Arihant Mechanics (Chapter 6, 7 and 8) Two by Two

Part 1 by 2

Understanding Physics for IIT JEE

by

D C Pandey

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Inspiring Minds

According to Current Test Pattern
at the Level of Class XI-XII

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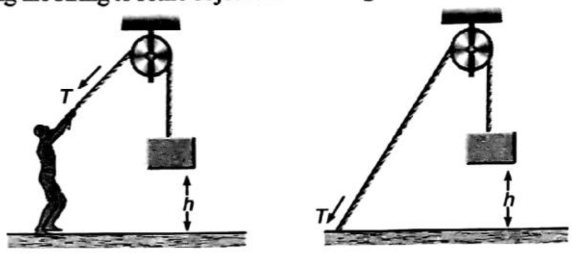
Work, Energy & Power

Chapter Contents

6.1 Introduction to Work

In our daily life 'work' has many different meanings. For example, Ram is working in a factory. The machine is in working order. Let us work out a plan for the next year, etc. In physics however, the term 'work' has a special meaning. In physics, work is always associated with a force and a displacement. We note that for work to be done, the force must act through a distance. Consider a person holding a weight a distance 'h' off the floor as shown in figure. In everyday usage, we might say that the man is doing a work, but in our scientific definition, no work is done by a force acting on a stationary object. We could eliminate the effort of holding the weight by merely tying the string to some object and the weight could be supported with no help from us.



No work is done by the man holding the weight at a fixed position. The same task could be accomplished by tying the rope to a fixed point.

Fig. 6.1

Let us now see what does 'work' mean in the language of physics.

6.2 Work Done

There are mainly three methods of finding work done.

(i) Work done by a constant force ($W = \vec{\mathbf{F}} \cdot \vec{\mathbf{S}} = FS \cos \theta$).

(ii) Work done by a variable force ($W = \int \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{S}}$).

(iii) Work done by area under F-S graph.

(i) Work done by a constant force

Let us first consider the simple case of a constant force $\vec{\mathbf{F}}$ acting on a body. Further, let us also assume that the body moves in a straight line; in the direction of force. In this case we define the work done by the force on the body as the product of the magnitude of the force $\vec{\mathbf{F}}$ and the distance S through which the body moves.

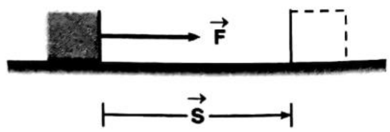


Fig. 6.2

That is, the work W is given by

$$W = F \cdot S$$

On the other hand, in a situation when the constant force does not act along the same direction as the displacement of the body, the component of force $\vec{\mathbf{F}}$ along the displacement $\vec{\mathbf{S}}$ is effective in doing work.

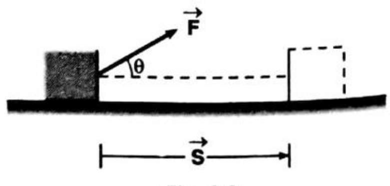


Fig. 6.3

Thus, in this case, work done by a constant force $\vec{\mathbf{F}}$ is given by

$$W = (\text{component of force along the displacement}) \times (\text{displacement})$$

or $$W = (F\cos\theta)(S)$$

or $$W = \vec{F}\cdot\vec{S}$$ (from the definition of dot product)

So, work done is a scalar or dot product of $\vec{F}$ and $\vec{S}$.

Regarding work it is worthnoting that :

1. Work can be positive, negative or even zero also, depending on the angle (θ) between the force vector $\vec{F}$ and displacement vector $\vec{S}$. Work done by a force is zero when $\theta = 90^\circ$, it is positive when $\theta < 90^\circ$ and negative when $\theta > 90^\circ$. For example, when a person lifts a body, the work done by the lifting force is positive (as $\theta = 0^\circ$) but work done by the force of gravity is negative (as $\theta = 180^\circ$). Similarly work done by centripetal force is always zero (as $\theta = 90^\circ$).
2. Work depends on frame of reference. With change of frame of reference inertial force does not change while displacement may change. So, the work done by a force will be different in different frames. For example, if a person is pushing a box inside a moving train, then work done as seen from the frame of reference of train is $\vec{F}\cdot\vec{S}$ while as seen from the ground it is $\vec{F}\cdot(\vec{S}+\vec{S}_0)$. Here $\vec{S}_0$, is the displacement of train relative to ground.
3. Suppose a body is displaced from point A to point B, then

$$\vec{S} = \vec{r}_B - \vec{r}_A$$
$$= (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$$

Sample Example 6.1 *A body is displaced from $A = (2\,m, 4\,m, -6\,m)$ to $\vec{r}_B = (6\hat{i} - 4\hat{j} + 4\hat{k})\,m$ under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} - \hat{k})\,N$. Find the work done.*

Solution $$\vec{r}_A = (2\hat{i} + 4\hat{j} - 6\hat{k})\,\text{m}$$

$\therefore$ $$\vec{S} = \vec{r}_B - \vec{r}_A$$
$$= (6\hat{i} - 4\hat{j} + 2\hat{k}) - (2\hat{i} + 4\hat{j} - 6\hat{k})$$
$$= 4\hat{i} - 8\hat{j} + 8\hat{k}$$

$$W = \vec{F}\cdot\vec{S} = (2\hat{i} + 3\hat{j} - \hat{k})\cdot(4\hat{i} - 8\hat{j} + 8\hat{k}) = 8 - 24 - 8 = -24\text{ J}$$ **Ans.**

Sample Example 6.2 *A block of mass $m = 2\,kg$ is pulled by a force $F = 40\,N$ upwards through a height $h = 2\,m$. Find the work done on the block by the applied force F and its weight mg. $(g = 10\,m/s^2)$*

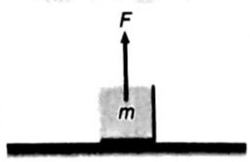


Fig. 6.4

Solution Weight $mg = (2)(10) = 20\,\text{N}$

Work done by the applied force $W_F = Fh\cos 0^\circ$

As the angle between force and displacement is 0°

or $$W_F = (40)(2)(1) = 80\text{ J}$$ **Ans.**

Similarly, work done by its weight

$$W_{mg} = (mg)(h)\cos 180^\circ$$

or $$W_{mg} = (20)(2)(-1) = -40\text{ J}$$ **Ans.**

Sample Example 6.3 *Two unequal masses of 1 kg and 2 kg are attached at the two ends of a light inextensible string passing over a smooth pulley as shown in figure. If the system is released from rest, find the work done by string on both the blocks in 1 s.*
(Take $g = 10\text{ m/s}^2$).

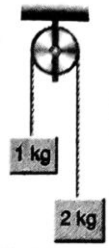


Fig. 6.5

Solution Net pulling force on the system is

$$F_{net} = 2g - 1g = 20 - 10 = 10\text{ N}$$

Total mass being pulled $$m = (1 + 2) = 3\text{ kg}$$

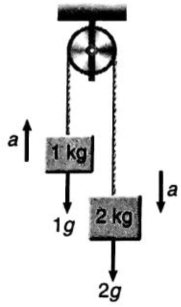


Fig. 6.6 (a)

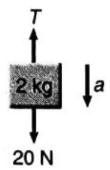


Fig. 6.6 (b)

Therefore, acceleration of the system will be

$$a = \frac{F_{net}}{m} = \frac{10}{3}\text{ m/s}^2$$

Displacement of both the blocks in 1 s is

$$S = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{10}{3}\right)(1)^2 = \frac{5}{3}\text{ m}$$

Free body diagram of 2 kg block is shown in Fig. 6.6 (b).

Using $\Sigma F = ma$, we get

$$20 - T = 2a = 2\left(\frac{10}{3}\right)$$

or $$T = 20 - \frac{20}{3} = \frac{40}{3} \text{ N}$$

$\therefore$ Work done by string (tension) on 1 kg block in 1 s is

$$W_1 = (T)(S)\cos 0°$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(1) = \frac{200}{9} \text{ J}$$ **Ans.**

Similarly, work done by string on 2 kg block in 1 s will be

$$W_2 = (T)(S)(\cos 180°)$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(-1) = -\frac{200}{9} \text{ J}$$ **Ans.**

(ii) Work done by a variable force

So far we have considered the work done by a force which is constant both in magnitude and direction. Let us now consider a force which acts always in one direction but whose magnitude may keep on varying. We can choose the direction of the force as x-axis. Further, let us assume that the magnitude of the force is also a function of x or say $F(x)$ is known to us. Now, we are interested in finding the work done by this force in moving a body from x_1 to x_2.

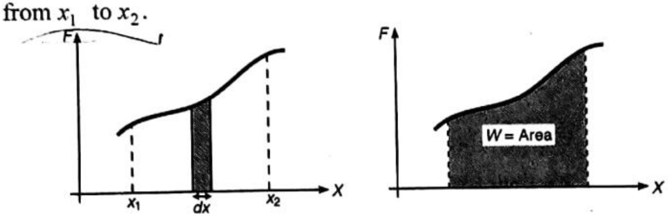


Fig. 6.7

Work done in a small displacement from x to $x + dx$ will be

$$dW = F \cdot dx$$

Now, the total work can be obtained by integration of the above elemental work from x_1 to x_2 or

$$W = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} F \cdot dx$$

It is important to note that $\int_{x_1}^{x_2} F\,dx$ is also the area under F-x graph between $x = x_1$ to $x = x_2$.

Spring Force

An important example of the above idea is a spring that obeys Hooke's law. Consider the situation shown in figure. One end of a spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x = 0$ denote the position of the block when the spring is in its natural length. When the block is displaced by an amount x (either compressed or elongated) a restoring force (F) is applied by the spring on the block. The direction of this force F is always towards its mean position ($x = 0$) and the magnitude is directly proportional to x or

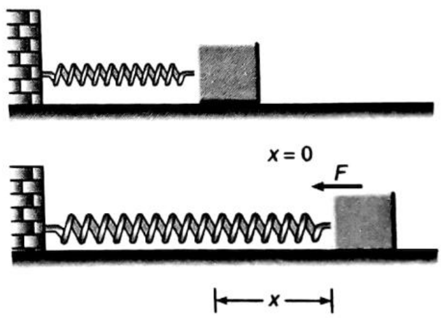


Fig. 6.8

$$F \propto x \quad \text{(Hooke's law)}$$

$$\therefore \quad F = -kx \quad \ldots\text{(i)}$$

Here, k is a constant called force constant of spring and depends on the nature of spring. From Eq. (i) we see that F is a variable force and F-x graph is a straight line passing through origin with slope $= -k$. Negative sign in Eq. (i) implies that the spring force F is directed in a direction opposite to the displacement x of the block.

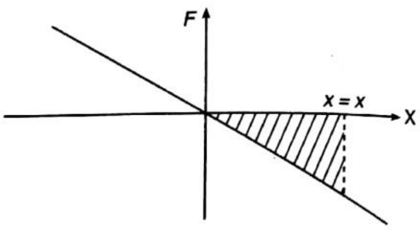


Fig. 6.9

Let us now find the work done by this force F when the block is displaced from $x = 0$ to $x = x$. This can be obtained either by integration or the area under F-x graph.

Thus, $$W = \int dW = \int_0^x F dx = \int_0^x -kx\, dx = -\frac{1}{2} kx^2$$

Here, work done is negative because force is in opposite direction of displacement.

Similarly, if the block moves from $x = x_1$ to $x = x_2$. The limits of integration are x_1 and x_2 and the work done is

$$W = \int_{x_1}^{x_2} -kx\, dx = \frac{1}{2} k (x_1^2 - x_2^2)$$

Sample Example 6.4 *A force $F = (2 + x)$ acts on a particle in x-direction where F is in newton and x in metre. Find the work done by this force during a displacement from $x = 1.0$ m to $x = 2.0$ m*

Solution As the force is variable, we shall find the work done in a small displacement from x to $x + dx$ and then integrate it to find the total work. The work done in this small displacement is

$$dW = F\, dx = (2 + x)\, dx$$

Thus, $$W = \int_{1.0}^{2.0} dW = \int_{1.0}^{2.0} (2 + x)\, dx$$

$$= \left[2x + \frac{x^2}{2}\right]_{1.0}^{2.0} = 3.5 \text{ J}$$ **Ans.**

Sample Example 6.5 *A force $F = -\frac{k}{x^2}$ $(x \neq 0)$ acts on a particle in x-direction. Find the work done by this force in displacing the particle from. $x = +a$ to $x = +2a$. Here, k is a positive constant.*

Solution $$W = \int F\, dx = \int_{+a}^{+2a} \left(\frac{-k}{x^2}\right) dx = \left[\frac{k}{x}\right]_{+a}^{+2a} = -\frac{k}{2a}$$ **Ans.**

Note *It is important to note that work comes out to be negative which is quite obvious as the force acting on the particle is in negative x-direction $\left(F = -\frac{k}{x^2}\right)$ while displacement is along positive x-direction. (from $x = a$ to $x = 2a$)*

(iii) Work done by area under *F-S* or *F-x* graph

This is applicable in one dimensional motion. When force and displacement are either parallel or antiparallel. Care has to be taken in the signs of F and x (force and displacement). If both have same signs work done will be + (Area) and if both have opposite signs, work done is – (Area). Let us take the following example.

Sample Example 6.6 *A force F acting on a particle varies with the position x as shown in figure. Find the work done by this force in displacing the particle from*

(a) $x = -2\,m$ *to* $x = 0$

(b) $x = 0$ *to* $x = 2\,m$

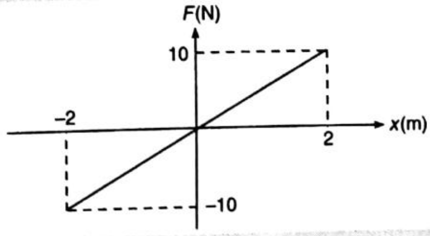


Fig. 6.10

Solution (a) From $x = -2\,\text{m}$ to $x = 0$, displacement of the particle is along positive x-direction while force acting on the particle is along negative x-direction. Therefore, work done is negative and given by the area under F-x graph.

$$\therefore \qquad W = -\frac{1}{2}(2)(10) = -10\ \text{J} \qquad \textbf{Ans.}$$

(b) From $x = 0$ to $x = 2$ m, displacement of particle and force acting on the particle both are along positive x-direction. Therefore, work done is positive and given by the area under F-x graph, or

$$W = \frac{1}{2}(2)(10) = 10\ \text{J} \qquad \textbf{Ans.}$$

6.3 Conservative & Non-Conservative Force Field

In the above article we considered the forces which were although variable but always directed in one direction. However, the most general expression for work done is

$$dW = \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} \quad \text{and} \quad W = \int_{\vec{r}_i}^{\vec{r}_f} dW = \int_{\vec{r}_i}^{\vec{r}_f} \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$$

Here, $\vec{\mathbf{dr}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$

$\vec{\mathbf{r}}_i$ = initial position vector and $\vec{\mathbf{r}}_f$ = final position vector

Conservative and non-conservative forces can be better understood after going through the following two examples.

Sample Example 6.7 *An object is displaced from point A(2 m, 3 m, 4 m) to a point B (1 m, 2 m, 3 m) under a constant force* $\vec{\mathbf{F}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ *N. Find the work done by this force in this process.*

Solution

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = \int_{(2\,\text{m},\,3\,\text{m},\,4\,\text{m})}^{(1\,\text{m},\,2\,\text{m},\,3\,\text{m})} (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}})$$

$$= [2x + 3y + 4z]_{(2\,\text{m},\,3\,\text{m},\,4\,\text{m})}^{(1\,\text{m},\,2\,\text{m},\,3\,\text{m})} = -9\ \text{J} \qquad \textbf{Ans.}$$

Alternate Solution

Since, $\vec{F}$ = constant, we can also use.

$$W = \vec{F} \cdot \vec{S}$$

Here,
$$\vec{S} = \vec{r}_f - \vec{r}_i = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})$$
$$= (-\hat{i} - \hat{j} - \hat{k})$$

$\therefore$
$$W = (2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (-\hat{i} - \hat{j} - \hat{k})$$
$$= -2 - 3 - 4 = -9\text{J}$$

Ans.

Sample Example 6.8 *An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})$ m to $\vec{r}_2 = (4\hat{i} + 6\hat{j})$ m under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})$ N. Find the work done by this force.*

Solution
$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\, dx + 2y\, dy) = [x^3 + y^2]_{(2,\,3)}^{(4,\,6)}$$
$$= 83 \text{ J}$$

Ans.

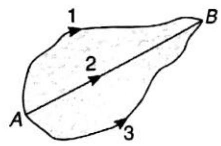


Fig. 6.11

In the above two examples, we saw that while calculating the work done we did not mention the path through which the object was displaced. Only initial and final coordinates were required. It shows that in both the examples, the work done is path independent or work done will be equal on whichever path we follow. Such forces in which work is path independent are known as **conservative forces.**

Thus, if a particle or an object is displaced from position A to position B through three different paths under a conservative force field. Then

$$W_1 = W_2 = W_3$$

Further, it can be shown that work done in a closed path is zero under a conservative force field. ($W_{AB} = -W_{BA}$ or $W_{AB} + W_{BA} = 0$). Gravitational force, Coulomb's force are few examples of conservative forces. On the other hand, if the work is path dependent or $W_1 \neq W_2 \neq W_3$, the force is called a **non-conservative.** Frictional forces, viscous forces are non-conservative in nature. Work done in a closed path is not zero in a non-conservative force field.

Introductory Exercise 6.1

1. A block is pulled a distance x along a rough horizontal table by a horizontal string. If the tension in the string is T, the weight of the block is W, the normal reaction is N and frictional force is F. Write down expressions for the work done by each of these forces.
2. A particle is pulled a distance l up a rough plane inclined at an angle α to the horizontal by a string inclined at an angle β to the plane ($\alpha + \beta < 90°$). If the tension in the string is T, the normal reaction between the particle and the plane is N, the frictional force is F and the weight of the particle is W. Write down expressions for the work done by each of these forces.

3. A bucket tied to a string is lowered at a constant acceleration of $g/4$. If the mass of the bucket is m and is lowered by a distance l then find the work done by the string on the bucket.

4. A 1.8 kg block is moved at constant speed over a surface for which coefficient of friction $\mu = \frac{1}{4}$. It is pulled by a force F acting at 45° with horizontal as shown in figure. The block is displaced by 2 m. Find the work done on the block by (a) the force F (b) friction (c) gravity.

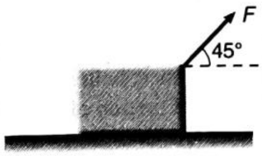


Fig. 6.12

5. A small block of mass 1 kg is kept on a rough inclined wedge of inclination 45° fixed in an elevator. The elevator goes up with a uniform velocity $v = 2$ m/s and the block does not slide on the wedge. Find the work done by the force of friction on the block in 1 s. ($g = 10$ m/s^2)
6. Two equal masses are attached to the two ends of a spring of force constant k. The masses are pulled out symmetrically to stretch the spring by a length $2x_0$ over its natural length. Find the work done by the spring on each mass.
7. Force acting on a particle varies with displacement as shown in figure. Find the work done by this force on the particle from $x = -4$ m to $x = +4$ m.

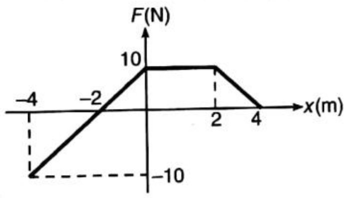


Fig. 6.13

6.4 Kinetic Energy

Kinetic energy (KE) is the capacity of a body to do work by virtue of its motion. If a body of mass m has a velocity v its kinetic energy is equivalent to the work which an external force would have to do to bring the body from rest upto its velocity v. The numerical value of the kinetic energy can be calculated from the formula.

$$\boxed{\text{KE} = \frac{1}{2}mv^2}$$

This can be derived as follows:

Consider a constant force F which acting on a mass m initially at rest, gives the mass a velocity v. If in reaching this velocity, the particle has been moving with an acceleration a and has been given a displacement s, then

$$F = ma \qquad \text{(Newton's law)}$$

$$v^2 = 2as$$

Work done by the constant force $= Fs$

or
$$W = (ma)\left(\frac{v^2}{2a}\right) = \frac{1}{2}mv^2$$

But the kinetic energy of the body is equivalent to the work done in giving the body this velocity.

Hence, $$KE = \frac{1}{2} mv^2$$

Regarding the kinetic energy the following two points are important to note.

1. Since, both m and v^2 are always positive. KE is always positive and does not depend on the direction of motion of the body.
2. Kinetic energy depends on the frame of reference. For example, the kinetic energy of a person of mass m sitting in a train moving with speed v is zero in the frame of train but $\frac{1}{2} mv^2$ in the frame of earth.

6.5 Work Energy Theorem

This theorem is a very important tool that relates the works to kinetic energy. According to this theorem:

Work done by all the forces (conservative or nonconservative, external or internal) acting on a particle or an object is equal to the change in kinetic energy of it.

$$\therefore \qquad W_{net} = \Delta KE = K_f - K_i$$

Let, $\vec{F}_1, \vec{F}_2 \ldots$ be the individual forces acting on a particle. The resultant force is $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots$ and the work done by the resultant force is

$$W = \int \vec{F} \cdot \vec{dr} = \int (\vec{F}_1 + \vec{F}_2 + \ldots) \cdot \vec{dr}$$

$$= \int \vec{F}_1 \cdot \vec{dr} + \int \vec{F}_2 \cdot \vec{dr} + \ldots$$

where $\int \vec{F}_1 \cdot \vec{dr}$ is the work done on the particle by $\vec{F}_1$ and so on. Thus, work energy theorem can also be written as: work done by the resultant force which is also equal to the sum of the work done by the individual forces is equal to change in kinetic energy.

Regarding the work-energy theorem it is worthnoting that:

(1) If W_{net} is positive then $K_f - K_i$ = positive,

i.e., $K_f > K_i$ or kinetic energy will increase and *vice-versa*.

(2) This theorem can be applied to non-inertial frames also. In a non-inertial frame it can be written as: work done by all the forces (including the pseudo forces) = change in kinetic energy in non-inertial frame. Let us take an example.

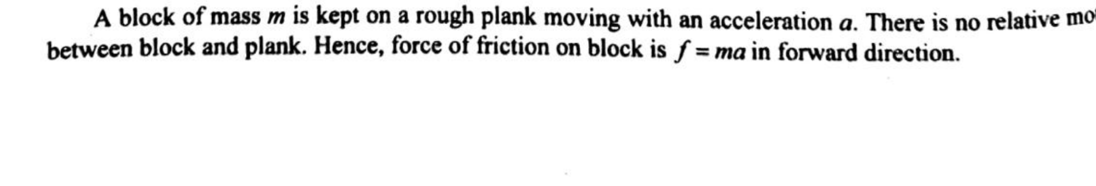


Fig. 6.14

Refer figure (a)

A block of mass m is kept on a rough plank moving with an acceleration a. There is no relative motion between block and plank. Hence, force of friction on block is $f = ma$ in forward direction.

Refer figure (b)

Horizontal forces on the block has been shown from ground (inertial) frame of reference.

If the plank moves a distance s on the ground the block will also move the same distance s as there is no slipping between the two. Hence, work done by friction on the block (w.r.t. ground) is

$$W_f = fs = mas$$

From work-energy principle if v is the speed of block (w.r.t. ground).

$$KE = W_f$$

or $$\frac{1}{2}mv^2 = mas \quad \text{or} \quad v = \sqrt{2as}$$

Thus velocity of block relative to ground is $\sqrt{2as}$.

Refer figure (c)

Free body diagram of the block has been shown from accelerating frame (plank).

Here, $$f_p = \text{pseudo force} = ma$$

Work done by all the forces,

$$W = W_f + W_{fp} = mas - mas = 0$$

From work-energy theorem,

$$\frac{1}{2}mv_r^2 = W = 0 \quad \text{or} \quad v_r = 0$$

Thus velocity of block relative to plank is zero.

Sample Example 6.9 *An object of mass m is tied to a string of length l and a variable force F is applied on it which brings the string gradually at angle θ with the vertical. Find the work done by the force F.*

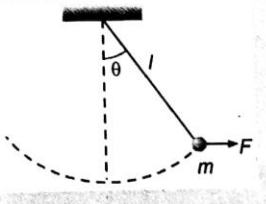


Fig. 6.15

Solution In this case three forces are acting on the object:

1. tension (T)
2. weight (mg) and
3. applied force (F)

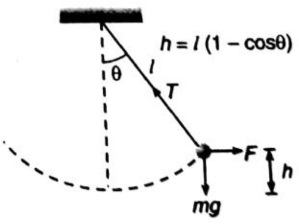


Fig. 6.16

Using work-energy theorem

$$W_{net} = \Delta KE$$

or $$W_T + W_{mg} + W_F = 0 \qquad \text{...(i)}$$

as $$\Delta KE = 0$$

because $$K_i = K_f = 0$$

Further, $W_T = 0$, as tension is always perpendicular to displacement.

$$W_{mg} = -mgh \quad \text{or} \quad W_{mg} = -mgl\,(1 - \cos\theta)$$

Substituting these values in Eq. (i), we get

$$W_F = mgl\,(1 - \cos\theta)$$ **Ans.**

Note *Here, the applied force F is variable. So, if we do not apply the work energy theorem we will first find the magnitude of F at different locations and then integrate dW* ($= \vec{F} \cdot \vec{dr}$) *with proper limits.*

Sample Example 6.10 *A body of mass m was slowly hauled up the hill as shown in the figure by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.*

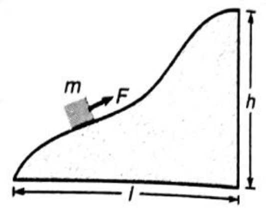


Fig. 6.17

Solution Four forces are acting on the body:

1. weight (mg)
2. normal reaction (N)
3. friction (f) and
4. the applied force (F)

Using work-energy theorem

$$W_{\text{net}} = \Delta KE$$

or $$W_{mg} + W_N + W_f + W_F = 0 \qquad \text{...(i)}$$

Here, $\Delta KE = 0$, because $K_i = 0 = K_f$

$$W_{mg} = -mgh$$

$$W_N = 0$$

(as normal reaction is perpendicular to displacement at all points)

W_f can be calculated as under:

$$f = \mu\, mg \cos\theta$$

∴ $$(dW_{AB})_f = -f\,ds$$

$$= -(\mu\, mg \cos\theta)\,ds$$

$$= -\mu\, mg\,(dl) \qquad (\text{as } ds\cos\theta = dl)$$

∴ $$W_f = -\mu\, mg\, \Sigma\, dl$$

$$= -\mu\, mgl$$

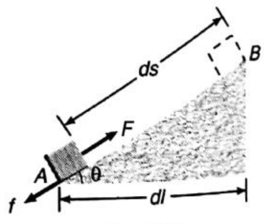


Fig. 6.18

Substituting these values in Eq. (i), we get

$$W_F = mgh + \mu mgl$$ **Ans.**

Note *Here again, if we want to solve this problem without using work-energy theorem we will first find magnitude of applied force* $\vec{F}$ *at different locations and then integrate dW* ($= \vec{F} \cdot \vec{dr}$) *with proper limits.*

Introductory Exercise 6.2

1. Velocity-time graph of a particle of mass 2 kg moving in a straight line is as shown in figure. Find the work done by all the forces acting on the particle.

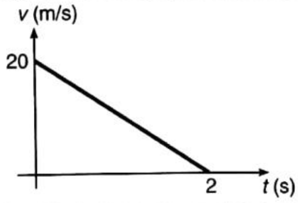


Fig. 6.19

2. Is work-energy theorem valid in a non-inertial frame?
3. A particle of mass m moves on a straight line with its velocity varying with the distance travelled according to the equation $v = \alpha\sqrt{x}$, where α is a constant. Find the total work done by all the forces during a displacement from $x = 0$ to $x = b$.
4. A 5 kg mass is raised a distance of 4 m by a vertical force of 80 N. Find the final kinetic energy of the mass if it was originally at rest. $g = 10 \text{ m/s}^2$.
5. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration $a = g$. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of angle θ as it slides down.
6. An object of mass m has a speed v_0 as it passes through the origin on its way out along the $+x$ axis. It is subjected to a retarding force given by $F_x = -Ax$. Here, A is a positive constant. Find its x-coordinate when it stops.
7. A block of mass M is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force F. The kinetic energy of the block increases by 40J in 1s. State whether the following statements are true or false:
   (a) The tension in the string is Mg
   (b) The work done by the tension on the block is 40J
   (c) The tension in the string is F
   (d) The work done by the force of gravity is 40J in the above 1s

6.6 Potential Energy

The energy possessed by a body or system by virtue of its position or configuration is known as the potential energy. For example, a block attached to a compressed or elongated spring possesses some energy called elastic potential energy. This block has a capacity to do work. Similarly, a stone when released from a certain height also has energy in the form of gravitational potential energy. Two charged particles kept at certain distance has electric potential energy.

Regarding the potential energy it is important to note that it is defined for a conservative force field only. For non-conservative forces it has no meaning. The change in potential energy (dU) of a system corresponding to a conservative internal force is given by

$$dU = -\vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = -dW \qquad \left(F = -\frac{dU}{dr}\right)$$

or $$\int_i^f dU = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

or $$U_f - U_i = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, *i.e.*, if we take $r_i = \infty$ (infinite) and $U_i = 0$ then we can write

$$U = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{r} = -W$$

or potential energy of a body or system is the negative of work done **by the conservative forces** in bringing it from infinity to the present position.

Regarding the potential energy it is worth noting that:

1. Potential energy can be defined only for conservative forces and it should be considered to be a property of the entire system rather than assigning it to any specific particle.
2. Potential energy depends on frame of reference.

Now, let us discuss three types of potential energies which we usually come across.

(a) Elastic Potential Energy

In Article 6.2, we have discussed the spring forces. We have seen there that the work done by the spring force (of course conservative for an ideal spring) is $-\frac{1}{2}kx^2$ when the spring is stretched or compressed by an amount x from its unstretched position. Thus,

$$U = -W = -\left(-\frac{1}{2}kx^2\right)$$

or $$U = \frac{1}{2}kx^2 \qquad (k = \text{spring constant})$$

Note that elastic potential energy is always positive.

(b) Gravitational Potential Energy

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by

$$U = -G\frac{m_1 m_2}{r}$$

Here, G = universal gravitation constant

$$= 6.67 \times 10^{-11} \frac{\text{N-m}^2}{\text{kg}^2}$$

If a body of mass m is raised to a height h from the surface of earth, the change in potential energy of the system (earth + body) comes out to be:

$$\Delta U = \frac{mgh}{\left(1 + \frac{h}{R}\right)} \qquad (R = \text{radius of earth})$$

or $$\Delta U \approx mgh \quad \text{if} \quad h << R$$

Thus, the potential energy of a body at height h, *i.e.*, mgh is really the change in potential energy of the system for $h << R$. So, be careful while using $U = mgh$, that h should not be too large. This we will discuss in detail in the chapter of Gravitation.

(c) Electric Potential Energy

The electric potential energy of two point charges q_1 and q_2 separated by a distance r in vacuum is given by

$$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r}$$

Here,
$$\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} = \text{constant}$$

6.7 Law of Conservation of Mechanical Energy

Suppose, only conservative forces operate on a system of particles and potential energy U is defined corresponding to these forces. There are either no other forces or the work done by them is zero. We have

$$U_f - U_i = -W$$

and
$$W = K_f - K_i \qquad \text{(from work energy theorem)}$$

then
$$U_f - U_i = -(K_f - K_i)$$

or
$$U_f + K_f = U_i + K_i \qquad \ldots\text{(i)}$$

The sum of the potential energy and the kinetic energy is called the total mechanical energy. We see from Eq. (i), that the **total mechanical energy of a system remains constant, if only conservative forces are acting on a system of particles and the work done by all other forces is zero.** This is called the conservation of mechanical energy.

The total mechanical energy is not constant, if non-conservative forces such as friction is acting between the parts of a system. However, the work energy theorem, is still valid. Thus, we can apply

$$W_c + W_{nc} + W_{\text{ext}} = K_f - K_i$$

Here,
$$W_c = -(U_f - U_i)$$

So, we get
$$W_{nc} + W_{\text{ext}} = (K_f + U_f) - (K_i + U_i)$$

or
$$W_{nc} + W_{\text{ext}} = E_f - E_i$$

Here, $E = K + U$ is the total mechanical energy.

• Problem Solving Technique

- If only conservative forces are acting on a system of particles and work done by any other external force is zero, then mechanical energy of the system will remain conserved. In this case some fraction of the mechanical energy will be decreasing while the other will be increasing. Problems can be solved by equating the magnitudes of the decrease and the increase. Let us see an example of this.
  In the arrangement shown in figure string is light and inextensible and friction is absent everywhere. Find the speed of both the blocks after the block A has ascended a height of 1 m. Given that $m_A = 1$ kg and $m_B = 2$ kg. ($g = 10$ m/s^2)

Fig. 6.20

Solution Friction is absent. Therefore, mechanical energy of the system will remain conserved. From constraint relations we see that speed of both the blocks will be same. Suppose it is v. Here gravitational potential energy of 2 kg block is decreasing while gravitational potential energy of 1 kg block is increasing. Similarly, kinetic energy of both the blocks is also increasing. So we can write:

Decrease in gravitational potential energy of 2 kg block = increase in gravitational potential energy of 1 kg block + increase in kinetic energy of 1 kg block + increase in kinetic energy of 2 kg block.

$\therefore \quad m_B gh = m_A gh + \frac{1}{2} m_A v^2 + \frac{1}{2} m_B v^2$

or $\quad (2)(10)(1) = (1)(10)(1) + \frac{1}{2}(1) v^2 + \frac{1}{2}(2) v^2$

or $\quad 20 = 10 + 0.5 v^2 + v^2$

or $\quad 1.5 v^2 = 10$

$\therefore \quad v^2 = 6.67 \text{ m}^2/\text{s}^2$

or $\quad v = 2.58$ m/s **Ans.**

- If some non-conservative forces such as friction are also acting on some parts of the system and work done by any other forces (excluding the conservative forces) is zero. Then we can apply

$$W_{nc} = E_f - E_i$$

or $\quad W_{nc} = (U_f - U_i) + (K_f - K_i) = \Delta U + \Delta K$

i.e., work done by non-conservative forces is equal to the change in mechanical (potential + kinetic) energy. But note that here all quantities are to be substituted with sign. Let us see an example of this.

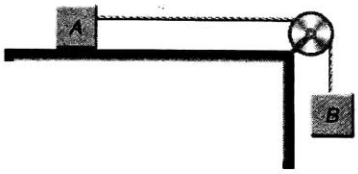


Fig. 6.21

In the arrangement shown in figure, $m_A = 1$ kg, $m_B = 4$ kg. String is light and inextensible while pulley is smooth. Coefficient of friction between block A and the table is $\mu = 0.2$. Find the speed of both the blocks when block B has descended a height $h = 1$ m. Take $g = 10 \text{ m/s}^2$.

Solution From constraint relation, we see that

$$v_A = v_B = v \text{ (say)}$$

Force of friction between block A and table will be

$$f = \mu m_A g = (0.2)(1)(10) = 2 \text{ N}$$

$\therefore \quad W_{nc} = \Delta U + \Delta K$

$\therefore \quad -fs = -m_B gh + \frac{1}{2}(m_A + m_B) v^2$

or $\quad (-2)(1) = -(4)(10)(1) + \frac{1}{2}(4 + 1) v^2$

$\quad -2 = -40 + 2.5 v^2$

or $\quad 2.5 v^2 = 38$

$\therefore \quad v^2 = 15.2 \text{ m}^2/\text{s}^2$

or $\quad v = 3.9$ m/s **Ans.**

Sample Example 6.11 *Consider the situation shown in figure. Mass of block A is m and that of block B is 2 m. The force constant of spring is K. Friction is absent everywhere. System is released from rest with the spring unstretched. Find :*

(a) the maximum extension of the spring x_m

(b) the speed of block A when the extension in the spring is $x = \frac{x_m}{2}$

(c) net acceleration of block B when extension in the spring is $x = \frac{x_m}{4}$

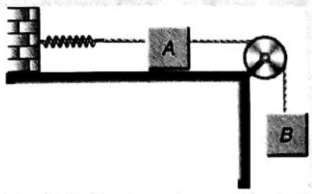


Fig. 6.22

Solution (a) At maximum extension in the spring

$$v_A = v_B = 0 \qquad \text{(momentarily)}$$

Therefore, applying conservation of mechanical energy:
decrease in gravitational potential energy of block B = increase in elastic potential energy of spring.

or
$$m_B g x_m = \frac{1}{2} K x_m^2$$

or
$$2\, mg x_m = \frac{1}{2} K x_m^2$$

$\therefore$
$$x_m = \frac{4\, mg}{K} \qquad \textbf{Ans.}$$

(b) At
$$x = \frac{x_m}{2} = \frac{2\, mg}{K}$$

Let
$$v_A = v_B = v \text{ (say)}$$

Then, decrease in gravitational potential energy of block B = increase in elastic potential energy of spring + increase in kinetic energy of both the blocks.

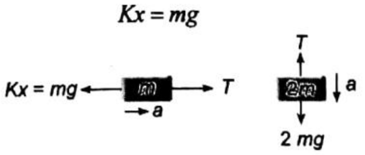
$\therefore$
$$m_B g x = \frac{1}{2} K x^2 + \frac{1}{2}(m_A + m_B) v^2$$

or
$$(2m)(g)\left(\frac{2\, mg}{K}\right) = \frac{1}{2} K \left(\frac{2\, mg}{K}\right)^2 + \frac{1}{2}(m + 2m) v^2$$

$\therefore$
$$v = 2g\sqrt{\frac{m}{3K}} \qquad \textbf{Ans.}$$

(c) At
$$x = \frac{x_m}{4} = \frac{mg}{K}$$

or
$$Kx = mg$$



$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{2\, mg - mg}{3\, m}$$

$$= \frac{g}{3} \text{ (downwards)} \qquad \textbf{Ans.}$$

Sample Example 6.12 *In the arrangement shown in figure $m_A = 4.0$ kg and $m_B = 1.0$ kg. The system is released from rest and block B is found to have a speed 0.3 m/s after it has descended through a distance of 1m. Find the coefficient of friction between the block and the table. Neglect friction elsewhere. (Take $g = 10$ m/s^2).*

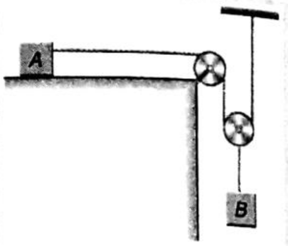


Fig. 6.23

Solution From constraint relations, we can see that

$$v_A = 2v_B$$

Therefore, $$v_A = 2(0.3) = 0.6 \text{ m/s}$$

as $$v_B = 0.3 \text{ m/s (given)}$$

Applying $$W_{nc} = \Delta U + \Delta K$$

we get $$-\mu\, m_A g S_A = -m_B g S_B + \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

Here, $$S_A = 2S_B = 2 \text{ m} \quad \text{as} \quad S_B = 1 \text{m (given)}$$

$\therefore$ $$-\mu(4.0)(10)(2) = -(1)(10)(1) + \frac{1}{2}(4)(0.6)^2 + \frac{1}{2}(1)(0.3)^2$$

or $$-80\mu = -10 + 0.72 + 0.045$$

or $$80\mu = 9.235 \quad \text{or} \quad \mu = 0.115$$ **Ans.**

Introductory Exercise 6.3

1. In the figure block *A* is released from rest when the spring is in its natural length. For the block *B* of mass *m* to leave contact with the ground at some stage what should be the minimum mass of block *A*?

Fig. 6.24

2. A chain of mass *m* and length *l* lies on a horizontal table. The chain is allowed to slide down gently from the side of the table. Find the speed of the chain at the instant when last link of the chain slides from the table. Neglect friction everywhere.

3. As shown in figure a smooth rod is mounted just above a table top. A 10 kg collar, which is able to slide on the rod with negligible friction is fastened to a spring whose other end is attached to a pivot at O. The spring has negligible mass, a relaxed length of 10 cm and a spring constant of 500 N/m. The collar is released from rest at point A. (a) What is its velocity as it passes point B ? (b) Repeat for point C.

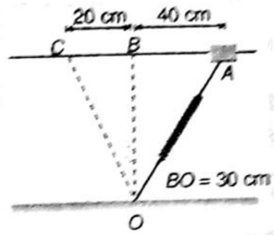


Fig. 6.25

4. A man pulls a bucket of water from a well of depth h. If mass of the rope and that of the bucket full of water are m and M respectively. Find the work done by the man.

5. A block of mass m is attached with a massless spring of force constant K. The block is placed over a rough inclined surface for which the coefficient of friction is $\mu = \frac{3}{4}$. Find the minimum value of M required to move the block up the plane. (Neglect mass of string and pulley. Ignore friction in pulley).

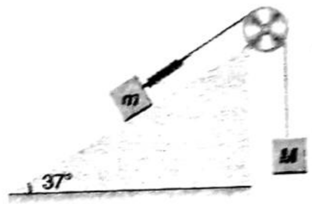


Fig. 6.26

6.8 Three Types of Equilibrium

A body is said to be in translatory equilibrium, if net force acting on the body is zero, *i.e.*,

$$\vec{F}_{net} = 0$$

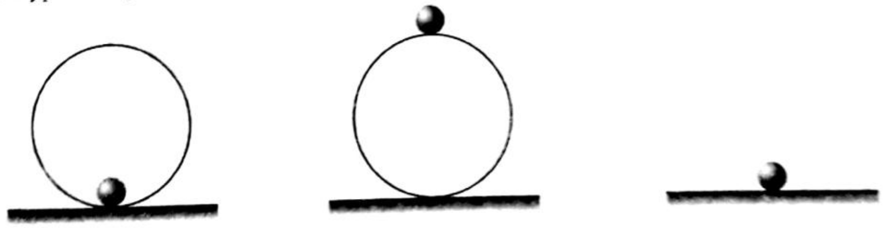
If the forces are conservative

$$F = -\frac{dU}{dr}$$

and for equilibrium $F = 0$.

So,
$$-\frac{dU}{dr} = 0, \quad \text{or} \quad \frac{dU}{dr} = 0$$

i.e., at equilibrium position slope of U-r graph is zero or the potential energy is optimum (maximum or minimum or constant). Equilibrium are of three types, *i.e.*, the situation where $F = 0$ and $\frac{dU}{dr} = 0$ can be obtained under three conditions. These are stable equilibrium, unstable equilibrium and neutral equilibrium. These three types of equilibrium can be better understood from the given three figures:



(a) (b) (c)

Fig. 6.27

Three identical balls are placed in equilibrium in positions as shown in figures (a), (b) and (c) respectively.

In Fig. (a) ball is placed inside a smooth spherical shell. This ball is in stable equilibrium position. In Fig. (b) the ball is placed over a smooth sphere. This is in unstable equilibrium position. In Fig. (c) the ball is placed on a smooth horizontal ground. This ball is in neutral equilibrium position.

The table given below explains what is the difference and what are the similarities between these three equilibrium positions in the language of physics.

Table 6.1

S. No.	Stable Equilibrium	Unstable Equilibrium	Neutral Equilibrium
1.	Net force is zero.	Net force is zero.	Net force is zero.
2.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.	$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.
3.	When displaced from its equilibrium position a net restoring force starts acting on the body which has a tendency to bring the body back to its equilibrium position.	When displaced from its equilibrium position, a net force starts acting on the body which moves the body in the direction of displacement or away from the equilibrium position.	When displaced from its equilibrium position the body has neither the tendency to come back nor to move away from the original position.
4.	Potential energy in equilibrium position is minimum as compared to its neighbouring points. or $\frac{d^2U}{dr^2} =$ positive	Potential energy in equilibrium position is maximum as compared to its neighbouring points. or $\frac{d^2U}{dr^2} =$ negative	Potential energy remains constant even if the body is displaced from its equilibrium position. or $\frac{d^2U}{dr^2} = 0$
5.	When displaced from equilibrium position the centre of gravity of the body goes up.	When displaced from equilibrium position the centre of gravity of the body comes down.	When displaced from equilibrium position the centre of gravity of the body remains at the same level.

• *Important Points Regarding* Equilibrium

- If we plot graphs between F and r or U and r, F will be zero at equilibrium while U will be maximum, minimum or constant depending on the type of equilibrium. This all is shown in Fig. 6.28

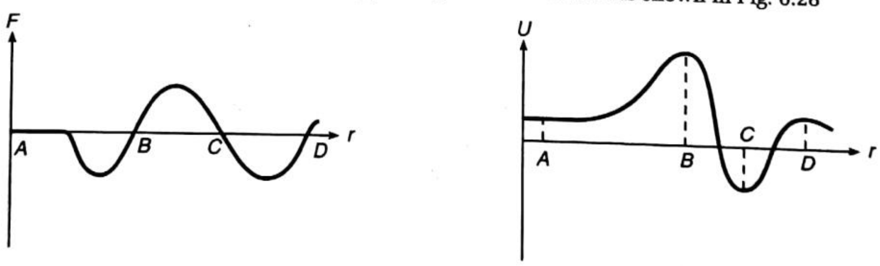
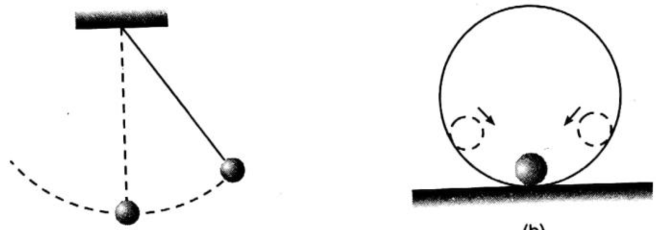


Fig. 6.28

At point A, $F = 0$, $\frac{dU}{dr} = 0$, but U is constant. Hence, A is neutral equilibrium position. At points B and D, $F = 0$, $\frac{dU}{dr} = 0$ but U is maximum. Thus, these are the points of unstable equilibrium.

At point C, $F = 0, \frac{dU}{dr} = 0$, but U is minimum. Hence, point C is in stable equilibrium position.



(a) (b)

Fig. 6.29

- Oscillations of a body take place about stable equilibrium position. For example, bob of a pendulum oscillates about its lowest point which is also the stable equilibrium position of bob. Similarly, in Fig. 6.27 (b), the ball will oscillate about its stable equilibrium position.
- If a graph between F and r is as shown in figure, then $F = 0$, at $r = r_1, r = r_2$ and $r = r_3$. Therefore, at these three points, body is in equilibrium. But these three positions are three different type of equilibriums. For example :

  at $r = r_1$, body is in unstable equilibrium. This is because, if we displace the body slightly rightwards (positive direction), force acting on the body is also positive, *i.e.*, away from $r = r_1$ position.

  at $r = r_2$, body is in stable equilibrium. Becaue if we displace the body rightwards (positive direction) force acting on the body is negative (or leftwards) or the force acting is restoring in nature.

  at $r = r_3$, equilibrium is neutral in nature. Because if we displace the body rightwards or leftwards force is again zero.

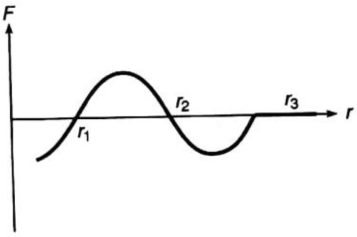


Fig. 6.30

Sample Example 6.13 *The potential energy of a conservative system is given by*

$$U = ax^2 - bx$$

where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

Solution In a conservative field

$$F = -\frac{dU}{dx}$$

$$\therefore \quad F = -\frac{d}{dx}(ax^2 - bx) = b - 2ax$$

For equilibrium $F = 0$

$$\text{or} \quad b - 2ax = 0 \quad \therefore \quad x = \frac{b}{2a}$$

From the given equation we can see that $\frac{d^2U}{dx^2} = 2a$ (positive), *i.e.*, U is minimum.

Therefore, $x = \frac{b}{2a}$ is the stable equilibrium position. **Ans.**

6.9 Power

Power is the rate at which a force does work. If a force does 20 J of work in 10 s, the average rate at which it is working is 2 J/s or the average power is 2 W.

The work done by a force $\vec{\mathbf{F}}$ in a small displacement $\vec{\mathbf{dr}}$ is $dW = \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$. Thus, the instantaneous power delivered by the force is

$$P = \frac{dW}{dt}$$

$$= \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{dr}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = Fv \cos \theta$$

Thus, power is equal to the scalar product of force with velocity. It is zero if force is perpendicular to velocity. For example, power of a centripetal force in a circular motion is zero.

Sample Example 6.14 *A train has a constant speed of* 40 *m/s on a level road against resistive force of magnitude* 3×10^4 *N. Find the power of the engine.*

Solution At constant speed, there is no acceleration, so the forces acting on the train are in equilibrium.

Therefore, $F = R$

$\therefore$ $F = 3 \times 10^4$ N

or $P = Fv$

We have, power $= 3 \times 10^4 \times 40 = 1.2 \times 10^6$ W **Ans.**

Sample Example 6.15 *A train of mass* 2.0×10^5 *kg has a constant speed of* 20 *m/s up a hill inclined at* $\theta = \sin^{-1}\left(\frac{1}{50}\right)$ *to the horizontal when the engine is working at* 8.0×10^5 *W. Find the resistance to motion of the train.* ($g = 9.8$ *m/s*^{2})

Solution Since, $P = Fv$

$$F = \frac{P}{v} = \frac{8.0 \times 10^5}{20} = 4.0 \times 10^4 \text{ N}$$

At constant speed, the forces acting on the train are in equilibrium. Resolving the forces parallel to the hill.

$$F = R + (2.0 \times 10^5)g \times \frac{1}{50}$$

$$4.0 \times 10^4 = R + 39200 \quad \text{or} \quad R = 800 \text{ N}$$

Therefore, the resistance is 800 N. **Ans.**

Sample Example Instance 6.16 *A block of mass m is pulled by a constant power P placed on a rough horizontal plane. The friction co-efficient between the block and surface is* μ. *Find the maximum velocity of the block.*

Solution Power $P = F \cdot v = \text{constant}$

$$F = \frac{P}{v} \quad \text{or} \quad F \propto \frac{1}{v}$$

as v increases, F decreases.

when $F = \mu mg$, net force on block becomes zero, *i.e.*, it has maximum or terminal velocity

$$\therefore \quad P = (\mu mg)\, v_{max}$$

or $$v_{max} = \frac{P}{\mu mg}$$ **Ans.**

Introductory Exercise 6.4

1. A ball of mass 1 kg is dropped from a tower. Find power of gravitational force at time $t = 2$ s. ($g = 10$ m/s^2)
2. A particle of mass m is lying on smooth horizontal table. A constant force F tangential to the surface is applied on it. Find :
   (a) average power over a time interval from $t = 0$ to $t = t$,
   (b) instantaneous power as function of time t.
3. A constant power P is applied on a particle of mass m. Find kinetic energy, velocity and displacement of particle as function of time t.
4. A time varying power $P = 2t$ is applied on a particle of mass m. Find :
   (a) kinetic energy and velocity of particle as function of time,
   (b) average power over a time interval from $t = 0$ to $t = t$.
5. Potential energy of a particle along x-axis, varies as, $U = -20 + (x-2)^2$, where U is in joule and x in meter. Find the equilibrium position and state whether it is stable or unstable equilibrium.
6. Force acting on a particle constrained to move along x-axis is $F = (x-4)$. Here, F is in newton and x in meter. Find the equilibrium position and state whether it is stable or unstable equilibrium.

Extra Points

- Work is a scalar quantity. It can be positive, negative or zero. The angle between $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$ decides whether the work done is positive, negative or zero.

  If $0° \le \theta < 90°$, W = positive

  If $\theta = 90°$, $W = 0$ and

  If $90° < \theta \le 180°$, W = negative
- The CGS unit of work is erg.

  $$1\ \text{J} = 10^7\ \text{erg}$$
- Power is also measured in horse power (HP).

  $$1\ \text{HP} = 746\ \text{watt}$$
- If only conservative forces are acting on a system, the mechanical energy of the system remains constant. Mechanical energy comprises of kinetic and potential. In potential usually gravitational and elastic comes in the question (as far as problems of work, power and energy are concerned). Now it may happen that some part of the energy might be decreasing while other part might be increasing. Energy conservation equation now can be written in two ways.

  First method : Magnitude of decrease of energy = magnitude of increase of energy.

Second method : $E_i = E_f$ ($i \to$ initial and $f \to$ final)

i.e., write down total initial mechanical energy on one side and total final mechanical energy on the other side. While writing gravitational potential energy we choose some reference point (where $h = 0$), but throughout the question this reference point should not change. Let us take a simple example.

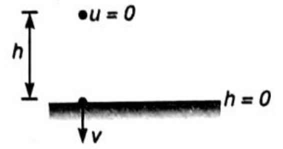


Fig. 6.31

A ball of mass 'm' is released from a height h as shown in figure. The velocity of particle at the instant when it strikes the ground can be found using energy conservation principle by following two methods.

Method 1 : Decrease in gravitational PE = increase in KE

or $$mgh = \frac{1}{2}mv^2$$

or $$v = \sqrt{2gh}$$

Method 2: $(PE + KE)_i = (PE + KE)_f$

For gravitational PE we take ground as the reference point.

$\therefore$ $$mgh + 0 = 0 + \frac{1}{2}mv^2$$

or $$v = \sqrt{2gh}$$

- If the system consists of frictional forces as well. Then, some mechanical energy will be lost in doing work against friction or

$$E_f < E_i$$

Now, suppose work done by friction is asked in the question, then find $E_f - E_i$ and if work done against friction is asked then write down $E_i - E_f$.

- Change in potential energy is equal to the negative of work done by the conservative force ($\Delta U = -\Delta W$). If work done by the conservative force is negative change in potential energy will be positive or potential energy of the system will increase and *vice-versa*.

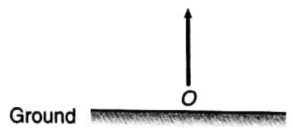


Fig. 6.32

This can be understood by a simple example. Suppose a ball is taken from the ground to some height, work done by gravity is negative, *i.e.*, change in potential energy should increase or potential energy of the ball will increase. Which happens so,

$$\Delta W_{gravity} = -\text{ve}$$

$\therefore$ $$\Delta U = +\text{ve} \qquad (\Delta U = -\Delta W)$$

or $$U_f - U_i = +\text{ve}$$

- $F = -\dfrac{dU}{dr}$, *i.e.*, conservative forces always act in a direction where potential energy of the system is decreased. This can also be shown as in Fig 6.33.

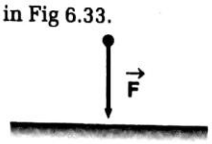


Fig. 6.33

If a ball is dropped from a certain height. The force on it (its weight) acts in a direction in which its potential energy decreases.

- Suppose a particle is released from point A with $u = 0$.

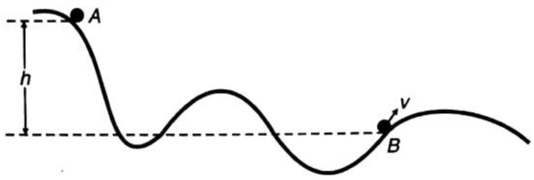


Fig. 6.34

Friction is absent everywhere. Then velocity at B will be

$$v = \sqrt{2gh}$$

(irrespective of the track it follows from A to B)

Here, $$h = h_A - h_B$$

- In circular motion, centripetal force acts towards centre. This force is perpendicular to small displacement $\vec{dS}$ and velocity $\vec{v}$. Hence, work done by it is zero and power of this force is also zero.
- For increase or decrease in gravitational potential energy of a particle (for small heights) we write,

$$\Delta U = mgh$$

Here, h is the change in height of particle. In case of a rigid body, h of centre of mass of the rigid body is seen.

Solved Examples

Level 1

Example 1 *An object of mass 5 kg falls from rest through a vertical distance of 20 m and attaches a velocity of 10 m/s. How much work is done by the resistance of the air on the object ? ($g = 10\ m/s^2$)*

Solution Applying work-energy theorem,

work done by all the forces = change in kinetic energy

or
$$W_{mg} + W_{air} = \frac{1}{2}mv^2$$

$\therefore$
$$W_{air} = \frac{1}{2}mv^2 - W_{mg}$$
$$= \frac{1}{2}mv^2 - mgh$$
$$= \frac{1}{2} \times 5 \times (10)^2 - (5) \times (10) \times (20)$$
$$= -750\ \text{J}$$

Ans.

Example 2 *A rod of length 1.0 m and mass 0.5 kg fixed at one end is initially hanging, vertical. The other end is now raised until it makes an angle 60° with the vertical. How much work is required ?*

Solution For increase in gravitational potential energy of a rod we see the centre of the rod.

$$W = \text{change in potential energy}$$
$$= mg\frac{l}{2}(1 - \cos\theta)$$

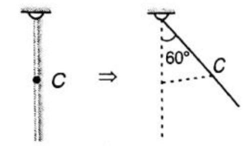


Fig. 6.35

Substituting the values, we have

$$W = (0.5)(9.8)\left(\frac{1.0}{2}\right)(1 - \cos 60°)$$
$$= 1.225\ \text{J}$$

Ans.

Example 3 *A smooth narrow tube in the form of an arc AB of a circle of centre O and radius r is fixed so that A is vertically above O and OB is horizontal. Particles P of mass m and Q of mass 2 m with a light inextensible string of length ($\pi r/2$) connecting them are placed inside the tube with P at A and Q at B and released from rest. Assuming the string remains taut during motion, find the speed of particles when P reaches B.*

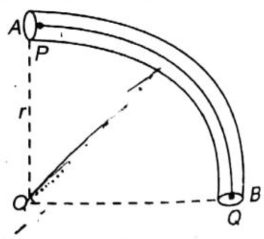


Fig. 6.36

Solution All surfaces are smooth. Therefore, mechanical energy of the system will remain conserved.

$\therefore$ Decrease in PE of both the blocks = increase in KE of both the blocks

$\therefore \qquad (mgr) + (2mg)\left(\frac{\pi r}{2}\right) = \frac{1}{2}(m + 2m)v^2$

or $\qquad v = \sqrt{\frac{2}{3}(1 + \pi)gr}$ **Ans.**

Example 4 *A small body of mass m is located on a horizontal plane at the point O. The body acquires a horizontal velocity v_0 due to friction. Find, the mean power developed by the friction force during the motion of the body, if the frictional coefficient $\mu = 0.27, m = 1.0\,kg$ and $v_0 = 1.5\,m/s$.*

Solution The body gains velocity due to friction. The acceleration due to friction.

$$a = \frac{\text{force of friction}}{\text{mass}} = \frac{\mu mg}{m} = \mu g$$

Further, $\qquad v_0 = at$

Therefore, $\qquad t = \frac{v_0}{a} = \frac{v_0}{\mu g} \qquad \ldots(i)$

From work energy theorem,

work done by force of friction = change in kinetic energy

or $\qquad W = \frac{1}{2}mv_0^2 \qquad \ldots(ii)$

$$\text{Mean power} = \frac{W}{t}$$

From Eqs. (i) and (ii)

$$P_{mean} = \frac{1}{2}\mu mgv_0$$

Substituting the values, we have

$$P_{mean} = \frac{1}{2} \times 0.27 \times 1.0 \times 9.8 \times 1.5$$

$$\approx 2.0\ \text{W}$$ **Ans.**

Example 5 *A small mass m starts from rest and slides down the smooth spherical surface of R. Assume zero potential energy at the top. Find :*

(a) the change in potential energy,

(b) the kinetic energy,

(c) the speed of the mass as a function of the angle θ made by the radius through the mass with the vertical.

Solution In the figure $h = R(1 - \cos\theta)$

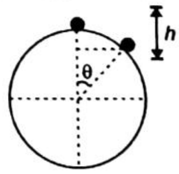


Fig. 6.37

(a) As the mass comes down, potential energy will decrease. Hence,

$$\Delta U = -mgh = -mgR(1 - \cos\theta)$$

(b) Magnitude of decrease in potential energy = increase in kinetic energy

$\therefore \qquad \text{Kinetic energy} = mgh$

$$= mgR(1 - \cos\theta)$$ **Ans.**

(c) $$\frac{1}{2}mv^2 = mgR\,(1-\cos\theta)$$

$\therefore$ $$v = \sqrt{2gR\,(1-\cos\theta)}$$ **Ans.**

Example 6 *The displacement x of a particle moving in one dimension, under the action of a constant force is related to time t by the equation*

$$t = \sqrt{x} + 3$$

where x is in metre and t in second. Calculate: (a) the displacement of the particle when its velocity is zero, (b) the work done by the force in the first 6 s.

Solution As $$t = \sqrt{x} + 3 \quad i.e., \quad x = (t-3)^2 \qquad \ldots\text{(i)}$$

So, $$v = (dx/dt) = 2(t-3) \qquad \ldots\text{(ii)}$$

(a) v will be zero when $2(t-3) = 0$ *i.e.*, $t = 3$

Substituting this value of t in Eq. (i), $x = (3-3)^2 = 0$

i.e., when velocity is zero, displacement is also zero. **Ans.**

(b) From Eq. (ii),

$$(v)_{t=0} = 2(0-3) = -6 \text{ m/s}$$

and $$(v)_{t=6} = 2(6-3) = 6 \text{ m/s}$$

So, from work-energy theorem

$$W = \Delta\text{KE} = \frac{1}{2}m[v_f^2 - v_i^2] = \frac{1}{2}m[6^2 - (-6)^2] = 0$$

i.e., work done by the force in the first 6 s is zero. **Ans.**

Level 2

Example 1 *A smooth track in the form of a quarter-circle of radius 6 m lies in the vertical plane. A ring of weight 4 N moves from P_1 and P_2 under the action of forces F_1, F_2 and F_3. Force F_1 is always towards. P_2 and is always 20 N in magnitude; force $\vec{F}_2$ always acts horizontally and is always 30 N in magnitude; force $\vec{F}_3$ always acts tangentially to the track and is of magnitude $(15 - 10\,s)$ N, where s is in metre. If the particle has speed 4 m/s at P_1, what will its speed be at P_2?*

Solution The work done by $\vec{F}_1$ is

$$W_1 = \int_{P_1}^{P_2} F_1 \cos\theta \, ds$$

From figure; $$s = R\left(\frac{\pi}{2} - 2\theta\right)$$

or $$ds = (6\text{ m})\,d(-2\theta) = -12\,d\theta$$

and $$F_1 = 20$$

Hence, $$W_1 = -240\int_{\pi/4}^{0} \cos\theta \, d\theta$$

$$= 240 \sin\frac{\pi}{4} = 120\sqrt{2}\text{ J}$$

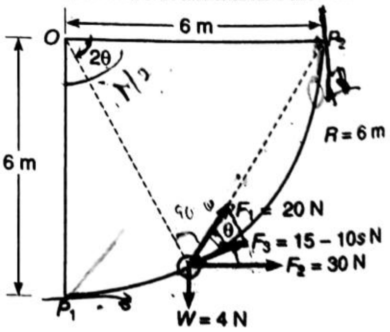


Fig. 6.38

The work done by $\vec{F}_3$ is

$$W_3 = \int F_3\, ds = \int_0^{6(\pi/2)} (15 - 10s)\, ds$$

$$= [15s - 5s^2]_0^{3\pi} = -302.8 \text{ J}$$

To calculate the work done by $\vec{F}_2$ and by W, it is convenient to take the projection of the path in the direction of the force, instead of *vice versa*. Thus,

$$W_2 = F_2(\overline{OP_2}) = 30(6) = 180 \text{ J}$$

$$W = (-W)(\overline{P_1O}) = (-4)(6) = -24 \text{ J}$$

The total work done is

$$W_1 + W_3 + W_2 + W = 23 \text{ J}$$

Then, by the work-energy principle.

$$K_{P_2} - K_{P_1} = 23 \text{ J}$$

$$= \frac{1}{2}\left(\frac{4}{9.8}\right)v_2^2 - \frac{1}{2}\left(\frac{4}{9.8}\right)(4)^2 = 23$$

$$v_2 = 11.3 \text{ m/s}$$ **Ans.**

Example 2 *One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially the spring makes an angle of 37° with the horizontal as shown in Fig. 6.39. When the system is released from rest, find the speed of the ring when the spring becomes horizontal.* [sin 37° = 3/5]

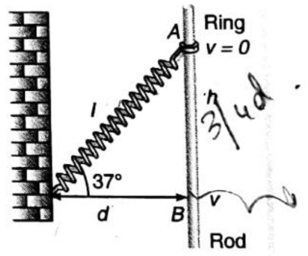


Fig. 6.39

Solution If l is the stretched length of the spring, then from figure

$$\frac{d}{l} = \cos 37° = \frac{4}{5}, \quad i.e., \quad l = \frac{5}{4}d$$

So, the stretch $$y = l - d = \frac{5}{4}d - d = \frac{d}{4}$$

and $$h = l \sin 37° = \frac{5}{4}d \times \frac{3}{5} = \frac{3}{4}d$$

Now, taking point B as reference level and applying law of conservation of mechanical energy between A and B,

$$E_A = E_B$$

or $$mgh + \frac{1}{2}ky^2 = \frac{1}{2}mv^2$$ [as for B, $h = 0$ and $y = 0$]

or $$\frac{3}{4}mgd + \frac{1}{2}k\left(\frac{d}{4}\right)^2 = \frac{1}{2}mv^2$$ [as for A, $h = \frac{3}{4}d$ and $y = \frac{1}{4}d$]

or $$v = d\sqrt{\frac{3g}{2d} + \frac{k}{16m}}$$ **Ans.**

Example 3 *A single conservative force $F(x)$ acts on a 1.0 kg particle that moves along the x-axis. The potential energy $U(x)$ is given by:*

$$U(x) = 20 + (x-2)^2$$

where x is in meters. At $x = 5.0$ m the particle has a kinetic energy of 20 J.

