periods of 50 minutes and 100 minutes respectively. Initially both of them contain equal number of atoms. Find the ratio of atoms left N_X/N_Y after 200 minutes.

25.3 SUCCESSIVE DISINTEGRATION

Suppose a parent radioactive nucleus A (decay constant $= \lambda_a$) has number of atoms N_0 at time $t = 0$. After disintegration it converts into a nucleus B (decay constant $= \lambda_b$) which is further radioactive. Initially ($t = 0$), number of atoms of B are zero. We are interested in finding N_b, the number of atoms of B at time t.

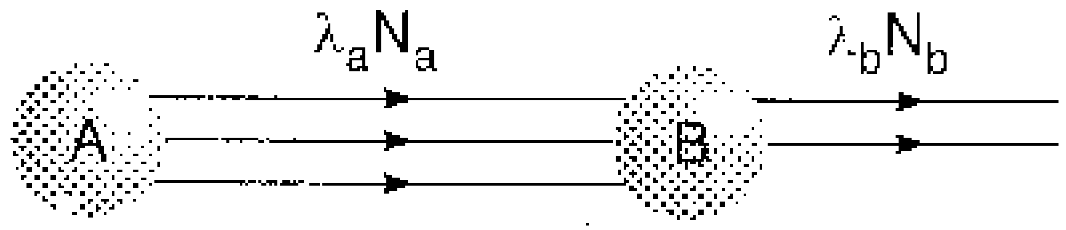


Fig. 25.9

At $t = 0$	N_0	0
At $t = t$	$N_a = N_0 e^{-\lambda_a t}$	$N_b = ?$

At time t, net rate of formation of B = rate of disintegration of A – rate of disintegration of B.

$$\therefore \quad \frac{dN_b}{dt} = \lambda_a N_a - \lambda_b N_b$$

or $$\frac{dN_b}{dt} = \lambda_a N_0 e^{-\lambda_a t} - \lambda_b N_b \qquad (\text{as } N_a = N_0 e^{-\lambda_a t})$$

or $$dN_b + \lambda_b N_b\, dt = \lambda_a N_0 e^{-\lambda_a t}$$

Multiplying this equation by $e^{\lambda_b t}$, we have

$$e^{\lambda_b t} dN_b + e^{\lambda_b} \lambda_b N_b \, dt = \lambda_a N_0 e^{(\lambda_b - \lambda_a)t}$$

$$\therefore \qquad d\{N_b e^{\lambda_b t}\} = \lambda_a N_0 e^{(\lambda_b - \lambda_a)t} dt$$

Integrating both sides, we get

$$N_b e^{\lambda_b t} = \left(\frac{\lambda_a}{\lambda_b - \lambda_a}\right) N_0 e^{(\lambda_b - \lambda_a)t} + c \qquad \text{...(i)}$$

where c is the constant of integration, which can be found as under.

At time, $t = 0$, $N_b = 0$

$$\therefore \qquad c = -\left(\frac{\lambda_a}{\lambda_b - \lambda_a}\right) N_0$$

Substituting this value in Eq. (i), we have

$$N_b = \frac{N_0 \lambda_a}{\lambda_b - \lambda_a} (e^{-\lambda_a t} - e^{-\lambda_b t}) \qquad \text{...(ii)}$$

Now following conclusions may be drawn from the above discussion.

(1) From Eq. (ii) we can see that $N_b = 0$ at time $t = 0$ (it was given) and at $t = \infty$ (because B is also radioactive)

(2) N_a will continuously decrease while N_b will first increase (until $\lambda_a N_a > \lambda_b N_b$), reaches to a maximum value (when $\lambda_a N_a = \lambda_b N_b$) and then decreases (when $\lambda_b N_b > \lambda_a N_a$). The two graphs for N_a and N_b with time are shown below.

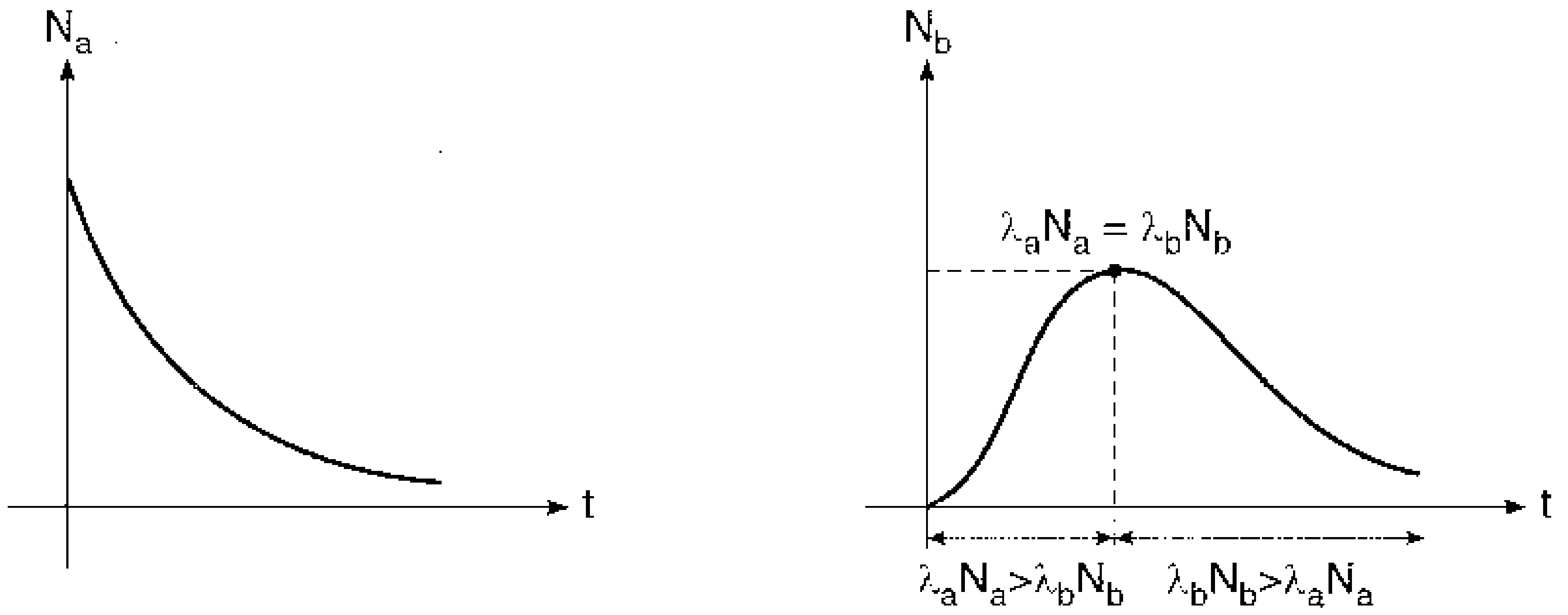


Fig. 25.10

(3) From equation number (ii) it seems as if λ_b should be greater than λ_a for this equation to hold good but it is not so. Because if $\lambda_b > \lambda_a$ then $e^{-\lambda_a t} > e^{-\lambda_b t}$ and N_b will be positive and if $\lambda_a > \lambda_b$ then $e^{-\lambda_a t} < e^{-\lambda_b t}$ and again N_b is positive.

EXAMPLE 25.5 *A radionuclide X is produced at constant rate α. At time $t = 0$, number of nuclei of X are zero. Find*

(a) the maximum number of nuclei of X.

(b) the number of nuclei at time t.

Decay constant of X is λ.

SOLUTION (a) Let N be the number of nuclei of X at time t.

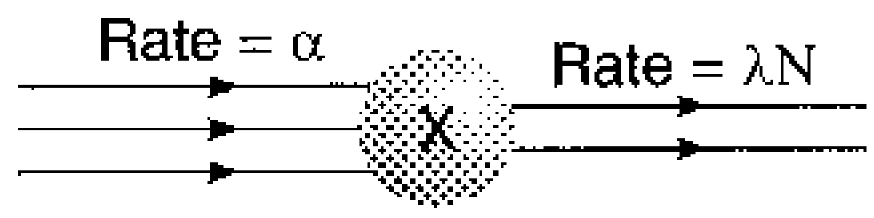


Fig. 25.11

Rate of formation of $X = \alpha$ (given)

Rate of disintegration $= \lambda N$

Number of nuclei of X will increase until both the rates will become equal. Therefore,

$$\alpha = \lambda N_{max}$$

$$\therefore \quad N_{max} = \frac{\alpha}{\lambda} \qquad \textbf{Ans.}$$

(b) Net rate of formation of X at time t is,

$$\frac{dN}{dt} = \alpha - \lambda N$$

$$\therefore \quad \frac{dN}{\alpha - \lambda N} = dt$$

Integrating with proper limits, we have

$$\int_0^N \frac{dN}{\alpha - \lambda N} = \int_0^t dt$$

or

$$N = \frac{\alpha}{\lambda}(1 - e^{-\lambda t}) \qquad \textbf{Ans.}$$

This expression shows that number of nuclei of X are increasing exponentially from 0 to $\frac{\alpha}{\lambda}$.

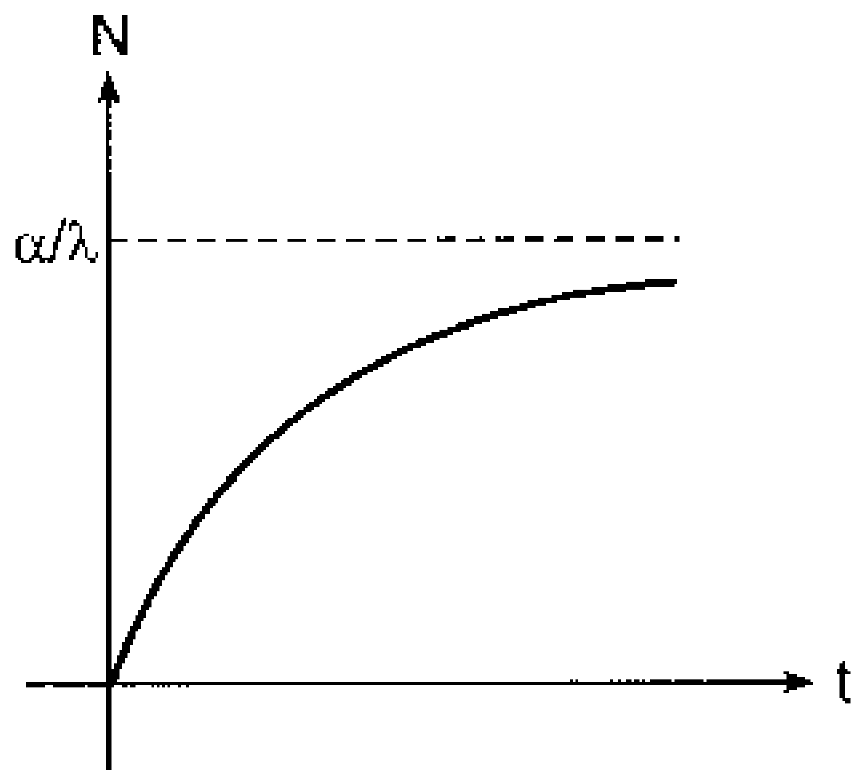


Fig. 25.12

EXAMPLE 25.6 *A radioactive nucleus X decays to a nucleus Y with a decay constant* $\lambda_X = 0.1\, s^{-1}$. *Y further decays to a stable nucleus Z with a decay constant* $\lambda_Y = 1/30\, s^{-1}$. *Initially, there are only X nuclei and their number is* $N_0 = 10^{20}$. *Setup the rate equations for the populations of X, Y and Z. The population of the Y nucleus as a function of time is given by*

$$N_Y\ (t) = \{N_0 \lambda_X / (\lambda_X - \lambda_Y)\}\ \{exp\,(-\lambda_Y t) - exp\,(-\lambda_X t)\}.$$

Find the time at which N_Y *is maximum and determine the population of X and Z at that instant.* **(JEE 2001)**

SOLUTION (i) Let at time $t = t$, number of nuclei of Y and Z are N_Y and N_Z. Then

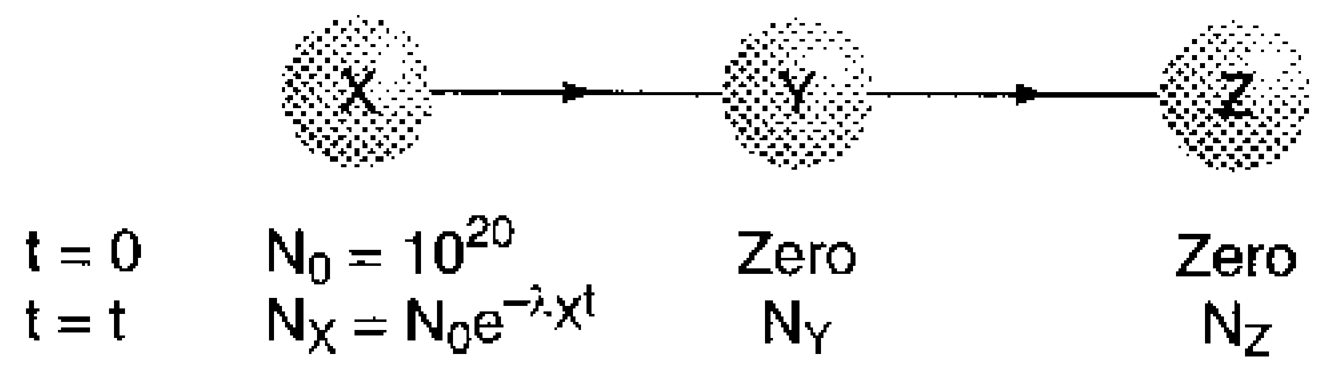


Fig. 25.13

Rate equation of the populations of X, Y and Z are

$$\left(\frac{dN_X}{dt}\right) = -\lambda_X N_X \qquad \text{...(i)}$$

$$\left(\frac{dN_Y}{dt}\right) = \lambda_X N_X - \lambda_Y N_Y \qquad \text{...(ii)}$$

and

$$\left(\frac{dN_Z}{dt}\right) = \lambda_Y N_Y \qquad \text{...(iii)}$$

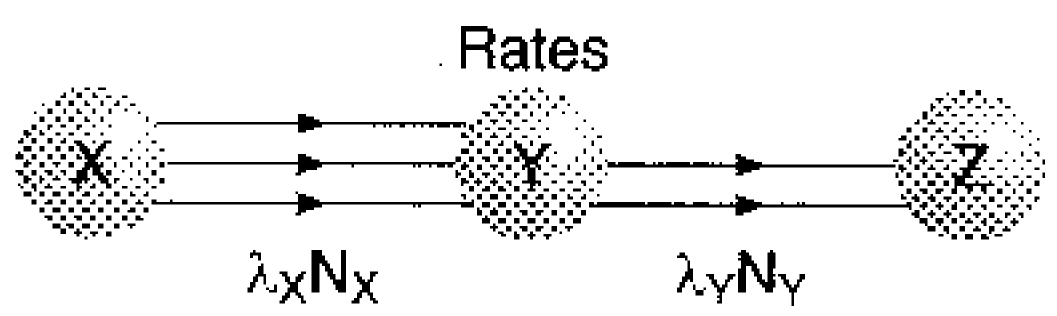


Fig. 25.14

(ii) Given $N_Y(t) = \frac{N_0 \lambda_X}{\lambda_X - \lambda_Y}[e^{-\lambda_Y t} - e^{-\lambda_X t}]$

For N_Y to be maximum $\quad \frac{dN_Y(t)}{dt} = 0$

i.e.,

$$\lambda_X N_X = \lambda_Y N_Y \qquad \text{[From Eq. (ii)]} \qquad \text{...(iv)}$$

or

$$\lambda_X (N_0 e^{-\lambda_X t}) = \lambda_Y \frac{N_0 \lambda_X}{\lambda_X - \lambda_Y}[e^{-\lambda_Y t} - e^{-\lambda_X t}]$$

or

$$\frac{\lambda_X - \lambda_Y}{\lambda_Y} = \frac{e^{-\lambda_Y t}}{e^{-\lambda_X t}} - 1 \quad \text{or} \quad \frac{\lambda_X}{\lambda_Y} = e^{(\lambda_X - \lambda_Y)t}$$

or

$$(\lambda_X - \lambda_Y)\, t \ln(e) = \ln\left(\frac{\lambda_X}{\lambda_Y}\right) \quad \text{or} \quad t = \frac{1}{(\lambda_X - \lambda_Y)} \ln\left(\frac{\lambda_X}{\lambda_Y}\right)$$

Substituting the values of λ_X and λ_Y, we have

$$t = \frac{1}{(0.1 - 1/30)} \ln\left(\frac{0.1}{1/30}\right) = 15 \ln(3)$$

or

$$t = 16.48 \text{ s}$$ **Ans.**

(iii) The population of X at this moment

$$N_X = N_0 e^{-\lambda_X t} = (10^{20})\, e^{-(0.1)(16.48)}$$

$$N_X = 1.92 \times 10^{19}$$

$$N_Y = \frac{N_X \lambda_X}{\lambda_Y} \qquad \text{[From Eq. (iv)]}$$

$$= (1.92 \times 10^{19}) \frac{(0.1)}{1/30} = 5.76 \times 10^{19}$$

$$\therefore \quad N_Z = N_0 - N_X - N_Y$$

$$= 10^{20} - 1.92 \times 10^{19} - 5.76 \times 10^{19}$$

or

$$N_Z = 2.32 \times 10^{19}$$

25.4 EQUIVALENCE OF MASS AND ENERGY

In 1905, while developing his special theory of relativity, Einstein made the suggestion that energy and mass are equivalent. He predicted that if the energy of a body changes by an amount E, its mass changes by an amount m given by the equation,

$$E = mc^2$$

where c is the speed of light. Everyday examples of energy gain are much too small to produce detectable changes of mass. But in nuclear physics this plays an important role. Mass appears as energy and the two can be regarded as equivalent. In nuclear physics mass is measured in **unified atomic mass units (u)**, 1 u being one-twelfth of the mass of carbon-12 atom and equals 1.66×10^{-27} kg. It can readily be shown using $E = mc^2$ that, 1 u mass has energy 931.5 MeV

Thus,

$$1 \text{ u} \equiv 931.5 \text{ MeV}$$

or

$$c^2 = 931.5 \text{ MeV/u}$$

A unit of energy may therefore be considered to be a unit of mass. For example, the electron has a rest mass of about 0.5 MeV.

If the principle of conservation of energy is to hold for nuclear reactions it is clear that mass and energy must be regarded as equivalent. The implication of $E = mc^2$ is that any reaction producing an appreciable mass decrease is a possible source of energy.

EXAMPLE 25.7 *Find the increase in mass of water when* 1.0 *kg of water absorbs* 4.2×10^3 *J of energy to produce a temperature rise of* 1 *K.*

SOLUTION

$$m = \frac{E}{c^2} = \frac{4.2 \times 10^3}{(3.0 \times 10^8)^2} \text{ kg}$$

$$= 4.7 \times 10^{-14} \text{ kg}$$

Ans.

25.5 BINDING ENERGY AND NUCLEAR STABILITY

The existence of a stable nucleus means that the nucleons (protons and neutrons) are in a bound state. Since the protons in a nucleus experience strong electrical repulsion, there must exist a stronger attractive force that holds the nucleus together. The **nuclear force** is a short range interaction that extends only to about 2 fm. (In contrast, the electromagnetic interaction is a long-range interaction). An important feature of the nuclear force is that it is essentially the same for all nucleons, independent of charge.

The **binding energy** (E_b) of a nucleus is the energy required to completely separate the nucleons. The origin of the binding energy may be understood with the help of mass-energy relation, $\Delta E = \Delta mc^2$, where Δm is the difference between the total mass of the separated nucleons and the mass of the stable nucleus. The mass of the stable nucleus is less than the sum of the mass of its nucleons. The binding energy of a nuclide $_Z X^A$ is thus,

$$E_b = [zm_p + (A - z)\, m_N - m_X]\, c^2 \qquad \text{...(i)}$$

where m_P = mass of proton, m_N = mass of neutron and m_X = mass of nucleus.

Note : (1) $\Delta m = [zm_p + (A - z)\, m_N - m_X]$ is called the **mass defect**. This much mass is lost during the formation of a nucleus. Energy $\Delta E = (\Delta m)\, c^2$ is liberated during the making of the nucleus. This is the energy due to which nucleons are bound together. So, to break the nucleus in its constituent nucleons this much energy has to be given to the nucleus.

(2) It is better to write Eq. (i) as under,

$$E_b = [zm_H + (A - z)\, m_N - m_A]\, c^2 \qquad \text{...(ii)}$$

where m_H is the mass of H atom and m_A the atomic mass. By using the masses of H atoms rather than protons, masses of the electrons in the atom cancel out. We do this because it is atomic masses that are measured directly by mass spectrometer. A slight error is made by doing so but that is negligible.

(3) **Stability :** Although nuclides with z values upto $z = 92$ (Uranium) occur naturally, not all of these are stable. The nuclide $^{209}_{83}\text{Bi}$ is the heaviest stable nucleus. Even though uranium is not stable, however, its long lived isotope ^{238}U, has a half life of some 4 billion year.

Binding energy per nucleon : If the binding energy of a nucleus is divided by its mass number, the binding energy per nucleon is obtained. A plot of binding energy per nucleon E_b / A as a function of mass number A for various stable nuclei is shown in figure.

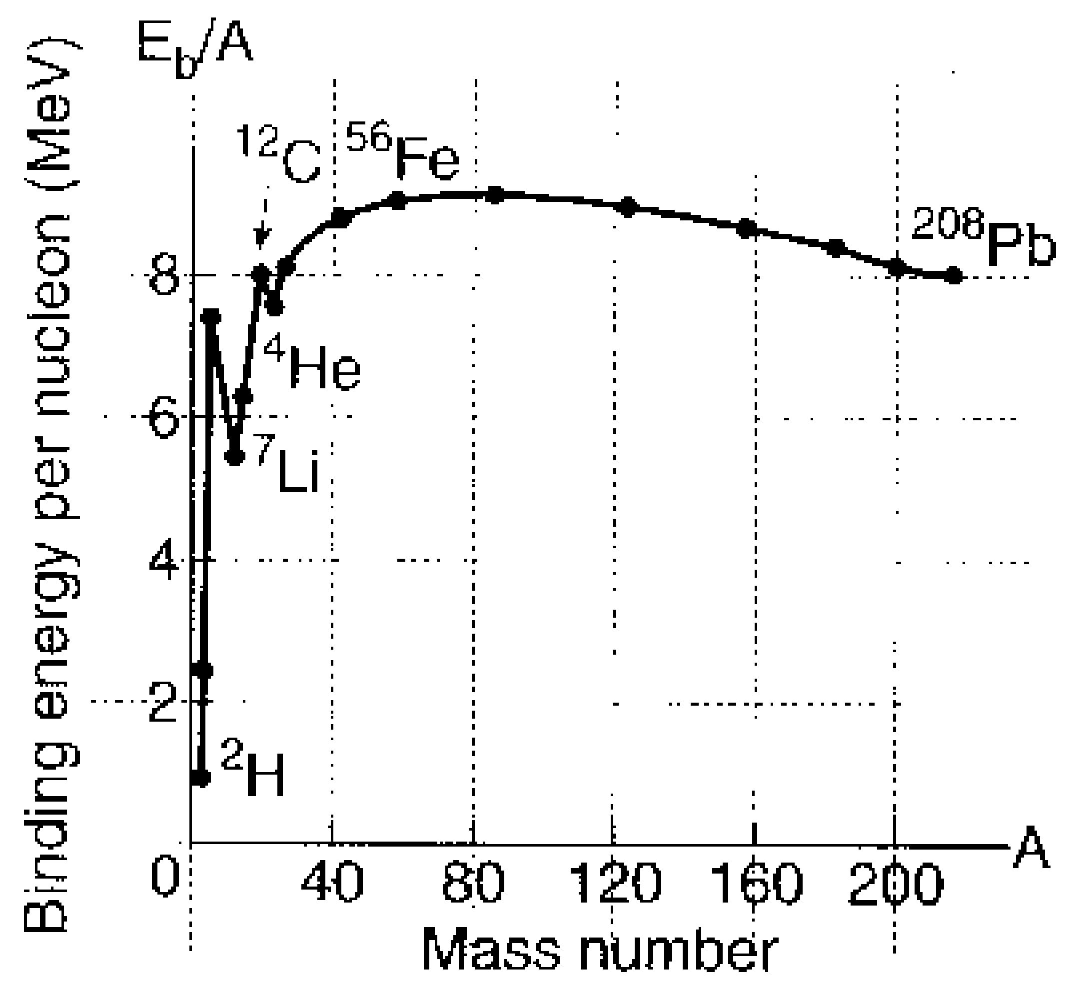


Fig. 25.15 The binding energy per nucleon, E_b/A, as a function of the mass number A.

