IMPORTANT FACTS AND FORMULAE FOR JEE

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AIEEE – MATHEMATICS

UNITS – 1: SETS, RELATIONS & FUNCTIONS

Algebraic properties of sets:

1. $A \cup (B \cup C) = (A \cup B) \cup C$

2. $A \cap B = B \cap A$

3. $A \cap \phi = \phi$

4. $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$

5. $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$ **(De Morgan's Laws)**

Formulae for domains of functions:

1. $\text{dom}(f + g) = \text{dom}f \cap \text{dom}g$

2. $\text{dom}(fg) = \text{dom}f \cap \text{dom}g$

3. $\text{dom}(f/g) = \text{dom}f \cap \text{dom}g \cap \{x : g(x) \neq 0\}$

4. $\text{dom}\left(\sqrt{f}\right) = \text{dom}f \cap \{x : f(x) \geq 0\}$

Type of functions:

1. **Surjective function:** If a function $f : A \to B$ is such that each element in B is an image of atleast one element in A, then f is a function of A 'onto' B or f is a *surjective function* (also called onto *function*) from A to B.

2. **Injective function:** If function does not take the same value at two distinct points in its domain, then the function is said to be an *injective function* (also called one-to-one *function*)

3. **Bijective function:** If a function f is both one-to-one and onto, the f is said to be a *bijective function.*

UNITS – 2: COMPLEX NUMBERS

1. Property of Order

It states that $(a + ib) < (\text{or} >)\ c + id$ is not defined for complex numbers. For example, the statement $9 + 6i < 2 - i$ make no sense.

i) The sum of four consecutive powers of i is zero.

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0,\ n \in \text{I}$$

ii) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only if atleast one of a and b is non-negative.

If a and b are both negative then

$$\sqrt{a}\sqrt{b} = -\sqrt{ab}$$

2. Properties of Conjugate Complex Numbers

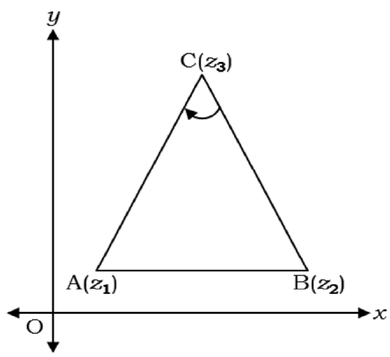
(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

(iii) $z_1\overline{z_2} - \overline{z_1}z_2 = 2\,\text{Re}\left(z_1\overline{z_2}\right)$

3. Rotational approach

If z_1, z_2, z_3 be vertices of a triangle ABC described in counter-clockwise sense (see Fig.) then:



$$\frac{(z_2 - z_3)}{|z_2 - z_3|} = \frac{(z_1 - z_3)}{|z_1 - z_3|} e^{i\alpha}$$

or

$$\text{amp}\,\frac{z_2 - z_3}{z_1 - z_3} = \alpha = \angle\text{BCA}$$

4. Properties of Modulus

(i) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(ii) $z\bar{z} = |z|^2$

(iii) $|z^n| = |z|^n$

(iv) $|z_1 + z_2| \le |z_1| + |z_2|$

(v) $|z_1 - z_2| \ge |z_1| - |z_2|$

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$

(ix) $\left||z_1| - |z_2|\right| \le |z_1 - z_2|$

(x) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \left(a^2 + b^2\right)\left(|z_1|^2 + |z_2|^2\right)$, where a, b$\in$R.

5. Properties of Argument of Complex Numbers

(i) Arg (z_1z_2) = Arg (z_1) + Arg (z_2)

(ii) $\text{Arg}\left(\frac{z_1}{z_2}\right)$ = Arg (z_1) - Arg (z_3)

(iii) Arg ($z_1z_2z_3....z_n$) = Arg (z_1) + Arg (z_2) + Arg (z_3).... + Arg (z_n)

(iv) Arg (z^n) = n Arg (z)

(v) If $\text{Arg}\left(\frac{z_2}{z_1}\right) = \theta$, then $\text{Arg}\frac{z_1}{z_2} = 2k\pi - \theta, k \in l$

6. Demoivre's theorem

(a) If n is a positive or negative integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(b) If n is a positive integer, then

$$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$

Demoivre's theorem is valid if n is any rational number.

7. Some important Results

(i) If z_1 and z_2 are two complex numbers, then the distance between z_1 and z_2 is $|z_1 - z_2|$.

(ii) Segment joining points A(z_1) and B(z_2) is divided by point P(z) in the ratio $m_1 : m_2$ then $z = \frac{m_1 z_2 + m_2 z_1}{(m_1 + m_2)}$, m_1 and m_2 are real.

(iii) The equation of the line joining z_1 and z_2 is given by

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z_1} & 1 \\ z_2 & \bar{z_2} & 1 \end{vmatrix} = 0 \qquad \text{(non-parametric form)}$$

(iv) Three points z_1, z_2 and z_3 are collinear if

$$\begin{vmatrix} z_1 & \bar{z_1} & 1 \\ z_2 & \bar{z_2} & 1 \\ z_3 & \bar{z_3} & 1 \end{vmatrix} = 0$$

(v) $\bar{a}z + a\bar{z} + b = 0,\ b \in \mathbb{R}$ describes equation of a straight line

Slope of line: The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is

$$-\frac{a}{b} = -\frac{\text{coeffi. of } \bar{z}}{\text{coeffi. of } z}$$

(vi) $|z - z_0| = r$ is equation of a circle, whose centre is z_0 and radius is r

(vii) If $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$ then point z describes an ellipse.

(viii) $\frac{|z - z_1|}{|z - z_2|} = k$ is a circle if $k \neq 1$, and is a line if $k = 1$.

(ix) If $\text{Arg}\frac{(z_1 - z_3)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_3)} = \pm\pi, 0$, the points z_1, z_2, z_3, z_4 are concyclic.

UNIT – 3: MATRICES & DETERMINANTS

1. Some Important Terms

A matrix $A = (a_{ij})_{m \times n}$ is said to be a

(i)	Row matrix	if $m = 1$
(ii)	Column matrix	if $n = 1$
(iii)	Null or zero matrix	if $a_{ij} = 0$, $\forall$ i and j
(iv)	Square matrix	if $m = n$
(v)	Diagonal matrix	if $m = n$ and $a_{ij} = 0$, $\forall i \neq j$
(vi)	Scalar matrix	if $m = n$ and $a_{ij} = 0$, $\forall i \neq j$ and $a_{ij} = \lambda, \forall i = J$
(vii)	Unit or identity matrix	if $m = n$ and $a_{ij} = 0$, $\forall i \neq j$ and $a_{ij} = 1, \forall i = J$

2. Properties of matrix multiplication

i) If A is a square matrix, then

$A^m A^n = A^{m+n}$, $\forall$ $m, n \in N$

$(A^m)^n = A^{mn}$, $\forall$ $m, n \in N$

ii) If A is an invertible matrix then

$(A^{-1}BA)^m = A^{-1} B^m A$

and $A^{-m} = (A^{-1})^m$, $\forall$ $m \in$ N

3. Transpose of a Matrix

Definition: The transpose of a matrix $A = (a_{ij})_{m \times n}$, denoted by A'

(or by A') is the matrix $A' = (b_{ij})_{n \times m}$, where $b_{ij} = a_{ji}$, $\forall$ i and j.

By $\overline{A}$ we mean a matrix $B = (b_{ij})_{m \times n}$, where $b_{ij} = \overline{a}_{ij}$, where $\overline{a}$ denotes conjugate of a and by A* we mean

$$A^* = (\overline{A})' = (\overline{A}')$$

Properties of Transpose of a Matrix

i. $(A + B)' = A' + B'$

ii. $(kA)' = kA'$ where k is a scalar.

iii. $(AB)' = B'A'$ [Reversal law]

iv. If A is an invertible matrix, then $(A^{-1})' = (A')^{-1}$

4. Adjoint and Inverse of a Matrix

The adjoint of a square matrix $A = (a_{ij})_{m\times n}$ is defined to be the matrix adj. $A = (b_{ij})_{n\times m}$, where b_{ij} = A_{ji} where, A_{ji} is the cofactor of (j, i)th element of A.

Properties of Adjoint

i) A(adj A) = (adj A) A = $|A| I_n$

ii) adj $(kA) = k^{n-1}$ (adj A)

iii) adj(AB) = (adj B) (adj A)

Properties of Inverse

i) $AA^{-1} = A^{-1}A = I_n$

ii) $(A^{-1})^{-1} = A$

iii) $(kA)^{-1} = k^{-1} A^{-1}$ if $k \neq 0$

iv) $(AB)^{-1} = B^{-1}A^{-1}$ [reversal low]

v) Inverse of a matrix if it exists is unique.

vi) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, Adj. $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

and $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} if ad - bc \neq 0$

5. Properties of Determinants

i) The interchange of any two rows (or columns) in Δ changes its sign.

ii) If all the elements of a row (or column) in Δ are zero or if two rows (or columns) are identical (or proportional), then the value of Δ is zero.

iii) If all the elements of one row (or column) is multiplied by a non-zero number k, then the value of the new determinant is k times the value of the original determinant.

iv) If Δ becomes zero on putting $x = \alpha$, then we say that $(x-\alpha)$ is factor of Δ.

v) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3 = \begin{vmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6. Systems of Linear Equations

The system of homogeneous linear equations

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_3z = 0$

and $a_3x + b_3y + c_3z = 0$

has a non-trivial solution (i.e., at least one of x, y, z is non-zero) if

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

and if $\Delta \neq 0$, then $x = y = z = 0$ is the only solution of above system (Trivial solution).

Cramer's Rule: Let us consider a system of equations

$a_1x + b_1y + c_1z = \text{d}_1$;

$a_2x + b_2y + c_2z = \text{d}_2$;

$a_3x + b_3y + c_3z = \text{d}_3$;

Here $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$,

By Cramer's Rule, we have $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, and $z = \frac{\Delta_3}{\Delta}$

7. Differentiation of Determinant Function

If $F(x) = \begin{vmatrix} f_1 & f_1 & f_1 \\ g_2 & g_2 & g_2 \\ h_3 & h_3 & h_3 \end{vmatrix}$ where $f_1, f_2, f_3; g_1, g_2, g_3; h_1, h_2, h_3;$ are the functions of x, then

$$\therefore \quad F'(x) = \begin{vmatrix} f_1' & f_1' & f_1' \\ g_2 & g_2 & g_2 \\ h_3 & h_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_1 & f_1 \\ g_2' & g_2' & g_2' \\ h_3 & h_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_1 & f_1 \\ g_2 & g_2 & g_2 \\ h_3' & h_3' & h_3' \end{vmatrix}$$

8. Cyclic Order

If elements of the rows (or columns) are in cyclic order,

i.e., i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \left(a^2+b^2+c^2-3abc\right)$

UNIT – 4: QUADRATIC EQUATIONS

1. Roots of the equation

$ax^2+bx+c=0 \quad \Rightarrow \quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ are α and β

2. Sum and product of the roots:

$S = \alpha + \beta = -\dfrac{b}{a}$; $\quad P = \alpha\beta = \dfrac{c}{a}$.