(a) What is the mechanical energy of the system?

(b) Make a plot of $U(x)$ as a function of x for $-10\,m \le x \le 10\,m$, and on the same graph draw the line that represents the mechanical energy of the system.
Use part (b) to determine.

(c) The least value of x and

(d) The greatest value of x between which the particle can move.

(e) The maximum kinetic energy of the particle and

(f) The value of x at which it occurs.

(g) Determine the equation for $F(x)$ as a function of x.

(h) For what (finite) value of x does $F(x) = 0$?

Solution (a) Potential energy at $x = 5.0$ m is

$$U = 20 + (5-2)^2 = 29\text{ J}$$

∴ Mechanical energy

$$E = K + U = 20 + 29 = 49\text{ J}$$

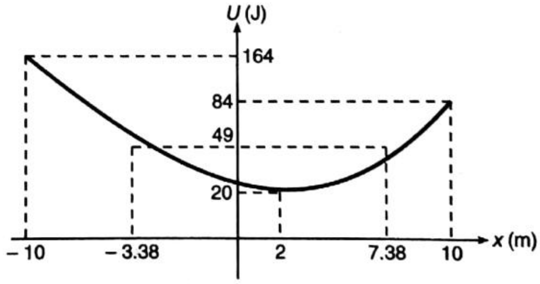


Fig. 6.40

(b) At $x = 10$m, $U = 84$ J at $x = -10$m, $U = 164$ J

and at $x = 2$m, U = minimum = 20 J

(c) and (d) : Particle will move between the points where its kinetic energy becomes zero or its potential energy is equal to its mechanical energy.

Thus, $$49 = 20 + (x-2)^2$$

or $$(x-2)^2 = 29$$

or $$x - 2 = \pm\sqrt{29} = \pm 5.38\text{ m}$$

∴ $$x = 7.38\text{ m and } -3.38\text{ m}$$

or the particle will move between $x = -3.38$ m and $x = 7.38$ m

(e) and (f) : Maximum kinetic energy is at $x = 2$ m, where the potential energy is minimum and this maximum kinetic energy is,

$$K_{max} = E - U_{min} = 49 - 20$$
$$= 29\text{ J}$$

(g) $$F = -\frac{dU}{dx} = -2(x-2) = (2-x)$$

(h) $$F(x) = 0, \quad \text{at} \quad x = 2.0\,\text{m}$$

where potential energy is minimum (the position of stable equilibrium).

Example 4 *A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion (Fig. 6.41). What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?*

Solution In order to obtain the velocity at point B, we apply the law of conservation of energy. So,

$$\text{Loss in PE} = \text{Gain in KE}$$

$$mg\,(H-h) = \frac{1}{2}mv^2$$

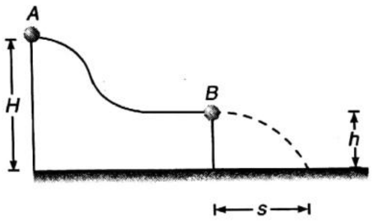


Fig. 6.41

$\therefore$ $$v = \sqrt{[2g\,(H-h)]}$$

Further $$h = \frac{1}{2}gt^2$$

$\therefore$ $$t = \sqrt{(2h/g)}$$

Now, $$s = v \times t = \sqrt{[2g(H-h)]} \times \sqrt{(2h/g)}$$

or $$s = \sqrt{[4h(H-h)]} \quad \ldots\text{(i)}$$

For maximum value of s $$ds/dh = 0$$

$\therefore$ $$\frac{1}{2\sqrt{[4h(H-h)]}} \times 4(H-2h) = 0$$

or $$h = \frac{H}{2}$$ **Ans.**

Substituting $h = H/2$, in Eq. (i), we get

$$s = \sqrt{[4(H/2)(H-H/2)]} = \sqrt{H^2} = H$$ **Ans.**

Example 5 *A particle slides along a track with elevated ends and a flat central part as shown in Fig. 6.42. The flat portion BC has a length $l = 3.0$ m. The curved portions of the track are frictionless. For the flat part the coefficient of kinetic friction is $\mu_k = 0.20$, the particle is released at point A which is at height $h = 1.5$ m above the flat part of the track. Where does the particle finally comes to rest?*

Solution As initial mechanical energy of the particle is mgh and final is zero, so loss in mechanical energy $= mgh$. This mechanical energy is lost in doing work against friction in the flat part,

So, $$\text{loss in mechanical energy} = \text{work done against friction}$$

or $$mgh = \mu mgs \quad \textit{i.e.,} \quad s = \frac{h}{\mu} = \frac{1.5}{0.2} = 7.5\,\text{m}$$

After starting from B the particle will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend and at C will have the same value as it had when ascending, but now it will move from C to B. The same will be repeated and finally the particle will come to rest at E such that

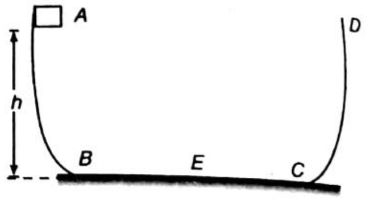


Fig. 6.42

$$BC + CB + BE = 7.5$$

or $$3 + 3 + BE = 7.5$$

i.e., $$BE = 1.5$$

So, the particle comes to rest at the centre of the flat part. **Ans.**

Example 6 *A 0.5 kg block slides from the point A on a horizontal track with an initial speed 3 m/s towards a weightless horizontal spring of length 1 m and force constant 2 N/m. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as 0.22 and 0.20 respectively. If the distance AB and BD are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely.* $[g = 10\ m/s^2]$

Solution As the track AB is frictionless, the block moves this distance without loss in its initial $\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 3^2 = 2.25\ \text{J}$. In the path BD as friction is present, so work done against friction

$$= \mu_k mgs = 0.2 \times 0.5 \times 10 \times 2.14 = 2.14\ \text{J}$$

So, at D the KE of the block is $= 2.25 - 2.14 = 0.11\ \text{J}$.

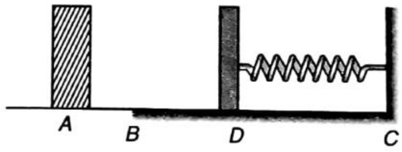


Fig. 6.43

Now, if the spring is compressed by x

$$0.11 = \frac{1}{2} \times k \times x^2 + \mu_k mgx$$

i.e., $$0.11 = \frac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x$$

or $$x^2 + x - 0.11 = 0$$

which on solving gives positive value of $x = 0.1\ \text{m}$

After moving the distance $x = 0.1\ \text{m}$ the block comes to rest. Now the compressed spring exerts a force:

$$F = kx = 2 \times 0.1 = 0.2\ \text{N}$$

on the block while limiting frictional force between block and track is $f_L = \mu_s mg = 0.22 \times 0.5 \times 10 = 1.1\ \text{N}$. Since, $F < f_L$. The block will not move back. So, the total distance moved by the block

$$= AB + BD + 0.1$$
$$= 2 + 2.14 + 0.1$$
$$= 4.24\ \text{m}$$

Ans.

Work done = change in KE

Example 7 *A small disc of mass m slides down a smooth hill of height h without initial velocity and gets onto a plank of mass M lying on a smooth horizontal plane at the base of hill Fig. 6.44. Due to friction between the disc and the plank, disc slows down and finally moves as one piece with the plank. (a) Find out total work performed by the friction forces in this process. (b) can it be stated that the result obtained does not depend on the choice of the reference frame.*

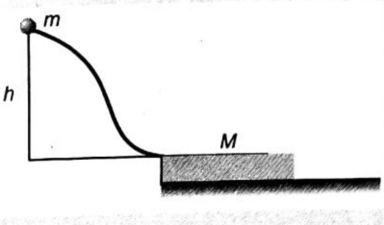


Fig. 6.44

Solution (a) When the disc slides down and comes onto the plank, then

$$m g h = \frac{1}{2} m v^2$$

$$\therefore \quad v = \sqrt{(2 g h)} \quad \ldots \text{(i)}$$

Let v_1 be the common velocity of both the disc and plank when they move together. From law of conservation of linear momentum,

$$m v = (M + m) v_1$$

$$\therefore \quad v_1 = \frac{m v}{(M + m)} \quad \ldots \text{(ii)}$$

Now, $\quad$ change in KE $= (K)_f - (K)_i = (\text{work done})_{\text{friction}}$

$$\therefore \quad \frac{1}{2}(M + m) v_1^2 - \frac{1}{2} m v^2 = (\text{work done})_{\text{friction}}$$

or

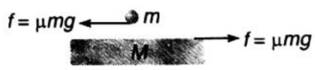
$$W_{\text{fr}} = \frac{1}{2}(M + m)\left[\frac{m v}{M + m}\right]^2 - \frac{1}{2} m v^2$$

$$= \frac{1}{2} m v^2 \left[\frac{m}{M + m} - 1\right]$$

as $\quad \frac{1}{2} m v^2 = mgh \quad \therefore \quad W_{fr} = - m g h \left[\frac{M}{M + m}\right]$ **Ans.**

(b) In part (a) we have calculated work done from the ground frame of reference. Now, let us take plank as the reference frame.

$$\text{Acceleration of plank } a_0 = \frac{f}{M} = \frac{\mu mg}{M}$$



FBD with respect to ground.

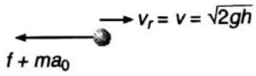


Fig. 6.45

Free body diagram of disc with respect to plank is shown in figure.
Here, ma_0 = pseudo force.
$\therefore$ Retardation of disc w.r.t. plank.

$$a_r = \frac{f + ma_0}{m} = \frac{\mu mg + \dfrac{\mu m^2 g}{M}}{m} = \mu g + \frac{\mu mg}{M}$$

$$= \left(\frac{M + m}{M}\right) \mu g$$

The disc will stop after travelling a distance S_r relative to plank, where

$$S_r = \frac{v_r^2}{2a_r} \qquad (0 = v_r^2 - 2a_r S_r)$$

$$= \frac{Mgh}{(M+m)\mu g}$$

$\therefore$ Work done by friction in this frame of reference

$$W_{fr} = -fS_r = -(\mu mg)\left[\frac{Mgh}{(M+m)\mu g}\right]$$

$$= -\frac{M\,mgh}{(M+m)}$$ which is same as part (a)

Note *Work done by friction in this problem does not depend upon the frame of reference, otherwise in general work depends upon reference frame.*

Example 8 *Two blocks A and B are connected to each other by a string and a spring. The string passes over a frictionless pulley as shown in Fig. 6.46. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2. The force constant of the spring is 1960 Nm^{-1}. If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring. ($g = 9.8\ m/s^2$)*

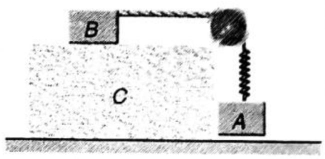


Fig. 6.46

Solution Let m be the mass of B. From its free-body diagram

$$T - \mu N = m \times 0 = 0$$

where T = tension of the string and $N = mg$

$$\therefore \qquad T = \mu mg$$

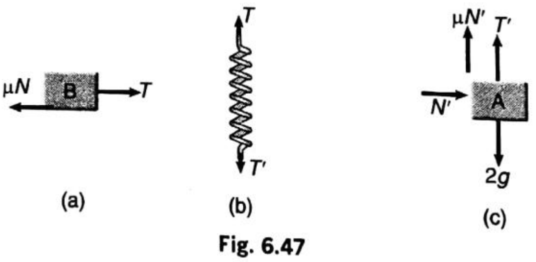


Fig. 6.47

From the free-body diagram of the spring

$$T - T' = 0$$

where T' is the force exerted by A on the spring

or $$T = T' = \mu mg$$

From the free-body diagram of A

$$2g - (T' + \mu N') = 2 \times 0 = 0$$

where N' is the normal reaction of the vertical wall of C on A and $N' = 2\times 0$ (as there is no horizontal acceleration of A)

$$\therefore \quad 2g = T' = \mu mg \quad \text{or} \quad m = \frac{2g}{\mu g} = \frac{2}{0.2} = 10\,\text{kg}$$ **Ans.**

Tensile force on the spring $= T$ or $T' = \mu mg = 0.2\times 10\times 9.8 = 19.6\,\text{N}$

Now, in a spring tensile force = force constant × extension

$$\therefore \quad 19.6 = 1960x \quad \text{or} \quad x = \frac{1}{100}\,\text{m}$$

or

$$U\text{ (energy of a spring)} = \frac{1}{2}kx^2$$

$$= \frac{1}{2}\times 1960\times\left(\frac{1}{100}\right)^2 = 0.098\,\text{J}$$ **Ans.**

Example 9 *A particle of mass m is moving in a circular path of constant radius r, such that its centripetal acceleration a_c is varying with time t as $a_c = k^2rt^2$ where k is a constant. What is the power delivered to the particle by the forces acting on it?* **[IIT JEE 1994]**

Solution As $a_c = (v^2/r)$ so $(v^2/r) = k^2rt^2$

$$\therefore \quad \text{Kinetic energy } K = \frac{1}{2}mv^2 = \frac{1}{2}mk^2r^2t^2$$

Now, from work-energy theorem $W = \Delta K = \frac{1}{2}mk^2r^2t^2 - 0$ [as at $t = 0, K = 0$]

So,

$$P = \frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}mk^2r^2t^2\right) = mk^2r^2t$$ **Ans.**

Alternate solution : Given that $a_c = k^2rt^2$, so that

$$F_c = ma_c = mk^2rt^2$$

Now, as $a_c = (v^2/r)$, so $(v^2/r) = k^2rt^2$ or $v = krt$

So, that $a_t = (dv/dt) = kr$

i.e., $F_t = ma_t = mkr$

Now, as $\vec{\mathbf{F}} = \vec{\mathbf{F}}_c + \vec{\mathbf{F}}_t$

So,

$$P = \frac{dW}{dt} = \vec{\mathbf{F}}\cdot\vec{\mathbf{v}} = (\vec{\mathbf{F}}_c + \vec{\mathbf{F}}_t)\cdot\vec{\mathbf{v}}$$

Now, in circular motion $\vec{\mathbf{F}}_c$ is perpendicular to $\vec{\mathbf{v}}$ while $\vec{\mathbf{F}}_t$ parallel, so

$$P = F_t v$$ [as $\vec{\mathbf{F}}_c\cdot\vec{\mathbf{v}} = 0$]

$$\therefore \quad P = mk^2r^2t$$ **Ans.**

EXERCISES

AIEEE Corner

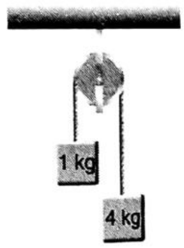
Subjective Questions (Level 1)

Work Done

(a) By a constant force

1. A block of mass 2 kg is pulled upwards by a force $F = 40$ N as shown in figure. Block is displaced by 2 m. Find work done by the applied force and also due to the force of gravity on the block. (Take $g = 10 \text{ m/s}^2$)

2. A block is displaced from (1m, 4m, 6m) to $(2\hat{i} + 3\hat{j} - 4\hat{k})$ m under a constant force $\vec{F} = (6\hat{i} - 2\hat{j} + \hat{k})$ N. Find the work done by this force.

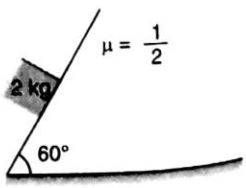


3. The system shown in figure is released from rest. String is massless and pulley is smooth. Find :
   (a) the work done by gravity on 4 kg block in 2 s,
   (b) the work done by string on 1 kg block in the same interval of time.
   (Take $g = 10 \text{ m/s}^2$)

4. A block of mass 2.5 kg is pushed 2.20 m along a frictionless horizontal table by a constant 16 N force directed 45° above the horizontal. Determine the work done by :
   (a) the applied force,
   (b) the normal force exerted by the table,
   (c) the force of gravity and
   (d) determine the total work done on the block.

5. A block of mass 2 kg is released from rest on a rough inclined ground as shown in figure. Find the work done on the block by :
   (a) gravity,
   (b) force of friction.

   when the block is displaced downwards along the plane by 2 m.
   (Take $g = 10 \text{ m/s}^2$)



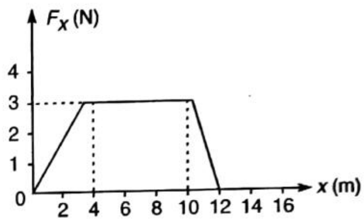
(b) By a variable force

6. A block is constrained to move along x-axis under a force $F = -2x$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 2$ m to $x = -4$ m.

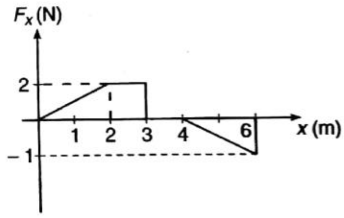
7. A block is constrained to move along x-axis under a force $F = \frac{4}{x^2}$ $(x \neq 0)$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 4$ m to $x = 2$ m.

(c) By area under *F*-*x* graph

8. A particle is subjected to a force F_x that varies with position as shown in figure. Find the work done by the force on the body as it moves :
   (a) from $x = 10.0$ m to $x = 5.0$ m,
   (b) from $x = 5.0$ m to $x = 10.0$ m,
   (c) from $x = 10.0$ m to $x = 15.0$ m,
   (d) what is the total work done by the force over the distance $x = 0$ to $x = 15.0$ m ?



9. A child applies a force $\vec{F}$ parallel to the x-axis to a block moving on a horizontal surface. As the child controls the speed of the block, the x-component of the force varies with the x-coordinate of the block as shown in figure. Calculate the work done by the force $\vec{F}$ when the block moves :
   (a) from $x = 0$ to $x = 3.0$ m
   (b) from $x = 3.0$ m to $x = 4.0$ m
   (c) from $x = 4.0$ m to $x = 7.0$ m
   (d) from $x = 0$ to $x = 7.0$ m



Conservative Force Field and Potential Energy

10. The potential energy of a two particle system separated by a distance r is given by $U(r) = \frac{A}{r}$ where A is a constant. Find the radial force F_r, that each particle exerts on the other.

11. A single conservative force F_x acts on a 2 kg particle that moves along the x-axis. The potential energy is given by :

$$U = (x-4)^2 - 16$$

Here, x is in metre and U in joule. At $x = 6.0$ m kinetic energy of particle is 8 J. Find :
   (a) total mechanical energy
   (b) maximum kinetic energy
   (c) values of x between which particle moves
   (d) the equation of F_x as a function of x.
   (e) the value of x at which F_x is zero.

Kinetic Energy and Work-Energy Theorem

12. Momentum of a particle is increased by 50%. By how much percentage kinetic energy of particle will increase ?

13. Kinetic energy of a particle is increased by 1%. By how much percentage momentum of the particle will increase ?

14. Displacement of a particle of mass 2 kg varies with time as : $s = (2t^2 - 2t + 10)$ m. Find total work done on the particle in a time interval from $t = 0$ to $t = 2$ s.

15. A block of mass 30 kg is being brought down by a chain. If the block acquires a speed of 40 cm/s in dropping down 2 m. Find the work done by the chain during the process. ($g = 10\ \text{m/s}^2$)

16. An object of mass 5 kg falls from rest through a vertical distance of 20 m and reaches a velocity of 10 m/s. How much work is done by the push of the air on the object ? ($g = 10\ \text{m/s}^2$)

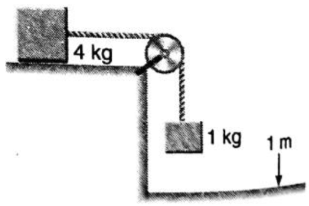
17. A 1.5 kg block is initially at rest on a horizontal frictionless surface when a horizontal force in the positive direction of x-axis is applied to the block. The force is given by $\vec{\mathbf{F}}(x) = (2.5 - x^2)\hat{\mathbf{i}}$ N, where x is in metre and the initial position of the block is $x = 0$.
    (a) What is the kinetic energy of the block as it passes through $x = 2.0$ m ?
    (b) What is the maximum kinetic energy of the block between $x = 0$ and $x = 2.0$ m ?

18. A helicopter lifts a 72 kg astronaut 15 m vertically from the ocean by means of a cable. The acceleration of the astronaut is $g/10$. How much work is done on the astronaut by ($g = 9.8\ \text{m/s}^2$)
    (a) the force from the helicopter and
    (b) the gravitational force on her ?
    (c) What are the kinetic energy and
    (d) the speed of the astronaut just before she reaches the helicopter ?

Mechanical Energy

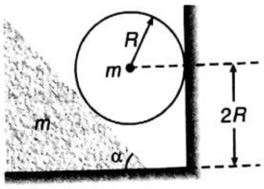
(a) Without friction (when mechanical energy is conserved)

19. A 4 kg block is on a smooth horizontal table. The block is connected to a second block of mass 1 kg by a massless flexible taut cord that passes over a frictionless pulley. The 1 kg block is 1 m above the floor. The two blocks are released from rest. With what speed does the 1 kg block hit the ground ?



20. Block A has a weight of 300 N and block B has a weight of 50 N. Determine the distance that A must descend from rest before it obtains a speed of 2.5 m/s. Neglect the mass of the cord and pulleys.

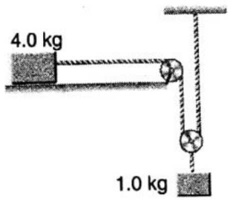
21. A sphere of mass m held at a height $2R$ between a wedge of same mass m and a rigid wall, is released from rest. Assuming that all the surfaces are frictionless. Find the speed of both the bodies when the sphere hits the ground.



22. The system is released from rest with the spring initially stretched 75 mm. Calculate the velocity v of the block after it has dropped 12 mm. The spring has a stiffness of 1050 N/m. Neglect the mass of the small pulley.

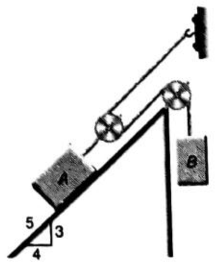
(b) With friction (when mechanical energy does not remain conserved)

23. Consider the situation shown in figure. The system is released from rest and the block of mass 1 kg is found to have a speed 0.3 m/s after it has descended through a distance of 1 m. Find the coefficient of kinetic friction between the block and the table.

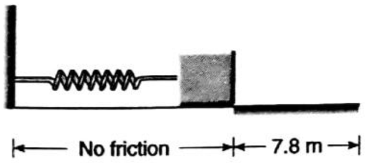


24. A disc of mass 50 g slides with zero initial velocity down an inclined plane set at an angle 30° to the horizontal. Having traversed a distance of 50 cm along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient 0.15 for both inclined and horizontal planes. ($g = 10 \text{m/s}^2$)

25. Block A has a weight of 300 N and block B has a weight of 50 N. If the coefficient of kinetic friction between the incline and block A is $\mu_k = 0.2$. Determine the speed of block A after it moves 1 m down the plane, starting from rest. Neglect the mass of the cord and pulleys.

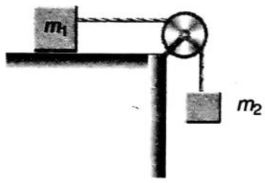


26. Figure shows, a 3.5 kg block accelerated by a compressed spring whose spring constant is 640 N/m. After leaving the spring at the spring's relaxed length, the block travels over a horizontal surface, with a coefficient of kinetic friction of 0.25, for a distance of 7.8 m before stopping. ($g = 9.8\,m/s^2$)



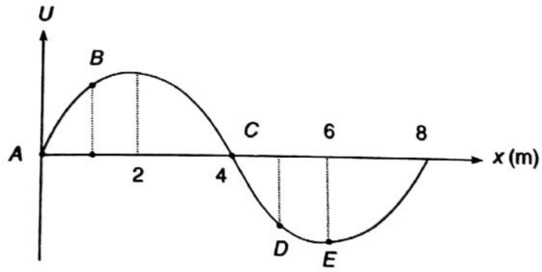
(a) What is the increase in the thermal energy of the block-floor system ?
(b) What is the maximum kinetic energy of the block ?
(c) Through what distance is the spring compressed before the block begins to move ?

27. Two masses $m_1 = 10$ kg and $m_2 = 5$ kg are connected by an ideal string as shown in the figure. The coefficient of friction between m_1 and the surface is $\mu = 0.2$. Assuming that the system is released from rest. Calculate the velocity of blocks when m_2 has descended by 4 m. ($g = 10\,m/s^2$)



Three Types of Equilibrium

28. For the potential energy curve shown in figure.



(a) Determine whether the force F is positive, negative or zero at the five points A, B, C, D and E.
(b) Indicate points of stable, unstable and neutral equilibrium.

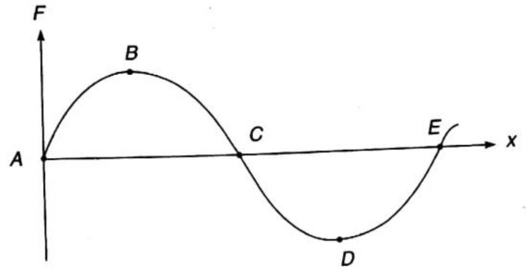
29. Potential energy of a particle moving along x-axis is given by :

$$U = \left(\frac{x^3}{3} - 4x + 6\right)$$

Here, U is in joule and x in metre. Find position of stable and unstable equilibrium.

30. Force acting on a particle moving along x-axis is as shown in figure. Find points of stable and unstable equilibrium.

31. Two point charges $+q$ and $+q$ are fixed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third point charge $-q$ is at origin. State whether its equilibrium is stable, unstable or neutral if it is slightly displaced :



(a) along x-axis. (b) along y-axis.

Power

32. A block of mass 1 kg starts moving with constant acceleration $a = 4\ \text{m/s}^2$. Find :

(a) average power of the net force in a time interval from $t = 0$ to $t = 2$ s,

(b) instantaneous power of the net force at $t = 4$ s.

33. An engine working at a constant power P draws a load of mass m against a resistance r. Find the maximum speed of the load and the time taken to attain half this speed.

Objective Questions (Level 1)

Single Correct Option

1. Identify, which of the following energies can be positive only

(a) kinetic energy (b) potential energy

(c) mechanical energy (d) both kinetic and mechanical energies

2. The total work done on a particle is equal to the change in its kinetic energy

(a) always

(b) only if the force acting on the body are conservative

(c) only in the inertial frame

(d) only if no external force is acting

3. Work done by force of static friction

(a) can be positive (b) can be negative (c) can be zero (d) All of these

4. Work done when a force $\vec{\mathbf{F}} = (\vec{\mathbf{i}} + 2\vec{\mathbf{j}} + 3\vec{\mathbf{k}})$ N acting on a partricle takes it from the point $\vec{\mathbf{r}}_1 = (\vec{\mathbf{i}} + \vec{\mathbf{j}} + \vec{\mathbf{k}})$ to the point $\vec{\mathbf{r}}_2 = (\vec{\mathbf{i}} - \vec{\mathbf{j}} + 2\vec{\mathbf{k}})$ is

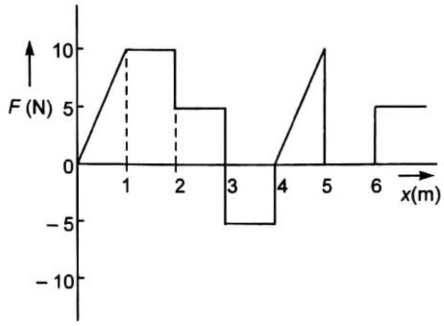
(a) -3 J (b) -1 J (c) zero (d) 2 J

5. A particle moves along the x-axis from $x = 0$ to $x = 5$ m under the influence of a force given by $F = 7 - 2x + 3x^2$. The work done in the process is

(a) 360 J (b) 85 J (c) 185 J (d) 135 J

6. A particle moves with a velocity $\vec{v} = (5\vec{i} - 3\vec{j} + 6\vec{k})\ \text{ms}^{-1}$ under the influence of a constant force $\vec{F} = (10\hat{i} + 10\hat{j} + 20\hat{k})$ N. The instantaneous power applied to the particle is
(a) 200 W (b) 320 W (c) 140 W (d) 170 W

7. The relationship between the force F and position x of a body is as shown in figure. The work done in displacing the body from $x = 1\,\text{m}$ to $x = 5\,\text{m}$ will be



(a) 30 J (b) 15 J (c) 25 J (d) 20 J

8. A pump is required to lift 800 kg of water per minute from a 10 m deep well and eject it with speed of 20 m/s. The required power in watts of the pump will be
(a) 6000 (b) 4000 (c) 5000 (d) 8000

9. Under the action of a force, a 2 kg body moves such that its position x as a function of time is given by $x = \frac{t^3}{3}$, where x is in metre and t in second. The work done by the force in the first two seconds is
(a) 1600 J (b) 160 J (c) 16 J (d) 1.6 J

10. The kinetic energy of a projectile at its highest position is K. If the range of the projectile is four times the height of the projectile, then the initial kinetic energy of the projectile is
(a) $\sqrt{2}K$ (b) $2K$ (c) $4K$ (d) $2\sqrt{2}K$

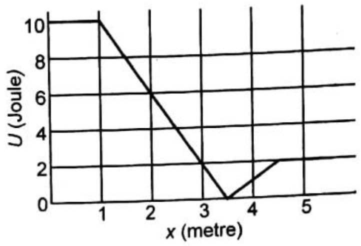
11. Power applied to a particle varies with time as $P = (3t^2 - 2t + 1)$ watt, where t is in second. Find the change in its kinetic energy between time $t = 2$ s and $t = 4$ s
(a) 32 J (b) 46 J (c) 61 J (d) 102 J

12. A block of mass 10 kg is moving in x-direction with a constant speed of 10 m/s. It is subjected to a retarding force $F = -0.1x$ J/m during its travel from $x = 20\,\text{m}$ to $x = 30\,\text{m}$. Its final kinetic energy will be
(a) 475 J (b) 450 J (c) 275 J (d) 250 J

13. A ball of mass 12 kg and another of mass 6 kg are dropped from a 60 feet tall building. After a fall of 30 feet each, towards earth, their kinetic energies will be in the ratio of
(a) $\sqrt{2}:1$ (b) $1:4$ (c) $1:2$ (d) $1:\sqrt{2}$

14. A spring of spring constant 5×10^3 N/m is stretched initially by 5 cm from the unstretched position. The work required to further stretch the spring by another 5 cm is
(a) 6.25 N-m (b) 12.50 N-m (c) 18.75 N-m (d) 25.00 N-m

15. A body with mass 1 kg moves in one direction in the presence of a force which is described by the potential energy graph. If the body is released from rest at $x = 2$m, than its speed when it crosses $x = 5$ m is (Neglect dissipative forces)



(a) $2\sqrt{2}\ \text{ms}^{-1}$ (b) $1\ \text{ms}^{-1}$ (c) $2\ \text{ms}^{-1}$ (d) $3\ \text{ms}^{-1}$

16. A body has kinetic energy E when projected at angle of projection for maximum range. Its kinetic energy at the highest point of its path will be

(a) E (b) $\frac{E}{2}$ (c) $\frac{E}{\sqrt{2}}$ (d) zero

17. A person pulls a bucket of water from a well of depth h. If the mass of uniform rope is m and that of the bucket full of water is M, then work done by the person is

(a) $\left(M + \frac{m}{2}\right)gh$ (b) $\frac{1}{2}(M + m)gh$ (c) $(M + m)gh$ (d) $\left(\frac{M}{2} + m\right)gh$

18. The minimum stopping distance of a car moving with velocity v is x. If the car is moving with velocity $2v$, then the minimum stopping distance will be

(a) $2x$ (b) $4x$ (c) $3x$ (d) $8x$

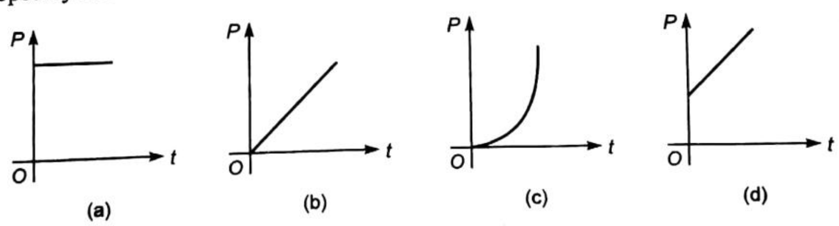
19. A projectile is fired from the origin with a velocity v_0 at an angle θ with the x-axis. The speed of the projectile at an altitude h is

(a) $v_0 \cos\theta$ (b) $\sqrt{v_0^2 - 2gh}$ (c) $\sqrt{v_0^2 \sin^2\theta - 2gh}$ (d) None of these

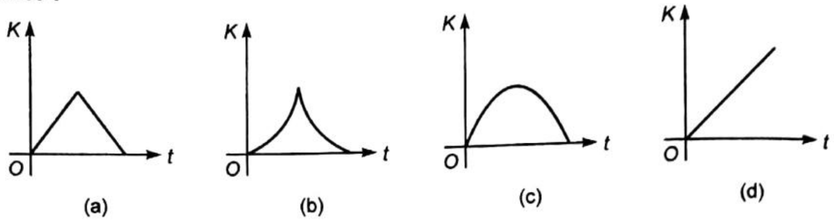
20. A particle of mass m moves from rest under the action of a constant force F which acts for two seconds. The maximum power attained is

(a) $2Fm$ (b) $\frac{F^2}{m}$ (c) $\frac{2F}{m}$ (d) $\frac{2F^2}{m}$

21. A body moves under the action of a constant force along a straight line. The instantaneous power developed by this force with time t is correctly represented by



22. A ball is dropped at $t = 0$ from a height on a smooth elastic surface. Identify the graph which correctly represents the variation of kinetic energy K with time t



23. A block of mass 5 kg is raised from the bottom of the lake to a height of 3 m without change in kinetic energy. If the density of the block is 3000 kg m^{-3}, then the work done is equal to

(a) 100 J (b) 150 J (c) 50 J (d) 75 J

24. A body of mass m is projected at an angle θ with the horizontal with an initial velocity v_0. The average power of gravitational force over the whole time of flight is

(a) $mg \cos\theta$ (b) $\frac{1}{2} mg \sqrt{u \cos\theta}$

(c) $\frac{1}{2} mgu \sin\theta$ (d) zero

25. A spring of force constant k is cut in two parts at its one-third length. When both the parts are stretched by same amount. The work done in the two parts will be :

Note *Spring constant of a spring is inversely proportional to length of spring*

(a) equal in both (b) greater for the longer part

(c) greater for the shorter part (d) data insufficient

26. A particle moves under the action of a force $\vec{F} = 20\hat{i} + 15\hat{j}$ along a straight line $3y + \alpha x = 5$ where α is a constant. If the work done by the force F is zero, then the value of α is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) 3 (d) 4

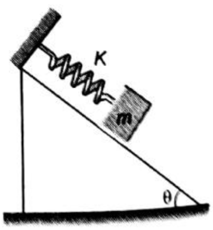
27. A system of wedge and block as shown in figure, is released with the spring in its natural length. All surfaces are frictionless. Maximum elongation in the spring will be

(a) $\frac{2\,mg \sin\theta}{K}$

(b) $\frac{mg \sin\theta}{K}$

(c) $\frac{4\,mg \sin\theta}{K}$

(d) $\frac{mg \sin\theta}{2K}$



28. A force $\vec{F} = (3t\,\hat{i} + 5\,\hat{j})$N acts on a body due to which its displacement varies as $\vec{S} = (2t^2\,\hat{i} - 5\,\hat{j})$m. Work done by this force in 2 second is

(a) 23 J (b) 32 J (c) 46 J (d) 20 J

29. An open knife of mass m is dropped from a height h on a wooden floor. If the blade penetrates up to the depth d into the wood, the average resistance offered by the wood to the knife edge is

(a) $mg\left(1 + \frac{h}{d}\right)$ (b) $mg\left(1 + \frac{h}{d}\right)^2$ (c) $mg\left(1 - \frac{h}{d}\right)$ (d) $mg\left(1 + \frac{d}{h}\right)$

30. Two springs have force constants k_A and k_B such that $k_B = 2k_A$. The four ends of the springs are stretched by the same force. If energy stored in spring A is E, then energy stored in spring B is

(a) $\frac{E}{2}$ (b) $2E$ (c) E (d) $4E$

31. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a nearly weightless spring of force constant $k = 50$ N/m. The maximum compression of the spring would be

(a) 0.15 m (b) 0.12 m (c) 0.5 m (d) 0.25 m

32. A bullet moving with a speed of $100\ \text{ms}^{-1}$ can just penetrate into two planks of equal thickness. Then the number of such planks, if speed is doubled will be

(a) 6 (b) 10 (c) 4 (d) 8

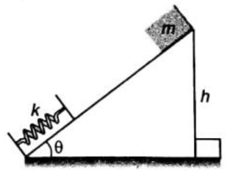
33. A body of mass 100 g is attached to a hanging spring whose force constant is 10 N/m. The body is lifted until the spring is in its unstretched state and then released. Calculate the speed of the body when it strikes the table 15 cm below the release point

(a) 1 m/s (b) 0.866 m/s (c) 0.225 m/s (d) 1.5 m/s

34. An ideal massless spring S can be compressed 1.0 m in equilibrium by a force of 100 N. This same spring is placed at the bottom of a friction less inclined plane which makes an angle $\theta = 30°$ with the horizontal. A 10 kg mass m is released from the rest at the top of the inclined plane and is brought to rest momentarily after compressing the spring by 2.0 m. The distance through which the mass moved before coming to rest is

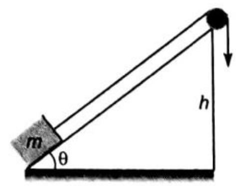
(a) 8 m (b) 6 m (c) 4 m (d) 5 m

35. A body of mass m is released from a height h on a smooth inclined plane that is shown in the figure. The following can be true about the velocity of the block knowing that the wedge is fixed



(a) v is highest when it just touches the spring
(b) v is highest when it compresses the spring by some amount
(c) v is highest when the spring comes back to natural position
(d) data insufficient

36. A block of mass m is directly pulled up slowly on a smooth inclined plane of height h and inclination θ with the help of a string parallel to the incline. Which of the following statement is incorrect for the block when it moves up from the bottom to the top of the incline?

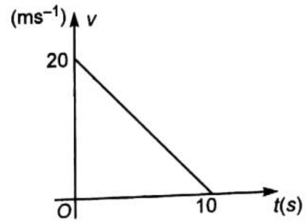


(a) Work done by the normal reaction force is zero.
(b) Work done by the string is mgh
(c) Work done by gravity is mgh
(d) Net work done on the block is zero

37. A spring of natural length l is compressed vertically downward against the floor so that its compressed length becomes $\frac{l}{2}$. On releasing, the spring attains its natural length. If k is the stiffness constant of spring, then the work done by the spring on the floor is

(a) zero (b) $\frac{1}{2}kl^2$ (c) $\frac{1}{2}k\left(\frac{l}{2}\right)^2$ (d) kl^2

38. The velocity of a particle decreases uniformly from 20 ms^{-1} to zero in 10 s as shown in figure. If the mass of the particle is 2 kg, then identify the correct statement.



(a) The net force acting on the particle is opposite to the direction of motion
(b) The work done by friction force is −400 J
(c) The magnitude of friction force acting on the particle is 4 N
(d) All of the above

39. A ball is dropped onto a floor from a height of 10 m. If 20% of its initial energy is lost, then the height of bounce is
(a) 2 m (b) 4 m (c) 8 m (d) 6.4 m

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***
(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***
(*c*) *If **Assertion** is true, but the **Reason** is false.*
(*d*) *If **Assertion** is false but the **Reason** is true.*

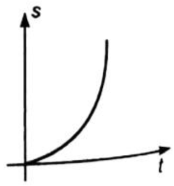
1. Assertion : Power of a constant force is also constant.
Reason : Net constant force will always produce a constant acceleration.

2. Assertion : A body is moved from $x = 2$ to $x = 1$, under a force $F = 4x$, the work done by this force is negative.
Reason : Force and displacement are in opposite directions.

3. Assertion : If work done by conservative forces is positive, kinetic energy will increase.
Reason : Because potential energy will decrease.

4. Assertion : In circular motion work done by all the forces acting on the body is zero.
Reason : Centripetal froce and velocity are mutually perpendicular.

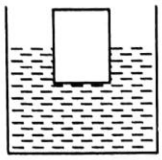
5. Assertion : Corresponding to displacement- time graph of a particle moving in a straight line we can say that total work done by all the forces acting on the body is positive.
Reason : Speed of particle is increasing.



6. **Assertion :** Work done by a constant force is path independent.
Reason : All constant forces are conservative in nature.

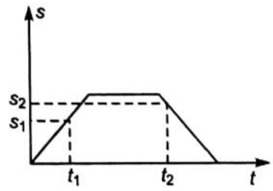
7. **Assertion :** Work-energy theorem can be applied for non-inertial frames also.
Reason : Earth is a non-inertial frame.

8. **Assertion :** A wooden block is floating in a liquid as shown in figure. In vertical direction equilibrium of block is stable.



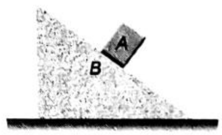
Reason : When depressed in downward direction is starts oscillating.

9. **Assertion :** Displacement-time graph of a particle moving in a straight line is shown in figure. Work done by all the forces between time interval t_1 and t_2 is definitely zero.

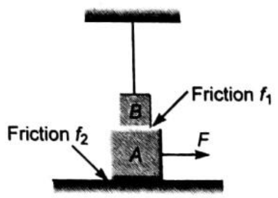


Reason : Work done by all the forces is equal to change in kinetic energy.

10. **Assertion :** All surfaces shown in figure are smooth. Block A comes down along the wedge B. Work done by normal reaction (between A and B) on B is positive while on A it is negative.
Reason : Angle between normal reaction and net displacement of A is greater than 90° while between normal reaction and net displacement of B is less than 90°.



11. **Assertion :** A plank A is placed on a rough surface over which a block B is placed. In the shown situation, elastic cord is unstretched. Now a gradually increasing force F is applied slowly on A until the moment relative motion between the block and plank starts.
At this moment cord is making an angle θ with the vertical. Work done by force F is equal to energy lost against friction f_2, plus potential energy stored in the cord.
Reason : Work done by static friction f_1 on the system as a whole is zero.

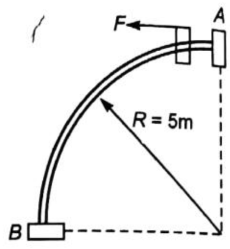


12. **Assertion :** A block of mass m starts moving on a rough horizontal surface with a velocity v. It stops due to friction between the block and the surface after moving through a ceratin distance. The surface is now tilted to an angle of 30° with the horizontal and the same block is made to go up on the surface with the same initial velocity v. The decrease in the mechanical energy in the second situation is smaller than that in the first situation.
Reason : The coefficient of friction between the block and the surface decreases with the increase in the angle of inclination.

Objective Questions (Level 2)

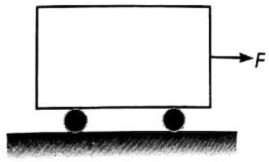
Single Correct Option

1. A bead of mass $\frac{1}{2}$ kg starts from rest from A to move in a vertical plane along a smooth fixed quarter ring of radius 5 m, under the action of a constant horizontal force $F = 5$ N as shown. The speed of bead as it reaches the point B is [Take $g = 10\,\text{ms}^{-2}$]



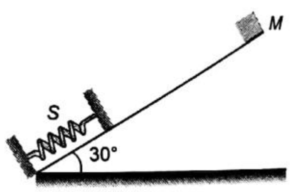
(a) 14.14 ms^{-1}
(b) 7.07 ms^{-1}
(c) 4 ms^{-1}
(d) 25 ms^{-1}

2. A car of mass m is accelerating on a level smooth road under the action of a single force F. The power delivered to the car is constant and equal to P. If the velocity of the car at an instant is v, then after travelling how much distance it becomes double?



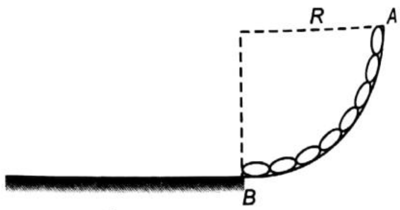
(a) $\frac{7mv^3}{3P}$
(b) $\frac{4mv^3}{3P}$
(c) $\frac{mv^3}{P}$
(d) $\frac{18mv^3}{7P}$

3. An ideal massless spring S can be compressed 1 m by a force of 100 N in equilibrium. The same spring is placed at the bottom of a frictionless plane inclined at 30° to the horizontal. A 10 kg block M is released from rest at the top of the incline and is brought to rest momentarily after compressing the spring by 2 m. If g = 10 ms^{-2}, what is the speed of mass just before it touches the spring?



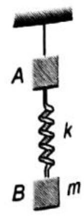
(a) $\sqrt{20}\,\text{ms}^{-1}$
(b) $\sqrt{30}\,\text{ms}^{-1}$
(c) $\sqrt{10}\,\text{ms}^{-1}$
(d) $\sqrt{40}\,\text{ms}^{-1}$

4. A smooth chain AB of mass m rests against a surface in the form of a quarter of a circle of radius R. If it is released from rest, the velocity of the chain after it comes over the horizontal part of the surface is



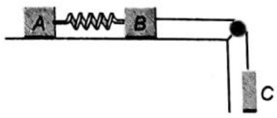
(a) $\sqrt{2gR}$
(b) $\sqrt{gR}$
(c) $\sqrt{2gR\left(1-\frac{2}{\pi}\right)}$
(d) $\sqrt{2gR\,(2-\pi)}$

5. Initially the system shown in figure is in equilibrium. At the moment, the string is cut the downward acceleration of blocks A and B are respectively a_1 and a_2. The magnitudes of a_1 and a_2 are



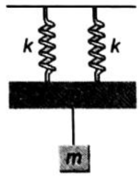
(a) zero and zero
(b) $2g$ and zero
(c) g and zero
(d) None of the above

6. In the diagram shown, the blocks A and B are of the same mass M and the mass of the block C is M_1. Friction is present only under the block A. The whole system is suddenly released from the state of rest. The minimum coefficient of friction to keep the block A in the state of rest is equal to-



(a) $\frac{M_1}{M}$ (b) $\frac{2M_1}{M}$
(c) $\frac{M_1}{2M}$ (d) None of these

7. System shown in figure is in equilibrium. Find the magnitude of net change in the string tension between two masses just after, when one of the springs is cut. Mass of both the blocks is same and equal to m and spring constant of both the springs is k

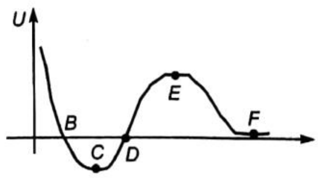


(a) $\frac{mg}{2}$ (b) $\frac{mg}{4}$
(c) $\frac{mg}{3}$ (d) $\frac{3mg}{2}$

8. A body is moving down an inclined plane of slope 37°. The coefficient of friction between the body and the plane varies as $\mu = 0.3\,x$, where x is the distance traveled down the plane by the body. The body will have maximum speed. $\left(\sin 37° = \frac{3}{5}\right)$

(a) at $x = 1.16\,\text{m}$ (b) at $x = 2\,\text{m}$
(c) at bottommost point of the plane (d) at $x = 2.5\,\text{m}$

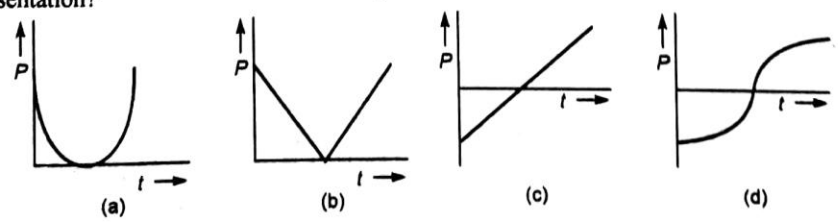
9. The given plot shows the variation of U, the potential energy of interaction between two particles with the distance separating them r.



1. B and D are equilibrium points
2. C is a point of stable equilibrium
3. The force of interaction between the two particles is attractive between points C and D and repulsive between D and E
4. The force of interaction between particles is repulsive between points E and F. Which of the above statements are correct?

(a) 1 and 2 (b) 1 and 4 (c) 2 and 4 (d) 2 and 3

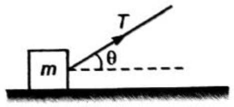
10. A particle is projected at $t = 0$ from a point on the ground with certain velocity at an angle with the horizontal. The power of gravitation force is plotted against time. Which of the following is the best representation?



11. A block of mass m is attached to one end of a mass less spring of spring constant k. The other end of spring is fixed to a wall. The block can move on a horizontal rough surface. The coefficient of friction between the block and the surface is μ. Then the compression of the spring for which maximum extension of the spring becomes half of maximum compression is

(a) $\frac{2mg\mu}{k}$ (b) $\frac{mg\mu}{k}$ (c) $\frac{4mg\mu}{k}$ (d) None of these

12. A block of mass m slides along the track with kinetic friction μ. A man pulls the block through a rope which makes an angle θ with the horizontal as shown in the figure. The block moves with constant speed V. Power delivered by man is

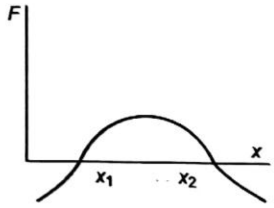


(a) TV (b) $TV\cos\theta$ (c) $(T\cos\theta - \mu mg)V$ (d) zero

13. The potential energy ϕ in joule of a particle of mass 1 kg moving in x-y plane obeys the law, $\phi = 3x + 4y$. Here x and y are in metres. If the particle is at rest at (6m, 8m) at time 0, then the work done by conservative force on the particle from the initial position to the instant when it crosses the x-axis is

(a) 25 J (b) – 25 J (c) 50 J (d) – 50 J

14. The force acting on a body moving along x-axis varies with the position of the particle shown in the figure. The body is in stable equilibirum at

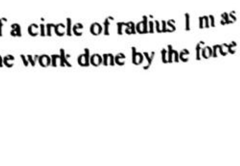


(a) $x = x_1$ (b) $x = x_2$ (c) both x_1 and x_2 (d) neither x_1 nor x_2

15. A small mass slides down an inclined plane of inclination θ with the horizontal. The co-efficient of friction is $\mu = \mu_0 x$ where x is the distance through which the mass slides down and μ_0 a positive constant. Then the distance covered by the mass before it stops is

(a) $\frac{2}{\mu_0}\tan\theta$ (b) $\frac{4}{\mu_0}\tan\theta$ (c) $\frac{1}{2\mu_0}\tan\theta$ (d) $\frac{1}{\mu_0}\tan\theta$

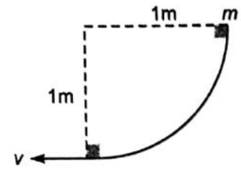
16. Two light vertical springs with spring constants k_1 and k_2 are separated by a distance l. Their upper ends are fixed to the ceiling and their lower ends to the ends A and B of a light horizontal rod AB. A vertical downward force F is applied at point C on the rod. AB will remain horizontal in equilibrium if the distance AC is



(a) $\frac{lk_1}{k_2}$ (b) $\frac{lk_1}{k_2 + k_1}$

(c) $\frac{lk_2}{k_1}$ (d) $\frac{lk_2}{k_1 + k_2}$

17. A block of mass 1 kg slides down a curved track which forms one quadrant of a circle of radius 1 m as shown in figure. The speed of block at the bottom of the track is $v = 2\,ms^{-1}$. The work done by the force of friction is

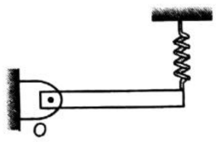


(a) + 4 J (b) − 4 J (c) − 8 J (d) + 8 J

18. The potential energy function for a diatomic molecule is $U(x)=\frac{a}{x^{12}}-\frac{b}{x^6}$. In stable equilibrium, the distance between the particles is

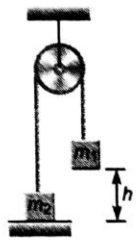
(a) $\left(\frac{2a}{b}\right)^{1/6}$ (b) $\left(\frac{a}{b}\right)^{1/6}$ (c) $\left(\frac{b}{2a}\right)^{1/6}$ (d) $\left(\frac{b}{a}\right)^{1/6}$

19. A rod of mass M hinged at O is kept in equilibrium with a spring of stiffness k as shown in figure. The potential energy stored in the spring is



(a) $\frac{(mg)^2}{4k}$ (b) $\frac{(mg)^2}{2k}$ (c) $\frac{(mg)^2}{8k}$ (d) $\frac{(mg)^2}{k}$

20. In the figure m_1 and m_2 ($<m_1$) are joined together by a pulley. When the mass m_1 is released from the height h above the floor, it strikes the floor with a speed



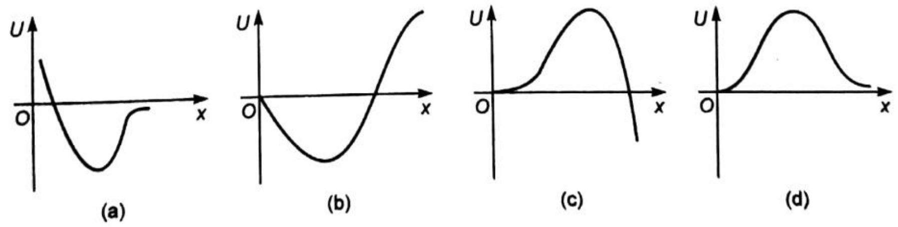
(a) $\sqrt{2gh\left(\frac{m_1-m_2}{m_1+m_2}\right)}$

(b) $\sqrt{2gh}$

(c) $\sqrt{\frac{2m_2gh}{m_1+m_2}}$

(d) $\sqrt{\frac{2m_1gh}{m_1+m_2}}$

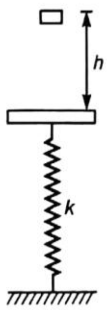
21. A particle free to move along x- axis is acted upon by a force $F=-ax+bx^2$ where a and b are positive constants. For $x\geq 0$, the correct variation of potential energy function U(x) is best represented by



22. Equal net forces act on two different blocks A and B of masses m and 4m respectively. For same displacement, identify the correct statement.
(a) Their kinetic energies are in the ratio $\frac{K_A}{K_B} = \frac{1}{4}$
(b) Their speeds are in the ratio $\frac{v_A}{v_B} = \frac{1}{1}$
(c) Work done on the blocks are in the ratio $\frac{W_A}{W_B} = \frac{1}{1}$
(d) All of the above

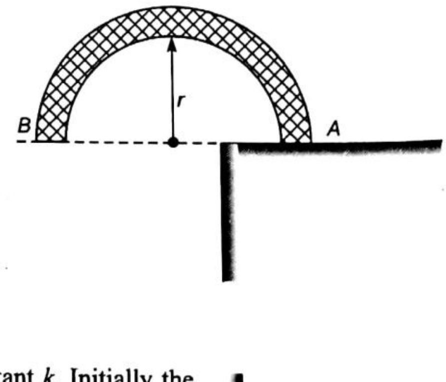
23. The potential energy function of a particle in the x-y plane is given by $U = k(x + y)$, where k is a constant. The work done by the conservative force in moving a particle from (1, 1) to (2, 3) is
(a) $-3k$ (b) $+3k$ (c) k (d) None of these

24. A vertical spring is fixed to one of its end and a massless plank fitted to the other end. A block is released from a height h as shown. Spring is in relaxed position. Then choose the correct statement.



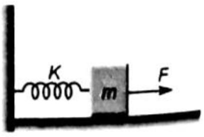
(a) The maximum compression of the spring does not depend on h.
(b) The maximum kinetic energy of the block does not depend on h.
(c) The compression of the spring at maximum KE of the block does not depend on h.
(d) The maximum compression of the spring does not depend on k.

25. A uniform chain of length πr lies inside a smooth semicircular tube AB of radius r. Assuming a slight disturbance to start the chain in motion, the velocity with which it will emerge from the end B of the tube will be



(a) $\sqrt{gr\left(1+\frac{2}{\pi}\right)}$
(b) $\sqrt{2gr\left(\frac{2}{\pi}+\frac{\pi}{2}\right)}$
(c) $\sqrt{gr(\pi+2)}$
(d) $\sqrt{\pi gr}$

26. A block of mass m is connected to a spring of force constant k. Initially the block is at rest and the spring has natural length. A constant force F is applied horizontally towards right. The maximum speed of the block will be (there is no friction between block and the surface)



(a) $\frac{F}{\sqrt{2mk}}$ (b) $\frac{F}{\sqrt{mk}}$
(c) $\frac{\sqrt{2}F}{\sqrt{mk}}$ (d) $\frac{2F}{\sqrt{mk}}$

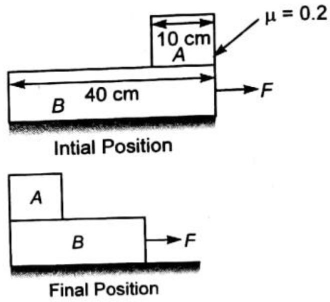
27. Two blocks are connected to an ideal spring of stiffness 200N/m. At a certain moment, the two blocks are moving in opposite directions with speeds 4 ms^{-1} and 6 ms^{-1}, and the instantaneous elongation of the spring is 10 cm. The rate at which the spring energy $\left(\frac{kx^2}{2}\right)$ is increasing is

(a) 500 J/s (b) 400 J/s (c) 200 J/s (d) 100 J/s

28. A mass less spring of stiffness k connects two blocks of masses m and $3m$. The system is lying on a frictionless horizontal surface. A constant horizontal force F starts acting on the block of mass m, directed towards the other block. Then the maximum compression of the spring will be

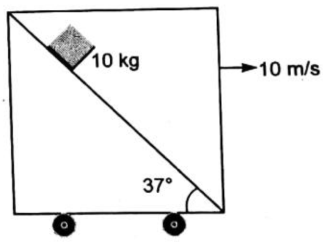
(a) $\frac{3F}{4k}$ (b) $\frac{9F}{8k}$ (c) $\frac{3F}{2k}$ (d) $\frac{4F}{3k}$

29. A block A of mass 45 kg is placed on another block B of mass 123 kg. Now block B is displaced by external agent by 50 cm horizontally towards right. During the same time block A just reaches to the left end of block B. Initial and final positions are shown in figures. The work done on block A in ground frame is



(a) – 18 J
(b) 18 J
(c) 36 J
(d) – 36 J

30. A block of mass 10 kg is released on a fixed wedge inside a cart which is moved with constant velocity 10 ms^{-1} towards right. Take initial velocity of block with respect to cart zero. Then work done by normal reaction on block in two seconds from ground frame will be ($g = 10\,ms^{-2}$)



(a) 1320 J
(b) 960 J
(c) 1200 J
(d) 240 J

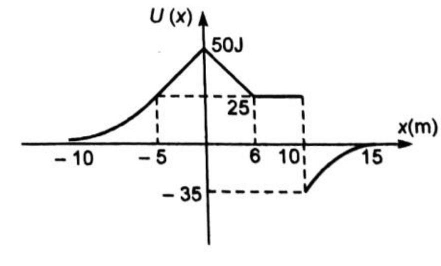
31. A block tied between two springs is in equilibrium. If upper spring is cut, then the acceleration of the block just after cut is 5 ms^{-2}. Now if instead of upper string lower spring is cut, then the acceleration of the block just after the cut will be : (Take $g = 10\,m/s^2$)

(a) 1.25 ms^{-2} (b) 5 ms^{-2}
(c) 10 ms^{-2} (d) 2.5 ms^{-2}

Passage (Q. No. 32 to 33)

The figure shows the variation of potential energy of a particle as a function of x, the x-coordinate of the region. It has been assumed that potential energy depends only on x. For all other values of x, U is zero, *i.e.*, for $x < -10$ and $x > 15, U = 0$.

Based on above information answer the following questions.



32. If total mechanical energy of the particle is 25 J, then it can be found in the region
(a) $-10 < x < -5$ and $6 < x < 15$
(b) $-10 < x < 0$ and $6 < x < 10$
(c) $-5 < x < 6$
(d) $-10 < x < 10$

33. If total mechanical energy of the particle is -40 J, then it can be found in region
(a) $x < -10$ and $x > 15$
(b) $-10 < x < -5$ and $6 < x < 15$
(c) $10 < x < 15$
(d) It is not possible

More than One Correct Options

1. The potential energy of a particle of mass 5 kg moving in xy plane is given as $U = 7x + 24y$ joules, X and Y being in metre. Initially at $t = 0$ the particle is at the origin (0, 0) moving with a velocity of $(8.6\hat{i} + 23.2\hat{j})\ ms^{-1}$. Then
(a) The velocity of the particle at $t = 4$ s, is $5\ ms^{-1}$
(b) The acceleration of the particle is $5\ ms^{-2}$
(c) The direction of motion of the particle initially (at $t = 0$) is at right angles to the direction of acceleration
(d) The path of the particle is circle

2. The potential energy of a particle is given by formula $U = 100 - 5x + 100x^2$, U and x are in SI units. If mass of the particle is 0.1 kg then magnitude of it's acceleration
(a) At 0.05 m from the origin is $50\ ms^{-2}$
(b) At 0.05 m from the mean position is $100\ ms^{-2}$
(c) At 0.05 m from the origin is $150\ ms^{-2}$
(d) At 0.05 m from the mean position is $200\ ms^{-2}$

3. One end of a light spring of spring constant k is fixed to a wall and the other end is tied to a block placed on a smooth horizontal surface. In a displacement, the work done by the spring is $+\left(\frac{1}{2}\right)kx^2$. The possible cases are
(a) The spring was initially compressed by a distance x and was finally in its natural length
(b) It was initially stretched by a distance x and finally was in its natural length
(c) It was initially in its natural length and finally in a compressed position
(d) It was initially in its natural length and finally in a stretched position

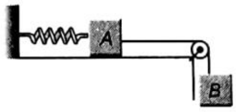
4. Identify the correct statement about work energy theorem
(a) work done by all the conservative forces is equal to the decrease in potential energy.
(b) work done by all the forces except the conservative forces is equal to the change in mechanical energy.
(c) work done by all the forces is equal to the change in kinetic energy.
(d) work done by all the forces is equal to the change in potential energy.

5. A disc of mass 3m and a disc of mass m are connected by a mass less spring of stiffness k. The heavier disc is placed on the ground with the spring vertical and lighter disc on top. From its equilibrium position the upper disc is pushed down by a distance δ and released. Then
(a) if $\delta > \frac{3\,mg}{k}$, the lower disc will bounce up
(b) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = $6\,mg$.

(c) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = 4 mg.

(d) if $\delta > \frac{4\,mg}{k}$, the lower disc will bounce up

6. In the adjoining figure block A is of mass m and block B is of mass 2 m. The spring has force constant k. All the surfaces are smooth and the system is released from rest with spring unstretched.



(a) The maximum extension of the spring is $\frac{4mg}{k}$

(b) The speed of block A when extension in spring is $\frac{2mg}{k}$, is $2g\sqrt{\frac{2m}{3k}}$

(c) Net acceleration of block B when the extension in the spring is maximum, is $\frac{2}{3}g$.