Note : that it is the binding energy per nucleon which is more important for stability of a nucleus rather than the total binding energy.

Following conclusions can be drawn from the above graph.

(1) The greater the binding energy per nucleon the more stable is the nucleus. The curve reaches a maximum of about 8.75 MeV in the vicinity of $^{56}_{26}\text{Fe}$ and then gradually falls to 7.6 MeV for $^{238}_{92}\text{U}$.

(2) In a nuclear reaction energy is released if total binding energy is increasing. Let us take an example.

Suppose a nucleus X, which has total binding energy of 100 MeV converts into some another nucleus Y which has total binding energy 120 MeV. Then in this process 20 MeV energy will be released. This is bacause 100 MeV energy has already been released during the formation of X while in case of Y it is 120 MeV. So the remaining 20 MeV will be released now.

Energy is released if ΣE_b is increasing.

(3) ΣE_b in a nuclear process is increased if binding energy per nucleon of the daughter products gets increased. Let us take an example. Consider a nucleus $X\,(A_X = 100)$ breaks in two lighter nuclei $Y\,(A_Y = 60)$ and $Z(A_Z = 40)$.

$$X \rightarrow Y + Z$$

Binding energy per nucleon of these three are say, 7 MeV, 7.5 MeV and 8.0 MeV. Then total binding energy of X is $100 \times 7 = 700$ MeV and that of $Y + Z$ is $(60 \times 7.5) + (40 \times 8.0) = 770$ MeV. So in this process 70 MeV energy will be released.

(4) Binding energy per nucleon is increased if two or more lighter nuclei combine to form a heavier nucleus. This process is called **nuclear fusion**.

Fig. 25.16

In **nuclear fission** a heavy nucleus splits into two or more lighter nuclei of almost equal mass.

In both the processes E_b / A is increasing. Thus, energy will be released.

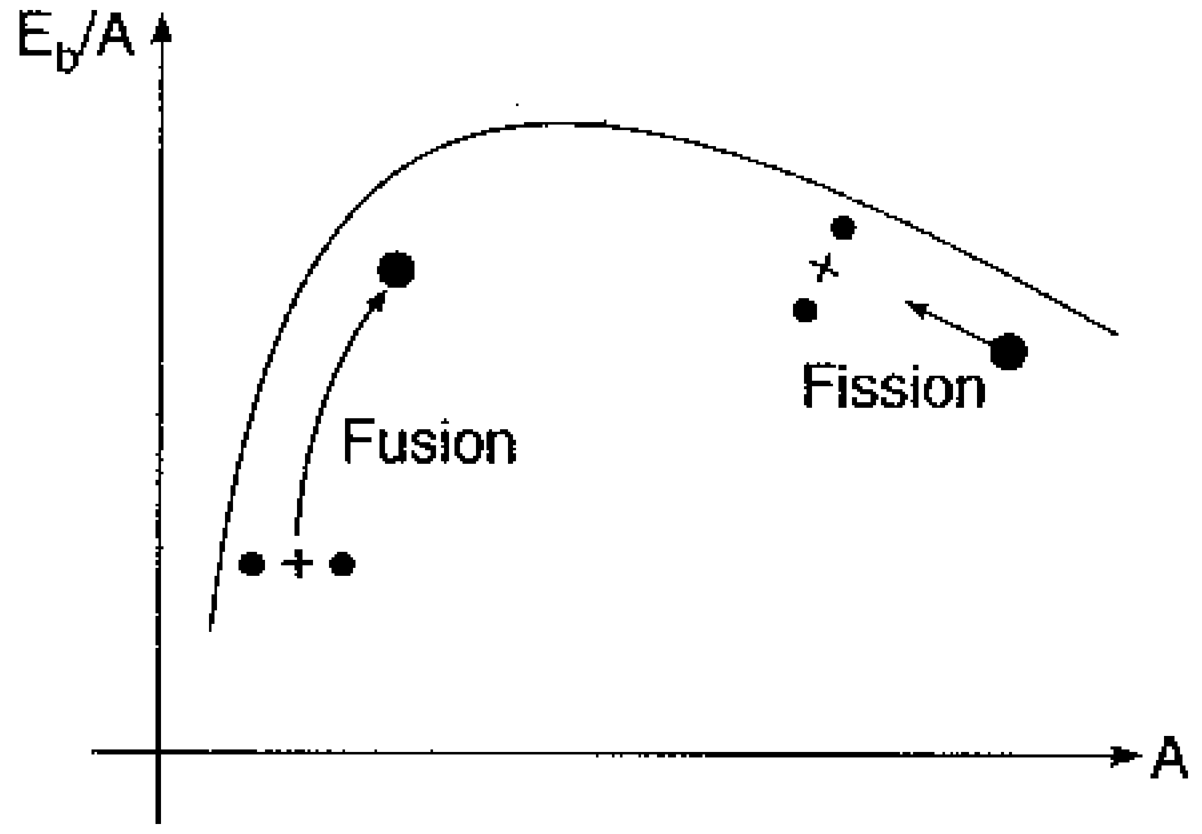


Fig. 25.17

25.6 NUCLEAR FISSION (DIVIDE AND CONQUER)

As we saw in the above article nuclear fission occurs when a heavy nucleus such as ^{235}U, splits into two lighter nuclei. In nuclear fission the combined mass of the daughter nuclei is less than the mass of the parent nucleus. The difference is called the mass defect. Fission is initiated when a heavy nucleus captures a thermal neutron (slow neutrons). Multiplying the mass defect by c^2 gives the numerical value of the released energy. Energy is released because the binding energy per nucleon of the daughter nuclei is about 1 MeV greater than that of the parent nucleus.

The fission of ^{235}U by thermal neutrons can be represented by the equation,

$$^{1}_{0}n + {}^{235}_{92}\text{U} \longrightarrow {}^{236}_{92}\text{U}^* \longrightarrow X + Y + \text{neutrons}$$

where $^{236}\text{U}^*$ is an intermediate excited state that lasts only for 10^{-12} s before breaking into nuclei X and Y, which are called fission fragments. In any fission equation there are many combinations of X and Y that satisfy the requirements of conservation of energy and charge with uranium, for example, there are about 90 daughter nuclei that can be formed.

Fission also results in the production of several neutrons, typically two or three. On the average, about 2.5 neutrons are released per event. A typical fission reaction for uranium is

$$^{1}_{0}n + {}^{235}_{92}\text{U} \longrightarrow {}^{141}_{56}\text{Ba} + {}^{92}_{36}\text{Kr} + 3\,({}^{1}_{0}n)$$

About 200 MeV is released in the fission of a heavy nucleus. The fission energy appears mostly as kinetic energy of the fission fragments (*e.g.*, barium and krypton nuclei) which fly apart at great speed. The kinetic energy of the fission neutrons also makes a slight contribution. In addition one or both of the large fragments are highly radioactive and small amount of energy takes the form of beta and gamma radiation.

Chain Reaction : Shortly after nuclear fission was discovered, it was realized that, the fission neutrons can cause further fission of ^{235}U and a chain reaction can be maintained.

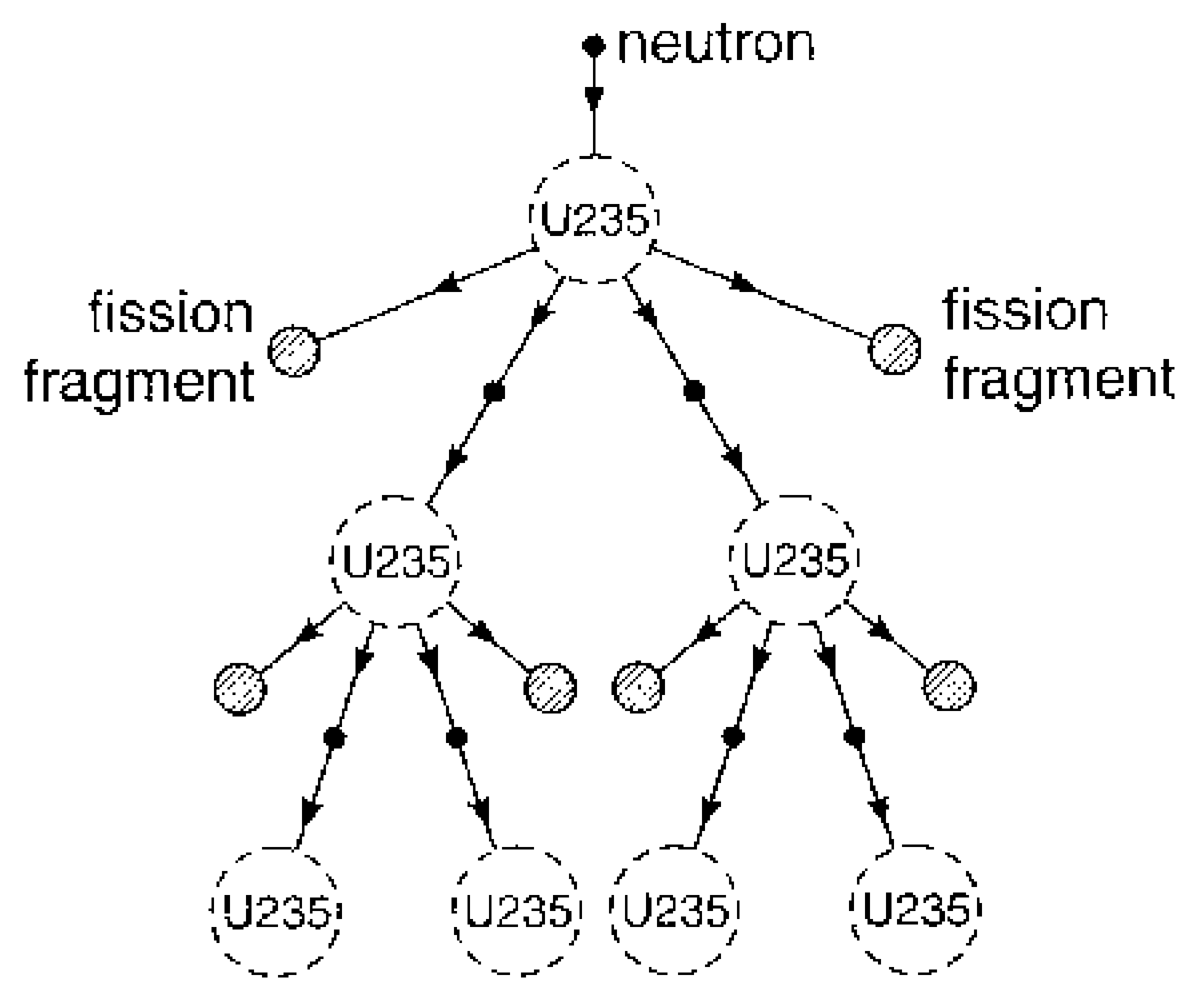


Fig. 25.18 A chain reaction

In practice only a proportion of the fission neutrons is available for new fissions since, some are lost by escaping from the surface of the uranium before colliding with another nucleus. The ratio of neutrons escaping to those causing fission decreases as the size of the piece of uranium-235 increases and there is a **critical size** (about the size of a cricket ball) which must be attained before a chain reaction can start.

In the **'atomic bomb'** an increasing uncontrolled chain reaction occurs in a very short time when two pieces of Uranium-235 are rapidly brought together to form a mass greater than the critical size.

Nuclear Reactors : In a nuclear reactor the chain reaction is steady and controlled so that on average only one neutron from each fission produces another fission. The reaction rate is adjusted by inserting neutron-absorbing rods of boron steel into the Uranium 235.

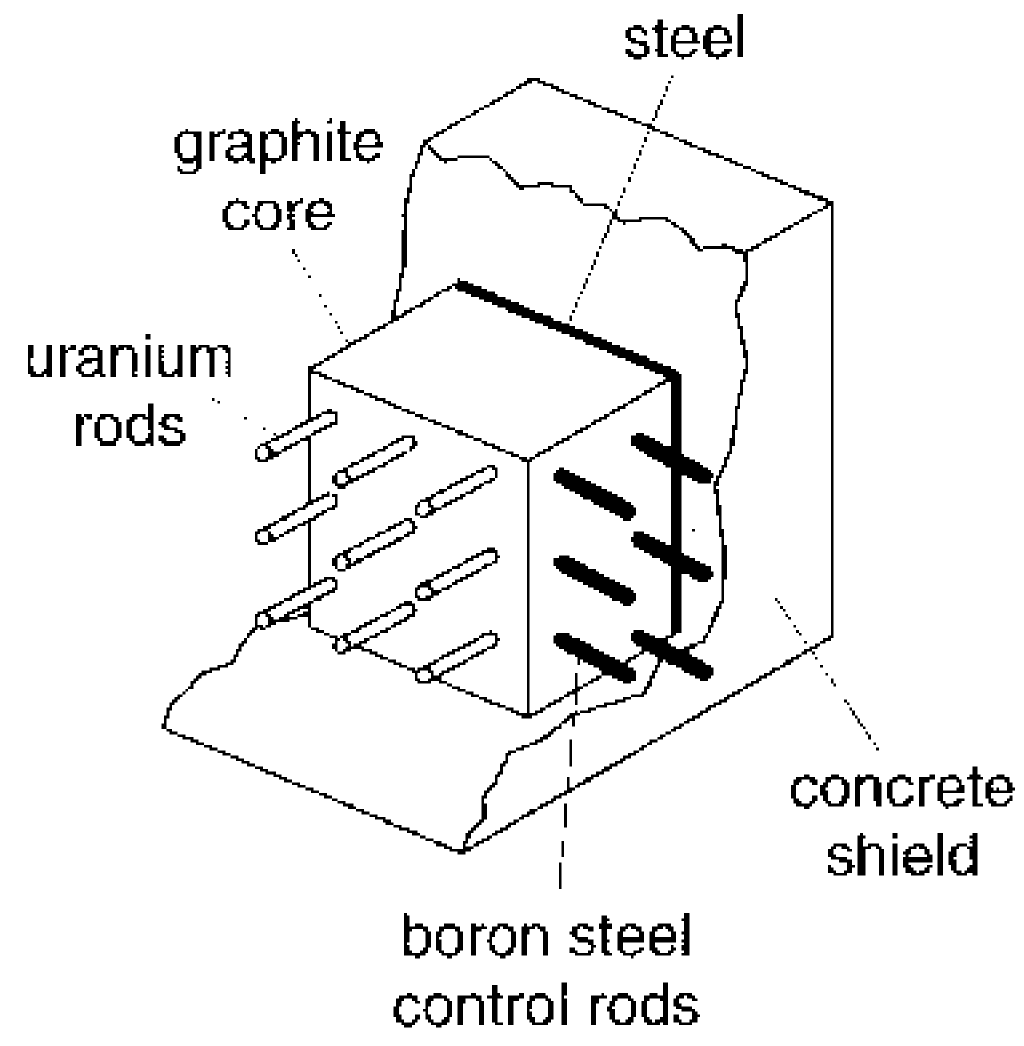


Fig. 25.19 Nuclear reactor

Graphite core is used as a **moderator** to slow down the neutrons.

Natural Uranium contains over 99% of ^{238}U and less than 1% of ^{235}U. The former captures the medium speed fission neutrons without fissioning. It fissions with very fast neutrons. On the other hand ^{235}U (and plutonium-239) fissions with slow neutrons and the job of moderator is to slow down the fission neutrons very quickly so that most escape capture by ^{238}U and then cause the fission of ^{235}U.

A bombarding particle gives up most energy when it has an elastic collision with a particle of similar mass. For neutrons, hydrogen atoms would be most effective but they absorb the neutrons. But deuterium (in heavy water) and carbon (as graphite) are both suitable as moderator.

To control the power level **control rods** are used. These rods are made of materials such as cadmium, that are very efficient in absorbing neutrons.

The first nuclear reactor was built by Enrico Fermi and his team at the University of Chicago in 1942.

25.7 NUCLEAR FUSION

Binding energy for light nuclei ($A < 20$) is much smaller than the binding energy for heavier nuclei. This suggests a process that is the reverse of fission. When two light nuclei combine to form a heavier nucleus, the process is called nuclear fusion. The union of light nuclei into heavier nuclei also lead to a transfer of mass and a consequent liberation of energy. Such a reaction has been achieved in **'hydrogen bomb'** and it is believed to be the principal source of the sun's energy.

A reaction with heavy hydrogen or deuterium which yields 3.3 MeV per fusion is,

$$^2_1\text{H} + ^2_1\text{H} \rightarrow ^3_2\text{He} + ^1_0n$$

By comparison with the 200 MeV per fission of ^{235}U this seems small, but per unit mass of material it is not. Fusion of two deuterium nuclei, *i.e.*, deuterons, will only occur if they overcome their mutual electrostatic repulsion. This may happen, if they collide at very high speed when, for example, they are raised to a very high temperature ($10^8 - 10^9$ K). So, much high temperature is obtained by using an atomic (fission) bomb to trigger off fusion.

If a controlled fusion reaction can be achieved an almost unlimited supply of energy will become available from deuterium in the water of the oceans.

25.8 Q-VALUE OF A NUCLEAR REACTION

Consider a nuclear reaction in which a target nucleus X is bombarded by a particle 'a' resulting in a daughter nucleus Y and a particle b.

$$a + X \rightarrow Y + b$$

Sometimes this reaction is written as,

$$X\ (a, b)Y$$

The reaction energy Q associated with a nuclear reaction is defined as the total energy released as a result of the reaction. Thus,

$$Q = (M_a + M_X - M_Y - M_b)c^2$$

A reaction for which Q is positive is called **exothermic**. A reaction for which Q is negative is called **endothermic**.

In an exothermic reaction, the total mass of incoming particles is greater than that of the outgoing particles and the Q value is positive. If the total mass of the incoming particles is less than that of the outgoing particles, energy is required for reaction to take-place and the reaction is said to be endothermic. Thus, an endothermic reaction does not occur unless the bombarding particle has a kinetic energy greater than $|Q|$. The minimum energy necessary for such a reaction to occur is called **threshold energy K_{th}**. The threshold energy is somewhat greater than $|Q|$ because the outgoing particles must have some kinetic energy to conserve momentum.

Thus, $K_{th} > |Q|$ (in endothermic reaction)

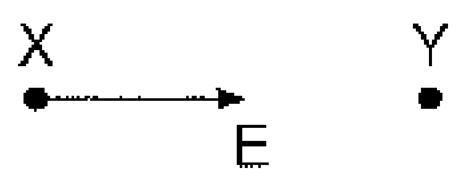


Fig. 25.20

Consider a bombarding particle X of mass m_1 and a target Y of mass m_2 (at rest). The threshold energy of X for endothermic reaction (negative value of Q) to take place is,

$$K_{th} = |Q|\left(\frac{m_1}{m_2} + 1\right)$$

EXAMPLE 25.8 *Find the minimum kinetic energy of an α-particle to cause the reaction $^{14}N(\alpha, p)^{17}O$. The masses of ^{14}N, ^{4}He, ^{1}H and ^{17}O are respectively 14.00307 u, 4.00260 u, 1.00783 u and 16.99913 u.*

SOLUTION Since, the masses are given in atomic mass units, it is easiest to proceed by finding the mass difference between reactants and products in the same units and then multiplying by 931.5 MeV/u. Thus, we have

$$Q = (14.00307\text{ u} + 4.00260\text{ u} - 1.00783\text{ u} - 16.99913\text{ u})\left(931.5\,\frac{\text{MeV}}{\text{u}}\right)$$

$$= -1.20\text{ MeV}$$

Q value is negative. It means reaction is endothermic.

So, the minimum kinetic energy of α-particle to initiate this reaction would be,

$$K_{\min} = |Q|\left(\frac{m_\alpha}{m_N} + 1\right)$$

$$= (1.20)\left(\frac{4.00260}{14.00307} + 1\right)$$

$$= 1.54\text{ MeV}$$ **Ans.**

EXAMPLE 25.9 *Neon-23 decays in the following way*

$$^{23}_{10}\text{Ne} \longrightarrow {}^{23}_{11}\text{Na} + {}^{0}_{-1}e + \bar{\nu}$$

Find the minimum and maximum kinetic energy that the beta particle $({}^{0}_{-1}e)$ can have. The atomic masses of ^{23}Ne and ^{23}Na are 22.9945 u and 22.9898 u, respectively.

SOLUTION Here, atomic masses are given (not the nuclear masses), but still we can use them for calculating the mass defect because mass of electrons get cancelled both sides. Thus,

Mass defect $\Delta m = (22.9945 - 22.9898) = 0.0047\text{ u}$

$\therefore$ $Q = (0.0047\text{ u})(931.5\text{ MeV/u})$

$= 4.4\text{ MeV}$

Hence, the energy of beta particles can range from 0 to 4.4 MeV. **Ans.**

INTRODUCTORY EXERCISE 25.2

1. When fission occurs, several neutrons are released and the fission fragments are beta radioactive. why?
2. (a) How much mass is lost per day by a nuclear reactor operated at a 10^9 watt power level.
   (b) If each fission releases 200 MeV, how many fissions occur per second to yield this power level.
3. Find energy released in the alpha decay

$$^{238}_{92}\text{U} \longrightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He}$$

Given
$$M\,({}^{238}_{92}U) = 238.050784\ \text{u}$$
$$M\,({}^{234}_{90}Th) = 234.043593\ \text{u}$$
$$M\,({}^{4}_{2}He) = 4.002602\ \text{u}$$

4. Complete the nuclear reactions.

(a) ${}^{6}_{3}Li + ? \longrightarrow {}^{7}_{4}Be + {}^{1}_{0}n$

(b) ${}^{35}_{13}Cl + ? \longrightarrow {}^{32}_{16}S + {}^{4}_{2}He$

(c) ${}^{9}_{4}Be + {}^{4}_{2}He \longrightarrow 3\,({}^{4}_{2}He) + ?$

(d) ${}^{79}_{35}Br + {}^{2}_{1}H \longrightarrow ? + 2({}^{1}_{0}n)$

MISCELLANEOUS EXAMPLES

EXAMPLE 1 *A ^{118}Cd radionuclide goes through the transformation chain.*

$$^{118}\text{Cd} \xrightarrow[30\text{ min}]{} {}^{118}\text{In} \xrightarrow[45\text{ min}]{} {}^{118}\text{Sn} \ (stable)$$

The half lives are written below the respective arrows. A time $t=0$ only Cd was present. Find the fraction of nuclei transformed into stable over 60 minutes.

SOLUTION At time $t = t$ $\quad N_1 = N_0 e^{-\lambda_1 t}$ and $N_2 = \dfrac{N_0\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$ (see Article 25.3)

$$\therefore \quad N_3 = N_0 - N_1 - N_2$$

$$= N_0\left[1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})\right]$$

$$\therefore \quad \frac{N_3}{N_0} = 1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\lambda_1 = \frac{0.693}{30} = 0.0231 \text{ min}^{-1}$$

$$\lambda_2 = \frac{0.693}{45} = 0.0154 \text{ min}^{-1}$$

and $\quad t = 60$ minutes

$$\therefore \quad \frac{N_3}{N_0} = 1 - e^{-0.0231\times 60} - \frac{0.0231}{0.0154 - 0.0231}(e^{-0.0231\times 60} - e^{-0.0154\times 60})$$

$$= 1 - 0.25 + 3\,(0.25 - 0.4)$$

$$= 0.31$$

Ans.

EXAMPLE 2 *A ^{7}Li target is bombarded with a proton beam current of 10^{-4} A for 1 hour to produce ^{7}Be of activity 1.8×10^8 disintegrations per second. Assuming that one ^{7}Be radioactive nucleus is produced by bombarding 1000 protons, determine its half-life.*

SOLUTION At time t, let say there are N atoms of ^{7}Be (radioactive). Then net rate of formation of ^{7}Be nuclei at this instant is,

$$\frac{dN}{dt} = \frac{10^{-4}}{1.6\times 10^{-19}\times 1000} - \lambda N$$

or

$$\frac{dN}{dt} = 6.25\times 10^{11} - \lambda N$$

or

$$\int_0^{N_0} \frac{dN}{6.25\times 10^{11} - \lambda N} = \int_0^{3600} dt$$

where N_0 are the number of nuclei at $t = 1$ hr or 3600 second.

$$\therefore \quad -\frac{1}{\lambda}\ln\left(\frac{6.25\times10^{11}-\lambda N_0}{6.25\times10^{11}}\right)=3600$$

λN_0 = activity of ^{7}Be at $t = 1$ hr $= 1.8\times10^8$ disintegrations/second

$$\therefore \quad -\frac{1}{\lambda}\ln\left(\frac{6.25\times10^{11}-1.8\times10^{8}}{6.25\times10^{11}}\right)=3600$$

$$\therefore \quad \lambda = 8.0\times10^{-8}\ \text{sec}^{-1}$$

Therefore, half life

$$t_{1/2}=\frac{0.693}{8.0\times10^{-8}}=8.66\times10^{6}\ \text{sec}$$

$$=100.26\ \text{days}$$

Ans.