3. To find the equation whose roots are α and β

$x^2 - Sx + P = 0$

where S is sum and P is product of roots.

4. Nature of the roots

$D = b^2 - 4ac$ where D is called discriminant.

(a) If $b^2 - 4ac \geq 0$, roots are real

i) $b^2 - 4ac > 0$, roots are real and unequal.

ii) $b^2 - 4ac = 0$, roots are real an equal. In this case, each root = $-\dfrac{b}{2a}$

iii) $b^2 - 4ac$ = perfect square, roots are rational.

iv) $b^2 - 4ac$ = not a perfect square, roots are irrational.

(b) if $b^2 - 4ac < 0$, i.e., -ve, then $\sqrt{b^2-4ac}$ is imaginary.

Therefore the roots are imaginary and unequal.

5. Symmetric function of the roots

If α and β are the roots of $ax^2 + bx + c = 0$, then

i) $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

iii) $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

iv) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

v) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

vi) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

6. Condition for Common roots

Equations: $a_1x^2 + b_1x + c_1 = 0$

Roots: $a_2x^2 + b_2x + c_2 = 0$

Condition for one common root:

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

Condition for both the roots to be common:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

⇒ **Note:** If the coefficient of x^2 in both the equations be unity, then the value of common root is obtained by subtracting their equations.

7. Inequalities

a) $x^2 > a^2$ is positive if $< -a$ or $x > a$

b) $x^2 < a^2$ if $-a < x < a$

c) $a^2 < x^2 < b^2$ double inequality $\therefore$ $a < x < b$ or $-b < x < -a$.

8. In an inequality you can always multiply or divide by a +ve quantity but not by a –ve quantity. Multiplying by a –ve quantity or taking reciprocal will reverse the inequality.

e.g., $a > b \Rightarrow -a < -b$ or $\frac{1}{a} < \frac{1}{b}$.

UNIT – 5: PERMUTATIONS & COMBINATIONS

1. Fundamental Principle of Counting

(i) ***Multiplication Principle*:** There are m ways doing one work and n way doing another work then way of doing both work together $= m.n.$

(ii) ***Addition Principle*:** There are m way doing one work and n ways doing another work then ways doing either m ways or n ways $= m + n.$

2. The number of ways of arranging n distinct objects in a row taking $r(0 \le r \le n)$ at a time is denoted by ${}^nP_r$ or $P(n, r)$

and $${}^nP_r = n(n-1)(n-2)....(n-r+1) = \frac{n!}{(n-r)!}$$

Note that, ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$

3. The number of ways of arranging n distinct objects along a circle is $(n-1)!$

4. The number of ways of arranging n beads along a circular wire is $\frac{(n-1)!}{2}$.

5. The number of permutations of n things taken all at a time, p are alike of one kind, q are alike of another kind and r are alike of a third kind and the rest $n - (p + q + r)$ are all different is $\frac{n!}{p!q!r!}$.

6. The number of ways of arranging n distinct objects taking r of them at a time where any object may be repeated any number of times is n^r.

7. The number of selecting at least one object out of 'n' distinct object $= 2^n - 1$

8. Derangements: The number of derangements (No object goes to its scheduled place) of n objects.

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - .... + (-1)^n \frac{1}{n!}\right)$$

9. The coefficient of x^r in the expansion of $(1-x)^{-n} = {}^{n+r-1}C_r$

10. Some Important Results

(i) ${}^nC_0 = {}^nC_n = 1, {}^nC_1 = n$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$

(iv) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots\ldots + {}^{2n+1}C_n = 2^{2n}$

(v) ${}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}$

(vi) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(vii) ${}^nC_r + {}^{n+1}C_n + {}^{n+2}C_n + \ldots\ldots + {}^{2n+1}C_n = {}^{2n}C_{n+1}$

UNIT – 6: MATHEMATICAL INDUCTION AND ITS APPLICTIONS

It is often used to prove a statement depending upon a natural number *n*.

I[st] Principle of Induction

If P(*n*) is a statement depending upon *n*, then to prove it by induction, we proceed as follow:

i) Verify the validity of P(*n*) for $n = 1$

ii) Assume that P(*n*) is true for some positive integer *m* and then using it establish the validity of p(*n*) for $n = m + 1$.

Then, P(*n*) is true for each $n \in N$

II[nd] Principle of Induction:

In P(*n*) is a statement depending upon *n* but beginning with some positive integer *k*, then to prove P(*n*), we proceed as follows:

i) Verify the validity of P(*n*) for $n = k$

ii) Assume that the statement is true for $n = m \geq k$. Then, using it establish the validity of P(*n*) for $n = m + 1$.

Then, P(*n*) is true for each $n \geq k$.

UNIT – 7: BINOMIAL THEOREM & ITS APPLICATIONS

1. Binomial Theorem

It *n* is a positive integer and $x, y, \in C$ then

$$(x+y)^n = {}^nC_0\, x^{n-0}\, y^0 + {}^nC_1\, x^{n-1}\, y^1 + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots + {}^nC_n\, y^n$$

Here ${}^nC_0, {}^nC_1, {}^nC_2 \ldots\ldots {}^nC_n$ are called binomial coefficient.

2. Some Important Points to Remember

(i) ***General term:*** General term = (r + 1)th term

$\therefore \quad \Rightarrow \quad T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$, where r = 0, 1, 2,....., n.

(ii) ***Middle term:*** The middle term depends upon the value of n.

(a) If n is even,

i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term.

(b) If n is odd, there are two middle terms,

i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th are two middle terms.

(iii) ***Greatest Term:***

(a) If p is an integer, then Tp and T_{p+1} are equal and both are greatest term

(b) If p is not an integer, then $T_{[p]+1}$ is the greatest term, where [.] denotes the greatest integral part.

(iv) ***Greatest Coefficient:***

(a) If n is even, then greatest coefficient ${}^nC_{n/2}$

(b) If n is odd, then greatest coefficients are ${}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$.

(v) ***Important Formulae:***

(a) $C_0 + C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots C_n = 2^n$

(b) $C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$

$\Rightarrow$ sum of odd binomial coefficients = sum of even binomial coefficients

(c) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n$

(d) $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots\ C_{n-r}\, C_n = {}^{2n}C_{n-r}$

where C_0, C_1, C_2, C_3,, ... represent ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \ldots\ldots$

(vi) ***General Binomial theorem:***

If $n \in R, -1 < x < 1$, then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+....+\frac{n(n-1)(n-2)}{r!}x^r+....\infty$$

(vii) ***Important Expansions:***

(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...+ (-1)^r x^r + ...$

(b) $(1-x)^{-1} = 1 + x^1 + x^2 + x^3 + ...+ x^r + ...$

(c) $(1+x)^{-2} = 1 - {}^2C_1 x + {}^3C_2 x^2 - {}^4C_3 x^3 + ... + (-1)^r\, {}^{r+1}C_r\, x^r + ...$

(d) $(1+x)^{-2} = 1 + {}^2C_1 x + {}^3C_2 x^2 + {}^4C_3 x^3 + ... + {}^{r+1}C_r\, x^r + ...$

UNIT – 8: SEQUENCES & SERIES

1. Arithmetic Progression

(i) $T_n = a+(n-1)d, n \in I$

(ii) $S_n = \frac{n}{2}\left[2a+(n-1)d\right], n \in I$

where T_n = nth term of an AP

S_n = sum of n terms of an AP

a = first term of an AP

d = Common difference

(iii) Arithmetic Mean (AM) of two quantities a and b,

$$A = \frac{a+b}{2}$$

(iv) AM of n gives quantities,

$$A_n = \frac{x_1 + x_2 + ...x_n}{n}$$

(v) $T_n = S_n - S_{n-1}\,(n \ge 2)$

2. Geometric Progression (GP)

(i) $T_n = ar^{n-1}$ where a = first term of the GP, r = common ratio of the GP and T_n = n^{th} term of the GP

(ii) Sum of n terms of a GP

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

(iii) Sum of an infinite number of terms of a GP, when $|r| < 1$

$$S_\infty = \frac{a}{1-r}$$

(iv) Geometric mean of two quantities a and b,

$$G^2 = ab,\ ab > 0$$

(v) Geometric mean of n given quantities, a_1, a_2, ..., a_n.

$$G_n = (a_1 a_2 a_3 \ldots\ldots a_{n-1} a_n)^{1/n}$$

(vi) Common ratio $= \dfrac{T_n}{T_{n-1}}$.

3. Harmonic Progression (HP)

(i) nth term of a HP,

$$T_n = \frac{1}{a+(n-1)d}$$

(ii) Harmonic mean of a and b,

$$H = \frac{2ab}{a+b}$$

(iii) Relation between A, G and H

$$AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right)$$

$$= ab = G^2$$

$\Rightarrow$ $AH = G^2$

(iv) $A > G > H$, i.e., A, G, H are in descending order of magnitude.

4. Arithmetico-Geometric Sequence

Sum of an infinite number of terms of the sequence ab, $(a + d)br$, $(a + 2d)br^2$,... is

$$S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

5. Series of natural numbers

(i) $\sum n = 1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$

(ii) $\sum n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iv) $\sum a = a + a + ... + a(n \text{ terms}) = na$

6. Exponential and Logarithmic series

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(ii) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

(iii) $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(iv) $a^x = 1 + x \ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + ...$

(v) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ......|x| < 1$

(vi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...n \in R$

UNIT -: 9 DIFFERENTIAL CALCULUS

Functions

Formulae for domain of a function

1. Domain ($f(x) + g(x)$) = Domain $f(x) \cap$ Domain $g(x)$

2. Domain ($f(x) \,.\, g(x)$) = Domain $f(x) \cap$ Domain $g(x)$

3. Domain $\left(\frac{f(x)}{g(x)}\right)$ = Domain $f(x) \cap$ Domain $g(x)$ $\cap \{x : g(x) \neq 0\}$

4. Domain $\sqrt{f(x)}$ = Domain $f(x) \cap \{x : f(x) \geq 0\}$

5. Domain (fog) = Domain $g(x)$, where fog is defined by fog(x) = $f\{g(x)\}$

Odd and Even Function

(i) A function is an odd function if $f(-x) = -f(x)$ for all x.

(ii) A function is an even function if $f(-x) = f(x)$ for all x.

Periodic function

A function $f(x)$ is said to be a periodic function of x, if there exist a positive real number T such that $f(x + T) = f(x)$ for all x.

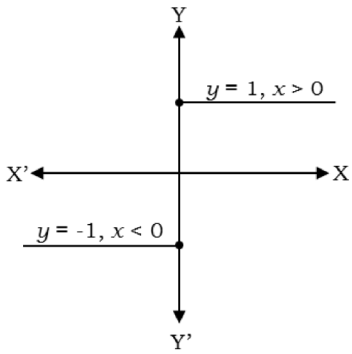
If positive value of T independent of x then $f(x)$ is periodic function and if the value of T depends upon x, then $f(x)$ is non-periodic.