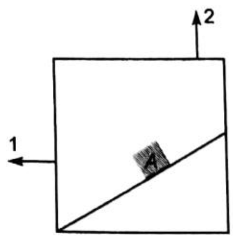
(d) Tension in the thread for extension of $\frac{2mg}{k}$ in spring is mg

7. Mark out the correct statement (s).
(a) Total work done by internal forces on a system is always zero
(b) Total work done by internal forces on a system may sometimes be zero
(c) Total work done by internal forces acting between the particles of a rigid body is always zero
(d) Total work done by internal forces acting between the particles of a rigid body may sometimes be zero

8. If kinetic energy of a body is increasing then :
(a) work done by conservative forces must be positive
(b) work done by conservative forces may be positive
(c) work done by conservative forces may be zero
(d) work done by non conservative forces may be zero

9. At two positions kinetic energy and potential energy of a particle are
$K_1 = 10\,\text{J} : U_1 = -20\,\text{J}, K_2 = 20\,\text{J}, U_2 = -10\,\text{J}$
In moving from 1 to 2
(a) work done by conservative forces is positive
(b) work done by conservative forces is negative
(c) work done by all the forces is positive
(d) work done by all the forces is negative

10. Block A has no relative motion with respect to wedge fixed to the lift as shown in figure during motion-1 or motion-2



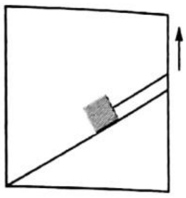
(a) work done by gravity on block A in motion-2 is less than in motion-1
(b) work done by normal reaction on block A in both the motions will be positive
(c) work done by force of friction in motion-1 may be positive
(d) work done by force of friction in motion-1 may be negative

Match the Columns

1. A body is displaced from $x = 4\,\text{m}$ to $x = 2\,\text{m}$ along the x-axis. For the forces mentioned in column I, match the corresponding work done is column II.

Column I	Column II
(a) $\vec{\mathbf{F}} = 4\hat{\mathbf{i}}$	(p) positive
(b) $\vec{\mathbf{F}} = (4\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$	(q) negative
(c) $\vec{\mathbf{F}} = -4\hat{\mathbf{i}}$	(r) zero
(d) $\vec{\mathbf{F}} = (-4\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$	(s) $\lvert W \rvert = 8$ units

2. A block is placed on a rough wedge fixed on a lift as shown in figure. A string is also attached with the block. The whole system moves upwards. Block does not lose contact with wedge on the block. Match the following two columns regarding the work done.

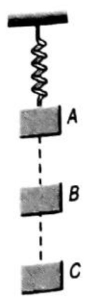


Column I	Column II
(a) Work done by normal reaction	(p) positive
(b) Work done by gravity	(q) negative
(c) Work done by friction	(r) zero
(d) Work done by tension	(s) Can't say anything

3. Two positive charges $+q$ each are fixed at points $(-a, 0)$ and $(a, 0)$. A third charge $+Q$ is placed at origin. Corresponding to small displacement of $+Q$ in the direction mentioned in column I, match the corresponding equilibrium of column II.

Column I	Column II
(a) Along positive x-axis	(p) stable equilibrium
(b) Along positive y-axis	(q) unstable equilibrium
(c) Along positive z-axis	(r) neutral equilibrium
(d) Along the line $x = y$	(s) no equilibrium

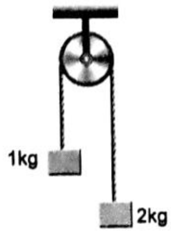
4. A block attached with a spring is released from A. Position-B is the mean position and the block moves to point C. Match the following two columns.



Column I	Column II
(a) From A to B decrease in gravitational potential energy is........ the increase in spring potential energy.	(p) less than
(b) From A to B increase in kinetic energy of block is......... the decrease in gravitational potential energy.	(q) more than
(c) From B to C decrease in kinetic energy of block is..... the increase in spring potential energy.	(r) equal to
(d) From B to C decrease in gravitational potential energy is .... the increase in spring potential energy.	

5. System shown in figure is released from rest. Friction is absent and string is massless. In time $t = 0.3$ s. Take $g = 10\,\text{ms}^{-2}$

Column I	Column II
(a) Work done by gravity on 2 kg block	(p) -1.5 J
(b) Work done by gravity on 1 kg block	(q) 2 J
(c) Work done by string on 2 kg block	(r) 3 J
(d) Work done by string on 1 kg block	(s) -2 J



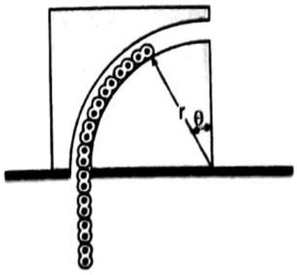
6. In column I some statements are given related to work done by a force on an object while in column II the sign and information about value of work done is given. Match the entries of Column I with the entries of Column II.

Column I	Column II
(a) Work done by friction force on the block as it slides down a rigid fixed incline with respect to ground.	(p) Positive
(b) In above case work done by friction force on incline with respect to ground.	(q) Negative
(c) Work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket with respect to ground.	(r) Zero
(d) Total work done by friction force in (a) with respect to ground.	(s) may be positive, negative or zero.

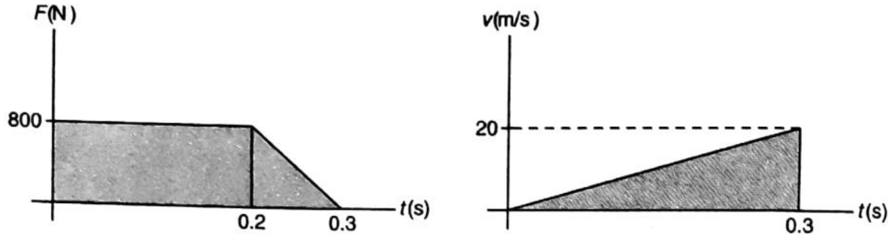
Subjective Questions (Level 2)

1. Two blocks of masses m_1 and m_2 connected by a light spring rest on a horizontal plane. The coefficient of friction between the blocks and the surface is equal to μ. What minimum constant force has to be applied in the horizontal direction to the block of mass m_1 in order to shift the other block?

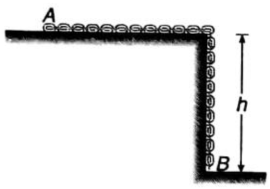
2. The flexible bicycle type chain of length $\frac{\pi r}{2}$ and mass per unit length ρ is released from rest with $\theta = 0°$ in the smooth circular channel and falls through the hole in the supporting surface. Determine the velocity v of the chain as the last link leaves the slot.



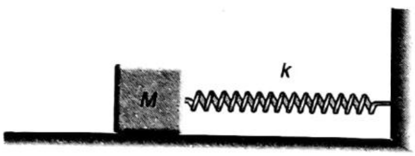
3. A baseball having a mass of 0.4 kg is thrown such that the force acting on it varies with time as shown in the first graph. The corresponding velocty time graph is shown in the second graph. Determine the power applied as a function of time and the work done till $t = 0.3$ s.



4. A chain AB of length l is loaded in a smooth horizontal table so that its fraction of length h hangs freely and touches the surface of the table with its end B. At a certain moment, the end A of the chain is set free, with what velocity will this end of the chain slip out of the table?

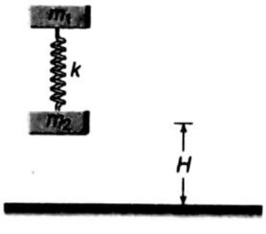


5. The block shown in the figure is acted on by a spring with spring constant k and a weak frictional force of constant magnitude f. The block is pulled a distance x_0 from equilibrium position and then released. It oscillates many times and ultimately comes to rest.



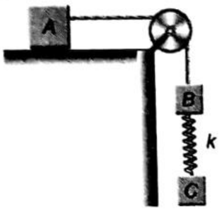
(a) Show that the decrease of amplitude is the same for each cycle of oscillation.
(b) Find the number of cycles the mass oscillates before coming to rest.

6. A spring mass system is held at rest with the spring relaxed at a height H above the ground. Determine the minimum value of H so that the system has a tendency to rebound after hitting the ground. Given that the coefficient of restitution between m_2 and ground is zero.



7. A block of mass m moving at a speed v compresses a spring through a distance x before its speed is halved. Find the spring constant of the spring.

8. In the figure shown masses of the blocks A, B and C are 6 kg, 2 kg and 1 kg respectively. Mass of the spring is negligibly small and its stiffness is 1000 N/m. The coefficient of friction between the block A and the table is $\mu = 0.8$. Initially block C is held such that spring is in relaxed position. The block is released from rest. Find : ($g = 10\,\text{m/s}^2$)



(a) the maximum distance moved by the block C.
(b) the acceleration of each block, when elongation in the spring is maximum.

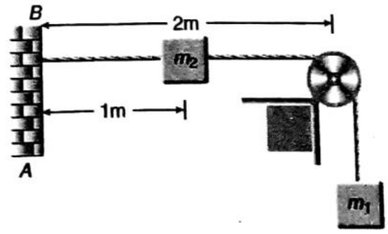
9. A body of mass m slides down a plane inclined at an angle α. The coefficient of friction is μ. Find the rate at which kinetic plus gravitational potential energy is dissipated at any time t.

10. A particle moving in a straight line is acted upon by a force which works at a constant rate and changes its velocity from u and v over a distance x. Prove that the time taken in it is

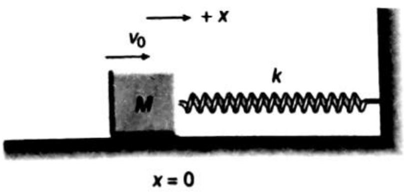
$$\frac{3}{2}\frac{(u+v)x}{u^2+v^2+uv}$$

11. A chain of length l and mass m lies on the surface of a smooth sphere of radius $R > l$ with one end tied to the top of the sphere. (a) Find the gravitational potential energy of the chain with reference level at the centre of the sphere. (b) Suppose the chain is released and slides down the sphere. Find the kinetic energy of the chain, when it has slid through an angle θ. (c) Find the tangential acceleration $\frac{dv}{dt}$ of the chain when the chain starts sliding down.

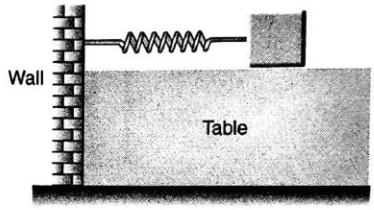
12. Find the speed of both the blocks arrangement at the moment the block m_2 hits the wall AB, after the blocks are released from rest. Given that $m_1 = 0.5$ kg and $m_2 = 2$ kg, ($g = 10\,\text{m/s}^2$)



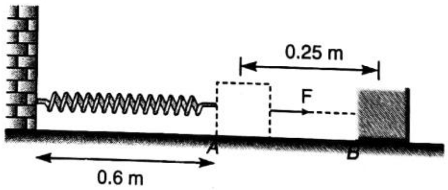
13. A block of mass M slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$ where b is a positive constant. Find the loss in mechanical energy when the block has first come momentarily to rest.



14. A small block of ice with mass 0.120 kg is placed against a horizontal compressed spring mounted on a horizontal table top that is 1.90 m above the floor. The spring has a force constant $k = 2300\,\text{N/m}$ and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the ice and the table, what is the speed of the block of ice when it reaches the floor. ($g = 9.8\,\text{m/s}^2$)



15. A 0.500 kg block is attached to a spring with length 0.60 m and force constant $k = 40.0\,\text{N/m}$. The mass of the spring is negligible. You pull the block to the right along the surface with a constant horizontal force $F = 20.0\,\text{N}$. (a) What is the block's speed when the block reaches point B, which is 0.25 m to the right of point A? (b) When the block reaches point B, you let go off the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached? Neglect size of block and friction.



ANSWERS

Introductory Exercise 6.1

1. $Tx, 0, 0, -Fx$ 2. $Tl\cos\beta$, 0, $-Fl$, $-Wl\sin\alpha$ 3. $-\frac{3}{4}mgl$ 4. (a) 7.2 J (b) − 7.2 J (c) zero
5. 10 J 6. $-Kx_0^2$ 7. 30 J

Introductory Exercise 6.2

1. − 400 J 2. Yes 3. $\frac{1}{2}m\alpha^2 b$ 4. 120 J 5. $\sqrt{2Rg(1+\sin\theta-\cos\theta)}$ 6. $v_0\sqrt{\frac{m}{A}}$
7. (a) False (b) False (c) True (d) False

Introductory Exercise 6.3

1. $\frac{m}{2}$ 2. $\sqrt{gl}$ 3. (a) 2.45 m/s (b) 2.15 m/s 4. $\left(M+\frac{m}{2}\right)gh$ 5. $\frac{3}{5}$ m

Introductory Exercise 6.4

1. 200 W 2. (a) $\frac{F^2t}{2m}$ (b) $\frac{F^2t}{m}$ 3. $K = Pt$, $v=\sqrt{\frac{2Pt}{m}}$, $S=\sqrt{\frac{8P}{9m}}t^{3/2}$
4. (a) $K=t^2$, $v=\sqrt{\frac{2}{m}}t$ (b) $P_{av}=t$ 5. $x = 2$ m, stable 6. $x = 4$ m, unstable

AIEEE Corner

Subjective Questions (Level 1)

1. 80 J, − 40 J 2. −2 J 3. (a) 480 J (b) 192 J 4. (a) 24.9 J (b) zero (c) zero (d) 24.9 J
5. (a) 34.6 J (b) −10 J 6. −12 J 7. −1 J 8. (a) −15 J (b) + 15 J (c) 3 J (d) 27 J
9. (a) 4.0 J (b) zero (c) −1.0 J (d) 3.0 J 10. $F_r=\frac{A}{r^2}$
11. (a) −4 J (b) 12 J (c) $x=(4-2\sqrt{3})$ m to $x=(4+2\sqrt{3})$ m (d) $F_x = 8-2x$ (e) $x = 4$ m
12. 125% 13. 0.5% 14. 32 J 15. −597.6 J 16. −750 J 17. (a) 2.33 J (b) 2.635 J
18. (a) 11642 J (b) −10584 J (c) 1058 J (d) 5.42 ms^{-1} 19. 2 ms^{-1} 20. 0.796 m
21. $v_w=\sqrt{2gR}\cos\alpha$, $v_s=\sqrt{2gR}\sin\alpha$ 22. $v = 0.371\ ms^{-1}$ 23. $\mu_k = 0.12$ 24. − 0.051 J
25. 1.12 ms^{-1} 26. (a) 66.88 J (b) 66.88 J (c) 45.7 cm 27. 4 m/s
28. (a) F_A and F_B are negative, F_C and F_D are positive and F_E is zero
(b) $x = 2$ m is unstable equilibrium position
$x = 6$ m is stable equilibrium position
There is no point of neutral equilibrium.
29. $x = 2$ m is position of stable equilibrium. $x = -2$ m is position of unstable equilibrium.
30. Points A and E are unstable equilibrium positions. Point C is stable equilibrium position.
31. (a) unstable (b) stable 32. (a) 16 W (b) 64 W 33. $\frac{P}{r}, \frac{Pm}{8r^2}$

Objective Questions (Level 1)

1. (a)	2. (a)	3. (d)	4. (b)	5. (d)	6. (c)	7. (b)	8. (b)	9. (c)	10. (b)
11. (b)	12. (a)	13. (c)	14. (c)	15. (a)	16. (b)	17. (a)	18. (b)	19. (b)	20. (d)
21. (b)	22. (b)	23. (a)	24. (d)	25. (c)	26. (d)	27. (a)	28. (b)	29. (a)	30. (a)
31. (a)	32. (d)	33. (b)	34. (c)	35. (b)	36. (c)	37. (a)	38. (a)	39. (c)	

JEE Corner

Assertion and Reason

1. (d) 2. (a) 3. (d) 4. (d) 5. (a) 6. (c) 7. (b) 8. (a) 9. (d) 10. (a)
11. (a) 12. (c)

Objective Questions (Level 2)

1. (a) 2. (a) 3. (a) 4. (c) 5. (b) 6. (b) 7. (a) 8. (d) 9. (c) 10. (c)
11. (c) 12. (b) 13. (c) 14. (b) 15. (a) 16. (d) 17. (c) 18. (a) 19. (c) 20. (a)
21. (c) 22. (c) 23. (a) 24. (c) 25. (b) 26. (b) 27. (c) 28. (c) 29. (b) 30. (b)
31. (b) 32. (a) 33. (d)

More than One Correct Options

1. (a,b) 2. (a,b,c) 3. (a,b) 4. (b,c) 5. (b,d) 6. (a) 7. (b,c)
8. (b,c,d) 9. (b,c) 10. (all)

Match the Columns

1. (a) → (q,s) (b) → (q,s) (c) → (p,s) (d) → (p,s)
2. (a) → (p) (b) → (q) (c) → (r) (d) → (p)
3. (a) → (p) (b) → (q) (c) → (q) (d) → (s)
4. (a) → (q) (b) → (p) (c) → (p) (d) → (p)
5. (a) → (r) (b) → (p) (c) → (s) (d) → (q)
6. (a) → (q) (b) → (r) (c) → (p) (d) → (q)

Subjective Questions (Level 2)

1. $\left(m_1 + \frac{m_2}{2}\right)\mu g$ 2. $\sqrt{gr\left(\frac{\pi}{2} + \frac{4}{\pi}\right)}$

3. For $t \le 0.2$ s, $P = (53.3\,t)$ kW, for $t > 0.2$ s, $P = (160t - 533t^2)$ kW, 1.69 kJ. 4. $\sqrt{2gh \ln\left(\frac{l}{h}\right)}$

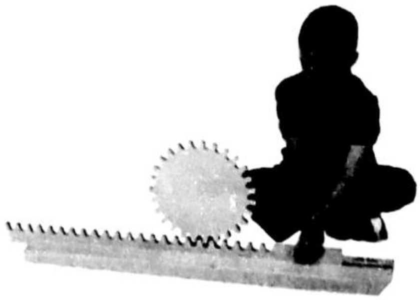
5. (b) $\frac{1}{4}\left[\frac{kx_0}{f} - 1\right]$ 6. $H_{min} = \frac{m_2 g}{k}\left[\frac{m_2 + 2m_1}{2m_1}\right]$ 7. $\frac{3mv^2}{4x^2}$

8. (a) 2×10^{-2}m (b) $a_A = a_B = 0$, $a_C = 10$ m/s^2 9. $\mu m g^2 \cos\alpha(\sin\alpha - \mu\cos\alpha)t$

11. (a) $\frac{mR^2 g}{l}\sin\left(\frac{l}{R}\right)$ (b) $\frac{mR^2 g}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$ (c) $\frac{Rg}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right]$

12. $v_1 = 3.03$ ms^{-1}, $v_2 = 3.39$ ms^{-1} 13. $\frac{bgV_0^2 M^2}{2(k + bMg)}$ 14. 8.72 ms^{-1} 15. (a) 3.87 ms^{-1} (b) 0.10 m

Chapter 7 – Circular Motion



7

Circular Motion

Chapter Contents

7.1 Kinematics of Circular Motion

Angular Variables

Circular motion is a two dimensional motion or motion in a plane.

Suppose a particle P is moving in a circle of radius r and centre O.

The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle. As the particle moves on the circle its angular position θ changes. Suppose the point rotates an angle $\Delta\theta$ in time Δt. The rate of change of angular position is known as the **angular velocity** (ω). Thus,

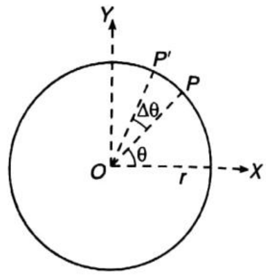


Fig. 7.1

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The rate of change of angular velocity is called the **angular acceleration** (α). Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t \quad \text{and} \quad \omega^2 = \omega_0^2 + 2\alpha\theta$$

Here, ω_0 and ω are the angular velocities at time $t = 0$ and t and θ the angular position at time t. The linear distance PP' travelled by the particle in time Δt is

$$\Delta s = r\Delta\theta$$

or $$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

or $$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

or $$v = r\omega$$

Here, v is the linear speed of the particle.

Differentiating again with respect to time, we have

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \quad \text{or} \quad a_t = r\alpha$$

Here, $a_t = \dfrac{dv}{dt}$ is the rate of change of speed (not the rate of change of velocity). This is called the tangential acceleration of the particle. Later, we will see that a_t is the component of net acceleration $\vec{a}$ of the particle moving in a circle along the tangent.

Unit vectors along the radius and the tangent

Consider a particle P moving in a circle of radius r and centre at origin O. The angular position of the particle at some instant is say θ. Let us here define two unit vectors, one is $\hat{e}_r$ (called radial unit vector) which is along OP and the other is $\hat{e}_t$ (called the tangential unit vector) which is perpendicular to OP. Now, since

$$|\hat{e}_r| = |\hat{e}_t| = 1$$

We can write these two vectors as $$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

and $$\hat{e}_t = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Velocity and acceleration of particle in circular motion: The position vector of particle P at the instant shown in figure can be written as

$$\vec{\mathbf{r}} = \overrightarrow{\mathbf{OP}} = r\hat{\mathbf{e}}_r$$

or

$$\vec{\mathbf{r}} = r(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$$

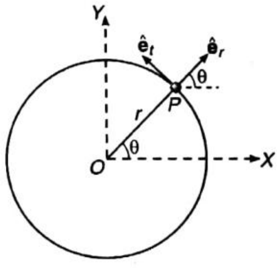


Fig. 7.2

The velocity of the particle can be obtained by differentiating $\vec{\mathbf{r}}$ with respect to time t. Thus,

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}[r(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})]$$

$$= r\left[\left(-\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}}\right]$$

or

$$\vec{\mathbf{v}} = r\omega[-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}] \qquad \left(\because \frac{d\theta}{dt} = \omega\right) \quad \text{...(i)}$$

∴

$$\vec{\mathbf{v}} = r\omega\,\hat{\mathbf{e}}_t$$

Thus, we see that velocity of the particle is $r\omega$ along $\hat{\mathbf{e}}_t$ or in tangent direction. Acceleration of the particle can be obtained by differentiating Eq. (i) with respect to time t. Thus,

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}[r\omega(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})]$$

$$= r\left[\omega\frac{d}{dt}(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) + \frac{d\omega}{dt}(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\right]$$

$$= \omega r\left[-\cos\theta\frac{d\theta}{dt}\hat{\mathbf{i}} - \sin\theta\frac{d\theta}{dt}\hat{\mathbf{j}}\right] + r\frac{d\omega}{dt}\hat{\mathbf{e}}_t$$

$$= -\omega^2 r[\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}] + r\frac{d\omega}{dt}\hat{\mathbf{e}}_t$$

$$\vec{\mathbf{a}} = -\omega^2 r\,\hat{\mathbf{e}}_r + \frac{dv}{dt}\hat{\mathbf{e}}_t$$

Thus, acceleration of a particle moving in a circle has two components one is along $\hat{\mathbf{e}}_t$ (along tangent) and the other along $-\hat{\mathbf{e}}_r$ (or towards centre). Of these the first one is called the tangential acceleration. (a_t) and the other is called radial or centripetal acceleration (a_r). Thus,

$$a_t = \frac{dv}{dt} = \text{rate of change of speed}$$

and

$$a_r = r\omega^2 = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$$

Here, the two components are mutually perpendicular. Therefore, net acceleration of the particle will be:

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(r\omega^2)^2 + \left(\frac{dv}{dt}\right)^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Following three points are important regarding the above discussion:

1. In uniform circular motion, speed (v) of the particle is constant, *i.e.*, $\frac{dv}{dt} = 0$. Thus,

$$a_t = 0 \quad \text{and} \quad a = a_r = r\omega^2$$

2. In accelerated circular motion, $\frac{dv}{dt}$ = positive, *i.e.*, a_t is along $\hat{\mathbf{e}}_t$ or tangential acceleration of particle is parallel to velocity $\vec{\mathbf{v}}$ because $\vec{\mathbf{v}} = r\omega\,\hat{\mathbf{e}}_t$ and $\vec{\mathbf{a}}_t = \frac{dv}{dt}\,\hat{\mathbf{e}}_t$.

3. In decelerated circular motion, $\frac{dv}{dt}$ = negative and hence, tangential acceleration is anti-parallel to velocity $\vec{\mathbf{v}}$.

Sample Example 7.1 *A particle moves in a circle of radius 0.5 m with a linear speed of 2 m/s. Find its angular speed.*

Solution The angular speed is

$$\omega = \frac{v}{r} = \frac{2}{0.5} = 4 \text{ rad/s}$$

Sample Example 7.2 *A particle moves in a circle of radius 0.5 m at a speed that uniformly increases. Find the angular acceleration of particle if its speed changes from 2.0 m/s to 4.0 m/s in 4.0 s.*

Solution The tangential acceleration of the particle is

$$a_t = \frac{dv}{dt} = \frac{4.0 - 2.0}{4.0}$$

$$= 0.5 \text{ m/s}^2$$

The angular acceleration is :

$$\alpha = \frac{a_t}{r} = \frac{0.5}{0.5} = 1 \text{ rad/s}^2$$

Sample Example 7.3 *The speed of a particle moving in a circle of radius $r = 2$ m varies with time t as $v = t^2$ where t is in second and v in m/s. Find the radial, tangential and net acceleration at $t = 2$ s.*

Solution Linear speed of particle at $t = 2$ s is

$$v = (2)^2 = 4 \text{ m/s}$$

$\therefore$ Radial acceleration $\quad a_r = \frac{v^2}{r} = \frac{(4)^2}{2} = 8 \text{ m/s}^2$

The tangential acceleration is $\quad a_t = \frac{dv}{dt} = 2t$

$\therefore$ Tangential acceleration at $t = 2$ s is

$$a_t = (2)(2) = 4 \text{ m/s}^2$$

$\therefore$ Net acceleration of particle at $t = 2$s is

$$a = \sqrt{(a_r)^2 + (a_t)^2} = \sqrt{(8)^2 + (4)^2}$$

or $$a = \sqrt{80} \text{ m/s}^2$$

Note *On any curved path (not necessarily a circular one) the acceleration of the particle has two components a_t and a_n in two mutually perpendicular directions. Component of $\vec{\mathbf{a}}$ along $\vec{\mathbf{v}}$ is a_t and perpendicular to $\vec{\mathbf{v}}$ is a_n. Thus,*

$$|\vec{\mathbf{a}}| = \sqrt{a_t^2 + a_n^2}$$

Introductory Exercise 7.1

1. Is the acceleration of a particle in uniform circular motion constant or variable?
2. Is it necessary to express all angles in radian while using the equation $\omega = \omega_0 + \alpha t$?
3. Which of the following quantities may remain constant during the motion of an object along a curved path?
   (i) Velocity (ii) Speed (iii) Acceleration (iv) Magnitude of acceleration
4. A particle moves in a circle of radius 1.0 cm with a speed given by $v = 2t$ where v is in cm/s and t in seconds.
   (a) Find the radial acceleration of the particle at $t = 1$ s.
   (b) Find the tangential acceleration at $t = 1$ s.
   (c) Find the magnitude of net acceleration at $t = 1$ s.
5. A particle is moving with a constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \left(\frac{\pi}{2}\right)$.
6. A particle is moving with a constant angular acceleration of 4 rad/s^2 in a circular path. At time $t = 0$, particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

7.2 Dynamics of Circular Motion

If a particle moves with constant speed in a circle, motion is called uniform circular. In uniform circular motion a resultant non-zero force acts on the particle. This is because a particle moving in a circle is accelerated even if speed of the particle is constant. This acceleration is due to the change in direction of the velocity vector. As we have seen in Article 7.1 that in uniform circular motion tangential acceleration (a_t) is zero. The acceleration of the particle is towards the centre and its magnitude is $\frac{v^2}{r}$. Here, v is the speed of the particle and r the radius of the circle. The direction of the resultant force F is, therefore, towards centre and its magnitude is

$$F = ma \quad \text{or} \quad F = \frac{mv^2}{r}$$

or $$F = mr\omega^2 \qquad \text{(as } v = r\omega\text{)}$$

Here, ω is the angular speed of the particle. This force F is called the centripetal force. Thus, a centripetal force of magnitude $\frac{mv^2}{r}$ is needed to keep the particle moving in a circle with constant speed. This force is provided by some external source such as friction, magnetic force, Coulomb force, gravitation, tension, etc.

Note *I have found students often confused over the centripetal force. They think that this force acts on a particle moving in a circle. This force does not act but required for moving in a circle which is being provided by the other forces acting on the particle. Let, us take an example. Suppose a particle of mass 'm' is moving in a vertical circle with the help of a string of length l fixed at point O. Let v be the speed of the particle at lowest position. When I ask the students what forces are acting on the particle in this position? They immediately say, three forces are acting on the particle (1) tension T (2) weight mg and (3) centripetal force* $\frac{mv^2}{l}$ *(r = l). However, they are wrong. Only first two forces T and mg are acting on the particle. Third force* $\frac{mv^2}{l}$ *is required for circular motion which is being provided by T and mg. Thus, the resultant of these two forces is* $\frac{mv^2}{l}$ *towards O. Or we can write*

$$T - mg = \frac{mv^2}{l}$$

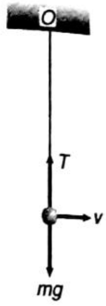


Fig. 7.3

Circular Turning of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways:

1. By friction only.
2. By banking of roads only.
3. By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both. Now, let us write equations of motion in each of the three cases separately and see what are the constraints in each case.

1. By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards centre.

Thus, $$f = \frac{mv^2}{r}$$

Further, limiting value of f is μN.

or $$f_L = \mu N = \mu mg \qquad (N = mg)$$

Therefore, for a safe turn without sliding

$$\frac{mv^2}{r} \le f_L \quad \text{or} \quad \frac{mv^2}{r} \le \mu mg$$

or $$\mu \ge \frac{v^2}{rg} \quad \text{or} \quad v \le \sqrt{\mu rg}$$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

Note *You might have seen that if the speed of the car is too high, car starts skidding outwards. With this radius of the circle increases or the necessary centripetal force is reduced (centripetal force* $\propto \frac{1}{r}$*).*

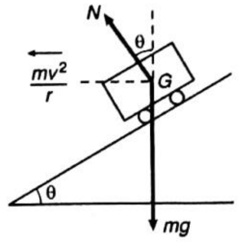


Fig. 7.4

2. By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

Applying Newton's second law along the radius and the first law in the vertical direction.

$$N \sin\theta = \frac{mv^2}{r} \quad \text{and} \quad N\cos\theta = mg$$

From these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \qquad \text{...(i)}$$

or

$$v = \sqrt{rg\tan\theta} \qquad \text{...(ii)}$$

Note *This is the speed at which car does not slide down even if track is smooth. If track is smooth and speed is less than $\sqrt{rg\tan\theta}$, vehicle will move down so that r gets decreased and if speed is more than this vehicle will move up.*

3. By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, *i.e.*, weight (mg) is fixed both in magnitude and direction. The direction of

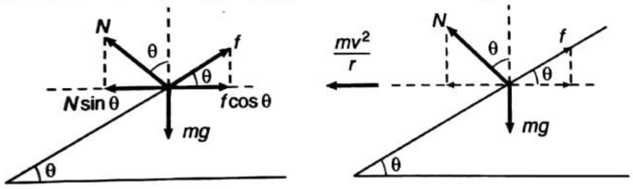


Fig. 7.5

second force, *i.e.*, normal reaction N is also fixed (perpendicular to road) while the direction of the third force, *i.e.*, friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit $(f_L = \mu N)$. So, the magnitude of normal reaction N and direction plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre. Of these m and r are also constant. Therefore, magnitude of N and direction plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that:

(i) Friction f is outwards if the vehicle is at rest or $v = 0$. Because in that case the component of weight $mg\sin\theta$ is balanced by f.

(ii) Friction f is inwards if $\quad v > \sqrt{rg\tan\theta}$

(iii) Friction f is outwards if $\quad v < \sqrt{rg\tan\theta}$

(iv) Friction f is zero if $\quad v = \sqrt{rg\tan\theta}$

Let us now see how the force of friction and normal reaction change as speed is gradually increased.

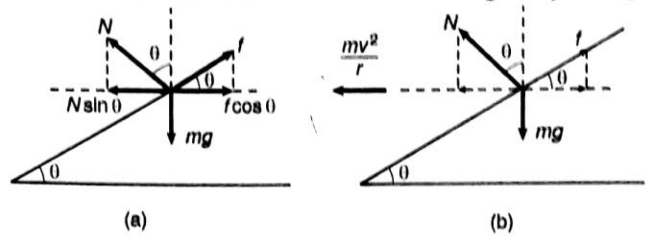


Fig. 7.6

In figure (a) : When the car is at rest force of friction is upwards. We can resolve the forces in any two mutually perpendicular directions. Let us resolve them in horizontal and vertical directions.

$$\Sigma F_H = 0 \quad \therefore \quad N \sin \theta - f \cos \theta = 0 \qquad \text{...(i)}$$

$$\Sigma F_V = 0 \quad \therefore \quad N \cos \theta + f \sin \theta = mg \qquad \text{...(ii)}$$

In figure (b) : Now, the car is given a small speed v, so that a centripetal force $\frac{mv^2}{r}$ is now required in horizontal direction towards centre. So, Eq. (i) will now become,

$$N \sin \theta - f \cos \theta = \frac{mv^2}{r}$$

or we can say, in case (a) $N \sin \theta$ and $f \cos \theta$ were equal while in case (b) their difference is $\frac{mv^2}{r}$. This can occur in following three ways:

(i) N increases while f remains same.
(ii) N remains same while f decreases or
(iii) N increases and f decreases.

But only third case is possible, *i.e.*, N will increase and f will decrease. This is because equation number (ii), $N \cos \theta + f \sin \theta = mg$ is still has to be valid.

So, to keep $N \cos \theta + f \sin \theta$ to be constant $(= mg)$ N should increase and f should decrease (as θ = constant)

Now, as speed goes on increasing, force of friction first decreases. Becomes zero at $v = \sqrt{rg \tan \theta}$ and then reverses it direction.

Let us take an example which illustrates the theory.

Sample Example 7.4 *A turn of radius 20 m is banked for the vehicle of mass 200 kg going at a speed of 10 m/s. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed*

(a) 5 m/s

(b) 15 m/s. Assume that friction is sufficient to prevent slipping. $(g = 10 \text{ m/s}^2)$

Solution (a) The turn is banked for speed $v = 10 \text{ m/s}$

Therefore, $$\tan \theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$$

Now, as the speed is decreased, force of friction f acts upwards.

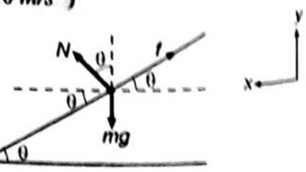


Fig. 7.7

Using the equations

$$\Sigma F_x = \frac{mv^2}{r}$$

and $\Sigma F_y = 0$, we get

$$N \sin\theta - f\cos\theta = \frac{mv^2}{r} \quad \text{...(i)}$$

$$N\cos\theta + f\sin\theta = mg \quad \text{...(ii)}$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 5$ m/s, $m = 200$ kg and $r = 20$ m, in the above equations, we get

$$f = 300\sqrt{5}\text{ N} \quad \text{(outwards)}$$

(b) In the second case force of friction f will act downwards.

Using $\Sigma F_x = \frac{mv^2}{r}$

and $\Sigma F_y = 0$, we get

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \text{...(iii)}$$

$$N\cos\theta - f\sin\theta = mg \quad \text{...(iv)}$$

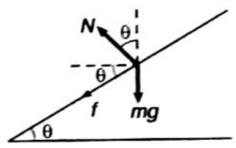


Fig. 7.8

Substituting $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 15$ m/s, $m = 200$ kg and $r = 20$ m in the above equations, we get

$$f = 500\sqrt{5}\text{ N} \quad \text{(downwards)}$$

Conical Pendulum

If a small particle of mass m tied to a string is whirled in a horizontal circle, as shown in Fig. 7.9 The arrangement is called the 'conical pendulum'. In case of conical pendulum the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force. Thus,

$$T\sin\theta = \frac{mv^2}{r} \quad \text{...(i)}$$

and

$$T\cos\theta = mg \quad \text{...(ii)}$$

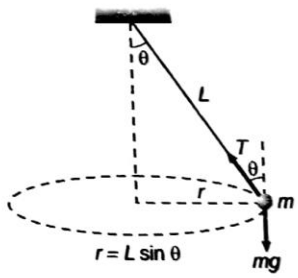


Fig. 7.9

From these two equations, we can find

$$v = \sqrt{rg\tan\theta}$$

∴ Angular speed

$$\omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}}$$

So, the time period of pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

$$= 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

or $$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Note *This is similar to the case, when necessary centripetal force to vehicles is provided by banking. The only difference is that the normal reaction is being replaced by the tension.*

'Death Well' or Rotor

In case of 'death well' a person drives a bicycle on a vertical surface of a large wooden well while in case of a rotor at a certain angular speed of rotor a person hangs resting against the wall without any support from the bottom. In death well walls are at rest and person revolves while in case of rotor person is at rest and the walls rotate.

In both cases friction balances the weight of person while reaction provides the centripetal force for circular motion, *i.e.*,

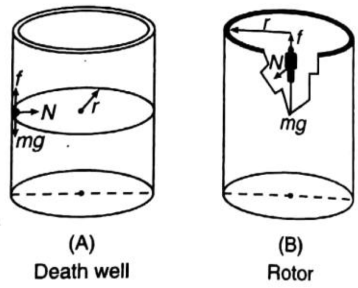


Fig. 7.10

$$f = mg$$

and $$N = \frac{mv^2}{r} = mr\omega^2 \qquad (v = r\omega)$$

A cyclist on the bend of a road

In figure,

$$F = \sqrt{N^2 + f^2}$$

When the cyclist is inclined to the centre of the rounding of its path, the resultant of N, f and mg is directed horizontally to the centre of the circular path of the cycle. This resultant force imparts a centripetal acceleration to the cyclist.

Resultant of N and f, *i.e.*, F should pass through G, the centre of gravity of cyclist (for complete equilibrium, rotational as well as translational). Hence,

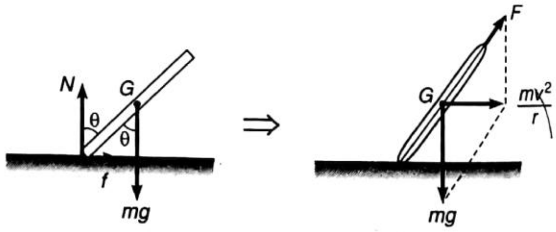


Fig. 7.11

$$\tan\theta = \frac{f}{N}$$

where $$f = \frac{mv^2}{r} \text{ and } N = mg$$

$\therefore$ $$\tan\theta = \frac{v^2}{rg}$$

Centrifugal Force

Newton's laws are valid only in inertial frames. In non-inertial frames a pseudo force $-m\vec{a}$ has to be applied on a particle of mass m ($\vec{a}$ = acceleration of frame of reference). After applying the pseudo force one can apply Newton's laws in their usual form. Now, suppose a frame of reference is rotating with constant angular velocity ω in a circle of radius 'r'. Then it will become a non-inertial frame of acceleration $r\omega^2$ towards the centre. Now, if we see an object of mass 'm' from this frame then obviously a pseudo force of magnitude

$mr\omega^2$ will have to be applied to this object in a direction away from the centre. This pseudo force is called the centrifugal force. After applying this force we can now apply Newton's laws in their usual form. Following example will illustrate the concept more clearly.

Sample Example 7.5 *A particle of mass m is placed over a horizontal circular table rotating with an angular velocity ω about a vertical axis passing through its centre. The distance of the object from the axis is r. Find the force of friction f between the particle and the table.*

Solution Let us solve this problem from both frames. The one is a frame fixed on ground and the other is a frame fixed on table itself.

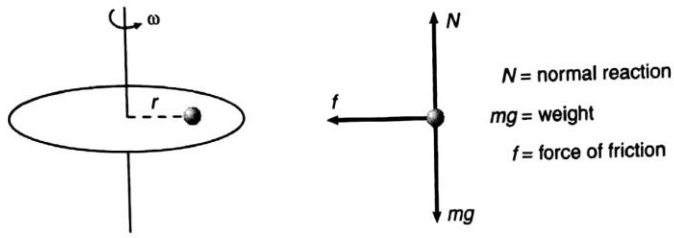


Fig. 7.12

From frame of reference fixed on ground (inertial)

Here, N will balance its weight and the force of friction f will provide the necessary centripetal force.

Thus, $f = mr\omega^2$ **Ans.**

From frame of reference fixed on table itself (non-inertial)

In the free body diagram of particle with respect to table, in addition to above three forces (N, mg and f) a pseudo force of magnitude $mr\omega^2$ will have to be applied in a direction away from the centre. But one thing should be clear that in this frame the particle is in equilibrium, *i.e.*, N will balance its weight in vertical direction while f will balance the pseudo force in horizontal direction.

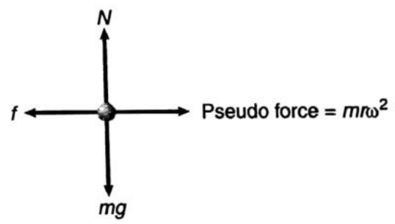


Fig. 7.13

or $f = mr\omega^2$ **Ans.**

Thus, we see that 'f' comes out to be $mr\omega^2$ from both the frames.

Now, let us take few more examples of circular motion.

Sample Example 7.6 *A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Find the tension in the string and the magnitude of net force on the bob at that instant.*

Solution (i) The forces acting on the bob are :

(a) the tension T

(b) the weight mg

As the bob moves in a circle of radius L with centre at O. A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O. This force will be provided by the resultant of T and $mg \cos \alpha$. Thus,

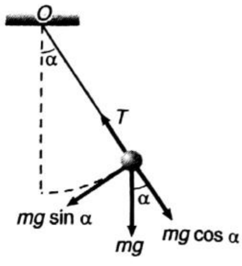


Fig. 7.14

or $$T - mg \cos \alpha = \frac{mv^2}{L}$$

$$T = m\left(g \cos \alpha + \frac{v^2}{L}\right)$$

(ii) $$|\vec{\mathbf{F}}_{net}| = \sqrt{(mg \sin \alpha)^2 + \left(\frac{mv^2}{L}\right)^2} = m\sqrt{g^2 \sin^2 \alpha + \frac{v^4}{L^2}}$$

Sample Example 7.7 *A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.*

Solution Let ω be the angular speed of rotation of the bowl. Two forces are acting on the ball.

1. normal reaction N
2. weight mg

The ball is rotating in a circle of radius $r (= R \sin \alpha)$ with centre at A at an angular speed ω. Thus,

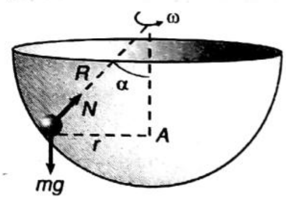


Fig. 7.15

$$N \sin \alpha = mr\omega^2$$

$$= mR\omega^2 \sin \alpha \qquad ...(i)$$

and $$N \cos \alpha = mg \qquad ...(ii)$$

Dividing Eq. (i) by (ii), we get

$$\frac{1}{\cos \alpha} = \frac{\omega^2 R}{g}$$

$$\therefore \qquad \omega = \sqrt{\frac{g}{R \cos \alpha}}$$

Introductory Exercise 7.2

1. Is a body in uniform circular motion in equilibrium?
2. Find the maximum speed at which a truck can safely travel without toppling over, on a curve of radius 250 m. The height of the centre of gravity of the truck above the ground is 1.5 m and the distance between the wheels is 1.5 m, the truck being horizontal.
3. (a) How many revolutions per minute must the apparatus shown in figure make about a vertical axis so that the cord makes an angle of 45° with the vertical?
   (b) What is the tension in the cord then. Given, $l = \sqrt{2}$ m, $a = 20$ cm and $m = 5.0$ kg?

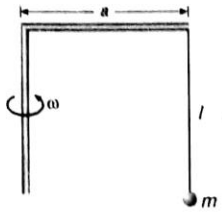


Fig. 7.16

4. A car moves at a constant speed on a straight but hilly road. One section has a crest and dip of the same 250 m radius.
   (a) As the car passes over the crest the normal force on the car is one half the 16 kN weight of the car. What will be the normal force on the car as its passes through the bottom of the dip?
   (b) What is the greatest speed at which the car can move without leaving the road at the top of the hill?
   (c) Moving at a speed found in part (b) what will be the normal force on the car as it moves through the bottom of the dip? (Take $g = 10\ \text{m/s}^2$)
5. A car driver going at speed v suddenly finds a wide wall at a distance r. Should he apply brakes or turn the car in a circle of radius r to avoid hitting the wall.
6. Show that the angle made by the string with the vertical in a conical pendulum is given by $\cos\theta = \frac{g}{L\omega^2}$ where L is the length of the string and ω is the angular speed.

7.3 Motion in a Vertical Circle

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point B of the circle as shown in figure. Here,

$$h = R(1 - \cos\theta) \qquad \ldots\text{(i)}$$

From conservation of mechanical energy

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

or

$$v^2 = u^2 - 2gh \qquad \ldots\text{(ii)}$$

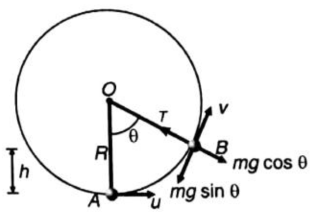


Fig. 7.17

The necessary centripetal force is provided by the resultant of tension T and $mg\cos\theta$

$\therefore$
$$T - mg\cos\theta = \frac{mv^2}{R} \qquad \ldots\text{(iii)}$$

Now, following three conditions arise depending on the value of u.

Condition of Looping the Loop ($u \geq \sqrt{5gR}$)

The particle will complete the circle if the string does not slack even at the highest point ($\theta = \pi$). Thus, tension in the string should be greater than or equal to zero ($T \geq 0$) at $\theta = \pi$. In critical case substituting $T = 0$ and $\theta = \pi$ in Eq. (iii), we get

$$mg = \frac{mv_{\text{min}}^2}{R}$$

or

$$v_{\text{min}}^2 = gR$$

or $$v_{min} = \sqrt{gR} \quad \text{(at highest point)}$$

Substituting $\theta = \pi$ in Eq. (i), $$h = 2R$$

Therefore, from Eq. (ii)

$$u_{min}^2 = v_{min}^2 + 2gh$$

or $$u_{min}^2 = gR + 2g(2R) = 5\,gR$$

or $$u_{min} = \sqrt{5\,gR}$$

Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At $u = \sqrt{5gR}$, velocity at highest point is $v = \sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0°$ and $v = \sqrt{5gR}$ in Eq. (iii), we get $T = 6\,mg$ or in the critical condition tension in the string at lowest position is $6\,mg$. This is shown in figure.

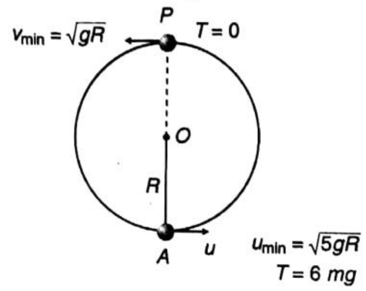


Fig. 7.18

If $u < \sqrt{5gR}$, following two cases are possible.

Condition of Leaving the Circle ($\sqrt{2gR} < u < \sqrt{5gR}$)

If $u < \sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From Eq. (iii), tension in the string becomes zero ($T = 0$)

where, $$\cos\theta = \frac{-v^2}{Rg}$$

or $$\cos\theta = \frac{2gh - u^2}{Rg}$$

Substituting, this value of $\cos\theta$ in Eq. (i), we get

$$\frac{2gh - u^2}{Rg} = 1 - \frac{h}{R}$$

or $$h = \frac{u^2 + Rg}{3g} = h_1 \text{ (say)} \qquad \text{...(iv)}$$

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle becomes zero when

$$0 = u^2 - 2gh$$

or $$h = \frac{u^2}{2g} = h_2 \text{ (say)} \quad \ldots(v)$$

i.e., at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero. or $T = 0$ but $v \neq 0$. This is possible only when

$$h_1 < h_2$$

or $$\frac{u^2 + Rg}{3g} < \frac{u^2}{2g}$$

or $$2u^2 + 2Rg < 3u^2$$

or $$u^2 > 2Rg$$

or $$u > \sqrt{2Rg}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

From Eq. (iv), we can see that $h > R$ if $u^2 > 2gR$. Thus, the particle, will leave the circle when $h > R$ or $90° < \theta < 180°$. This situation is shown in the figure.

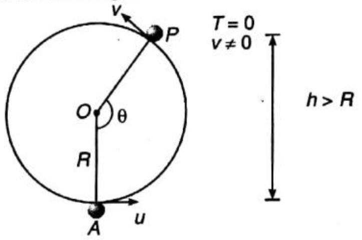


Fig. 7.19

$$\sqrt{2gR} < u < \sqrt{5gR} \quad \text{or} \quad 90° < \theta < 180°$$

Note *That after leaving the circle, the particle will follow a parabolic path.*

Condition of Oscillation ($0 < u \leq \sqrt{2gR}$)

The particle will oscillate, if velocity of the particle becomes zero but tension in the string is not zero. or $v = 0$, but $T \neq 0$. This is possible when

$$h_2 < h_1$$

or $$\frac{u^2}{2g} < \frac{u^2 + Rg}{3g} \quad \text{or} \quad 3u^2 < 2u^2 + 2Rg$$

or $$u^2 < 2Rg \quad \text{or} \quad u < \sqrt{2Rg}.$$

Moreover, if $h_1 = h_2, u = \sqrt{2Rg}$ and tension and velocity both becomes zero simultaneously.

Further, from Eq. (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$. Thus, for $0 < u \leq \sqrt{2gR}$, particle oscillates in lower half of the circle ($0° < \theta \leq 90°$). This situation is shown in the figure.

$$0 < u \leq \sqrt{2gR} \quad \text{or} \quad 0° < \theta \leq 90°$$

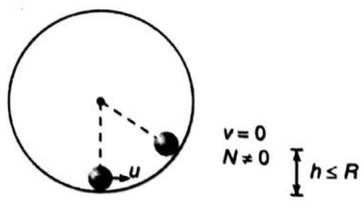


Fig. 7.20

Note *The above three conditions have been derived for a particle moving in a vertical circle attached to a string. The same conditions apply, if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.*

Condition of looping the loop is $u \geq \sqrt{5\,gR}$

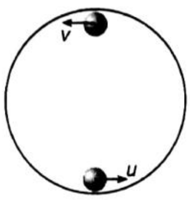


Fig. 7.21

Condition of leaving the circle $\sqrt{2gR} < u < \sqrt{5gR}$

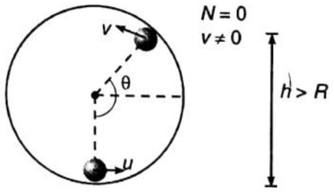


Fig. 7.22

Condition of oscillation is $0 < u \leq \sqrt{2\,gR}$

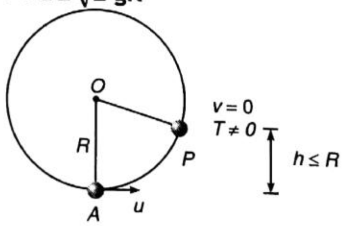


Fig. 7.23

Sample Example 7.8 *A heavy particle hanging from a fixed point by a light inextensible string of length l is projected horizontally with speed $\sqrt{gl}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.*

Solution Let $T = mg$ at angle θ as shown in figure.

$$h = l(1 - \cos\theta) \qquad \ldots\text{(i)}$$

Applying conservation of mechanical energy between points A and B, we get

$$\frac{1}{2} m(u^2 - v^2) = mgh$$

Here, $$u^2 = gl \qquad \ldots\text{(ii)}$$

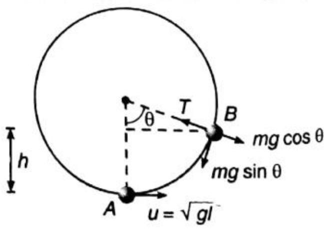


Fig. 7.24

and $\quad v$ = speed of particle in position B

$\therefore$ $$v^2 = u^2 - 2gh \quad \text{...(iii)}$$

Further, $$T - mg\cos\theta = \frac{mv^2}{l}$$

or $$mg - mg\cos\theta = \frac{mv^2}{l} \quad (T = mg)$$

or $$v^2 = gl(1 - \cos\theta) \quad \text{...(iv)}$$

Substituting values of v^2, u^2 and h from Eqs. (iv), (ii) and (i) in Eq. (iii), we get

$$gl(1 - \cos\theta) = gl - 2gl(1 - \cos\theta)$$

or $$\cos\theta = \frac{2}{3}$$

or $$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Substituting $\cos\theta = \frac{2}{3}$ in Eq. (iv), we get

$$v = \sqrt{\frac{gl}{3}}$$

Introductory Exercise 7.3

1. A stone tied to a string of length L is whirled in a vertical circle with the other end of the string at the centre. At a certain instant of time the stone is at it lowest position and has a speed u. Find the magnitude of the change in its velocity as it reaches a position, where the string is horizontal.

2. With what minimum speed v must a small ball should be pushed inside a smooth vertical tube from a height h so that it may reach the top of the tube? Radius of the tube is R.

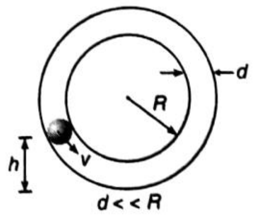


Fig. 7.25

3. A bob is suspended from a crane by a cable of length $l = 5$ m. The crane and load are moving at a constant speed v_0. The crane is stopped by a bumper and the bob on the cable swings out an angle of 60°. Find the initial speed v_0. (g = 9.8 m/s^2)

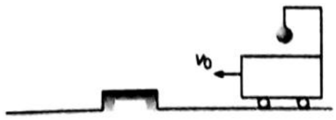


Fig. 7.26

Extra Points

- If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottommost point is not $\sqrt{5gR}$. Because in this case, velocity of the particle at the topmost point can be zero also. Using conservation of mechanical energy between points A and B as shown in Fig. 7.27, we get

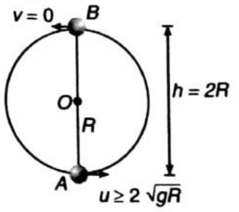


Fig. 7.27

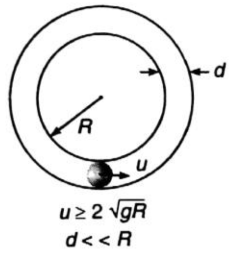


Fig. 7.28

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

or $$\frac{1}{2}mu^2 = mg\,(2R) \qquad (\text{as } v = 0)$$

$\therefore$ $$u = 2\sqrt{gR}$$

Therefore, the minimum value of u in this case is $2\sqrt{gR}$.

Same is the case when a particle is compelled to move inside a smooth vertical tube as shown in Fig. 7.28.

- In uniform circular motion although the speed of the particle remains constant yet the particle is accelerated due to change in direction of velocity. Therefore, the forces acting on the particle in uniform circular motion can be resolved in two directions one along the radius (parallel to acceleration) and another perpendicular to radius (perpendicular to acceleration). Along the radius net force should be equal to $\frac{mv^2}{R}$ and perpendicular to it net force should be zero.
- Oscillation of a pendulum is part of a circular motion. At point A and C since velocity is zero, net centripetal force will be zero. Only tangential force is present. From A and B or C to B speed of the bob increases. Therefore, tangential force is parallel to velocity. From B to A or B to C speed of the bob decreases. Hence, tangential force is antiparallel to velocity.

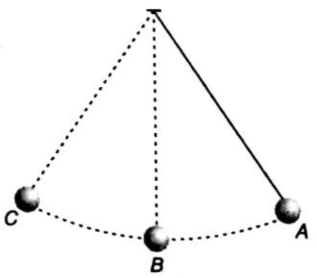


Fig. 7.29

- **In circular motion a particle has two speeds:**

  (i) angular speed $\omega = \frac{d\theta}{dt}$ **and**

  (ii) linear speed $v = \frac{ds}{dt}$

  They are related to each other by the relation $v = R\omega$. **Here,** R **is the radius of the circle.**

- In circular motion acceleration of the particle has two components :
  (i) tangential acceleration $a_t = \frac{dv}{dt} = R\alpha$
  (ii) normal or radial acceleration $a_n = \frac{v^2}{R} = R\omega^2$

  a_t and a_n are two perpendicular components of $\vec{a}$. Hence, we can write $a = \sqrt{a_t^2 + a_n^2}$

  Since, circular motion, is a 2-*D* motion we can write

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{r}\right)^2}$$

Here, $$v = \sqrt{v_x^2 + v_y^2} \quad \text{or} \quad v^2 = v_x^2 + v_y^2$$

- **Condition of toppling of a vehicle on circular tracks :** While moving in a circular track normal reaction on the outer wheels (N_1) is more than the normal reaction on inner wheels (N_2).

or $$N_1 > N_2$$

This can be shown as:

Distance between two wheels = $2a$

Height of centre of gravity of car from road = h

For translational equilibrium of car:

$$N_1 + N_2 = mg \qquad \ldots\text{(i)}$$

and $$f = \frac{mv^2}{r} \qquad \ldots\text{(ii)}$$

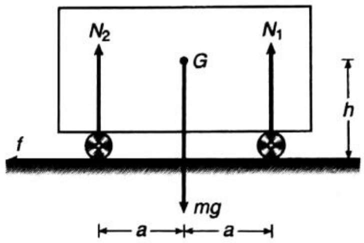


Fig. 7.30

and for rotational equilibrium of car, net torque about centre of gravity should be zero.

or $$N_1(a) = N_2(a) + f(h) \qquad \ldots\text{(iii)}$$

From Eq. (iii), we can see that

$$N_2 = N_1 - \left(\frac{h}{a}\right)f = N_1 - \left(\frac{mv^2}{r}\right)\left(\frac{h}{a}\right) \qquad \ldots\text{(iv)}$$

or $$N_2 < N_1$$

From Eq. (iv), we see that N_2 decreases as v is increased.

In critical case, $$N_2 = 0$$

or $$N_1 = mg \qquad \text{[From Eq. (i)]}$$

∴ $$N_1(a) = f(h) \qquad \text{[From Eq. (iii)]}$$

or $$(mg)(a) = \left(\frac{mv^2}{r}\right)(h)$$

or $$v = \frac{gra}{h}$$

Now, if $v > \sqrt{\frac{gra}{h}}$, $N_2 = 0$, and the car topples outwards.

Therefore, for a safe turn without toppling $v \le \sqrt{\frac{gra}{h}}$.

- From the above discussion we can conclude that while taking a turn on a level road there are two critical speeds, one is the maximum speed for sliding ($= \sqrt{\mu rg}$) and another is maximum speed for toppling $\left(= \sqrt{\frac{gra}{h}}\right)$. One should keep his car's speed less than both of them for neither to slide nor to overturn.

- **Motion of a ball over a smooth solid sphere**

Suppose a small ball of mass m is given a velocity v over the top of a smooth sphere of radius R. The equation of motion for the ball at the topmost point will be.

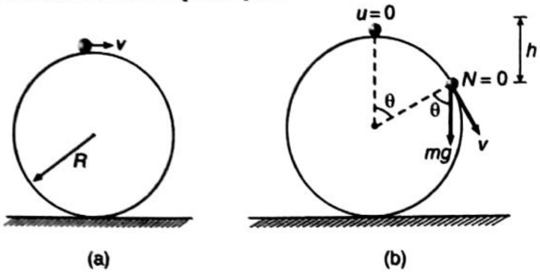


Fig. 7.31

$$mg - N = \frac{mv^2}{R}$$

or
$$N = mg - \frac{mv^2}{R}$$

From this equation we see that value of N decreases as v increases. Minimum value of N can be zero. Hence,

$$0 = mg - \frac{mv_{max}^2}{R}$$

or
$$v_{max} = \sqrt{Rg}$$

From here we can conclude that ball will lose contact with the sphere right from the beginning if velocity of ball at topmost point $v > \sqrt{Rg}$. If $v < \sqrt{Rg}$ it will definitely lose contact but after moving certain distance over the sphere. Now let us find the angle θ where the ball loses contact with the sphere if velocity at topmost point is just zero. Fig. 7.31 (b)

$$h = R(1 - \cos\theta) \quad \text{...(i)}$$

$$v^2 = 2gh \quad \text{...(ii)}$$

$$mg\cos\theta = \frac{mv^2}{R} \quad (\text{as } N = 0) \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

Thus the ball can move on the sphere maximum upto $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Exercise : Find angle θ where the ball will lose contact with the sphere, if velocity at topmost point is $u = \frac{v_{max}}{2} = \frac{\sqrt{gR}}{2}$.

$$\theta = \cos^{-1}\left(\frac{3}{4}\right) = 41.4°$$ **Ans.**

Hint: Only equation (ii) will change as,

$$v^2 = u^2 + 2gh \quad (u \neq 0)$$

- **Relation between angular velocity vector $\vec{\omega}$, velocity vector $\vec{v}$ and position vector of the particle with respect to centre, $\vec{r}$ is given by:**

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Solved Examples

Example 1 *If a point moves along a circle with constant speed, prove that its angular speed about any point on the circle is half of that about the centre.*

Solution Let, O be a point on a circle and P be the position of the particle at any time t, such that

$$\angle POA = \theta. \quad \text{Then,} \quad \angle PCA = 2\theta$$

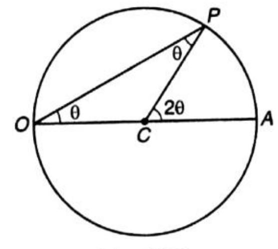


Fig. 7.32

Here, C is the centre of the circle.

Angular velocity of P about O is

$$\omega_O = \frac{d\theta}{dt}$$

and angular velocity of P about C is,

$$\omega_C = \frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt}$$

or $\omega_C = 2\omega_0$ **Proved.**

Example 2 *A car is travelling along a circular curve that has a radius of* 50 *m. If its speed is* 16 *m/s and is increasing uniformly at* 8 *m/s*^{2}*, determine the magnitude of its acceleration at this instant.*

Solution

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{R}\right)^2} = \sqrt{(8)^2 + \left(\frac{256}{50}\right)^2} = 9.5 \text{ m/s}^2$$

Example 3 *A particle moves in a circle of radius* 2.0 *cm at a speed given by* $v = 4t$*, where v is in cm/s and t in seconds.*

(a) Find the tangential acceleration at $t = 1$ s.

(b) Find total acceleration at $t = 1$ s.

Solution (a) Tangential acceleration

$$a_t = \frac{dv}{dt} \quad \text{or} \quad a_t = \frac{d}{dt}(4t) = 4 \text{ cm/s}^2$$

i.e., a_t is constant or tangential acceleration at $t = 1$ s is 4 cm/s^2.

