Physics for IIT-JEE VOL. II

HALLIDAY / RESNICK / WALKER

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Manish K. Singhal

Wiley India Pvt. Ltd.



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Dedication

This book is dedicated to my loving teacher,
Prof. P.V. Krishnan *(retired professor, IIT Delhi), and his wife,*
mother ***Krishangi Devi****, whose teachings gave me life.*
Whatever I am today, I owe it to them.

Preface

Having experienced teaching at various levels and with a varied group of students, I have experienced the shortcomings faced by most of the students. I can also remember my own experiences of failure and dejection when I used to study. Often, I did not find answers to my questions, or I could not find a book that could help me think, nor did I find a collection of problems that stimulate the mind. In this book, finally I have come to fulfill a vision of a book that has *an international collection of problems, the theory to such depth as we cover at Bansal Classes, Kota, and American quality of printing*. This book is not a substitute for a teacher. I strictly believe and this has been proven over the years: To learn, you need a teacher. But learning under a qualified teacher will help you understand this book better.

Often students find Physics tough or at least challenging. Perhaps it is both because it tries to encourage you to think and it does not follow any stereotyped pattern. A thinking individual will do well in physics. In the modern learning atmosphere, perhaps we are never even encouraged to think. I have made an attempt to cross this barrier and challenge you to think at numerous places. For this purpose, I have scattered numerous checkpoints throughout the chapters. The problem section begins with a set of assertions and reason problems, and the student is required to find whether those statements are correct or not. And if they are, what are the reasons.

Those of you who are accustomed to the original Halliday, Resnick, and Walker would be in for a surprise because although this is supposed to be an adaptation, I have freely made the changes as per the needs of the students who are appearing in competitive examinations like Physics Olympiads, IIT JEE, and so on. So the net result is that you have an altogether new book, my style. Three new chapters have been added and many have been deleted. All over the world, slowly objective type examinations are replacing subjective papers. So we have added extensive collection of objective problems on each and every concept covered in chapter. Each chapter has been started with a chapter opener which illustrates real world phenomena based on that chapter. Some of them are fun, and some of them are real phenomena which should set your mind thinking.

The book follows a particular format. I introduce a concept and try to illustrate it by the means of an example. In the spirit of what we did in Volume I of this book, we have explained each illustration in four sections: Conceptualize, classify, compute, and conclude.

- **Conceptualize** explains the problem with a clue to the solution.
- **Classify** contains details of the principles of physics used to solve the problem.
- **Compute** gives the detailed solution.
- The real punch lies in the **conclude** section where many a times the student has to think on the lines of great physicists.

In the exercises that follow at the end of each chapter, I am trying to take the students through the same ideas. The spirit of the book will be marred if the student just focuses on the theory and does not try out the exercises that follow. I have tried to arrange the exercises in increasing order of difficulty. The answers are given at the end of the chapter. If one is not able to solve a particular question, then it will do him good to go back to the illustration and try it out himself.

How to Approach the Question Bank

To gain the most, I would suggest the following sequence:

- **Step 1:** First, go through the text of a section thoroughly with the help of a qualified teacher. Try solving the illustrations yourself and feel the joy of discovering physics.
- **Step 2:** Then go through the multiple choice questions and the subjective exercises pertaining to that section.
- **Step 3:** At the end of that, when you are confident of the basic concepts of the entire chapter, there are additional problems given at the end of problems section.
- **Step 4:** After doing all these, you should go to the **Assertion and Reasons** (Reasoning Type Questions) section and solve them to see if you do understand the principles completely.

These problems take you to the realm of IIT JEE, Physics Olympiad, and even beyond. Do not feel discouraged if you have not been able to solve the problems at the first go. Remember, even Einstein and Newton had faced failures in their lives. Failure is a stepping stone to success. I wish you all the best in your endeavors.

My team and I have tried our level best to offer you an error-free book, even going through the whole manuscript three or four times. All the questions that have been set in the problem section has been solved twice, once by me personally and then by my colleague. Yet, it may be possible that errors may remain in the book. Comments or suggestions on the content and presentation of the book will be received with great interest and highly appreciated.

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Acknowledgments

I have dedicated this book to my loving teacher Prof. P.V. Krishnan and his wife, mother Krishangi Devi. Truly, if they did not give me a spiritual vision of life, I would never even have the boldness of taking up teaching as a profession, much less to venture writing this book. He taught me the real meaning of Science and opened my eyes to a Science which is much more profound and vast than what we are studying here. I encourage all my readers to study his books, "Science and Nescience" and "Wake up! Intellectuals." They gave me an entirely new perspective to life, they will surely change yours.

This book was made possible only by an extensive help rendered by a big group of friends. The leader among them was my friend and colleague Ajit Aggarwal who laboriously went through the entire book and solved each problem personally in detail. My friend Dr. G. Partheepan at IIT Delhi was extremely prompt with his valuable help with the research papers. I am indeed very grateful to Mr. Jearl Walker for allowing me to edit the book the way I wanted. Thanks!

Thanks are also due to Amit Gupta, author of Vol. I, who introduced me to the idea. Prominent among those who helped me with the books are Kishore Rane of Paravyoma classes, Vishakhapatnam; Anurag Mishra, Bansal Classes, Kota; and Ambarish Srivastava, FIITJEE, Delhi. The kind of open heartedness which these three individuals showed in sharing their collection of books with me is a rare example in this age of hypocrisy and cynicism.

I am also deeply indebted to Mr. V.K. Bansal of Bansal classes, Kota, who gave us such an open learning atmosphere at Kota. His dedication to teaching excellence is something which is unparalleled and would be hard to beat in years to come. My editor Meenakshi Sehrawat and her entire team deserve a gold medal for keeping up with my vision of giving the students gem of a book. My typist Prashant Gupta deserves a special applause for making figures that match the quality of the original book. Special thanks are due to my students of today and past whose thought-provoking questions helped me to explore Physics deeper and deeper. Last but not the least, I owe a special thanks to Paras Bansal and Vikas Gupta at Wiley India for putting so much faith in me and risking a big investment for bringing out their first ever textbook in India in color. And that too at Indian prices! This project would have never been successful without these people supporting me.

Manish Kumar Singhal
April 2010

Note to the Student

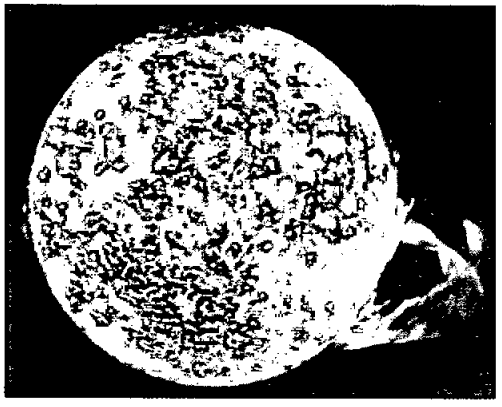
The IIT-JEE is one of the hardest exams to crack for students. For a very simple reason – concepts cannot be learned by rote, they have to be absorbed, and IIT believes in strong concepts. Each question in the IIT-JEE entrance exam is meant to push the analytical ability of the student to its limit. That is why the questions are called brainteasers!

This is where Halliday, Resnick and Walker's *Fundamentals of Physics* comes in. This text has been the definitive text for learning Physics concepts and solving problems conceptually for the last 33 years. Its unparalleled approach to teaching physics has been appreciated globally by students and teachers alike. And that is why it is probably the best resource for an IIT aspirant like you today.

In collaboration with experts from IIT-JEE coaching, the original Halliday, Resnick and Walker edition has been customized to give you the tools that will help you crack the toughest entrance exam in India.

If you take full advantage of the unique features and elements of this textbook, we believe your experience will be fulfilling and enjoyable. Let's walk through some of the special book features that will help you in your efforts to crack IIT-JEE.

PEDAGOGY



Courtesy NASA/JSC

At 2:45 A.M. on March 13, 1989, the entire power-grid system for the Canadian province of Quebec failed, leaving millions of people without power on that cold night. In fact, many power-grid systems in the Northern Hemisphere malfunctioned that night, creating a nightmare situation for the engineers who maintained the systems. The cause was not a sudden overtaxing demand for power or a failure of aging equipment. Rather, the cause was an explosion that had occurred on the Sun's surface three days earlier.

How can a solar explosion shut down a power-grid system?

The answer is in this chapter.

OPENING TEASER

The **opening teaser** and the related image at the beginning are meant to serve two purposes: First, they are real-life phenomena and thus make the concept something real and not just a bookish detail. And second, they make the student curious and incite him/her to get deeper into the chapter and its contents.

CONCEPT EXPLANATION

The **concept explanation** follows. This will lead you to the answer of the teaser in the beginning. The inimitable style of Halliday, Resnick and Walker is obvious in the simple language and real-life examples.

24-1 | WHAT IS PHYSICS?

One goal of physics is to identify basic forces in our world, such as the electric force we discussed in Chapter 22. A related goal is to determine whether a force is conservative—that is, whether a potential energy can be associated with it. The motivation for associating a potential energy with a force is that we can then apply the principle of the conservation of mechanical energy to closed systems involving the force. This extremely powerful principle allows us to calculate the results of experiments for which force calculations alone would be very difficult. Experimentally, physicists and engineers discovered that the electrostatic force is conservative and thus has an associated electric potential energy. In this chapter we first define this type of potential energy and then put it to use.

24-2 | Electric Potential Energy

Any charge in an electric field experiences the action of a force. Consequently, a certain work is done when a charge moves in the field. This work depends on the field strength at different points and on the charge displacement. However, *if a charge describes a closed curve, that is, returns to the original position, the work done by the field is equal to zero* irrespective of the field configuration and the shape of the path along which the charge has moved.

Electric potential is a scalar property associated with an electric field, regardless of whether a charged object has been placed in that field; it is measured in joules per coulomb, or volts.

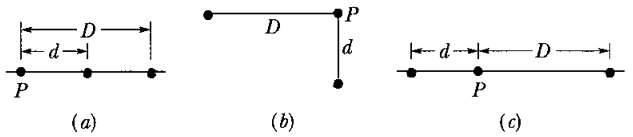
IMPORTANT POINTS

Throughout the book, important tips will be given in a box marked with a **pointed finger**. This technique highlights the tips and also aids retention.

CHECKPOINTS

These are important points after each section in the chapter that help check your understanding of what has been covered till then.

CHECKPOINT 4 The figure here shows three arrangements of two protons. Rank the arrangements according to the net electric potential produced at point P by the protons, greatest first.



PROBLEM-SOLVING TACTICS

Tactic 1: *The Symbol V and Potential Difference* In previous chapters, the symbol V represents an electric potential at a point or along an equipotential surface. However, in matters concerning electrical devices, V often represents a *potential difference* between two points or two equipotential surfaces. Equation 26-1 is an example of this second use of the symbol. In Section 26-3, you will see a mixture of the two meanings of V. There and in later chapters, you need to be alert as to the intent of this symbol.

You will also be seeing, in this book and elsewhere, a variety of phrases regarding potential difference. A potential difference or a "potential" or a "voltage" may be *applied* to a device, or it may be *across* a device. A capacitor can be charged to a potential difference, as in "a capacitor is charged to 12 V." Also, a battery can be characterized by the potential difference across it, as in "a 12 V battery." Always keep in mind what is meant by such phrases: There is a potential difference between two points, such as two points in a circuit or at the terminals of a device such as a battery.

PROBLEM-SOLVING TACTICS

These contain helpful instructions to guide the beginning physics student as to how to solve problems and avoid common errors.