Signum function

The signum function f is defined as

$$\text{S}gn\, f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

As x shown in figure



Greatest Integer function

If $f(x) = [x + n]$, where $n \in \text{I}$ and [.] denotes the greatest integer function, then $f(x) = n + [x]$.

a) $x - 1 < [x] \le x$

b) $[-x] = -[x]$, if $x \in \text{I}$

c) $[-x] = -[x] - 1$, if $x \notin \text{I}$

d) $[x] \ge n \Rightarrow x \ge n, n \in \text{I}$

e) $[x] \le n \Rightarrow x < n + 1, n \in \text{I}$

f) $[x] > n \Rightarrow x \ge n + 1, n \in \text{I}$

g) $[x] < n \Rightarrow x < n, n \in \text{I}$

Modulus function (or Absolut-Value-function)

Modulus function is given by

$$y = |x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

Properties of Modulus Function

a) $|x| \le a \Rightarrow -a \le x \le a; (a \ge 0)$

b) $|x| \geq a \Rightarrow -a$ or $x \geq a; (a \geq 0)$

c) $|x \pm y| \leq |x| + |y|$

d) $|x \pm y| \geq |x| - |y|$

Limits and Continuity

Frequently used Limits

(i) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e = \lim_{h \to 0} (1+h)^{1/h}$

(ii) $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x} = e^{a}$

(iii) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where $n \in Q$

(iv) $\lim_{\theta \to 0} \frac{\sin \theta^c}{\theta^c} = 1$ and $\lim_{\theta \to 0} \frac{\tan \theta^c}{\theta^c} = 1$

(v) $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \, (a > 0)$

(vi) $\lim_{x \to 0} \frac{e^x - 1}{x} = l$

(vii) $\lim_{x \to 0} \frac{\log_a (1+x)}{x} = \log_a e \, (a > 0, a \neq 1)$

Expansions of Trigonometric Functions

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - .....$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .....$

(iii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + ...$

Sandwich Theorem

If *f*, *g*, *h* are functions, such that $f(x) \leq g(x) \leq h(x)$ then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)$$

Indeterminate Forms

If a function $f(x)$ takes any of the following forms at $x = a$

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^{0}, \infty^{0}, 1^{\infty}$$

then $f(x)$ is said to be indeterminate at $x = a$.

L'Hospital's Rule

Let $f(x)$ and $g(x)$ be two functions, such that $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Provided $f'(a)$ and $g'(a)$ are both non-zero.

Derivatives

Differentiability and Continuity

For a function $f(x)$

(a)	Differentiable	$\Rightarrow$	Continuous
(b)	Continuous	$\not\Rightarrow$	Differentiable
(c)	Not Continuous	$\Rightarrow$	Not Differentiable

Theorems on Derivatives

(i) $\frac{d}{dx}\{f_1(x) \pm f_2(x)\} = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$

(ii) $\frac{d}{dx}(k(x)) = k\frac{d}{dx} f_1(x)$, where k is any constant.

(iii) $\frac{d}{dx}\{f_1(x) \cdot f_2(x)\} = f_1(x)\frac{d}{dx} f_2(x) + f_2(x)\frac{d}{dx} f_1(x)$

(iv) If $y = f_1(u)$, $u = f_2(v)$ and $v = f_3(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Some Standard differentials

(i) $\frac{d}{dx} = x^{n} = nx^{n-1}, x \in \mathrm{R}, n \in \mathrm{R}, x > 0$

(ii) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(iii) $\frac{d}{dx}\left(e^{x}\right) = e^{x}$

(iv) $\frac{d}{dx}\left(a^{x}\right) = a^{x} \ln a$

(v) $\frac{d}{dx}(ln\,x) = \frac{1}{x}$

(vi) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(vii) $\frac{d}{dx}(\sin x) = \cos x$

(viii) $\frac{d}{dx}(\cos x) = -\sin x$

(ix) $\frac{d}{dx}(\tan x) = \sec^2 x$

(x) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(xi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xii) $\frac{d}{dx}(\text{cosec}\, x) = \text{cosec}\, x \cot x$

(xiii) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{\left(1-x^2\right)}}$

(xiv) $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{\left(1-x^2\right)}}$

(xv) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$

(xvii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{\left(x^2-1\right)}}$

(xviii) $\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{\left(x^2-1\right)}}$

Some Standard Substitutions

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos 2\theta$
$\sqrt{\left(2ax - x^2\right)}$	$x = a\left(1-\cos\theta\right)$

Critical Points

The points on the curve $y = f(x)$ at which $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist are known as the critical points.

Rolle's Theorem

If a function $f(x)$ is defined on $[a,b]$ satisfying

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

(iii) $f(a) = f(b)$ then $c \in (a,b)$ such that $f'(c) = 0$

Lagrange's mean value Theorem

If a function $f(x)$ is defined on $[a, b]$ satisfying

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b) then $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Test of Monotonicity

(i) The function $f(x)$ is ***monotonically increasing*** in the interval $[a, b]$, if $f'(x) \geq 0$ in $[a, b]$

(ii) The function $f(x)$ is ***strictly increasing*** in the interval $[a, b]$, if $f'(x) > 0$ in $[a, b]$

(iii) The function $f(x)$ is ***monotonically decreasing*** in the interval $[a, b]$, if $f'(x) \leq 0$ in $[a, b]$

(iv) The function $f'(x)$ is ***strictly decreasing*** in the interval $[a, b]$, if $f'(x) < 0$ in $[a, b]$

Working Rule for Finding Maxima and Minima

(a) First Derivative Test

To check the maxima or minima at $x = a$

(i) If $f'(x) > 0$ at $x < a$ and $f'(x) < 0$ at $x > a$ i.e., the sign of $f'(x)$ changes from +ve to –ve, then $f(x)$ has a local maximum at $x = a$.

(ii) If $f'(x) < 0$ at $x < a$ and $f'(x) > 0$ at $x > a$ i.e., the sign of $f'(x)$ changes from -ve to +ve, then $f(x)$ has a local minimum at $x = a$.

(iii) If the sign of $f'(x)$ does not change, then $f(x)$ has neither local maximum nor local minimum at $x = a$, then point 'a' is called a point of ***inflection***.

(b) Second Derivative Test

(i) If $f''(a) < 0$ and $f'(a) = 0$, then 'a' is a point of local maximum.

(ii) If $f''(a) > 0$ and $f'(a) = 0$, then 'a' is a point of local minimum.

(iii) If $f''(a) = 0$ and $f'(a) = 0$, then further differentiate and obtain $f'''(a)$

(iv) If $f'(a) = f''(a) = f'''(a) = \ldots\ldots = f^{n-1}(a) = 0$ and $f^{n}(a) \neq 0$

In n is odd then $f(x)$ has neither local maximum nor local minimum at $x = a$, then point 'a' is called a point of ***inflection***.

If n is even, then if $f^{n}(a) < 0$ then $f(x)$ has a local maximum at $x = a$ and if $f^{n}(a) > 0$ then $f(x)$ has a local minimum at $x = a$.

Note: Maximum or Minimum values are also called local extremum values. For the points of local extremum either $f'(x) = 0$ or $f'(x)$ does not exist.

UNIT – 10: INTEGRAL CALCULUS

Methods of Integration

List of Basic Forms of Integrals

1. $\int f(\varphi(x)\varphi'(x))\,dx$

Substitution $\varphi(x) = t$

2. $\int f(x)\varphi'(x)\,dx$

Integration by parts

$\int f(x)\varphi'(x)\,dx = f(x)\varphi(x) - \int \varphi(x) f'(x)\,dx$

3. $\int e^{ax}\,p_n(x)\,dx$, where $p_n(x)$ is polynomial of degree n.

Applying the formula for multiple integration by parts (see above), we get

$$\int e^{ax}\,p_n(x)\,dx = e^{ax}\left[\frac{p_n(x)}{\alpha} - \frac{p_n'(x)}{\alpha^2} + \frac{p_n''(x)}{\alpha^2} - \ldots\ldots + (-1)^n \frac{p_n^{(n)}(x)}{\alpha^{n+1}}\right] + C$$

4. $\int \frac{Mx + N}{x^2 + px + q}\,dx.\ p^2 - 4q < 0$

Substitution, $x + \frac{p}{2} = t$

5. $I_n = \int \frac{dx}{\left(x^2 + 1\right)^n}$

Reduction formula is used

$$I_n = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} I_{n-1}$$

6. $\int \frac{P(x)}{Q(x)} dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational fraction

$$Q(x) = (x - x_1)^1 \, (x - x_2)^m \ldots\ldots (x^2 + px + q) \ldots\ldots$$

Integrand is expressed in the form of a sum of partial fractions

$$\int \frac{P(x)}{Q(x)} = \frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_1)^2} + \ldots\ldots \frac{A_1}{(x-x_1)^1} + \frac{B_1}{(x-x_1)} + \frac{B_2}{(x-x_1)^2} + \ldots.$$

$$\ldots + \frac{B_m}{(x-x_2)^m} + \ldots\ldots \frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \ldots\ldots + \frac{M_kx+N_k}{(x^2+px+q)^k} + \ldots.$$

7. $\int \frac{Mx+N}{\sqrt{ax^2+bx+c}} dx$

By the substitution $x + \frac{b}{2a} = t$ the integral is reduced to a sum of two integrals:

$$\int \frac{Mx+N}{\sqrt{ax^2+bx+c}} dx = M_1 \int \frac{t\,dt}{\sqrt{at^2+m}} + N_1 \int \frac{dt}{\sqrt{at^2+m}}$$

The first integral is reduced to the integral of a power function and the second one is a tabular integral.

8. $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$, where R is a rational function of x and $\sqrt{ax^2+bx+c}$.

Reduced to an integral of rational fraction by the Euler substitutions:

$$\sqrt{ax^2+bx+c} = t \pm x\sqrt{a} \;\; (a > 0)$$

$$\sqrt{ax^2+bx+c} = tx \pm x\sqrt{c} \; (c > 0)$$

$$\sqrt{ax^2+bx+c} = t(x - x_1) \; (4\alpha - b^2 < 0)$$

Where x_1 is the root of the trinomial $ax^2 + bx + c$.

The indicated integral can also be evaluated by the trigonometric substitutions:

$$x+\frac{b}{2a}=\begin{cases}\dfrac{\sqrt{b^2-4ac}}{2a}\sin t\\ \dfrac{\sqrt{b^2-4ac}}{2a}\cot t \quad (a<0, ac-b^2<0)\end{cases}$$

$$x+\frac{b}{2a}=\begin{cases}\dfrac{\sqrt{b^2-4ac}}{2a}\sec t\\ \dfrac{\sqrt{b^2-4ac}}{2a}\text{cosec } t \quad (a>0, 4ac-b^2<0)\end{cases}$$

$$x+\frac{b}{2a}=\begin{cases}\dfrac{\sqrt{4ac-b^2}}{2a}\tan t\\ \dfrac{\sqrt{4a-b^2c}}{2a}\cot t \quad (a>0, 4ac-b^2<0)\end{cases}$$

9. $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$, where $P_n(x)$ is a polynomial of degreed n.

write the equality

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c}+k\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

Where $Q_{n-1}(x)$ is a polynomial of degree $n-1$. Differentiating both parts of this equality and multiplying by $\sqrt{ax^2+bx+c}$, we get the identity

$$P_n(x) = Q_{n-1}(x)(ax^2+bx+c)+\frac{1}{2}Q_{n-1}(x)(2ax+b)+k,$$

Which gives a system of $n+1$ linear equations for determining the coefficients of the polynomial $Q_{n-1}(x)$ and factor k.