(b) Normal acceleration

$$a_n = \frac{v^2}{R} = \frac{(4t)^2}{R}$$

or

$$a_n = \frac{16t^2}{2.0} = 8.0t^2$$

At $t = 1$ s, $a_n = 8.0 \text{ cm/s}^2$

$\therefore$ Total acceleration $a = \sqrt{a_t^2 + a_n^2}$

or

$$a = \sqrt{(4)^2 + (8)^2} = \sqrt{80} = 4\sqrt{5} \text{ cm/s}^2$$

Example 4 *A boy whirls a stone in a horizontal circle of radius* 1.5 *m and at height* 2.0 *m above level ground. The string breaks, and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of* 10 *m. What is the magnitude of the centripetal acceleration of the stone while in circular motion?*

Solution

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2}{9.8}} = 0.64 \text{ s}$$

$$v = \frac{10}{t} = 15.63 \text{ m/s}$$

$$\therefore \quad a = \frac{v^2}{R} = 163 \text{ m/s}^2$$

Example 5 *Two particles A and B start at the origin O and travel in opposite directions along the circular path at constant speeds* $v_A = 0.7$ *m/s and* $v_B = 1.5$ *m/s, respectively. Determine the time when they collide and the magnitude of the acceleration of B just before this happens.*

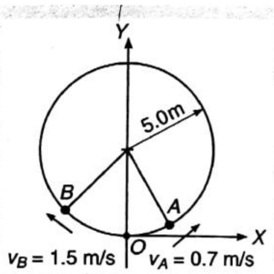


Fig. 7.33

Solution $1.5t + 0.7t = 2\pi R = 10\pi$

$$\therefore \quad t = \frac{10\pi}{2.2} = 14.3 \text{ s}$$

$$a = \frac{v_B^{\;2}}{R} = 0.45 \text{ m/s}^2$$

Example 6 *A particle is projected with a speed u at an angle* θ *with the horizontal. What is the radius of curvature of the parabola traced out by the projectile at a point where the particle velocity makes an angle* $\frac{\theta}{2}$ *with the horizontal.*

Solution Let v be the velocity at the desired point. Horizontal component of velocity remains unchanged. Hence,

$$v \cos \frac{\theta}{2} = u \cos \theta$$

$$\therefore \quad v = \frac{u \cos \theta}{\cos \frac{\theta}{2}} \qquad \ldots\text{(i)}$$

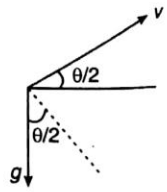


Fig. 7.34

Radial acceleration is the component of acceleration perpendicular to velocity or

$$a_n = g \cos\left(\frac{\theta}{2}\right)$$

$\therefore$ $$\frac{v^2}{R} = g\cos\left(\frac{\theta}{2}\right) \quad \ldots\text{(ii)}$$

Substituting value of v from Eq. (i) in Eq. (ii), we have radius of curvature

$$R = \frac{\left[\dfrac{u\cos\theta}{\cos\left(\dfrac{\theta}{2}\right)}\right]^2}{g\cos\left(\dfrac{\theta}{2}\right)} = \frac{u^2\cos^2\theta}{g\cos^3\left(\dfrac{\theta}{2}\right)}$$

Example 7 *A point moves along a circle with a velocity $v = kt$, where $k = 0.5$ m/s^2. Find the total acceleration of the point at the moment when it has covered the n^{th} fraction of the circle after the beginning of motion, where $n = \frac{1}{10}$.*

Solution $v = \frac{ds}{dt} = kt$ or $\int_0^s ds = k\int_0^t t\,dt$

$\therefore$ $$s = \frac{1}{2}kt^2$$

For completion of nth fraction of circle, $s = 2\pi rn$

or $$t^2 = (4\pi nr)/k \quad \ldots\text{(i)}$$

$$\text{Tangential acceleration} = a_T = \frac{dv}{dt} = k \quad \ldots\text{(ii)}$$

$$\text{Normal acceleration} = a_N = \frac{v^2}{r} = \frac{k^2t^2}{r} \quad \ldots\text{(iii)}$$

or $$a_N = 4\pi nk$$

$\therefore$ $$a = \sqrt{(a_T^2 + a_N^2)} = [k^2 + 16\pi^2 n^2 k^2]^{1/2}$$

$$= k[1 + 16\pi^2 n^2]^{1/2}$$

$$= 0.50\,[1 + 16 \times (3.14)^2 \times (0.10)^2]^{1/2}$$

$$= 0.8 \text{ m/s}^2$$

Example 8 *In a two dimensional motion of a body prove that tangential acceleration is nothing but component of acceleration along velocity.*

Solution Let velocity of the particle be,

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

Acceleration $$\vec{\mathbf{a}} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}}$$

Component of $\vec{\mathbf{a}}$ along $\vec{\mathbf{v}}$ will be,

$$\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = v_x \cdot \frac{dv_x}{dt} + v_y \cdot \frac{dv_y}{dt} \qquad \ldots\text{(i)}$$

Further, tangential acceleration of particle is rate of change of speed.

or
$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{v_x^2 + v_y^2}\right)$$

or
$$a_t = \frac{1}{2\sqrt{v_x^2 + v_y^2}}\left[2v_x \cdot \frac{dv_x}{dt} + 2v_y \frac{dv_y}{dt}\right]$$

or
$$a_t = \frac{v_x \cdot \frac{dv_x}{dt} + v_y \cdot \frac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we can see that

$$a_t = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

or Tangential acceleration = component of acceleration along velocity. **Hence proved.**

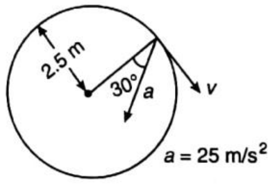
EXERCISES

AIEEE Corner

Subjective Questions (Level 1)

Kinematics of Circular Motion

1. A particle rotates in a circular path of radius 54 m with varying speed $v = 4t^2$. Here v is in m/s and t in second. Find angle between velocity and acceleration at $t = 3$ s.
2. A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 m/s and is increasing uniformly at 8 m/s^2. Determine the magnitude of its acceleration at this instant.
3. A particle is projected with a speed u at an angle θ with the horizontal. Consider a small part of its path near the highest position and take it approximately to be a circular arc. What is the radius of this circle? This radius is called the radius of curvature of the curve at the point.
4. Figure shows the total acceleration and velocity of a particle moving clockwise in a circle of radius 2.5 m at a given instant of time. At this instant, find :



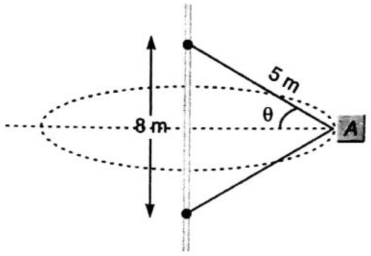
   (a) the radial acceleration,
   (b) the speed of the particle and
   (c) its tangential acceleration.
5. A particle moves in a circle of radius 1.0 cm at a speed given by $v = 2.0t$, where v is in cm/s and t in seconds.
   (a) Find the radial acceleration of the particle at $t = 1$s.
   (b) Find the tangential acceleration at $t = 1$ s.
   (c) Find the magnitude of the total acceleration at $t = 1$ s.
6. A boy whirls a stone of small mass in a horizontal circle of radius 1.5 m and at height 2.9 m above level ground. The string breaks and the stone flies off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion ?

Dynamics of Circular Motion

7. A turn has a radius of 10 m. If a vehicle goes round it at an average speed of 18 km/h, what should be the proper angle of banking?
8. If the road of the previous problem is horizontal (no banking), what should be the minimum friction coefficient so that a scooter going at 18 km/h does not skid?
9. A circular road of radius 50 m has the angle of banking equal to 30°. At what speed should a vehicle go on this road so that the friction is not used?

10. A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed?

11. A 4 kg block is attached to a vertical rod by means of two strings of equal length. When the system rotates about the axis of the rod, the strings are extended as shown in figure.



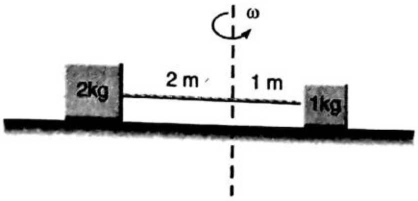
(a) How many revolutions per minute must the system make in order for the tension in the upper string to be 200 N?
(b) What is the tension in the lower string then ?

12. A block of mass m is kept on a horizontal ruler. The friction coefficient between the ruler and the block is μ. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end.
(a) What can the maximum angular speed be for which the block does not slip ?
(b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration α, at what angular speed will the block slip ?

13. Three particles, each of mass m are situated at the vertices of an equilateral triangle of side a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original mutual separation a. Find the initial velocity that should be given to each particle and also the time period of the circular motion. $\left(F = \dfrac{Gm_1 m_2}{r^2}\right)$

14. A thin circular wire of radius R rotates about its vertical diameter with an angular frequency ω. Show that a small bead on the wire remains at its lowermost point for $\omega \le \sqrt{g/R}$. What is angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

15. Two blocks tied with a massless string of length 3 m are placed on a rotating table as shown. The axis of rotation is 1 m from 1 kg mass and 2 m from 2 kg mass. The angular speed $\omega = 4$ rad/s. Ground below 2 kg block is smooth and below 1 kg block is rough. ($g = 10 \text{m/s}^2$)



(a) Find tension in the string, force of friction on 1 kg block and its direction.

(b) If coefficient of friction between 1 kg block and ground is $\mu = 0.8$. Find maximum angular speed so that neither of the blocks slips.

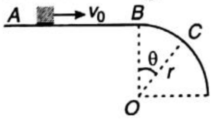
(c) If maximum tension in the string can be 100 N, then find maximum angular speed so that neither of the blocks slips.

Note *Assume that in part (b) tension can take any value and in parts (a) and (c) friction can take any value.*

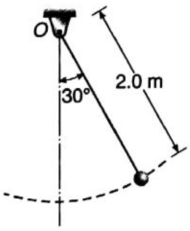
16. What is the maximum speed at which a railway carriage can move without toppling over along a curve of radius $R = 200$ m if the distance from the centre of gravity of the carriage to the level of the rails is $h = 1.5$ m, the distance between the rails is $l = 1.5$ m and the rails are laid horizontally ? (Take $g = 10 \text{ m/s}^2$)

Motion in Vertical Circle

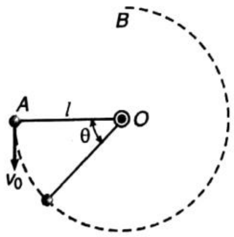
17. A small block slides with velocity $0.5\sqrt{gr}$ on the horizontal frictionless surface as shown in the figure. The block leaves the surface at point C. Calculate angle θ in the figure.



18. The bob of the pendulum shown in figure describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown. Find the velocity and the acceleration of the bob in that position.

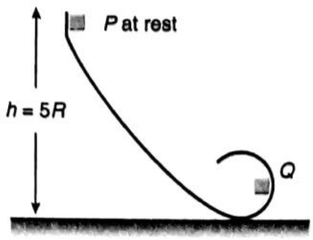


19. The sphere at A is given a downward velocity v_0 of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length $l = 2$ m attached to a support at O. Determine the angle θ at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.



20. A particle is suspended from a fixed point by a string of length 5 m. It is projected from the equilibrium position with such a velocity that the string slackens after the particle has reached a height 8 m above the lowest point. Find the velocity of the particle, just before the string slackens. Find also, to what height the particle can rise further ?

21. A small block of mass m slides along a smooth circular track of radius R as shown in the figure.



(a) If it starts from rest at P, what is the resultant force acting on it at Q ?
(b) At what height above the bottom of the loop should the block be released so that the force it exerts against the track at the top of the loop equals its weight ?

Objective Questions (Level 1)

Single Correct Option

1. A particle is revolving in a circle with increasing its speed uniformly. Which of the following is constant ?
   (a) Centripetal acceleration (b) Tangential acceleration
   (c) Angular acceleration (d) None of these
2. A particle is moving in a circular path with a constant speed. If θ is the angular displacement, then starting from $\theta = 0$, the maximum and minimum change in the linear momentum will occur when value of θ is respectively
   (a) 45° and 90° (b) 90° and 180° (c) 180° and 360° (d) 90° and 270°
3. A simple pendulum of length l has maximum angular displacement θ. Then maximum kinetic energy of a bob of mass m is
   (a) $\frac{1}{2}mgl$ (b) $\frac{1}{2}mgl\cos\theta$ (c) $mgl(1-\cos\theta)$ (d) $\frac{1}{2}mgl\sin\theta$
4. A particle of mass m is fixed to one end of a light rigid rod of length l and rotated in a vertical circular path about its other end. The minimum speed of the particle at its highest point must be
   (a) zero (b) $\sqrt{gl}$ (c) $\sqrt{1.5\,gl}$ (d) $\sqrt{2\,gl}$
5. A simple pendulum of length l and mass m is initially at its lowest position. It is given the minimum horizontal speed necessary to move in a circular path about the point of suspension. The tension in the string at the lowest position of the bob is
   (a) 3 mg (b) 4 mg (c) 5 mg (d) 6 mg
6. A point moves along a circle having a radius 20 cm with a constant tangential acceleration 5 cms^{-2}. How much time is needed after motion begins for the normal acceleration of the point to be equal to tangential acceleration?
   (a) 1 s (b) 2 s (c) 3 s (d) 4 s
7. A ring of mass 2π kg and of radius 0.25 m is making 300 rpm about an axis through its perpendicular to its plane. The tension in newton developed in ring is approximately
   (a) 50 (b) 100 (c) 175 (d) 250

8. A car is moving on a circular level road of curvature 300 m. If the coefficient of friction is 0.3 and acceleration due to gravity is 10 m/s^2, the maximum speed of the car can be
(a) 90 km/h (b) 81 km/h (c) 108 km/h (d) 162 km/h

9. A string of length 1 m is fixed at one end with a bob of mass 100 g and the string makes $\frac{2}{\pi}$ rev s^{-1} around a vertical axis through a fixed point. The angle of inclination of the string with vertical is
(a) $\tan^{-1}\left(\frac{5}{8}\right)$ (b) $\tan^{-1}\left(\frac{3}{5}\right)$ (c) $\cos^{-1}\left(\frac{8}{5}\right)$ (d) $\cos^{-1}\left(\frac{5}{8}\right)$

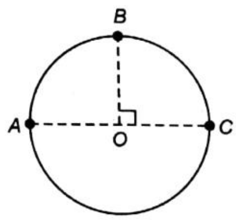
10. In the previous question, the tension in the string is
(a) $\frac{5}{8}$ N (b) $\frac{8}{5}$ N (c) $\frac{50}{8}$ N (d) $\frac{80}{5}$ N

11. A small particle of mass 0.36 g rests on a horizontal turntable at a distance 25 cm from the axis of spindle. The turntable is accelerated at a rate of $\alpha = \frac{1}{3}$ rad s^{-2}. The frictional force that the table exerts on the particle 2 s after the startup is
(a) 40 μN (b) 30 μN (c) 50 μN (d) 60 μN

12. A simple pendulum of length l and bob of mass m is displaced from its equilibrium position O to a position P so that height of P above O is h. It is then released. What is the tension in the string when the bob passes through the equilibrium position O? Neglect friction. v is the velocity of the bob at O.
(a) $m\left(g + \frac{v^2}{l}\right)$ (b) $\frac{2\,mgh}{l}$ (c) $mg\left(1 + \frac{h}{l}\right)$ (d) $mg\left(1 + \frac{2h}{l}\right)$

13. Two particles revolve concentrically in a horizontal plane in the same direction. The time required to complete one revolution for particle A is 3 min, while for particle B is 1 min. The time required for A to complete one revolution relative to B is
(a) 2 min (b) 1 min (c) 1.5 min (d) 1.25 min

14. Three particles A, B and C move in a circle in anticlockwise direction with speeds 1 ms^{-1}, 2.5 ms^{-1} and 2 ms^{-1} respectively. The initial positions of A, B and C are as shown in figure. The ratio of distance travelled by B and C by the instant A, B and C meet for the first time is
(a) 3 : 2 (b) 5 : 4
(c) 3 : 5 (d) data insufficient



JEE Corner

Assertion and Reason

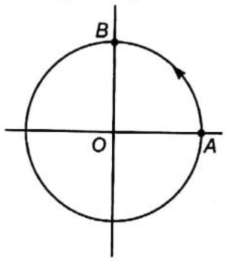
Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A car moving on a horizontal rough road with velocity v can be stopped in a minimum distance d. If the same car, moving with same speed v takes a circular turn, then minimum safe radius can be $2d$.

   Reason : $d = \frac{v^2}{2\mu g}$ and minimum safe radius $= \frac{v^2}{\mu g}$

2. **Assertion :** A particle is rotating in a circle with constant speed as shown. Between points A and B ratio of average acceleration and average velocity is angular velocity of particle about point O.



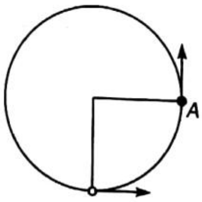
   Reason : Since speed is constant, angular velocity is also constant.

3. **Assertion :** A frame moving in a circle with constant speed can never be an inertial frame.
   Reason : It has a constant acceleration.

4. **Assertion :** In circular motion, dot product of velocity vector $(\vec{v})$ and acceleration vector $(\vec{a})$ may be positive, negative or zero.
   Reason : Dot product of angular velocity vector and linear velocity vector is always zero.

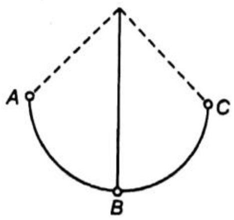
5. **Assertion :** Velocity and acceleration of a particle in circular motion at some instant are: $\vec{v} = (2\hat{i})\,ms^{-1}$ and $\vec{a} = (-\hat{i} + 2\hat{j})\,ms^{-2}$, then radius of circle is 2 m.
   Reason : Speed of particle is decreasing at a rate of 1 ms^{-2}.

6. **Assertion :** In vertical circular motion, acceleration of bob at position A is greater than 'g'.



   Reason : Net acceleration at A is resultant of tangential and radial components of acceleration.

7. **Assertion :** A pendulum is oscillating between points A, B and C. Acceleration of bob at points A or C is zero.

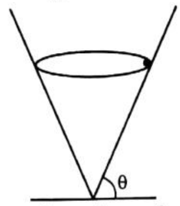


   Reason : Velocity at these points is zero.

8. **Assertion :** Speed of a particle moving in a circle varies with time as, $v = (4t - 12)$. Such type of circular motion is not possible.
Reason : Speed cannot be change linearly with time.

9. **Assertion :** Circular and projectile both are two dimensional motion. But in circular motion we cannot apply $\vec{v} = \vec{u} + \vec{a}\, t$ directly, whereas in projectile motion we can.
Reason : Projectile motion takes place under gravity, while in circular motion gravity has no role.

10. **Assertion :** A particle of mass m takes uniform horizontal circular motion inside a smooth funnel as shown. Normal reaction in this case is not $mg \cos \theta$.



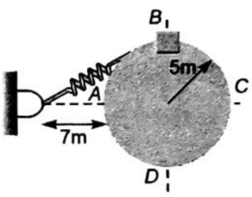
Reason : Acceleration of particle is not along the surface of funnel.

11. **Assertion :** When water in a bucket is whirled fast overhead, the water does not fall out at the top of the circular path.
Reason : The centripetal force in this position on water is more than the weight of water.

Objective Questions (Level 2)

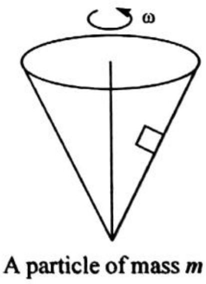
Single Correct Option

1. A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m. The spring lying in the plane of the circular track and having spring constant 200 Nm^{-1} is undeformed when the collar is at A. If the collar starts from rest at B, the normal reaction exerted by the track on the collar when it passes through A is



(a) 360 N (b) 720 N
(c) 1440 N (d) 2880 N

2. A particle is at rest with respect to the wall of an inverted cone rotating with uniform angular velocity ω about its central axis. The surface between the particle and the wall is smooth. Regarding the displacement of particle along the surface up or down, the equilibrium of particle is

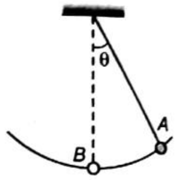


(a) stable (b) unstable
(c) neutral (d) None of these

3. A rough horizontal plate rotates with angular velocity ω about a fixed vertical axis. A particle of mass m lies on the plate at a distance $\frac{5a}{4}$ from this axis. The coefficient of friction between the plate and the particle is $\frac{1}{3}$. The largest value of ω^2 for which the particle will continue to be at rest on the revolving plate is

(a) $\frac{g}{3a}$ (b) $\frac{4g}{5a}$ (c) $\frac{4g}{9a}$ (d) $\frac{4g}{15a}$

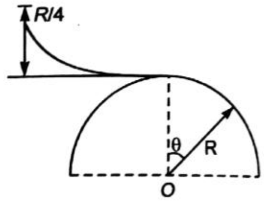
4. A ball attached to one end of a string swings in a vertical plane such that its acceleration at point A (extreme position) is equal to its acceleration at point B (mean position). The angle θ is



(a) $\cos^{-1}\left(\frac{2}{5}\right)$ (b) $\cos^{-1}\left(\frac{4}{5}\right)$

(c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) None of these

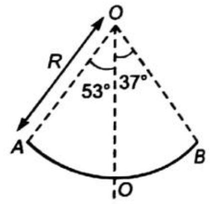
5. A skier plans to ski a smooth fixed hemisphere of radius R. He starts from rest from a curved smooth surface of height $\left(\frac{R}{4}\right)$. The angle θ at which he leaves the hemisphere is



(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\frac{5}{\sqrt{3}}$

(c) $\cos^{-1}\left(\frac{5}{6}\right)$ (d) $\cos^{-1}\left[\frac{5}{2\sqrt{3}}\right]$

6. A section of fixed smooth circular track of radius R in vertical plane is shown in the figure. A block is released from position A and leaves the track at B. The radius of curvature of its trajectory just after it leaves the track at B is ?



(a) R (b) $\frac{R}{4}$

(c) $\frac{R}{2}$ (d) None of these

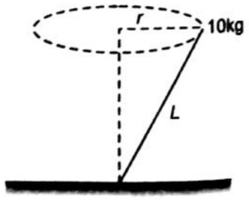
7. A particle is projected with velocity u horizontally from the top of a smooth sphere of radius a so that it slides down the outside of the sphere. If the particle leaves the sphere when it has fallen a height $\frac{a}{4}$, the value of u is

(a) $\sqrt{ag}$ (b) $\frac{\sqrt{ag}}{4}$ (c) $\frac{\sqrt{ag}}{2}$ (d) $\frac{\sqrt{ag}}{3}$

8. A particle of mass m describes a circle of radius r. The centripetal acceleration of the particle is $\frac{4}{r^2}$. What will be the momentum of the particle ?

(a) $2\frac{m}{r}$ (b) $2\frac{m}{\sqrt{r}}$ (c) $4\frac{m}{\sqrt{r}}$ (d) None of these

9. A 10 kg ball attached at the end of a rigid massless rod of length 1 m rotates at constant speed in a horizontal circle of radius 0.5 m and period of 1.58 s, as shown in the figure. The force exerted by the rod on the ball is $(g = 10\,\text{ms}^{-2})$



(a) 158 N (b) 128 N

(c) 110 N (d) 98 N

10. A particle is moving in a circle of radius R with initial speed v. It starts retarding with constant retardation $\frac{v^2}{4\pi R}$. The number of revolutions it makes in time $\frac{8\pi R}{v}$ is

(a) 3 (b) 4 (c) 5 (d) 2

11. A disc is rotating in a room. A boy standing near the rim of the disc of radius R finds the water droplet falling from the ceiling is always falling on his head. As one drop hits his head, other one starts from the ceiling. If height of the roof above his head is H, then angular velocity of the disc is

(a) $\pi\sqrt{\frac{2gR}{H^2}}$ (b) $\pi\sqrt{\frac{2gH}{R^2}}$ (c) $\pi\sqrt{\frac{2g}{H}}$ (d) None of these

12. In a clock, what is the time period of meeting of the minute hand and the second hand?

(a) 59 s (b) $\frac{60}{59}$ s (c) $\frac{59}{60}$ s (d) $\frac{3600}{59}$ s

13. When a driver of car A sees a car B moving towards his car and at distance 30 m, driver of car A, takes a left turn of 30°. At the same instant the driver of the car B takes a turn to his right at an angle 60°. The two cars collide after two seconds, then the velocity (in ms^{-1}) of the car A and B respectively will be (assume both cars are moving along same line with constant speed)

(a) 7.5, $7.5\sqrt{3}$ (b) 7.5, 7.5 (c) $7.5\sqrt{3}$, 7.5 (d) None of these

14. A particle of mass m starts to slide down from the top of the fixed smooth sphere. What is the acceleration when it breaks off the sphere ?

(a) $\frac{2g}{3}$ (b) $\frac{\sqrt{5}g}{3}$ (c) g (d) $\frac{g}{3}$

15. An automobile enters a turn of radius r. If the road is banked at an angle of 45° and the coefficient of friction is 1, the minimum and maximum speed with which the automobile can negotiate the turn without skidding is?

(a) $\sqrt{\frac{rg}{2}}$ and $\sqrt{rg}$

(b) $\frac{\sqrt{rg}}{2}$ and $\sqrt{rg}$

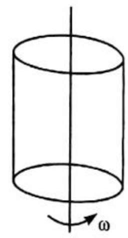
(c) $\frac{\sqrt{rg}}{2}$ and $2\sqrt{rg}$

(d) zero and infinite provided plane does not break

16. A particle is given an initial speed u inside a smooth spherical shell of radius R so that it is just able to complete the circle. Acceleration of the particle, when its velocity is vertical, is

(a) $g\sqrt{10}$ (b) g (c) $g\sqrt{2}$ (d) $g\sqrt{6}$

17. An insect of mass $m = 3$ kg is inside a vertical drum of radius 2 m that is rotating with an angular velocity of 5 rad s^{-1}. The insect doesn't fall off. Then the minimum coefficient of friction required is



(a) 0.5
(b) 0.4
(c) 0.2
(d) None of the above

18. A rod is given an angular acceleration α from rest so that it rotates in horizontal plane about a vertical axis. It has a ring at a distance r from the axis of rotation. The friction coefficient between the ring and the rod is μ. Neglecting gravity find the time after witch the ring will start to slip on the rod is.
(Take $\alpha = 3$ rad s^{-2} and $\mu = 1/3$)

(a) 1 s (b) $\frac{1}{3}$ s (c) $\frac{1}{3\sqrt{3}}$ s (d) $3\sqrt{3}$ s

19. A simple pendulum is released from rest with the string in horizontal position. The vertical component of the velocity of the bob becomes maximum, when the string makes an angle θ with the vertical. The angle θ is equal to

(a) $\frac{\pi}{4}$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{3}$

20. A particle is moving in a circle of radius R in such a way that at any instant the normal and tangential component of its acceleration are equal. If its speed at $t = 0$ is v_0. The time taken to complete the first revolution is

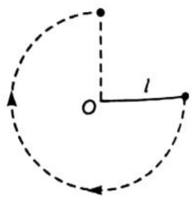
(a) $\frac{R}{v_0}$ (b) $\frac{R}{v_0}e^{-2\pi}$ (c) $\frac{R}{v_0}(1 - e^{-2\pi})$ (d) $\frac{R}{v_0}(1 + e^{-2\pi})$

21. A particle is moving in a circular path in the vertical plane. It is attached at one end of a string of length l whose other end is fixed. The velocity at lowest point is u. The tension in the string is $\vec{T}$ and acceleration of the particle is $\vec{a}$ at any position. Then $\vec{T}.\vec{a}$ is zero at highest point if ?
(a) $u > \sqrt{5gl}$ (b) $u = \sqrt{5gl}$
(c) Both (a) and (b) are correct (d) Both (a) and (b) are wrong

22. In the above question, $\vec{T}.\vec{a}$ is positive at the lowest point for
(a) $u \le \sqrt{2gl}$ (b) $u = \sqrt{2gl}$ (c) $u < \sqrt{2gl}$ (d) any value of u

Passage-1 (Q.No. 23 to 24)

A ball with mass m is attached to the end of a rod of mass M and length l. The other end of the rod is pivoted so that the ball can move in a vertical circle. The rod is held in the horizontal position as shown in the figure and then given just enough a downward push so that the ball swings down and just reaches the vertical upward position having zero speed there. Now answer the following questions.



23. The change in potential energy of the system (ball + rod) is

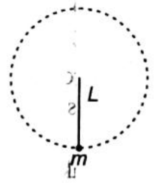
(a) mgl (b) $(M + m)gl$ (c) $\left(\frac{M}{2} + m\right)gl$ (d) $\frac{(M + m)}{2}gl$

24. The initial speed given to the ball is

(a) $\sqrt{\frac{Mgl+2mgl}{m}}$ (b) $\sqrt{2gl}$ (c) $\sqrt{\frac{2Mgl+mgl}{m}}$ (d) None of these

Note *Attempt the above question after studying chapter of rotational motion.*

Passage-2 (Q.No 25 to 27)



A small particle of mass m attached with a light inextensible thread of length L is moving in a vertical circle. In the given case particle is moving in complete vertical circle and ratio of its maximum to minimum velocity is 2 : 1.

25. Minimum velocity of the particle is

(a) $4\sqrt{\frac{gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$ (c) $\sqrt{\frac{gL}{3}}$ (d) $3\sqrt{\frac{gL}{3}}$

26. The kinetic energy of particle at the lower most position is

(a) $\frac{4mgL}{3}$ (b) $2mgL$ (c) $\frac{8mgL}{3}$ (d) $\frac{2mgL}{3}$

27. Velocity of particle when it is moving vertically downward is

(a) $\sqrt{\frac{10gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$ (c) $\sqrt{\frac{8gL}{3}}$ (d) $\sqrt{\frac{13gL}{3}}$

More than One Correct Options

1. A ball tied to the end of the string swings in a vertical circle under the influence of gravity.

(a) When the string makes an angle 90° with the vertical, the tangential acceleration is zero and radial acceleration is somewhere between minimum and maximum.

(b) When the string makes an angle 90° with the vertical, the tangential acceleration is maximum and radial acceleration is somewhere between maximum and minimum.

(c) At no place in circular motion, tangential acceleration is equal to radial acceleration.

(d) When radial acceleration has its maximum value, the tangential acceleration is zero.

2. A small spherical ball is suspended through a string of length l. The whole arrangement is placed in a vehicle which is moving with velocity v. Now, suddenly the vehicle stops and ball starts moving along a circular path. If tension in the string at the highest point is twice the weight of the ball then (Assume that the ball completes the vertical circle)

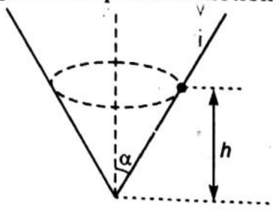
(a) $v=\sqrt{5gl}$

(b) $v=\sqrt{7gl}$

(c) velocity of the ball at highest point is $\sqrt{gl}$

(d) velocity of the ball at the highest point is $\sqrt{3gl}$

3. A particle is describing circular motion in a horizontal plane in contact with the smooth surface of a fixed right circular cone with its axis vertical and vertex down. The height of the plane of motion above the vertex is h and the semi-vertical angle of the cone is α. The period of revolution of the particle



(a) increases as h increases

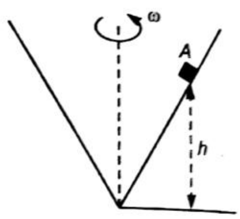
(b) decreases as h decreases

(c) increases as α increases

(d) decreases as α increases

4. In circular motion of a particle,
   (a) particle cannot have uniform motion
   (b) particle cannot have uniformly accelerated motion
   (c) particle cannot have net force equal to zero
   (d) particle cannot have any force in tangential direction

5. A smooth cone is rotated with an angular velocity ω as shown. A block A is placed at height h. A block has no motion relative to cone. Choose the correct options, when ω is increased.



   (a) net force acting on block will increase
   (b) normal reaction acting on block will increase
   (c) h will increase
   (d) normal reaction will remain unchanged

Match the Columns

1. A bob of mass m is suspended from point O by a massless string of length l as shown. At the bottommost point it is given a velocity $u = \sqrt{12gl}$ for $l = 1\,\text{m}$ and $m = 1\,\text{kg}$, match the following two columns when string becomes horizontal ($g = 10\,\text{ms}^{-2}$)

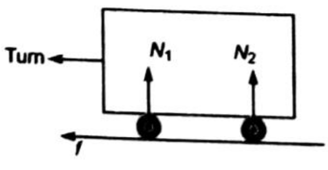
Column I	Column II (SI units)
(a) Speed of bob	(p) 10
(b) Acceleration of bob	(q) 20
(c) Tension in string	(r) 100
(d) Tangential acceleration of bob	(s) None

2. Speed of a particle moving in a circle of radius 2 m varies with time as $v = 2t$ (SI units). At $t = 1\,\text{s}$ match the following two columns.

Column I	Column II (SI units)
(a) $\vec{a} \cdot \vec{v}$	(p) $2\sqrt{2}$
(b) $\lvert \vec{a} \times \vec{\omega} \rvert$	(q) 2
(c) $\vec{v} \cdot \vec{\omega}$	(r) 4
(d) $\lvert \vec{v} \times \vec{a} \rvert$	(s) None

Here, symbols have their usual meanings.

3. A car is taking turn on a rough horizontal road without slipping as shown in figure. Let F is centripetal force : f the force of friction, N_1 and N_2 are two normal reactions. As the speed of car is increased. match the following two columns.



Column I	Column II
(a) N_1	(p) will increase
(b) N_2	(q) will decrease
(c) F/f	(r) will remain unchanged
(d) f	(s) cannot say anything

4. Position vector (with respect to centre) velocity vector and acceleration vector of *a* particle in circular motion are $\vec{r} = (3\hat{i} - 4\hat{j})$ m, $\vec{v} = (4\hat{i} - a\hat{j})\,ms^{-1}$ and $\vec{a} = (-6\hat{i} + b\hat{j})\,ms^{-2}$. Speed of particle is constant. Match the following two columns.

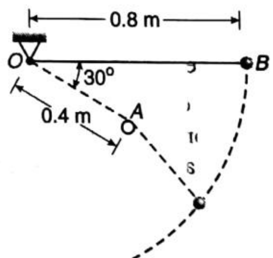
Column I	Column II (SI units)
(a) Value of a	(p) 8
(b) Value of b	(q) 3
(c) Radius of circle	(r) 5
(d) $\vec{r}\cdot(\vec{v}\times\vec{a})$	(s) None

5. A particle is rotating in a circle of radius $R = \frac{2}{\pi}$ m, with constant speed 1 ms^{-1}. Match the following two columns for the time interval when it completes $\frac{1}{4}$th of the circle.

Column I	Column II (SI units)
(a) Average speed	(p) $\frac{\sqrt{2}}{\pi}$
(b) Average velocity	(q) $2\frac{\sqrt{2}}{\pi}$
(c) Average acceleration	(r) $\sqrt{2}$
(d) Displacement	(s) 1

Subjective Questions (Level 2)

1. Bob B of the pendulum AB is given an initial velocity $\sqrt{3Lg}$ in horizontal direction. Find the maximum height of the bob from the starting point :
   (a) if AB is a massless rod,
   (b) if AB is a massless string.



2. A small sphere B of mass m is released from rest in the position shown and swings freely in a vertical plane, first about O and then about the peg A after the cord comes in contact with the peg. Determine the tension in the cord :
   (a) just before the sphere comes in contact with the peg.
   (b) just after it comes in contact with the peg.

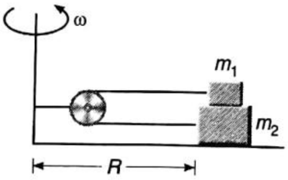
3. A particle of mass m is suspended by a string of length l from a fixed rigid support. A sufficient horizontal velocity $v_0 = \sqrt{3gl}$ is imparted to it suddenly. Calculate the angle made by the string with the vertical when the acceleration of the particle is inclined to the string by 45°.

4. A turn of radius 20 m is banked for the vehicles going at a speed of 36 km/h. If the coefficient of static friction between the road and the tyre is 0.4. What are the possible speeds of a vehicle so that it neither slips down nor skids up ? ($g = 9.8\,\text{m/s}^2$)

5. A particle is projected with a speed u at an angle θ with the horizontal. Find the radius of curvature of the parabola traced out by the projectile at a point, where the particle velocity makes an angle $\theta/2$ with the horizontal.

6. A particle is projected with velocity $20\sqrt{2}$ m/s at 45° with horizontal. After 1 s find tangential and normal acceleration of the particle. Also, find radius of curvature of the trajectory at that point. (Take $g = 10\,\text{m/s}^2$)

7. If the system shown in the figure is rotated in a horizontal circle with angular velocity ω. Find : ($g = 10\,\text{m/s}^2$)

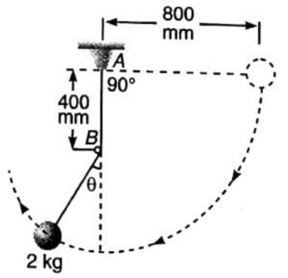


(a) the minimum value of ω to start relative motion between the two blocks.

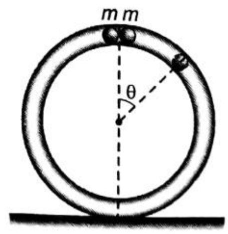
(b) tension in the string connecting m_1 and m_2 when slipping just starts between the blocks.

The coefficient of friction between the two masses is 0.5 and there is no friction between m_2 and ground. The dimensions of the masses can be neglected. (Take $R = 0.5\,\text{m}$, $m_1 = 2\,\text{kg}$, $m_2 = 1\,\text{kg}$)

8. The simple 2 kg pendulum is released from rest in the horizontal position. As it reaches the bottom position, the cord wraps around the smooth fixed pin at B and continues in the smaller arc in the vertical plane. Calculate the magnitude of the force R supported by the pin at B when the pendulum passes the position $\theta = 30°$. ($g = 9.8\,\text{m/s}^2$)

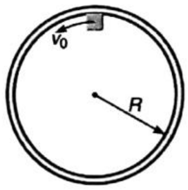


9. A circular tube of mass M is placed vertically on a horizontal surface as shown in the figure. Two small spheres, each of mass m, just fit in the tube, are released from the top. If θ gives the angle between radius vector of either ball with the vertical, obtain the value of the ratio M/m if the tube breaks its contact with ground when $\theta = 60°$. Neglect any friction.

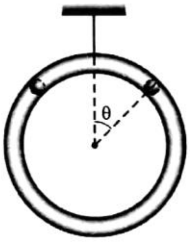


10. A table with smooth horizontal surface is turning at an angular speed ω about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the centre. Find the speed of the particle with respect to the table as its distance from the centre becomes L.

11. A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius R. At time $t = 0$, block is moving along the inside of the ring (*i.e.*, in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ. Find the speed of the block at time t.



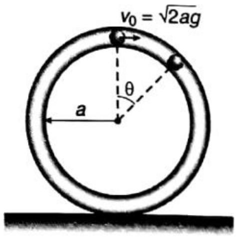
12. A ring of mass M hangs from a thread and two beads of mass m slides on it without friction. The beads are released simultaneously from the top of the ring and slides down in opposite sides. Show that the ring will start to rise, if $m > \frac{3M}{2}$.



13. A smooth circular tube of radius R is fixed in a vertical plane. A particle is projected from its lowest point with a velocity just sufficient to carry it to the highest point. Show that the time taken by the particle to reach the end of the horizontal diameter is $\sqrt{\frac{R}{g}} \ln (1+\sqrt{2})$.

Hint : $\int \sec\theta \cdot d\theta = \ln(\sec\theta + \tan\theta)$

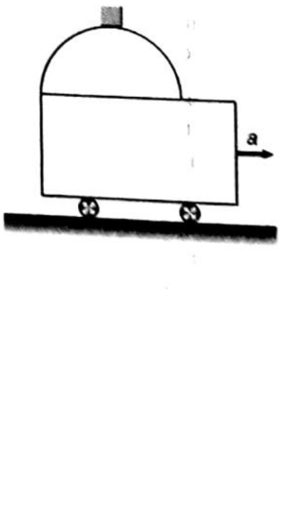
14. A heavy particle slides under gravity down the inside of a smooth vertical tube held in vertical plane. It starts from the highest point with velocity $\sqrt{2ag}$ where a is the radius of the circle. Find the angular position θ (as shown in figure) at which the vertical acceleration of the particle is maximum.



15. A vertical frictionless semicircular track of radius 1 m is fixed on the edge of a movable trolley (figure). Initially the system is rest and a mass m is kept at the top of the track. The trolley starts moving to the right with a uniform horizontal accelertion $a = 2g/9$. The mass slides down the track, eventually losing contact with it and dropping to the floor 1.3 m below the trolley. This 1.3 m is from the point where mass loses contact. $(g = 10\,\text{m/s}^2)$

(a) Calculate the angle θ at which it loses contact with the trolley and

(b) the time taken by the mass to drop on the floor, after losing contact.



ANSWERS

Introductory Exercise 7.1

1. Variable **2.** No **3.** speed, acceleration, magnitude of acceleration

4. (a) 4.0 cms^{-2} (b) 2.0 cms^{-2} (c) $2\sqrt{5}$ cms^{-2} **5.** $\frac{2\sqrt{2}}{\pi}$ **6.** $\frac{1}{2}$ s

Introductory Exercise 7.2

1. No **2.** 35 ms^{-1} **3.** (a) 27.6 (b) 69.3 N **4.** (a) 24 kN (b) 50 ms^{-1} (c) 32 kN

5. He should apply the brakes

Introductory Exercise 7.3

1. $\sqrt{2(u^2 - gL)}$ **2.** $\sqrt{2g(2R - h)}$ **3.** 7 ms^{-1}

AIEEE Corner

Subjective Questions (Level 1)

1. 45° **2.** 9.5 ms^{-2} **3.** $\frac{u^2 \cos^2 \theta}{g}$ **4.** (a) 21.65 ms^{-2} (b) 7.35 ms^{-1} (c) 12.5 ms^{-2}

5. (a) 4.0 cms^{-2} (b) 2.0 cms^{-2} (c) 4.47cms^{-2} **6.** 113 ms^{-2} **7.** $\tan^{-1}(1/4)$ **8.** 0.25

9. 17 ms^{-1} **10.** 4.7 $rads^{-1}$ **11.** (a) 39.6 rpm (b) 150 N **12.** (a) $\sqrt{\mu g/L}$ (b) $\left[\left(\frac{\mu g}{L}\right)^2 - \alpha^2\right]^{1/4}$

13. $v = \sqrt{\frac{Gm}{a}}$, $T = 2\pi\sqrt{\frac{a^3}{3Gm}}$ **14.** 60°

15. (a) $T = 64$ N, $f = 48$ N (outwards) (b) 1.63 rad/s (c) 5 rad/s **16.** 31.6 ms^{-1} **17.** $\theta = \cos^{-1}\left(\frac{3}{4}\right)$

18. 5.66 ms^{-1}, 16.75 ms^{-2} **19.** $\sin^{-1}\left(\frac{1}{4}\right)$ **20.** 5.42 ms^{-1}, 0.96 m **21.** (a) $F = \sqrt{65}\, mg$ (b) $h = 3R$

Objective Questions (Level 1)

1. (b) **2.** (c) **3.** (c) **4.** (a) **5.** (d) **6.** (b) **7.** (d) **8.** (c) **9.** (d) **10.** (b)
11. (c) **12.** (d) **13.** (c) **14.** (b)

JEE Corner

Assertion and Reason

1. (a) **2.** (b) **3.** (c) **4.** (b) **5.** (b) **6.** (a) **7.** (d) **8.** (c) **9.** (c) **10.** (a)
11. (a)

Objective Questions (Level 2)

1. (c) **2.** (b) **3.** (d) **4.** (c) **5.** (c) **6.** (c) **7.** (c) **8.** (b) **9.** (b) **10.** (d)
11. (c) **12.** (d) **13.** (c) **14.** (b) **15.** (d) **16.** (a) **17.** (c) **18.** (b) **19.** (b) **20.** (c)
21. (b) **22.** (d) **23.** (c) **24.** (d) **25.** (b) **26.** (c) **27.** (a)

More than One Correct Options

1. (b,d) **2.** (b,d) **3.** (a,c) **4.** (all) **5.** (a,b,c)

Match the Columns

1. (a) → (p) (b) → (s) (c) → (r) (d) → (p)
2. (a) → (r) (b) → (p) (c) → (s) (d) → (r)
3. (a) → (q) (b) → (p) (c) → (r) (d) → (p)
4. (a) → (s) (b) → (s) (c) → (r) (d) → (s)
5. (a) → (s) (b) → (q) (c) → (r) (d) → (q)

Subjective Questions (Level 2)

1. (a) $\frac{3L}{2}$ (b) $\frac{40L}{27}$ **2.** (a) $\frac{3mg}{2}$ (d) $\frac{5mg}{2}$ **3.** $\theta = \frac{\pi}{2}$ **4.** $4.2 \text{ ms}^{-1} \le v \le 15 \text{ ms}^{-1}$

5. $\frac{u^2 \cos^2 \theta}{g \cos^3(\theta/2)}$ **6.** $a_t = -2\sqrt{5} \text{ ms}^{-2}$, $a_n = 4\sqrt{5} \text{ ms}^{-2}$, $R = 25\sqrt{5}$ m **7.** (a) $\omega_{min} = 6.32$ rad/s (b) $T = 30$ N

8. 45 N **9.** $\frac{M}{m} = \frac{1}{2}$ **10.** $v = \omega\sqrt{L^2 - a^2}$ **11.** $\frac{v_0}{1 + \frac{\mu v_0 t}{R}}$ **14.** $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

15. (a) 37° (b) 0.38 s

Chapter 8 – Center of Mass Conservation of Linear Momentum Impulse and Collision

8

Centre of Mass, Conservation of Linear Momentum, Impulse and Collision

Chapter Contents

8.1 Centre of Mass

When we consider the motion of a system of particles, there is one point in it which behaves as though the entire mass of the system (*i.e.*, the sum of the masses of all the individual particles) is concentrated there and its motion is the same as would ensue if the resultant of all the forces acting on all the particles were applied directly to it. This point is called the centre of mass (COM) of the system. The concept of COM is very useful in solving many problems, in particular, those concerned with collision of particles.

Position of centre of mass

First of all we find the position of COM of a system of particles. Just to make the subject easy we classify a system of particles in three groups:

1. System of two particles.
2. System of a large number of particles and
3. Continuous bodies.

Now, let us take them separately.

Position of COM of Two Particles

Centre of mass of two particles of mass m_1 and m_2 separated by a distance of d lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

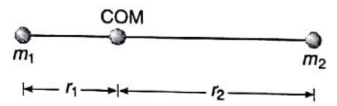


Fig. 8.1

i.e., $$r \propto \frac{1}{m}$$

or $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$m_1 r_1 = m_2 r_2$$

or $$r_1 = \left(\frac{m_2}{m_2 + m_1}\right) d \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) d$$

Here, r_1 = distance of COM from m_1

and r_2 = distance of COM from m_2

From the above discussion, we see that

$r_1 = r_2 = \frac{d}{2}$ if $m_1 = m_2$, *i.e.*, COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$, *i.e.*, COM is nearer to the particle having larger mass.

Sample Example 8.1 *Two particles of mass 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.*

Solution Since, both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at $x = x$, then

r_1 = distance of COM from the particle of mass 1 kg $= x$

and r_2 = distance of COM from the particle of mass 2 kg
$= (3 - x)$

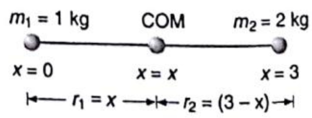


Fig. 8.2

Using $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$\frac{x}{3-x} = \frac{2}{1} \quad \text{or} \quad x = 2\,\text{m}$$

Thus, the COM of the two particles is located at $x = 2$ m.

Position of COM of a Large Number of Particles

If we have a system consisting of n particles, of mass $m_1, m_2, \ldots, m_n$ with $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \ldots, \vec{\mathbf{r}}_n$ as their position vectors at a given instant of time. The position vector $\vec{\mathbf{r}}_{COM}$ of the COM of the system at that instant is given by:

$$\vec{\mathbf{r}}_{COM} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \ldots + m_n\vec{\mathbf{r}}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{\mathbf{r}}_i}{\sum_{i=1}^{n} m_i}$$

or $$\vec{\mathbf{r}}_{COM} = \frac{\sum_{i=1}^{n} m_i\vec{\mathbf{r}}_i}{M}$$

Here, $M = m_1 + m_2 + \ldots + m_n$ and $\Sigma m_i \vec{\mathbf{r}}_i$ is called the first moment of the mass.

Further, $$\vec{\mathbf{r}}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

and $$\vec{\mathbf{r}}_{COM} = x_{COM}\hat{\mathbf{i}} + y_{COM}\hat{\mathbf{j}} + z_{COM}\hat{\mathbf{k}}$$

So, the cartesian co-ordinates of the COM will be

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\Sigma m_i}$$

or $$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}$$

Similarly, $$y_{COM} = \frac{\sum_{i=1}^{n} m_iy_i}{M}$$

and $$z_{COM} = \frac{\sum_{i=1}^{n} m_iz_i}{M}$$

Sample Example 8.2 *The position vector of three particles of mass $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg are $\vec{\mathbf{r}}_1 = (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$ m, $\vec{\mathbf{r}}_2 = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ m and $\vec{\mathbf{r}}_3 = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ m respectively. Find the position vector of their centre of mass.*

Solution The position vector of COM of the three particles will be given by

$$\vec{r}_{COM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

Substituting the values, we get

$$\vec{r}_{COM} = \frac{(1)(\hat{i} + 4\hat{j} + \hat{k}) + (2)(\hat{i} + \hat{j} + \hat{k}) + 3(2\hat{i} - \hat{j} - 2\hat{k})}{1 + 2 + 3}$$

$$= \frac{9\hat{i} + 3\hat{j} - 3\hat{k}}{6}$$

$$\vec{r}_{COM} = \frac{1}{2}(3\hat{i} + \hat{j} - \hat{k})\,\text{m}$$

Sample Example 8.3 *Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find the position of centre of mass of the particles.*

Solution Assuming D as the origin, DC as x-axis and DA as y-axis, we have

$$m_1 = 1\,\text{kg}, (x_1, y_1) = (0, 1\,\text{m})$$

$$m_2 = 2\,\text{kg}, (x_2, y_2) = (1\,\text{m}, 1\,\text{m})$$

$$m_3 = 3\,\text{kg}, (x_3, y_3) = (1\,\text{m}, 0)$$

and

$$m_4 = 4\,\text{kg}, (x_4, y_4) = (0, 0)$$

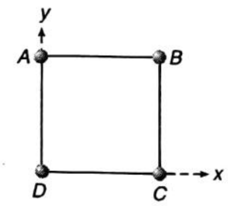


Fig. 8.3

Coordinates of their COM are

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{5}{10} = \frac{1}{2}\,\text{m} = 0.5\,\text{m}$$

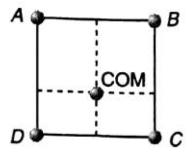


Fig. 8.4

Similarly,

$$y_{COM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{3}{10}\,\text{m} = 0.3\,\text{m}$$

$$\therefore \quad (x_{COM}, y_{COM}) = (0.5\,\text{m}, 0.3\,\text{m})$$

Thus, position of COM of the four particles is as shown in figure.

Position of COM of Continuous Bodies

If we consider the body to have continuous distribution of matter the summation in the formula of COM is replaced by integration. Suppose x, y and z are the co-ordinates of a small element of mass dm, we write the co-ordinates of COM as

$$x_{\text{COM}} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}$$

$$y_{\text{COM}} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

and $$z_{\text{COM}} = \frac{\int z\,dm}{\int dm} = \frac{\int z\,dm}{M}$$

Let us take an example.

Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

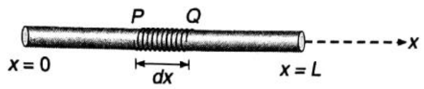


Fig. 8.5

$$\text{Mass per unit length of the rod} = \frac{M}{L}$$

Hence, the mass of the element PQ of length dx situated at $x = x$ is $dm = \frac{M}{L}\,dx$

The coordinates of the element PQ are $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{\text{COM}} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L (x)\left(\frac{M}{L}\,dx\right)}{M} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{\text{COM}} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $$z_{\text{COM}} = 0$$

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$. Or it lies at the centre of the rod.

Proceeding in the similar manner, we can find the COM of certain rigid bodies. Centre of mass of some well known rigid bodies are given below:

1. Centre of mass of a uniform rectangular, square or circular plate lies at its centre.

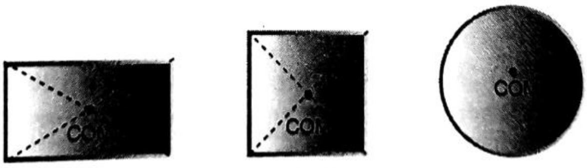


Fig. 8.6

2. Centre of mass of a uniform **semicircular ring** lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry where R is the radius of the ring.

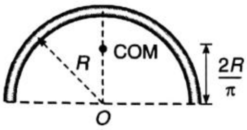


Fig. 8.7

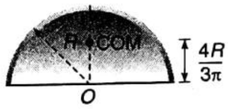


Fig. 8.8

3. Centre of mass of a uniform **semicircular disc** of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in Fig. 8.8.

4. Centre of mass of a **hemispherical shell** of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure.

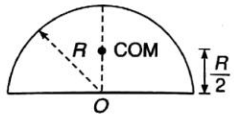


Fig. 8.9

5. Centre of mass of a **solid hemisphere** of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry.

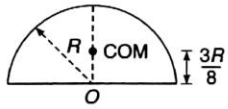


Fig. 8.10

Sample Example 8.4 *A rod of length L is placed along the x-axis between x = 0 and x = L. The linear density (mass/length) ρ of the rod varies with the distance x from the origin as ρ = a + bx. Here, a and b are constants. Find the position of centre of mass of this rod.*

Solution Mass of element PQ of length dx situated at $x = x$ is

$$dm = \rho\, dx = (a + bx)\, dx$$

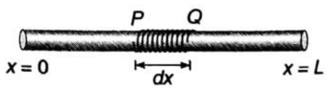


Fig. 8.11

The COM of the element has co-ordinates $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\, dm}{\int_0^L dm} = \frac{\int_0^L (x)(a + bx)\, dx}{\int_0^L (a + bx)\, dx}$$

$$= \frac{\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^L}{\left[ax + \frac{bx^2}{2}\right]_0^L}$$

or

$$x_{COM} = \frac{3aL + 2bL^2}{6a + 3bL}$$

The y-coordinate of COM of the rod is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly,

$$z_{COM} = 0$$

Hence, the centre of mass of the rod lies at $\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$ **Ans.**

Important Points Regarding Position of Centre of Mass

- For a laminar type (2-dimensional) body the formulae for finding the position of centre of mass are as follows:

(i) $$\vec{r}_{COM} = \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + \ldots + A_n\vec{r}_n}{A_1 + A_2 + .. + A_n}$$

(ii) $$x_{COM} = \frac{A_1x_1 + A_2x_2 + \ldots + A_nx_n}{A_1 + A_2 + \ldots + A_n}$$

$$y_{COM} = \frac{A_1y_1 + A_2y_2 + \ldots + A_ny_n}{A_1 + A_2 + \ldots + A_n}$$

and $$z_{COM} = \frac{A_1z_1 + A_2z_2 + \ldots + A_nz_n}{A_1 + A_2 + \ldots + A_n}$$

Here, A stands for the area.

- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(i) $$\vec{r}_{COM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$$

or $$\vec{r}_{COM} = \frac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$$

(ii) $$x_{COM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2}$$

or $$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$y_{COM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2}$$

or $$y_{COM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

and $$z_{COM} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2}$$

or $$z_{COM} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$$

Here, m_1, A_1, $\vec{r}_1$ x_1, y_1 and z_1 are the values for the whole mass while m_2, A_2, $\vec{r}_2$, $\vec{x}_2$, y_2 and z_2 are the values for the mass which has been removed. Let us see two examples in support of the above theory.

Sample Example 8.5 *Find the position of centre of mass of the uniform lamina shown in figure.*

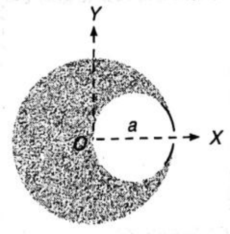


Fig. 8.12

Solution Here,

$$A_1 = \text{area of complete circle} = \pi a^2$$

$$A_2 = \text{area of small circle} = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

$$(x_1, y_1) = \text{coordinates of centre of mass of large circle} = (0, 0)$$

$$(x_2, y_2) = \text{coordinates of centre of mass of small circle} = \left(\frac{a}{2}, 0\right)$$

Using

$$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

we get

$$x_{COM} = \frac{-\frac{\pi a^2}{4}\left(\frac{a}{2}\right)}{\pi a^2 - \frac{\pi a^2}{4}} = \frac{-\left(\frac{1}{8}\right)}{\left(\frac{3}{4}\right)} a = -\frac{a}{6}$$

and $y_{COM} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$.

Introductory Exercise 8.1

1. What is the difference between centre of mass and centre of gravity?
2. The centre of mass of a rigid body always lies inside the body. Is this statement true or false ?
3. The centre of mass always lies on the axis of symmetry if it exists. Is this statement true or false?
4. If all the particles of a system lie in y-z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or false?
5. What can be said about the centre of mass of a solid hemisphere of radius r without making any calculation. Will its distance from the centre be more than $r/2$ or less than $r/2$?
6. All the particles of a body are situated at a distance R from the origin. The distance of the centre of mass of the body from the origin is also R. Is this statement true or false?

7. Three particles of mass 1 kg, 2 kg and 3 kg are placed at the corners *A, B* and *C* respectively of an equilateral triangle *ABC* of edge 1 m. Find the distance of their centre of mass from *A*.

8. Find the distance of centre of mass of a uniform plate having semicircular inner and outer boundaries of radii *a* and *b* from the centre *O*.

Hint : Distance of COM of semicircular plate from centre is $\frac{4r}{3\pi}$.

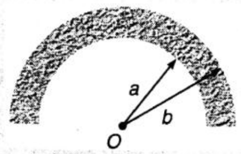


Fig. 8.13

9. Find the position of centre of mass of the section shown in figure.

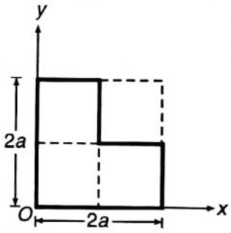


Fig. 8.14

Note *Solve the problem by using both the formulae:*

(i) $x_{COM} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2}$ *and*

(ii) $x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$

Motion of the Centre of Mass

Let us consider the motion of a system of *n* particles of individual masses $m_1, m_2, \ldots, m_n$ and total mass *M*. It is assumed that no mass enters or leaves the system during its motion, so that *M* remains constant. Then, as we have seen, we have the relation

$$\vec{\mathbf{r}}_{COM} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \ldots + m_n\vec{\mathbf{r}}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \ldots + m_n\vec{\mathbf{r}}_n}{M}$$

or $$M\vec{\mathbf{r}}_{COM} = m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \ldots + m_n\vec{\mathbf{r}}_n$$

Differentiating this expression with respect to time *t*, we have

$$M\frac{d\vec{\mathbf{r}}_{COM}}{dt} = m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt} + \ldots + m_n\frac{d\vec{\mathbf{r}}_n}{dt}$$

Since, $$\frac{d\vec{\mathbf{r}}}{dt} = \text{velocity}$$

Therefore, $$M\vec{\mathbf{v}}_{COM} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \ldots + m_n\vec{\mathbf{v}}_n \quad \ldots\text{(i)}$$

or velocity of the COM is

$$\vec{v}_{COM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + .. + m_n \vec{v}_n}{M}$$

or
$$\vec{v}_{COM} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{M}$$

Further, $m\vec{v}$ = momentum of a particle $\vec{p}$. Therefore, Eq. (i) can be written as

$$\vec{p}_{COM} = \vec{p}_1 + \vec{p}_1 + \ldots + \vec{p}_n$$

or
$$\vec{p}_{COM} = \sum_{i=1}^{n} \vec{p}_i$$

Differentiating Eq. (i) with respect to time t, we get

$$M\frac{d\vec{v}_{COM}}{dt} = m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + \ldots + m_n \frac{d\vec{v}_n}{dt}$$

or
$$M\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n \quad \ldots(ii)$$

or
$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n}{M}$$

or
$$\vec{a}_{COM} = \frac{\sum_{i=1}^{n} m_i \vec{a}_i}{M}$$

Further, in accordance with Newton's second law of motion $\vec{F} = m\vec{a}$. Hence, Eq. (ii) can be written as

$$\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

or
$$\vec{F}_{COM} = \sum_{i=1}^{n} \vec{F}_i$$

Thus, as pointed out earlier also, the centre of mass of a system of particles moves as though it were a particle of mass equal to that of the whole system with all the external forces acting directly on it.

Important Points Regarding Motion of Centre of Mass

- Students are often confused over the problems of centre of mass. They cannot answer even the basic problems of COM. For example, let us take a simple problem: two particles one of mass 1 kg and the other of 2 kg are projected simultaneously with the same speed from the roof of a tower, the one of mass 1 kg vertically upwards and the other vertically downwards. What is the acceleration of centre of mass of these two particles? When I ask this question in my first class of centre of mass, three answers normally come among the students g, $\frac{g}{3}$ and zero. The correct answer is g. Because

$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

Here,
$$\vec{a}_1 = \vec{a}_2 = g \quad \text{(downwards)}$$

$\therefore$ $$\vec{\mathbf{a}}_{COM} = \frac{(1)(g) + (2)(g)}{1+2} = g \qquad \text{(downwards)}$$

The idea behind this is that apply the basic equations when asked anything about centre of mass. Just as a revision I am writing below all the basic equations of COM at one place.

$$\vec{\mathbf{r}}_{COM} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2 + \ldots + m_n \vec{\mathbf{r}}_n}{m_1 + m_2 + \ldots + m_n}$$

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}$$

$$y_{COM} = \frac{m_1 y_1 + m_2 y_2 + \ldots + m_n y_n}{m_1 + m_2 + \ldots + m_n}$$

$$z_{COM} = \frac{m_1 z_1 + m_2 z_2 + \ldots + m_n z_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{\mathbf{v}}_{COM} = \frac{m_1 \vec{\mathbf{v}}_1 + m_2 \vec{\mathbf{v}}_2 + \ldots + m_n \vec{\mathbf{v}}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{\mathbf{P}}_{COM} = \vec{\mathbf{P}}_1 + \vec{\mathbf{P}}_2 + \ldots + \vec{\mathbf{P}}_n$$

$$\vec{\mathbf{a}}_{COM} = \frac{m_1 \vec{\mathbf{a}}_1 + m_2 \vec{\mathbf{a}}_2 + \ldots + m_n \vec{\mathbf{a}}_n}{m_1 + m_2 + \ldots + m_n}$$

and $$\vec{\mathbf{F}}_{COM} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \ldots + \vec{\mathbf{F}}_n$$

Sample Example 8.6 *Two particles A and B of mass 1 kg and 2 kg respectively are projected in the directions shown in figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially they were 90 m apart. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant.* $(g = 10\ m/s^2)$

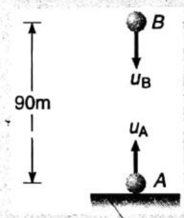


Fig. 8.15

Solution Using $$m_A r_A = m_B r_B$$

or $$(1)(r_A) = (2)(r_B)$$

or $$r_A = 2r_B \qquad \ldots(i)$$

and $$r_A + r_B = 90\text{ m} \qquad \ldots(ii)$$

Solving these two equations, we get

$$r_A = 60\text{ m} \quad \text{and} \quad r_B = 30\text{ m}$$

i.e., COM is at height 60 m from the ground at time $t = 0$.

Further, $$\vec{\mathbf{a}}_{COM} = \frac{m_A \vec{\mathbf{a}}_A + m_B \vec{\mathbf{a}}_B}{m_A + m_B}$$

$$= g = 10\text{ m/s}^2 \qquad \text{(downwards)}$$

as $$\vec{\mathbf{a}}_A = \vec{\mathbf{a}}_B = g \qquad \text{(downwards)}$$

$$\vec{u}_{COM} = \frac{m_A \vec{u}_A + m_B \vec{u}_B}{m_A + m_B}$$

$$= \frac{(1)(200) - (2)(50)}{1+2} = \frac{100}{3} \text{ m/s} \qquad \text{(upwards)}$$

Let, h be the height attained by COM beyond 60 m. Using,

$$v_{COM}^2 = u_{COM}^2 + 2a_{COM} h$$

or
$$0 = \left(\frac{100}{3}\right)^2 - (2)(10)h$$

or
$$h = \frac{(100)^2}{180} = 55.55 \text{ m}$$

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 55.55$$
$$= 115.55 \text{ m}$$

Sample Example 8.7 *In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.*

Fig. 8.16

Solution Net pulling force on the system is $(m_A - m_B)g$

or
$$(2-1)g = g$$

Total mass being pulled is $m_A + m_B$ or 3 kg

$\therefore$
$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{g}{3}$$

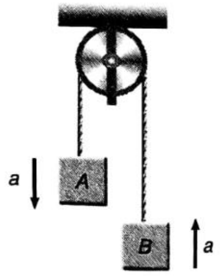


Fig. 8.17

Now,
$$\vec{a}_{COM} = \frac{m_A \vec{a}_A + m_B \vec{a}_B}{m_A + m_B}$$

$$= \frac{(2)(a) - (1)(a)}{1+2} = \frac{a}{3}$$

$$= \frac{g}{9} \text{ downwards}$$

Alternate Method

Free body diagram of block A is shown in figure.

$$2g - T = m_A(a)$$

or
$$T = 2g - m_A a$$

$$= 2g - (2)\left(\frac{g}{3}\right) = \frac{4g}{3}$$

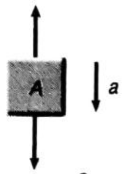


Fig. 8.18

Free body diagrams of A and B both are as shown in Fig. 8.19.

$$\vec{a}_{COM} = \frac{\text{Net force on both the blocks}}{m_A + m_B}$$

$$= \frac{(m_A + m_B)g - 2T}{2+1}$$

$$= \frac{3g - \frac{8g}{3}}{3}$$

$$= \frac{g}{9} \text{ downwards}$$

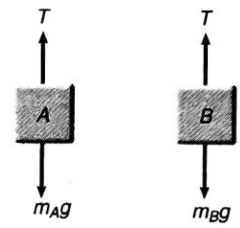


Fig. 8.19

Introductory Exercise 8.2

1. Two particles of mass 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$ s.
2. Two blocks A and B of equal masses are attached to a string passing over a smooth pulley fixed to a wedge as shown in figure. Find the magnitude of acceleration of centre of mass of the two blocks when they are released from rest. Neglect friction.