EXAMPLE 3 *A nucleus X-initially at rest, undergoes alpha-decay, according to the equation*

$$^{A}_{92}X \longrightarrow\ ^{228}_{z}Y+\alpha$$

(*a*) *Find the value of A and z in the above process.*

(*b*) *The α-particle in the above process is found to move in a circular track of radius* 1.1×10^{2} *m in a uniform magnetic field of* 3.0×10^{3} *T. Find the energy (in MeV) released during the process and binding energy of the parent nucleus X.*

Given : $m_y = 228.03$ *amu* $\quad m_\alpha = 4.003$ *amu* $\quad m\,(^1_0n) = 1.009$ *amu* $\quad m(^1_1\text{H}) = 1.008$ *amu*

$1\ amu = 1.66\times10^{-27}\ kg \equiv 931.5\ MeV/c^2$.

SOLUTION (a) The given equation is,

$$^{A}_{92}X \longrightarrow\ ^{228}_{z}Y+\ ^{4}_{2}\text{He}$$

$$A = 228+4 = 232$$ **Ans.**

and

$$92 = z+2 \qquad \therefore\ z = 90$$ **Ans.**

(b)

$$\frac{m_\alpha v_\alpha^2}{r}=qv_\alpha B$$

$$\therefore \quad v_\alpha=\sqrt{\frac{rqB}{m_\alpha}}$$

$$=\sqrt{\frac{1.1\times10^{2}\times2\times1.6\times10^{-19}\times3\times10^{3}}{4.003\times1.66\times10^{-27}}}$$

$$=4.0\times10^{6}\ \text{m/s}$$

From conservation of linear momentum,

$$m_\alpha v_\alpha = m_y v_y$$

$$\therefore \quad v_y=\frac{m_\alpha v_\alpha}{m_y}=\frac{(4.003)\,(4.0\times10^{6})}{(228.03)}$$

$$=7.0\times10^{4}\ \text{m/s}$$

Therefore, energy released during the process

$$= \frac{1}{2}\,[m_\alpha\, v_\alpha^2 + m_y\, v_y^2]$$

$$= \frac{(1.66\times10^{-27})}{(2\times1.6\times10^{-13})}\,[(4.003)\,(4.0\times10^6)^2 + (228.03)\,(7.0\times10^4)^2]\,\text{MeV}$$

$$= 0.34\ \text{MeV} = \frac{0.34}{931.5}\ \text{amu} = 0.000365\ \text{amu}$$ **Ans.**

Therefore, mass of $\quad {}^{232}_{92}X = m_y + m_\alpha + 0.000365 = 232.033365\ \text{u}$

Mass defect $\quad \Delta m = 92\,(1.008) + (232 - 92)\,(1.009) - 232.033365$

$$= 1.962635\ \text{amu}$$

$\therefore$ $\quad$ Binding energy $= 1.962635 \times 931.5$ MeV

$$= 1828.2\ \text{MeV}$$ **Ans.**

EXAMPLE 4 *In the fusion reaction* ${}^2_1\text{H} + {}^2_1\text{H} \longrightarrow {}^3_2\text{He} + {}^1_0n$, *the masses of deuteron, helium and neutron expressed in amu are* 2.015, 3.017 *and* 1.009 *respectively. If* 1 *kg of deuterium undergoes complete fusion, find the amount of total energy released.* 1 *amu* $\equiv 931.5\ MeV/c^2$.

SOLUTION $\quad \Delta m = 2\,(2.015) - (3.017 + 1.009) = 0.004$ amu

$\therefore$ $\quad$ Energy released $= (0.004 \times 931.5)$ MeV $= 3.726$ MeV

Energy released per deuteron $= \dfrac{3.726}{2} = 1.863$ MeV

Number of deuterons in 1 kg $= \dfrac{6.02\times10^{26}}{2} = 3.01\times10^{26}$

$\therefore$ $\quad$ Energy released per kg of deuterium fusion $= (3.01\times10^{26} \times 1.863) = 5.6\times10^{26}$ MeV

$$\approx 9.0\times10^{13}\ \text{J}$$ **Ans.**

EXAMPLE 5 *The mean lives of a radio-active substances are* 1620 *years and* 405 *years for* α*-emission and* β *emission respectively. Find out the time during which three-fourth of a sample will decay if it is decaying both by* α*-emission and* β*-emission simultaneously.*

SOLUTION Let at some instant of time t, number of atoms of the radioactive substance are N. It may decay either by α-emission or by β-emission. So, we can write,

$$\left(\frac{-dN}{dt}\right)_{\text{net}} = \left(\frac{-dN}{dt}\right)_\alpha + \left(\frac{-dN}{dt}\right)_\beta$$

If the effective decay constant is λ, then

$$\lambda N = \lambda_\alpha N + \lambda_\beta N$$

or $\quad \lambda = \lambda_\alpha + \lambda_\beta = \dfrac{1}{1620} + \dfrac{1}{405}$

$$= \frac{1}{324}\ \text{year}^{-1}$$

Now, $$\frac{N_0}{4} = N_0 e^{\lambda t}$$

∴ $$-\lambda t = \ln\left(\frac{1}{4}\right) = -1.386$$

or $$\left(\frac{1}{324}\right)t = 1.386$$

∴ $$t = 449 \text{ years}$$ **Ans.**

EXAMPLE 6 *In the chemical analysis of a rock the mass ratio of two radioactive isotopes is found to be* $100:1$. *The mean lives of the two isotopes are* 4×10^9 *years and* 2×10^9 *years respectively. If it is assumed that at the time of formation the atoms of both the isotopes were in equal proportional, calculate the age of the rock. Ratio of the atomic weights of the two isotopes is* $1.02:1$.

SOLUTION At the time of observation ($t = t$),

$$\frac{m_1}{m_2} = \frac{100}{1} \quad \text{(given)}$$

Further it is given that $$\frac{A_1}{A_2} = \frac{1.02}{1}$$

Number of atoms $$N = \frac{m}{A}$$

∴ $$\frac{N_1}{N_2} = \frac{m_1}{m_2} \times \frac{A_2}{A_1} = \frac{100}{1.02} \quad \text{...(i)}$$

Let N_0 be the number of atoms of both the isotopes at the time of formation, then

$$\frac{N_1}{N_2} = \frac{N_0 e^{-\lambda_1 t}}{N_0 e^{-\lambda_2 t}} = e^{(\lambda_2 - \lambda_1)t} \quad \text{...(ii)}$$

Equating (i) and (ii), we have

$$e^{(\lambda_2 - \lambda_1)t} = \frac{100}{1.02} \quad \text{or} \quad (\lambda_2 - \lambda_1)t = \ln(100) - \ln(1.02)$$

∴ $$t = \frac{\ln(100) - \ln(1.02)}{\left(\frac{1}{2\times10^9} - \frac{1}{4\times10^9}\right)}$$

Substituting the values, we have

$$t = 1.834 \times 10^{10} \text{ years}$$ **Ans.**

EXAMPLE 7 *A radionuclide with half life* 1620 sec *is produced in a reactor at a constant rate* 1000 *nuclei per second. During each decay energy* 200 *MeV is released. If production of radio nuclides started started at* $t = 0$, *calculate*

(*a*) *rate of release of energy at* $t = 3240$ *sec.*
(*b*) *total energy released upto* $t = 405$ *sec.*

SOLUTION (a) Let N be the number of nuclei at time t, then net rate of increase of nuclei at instant t is,

$$\frac{dN}{dt} = \alpha - \lambda N \qquad \text{(where } \alpha = \text{rate of production of nuclei)}$$

$$\therefore \quad \int_0^N \frac{dN}{\alpha - \lambda N} = \int_0^t dt$$

$$\therefore \quad N = \frac{\alpha}{\lambda}(1 - e^{-\lambda t}) \qquad \text{...(i)}$$

Rate of decay at this instant $\quad R = \lambda N = \alpha\ (1 - e^{-\lambda t})$

Hence, rate of release of energy at this time = R (energy released in each decay)

$$= \alpha\ (1 - e^{-\lambda t})\ (200)\ \text{MeV/sec}$$

Substituting the values, we have

$$\text{rate of release of energy } = 1000\ (1 - e^{-\frac{0.693}{1620} \times 3240})\ (200)$$

$$= 1.5 \times 10^5\ \text{MeV/second}$$ **Ans.**

(b) Total number of nuclei decayed upto time $t = \alpha t - N$

$$= \alpha t - \frac{\alpha}{\lambda}\ (1 - e^{-\lambda t})$$

Hence, total energy released upto this instant

$$E = [\alpha t - \frac{\alpha}{\lambda}(1 - e^{-\lambda t})]\ (200)\ \text{MeV}$$

Substituting the values, we have

$$E = \left[1000 \times 405 - \frac{1000}{0.693/1620}(1 - e^{-\frac{0.693}{1620} \times 405})\right] \times 200\ \text{MeV}$$

$$= 6.63 \times 10^6\ \text{MeV}$$ **Ans.**

EXAMPLE 8 *Natural uranium is a mixture of three isotopes* $^{234}_{92}\text{U}$, $^{235}_{92}\text{U}$ *and* $^{238}_{92}\text{U}$ *with mass percentage* 0.01% 0.71% *and* 99.28% *respectively. The half life of three isotopes are* 2.5×10^5 *years,* 7.1×10^8 *years and* 4.5×10^9 *years respectively. Determine the share of radioactivity of each isotope into the total activity of the natural uranium.*

SOLUTION Let R_1, R_2 and R_3 be the activities of U^{234}, U^{235} and U^{238} respectively.

Total activity $\quad R = R_1 + R_2 + R_3$

Share of U^{234}, $\quad \dfrac{R_1}{R} = \dfrac{\lambda_1 N_1}{\lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3}$

Let m be the total mass of natural uranium.

Then $m_1 = \dfrac{0.01}{100} m \qquad m_2 = \dfrac{0.71}{100} m \qquad$ and $\qquad m_3 = \dfrac{99.28}{100} m$

Now, $N_1 = \frac{m_1}{M_1}$ $\quad N_2 = \frac{m_2}{M_2}$ $\quad$ and $\quad N_3 = \frac{m_3}{M_3}$

where M_1, M_2 and M_3 are atomic weights.

$$\therefore \quad \frac{R_1}{R} = \frac{\left(\frac{m_1}{M_1}\right)\frac{1}{T_1}}{\frac{m_1}{M_1}\cdot\frac{1}{T_1} + \frac{m_2}{M_2}\cdot\frac{1}{T_2} + \frac{m_3}{M_3}\cdot\frac{1}{T_3}}$$

$$= \frac{\frac{(0.01/100)}{234} \times \frac{1}{2.5\times 10^5 \text{ years}}}{\left(\frac{0.01/100}{234}\right)\left(\frac{1}{2.5\times 10^5}\right) + \left(\frac{0.71/100}{235}\right)\left(\frac{1}{7.1\times 10^8}\right) + \left(\frac{99.28/100}{238}\right)\left(\frac{1}{4.5\times 10^9}\right)}$$

$$= 0.648 \approx 64.8\%$$

Similarly, share of $\quad U^{235} = 0.016\%$

and of $\quad U^{238} = 35.184\%$ **Ans.**

EXAMPLE 9 *A proton is bombarded on a stationary lithium nucleus. As a result of the collision two α-particles are produced. If the direction of motion of the α-particles with the initial direction of motion makes an angle* $\cos^{-1}(1/4)$, *find the kinetic energy of the striking proton. Given binding energies per nucleon of* Li^7 *and* He^4 *are* 5.60 *and* 7.06 *MeV respectively.*

(*Assume mass of proton ≈ mass of neutron*).

SOLUTION Q value of the reaction is,

$$Q = (2\times 4\times 7.06 - 7\times 5.6) \text{ MeV} = 17.28 \text{ MeV}$$

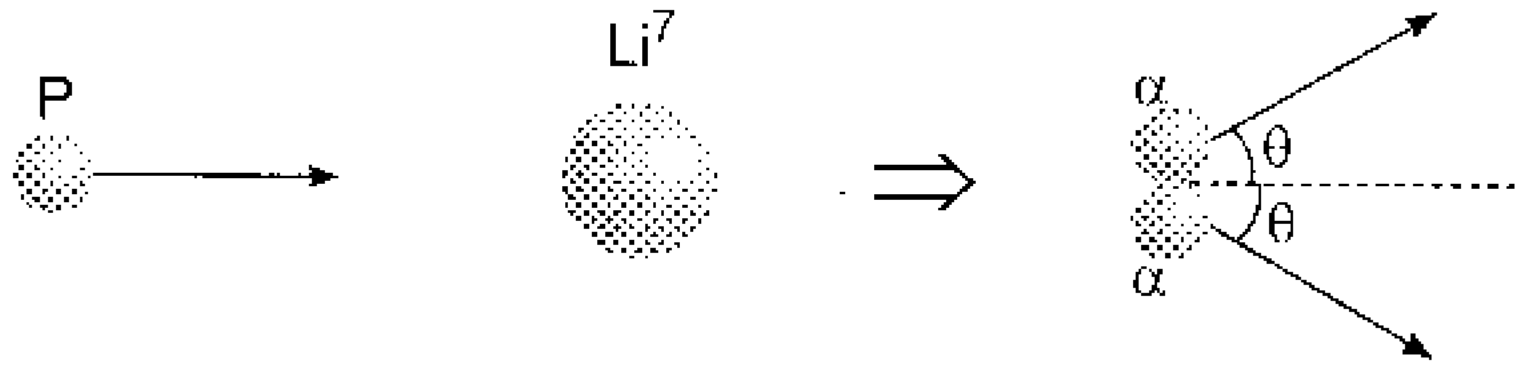


Fig. 25.21

Applying conservation of energy for collision,

$$K_p + Q = 2K_\alpha \quad \text{...(i)}$$

(Here K_p and K_α are the kinetic energies of proton and α-particle respectively)

From conservation of linear momentum,

$$\sqrt{2m_pK_p} = 2\sqrt{2m_\alpha K_\alpha}\cos\theta \quad \text{...(ii)}$$

$$\therefore \quad K_p = 16K_\alpha\cos^2\theta = (16K_\alpha)\left(\frac{1}{4}\right)^2 \quad (\text{as } m_\alpha = 4m_p)$$

$$\therefore \quad K_\alpha = K_p \quad \text{...(iii)}$$

Solving Eqs. (i) and (iii) with $Q = 17.28$ MeV

we get $$K_p = 17.28 \text{ MeV}$$ **Ans.**

EXAMPLE 10 *Suppose a nucleus initially at rest undergoes α-decay according to equation*

$$^{225}_{92}X \longrightarrow Y + \alpha$$

At $t = 0$, the emitted α-particle enters in a region of space where a uniform magnetic field $\vec{\mathbf{B}} = B_0\hat{\mathbf{i}}$ and electric field $\vec{\mathbf{E}} = E_0\hat{\mathbf{i}}$ exist. The α-particle enters in the region with velocity $\vec{\mathbf{v}} = v_0\hat{\mathbf{j}}$ from $x = 0$. At time $t = \sqrt{3} \times 10^7 \dfrac{m_\alpha}{q_\alpha E_0}$ sec, the particle was observed to have speed twice the initial speed v_0, then find

(*a*) *the velocity of α-particle at time t*
(*b*) *the initial velocity v_0 of the α-particle*
(*c*) *the binding energy per nucleon of α-particle*

Given that $m(Y) = 221.03\,u$ $m(\alpha) = 4.003\,u$ $m(n) = 1.009\,u$ $m(p) = 1.008\,u$

$$\text{mass of } \alpha\text{-particle } m_\alpha = \frac{2}{3} \times 10^{-26} \text{ kg}$$

$$\text{charge on } \alpha \text{ particle } q_\alpha = 3.2 \times 10^{-19} \text{ C} \quad \text{and} \quad 1\,u = 931 \text{ MeV}/c^2$$

SOLUTION (a) Magnetic force on α-particle, (at $t = 0$)

$$\vec{\mathbf{F}}_m = q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) = q_\alpha[(v_0\hat{\mathbf{j}}) \times (B_0\hat{\mathbf{i}})]$$
$$= -q_\alpha v_0 B_0 \hat{\mathbf{k}}$$

Force due to electric field (at any time t)

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = q_\alpha E_0 \hat{\mathbf{i}}$$

Hence, the particle will move in a circular path in y-z plane due to magnetic field and at the same time it will move along x-direction. The resultant path is therefore, a helix with increasing pitch.

Hence, velocity of particle at any time t can be written as,

$$\vec{\mathbf{v}} = \left(\frac{q_\alpha E_0}{m_\alpha} t\right)\hat{\mathbf{i}} + v_0\cos\theta\,\hat{\mathbf{j}} - v_0\sin\theta\,\hat{\mathbf{k}}$$ **Ans.**

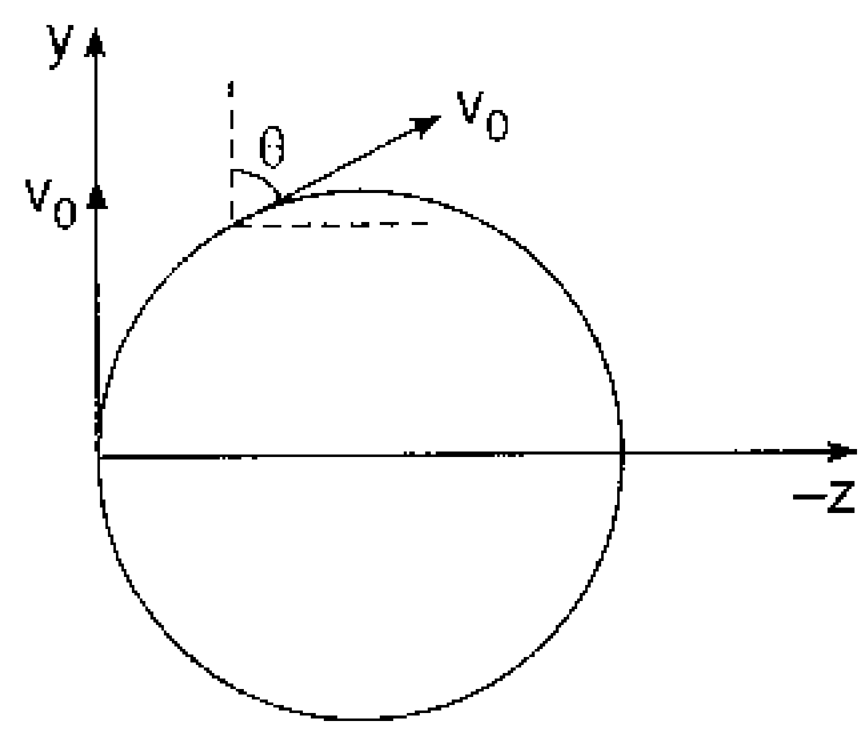


Fig. 25.22

Here $$\theta = \omega t = \frac{B_0 q_\alpha}{m_\alpha} t$$

(b) Speed of particle at any time t would be,

$$v = \sqrt{\left(\frac{q_\alpha E_0 t}{m_\alpha}\right)^2 + v_0^2}$$

Given $v = 2v_0$ at $t = (\sqrt{3} \times 10^7)\dfrac{m_\alpha}{q_\alpha E_0}$, we get

$$(2v_0)^2 = (\sqrt{3} \times 10^7)^2 + v_0^2$$

$\therefore$ $$v_0 = 10^7 \text{ m/s}$$ **Ans.**

when an α-particle is emitted with velocity v_0 from a stationary nucleus X, decay product (nucleus Y) recoils. Hence,

$$m_y v_y = m_\alpha v_0$$

$\therefore$ $$v_y = \frac{m_\alpha v_0}{m_y} = \left(\frac{4.003}{221.03}\right)(10^7)$$

$$= 1.81 \times 10^5 \text{ m/s}$$

$\therefore$ Total energy released during α-decay of nucleus X,

$$E = \text{K.E of nucleus } Y + \text{K.E of } \alpha\text{-particle}$$

$$= \frac{1}{2} m_y v_y^2 + \frac{1}{2} m_\alpha v_0^2$$

$$= \frac{1.66 \times 10^{-27}}{2 \times 1.6 \times 10^{-13}} [(221.03)(1.81 \times 10^5)^2 + (4.003)(10^7)^2]$$

$$= 2.11 \text{ MeV}$$

Hence, Mass lost during α-decay $= \dfrac{2.11}{931.5}$ u

$$= 0.0023 \text{ u}$$

Mass of nucleus X, $$m_x = (m_y + m_\alpha + 0.0023) \text{ u}$$

$$= 225.0353 \text{ u}$$

Mass defect in nucleus X, $$\Delta m = 92\, m_p + (225 - 92)\, m_n - m_x = 1.898 \text{ u}$$

$\therefore$ Binding energy per nucleon $= \dfrac{1.898 \times 931.5}{225}$ MeV

$$= 7.86 \text{ MeV}$$ **Ans.**

ASSIGNMENT

LEVEL-I

1. The radioactivity of a uranium specimen with mass number 238 is $2.5 \times 10^4\ s^{-1}$, the specimen's mass is 2.0 g. Find the half-life.

2. The disintegration rate of a certain radioactive sample at any instant is 4750 disintegrations per minute. Five minutes later the rate becomes 2700 per minute. Calculate

(a) decay constant and

(b) half-life of the sample.

3. There is a stream of neutrons with a kinetic energy of 0.0327 eV. If the half life of neutrons be 700 s, what fraction of neutrons will decay before they travel a distance of 10 m? Mass of neutron equal to 1.675×10^{-27} kg.

4. What is the probability that a radioactive atom having a mean life of 10 days decays during the fifth day?

5. A sample of 1 gm of ${}^{109}_{83}Bi$ with a half life of 2.7×10^7 years decays into a stable isotope of thallium by emitting α-particles.

(a) What is the activity of the sample?

(b) What will be the activity of the sample after 2 yrs?

(c) After what time does the activity reduces to 25% of the original activity?

6. Consider two decay reactions.

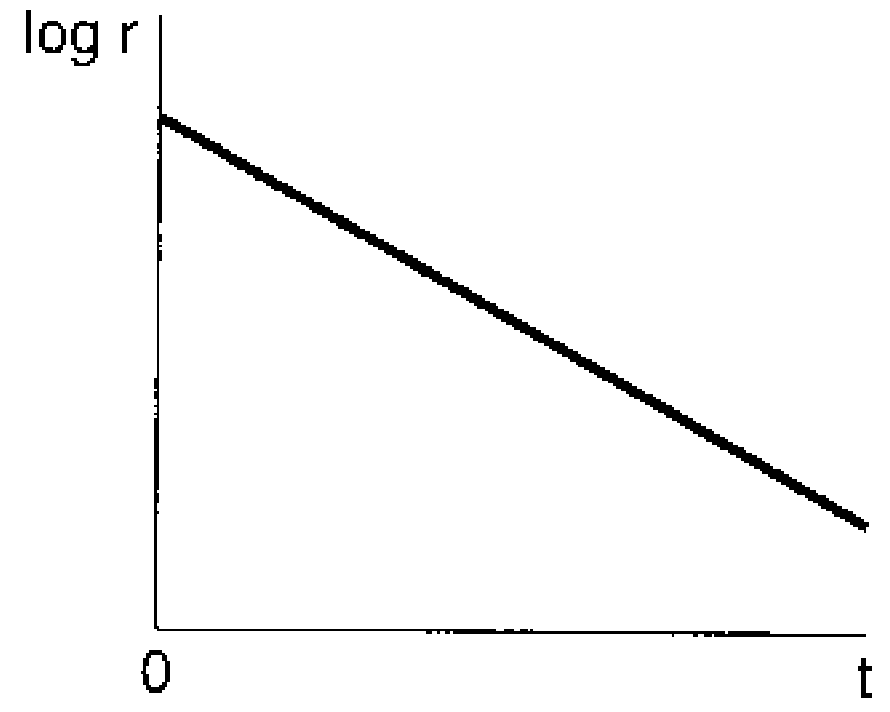
(a) ${}^{238}_{92}U \longrightarrow {}^{206}_{82}Pb + 10$ protons $+ 22$ neutrons

(b) ${}^{238}_{92}U \longrightarrow {}^{206}_{82}Pb + 8\,{}^{4}_{2}He + 6$ electrons

Are both the reactions possible?

Given : Average binding energy of ${}^{238}_{92}U = 7.57$ MeV, that of ${}^{206}_{82}Pb = 7.83$ MeV and that of ${}^{4}_{2}He = 7$ MeV per nucleon.