And the integral $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ is taken by the method considered in No. 10(M = 0; N = 0)

10. $\int \frac{dx}{(x-x_1)^m\sqrt{ax^2+bx+c}}$ 3

This integral is reduced to the above–considered integral by the substitution

$$x - x_1 = \frac{1}{t}$$

11. $\int R(\sin x, \cos x)\,dx$

Universal substitution $\tan \frac{x}{2} = t$

If R(-sin x, cos x) = -R(sin$x$, cos x), then the substitution cos x = t is applied

If R(sin x, -cos x) = -R(sin$x$, cos x), then the substitution sin x = t is applied

If R(-sin x, -cos x) = -R(sin$x$, cos x), then the substitution tan x = t is applied

12. $\int \sin ax \sin bx\, dx$

$\int \sin ax \cos bx\, dx$

$\int \cos ax \cos bx\, dx$

Transform the product of trigonometric function into a sum or difference, using on of the following formulas;

$$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$$

$$\cos ax \cos bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$$

$$\sin ax \cos bx = \frac{1}{2}[\sin(a-b)x - \sin(a+b)x]$$

13. $\int \sin^{\boldsymbol{m}} x \cos^{\boldsymbol{n}} x\, dx$

Where m and n are integers.

If m is an odd positive number, then apply the substitution cos x = t

If n is an odd positive number, apply the substitution sin x = t

If m + n is an negative number, apply the substitution tan x = 1

If m and n are even non-negative numbers, use the formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}; \cos^2 x = \frac{1+\cos 2x}{2}$$

14. $\int \sin^p x \cos^q x \, dx \quad (0 < x < \pi/2)$

p and *q* –rational numbers.

Reduce to the integral of the binomial differential by the substitution $\sin x = t$

$$\int \sin^p x \cos^q x \, dx = \int tp\left(1-t^2\right)^{q-1} dt \qquad \text{(see No. 14)}$$

15. $\int R(e^{ax}) \, dx$.

Transform into an integral of a rational function by the substitution $e^{ax} = t$

Indefinite Integration

1. Properties of Indefinite Integration

(i) $\int \left[f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_n(x)\right] dx$

$$= \int f_1(x) \, dx \pm \int f_2(x) \, dx \pm \ldots\ldots \pm \int f_n(x) \, dx + C$$

(ii) $= \int k \, f(x) \, dx = k \int f(x) \, dx + C$

(iii) $\int [f(x)]^n \, f'(x) \, dx = \frac{[f(x)]^{n+1}}{(n+1)} + C$

(iv) $\int \frac{f'(x)}{f'(x)} \, dx = ln|f(x)| + C$

(v) $\int \frac{f'(x)}{f'(x)} \, dx = 2\sqrt{f(x)} + C$

2. Standard Results in Integration

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$

In particular $\int 1.dx = \int x^0 dx = x + C$

(ii) $\int \frac{1}{x} dx = ln|x| + C$

(iii) $\int e^x \, dx = e^x + C$

(iv) $\int a^x \, dx = \frac{a^x}{ln\ a} + C; a \neq 1, a > 0$

(v) $\int \sin x \, dx = -\cos x + C$

(vi) $\int \cos x \, dx = \sin x + C$

(vii) $\int \tan x \, dx = ln|\sec x| + C$

(viii) $\int \cot x \, dx = ln|\sin x| + C$

(ix) $\int \sec x \, dx = ln|\sec x + \tan x| + C = -ln|\sec x - \tan x| + C$

$$= ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

(x) $\int \text{cosec}\, x \, dx = ln|\text{cosec}\, x - \cot x| + C = ln|\text{cosec}\, x + \cot x| + C$

$$= ln\left|\tan\frac{x}{2}\right| + C$$

(xi) $\int \sec^2 x \, dx = \tan x + C$

(xii) $\int \text{cosec}^2 x \, dx = -\cot x + C$

(xiii) $\int \sec x \tan x \, dx = \sec x + C$

(xiv) $\int \cos\sec x \cot x \, dx = -\text{cosec}\, x + C$

(xv) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} ln\left|\frac{x - a}{x + a}\right| + C$

(xvii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} ln\left|\frac{a + x}{a - x}\right| + C$

(xviii) $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = ln\left|x + \sqrt{(x^2 - a^2)}\right| + C$

(xx) $\int \frac{dx}{\sqrt{(x^2 + a^2)}} = ln\left|x + \sqrt{(x^2 + a^2)}\right| + C$

(xxi) $\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xxii) $\int \sqrt{(a^2 - x^2)}\,dx = \frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(xxiii) $\int \sqrt{(a^2 + x^2)}\,dx = \frac{x}{2}\sqrt{(a^2 + x^2)} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C$

(xxiv) $\int \sqrt{(x^2 - a^2)}\,dx = \frac{x}{2}\sqrt{(x^2 - a^2)} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$

(xxv) $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$

(xxvi) $\int \frac{1}{x^n}\,dx = \frac{-1}{(n-1)\,x^{n-1}} + C$

Definite Integration

Properties of Definite Integrals

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Where c, is a point within or out of the interval $[a, b]$.

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(v) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ or 0

According as $f(x)$ is even or odd function of x.

(vi) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

(vii) $\int_a^b f(x)\,dx = (b-a)\int_0^1 f((b-a)x+a)\,dx$

(viii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(ix) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$

where $f(x)$ is periodic with period T and $n \in I$.

(x) If $f(a+x) = f(x)$, then $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(xi) If $f(x) \le \phi$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

(xii) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(xiii) **Leibnitz's rule:**

If f continuous on $[a, b]$ and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$ then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}.\frac{dv}{dx} - f\{u(x)\}.\frac{du}{dx}$$

UNIT – 11: DIFFERENTIAL EQUATIONS

(i) $d\,(xy) = x\,dy + y\,dx$

(ii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iii) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$

(iv) $d\,[ln(xy)] = \frac{x\,dy - y\,dx}{xy}$

(v) $d\left(ln\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$

(vi) $d\left[\frac{1}{2}ln\left(x^2 + y^2\right)\right] = \frac{x\,dx + y\,dy}{x^2 + y^2}$

(vii) $d\left(x^p y^q\right) = x^{p-1}y^{q-1}(py\,dx + qx\,dy)$

UNITS – 17: TRIGONOMETRY

Trigonometric Functions and Identities

1. Measurement of an Angle

(i) 1^o = 60 minutes = $60'$

(ii) $1'$ = 60 seconds = $60''$

(iii) Each interior angle of a regular polygon of n sides is equal to $\frac{(n-2)\pi}{n}, n > 2$

2. Some Important Formulae and Identities

sin(A+B) = sin A cos B + cos A sin B   sin(A-B) = sin A cos B - cos A sin B

cos(A+B) = cos A cos B - sin A sin B   cos(A-B) = cos A cos B + sin A sin B

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$; $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan(45^0 + A) = \dfrac{1 + \tan A}{1 - \tan A}$; $\tan(45^0 - A) = \dfrac{1 - \tan A}{1 + \tan A}$

$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

$\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

2 sin A cos B = sin (A+B) + sin (A-B)   2 cos A sin B = sin (A+B) - sin (A-B)

2 cos A cos B = cos (A+B) + cos (A-B)   2 sin A sin B = cos (A-B) - cos (A+B)

$\sin C + \sin D = 2\sin\dfrac{C + D}{2}\cos\dfrac{C - D}{2}$ $\sin C - \sin D = 2\cos\dfrac{C + D}{2}\sin\dfrac{C - D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C + D}{2}\cos\dfrac{C - D}{2}$ $\cos C - \cos D = 2\sin\dfrac{C + D}{2}\cos\dfrac{D - C}{2}$

$\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1 - \tan^2 A}$ $\cos 2A = \begin{cases} 1 - \sin^2 A \\ \cos^2 A - \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A} \\ 2\cos^2 A - A \end{cases}$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ $\sin^2 A = \dfrac{1 - \cos 2A}{2}$ and $\cos^2 A = \dfrac{1 + \cos 2A}{2}$

$\sin 3A = 3\sin A - 4\sin^3 A$; $\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

3. Conditional Identities

If A + B + C = 180⁰, then

(i) sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C

(ii) cos 2 A + cos 2B + cos 2C = - 1 - 4 cos A cos B cos C

(iii) sin A + sin B + sin C = $4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) tan A + tan B + tan C = tan A tan B tan c

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

4. Trigonometrical Ratios

(i) $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

(ii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

(iii) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta = 1 - \sin^2\theta\cos^2\theta$

(iv) $\sin^2\theta + \text{cosec}^2\theta \geq 2, \cos^2\theta + \sec^2\theta \geq 2$ and $\sec^2\theta + \text{cosec}^2\theta \geq 4$

5. Some standard Results

(i) $\tan\theta + \cot\theta = 2\,\text{cosec}\,2\theta$

(ii) $\cot\theta - \tan\theta = 2\cot 2\theta$

(iii) $\sin\theta + \cos\theta$ has the same sign as that of sin $\left(\theta + \frac{\pi}{4}\right)$

(iv) $\sin\theta - \cos\theta$ has the same sign as that of sin $\left(\theta - \frac{\pi}{4}\right)$

(v) Maximum value of $a \sin x + b \cos x$ is $\sqrt{a^2 + b^2}$ and its minimum value is $-\sqrt{a^2 + b^2}$.

(vi) The equation $a \sin x + b \cos x = c$ has real solutions only if $|c| \leq \sqrt{a^2 + b^2}$, i.e., if $c^2 \leq a^2 + b^2$

(vii) $\tan(x_1 + x_2 + x_3 + \ldots\ldots + x_n) = \dfrac{S_1 - S_3 + S_5 - \ldots\ldots}{1 - S_2 + S_4 - \ldots\ldots}$,

UNIT – 12 : TWO DIMENSIONAL GEOMETRY

Straight Lines

1. Distance Formula

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : Distance of (x_1, y_1) from origin $= \sqrt{x_1^2 + y_1^2}$

2. Section formula

If R (x, y) divides the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio m_1 : m_2 $(m_1, m_2 > 0)$ then

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}; \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \qquad \text{(divides internally)}$$

and $$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}; \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \qquad \text{(divides externally)}$$

The mid point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

3. Area of a Triangle

The area of a ΔABC with vertices A (x_1, y_1); B (x_2, y_2) and C (x_3, y_3) is denoted by Δ and is given as:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

4. Standard points of a Triangle

(i) The centroid of a triangle with vertices A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(ii) The incentre of the triangle ABC is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

a, b, c being the sides BC, CA, AB of the triangle respectively.