B. THE 4Cs CONCEPT

It is normal to be nervous about solving tough IIT-JEE problems. But not if you know how to solve them using the 4Cs. The 4Cs is a unique and logical technique of solving problems conceptually. It is a step-based approach intended to simplify the process of applying formulas and concepts in the computation of the solution. See the Halliday way of 4Cs for solving the problem with explanations provided for each step. The purpose of each C is explained below.

CONCEPTUALIZE
First, the concept used in the problem is identified and a brief explanation of the problem is given.

CLASSIFY
Based on what value needs to be obtained and what values are given in the question, the formulae used are listed.

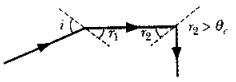
COMPUTE
The actual solution comes next, including the application of formulae and the resultant equations.

CONCLUDE
The problem ends with the explanation of the values derived as the answer.

Sample Problem 35-26

What is the relationship between A and n, so that no rays come out of second face (Fig. 35-58)?

FIG. 35-58 The prism has such a refractive index and an angle of prism combination that all the incident rays undergo total internal reflection.



Conceptualize: Here, we want to choose the refractive index of the prism such that a ray will always undergo total internal reflection. Note that this total internal reflection can occur only when the light ray goes from denser to rarer medium. In other words, this will occur only when the ray is incident on the second surface.

The situation implies that for the smallest angle r_2 also, the total internal reflection should occur.

Classify: $r_1 + r_2 = A$. So when r_2 is minimum, r_1 is maximum. The angle of incidence i is also maximum by Snell's law. But the maximum angle of incidence can be 90°. This problem implies that if a total internal reflection occurs when angle of incidence is 90°, then total internal reflection will occur at all the angles.

Compute: Applying Snell's law at the first surface:

$$1 \times \sin 90° = n \sin r_1$$

$$r_1 = \sin^{-1}\left(\frac{1}{n}\right)$$

$$r_2 = A - \sin^{-1}\left(\frac{1}{n}\right).$$

But for total internal reflection at the second surface:

$$r_2 > \theta_c$$

$$\sin r_2 > 1 \sin \theta_c \left(\frac{1}{n}\right)$$

$$A - \sin^{-1}\left(\frac{1}{n}\right) > \sin^{-1}\left(\frac{1}{n}\right).$$

Thus, the condition becomes

$$n > \operatorname{cosec}\left(\frac{A}{2}\right).$$

Conclude: Many optical instruments, such as binoculars, periscopes, and telescopes, use glass prisms and total internal reflection to turn a beam of light through 90° or 180°.

ASSESSMENT – AS PER IIT-JEE PATTERN

Mere theory is not enough. It is also important to test what has been taught. For the test to be effective, the assessment technique should be comprehensive. In the context of this book, the assessment also needs to be in resonance with the IIT-JEE paper pattern because unless the student practices the IIT-JEE way, he/she will not be sufficiently equipped to crack the exam. Keeping this in mind, the assessment has been divided into three main parts – *Reasoning Type Questions, Multiple Choice Questions, and Problems* – plus a section of *Additional Problems* for students interested in testing their understanding of the whole chapter. Each part contributes to the preparation of IIT-JEE because each part is modeled on the actual IIT-JEE paper.

REASONING TYPE

The first section of the assessment is reasoning-type questions, similar to the ones asked in the IIT-JEE paper. Practicing these would mean effective preparation for the exam.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: It is less dangerous to touch a faulty electrical appliance at 200 V than a Van de Graff generator at 20 000 V.

Statement 2: If we have two similar bodies at the different potential, the body with a higher potential will have more charge on it.

Statement 2: x component of electric field is given by

$$E_x = -\frac{\partial V}{\partial x},$$

and on equipotential surface potential V is constant.

4. Statement 1: If the distance between two point charges increases, their potential energy decreases.

Statement 2: The potential energy of two point charges is inversely proportional to the distance between the two charges.

5. Statement 1: If the electric potential at all the points in space is zero, the electric field at all the points in space must be zero.

Statement 2: If the electric field at all the points in space is zero, the electric potential at all the points in space must be zero.

MULTIPLE CHOICE QUESTIONS

sec. 24-2 Electric Potential Energy

1. A point charge Q is moved along a circular path around another fixed point charge. The work done by the electric field for one complete circle is zero

(a) If fixed charge is at the center of the circular path
(b) If the two charges have the same magnitude
(c) If the two charges have the same magnitude and opposite signs
(d) All above cases

2. Three charged particles are initially in a position 1. They are free to move and they come to another position 2 after some time. Let U_1 and U_2 be the electrostatic potential energies in position 1 and 2. Then

(a) $U_1 > U_2$
(b) $U_2 > U_1$
(c) $U_1 = U_2$

3. When a negative charge moves in a direction opposite to the direction of an electric field,

(a) The field does work on the charge
(b) The charge does work on the field
(c) The charge gains potential energy
(d) The charge loses potential energy

4. A point charge q moves from point P to point S along the path P, Q, R, and S in a uniform electric field pointing parallel to the positive direction of the x axis. The coordinates of the points P, Q, R, and S are $(a, b, 0)$, $(2a, 0, 0)$, $(a, -b, 0)$, and $(0, 0, 0)$, respectively (Fig. 24-37). The work done by the field in the above proc-

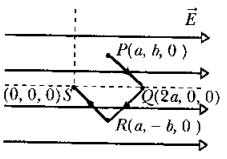


FIG. 24-37 Multiple choice question 4.

MULTIPLE CHOICE QUESTIONS

This form of Assessment forms the basis of the IIT-JEE paper. If a student becomes adept at attempting these types of questions, he/she has a good chance of doing well in the exam.

PROBLEMS

A number of numerical problems testing different concepts covered within the chapter are given here. All problems are of different levels of difficulty and meant to help students apply the concepts. Sections are mentioned for students to know which section to refer to in case of doubt.

PROBLEMS

sec. 24-2 Electric Potential Energy

1. Electrostatic work w is done on a charged particle, going from point A to point B, without any other external force acting on it. You next apply a force to move the particle back to point A, increasing its kinetic energy by an amount equal to $2w$. How much work did you do?

2. How much work is required to set up the arrangement of Fig. 24-48 if $q = 2.0\ \mu C$, $a = 9.0$ cm, and the particles are initially infinitely far apart and at rest?

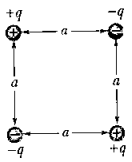


FIG. 24-48 Problem 2.

3. A particle of charge $+7.5\ \mu C$ is released from rest at the point $x = 60$ cm on an x axis. The particle begins to move due to the presence of a charge Q that remains fixed at the origin. What is the kinetic energy of the particle at the instant it has moved 40 cm if $Q = +20\ \mu C$?

4. Two charged objects are held a distance r apart. The first object has a mass m and a charge $+2q$, while the second object has a mass $2m$ and a charge $+q$. The objects are released from rest. Assume that the only force acting on either charge is the electrostatic force from the other charge. (a) When the objects are in motion what is the ratio of the first object's kinetic energy to that of the second object? What is the speed of (b) the first object and (c) the second object when they are at a large disctance from each other?

5. Two electrons are fixed 2.0 cm apart. Another electron is shot from infinity and stops midway between the two. What is its initial speed?

7. A positron (charge $+e$, mass equal to the electron mass) is moving at $1.0\ \ 10^7$ m/s in the positive direction of an x axis when, at $x = 0$, it encounters an electric field directed along the x axis. The electric potential V associated with the field is given in Fig. 24-49. The scale of the vertical axis is set by $V_s = 500.0$ V. (a) Does the positron emerge from the field at $x = 0$ (which means its motion is reversed) or at $x = 0.50$ m (which means its motion is not reversed)? (b) What is its speed when it emerges?

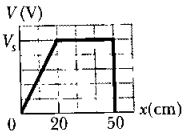


FIG. 24-49 Problem 7.

sec. 24-5 Calculating the Potential from the Field

8. When an electron moves from A to B along an electric field line in Fig. 24-50, the electric field does $3.84\ \ 10^{-19}$ J of

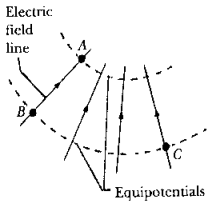


FIG. 24-50 Problem 8.

ADDITIONAL PROBLEMS

Some bonus extra problems are given as part of Assessment at the end of each chapter. These problems are not ordered or sorted in any way so that a studentmust determine which parts of the chapter apply to any given problem.

ADDITIONAL PROBLEMS

41. A nonconducting sphere has radius $R = 2.31$ cm and uniformly distributed charge $q = +3.50$ fC. Take the electric potential at the sphere's center to be $V_0 = 0$. What is V at radial distance (a) $r = 1.45$ cm and (b) $r = R$.

42. Two charged particles are shown in Fig. 24-64*a*. Particle 1, with charge q_1, is fixed in place at distance d. Particle 2, with charge q_2, can be moved along the x axis. Figure 24-64*b* gives the net electric potential V at the origin due to the two particles as a function of the x coordinate of particle 2. The scale of the x axis is set by $x_s = 16.0$ cm. The plot has an asymptote of

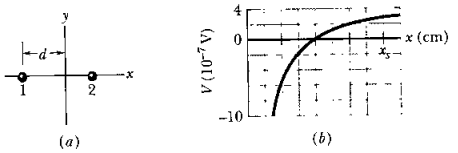


FIG. 24-64 Problem 42.

$V = 5.76 \times 10^{-7}$ V as $x \to \infty$. What is q_2 in terms of e?

43. A nonuniform linear charge distribution given by $\lambda = bx$, where b is a constant, is located along an x axis from $x = 0$ to $x = 0.20$ m. If $b = 20$ nC/m^2 and $V = 0$ at infinity, what is the electric potential at (a) the origin and (b) the point $y = 0.15$ m on the y axis?

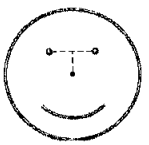
44. *Proton in a well.* Figure 24-65 shows electric potential V along an x axis. The scale of the vertical axis is set by $V_s = 10.0$ V. A proton is to be released at $x = 3.5$ cm with initial kinetic

45. The smiling face of Fig. 24-66 consists of three items:

(a) a thin rod of charge $-3.0\ \mu C$ that forms a full circle of radius 6.0 cm;

(b) a second thin rod of charge $2.0\ \mu C$ that forms a circular arc of radius 4.0 cm, subtending an angle of 90° about the center of the full circle;

(c) an electric dipole with a dipole moment that is perpendicular to a radial line and has magnitude 1.28×10^{-21} C · m.


FIG. 24-66
Problem 45.

What is the net electric potential at the center?

46. In Fig. 24-67, we move a particle of charge $+2e$ from infinity to the x axis. How much work do we do? Distance D is 4.00 m.

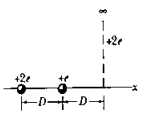


FIG. 24-67
Problem 46.

47. A particle of positive charge Q is fixed at point P. A second particle of mass m and negative charge $-q$ moves at constant speed in a circle of radius r_1, centered at P. Derive an expression for the work W that must be done by an external agent on the second particle to increase the radius of the circle of motion to r_2.

ANSWERS

The Answer key at the end of each chapter contains answers to all questions including checkpoints, reasoning questions, multiple choice questions, problems and additional problems.