7. A counter registers the rate of radioactive decay, that is, the number of radioactive decay acts taking place every second. The results obtained in such measurements are plotted in the form of a diagram in which the time interval from the beginning of counting is laid off on the horizontal axis and the logarithm of the decay rate, on the vertical axis. How to find the half-life of the radioactive element from such a diagram?



8. Determine the amount of $_{84}Po^{210}$ (polonium) necessary to provide a source of α-particles of 5 millicuries strength, if half life of polonium is 138 days. (given 1 curie $= 3.7 \times 10^{10}$ disintegrations/sec)

9. 8 protons and 8 neutrons are separately at rest. How much energy will be released if we form $^{16}_{8}O$ nucleus?

Given : Mass of $^{16}_{8}O$ atom $= 15.994915$ u
Mass of neutron $= 1.008665$ u
Mass of hydrogen atom $= 1.007825$ u.

10. A F^{32} radio nuclide with half life $T = 14.3$ days is produced in a reactor at a constant rate $q = 2 \times 10^9$ nuclei per second. How soon after the beginning of production of that radio nuclide will its activity be equal to $R = 10^9$ disintegrations per second?

11. The half lives of radioisotopes P^{32} and P^{33} are 14 days and 25 days respectively. These radioisotopes are mixed in the ratio of 4 : 1 of their atoms. If the initial activity of the mixed sample is 3.0 mCi, find the activity of the mixed isotopes after 60 days.

12. Assuming the splitting of U^{235} nucleus liberates 200 MeV energy, find

(a) the energy liberated in the fission of 1 kg of U^{235} and

(b) the mass of coal with calorific value of 30 kJ/gm which is equivalent to 1 kg of U^{235}.

13. The specific activity of a preparation consisting of radioactive Co^{58} and non-radioactive Co^{59} is equal to 2.2×10^{12} dis/sec gm. The half-life of Co^{58} is 71.3 days. Find the ratio of the mass of radioactive cobalt in that preparation to the total mass of the preparation.

14. Old wood from an Egyptian tomb, 4500 years old has C-14 activity of 7.3 dis. min^{-1} g^{-1}. Old wood known to be 2500 years old has a C-14 activity 9.3 dis. min^{-1} g^{-1}.

(a) What is half life for C-14?

(b) What is the activity of fresh wood?

15. The radionuclide ^{11}C decays according to

$$^{11}C \longrightarrow {}^{11}B + e^+ + \nu.$$

The maximum energy of the emitted positrons is 0.961 MeV.

Given that atomic mass of $^{11}C = m_c = 11.011434\,u$, atomic mass of $^{11}B = m_B = 11.009305$ u, and the mass of positron $= m_p = 0.0005486$ u, calculate disintegration energy Q and compare it with the maximum energy of the emitted positron given above. (1 u = 931 MeV)

16. Consider a radioactive disintegration according to the equation $A \to B \to C$. Decay constant of A and B is same and equal to λ. Number of nuclei of A, B and C are N_0, 0, 0 respectively at $t = 0$. Find

(a) Number of nuclei of B as function of time t.

(b) Time t at which the activity of B is maximum and the value of maximum activity of B.

17. $^{212}_{83}$Bi decay as per following equation.

$$^{212}_{83}\text{Bi} \longrightarrow {}^{208}_{81}\text{Tl} + {}^{4}_{2}\text{He}$$

The kinetic energy of α-particle emitted is 6.802 MeV. Calculate the kinetic energy of Tl recoil atoms.

18. Polonium ($^{210}_{84}$Po) emits $^4_2$He particles and is converted into lead ($^{206}_{82}$Pb). This reaction is used for producing electric power in a space mission. Po^{210} has half-life of 138.6 days. Assuming an efficiency of 10% for the thermoelectric machine, how much ^{210}Po is required to produce 1.2×10^7 J of electric energy per day at the end of 693 days. Also find the initial activity of the material.

Given : Masses of nuclei

^{210}Po = 209.98264 amu, ^{206}Pb = 205.97440 amu,

$^4_2$He = 4.00260 amu.

1 amu = 931 MeV and Avogadro number $= 6 \times 10^{23}$ /mol.

19. A number N_0 of atoms of a radioactive element are placed inside a closed volume. The radioactive decay constant for the nuclei of this element is λ_1. The daughter nuclei that form as a result of the decay process are assumed to be radioactive, too, with a radioactive decay constant λ_2. Determine the time variation of the number of such nuclei. Consider two limiting cases: $\lambda_1 >> \lambda_2$ and $\lambda_1 << \lambda_2$.

20. In a neutron induced fission of $_{92}U^{235}$ nucleus, usable energy of 185 MeV is released. If a $_{92}U^{235}$ reactor is continuously operating it at a power level of 100 MW how long will it take for 1 kg of uranium to be consumed in this reactor?

LEVEL II

1. A radionuclide consists of two isotopes. One of the isotopes decays by α-emission and other by β-emission with half lives $T_1 = 405$ sec, $T_2 = 1620$ sec, respectively. At $t = 0$, probabilities of getting α and β-particles from the radionuclide are equal. Calculate their respective probabilities at $t = 1620$ sec. If at $t = 0$, total number of nuclei in the radio nuclide are N_0. Calculate the time 't' when total number of nuclei remained undecayed becomes equal to $N_0/2$.

$\log_{10} 2 = 0.3010$, $\log_{10} 5.94 = 0.7742$
and $x^4 + 4x - 2.5 = 0$, $x = 0.594$

2. In a nuclear reaction

$$\alpha + {}_7N^{14} \longrightarrow {}_8O^{17} + p$$

when α-particles of KE 7.7 MeV were bombarded on nitrogen atom protons were ejected with a kinetic energy of 5.5 MeV.

(a) Find the Q-value of the reaction

(b) Find the angle ϕ between the direction of motion of proton and α-particle.

Given that atomic mass of ${}_1H^1 = 1.00814$ amu., atomic mass of ${}_7N^{14} = 14.00752$ amu, Atomic mass of ${}_8O^{17} = 17.00453$ and atomic mass of ${}_2He^4 = 4.00388$ amu.

3. A neutron with kinetic energy $K = 10$ MeV activates a nuclear reaction

$$n + {}^{12}C \longrightarrow {}^9Be + \alpha$$

Find the kinetic energy of the alpha particles outgoing at right angle to the direction of incoming neutrons.

[Take u $= 931.5$ MeV and threshold energy of reaction $(E_{th}) = 6.17$ MeV]

4. Find the amount of heat generated by 1 mg of Po^{210} preparation during the mean-life period of these nuclei if the emitted alpha particles are known to possess kinetic energy 5.3 MeV and practically all daughter nuclei are formed directly in the ground state.

5. Ac^{227} has a half life of 21.8 years with respect to radioactive decay. The decay follows two parallel paths, one leading the Th^{227} and the other leading to Fr^{223}. The percentage yields of these two daughters nuclides are 1.2% and 98.8% respectively. What is the rate constant in yrs^{-1}, for each of the separate paths?

6. In an agricultural experiment, a solution containing 1 mole of a radioactive material ($T_{1/2} = 14.3$ days) was injected into the roots of a plant. The plant was allowed 70 hours to settle down and then activity was measured in its fruit. If the activity measured was 1 μCi, what percentage of activity is transmitted from the root to the fruit in steady state?

7. It is proposed to use the nuclear fusion reaction ${}_1H^2 + {}_1H^2 \longrightarrow {}_2He^4$ in a nuclear reactor of 200 MW rating. If the energy from the above reaction is used with a 25% efficiency in the reactor, Their optical path difference $= \frac{\mu d^2}{2D}$

Therefore, net path difference $= (\sqrt{d^2 + l^2} - l) + \frac{\mu d^2}{2D}$

or phase difference $\phi = \frac{2\pi}{\lambda}\left[\left(\sqrt{d^2 + l^2} - d\right) + \frac{\mu d^2}{2D}\right]$ **Ans.**

(b) When liquid is filled between slits and source S.

$$\Delta x = \mu\left((\sqrt{d^2 + l^2} - l) + \frac{d^2}{2D}\right)$$

$\therefore$ $\phi = \frac{2\pi}{\lambda}\left[\mu(\sqrt{d^2 + l^2} - l) + \frac{d^2}{2D}\right]$ **Ans.**

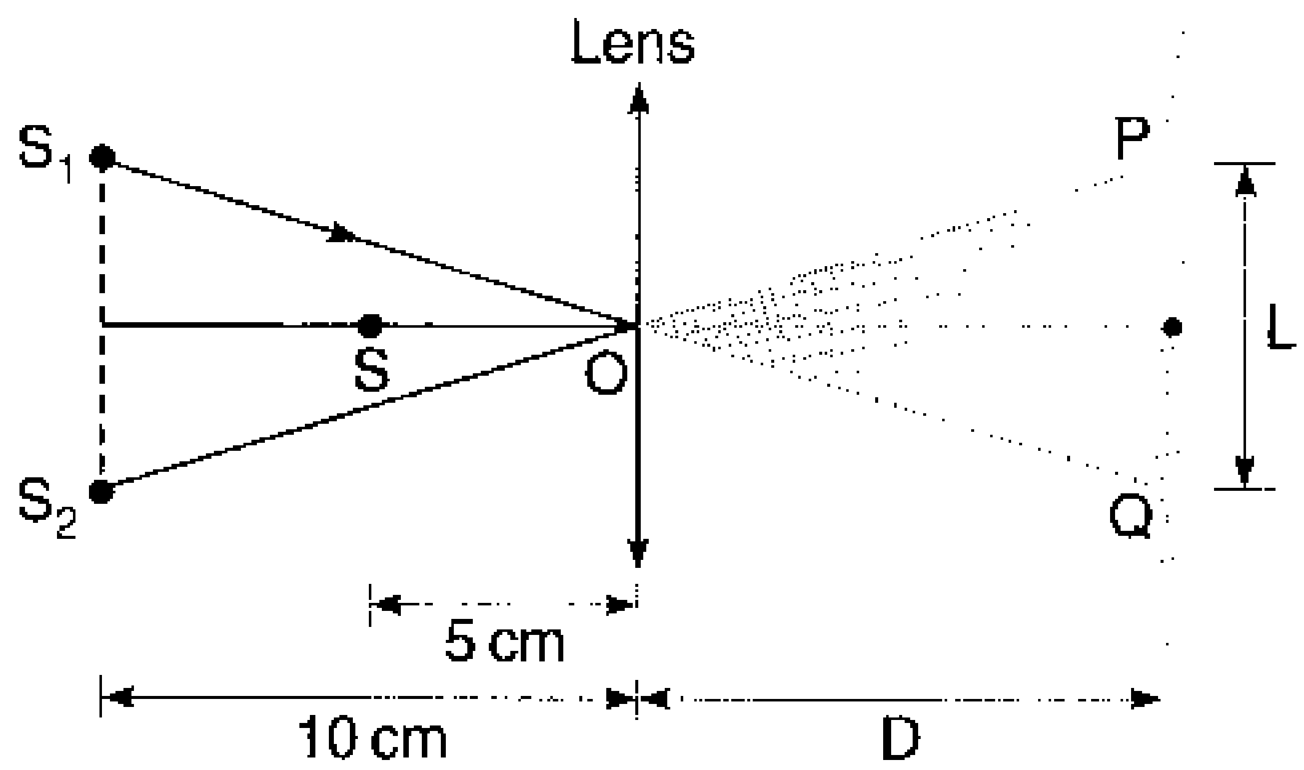
5. Applying the lens formula, $\frac{1}{v} + \frac{1}{5} = \frac{1}{10}$

$\therefore$ $v = -10$ cm

$$m = \frac{v}{u} = \frac{-10}{-5} = 2$$

i.e., two virtual sources are formed with distance between them

$d = 0.5$ mm (see the hint of Q. No. 2)



Fringe width $\omega = \frac{\lambda(D + 10)}{d}$

Fringes are observed between the region P and Q (waves interfere in this region only), where

$$\frac{L}{D} = \frac{d}{10} \quad \text{or} \quad L = \frac{dD}{10}$$

Number of fringes that can be observed on the screen,

$$n = \frac{L}{\omega} = \frac{d^2 D}{10\lambda(D + 10)}$$

Substituting the values, $3 = \frac{(0.05)^2 D}{10 \times 5.0 \times 10^{-5}(D + 10)}$ ...(i)

Solving this equation, we get

$D = 15$ cm **Ans.**

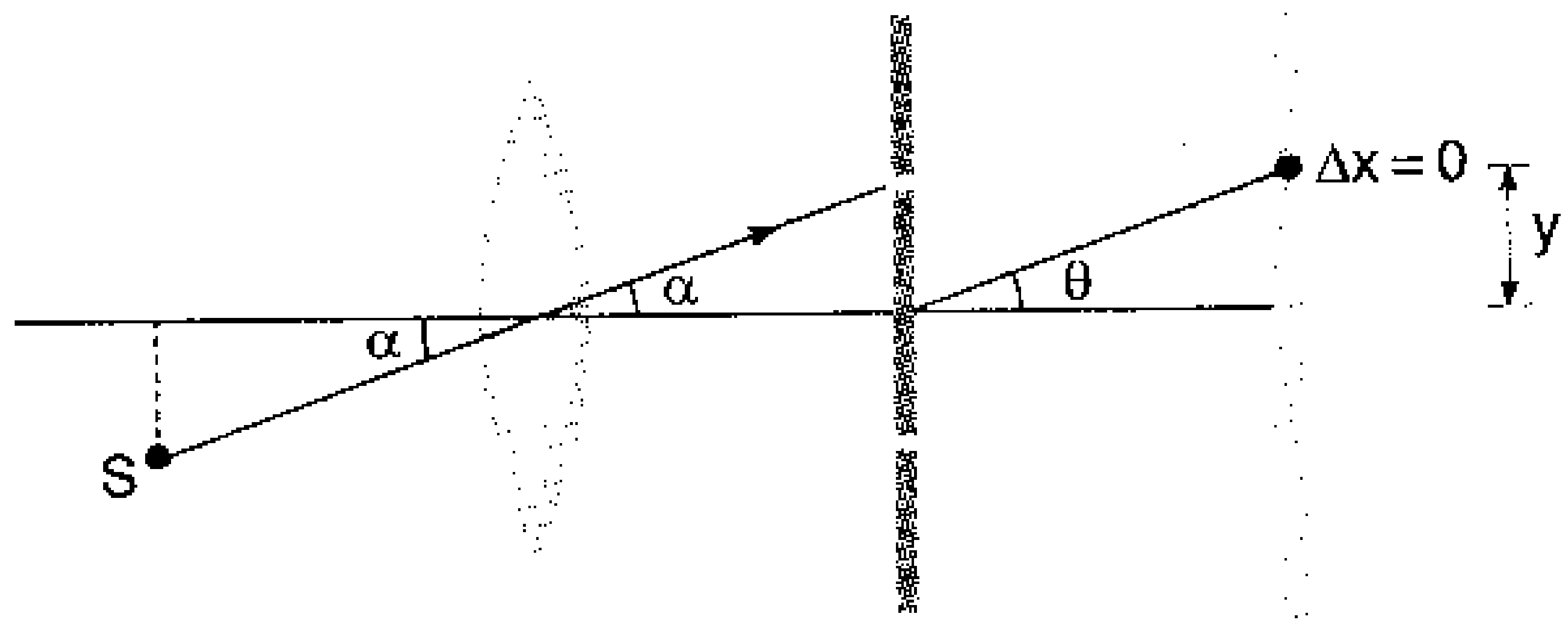
From Eq. (i) $n = \frac{d^2}{10\lambda\left(1 + \frac{10}{D}\right)}$

n will be maximum as $D \to \infty$ or $\frac{10}{D} \to 0$

$$\therefore \quad n_{max} = \frac{d^2}{10\lambda} = \frac{(0.05)^2}{10 \times 5.0 \times 10^{-5}}$$
$$= 5$$ **Ans.**

6. $\frac{1}{f} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{20} - \frac{1}{-20}\right)$ or $f = 20$ cm

Since source lies in focal plane of lens. So, all the emergent rays will be parallel.



$$\tan\alpha = \frac{d/2}{20} = \frac{d}{40} = \frac{1}{400} \approx \sin\alpha$$

Initial path difference = $d \sin\alpha$

Let the central maxima is obtained at angle θ. Then

$$d \sin\theta = d \sin\alpha$$

or $$\sin\theta = \sin\alpha \quad \text{or} \quad \tan\theta = \tan\alpha$$

$$\therefore \quad \frac{y}{D} = \frac{d}{40}$$

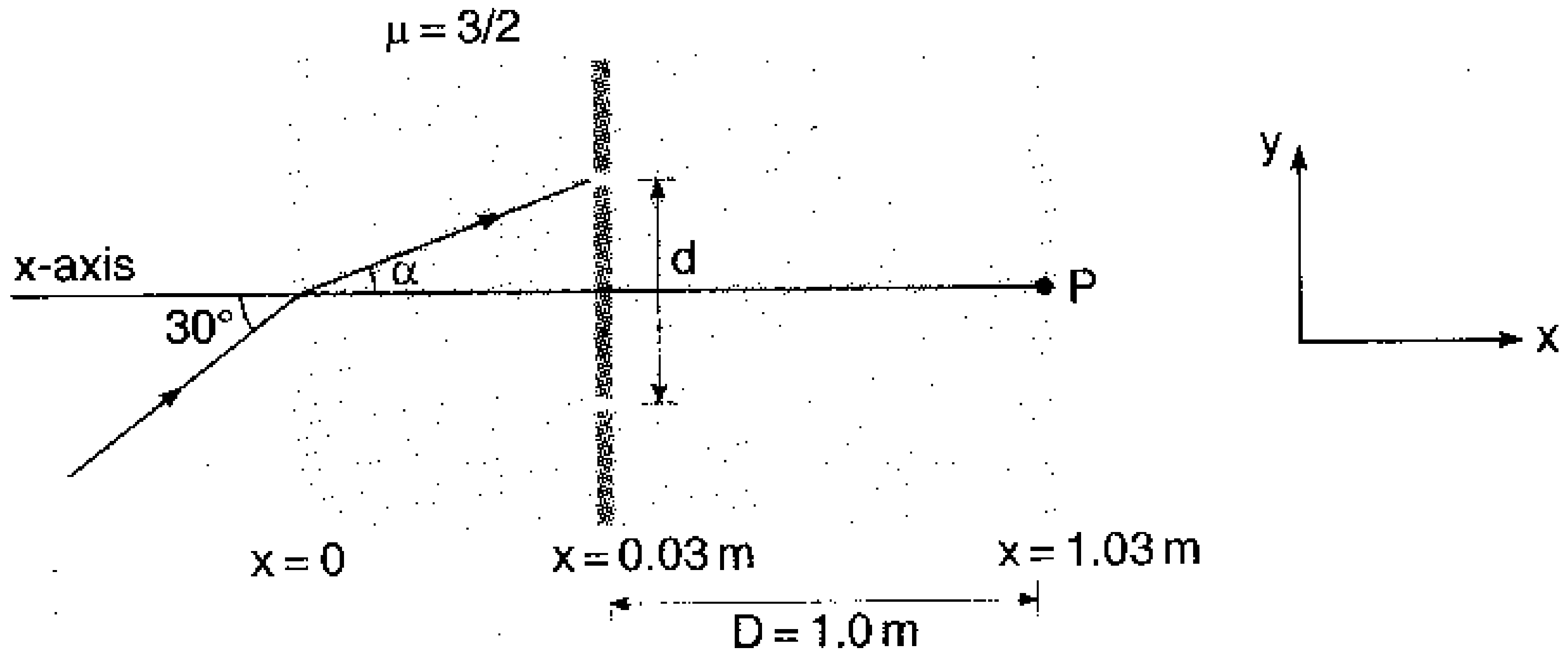
$$\therefore \quad y = \frac{Dd}{40} = \frac{100}{40} d = 2.5d = 2.5 \text{ mm}$$ **Ans.**

At O, net path difference $\Delta x = d \sin\alpha = \frac{1}{400}$ mm.

$$\therefore \quad \phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{5000 \times 10^{-10}} \times \frac{1}{400} \times 10^{-3}$$

$$I = I_{max} \cos^2\frac{\phi}{2} = I_{max}$$ **Ans.**

7. (a) $$d = 2 \times 0.9 = 1.8 \text{ mm}$$
$$\lambda' = \frac{\lambda}{\mu} = \frac{900}{3/2} = 600 \text{ nm}$$



$$\frac{3}{2} = \frac{\sin 30°}{\sin \alpha} \qquad \therefore \quad \alpha = 20°$$

Initial path difference $\Delta x = d \sin \alpha = (1.8)\left(\frac{1}{3}\right) = 0.6$ mm

$$\therefore \quad \phi = \frac{2\pi}{\lambda'} \Delta x = \frac{2\pi}{600 \times 10^{-9}} \times 0.6 \times 10^{-3} = 2000\,\pi$$

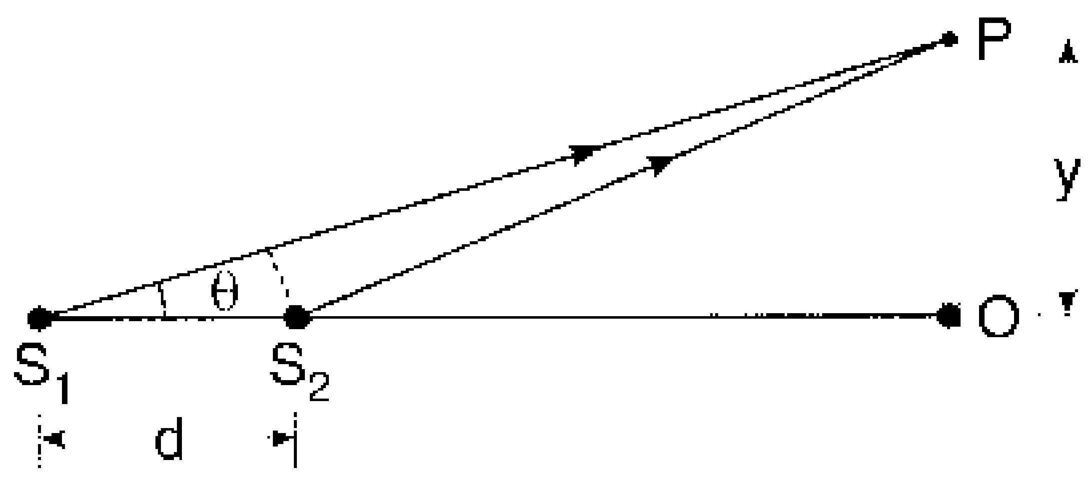
$$\therefore \quad I = I_{max} \cos^2\left(\frac{\phi}{2}\right) = I_{max}$$ **Ans.**

(b) Net path difference at P is now, $\Delta x = (0.6 \text{ mm}) + \left(\frac{1.5}{1.4} - 1\right)(4.2 \text{ mm})$

$= 0.3$ mm $\qquad \therefore \quad \phi = 1000\pi$ and $I = I_{max}$

Ans.

8. (a) $\Delta x = d \cos \theta$



$$\cos \theta = 1 - \frac{\theta^2}{2} \quad \text{(when } \theta \text{ is small)}$$

$$\therefore \quad \Delta x = d\left(1 - \frac{\theta^2}{2}\right) = d\left(1 - \frac{y^2}{2D^2}\right)$$

For n^{th} maxima $\qquad \Delta x = n\lambda$

$$\therefore \quad y = \text{radius of } n^{th} \text{ bright ring} = D\sqrt{2\left(1 - \frac{n\lambda}{d}\right)}$$ **Ans.**

(b) $\qquad d = 1000\lambda$

At O, $\qquad \Delta x = d = 1000\lambda$

i.e., at O, 1000^{th} order maxima is obtained.

Substituting $n = 998$ in $\qquad y = D\sqrt{2\left(1 - \frac{n\lambda}{d}\right)}$

We get the radius of second closest ring

$$r = 6.32 \text{ cm}$$ **Ans.**

(c) $n = 998$ **Ans.**

9. (a) The optical path difference between the two waves arriving at P is

$$\Delta x = \frac{y_1 d}{D_1} + \frac{y_2 d}{D_2} = \frac{(1)(10)}{10^3} + \frac{(5)(10)}{2 \times 10^3}$$

$$= 3.5 \times 10^{-2} \text{ mm} = 0.035 \text{ mm}$$

As, $\qquad \Delta x = 70\lambda$

$\therefore$ 70^{th} order maxima is obtained at P.