(iii) The orthocenter of the triangle ABC is

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

(iv) The circumcentre of the triangle ABC is

$$\left(\frac{x_1 \tan 2A + x_2 \tan 2B + x_3 \tan 2C}{\tan 2A + \tan 2B + \tan 2C}, \frac{y_1 \tan 2A + y_2 \tan 2B + y_3 \tan 2C}{\tan 2A + \tan 2B + \tan 2C}\right)$$

5. Collinearity of three Given Points

The three given points are collinear i.e., lie on the same straight line if

(i) Area of triangle ABC is zero.

(ii) Slope of AB = slope of BC = slope of AC

(iii) Distance between A and B + distance between B and C = Distance between A and C

(iv) Find the equation of the line passing through any two points, if third point satisfied the equation of the line then three points are collinear.

6. Slope of a line

The slope of a line joining two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta \left(x_1 \neq x_2\right)$$

where θ is angle which the line makes with the positive direction of x-axis.

7. Equation of Straight line in Various Forms

(i) ***General Form :*** The general equation of the first degree in x and y is $ax + by + c = 0$,where a and b can not be zero at the same time.

Its slope is $= -\frac{a}{b}$

Intercept on the x-axis is $= -\frac{c}{a}$

Intercept on the y-axis $= -\frac{c}{b}$

(ii) ***Intercepts form :*** The equation of a line making intercepts 'a' and 'b' upon x and y axes respectively is given by

$$\frac{x}{a} + \frac{y}{b} = 1$$

(iii) ***Slope-intercept form :*** The equation of line which has slope m and cuts off an intercept c upon y-axis is given by $y = mx + c$ where $m = \tan\theta$

(iv) ***Point-Slope form :*** The equation of a line passing through the point (x_1, y_1) and having slope m is given by

$$y - y_1 = m(x - x_1)$$

(v) ***Two point Form :*** The equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is given by $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

(vi) ***Parametric Form :*** The equation of a line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$

Where r is the distance of the point (x, y) from the point (x_1, y_1). If r is positive, then the point (x, y) is on the right of (x_1, y_1) but if r is negative then (x, y) is on the left of (x_1, y_1).

(vii) ***Normal Form*** **:** The equation of a line on which the perpendicular from origin is of length p and the perpendicular makes an angle α with the positive direction of x-axis is given by, $x\cos\alpha + y\sin\alpha = p$

8. Positions of points (x_1, y_1) and (x_2, y_2) relative to a given line

If the points (x_1, y_1) and (x_2, y_2) are on the same side of the line

$ax + by + c = 0$, then $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ both are of the same sign and hence $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$, and if the points ($x_1$, y_1) and (x_2, y_2) are on the opposite of the line $ax + by + c = 0$, then $ax_1 + by_1 + c$ *and* $ax_2 + by_2 + c$ both are of signs opposite to each other and hence

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

9. Angle between two Lines

Angle between two lines whose slopes are m_1 and m_2 is $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Corollary 1 : If two lines whose slopes are m_1 and m_2 are parallel if

$\theta = 0(\text{or } \pi) \Leftrightarrow \tan\theta = 0 \Leftrightarrow m_1 = m_2$.

Corollary 2 : If two lines whose slopes are m_1 and m_2 are perpendicular if $\theta = \frac{\pi}{2}\left(\text{or} - \frac{\pi}{2}\right)$

$\Rightarrow \cot\theta = 0 \Rightarrow m_1 . m_2 = -1$.

Note : Two lines given by the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

(ii) Perpendicular if $a_1 a_2 + 4b_1 b_2 = 0$

(iii) Identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

10. Length of perpendicular from a Point on a Line

The length of perpendicular from (x_1, y_1) on $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}\right|$$

11. Distance between Two Parallel Line.

The perpendicular distance between the parallel lines $ax + by + c = 0$ and $ax + by + c_1 = 0$ is

$$\left|\frac{c_1 - c}{\sqrt{(a^2 + b^2)}}\right|$$

12. Family of Lines

Any line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ can be represented by the equation $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$.

13. Concurrent Lines

The three lines $a_ix + b_iy + c_i = 0, i = 1, 2, 3$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

14. Equations of the bisectors of the angles between two Lines

Equations of the bisectors of the lines

$L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$

$(a_1b_2 \neq a_2b_1)$ where $c_1 > 0$ and $c_2 > 0$ are

$$\frac{(a_1x+b_1y+c_1)}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{(a_2x+b_2y+c_2)}{\sqrt{(a_2^2+b_2^2)}}$$

Conditions	**Obtuse angle bisector**	**Acute angle bisector**
$a_1a_2+b_1b_2<0$	-	+
$a_1a_2+b_1b_2>0$	+	-

Pair of Straight Lines

1. Homogeneous Equation of Second Degree

An equation of the form $ax^2+2h\,xy+by^2=0$ is called a homogeneous equation of second degree. It represent two straight lines through the origin.

(i) The lines are real and distinct if $h^2-ab>0$.

(ii) The lines are coincident if $h^2-ab=0$.

(iii) The lines are imaginary if $h^2-ab<0$.

(iv) If the lines represented by $ax^2+2h\ xy+by^2=0$ be $y-m_1x=0$ and $y-m_2x=0$ then $(y-m_1x)(y-m_2x)=y^2+\frac{2h}{b}xy+\frac{a}{b}x^2=0$

$\Rightarrow\ m_1+m_2=-\frac{2h}{b},\ m_1m_2=\frac{a}{b}$

2. Angle between two Lines

If $a+b\neq 0$ and θ is the actual angle between the lines $ax^2+2h\,xy+by^2=0$, then

$$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

Note: Lines $ax^2+2h\,xy+by^2=0$ are mutually perpendicular iff $a+b=0$

3. Equation of the Bisectors of the Angles between the Lines $ax^2 + 2hxy + by^2 = 0$

The equation of bisectors is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

4. General equation of Second Degree

The equation, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, is the general second degree equation.

It represents a pair of straight lines if

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e., if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

otherwise it represents a conic (i.e., if $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$)

Circle and Family of circles

1. The equation of a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$. If the centre is at the origin, the equation of circle is $x^2 + y^2 = r^2$.

2. Equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

3. The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Where g, f, c are constants. The centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}, \left(g^2 + f^2 \geq c\right)$.

Note : A general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

In x, y represents a circle if

(i) Coefficient of x^2 = coefficient of y^2 i.e., $a = b \neq 0$

(ii) Coefficient of xy is zero, i.e. $h = 0$.

4. The equation of the circle through three non-collinear points A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

5. The point P (x_1, y_1) lies outside, on or inside the circle

$S \equiv x^2+y^2+2gx+2fy+c=0.$ according as

$S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c >= \text{or} < 0.$

6. The parametric co-ordinates of any points on the circle

$(x-h)^2+(y-k)^2=r^2$ are given by

$(h+r\cos\theta, k+r\sin\theta), (0 \le \theta < 2\pi)$

7. The equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at the point (x_1, y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$ and that of the normal is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$

The equation of tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) is

$xx_1+yy_1=r^2$ and that of the normal $\frac{x}{x_1}=\frac{y}{y_1}$.

8. The general equation of a line with slope m and which is tangent to a circle

$x^2+y^2=a^2$ is $y=mx \pm a\sqrt{(1+m^2)}$.

9. The locus of point of intersection of two perpendicular tangents is called the director circle. The director circle of the circle

$x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

10. Equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

in terms of its middle point (x_1, y_1) is

$T = S_1$

Where

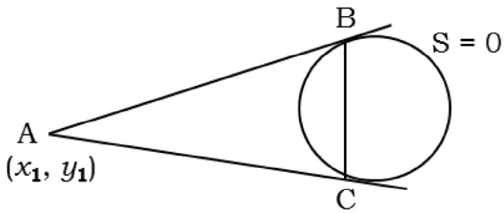
$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$; $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

11. Equation of the Chord of contact

Equation of the chord of contact of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

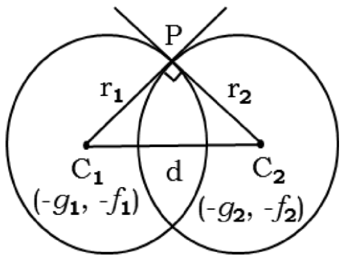
is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$



12. Length of tangent $= \sqrt{\left(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c\right)} = \sqrt{S_1}$

13. Condition of orthogonality of two circles

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$ OR $r_1^2 + r_2^2 = d^2$



14. Pair of tangents

Tangents are drawn from P(x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

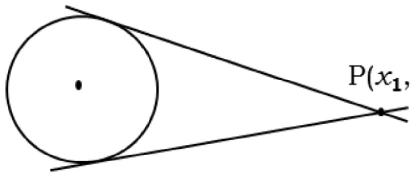
Then equation of pair of tangents is

$SS_1 = T^2$

Where $S \equiv x^2 + y^2 + 2gs + 2fy + c = 0$

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$



15. Radical Axis : The equation of radical-axis if two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$ (coefficient of x^2, y^2 in S_1 and S_2 are 1).

16. Family of Circles

Let $S \equiv x^2 + y^2 + 2gx + 2fy + C = 0$

$S' \equiv x^2 + y^2 + 2g'x + 2f'y + C' = 0$

and $L = px + qy + r = 0$, then

(i) If S = 0 and S' = 0 intersect in real and distinct points, $S + \lambda S' = 0 (\lambda \neq -1)$ represents a family of circles passing through these points. S - S' = 0 $(\text{for } \lambda = -1)$ represents the common chord of the circles S = 0 and S' = 0.

Conic Section

Conic is the locus of a point moving in a plane so that the ratio of its distance from a fixed point (known as focus) to its distance from a fixed line (known as directrix) is constant. This ratio is known as Eccentricity and is denoted by *e*.

If $e = 1$, then locus is a **Parabola.**

If $e < 1$, then locus is an **Ellipse.**

If $e > 1$, then locus is a **Hyperbola.**

Nature of Conics

The equation of conics is represented by the general equation of second degree

$$ax^2 + 2h\,xy + by^2 + 2gx + 2fy + c = 0, \text{ when its discriminant, } \Delta \neq 0.$$

The discriminant of the above equation is given by

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

The following table shows the nature of conic for different condition on a, b and h.

Condition	Nature of Conic
$h = 0,\ a = b$	a Circle
$ab - h^2 = 0$	a Parabola
$ab - h^2 > 0$	an Ellipse
$ab - h^2 < 0$	a Hyperbola
$ab - h^2$ and $a + b = 0$	a Rectangular hyperbola.

Parabola

1. Standard form of a Parabola

The general form of standard parabola is $y^2 = 4ax$, where a is constant.

2. Important Properties

(i) SP = PM and AS = AO

(ii) Vertex is at origin $A \equiv (0,0)$

(iii) Focus is at $S \equiv (a,0)$

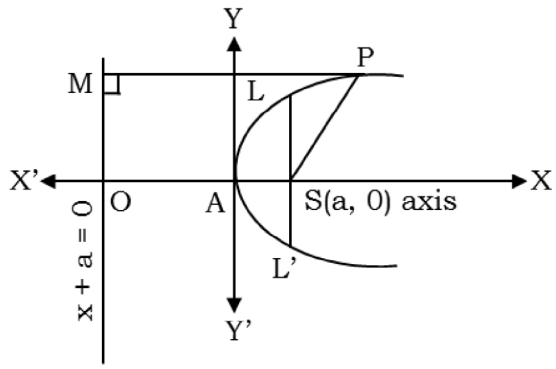
(iv) Directrix is $x + a = 0$

(v) Axis is $y = 0$ (x-axis)

(vi) Length of latus rectum = LL' = $4a$

(vii) Ends of the latus rectum are $L \equiv (a, 2a)$ and $L'(a, -2a)$.