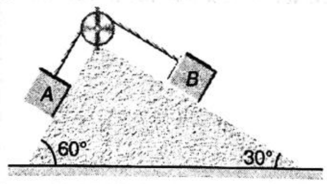


Fig. 8.20

8.2 Law of Conservation of Linear Momentum

The product of mass and the velocity of a particle is defined as its linear momentum ($\vec{p}$). So,

$$\vec{p} = m\vec{v}$$

The magnitude of linear momentum may be written as

$$p = mv$$

or $$p^2 = m^2v^2 = 2m\left(\frac{1}{2}mv^2\right) = 2mK$$

Thus, $$p = \sqrt{2Km} \quad \text{or} \quad K = \frac{p^2}{2m}$$

Here, K is the kinetic energy of the particle. In accordance with Newton's second law,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

Thus, $$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$$

In case the external force applied to a particle (or a body) be zero, we have

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = 0 \quad \text{or} \quad \vec{\mathbf{p}} = \text{constant}$$

showing that in the absence of an external force, the linear momentum of a particle (or the body) remains constant. This is called the law of conservation of linear momentum. The law may be extended to a system of particles or to the centre of mass of a system of particles. For example, for a system of particles it takes the form.

If net force (or the vector sum of all the forces) on a system of particles is zero, the vector sum of linear momentum of all the particles remain conserved, or

If $$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \ldots + \vec{\mathbf{F}}_n = 0$$

Then, $$\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3 + \ldots + \vec{\mathbf{p}}_n = \text{constant}$$

The same is the case for the centre of mass of a system of particles, *i.e.,* if

$$\vec{\mathbf{F}}_{COM} = 0, \vec{\mathbf{p}}_{COM} = \text{constant.}$$

Thus, the law of conservation of linear momentum can be applied to a single particle, to a system of particles or even to the centre of mass of the particles.

The law of conservation of linear momentum enables us to solve a number of problems which can not be solved by a straight application of the relation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.

For example, suppose a particle of mass m intially at rest, suddenly explodes into two fragments of masses m_1 and m_2 which fly apart with velocities $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ respectively. Obviously, the forces resulting in the explosion of the particle must be internal forces, since no external force has been applied. In the absence of the external forces, therefore, the momentum must remain conserved and we should have

$$m\vec{\mathbf{v}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2$$

Since, the particle was initially at rest, $\vec{\mathbf{v}} = 0$ and therefore,

$$m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = 0$$

$$\vec{\mathbf{v}}_1 = -\frac{m_2}{m_1}\vec{\mathbf{v}}_2 \quad \text{or} \quad \frac{|\vec{\mathbf{v}}_1|}{|\vec{\mathbf{v}}_2|} = \frac{m_2}{m_1}$$

Showing at once that the velocities of the two fragments must be inversely proportional to their masses and in opposite directions along the same line. This result could not possibly be arrived at from the relation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, since we know nothing about the forces that were acting during the explosion. Nor, could we derive it from the law of conservation of energy.

Important Points Regarding Net Force Equal to Zero on Centre of Mass

- If a projectile explodes in air in different parts, the path of the centre of mass remains unchanged. This is because during explosion no external force (except gravity) acts on the centre of mass. The situation is as shown in figure.
  Path of COM is *ABC*, even though the different parts travel in different directions after explosion.

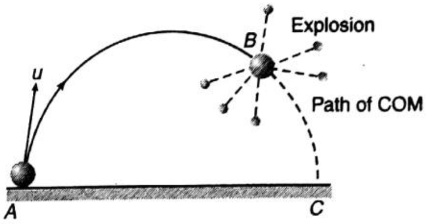


Fig. 8.21

- Suppose a system consists of more than one particle (or bodies). Net external force on the system in a particular direction is zero. Initially the centre of mass of the system is at rest, then obviously the centre of mass will not move along that particular direction even though some particles (or bodies) of the system may move along that direction. The following example will illustrate the above theory.

Sample Example 8.8 *A projectile of mass 3 m is projected from ground with velocity $20\sqrt{2}$ m/s at 45°. At highest point it explodes into two pieces. One of mass 2 m and the other of mass m. Both the pieces fly off horizontally in opposite directions. Mass 2 m falls at a distance of 100 m from point of projection. Find the distance of second mass from point of projection where it strikes the ground. ($g = 10\ m/s^2$)*

Solution Range of the projectile in the absence of explosion

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$= \frac{(20\sqrt{2})^2 \sin 90°}{10}$$

$$= 80\ \text{m}$$

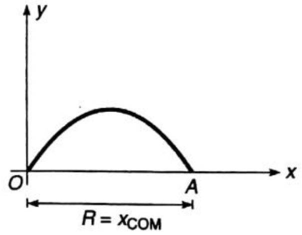


Fig. 8.22

The path of centre of mass of projectile will not change, *i.e.*, x_{COM} is still 80 m. Now, from the definition of centre of mass

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or

$$80 = \frac{(m)(x_1) + (2m)(100)}{m + 2m}$$

Solving this equation, we get

$$x_1 = 40\ \text{m}$$

Therefore, the mass m will fall at a distance $x_1 = 40$ cm from point of projection. **Ans.**

Sample Example 8.9 *A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.*

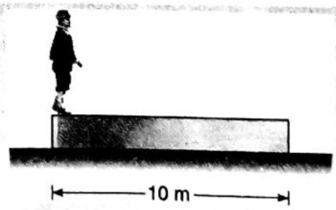


Fig. 8.23

Solution Here, the system is man + plank. Net force on this system in horizontal direction is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in horizontal direction.

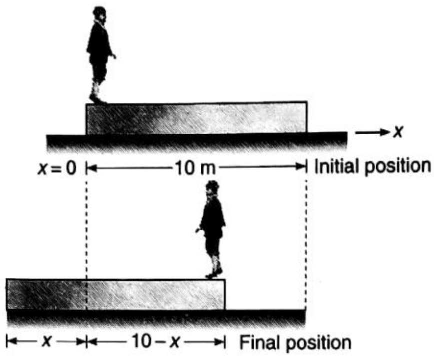


Fig. 8.24

Let x be the displacement of the Plank. Assuming the origin, *i.e.*, $x = 0$ at the position shown in figure.

As we said earlier also, the centre of mass will not move in horizontal direction (x-axis). Therefore, for centre of mass to remain stationary,

$$x_i = x_f$$

$$\frac{(60)(0) + 20\left(\frac{10}{2}\right)}{60 + 20} = \frac{(60)(10 - x) + 20\left(\frac{10}{2} - x\right)}{60 + 20}$$

or $$\frac{5}{4} = \frac{6(10 - x) + 2\left(\frac{10}{2} - x\right)}{8} = \frac{60 - 6x + 10 - 2x}{8}$$

or $$5 = 30 - 3x + 5 - x \quad \text{or} \quad 4x = 30$$

or $$x = \frac{30}{4}\text{ m} \quad \text{or} \quad x = 7.5\text{ m}$$

Note *The centre of mass of the plank lies at its centre.*

Sample Example 8.10 *A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.*

Solution Absolute velocity of man $= v_r - v$ where $v =$ recoil velocity of platform. Taking the platform and the man as a system, net external force on the system in horizontal direction is zero. The linear momentum of the system remains constant. Initially both the man and the platform were at rest.

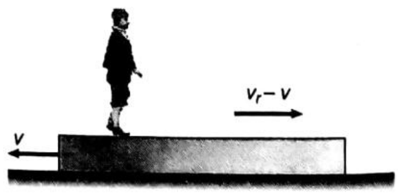


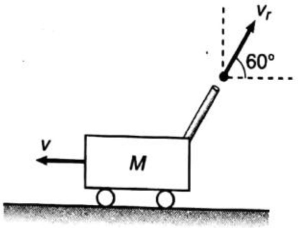
Fig. 8.25

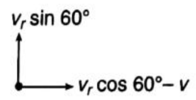
Hence, $$0 = m_1(v_r - v) - m_2 v$$

$\therefore$ $$v = \frac{m_1 v_r}{m_1 + m_2}$$

Sample Example 8.11 *A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun.*

Solution Let the recoil speed of gun is v. Taking gun + bullet as the system. Net external force on the system in horizontal direction is zero. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum in horizontal direction, we get





Components of velocity of bullet relative to ground

Fig. 8.26

$$Mv - m(v_r \cos 60° - v) = 0$$

$$\therefore \qquad v = \frac{mv_r \cos 60°}{M + m}$$

or

$$v = \frac{mv_r}{2(M + m)}$$

Introductory Exercise 8.3

1. A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find velocity of trolley and total energy produced by man.

2. Two blocks A and B of mass m_A and m_B are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Show that the kinetic energies of the blocks are, at any instant inversely proportional to their masses.

3. Three particles of mass 20 g, 30 g and 40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s. When due to their mutual interaction, the first particle comes to rest, the second acquires a velocity $10\hat{i} + 20\hat{k}$. What is then the velocity of the third particle?

4. A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal mass. One fragment, whose initial speed is zero falls vertically. How far from the gun does the other fragment land, assuming a level terrain? Take $g = 10 \text{ m/s}^2$?

5. A particle of mass 2 m is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces. As a result of explosion one part comes to rest. Find the maximum height attained by the other part. Take $g = 10 \text{ m/s}^2$.

6. A boy of mass 60 kg is standing over a platform of mass 40 kg placed over a smooth horizontal surface. He throws a stone of mass 1 kg with velocity $v = 10$ m/s at an angle of 45° with respect to the ground. Find the displacement of the platform (with boy) on the horizontal surface when the stone lands on the ground. Take $g = 10 \text{ m/s}^2$.

7. A gun fires a bullet. The barrel of the gun is inclined at an angle of 45° with horizontal. When the bullet leaves the barrel it will be travelling at an angle greater than 45° with the horizontal. Is this statement true or false?

8.3 Variable Mass

In our discussion of the conservation of linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those systems whose mass is variable, *i.e.*, those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping, thereby continuously decreasing its mass.

In such problems you have nothing to do but apply a thrust force ($\vec{\mathbf{F}}_t$) to the main mass in addition to the all other forces acting on it. This thrust force is given by,

$$\vec{\mathbf{F}}_t = \vec{\mathbf{v}}_{\text{rel}} \left(\frac{dm}{dt}\right)$$

Here, $\vec{\mathbf{v}}_{\text{rel}}$ is the velocity of the mass gained or mass ejected relative to the main mass. In case of rocket this is sometimes called the exhaust velocity of the gases. $\frac{dm}{dt}$ is the rate at which mass is increasing or decreasing.

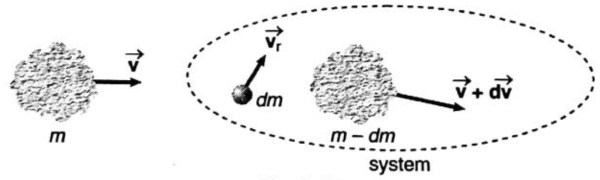


Fig. 8.27

The expression for the thrust force can be derived from the conservation of linear momentum in the absence of any external forces on a system as follows:

Suppose at some moment $t = t$ mass of a body is m and its velocity is $\vec{\mathbf{v}}$. After some time at $t = t + dt$ its mass becomes $(m - dm)$ and velocity becomes $\vec{\mathbf{v}} + \vec{\mathbf{dv}}$. The mass dm is ejected with relative velocity $\vec{\mathbf{v}}_r$. Absolute velocity of mass 'dm' is therefore $(\vec{\mathbf{v}}_r + \vec{\mathbf{v}} + \vec{\mathbf{dv}})$. If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f$$

or $$m\vec{\mathbf{v}} = (m - dm)(\vec{\mathbf{v}} + \vec{\mathbf{dv}}) + dm(\vec{\mathbf{v}}_r + \vec{\mathbf{v}} + \vec{\mathbf{dv}})$$

or $$m\vec{\mathbf{v}} = m\vec{\mathbf{v}} + md\vec{\mathbf{v}} - dm\,\vec{\mathbf{v}} - (dm)(\vec{\mathbf{dv}}) + dm\,\vec{\mathbf{v}} + \vec{\mathbf{v}}_r\,dm + (dm)(\vec{\mathbf{dv}})$$

∴ $$m\vec{\mathbf{dv}} = -\vec{\mathbf{v}}_r\,dm$$

or $$m\left(\frac{\vec{\mathbf{dv}}}{dt}\right) = \vec{\mathbf{v}}_r\left(-\frac{dm}{dt}\right)$$

Here, $$m\left(\frac{\vec{\mathbf{dv}}}{dt}\right) = \text{thrust force } (\vec{\mathbf{F}}_t)$$

and $$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

Problems Related to Variable Mass can be Solved in Following Three Steps

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt} \qquad (m = \text{mass at that particular instant})$$

Rocket Propulsion

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially let us suppose that the velocity of the rocket is u.

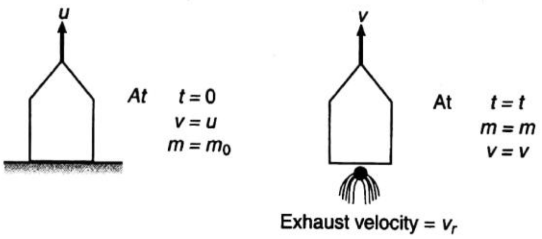


Fig. 8.28

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket

$$F_t = v_r\left(-\frac{dm}{dt}\right) \qquad \text{(upwards)}$$

2. Weight of the rocket

$$W = mg \qquad \text{(downwards)}$$

3. Net force on the rocket

$$F_{net} = F_t - W \qquad \text{(upwards)}$$

or

$$F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or

$$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

or $$\int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$$

or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$$

Thus, $$\mathbf{v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)} \qquad \ldots(i)$$

Note 1. $F_t = v_r\left(-\frac{dm}{dt}\right)$ *is upwards, as* v_r *is downwards and* $\frac{dm}{dt}$ *is negative.*

2. *If gravity is ignored and initial velocity of the rocket* $u = 0$, *Eq. (i) reduces to* $v = v_r \ln\left(\frac{m_0}{m}\right)$.

Sample Example 8.12 *(a) A rocket set for vertical firing weighs* 50 kg *and contains* 450 kg *of fuel. It can have a maximum exhaust velocity of* 2 km/s. *What should be its minimum rate of fuel consumption*

(i) to just lift it off the launching pad?

(ii) to give it an acceleration of 20 m/s^2?

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s *after whole of the fuel is consumed? (Take* $g = 9.8\ m/s^2$*)*

Solution (a) (i) To just lift it off the launching pad

$$\text{weight} = \text{thrust force}$$

or $$mg = v_r\left(\frac{-dm}{dt}\right)$$

or $$\left(\frac{-dm}{dt}\right) = \frac{mg}{v_r}$$

Substituting the values, we get

$$\left(\frac{-dm}{dt}\right) = \frac{(450 + 50)(9.8)}{2 \times 10^3}$$

$$= 2.45\ \text{kg/s}$$

(ii) Net acceleration $a = 20\ \text{m/s}^2$

$\therefore$ $$ma = F_t - mg$$

or $$a = \frac{F_t}{m} - g$$

or $$a = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

This gives $$\left(-\frac{dm}{dt}\right) = \frac{m(g + a)}{v_r}$$

Substituting the values, we get

$$\left(-\frac{dm}{dt}\right)=\frac{(450+50)(9.8+20)}{2\times 10^3}$$

$$=7.45 \text{ kg/s}$$

(b) The rate of fuel consumption is 10 kg/s. So, the time for the consumption of entire fuel is

$$t=\frac{450}{10}=45 \text{ s}$$

Using Eq. (i), *i.e.*, $v=u-gt+v_r \ln\left(\frac{m_0}{m}\right)$

Here, $u=0$, $v_r=2\times 10^3$ m/s, $m_0=500$ kg and $m=50$ kg

Substituting the values, we get

$$v=0-(9.8)(45)+(2\times 10^3)\ln\left(\frac{500}{50}\right)$$

or $$v=-441+4605.17$$

or $$v=4164.17 \text{ m/s}$$

or $$v=4.164 \text{ km/s}$$

Introductory Exercise 8.4

1. A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/s. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also, calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is (g = 9.8 m/s^2)
   (i) 2 kg/s (ii) 20 kg/s
2. A rocket is moving vertically upward against gravity. Its mass at time t is $m=m_0-\mu t$ and it expels burnt fuel at a speed u vertically downward relative to the rocket. Derive the equation of motion of the rocket but do not solve it. Here, μ is constant.
3. A rocket of initial mass m_0 has a mass $m_0(1-t/3)$ at time t. The rocket is launched from rest vertically upwards under gravity and expels burnt fuel at a speed u relative to the rocket vertically downward. Find the speed of rocket at $t=1$.

8.4 Impulse

Consider a constant force $\vec{F}$ which acts for a time t on a body of mass m, thus, changing its velocity from $\vec{u}$ to $\vec{v}$. Because the force is constant, the body will travel with constant acceleration $\vec{a}$ where

$$\vec{F}=m\vec{a}$$

and $$\vec{a}t=\vec{v}-\vec{u}$$

hence, $$\frac{\vec{F}}{m}t=\vec{v}-\vec{u}$$

or $$\vec{F}t=m\vec{v}-m\vec{u}$$

The product of constant force $\vec{\mathbf{F}}$ and the time t for which it acts is called the **impulse** ($\vec{\mathbf{J}}$) of the force and this is equal to the change in linear momentum which it produces.

Thus, $$\text{Impulse } (\vec{\mathbf{J}}) = \vec{\mathbf{F}}\,t = \Delta\,\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$$

Instantaneous Impulse : There are many occasions when a force acts for such a short time that the effect is instantaneous, *e.g.*, a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (*i.e.*, impulse) can be known by measuring the initial and final momenta. Thus, we can write

$$\vec{\mathbf{J}} = \int \vec{\mathbf{F}}\,dt = \Delta\,\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$$

Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force-time (F-t) graph in the same time interval.

Sample Example 8.13 *A truck of mass 2×10^3 kg travelling at 4 m/s is brought to rest in 2 s when it strikes a wall. What force (assume constant) is exerted by the wall?*

Solution Using impulse = change in linear momentum

We have, $$F \cdot t = mv_f - mv_i = m(v_f - v_i)$$

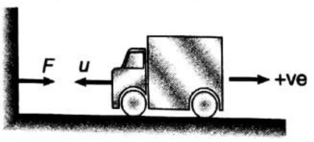


Fig. 8.29

or $$F(2) = 2 \times 10^3\,[0 - (-4)]$$

or $$2F = 8 \times 10^3$$

or $$F = 4 \times 10^3 \text{ N}$$

Sample Example 8.14 *A ball of mass m, travelling with velocity $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ receives an impulse $-3m\hat{\mathbf{i}}$. What is the velocity of the ball immediately afterwards?*

Solution Using $$\vec{\mathbf{J}} = m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)$$

$$-3m\,\hat{\mathbf{i}} = m[\,\vec{\mathbf{v}}_f - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})]$$

or $$\vec{\mathbf{v}}_f = -3\hat{\mathbf{i}} + (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$$

or $$\vec{\mathbf{v}}_f = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

Note *The velocity component in the direction of $\hat{\mathbf{j}}$ is unchanged. This is because there is no impulse component in this direction.*

Sample Example 8.15 *A bullet of mass 10^{-3} kg strikes an obstacle and moves at 60° to its original direction. If its speed also changes from 20 m/s to 10 m/s. Find the magnitude of impulse acting on the bullet.*

Solution Mass of the bullet $m = 10^{-3}$ kg

Consider components parallel to J_1.

$$J_1 = 10^{-3}\,[-10\cos 60° - (-20)]$$

or $$J_1 = 15 \times 10^{-3} \text{ N-s}$$

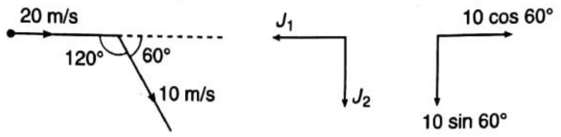


Fig. 8.30

Similarly, praralIel to J_2, we have

$$J_2 = 10^{-3}[10\sin 60° - 0] = 5\sqrt{3} \times 10^{-3} \text{ N-s}$$

The magnitude of resultant impulse is given by

$$J = \sqrt{J_1^2 + J_2^2} = 10^{-3}\sqrt{(15)^2 + (5\sqrt{3})^2}$$

or $$J = \sqrt{3} \times 10^{-2} \text{ N-s}$$

Sample Example 8.16 *A particle of mass 2 kg is initially at rest. A force starts acting on it in one direction whose magnitude changes with time. The force time graph is shown in figure.*

Find the velocity of the particle at the end of 10 s.

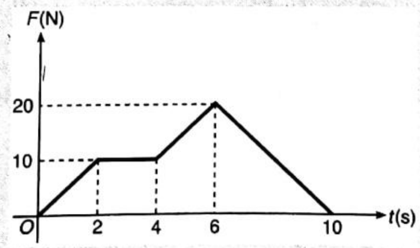


Fig. 8.31

Solution Using impulse = Change in linear momentum (or area under F-t graph)

We have, $$m(v_f - v_i) = \text{Area}$$

or $$2(v_f - 0) = \frac{1}{2} \times 2 \times 10 + 2 \times 10 + \frac{1}{2} \times 2 \times (10 + 20) + \frac{1}{2} \times 4 \times 20$$

$$= 10 + 20 + 30 + 40$$

or $$2v_f = 100$$

$\therefore$ $$v_f = 50 \text{ m/s}$$ **Ans.**

Introductory Exercise 8.5

1. A particle of mass 1 kg is projected from the ground at an angle of 60° with horizontal at a velocity of 20 m/s. Find the magnitude of change in its velocity in 1 s. ($g = 10\ \text{m/s}^2$)

2. Velocity of a particle of mass 2 kg varies with time t according to the equation $\vec{\mathbf{v}} = (2t\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ m/s. Here, t is in seconds. Find the impulse imparted to the particle in the time interval from $t = 0$ to $t = 2$ s.

3. A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in figure. Find the impulse imparted by the string to the ball immediately after the string becomes taut. (Take $g = 10\ \text{m/s}^2$)

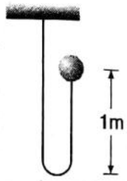


Fig. 8.32

8.5 Collision

Contrary to the meaning of the term 'collision' in our everyday life, in physics it does not necessarily mean one particle 'striking' against other. Indeed two particles may not even touch each other and may still be said to collide. All that is implied is that as the particles approach each other,

(i) an impulse (a large force for a relatively short time) acts on each colliding particles.

(ii) the total momentum of the particles remain conserved.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensible in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

Two blocks of masses m_1 and m_2 are moving with velocities v_1 and $v_2\ (< v_1)$ along the same straight line in a smooth horizontal surface. A spring is attached to the block of mass m_2. Now, let us see what happens during the collision between two particles.

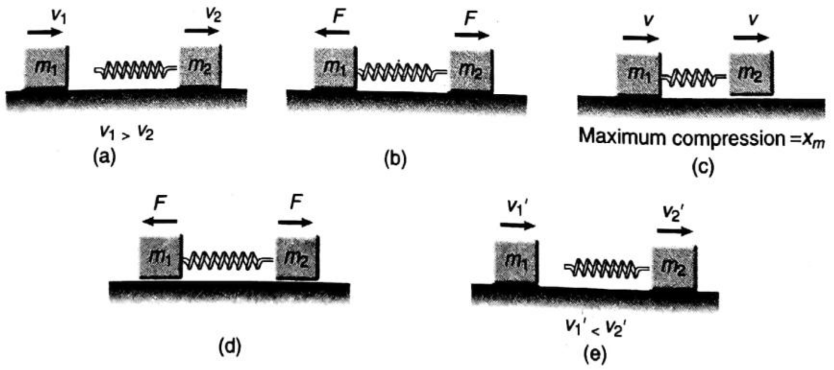


Fig. 8.33

Figure (a) Block of mass m_1 is behind m_2. Since, $v_1 > v_2$, the blocks will collide after some time.

Figure (b) The spring is compressed. The spring force $F\ (= kx)$ acts on the two blocks in the directions shown in figure. This force decreases the velocity of m_1 and increases the velocity of m_2.

Figure (c) The spring will compress till velocity of both the blocks become equal. So, at maximum compression (say x_m) velocities of both the blocks are equal (say v).

Figure (d) Spring force is still in the directions shown in figure, *i.e.*, velocity of block m_1 is further decreased and that of m_2 is increased. The spring now starts relaxing.

Figure (e) The two blocks are separated from one another. Velocity of block m_2 becomes more than the velocity of block m_1, *i.e.*, $v_2' > v_1'$.

Equations Which can be Used in the Above Situation

Assuming spring to be perfectly elastic following two equations can be applied in the above situation.

(i) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, *i.e.*,

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)v = m_1 v_1' + m_2 v_2' \qquad \ldots\text{(i)}$$

(ii) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, *i.e.*,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2} kx_m^2$$

$$= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad \ldots\text{(ii)}$$

Note *In the above situation we have assumed spring to be perfectly elastic, i.e., it regains its original shape and size after the two blocks are separated. In actual practice there is no such spring between the two blocks. During collision both the blocks (or bodies) are a little bit deformed. This situation is similar to the compression of the spring. Due to deformation two equal and opposite forces act on both the blocks. These two forces redistribute their linear momentum in such a manner that both the blocks are separated from one another. The collision is said to be elastic if both the blocks regain their original shape and size completely after they are separated. On the other hand if the blocks do not return to their original form the collision is said to be inelastic. If the deformation is permanent and the blocks move together with same velocity after the collision, the collision is said to be perfectly inelastic.*

Sample Example 8.17 *Two blocks A and B of equal mass $m = 1.0$ kg are lying on a smooth horizontal surface as shown in figure. A spring of force constant $k = 200$ N/m is fixed at one end of block A. Block B collides with block A with velocity $v_0 = 2.0$ m/s. Find the maximum compression of the spring.*

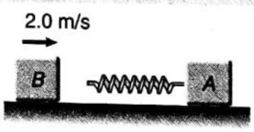


Fig. 8.34

Solution At maximum compression (x_m) velocity of both the blocks is same, say it is v. Applying conservation of linear momentum, we have

$$(m_A + m_B)v = m_B v_0$$

or

$$(1.0 + 1.0)v = (1.0)\, v_0$$

or

$$v = \frac{v_0}{2} = \frac{2.0}{2} = 1.0 \text{ m/s}$$

Using conservation of mechanical energy, we have

$$\frac{1}{2} m_B v_0^2 = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2} kx_m^2$$

Substituting the values, we get

$$\frac{1}{2}\times(1)\times(2.0)^2=\frac{1}{2}\times(1.0+1.0)\times(1.0)^2+\frac{1}{2}\times(200)\times x_m^2$$

or $$2=1.0+100x_m^2$$

or $$x_m=0.1\,\text{m}=10.0\,\text{cm}$$

Types of Collision

Collision between two bodies may be classified in two ways:

1. Elastic collision and inelastic collision.
2. Head on collision or oblique collision.

As discussed earlier also collision between two bodies is said to be **elastic** if both the bodies come to their original shape and size after the collision, *i.e.*, no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision. On the other hand, in an **inelastic** collision, the colliding bodies do not return to their original shape and size completely after collision and some part of the mechanical energy of the system goes to the deformation potential energy. Thus, only linear momentum remains conserved in case of an inelastic collision.

Further, a collision is said to be **head on (or direct)** if the directions of the velocity of colliding objects are along the line of action of the impulses, acting at the instant of collision. If just before collision, at least one of the colliding objects was moving in a direction different from the line of action of the impulses, the collision is called **oblique or indirect.**

Problems related to oblique collision are usually not asked in any medical entrance test. Hence, only head on collision is discussed below.

(i) Head on Elastic Collision

Let the two balls of mass m_1 and m_2 collide each other elastically with velocities v_1 and v_2 in the directions shown in Fig. 8.35(a). Their velocities become v_1' and v_2' after the collision along the same line. Applying conservation of linear momentum, we get

(a) Before collision (b) After collision

Fig. 8.35

$$m_1v_1+m_2v_2=m_1v_1'+m_2v_2' \quad \text{...(i)}$$

In an elastic collision kinetic energy before and after collision is also conserved. Hence,

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii) for v_1' and v_2', we get

$$v_1'=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_1+\left(\frac{2m_2}{m_1+m_2}\right)v_2 \quad \text{...(iii)}$$

and $$v_2'=\left(\frac{m_2-m_1}{m_1+m_2}\right)v_2+\left(\frac{2m_1}{m_1+m_2}\right)v_1 \quad \text{...(iv)}$$

Special Cases

1. If $m_1 = m_2$, then from Eqs. (iii) and (iv), we can see that

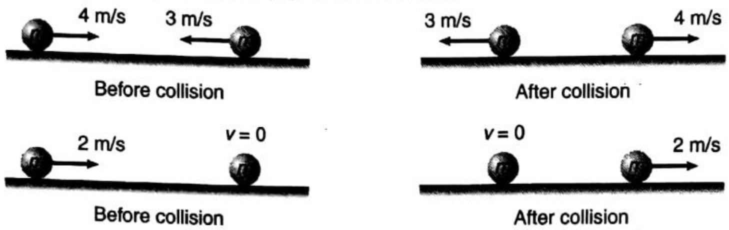


Fig. 8.36

$$v_1' = v_2 \text{ and } v_2' = v_1$$

i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., *e.g.*,

2. If $m_1 >> m_2$ and $v_1 = 0$.

Fig. 8.37

Then $$\frac{m_2}{m_1} \approx 0$$

with these two substitutions $\left(v_1 = 0 \text{ and } \frac{m_2}{m_1} = 0\right)$

we get the following two results:

$$v_1' \approx 0 \quad \text{and} \quad v_2' \approx -v_2$$

i.e., the particle of mass m_1 remains at rest while the particle of mass m_2 bounces back with same speed v_2.

3. If $$m_2 >> m_1 \quad \text{and} \quad v_1 = 0$$

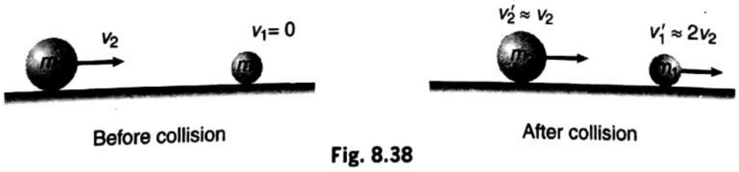


Fig. 8.38

with the substitution $\frac{m_1}{m_2} \approx 0$ and $v_1 = 0$, we get the results

$$v_1' \approx 2v_2 \quad \text{and} \quad v_2' \approx v_2$$

i.e., the mass m_1 moves with velocity $2v_2$ while the velocity of mass m_2 remains unchanged.

Note *It is important to note that Eqs. (iii) and (iv) and their three special cases can be used only in case of a head on elastic collision between two particles. I have found that many students apply these two equations even if the collision is inelastic and do not apply these relations where clearly a head on elastic collision is given in the problem.*

Sample Example 8.18 *Two particles of mass m and 2 m moving in opposite directions collide elastically with velocities v and 2v. Find their velocities after collision.*

Solution Here, $v_1 = -v, v_2 = 2v, m_1 = m$ and $m_2 = 2m.$

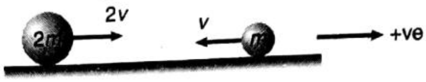


Fig. 8.39

Substituting these values in Eqs. (iii) and (iv), we get

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-v) + \left(\frac{4m}{m+2m}\right)(2v)$$

or $$v_1' = \frac{v}{3} + \frac{8v}{3} = 3v$$

and $$v_2' = \left(\frac{2m-m}{m+2m}\right)(2v) + \left(\frac{2m}{m+2m}\right)(-v)$$

or $$v_2' = \frac{2}{3}v - \frac{2}{3}v = 0$$

i.e., the second particle (of mass $2m$) comes to a rest while the first (of mass m) moves with velocity $3v$ in the direction shown in Fig. 8.40.

Fig. 8.40

Sample Example 8.19 *Two pendulum bobs of mass m and 2m collide elastically at the lowest point in their motion. If both the balls are released from a height H above the lowest point, to what heights do they rise for the first time after collision?*

Solution Given, $m_1 = m, m_2 = 2m, v_1 = -\sqrt{2gH}$ and $v_2 = \sqrt{2gH}$

Since, the collision is elastic. Using Eqs. (iii) and (iv) discussed in the theory the velocities after collision are

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-\sqrt{2gH}) + \left(\frac{4m}{m+2m}\right)\sqrt{2gH}$$

$$= \frac{\sqrt{2gH}}{3} + \frac{4\sqrt{2gH}}{3} = \frac{5}{3}\sqrt{2gH}$$

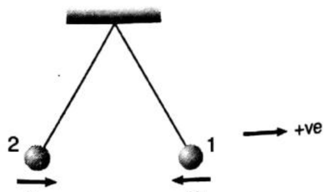


Fig. 8.41

and $$v_2' = \left(\frac{2m-m}{m+2m}\right)(\sqrt{2gH}) + \left(\frac{2m}{m+2m}\right)(-\sqrt{2gH})$$

$$= \frac{\sqrt{2gH}}{3} - \frac{2\sqrt{2gH}}{3} = -\frac{\sqrt{2gH}}{3}$$

i.e., the velocities of the balls after the collision are as shown in figure.

Therefore, the heights to which the balls rise after the collision are:

$$h_1 = \frac{(v_1')^2}{2g} \quad \text{(using } v^2 = u^2 - 2gh\text{)}$$

or

$$h_1 = \frac{\left(\frac{5}{3}\sqrt{2gH}\right)^2}{2g} \quad \text{or} \quad h_1 = \frac{25}{9}H$$

and

$$h_2 = \frac{(v_2')^2}{2g} \quad \text{or} \quad h_2 = \frac{\left(\frac{\sqrt{2gH}}{3}\right)^2}{2g}$$

or

$$h_2 = \frac{H}{9}$$

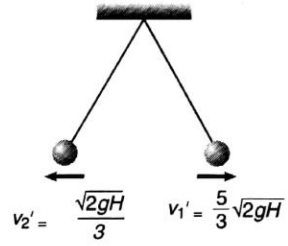


Fig. 8.42

Note *Since the collision is elastic, mechanical energy of both the balls will remain conserved, or*

$$E_i = E_f$$

$\Rightarrow$ $$(m + 2m)\,gH = mgh_1 + 2mgh_2$$

$\Rightarrow$ $$3mgH = (mg)\left(\frac{25}{9}H\right) + (2mg)\left(\frac{H}{9}\right)$$

$\Rightarrow$ $$3mgH = 3mgH$$

Introductory Exercise 8.6

1. Two blocks of mass 3 kg and 6 kg respectively are placed on a smooth horizontal surface. They are connected by a light spring of force constant $k = 200$ N/m. Initially the spring is unstretched. The indicated velocities are imparted to the blocks. Find the maximum extension of the spring.

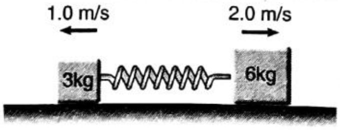


Fig. 8.43

2. A particle moving with kinetic energy K makes a head on elastic collision with an identical particle at rest. Find the maximum elastic potential energy of the system during collision.

3. Show that in a head on elastic collision between two particles, the transference of energy is maximum when their mass ratio is unity.

4. What is the fractional decrease in kinetic energy of a particle of mass m_1 when it makes a head on elastic collision with a particle of mass m_2 kept at rest?

5. A moving particle of mass m makes a head on elastic collision with a particle of mass $2m$ which is initially at rest. Find the fraction of kinetic energy lost by the colliding particle after collision.

6. Three balls A, B and C are placed on a smooth horizontal surface. Given that $m_A = m_C = 4m_B$. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls. All collisions are elastic.

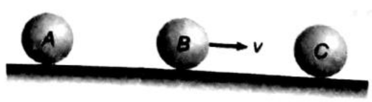


Fig. 8.44

7. In one dimensional elastic collision of equal masses, the velocities are interchanged. Can velocities in a one dimensional collision be interchanged if the masses are not equal.

8. Two balls shown in figure are identical. Ball A is moving towards right with a speed v and the second ball is at rest. Assume all collisions to be elastic. Show that the speeds of the balls remain unchanged after all the collisions have taken place.

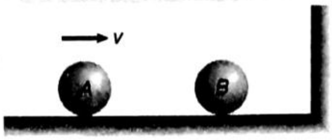


Fig. 8.45

(ii) Head on Inelastic Collision

As we have discussed earlier also, in an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

Fig. 8.46

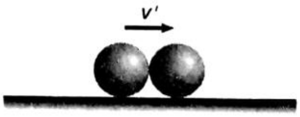


Fig. 8.47

Suppose the velocities of two particles of mass m_1 and m_2 before collision be v_1 and v_2 in the directions shown in figure. Let v_1' and v_2' be their velocities after collision. The law of conservation of linear momentum gives

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \quad \text{...(v)}$$

Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity, say v' as shown in figure. In this case, Eq. (v) can be written as

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v'$$

or

$$v' = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \quad \text{...(vi)}$$

Newton's Law of Restitution

When two objects are in direct (head on) impact, the speed with which they separate after impact is usually less than or equal to their speed of approach before impact.

Experimental evidence suggests that the ratio of these relative speeds is constant for two given set of objects. This property formulated by Newton, is known as the law of restitution and can be written in the form

$$\frac{\text{separation speed}}{\text{approach speed}} = e \quad \text{...(vii)}$$

The ratio e is called the coefficient of restitution and is constant for two particular objects.

In general $\quad 0 \le e \le 1$

$e = 0$, for completely inelastic collision, as both the objects stick together. So, their separation speed is zero or $e = 0$ from Eq. (vii).

$e = 1$, for an elastic collision, as we can show from Eq. (iii) and (iv), that

$$v_1' - v_2' = v_2 - v_1$$

or
$$\text{separation speed} = \text{approach speed}$$

or
$$e = 1$$

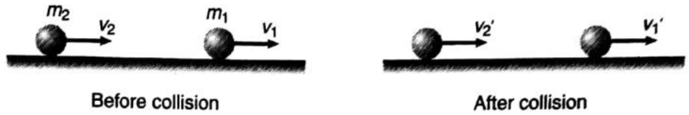


Fig. 8.48

Let us now find the velocities of two particles after collision if they collide directly and the coefficient of restitution between them is given as e.

Applying conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{...(viii)}$$

Further,
$$\text{separation speed} = e\,(\text{approach speed})$$

or
$$v_1' - v_2' = e\,(v_2 - v_1) \qquad \text{...(ix)}$$

Solving Eqs. (viii) and (ix), we get

$$v_1' = \left(\frac{m_1 - em_2}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right) v_2 \qquad \text{...(x)}$$

and
$$v_2' = \left(\frac{m_2 - em_1}{m_1 + m_2}\right) v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right) v_1 \qquad \text{...(xi)}$$

Special Cases

1. If collision is elastic, *i.e.*, $e = 1$, then

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

and
$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1$$

which are same as Eqs. (iii) and (iv).

2. If collision is perfectly inelastic, *i.e.*, $e = 0$, then

$$v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = v' \text{ (say)}$$

which is same as Eq. (vi).

3. If $m_1 = m_2$ and $v_1 = 0$, then

Fig. 8.49

$$v_1' = \left(\frac{1+e}{2}\right)v_2 \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right)v_2$$

Note (i) *If mass of one body is very-very greater than that of the other, then after collision velocity of heavy body does not change appreciably. (Whether the collision is elastic or inelastic).*

(ii) *In the situation shown in figure if e is the coefficient of restitution between the ball and the ground, than after nth collision with the floor the speed of ball will remain* $e^n v_0$ *and it will go upto a height* $e^{2n}h$ *or,*

$$v_n = e^n v_0 = e^n\sqrt{2gh} \quad \text{and} \quad h_n = e^{2n}h$$

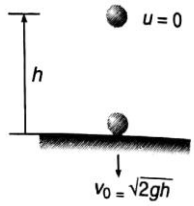


Fig. 8.50

EXERCISE *Derive the above two relations.*

Sample Example 8.20 *A ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4th of the original. Find the coefficient of restitution.*

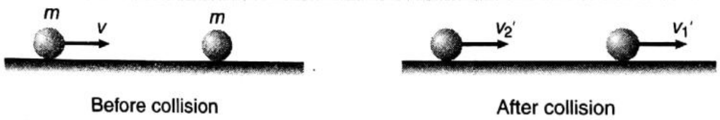


Fig. 8.51

Solution As we have seen in the above discussion, that under the given conditions:

$$v_1' = \left(\frac{1+e}{2}\right)v \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right)v$$

Given that

$$K_f = \frac{3}{4}K_i$$

or

$$\frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or

$$(1+e)^2 + (1-e)^2 = 3$$

or

$$2 + 2e^2 = 3$$

or

$$e^2 = \frac{1}{2}$$

or

$$e = \frac{1}{\sqrt{2}}$$

Sample Example 8.21 *A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in figure. Assuming collision to be elastic, find the velocity of ball immediately after the collision.*

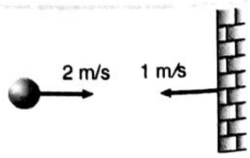


Fig. 8.52

Solution The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since, collision is elastic ($e = 1$).

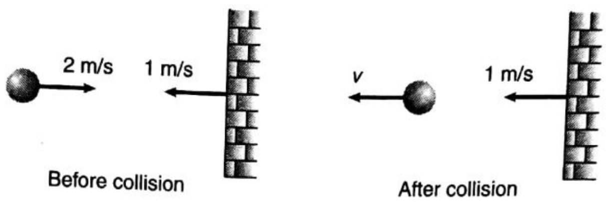


Fig. 8.53

$$\text{separation speed} = \text{approach speed}$$

or $$v - 1 = 2 + 1$$

or $$v = 4 \text{ m/s}$$

Sample Example 8.22 *After perfectly inelastic collision between two identical particles moving with same speed in different directions, the speed of the particles becomes half the initial speed. Find the angle between the two before collision.*

Solution Let θ be the desired angle. Linear momentum of the system will remain conserved. Hence

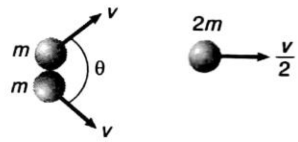


Fig. 8.54

$$P^2 = P_1^2 + P_2^2 + 2P_2P_2 \cos\theta$$

or $$\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$$

or $$1 = 1 + 1 + 2\cos\theta$$

or $$\cos\theta = -\frac{1}{2}$$

$\therefore$ $$\theta = 120°$$

Introductory Exercise 8.7

1. Ball 1 collides directly with another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls?
2. A particle of mass 0.1 kg moving at an initial speed v collides with another particle of same mass kept initially at rest. If the total energy becomes 0.2 J after the collision, what would be the minimum and maximum values of v ?
3. A particle of mass m moving with a speed v hits elastically another stationary particle of mass $2m$ on a smooth horizontal circular tube or radius r. Find the time when the next collision will take place?
4. In a one-dimensional collision between two identical particles A and B, B is stationary and A has momentum p before impact. During impact B gives an impulse J to A. Find the coefficient of restitution between A and B ?

Oblique Collision

During collision between two objects a pair of equal and opposite impulses act at the moment of impact. If just before impact at least one of the objects was moving in a direction different from the line of action of these impulses the collision is said to be oblique.

In the figure, two balls collide obliquely. During collision impulses act in the direction xx. Henceforth, we will call this direction as common normal direction and a direction perpendicular to it (*i.e.*, yy) as common tangent. Following four points are important regarding an oblique collision.

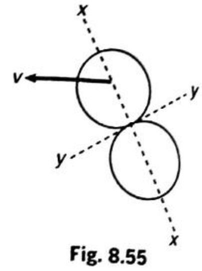


Fig. 8.55

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, *i.e.*, along common normal direction we can apply

$$\text{Relative speed of separation} = e\,(\text{relative speed of approach})$$

Here, e is the coefficient of restitution between the particles. Here, are few examples in support of the above theory.

Sample Example 8.23 *A ball of mass m hits a floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.*

Solution The component of velocity v_0 along common tangent direction $v_0 \sin\alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying

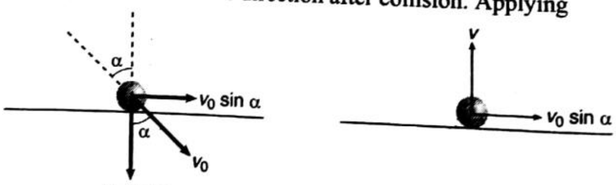


Fig. 8.56

$$\text{Relative speed of separation} = e\,(\text{relative speed of approach})$$

along common normal direction, we get

$$v = ev_0 \cos\alpha$$

Thus, after collision components of velocity v' are $v_0 \sin\alpha$ and $ev_0 \cos\alpha$

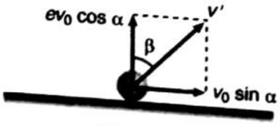


Fig. 8.57

$\therefore$ $$v' = \sqrt{(v_0 \sin\alpha)^2 + (ev_0 \cos\alpha)^2}$$

and
$$\tan\beta = \frac{v_0 \sin\alpha}{e v_0 \cos\alpha}$$

or
$$\tan\beta = \frac{\tan\alpha}{e}$$

Note *For elastic collision,* $e = 1$

$\therefore \; v' = v_0$ *and* $\beta = \alpha$.

Introductory Exercise 8.8

1. A ball falls vertically on an inclined plane of inclination α with speed v_0 and makes a perfectly elastic collision. What is angle of velocity vector with horizontal after collision.
2. A ball falls on the ground from a height h. The coefficient of restitution between the ball and the ground is e. Find the speed and height upto which it rebounds after nth impact of the ball with the ground?
3. A sphere A of mass m, travelling with speed v, collides directly with a stationary sphere B. If A is brought to rest and B is given a speed V, find (a) the mass of B (b) the coefficient of restitution between A and B?
4. Two billiard balls of same size and mass are in contact on a billiard table. A third ball of same mass and size strikes them symmetrically and remains at rest after the impact. Find the coefficient of restitution between the balls?
5. A smooth sphere is moving on a horizontal surface with velocity vector $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ immediately before it hits a vertical wall. The wall is parallel to $\hat{\mathbf{j}}$ and the coefficient of restitution of the sphere and the wall is $e = \frac{1}{2}$. Find the velocity of the sphere after it hits the wall?
6. A ball is projected from the ground at some angle with horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e?

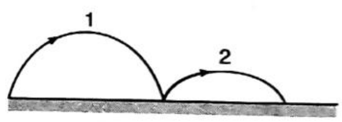


Fig. 8.58

Extra Points

- During collision if mass of one body is very much greater than the mass of the other body then the velocity of heavy body remains almost unchanged after collision, whether the collision is elastic or inelastic.
- Net force on a system is zero, it does not mean that centre of mass is at rest. It might be moving with constant velocity.
- Centre of mass of a rigid body is not necessarily the geometric centre of the rigid body.
- If two many particles are in air, then relative acceleration between any two is zero but acceleration of their centre of mass is g downwards.
- Coefficient of restitution is the mutual intrinsic properties of two bodies. Its value varies from 0 to 1.
- **Centre of mass frame of reference or *C*-frame of reference or zero momentum frame :** A frame of reference carried by the centre of mass of an isolated system of particles (*i.e.*, a system not subjected to any external forces) is called the centre of mass or *C*-frame of reference. In this frame of reference.

  (i) Position vector of centre of mass is zero.

  (ii) Velocity and hence momentum of centre of mass is also zero.
- In the situation discussed above we can also apply

$$\Sigma m_R x_R = \Sigma m_L x_L$$

Here, $\Sigma m_R\, x_R$ is the summation of product of x and m of the particles (or bodies) which are moving towards right and $\Sigma m_L x_L$ is the summation of product of x and m of the particles (or bodies) which are moving towards left. But remember the following three conditions while using the above equation.

(i) This equation can be applied when centre of mass does not move in x-direction.

(ii) In the above equation x is the displacement of particle relative to ground.

(iii) Apply the above equation while solving the objective problems only. Solve the subjective problems by the method discussed earlier.

Let us solve example 5.10 using the above method.

Here, x_L = displacement of plank towards left = x

m_L = mass of plank = 20 kg

x_R = displacement of man relative to ground towards right = $10 - x$

and m_R = mass of man = 60 kg

Applying $x_R m_R = x_L m_L$, we get

$$(10 - x)(60) = 20x$$

or $$x = 30 - 3x$$

or $$4x = 30$$

$\therefore$ $$x = \frac{30}{4} = 7.5 \text{ m}$$

- A liquid of density ρ is filled in a container as shown in figure. The liquid comes out from the container through a orifice of area 'a' at a depth 'h' below the free surface of the liquid with a velocity v. This exerts a thrust force in the container in the backward direction. This thrust force is given by

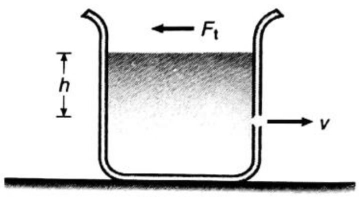


Fig. 8.59

$$F_t = v_r\left(-\frac{dm}{dt}\right)$$

Here, $v_r = v$ (in forward direction)

and $$\left(-\frac{dm}{dt}\right) = \rho a v$$

as $$\left(\frac{dV}{dt}\right) = \text{Volume of liquid flowing per second}$$

$$= av$$

$\therefore$ $$\left(-\frac{dm}{dt}\right) = \rho\left(\frac{dV}{dt}\right) = \rho a v$$

$\therefore$ $$F_t = v(\rho a v)$$

or $$F_t = \rho a v^2 \text{ (in backward direction)}$$

Further, we will see in the chapter of fluid mechanics that $v = \sqrt{2gh}$.

- Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in Fig. 8.60(a) or the end of the chain piled on the platform is lifted vertically as in Fig. 8.60(b). In both the cases, due to

Fig. 8.60

increase of mass in the portion of the chain which is moving with a velocity v at certain moment of time a thrust force acts on this part of the chain which is given by

$$F_t = v_r\left(\frac{dm}{dt}\right)$$

Here, $$v_r = v \quad \text{and} \quad \frac{dm}{dt} = \lambda v$$

Here, v_r is upwards in case (a) and downwards in case (b). Thus,

$$F_t = \lambda v^2$$

The direction of F_t is upwards in case (a) and downwards in case (b).

- Suppose a ball is a projected with speed u at an angle θ with horizontal. It collides at some distance with a wall parallel to y-axis as shown in figure. Let v_x and v_y be the components of its velocity along x and y-directions at the time of impact with wall. Coefficient of restitution between the ball and the wall is e. Component of its velocity along y-direction (common tangent) v_y will remain unchanged while component of its velocity along x-direction (common normal) v_x will become ev_x is opposite direction.

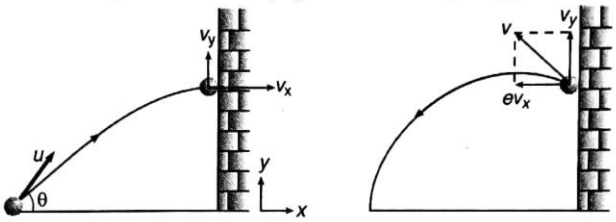


Fig. 8.61

Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and maximum height attained by the ball will remain same as it would had been in the absence of collision with the wall. Thus,

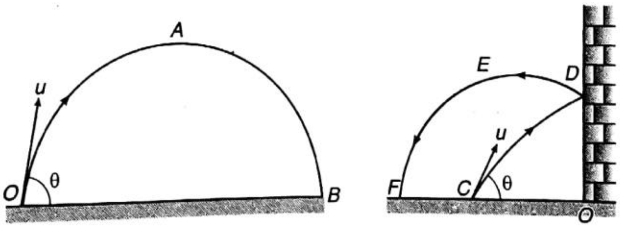


Fig. 8.62

$$t_{OAB} = t_{CD} + t_{DEF} = T = \frac{2u\sin\theta}{g}$$

and $$h_A = h_E = \frac{u^2\sin^2\theta}{2g}$$

Further, $$CO + OF \le \text{Range or } OB$$

It collision is elastic, then $$CO + OF = \text{Range} = \frac{u^2\sin 2\theta}{g}$$

and if it is inelastic, $$CO + OF < \text{Range}$$

Solved Examples

Level 1

Example 1 *The friction coefficient between the horizontal surface and each of the block shown in the figure is 0.2. The collision between the blocks is perfectly elastic. Find the separation between them when they come to rest.*
(Take $g = 10\, m/s^2$).

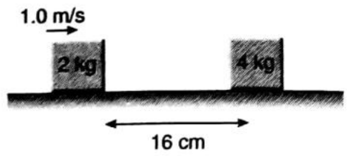


Fig. 8.63

Solution Velocity of first block before collision,

$$v_1^2 = 1^2 - 2\,(2)\times 0.16 = 1 - 0.64$$

$$v_1 = 0.6 \text{ m/s}$$

By conservation of momentum, $2\times 0.6 = 2v_1' + 4v_2'$

also $v_2' - v_1' = v_1$ for elastic collision

It gives $v_2' = 0.4$ m/s

$v_1' = -0.2$ m/s

Now distance moved after collision

$$s_1 = \frac{(0.4)^2}{2\times 2} \quad \text{and} \quad s_2 = \frac{(0.2)^2}{2\times 2}$$

$\therefore$ $$s = s_1 + s_2 = 0.05 \text{ m} = 5 \text{ cm}$$

Example 2 *A pendulum bob of mass 10^{-2} kg is raised to a height 5×10^{-2} m and then released. At the bottom of its swing, it picks up a mass 10^{-3} kg. To what height will the combined mass rise ?*

Solution Velocity of pendulum bob in mean position

$$v_1 = \sqrt{2gh} = \sqrt{2\times 10\times 5\times 10^{-2}} = 1 \text{m/s}$$

When the bob picks up a mass 10^{-3} kg at the bottom, then by conservation of linear momentum the velocity of coalesced mass is given by

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v$$

$$10^{-2} + 10^{-3}\times 0 = (10^{-2} + 10^{-3})\, v$$

or $$v = \frac{10^{-2}}{1.1\times 10^{-2}} = \frac{10}{11} \text{ m/s}$$

Now, $$h = \frac{v^2}{2g} = \frac{(10/11)^2}{2\times 10} = 4.1\times 10^{-2} \text{ m}$$

Example 3 *Three identical balls, ball I, ball II and ball III are placed on a smooth floor on a straight line at the separation of 10 m between balls as shown in figure. Initially balls are stationary. Ball I is given velocity of 10 m/s towards ball II, collision between ball I and II is inelastic with coefficient of restitution 0.5 but collision between ball II and III is perfectly elastic. What is the time interval between two consecutive collisions between ball I and II ?*

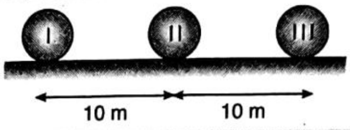


Fig. 8.64

Solution Let velocity of I ball and II ball after collision be v_1 and v_2

$$v_2 - v_1 = 0.5 \times 10 \quad \ldots(i)$$
$$mv_2 + mv_1 = m \times 10 \quad \ldots(ii)$$

$\Rightarrow$
$$v_2 + v_1 = 10$$

Solving Eqs. (i) and (ii)

$$v_1 = 2.5 \text{ m/s}, \quad v_2 = 7.5 \text{ m/s}$$

Ball II after moving 10 m collides with ball III elastically and stops. But ball I moves towards ball II. Time taken between two consecutive collisions

$$\frac{10}{7.5} = \frac{10 - 10 \times \frac{2.5}{7.5}}{2.5} = 4 \text{ s}$$

Example 4 *A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. They system is now released from the rest. What is the velocity of the plank when block leaves the plank ? (The stiffness constant of spring is 100 N/m)*

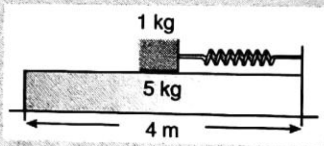


Fig. 8.65

Solution Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$

[M = mass of the plank, m = mass of the block]

$\Rightarrow$
$$100 = u^2 + 5v^2 \quad \ldots(i)$$

By conservation of momentum

$$mu + Mv = 0$$

$\Rightarrow$
$$u = -5v \quad \ldots(ii)$$

Solving Eqs (i) and (ii)

$$30v^2 = 100 \Rightarrow v = \sqrt{\frac{10}{3}} \text{ m/s}$$

From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls, velocity of plank $= \sqrt{\frac{10}{3}}$ m/s.

Example 5 *Two identical blocks each of mass $M = 9$ kg are placed on a rough horizontal surface of frictional coefficient $\mu = 0.1$. The two blocks are joined by a light spring and block B is in contact with a vertical fixed wall as shown in figure. A bullet of mass $m = 1$ kg and $v_0 = 10$ m/s hits block A and gets embedded in it.*

Find the maximum compression of spring. (Spring constant $= 240$ N/m, $g = 10$ m/s^2)

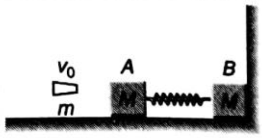


Fig. 8.66

Solution For the collision

$$1 \times 10 = 10 \times v \quad \Rightarrow \quad v = 1 \text{ m/s}$$

If x be the maximum compression

$$\frac{1}{2} \times 10 \times 1^2 = \mu\,(m + M)\,gx + \frac{1}{2}kx^2$$

$$5 = 10x + 120x^2 \quad \Rightarrow \quad x = \frac{1}{6}\,m$$

Example 6 *A particle of mass 2 kg moving with a velocity $5\hat{\mathbf{i}}$ m/s collides head-on with another particle of mass 3 kg moving with a velocity $-2\hat{\mathbf{i}}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction. Find :*

(a) velocity of the centre of mass after the collision,
(b) velocity of the second particle after the collision,
(c) coefficient of restitution.

Solution (a) $\vec{\mathbf{v}}_c = \dfrac{m_1\vec{\mathbf{u}}_1 + m_2\vec{\mathbf{u}}_2}{m_1 + m_2} = 0.8\,\hat{\mathbf{i}}$ m/s

(b) $\vec{\mathbf{v}}_1 = -1.6\,\hat{\mathbf{i}}$ m/s

From COM, $$m_1\vec{\mathbf{u}}_1 + m_2\vec{\mathbf{u}}_2 = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2$$

$\Rightarrow$ $$v_2 = 2.4\,\hat{\mathbf{i}} \text{ m/s}$$

(c) $e = \dfrac{v_2 - v_1}{u_1 - u_2} = \dfrac{4}{7}$

Example 7 *Two blocks A and B of equal mass are released on two sides of a fixed wedge C as shown in figure. Find the acceleration of centre of mass of blocks A and B. Neglect friction.*

Fig. 8.67

Solution Acceleration of both the blocks will be $g \sin 45°$ or $\frac{g}{\sqrt{2}}$ at right angles to each other. Now,

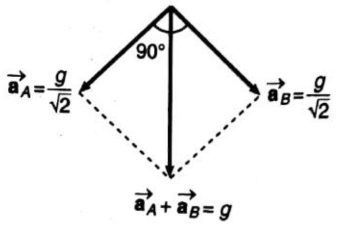


Fig. 8.68

$$\vec{a}_{COM} = \frac{m_A \vec{a}_A + m_B \vec{a}_B}{m_A + m_B}$$

Here, $m_A = m_B$

$\therefore$ $$\vec{a}_{COM} = \frac{1}{2}(\vec{a}_A + \vec{a}_B) = \frac{1}{2} g \quad \textbf{(downwards)}$$

Example 8 *A block of mass m is released from the top of a wedge of mass M as shown in figure. Find the displacement of wedge on the horizontal ground when the block reaches the bottom of the wedge. Neglect friction everywhere.*

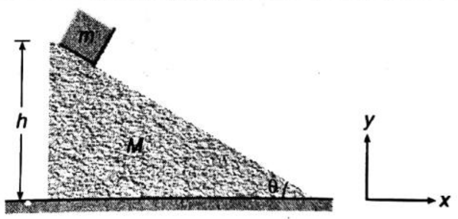


Fig. 8.69

Solution Here, the system is wedge + block. Net force on the system in horizontal direction (x-direction) is zero, therefore, the centre of mass of the system will not move in x-direction so we can apply,

$$x_R m_R = x_L m_L \quad \text{...(i)}$$

Let x be the displacement of wedge. Then,

x_L = displacement of wedge towards left $= x$

m_L = mass of wedge $= M$

x_R = displacement of block with respect to ground towards right $= h \cot \theta - x$

and m_R = mass of block $= m$

Substituting in Eq. (i), we get

$$m(h \cot \theta - x) = xM$$

$\therefore$ $$x = \frac{mh \cot \theta}{M + m}$$

Example 9 *A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.*

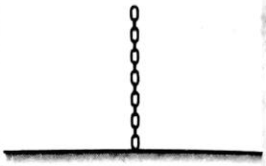


Fig. 8.70

Solution **Force exerted by the chain on the table.** It consists of two parts:

1. Weight of the portion BC of the chain lying on the table,

$$W = \frac{mg}{2} \quad \text{(downwards)}$$

2. Thrust force $F_t = \lambda v^2$

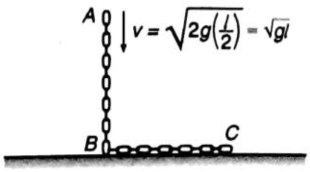


Fig. 8.71

Here, λ = mass per unit length of chain $= \frac{m}{l}$

$$v^2 = (\sqrt{gl})^2 = gl$$

$\therefore$ $$F_t = \left(\frac{m}{l}\right)(gl) = mg \quad \text{(downwards)}$$

$\therefore$ Net force exerted by the chain on the table is

$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2}mg \quad \text{(downwards)}$$

So, from Newton's third law the force exerted by the table on the chain will be $\frac{3}{2}mg$ (vertically upwards).

Note *Here, the thrust force (F_t) applied by the chain on the table will be vertically downwards, as $F_t = v_r\left(\frac{dm}{dt}\right)$ and in this expression v_r is downwards plus $\frac{dm}{dt}$ is positive. So, F_t will be downwards.*

Example 10 *A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.*

Solution In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v\cos\theta$ becomes zero after collision, while that of 2 becomes $v\cos\theta$. While the components along common tangent direction of both the particles remain

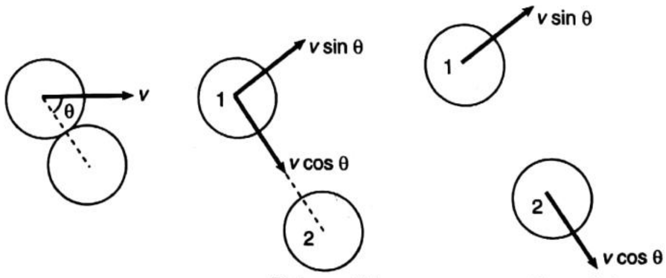


Fig. 8.72

unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given a head :

Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin \theta$	$v \sin \theta$	$v \cos \theta$	0
2	0	0	0	$v \cos \theta$

From the above table and figure, we see that both the balls move at right angles after collision with velocities $v \sin \theta$ and $v \cos \theta$.

Example 11 *A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning ceases after one minute, find the maximum velocity of the rocket. (Take g as constant at 10 ms^{-2})*

Solution Using the velocity equation as derived in theory.

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Here, $u = 0, \quad t = 60\,\text{s}, \quad g = 10 \text{ m/s}^2, \quad v_r = 2000 \text{ m/s}, \quad m_0 = 1000\,\text{kg}$

and $$m = 1000 - 10 \times 60 = 400\,\text{kg}$$

we get, $$v = 0 - 600 + 2000 \ln\left(\frac{1000}{400}\right)$$

or $$v = 2000 \ln 2.5 - 600$$

The maximum velocity of the rocket is $200\,(10 \ln 2.5 - 3) = 1232.6 \text{ ms}^{-1}$.