(b) At O, $\qquad \Delta x = \frac{y_1 d}{D_1} = 10^{-2}$ mm $= 0.01$ mm

As $\qquad \Delta x = 20\lambda$

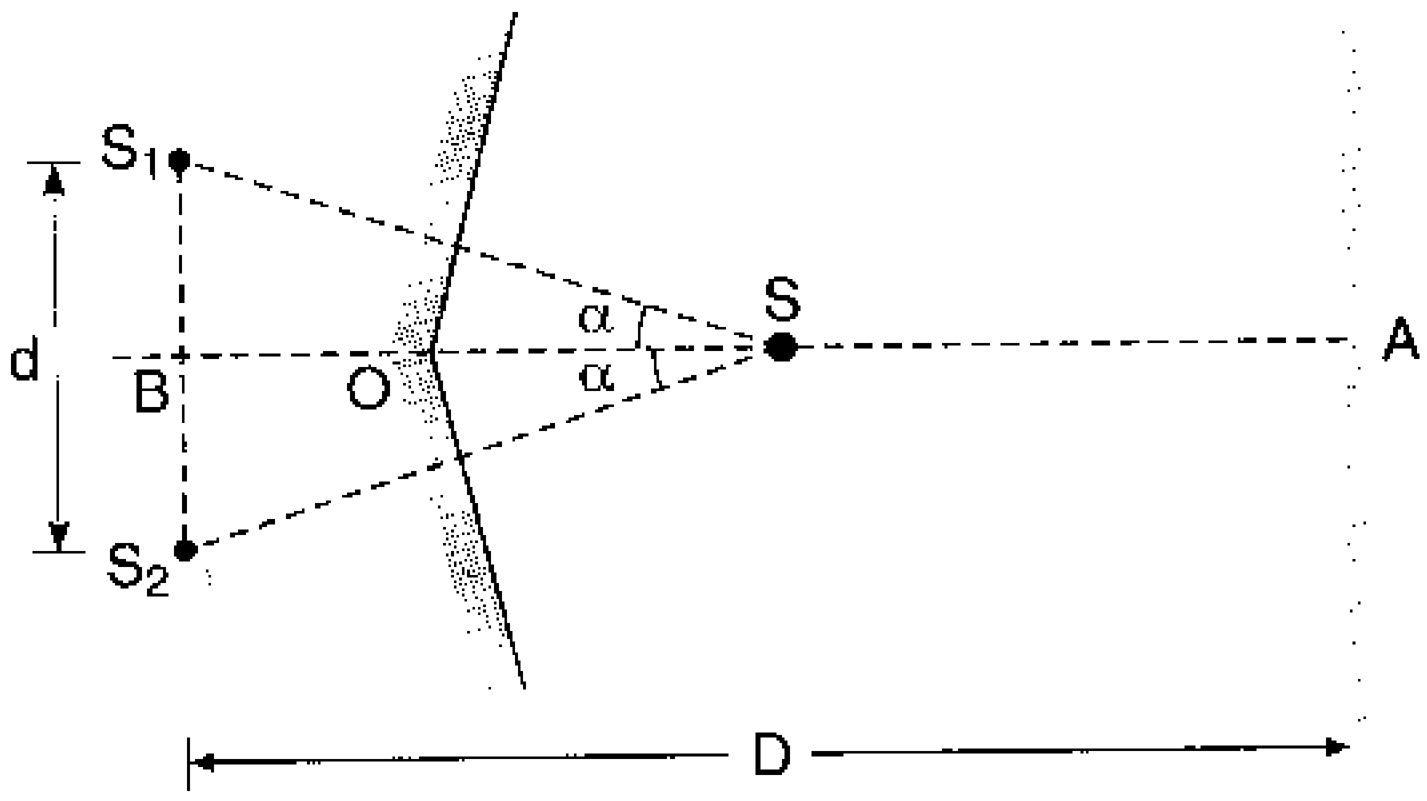
$\therefore$ 20^{th} order maxima is obtained at O.

(c) $\qquad (\mu - 1)\,t = 0.01$ mm

$$\therefore \quad t = \frac{0.01}{1.5 - 1} = 0.02 \text{ mm} = 20\,\mu\text{m}$$ **Ans.**

Since the pattern has to be shifted upwards, therefore, the film must be placed infront of S_1.

10. $$\omega = \frac{\lambda D}{d}$$



$D = AB \approx a + b$ and $d = S_1S_2$

In ΔS_1SB, $\frac{d}{2} = 2b\frac{\alpha}{2}$ or $d = 2b\alpha$

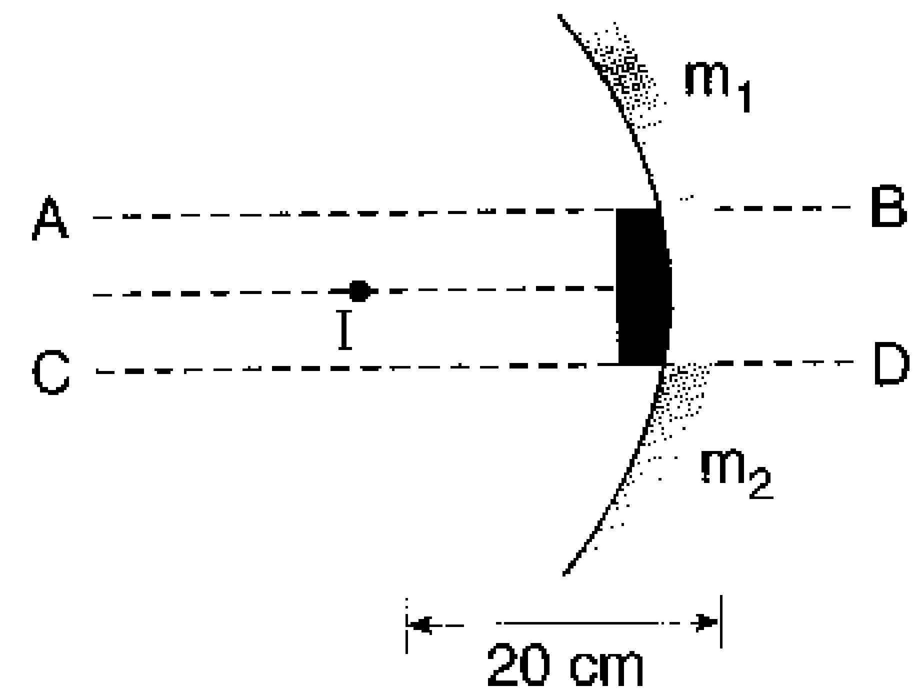
$$\therefore \quad \omega = \frac{\lambda(a+b)}{2b\alpha}$$ **Ans.**

11. (a) $$\frac{1}{10} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{R} + \frac{1}{R}\right) \quad \therefore \quad R = 10 \text{ cm}$$

Now applying $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ twice

We have,
$$\frac{3/2}{v_1} - \frac{1}{-20} = \frac{3/2 - 1}{10} \quad \text{...(i)}$$
$$\frac{4/3}{v_2} - \frac{3/2}{v_1} = \frac{4/3 - 3/2}{-10} \quad \text{...(ii)}$$

Adding Eq. (i) and Eq. (ii), we have $v_2 = 80$ cm **Ans.**



(b) The image formed by (lens + water) system will act as an object for the mirror.

This is below the axis of m_1 and at the same distance as the centre is, therefore, its image will be formed vertically above at 1 mm from AB. Similarly m_2 will form an image of I_1 1 mm below CD.

Therefore,
$$I_1I_2 = 1 + 1 + 1 + 1 = 4 \text{ mm}$$ **Ans.**

(c) $d = I_1I_2 = 4$ mm, $D = 80$ cm

$$\lambda = \frac{c}{\mu_w f} = \left(\frac{3.0 \times 10^8}{7.5 \times 10^{14}} \times \frac{3}{4}\right)$$

$$\therefore \quad \omega = \frac{\lambda D}{d} = \frac{3}{4}\left(\frac{3.0 \times 10^8}{7.5 \times 10^{14}}\right)\left(\frac{80 \times 10^{-2}}{4 \times 10^{-3}}\right) \text{m} = 6 \times 10^{-5} \text{ m}$$

$$= 60\,\mu\text{m}$$ **Ans.**

12. (a) $$(\Delta x)_{\text{net}} = 0$$

$$\therefore \quad \frac{y_1 d}{D_1} = \frac{y_2 d}{D_2}$$

$$\therefore \quad \frac{d/2}{1.5} = \frac{y}{2.0}$$

or $$y = \frac{d}{1.5} = \frac{6}{1.5} = 4 \text{ mm}$$ **Ans.**

(b) At O, net path difference,

$$\Delta x = \frac{y_1 d}{D_1} = \frac{(d/2)(d)}{D_1} = \frac{(6\times10^{-3})^2}{2\times1.5}$$

$$= 12\times10^{-6}\text{ m}$$

$$= 120\times10^{-7}\text{ m}$$

$$\lambda = 6000\text{ Å} = 6\times10^{-7}\text{ m}$$

As $\Delta x = 20\lambda$, therefore at O bright fringe of order 20 will be obtained.

(c) $$I = I_{max}\cos^2\left(\frac{\phi}{2}\right)$$

$$\frac{3}{4}I_{max} = I_{max}\cos^2\left(\frac{\phi}{2}\right)$$

$\therefore$ $$\frac{\phi}{2} = \frac{\pi}{6}$$

$$\phi = \frac{\pi}{3} = \left(\frac{2\pi}{\lambda}\right)(\mu-1)t$$

$\therefore$ $$t = \frac{\lambda}{6(\mu-1)} = \frac{6000}{6(1.5-1)}$$

$$= 2000\text{ Å}$$ **Ans.**

13. (a) $$\left(1-\frac{1}{\mu}\right)t = \frac{3\lambda}{\mu}$$

$\therefore$ $$t = \frac{3\lambda}{(\mu-1)} = \frac{3\times0.78}{1.3-1} = 7.8\ \mu\text{m}$$ **Ans.**

(b) **Upwards:** $$\frac{yd}{D} - \left(1-\frac{1}{\mu}\right)t = \frac{4\lambda}{\mu}$$

Solving, we get $y = 4.2$ mm **Ans.**

Downwards: $$t\left(1-\frac{1}{\mu}\right) + \frac{yd}{D} = \frac{4\lambda}{\mu}$$

Solving, we get $y = 0.6$ mm **Ans.**

MODERN PHYSICS-I

LEVEL-I

1. $$\lambda = 280\times10^{-9}\text{ m}$$

$$E = \frac{hc}{\lambda} = \frac{(4.316\times10^{-15}\text{ eV-sec})(3.0\times10^8\text{ m/s})}{(280\times10^{-9}\text{ m})} = 4.6\text{ eV}$$ **Ans.**

$$m = \frac{E}{c^2} = \frac{4.6\times1.6\times10^{-19}}{(3.0\times10^8)^2} = 8.2\times10^{-36}\text{ kg}$$ **Ans.**

$$P = \frac{E}{c} = \frac{4.6\times1.6\times10^{-19}}{3.0\times10^8} = 2.45\times10^{-27}\text{ kg-m/s}$$ **Ans.**

2. Ionization energy for hydrogen is 13.6 eV.

Therefore, ionization energy for doubly ionized lithium atom ($Z = 3$) will be, $(13.6)(Z^2)$ eV or 122.4 eV **Ans.**

3. Potential energy in ground state is otherwise −27.2 eV. It is taken as zero, *i.e.*, potential energy is increased by 27.2 eV. Kinetic energy in all energy states will remain unchanged while potential energy and hence, the total energy in all states will increase by 27.2 eV. First excited state means $n = 2$.

$$E_2 = -3.4 \text{ eV}$$

$$\therefore \quad E_2' = -3.4 + 27.2 = 23.8 \text{ eV}$$ **Ans.**

4. After removing the first electron it will become He^+ ion. The ionization energy of single electron in He^+ ion ($Z = 2$) is,

$$13.6(Z^2) = 54.4 \text{ eV}$$

Therefore, total energy required to remove both the electrons,

$$E = (24.6 + 54.4) \text{ eV} = 79 \text{ eV}$$ **Ans.**

5.
$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.31 \times 293}{2 \times 10^{-3}}} = 1911 \text{ m/s}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv_{\text{rms}}}$$

Mass of one hydrogen molecule
$$m = \frac{2}{6.02 \times 10^{26}} \text{ kg} = 3.32 \times 10^{-27} \text{ kg}$$

$$\therefore \quad \lambda = \frac{6.63 \times 10^{-34}}{3.32 \times 10^{-27} \times 1911} \text{ m}$$

$$= 1.04 \times 10^{-10} \text{ m} = 1.04 \text{ Å}$$ **Ans.**

6.
$$\frac{1}{\lambda} = R(Z-1)^2 \left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$

$$\therefore \quad \frac{1}{0.76 \times 10^{-10}} = (1.09 \times 10^7)(Z-1)^2 \left(\frac{3}{4}\right)$$

$$\therefore \quad Z - 1 \approx 40$$

$$\therefore \quad Z \approx 41$$ **Ans.**

7.
$$E = hf = (4.316 \times 10^{-15} \text{ eV-sec})(1.5 \times 10^{15} \text{ sec}^{-1}) \approx 6.5 \text{ eV}$$

$$\therefore \quad K_{\text{max}} = E - W = (6.5 - 3.7) \text{ eV} = 2.8 \text{ eV}$$ **Ans.**

8.
$$\lambda_1 - \lambda_2 = \frac{12375}{V_1} - \frac{12375}{V_2} = 12375 \left(\frac{1}{V} - \frac{1}{1.5V}\right)$$

or
$$0.26 = (12375)\left(\frac{1}{3V}\right)$$

or
$$V = 15865 \text{ volt}$$ **Ans.**

9.
$$eV_1 = E_1 - W \quad \text{...(i)}$$

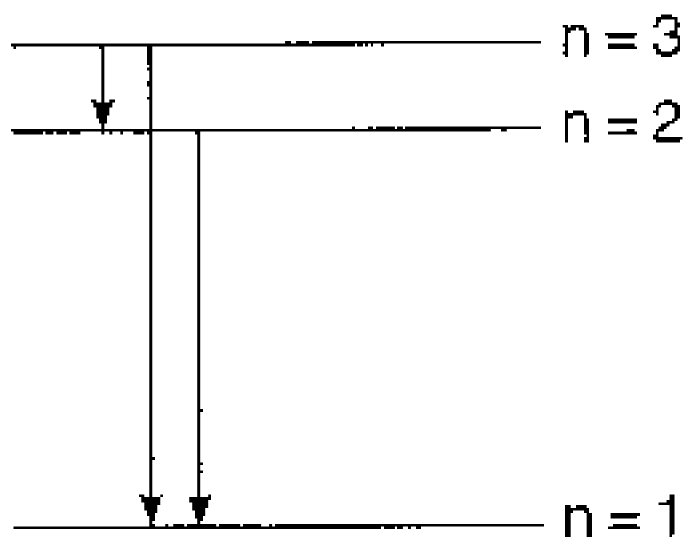
$$eV_2 = E_2 - W \quad \text{...(ii)}$$

Eqs. (i) − (ii), gives
$$e(V_1 - V_2) = (E_1 - E_2)$$

$$V_1 - V_2 = \frac{hc}{e}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

$$= \frac{6.6 \times 10^{-34} \times 3.0 \times 10^{8}}{1.6 \times 10^{-19} \times 10^{-10}} \left(\frac{1}{3000} - \frac{1}{3010} \right) = 0.012 \text{ volt}$$ **Ans.**

10. $$E = \frac{12375}{1023} \approx 12.1 \text{ eV} = E_3 - E_1$$



So, in the resulting spectrum three lines are possible.

$$\lambda_{max} = \frac{12375}{E_3 - E_1} = 1023 \text{ Å}$$ **Ans.**

11. $$\text{Fringe width } \omega = \frac{\lambda D}{d}$$

$\therefore$ $$d = \frac{\lambda D}{\omega} = \frac{(0.01 \times 10^{-10})(40 \times 10^{-2})}{(0.1 \times 10^{-6})}$$

$$= 4.0 \times 10^{-6} \text{ m}$$

$$= 4 \text{ μm}$$ **Ans.**

12. $$K_{max} = E - W$$

$\therefore$ $$3 \text{ eV} = \frac{12375}{\lambda} - \frac{12375}{5000}$$

or $$\lambda = 2260 \text{ Å}$$ **Ans.**

13. (a) $$hf_0 = W = 3.75 \times 1.6 \times 10^{-19} \text{ J}$$

$\therefore$ $$f_0 = \frac{3.75 \times 1.6 \times 10^{-19}}{6.6 \times 10^{-34}} \approx 10^{15} \text{ Hz}$$ **Ans.**

(b) $$E = \frac{12375}{1980} \text{ eV} = 6.25 \text{ eV}$$

$$K_{max} = 2.5 \text{ eV}$$

$\therefore$ $$W = E - K_{max} = 3.75 \text{ eV}$$ **Ans.**

(c) $$K_{max} = 2 \text{ eV}$$

14. $$\frac{1}{2} mv_1^2 = \frac{hc}{\lambda_1} - W \quad \text{...(i)}$$

$$\frac{1}{2} mv_2^2 = \frac{hc}{\lambda_2} - W \quad \text{...(ii)}$$

Dividing Eq. (i) with Eq. (ii), with $v_1 = 2v_2$, we have $4 = \dfrac{\frac{hc}{\lambda_1} - W}{\frac{hc}{\lambda_2} - W}$

$\therefore$ $$3W = 4\left(\frac{hc}{\lambda_2}\right) - \left(\frac{hc}{\lambda_1}\right) = \frac{4 \times 12375}{5400} - \frac{12375}{3500} = 5.63 \text{ eV}$$

$\therefore$ $$W = 1.9 \text{ eV}$$ **Ans.**

15. $E_1 = \dfrac{12375}{4000} = 3.1 \text{ eV}$, $\quad E_2 = \dfrac{12375}{4800} = 2.57 \text{ eV}$

$$E_3 = \frac{12375}{6000} = 2.06 \text{ eV} \quad \text{and} \quad E_4 = \frac{12375}{7000} = 1.77 \text{ eV}$$

Therefore, light of wavelengths 4000 Å, 4800 Å and 6000 Å can only emit photoelectrons.

$\therefore$ Number of photoelectrons emitted per second $= \dfrac{I_1A_1}{E_1} + \dfrac{I_2A_2}{E_2} + \dfrac{I_3A_3}{E_3}$

$$= IA\left(\frac{E_1E_2 + E_2E_3 + E_1E_3}{E_1E_2E_3}\right)$$

$$= \frac{(1.5\times10^{-3})(10^{-4})}{1.6\times10^{-19}}\left[\frac{3.1\times2.57 + 2.57\times2.06 + 3.1\times2.06}{3.1\times2.57\times2.06}\right]$$

$$= 1.12\times10^{12}$$ **Ans.**

16. (a) $$E = hf = (6.6\times10^{-34})(5.5\times10^{14}) = 36.3\times10^{-20}\text{ J}$$

$$= 2.27\text{ eV}$$ **Ans.**

(b) Number of photons leaving the source per second,

$$n = \frac{P}{E} = \frac{0.1}{36.3\times10^{-20}} = 2.75\times10^{17}$$ **Ans.**

(c) Number of photons falling on cathode per sec

$$n_1 = \frac{0.15}{100}\times2.75\times10^{17} = 4.125\times10^{14}$$

Number of photoelectrons emitting per second

$$n_2 = \frac{6\times10^{-6}}{1.6\times10^{-19}} = 3.75\times10^{13}$$

$\therefore$ $$\% = \frac{n_2}{n_1}\times100$$

$$= \frac{3.75\times10^{13}}{4.125\times10^{14}}\times100 = 9\%$$ **Ans.**

17. $$K_{max} = E - W = \frac{hc}{\lambda} - \frac{hc}{\lambda_0}$$

$$0.18\times1.6\times10^{-19} = \frac{h(3.0\times10^8)}{5461\times10^{-10}} - \frac{h(3.0\times10^8)}{\lambda_0} \quad \text{...(i)}$$

Further, $$eV_0 = E - W = \frac{hc}{\lambda} - \frac{hc}{\lambda_0}$$

$\therefore$ $$4.6\times1.6\times10^{-19} = \frac{h(3.0\times10^8)}{1849\times10^{-10}} - \frac{h(3.0\times10^8)}{\lambda_0} \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), for h and λ_0, we get

$$h = 6.6\times10^{-34}\text{ J-s} \quad \text{and} \quad \lambda_0 = 5990.25\text{ Å}$$ **Ans.**

18. $$E = \frac{12375}{2480} \approx 5.0\text{ eV} = 8.0\times10^{-19}\text{ J}$$

Number of photons emitted per second from the source,

$$n_1 = \frac{P}{E} = \frac{40}{8.0\times10^{-19}} = 5.0\times10^{19}$$ **Ans.**

$$n_2 = \frac{n_1}{4\pi r^2} = \frac{5.0 \times 10^{19}}{4\pi(2)^2} \approx 10^{18}$$ **Ans.**

$$K_{max} = E - W = 5.0 - 3.68 = 1.32 \text{ eV}$$ **Ans.**

$$\lambda_0 = \frac{12375}{W} = \frac{12375}{3.68} = 3363 \text{ Å}$$ **Ans.**

19.

$$E_1 = \frac{12375}{5000} = 2.475 \text{ eV}$$

$$E_2 = \frac{12375}{6000} = 2.06 \text{ eV} \quad \text{and} \quad E_3 = \frac{12375}{7000} = 1.77 \text{ eV}$$

Since, W is 1.9 eV, photons of energy E_1 and E_2 can only emit photoelectrons. Charge emitted per second.

$$= (1.6 \times 10^{-19})\left(\frac{1}{3}\right)\frac{(10^{-3})(10^{-4})}{\pi \times (10^{-3})^2 \times 2.475 \times 1.6 \times 10^{-19}} + (1.6 \times 10^{-19})\left(\frac{1}{3}\right)\frac{(10^{-3})(10^{-4})}{\pi \times (10^{-3})^2 \times 2.06 \times 1.6 \times 10^{-19}}$$

$$= 9.28 \times 10^{-3} \text{ C}$$

20.

$$E = \frac{hc}{\lambda} = \frac{(6.6 \times 10^{-34})(3.0 \times 10^8)}{(4.8 \times 10^{-7})} = 4.125 \times 10^{-19} \text{ J}$$

Number of photons striking the metal plate per square meter per second

$$= \left(\frac{P}{E}\right)\left(\frac{1}{4\pi r^2}\right)$$

$$= \left(\frac{1.0}{4.125 \times 10^{-19}}\right)\frac{1}{(4\pi)(2)^2} = 4.82 \times 10^{16}$$ **Ans.**

21. $\lambda = 180 \text{ nm} = 1800 \text{ Å}$

$\therefore$ $$E = \frac{12375}{1800} = 6.875 \text{ eV}$$

$\therefore$ $$K_{max} = E - W = 4.875 \text{ eV}$$

Now, $$r = \frac{\sqrt{2Km}}{Bq}$$

Substituting the values, $$r = \frac{\sqrt{2 \times 4.875 \times 1.6 \times 10^{-19} \times 9.1 \times 10^{-31}}}{5 \times 10^{-5} \times 1.6 \times 10^{-19}}$$

$$= 0.15 \text{ m} = 15 \text{ cm}$$ **Ans.**

22. Total number of electrons striking the target per second

$$n = \frac{10 \times 10^{-3}}{1.6 \times 10^{-19}} = 6.25 \times 10^{16}$$

Kinetic energy of one electron $= 40 \times 10^3 \text{ eV} = 40 \times 10^3 \times 1.6 \times 10^{-19} = 6.4 \times 10^{-15}$ J

$\therefore$ Total energy of electrons striking the target (per second) $= 6.25 \times 10^{16} \times 6.4 \times 10^{-15} = 400$ J

(a) Total power emitted as X-rays = 1% of 400 = 4 watt **Ans.**

(b) Heat produced per second = (400 − 4) J/s = 396 J/s **Ans.**

23.

$$E_n - E_1 = \frac{12375}{1085} + \frac{12375}{304}$$

$\therefore$ $$(13.6)(Z^2)\left(1 - \frac{1}{n^2}\right) = 52.1 \text{ eV}$$

or $$1-\frac{1}{n^2}=0.96 \qquad (Z=2)$$

$\therefore$ $$n\approx 5$$ **Ans.**

24. (a) $$\frac{n(n-1)}{2}=6$$

$\therefore$ $$n=4$$

i.e., if $n_1=n$, then $n_2=n-4$

Now, $$\frac{(13.6)Z^2}{n^2}=0.544 \qquad \text{...(i)}$$

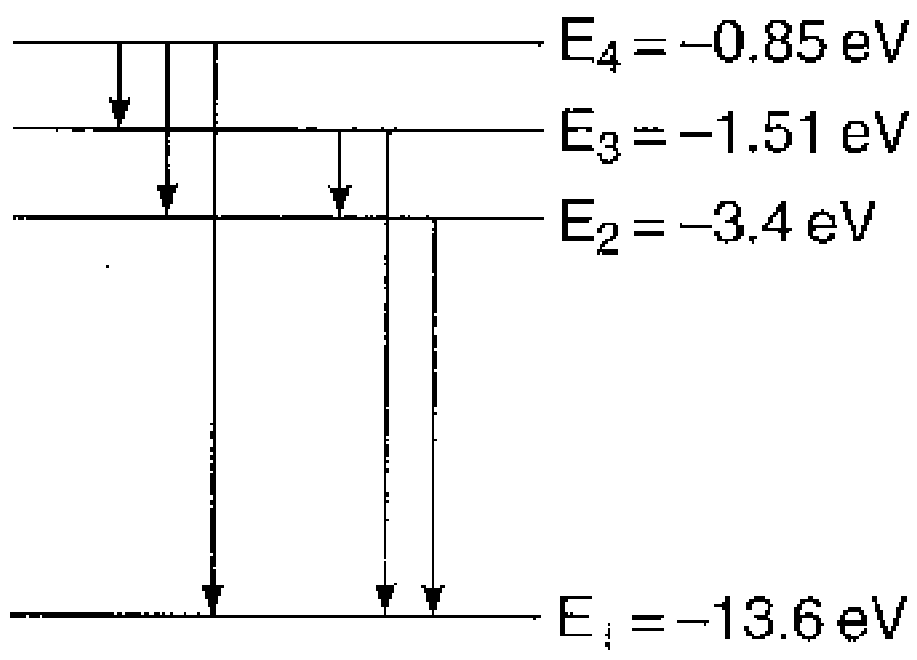
and $$\frac{(13.6)Z^2}{(n-4)^2}=0.85 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii),

$$Z=4 \quad \text{and} \quad n=20$$ **Ans.**

(b) $$\lambda_{min}=\frac{12375}{-0.544-(0.85)}=40441 \text{ Å}$$ **Ans.**

25. $$\frac{K_1}{K_2}=5$$



$\therefore$ $$\frac{\Delta E_1-W}{\Delta E_2-W}=5 \qquad \text{...(i)}$$

Here, $\Delta E_1=E_4-E_1=12.75$ eV and $\Delta E_2=E_3-E_1=12.09$ eV

Substituting in Eq. (i) and solving we get

$$W=11.93 \text{ eV}$$ **Ans.**

26. (b) $$\lambda_{min}=\frac{12375}{5.3}=2335 \text{ Å}$$

(c) $$\Delta E_{31}=-3.08-(-15.6)=12.52 \text{ eV}$$

Therefore, excitation potential for state $n=3$ is 12.52 volt.