(viii) The parametric equation is : $x = at^2$, $y = 2at$.



3. General equation of a parabola

$$(x-a)^2+(y-b)^2=\frac{(lx+my+n)^2}{l^2+m^2}$$

This equation is of the form

$$(mx-ly)^2+2gx+2fy+C=0$$

Which is the general equation of a parabola.

4. Position of a Point (*h*, *k*) with respect to a Parabola $y^2 = 4ax$

Let P be any point (h, k).

Now P will lie outside, on or inside the parabola according as $\left(k^2-4ah\right)>=0$.

5. Equation of the tangent

Equation of tangents in terms of slope, $y=mx+\frac{a}{m}, m\neq 0$.

The equation of the tangent at any point (x_1, y_1) on the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

Corollary 1 : Equation of tangent at any point having parameter 't' is

$ty = x + at^2$ Slope of tangent is $\frac{1}{t}$

Corollary 2 : Co – ordinates of the point of intersection of tangents at 't_1' and 't_2' is $\{at_1t_2, a(t_1+t_2)\}$.

Corollary 3 : If the chord joining 't_1' and 't_2' is a focal chord, then

$t_1 t_2 = -1.$

$$\Rightarrow t_2 = \frac{-1}{t_1}$$

Hence if one extremity of a focal chord is $\left(at_1^2, 2at_1\right)$, then the other extremity $\left(at_2^2, at_2\right)$ becomes $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.

6. Equation of the Normal

Equation of normal at any point 't' is

$$y = -tx + 2at + at^3$$

Slope of normal is $-t$.

7. Equation of the Normal in terms of slope

$$y = mx - 2am - am^3$$

at the point $(am^2, -2am)$

Hence any line $y = mx + c$ will be a normal to the parabola if $c = -2am - am^3$.

8. Equation of chord with mid point (x_1, y_1)

The Equation of the chord of parabola $y^2 = 4ax$, whose mid point be (x_1, y_1) is

$$T = S_1$$

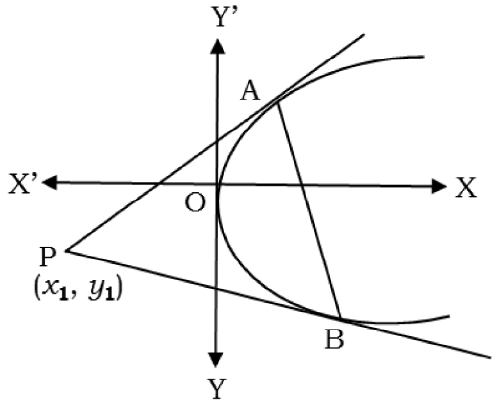
Where $T \equiv yy_1 - 2a\left(x + x_1\right) = 0$

and $S_1 = y_1^2 - 4ax_1 = 0$

9. Chord of Contact

If PA and PB be the tangents through point $P(x_1, y_1)$ (see Figure) to the parabola $y^2 = 4ax$, then the equation of the chord of contact AB is

$$yy_1 = 2a(x + x_1) \text{ or } T = 0 \text{ at } (x_1, y_1)$$



10. Pair of Tangents

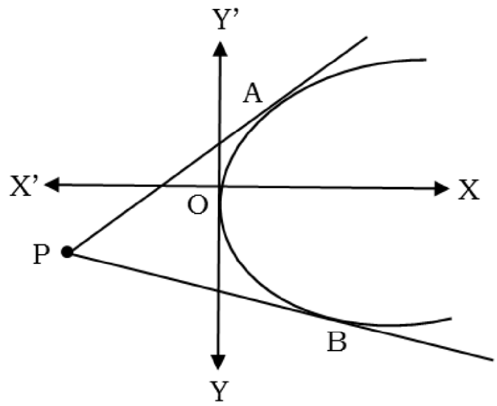
If P (x_1, y_1) be any point lies outside the parabola $y^2 = 4\ ax$, and a pair of tangents PA, PB can be draw to it from P, (see Figure) then the equation of pair of tangents PA and PB is

$$SS_1 = T^2$$

Where, $S \equiv y^2 - 4ax = 0$

$$S_1 \equiv y_1^2 - 4ax_1 = 0$$

$$T \equiv yy_1 - 2a(x + x_1) = 0$$

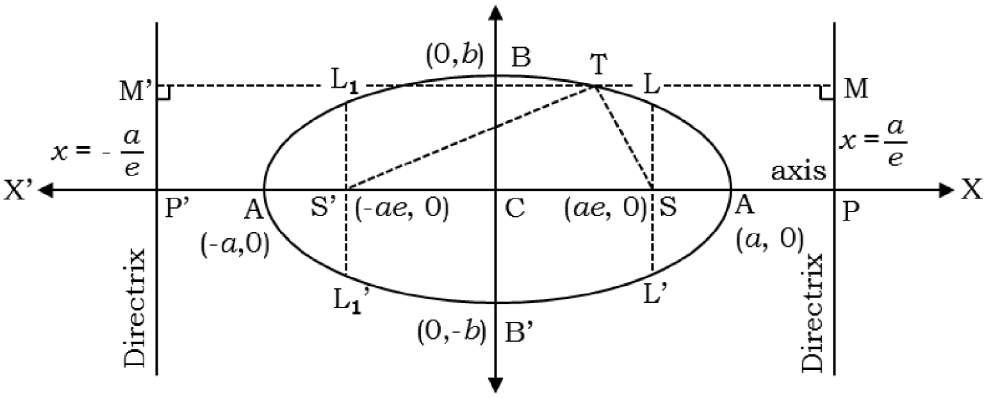


Ellipse

1. Standard form of an Ellipse

The general form of standard ellipse is :

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b)$, where a and b are constants. (see Figure below)



2. Important Properties

(i) ST = e TM

(ii) Co-ordinate of centre C (0,0)

(iii) Equation of directrix $x=\pm a/e$

(iv) Equation of latus rectum $x=\pm\ ae$

3. General Equation of an Ellipse

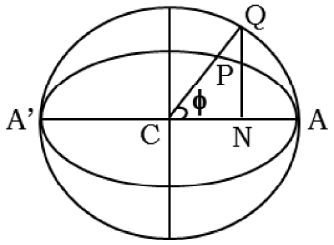
$$\left(l^2+m^2\right)\left\{(x-a)^2+(y-b)^2\right\}=e^2\,(lx+my+n)^2$$

4. Parametric Equation of an Ellipse

The parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $x=a\cos\phi,\ y=b\sin\phi$, where ϕ is the parameter.

5. Auxiliary Circle and Eccentric angle

The circle described on the major axis of an ellipse as diameter is called its auxiliary circle (see Figure) .



The equation of the auxiliary circle is

$$x^2 + y^2 = a^2$$

$\therefore$ Q $\equiv (a\cos\phi, a\sin\phi)$ and P $\equiv (a\cos\phi, b\sin\phi)$

$(\phi = \textit{eccentric angle})$

6. Condition for tangency

A line $y = mx + c$ is tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c = \pm\sqrt{(a^2m^2 + b^2)}$.

Corollary 1 : $x\cos\alpha + y\sin\alpha = p$ is a tangent if

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.$$

Corollary 2 : $lx + my + n = 0$ is a tangent if $n^2 = a^2l^2 + b^2m^2$.

7. Equation of the Tangent

(i) The equation of the tangent at any point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

(ii) The equation of tangent at any point 'ϕ' is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$.

8. Equation of the Normal

(i) The equation of the normal at any point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(ii) The equation of the normal at any point 'ϕ' is

$$ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2$$

9. Equation of chord with mid point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point be (x_1, y_1) is T = S_1

Where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0;\ S_1 \equiv \frac{x_1^2}{a^2}+\frac{y^2}{b^2}=1$

10. Chord of Contact

If PA and PB be the tangents through point P(x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

11. Pair of tangents

Let P(x_1, y_1) be any point lying outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a pair of tangents PA, PB can be drawn to it from P. then the equation of pair of tangents of PA and PB is

$SS_1 = T^2$

12. Important properties of an Ellipse

(i) If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse then $\alpha+\beta+\gamma+\delta=2n\pi, n \in I$

(ii) The necessary and sufficient condition for the normal at three α, β, γ points on the ellipse to be concurrent is

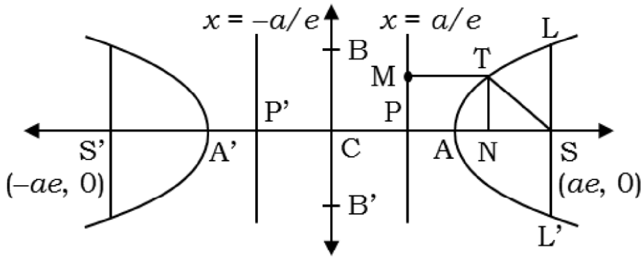
$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$$

Hyperbola

1. Standard form of a Hyperbola

If (a, b) is the focus S, and $lx + my + n = 0$ is the equation of directrix, then the standard equation of a hyperbola is

$$\left(l^2+m^2\right)\left\{(x-a)^2+(y-b)^2\right\}=e^2(lx+my+n)^2$$



2. Important Properties

(i) ST = e TM

(ii) Co-ordinates of vertices A and A' are $(\pm a, 0)$.

(iii) Co-ordinates of the foci S and S' are $(\pm ae, 0)$.

(iv) Equation of directrix $x = \pm a / e$.

(v) Equation of latus rectum $x = \pm ae$ and length $LL' = L_1L_1 = \frac{2b^2}{a}$.

3. Parametric Equation of the Hyperbola

The parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x = a \sec \phi$, $y = b \tan \phi$, where ϕ is the parameter.

4. Condition for Tangency

A line $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ iff $c^2 = a^2m^2 - b^2$.

5. Equation of the Tangent

(i) The equation of the tangent at any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(ii) The equation of tangent at any point 'ϕ' is $\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1$

6. Equation of Chord with Midpoint (x_1, y_1)

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid point be (x_1, y_1) is T = S_1

Where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 = 0$, $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$

7. Chord of Contact

If two tangents are drawn through a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of the chord of contact is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \text{ or } T = 0 \text{ at} (x_1, y_1)$$

8. Pair of Tangents

If a pair of tangents is drawn from any point (x_1, y_1) outside the hyperbola, then the equation of pair of tangent is

$$SS_1 = T^2$$

Where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$; $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0$; $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

9. Asymptotes of Hyperbola

A hyperbola has two asymptotes passing through its centre. Asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Note : Angle between asymptotes $= 2 \tan^{-1}\left(\frac{b}{a}\right)$.

10. Rectangular Hyperbola

If asymptotes of the standard hyperbola are perpendicular to each other it is known as rectangular hyperbola.