ANSWERS

Checkpoints

1. a
2. (a) $11h/6$ from fish itself; (b) 17 $h/6$
3. The straw is not a ray! Note that the straw has a lateral shift as well as a bend. Actually, the rays from the straw reach our eyes after refraction from the water and the apparent depth of different parts of the straw is different from its actual value. So the straw appears to be bent.
4. $d_a = \dfrac{d\cos^3\theta}{n\cos^3\phi}$
5. (a) e; (b) virtual, same
7. Virtual, same as object, diverging

Reasoning Type Questions

1. (a)
2. (a)
3. (d)
4. (d)
5. (a)
6. (a)
7. (b)
8. (c)
9. (b)
10. (b)

Multiple Choice Questions

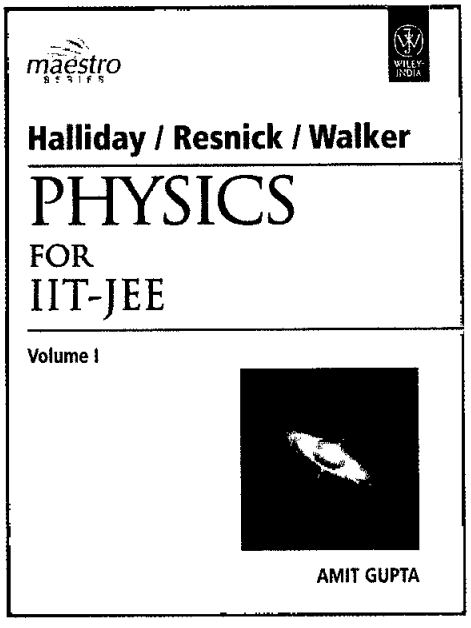
1. (d)
2. (b)
3. (c)
4. (a)
5. (c)
6. (b)
7. (c), (d)
8. (a)
9. (b)
10. (a)
11. (b)
12. (c)
13. (c)
14. (d)
15. (d)
16. (b)
17. (c)
18. (b)
19. (c)
20. (a)
21. (b)
22. (d)

ABOUT VOLUME I

The topics covered in the first volume are:

- General Physics
- Mechanics
- Thermal Physics

It has been adapted by an expert IIT-JEE teacher – Mr. Amit Gupta.



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Coulomb's Law and Electric Field

22



Samuel Ashfield/Taxi/Getty Images, Inc.

Hospital personnel go to extraordinary lengths to avoid bacterial infection of a patient. Surfaces are scrubbed, masks are donned, hands are meticulously cleaned and then gloved, and instruments are sanitized at high temperature and in alcohol baths. But there are still subtle sources of bacteria, such as possibly in this photograph.

Can you find the bacterial source?

The answer is in this chapter.

22-1 WHAT IS PHYSICS?

You are surrounded by devices that depend on the physics of electromagnetism, which is the combination of electric and magnetic phenomena. This physics is at the root of computers, television, radio, telecommunications, household lighting, and even the ability of food wrap to cling to a container. This physics is also the basis of the natural world. Not only does it hold together all the atoms and molecules in the world, it also produces lightning, auroras, and rainbows.

The physics of electromagnetism was first studied by the early Greek philosophers, who discovered that if a piece of amber is rubbed and then brought near bits of straw, the straw will jump to the amber. We now know that the attraction between amber and straw is due to an electric force. The Greek philosophers also discovered that if a certain type of stone (a naturally occurring magnet) is brought near bits of iron, the iron will jump to the stone. We now know that the attraction between magnet and iron is due to a magnetic force.

From these modest origins with the Greek philosophers, the sciences of electricity and magnetism developed separately for centuries—until 1820, in fact, when Hans Christian Oersted found a connection between them: an electric current in a wire can deflect a magnetic compass needle. Interestingly enough, Oersted made this discovery, a big surprise, while preparing a lecture demonstration for his physics students.

The new science of electromagnetism was developed further by workers in many countries. One of the best was Michael Faraday, a truly gifted experimenter with a talent for physical intuition and visualization. That talent is attested to by the fact that his collected laboratory notebooks do not contain a single equation. In the mid-nineteenth century, James Clerk Maxwell put Faraday's ideas into mathematical form, introduced many new ideas of his own, and put electromagnetism on a sound theoretical basis.

One question that all these workers pondered over was: How does a charged particle "know" the presence of another one? That is, since the particles do not touch, how can particle 2 push on particle 1—how can there be such an *action at a distance* with no visible connection between the particles?

One purpose of physics is to record observations about our world, such as the magnitude and direction of the push on particle 1. Another purpose is to provide a deeper explanation of what is recorded. One purpose of this chapter is also to provide such a deeper explanation to our nagging questions about electric force at a distance. We can answer those questions by saying that particle 2 sets up an *electric field* in the space surrounding itself. If we place particle 1 at any given point in that space, the particle "knows" of the presence of particle 2 because it is affected by the electric field that particle 2 has already set up at that point. Thus, particle 2 pushes on particle 1 not by touching it but by means of the electric field produced by particle 2.

Our goal in this chapter is also to define electric field and discuss how to calculate it for various arrangements of charged particles.

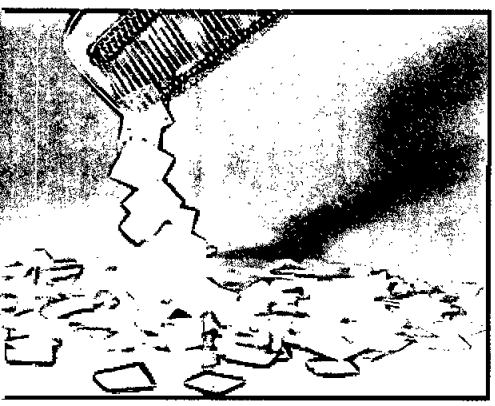


FIG. 22-1 Static cling, an electrical phenomenon that accompanies dry weather, causes these pieces of paper to stick to one another and to the plastic comb, and your clothing to stick to your body. *(Fundamental Photographs)*

22-2 | Electric Charge

In dry weather, you can produce a spark by walking across certain types of carpet and then bringing one of your fingers near a metal doorknob, metal faucet, or even a friend. You can also produce multiple sparks when you pull, say, a sweater from your body or clothes from a dryer. Sparks and the "static cling" of clothing (similar to what is seen in Fig. 22-1) are usually just annoying. However, if you happen to pull off a sweater and then spark to a computer, the results are more than just annoying.

These examples reveal that we have electric charge in our bodies, sweaters, carpets, doorknobs, faucets, and computers. In fact, every object contains a vast amount of electric charge. **Electric charge** is an intrinsic characteristic of the fundamental particles making up those objects; that is, it is a property that comes automatically with those particles wherever they exist.

The vast amount of charge in an everyday object is usually hidden because the object contains equal amounts of the two kinds of charge: *positive charge* and *negative charge*. With such an equality—or *balance*—of charge, the object is said to be *electrically neutral*; that is, it contains no *net* charge. If the two types of charge are not in balance, then there *is* a net charge. We say that an object is *charged* to indicate that it has a charge imbalance, or net charge. The imbalance is always much smaller than the total amounts of positive charge and negative charge contained in the object.

Charged objects interact by exerting forces on one another. To show this, we first charge a glass rod by rubbing one end with silk. At points of contact between the rod and the silk, tiny amounts of charge are transferred from one to the other, slightly upsetting the electrical neutrality of each. (We *rub* the silk over the rod to increase the number of contact points and thus the amount, still tiny, of transferred charge.)

Suppose we now suspend the charged rod from a thread to *electrically isolate* it from its surroundings so that its charge cannot change. If we bring a second, similarly charged, glass rod nearby (Fig. 22-2*a*), the two rods *repel* each other; that is, each rod experiences a force directed away from the other rod. However, if we rub a *plastic* rod with fur and then bring the rod near the suspended glass rod (Fig. 22-2*b*), the two rods *attract* each other; that is, each rod experiences a force directed toward the other rod.

We can understand these two demonstrations in terms of positive and negative charges. When a glass rod is rubbed with silk, the glass loses some of its negative charge and then has a small unbalanced positive charge (represented by the plus signs in Fig. 22-2*a*). When the plastic rod is rubbed with fur, the plastic gains a small unbalanced negative charge (represented by the minus signs in Fig. 22-2*b*).

The charging done by the way described above is called charging by friction. The charges are transmitted from one place to another depending on the tendency of the substance to acquire electrons. For example, in this case, silk has a greater tendency to acquire electrons than glass. So it becomes negatively charged whereas glass becomes positively charged.

Suppose you charge a rubber rod and then touch it to a neutral object. Some charge, repelled by the negative charge on the rod, will be transferred to the originally neutral object. In the case of some materials, the charge will stay on the spot where the originally neutral object is touched by the charged rod. Such materials are referred to as *insulators*, materials through which charge cannot move, or, through which the movement of charge is very limited. Examples of good insulators are quartz, glass, and air. In the case of other materials, the charge almost instantly spreads out all over the material in question. Materials in which the charge is free to move about are referred to as *conductors*. Examples of good conductors are metals and saltwater.

Let us perform another experiment. (1) Bring two metal spheres, A and B, supported on insulating stands, in contact as shown in Fig. 22-3*a*. (2) Bring a positively charged rod near one of the spheres, say A, taking care that it does not touch the sphere. The free electrons in the spheres are attracted toward the rod. This leaves an excess of positive charge on the rear surface of sphere B. Both kinds of charges are bound in the metal spheres and cannot escape. They, therefore, reside on the surfaces, as shown in Fig. 22-3*b*. The left surface of sphere A has an excess of negative charge and the right surface of sphere B has an excess of positive charge. However, not all the electrons in the spheres have

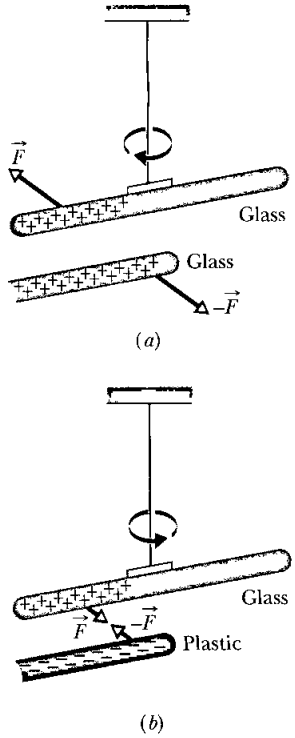


FIG. 22-2 (*a*) Two charged rods of the same sign repel each other. (*b*) Two charged rods of opposite signs attract each other. Plus signs indicate a positive net charge, and minus signs indicate a negative net charge.

accumulated on the left surface of A. The process is called *induction of charge* and it happens almost instantly. The accumulated charges remain on the surface as shown, until the glass rod is held near the sphere. If the rod is removed, the charges are not acted by any outside force and they redistribute to their original neutral state. (3) Separate the spheres by a small distance while the glass rod is still held near sphere A, as shown in Fig. 22-3*c*. The two spheres are found to be oppositely charged and attract each other. (4) Remove the rod. The charges on spheres rearrange themselves as shown in Fig. 22-3*d*. Now, separate the spheres quite apart. The charges on them get uniformly distributed over them, as shown in Fig. 22-3*e*.

In this process, the metal spheres will each be equal and oppositely charged. This is *charging by induction*. The positively charged glass rod does not lose any of its charge, contrary to the process of charging by friction.

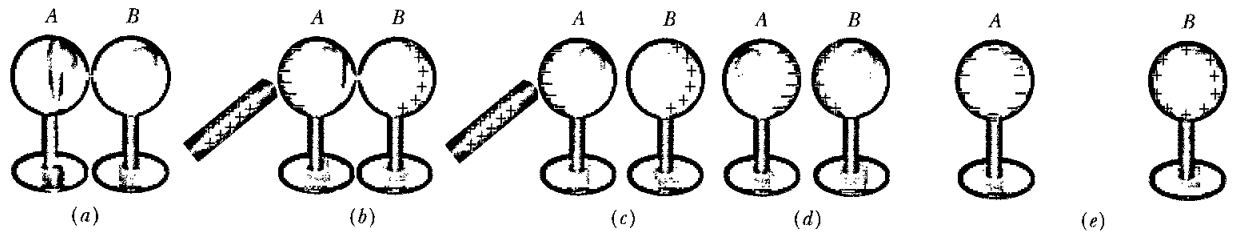


FIG. 22-3 Charging by induction.