(d) $$\frac{1}{\lambda_{31}}=\frac{\Delta E_{31}}{12375}\text{ Å}^{-1}=\frac{12.52}{12375}\text{ Å}^{-1}$$

$$\approx 1.01\times 10^7 \text{ m}^{-1}$$ **Ans.**

(e) (i) $$E_2-E_1=10.3 \text{ eV} > 6 \text{ eV}$$

Hence, the striking electron cannot excite the hypothetical atoms. So the electron will keep its energy with itself.

$\therefore$ $$K_{min}=6 \text{ eV}$$ **Ans.**

(ii) $$E_2-E_1=10.3 \text{ eV} < 11 \text{ eV}$$

So, the electron can excite the atom.

$$K_{min}=(11-10.3)\text{ eV}=0.7 \text{ eV}$$ **Ans.**

27. (a) $$E_3-E_1=\frac{(-13.6)(3)^2}{(3)^2}-\left[\frac{(-13.6)(3)^2}{(1)^2}\right]=108.8 \text{ eV}$$

$\therefore$ $$\lambda=\frac{12375}{108.8}\text{ Å}=113.74 \text{ Å}$$ **Ans.**

(b) Number of lines in emission spectrum $= \dfrac{n(n-1)}{2}$

$$= \frac{(3)(3-1)}{2} = 3$$

Ans.

28. For shorter wavelength: $\Delta E = E_4 - E_3 = \dfrac{(-13.6)(3)^2}{(4)^2} - \left[\dfrac{(-13.6)(3)^2}{(3)^2}\right] = 5.95 \text{ eV}$

$$W = E - K_{max} = (5.95 - 3.95) \text{ eV} = 2 \text{ eV}$$

For longer wavelength:

$$\Delta E = E_5 - E_4 = \frac{(-13.6)(3)^2}{(5)^2} - \left[\frac{(-13.6)(3)^2}{(4)^2}\right] = 2.754 \text{ eV}$$

$\therefore$ $K_{max} = E - W = 0.754$ eV

or stopping potential is 0.754 volt

Ans.

29. $\lambda_1 = 4102$ Å, $\lambda_2 = 4861$ Å

or $$\frac{1}{\lambda_1} = R\left(\frac{1}{2^2} - \frac{1}{n_1^2}\right)$$

Substituting values of λ_1 and R, we get $n_1 = 6$

Similarly, $$n_2 = 4 \text{ for } \frac{1}{\lambda_2} = R\left(\frac{1}{2^2} - \frac{1}{n_2^2}\right)$$

Now, $$k = k_1 - k_2$$

or $$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

Transition $n_2 \to n_1$ or $6 \to 4$ corresponds to second line of Brackett series, whose wavelength is

$$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{6^2}\right)$$

Substituting the values we have $\lambda = 26206$ Å

Ans.

30. $$E = hcR\left(\frac{1}{1^2} - \frac{1}{5^2}\right) = \frac{24}{25} hcR \qquad \text{...(i)}$$

This energy will be shared by photon and the atom.

Thus, $$E = hf + E_0 \qquad \text{...(ii)}$$

where $E_0 = \dfrac{p^2}{2M}$ is the atom's recoil energy and P is the momentum due to emission of a photon. In accordance with the law of conservation of linear momentum.

$$p = p_{Ph} = \frac{hf}{c} \qquad \text{...(iii)}$$

From above equations, we get $$E_0 = \frac{h^2 f^2}{2Mc^2}$$

and $$hf = \frac{2E}{1 + \sqrt{1 + \dfrac{2E}{Mc^2}}}$$

Since, the transition energy in hydrogen atom is below 13.6 eV, $\frac{2E}{Mc^2} \approx 10^{-8}$.

$$\therefore \qquad hf \approx E = \frac{24}{25}\, hcR$$

$\therefore$ Recoil energy of atom is $\qquad \dfrac{24^2 h^2 R^2}{2 \times 25^2 M} = \dfrac{h^2 R^2}{2.17M}$

and the velocity of the atom is $\dfrac{24hR}{25M}$

31. Magnetic moment $\mu = NiA = \left(\frac{e}{T}\right)(\pi r^2)$

or $$\mu = \left(\frac{e}{2\pi r/v}\right)(\pi r^2) = \frac{evr}{2} \qquad \ldots\text{(i)}$$

We know that $$mvr = \frac{nh}{2\pi} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), $$\mu = \frac{neh}{4\pi m} \qquad \textbf{Ans.}$$

Magnetic induction, $$B = \frac{\mu_0 i}{2r} = \frac{\mu_0 e}{2rT}$$

or $$B = \frac{\mu_0 ev}{(2r)(2\pi r)} = \frac{\mu_0 ev}{4\pi r^2} \qquad \ldots\text{(iii)}$$

From Newton's second law, $$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r} \quad \text{or} \quad v^2 = \frac{e^2}{4\pi\varepsilon_0 mr} \qquad \ldots\text{(iv)}$$

Solving all these equations, we get $$B = \frac{\mu_0 \pi m^2 e^7}{8\varepsilon_0 h^5 n^5} \qquad \textbf{Ans.}$$

32. (a) Energy required to just eject a photoelectron from the potassium surface is equal to the work function, *i.e.*,

$$2.22 \text{ eV} = 3.552 \times 10^{-19} \text{ J}$$

As per the classical theory, energy flow is a continuous process and a photoelectron will be ejected if a potassium atom receives this amount of energy over a length of time.

The potassium atom is at a distance of 1.0 m from the source. Hence, intensity of ultraviolet radiation on the potassium surface is,

$$I = \frac{P}{4\pi r^2} = \frac{0.2}{4\pi(1)^2} = \frac{0.2}{4\pi} \text{ watt/m}^2$$

Cross-sectional area of the potassium atom is,

$$A = \pi(2 \times 10^{-10})^2 = 4\pi \times 10^{-20} \text{ m}^2$$

Hence, the exposure time is given by,

$$t = \frac{3.552 \times 10^{-19}}{\left(\frac{0.2}{4\pi}\right)(4\pi \times 10^{-20})} = 177.6 \text{ sec} \qquad \textbf{Ans.}$$

(b) $$E = \frac{12375}{2537} = 4.87 \text{ eV} \qquad \textbf{Ans.}$$

(c) Let N be the number of photons reaching the cathode per second. Then intensity at cathode is,

$$I' = N \times E$$

But energy falling per second on the cathode is $\left(\frac{0.2}{4\pi}\right)(4 \times 10^{-4})$

$\therefore$ $$N = \frac{0.2 \times 4 \times 10^{-4}}{4\pi E} = \frac{0.2 \times 4 \times 10^{-4}}{4\pi \times 4.87 \times 1.6 \times 10^{-19}} = 8.12 \times 10^{12} \text{ photons/sec}$$ **Ans.**

10% of these photons are able to eject electrons.

Hence, $$i = 0.1 \times N \times 1.6 \times 10^{-19} \text{ A} = 65 \text{ nA}$$ **Ans.**

(d) $$K_{max} = E - W = (4.87 - 2.22) \text{ eV} = 2.65 \text{ eV}$$

$\therefore$ Stopping potential = 2.65 volt

LEVEL-II

1. Energy of electron in ground state of hydrogen atom is – 13.6 eV. Earlier it had a kinetic energy of 2 eV. Therefore, energy of photon released during formation of hydrogen atom,

$$\Delta E = 2 - (-13.6) = 15.6 \text{ eV}$$

$\therefore$ $$\lambda = \frac{12375}{\Delta E} = \frac{12375}{15.6} = 793.3 \text{ Å}$$ **Ans.**

2. Force of interaction between electron and proton is

$$F = -\frac{dU}{dr} = \frac{-k}{r}$$

Force is negative. It means there is attraction between the particles and they are bound to each other. This force provides the necessary centripetal force for the electron.

$\therefore$ $$\frac{mv^2}{r} = \frac{k}{r} \quad \text{...(i)}$$

According to Bohr's assumption,

$$mvr = n\frac{h}{2\pi} \quad \text{...(ii)}$$

Solving equations (i) and (ii), we get

$$r = \frac{nh}{2\pi\sqrt{mk}} \quad \text{and} \quad v = \sqrt{\frac{k}{m}}$$

$\therefore$ $$E = U + \frac{1}{2}mv^2 = k\ln r - \frac{k}{2} + \frac{k}{2} = k\ln r$$

Thus, $$r_n = \frac{nh}{2\pi\sqrt{mk}} \quad \text{and} \quad E_n = k\ln\left\{\frac{nh}{2\pi\sqrt{mk}}\right\}$$ **Ans.**

3. (a) $$r = \frac{mv}{Be} \quad \text{...(i)}$$

$$mvr = \frac{nh}{2\pi} \quad \text{...(ii)}$$

Solving these two equations, we get

$$r = \sqrt{\frac{nh}{2\pi Be}} \quad \text{and} \quad v = \sqrt{\frac{nhBe}{2\pi m^2}}$$

(b) $$K = \frac{1}{2}mv^2 = \frac{nhBe}{4\pi m}$$ **Ans.**

(c) $$M = iA = \left(\frac{e}{T}\right)(\pi r^2) = \frac{e}{\left(\frac{2\pi r}{v}\right)}(\pi r^2) = \frac{evr}{2}$$

$$= \frac{e}{2}\sqrt{\frac{nh}{2\pi Be}}\sqrt{\frac{nhBe}{2\pi m^2}} = \frac{nhe}{4\pi m}$$

$$U = -MB\cos 180° = \frac{nheB}{4\pi m}$$

Note : Angle between $\vec{\mathbf{M}}$ and $\vec{\mathbf{B}}$ will be 180°. Think why?

(d) $$E = U + K = \frac{nheB}{2\pi m}$$

(e) $$|\phi| = B\pi r^2 = \frac{nh}{2e}$$

4. Let m_1 and m_2 be the mass of α-particle and hydrogen atom. From conservation of momentum,

$$m_1u_1 = (m_1 + m_2)v$$

where u_1 is the initial velocity of the incident α-particle and v is the final common velocity (or velocity of centre of mass) of the particles.

From conservation of energy $$\frac{1}{2}m_1v_1^2 = \frac{1}{2}(m_1 + m_2)v^2 + \Delta E_0$$

where ΔE_0 = ionization energy

$\therefore$ $$\frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)v_1^2 = \Delta E_0$$

or $$K_1 = \frac{1}{2}m_1v_1^2 = \left(\frac{m_1 + m_2}{m_2}\right)\Delta E_0$$

$$= \left(1 + \frac{m_1}{m_2}\right)\Delta E_0$$

$$= \left(1 + \frac{4}{1}\right)(13.6)\text{ eV}$$

$$= 68\text{ eV}$$ **Ans.**

5. $$5.667 = 13.6\,(Z_B^2 - Z_A^2)\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$$

$\therefore$ $$Z_B^2 - Z_A^2 = 3 \qquad \text{...(i)}$$

From conservation of linear momentum,

$$m_Au = Mv_1 - m_Au$$

or $$2m_Au = Mv_1$$

Similarly $$2m_Bu = Mv_2$$

Given $$Mv_2 = 2Mv_1$$

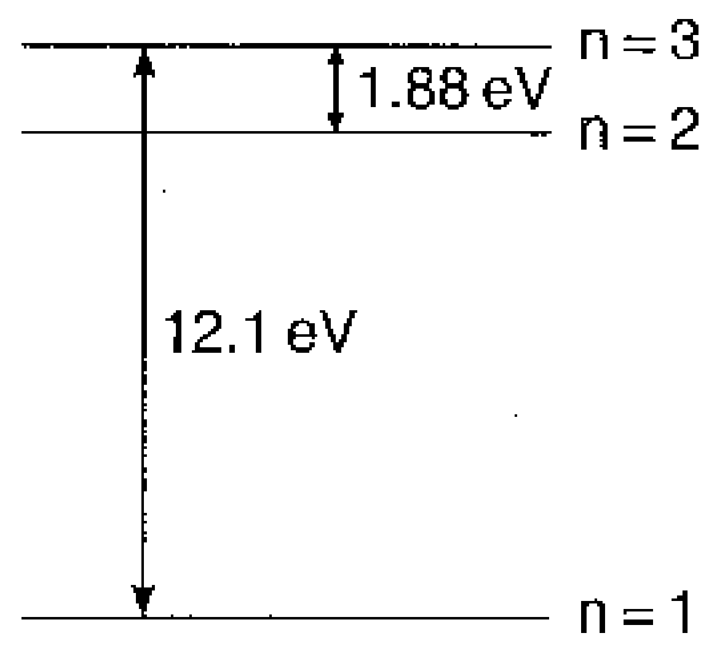
$\therefore$ $$m_B = 2m_A \qquad \text{...(ii)}$$

Since both A and B contain equal number of protons and neutrons we can write

$$\frac{m_A}{m_B} = \frac{2Z_A}{2Z_B} = \frac{Z_A}{Z_B} \quad \text{...(iii)}$$

From these equations we get $Z_A = 1$ and $Z_B = 2$ *i.e.*, A is ${}_1H^2$ and B is ${}_2He^4$ (both having single electron).

6. Energy of photon corresponding to $\lambda = 6563$ Å is, $\frac{12375}{6563}$ eV or 1.88 eV which is the difference in energy between $n = 3$ and $n = 2$. Hence the single electron in the hydrogen atom should excite at least upto $n = 3$. For this minimum energy of the striking electron should be 12.1 eV.



7. (a) As the atoms finally emit radiation of only 3 different photon energies final excited state corresponds to $n = 3$.

$\therefore$ Initial excited state corresponds to $n = 2$.

$$\therefore \quad Z^2 (13.6)\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{12375}{1654}$$

or $$Z = 2 \quad \textbf{Ans.}$$

Therefore, it is helium atom.

(b) $$\text{Ionization energy} = Z^2(13.6 \text{ eV}) = (2)^2\,(13.6 \text{ eV})$$

$$= 54.4 \text{ eV} \quad \textbf{Ans.}$$

(c) $$\frac{n(n-1)}{2} = 6 \qquad \therefore \; n = 4$$

$$E = E_4 - E_2 = (13.6)(4)\left(\frac{1}{4} - \frac{1}{16}\right)$$

$$= 10.2 \text{ eV} \quad \textbf{Ans.}$$

8. (a) Number of photoelectrons emitted per second $= \left(\frac{1}{10^6}\right)\left(\frac{P}{E}\right)\left(\frac{\pi r^2}{4\pi R^2}\right)$

or $$n = \left(\frac{1}{10^6}\right)\left(\frac{3.2 \times 10^{-3}}{5.0 \times 1.6 \times 10^{-19}}\right)\left(\frac{1}{4\pi \times 0.8 \times 0.8}\right)(\pi \times 8.0 \times 10^{-3} \times 8.0 \times 10^{-3})$$

$$n = 10^5 \text{ sec}^{-1} \quad \textbf{Ans.}$$

(b) $$K_{max} = E - W = 2.0 \text{ eV}$$

$$\lambda_2 = \sqrt{\frac{150}{\text{KE (in eV)}}} \text{ Å} \quad \text{(for an electron)}$$

$$= \sqrt{\frac{150}{2}} = 8.66 \text{ Å}$$

$$\lambda_1 = \text{wavelength of incident photon} = \frac{12375}{5.0} = 2475 \text{ Å}$$

$$\frac{\lambda_1}{\lambda_2} \approx 286 \quad \textbf{Ans.}$$

(c) Photoemission will stop when potential on the sphere becomes equal to the stopping potential.

(d) $K_{max} = 2\,eV_0$. Therefore, the stopping potential V_0 is 2 volt. Let t be the desired time. Then

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{1}{(4\pi\varepsilon_0)}\frac{(\text{net})}{r}$$

$$\therefore \quad t = \frac{V_0 r}{\left(\frac{1}{4\pi\varepsilon_0}\right)(ne)} = \frac{2\times 8.0\times 10^{-3}}{9.0\times 10^9\times 10^5\times 1.6\times 10^{-19}}$$

$$= 111 \text{ sec}$$

Ans.

9. Using classical mechanics, we obtain

$$\frac{\Delta K}{K_i} = \frac{4m_1m_2}{(m_1+m_2)^2} = \frac{4\,(1)\,(2)}{(1+2)^2} = \frac{8}{9}$$

where K_i is the initial kinetic energy of neutron and ΔK is the energy loss.

After first collision $\Delta K_1 = \frac{8}{9}K_0$

After second collision $\Delta K_2 = \frac{8}{9}K_1$ and so on

$\therefore$ Total energy loss $\quad \Delta K = \Delta K_1 + \Delta K_2 + \ldots + \Delta K_n = \frac{8}{9}(K_0 + K_1 + \ldots + K_{n-1})$

As, $\quad K_1 = K_0 - \Delta K_1 = \frac{K_0}{9}, \quad E_2 = \frac{K_1}{9} = \left(\frac{1}{9}\right)^2 K_0$

$$K_{n-1} = \left(\frac{1}{9}\right)^{n-1} K_0$$

$$\therefore \quad \Delta K = \frac{8}{9}K_0\left[1 + \frac{1}{9} + \ldots \left(\frac{1}{9}\right)^{n-1}\right]$$

$$\frac{\Delta K}{K_0} = \frac{8}{9}\left[\frac{1-\frac{1}{9^n}}{1-\frac{1}{9}}\right] = 1 - \frac{1}{9^n}$$

Here, $\quad K_0 = 10^6$ eV, $\quad \Delta K = (10^6 - 0.025)$ eV

$$\therefore \quad \frac{1}{9^n} = \frac{K_0 - \Delta K}{K_0} = \frac{0.025}{10^6} \quad \text{or} \quad 9^n = 4\times 10^7$$

Taking log both sides and solving, we get

$$n = 8$$

Ans.

10. $U = -1.7$ eV

$$\therefore \quad E = \frac{U}{2} = -0.85 \text{ eV} = \frac{-13.6}{n^2}$$

$$\therefore \quad n = 4$$

Ejected photoelectron will have minimum de-Broglie wavelength corresponding to transition from $n = 4$ to $n = 1$

$$\Delta E = E_4 - E_1 = -0.85 - (-13.6) = 12.75 \text{ eV}$$

$$K_{max} = \Delta E - W = 10.45 \text{ eV}$$

$$\therefore \quad \lambda = \sqrt{\frac{150}{10.45}} \text{ Å} \qquad \text{(for an electron)}$$

$$= 3.8 \text{ Å}$$

Ans.

11. **For Li^{2+}:** $Z = 3$

$$\therefore \qquad E_2 = \frac{-13.6\,(3)^2}{(2)^2} = -30.6\text{ eV}$$

$$E_3 = \frac{-13.6\,(3)^2}{(3)^2} = -13.6\text{ eV}$$

Energy required for Li^{2+}, ion to go from first excited state $(n = 2)$ to second excited state $(n = 3) = -13.6 - (-30.6) = 17$ eV

Energy released by hydrogen atom to go from first excited state to ground state $= -3.4 - (-13.6) = 10.2$ eV

So, $17 - 10.2 = 6.8$ eV energy should come from loss in KE in collision. From conservation of linear momentum, velocity of combined mass

$$v = \frac{m_1 u_1}{m_1 + m_2} \qquad (m_2 = 7m_1)$$

$$\Delta KE = \frac{1}{2} m_1 u_1^2 - \frac{1}{2}(m_1 + m_2)\, v^2$$

This should be equal to 6.8 eV. Solving these equations, we get

$$u_1 = 3.9 \times 10^4 \text{ m/s}$$ **Ans.**

12. Frequency of electron in n^{th} orbit is given by

$$f_n = \frac{K^2 m e^4}{2\pi n^3 \hbar^3} \qquad (\text{for } Z = 1)$$

where $$\hbar = \frac{h}{2\pi} \quad \text{and} \quad K = \frac{1}{4\pi\varepsilon_0}$$

Further, $$E_n = -\frac{mK^2e^4}{2n^2\hbar^2}$$

and $$E_{n-1} = -\frac{mK^2e^4}{2\,(n-1)^2\hbar^2}$$

We can show that for large values of n or $\left(\frac{1}{n} \to 0\right)$

$$f_n = \frac{E_n - E_{n-1}}{h}$$

13. **(a) and (b)** When hydrogen atom is excited.

$$eV = E_0\left(\frac{1}{1} - \frac{1}{n^2}\right) \qquad \ldots\text{(i)}$$

When ion is excited,

$$eV = E_0 Z^2\left[\frac{1}{2^2} - \frac{1}{n_1^2}\right] \qquad \ldots\text{(ii)}$$

Wavelength of emitted light

$$\frac{hc}{\lambda_1} = E_0\left(\frac{1}{1} - \frac{1}{n^2}\right) \qquad \ldots\text{(iii)}$$

$$\frac{hc}{\lambda_2} = E_0 Z^2\left(\frac{1}{1} - \frac{1}{n_1^2}\right) \qquad \ldots\text{(iv)}$$

Further it is given that

$$\frac{\lambda_1}{\lambda_2} = \frac{5}{1} \qquad \text{...(v)}$$

Solving the above equations, we get

$$Z = 2, \quad n = 2, \; n_1 = 4 \quad \text{and} \quad V = 10.2 \text{ volt}$$ **Ans.**

(c) Energy of emitted photon by the hydrogen atom $= E_2 - E_1$

$$= 10.2 \text{ eV}$$ **Ans.**

and by the ion $= E_4 - E_1 = (13.6)\,(2)^2 \left(1 - \frac{1}{16}\right)$

$$= 51 \text{ eV}$$ **Ans.**

14.

$$0.6 = \frac{12375}{4950} - W \qquad \text{...(i)}$$

$$1.1 = \frac{12375}{\lambda} - W \qquad \text{...(ii)}$$

Solving above two equations, we get

$$W = 1.9 \text{ eV} \quad \text{and} \quad \lambda = 4125 \text{ Å}$$ **Ans.**

15.

$$E = \frac{12375}{4000} = 3.1 \text{ eV}$$

Number of photoelectrons emitted per second

$$n = \left(\frac{1}{10^6}\right)\left(\frac{5}{3.1 \times 1.6 \times 10^{-19}}\right) = 1.0 \times 10^{13} \text{ per second}$$

$\therefore$

$$i = (ne) = 1.0 \times 10^{13} \times 1.6 \times 10^{-19} = 1.6 \times 10^{-6} \text{ A}$$

$$= 1.6 \; \mu\text{A}$$ **Ans.**

16. (a)

$$E = \frac{12375}{4000} = 3.1 \text{ eV}$$

Energy of electron after first collision

$$E_1 = 90\% \text{ of } E = 2.79 \text{ eV} \qquad \text{(as 10\% is lost)}$$

Energy of electron after second collision

$$E_2 = 90\% \text{ of } E_1 = 2.51 \text{ eV}$$

KE of this electron after emitting from the metal surface = (2.51 – 2.2) eV = 0.31 eV **Ans.**

(b) Energy after third collision,

$$E_3 = 90\% \text{ of } E_2 = 2.26 \text{ eV}$$

Similarly,

$$E_4 = 90\% \text{ of } E_3 = 2.03 \text{ eV}$$

So, after four collisions it becomes unable to come out of the metal.