Level 2

Example 1 *A ball is projected from the ground with speed u at an angle α with horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.*

Solution As we have discussed in the theory, the horizontal component of the velocity of ball during the path OAB is $u \cos \alpha$ while in its return journey BCO it is $eu \cos \alpha$. The time of flight T also remains unchanged. Hence,

$$T = t_{OAB} + t_{BCO}$$

or $$\frac{2u \sin \alpha}{g} = \frac{a}{u \cos \alpha} + \frac{a}{eu \cos \alpha}$$

or $$\frac{a}{eu \cos \alpha} = \frac{2u \sin \alpha}{g} - \frac{a}{u \cos \alpha}$$

or $$\frac{a}{eu \cos \alpha} = \frac{2u^2 \sin \alpha \cos \alpha - ag}{gu \cos \alpha}$$

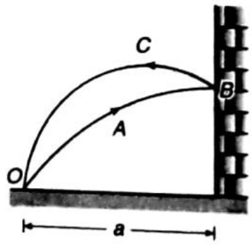


Fig. 8.73

$$\therefore \qquad e = \frac{ag}{2u^2 \sin\alpha\cos\alpha - ag}$$

or
$$e = \frac{1}{\left(\dfrac{u^2 \sin 2\alpha}{ag} - 1\right)}$$

Note *The concept which we have discussed in the oblique collision can also be applied when a ball collides with a wedge. This can be understood with the help of an Problem given below.*

Example 2 *A ball of mass $m = 1$ kg falling vertically with a velocity $v_0 = 2$ m/s strikes a wedge of mass $M = 2$ kg kept on a smooth, horizontal surface as shown in figure. The coefficient of restitution between the ball and the wedge is $e = \frac{1}{2}$. Find the velocity of the wedge and the ball immediately after collision.*

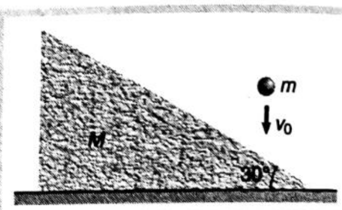


Fig. 8.74

Solution Given $M = 2$ kg and $m = 1$ kg

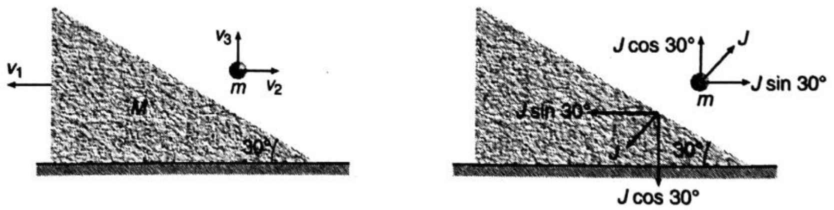


Fig. 8.75

Let, J be the impulse between ball and wedge during collision and v_1, v_2 and v_3 be the components of velocity of the wedge and the ball in horizontal and vertical directions respectively.

Applying impulse = change in momentum

we get
$$J \sin 30° = Mv_1 = mv_2$$

or
$$\frac{J}{2} = 2v_1 = v_2 \qquad \text{...(i)}$$

$$J \cos 30° = m(v_3 + v_0)$$

or
$$\frac{\sqrt{3}}{2} J = (v_3 + 2) \qquad \text{...(ii)}$$

Applying, relative speed of separation $= e$ (relative speed of approach) in common normal direction, we get

$$(v_1 + v_2)\sin 30° + v_3 \cos 30° = \frac{1}{2}(v_0 \cos 30°)$$

or
$$v_1 + v_2 + \sqrt{3}v_3 = \sqrt{3} \qquad \text{...(iii)}$$

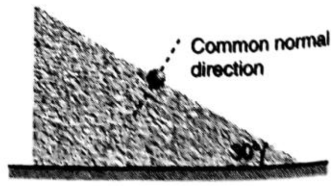


Fig. 8.76

Solving Eqs. (i), (ii) and (iii), we get

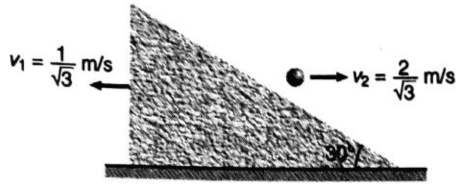


Fig. 8.77

$$v_1 = \frac{1}{\sqrt{3}} \text{ m/s}$$

$$v_2 = \frac{2}{\sqrt{3}} \text{ m/s} \quad \text{and} \quad v_3 = 0$$

Thus, velocities of wedge and ball are $v_1 = \frac{1}{\sqrt{3}}$ m/s and $v_2 = \frac{2}{\sqrt{3}}$ m/s in horizontal direction as shown in figure.

Note *If a particle (or a body) can move in a straight line and we want to find its velocity from the given conditions we take only one unknown v. If the particle can move in a plane we take two unknowns v_x and v_y (with $x \perp y$). Similarly if it can move in space we take three unknowns v_x, v_y and v_z.*
For instance in the above Problem, the wedge can move only in horizontal line, so we took only one unknown v_1. The ball can move in a plane, so we took two unknowns v_2 and v_3. Further, note that x and y axis should be perpendicular to each other. They may be along horizontal and vertical or along common tangent (along the plane in this case) and common normal (perpendicular to plane).

Example 3 *Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure. (a) Find the displacement of the centre of mass at time t. (b) If the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.*

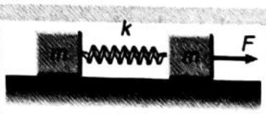


Fig. 8.78

Solution (a) The acceleration of the centre of mass is

$$a_{COM} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2} a_{COM} t^2 = \frac{Ft^2}{4m}$$

(b) Suppose the displacement of the first block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m}$$

or

$$\frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or
$$x_1 + x_2 = \frac{Ft^2}{2m} \quad \text{...(i)}$$

Further, the extension of the spring is $x_1 - x_2$. Therefore,
$$x_1 - x_2 = x_0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),
$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right)$$

and
$$x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right).$$

Example 4 *A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.*

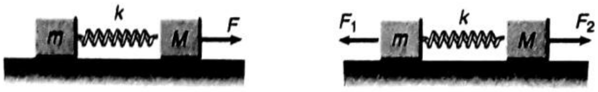


Fig. 8.79

Solution The centre of mass of the system (two blocks + spring) moves with an acceleration $a = \frac{F}{m+M}$. Let us solve the problem in a frame of reference fixed to the centre of mass of the system. As this frame is accelerated with respect to the ground, we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is
$$F_1 = ma = \frac{mF}{m+M} \quad \text{(towards left)}$$

and the net external force on M is
$$F_2 = F - Ma = F - \frac{MF}{m+M} = \frac{mF}{m+M} \quad \text{(towards right)}$$

As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The extension of the spring will be maximum at this instant. Suppose, the left block is displaced through a distance x_1 and the right block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are
$$W = F_1x_1 + F_2x_2 = \frac{mF}{m+M}(x_1 + x_2)$$

This should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy. Thus,
$$\frac{mF}{m+M}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2$$

or
$$x_1 + x_2 = \frac{2mF}{k(m+M)}$$

This is the maximum extension of the spring.

Example 5 *The coefficient of restitution between a snooker ball and the side cushion is $\frac{1}{3}$. If the ball hits the cushion and then rebounds at right angles to its original direction, show that the angles made with the side cushion by the direction of motion before and after impact are 60° and 30° respectively.*

Solution Let the original speed be u, in a direction making an angle θ with the side cushion. Using the law of restitution

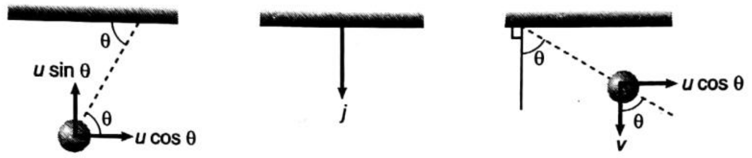


Fig. 8.80

$$v = \frac{1}{3}(u \sin \theta)$$

After impact,
$$\tan \theta = \frac{u \cos \theta}{v} = \frac{3 \cos \theta}{\sin \theta}$$

$\Rightarrow$
$$\tan^2 \theta = 3$$

$\Rightarrow$
$$\tan \theta = \sqrt{3}$$

$\Rightarrow$
$$\theta = 60°$$

Therefore, the directions of motion before and after impact are at 60° and 30° to the cushion.

Example 6 *Two blocks A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third block C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0. Determine (i) the common velocity of A and B at time t_0, and (ii) the spring constant.*

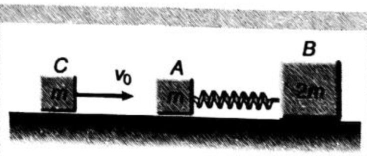


Fig. 8.81

Solution Initially, the blocks A and B are at rest and C is moving with velocity v_0 to the right.

As masses of C and A are same and the collision is elastic the body C transfers its whole momentum mv_0 to body A and as a result the body C stops and A starts moving with velocity v_0 to the right. At this instant the spring is uncompressed and the body B is still at rest.

The momentum of the system at this instant $= mv_0$

Now, the spring is compressed and the body B comes in motion. After time t_0, the compression of the spring is x_0 and common velocity of A and B is v (say).

As external force on the system is zero, the law of conservation of linear momentum gives

$$mv_0 = mv + (2m)v$$

or

$$v = \frac{v_0}{3}$$

The law of conservation of energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2$$

or

$$\frac{1}{2}mv_0^2 = \frac{3}{2}mv^2 + \frac{1}{2}kx_0^2 \quad \text{...(ii)}$$

$$\frac{1}{2}mv_0^2 = \frac{3}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

∴

$$\frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 - \frac{1}{6}mv_0^2$$

or

$$\frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$$

∴

$$k = \frac{2}{3}\frac{mv_0^2}{x_0^2}$$

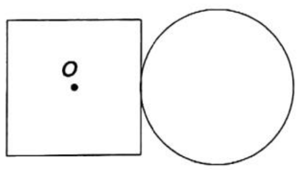
EXERCISES

AIEEE Corner

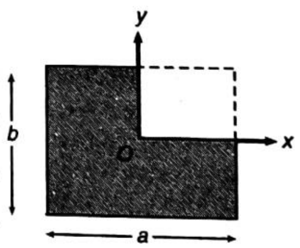
Subjective Questions (Level 1)

Centre of Mass

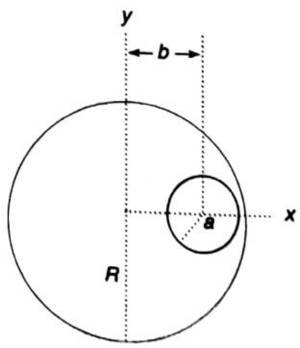
1. Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find square of distance of their centre of mass from A.
2. A square lamina of side a and a circular lamina of diameter a are placed touching each other as shown in figure. Find distance of their centre of mass from point O, the centre of square.



3. Consider a rectangular plate of dimensions $a \times b$. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be.



4. The uniform solid sphere shown in the figure has a spherical hole in it. Find the position of its centre of mass.



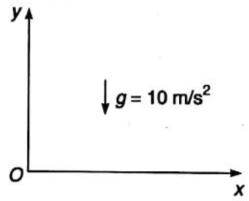
5. There are two masses m_1 and m_2 placed at a distance l apart. Let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C. Find the distance, from C where new centre of mass will be located.

6. The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 \frac{x^2}{l^2}$. Find the position of centre of mass of rod.

7. A block of mass 1 kg is at $x = 10$ m and moving towards negative x-axis with velocity 6 m/s. Another block of mass 2 kg is at $x = 12$ m and moving towards positive x-axis with velocity 4 m/s at the same instant. Find position of their centre of mass after 2 s.

8. Two particles of mass 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$ s.

9. x-y is the vertical plane as shown in figure. A particle of mass 1 kg is at (10 m, 20 m) at time $t = 0$. It is released from rest. Another particle of mass 2 kg is at (20 m, 40 m) at the same instant. It is projected with velocity $(10\hat{i} + 10\hat{j})$ m/s. After 1 s. Find :



(a) acceleration, (b) velocity and (c) position of their centre of mass.

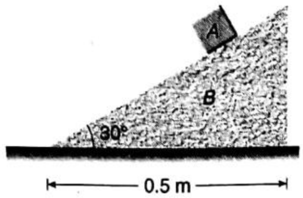
10. At one instant, the centre of mass of a system of two particles is located on the x-axis at $x = 3.0$ m and has a velocity of $(6.0 \text{ m/s})\hat{j}$. One of the particles is at the origin, the other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 12.0$ m.
(a) What is the mass of the particle at the origin ?
(b) Calculate the total momentum of this system.
(c) What is the velocity of the particle at the origin ?

11. A stone is dropped at $t = 0$. A second stone, with twice the mass of the first, is dropped from the same point at $t = 100$ ms.
(a) How far below the release point is the centre of mass of the two stones at $t = 300$ ms ? (Neither stone has yet reached the ground).
(b) How fast is the centre of mass of the two-stone system moving at that time ?

12. A system consists of two particles. At $t = 0$, one particle is at the origin; the other, which has a mass of 0.60 kg, is on the y-axis at $y = 80$ m. At $t = 0$ the centre of mass of the system is on the y-axis at $y = 24$ m and has a velocity given by $(6.0 \text{ m/s}^3)t^2\hat{j}$.
(a) Find the total mass of the system.
(b) Find the acceleration of the centre of mass at any time t.
(c) Find the net external force acting on the system at $t = 3.0$ s.

13. A straight rod of length L has one of its end at the origin and the other at $x = L$. If the mass per unit length of the rod is given by Ax where A is a constant, where is its centre of mass?

14. Block A has a mass of 5 kg and is placed on top of a smooth triangular block, B having a mass of 30 kg. If the system is released from rest, determine the distance, B moves when A reaches the bottom. Neglect the size of block A.



Conservation of Linear Momentum

15. Two blocks A and B of mass 1 kg and 2 kg are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Find the ratio of their :
    (a) speed (b) momentum and (c) kinetic energy at any instant.

16. A trolley was moving horizontally on a smooth ground with velocity v with respect to the earth. Suddenly a man starts running from rear end of the trolley with a velocity $\frac{3}{2}v$ with respect to the trolley. After reaching the other end, the man turns back and continues running with a velocity $\frac{3}{2}v$ with respect to trolley in opposite direction. If the length of the trolley is L, find the displacement of the man with respect to earth when he reaches the starting point on the trolley. Mass of the trolley is equal to the mass of the man.

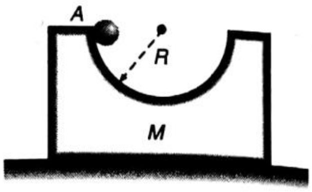
17. A man of mass m climbs to a rope ladder suspended below a balloon of mass M. The balloon is stationary with respect to the ground.
    (a) If the man begins to climb the ladder at speed v (with respect to the ladder), in what direction and with what speed (with respect to the ground) will the balloon move ?
    (b) What is the state of the motion after the man stops climbing ?

18. A 4.00 g bullet travelling horizontally with a velocity of magnitude 500 m/s is fired into a wooden block with a mass of 1.00 kg, initially at rest on a level surface. The bullet passes through the block and emerges with speed 100 m/s. The block slides a distance of 0.30 m along the surface from its initial position.
    (a) What is the coefficient of kinetic friction between block and surface ?
    (b) What is the decrease in kinetic energy of the bullet ?
    (c) What is the kinetic energy of the block at the instant after the bullet has passed through it ?
    Neglect friction during collision of bullet with the block.

19. A bullet of mass 0.25 kg is fired with velocity 302 m/s into a block of wood of mass $m_1 = 37.5$ kg. It gets embedded into it. The block m_1 is resting on a long block m_2 and the horizontal surface on which it is placed is smooth. The coefficient of friction between m_1 and m_2 is 0.5. Find the displacement of m_1 on m_2 and the common velocity of m_1 and m_2. Mass $m_2 = 1.25$ kg.

20. A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length l. At the initial moment the wagon and the pendulum are at rest and the string is deflected through an angle α from the vertical. Find the velocity of the wagon when the pendulum passes through its mean position.

21. A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface shown in figure. A uniform cylinder of radius r and mass m is released from rest at the point A. The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reaches the bottom of the track ? How fast is the block moving when the cylinder reaches the bottom of the track ?

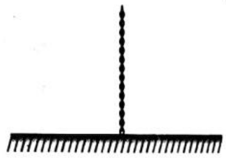


22. A ball of mass 50 g moving with a speed 2 m/s strikes a plane surface at an angle of incidence 45°. The ball is reflected by the plane at equal angle of reflection with the same speed. Calculate :
(a) The magnitude of the change in momentum of the ball.
(b) The change in the magnitude of the momentum of the wall.

Variable Mass

23. A rocket of mass 40 kg has 160 kg fuel. The exhaust velocity of the fuel is 2.0 km/s. The rate of consumption of fuel is 4 kg/s. Calculate the ultimate vertical speed gained by the rocket. ($g = 10 \text{m/s}^2$)

24. A uniform rope of mass m per unit length, hangs vertically from a support so that the lower end just touches the tabletop shown in figure. If it is released, show that at the time a length y of the rope has fallen, the force on the table is equivalent to the weight of a length $3y$ of the rope.



25. Sand drops from a stationary hopper at the rate of 5 kg/s on to a conveyor belt moving with a constant speed of 2 m/s. What is the force required to keep the belt moving and what is the power delivered by the motor, moving the belt ?

26. Find the mass of the rocket as a function of time, if it moves with a constant acceleration a, in absence of external forces. The gas escapes with a constant velocity u relative to the rocket and its mass initially was m_0.

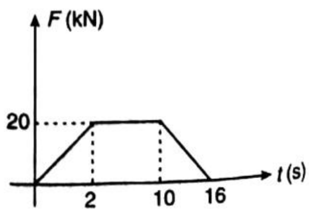
Impulse

27. A 5.0 g bullet moving at 100 m/s strikes a log. Assume that the bullet undergoes uniform deceleration and stops in 6.0 cm. Find (a) the time taken for the bullet to stop, (b) the impulse on the log and (c) the average force experienced by the log.

28. A 3.0 kg block slides on a frictionless horizontal surface, first moving to the left at 50 m/s. It collides with a spring as it moves left, compresses the spring and is brought to rest momentarily. The body continues to be accelerated to the right by the force of the compressed spring. Finally, the body moves to the right at 40 m/s. The block remains in contact with the spring for 0.020 s. What were the magnitude and direction of the impulse of the spring on the block? What was the spring's average force on the block?

29. Velocity of a particle of mass 2 kg varies with time t according to the equation $\vec{v} = (2t\,\hat{i} - 4\,\hat{j})$m/s. Here, t is in seconds. Find the impulse imparted to the particle in the time interval from $t = 0$ to $t = 2$s.

30. The net force *versus* time graph of a rocket is shown in figure. The mass of the rocket is 1200 kg. Calculate velocity of rocket, 16 seconds after starting from rest. Neglect gravity



Collision

31. After an elastic collision between two balls of equal masses, one is observed to have a speed of 3 m/s along the positive x-axis and the other has a speed of 2 m/s along the negative x-axis. What were the original velocities of the balls ?
32. A ball of mass 1 kg moving with 4 ms^{-1} along $+x$-axis collides elastically with an another ball of mass 2 kg moving with 6 m/s is opposite direction. Find their velocities after collision.
33. Ball 1 collides directly with an another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls ?
34. A ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4 of the original KE. Calculate the coefficient of restitution.
35. Block A has a mass 3 kg and is sliding on a rough horizontal surface with a velocity $v_A = 2$ m/s when it makes a direct collision with block B, which has a mass of 2 kg and is originally at rest. The collision is perfectly elastic. Determine the velocity of each block just after collision and the distance between the blocks when they stop sliding. The coefficient of kinetic friction between the blocks and the plane is $\mu_k = 0.3$. (Take $g = 10 \text{m/s}^2$)

Objective Questions (Level 1)

Single Correct Option

1. A ball is dropped from a height of 10 m. Ball is embedded in sand through 1 m and stops.
   (a) only momentum remains conserved
   (b) only kinetic energy remains conserved
   (c) both momentum and kinetic energy are conserved
   (d) neither kinetic energy nor momentum is conserved
2. If no external force acts on a system
   (a) velocity of centre of mass remains constant
   (b) position of centre of mass remains constant
   (c) acceleration of centre of mass remains non-zero and constant
   (d) All of the above
3. When two blocks connected by a spring move towards each other under mutual interaction
   (a) their velocities are equal
   (b) their accelerations are equal
   (c) the force acting on them are equal and opposite
   (d) All of the above
4. If two balls collide in air while moving vertically, then momentum of the system is conserved because
   (a) gravity does not affect the momentum of the system
   (b) force of gravity is very less compared to the impulsive force
   (c) impulsive force is very less than the gravity
   (d) gravity is not acting during collision
5. When a cannon shell explodes in mid air, then identify the incorrect statement
   (a) the momentum of the system is conserved at the time of explosion
   (b) the kinetic energy of the system always increases
   (c) the trajectory of centre of mass remains unchanged
   (d) None of the above
6. In an inelastic collision
   (a) momentum of the system is always conserved.

(b) velocity of separation is less than the velocity of approach
(c) the coefficient of restitution can be zero .
(d) All of the above.

7. The momentum of a system is defined
(a) as the product of mass of the system and the velocity of centre of mass .
(b) as the vector sum of the momentum of individual particles .
(c) for bodies undergoing translational, rotational and oscillatory motion .
(d) All of the above

8. The momentum of a system with respect to centre of mass
(a) is zero only if the system is moving uniformly.
(b) is zero only if no external force acts on the system.
(c) is always zero
(d) can be zero in certain conditions

9. Three identical particles are located at the vertices of an equilateral triangle. Each particle moves along a meridian with equal speed towards the centroid and collides inelastically.
(a) all the three particles will bounce back along the meridians with lesser speed.
(b) all the three particles will become stationary
(c) all the particles will continue to move in their original directions but with lesser speed
(d) nothing can be said

10. The average resisting force that must act on a 5 kg mass to reduce its speed from 65 to 15 ms^{-1} in 2s is
(a) 12.5 N (b) 125 N (c) 1250 N (d) None of these

11. In a carbon monoxide molecule, the carbon and the oxygen atoms are separated by a distance 1.2×10^{-10} m. The distance of the centre of mass from the carbon atom is
(a) 0.48×10^{-10} m (b) 0.51×10^{-10} m (c) 0.74×10^{-10} m (d) 0.64×10^{-10} m

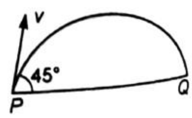
12. A bomb of mass 9 kg explodes into two pieces of masses 3 kg and 6 kg. The velocity of mass 3 kg is 16 ms^{-1}. The kinetic energy of mass 6 kg is
(a) 96 J (b) 384 J (c) 192 J (d) 768 J

13. A heavy ball moving with speed v collides with a tiny ball. The collision is elastic, then immediately after the impact, the second ball will move with a speed approximately equal to
(a) v (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{3}$

14. A loaded 20, 000 kg coal wagon is moving on a level track at 6 ms^{-1}. Suddenly 5000 kg of coal is dropped out of the wagon. The final speed of the wagon is
(a) 6 ms^{-1} (b) 8 ms^{-1} (c) 4.8 ms^{-1} (d) 4.5 ms^{-1}

15. A machine gun shoots a 40 g bullet at a speed of 1200 ms^{-1}. The man operating the gun can bear a maximum force of 144 N. The maximum number of bullets shot per second is
(a) 3 (b) 5 (c) 6 (d) 9

16. A projectile of mass m is fired with a velocity v from point P at an angle 45°. Neglecting air resistance, the magnitude of the change in momentum leaving the point P and arriving at Q is



(a) $mv\sqrt{2}$ (b) $2mv$
(c) $\frac{mv}{2}$ (d) $\frac{mv}{\sqrt{2}}$

17. A ball after freely falling from a height of 4.9 m strikes a horizontal plane. If the coefficient of restitution is 3/4, the ball will strike second time with the plane after

(a) $\frac{1}{2}$ s (b) 1 s (c) $\frac{3}{2}$ s (d) $\frac{3}{4}$ s

18. The centre of mass of a non uniform rod of length L, whose mass per unit length varies as $\rho = \frac{k \cdot x^2}{L}$ where k is a constant and x is the distance of any point from one end is (from the same end)

(a) $\left(\frac{3}{4}\right)L$ (b) $\left(\frac{1}{4}\right)L$ (c) $\left(\frac{1}{6}\right)L$ (d) $\left(\frac{2}{3}\right)L$

19. A boat of length 10 m and mass 450 kg is floating without motion in still water. A man of mass 50 kg standing at one end of it walks to the other end of it and stops. The magnitude of the displacement of the boat in metres relative to ground is

(a) zero (b) 1 m (c) 2 m (d) 5 m

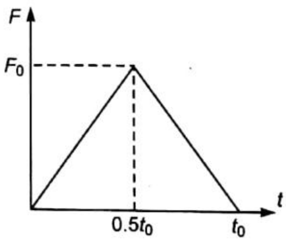
20. A man of mass M stands at one end of a stationary plank of length L, lying on a smooth surface. The man walks to the other end of the plank. If the mass of the plank is $\frac{M}{3}$, the distance that the man moves relative to the ground is

(a) $\frac{3L}{4}$ (b) $\frac{L}{4}$ (c) $\frac{4L}{5}$ (d) $\frac{L}{3}$

21. A ball of mass m moving at a speed v collides with another ball of mass $3m$ at rest. The lighter block comes to rest after collision. The coefficient of restitution is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) None of these

22. A particle of mass m moving with velocity u makes an elastic one-dimensional collision with a stationary particle of mass m. They come in contact for a very small time t_0. Their force of interaction increases from zero to F_0 linearly in time $0.5t_0$, and decreases linearly to zero in further time $0.5\,t_0$ as shown in figure. The magnitude of F_0 is



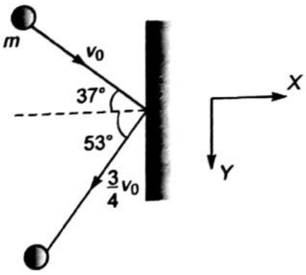
(a) $\frac{mu}{t_0}$ (b) $\frac{2mu}{t_0}$

(c) $\frac{mu}{2t_0}$ (d) None of these

23. Two identical blocks A and B of mass m joined together with a massless spring as shown in figure are placed on a smooth surface. If the block A moves with an acceleration a_0, then the acceleration of the block B is

(a) a_0 (b) $-a_0$ (c) $\frac{F}{m} - a_0$ (d) $\frac{F}{m}$

24. A ball of mass m moving with velocity v_0 collides a wall as shown in figure. After impact it rebounds with a velocity $\frac{3}{4}v_0$. The impulse acting on ball during impact is

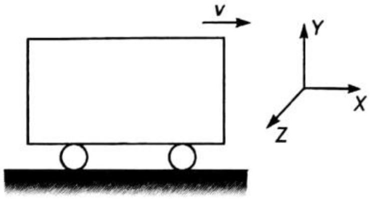


(a) $-\frac{m}{2}v_0\,\hat{\mathbf{j}}$ (b) $-\frac{3}{4}mv_0\,\hat{\mathbf{i}}$ (c) $\frac{-5}{4}mv_0\,\hat{\mathbf{i}}$ (d) None of these

25. A steel ball is dropped on a hard surface from a height of 1 m and rebounds to a height of 64 cm. The maximum height attained by the ball after n^{th} bounce is (in m)

(a) $(0.64)^{2n}$ (b) $(0.8)^{2n}$ (c) $(0.5)^{2n}$ (d) $(0.8)^n$

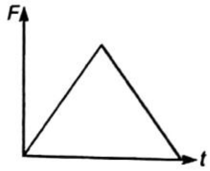
26. A car of mass 500 kg (including the mass of a block) is moving on a smooth road with velocity $1.0\ \text{ms}^{-1}$ along positive x-axis. Now a block of mass 25 kg is thrown outside with absolute velocity of $20\ \text{ms}^{-1}$ along positive z-axis. The new velocity of the car is (ms^{-1})

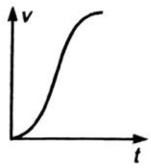


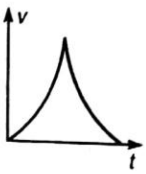
(a) $10\hat{\mathbf{i}}+20\hat{\mathbf{k}}$ (b) $10\hat{\mathbf{i}}-20\hat{\mathbf{k}}$

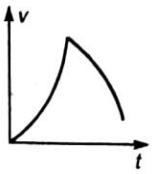
(c) $\frac{20}{19}\hat{\mathbf{i}}-\frac{20}{19}\hat{\mathbf{k}}$ (d) $10\hat{\mathbf{i}}-\frac{20}{19}\hat{\mathbf{k}}$

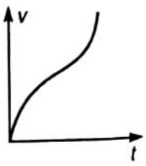
27. The net force acting on a particle moving along a straight line varies with time as shown in the diagram. Force is parallel to velocity. Which of the following graph is best representative of its speed with time? (Initial velocity of the particle is zero)



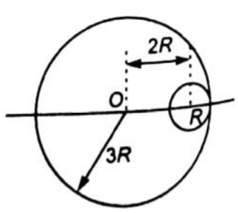
(a)


(b)


(c)


(d)


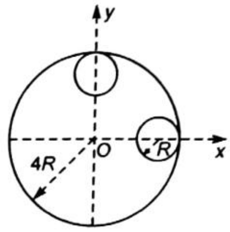
28. In the figure shown, find out centre of mass of a system of a uniform circular plate of radius $3R$ from O in which a hole of radius R is cut whose centre is at $2R$ distance from the centre of large circular plate



(a) $\frac{R}{2}$ (b) $\frac{R}{5}$

(c) $\frac{R}{4}$ (d) None of these

29. From the circular disc of radius $4R$ two small discs of radius R are cut off. The centre of mass of the new structure will be at



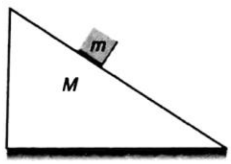
(a) $\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(b) $-\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(c) $-\hat{\mathbf{i}}\frac{R}{5}-\hat{\mathbf{j}}\frac{R}{5}$

(d) None of the above

30. A block of mass m rests on a stationary wedge of mass M. The wedge can slide freely on a smooth horizontal surface as shown in figure. If the block starts from rest



(a) the position of the centre of mass of the system will change

(b) the position of the centre of mass of the system will change along the vertical but not along the horizontal

(c) the total energy of the system will remain constant.

(d) All of the above

31. A bullet of mass m hits a target of mass M hanging by a string and gets embedded in it. If the block rises to a height h as a result of this collision, the velocity of the bullet before collision is

(a) $v=\sqrt{2gh}$ (b) $v=\sqrt{2gh}\left[1+\frac{m}{M}\right]$ (c) $v=\sqrt{2gh}\left[1+\frac{M}{m}\right]$ (d) $v=\sqrt{2gh}\left[1-\frac{m}{M}\right]$

32. A loaded spring gun of mass M fires a bullet of mass m with a velocity v at an angle of elevation θ. The gun is initially at rest on a horizontal smooth surface. After firing, the centre of mass of the gun and bullet system

(a) moves with velocity $\frac{v}{M}m$

(b) moves with velocity $\frac{vm}{M\cos\theta}$ in the horizontal direction

(c) remains at rest

(d) moves with velocity $\frac{v(M-m)}{M+m}$ in the horizontal direction

33. Two bodies with masses m_1 and m_2 $(m_1 > m_2)$ are joined by a string passing over fixed pulley. Assume masses of the pulley and thread negligible. Then the acceleration of the centre of mass of the system (m_1+m_2) is

(a) $\left(\frac{m_1-m_2}{m_1+m_2}\right)g$ (b) $\left(\frac{m_1-m_2}{m_1+m_2}\right)^2 g$ (c) $\frac{m_1 g}{m_1+m_2}$ (d) $\frac{m_2 g}{m_1+m_2}$

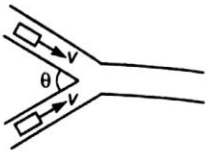
34. A rocket of mass m_0 has attained a speed equal to its exhaust speed and at that time the mass of the rocket is m. Then the ratio $\frac{m_0}{m}$ is (neglect gravity)

(a) 2.718 (b) 7.8 (c) 3.14 (d) 4

35. A jet of water hits a flat stationary plate perpendicular to its motion. The jet ejects 500g of water per second with a speed of $1\ \text{ms}^{-1}$. Assuming that after striking, the water flows parallel to the plate, then the force exerted on the plate is

(a) 500 N (b) 1.0 N (c) 0.5 N (d) 1000 N

36. Two identical vehicles are moving with same velocity v towards an intersection as shown in figure. If the collision is completely inelastic, then



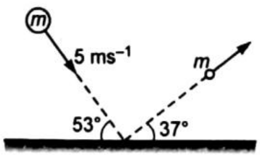
(a) the velocity of separation is zero

(b) the velocity of approach is $2v\sin\dfrac{\theta}{2}$

(c) the common velocity after collision is $v\cos\dfrac{\theta}{2}$

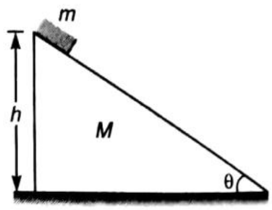
(d) All of the above

37. A ball of mass $m = 1\,\text{kg}$ strikes smooth horizontal floor as shown in figure. The impulse exerted on the floor is



(a) 6.25 Ns (b) 1.76 Ns (c) 7.8 Ns (d) 2.2 Ns

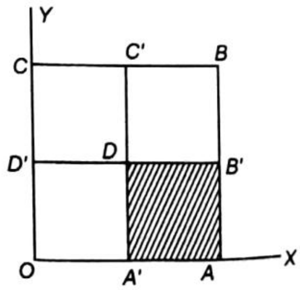
38. A small block of mass m is placed at rest on the top of a smooth wedge of mass M, which in turn is placed at rest on a smooth horizontal surface as shown in figure. It h be the height of wedge and θ is the inclination, then the distance moved by the wedge as the block reaches the foot of the wedge is



(a) $\dfrac{Mh\cot\theta}{M+m}$ (b) $\dfrac{mh\cot\theta}{M+m}$

(c) $\dfrac{Mh\,\text{cosec}\theta}{M+m}$ (d) $\dfrac{mh\,\text{cosec}\theta}{M+m}$

39. A square of side 4 cm and uniform thickness is divided into four squares. The square portion $A'\,AB'\,D$ is removed and the removed portion is placed over the portion $DB'\,BC'$. The new position of centre of mass is



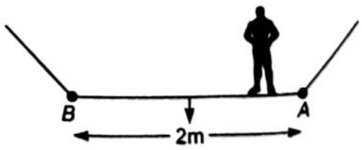
(a) (2 cm, 2 cm)

(b) (2 cm, 3 cm)

(c) (2 cm, 2.5 cm)

(d) (3 cm, 3 cm)

40. A boy having a mass of 40 kg stands at one end A of a boat of length 2 m at rest. The boy walks to the other end B of the boat and stops. What is the distance moved by the boat? Friction exists between the feet of the boy and the surface of the boat. But the friction between the boat and the water surface may be neglected. Mass of the boat is 15 kg.



(a) 0.49 m (b) 2.46 m (c) 1.46 m (d) 3.2 m

41. Three identical particle with velocities $v_0\,\hat{\mathbf{i}}$, $-3v_0\,\hat{\mathbf{j}}$ and $5v_0\,\hat{\mathbf{k}}$ collide successively with each other in such a way that they form a single particle. The velocity vector of resultant particle is

(a) $\dfrac{v_0}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (c) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (d) $\dfrac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

42. A mortar fires a shell of mass M which explodes into two pieces of mass $\frac{M}{5}$ and $\frac{4M}{5}$ at the top of the trajectory. The smaller mass falls very close to the mortar. In the same time the bigger piece lands a distance D from the mortar. The shell would have fallen at a distance R from the mortar if there was no explosion. The value of D is (neglect air resistance)

(a) $\frac{3R}{2}$ (b) $\frac{4R}{3}$ (c) $\frac{5R}{4}$ (d) None of these

43. A moving particle of mass m makes a head on elastic collision with a particle of mass $2m$ which is initially at rest. The fraction of the kinetic energy lost by the colliding particle is

(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{8}{9}$

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

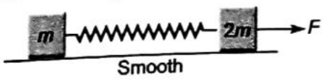
(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*

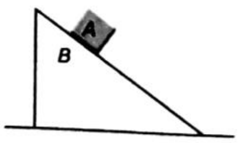
(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Centre of mass of a rigid body always lies inside the body.
   Reason : Centre of mass and centre of gravity coincide if gravity is unifrom.
2. **Assertion :** A constant force F is applied on two blocks and one spring system as shown in figure. Velocity of centre of mass increases linearly with time.



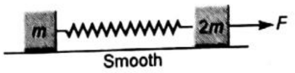
   Reason : Acceleration of centre of mass is constant.
3. **Assertion :** To conserve linear momentum of a system, no force should act on the system.
   Reason : If net force on a system is zero, its linear momentum should remain constant.
4. **Assertion :** A rocket moves forward by pushing the surrounding air backwards.
   Reason : It derives the necessary thrust to move forward according to Newton's third law of motion.
5. **Assertion :** Internal forces cannot change linear momentum.
   Reason : Internal forces can change the kinetic energy of a system.
6. **Assertion :** In case of bullet fired from gun, the ratio of kinetic energy of gun and bullet is equal to ratio of mass of bullet and gun.
   Reason : Kinetic energy $\propto \frac{1}{\text{mass}}$; if momentum is constant.
7. **Assertion :** All surfaces shown in figure are smooth. System is released from rest. Momentum of system in horizontal direction is constant but overall momentum is not constant.



   Reason : A net vertically upward force is acting on the system.

8. **Assertion :** During head on collision between two bodies let Δp_1 is change in momentum of first body and Δp_2 the change in momentum of the other body, then $\Delta p_1 = \Delta p_2$.
Reason : Total momentum of the system should remain constant.

9. **Assertion :** In the system shown in figure spring is first stretched then left to oscillate. At some instant kinetic energy of mass m is K. At the same instant kinetic energy of mass $2m$ should be $\frac{K}{2}$.



Reason : Their linear momenta are equal and opposite and $K = \frac{p^2}{2m}$ or $K \propto \frac{1}{m}$.

10. **Assertion :** Energy can not be given to a system without giving it momentum.
Reason : If kinetic energy is given to a body it means it has acquired momentum.

11. **Assertion :** The centre mass of an electron and proton, when released moves faster towards proton.
Reason : Proton is heavier than electron.

12. **Assertion :** The relative velocity of the two particles in head-on elastic collision is unchanged both in magnitude and direction.
Reason : The relative velocity is unchanged in magnitude but gets reversed in direction.

13. **Assertion :** An object of mass m_1 and another of mass m_2 $(m_2 > m_1)$ are released from certain distance. The objects move towards each other under the gravitational force between them. In this motion, centre of mass of their system will continuously move towards the heavier mass m_2.
Reason : In a system of a heavier and a lighter mass, centre of mass lies closer to the heavier mass.

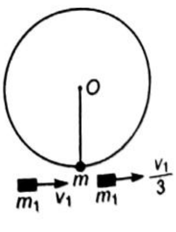
14. **Assertion :** A given force applied in turn to a number of different masses may cause the same rate of change in momentum in each but not the same acceleration to all.
Reason : $\vec{\mathbf{F}} = \frac{\vec{\mathbf{dp}}}{dt}$ and $\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}}{m}$

15. **Assertion :** In an elastic collision between two bodies, the relative speed of the bodies after collision is equal to the relative speed before the collision.
Reason : In an elastic collision, the linear momentum of the system is conserved.

Objective Questions (Level 2)

Single Correct Option

1. A pendulum consists of a wooden bob of mass m and length l. A bullet of mass m_1 is fired towards the pendulum with a speed v_1 and it emerges from the bob with speed $\frac{v_1}{3}$. The bob just completes motion along a vertical circle. Then v_1 is



(a) $\frac{m}{m_1}\sqrt{5gl}$
(b) $\frac{3m}{2m_1}\sqrt{5gl}$
(c) $\frac{2}{3}\left(\frac{m}{m_1}\right)\sqrt{5gl}$
(d) $\left(\frac{m_1}{m}\right)\sqrt{gl}$

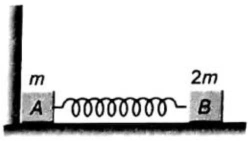
2. A bob of mass m attached with a string of length l tied to a point on ceiling is released from a position when its string is horizontal. At the bottom most point of its motion, an identical mass m gently stuck to it. Find the maximum angle from the vertical to which it rotates

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{1}{4}\right)$ (d) $60°$

3. A train of mass M is moving on a circular track of radius R with constant speed v. The length of the train is half of the perimeter of the track. The linear momentum of the train will be

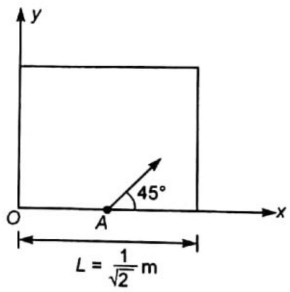
(a) zero (b) $\frac{2Mv}{\pi}$ (c) MvR (d) Mv

4. Two blocks A and B of mass m and $2m$ are connected together by a light spring of stiffness k. The system is lying on a smooth horizontal surface with the block A in contact with a fixed vertical wall as shown in the figure. The block B is pressed towards the wall by a distance x_0 and then released. There is no friction anywhere. If spring takes time Δt to acquire its natural length then average force on the block A by the wall is



(a) zero (b) $\frac{\sqrt{2mk}}{\Delta t}x_0$ (c) $\frac{\sqrt{mk}}{\Delta t}x_0$ (d) $\frac{\sqrt{3mk}}{\Delta t}x_0$

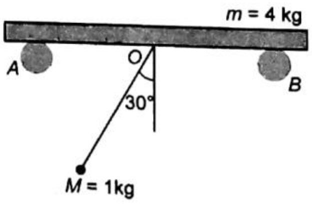
5. A striker is shot from a square carom board from a point A exactly at midpoint of one of the walls with a speed of 2 ms^{-1} at an angle of 45° with the x-axis as shown in the figure. The collisions of the striker with the walls of the fixed carom are perfectly elastic. The coefficient of kinetic friction between the striker and board is 0.2. The coordinate of the striker when it stops (taking point O to be the origin) is (in SI units)



(a) $\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}$ (b) $0, \frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}, 0$ (d) $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$

6. A ball of mass 1 kg is suspended by an inextensible string 1 m long attached to a point O of a smooth horizontal bar resting on fixed smooth supports A and B. The ball is released from rest from the position when the string makes an angle 30° with the vertical. The mass of the bar is 4 kg. The displacement of bar when ball reaches the other extreme position (in m) is



(a) 0.4 (b) 0.2

(c) 0.25 (d) 0.5

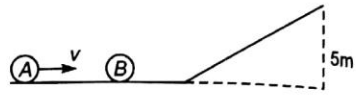
7. A ball falls vertically onto a floor with momentum p and then bounces repeatedly. If coefficient of restitution is e, then the total momentum imparted by the ball to the floor is

(a) $p(1+e)$ (b) $\frac{p}{1-e}$ (c) $p\left(\frac{1-e}{1+e}\right)$ (d) $p\left(\frac{1+e}{1-e}\right)$

8. A bullet of mass m penetrates a thickness h of a fixed plate of mass M. If the plate was free to move, then the thickness penetrated will be

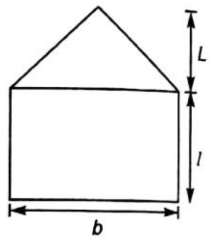
(a) $\frac{Mh}{M+m}$ (b) $\frac{2Mh}{M+m}$ (c) $\frac{mh}{2(M+m)}$ (d) $\frac{Mh}{2(M+m)}$

9. Two identical balls of equal masses A and B, are lying on a smooth surface as shown in the figure. Ball A hits the ball B (which is at rest) with a velocity $v = 16\,\text{ms}^{-1}$. What should be the minimum value of coefficient of restitution e between A and B so that B just reaches the highest point of inclined plane. ($g = 10\,\text{ms}^{-2}$)



(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

10. The figure shows a metallic plate of uniform thickness and density. The value of l in terms of L so that the centre of mass of the system lies at the interface of the triangular and rectangular portion is



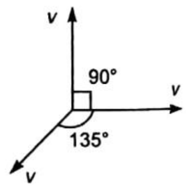
(a) $l = \frac{L}{3}$ (b) $l = \frac{L}{2}$

(c) $l = \frac{L}{\sqrt{3}}$ (d) $l = \sqrt{\frac{2}{3}}\,L$

11. Particle A makes a head on elastic collision with another stationary particle B. They fly apart in opposite directions with equal velocities. The mass ratio will be

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

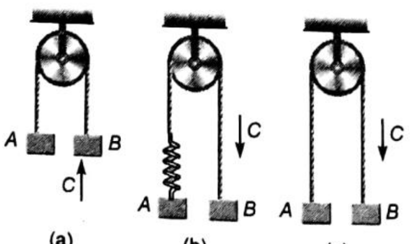
12. A particle of mass $4m$ which is at rest explodes into four equal fragments. All four fragments scattered in the same horizontal plane. Three fragments are found to move with velocity v as shown in the figure. The total energy released in the process is



(a) $mv^2(3 - \sqrt{2})$ (b) $\frac{1}{2}mv^2(3 - \sqrt{2})$

(c) $2mv^2$ (d) $\frac{1}{2}mv^2(1 + \sqrt{2})$

13. In figures (a), (b) and (c) shown, the objects A, B and C are of same mass. String, spring and pulley are massless. C strikes B with velocity u in each case and sticks it. The ratio of velocity of B in case (a) to (b) to (c) is



(a) 1 : 1 : 1 (b) 3 : 3 : 2 (c) 3 : 2 : 2 (d) 1 : 2 : 3

14. A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\theta = 30°$ with horizontal. Then the speed of the ladder's centre of mass must be

(a) $\frac{\sqrt{3}}{2}v$ (b) $\frac{v}{2}$ (c) v (d) $2v$

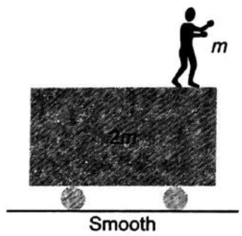
15. A body of mass 2 g, moving along the positive *x*-axis in gravity free space with velocity $20\,\text{cms}^{-1}$ explodes at $x = 1\,\text{m}$, $t = 0$ into two pieces of masses 2/3 g and 4/3 g. After 5s, the lighter piece is at the point (3m, 2m, − 4 m). Then the position of the heavier piece at this moment, in metres is

(a) (1.5, − 1, − 2) (b) (1.5, − 2, − 2) (c) (1.5, − 1, − 1) (d) None of these

16. A body of mass m is dropped from a height of h. Simultaneously another body of mass $2m$ is thrown up vertically with such a velocity v that they collide at height $\frac{h}{2}$. If the collision is perfectly inelastic, the velocity of combined mass at the time of collision with the ground will be

(a) $\sqrt{\frac{5gh}{4}}$ (b) $\sqrt{gh}$ (c) $\sqrt{\frac{gh}{4}}$ (d) None of these

17. A man is standing on a cart of mass double the mass of man. Initially cart is at rest. Now, man jumps horizontally with velocity u relative to cart. Then work done by man during the process of jumping will be



(a) $\frac{mu^2}{2}$

(b) $\frac{3mu^2}{4}$

(c) mu^2

(d) None of the above

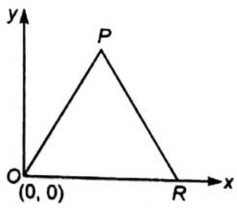
18. Two balls of equal mass are projected upwards simultaneously, one from the ground with initial velocity $50\,\text{ms}^{-1}$ and the other from a 40m tower with initial velocity of $30\,\text{ms}^{-1}$. The maximum height attained by their COM will be

(a) 80 m (b) 60 m (c) 100 m (d) 120 m

19. A particle of mass m and momentum $\vec{p}$ moves on a smooth horizontal table and collides directly and elastically with a similar particle (of mass m) having momentum $-2\vec{p}$. The loss (−) or gain (+) in the kinetic energy of the first particle in the collision is

(a) $+\frac{p^2}{2m}$ (b) $-\frac{p^2}{4m}$ (c) $+\frac{3p^2}{2m}$ (d) zero

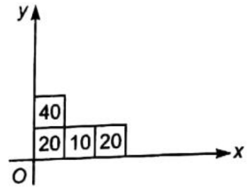
20. An equilateral triangular plate of mass $4m$ of side a is kept as shown. Consider two cases : (i) a point mass $4m$ is placed at the vertex P of the plate (ii) a point mass m is placed at the vertex R of the plate. In both cases the x coordinate of centre of mass remains the same. Then x coordinate of centre of mass of the plate is



(a) $\frac{a}{3}$ (b) $\frac{a}{6}$

(c) $\frac{6a}{7}$ (d) $\frac{2a}{3}$

21. Four cubes of side a each of mass 40 g, 20 g, 10 g and 20 g are arranged in XY plane as shown in the figure. The coordinates of COM of the combination with respect to point O is

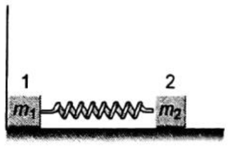


(a) $\frac{19a}{18}, \frac{17a}{18}$ (b) $\frac{17a}{18}, \frac{11a}{18}$ (c) $\frac{17a}{18}, \frac{13a}{18}$ (d) $\frac{13a}{18}, \frac{17a}{18}$

22. A particle of mass m_0, travelling at speed v_0, strikes a stationary particle of mass $2m_0$. As a result, the particle of mass m_0 is deflected through 45° and has a final speed of $\frac{v_0}{\sqrt{2}}$. Then the speed of the particle of mass $2m_0$ after this collision is

(a) $\frac{v_0}{2}$ (b) $\frac{v_0}{2\sqrt{2}}$ (c) $\sqrt{2}v_0$ (d) $\frac{v_0}{\sqrt{2}}$

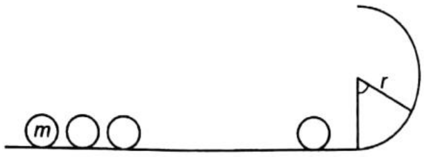
23. Two bars of masses m_1 and m_2, connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Bar 2 is shifted by a small distance x_0 to the left and released. The velocity of the centre of mass of the system when bar 1 breaks off the wall is



(a) $x_0\sqrt{\frac{km_2}{m_1+m_2}}$ (b) $\frac{x_0}{m_1+m_2}\sqrt{km_2}$

(c) $x_0k\frac{m_1+m_2}{m_2}$ (d) $x_0\frac{\sqrt{km_1}}{(m_1+m_2)}$

24. n elastic balls are placed at rest on a smooth horizontal plane which is circular at the ends with radius r as shown in the figure. The masses of the balls are m, $\frac{m}{2}, \frac{m}{2^2}$ .... $\frac{m}{2^{n-1}}$ respectively. What is the minimum velocity which should be imparted to the first ball of mass m such that n^{th} ball completes the vertical circle



(a) $\left(\frac{3}{4}\right)^{n-1}\sqrt{5gr}$ (b) $\left(\frac{4}{3}\right)^{n-1}\sqrt{5gr}$

(c) $\left(\frac{3}{2}\right)^{n-1}\sqrt{5gr}$ (d) $\left(\frac{2}{3}\right)^{n-1}\sqrt{5gr}$

Passage-1 (Q. No. 25 to 26)

A block of mass 2 kg is attached with a spring of spring constant 4000 Nm^{-1} and the system is kept on smooth horizontal table. The other end of the spring is attached with a wall. Initially spring is stretched by 5 cm from its natural position and the block is at rest. Now suddenly an impulse of 4 kg-ms^{-1} is given to the block towards the wall.

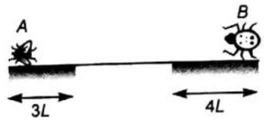
25. Find the velocity of the block when spring acquires its natural length

(a) 5 ms^{-1} (b) 3 ms^{-1} (c) 6 ms^{-1} (d) None of these

26. Approximate distance travelled by the block when it comes to rest for a second time (not including the initial one) will be (Take $\sqrt{45} = 6.70$)
(a) 30 cm (b) 25 cm (c) 40 cm (d) 20 cm

Passage-2 (Q. No. 27 to 31)

A uniform bar of length $12L$ and mass $48\,m$ is supported horizontally on two smooth tables as shown in figure. A small moth (an insect) of mass $8m$ is sitting on end A of the rod and a spider (an insect) of mass $16m$ is sitting on the other end B. Both the insects moving towards each other along the rod with moth moving at speed $2v$ and the spider at half this speed (absolute). They meet at a point P on the rod and the spider eats the moth. After this the spider moves with a velocity $\frac{v}{2}$ relative to the rod towards the end A. The spider takes negligible time in eating on the other insect. Also, let $v = \frac{L}{T}$ where T is a constant having value 4 s.



27. Displacement of the rod by the time the insect meet the moth is
(a) $\frac{L}{2}$ (b) L (c) $\frac{3L}{4}$ (d) zero

28. The point P is at
(a) the centre of the rod (b) the edge of the table supporting the end B
(c) the edge of the table supporting end A (d) None of these

29. The speed of the bar after the spider eats up the moth and moves towards A is
(a) $\frac{v}{2}$ (b) v (c) $\frac{v}{6}$ (d) $2v$

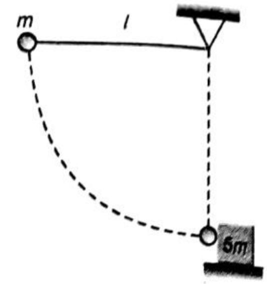
30. After starting from end B of the rod the spider reaches the end A at a time
(a) 40 s (b) 30 s (c) 80 s (d) 10 s

31. By what distance the centre of mass of the rod shifts during this time?
(a) $\frac{8L}{3}$ (b) $\frac{4L}{3}$ (c) L (d) $\frac{L}{3}$

More than One Correct Options

1. A particle of mass m, moving with velocity v collides a stationary particle of mass $2m$. As a result of collision, the particle of mass m deviates by 45° and has final speed of $\frac{v}{2}$. For this situation mark out the correct statement (s).
(a) The angle of divergence between particles after collision is $\frac{\pi}{2}$
(b) The angle of divergence between particles after collision is less than $\frac{\pi}{2}$
(c) Collision is elastic
(d) Collision is inelastic

2. A pendulum bob of mass m connected to the end of an ideal string of length l is released from rest from horizontal position as shown in the figure. At the lowest point the bob makes an elastic collision with a stationary block of mass $5m$, which is kept on a frictionless surface. Mark out the correct statement(s) for the instant just after the impact.



(a) tension in the string is $\frac{17}{9} mg$

(b) tension in the string is 3 mg..

(c) the velocity of the block is $\frac{\sqrt{2gl}}{3}$

(d) the maximum height attained by the pendulum bob after impact is (measured from the lowest position) $\frac{4l}{9}$

3. A particle of mass m strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebound with velocity v making an angle ϕ with the floor. The coefficient of restitution between the particle and the floor is e. Then

(a) the impulse delivered by the floor to the body is $mv(1+e)\sin\theta$

(b) $\tan\phi = e\tan\theta$

(c) $v = u\sqrt{1-(1-e^2)\sin^2\theta}$

(d) the ratio of the final kinetic energy to the initial kinetic energy is $\cos^2\theta + e^2\sin^2\theta$

4. A particle of mass m moving with a velocity $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\,\text{ms}^{-1}$ collides with another body of mass M and finally moves with velocity $(-2\hat{\mathbf{i}} + \hat{\mathbf{j}})\,\text{ms}^{-1}$. Then during the collision

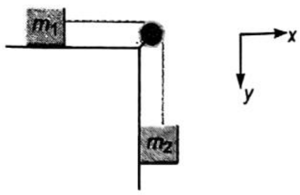
(a) impulse received by m is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(b) impulse received by m is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(c) impulse received M is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(d) impulse received by M is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

5. All surfaces shown in figure are smooth. System is released from rest. X and Y components of acceleration of COM are



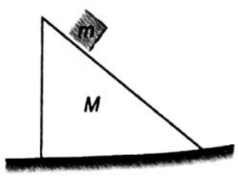
(a) $(a_{cm})_x = \frac{m_1 m_2 g}{m_1 + m_2}$

(b) $(a_{cm})_x = \frac{m_1 m_2 g}{(m_1 + m_2)^2}$

(c) $(a_{cm})_y = \left(\frac{m_2}{m_1 + m_2}\right)^2 g$

(d) $(a_{cm})_y = \left(\frac{m_2}{m_1 + m_2}\right) g$

6. A block of mass m is placed at rest on a smooth wedge of mass M placed at rest on a smooth horizontal surface.As the system is released



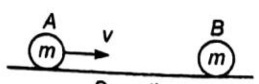
(a) the COM of the system remains stationary

(b) the COM of the system has an acceleration g vertically downward

(c) momentum of the system is conserved along the horizontal direction

(d) acceleration of COM is vertically downward $(a < g)$

7. In the figure shown, coefficient of restitution between A and B is $e = \frac{1}{2}$, then



(a) velocity of B after collision is $\frac{v}{2}$

(b) impulse between two during collision is $\frac{3}{4}mv$

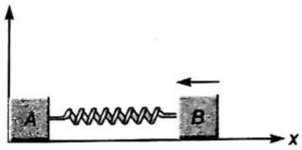
(c) loss of kinetic energy during the collision is $\frac{3}{16}mv^2$

(d) loss of kinetic energy during the collision is $\frac{1}{4}mv^2$

8. In case of rocket propulsion choose the correct options.
(a) Momentum of system always remains constant
(b) Newton's third law is applied
(c) If exhaust velocity and rate of burning of mass is kept constant, then acceleration of rocket will go on increasing
(d) Newton's second law can be applied

Match the Columns

1. Two identical blocks A and B are connected by a spring as shown in figure. Block A is not connected to the wall parallel to y-axis. B is compressed from the natural length of spring and then left. Neglect friction. Match the following two columns.



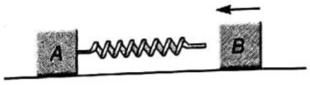
Column I	Column II
(a) Acceleration of centre of mass of two blocks	(p) remains constant
(b) Velocity of centre of mass of two blocks	(q) first increases then becomes constant
(c) x-coordinate of centre of mass of two blocks	(r) first decreases then becomes zero
(d) y-coordinate of centre of mass of two blocks	(s) continuously increases

2. One particle is projected from ground upwards with velocity 20 ms^{-1}. At the same time another indentical particle is dropped from a height of 180 m but not along the same vertical line. Assume that collision of first particle with ground is perfectly inelastic. Match the following two columns for centre of mass of the two particles ($g = 10\,ms^{-2}$)

Column I	Column II
(a) Initial acceleration	(p) 5 SI units
(b) Initial velocity	(q) 10 SI units
(c) Acceleration at $t = 5$ s	(r) 20 SI units
(d) Velocity at $t = 5$ s	(s) 25 SI units

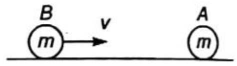
Note *Only magnitudes are given in column-II.*

3. Two identical blocks of mass 0.5 kg each are shown in figure. A massless elastic spring is connected with A. B is moving towards A with kinetic energy of 4 J. Match following two columns. Neglect friction.



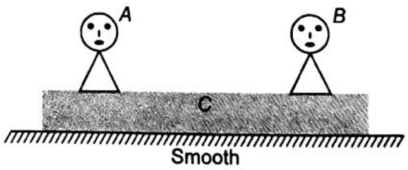
Column I	Column II
(a) Initial momentum of B	(p) zero
(b) Momentum of centre of mass of two blocks	(q) 1 kg-ms^{-1}
(c) Momentum of A at maximum compression	(r) 2 kg-ms^{-1}
(d) Momentum of B when spring is relaxed after compression	(s) 4 kg-ms^{-1}

4. Two identical balls A and B are kept on a smooth table as shown. B collides with A with speed v. For different conditions mentioned in column I, match with speed of A after collision given in column II.



Column I	Column II
(a) Elastic collision	(p) $\frac{3}{4}v$
(b) Perfectly inelastic collision	(q) $\frac{5}{8}v$
(c) Inelastic collision with $e = \frac{1}{2}$	(r) v
(d) Inelastic collision with $e = \frac{1}{4}$	(s) $\frac{v}{2}$

5. Two boys A and B of masses 30 kg and 60 kg are standing over a plank C of mass 30 kg as shown. Ground is smooth. Match the displacement of plank of column II with the conditions given in column I.

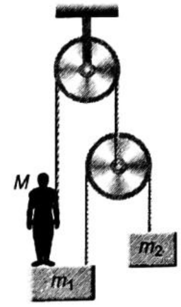


Column I	Column II
(a) A moves x towards right	(p) x towards right
(b) B moves x towards left	(q) $2x$ towards left
(c) A moves x towards right and B moves x towards left	(r) $\frac{x}{3}$ towards left
(d) A and B both move x towards right	(s) None

Note *All displacements mentioned in two columns are with respect to ground.*

6. A man of mass M is standing on a platform of mass m_1 and holding a string passing over a system of ideal pulleys. Another mass m_2 is hanging as shown.
$(m_2 = 20\,\text{kg}, m_1 = 10\,\text{kg}, g = 10\,\text{ms}^{-2})$

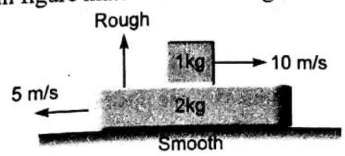
Column I	Column II
(a) Weight of man for equilibrium	(p) 100 N
(b) Force exerted by man on string to accelerate the COM of system upwards	(q) 150 N
(c) Force exerted by man on string to accelerate the COM of system downwards	(r) 500 N
(d) Normal reaction of platform on man in equilibrium	(s) 600 N



7. Two blocks of masses 3 kg and 6 kg are connected by an ideal spring and are placed on a frictionless horizontal surface. The 3 kg block is imparted a speed of 2 ms^{-1} towards left. (consider left as positive direction)

Column I	Column II
(a) When the velocity of 3 kg block is $\frac{2}{3}$ ms^{-1}	(p) velocity of centre of mass is $\frac{2}{3}$ ms^{-1}
(b) When the velocity of 6 kg block is $\frac{2}{3}$ ms^{-1}	(q) deformation of the spring is zero
(c) When the speed of 3 kg block is minimum	(r) deformation of the spring is maximum
(d) When the speed of 6 kg block is maximum	(s) both the blocks are at rest with respect to each other

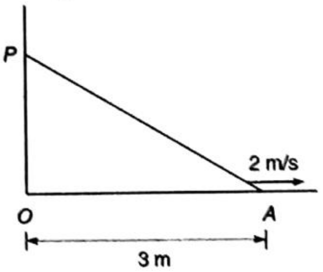
8. In a two block system shown in figure match the following



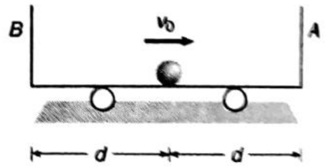
Column I	Column II
(a) Velocity of centre of mass	(p) Keep on changing all the time
(b) Momentum of centre of mass	(q) First decreases then become zero
(c) Momentum of 1 kg block	(r) Zero
(d) Kinetic energy of 2 kg block	(s) Constant

Subjective Questions (Level 2)

1. A ladder AP of length 5 m inclined to a vertical wall is slipping over a horizontal surface with velocity of 2 m/s, when A is at distance 3 m from ground. What is the velocity of COM at this moment?



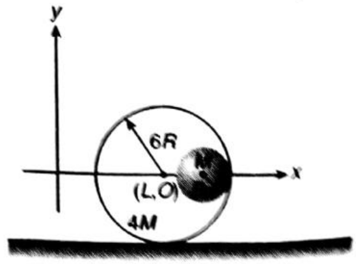
2. A ball of negligible size and mass m is given a velocity v_0 on the centre of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e. Determine the velocity of the ball and the cart just after the ball strikes A. Also, determine the total time needed for the ball to strike A, rebound, then strike B, and rebound and then return to the centre of the cart. Neglect friction.



3. Two point masses m_1 and m_2 are connected by a spring of natural length l_0. The spring is compressed such that the two point masses touch each other and then they are fastened by a string. Then the system is moved with a velocity v_0 along positive x-axis. When the system reached the origin the string breaks ($t = 0$). The position of the point mass m_1 is given by $x_1 = v_0 t - A(1 - \cos \omega t)$ where A and ω are constants.

Find the position of the second block as a function of time. Also, find the relation between A and l_0.

4. A small sphere of radius R is held against the inner surface of larger sphere of radius $6R$ (as shown in figure). The masses of large and small spheres are $4M$ and M respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the coordinates of the centre of the large sphere, when the smaller sphere reaches the other extreme position.



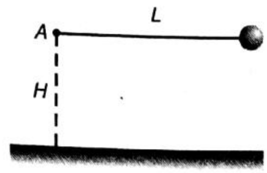
5. A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v_0, express in terms of the length y of chain which is off the floor at any given instant.