If you do not have two spheres, but have only one, then also you can do this induction experiment. Connect the sphere to Earth while you bring the positively charged glass rod near the conducting sphere. What will happen is that the conducting sphere will be negatively charged by gaining some electrons from the Earth. The Earth, as you will soon realize, is a gigantic conductor. Our demonstrations reveal the following:

☞ Charges with the same electrical sign repel each other, and charges with opposite electrical signs attract each other.

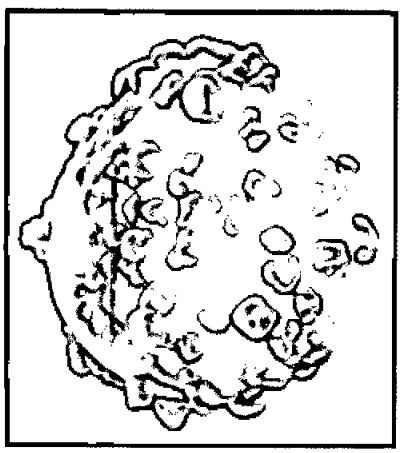


FIG. 22-4 A carrier bead from a photocopying machine; the bead is covered with toner particles that cling to it by electrostatic attraction. The diameter of the bead is about 0.3 mm. *(Courtesy Xerox)*

In Section 22-4, we shall put this rule into quantitative form as Coulomb's law of *electrostatic force* (or *electric force*) between charges. The term *electrostatic* is used to emphasize that, relative to each other, the charges are either stationary or moving only very slowly.

The "positive" and "negative" labels and signs for electric charge were chosen arbitrarily by Benjamin Franklin. He could easily have interchanged the labels or used some other pair of opposites to distinguish the two kinds of charge. (Franklin was a scientist of international reputation. It has even been said that Franklin's triumphs in diplomacy in France during the American War of Independence were facilitated, and perhaps even made possible, because he was so highly regarded as a scientist.)

The attraction and repulsion between charged bodies have many industrial applications, including electrostatic paint spraying and powder coating, fly-ash collection in chimneys, nonimpact ink-jet printing, and photocopying. Figure 22-4 shows a tiny carrier bead in a photocopying machine, covered with particles of black powder called *toner*, which stick to it by means of electrostatic forces. The negatively charged toner particles are eventually attracted from the carrier bead to a rotating drum, where a positively charged image of the document being copied has formed. A charged sheet of paper then attracts the toner particles from the drum to itself, after which they are heat-fused permanently in place to produce the copy.

A simple device that can be used to test the presence of charge is the *electroscope* (Fig. 22-5). A pair of thin strips made up of gold is suspended from a small metallic rod in an insulating enclosure; the rod is connected to a metal ball on top of the enclosure. When the metal ball is charged, the strips separate because the charge spreads out over the metallic parts and so the two strips which now carry like charges repel.

However, when a charged rod is brought near the ball of the electroscope, the strips separate even without any contact between the rod and the electroscope. As the distance between the rod and the electroscope is increased again, the strips drop back, showing that no charge has been transferred from the rod to the electroscope (Fig. 22-6).

Why do the strips separate even though the electroscope remains neutral? The negative charge on the rod repels the free electrons in the metallic parts of the electroscope—they are pushed as far away as possible from the rod and pile up in the strips. This redistribution of charge is nearly instantaneous. The top of the electroscope thus ends up with a deficit of electrons—a positive charge—and the strips end up with a surplus of a negative charge. The negative charge on the strips is responsible for the repulsion between them. When the rod is removed, the electrons, being repelled by each other and attracted to the positive charge on the electroscope ball, immediately flow back to their normal position, evening out the uneven distribution of positive and negative charges.

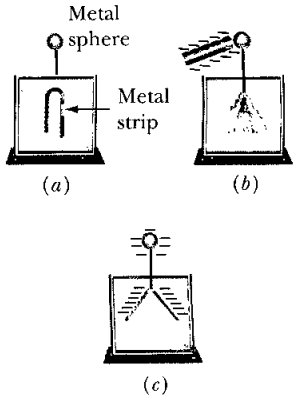


FIG. 22-5 (*a*) Electroscope. (*b*) When the ball of an electroscope is charged. (*c*) The two metal strips of the electroscope move apart.

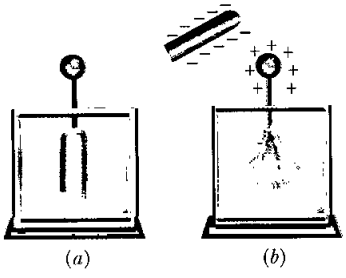


FIG. 22-6 The leaves of a neutral electroscope (*a*) ordinarily hang vertically down, but when a charged rod is held nearby, (*b*) they separate.

Bacterial Contamination and Electrostatics

Electrostatic forces can play a subtle role in bacterial contamination in a hospital, such as during endoscopic surgery. In such a procedure, a surgeon sees the interior of a patient's body on the viewing screen of a video monitor. In a traditional (tube) monitor (not a "flat-screen monitor"), the screen image is produced by electrons directed toward a positively charged screen. The charged screen also attracts airborne particles floating around in the operating room, such as lint, dust, and skin cells. If an airborne particle is negatively charged, it is pulled onto the screen's exterior surface. If, instead, it is electrically neutral, some of its conduction electrons can be pulled to the side of the particle nearest the screen, giving the particle an induced charge (Fig. 22-7*a*). Such a particle is then pulled to the screen's exterior surface.

Because many of the particles collected on the screen's exterior surface carry bacteria, the screen becomes contaminated with bacteria. Suppose a surgeon's gloved fingers come within a few centimeters of the screen, pointing to a particular part of the image, say, in explaining a surgical concern to other medical staff. The positively charged screen pulls electrons from inside the fingers to the fingertips (Fig. 22-7*b*). The negatively charged fingertips then cause particles (airborne or on the screen) to collect on the gloves at the tips. When the surgeon next touches the patient with the contaminated gloves, the bacteria end up on or

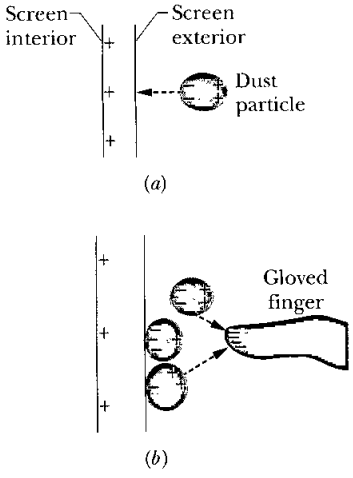
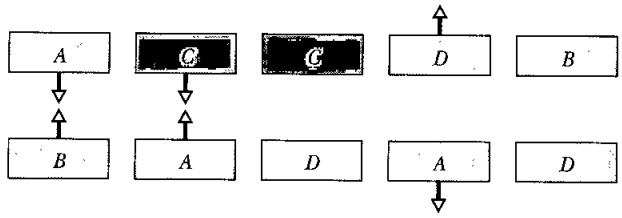


FIG. 22-7 (*a*) A cross section of the viewing screen on a traditional (tube) video monitor. The positively charged screen produces induced charge on a nearby neutral dust particle. (*b*) A gloved finger (not to scale) brought near the screen has an induced charge and can attract dust particles from the air and from the screen.

(worse) inside the patient's body. To avoid this risk, surgeons are now warned not to bring fingers near a video monitor.

Similar contamination can occur with the plastic aprons commonly worn by hospital staff to protect them from a patient's blood. Those aprons can become highly charged when peeled from a dispenser or when repeatedly rubbed against underlying clothing or skin, especially in a dry environment. Once an apron becomes charged, it can pull bacteria and contaminated dust out of the surrounding air. Because a staff member occasionally touches the apron, the bacteria can easily be transferred to a patient during an examination or surgery.

CHECKPOINT 1 The figure shows five pairs of plates: A, B, and D are charged plastic plates and C is an electrically neutral copper plate. The electrostatic forces between the pairs of plates are shown for three of the pairs. For the remaining two pairs, do the plates repel or attract each other?



22-3 | Coulomb's Law

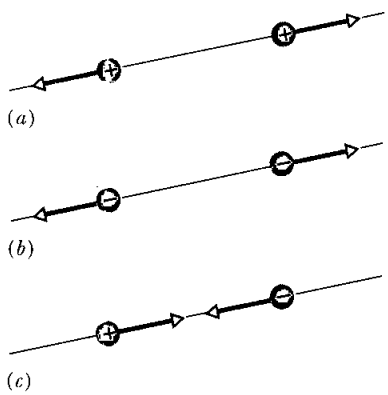


FIG. 22-8 Two charged particles repel each other if they have the same sign of charge, either (*a*) both positive or (*b*) both negative. (*c*) They attract each other if they have opposite signs of charge.

If two charged particles are brought near each other, they each exert a force on the other. If the particles have the same sign of charge, they repel each other (Figs. 22-8*a* and *b*). That is, the force on each particle is directed away from the other particle, and if the particles can move, they move away from each other. If, instead, the particles have opposite signs of charge, they attract each other (Fig. 22-8*c*) and, if free to move, they move closer to each other.

This force of repulsion or attraction due to the charge properties of objects is called an **electrostatic force**. The equation giving the force for charged *particles* is called **Coulomb's law** after Charles-Augustin de Coulomb, whose experiments in 1785 led him to it. In terms of the particles in Fig. 22-9, where particle 1 has charge q_1 and particle 2 has charge q_2, the force on particle 1 is

$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{\mathrm{r}} \quad \text{(Coulomb's law)}, \tag{22-1}$$

in which $\hat{\mathrm{r}}$ is a unit vector along an axis extending through the two particles, r is the distance between them, and k is a constant. (As with other unit vectors, $\hat{\mathrm{r}}$ has a magnitude of exactly 1 and no dimension or unit; its purpose is to point.) If the particles have the same signs of charge, the force on particle 1 is in the direction of $\hat{\mathrm{r}}$; if they have opposite signs, the force is opposite $\hat{\mathrm{r}}$.

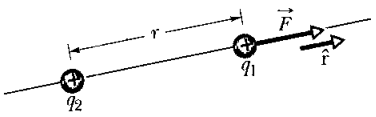


FIG. 22-9 The electrostatic force on particle 1 can be described in terms of a unit vector $\hat{\mathrm{r}}$ along an axis through the two particles.

For using the formula in the vector form, we should substitute the values of charges with sign. The answer will automatically give the direction of the forces. For example, in Fig. 22-9 if the charge q_1 is positive and q_2 is positive too, then the force on q_1 is repulsive and hence the charge q_1 should experience a force in the positive direction. This can be seen from the Coulomb's law too. If we are considering force on q_1, $\hat{\mathrm{r}}$ is in positive direction (along the first quadrant), then products of q_1 and q_2 will be positive in this case. So $\vec{F}$ is positive. In this form, we can easily see that like charges attract and unlike charges repel (Figs. 22-8 *a*, *b*, and *c*).

Many times, we will use the Coulomb's law in vector form. But most of the times, we take only its magnitude and decide the direction by the fact that the like charges repel and unlike charges attract.

Coulomb's law has survived every experimental test; no exceptions to it have ever been found. It holds even within the atom, correctly describing the

force between the positively charged nucleus and each of the negatively charged electrons, even though classical Newtonian mechanics fails in that realm and is replaced there by quantum physics. This simple law also correctly accounts for the forces that bind atoms together to form molecules, and for the forces that bind atoms and molecules together to form solids and liquids.