17. Energy of photon of the first line of Lymen series

$$E = E_2 - E_1 = (13.6)\,(2)^2 \left(1 - \frac{1}{4}\right)$$

$$= 40.8 \text{ eV}$$

Energy required to ionize the hydrogen atom is 13.6 eV. Therefore, kinetic energy of electron emitted from the hydrogen atom is,

$$K = (40.8 - 13.6)\text{ eV} = 27.2\text{ eV}$$
$$= 4.352 \times 10^{-18}\text{ J}$$
$$= \frac{1}{2}mv^2$$

$\therefore$
$$v = \sqrt{\frac{2 \times 4.352 \times 10^{-18}}{9.1 \times 10^{-31}}}$$
$$= 3.1 \times 10^6\text{ m/s}$$ **Ans.**

18.
$$\text{Reduced mass } \mu = \frac{(m)(m)}{(m+m)} = \frac{m}{2}$$
$$r \propto \frac{1}{m}$$
$$r_1^H = 0.53\text{ Å}$$

$\therefore$
$$r = (2)(0.53) = 1.06\text{ Å}$$ **Ans.**

$$E \propto m$$ (binding energy)
$$E_H = 13.6\text{ eV}$$

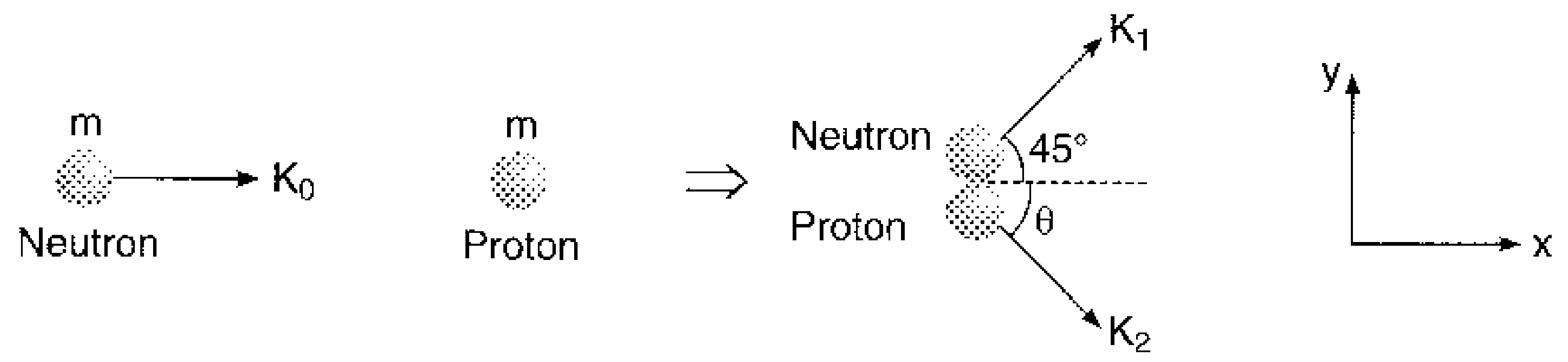
$\therefore$
$$E = \frac{13.6}{2} = 6.8\text{ eV}$$ **Ans.**

Further,
$$(E_{21})_H = 10.2\text{ eV}$$ (First line of Lymen series)

$\therefore$
$$E_{21} = \frac{10.2}{2} = 5.1\text{ eV}$$

$\therefore$
$$\lambda = \frac{12375}{5.1}\text{ Å} = 2426\text{ Å}$$ **Ans.**

19. Refer hint of Q. No. 9 : Mass of neutron $\approx$ mass of proton $= m$



From conservation of momentum in y-direction
$$\sqrt{2mK_1}\sin 45° = \sqrt{2mK_2}\sin\theta \quad \text{...(i)}$$

in x-direction
$$\sqrt{2mK_0} - \sqrt{2mK_1}\cos 45° = \sqrt{2mK_2}\cos\theta \quad \text{...(ii)}$$

Squaring and adding equations (i) and (ii), we have
$$K_2 = K_1 + K_0 - \sqrt{2K_0K_1} \quad \text{...(iii)}$$

From conservation of energy
$$K_2 = K_0 - K_1 \quad \text{...(iv)}$$

Solving equations (iii) and (iv), we get
$$K_1 = \frac{K_0}{2}$$

$$\therefore \quad \Delta K = K_0 - K_1 + \frac{K_0}{2}$$

i.e., after each collision energy remains half. Therefore, after n collisions,

$$K_n = K_0\left(\frac{1}{2}\right)^n$$

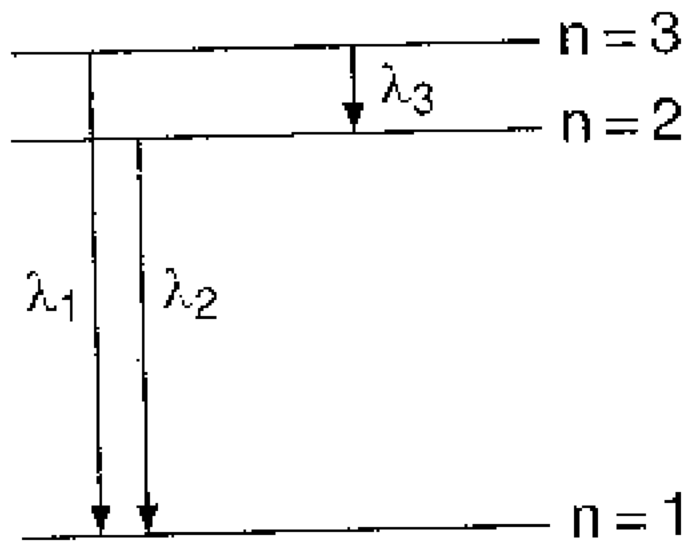
$$\therefore \quad 0.23 = (4.6 \times 10^6)\left(\frac{1}{2}\right)^n$$

$$2^n = \frac{4.6 \times 10^6}{0.23}$$

Taking log and solving, we get

$$n \approx 24$$ **Ans.**

20. Given $E_3 = 0$ $\lambda_1 = 460$ Å



$$E_3 - E_1 = \frac{12375}{460} = 26.9 \text{ eV}$$

$$\therefore \quad 0 - E_1 = 26.9 \text{ eV}$$

or $$E_1 = -26.9 \text{ eV}$$ **Ans.**

Further, $$\lambda_3 = 1035 \text{ Å} \quad \text{or} \quad E_3 - E_2 = \frac{12375}{1035} \text{ Å} = 12.0 \text{ eV}$$

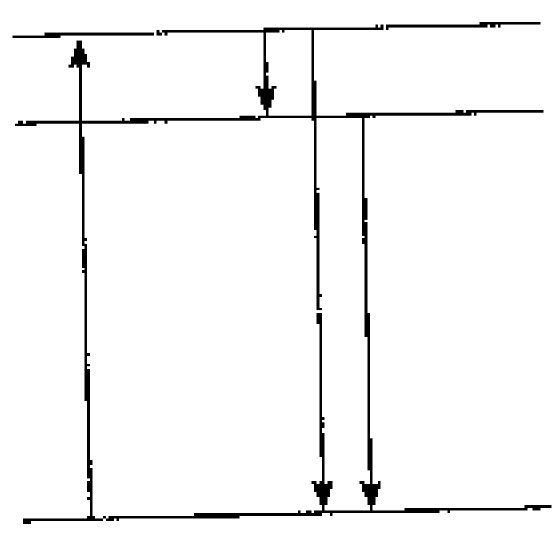
or $$E_2 = -12.0 \text{ eV}$$ **Ans.**

21. (a) $$\frac{n(n-1)}{2} = 3$$

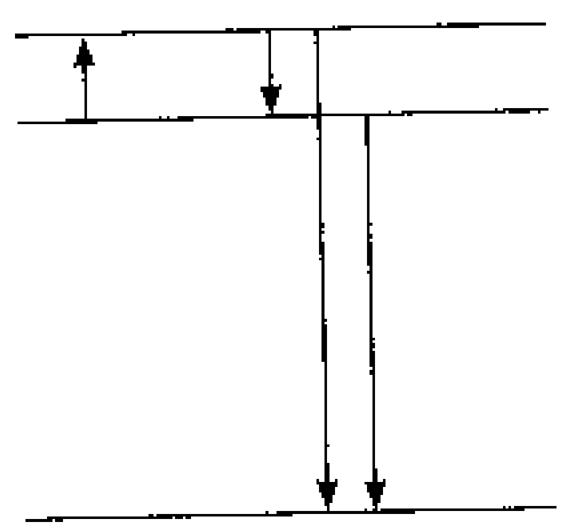
$$\therefore \quad n = 3$$

i.e., after excitation atom jumps to second excited state. Hence $n_f = 3$. So n_i can be 1 or 2.

If $n_i = 1$ then energy emitted is either equal to, greater than or less than the energy absorbed. Hence the emitted wavelength is either equal to, less than or greater than the absorbed wavelength. Hence $n_i \neq 1$.



$n_i = 1$



$n_i = 2$

If $n_i = 2$, then $E_e \geq E_a$. Hence $\lambda_e \leq \lambda_b$

$$\therefore \quad n_i = 2$$ **Ans.**

(b) $$E_3 - E_2 = 68 \text{ eV}$$

$$\therefore \quad (13.6)(Z^2)\left(\frac{1}{4} - \frac{1}{9}\right) = 68$$

$$\therefore \quad Z = 6$$ **Ans.**

(c) $$\lambda_{min} = \frac{12375}{E_3 - E_1} = \frac{12375}{(13.6)(6)^2\left(1 - \frac{1}{9}\right)} = 28.43 \text{ Å}$$ **Ans.**

(d) Ionization energy $= (13.6)(6)^2 = 489.6$ eV **Ans.**

$$\lambda = \frac{12375}{489.6} = 25.3 \text{ Å}$$ **Ans.**

22. (a)
$$2\left\{\frac{12375\times 10^{-10}}{34.3\times 10^{3}}\right\}=\frac{1}{1.09\times 10^{7}\,(Z-1)^{2}\left(1-\frac{1}{4}\right)}$$

$\therefore$ $$Z = 42$$ **Ans.**

(b) Energy of K_2 line = 100 – 24 = 76 keV

$\therefore$ $$\lambda_{K_\alpha} = \frac{12375}{76\times 10^{3}}\ \text{Å}$$

$$= 0.163\ \text{Å}$$ **Ans.**

23. Pitch of helical path $p = (v\cos\theta)\,T = \frac{vT}{2}$. (as $\theta = 60°$)

$$T = \frac{2\pi m}{Bq} = \frac{2\pi}{B\alpha} \qquad \left(\alpha = \frac{q}{m}\right)$$

$\therefore$ $$p = \frac{\pi v}{B\alpha}$$

or $$v = \frac{B\alpha p}{\pi} \qquad \text{...(i)}$$

$$\text{KE} = \frac{1}{2}mv^2 = E - W$$

$\therefore$ $$W = E - \frac{1}{2}mv^2 \qquad \text{...(ii)}$$

Substituting value of v from Eq. (i) in Eq. (ii) we get

$$W = 4.9 - \frac{1}{2}\times\frac{9.1\times 10^{-31}\times(2.5\times 10^{-3})^2\,(1.76\times 10^{11})^2\,(2.7\times 10^{-3})^2}{\pi^2\times 1.6\times 10^{-19}}$$

$$= (4.9 - 0.4)\ \text{eV}$$
$$= 4.5\ \text{eV}$$ **Ans.**

24.
$$f = \frac{v_1}{2\pi r_1} = \frac{c}{\lambda}$$

$\therefore$ $$\lambda = \frac{2\pi c r_1}{v_1} = \frac{(2\pi)\,(3.0\times 10^{8})\,(0.529\times 10^{-10})\,(10^{10})}{(2.2\times 10^{6})}\ \text{Å}$$

$$= 453\ \text{Å}$$ **Ans.**

25. (a)
$$\lambda = 1500\left(\frac{1}{1-1/p^2}\right)$$

λ_{max} corresponds to least energetic photon with $p = 2$.

$\therefore$ $$\lambda_{max} = 1500\left(\frac{1}{1-1/4}\right) = 2000\ \text{Å}$$ **Ans.**

λ_{min} corresponds to most energetic photon with $p = \infty$

$\therefore$ $$\lambda_{min} = 1500\ \text{Å}$$ **Ans.**

(b) $$\lambda_{\alpha\,1} = 1500\ \text{Å}$$

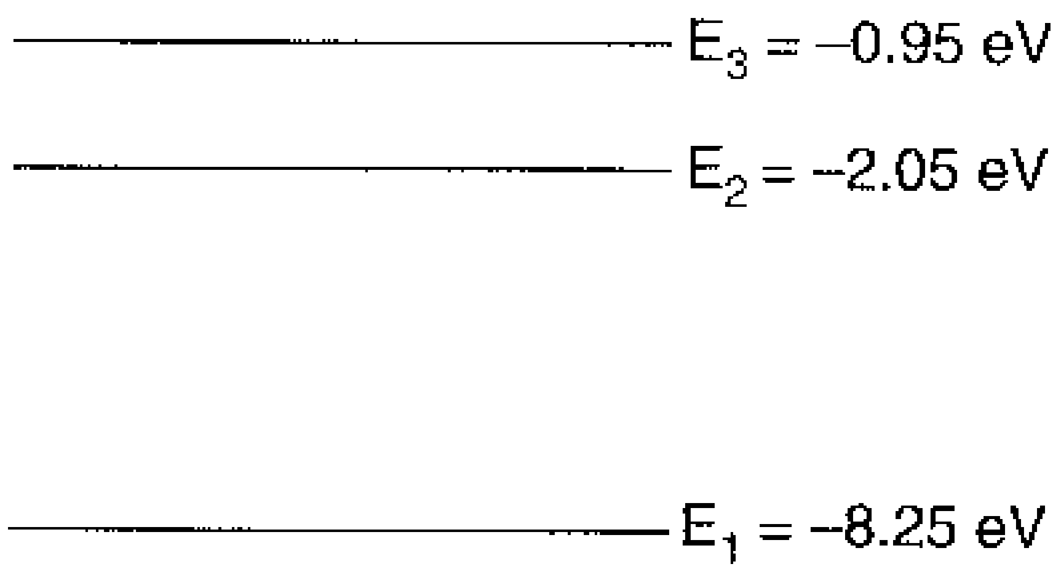
$\therefore$ $$E_\alpha - E_1 = \frac{12375}{1500}\ \text{eV} = 8.25\ \text{eV}$$

$\therefore \quad E_1 = -8.25 \text{ eV}$ (as $E_\alpha = 0$)

$$\lambda_{2-1} = 2000 \text{ Å}$$

$$\therefore \quad E_2 - E_1 = \frac{12375}{2000} \text{ eV} = 6.2 \text{ eV}$$

$$\therefore \quad E_2 = -2.05 \text{ eV}$$



Similarly $\lambda_{31} = 1500\left(\frac{1}{1-1/9}\right) = 1687.5 \text{ Å}$

$$\therefore \quad E_3 - E_1 = \frac{12375}{1687.5} \text{ eV} = 7.3 \text{ eV}$$

$$\therefore \quad E_3 = -0.95 \text{ eV}$$

(c) Ionization potential = 8.25 volt **Ans.**

26. (a) $K_1 = \frac{12375}{3000} - W$ ...(i)

$$K_2 = \frac{12375}{1650} - W \quad \text{...(ii)}$$

$$v_2 = 2v_1 \quad \therefore \quad K_2 = 4K_1 \quad \text{...(iii)}$$

Solving these equations, we get $W = 3 \text{ eV}$

$\therefore$ Threshold wavelength $\lambda_0 = \frac{12375}{3} = 4125 \text{ Å}$ **Ans.**

(b) $E_2 = \frac{12375}{1650} = 7.5 \text{ eV} = 12 \times 10^{-19} \text{ J}$

Therefore, number of photons incident per second

$$n_2 = \frac{P_2}{E_2} = \frac{5.0 \times 10^{-3}}{12 \times 10^{-19}} = 4.17 \times 10^{15} \text{ per second}$$

Number of electrons emitted per second ($\gamma = 5.1\%$) $= \frac{5.1}{100} \times 4.17 \times 10^{15}$

$$= 2.13 \times 10^{14} \text{ per second}$$

$\therefore$ Saturation current in second case $i = (2.13 \times 10^{14})(1.6 \times 10^{-19})$ amp

$$= 3.4 \times 10^{-5} \text{ A}$$

$$= 34 \ \mu\text{A}$$ **Ans.**

(c) Energy of photon in first case $= \frac{12375}{3000} = 4.125 \text{ eV}$

or $E_1 = 6.6 \times 10^{-19} \text{ J}$

Rate of incident photons $= \frac{P_1}{E_1} = \frac{10^{-3}}{6.6 \times 10^{-19}} = 1.52 \times 10^{15}$ per second

Number of electrons ejected $= \frac{4.8 \times 10^{-3}}{1.6 \times 10^{-19}}$ per second

$$= 3.0 \times 10^{16} \text{ per second}$$

$\therefore$ Efficiency of photoelectron generation $= \frac{1.52 \times 10^{15}}{3.0 \times 10^{16}} \times 100 = 5.1\%$ **Ans.**

MODERN PHYSICS-II

LEVEL-I

1. $N = \frac{2}{238} \times 6.02 \times 10^{23} = 5.06 \times 10^{21}, \quad R = \lambda N$

$\therefore$ $$\lambda = \frac{R}{N} = \frac{2.5 \times 10^4}{5.06 \times 10^{21}} = 4.94 \times 10^{-18} \text{ s}^{-1}$$

$$t_{1/2} = \frac{0.693}{\lambda} = 1.4 \times 10^{17} \text{ s}$$

2. (a) $$R = R_0 e^{-\lambda t}$$

$\therefore$ $$2700 = 4750 e^{-5\lambda}$$

$$\lambda = 0.113 \text{ min}^{-1}$$ **Ans.**

(b) $$t_{1/2} = \frac{0.693}{\lambda} = 6.132 \text{ min}$$ **Ans.**

3. $$v = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2 \times 0.0327 \times 1.6 \times 10^{-19}}{1.675 \times 10^{-27}}} = 2.5 \times 10^3 \text{ m/s}$$

$$t = \frac{s}{v} = \frac{10}{2.5 \times 10^3} = 4.0 \times 10^{-3} \text{ s}$$

$$\text{Fraction decayed} = \frac{N_0(1 - e^{-\lambda t})}{N_0} = 1 - e^{-\lambda t}$$

$$= 1 - e^{-0.6930 \times 4 \times 10^{-3}/700}$$

$$= 3.96 \times 10^{-6}$$ **Ans.**

4. Probability of a nucleus to decay in time $t = 1 - e^{-\lambda t} = 1 - e^{-(1/10)(5)} = 0.39$ **Ans.**

5. (a) $$N = \frac{1}{109} \times 6.02 \times 10^{23}$$

$\therefore$ $$R = \lambda N = \frac{0.693}{2.7 \times 10^7} \times \frac{1}{109} \times 6.02 \times 10^{23}$$

$$= 1.42 \times 10^{14} \text{ per year}$$ **Ans.**

(b) After 2 years, $$R = R_0 e^{-\lambda t} = (1.42 \times 10^{14})\, e^{[-0.693/2.7 \times 10^7](2)}$$

$$= 1.41 \times 10^{14} \text{ per year}$$ **Ans.**

(c) After 2 half lives activity reduces to 25% of the original value.

$\therefore$ $$t = 2t_{1/2} = 5.4 \times 10^7 \text{ years}$$ **Ans.**

6. The reaction is possible (spontaneously) if the binding energy of products is larger than that of $^{238}_{92}\text{U}$.

(a) Total binding energy of $^{238}_{92}\text{U} = 238 \times 7.57 = 1801.66$ MeV and binding energy of $^{206}_{82}\text{Pb} = 206 \times 7.83 = 1612.981$ MeV the binding energy of products is less than that of $^{238}_{92}\text{U}$. Note that the protons and neutrons are free and do not have any binding energy. Hence, this reaction is not possible spontaneously. It can take place only if $1801.66 - 1612.98 = 188.68$ MeV of energy is supplied from outside.

(b) In the second case binding energy of products is larger than the binding energy of the parent nucleus. Hence, reaction is possible spontaneously.

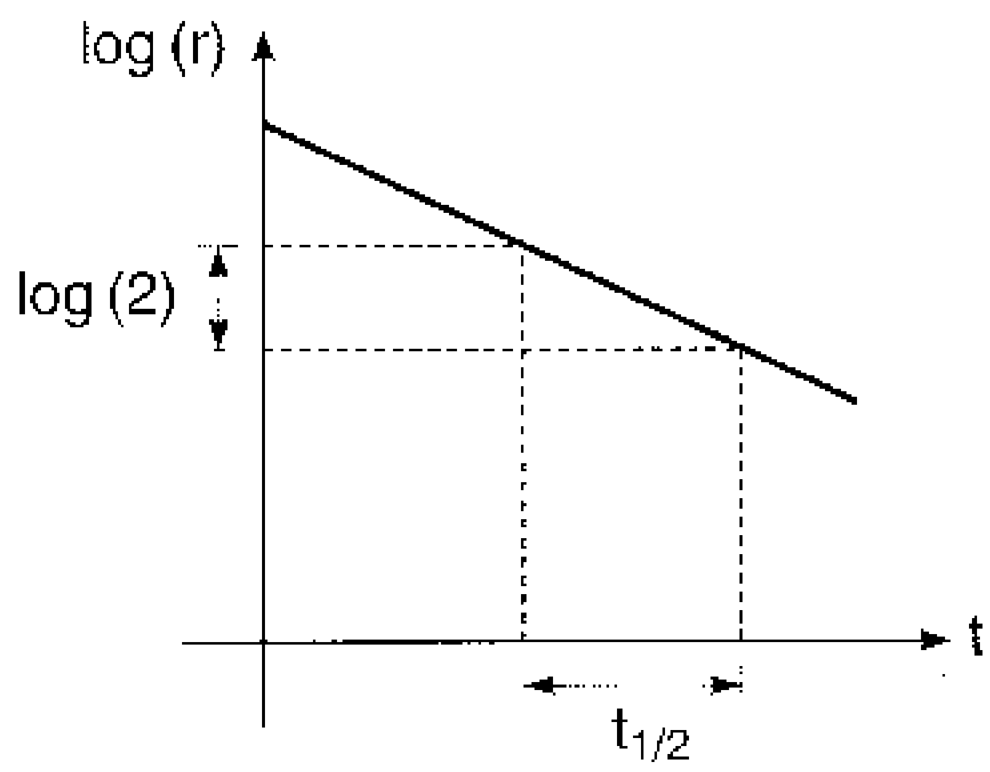
7. Let N_0 is the number of radioactive atoms in the radioactive sample at the beginning of counting and λ is the decay constant, then at time t after the beginning of counting the number of atoms will be,

$$N = N_0 e^{-\lambda t}$$

The rate with which this number changes is,

$$\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t} = -\lambda N$$

A counter registers only the radioactive particles that fly in its direction. The fraction of such particles in the overall number of radioactive particles emitted by the sample depends on the size and position of the counter and can be characterized by a factor a (with $a < 1$). Thus, the counting rate can be expressed in the form,



$$r = a\left|\frac{dN}{dt}\right| = aN_0\lambda e^{-\lambda t}$$

or $$ln(r) = ln\,(aN_0\lambda) - \lambda t$$

To determine the half life of the radioactive element there is no need to measure the slope and find the λ vs t dependence and using the well known formula to calculate $t_{1/2}$. Suffice is to lay off in any place on the vertical axis a segment equal to logarithm of two (irrespective of base, whether it is 10 or e) and draw through the end points of this segments straight line parallel to the horizontal axis. The points at which these straight lines intersect the experiments straight line that represents the variation in the rate of counting determine the boundaries of the time interval in the course of which the counting rate decreases by a factor 2.