Properties of Rectangular Hyperbola $xy = c^2$

(i) Eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}.$

(ii) Since $x = ct,\ y = \frac{c}{t}$ satisfies $xy = c^2, (x, y) = \left(ct, \frac{c}{t}\right)$ is called a 't' point with parameter t.

(iii) Equation of the chord joining t_1 and t_2 is

$$x + y\,t_1 t_2 - c(t_1 + t_2) = 0$$

(iv) Equation of tangent at 't' is

$$x + yt^2 - 2ct = 0$$

(v) Equation of normal at 't' is

$$xt^2 - yt - ct^4 + c = 0$$

(vi) Equation of tangent at (x_1, y_1) is

$$xy_1 + yx_1 = 2c^2$$

(vii) Equation of normal at (x_1, y_1) is

$$xx_1 - yy_1 = x_1^2 - y_1^2$$

UNIT – 13 : THREE DIMENSIONAL GEOMETRY

1. **Distance formula:** The distance between two points P(x_1, y_1, z_1) and Q (x_1, y_2, z_2) in space is given by $\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

2. **Section formula:** If R (x, y, z) divides the join of P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) in the ratio $m_1 : m_2$ $(m_1, m_2 > 0)$ then

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2};\ y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2};\ z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

(divides internally), and

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2};\ y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2};\ z = \frac{m_1z_2 - m_2z_1}{m_1 - m_2}$$ (divides externally)

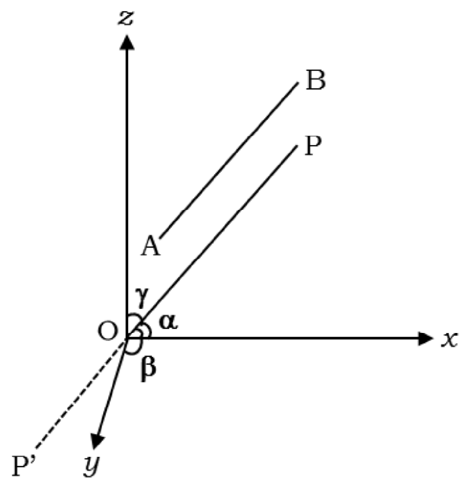
3. Centroid of a triangle: The centroid of a triangle ABC whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

4. Direction Cosines (d.c.'s):

If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of a given line AB, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



5. Direction ratios (d.r.'s): direction ratios of a line are numbers which are proportional to the direction cosines of a line. Direction ratios of a line PQ, where P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

6. Relation between the d.c.'s and d.r.'s: If a, b, c are the d.r.'s and l, m, n are the d.c.'s then

$$l = \pm\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \; m = \pm\frac{b}{\sqrt{(a^2+b^2+c^2)}}, \; n = \pm\frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

7. Angle between two Lines

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)}\sqrt{\left(a_2^2 + b_2^2 + c_2^2\right)}}$$

and $$\sin\theta = \pm\frac{\sqrt{\Sigma\left(b_1c_2 - b_2c_1\right)^2}}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)}\sqrt{\left(a_2^2 + b_2^2 + c_2^2\right)}}$$

8. Angle between two Planes

If θ be the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then

$$\theta = \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)}\sqrt{\left(a_2^2 + b_2^2 + c_2^2\right)}}\right)$$

Also, **1.** If planes are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

2. If planes are parallel then, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

9. Length of Perpendicular from a point to a Plane

The length of perpendicular from (x_1, y_1, z_1) on $ax + by + cz + d = 0$ is

$$\frac{\left|ax_1 + by_1 + cz_1 + d\right|}{\sqrt{\left(a^2 + b^2 + c^2\right)}}$$

10. Family of Planes

Any plane passing through the line of intersection of the planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ can be represented by the equation

$$\left(ax + by + cz + d\right) + \lambda\left(a_1x + b_1y + c_1z + d_1\right) = 0$$

UNITS – 14 : VECTOR ALGEBRA

1. Position Vector of R($\bar{r}$) dividing $\overline{PQ}$ in the ratio $m : n$ is

$\bar{r} = \dfrac{m\bar{q} + n\bar{p}}{m+n}$ [internal division] and

$\bar{r} = \dfrac{m\bar{q} - n\bar{p}}{m-n}$ [external division]

2. If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$,

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta$; also $\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,.\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

3. $\bar{a}.\bar{b} = 0 \quad \Rightarrow \; a = 0$ or $b = 0$ or $a \perp b$

4. $\bar{a} \times \bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\bar{e}$; $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}|\sin\theta$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

6. $$(\bar{a} \times \bar{b}).\bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \bar{a}.(\bar{b} \times \bar{c}) = [\bar{a}\,\bar{b}\,\bar{c}]$$

7. $\bar{a}, \bar{b}, \bar{c}$ are coplanar if and only if $\bar{a}.(\bar{b} \times \bar{c}) = 0$.

8. Area of parallelogram OACB $= |\bar{a} \times \bar{b}|$

Area of ΔOAB $= \dfrac{1}{2}|\bar{a} \times \bar{b}|$

9. Volume of parallelepiped = $[\bar{a}\,\bar{b}\,\bar{c}]$

10. For any 3 vectors $\bar{a}, \bar{b}, \bar{c}, [\bar{a}\,\bar{b}\,\bar{c}] = [\bar{b}\,\bar{c}\,\bar{a}] = [\bar{c}\,\bar{a}\,\bar{b}]$

11. Unit vector $\perp^r$ to $\bar{a}$ and $\bar{b}$ $= \pm \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$ and $\sin\theta = \frac{|\bar{a} \times \bar{b}|}{|\bar{a}||\bar{b}|}$

UNITS – 15 : MEASURES OF CENTRAL TENDENCY & DISPERSION

1. Arithmatic Mean (AM)

(i) Individual Series $\bar{x} = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$

(ii) Discrete Series $\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{f_1 + f_2 + \ldots\ldots + f_n} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i$

where $N = \sum_{i=1}^{n} f_i$

2. Geometric Mean (GM)

(i) Individual series

$G = (x_1, x_2, \ldots\ldots x_n)^{1/n}$

(ii) Discrete series

$G = (x_1^{f_1}\, x_2^{f_2} \ldots\ldots x_n^{f_n})^{1/N}$, where $N = \sum_{i=1}^{n} f_i$

3. Harmonic Mean (HM)

$$H = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots + \frac{1}{x_n}\right)}$$

or $\frac{1}{H} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$ (Individual series)

and $\frac{1}{H} = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i}}$ (Discrete series)

If $x_1, x_2, \ldots\ldots, x_n > 0$, then it is known that

AM ≥ GM ≥ HM

4. Median

$$M = l + \frac{h}{f}\left(\frac{N}{2} - C\right) \text{ where, } \quad N = \sum_{i=1}^{n} f_i$$

h = the width of the median class

C = the cumulative frequency (c.f.) of the preceding to the median class.

f = the frequency of the median class

l = the lower limit of the median class

5. Mode

$$\text{Mode (for continuous series)} = l + \frac{(f_1 - f_0)}{(2f_1 - f_2 - f_0)} \times h$$

where l = the lower limit of the modal class (the class having maximum frequency)

f_1 = frequency of the modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

h = width of the modal class

6. Mean - Deviation (MD)

$$MD = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - A|, \text{ where } N = \sum_{i=1}^{n} f_i$$

7. Standard Deviation (SD)

$$\alpha^2 \text{ (Variance) } = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

and Standard deviation = $\sqrt{\text{Variance}}$

UNITS – 16 : PROBABILITY

1. Definition

The probability of occurrence of an event is the ratio of the number of cases in its favour to the total number of cases (equally likely).

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of Favourable cases}}{\text{total number of cases}}$$

2. Types of Events

(i) **Equally likely Events :** The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

(ii) **Independent Events :** Two events are said to be independent if the occurrence of one does not depend upon the other. If a set of events $E_1, E_2, \ldots\ldots, E_n$ are independent events, then

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots\ldots \cap E_n) = P(E_1).P(E_2)\ldots\ldots P(E_n)$$

(iii) **Mutually Exclusive Events :** A set of events is said to be *mutually exclusive* if occurrence of one of them precludes the occurrence of any of the remaining events.

If a set of events $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive events, then

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots\ldots \cap E_n) = \phi$$

then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = P(E_1) + P(E_2) + \ldots\ldots P(E_n)$

(iv) **Exhaustive Events :** A set of events is said to be *Exhaustive* if the performance of the experiment results in the occurrence of at least one of them.

If a set of Events $E_1, E_2, \ldots\ldots, E_n$ are exhaustive events, then

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = 1$$

(v) **Compound Events :** If $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events, then if E is any event,

$$P(E) = \sum_{i=1}^{n} P(E \cap E_i) = \sum_{i=1}^{n} P(E_i).P\left(\frac{E}{E_i}\right), \text{ if } P(E_i) > 0.$$

3. Conditional Probability

The probability of occurrence of an events E_1, given that E_2 has already occurred is called the conditional probability of occurrence of E_1 on the condition that E_2 has already occurred. It is denoted by $P\left(\frac{E_1}{E_2}\right)$

i.e., $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, E_2 \neq \phi$

4. Bayes's Theorem or Inverse Probability

If $E_1, E_2, \ldots\ldots, E_n$ are n mutually exclusive and exhaustive events such that

$$P(E_i) > 0, (0 \le i \le n)$$

and E is any event, then for $1 \le k \le n$,

$$P\left(\frac{E_k}{E}\right) = \frac{P(E_k)P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n} P(E_k)P\left(\frac{E}{E_k}\right)}.$$

5. Important Results

(i) If E_1 and E_2 are arbitrary events, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

(ii) If E_1, E_2, E_3 are three events then

$$P(E_1 \cup E_2 \cup E_2) = P(E_1) + P(E_2) + P(E_3)$$
$$-P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

UNITS – 17 : TRIGONOMETRY

Trigonometric Functions and Identities

1. Measurement of an Angle

(i) 1^{o} = 60 minutes = 60' (ii) 1' = 60 seconds = 60"

(iii) Each interior angle of a regular polygon of n sides is equal to

$\frac{(n-2)\pi}{n}, n > 2$.