The SI unit of charge is the **coulomb**. For practical reasons having to do with the accuracy of measurements, the coulomb unit is derived from the SI unit *ampere* for *electric current i*. Current is the rate *dq/dt* at which charge moves past a point or through a region. In Chapter 27 we shall discuss current in detail. Until then we shall use the relation

$$i = \frac{dq}{dt} \quad \text{(electric current)}, \tag{22-2}$$

in which i is the current (in amperes) and dq (in coulombs) is the amount of charge moving past a point or through a region in time dt (in seconds). Rearranging Eq. 22-2 tells us that

$$1\ \text{C} = (1\ \text{A})(1\ \text{s}).$$

For historical reasons (and because doing so simplifies many other formulas), the electrostatic constant k of Eq. 22-1 is usually written $1/4\pi\varepsilon_0$. Then the magnitude of the force in Coulomb's law becomes

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \quad \text{(Coulomb's law)}. \tag{22-3}$$

The constants in Eqs. 22-1 and 22-3 have the value

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2. \tag{22-4}$$

The quantity ε_0, called the **permittivity constant**, sometimes appears separately in equations and is

$$\varepsilon_0 = 8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2. \tag{22-5}$$

The electrostatic force obeys the principle of superposition. If we have n charged particles, they interact independently in pairs, and the force on any one of them, let us say particle 1, is given by the vector sum

$$\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}, \tag{22-6}$$

in which, for example, $\vec{F}_{14}$ is the force acting on particle 1 due to the presence of particle 4. Most of the students as well as teachers take superposition as obvious consequence of law of vector addition. However, this is not the case. It states two things: one is that the net force on any one charge is unaffected by the presence of other charges. And two, the net force on it can be found by vector addition of the forces due to each of these charges as if they were acting alone. Note that in Newton's law, there was no requirement that the forces will not change once other forces act on the body (Fig. 22-10).

Take a simple example. Assume that a box is lying on the ground. The ground exerts a normal force on it. Now if you start pulling it with a force less than the weight, the normal force would change. The forces are normally dependent on the presence of other forces. But electrostatic force is such that it does not depend on the presence of other forces.

Note that the Coulomb's force between two point charges is independent of the medium between them. When the medium is other than vacuum, some

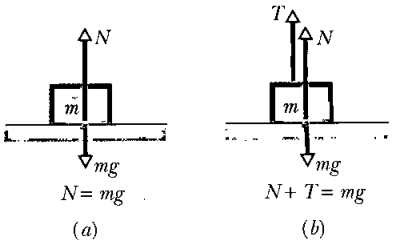


FIG. 22-10 (*a*) A body kept on ground experiences normal force from ground. (*b*) A force of T upward changes the normal force exerted by ground. N is not independent of other forces, hence, it does not obey superposition principle. However, mg is independent of other forces, therefore it obeys superposition principle.

charge distribution occurs in the medium due to which the net force on any of the charges decreases. But the force on 1 due to 2 alone remains the same.

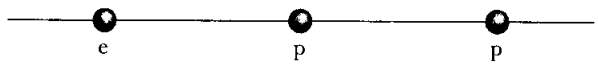
$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}.$$

If the medium is infinite in extent and liquid in nature, then

$$\vec{F}_{net} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1 q_2}{r^2}\hat{r}_{12},$$

where $\varepsilon_r > 1$ which in turn means that $F_{net} < F_{12}$. ε_r is known as the relative permittivity of the medium.

CHECKPOINT 2 The figure shows two protons (symbol p) and one electron (symbol e) on an axis. What is the direction of (a) the electrostatic force on the central proton due to the electron, (b) the electrostatic force on the central proton due to the other proton, and (c) the net electrostatic force on the central proton?



Sample Problem 22-1 Build your skill

(a) Figure 22-11*a* shows two positively charged particles fixed in place on an x axis. The charges are $q_1 = 1.60 \times 10^{-19}$ C and $q_2 = 3.20 \times 10^{-19}$ C, and the particle separation is $R = 0.0200$ m. What are the magnitude and direction of the electrostatic force $\vec{F}_{12}$ on particle 1 from particle 2?

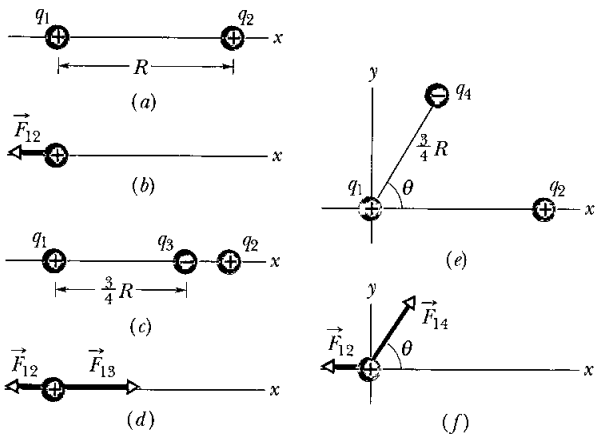


FIG. 22-11 (*a*) Two charged particles of charges q_1 and q_2 are fixed in place on an x axis. (*b*) The free-body diagram for particle 1, showing the electrostatic force on it from particle 2. (*c*) Particle 3 included. (*d*) Free-body diagram for particle 1. (*e*) Particle 4 included. (*f*) Free-body diagram for particle 1.

Conceptualize/Classify: Because both particles are positively charged, particle 1 is repelled by particle 2, with a force magnitude given by Eq. 22-3. Thus, the direction of force $\vec{F}_{12}$ on particle 1 is *away from* particle 2, in the negative direction of the x axis, as indicated in the free-body diagram of Fig. 22-11*b*.

Compute: Using Eq. 22-3 with separation R substituted for r, we can write the magnitude F_{12} of this force as

$$\begin{aligned} F_{12} &= \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{R^2} \\ &= (8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2) \\ &\quad\times\frac{(1.60\times10^{-19}\ \text{C})(3.20\times10^{-19}\ \text{C})}{(0.0200\ \text{m}^2)} \\ &= 1.15\times10^{-24}\ \text{N}. \end{aligned}$$

Thus, force $\vec{F}_{12}$ has the following magnitude and direction (relative to the positive direction of the x axis):

$$1.15\times10^{-24}\ \text{N} \quad \text{and} \quad 180°. \qquad \text{(Answer)}$$

We can also write $\vec{F}_{12}$ in unit-vector notation as

$$\vec{F}_{12} = -(1.15\times12^{-24}\ \text{N})\,\hat{i}. \qquad \text{(Answer)}$$

(b) Figure 22-11*c* is identical to Fig. 22-11*a* except that particle 3 now lies on the x axis between particles 1 and 2. Particle 3 has charge $q_3 = -3.20\times10^{-19}$ C and is at a distance $\frac{3}{4}$R from particle 1. What is the net electrostatic force $\vec{F}_{1,net}$ on particle 1 due to particles 2 and 3?

Conceptualize/Classify: The presence of particle 3 does not alter the electrostatic force on particle 1 from particle 2. Thus, force $\vec{F}_{12}$ still acts on particle 1. Similarly, the force $\vec{F}_{13}$ that acts on particle 1 due to particle 3 is not affected by the presence of particle 2. Because particles 1 and 3 have charge of opposite signs, particle 1 is attracted to particle 3. Thus, force $\vec{F}_{13}$ is directed *toward* particle 3, as indicated in the free-body diagram of Fig. 22-11*d*.

Compute: To find the magnitude of $\vec{F}_{13}$, we can rewrite Eq. 22-3 as

$$\begin{aligned} F_{13} &= \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_3|}{\left(\frac{3}{4}R\right)^2} \\ &= (8.99\times10^9\,\mathrm{N\cdot m^2/C^2}) \\ &\quad\times\frac{(1.60\times10^{-19}\,\mathrm{C})(3.20\times10^{-19}\,\mathrm{C})}{\left(\frac{3}{4}\right)^2(0.0200\,\mathrm{m})^2} \\ &= 2.05\times10^{-24}\,\mathrm{N}. \end{aligned}$$

We can also write $\vec{F}_{13}$ in unit-vector notation:

$$\vec{F}_{13} = (2.05\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}}.$$

The net force $\vec{F}_{1,\mathrm{net}}$ on particle 1 is the vector sum of $\vec{F}_{12}$ and $\vec{F}_{13}$; that is, from Eq. 22-6, we can write the net force $\vec{F}_{1,\mathrm{net}}$ on particle 1 in unit-vector notation as

$$\begin{aligned} \vec{F}_{1,\mathrm{net}} &= \vec{F}_{12} + \vec{F}_{13} \\ &= -(1.15\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}} + (2.05\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}} \\ &= (9.00\times10^{-25}\,\mathrm{N})\hat{\mathrm{i}}. \qquad \text{(Answer)} \end{aligned}$$

Thus, $\vec{F}_{1,\mathrm{net}}$ has the following magnitude and direction (relative to the positive direction of the x axis):

$$9.00\times10^{-25}\,\mathrm{N} \text{ and } 0°. \qquad \text{(Answer)}$$

(c) Figure 22-11*e* is identical to Fig. 22-11*a* except that particle 4 is now included. It has charge $q_4 = -3.20\times10^{-19}$ C, is at a distance $\frac{3}{4}R$ from particle 1, and lies on a line that makes an angle $\theta = 60°$ with the x axis. What is the net electrostatic force, $\vec{F}_{1,\mathrm{net}}$ on particle 1 due to particles 2 and 4?

Conceptualize/Classify: The net force $\vec{F}_{1,\mathrm{net}}$ is the vector sum of $\vec{F}_{12}$ and a new force $\vec{F}_{14}$ acting on particle 1 due to particle 4. Because particles 1 and 4 have charge of opposite signs, particle 1 is attracted to particle 4. Thus, force $\vec{F}_{14}$ on particle 1 is directed *toward* particle 4, at angle $\theta = 60°$, as indicated in the free-body diagram of Fig. 22-11*f*.

Compute: We can rewrite Eq. 22-3 as

$$\begin{aligned} F_{14} &= \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_4|}{\left(\frac{3}{4}R\right)^2} \\ &= (8.99\times10^9\,\mathrm{N\cdot m^2/C^2}) \\ &\quad\times\frac{(1.60\times10^{-19}\,\mathrm{C})(3.20\times10^{-19}\,\mathrm{C})}{\left(\frac{3}{4}\right)^2(0.0200\,\mathrm{m})^2} \\ &= 2.05\times10^{-24}\,\mathrm{N}. \end{aligned}$$

Then from Eq. 22-6, we can write the net force $\vec{F}_{1,\mathrm{net}}$ on particle 1 as

$$\vec{F}_{1,\mathrm{net}} = \vec{F}_{12} + \vec{F}_{14}.$$

Because the forces $\vec{F}_{12}$ and $\vec{F}_{14}$ are not directed along the same axis, we *cannot* sum simply by combining their magnitudes. Instead, we must add them as vectors, using one of the following methods.

Method 1. *Summing directly on a vector-capable calculator.* For $\vec{F}_{12}$, we enter the magnitude 1.15×10^{-24} and the angle 180°. For $\vec{F}_{14}$, we enter the magnitude 2.05×10^{-24} and the angle 60°. Then we add the vectors.