8. $$N = \frac{R}{\lambda} = \frac{5 \times 10^{-3} \times 3.7 \times 10^{10}}{\dfrac{0.693}{138 \times 24 \times 3600}} = 3.2 \times 10^{15}$$

$\therefore$ $$m = \left(\frac{3.2 \times 10^{15}}{6.02 \times 10^{23}}\right)(210)\text{ g}$$

$$= 1.12 \times 10^{-6}\text{ g}$$ **Ans.**

9. $$\Delta E = (8m_p + 8m_n - m_o) \times 931.5\text{ MeV}$$

$$= 127.6\text{ MeV}$$ **Ans.**

10. $$N = \frac{R}{\lambda} = \frac{10^9}{\dfrac{0.693}{14.3 \times 3600}} = 7.43 \times 10^{13}$$

Now, $$\frac{dN}{dt} = q - \lambda N \quad \text{or} \quad \int_0^N \frac{dN}{q - \lambda N} = \int_0^t dt$$

$\therefore$ $$N = \frac{q}{\lambda}(1 - e^{-\lambda t})$$

Substituting the values, $$7.43 \times 10^{13} = \frac{2 \times 10^9}{\dfrac{0.693}{14.3 \times 3600}}[1 - e^{-(0.693/14.3 \times 3600)t}]$$

Solving this equation we get $$t = 14.3\text{ Hr}$$ **Ans.**

11.
$$R = \lambda N = \frac{N ln(2)}{t_{1/2}}$$

$\therefore$
$$R \propto \frac{N}{t_{1/2}}$$

Given, $\frac{N_1}{N_2} = \frac{4}{1}$ and $\frac{(t_{1/2})_1}{(t_{1/2})_2} = \frac{14}{25}$

$\therefore$
$$\frac{R_1}{R_2} = \frac{N_1}{N_2} \times \frac{(t_{1/2})_2}{(t_{1/2})_1} = \frac{4}{1} \times \frac{25}{14} = \frac{100}{14}$$

or
$$R_1 = \frac{100}{114} \times 3.0 = 2.63 \text{ mCi}$$

and
$$R_2 = 3.0 - 2.63 = 0.37 \text{ mCi}$$

After 60 days,
$$R_1' = R_1 e^{-\lambda_1 t} = (2.63)\, e^{-(0.693/14) \times 60} = 0.135 \text{ mCi}$$
$$R_2' = R_2 e^{-\lambda_2 t} = (0.37)\, e^{-(0.693/25) \times 60} = 0.07 \text{ mCi}$$

$\therefore$ Total activity $R' = R_1' + R_2' = 0.205$ mCi **Ans.**

12. (a) Total number of atoms in 1 kg of U^{238}
$$N = \frac{1}{238} \times 6.02 \times 10^{26} = 2.53 \times 10^{24}$$

$\therefore$ Total energy released $= (200 \times 2.53 \times 10^{24})$ MeV

$= 8.09 \times 10^{13}$ J **Ans.**

(b)
$$m = \frac{8.09 \times 10^{13}}{30 \times 10^{3}} \text{ g} = 2.7 \times 10^{9} \text{ g} = 2.7 \times 10^{6} \text{ kg}$$

13. Let x the desired ratio.

then, mass of Co^{58} in 1 g $= x$

$\therefore$
$$N = \frac{x}{58} \times 6.02 \times 10^{23}$$

Given, $2.2 \times 10^{12} = \lambda N = \frac{0.693}{71.3 \times 24 \times 3600} \times \frac{x}{58} \times 6.02 \times 10^{23}$

$\therefore$
$$x = 1.88 \times 10^{-3}$$
Ans.

14. (a) $R = R_0 e^{-\lambda t}$
$$7.3 = 9.3 e^{-\lambda(4500 - 2500)}$$

$\therefore$
$$\lambda = 1.21 \times 10^{-4} \text{ year}^{-1}$$

$\therefore$
$$t_{1/2} = \frac{0.693}{\lambda} = 5724 \text{ years}$$
Ans.

(b) Further applying, $R = R_0 e^{-\lambda t}$

$\therefore$
$$R_0 = Re^{\lambda t} = (7.3)e^{(1.21 \times 10^{-4})(4500)}$$

or
$$R_0 = 12.58 \text{ dis/min-g}$$
Ans.

15.
$$Q = [m'_c - (m'_B + m_p)]\,c^2$$

where m_c' and m_B' are the nuclear mass of ^{11}C and ^{11}B.

$$m_B = m_B' + 5m_e$$
$$m_c = m_c' + 6\,m_e$$

$\therefore$
$$Q = [(m_c - 6m_e) - (m_B - 5m_e + m_p)]c^2$$
$$= [m_c - m_B - 2m_p\,]c^2 \qquad \text{(mass of electron = mass of positron)}$$
$$= (11.011434 - 11.009305 - 2 \times 0.0005486)931$$
$$= 0.961 \text{ MeV}$$ **Ans.**

The disintegration energy is equal to the maximum energy of the emitted photon.

16. (a)

	A	B	C
At $t = 0$	N_0	0	0
At t	N_1	N_2	N_3

Here,
$$N_1 = N_0 e^{-\lambda t} \qquad \text{...(i)}$$
$$\frac{dN_2}{dt} = \lambda\,(N_1 - N_2)$$

or
$$\frac{dN_2}{dt} = \lambda N_0 e^{-\lambda t} - \lambda N_2$$

or
$$dN_2 + \lambda N_2 dt = \lambda N_0 e^{-\lambda t}$$

$\therefore$
$$e^{\lambda t} dN_2 + \lambda N_2 e^{\lambda t} dt = \lambda N_0 dt$$

or
$$d(N_2 e^{\lambda t}) = \lambda N_0 dt$$

$\therefore$
$$N_2 e^{\lambda t} = \lambda N_0 t + c$$

At $t = 0$, $N_2 = 0$, $\therefore c = 0$

$\therefore$
$$N_2 = \lambda N_0 (te^{-\lambda t})$$

(b) Activity of B is,
$$R_2 = \lambda N_2 = \lambda^2 N_0 (te^{-\lambda t})$$

For maximum activity $\dfrac{dR_2}{dt} = 0$

$\therefore$
$$t = \frac{1}{\lambda}$$ **Ans.**

$\therefore$
$$R_{\max} = \frac{\lambda N_0}{e}$$ **Ans.**

17. $M_\alpha v_\alpha = mv \qquad \therefore \qquad v = \dfrac{M_\alpha v_\alpha}{M}$

Kinetic energy of Tl atom is

$$\text{K.E.} = \frac{1}{2} Mv^2 = \frac{1}{2} M \left(\frac{M_\alpha^2 v_\alpha^2}{M^2}\right) = \left(\frac{1}{2} M_\alpha v_\alpha^2\right)\left(\frac{M_\alpha}{M}\right)$$
$$= (\text{K.E.})_\alpha \left(\frac{M_\alpha}{M}\right)$$
$$= 6.082 \times \frac{4}{208} \text{ MeV}$$
$$= 0.1308 \text{ MeV}$$ **Ans.**

18. $^{210}_{84}\text{Po} \longrightarrow {}^{206}_{82}\text{Pb} - {}^{4}_{2}\text{He}$

$$\Delta m = 0.00564 \text{ amu}$$

Energy liberated per reaction $= (\Delta m)931$ MeV$= 8.4 \times 10^{-13}$ J

Electrical energy produced $= 8.4 \times 10^{-14}$ J

Let m g of ^{210}Po is required to produce the desired energy.

$$N = \frac{m}{210} \times 6 \times 10^{23}$$

$$\lambda = \frac{0.693}{t_{1/2}} = 0.005 \text{ per day}$$

$$\left(-\frac{dN}{dt}\right) = \lambda N = \frac{(0.005)(6 \times 10^{23} \text{ m})}{210} \text{ per day}$$

$\therefore$ Electrical energy produced per day

$$= \frac{(0.005)(6 \times 10^{23} \text{ m})}{210} \times 8.4 \times 10^{-14} \text{ J}$$

This is equal to 1.2×10^{7} J (given)

$\therefore$ $$m = 10 \text{ g}$$ **Ans.**

Activity at the end of 693 days is, $R = \dfrac{0.005 \times 6 \times 10^{23} \times 10}{210} = \dfrac{10^{21}}{7}$ per day $= R_0 \left(\dfrac{1}{2}\right)^n$

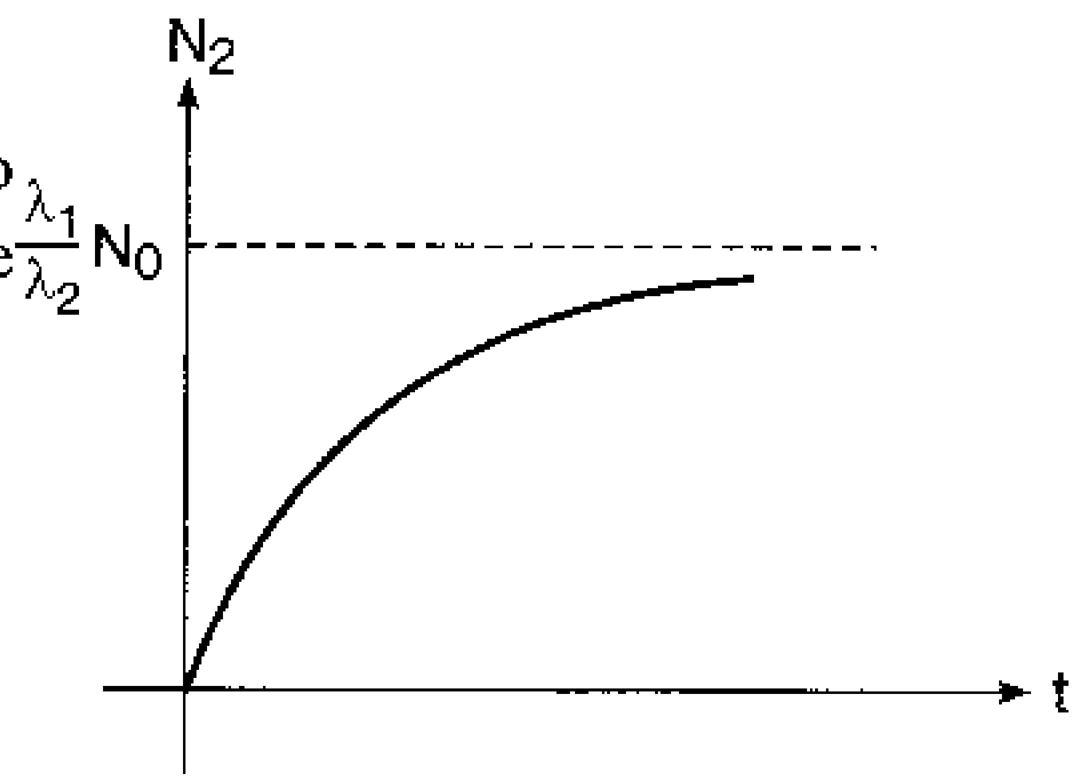
Here, n = number of half lives $= \dfrac{693}{138.6} = 5$

$\therefore$ $$R_0 = R(2)^5 = 32 \times \frac{10^{21}}{7} = 4.57 \times 10^{21} \text{ per day}$$ **Ans.**

19. $$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$ (derived in theory)

When $\lambda_1 >> \lambda_2$: $N_2 \approx N_0 e^{-\lambda_2 t}$

Physically this means that parent nuclei practically instantly transform into daughter nuclei, which then decay according to the law of radioactive decay with a certain decay constant.



$$N_2 \approx = \frac{\lambda_1}{\lambda_2} N_0 e^{-\lambda_1 t}$$

When $\lambda_1 << \lambda_2$, $$N_2 \approx \frac{\lambda_1}{\lambda_2} N_0 e^{-\lambda_1 t}$$

i.e., N_2 versus t graph in this case is as shown in the figure.

20. Total energy released $= \dfrac{1}{235} \times 6.02 \times 10^{26} \times 185 \times 1.6 \times 10^{-13}$ J $= 7.58 \times 10^{13}$ J

Power $= 10^{8}$ J/s

Therefore, $$\text{time} = \frac{7.58 \times 10^{13}}{10^{8}} \text{ sec} = 8.78 \text{ days}$$

LEVEL-II

1. (i) At $t = 0$, probabilities of getting α and β particles are same. This implies that initial activity of both is equal. Say it is R_0.

Activity after $t = 1620$ sec

$$R_1 = R_0\left(\frac{1}{2}\right)^{1620/405} = \frac{R_0}{16}$$

and
$$R_2 = R_0\left(\frac{1}{2}\right)^{1620/1620} = \frac{R_0}{2}$$

$$\text{Total activity } R = R_1 + R_2 = \frac{9}{16}R_0$$

$$\text{Probability of getting } \alpha\text{-particles} = \frac{R_1}{R} = \frac{1}{9}$$

and
$$\text{probability of getting } \beta\text{-particles} = \frac{R_2}{R} = \frac{8}{9}$$ **Ans.**

(ii) $R_{01} = R_{02}$

$$\therefore \quad \frac{N_{01}}{T_1} = \frac{N_{02}}{T_2}$$

$$\therefore \quad \frac{N_{01}}{N_{02}} = \frac{1}{4}$$

Let N_0 be the total number of nuclei at $t = 0$.

then,
$$N_{01} = \frac{N_0}{5} \quad \text{and} \quad N_{02} = \frac{4N_0}{5}$$

Given, that
$$N_1 + N_2 = \frac{N_0}{2}$$

or
$$\frac{N_0}{5}\left(\frac{1}{2}\right)^{t/405} + \frac{4N_0}{5}\left(\frac{1}{2}\right)^{t/1620} = \frac{N_0}{2}$$

Let
$$\left(\frac{1}{2}\right)^{t/1620} = x$$

Then, above equation becomes, $x^4 + 4x - 2.5 = 0$

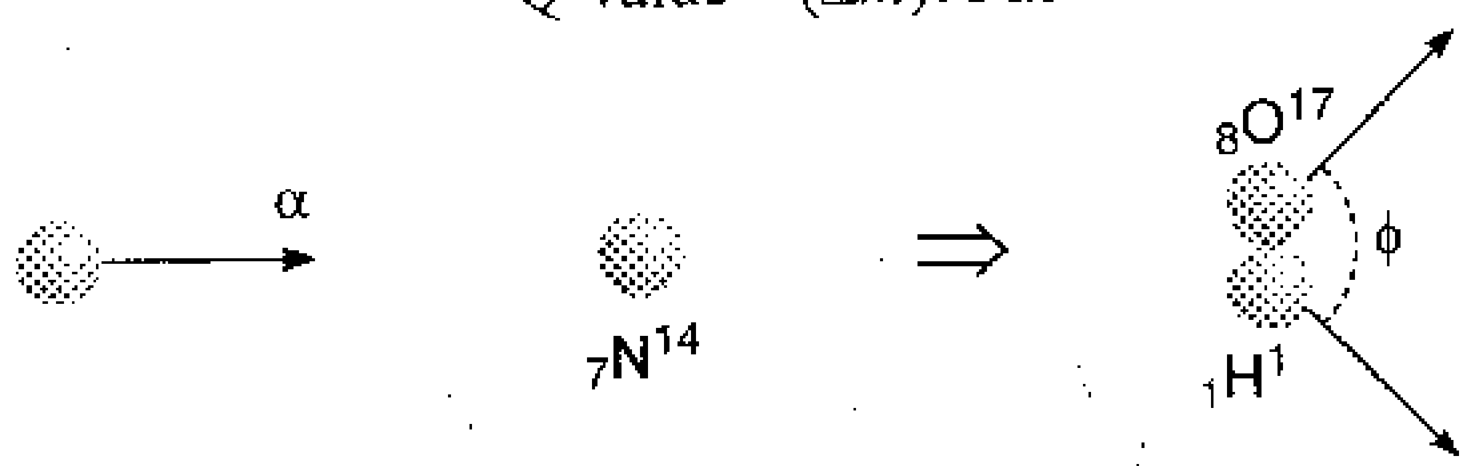
$$\therefore \quad x = 0.594$$

or
$$\left(\frac{1}{2}\right)^{t/1620} = 0.594$$

Solving it we get $t = 1215$ sec **Ans.**

2. (a) Q-value $= (\Delta m)931.5$ MeV $= -1.18$ MeV **Ans.**

(b)



From conservation of linear momentum

$$P_\alpha^2 = P_0^2 + P_H^2 + 2P_0P_H \cos\phi$$

$$\therefore \quad 2m_\alpha K_\alpha = 2m_0K_0 + 2m_HK_H + 2\sqrt{(2m_0K_0)(2m_HK_H)}\cos\phi \qquad \text{...(i)}$$

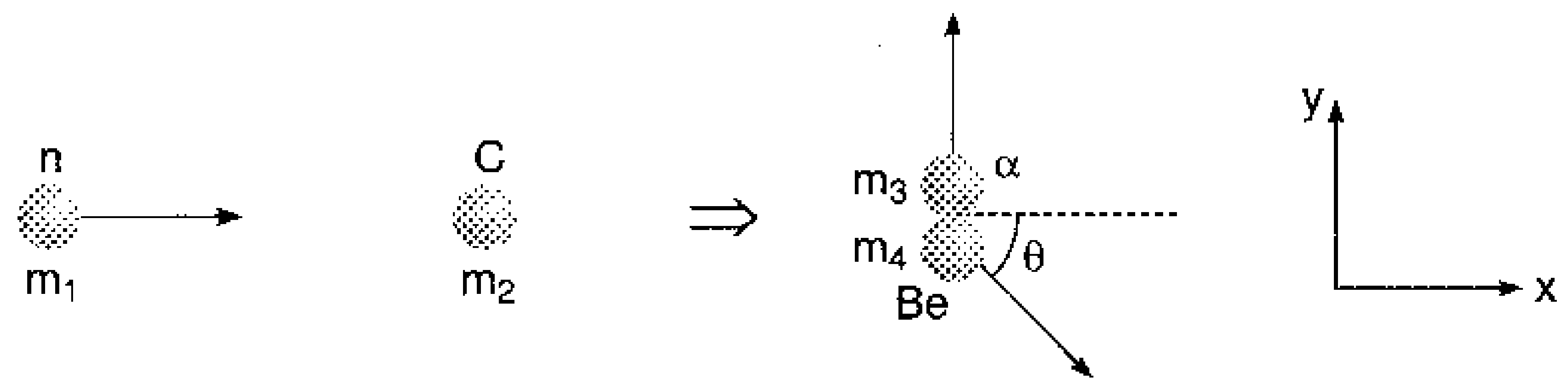
From conservation of energy

$$K_0 = Q + K_\alpha - K_H = (-1.18 + 7.7 - 5.5)\text{ MeV} = 1.02\text{ MeV}$$

$$\therefore \quad \cos\phi = -\left[\frac{m_0K_0 - m_\alpha K_\alpha - m_HK_H}{2\sqrt{m_\alpha m_H K_\alpha K_H}}\right] = 0.73$$

$$\therefore \quad \phi \approx 43°18'$$ **Ans.**

3.



$$Q + K_1 = K_3 + K_4 \qquad \text{...(i)}$$

Conservation of momentum along x-axis.

$$\sqrt{2m_1K_1} = \sqrt{2m_4K_4}\cos\theta \qquad \text{...(ii)}$$

Along y-axis,

$$\sqrt{2m_3K_3} = \sqrt{2m_4K_4}\sin\theta \qquad \text{...(iii)}$$

Squaring and adding Eqs. (ii) and (iii), $m_1K_1 + m_3K_3 = m_4K_4$

or

$$K_4 = \frac{m_1}{m_4}K_1 + \frac{m_3}{m_4}K_3$$

Substituting value of K_4 in Eq. (i), we get

$$Q + \left(1 - \frac{m_1}{m_4}\right)K_1 = \left(1 + \frac{m_3}{m_4}\right)K_3$$

Here,

$$Q = \frac{-E_{th}}{\left(1 + \frac{m_1}{m_2}\right)} = \frac{-6.17}{1 + \frac{1}{12}} = -5.69\text{ MeV}$$

Thus,

$$\left(1 + \frac{4}{9}\right)K_3 = -5.69 + \left(1 - \frac{1}{9}\right)(10)$$

or

$$K_3 = 2.21\text{ MeV}$$ **Ans.**

4.

$$N = \frac{10^{-3}}{210} \times 6.02 \times 10^{23} = 2.87 \times 10^{18}$$

During one mean life period 63.8% nuclei are decayed. Hence, energy released

$$E = 0.638 \times 2.87 \times 10^{18} \times 5.3 \times 1.6 \times 10^{-13}\text{ J} = 1.55 \times 10^6\text{ J}$$ **Ans.**

5.

$$\frac{R_1}{R_2} = \frac{\lambda_1 N}{\lambda_2 N} = \frac{1.2}{98.8} \quad \text{or} \quad \lambda_2 = 82.33\lambda_1 \qquad \text{...(i)}$$

Further,

$$\lambda = \frac{0.693}{21.8}\text{ year}^{-1} = 0.0318\text{ year}^{-1} \qquad \text{...(ii)}$$

Also,

$$\left(-\frac{dN}{dt}\right) = \left(-\frac{dN_1}{dt}\right) + \left(-\frac{dN_2}{dt}\right)$$

or

$$\lambda N = \lambda_1 N + \lambda_2 N \quad \text{or} \quad \lambda = \lambda_1 + \lambda_2 \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\lambda_1 = 3.81 \times 10^{-4} \text{ year}^{-1} \quad \text{and} \quad \lambda_2 = 3.14 \times 10^{-2} \text{ year}^{-1}$$ **Ans.**

6. $$R_0 = \lambda N = \frac{0.693}{14.3 \times 3600 \times 24} \times 6.02 \times 10^{23} \text{ per sec} = 3.37 \times 10^{17} \text{ per sec}$$

After 70 hours activity, $R = R_0 e^{-\lambda t} = (3.37 \times 10^{17}) e^{-(0.693/14.3 \times 24)(70)} = 2.92 \times 10^{17}$ per sec

In fruits activity was observed 1 μCi or 3.7×10^4 per sec. Therefore, percentage of activity transmitted from root to the fruit.

$$= \frac{3.7 \times 10^4}{2.92 \times 10^{17}} \times 100 = 1.26 \times 10^{-11} \%$$ **Ans.**

7. $$\Delta m = 2(\text{mass of } {}_1H^2) - (\text{mass of } {}_2He^4) = 0.0256\, u$$

$\therefore$ $$E = 0.0256 \times 931.5 \text{ MeV} = 23.85 \text{ MeV}$$

Total energy required per day $= 200 \times 10^6 \times 24 \times 3600 \text{ J} = 1.728 \times 10^{13}$ J

Let m be the mass of deuterium required. Then energy required for reactor.

$$= \left(\frac{25}{100}\right)\left(\frac{m/2}{2}\right)(6.02 \times 10^{23})(23.85 \times 1.6 \times 10^{-13})$$

This should be equal to 1.728×10^{13} J

$\therefore$ $$m = \frac{4 \times 100 \times 1.728 \times 10^{13}}{25 \times 6.02 \times 10^{23} \times 23.85 \times 1.6 \times 10^{-13}} \text{ g} = 120.35 \text{ g}$$ **Ans.**

8. See the hint of Q. No. 4.

9. (a) $$\Delta m = 0.00564 \text{ amu} \equiv 5.25 \text{ MeV} = 8.4 \times 10^{-13} \text{ J}$$

$$\lambda = \frac{0.693}{t_{1/2}} = 0.005 \text{ per day}$$

Let m g of Po^{210} are required per day for the reactor

$$n = \frac{(6.02 \times 10^{23})m}{210}$$

$$\left(-\frac{dN}{dt}\right) = \lambda N = \frac{0.005 \times 6.02 \times 10^{23} \times m}{210} \text{ per day}$$

So, Energy produced per day $= \dfrac{0.005 \times 6.02 \times 10^{23} \times m}{210} \times 8.4 \times 10^{-13} \text{ J} = (12 \times 10^6)m$ J

Now, 10% of $(12 \times 10^6)m$ should be equal to $2 \times 10^3 \times 24 \times 3600$ J.

Hence, $$m = \frac{2 \times 10^3 \times 24 \times 3600}{1.2 \times 10^6} = 144 \text{ g}$$ **Ans.**

(b) $$R = \lambda N = (0.005)\left(\frac{144}{210}\right)(6.02 \times 10^{23}) = 2.064 \times 10^{21} \text{ per day}$$

Now, $$R = R_0 \left(\frac{1}{2}\right)^{10} \qquad \left(n = \frac{1386}{138.6} = 10\right)$$

$\therefore$ $$R_0 = (2)^{10} R = 2.11 \times 10^{24} \text{ per day}$$ **Ans.**

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