(a) The magnitude of the force P applied to end A.

(b) Energy lost during the lifting of the chain.

6. A is a fixed point at a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length $L (> H)$ has one end attached to A and other to a heavy particle. The particle is held at the

level of A with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



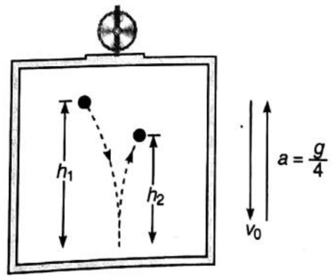
7. A particle of mass $2m$ is projected at an angle of $45°$ with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces.

   As a result of explosion one part comes to rest. Find the maximum height attained by the other part. (Take $g = 10\,\text{m/s}^2$)

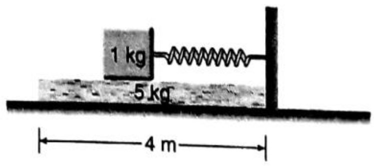
8. A sphere of mass m, impinges obliquely on a sphere, of mass M, which is at rest. Show that, if $m = eM$, the directions of motion of the spheres after impact are at right angles.

9. A gun of mass M (including the carriage) fires a shot of mass m. The gun along with the carriage is kept on a smooth horizontal surface. The muzzle speed of the bullet v_r is constant. Find :
   (a) The elevation of the gun with horizontal at which maximum range of bullet with respect to the ground is obtained.
   (b) The maximum range of the bullet.

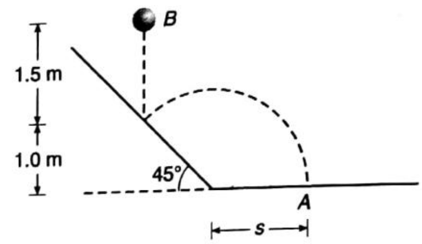
10. A ball is released from rest relative to the elevator at a distance h_1 above the floor. The speed of the elevator at the time of ball release is v_0. Determine the bounce height h_2 relative to elevator of the ball (a) if v_0 is constant and (b) if an upward elevator acceleration $a = g/4$ begins at the instant the ball is released. The coefficient of restitution for the impact is e.



11. A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. The system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is 100 N/m).

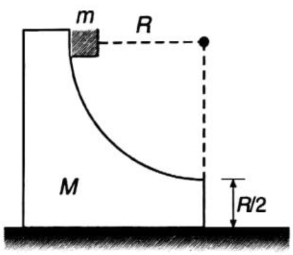


12. To test the manufactured properties of 10 N steel balls, each ball is released from rest as shown and strikes a $45°$ inclined surface. If the coefficient of restitution is to be $e = 0.8$, determine the distance s to where the ball must strike the horizontal plane at A. At what speed does the ball strike at A? ($g = 9.8\,\text{m/s}^2$)

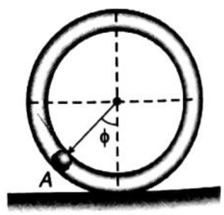


13. Two particles A and B of equal masses lie close together on a horizontal table and are connected by a light inextensible string of length l. A is projected vertically upwards with a velocity $\sqrt{10gl}$. Find the velocity with which it reaches the table again.

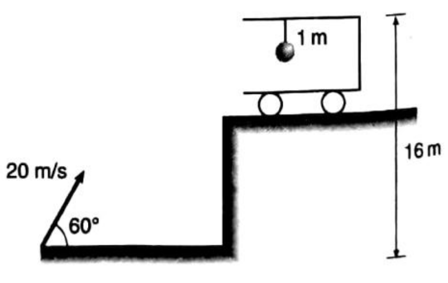
14. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown in figure. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path. Find the horizontal distance from the bottom of block when cube hits the table.



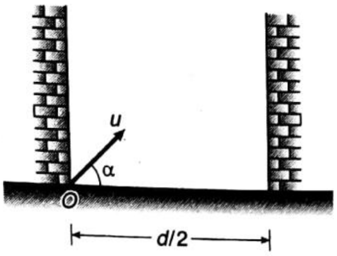
15. A thin hoop of mass M and radius r is placed on a horizontal plane. At the initial instant, the hoop is at rest. A small washer of mass m with zero initial velocity slides from the upper point of the hoop along a smooth groove in the inner surface of the hoop. Determine the velocity u of the centre of the hoop at the moment when the washer is at a certain point A of the hoop, whose radius vector forms an angle ϕ with the vertical (figure). The friction between the hoop and the plane should be neglected.



16. A shell of mass 1 kg is projected with velocity 20 m/s at an angle 60° with horizontal. It collides inelastically with a ball of mass 1 kg which is suspended through a thread of length 1 m. The other end of the thread is attached to the ceiling of a trolley of mass $\frac{4}{3}$ kg as shown in figure. Initially the trolley is stationary and it is free to move along horizontal rails without any friction. What is the maximum deflection of the thread with vertical? String does not slack. Take $g = 10\ \text{m/s}^2$.

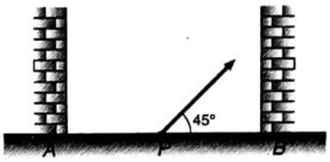


17. A small ball is projected at an angle α between two vertical walls such that in the absence of the wall its range would have been $5d$. Given that all the collisions are perfectly elastic, find

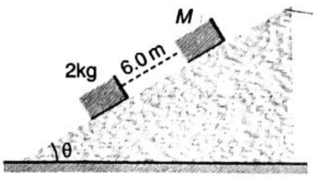


(a) maximum height attained by the ball.
(b) total number of collisions with the walls before the ball comes back to the ground, and
(c) point at which the ball finally falls. The walls are supposed to be very tall.

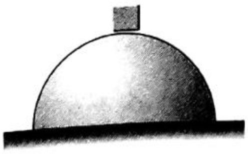
18. Two large rigid vertical walls A and B are parallel to each other and separated by 10 metres. A particle of mass 10 g is projected with an initial velocity of 20 m/s at 45° to the horizontal from point P on the ground, such that $AP = 5$ m. The plane of motion of the particle is vertical and perpendicular to the walls. Assuming that all the collisions are perfectly elastic, find the maximum height attained by the particle and the total number of collisions suffered by the particle with the walls before it hits ground. Take $g = 10\,\text{m/s}^2$.



19. Two blocks of mass 2 kg and M are at rest on an inclined plane and are separated by a distance of 6.0 m as shown. The coefficient of friction between each block and the inclined plane is 0.25. The 2 kg block is given a velocity of 10.0 m/s up the inclined plane. It collides with M, comes back and has a velocity of 1.0 m/s when it reaches its initial position. The other block M after the collision moves 0.5 m up and comes to rest. Calculate the coefficient of restitution between the blocks and the mass of the block M.
[Take $\sin\theta \approx \tan\theta = 0.05$ and $g = 10\,\text{m/s}^2$]



20. A small block of mass m is placed on top of a smooth hemisphere also of mass m which is placed on a smooth horizontal surface. If the block begins to slide down due to a negligible small impulse, show that it will loose contact with the hemisphere when the radial line through vertical makes an angle θ given by the equation $\cos^3\theta - 6\cos\theta + 4 = 0$.



21. A ball is projected from a given point with velocity u at some angle with the horizontal and after hitting a vertical wall returns to the same point. Show that the distance of the point from the wall must be less than $\dfrac{eu^2}{(1+e)g}$, where e is the coefficient of restitution.

ANSWERS

Introductory Exercise 8.1

1. $\vec{r}_{COM} = \dfrac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$ while $\vec{r}_{CG} = \dfrac{\sum_{i=1}^{n} w_i \vec{r}_i}{\sum_{i=1}^{n} w_i}$ Here, w = weight (mg), $\vec{r}_{COM} = \vec{r}_{CG}$ when $\vec{g}$ = constant 2. False

3. True 4. True 5. less than $\frac{r}{2}$ 6. False 7. $\frac{\sqrt{19}}{6}$ m 8. $\frac{4}{3\pi}\left\{\frac{a^2+ab+b^2}{a+b}\right\}$ 9. $\left(\frac{5a}{6}, \frac{5a}{6}\right)$

Introductory Exercise 8.2

1. zero 2. $\left(\frac{\sqrt{3}-1}{4\sqrt{2}}\right) g$

Introductory Exercise 8.3

1. 9 ms^{-1}, 1.08 kJ 3. $2.5\,\hat{i} + 15\hat{j} + 5\hat{k}$ 4. 60 m 5. 35 m 6. 10 cm 7. True

Introductory Exercise 8.4

1. 1.225 kgs^{-1} (i) 2.8 kms^{-1} (ii) 3.6 kms^{-1} 2. $(m_0 - \mu t)\frac{d^2x}{dt^2} = \mu u - (m_0 - \mu t)\, g$ 3. $u \ln\left(\frac{3}{2}\right) - g$,

Introductory Exercise 8.5

1. 10 m/s (downwards) 2. $(8\,\hat{i})$ N-s 3. $2\sqrt{10}$ N-s

Introductory Exercise 8.6

1. 30 cm 2. $\frac{K}{2}$ 4. $\frac{4m_1m_2}{(m_1+m_2)^2}$ 5. $\frac{8}{9}$ 6. Two 7. No

Introductory Exercise 8.7

1. $\frac{1}{3}$ 2. 2 ms^{-1}, $2\sqrt{2}$ ms^{-1} 3. $\frac{2\pi r}{v}$ 4. $\frac{2J}{P} - 1$

Introductory Exercise 8.8

1. $90° - 2\alpha$ 2. $e^n\sqrt{2gh}, e^{2n} h$ 3. (a) $\frac{mv}{V}$ (b) $\frac{V}{v}$ 4. $\frac{2}{3}$ 5. $-\hat{i} + 2\hat{j}$ 6. $\frac{1}{e}, \frac{1}{e}, \frac{1}{e^2}$

AIEEE Corner

Subjective Questions (Level 1)

1. 0.74 m^2 2. $\left(\frac{\pi}{\pi+4}\right) a$ 3. $\left(-\frac{a}{12}, -\frac{b}{12}\right)$ 4. $x_{COM} = -\frac{a^3 b}{R^3 - a^3}$ 5. $\frac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$

6. $x_{COM} = \frac{3l}{4}$ 7. x_{CM} = 12.67 m 8. zero 9. (a) $(-10\hat{j})\,ms^{-2}$ (b) $\frac{10}{3}(2\hat{i} - \hat{j})\,ms^{-1}$ (c) $\left(\frac{70}{3}\hat{i} + 35\hat{j}\right)$ m

10. (a) 0.30 kg (b) $(2.4\ kg\text{-}ms^{-1})\hat{j}$ (c) $(8.0\ ms^{-1})\hat{j}$ 11. (a) 28 cm (b) 2.3 ms^{-1}

12. (a) 2.0 kg (b) $(12.0\ ms^{-2})t\,\hat{j}$ (c) $(72.0\ N)\hat{j}$ 13. $\frac{2}{3}L$ 14. 71.4 mm 15. (a) 2 (b) 1 (c) 2

16. $\frac{4L}{3}$ 17. (a) $\frac{mv}{M+m}$ (b) balloon will also stop moving 18. (a) 0.435 (b) 480 J (c) 1.28 J

19. 0.013, 1.94 m/s 20. $2m \sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{gl}{M(M+m)}}$ 21. $\frac{m(R-r)}{M+m}, m\sqrt{\frac{2g(R-r)}{M(M+m)}}$

22. (a) 0.14 kg-ms^{-1} (b) zero **23.** 2.82 kms^{-1} **25.** 10 N, 20 W **26.** $m = m_0 e^{-(a/u)t}$

27. (a) 1.2×10^{-3} s (b) 0.5 N-s (c) 417 N **28.** 270 N-s (to the right), 13.5 kN (to the right)

29. $(8\hat{i})$kg-ms^{-1} **30.** 200 ms^{-1} **31.** 2 ms^{-1} in negative x-axis, 3 m/s in positive x-axis.

32. $v_1 = \dfrac{28}{3}$ ms^{-1} (in negative x-direction) and $v_2 = \dfrac{2}{3}$ ms^{-1} (in positive x-direction) **33.** $e = \dfrac{1}{3}$ **34.** $e = \dfrac{1}{\sqrt{2}}$

35. 0.4 ms^{-1}, 2.4 ms^{-1}, 0.933 m

Objective Questions (Level 1)

1. (a)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (b)
11. (d)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (d)	**30.** (d)
31. (c)	**32.** (c)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (c)
41. (d)	**42.** (c)	**43.** (d)							

JEE Corner

Assertion and Reason

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (d)
11. (d)	**12.** (d)	**13.** (d)	**14.** (a)	**15.** (d)					

Objective Questions (Level 2)

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (d)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (a)									

More than One Correct Options

1. (b,d) **2.** (a,c,d) **3.** (all) **4.** (b,d) **5.** (b,c) **6.** (c,d) **7.** (b,c) **8.** (b,c,d)

Match the Columns

1. (a) → (r) (b) → (q) (c) → (s) (d) → (p)

2. (a) → (q) (b) → (q) (c) → (p) (d) → (s)

3. (a) → (r) (b) → (q) (c) → (q) (d) → (p)

4. (a) → (r) (b) → (s) (c) → (p) (d) → (q)

5. (a) → (r) (b) → (p) (c) → (p) (d) → (s)

6. (a) → (r) (b) → (r,s) (c) → (p,q) (d) → (p)

7. (a) → (p,r,s) (b) → (p,r,s) (c) → (p) (d) → (p,q)

8. (a) → (r,s) (b) → (r,s) (c) → (q) (d) → (q)

Subjective Questions (Level 2)

1. 1.25 ms^{-1} **2.** $v_{ball} = \left(\dfrac{eM - m}{M + m}\right) v_0$ (leftwards), $v_{cart} = \left(\dfrac{e + 1}{M + m}\right) mv_0$ (rightwards); $t = \dfrac{d}{v_0}\left(1 + \dfrac{2}{e} + \dfrac{1}{e^2}\right)$

3. $x_2 = v_0 t + \dfrac{m_1}{m_2} A(1 - \cos \omega t)$, $l_0 = \left(\dfrac{m_1}{m_2} + 1\right) A$ **4.** $(L + 2R, 0)$ **5.** (a) $\dfrac{m}{l}(gy + v_0^2)$ (b) $\dfrac{myv_0^2}{2l}$

6. $\dfrac{H^5}{L^4}$ **7.** 35 m **9.** (a) 45° (b) $\left(\dfrac{M}{M + m}\right)\dfrac{v_r^2}{g}$ **10.** (a) $e^2 h_1$ (b) $e^2 h_1$ **11.** $\sqrt{\dfrac{10}{3}}$ ms^{-1}

12. 0.93 m, 6.6 ms^{-1} **13.** $2\sqrt{gl}$ **14.** $R\sqrt{\dfrac{2(M + m)}{M}}$ **15.** $v = m \cos \phi \sqrt{\dfrac{2gr(1 + \cos \phi)}{(M + m)(M + m \sin^2 \phi)}}$

16. 60° **17.** (a) $\dfrac{u^2 \sin^2 \alpha}{2g}$ (b) nine (c) point O **18.** 10 m, Four **19.** 0.84, 15.12 kg

Hints & Solutions (Subjective Questions Level 2)

Hints & Solutions
(Subjective Questions : Level 2)

Chapter 3 — Motion in One Dimension

Level 2

1. $$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{\sqrt{2g\,h_f} + \sqrt{2g\,h_i}}{\Delta t}$$
$$= \frac{\sqrt{2 \times 9.8 \times 2} + \sqrt{2 \times 9.8 \times 4}}{12 \times 10^{-3}} = 1.26 \times 10^3 \text{ m/s}^2$$ **Ans.**

Note : v_f is upwards (+ve) and v_i is downwards (– ve).

2. $v\,dv = ads$

∴ $$\int_0^v v dv = \int_0^{12\text{ m}} a\,ds$$

∴ $$\frac{v^2}{2} = \text{area under } a\text{-}s \text{ graph from } s = 0 \text{ to } s = 12 \text{ m.}$$
$$= 2 + 12 + 6 + 4$$
$$= 24 \text{ m}^2/\text{s}^2$$

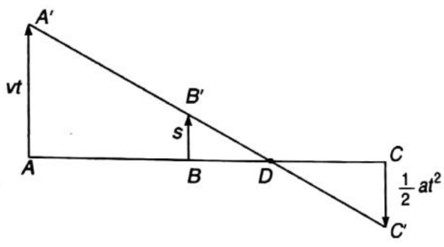
or $$v = \sqrt{48} \text{ m/s} = 4\sqrt{4} \text{ m/s}$$ **Ans.**

3. Let $AB = BC = d$

$BD = x$

and $BB' = s$ = displacement of point B.

From similar triangles we can write,



$$\frac{vt}{d + x} = \frac{s}{x} = \frac{\frac{1}{2}at^2}{d - x}$$

From first two equations we have,

$$1 + \frac{d}{x} = \frac{vt}{s}$$

or $$\frac{d}{x} = \frac{vt}{s} - 1 \qquad \text{...(i)}$$

From last two equations we have,

$$\frac{d}{x} - 1 = \frac{\frac{1}{2}at^2}{s}$$

or $$\frac{d}{x} = \frac{\frac{1}{2}at^2}{s} + 1 \qquad \text{...(ii)}$$

Equating (i) and (ii) we have,

$$\frac{vt}{s} - 1 = \frac{\frac{1}{2}at^2}{s} + 1$$

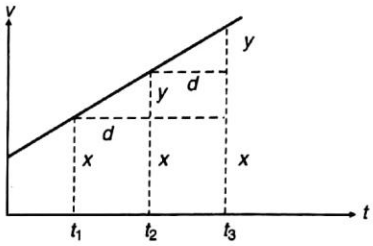
or $$\frac{vt - \frac{1}{2}at^2}{s} = 2$$

∴ $$s = \left(\frac{v}{2}\right)t - \frac{1}{2}\left(\frac{a}{2}\right)t^2$$

Comparing with $s = ut + \frac{1}{2}at^2$ we have,

Initial velocity of B is $+\frac{v}{2}$ and acceleration $-\frac{a}{2}$.

4. Let us draw v-t graph of the given situation, area of which will give the displacement and slope the acceleration.



$$s_2 - s_1 = x\,d + \frac{1}{2}yd \qquad \text{...(i)}$$

$$s_3 - s_2 = xd + yd + \frac{1}{2}yd \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we have

$$s_3 + s_1 - 2s_2 = yd$$

or

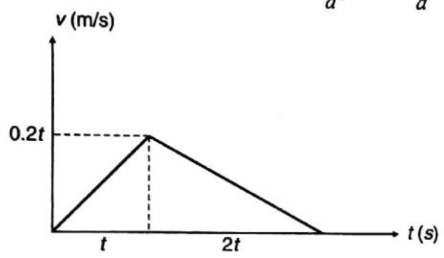
$$s_3 + s_1 - 2\sqrt{s_1 s_3} = yd \qquad (s_2 = \sqrt{s_1 s_3})$$

Dividing by d^2 both sides we have,

$$\frac{(\sqrt{s_1} - \sqrt{s_3})^2}{d^2} = \frac{y}{d} = \text{slope of } v\text{-}t \text{ graph} = a.$$

Hence proved.

5.



Area of $v - t$ graph = displacement

$\therefore$ $$\frac{1}{2}(3t)(0.2\,t) = 14$$ **Ans.**

Solving this equation we get $3t = 20.5$ sec.

Note *Maximum speed $0.2t$ is less than 2.5 m/s.*

6. Let t_0 be the breaking time and a the magnitude of deceleration.

80.5 km/h = 22.36 m/s, 48.3 km/h = 13.42 m/s.

In the first case,

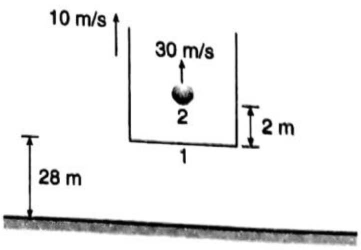
$$56.7 = (22.36 \times t_0) + \frac{(22.36)^2}{2a} \qquad \text{...(i)}$$

and

$$24.4 = (13.42\,t_0) + \frac{(13.42)^2}{2a} \qquad \text{...(ii)}$$

Solving these two equation we get $t_0 = 0.74$ s and $a = 6.2$ m/s^2 **Ans.**

7. Absolute velocity of ball = 30 m/s



(a) Maximum height of ball from ground $= 28 + 2 + \frac{(30)^2}{2 \times 9.8} = 76$ m

(b) Ball will return to the elevator floor when,

$$s_1 = s_2 + 2$$

or

$$10t = (30t - 4.9\,t^2) + 2$$

Solving we get $t = 4.2$ s **Ans.**

8. (a) Average velocity $= \dfrac{\text{displacement}}{\text{time}} = \dfrac{\int_0^5 v\,dt}{5} = \dfrac{\int_0^5 (3t - t^2)\,dt}{5}$

$$= -0.833 \text{ m/s}$$

(b) Velocity of particle = 0 at $t = 3$ s

i.e., at 3 s, particle changes its direction of motion.

Average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{(\text{distance from 0 to 3 s}) + (\text{distance from 3 s to 5 s})}{\text{time}}$

$$d_{0\text{-}3} = \int_0^3 (3t - t^2)dt = 4.5 \text{ m}$$

$$d_{3\text{-}5} = \int_3^5 (t^2 - 3t)\,dt = 8.67 \text{ m}$$

$\therefore$ Average speed $= \dfrac{4.5 + 8.67}{5} = 2.63$ m/s **Ans.**

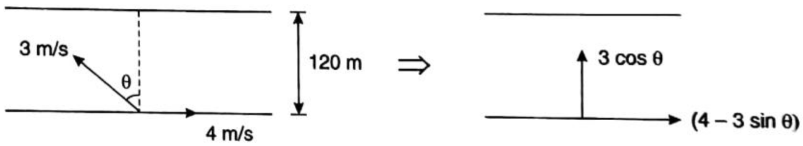
9. (a) $a = 2t - 2$ (from the graph)

Now $\int_0^v dv = \int_0^t a\,dt = \int_0^t (2t - 2)\,dt$

$\therefore$ $v = t^2 - 2t$

(b) $s = \int_2^4 v\,dt = \int_2^4 (t^2 - 2t)\,dt = 6.67$ m

10. (a)



Time to cross the river $t_1 = \dfrac{120}{3\cos\theta} = \dfrac{40}{\cos\theta} = 40\sec\theta$

Drift along the river $x = (4 - 3\sin\theta)\left(\dfrac{40}{\cos\theta}\right)$

$$= (160\sec\theta - 120\tan\theta)$$

To reach directly opposite, this drift will be covered by walking speed. Time taken in this,

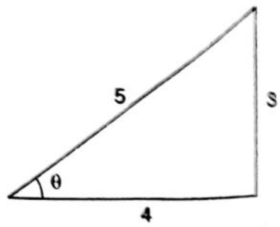
$$t_2 = \frac{160\sec\theta - 120\tan\theta}{1} = 160\sec\theta - 120\tan\theta$$

$\therefore$ Total time taken $t = t_1 + t_2 = (200\sec\theta - 120\tan\theta)$

For t to be minimum, $\dfrac{dt}{d\theta} = 0$

or $200\sec\theta\tan\theta - 120\sec^2\theta = 0$

or $\theta = \sin^{-1}(3/5)$



(where $\sin\theta = \frac{3}{5}$)

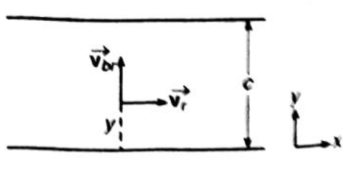
(b) $t_{min} = 200\sec\theta - 120\tan\theta$

$$= 200 \times \frac{5}{4} - 120 \times \frac{3}{4}$$

$$= 250 - 90 = 160 \text{ s} = 2 \text{ min } 40 \text{ s}$$ **Ans.**

11. Given that $|\vec{v}_{br}| = v_y = \dfrac{dy}{dt} = u$ ...(i)

$|\vec{v}_r| = v_x = \dfrac{dx}{dt} = \left(\dfrac{2v_0}{c}\right)y$ ...(ii)



From Eqs. (i) and (ii) we have, $\dfrac{dy}{dx} = \dfrac{uc}{2v_0 y}$

or $\int_0^y y\,dy = \dfrac{uc}{2v_0}\int_0^x dx$ or $y^2 = \dfrac{ucx}{v_0}$ **Ans.**

At $y = \dfrac{c}{2}, x = \dfrac{cv_0}{4u}$ or $x_{net} = 2x = \dfrac{cv_0}{2u}$ **Ans.**

12. $a = v\dfrac{dv}{ds} = v$ (slope of v-s graph)

At $s = 50$ m : $v = 20$ m/s and $\dfrac{dv}{ds} = \dfrac{40}{100} = 0.4$ per sec

$\therefore$ $a = 20 \times 0.4 = 8\ \text{m/s}^2$

At $s = 150$ m : $v = (40 + 5) = 45$ m/s and $\dfrac{dv}{ds} = \dfrac{10}{100} = 0.1$ per sec

$\therefore$ $a = 45 \times 0.1 = 4.5\ \text{m/s}^2$

a-s graph :

From $s = 0$ to $s = 100$ m : $v = 0.4\,s$ and $\dfrac{dv}{ds} = 0.4$

$\therefore$ $a = v.\dfrac{dv}{ds} = 0.16\,s$

i.e., a-s graph is a straight line passing though origin of slope 0.16 per $(\text{sec})^2$.

At $s = 100$ m, $a = 0.16 \times 100 = 16\ \text{m/s}^2$

From $s = 100$ m to $s = 200$ m :

$$v = 0.1s + 30$$

$$\frac{dv}{ds} = 0.1$$

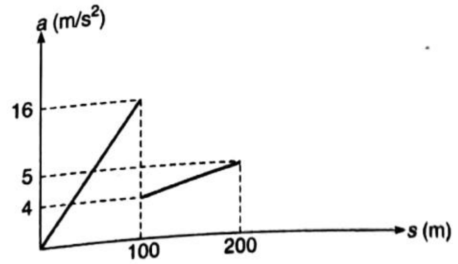
$\therefore$ $a = v\dfrac{dv}{ds} = (0.1s + 30)(0.1) = (0.01s + 3)$

i.e., a-s graph is straight line of slope 0.01 $(\text{sec})^{-2}$ and intercept 3 m/s^2.

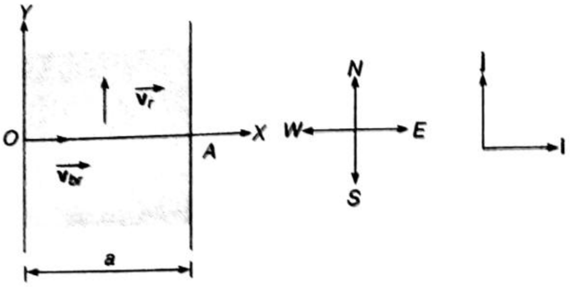
At $s = 100$ m, $a = 4\ \text{m/s}^2$

and at $s = 200$ m, $a = 5\ \text{m/s}^2$

Corresponding a-s graph is as shown in figure



13. (a) Let $\vec{v}_{br}$ be the velocity of boatman relative to river, $\vec{v}_r$ the velocity of river and $\vec{v}_b$ is the absolute velocity of boatman. Then



$$\vec{v}_b = \vec{v}_{br} + \vec{v}_r$$

Given, $|\vec{v}_{br}| = v$ and $|\vec{v}_r| = u$

Now $$u = v_y = \frac{dy}{dt} = x(a-x)\frac{v}{a^2} \quad \text{...(i)}$$

and $$v = v_x = \frac{dx}{dt} = v \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{dy}{dx} = \frac{x(a-x)}{a^2} \quad \text{or} \quad dy = \frac{x(a-x)}{a^2}\,dx$$

or $$\int_0^y dy = \int_0^x \frac{x(a-x)}{a^2}\,dx$$

or $$y = \frac{x^2}{2a} - \frac{x^3}{3a^2} \quad \text{...(iii)}$$

This is the desired equation of trajectory.

(b) Time taken to cross the river is

$$t = \frac{a}{v_x} = \frac{a}{v}$$

(c) When the boatman reaches the opposite side, $x = a$ or $v_y = 0$ [from Eq. (i)]
Hence, resultant velocity of boatman is v along positive x-axis or due east.

(d) From Eq. (iii)

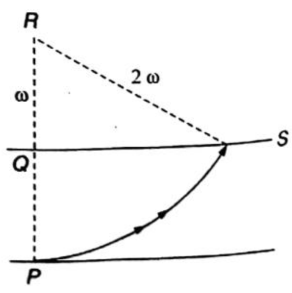
$$y = \frac{a^2}{2a} - \frac{a^3}{3a^2} = \frac{a}{6}$$

At $x = a$ (at opposite bank).
Hence, displacement of boatman will be

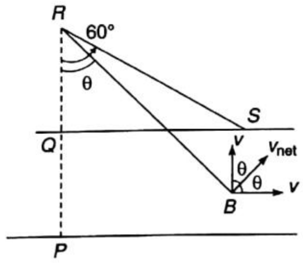
$$\vec{s} = x\,\hat{i} + y\,\hat{j} \quad \text{or} \quad \vec{s} = a\hat{i} + \frac{a}{6}\hat{j}$$

14. (a) Since the resultant velocity is always perpendicular to the line joining boat and R, the boat is moving in a circle of redius 2ω and centre at R.

(b) Drifting $= QS = \sqrt{4\omega^2 - \omega^2} = \sqrt{3}\omega$.



(c) Suppose at any arbitrary time, the boat is at point B.



$$V_{net} = 2v\cos\theta$$

$$\frac{d\theta}{dt} = \frac{V_{net}}{2\omega} = \frac{v\cos\theta}{\omega}$$

or $$\frac{\omega}{v}\sec\theta\, d\theta = dt$$

$\therefore$ $$\int_0^t dt = \frac{\omega}{v}\int_0^{60°}\sec\theta\, d\theta$$

$\therefore$ $$t = \frac{\omega}{v}[\ln(\sec\theta + \tan\theta)]_0^{60°}$$

or $$t = \frac{1.317\omega}{v}$$ **Ans.**

15. For $0 < s \le 60$ m $$v = \frac{12}{60}s + 3 = 3 + \frac{s}{5}$$

$$\frac{dv}{dt} = \left(\frac{1}{5}\right)\cdot\frac{ds}{dt} = \frac{1}{5}(v) = \frac{1}{5}\left(3 + \frac{s}{5}\right) = \frac{3}{5} + \frac{s}{25} \quad \text{...(i)}$$

or $$a = \frac{3}{5} + \frac{s}{25}$$

i.e., *a-s* graph is a straight line.

At $s = 0$, $$a = \frac{3}{5}\ \text{m/s}^2 = 0.6\ \text{m/s}^2$$

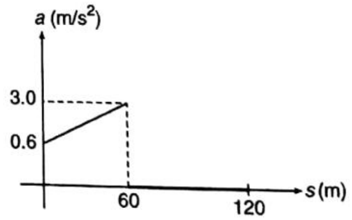
and at $s = 60$ m, $a = 3.0\ \text{m/s}^2$

For $s > 60$ m

v = constant

$\therefore$ $a = 0$

Therefore the corresponding *a-s* graph is shown in figure.



From Eq. (i), $$\frac{dv}{dt} = \frac{v}{5} \quad \text{or} \quad \int_3^v \frac{dv}{v} = \frac{1}{5}\int_0^t dt$$

$$\ln\left(\frac{v}{3}\right) = \frac{t}{5}$$

$\therefore$ $$v = 3e^{t/5} \quad \text{or} \quad \int_0^{60} ds = 3\int_0^{t_1} e^{t/5}\, dt$$

$$60 = 15(e^{t_1/5} - 1) \quad \text{or} \quad t_1 = 8.0\ \text{s}$$

Time taken to travel next 60 m with speed 15m/s will be $\frac{60}{15} = 4$ s

$\therefore$ Total time = 12.0 s **Ans.**

16. From the graph, $$a = 22.5 - \frac{22.5}{150}\cdot s \quad \text{or} \quad \int_0^v v\cdot dv = \int_0^{60}\left(22.5 - \frac{22.5}{150}\times s\right)ds$$

$\therefore$ $$\frac{v^2}{2} = 22.5\times 60 - \frac{22.5}{150}\times\frac{(60)^2}{2}$$

$\therefore$ $$v = 46.47\ \text{m/s}$$ **Ans.**

17. (a) $$u_x = 3\ \text{m/s}$$
$$a_x = -1.0\ \text{m/s}^2$$

Maximum x coordinate is attained after time $t = \left|\frac{u_x}{a_x}\right| = 3$ s

At this instant $v_x = 0$ and $v_y = u_y + a_y t = 0 - 0.5 \times 3 = -1.5$ m/s

$\therefore$ $\vec{v} = (-1.5\,\hat{j})$ m/s **Ans.**

(b) $x = u_x t + \frac{1}{2} a_x t^2 = 3 \times 3 + \frac{1}{2}(-1.0)(3)^2 = 4.5$ m

$y = u_y t + \frac{1}{2} a_y t^2 = 0 - \frac{1}{2}(0.5)(3)^2 = -2.25$ m

$\therefore$ $\vec{r} = (4.5\,\hat{i} - 2.25\,\hat{j})$ m **Ans.**

18. (b) $\vec{v} = v_x\,\hat{i} + v_y\,\hat{j}$ and $\vec{a} = a_x\,\hat{i} + a_y\,\hat{j}$

$$\vec{v} \cdot \vec{a} = v_x a_x + v_y a_y$$

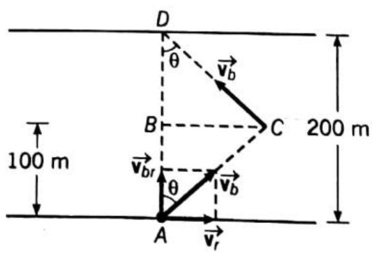
Further $v = \sqrt{v_x^2 + v_y^2}$

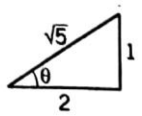
$\therefore$ $\frac{\vec{v} \cdot \vec{a}}{v} = \frac{v_x a_x + v_y a_y}{\sqrt{v_x^2 + v_y^2}} = \frac{dv}{dt} = a_t$

or component of $\vec{a}$ parallel to $\vec{v}$ = tangential acceleration.

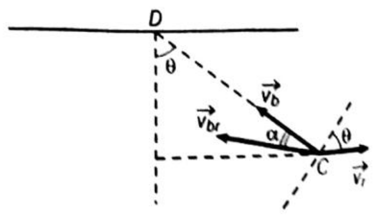
19. (a) $v_{br} = 4$ m/s, $v_r = 2$ m/s

$$\tan\theta = \frac{BC}{AB} = \frac{|\vec{v}_r|}{|\vec{v}_{br}|} = \frac{2}{4} = \frac{1}{2}$$





In this case $\vec{v}_b$ should be along CD



$\therefore$ $v_r \cos\theta = v_{br} \sin\alpha$

$$2\left(\frac{2}{\sqrt{5}}\right) = 4\sin\alpha \quad \text{or} \quad \sin\alpha = \frac{1}{\sqrt{5}}$$

$\therefore$ $\alpha = \theta = \tan^{-1}\left(\frac{1}{2}\right)$

(b) $t_1 = \frac{200}{|\vec{v}_{br}|} = \frac{200}{4} = 50$ s

$$DC = DB\sec\theta = (100)\frac{\sqrt{5}}{2} = 50\sqrt{5} \text{ m}$$

$$|\vec{v}_b| = |\vec{v}_{br}| \cos\alpha - |\vec{v}_r| \sin\theta = 4\left(\frac{2}{\sqrt{5}}\right) - 2\left(\frac{1}{\sqrt{5}}\right) = \frac{6}{\sqrt{5}} \text{ m/s}$$

$\therefore$
$$t_2 = \frac{t_1}{2} + \frac{DC}{|\vec{v}_b|} = 25 + \frac{50\sqrt{5}}{\frac{6}{\sqrt{5}}} = \frac{200}{3} \text{ s}$$

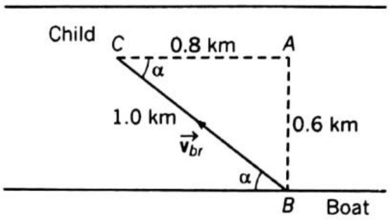
or
$$\frac{t_2}{t_1} = \frac{4}{3}$$
Ans.

20.
$$\vec{v}_b = \text{velocity of boatman} = \vec{v}_{br} + \vec{v}_r$$

And
$$\vec{v}_c = \text{velocity of child} = \vec{v}_r$$

$\therefore$
$$\vec{v}_{bc} = \vec{v}_b - \vec{v}_c = \vec{v}_{br}$$

$\vec{v}_{bc}$ should be along BC

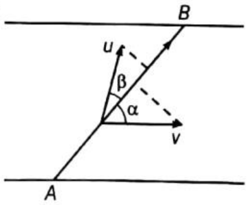


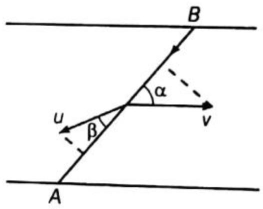
i.e., $\vec{v}_{br}$ should be along BC, where $\tan\alpha = \frac{0.6}{0.8} = \frac{3}{4}$

or
$$\alpha = 37°$$
Ans.

Further
$$t = \frac{BC}{|\vec{v}_{br}|} = \frac{1}{20} \text{ h} = 3 \text{ min}$$
Ans.

21. In order that the moving launch is always on the straight line AB, the components of velocity of the current and of the launch in the direction perpendicular to AB should be equal, *i.e.*,





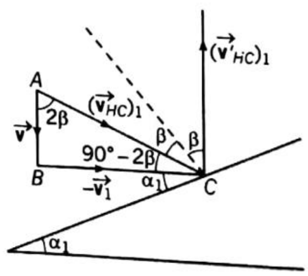
$$u \sin\beta = v \sin\alpha \qquad \text{...(i)}$$

$$S = AB = (u\cos\beta + v\cos\alpha)t_1 \qquad \text{...(ii)}$$

Further
$$BA = (u\cos\beta - v\cos\alpha)t_2 \qquad \text{...(iii)}$$
$$t_1 + t_2 = t \qquad \text{...(iv)}$$

Solving these equations after proper substitution, we get
$$u = 8 \text{ m/s} \quad \text{and} \quad \beta = 12°$$
Ans.

22. Here, absolute velocity of hail stones $\vec{v}$ before colliding with wind screens is vertically downwards and velocity of hail stones with respect to cars after collision $\vec{v}'_{HC}$ is vertically upwards. Collision is elastic, hence, velocity of hail stones with respect to cars before collision $\vec{v}_{HC}$ and after collision $\vec{v}'_{HC}$ will make equal angles with the normal to the wind screen.



$(\vec{v}_{HC})_1$ = velocity of hail stones – velocity of car 1

$= \vec{v} - \vec{v}_1$

From the figure, we can see that
$$\beta + 90° - 2\beta + \alpha_1 = 90°$$

or
$$\alpha_1 = \beta$$

or
$$2\beta = 2\alpha_1$$

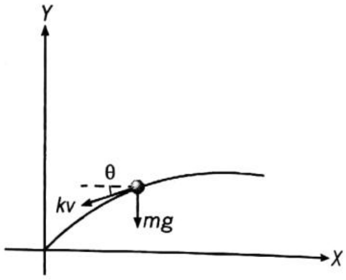
In ΔABC, $\tan 2\beta = \tan 2\alpha_1 = \dfrac{v_1}{v}$ ...(i)

Similarly, we can show that $\tan 2\alpha_2 = \dfrac{v_2}{v}$ ...(ii)

From Eq. (i) and (ii), we get $\dfrac{v_1}{v_2} = \dfrac{\tan 2\alpha_1}{\tan 2\alpha_2} = \dfrac{\tan 60°}{\tan 30°} = \dfrac{\sqrt{3}}{1/\sqrt{3}} = 3$

$\therefore$ $\dfrac{v_1}{v_2} = 3$ **Ans.**

23 $a_x = \dfrac{dv_x}{dt} = -\dfrac{kv\cos\theta}{m} = -\dfrac{k}{m}v_x$



$\therefore$ $\dfrac{dv_x}{v_x} = -\dfrac{k}{m}dt$ or $\displaystyle\int_{v_0\cos\theta_0}^{v_x} \frac{dv_x}{v_x} = -\frac{k}{m}\int_0^t dt$

or $v_x = v_0 \cos\theta_0 \, e^{-\frac{k}{m}t}$ ...(i)

Similarly $a_y = \dfrac{dv_y}{dt} = -\dfrac{kv\sin\theta}{m} - g = -\left(\dfrac{k}{m}v_y + g\right)$

or $\displaystyle\int_{v_0\sin\theta_0}^{v_y} \frac{dv_y}{\frac{k}{m}v_y + g} = -\int_0^t dt$ or $\dfrac{m}{k}\left[\ln\left(\dfrac{k}{m}v_y + g\right)\right]_{v_0\sin\theta_0}^{v_y} = -t$

or $\dfrac{\left(\dfrac{k}{m}v_y + g\right)}{\left(\dfrac{k}{m}v_0\sin\theta_0 + g\right)} = e^{-\frac{k}{m}t}$

or $v_y = \dfrac{m}{k}\left[\left(\dfrac{k}{m}v_0\sin\theta_0 + g\right)e^{-\frac{k}{m}t} - g\right]$ ...(ii)

(b) Eq. (i) can be written as: $\dfrac{dx}{dt} = v_0\cos\theta_0\, e^{-\frac{k}{m}t}$

or $\displaystyle\int_0^x dx = v_0\cos\theta_0 \int_0^t e^{-\frac{k}{m}t}dt$ or $x = \dfrac{mv_0\cos\theta_0}{k}[1 - e^{-\frac{k}{m}t}]$

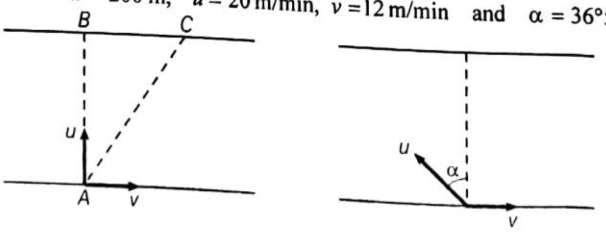
$x_m = \dfrac{mv_0\cos\theta_0}{k}$ at $t = \infty$. **Ans.**

24. In the first case $BC = vt_1$ and $w = ut_1$

In the second case $u\sin\alpha = v$ and $w = (u\cos\alpha)t_2$

Solving these four equations with proper substitution, we get

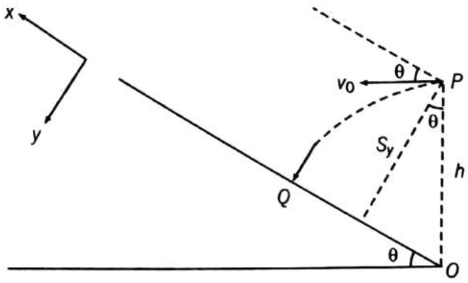
$w = 200$ m, $u = 20$ m/min, $v = 12$ m/min and $\alpha = 36°50'$ **Ans.**



Chapter 4 Projectile Motion

Level 2

1. $u_x = v_0 \cos\theta,\ u_y = v_0 \sin\theta, a_x = -g\sin\theta, a_y = g\cos\theta$
At $Q : v_x = 0 \quad \therefore\ u_x + a_x t = 0$



Or
$$t = \frac{v_0 \cos\theta}{g\sin\theta} \quad \text{...(i)}$$
$$S_y = h\cos\theta$$
$$\therefore \quad u_y t + \frac{1}{2} a_y t^2 = h\cos\theta$$
$$\therefore \quad (v_0 \sin\theta)\left(\frac{v_0\cos\theta}{g\sin\theta}\right) + \frac{1}{2}(g\cos\theta)\left(\frac{v_0\cos\theta}{g\sin\theta}\right)^2 = h\cos\theta$$

Solving this equation we get,
$$v_0 = \sqrt{\frac{2gh}{2+\cot^2\theta}} \quad \textbf{Ans.}$$

2. Let v_x and v_y be the compounts of v_0 along x and y directions.
$$(v_x)(2) = 2$$
$$\therefore \quad v_x = 1 \text{ m/s}$$
$$v_y(2) = 10 \quad \text{Or} \quad v_y = 5 \text{ m/s}$$
$$v_0 = \sqrt{v_x^2 + v_y^2} = \sqrt{26} \text{ m/s}$$
$$\tan\theta = v_y/v_x = 5/1$$
$$\therefore \quad \theta = \tan^{-1}(5) \quad \textbf{Ans.}$$

Note *We have seen relative motion between two particles.*
Relative acceleration between them is zero.

3.
$$\vec{V}_1 = (u\cos\alpha)\,\hat{i} + (u\sin\alpha - gt)\,\hat{j}$$
$$\vec{V}_2 = (v\cos\beta)\,\hat{i} + (v\sin\beta - gt)\,\hat{j}$$

These two velocity vectors will be parallel when, the ratio of coefficients of $\hat{i}$ and $\hat{j}$ are equal.

$$\therefore \qquad \frac{u\cos\alpha}{v\cos\beta}=\frac{u\sin\alpha - gt}{v\sin\beta - gt}$$

Solving we get,
$$t=\frac{uv\sin(\alpha-\beta)}{g(v\cos\beta - u\cos\alpha)}$$
Ans.

4. At height 2 m, projectile will be at two times, which are obtained from the equation,

$$+2=(10\sin 45°)t+\frac{1}{2}(-10)t^2$$

or $$2=5\sqrt{2}t-5t^2$$

or $$5t^2-5\sqrt{2}t+2=0$$

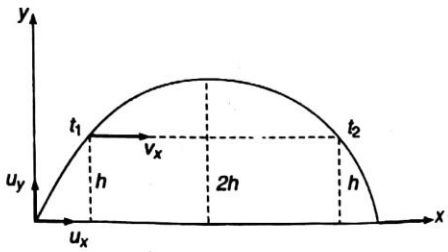
or $$t_1=\frac{5\sqrt{2}-\sqrt{50-40}}{10}=\frac{5\sqrt{2}-\sqrt{10}}{10}$$

and $$t_2=\frac{5\sqrt{2}+\sqrt{10}}{10}$$

Now $$d=(10\cos 45°)(t_2-t_1)$$
$$=\frac{10}{\sqrt{2}}\left(\frac{2\sqrt{10}}{10}\right)=4.47\text{ m}$$

Distance of point of projection from first hurdle $=(10\cos 45°)t_1$
$$=\frac{10}{\sqrt{2}}\left(\frac{5\sqrt{2}-\sqrt{10}}{10}\right)$$
$$=5-\sqrt{5}$$
$$=2.75\text{ m}$$
Ans.

5. $$2h=\frac{u_y^2}{2g}$$



or $$u_y=2\sqrt{gh}$$

Now $$(t_2-t_1)u_x=t_2v_x \text{ or } \frac{v_x}{u_x}=\frac{t_2-t_1}{t_2} \qquad \text{...(i)}$$

Further $$h=u_yt-\frac{1}{2}gt^2$$

or $$gt^2-2u_yt+h=0$$

or $$gt^2-4\sqrt{gh}\,t+2h=0$$

$$t_1=\frac{4\sqrt{gh}-\sqrt{16gh-8gh}}{2g}=(2-\sqrt{2})\sqrt{\frac{h}{g}} \quad \text{and} \quad t_2=(2+\sqrt{2})\sqrt{\frac{h}{g}}$$

Substituting in Eq. (i) we have,

$$\frac{v_x}{u_x}=\frac{2}{\sqrt{2}+1}$$
Ans.

6. (a) Time of descent $t=\sqrt{\frac{2H}{g}}=\sqrt{\frac{2\times 400}{10}}=8.94$ s

Now $$v_x=ay=\sqrt{5}\,y$$

or $$\frac{dx}{dt} = \sqrt{5}\left(\frac{1}{2}gt^2\right) = 5\sqrt{5}t^2$$

$\therefore$ $$\int_0^x dx = 5\sqrt{5}\int_0^t t^2\, dt$$

or horizontal drift $x = \frac{5\sqrt{5}}{3}(8.94)^3 = 2663 \text{ m} \approx 2.67 \text{ km}.$

(b) When particle strikes the ground :

$$v_x = \sqrt{5}y = (\sqrt{5})(400) = 400\sqrt{5} \text{ m/s}$$

$$v_y = gt = 89.4 \text{ m/s}$$

$\therefore$ $$\text{Speed} = \sqrt{v_x^2 + v_y^2} = 899 \text{ m/s} \approx 0.9 \text{ km/s}$$ **Ans.**

7. At $t = 0$, $\vec{v}_T = (10\hat{j})$ m/s $\quad \vec{v}_{ST} = 10\cos 37°\hat{k} - 10\sin 37°\hat{i} = (8\hat{k} - 6\hat{i})$ m/s

$\therefore$ $$\vec{v}_S = \vec{v}_{ST} + \vec{v}_T = (-6\hat{i} + 10\hat{j} + 8\hat{k}) \text{ m/s}$$

(a) At highest point vertical component $(\hat{k})$ of $\vec{v}_S$ will become zero. Hence, velocity of particle at highest point will become $(-6\hat{i} + 10\hat{j})$ m/s .

(b) Time of flight, $$T = \frac{2v_Z}{g} = \frac{2 \times 8}{10} = 1.6 \text{ s}$$

$$x = x_i + v_xT$$

$$= \frac{16}{\pi} - 6 \times 1.6 = -4.5 \text{ m}$$

$$y = (10)(1.6) = 16 \text{ m} \quad \text{and} \quad z = 0$$

Therefore coordinates of particle where it finally lands on the ground are (– 4.5 m, 16 m , 0).

At highest point $$t = \frac{T}{2} = 0.8 \text{ s}$$

$\therefore$ $$x = \frac{16}{\pi} - (6)(0.8) = 0.3 \text{ m}$$

$$y = (10)(0.8) = 8.0 \text{ m}$$

and $$z = \frac{v_Z^2}{2g} = \frac{(8)^2}{20} = 3.2 \text{ m}$$

Therefore, coordinates at highest point are, (0.3 m, 8.0 m, 3.2 m) **Ans.**

8. $|v_{21x}| = (v_1 + v_2)\cos 60° = 12$ m/s

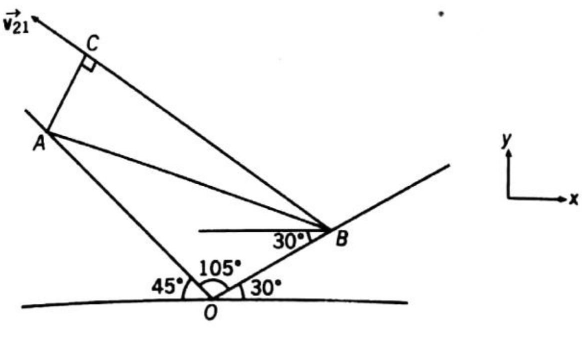
$|v_{21y}| = (v_2 - v_1)\sin 60° = 4\sqrt{3}$ m/s

$\therefore$ $$v_{21} = \sqrt{(12)^2 + (4\sqrt{3})^2} = \sqrt{192} \text{ m/s}$$

$BC = (v_{21})\,t = 240$ m

$AC = 70$ m (Given)

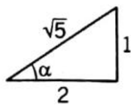
Hence, $AB = \sqrt{(240)^2 + (70)^2} = 250$ m **Ans.**



9. (a) Let, (x, y) be the coordinates of point C.

$$x = OD = OA + AD$$

$$\therefore \quad x = \frac{10}{3} + y \cot 37° = \frac{10 + 4y}{3} \quad \ldots(i)$$



As point C lies on the trajectory of a parabola, we have

$$y = x \tan \alpha - \frac{gx^2}{2u^2}(1 + \tan^2 \alpha) \quad \ldots(ii)$$

Given that, $\tan \alpha = 0.5 = \frac{1}{2}$

Solving Eqs. (i) and (ii), we get $x = 5$ m and $y = 1.25$ m.

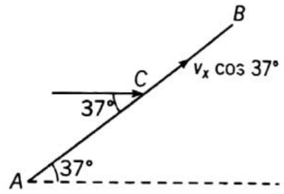
Hence, the coordinates of point C are (5 m, 1.25 m). **Ans.**

(b) Let v_y be the vertical component of velocity of the particle just before collision at C.

Using $v_y = u_y + a_y t$, we have

$$v_y = u \sin \alpha - g\,(x/u \cos \alpha) \qquad (\because\ t = x/u \cos \alpha)$$

$$= \frac{5\sqrt{5}}{\sqrt{5}} - \frac{10 \times 5}{(5\sqrt{5} \times 2/\sqrt{5})} = 0$$



Thus, at C, the particle has only horizontal component of velocity

$$v_x = u \cos \alpha = 5\sqrt{5} \times (2/\sqrt{5}) = 10 \text{ m/s}$$

Given, that the particle does not rebound after collision. So, the normal component of velocity (normal to the plane AB) becomes zero. Now, the particle slides up the plane due to tangential component $v_x \cos 37° = (10)\left(\frac{4}{5}\right) = 8$ m/s

Let h be the further height raised by the particle. Then

$$mgh = \frac{1}{2} m (8)^2 \quad \text{or} \quad h = 3.2 \text{ m}$$

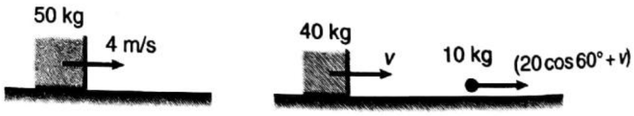
Height of the particle from the ground $= y + h$

$$\therefore \quad H = 1.25 + 3.2 = 4.45 \text{ m}$$ **Ans.**

10. For shell $u_z = 20 \sin 60° = 17.32$ m/s

$$\therefore \quad z = u_z t - \frac{1}{2} g t^2$$

$$= (17.32 \times 2) - \left(\frac{1}{2} \times 9.8 \times 4\right)$$



or

$$z = 15 \text{ m}$$

$$u_y = 0 \quad \therefore \quad y = 0$$

For u_x : conservation of linear momentum gives,

$$50 \times 4 = (40)(v) + 10(20 \cos 60° + v) \quad \text{or} \quad v = 2 \text{ m/s}$$

$$\therefore \quad u_x = (20 \cos 60°) + 2 = 12 \text{ m/s}$$

$$\therefore \quad x = u_x t = (12)(2) = 24 \text{ m}$$

$$\therefore \quad \vec{r} = (24\hat{i} + 15\hat{k}) \text{ m}$$ **Ans.**

Chapter 5 — Laws of Motion

Level 2

1. It is just like a projectile motion with g to be replaced by $g \sin 45°$.

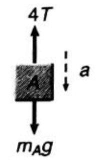
After 2 seconds, $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(10 \sin 45° - \frac{g}{\sqrt{2}} \times 2\right) + (10 \cos 45°)^2}$$

$$= 10 \text{ m/s}$$ **Ans.**

2. Suppose T be the tension in the string attached to block B. Then tension in the string connected to block A would be $4T$.

Similarly, if a be the acceleration of block A (downwards), then acceleration of block B towards right will be $4a$.

Equations of motion are:



For block A, $m_A g - 4T = m_A a$

or $50 - 4T = 5a$ ...(i)

For block B, $T - f = 10(4a)$

or $T - (0.1)(10)(10) = 40a$

or $T - 10 = 40a$ ...(ii)

Solving Eqs. (i) and (ii), we get $a = \frac{2}{33}$ m/s^2 **Ans.**

3. (a) When the truck accelerates eastward force of friction on mass is eastwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = (30 \times 1.8)$$
$$= 54 \text{ N}$$

Since it is less than $\mu_s mg$ $\therefore$ $f = 54$ N (eastwards)

(b) When the truck accelerates westwards, force of friction is westwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = 30 \times 3.8$$
$$= 114 \text{ N}$$

Since it is greater than $\mu_s mg$. Hence

$$f = f_k = \mu_k mg = 60 \text{ N (westwards)}$$ **Ans.**

4. Block B will fall vertically downwards and A along the plane.

Writing the equations of motion.

For block B, $m_B g - N = m_B a_B$

or $60 - N = 6a_B$ ...(i)

$(N + m_A g) \sin 30° = m_A a_A$

or $(N + 150) = 30\, a_A$ ...(ii)

Further $a_B = a_A \sin 30°$

or $a_A = 2a_B$ ...(iii)

Solving these three equations, we get

(a) $a_A = 6.36$ m/s^2 **Ans.**

(b) $a_{BA} = a_A \cos 30° = 5.5$ m/s^2 **Ans.**

5. Let acceleration of m be a_1 (absolute) and that of M be a_2 (absolute).
Writing equations of motion.

For m: $$mg \cos \alpha - N = ma_1 \quad \text{...(i)}$$

For M: $$N \sin \alpha = Ma_2 \quad \text{...(ii)}$$

Constraint equation can be written as,

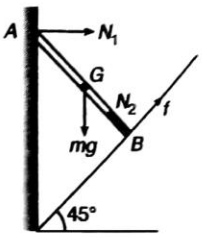
$$a_1 = a_2 \sin \alpha \qquad (a_2 \neq a_1 \sin \alpha, \text{ think why?})$$

Solving above three equations, we get

acceleration of rod, $$a_1 = \frac{mg \cos \alpha \sin \alpha}{\left(m \sin \alpha + \dfrac{M}{\sin \alpha}\right)}$$ **Ans.**

and acceleration of wedge $$a_2 = \frac{mg \cos \alpha}{m \sin \alpha + \dfrac{M}{\sin \alpha}}$$ **Ans.**

6. (a) N_2 and mg pass through G. N_1 has clockwise moment about G, so the ladder has a tendency to slip by rotating clockwise and the force of friction (f) at B is then up the plane.



(b) $$\Sigma M_A = 0$$

$\therefore$ $$fl = mg\left(\frac{l}{2} \sin 45°\right) \quad \text{...(i)}$$

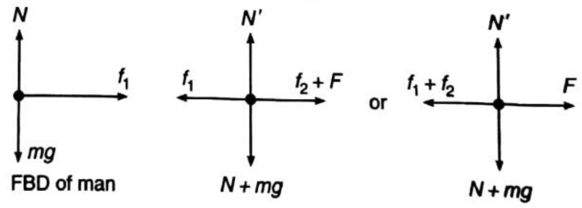
$\Sigma F_V = 0$

$\therefore$ $$mg = N_2 \cos 45° + f \sin 45° \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $$N_2 = \frac{3}{2\sqrt{2}} mg \quad \text{and} \quad f = \frac{mg}{2\sqrt{2}}$$

or $$\mu_{min} = \frac{f}{N_2} = \frac{1}{3}$$ **Ans.**

7.



FBD of man

FBD of plank

Here f_1 = force of friction between man and plank and f_2 = force of friction between plank and surface.
For the plank not to move

$$F - (f_2)_{max} \leq f_1 \leq F + (f_2)_{max}$$

or $$F - \mu (M + m)g \leq ma \leq F + \mu (M + m)g$$

or $$a \text{ should lie between } \frac{F}{m} - \frac{\mu (M + m)g}{m}$$

and $$\frac{F}{m} + \frac{\mu (M + m)g}{m}$$ **Ans.**

8. Writing equations of motion:

For M: $$5T - Mg = Ma_1 \quad \text{...(i)}$$

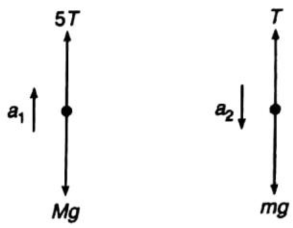
For m: $$mg - T = ma_2 \quad \text{...(ii)}$$

From constraint equation, $$a_2 = 5a_1 \quad \text{...(iii)}$$

Solving these equations, we get

acceleration of M, $$a_1 = \left(\frac{5m - M}{25m + M}\right) g$$

and of m, $$a_2 = 5\left(\frac{5m - M}{25m + M}\right) g$$



FBD of M FBD of m

9. $2a_1s_1 = 2a_2s_2$

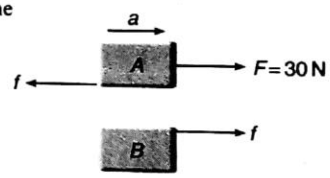
or $$\frac{a_1}{a_2} = \frac{s_2}{s_1} = \frac{n}{m} \quad \text{or} \quad \frac{g \sin \alpha}{\mu g \cos \alpha - g \sin \alpha} = \frac{m}{n}$$

Solving it, we get $$\mu = \left(\frac{m + n}{m}\right) \tan \alpha$$

Ans.

10. Limiting friction between A and B $\quad f_L = \mu N = 0.4 \times 100 = 40$ N

(a) Both the blocks will have a tendency to move together with same acceleration (say a). So, the force diagram is as shown.



Equations of motion are,

$$30 - f = 10 \times a \quad \text{...(i)}$$

$$f = 25 \times a \quad \text{...(ii)}$$

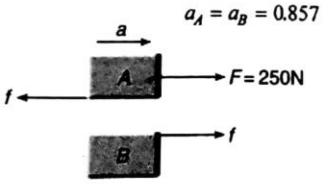
Solving these two equations, we get

$$a = 0.857 \text{ m/s}^2 \quad \text{and} \quad f = 21.42 \text{ N}$$

As this force is less than f_L, both the blocks will move together with same acceleration,

$$a_A = a_B = 0.857 \text{ m/s}^2$$

(b)



$$250 - f = 10a \quad \text{...(iii)}$$

$$f = 25a \quad \text{...(iv)}$$

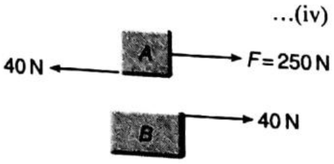
Solving Eqs. (iii) and (iv), we get $\quad f = 178.6$ N

As $f > f_L$, slipping will take place between the two blocks and

$$f = f_L = 40 \text{ N}$$

$$a_A = \frac{250 - 40}{10} = 21.0 \text{ m/s}^2$$

$$a_B = \frac{40}{25} = 1.6 \text{ m/s}^2$$



Ans.

11. Normal reaction between A and B would be $N = mg \cos \theta$. Its horizontal component is $N \sin \theta$. Therefore, tension in cord CD is equal to this horizontal component.

Hence, $$T = N \sin \theta = (mg \cos\theta)(\sin \theta) = \frac{mg}{2} \sin 2\theta$$

Ans.

12. Assuming that mass of truck >> mass of crate.

Retardation of truck $a_1 = (0.9)\,g = 9\ \text{m/s}^2$, retardation of crate $a_2 = (0.7)\,g = 7\ \text{m/s}^2$

or relative acceleration of crate $a_r = 2\,\text{m/s}^2$.

Truck will stop after time $t_1 = \dfrac{15}{9} = 1.67$ sec and crate will strike the wall at

$$t_2 = \sqrt{\frac{2s}{a_r}} = \sqrt{\frac{2 \times 3.2}{2}} = 1.78\ \text{s}$$

As $t_2 > t_1$, crate will come to rest after travelling a distance

$$s = \frac{1}{2}\,a_r t_1^2 = \frac{1}{2} \times 2.0 \times \left(\frac{15}{9}\right)^2 = 2.77\ \text{m}$$ **Ans.**

13. $\mu_k mg = 0.2 \times 10 \times 10 = 20\ \text{N}$

For t ≤ 0.2 sec :

Retardation $$a_1 = \frac{F + \mu_k mg}{m} = \frac{20 + 20}{10} = 4\ \text{m/s}^2$$

At the end of 0.2 sec,

$$v = u - a_1 t = 1.2 - 4 \times 0.2 = 0.4\ \text{m/s}$$

For t > 0.2 sec :

Retardation $$a_2 = \frac{10 + 20}{10} = 3\ \text{m/s}^2$$

Block will come to rest after time $t_0 = \dfrac{v}{a_2} = \dfrac{0.4}{3} = 0.13\ \text{s}$

∴ Total time $= 0.2 + 0.13 = 0.33\ \text{s}$ **Ans.**

14. Block will start moving at, $F = \mu\, mg$

or $$25t = (0.5)\,(10)\,(9.8) = 49\ \text{N}$$

∴ $$t = 1.96\ \text{s}$$

Velocity is maximum at the end of 4 second.