Method 2. *Summing in unit-vector notation.* First we rewrite $\vec{F}_{14}$ as

$$\vec{F}_{14} = (F_{14}\cos\theta)\hat{\mathrm{i}} + (F_{14}\sin\theta)\hat{\mathrm{j}}.$$

Substituting 2.05×10^{-24} N for F_{14} and 60° for θ, this becomes

$$\vec{F}_{14} = (1.025\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}} + (1.775\times10^{-24}\,\mathrm{N})\hat{\mathrm{j}}.$$

Then we sum:

$$\begin{aligned} \vec{F}_{1,\mathrm{net}} &= \vec{F}_{12} + \vec{F}_{14} \\ &= -(1.15\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}} \\ &\quad +(1.025\times10^{-24}\,\mathrm{N})\hat{\mathrm{i}} + (1.775\times10^{-24}\,\mathrm{N})\hat{\mathrm{j}} \\ &\approx (-1.25\times10^{-25}\,\mathrm{N})\hat{\mathrm{i}} + (1.78\times10^{-24}\,\mathrm{N})\hat{\mathrm{j}}. \end{aligned}$$

(Answer)

Method 3. *Summing components axis by axis.* The sum of the x components gives us

$$\begin{aligned} F_{1,\mathrm{net},x} &= F_{12,x} + F_{14,x} = F_{12} + F_{14}\cos 60° \\ &= -1.15\times10^{-24}\,\mathrm{N} + (2.05\times10^{-24}\,\mathrm{N})(\cos 60°) \\ &= -1.25\times10^{-25}\,\mathrm{N}. \end{aligned}$$

The sum of the y components gives us

$$\begin{aligned} F_{1,\mathrm{net},y} &= F_{12,y} + F_{14,y} = 0 + F_{14}\sin 60° \\ &= (2.05\times10^{-24}\,\mathrm{N})(\sin 60°) \\ &= 1.78\times10^{-24}\,\mathrm{N}. \end{aligned}$$

The net force $\vec{F}_{1,\mathrm{net}}$ has the magnitude

$$F_{1,\mathrm{net}} = \sqrt{F_{1,\mathrm{net},x}^2 + F_{1,\mathrm{net},y}^2} = 1.78\times10^{-24}\,\mathrm{N}. \qquad \text{(Answer)}$$

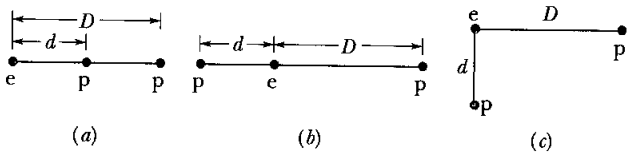
To find the direction of $\vec{F}_{1,\mathrm{net}}$, we take

$$\theta = \tan^{-1}\frac{F_{1,net,y}}{F_{1,net,x}} = -86.0°.$$

However, this is an unreasonable result because $\vec{F}_{1,\mathrm{net}}$ must have a direction between the directions of $\vec{F}_{12}$ and $\vec{F}_{14}$. To correct θ, we add 180°, obtaining

$$-86.0° + 180° = 94.0°. \qquad \text{(Answer)}$$

CHECKPOINT 3 The figure here shows three arrangements of an electron e and two protons p. (a) Rank the arrangements according to the magnitude of the net electrostatic force on the electron due to the protons, largest first. (b) In situation *c*, is the angle between the net force on the electron and the line labeled *d* less than or more than 45°?



PROBLEM-SOLVING TACTICS

Tactic 1: *Symbols Representing Charge* Here is a general guide to the symbols representing charge. If the symbol q, with or without a subscript, is used in a sentence when no electrical sign has been specified, the charge can be either positive or negative. Sometimes the sign is explicitly shown, as in the notation $+q$ or $-q$.

When more than one charged object is being considered, their charges might be given as multiples of a charge magnitude. As examples, the notation $+2q$ means a positive charge with magnitude twice that of some reference charge magnitude q, and $-3q$ means a negative charge with magnitude three times that of the reference charge magnitude q.

Sample Problem 22-2

Figure 22-12*a* shows two particles fixed in place: a particle of charge $q_1 = +8q$ at the origin and a particle of charge $q_2 = -2q$ at $x = L$. At what point (other than infinitely far away) can a proton be placed so that it is in *equilibrium* (the net force on it is zero)? Is that equilibrium *stable* or *unstable*?

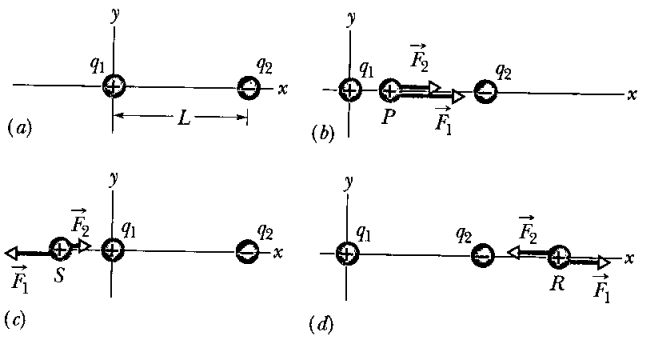


FIG. 22-12 (*a*) Two particles of charges q_1 and q_2 are fixed in place on an x axis, with separation L. (*b*)–(*d*) Three possible locations P, S, and R for a proton. At each location, $\vec{F}_1$ is the force on the proton from particle 1 and $\vec{F}_2$ is the force on the proton from particle 2.

Conceptualize: If $\vec{F}_1$ is the force on the proton due to charge q_1 and $\vec{F}_2$ is the force on the proton due to charge q_2, then the point we seek is where $\vec{F}_1 + \vec{F}_2 = 0$. Thus,

$$\vec{F}_1 = -\vec{F}_2 \qquad (22\text{-}7)$$

This tells us that at the point we seek, the forces acting on the proton due to the other two particles must be of equal magnitudes,

$$F_1 = F_2, \qquad (22\text{-}8)$$

and that the forces must have opposite directions.

Classify: Because a proton has a positive charge, the proton and the particle of charge q_1 are of the same sign, and force $\vec{F}_1$ on the proton must point away from q_1. Also, the proton and the particle of charge q_2 are of opposite signs, so force $\vec{F}_2$ on the proton must point toward q_2. "Away from q_1" and "toward q_2" can be in opposite directions only if the proton is located on the x axis.

If the proton is on the x axis at any point between q_1 and q_2, such as point P in Fig. 22-12*b*, then $\vec{F}_1$ and $\vec{F}_2$ are in the same direction and not in opposite directions as required. If the proton is at any point on the x axis to the left of q_1, such as point S in Fig. 22-12*c*, then $\vec{F}_1$ and $\vec{F}_2$ are in opposite directions. However, Eq. 22-3 tells us that $\vec{F}_1$ and $\vec{F}_2$ cannot have equal magnitudes there: F_1 must be greater than F_2, because F_1 is produced by a closer charge (with lesser r) of greater magnitude ($8q$ versus $2q$).

Finally, if the proton is at any point on the x axis to the right of q_2, such as point R in Fig. 22-12*d*, then $\vec{F}_1$ and $\vec{F}_2$ are again in opposite directions. However, because now the charge of greater magnitude (q_1) is *farther* away from the proton than the charge of lesser magnitude, there is a point at which F_1 is equal to F_2. Let x be the coordinate of this point, and let q_p be the charge of the proton.

Compute: With the aid of Eq. 22-3, we can now rewrite Eq. 22-8 (which says that the forces have equal magnitudes):

$$\frac{1}{4\pi\varepsilon_0}\frac{8qq_p}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{2qq_p}{(x-L)^2}. \qquad (22\text{-}9)$$

(Note that only the charge magnitudes appear in Eq. 22-9.) Rearranging Eq. 22-9 gives us

$$\left(\frac{x-L}{x}\right)^2 = \frac{1}{4}.$$

After taking the square roots of both sides, we have

$$\frac{x-L}{x} = \frac{1}{2},$$

which gives us

$$x = 2L. \qquad \text{(Answer)}$$

Conclude: The equilibrium at $x = 2L$ is unstable; that is, if the proton is displaced leftward from point R, then F_1 and F_2 both increase but F_2 increases more (because q_2 is closer than q_1), and a net force will drive the proton farther leftward. If the proton is displaced rightward, both F_1 and F_2 decrease but F_2 decreases more, and a net force will then drive the proton farther rightward. In a stable equilibrium, if the proton is displaced slightly, it returns to the equilibrium position.

PROBLEM-SOLVING TACTICS

Tactic 2: *Drawing Electrostatic Force Vectors* When you are given a diagram of charged particles, such as Fig. 22-11*a*, and are asked to find the net electrostatic force on one of them, you should usually draw a free-body diagram showing only the particle of concern and the forces that particle experiences, as in Fig. 22-11*b*. If, instead, you choose to superimpose those forces on the given diagram showing all the particles, be sure to draw the force vectors with either their tails (preferably) or their heads on the particle of concern. If you draw the vectors elsewhere in the diagram, you invite confusion—and confusion is guaranteed if you draw the vectors on the particles *causing* the forces on the particle of concern.

Sample Problem 22-3

Two identical, small, charged spheres, each having a mass of 4.0×10^{-2} kg hang in equilibrium as shown in Fig. 22-13*a*. The length of each string is 0.15 m and the angle θ is 37.0°. Find the magnitude of the charge on each sphere.

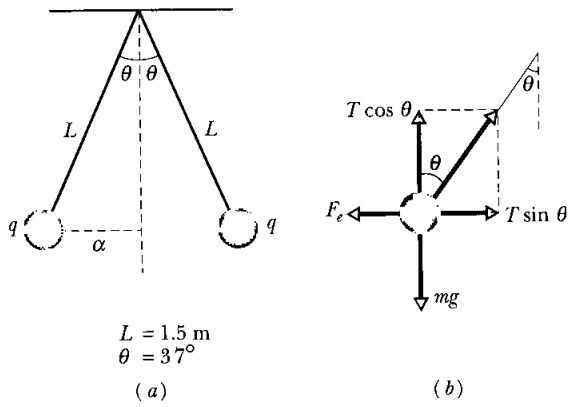


FIG. 22-13 (*a*) Two small charged spheres in equilibrium. (*b*) Free body diagram of left sphere.

Conceptualize: Figure 22-13*a* helps us to conceptualize this problem. The two spheres exert repulsive forces on each other. If they are held close to each other and released, they will move outward from the center and settle into the configuration in Fig. 22-13*a* after the damped oscillations due to air resistance have vanished.

Classify: The key phrase "in equilibrium" helps us to classify this as an equilibrium problem, which we approach as we did equilibrium problems in the chapter on Newton's Laws (in Volume 1) with the added feature that one of the forces on a sphere is an electric force. We analyze this problem by drawing the free-body diagram for the left-hand sphere in Fig. 22-13*b*. The sphere is in equilibrium under the application of forces T from the string, the electric force F_e from the other sphere, and the gravitational force mg.

Compute: Because the sphere is in equilibrium, the forces in the horizontal and vertical directions must separately add up to zero.

$$\sum F_x = T\sin\theta - F_e = 0. \tag{22-10}$$

$$\sum F_y = T\cos\theta - mg = 0. \tag{22-11}$$

From Eq. 22-11, we see that $T = mg/\cos\theta$; thus, T can be eliminated from Eq. 22-10 if we make this substitution. This gives a value for the magnitude of the electric force F_e:

$$F_e = mg\tan\theta = (4.0\times10^{-2}\ \text{kg})(10\ \text{m/s}^2)\tan(37°) = 0.3\ \text{N}.$$

Considering the geometry of the triangle in Fig. 22-13*a*, we see that $\sin\theta = \alpha/L$. Therefore,

$$a = L\sin\theta = (1.5\ \text{m})\sin(37°) = 0.9\ \text{m}.$$

The separation of the spheres is $2a = 1.8$ m. From Coulomb's law (Eq. 22-10), the magnitude of the electric force is

$$F_e = k_e\frac{|q|^2}{r^2},$$

where $r = 2a = 1.8$ m and $|q|$ is the magnitude of the charge on each sphere. (Note that the term $|q|^2$ arises here because the charge is the same on both spheres.) This equation can be solved for $|q|^2$ to give

$$|q|^2 = \frac{F_e r^2}{k_e} = \frac{0.3\times1.8}{9\times10^9\ \text{Nm}^2/\text{C}^2} = 1.08\times10^{-10}\ \text{C}^2,$$

$$|q| = 1.04\times10^{-5}\ \text{C}.$$

Conclude: In this problem, we found only the magnitude of the charge $|q|$ on the spheres. There is no way we could find the sign on the charge from the information given. In fact, the sign of the charge is not important. The

situation will be exactly the same whether both spheres are positively charged or negatively charged.