2. Some Important Formulae and Identities

sin (A + B) = sin A cos B + cos A sin B sin(A - B) = sin A cos B - cos A sin B

cos (A + B) = cos A cos B - sin A sin B cos (A - B) = cos A cos B + sin A sin B

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$; $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan(45^{o} + A) = \frac{1 + \tan A}{1 - \tan A}$; $\tan(45^{o} - A) = \frac{1 - \tan A}{1 + \tan A}$

sin (A + B) sin (A - B) = $\sin^2$ A - $\sin^2$ B = $\cos^2$ B - $\cos^2$ A

cos (A + B) cos (A - B) = $\cos^2$ A - $\sin^2$ B = $\cos^2$ B - $\sin^2$ A

2 sin A cos B = sin (A + B) + sin (A - B) 2 cos A sin B = sin (A + B) - sin (A - B)

2 cos A cos B = cos (A + B) + cos (A - B) 2 sin A sin B = cos (A - B) - cos (A + B)

sin C + sin D = $2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ sin C - sin D = $2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

cos C + cos D = $2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ cos C - cos D = $2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

sin 2A = 2 sin A cos A = $\frac{2\tan A}{1+\tan^2 A}$

$$\cos 2A = \begin{cases} 1 - 2\sin^2 A \\ \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A} \\ 2\cos^2 A - 1 \end{cases}$$

tan 2A = $\frac{2\tan A}{1-\tan^2 A}$

$\sin^2 A = \frac{1-\cos 2A}{2}$ and $\cos^2 A = \frac{1+\cos 2A}{2}$

$\sin 3A = 3 \sin A - 4 \sin^3 A$; $\cos 3A = 4 \cos^3 A - 3 \cos A$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

3. Conditional Identities

If $A + B + C = 180^o$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan c$

(vi) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

4. Trigonometrical Ratios

(i) $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

(ii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

(iii) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta = 1 - \sin^2\theta\cos^2\theta$

(iv) $\sin^2\theta + \text{cosec}^2\theta \geq 2, \cos^2\theta + \sec^2\theta \geq 2$ and $\sec^2\theta + \text{cosec}^2\theta \geq 4$

5. Some standard Results

(i) $\tan\theta + \cot\theta = 2\,\text{cosec}\,2\theta$

(ii) $\cot\theta - \tan\theta = 2\cot 2\theta$

(iii) $\sin\theta + \cos\theta$ has the same sign as that of $\sin\left(\theta + \frac{\pi}{4}\right)$

(iv) $\sin\theta - \cos\theta$ has the same sign as that of $\sin\left(\theta - \frac{\pi}{4}\right)$

(v) Maximum value of $a \sin x + b \cos x$ is $\sqrt{a^2 + b^2}$ and its minimum value is $-\sqrt{a^2 + b^2}$.

(vi) The equation $a \sin x + b \cos x = c$ has real solutions only if $|c| \le \sqrt{a^2 + b^2}$, i.e., if $c^2 \le a^2 + b^2$

(vii) $\tan(x_1 + x_2 + x_3 + \ldots\ldots + x_n) = \dfrac{S_1 - S_3 + S_5 - \ldots\ldots}{1 - S_2 + S_4 - \ldots\ldots}$,

where S_r stands for the sum of the products of $\tan x_1, \tan x_2, \tan x_3, \ldots\ldots \tan x_n$ taken r at a time. For example,

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\tan(A + B + C + D) = \frac{\sum \tan A - \sum \tan A \tan B \tan C}{1 - \sum \tan A \tan B + \tan A \tan B \tan C \tan D}$$

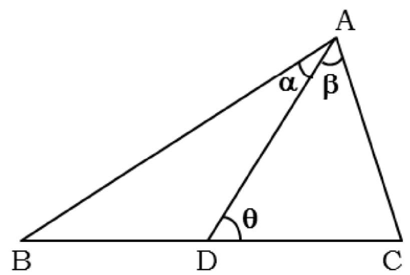
In particular, $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$

(viii) **"*m, n*" theorem**

If D is a point on the side BC of a triangle ABC such that BD : DC :: $m : n$, $\angle ADC = \theta, \angle BAD = \alpha, \angle CAD = \beta$ then

$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$ and

$(m + n)\cot\theta = n\cot B - m\cot C$



(ix) $\cos\theta \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^n\theta$

$$= \frac{\sin 2^{n+1}\theta}{2^{n+1}\sin\theta} \text{ for all } n \in N$$

(x) $(2\cos\theta-1)(2\cos 2\theta-1).(2\cos 2^2\theta-1)\ \ldots\ldots\ (2\cos 2^n\theta-1)$

$$= \frac{2\cos 2^{n+1}\theta+1}{2\cos\theta+1} \text{ for all } n \in \mathrm{N}$$

(xi) $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\sqrt{2+2\cos 2^n\theta}}}} = 2\cos\theta$

for all $n \in \mathrm{N}$, where there are n square root signs on the left hand side.

(xii) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots+\sin(\alpha+\overline{n-1}\,\beta)$

$$= \frac{\sin\left(\frac{\alpha+\alpha+\overline{n-1}\,\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\left(\frac{\beta}{2}\right)}, n \in \mathrm{N}$$

(xiii) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos(\alpha+\overline{n-1}\,\beta)$

$$= \frac{\cos\left(\frac{\alpha+\alpha+\overline{n-1}\,\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\left(\frac{\beta}{2}\right)}, n \in \mathrm{N}$$

Trignometric Equations

$\sin x = 0 \Leftrightarrow x = n\pi,\ n \in \mathrm{I}$

$\cos x = 0 \Leftrightarrow x = (2n+1)\frac{\pi}{2},\ n \in \mathrm{I}$

$\tan x = 0 \Leftrightarrow x = n\pi,\ n \in \mathrm{I}$

$\sin x = \sin\alpha \Leftrightarrow x = n\pi + (-1)^n\alpha,\ n \in \mathrm{I}$

$\cos x = \cos\alpha \Leftrightarrow x = 2n\pi \pm \alpha,\ n \in \mathrm{I}$

$\tan x = \tan\alpha \Leftrightarrow x = n\pi \pm \alpha,\ n \in \mathrm{I}$

$\sin^2 x = \sin^2\alpha \Leftrightarrow x = n\pi \pm \alpha,\ n \in \mathrm{I}$

$\cos^2 x = \cos^2 \alpha \Leftrightarrow x = n\pi \pm \alpha,\ n \in I$

$\tan^2 x = \tan^2 \alpha \Leftrightarrow x = n\pi \pm \alpha,\ n \in I$

Properties and Solution of Triangles

1. Relations between the Sides and Angles of a Triangle

(i) **Sine Formulae**

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of the triangle.

or $a = k \sin A$, $b = k \sin B$, $c = k \sin C$ where k is some non-zero constant.

or $\sin A = \frac{a}{k}, \sin B = \frac{b}{k}, \sin C = \frac{c}{k}$, k being some non-zero constant.

(ii) **Cosine Formulae**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \sin C = \frac{a^2 + b^2 - c^2}{2ab}$$

(iii) **Projection Formulae**

$a = b \cos C + c \cos B$; $b = c \cos A + a \cos C$; $c = a \cos B + b \cos A$

(iv) **Napier's Analogy**

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2};\ \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2};\ \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

(v) **Semi-sum Formulae**

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};\ \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}};\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}};\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}};\quad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};\ \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}};\ \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

where, $s = (a + b + c) / 2$

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$, where Δ = Area of ΔABC

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Hero's formula)

2. Circumradius (R) and Inradius (r) Formulae

(i) $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

(ii) $r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = 4R\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Inverse Trigonometric Functions

Important Formulae

$y = \sin^{-1}x$ iff $x = \sin y$, $|x| \le 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1}x$ iff $x = \cos y$, $|x| \le 1$, $y \in [0, \pi]$

$y = \tan^{-1}x$ iff $x = \tan y$, $x \in R$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \cot^{-1}x$ iff $x = \cot y$, $x \in R$, $y \in (0, \pi)$

$y = \sec^{-1}x$ iff $x = \sec y$, $|x| \ge 1$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

$y = \mathrm{cosec}^{-1}x$ iff $x = \mathrm{cosec}\ y$, $|x| \ge 1$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, |x| \le 1$ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R$

$\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, |x| \ge 1$ $\sin^{-1}(-x) = -\sin^{-1}x, |x| \le 1$

$\cos^{-1}(-x) = \pi - \cos^{-1}x,\ |x| \le 1$ $\tan^{-1}(-x) = -\tan^{-1}x,\ x \in R$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,\ |x| \ge 1$ $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x,\ 0 < |x| \le 1$

$\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x,\ |x| \ge 1$ $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x,\ 0 < |x| \le 1$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x,\ x > 0$ $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2},\ |x| \le 1$

$\sec(\text{cosec}^{-1}x) = \text{cosec}(\sec^{-1}x) = \frac{|x|}{\sqrt{x^2-1}};\ |x| > 1$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),\ xy < 1$ $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1}x,\ x < 1$

$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} - \tan^{-1}x,\ x > -1$ $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x, x \in R$

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x, x \ge 0$ $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x, |x| < 1$

$2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), |x| \le \frac{1}{\sqrt{2}}$ $2\cos^{-1}x = \cos^{-1}(2x^2-1),\ x \in [0, 1]$

$3\sin^{-1}x = \sin^{-1}(3x-4x^3), |x| \le \frac{1}{2}$ $3\cos^{-1}x = \cos^{-1}(4x^3-3x), \frac{1}{2} \le x \le 1$

$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

for those values of x and y in [-1, 1] for which LHS lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

where x, y are real numbers in [-1, 1] such that LHS lies in the interval $[0, \pi]$.

(i) $\sin^{-1} x + \sin^{-1} y = \pi$ iff $x = y = 1$

(ii) $\sin^{-1} x + \sin^{-1} y = -\pi$ iff $x = y = -1$

(iii) $\cos^{-1} x + \cos^{-1} y = 0$ iff $x = y = 1$

(iv) $\cos^{-1} x + \cos^{-1} y = 2\pi$ iff $x = y = -1$

(v) $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\cot^{-1} x = 2\tan^{-1}\left(\frac{1}{x}\right)$ for $x \geq 1$

(vi) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, where $xy > 1$, $x > 0$, $y > 0$.

UNITS – 18 : MATHEMATICAL REASONING

(i) $p \wedge q$ is true if p and q are both true.

(ii) $p \vee q$ is false if p and q are both false.

(iii) If p is true, then $\sim p$ is false.

If p is false, then $\sim p$ is true.

(iv) $p \Rightarrow q$ is true in all cases except when p is true and q is false.

(v) $p \Leftrightarrow q$

(a) $p \Leftrightarrow q$ is true if both p and q have same truth value.

(b) $p \Leftrightarrow q$ is false if p and q have opposite truth value.

(vi) If p is any statement, t is tautology and c is contradiction, then :

(a) $p \vee t \equiv t$ (b) $p \wedge t \equiv p$

(c) $p \vee c \equiv p$ (d) $p \wedge c \equiv c$

Complement Laws :

(a) $p \vee (\sim p) \equiv t$ (b) $p \wedge (\sim p) \equiv c$

(c) $\sim t \equiv c$ (d) $\sim c \equiv t$

Involution Laws :

(a) $\sim (\sim p) = p$

DeMorgan's Laws :

If p and q are any two statements, then :

(a) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$ (b) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

Laws of Contrapositive :

$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(vii) If a compound statement contains variables t (tautology) and c (contradiction), then its dual is obtained by replacing t by c and c by t in addition to the replacement of $\vee$ and $\wedge$ by $\wedge$ and $\vee$ respectively.

Types of Statements :

i) If a statement is always true, then the statement is called "tautology."

ii) If a statement is always false, then the statement is called "contradiction."

iii) If a statement is neither tautology nor a contradiction, then it is called "contingency."

Converse, Contrapositive, Inverse of a Statement

If $p \to q$ is a hypothesis, then

i) Converse $q \to p$.

ii) Contrapositive $\sim q \to \sim p$.

iii) Inverse $\sim p \to \sim q$.