∴ $$\frac{dv}{dt} = \frac{25t - 49}{10} = 2.5t - 4.9$$

∴ $$\int_0^{v_{max}} dv = \int_{1.96}^{4} (2.5\,t - 4.9)\,dt$$

∴ $$v_{max} = 5.2\ \text{m/s}$$ **Ans.**

For 4 s < t < 7 s

Net retardation $$a_1 = \frac{49 - 40}{10} = 0.9\ \text{m/s}^2$$

∴ $$v = v_{max} - a_1 t_1 = 5.2 - 0.9 \times 3 = 2.5\ \text{m/s}$$

For $t > 7$ s

$$\text{Retardation } a_2 = \frac{49}{10} = 4.9\ \text{m/s}^2$$

∴ $$t = \frac{v}{a_2} = \frac{2.5}{4.9} = 0.51\ \text{s}$$

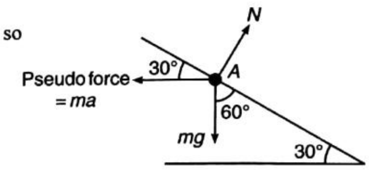
∴ $$\text{Total time} = (4 - 1.96) + (7 - 4) + (0.51)$$
$$= 5.55\ \text{s}$$ **Ans.**

15. Let B and C both move upwards (alongwith their pulleys) with speeds v_B and v_C then we can see that, A will move downward with speed, $2v_B + 2v_C$. So, with sign we can write,

$\therefore$
$$v_B = \frac{v_A}{2} - v_c$$

Substituting the values we have, $v_B = 0$ **Ans.**

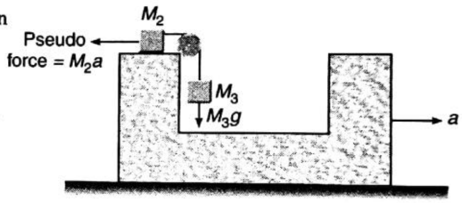
16. FBD of A with respect to frame is shown in figure..
A is in equilibrium under three concurrent forces shown in figure, so applying Lami's theorem



$$\frac{ma}{\sin(90+60)} = \frac{mg}{\sin(90+30)}$$

$\therefore$
$$a = \frac{g\cos 60°}{\cos 30°} = 5.66 \text{ m/s}^2$$

17. FBD of M_2 and M_3 in accelerated frame of reference is shown in figure.



Note: Only the necessary forces have been shown.

Mass M_3 will neither rise nor fall if net pulling force is zero.

i.e.,
$$M_2 a = M_3 g$$

or
$$a = \frac{M_3}{M_2} g$$

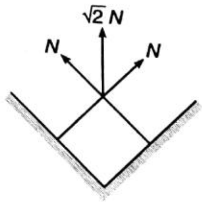
$\therefore$
$$F = (M_1 + M_2 + M_3)\, a = (M_1 + M_2 + M_3)\frac{M_3}{M_2} g$$ **Ans.**

18. Retardation $a = \mu_k g = 0.15 \times 9.8 = 1.47 \text{ m/s}^2$

Distance travelled before sliding stops is, $s = \frac{v^2}{2a} = \frac{(5)^2}{2 \times 1.47}$

$$\approx 8.5 \text{ m}$$ **Ans.**

19. $\sqrt{2}\, N = mg\cos\theta$



$\therefore$
$$N = \frac{mg\cos\theta}{\sqrt{2}}$$

$$a = \frac{mg\sin\theta - 2\mu_k N}{m} = g\sin\theta - \sqrt{2}\,\mu_k g\cos\theta$$

$$= g(\sin\theta - \sqrt{2}\mu_k\cos\theta)$$ **Ans.**

20. $v \cdot \frac{dv}{dx} = \frac{\text{Net force}}{\text{mass}} = \frac{F - \mu_k \rho (L-x) g}{\rho L}$

$\therefore$
$$\int_0^v v\, dv = \int_0^L \frac{F - \mu_k \rho (L-x) g}{\rho L} dx$$

$\therefore$
$$\frac{v^2}{2} = \frac{F}{\rho} - \mu_k gL + \frac{\mu_k gL}{2}$$

$\therefore$
$$v = \sqrt{\frac{2F}{\rho} - \mu_k gL}$$ **Ans.**

21. (a) $v = a_1 t_1 = 2.6 \text{ m/s}$

$$s_1 = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} \times 2 \times (1.3)^2 = 1.69 \text{ m}$$

$$s_2 = (2.2 - 1.69) = 0.51 \text{ m}$$

Now, $$s_2 = \frac{v^2}{2a_2}$$

$\therefore$ $$a_2 = \frac{v^2}{2s_1} = \frac{(2.6)^2}{2 \times 0.51} = 6.63 \text{ m/s}^2$$

and $$t_2 = \frac{v}{a_2} = 0.4 \text{ s}$$

(b) Acceleration of package will be 2m/s^2 while retardation will be $\mu_k g$ or 2.5 m/s^2 not 6.63 m/s^2.

For the package,

$$v = a_1 t_1 = 2.6 \text{ m/s}$$

$$s_1 = \frac{1}{2} a_1 t_1^2 = 1.69 \text{ m}$$

$$s_2 = v t_2 - \frac{1}{2} a_2' t_2^2 = 2.6 \times 0.4 - \frac{1}{2} \times 2.5 \times (0.4)^2$$

$$= 0.84 \text{ m}$$

$\therefore$ Displacement of package w.r.t. belt $= (0.84 - 0.51)$ m

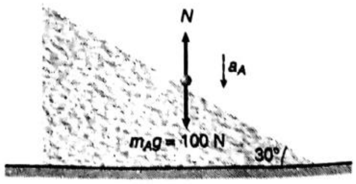
$= 0.33$ m **Ans.**

Alternate Sol : For last 0.4 seconds

$$|a_r| = 6.63 - 2.5 = 4.13 \text{ m/s}^2$$

$\therefore$ $$s_r = \frac{1}{2}|a_r| t^2 = \frac{1}{2} \times 4.13 \times (0.4)^2 = 0.33 \text{ m}$$

22. Free body diagram of crate A w.r.t ground is shown in figure.



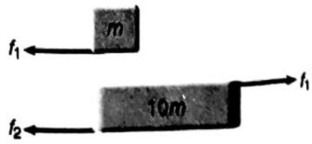
Equation of motion is: $100 - N = 10 a_A$ ...(i)

$$a_A = a \sin 30° = (2)\left(\frac{1}{2}\right)$$

or $$a_A = 1 \text{m/s}^2$$

Substituting in Eq. (i), we get $N = 90$ newton.

23. (a) Force of friction at different contacts are shown in figure.



Here, $f_1 = \mu_2 mg$ and $f_2 = \mu_1 (11\, mg)$

Given that $$\mu_2 > 11 \mu_1$$

$\therefore$ $$f_1 > f_2$$

Retardation of upper block $$a_1 = \frac{f_1}{m} = \mu_2 g$$

Acceleration of lower block $$a_2 = \frac{f_1 - f_2}{m} = \frac{(\mu_2 - 11\mu_1)g}{10}$$

Relative retardation of upper block $$a_r = a_1 + a_2$$

or $$a_r = \frac{11}{10}(\mu_2 - \mu_1)g$$

Now, $$0 = v_{min}^2 - 2a_r l$$

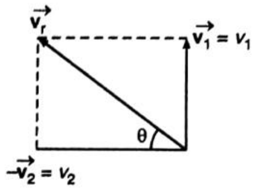
$\therefore$ $$v_{min} = \sqrt{2a_r l} = \sqrt{\frac{22(\mu_2 - \mu_1)gl}{10}}$$ **Ans.**

(b) $$0 = v_{min} - a_r t$$

or $$t = \frac{v_{min}}{a_r} = \sqrt{\frac{20l}{11(\mu_2 - \mu_1)g}}$$ **Ans.**

24. $v_r = \sqrt{v_1^2 + v_2^2}$

Retardation $a = \mu g$



$\therefore$ Time when slipping will stop is $t = \frac{v_r}{a}$ or $t = \frac{\sqrt{v_1^2 + v_2^2}}{\mu g}$ ...(i)

$$s_r = \frac{v_r^2}{2a} = \frac{v_1^2 + v_2^2}{2\mu g}$$

$$x_r = -s_r \cos\theta = -\left(\frac{v_1^2 + v_2^2}{2\mu g}\right)\left(\frac{v_2}{\sqrt{v_1^2 + v_2^2}}\right) = \frac{-v_2\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

$$y_r = s_r \sin\theta = \left(\frac{v_1^2 + v_2^2}{2\mu g}\right)\left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}\right) = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

In time t, belt will move a distance

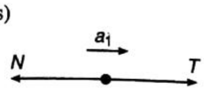
$s = v_2 t$ or $\frac{v_2\sqrt{v_1^2 + v_2^2}}{\mu g}$ in x-direction. Hence, coordinate of particle,

$$x = x_r + s = \frac{v_2\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

and $$y = y_r = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$$ **Ans.**

25. FBD of m_1 (showing only the horizontal forces)

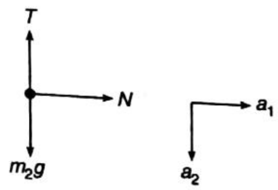
Equation of motion for m_1 is



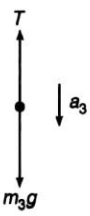
$$T - N = m_1 a_1 \quad ...(i)$$

Equations of motion for m_2 are $$N = m_2 a_1 \quad ...(ii)$$

and $$m_2 g - T = m_2 a_2 \quad ...(iii)$$



FBD of m_2



Equation of motion for m_3 are

$$m_3g - T = m_3a_3 \quad ...(iv)$$

Further from constraint equation we can find the relation,

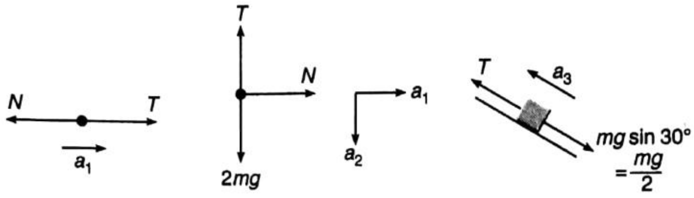
$$a_1 = a_2 + a_3 \quad ...(v)$$

We have five unknowns a_1, a_2, a_3, T and N solving, we get

$$a_1 = \frac{2m_1m_3\,g}{(m_2 + m_3)(m_1 + m_2) + m_2m_3}$$

Ans.

26.



FBD of $3m$ FBD of $2m$ FBD of m

Writing equations of motion,

$$T - N = 3ma_1 \quad ...(i)$$

$$N = 2ma_1 \quad ...(ii)$$

$$2mg - T = 2ma_2 \quad ...(iii)$$

$$T - \frac{mg}{2} = ma_3 \quad ...(iv)$$

From constraint equation,

$$a_1 = a_2 - a_3 \quad ...(v)$$

We have five unknowns. Solving the above five equations, we get

$$a_1 = \frac{3}{17}g, \quad a_2 = \frac{19}{34}g \quad \text{and} \quad a_3 = \frac{13}{34}g$$

Acceleration of $m = a_3 = \frac{13}{34}g$, acceleration of $2m = \sqrt{a_1^2 + a_2^2} = \frac{\sqrt{397}}{34}g$

and acceleration of $3m = a_1 = \frac{3}{17}g$

Ans.

27. $a = \dfrac{m_Ag}{m_A + M + m}$

For the equilibrium of B,

$$mg = \mu N = \mu(ma) = \frac{\mu m m_A g}{m_A + M + m}$$

$$\therefore \quad m_A = \frac{(M + m)m}{(\mu - 1)m}$$

$$m_A = \frac{(M + m)}{\mu - 1}$$

Ans.

Note $m_A > 0 \quad \therefore \quad \mu > 1$.

Chapter 6

Work, Energy & Power

Level 2

1.

$$(F - \mu m_1 g)x_m = \frac{1}{2}kx_m^2$$

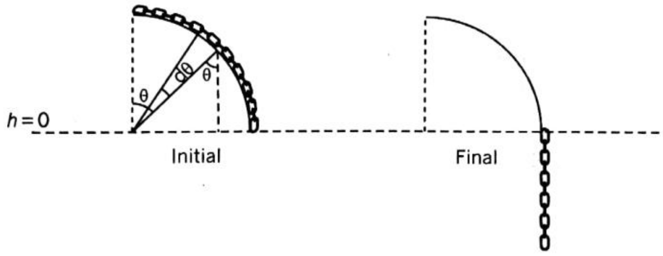
or $$kx_m = 2(F - \mu m_1 g)$$

Second block will shift if $kx_m \geq \mu m_2 g$

∴ $$2(F - \mu m_1 g) > \mu m_2 g$$

or $$F > \left(m_1 + \frac{m_2}{2}\right)\mu g$$ **Ans.**

2.



Initial PE, $$U_i = \int_{\theta=0°}^{\theta=\pi/2} (r\, d\theta)(\rho)(g)(r\cos\theta)$$

$$= (\rho g r^2)[\sin\theta]_0^{\pi/2} = \rho g r^2$$

Final PE

$$U_f = \left(\frac{\pi r}{2} \times \rho\right)(g)\left(-\frac{\pi r/2}{2}\right) = -\frac{\pi^2 r^2 \rho g}{8}$$

$$\Delta U = r^2 \rho g \left(1 + \frac{\pi^2}{8}\right)$$

$$\Delta U = \text{KE}$$

or $$r^2 \rho g\left(1 + \frac{\pi^2}{8}\right) = \frac{1}{2}\left(\frac{\pi r}{2}\right)(\rho) v^2$$

⇒ $$v = \sqrt{4rg\left(\frac{1}{\pi} + \frac{\pi}{8}\right)}$$

or $$v = \sqrt{rg\left(\frac{\pi}{2} + \frac{4}{\pi}\right)}$$ **Ans.**

3. For $t \le 0.2$ second

$$F = 800 \text{ N} \quad \text{and} \quad v = \left(\frac{20}{0.3}\right) t$$

$\therefore$ $$P = Fv = (53.3\, t) \text{ kW}$$ **Ans.**

For $t > 0.2$ second

$$F = 800 - \left(\frac{800}{0.1}\right)(t - 0.2)$$

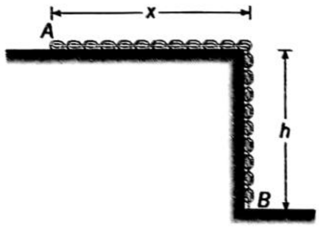
and $$v = \frac{20}{0.3} t$$

$\therefore$ $$P = Fv = (160\, t - 533\, t^2) \text{ kW}$$

$$W = \int_0^{0.2} (53.3\, t)\, dt + \int_{0.2}^{0.3} (160\, t - 533\, t^2)\, dt$$

$$= 1.69 \text{ kJ}$$ **Ans.**

4. At the instant shown in figure, net pulling force $= \frac{m}{l} gh$



Total mass being pulled $$= \frac{m}{l}(x + h)$$

$\therefore$ Acceleration $a = \dfrac{\text{net pulling force}}{\text{total mass being pulled}}$

$$= \frac{gh}{x + h}$$

$\therefore$ $$v\left(-\frac{dv}{dx}\right) = \frac{gh}{x + h}$$

or $$\int_0^v v \cdot dv = -gh \int_{(l-h)}^{0} \frac{dx}{x + h}$$

$\therefore$ $$\frac{v^2}{2} = gh\,[\ln(x + h)]_0^{l-h}$$

or $$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right)$$

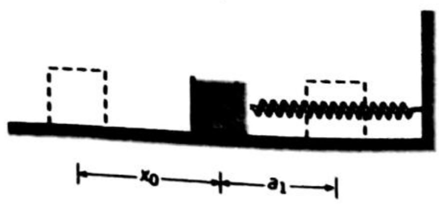
$\therefore$ $$v = \sqrt{2gh \ln(l/h)}$$ **Ans.**

5. (a) From energy conservation principle:

Work done against friction = decrease in elastic P.E.

or $$f(x_0 + a_1) = \frac{1}{2} k(x_0^2 - a_1^2)$$

or $$x_0 - a_1 = \frac{2f}{k} \quad \text{...(i)}$$



From Eq. (i), we see that decrease of amplitude $(x_0 - a_1)$ is $\frac{2f}{k}$, which is constant and same for each cycle of oscillation

(b) The block will come to rest when $$ka = f$$

or $$a = \frac{k}{f} \qquad \text{...(A)}$$

In the similar manner we can write

$$a_1 - a_2 = \frac{2f}{k} \qquad \text{...(ii)}$$

$$a_2 - a_3 = \frac{2f}{k} \qquad \text{...(iii)}$$

$$\dots\dots$$

$$a_{n-1} - a_n = \frac{2f}{k} \qquad \text{...(n)}$$

Adding Eqs. (i), (ii),... etc., we get $$x_0 - a_n = n\left(\frac{2f}{k}\right)$$

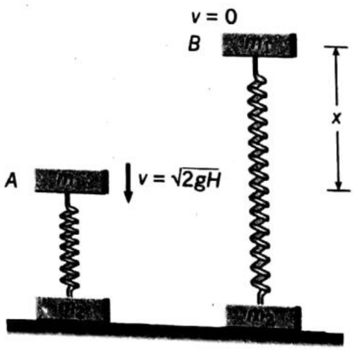
or $$a_n = x_0 - n\left(\frac{2f}{k}\right) \qquad \text{...(B)}$$

Equating Eq. (A) and (B), we get $$\frac{k}{f} = x_0 - n\left(\frac{2f}{k}\right)$$

or $$n = \frac{x_0 - \frac{k}{f}}{\frac{2f}{k}} = \frac{kx_0}{2f} - \frac{1}{2}$$

Number of cycles, $$m = \frac{n}{2} = \frac{kx_0}{4f} - \frac{1}{4} = \frac{1}{4}\left(\frac{kx_0}{f} - 1\right)$$ **Ans.**

6. Conservation of mechanical energy gives,



$$E_A = E_B$$

or $$\frac{1}{2} m_1 v^2 = \frac{1}{2} kx^2 + m_1 gx$$

or $$2m_1 gH = kx^2 + 2m_1 gx \qquad \text{...(i)}$$

The lower block will rebounce when

$$x > \frac{m_2 g}{k} \qquad (kx = m_2 g)$$

Substituting, $x = \frac{m_2 g}{k}$ in Eq. (i), we get

$$2m_1 gH = k\left(\frac{m_2 g}{k}\right)^2 + 2m_1 g\left(\frac{m_2 g}{k}\right)$$

or $$H = \frac{m_2 g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right)$$

Thus, $$H_{\min} = \frac{m_2 g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right)$$ **Ans.**

7. $E_i = E_f$

$\therefore$ $$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}kx^2$$

or $$k = \frac{3v^2m}{4x^2}$$ **Ans.**

8. $$\mu m_A g = 0.8 \times 6 \times 10 = 48\ \text{N}$$

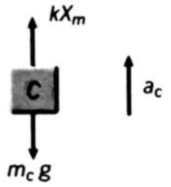
$$(m_B + m_C)g = (1 + 2) \times 10 = 30\ \text{N}$$

Since $$(m_B + m_C)g > \mu m_A g,\ a_A = a_B = 0.$$

From conservation of energy principle we can prove that maximum distance moved by C or maximum extension in the spring would be :

$$X_m = \frac{2m_C g}{k} = \frac{2 \times 1 \times 10}{1000} = 0.02\ \text{m}$$

At maximum extension :



$$a_C = \frac{kX_m - m_C g}{m_C}$$

Substituting the values we have, $a_C = 10\ \text{m/s}^2$. **Ans.**

9. Rate at which kinetic plus gravitational potential energy is dissipated at time t is actually the magnitude of power of frictional force at time t.

$$|P_f| = f.v = (\mu mg \cos\alpha)(at)$$
$$= (\mu mg \cos\alpha)[(g \sin\alpha - \mu g \cos\alpha)t]$$
$$= \mu mg^2 \cos\alpha(\sin\alpha - \mu \cos\alpha)t$$ **Ans.**

10. From work-energy principle, $$W = \Delta KE$$

$\therefore$ $$Pt = \frac{1}{2}m(v^2 - u^2)$$ (P = power)

or $$t = \frac{m}{2P}(v^2 - u^2) \quad \text{...(i)}$$

Further

$$F \cdot v = P$$

$\therefore$ $$m \cdot \frac{dv}{ds} \cdot v^2 = P$$

or $$\int_u^v v^2\, dv = \frac{P}{m}\int_0^x ds$$

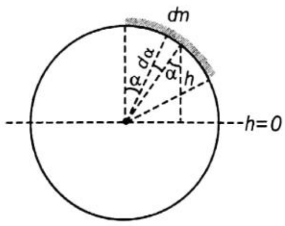
$\therefore$ $$(v^3 - u^3) = \frac{3P}{m} \cdot x$$

or $$\frac{m}{P} = \frac{3x}{v^3 - u^3}$$

Substituting in Eq. (i)

$$t = \frac{3x\,(u + v)}{2\,(u^2 + v^2 + uv)}$$ **Hence proved.**

11. (a) Mass per unit length $= \frac{m}{l}$



$$dm = \frac{m}{l} R\, d\alpha$$

$$h = R \cos \alpha$$

$$dU = (dm)\, gh = \frac{mgR^2}{l} \cos \alpha \cdot d\alpha$$

$\therefore$ $$U = \int_0^{l/R} dU = \frac{mgR^2}{l} \sin\left(\frac{l}{R}\right)$$

(b) $$\text{KE} = U_i - U_f$$

Here $$U_i = \frac{mgR^2}{l} \sin\left(\frac{l}{R}\right)$$

and $$U_f = \int_\theta^{l/R+\theta} dU = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}+\theta\right) - \sin\theta\right]$$

$\therefore$ $$\text{KE} = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$ **Ans.**

(c) $$\frac{1}{2} mv^2 = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

or $$v = \sqrt{\frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]}$$

$$v^2 = \frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

$\therefore$ $$2v \cdot \frac{dv}{dt} = \frac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right] \cdot \frac{d\theta}{dt}$$

$$\frac{dv}{dt} = \frac{\frac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]}{2v}\left(\frac{d\theta}{dt}\right) \quad \ldots\text{(i)}$$

Here $$\frac{\left(\frac{d\theta}{dt}\right)}{v} = \frac{\omega}{v} = \frac{1}{R}$$

Substituting in Eq. (i) $$\frac{dv}{dt} = \frac{gR}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]$$

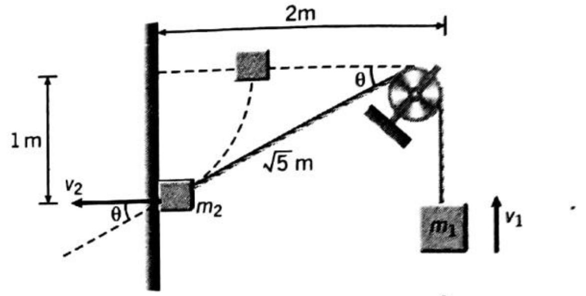
At $$t = 0,\ \theta = 0°$$

Hence, $$\frac{dv}{dt} = \frac{gR}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right]$$ **Ans.**

12. From conservation of energy,

$$m_2 g h_2 = m_1 g h_1 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Here, $$v_1 = v_2 \cos\theta$$



$\therefore$ $$2 \times 10 \times 1 = (0.5)(10)(\sqrt{5} - 1) + \frac{1}{2} \times 0.5 \times v_2^2 \times \left(\frac{2}{\sqrt{5}}\right)^2 + \frac{1}{2} \times 2 \times v_2^2$$

$\therefore$ $$20 = 6.18 + 0.2\, v_2^2 + v_2^2$$

$$v_2 = 3.39 \text{ m/s}$$ **Ans.**

and $$v_1 = v_2 \cos\theta = \frac{2}{\sqrt{5}} \times 3.39$$

or $$v_1 = 3.03 \text{ m/s}$$ **Ans.**

13. Net retarding force $= kx + bMgx$

$\therefore$ $$\text{Net retardation} = \left(\frac{k + bMg}{M}\right) \cdot x$$

So, we can write $$v \cdot \frac{dv}{dx} = -\left(\frac{k + bMg}{M}\right) \cdot x$$

or $$\int_{v_0}^{0} v \cdot dv = -\left(\frac{k + bMg}{M}\right) \int_0^x x\, dx$$

or $$x = \sqrt{\frac{M}{k + bMg}}\, v_0$$

Loss in mechanical energy $$\Delta E = \frac{1}{2} M v_0^2 - \frac{1}{2} kx^2$$

or $$\Delta E = \frac{1}{2} M v_0^2 - \frac{k}{2}\left(\frac{M}{k + bMg}\right) v_0^2$$

or $$\Delta E = \frac{v_0^2}{2}\left[M - k\left(\frac{M}{k + bMg}\right)\right] = \frac{v_0^2 b M^2 g}{2(k + bMg)}$$ **Ans.**

14. From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2} kx_i^2 + mgh_i = \frac{1}{2} m v_f^2$$

$\therefore$ $$v_f = \sqrt{2gh_i + \frac{k}{m} x_i^2}$$

Substituting the values we have,

$$v_f = \sqrt{2 \times 9.8 \times 1.9 + \frac{2300}{0.12}(0.045)^2}$$

$$= 8.72 \text{ m/s}$$ **Ans.**

15. (a) From work energy theorem,
Work done by all forces = change in kinetic energy

$\therefore$ $$Fx - \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$\therefore$ $$v = \sqrt{\frac{2Fx - kx^2}{m}}$$

Substituting the values we have,

$$v = \sqrt{\frac{2 \times 20 \times 0.25 - 40 \times 0.25 \times 0.25}{0.5}}$$

$$= \sqrt{15} \text{ m/s} = 3.87 \text{ m/s}$$ **Ans.**

(b) From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}kx_f^2$$

or $$x_f = \sqrt{\frac{mv_i^2}{k} + x_i^2}$$

$$= \sqrt{\frac{0.5 \times 15}{40} + (0.25)^2}$$

$$= 0.5 \text{ m (compression)}$$

$\therefore$ Distance of block from the wall = (0.6 – 0.5) m = 0.1 m **Ans.**

Chapter 7 Circular Motion

Level 2

1. (a) Applying conservation of energy

$$mgh = \frac{1}{2} m (\sqrt{3Lg})^2$$

$$\therefore \quad h = \frac{3L}{2} \quad \textbf{Ans.}$$

(b) Since $\sqrt{3Lg}$ lies between $\sqrt{2Lg}$ and $\sqrt{5Lg}$, the string will slack in upper half of the circle. Assuming that string slacks when it makes an angle θ with horizontal. We have

$$mg \sin \theta = \frac{mv^2}{L} \quad \text{...(i)}$$

$$v^2 = (\sqrt{3gL})^2 - 2gL (1 + \sin \theta) \quad \text{...(ii)}$$

Solving Eq. (i) and (ii), we get

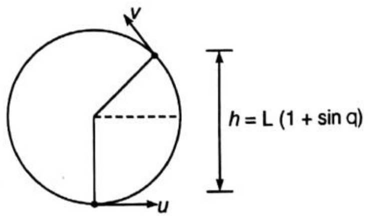
$$\sin \theta = \frac{1}{3} \quad \text{and} \quad v^2 = \frac{gL}{3}$$

Maximum height of the bob from starting point,

$$H = L (1 + \sin \theta) + \frac{v^2 \sin^2 (90° - \theta)}{2g}$$

$$= \frac{4L}{3} + \left(\frac{gL}{6g}\right) \cos^2 \theta = \frac{4L}{3} + \frac{4L}{27}$$

$$= \frac{40}{27} L \quad \textbf{Ans.}$$



Note *Maximum height in part (b) is less than that in part (a), think why?*

2. $$h = 0.8 \sin 30° = 0.4 \text{ m}$$

$$\therefore \quad v^2 = 2gh$$

(a) Just before,

$$T_1 - mg \sin 30° = \frac{mv^2}{R_1} \quad (R_1 = 0.8 \text{ m})$$

$$T_1 = \frac{mg}{2} + \frac{m (2g) (0.4)}{0.8} = \frac{3mg}{2} \quad \textbf{Ans.}$$

(b) Just after,

$$T_2 - mg \sin 30° = \frac{mv^2}{R_2} (R_2 = 0.4 \text{ m})$$

$$T_2 = \frac{mg}{2} + \frac{m (2g) (0.4)}{0.4}$$

or

$$T_2 = \frac{5mg}{2} \quad \textbf{Ans.}$$

3. $h = l(1 - \cos\theta)$

$$v^2 = v_0^2 - 2gh = 3gl - 2gl(1 - \cos\theta) = gl(1 + 2\cos\theta)$$

At 45° means radial and tangential components of acceleration are equal.

$\therefore$ $$\frac{v^2}{l} = g\sin\theta$$

or $$1 + 2\cos\theta = \sin\theta$$

Solving the equation we get, $\theta = 90°$ or $\frac{\pi}{2}$ **Ans.**

4. Banking angle, $\theta = \tan^{-1}\left(\frac{v^2}{Rg}\right)$

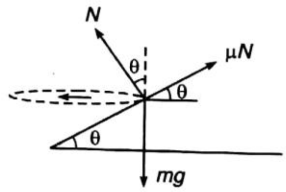
$$36\,\frac{\text{km}}{\text{h}} = 10 \text{ m/s}$$

$\therefore$ $$\theta = \tan^{-1}\left(\frac{100}{20 \times 9.8}\right) = 27°$$

Angle of repose, $\theta_r = \tan^{-1}(\mu) = \tan^{-1}(0.4) = 21.8°$

Since $\theta > \theta_r$, vehicle can not remain in the given position with $v = 0$. At rest it will slide down. To find minimum speed, so that vehicle does not slip down, maximum friction will act up the plane. To find maximum speed, so that the vehicle does not skid up, maximum friction will act down the plane.

Minimum Speed :



Equation of motion are,

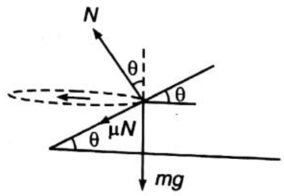
$$N\cos\theta + \mu N\sin\theta = mg \quad \text{...(i)}$$

$$N\sin\theta - \mu N\cos\theta = \frac{m}{R}v_{\min}^2 \quad \text{...(ii)}$$

Solving these two equations we get

$$v_{\min} = 4.2 \text{ m/s}$$ **Ans.**

Maximum speed :



Equations of motion are,

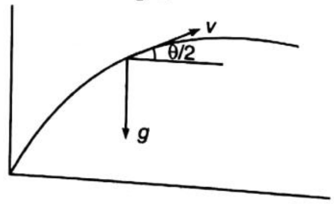
$$N\cos\theta - \mu N\sin\theta = mg \quad \text{...(iii)}$$

$$N\sin\theta + \mu N\cos\theta = \frac{m}{R}v_{\max}^2 \quad \text{...(iv)}$$

Solving these two equations, we have,

$$v_{\max} = 15 \text{ m/s}$$ **Ans.**

5. Let v be the velocity at that instant. Then, horizontal component of velocity remains unchanged.



$\therefore$ $$v\cos\frac{\theta}{2} = u\cos\theta$$

or $$v = \frac{u\cos\theta}{\cos\frac{\theta}{2}}$$

Tangential component of acceleration of this instant will be,

$$a_t = g\cos(\pi/2 + \theta/2) = -g\sin\theta/2$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{g^2 - g^2\sin^2\frac{\theta}{2}} = g\cos\frac{\theta}{2}$$

Since, $$a_n = \frac{v^2}{R}$$

or $$R = \frac{v^2}{a_n} = \frac{\left(\dfrac{u\cos\theta}{\cos\dfrac{\theta}{2}}\right)^2}{g\cos\dfrac{\theta}{2}} = \frac{u^2\cos^2\theta}{g\cos^3\left(\dfrac{\theta}{2}\right)}$$ **Ans.**

6. **After 1 sec :** $\vec{v} = \vec{u} + \hat{a}t = 20\,\hat{i} + 10\,\hat{j}$, $v = \sqrt{500}$ m/s $= 10\sqrt{5}$ m/s

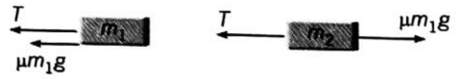
$$\vec{a} = -10\,\hat{j}$$

$$a_t = a\cos\theta = \frac{\vec{a}.\vec{v}}{v} = \frac{-100}{10\sqrt{5}} = -2\sqrt{5}\ \text{m/s}^2$$ **Ans.**

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(10)^2 - (2\sqrt{5})^3} = \sqrt{80}\ \text{m/s}^2 = 4\sqrt{5}\ \text{m/s}^2$$ **Ans.**

$$R = \frac{v^2}{a_n} = \frac{(10\sqrt{5})^2}{4\sqrt{5}} = 25\sqrt{5}\ \text{m}.$$ **Ans.**

7. (a) Force diagrams of m_1 and m_2 are as shown below:



(Only horizontal forces have been shown)
Equations of motion are :

$$T + \mu m_1 g = m_1 R\omega^2 \quad \text{...(i)}$$

$$T - \mu m_1 g = m_2 R\omega^2 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have

$$\omega = \sqrt{\frac{2m_1\mu g}{(m_1 - m_2)R}}$$

Substituting the values, we have

$$\omega_{min} = 6.32\ \text{rad/s}$$ **Ans.**

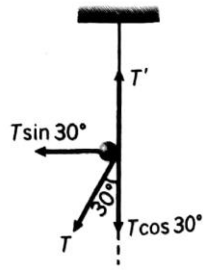
(b) $$T = m_2 R\omega^2 + \mu m_1 g$$
$$= (1)(0.5)(6.32)^2 + (0.5)(2)(10)$$
$$\approx 30\ \text{N}$$ **Ans.**

8. Speed of bob in the given position,

$$v = \sqrt{2gh}$$

Here, $$h = (400 + 400\cos 30°)\ \text{mm}$$
$$= 746\ \text{mm} = 0.746\ \text{m}$$

$\therefore$ $$v = \sqrt{2 \times 9.8 \times 0.746} = 3.82\ \text{m/s}$$



Now $$T - mg\cos\theta = \frac{mv^2}{r}$$

$$T = 2 \times 9.8 \times \cos 30° + \frac{2 \times (3.82)^2}{(0.4)}$$

or $$T = 90 \text{ N}$$

$\therefore$ $$R = T \sin 30° = 45 \text{ N}$$

$$T' = T \cos 30°$$ **Ans.**

9. Speed of each particle at angle θ is,

$$v = \sqrt{2gh}$$ (from energy conservation)

where $h = R(1 - \cos\theta)$

$\therefore$ $$v = \sqrt{2gR(1 - \cos\theta)}$$

$$N + mg\cos\theta = \frac{mv^2}{R}$$

or $$N + mg\cos\theta = 2mg(1 - \cos\theta)$$

or $$N = 2mg - 3mg\cos\theta \qquad \ldots\text{(i)}$$

The tube breaks its contact with ground when $2N\cos\theta > Mg$

Substituting, $$2N\cos\theta = Mg$$

or $$4mg\cos\theta - 6mg\cos^2\theta = Mg$$

Substituting, $$\theta = 60°$$

$$2mg - \frac{3mg}{2} = Mg$$

or $$\frac{M}{m} = \frac{1}{2}$$ **Ans.**

Note *Initially normal reaction on each ball will be radially outward and later it will be radially inward, so that normal reactions on tube is radially outward to break it off from the ground.*

10. At distance x from centre,
Centrifugal force = $mx\,\omega^2$
$\therefore$ acceleration $a = x\,\omega^2$

or $$v.\frac{dv}{dx} = x\,\omega^2$$

or $$\int_0^v v\,dv = \omega^2 \int_a^L x\,dx$$

or $$\frac{v^2}{2} = \frac{\omega^2}{2}(L^2 - a^2)$$

or $$v = \omega\sqrt{L^2 - a^2}$$ **Ans**

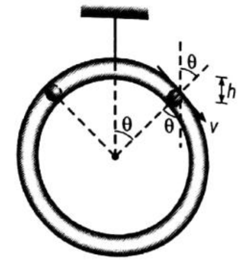
11. $$N = \frac{mv^2}{R}$$

$$f_{max} = \mu N = \frac{\mu m v^2}{R} \qquad \therefore \quad \text{Retardation } a = \frac{f_{max}}{m} = \frac{\mu v^2}{R}$$

$\therefore$ $$\left(-\frac{dv}{dt}\right) = \frac{\mu v^2}{R} \quad \text{or} \quad \int_{v_0}^{v} \frac{dv}{v^2} = -\frac{\mu}{R}\int_0^t dt$$

or $$v = \frac{v_0}{1 + \frac{\mu v_0 t}{R}}$$ **Ans.**

12. Let R be the radius of the ring



$$h = R(1 - \cos\theta)$$

$$v^2 = 2gh = 2gR(1 - \cos\theta)$$

$$\frac{mv^2}{R} = N + mg\cos\theta$$

or $$N = 2mg(1 - \cos\theta) - mg\cos\theta$$

$$N = 2mg - 3mg\cos\theta$$

In the critical condition, tension in the string is zero and net upward force on the ring :

$$F = 2N\cos\theta = 2mg(2\cos\theta - 3\cos^2\theta) \qquad \text{...(i)}$$

F is maximum when $\frac{dF}{d\theta} = 0$

or $$-2\sin\theta + 6\sin\theta\cos\theta = 0$$

or $$\cos\theta = \frac{1}{3}$$

Substituting in Eq. (i) $$F_{\max} = 2mg\left(2 \times \frac{1}{3} - 3 \times \frac{1}{9}\right) = \frac{2}{3}mg$$

$$F_{\max} > Mg$$

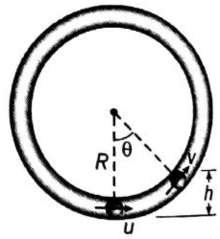
or $$\frac{2}{3}mg > Mg$$

or $$m > \frac{3}{2}M$$

Proved.

13. Minimum velocity of particle at the lowest position to complete the circle should be $\sqrt{4gR}$ inside a tube.

So, $$u = \sqrt{4gR}$$



$$h = R(1 - \cos\theta)$$

∴ $$v^2 = u^2 - 2gh$$

or $$v^2 = 4gR - 2gR(1 - \cos\theta)$$

$$= 2gR(1 + \cos\theta)$$

or $$v^2 = 2gR\left(2\cos^2\frac{\theta}{2}\right)$$

or $$v = 2\sqrt{gR}\cos\frac{\theta}{2}$$

From $$ds = v \cdot dt$$

We get $$R\,d\theta = 2\sqrt{gR}\cos\frac{\theta}{2} \cdot dt$$

or $$\int_0^t dt = \frac{1}{2}\sqrt{\frac{R}{g}}\int_0^{\pi/2}\sec\left(\frac{\theta}{2}\right)d\theta \quad \text{or} \quad t = \sqrt{\frac{R}{g}}\left[\ln\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

or $$t = \sqrt{\frac{R}{g}}\ln(1 + \sqrt{2})$$

Proved.

14 At position θ,

$$v^2 = v_0{}^2 + 2gh$$

where $h = a(1 - \cos\theta)$

$\therefore$ $v^2 = (\sqrt{2ag})^2 + 2ag(1 - \cos\theta)$

or $v^2 = 2ag(2 - \cos\theta)$ ...(i)

$$N + mg\cos\theta = \frac{mv^2}{a}$$

or $N + mg\cos\theta = 2mg(2 - \cos\theta)$

or $N = mg(4 - 3\cos\theta)$

Net vertical force, $F = N\cos\theta + mg = mg(4\cos\theta - 3\cos^2\theta + 1)$

This force (or acceleration) will be maximum when $\frac{dF}{d\theta} = 0$

or $-4\sin\theta + 6\sin\theta\cos\theta = 0$

So, either $\sin\theta = 0,\quad \theta = 0°$

or $\cos\theta = \frac{2}{3},\quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$

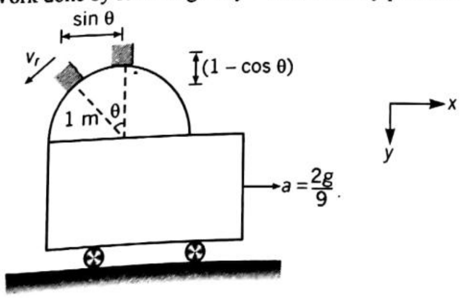
$\theta = 0°$ is unacceptable

Therefore, the desired position is at $\theta = \cos^{-1}\left(\frac{2}{3}\right)$ **Ans.**

15. (a) Let v_r be the velocity of mass relative to track at angular position θ.

From work energy theorem, KE of particle relative to track:

= Work done by force of gravity + work done by pseudo force



$\therefore$ $$\frac{1}{2}mv_r^2 = mg(1 - \cos\theta) + m\left(\frac{2g}{9}\right)\sin\theta$$

or $v_r^2 = 2g(1 - \cos\theta) + \frac{4g}{9}\sin\theta$ ...(i)

Particle leaves contact with the track where $N = 0$

or $mg\cos\theta - m\left(\frac{2g}{9}\right)\sin\theta = mv_r^2$

or $g\cos\theta - \frac{2g}{9}\sin\theta = 2g(1 - \cos\theta) + \frac{4g}{9}\sin\theta$

or $$3\cos\theta - \frac{6}{9}\sin\theta = 2$$

Solving this, we get $$\theta \approx 37^\circ$$ **Ans.**

(b) From Eq. (i), $$v_r = \sqrt{2g(1-\cos\theta) + \frac{4g}{9}\sin\theta}$$

or $$v_r = 2.58 \text{ m/s at } \theta = 37^\circ$$

Vertical component of its velocity is $$v_y = v_r \sin\theta = 2.58 \times \frac{3}{5}$$

$$= 1.55 \text{ m/s}$$

Now, $$1.3 = 1.55t + 5t^2 \qquad \left(s = ut + \frac{1}{2}gt^2\right)$$

or $$5t^2 + 1.55t - 1.3 = 0$$

or $$t = 0.38 \text{ s}$$ **Ans.**

Chapter 8 Centre of Mass, Conservation of Linear Momentum, Impulse and Collision

Level 2

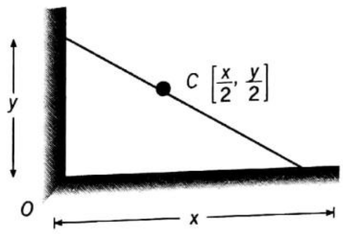
1. $$y = \sqrt{L^2 - x^2}$$

$\Rightarrow$ $$\frac{dy}{dt} = -\frac{x}{\sqrt{L^2 - x^2}}\frac{dx}{dt}$$

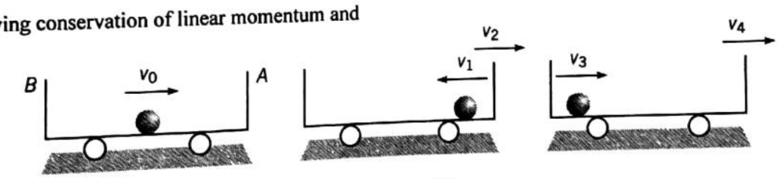
$$= -\frac{3 \times 2}{4} = -\frac{3}{2} \text{ m/s}$$

$$V_{CM} = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(1)^2 + \left(\frac{3}{4}\right)^2} = 1.25 \text{ m/s}$$ **Ans.**



2. Applying conservation of linear momentum and



$$e = \frac{\text{relative speed of separation}}{\text{relative speed of approach}},$$ we get

$$mv_0 = Mv_2 - mv_1 \quad \text{...(i)}$$

$$v_1 + v_2 = ev_0 \quad \text{...(ii)}$$

Solving these two equations, we get,

$$v_1 = \left(\frac{eM - m}{M + m}\right)v_0, \quad v_2 = m\left(\frac{e + 1}{M + m}\right)v_0$$ **Ans.**

The desired time is:

$$t = \frac{d}{v_0} + \frac{2d}{ev_0} + \frac{d}{e^2 v_0}$$

or

$$t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$$ **Ans.**

3. (i) $x_1 = v_0 t - A(1 - \cos \omega t)$

$$t\, x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = v_0 t$$

$\therefore$ $$x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos \omega t)$$

(ii) $a_1 = \frac{d^2 x_1}{dt^2} = -\omega^2 A \cos \omega t$

The separation $x_2 - x_1$ between the two blocks will be equal to l_0 when $a_1 = 0$ or $\cos \omega t = 0$

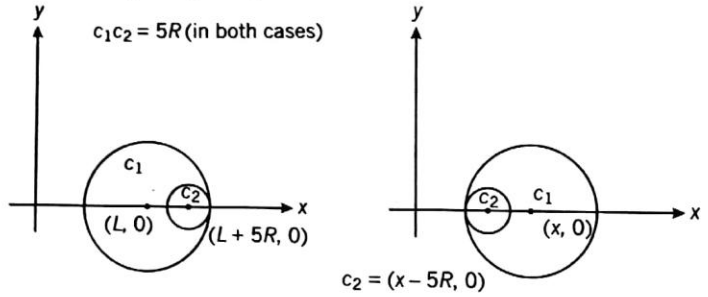
$$x_2 - x_1 = \frac{m_1}{m_2} A(1 - \cos \omega t) + A(1 - \cos \omega t)$$

or
$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A \qquad (\cos \omega t = 0)$$

Thus, the relation between l_0 and A is,

$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A \qquad \textbf{Ans.}$$

4. Since, all the surfaces are smooth, no external force is acting on the system in horizontal direction. Therefore, the centre of mass of the system in horizontal direction remains stationary.
x-coordinate of COM initially will given by



Initial

Final

$$x_i = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{(4M)(L) + M(L + 5R)}{4M + M} = (L + R) \qquad \text{...(i)}$$

Let $(x, 0)$ be the coordinates of the centre of large sphere in final position. Then, x-coordinate of COM finally will be

$$x_f = \frac{(4M)(x) + M(x - 5R)}{4M + M} = (x - R) \qquad \text{...(ii)}$$

Equating Eqs. (i) and (ii), we have

$$x = L + 2R$$

Therefore, coordinates of large sphere, when the smaller sphere reaches the other extreme position are $(L + 2R, 0)$.

Ans.

5. (a) Chain has a constant speed. Therefore, net force on it should be zero. Thus,

$$P = \text{Weight of length } y \text{ of chain} + \text{thrust force}$$

$$= \frac{m}{l} yg + \rho v_0^2 \qquad \left(\text{here } \rho = \frac{m}{l}\right)$$

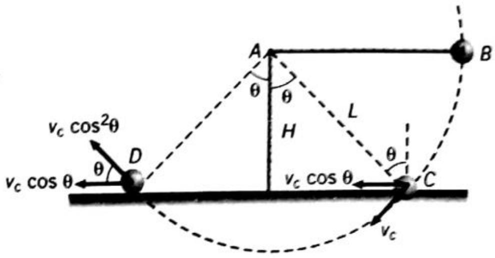
$$= \frac{m}{l}(gy + v_0^2) \qquad \textbf{Ans.}$$

(b) Energy lost during the lifting = work done by applied force – increase in mechanical energy of chain

$$= \int_0^y P \cdot dy - \left(\frac{m}{l} y\right) g \left(\frac{y}{2}\right) - \frac{1}{2}\left(\frac{m}{l} \cdot y\right) v_0^2$$

$$= \frac{myv_0^2}{2l} \qquad \textbf{Ans.}$$

6. In perfectly inelastic collision with the horizontal surface the component parallel to the surface will remain unchanged. Similarly when the string becomes taut again, the component perpendicular to its length will remain unchanged.



$$\cos\theta = \frac{H}{L}$$

$$v_c = \sqrt{2gH}$$

$$v_c \cos^2\theta = (\sqrt{2gH})\frac{H^2}{L^2} = v \text{ (say)}$$

$$\therefore \qquad h = \frac{v^2}{2g} = \frac{(2gH)\frac{H^4}{L^4}}{2g} = \frac{H^5}{L^4}$$ **Ans.**

7. Applying conservation of linear momentum at the time of collision, or at $t = 1$ s,

$$m\vec{v} + m(0) = 2m(20\hat{i} + 10\hat{j})$$

$$\therefore \qquad \vec{v} = 40\hat{i} + 20\hat{j}$$

At 1 sec, masses will be at height :

$$h_1 = u_y t + \frac{1}{2}v_y t^2 = (20)(1) + \frac{1}{2}(-10)(1)^2 = 15 \text{ m}$$

After explosion other mass will further rise to a height :

$$h_2 = \frac{u_y^2}{2g} = \frac{(20)^2}{2\times 10} = 20 \text{ m} : u_y = 20 \text{ m/s just after collisiion.}$$

$\therefore$ Total height $h = h_1 + h_2 = 35$ m **Ans.**

8. Let CT stands for common tangent direction and CN for common normal directions.

	Mass $m = eM$		Mass M	
	CT	CN	CT	CN
Befor collision	v_1 (let)	v_2 (let)	Zero (given)	Zero (given)
After collision	v_1	v_3 (suppose)	Zero	v_4 (suppose)

In the common tangent directions velocity components remain unchanged.
In common normal direction applying conservation of linear momentum and definition of e.

$$eMv_2 = eMv_3 + Mv_4 \qquad \text{...(i)}$$

From the definition of coefficient of restitution :

$$e = \frac{v_4 - v_3}{v_2} \qquad \text{...(ii)}$$

Solving these two equations we get,

$$v_3 = 0 \text{ but } v_4 \neq 0$$

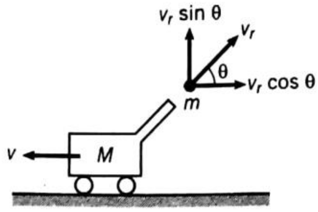
So, after collision velocity of m is along CT while that of M along CN or they are moving at right angles.

9. Muzzle velocity v_r is given to be constant
From conservation of linear momentum in horizontal direction we have,

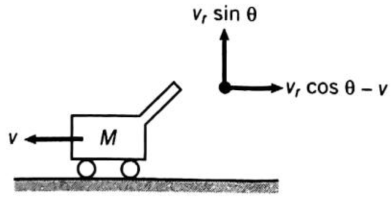
$$Mv = m(v_r \cos\theta - v)$$

$$v = \frac{mv_r \cos\theta}{M + m} \qquad \text{...(i)}$$

or



Components of velocity of bullet with respect to gun



Components of velocity of bullet with respect to ground

Further, range of bullet on horizontal ground

$$R = \frac{2v_r \sin\theta}{g}(v_r \cos\theta - v)$$

$$= \frac{2v_r \sin\theta}{g}\left(v_r \cos\theta - \frac{mv_r \cos\theta}{M + m}\right)$$

$$= \frac{2Mv_r^2 \sin\theta \cos\theta}{(M + m)g}$$

or $$R = \left(\frac{M}{M + m}\right)\frac{v_r^{\,2} \sin 2\theta}{g} \quad ...(ii)$$

(a) From Eq. (ii) we see that maximum range is at $\theta = 45°$ **Ans.**

(b) At $\theta = 45°$, $$R_{max} = \left(\frac{M}{M + m}\right)\frac{v_r^2}{g}$$ **Ans.**

10. (a) $u_r = 0, \quad a_r = g$

$\therefore$ $$v_r = \sqrt{2gh_1}$$

After collision relative velocity $v_r' = e\sqrt{2gh_1}$ and relative retardation is still g (downwards). Hence,

$$h_2 = \frac{(v_r')^2}{2g} = e^2 h_1$$ **Ans.**

(b) $$u_r = 0, \quad a_r = g + \frac{g}{4} = \frac{5g}{4}$$

$\therefore$ Just before collision $v_r = \sqrt{2\left(\frac{5g}{4}\right)h_1}$

Just after collision $v_r' = ev_r$.

Relative retardation is still $\frac{5g}{4}$.

Hence, $$h_2 = \frac{(v_r')^2}{2\left(\frac{5g}{4}\right)} = e^2 h_1$$ **Ans.**

11. Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$

[M = mass of the plank, m = mass of the block]

$\therefore$ $$100 = u^2 + 5v^2 \quad ...(i)$$

By conservation of momentum

$$mu + Mv = 0$$

$\Rightarrow$ $$u = -5v \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii)

$$30v^2 = 100$$

$\therefore$ $$v = \sqrt{\frac{10}{3}}\ \text{m/s}$$

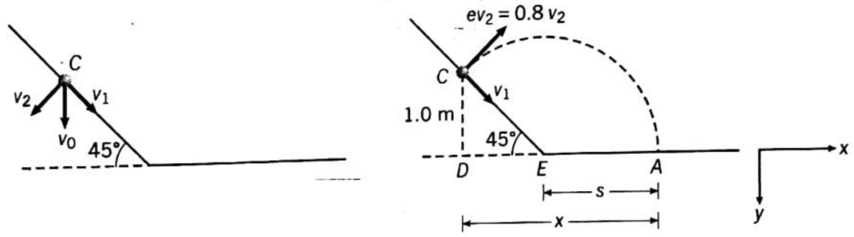
From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls velocity of plank $= \sqrt{\frac{10}{3}}$ m/s. **Ans.**

12. $v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 1.5} = 5.42$ m/s

Component of velocity parallel and perpendicular to plane at the time of collision.

$$v_1 = v_2 = \frac{v_0}{\sqrt{2}} = 3.83\ \text{m/s}$$



Component parallel to plane (v_1) remains unchanged, while component perpendicular to plane becomes ev_2, where

$$ev_2 = 0.8 \times 3.83 = 3.0\ \text{m/s}$$

$\therefore$ Component of velocity in horizontal direction after collision

$$v_x = \frac{(v_1 + ev_2)}{\sqrt{2}} = \frac{(3.83 + 3.0)}{\sqrt{2}} = 4.83\ \text{m/s}$$

While component of velocity in vertical direction after collision.

$$v_y = \frac{v_1 - ev_2}{\sqrt{2}} = \frac{3.83 - 3.0}{\sqrt{2}} = 0.59\ \text{m/s}$$

Let t be the time, the particle takes from point C to A, then

$$1.0 = 0.59t + \frac{1}{2} \times 9.8 \times t^2$$

$$t = 0.4\ \text{s} \qquad \text{(Positive value)}$$

Solving this we get,

$$DA = v_x t = (4.83)(0.4) = 1.93\ \text{m}$$

$\therefore$ $$S = DA - DE$$

$\therefore$ $$= 1.93 - 1.0$$

$$S = 0.93\,\text{m}$$ **Ans.**

$$v_{yA} = v_{yc} + gt = (0.59) + (9.8)(0.4) = 4.51\ \text{m/s}$$

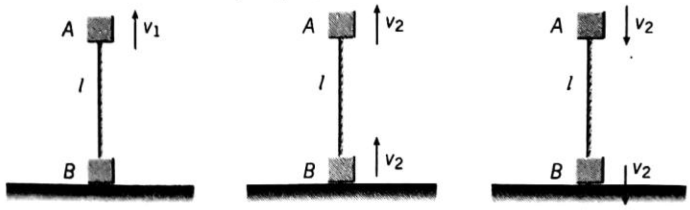
$$v_{xA} = v_{xC} = 4.83\ \text{m/s}$$

$$v_A = \sqrt{(v_{xA})^2 + (v_{yA})^2} = 6.6\ \text{m/s}$$ **Ans.**

$\therefore$

13. String becomes tight when A moves upwards by a distance l. Let v_1 be the velocity of A at this moment, then

$$v_1^2 = (\sqrt{10gl})^2 - 2gl = 8gl$$



or $$v_1 = \sqrt{8gl}$$

Let v_2 be the common velocities of both A and B just after string becomes tight. Then from conservation of linear momentum.

$$v_2 = \frac{v_1}{2} = \frac{\sqrt{8gl}}{2}$$

Both particles return to their original height with same speed v_2. String becomes loose after B strikes the ground and the speed v with which A strikes the ground is,

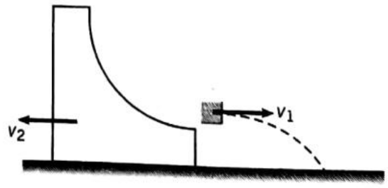
$$v^2 = v_2^2 + 2gl = \frac{8gl}{4} + 2gl$$

or $$v^2 = 4gl$$

or $$v = 2\sqrt{gl}$$ **Ans.**

14. $$mv_1 = Mv_2 \quad \text{...(i)}$$

$$mgR = \frac{1}{2} mv_1^2 + \frac{1}{2} Mv_2^2 \quad \text{...(ii)}$$



$$t = \sqrt{\frac{2(R/2)}{g}} = \sqrt{\frac{R}{g}} \quad \text{...(iii)}$$

The desired distance is $$S = (v_1 + v_2)t \quad \text{...(iv)}$$

Solving Eqs. (i) and (ii) for v_1 and v_2 and substituting in Eq. (iv), we get

$$S = R\sqrt{\frac{2(M+m)}{M}}$$ **Ans.**

15. Let v_r be the velocity of washer relative to centre of hoop and v the velocity of centre of hoop. Applying conservation of linear momentum and mechanical energy we have,

$$m(v_r \cos\phi - v) = Mv \quad \text{...(i)}$$

$$mgr(1 + \cos\phi) = \frac{1}{2} Mv^2 + \frac{1}{2} m(v_r^2 + v^2 - 2vv_r \cos\phi) \quad \text{...(ii)}$$

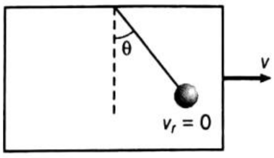
Solving Eqs. (i) and (ii), we have, $$v = m\cos\phi\sqrt{\frac{2gr(1+\cos\phi)}{(M+m)(M+m\sin^2\phi)}}$$ **Ans.**

16. $H = \dfrac{u^2 \sin^2 \theta}{2g} = \dfrac{(20)^2 \times \frac{3}{4}}{2 \times 10} = 15 \text{ m}$

i.e., the shell strikes the ball at highest point of its trajectory. Velocity of (ball + shell) just after collision,

$$v = \frac{u \cos 60°}{2} \quad \text{(from conservation of linear momentum)}$$

$$= \frac{20}{2 \times 2} = 5 \text{ m/s}$$



At highest point combined mass is at rest relative to the trolley. Let v be the velocity of trolley at this instant. From conservation of linear momentum we have,

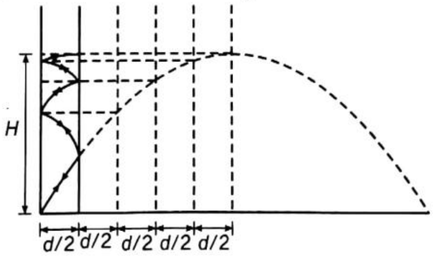
$$2 \times 5 = \left(2 + \frac{4}{3}\right)v \quad \text{or} \quad v = 3 \text{ m/s}$$

From conservation of energy, we have

$$\frac{1}{2} \times 2 \times (5)^2 - \frac{1}{2}\left(2 + \frac{4}{3}\right)(3)^2 = 2 \times 10(1 - \cos\theta)$$

solving we get $\cos\theta = \dfrac{1}{2}$ $\quad \therefore \; \theta = 60°$ **Ans.**

17. While colliding with the wall its vertical component (v_y) of velocity will remain unchanged (component along common tangent direction remains unchanged) while horizontal component (v_x) is reversed remaining same in magnitude. Thus, path of the particle will be as shown in figure.



(a) $H = \dfrac{u^2 \sin^2 \alpha}{2g}$

(b) Total number of collisions with the walls before the ball comes back to the ground are nine.

(c) Ball will return to point O (the starting point)

18. As the collisions are perfectly elastic, collision of the ball will not affect the vertical component of its velocity while the horizontal component will be simply reversed.

Hence, $\quad H_{max} = \dfrac{v_y^2}{2g} = \dfrac{[20 \times \sin 45°]^2}{2 \times 10} = 10 \text{ m}$

Total time of flight

$$T = \frac{2v_y}{g} = \frac{2 \times 20 \times \left(\frac{1}{\sqrt{2}}\right)}{10} = 2\sqrt{2} \text{ s}$$

Total horizontal distance travelled before striking the ground $x = v_x T = 40$ m

$$PB + BA + AB + BA + AB = 45 \text{ m}$$

Hence, total number of collision suffered by the particle with the walls before it hits ground = 4. **Ans.**

19. Let v_1 = velocity of block 2 kg just before collision

v_2 = velocity of block 2 kg just after collision

and v_3 = velocity of block M just after collision.

Applying work energy theorem

(change in kinetic energy = work done by all the forces) at different stages as shown in figure.

Figure 1.

$$\Delta KE = W_{friction} + W_{gravity}$$

$$\left[\frac{1}{2}m\{v_1^2 - (10)^2\}\right] = -6\mu\, mg\cos\theta - mgh_1 \qquad (m = 2\text{ kg})$$

or $$v_1^2 - 100 = 2[6\mu g\cos\theta + gh_1]$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(0.05)^2} \approx 0.99$$

$\therefore$ $$v_1^2 = 100 - 2[(6)(0.25)(10)(0.99) + (10)(0.3)]$$

$\Rightarrow$ $$v_1 \approx 8\text{ m/s}$$

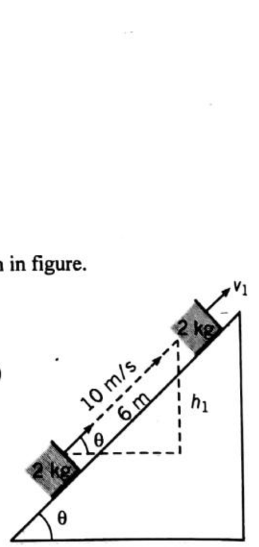


Fig. 1

Figure 2.

$$\Delta KE = W_{friction} + W_{gravity}$$

$$\frac{1}{2}m[(1)^2 - (v_2^2)] = -6\mu\, mg\cos\theta + mgh_1$$

or $$1 - v_1^2 = 2[-6\mu\, g\cos\theta + gh_1]$$

$$= 2[(-6)(0.25)(10)(0.99) + (10)(0.3)]$$

$$= -23.7$$

$\therefore$ $$v_2^2 = 24.7 \quad \text{or} \quad v_2 \approx 5\text{ m/s}$$

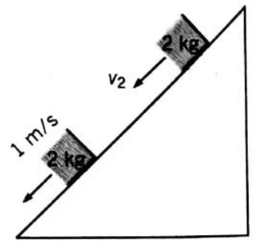


Fig. 2

Figure 3.

$$\Delta KE = W_{friction} + W_{gravity}$$

$$\frac{1}{2}M[0 - v_3^2] = -(0.5)(\mu)(M)\,g\cos\theta - Mgh_2$$

or $$-v_3^2 = -\mu\, g\cos\theta - 2gh_2$$

or $$v_3^2 = (0.25)(10)(0.99) + 2(10)(0.025)$$

or $$v_3^2 = 2.975$$

$\therefore$ $$v_3 \approx 1.72\text{ m/s}$$

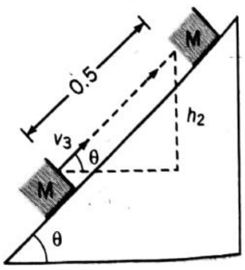


Fig. 3

Now

(i) $$\text{Coefficient of restitution} = \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}$$

$$= \frac{v_2 + v_3}{v_1}$$

$$= \frac{5 + 1.72}{8} = \frac{6.72}{8}$$

or $$e \approx 0.84$$

(ii) Applying conservation of linear momentum before and after collision

$$2v_1 = Mv_3 - 2v_2$$

$$\therefore \quad M = \frac{2(v_1 + v_2)}{v_3} = \frac{2(8+5)}{1.72} = \frac{26}{1.72}$$

$$M \approx 15.12 \text{ kg}$$ **Ans.**

20. Let v_r be the relative velocity of block as it leaves contact with the sphere ($N = 0$) and v the horizontal velocity of sphere at this instant.

Absolute components of velocity of block

Applying conservation of linear momentum in horizontal direction, we get

$$mv = m(v_r \cos\theta - v)$$

or $$2v = v_r \cos\theta \qquad \text{...(i)}$$

Conservation of mechanical energy gives,

$$mgr(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r\cos\theta)$$

or $$gr(1-\cos\theta) = v^2 + \frac{v_r^2}{2} - vv_r\cos\theta \qquad \text{...(ii)}$$

Equation of laws of motion gives,

$$mg\cos\theta = \frac{mv_r^2}{r}$$

or $$gr = \frac{v_r^2}{\cos\theta} \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\cos^3\theta - 6\cos\theta + 4 = 0$$ **Ans.**

Note *We have not considered pseudo force while writing the equation of motion. Think why?*

21. $$\frac{x}{u\cos\alpha} + \frac{x}{eu\cos\alpha} = T = \frac{2u\sin\alpha}{g}$$

or $$x = \frac{eu^2\sin 2\alpha}{(1+e)g}$$

$$x_{max} = \frac{eu^2}{(1+e)g} \quad \text{at} \quad 2\alpha = 90°$$ **Ans.**