Even if the charges on spheres are not equal, the symmetry is not destroyed and the angles remain the same. Newton's third law requires that the electric forces on the two charges be the same, regardless of the equality or nonequality of the charges. The solution to the example remains the same through the calculation of $|q|^2$. In this situation, the value of 1.08×10^{-10} C^2 corresponds to the product $q_1 q_2$, where q_1 and q_2 are the values of the charges on the two spheres. The symmetry of the problem would be destroyed if the masses of the spheres were not the same. In this case, the strings would make different angles with the vertical and the problem would be more complicated.

✓**CHECKPOINT 4** Assume that charges were initially different. If the charges on them were equally distributed would the angle of divergence increase, decrease, or stay the same?

22-4 | Charge Is Quantized

In Benjamin Franklin's day, electric charge was thought to be a continuous fluid—an idea that was useful for many purposes. However, we now know that fluids themselves, such as air and water, are not continuous but are made up of atoms and molecules; matter is discrete. Experiment shows that "electrical fluid" is also not continuous but is made up of multiples of a certain elementary charge. Any positive or negative charge q that can be detected can be written as

$$q = ne, \quad n = \pm 1, \pm 2, \pm 3, \ldots, \tag{22-12}$$

in which e, the **elementary charge**, has the approximate value

$$e = 1.602 \times 10^{-19} \text{ C}. \tag{22-13}$$

The elementary charge e is one of the important constants of nature. The electron and proton both have a charge of magnitude e (Table 22-1). (Quarks, the constituent particles of protons and neutrons, have charges of $\pm e/3$ or $\pm 2e/3$, but they apparently cannot be detected individually. For this and for historical reasons, we do not take their charges to be the elementary charge.)

TABLE 22-1
The Charges of Three Particles

Particle	Symbol	Charge
Electron	e or e^-	$-e$
Proton	p	$+e$
Neutron	n	0

You often see phrases—such as "the charge on a sphere," "the amount of charge transferred," and "the charge carried by the electron"—that suggest that charge is a substance. (Indeed, such statements have already appeared in this chapter.) You should, however, keep in mind what is intended: *Particles* are the substance and charge happens to be one of their properties, just as mass is.

When a physical quantity such as charge can have only discrete values rather than any value, we say that the quantity is **quantized**. It is possible, for example, to find a particle that has no charge at all or a charge of $+10e$ or $-6e$, but not a particle with a charge of, say, $3.57e$.

The quantum of charge is small. In an ordinary 100 W lightbulb, for example, about 10^{19} elementary charges enter the bulb every second and just as many leave. However, the graininess of electricity does not show up in such large-scale phenomena (the bulb does not flicker with each electron), just as you cannot feel the individual molecules of water with your hand.

In the formula

$$i = \frac{dq}{dt},$$

q has only discrete values; hence, q is a nondifferentiable function. But the charges we encounter in day-to-day life involve a large number of e^-; hence, it behaves almost like a continuous function.

If charge is very small then

$$i = \frac{\Delta q}{\Delta t}.$$

If charge is very large then

$$i = \frac{dq}{dt}.$$

CHECKPOINT 5 Initially, sphere A has a charge of $-50e$ and sphere B has a charge of $+20e$. The spheres are made of conducting material and are identical in size. If the spheres then touch, what is the resulting charge on sphere A?

Sample Problem 22-4

The nucleus in an iron atom has a radius of about 4.0×10^{-15} m and contains 26 protons.

(a) What is the magnitude of the repulsive electrostatic force between two of the protons that are separated by 4.0×10^{-15} m?

Conceptualize/Classify: The protons can be treated as charged particles, so the magnitude of the electrostatic force on one from the other is given by Coulomb's law.

Compute: Table 22-1 tells us that the charge of a proton is $+e$. Thus, Eq. 22-3 gives us

$$F = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$
$$= \frac{(8.99\times10^{9}\ \mathrm{N\cdot m^2/C^2})(1.602\times10^{-19}\ \mathrm{C})^2}{(4.0\times10^{-15}\ \mathrm{m})^2}$$
$$= 14\ \mathrm{N}. \qquad \text{(Answer)}$$

Conclude: This is a small force to be acting on a macroscopic object like a cantaloupe, but an enormous force to be acting on a proton. Such forces should explode the nucleus of any element but hydrogen (which has only one proton in its nucleus). However, they don't, not even in nuclei with a great many protons. Therefore, there must be some enormous attractive force to counter this enormous repulsive electrostatic force.

(b) What is the magnitude of the gravitational force between those same two protons?

Conceptualize/Classify: Because the protons are particles, the magnitude of the gravitational force on one from the other is given by Newton's equation for the gravitational force:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r} \qquad \text{(Newton's Law)}.$$

Compute: With m_p ($= 1.67 \times 10^{-27}$ kg) representing the mass of a proton, the above equation gives us

$$F = G\frac{m_p^2}{r^2}$$
$$= \frac{(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})(1.67\times10^{-27}\ \mathrm{kg})^2}{(4.0\times10^{-15}\ \mathrm{m})^2}$$
$$= 1.2\times10^{-35}\ \mathrm{N}. \qquad \text{(Answer)}$$

Conclude: This result tells us that the (attractive) gravitational force is far too weak to counter the repulsive electrostatic forces between protons in a nucleus. Instead, the protons are bound together by an enormous force called (aptly) the *strong nuclear force*—a force that acts between protons (and neutrons) when they are close together, as in a nucleus.

Although the gravitational force is many times weaker than the electrostatic force, it is more important in large-scale situations because it is always attractive. This means that it can collect many small bodies into huge bodies with huge masses, such as planets and stars, that then exert large gravitational forces. The electrostatic force, on the other hand, is repulsive for charges of the same sign, so it is unable to collect either positive charge or negative charge into large concentrations that would then exert large electrostatic forces.

22-5 | Charge Is Conserved

If you rub a glass rod with silk, a positive charge appears on the rod. Measurement shows that a negative charge of equal magnitude appears on the silk. This suggests that rubbing does not create charge but only transfers it from one body to another, upsetting the electrical neutrality of each body during the process. This hypothesis of **conservation of charge**, first put forward by Benjamin Franklin, has stood up under close examination, both for large-scale charged bodies and for atoms, nuclei, and elementary particles. No exceptions have ever been found. Thus, we add electric charge to our list of quantities—including energy and both linear and angular momentum—that obey a conservation law.

Important examples of the conservation of charge occur in the *radioactive decay* of nuclei, in which a nucleus transforms into (becomes) a different type of

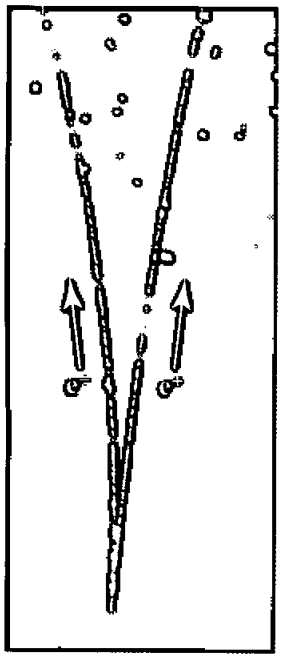


FIG. 22-14 A photograph of trails of bubbles left in a bubble chamber by an electron and a positron. The pair of particles was produced by a gamma ray that entered the chamber directly from the bottom. Being electrically neutral, the gamma ray did not generate a telltale trail of bubbles along its path, as the electron and positron did. *(Courtesy Lawrence Berkeley Laboratory)*

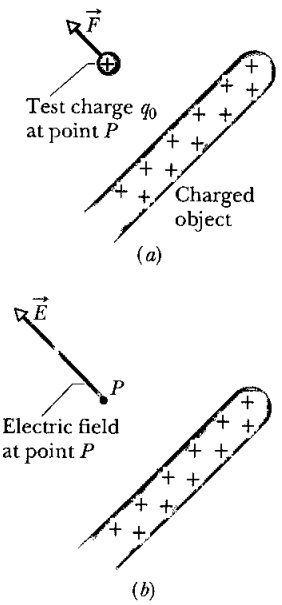


FIG. 22-15 (*a*) A positive test charge q_0 placed at point P near a charged object. An electrostatic force $\vec{F}$ acts on the test charge. (*b*) The electric field $\vec{E}$ at point P produced by the charged object.

nucleus. For example, a uranium-238 nucleus (^{238}U) transforms into a thorium-234 nucleus (^{234}Th) by emitting an *alpha particle.* Because that particle has the same makeup as a helium-4 nucleus, it has the symbol ^{4}He. The number used in the name of a nucleus and as a superscript in the symbol for the nucleus is called the *mass number* and is the total number of the protons and neutrons in the nucleus. For example, the total number in ^{238}U is 238. The number of protons in a nucleus is the *atomic number Z*, which is listed for all the elements in Appendix F. From that list we find that in the decay

$$^{238}\text{U} \rightarrow {}^{234}\text{Th} + {}^{4}\text{He}, \tag{22-14}$$

the *parent* nucleus ^{238}U contains 92 protons (a charge of $+92e$), the *daughter* nucleus ^{234}Th contains 90 protons (a charge of $+90e$), and the emitted alpha particle ^{4}He contains 2 protons (a charge of $+2e$). We see that the total charge is $+92e$ before and after the decay; thus, charge is conserved. (The total number of protons and neutrons is also conserved: 238 before the decay and 234 + 4 = 238 after the decay.)

Another example of charge conservation occurs when an electron e^- (whose charge is $-e$) and its antiparticle, the *positron* e^+ (whose charge is $+e$), undergo an *annihilation process* in which they transform into two *gamma rays* (high-energy light):

$$e^- + e^+ \rightarrow \gamma + \gamma \quad \text{(annihilation)}. \tag{22-15}$$

In applying the conservation-of-charge principle, we must add the charges algebraically, with due regard for their signs. In the annihilation process of Eq. 22-15 then, the net charge of the system is zero both before and after the event. Charge is conserved.

In *pair production*, the converse of annihilation, charge is also conserved. In this process a gamma ray transforms into an electron and a positron:

$$\gamma \rightarrow e^- + e^+ \quad \text{(pair production)}. \tag{22-16}$$

Figure 22-14 shows such a pair-production event that occurred in a bubble chamber. A gamma ray entered the chamber from the bottom and at one point transformed into an electron and a positron. Because those new particles were charged and moving, each left a trail of tiny bubbles. (The trails were curved because a magnetic field had been set up in the chamber.) The gamma ray, being electrically neutral, left no trail. Still, you can tell exactly where it underwent pair production—at the tip of the curved V, which is where the trails of the electron and positron begin.

22-6 | The Electric Field

The temperature at every point in a room has a definite value. You can measure the temperature at any given point or combination of points by putting a thermometer there. We call the resulting distribution of temperatures a *temperature field.* In much the same way, you can imagine a *pressure field* in the atmosphere; it consists of the distribution of air pressure values, one for each point in the atmosphere. These two examples are of *scalar fields* because temperature and air pressure are scalar quantities.

The electric field is a *vector field*; it consists of a distribution of *vectors,* one for each point in the region around a charged object, such as a charged rod. In principle, we can define the electric field at some point near the charged object, such as point P in Fig. 22-15*a*, as follows: We first place a *positive* charge q_0, called a *test charge,* at the point. We then measure the electrostatic force $\vec{F}$ that acts on the test charge. Finally, we define the electric field $\vec{E}$ at point P due to the charged object as

$$\vec{E} = \frac{\vec{F}}{q_0} \quad \text{(electric field)}. \tag{22-17}$$

Thus, the magnitude of the electric field $\vec{E}$ at point P is $E = F/q_0$, and the direction of $\vec{E}$ is that of the force $\vec{F}$ that acts on the *positive* test charge. As shown in Fig. 22-15*b*, we represent the electric field at P with a vector whose tail is at P. To define the electric field within some region, we must similarly define it at all points in the region.

The SI unit for the electric field is the newton per coulomb (N/C). Table 22-2 shows the electric fields that occur in a few physical situations. (We assume that in our defining procedure, the presence of the test charge does not affect the charge distribution on the charged object, and thus does not alter the electric field we are defining.)

To examine the role of an electric field in the interaction between charged objects, we have two tasks: (1) calculating the electric field produced by a given distribution of charge and (2) calculating the force that a given field exerts on a charge placed in it. We perform the first task in Sections 22-8 through 22-10 for several charge distributions. We perform the second task in Sections 22-11 and 22-12 by considering a point charge and a pair of point charges in an electric field. First, however, we discuss a way to visualize electric fields.

TABLE 22-2

Some Electric Fields

Field Location or Situation	Value (N/C)
At the surface of a uranium nucleus	3×10^{21}
Within a hydrogen atom, at a radius of 5.29×10^{-11} m	5×10^{11}
Electric breakdown occurs in air	3×10^{6}
Near the charged drum of a photocopier	10^{5}
Near a charged comb	10^{3}
In the lower atmosphere	10^{2}
Inside the copper wire of household circuits	10^{-2}

22-7 | Electric Field Lines

Michael Faraday, who introduced the idea of electric fields in the 19th century, thought of the space around a charged body as filled with *lines of force*. Although we no longer attach much reality to these lines, now usually called **electric field lines**, they still provide a nice way to visualize patterns in electric fields.

The relation between the field lines and electric field vectors is this: (1) At any point, the direction of a straight field line or the direction of the tangent to a curved field line gives the direction of $\vec{E}$ at that point, and (2) the field lines are drawn so that the number of lines per unit area, measured in a plane that is perpendicular to the lines, is proportional to the *magnitude* of $\vec{E}$. Thus, E is large where field lines are close together and small where they are far apart. The number of electric field lines drawn from any charge should be in proportion to the magnitude of the charge. This is obvious from the second rule. For a larger charge, the electric field should be stronger, and hence the number of field lines in space would be proportionately more.

Figure 22-16*a* shows a sphere of uniform negative charge. If we place a *positive* test charge anywhere near the sphere, an electrostatic force pointing *toward* the center of the sphere will act on the test charge as shown. In other words, the electric field vectors at all points near the sphere are directed radially toward the sphere. This pattern of vectors is neatly displayed by the field lines in Fig. 22-16*b*, which point in the same directions as the force and field vectors. Moreover, the spreading of the field lines with distance from the sphere tells us that the magnitude of the electric field decreases with distance from the sphere.

If the sphere of Fig. 22-16 were of uniform *positive* charge, the electric field vectors at all points near the sphere would be directed radially *away from* the sphere. Thus, the electric field lines would also extend radially away from the sphere. We then have the following rule:

> Electric field lines extend away from positive charge (where they originate) and toward negative charge (where they terminate).

The electric field lines in Fig. 22-16*b* are only in two dimensions. In reality these lines emanate in all the three dimensions. A correct representation is shown in Fig. 22-17.

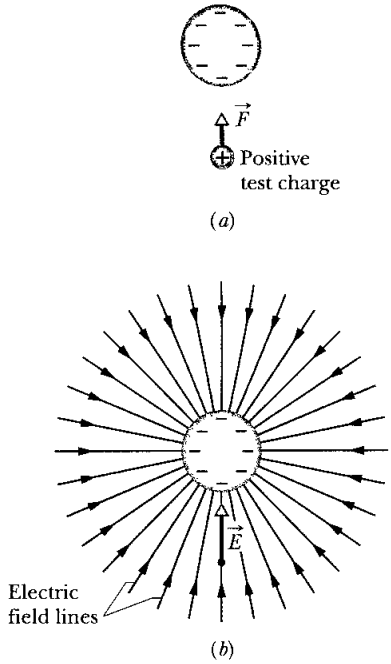


FIG. 22-16 (*a*) The electrostatic force $\vec{F}$ acting on a positive test charge near a sphere of uniform negative charge. (*b*) The electric field vector $\vec{E}$ at the location of the test charge, and the electric field lines in the space near the sphere. The field lines extend *toward* the negatively charged sphere. (They originate on distant positive charges.)

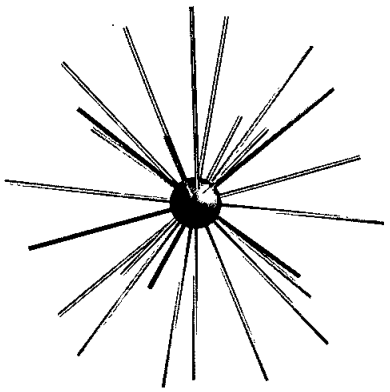


FIG. 22-17 Field lines emanate in three dimensions.

Figure 22-18*a* shows part of an infinitely large, nonconducting *sheet* (or plane) with a uniform distribution of positive charge on one side. If we were to place a positive test charge at any point near the sheet of Fig. 22-18*a*, the net electrostatic force acting on the test charge would be perpendicular to the sheet, because forces acting in all other directions would cancel one another as a result of the symmetry. Moreover, the net force on the test charge would point away from the sheet as shown. Thus, the electric field vector at any point in the space on either side of the sheet is also perpendicular to the sheet and directed away from it (Figs. 22-18*b* and *c*). Because the charge is uniformly distributed along the sheet, all the field vectors have the same magnitude. Such an electric field, with the same magnitude and direction at every point, is a *uniform electric field.*

Of course, no real nonconducting sheet (such as a flat expanse of plastic) is infinitely large, but if we consider a region that is near the middle of a real sheet and not near its edges, the field lines through that region are arranged as in Figs. 22-18*b* and *c*.

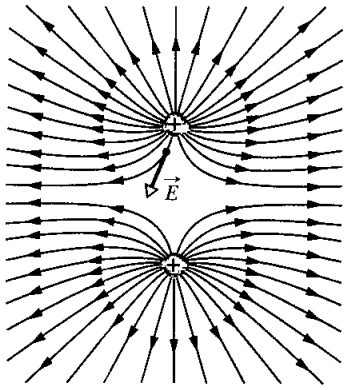


FIG. 22-19 Field lines for two equal positive point charges. The charges repel each other. (The lines terminate on distant negative charges.) To "see" the actual three-dimensional pattern of field lines, mentally rotate the pattern shown here about an axis passing through both charges in the plane of the page. The three-dimensional pattern and the electric field it represents are said to have *rotational symmetry* about that axis. The electric field vector at one point is shown; note that it is tangent to the field line through that point.

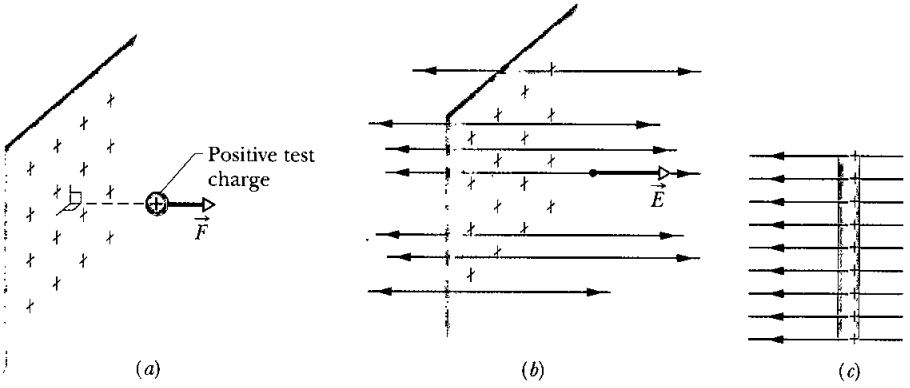
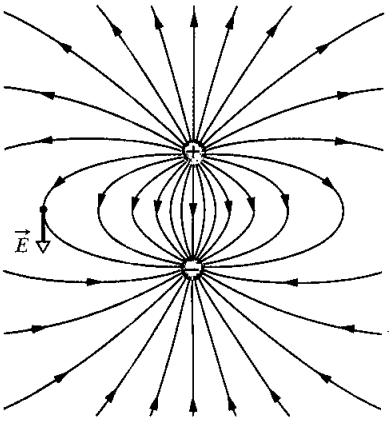


FIG. 22-18 (*a*) The electrostatic force $\vec{F}$ on a positive test charge near a very large, nonconducting sheet with uniformly distributed positive charge on one side. (*b*) The electric field vector $\vec{E}$ at the location of the test charge, and the electric field lines in the space near the sheet. The field lines extend *away from* the positively charged sheet. (*c*) Side view of (*b*).

Figure 22-19 shows the field lines for two equal positive charges. Figure 22-20 shows the pattern for two charges that are equal in magnitude but of opposite sign, a configuration that we call an **electric dipole**. Although we do not often use field lines quantitatively, they are very useful to visualize what is going on. Can you not almost "see" the charges being pushed apart in Fig. 22-19 and pulled together in Fig. 22-20?



22-8 | The Electric Field Due to a Point Charge

To find the electric field due to a point charge q (or charged particle) at any point a distance r from the point charge, we put a positive test charge q_0 at that point. From Coulomb's law (Eq. 22-1), the electrostatic force acting on q_0 is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{\mathrm{r}}. \qquad (22\text{-}18)$$

FIG. 22-20 Field lines for a positive point charge and a nearby negative point charge that are equal in magnitude. The charges attract each other. The pattern of field lines and the electric field it represents have rotational symmetry about an axis passing through both charges in the plane of the page. The electric field vector at one point is shown; the vector is tangent to the field line through the point.

The direction of $\vec{F}$ is directly away from the point charge if q is positive, and directly toward the point charge if q is negative. The electric field vector is, from Eq. 22-17,

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \text{(point charge)}. \tag{22-19}$$

The direction of $\vec{E}$ is the same as that of the force on the positive test charge: directly away from the point charge if q is positive, and toward it if q is negative.

Because there is nothing special about the point we chose for q_0, Eq. 22-19 gives the field at every point around the point charge q. The field for a positive point charge is shown in Fig. 22-21 in vector form (not as field lines).

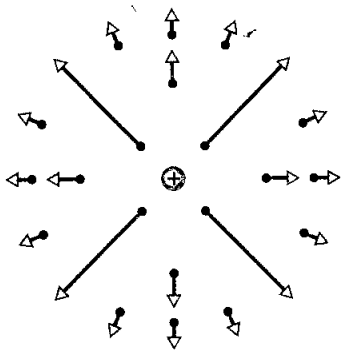


FIG. 22-21 The electric field vectors at various points around a positive point charge.

The variation of the electric field due to a point charge as we move away from it can be shown graphically as well. Figures 22-22*a* and *b* show the variation due to a positive and a negative charge, respectively.

We can quickly find the net, or resultant, electric field due to more than one point charge. If we place a positive test charge q_0 near n point charges $q_1, q_2, \ldots, q_n$, then, from Eq. 22-6, the net force $\vec{F}_0$ from the n point charges acting on the test charge is

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \cdots + \vec{F}_{0}.$$

Therefore, from Eq. 22-17, the net electric field at the position of the test charge is

$$\vec{E} = \frac{\vec{F}_0}{q_0} = \frac{\vec{F}_{01}}{q_0} + \frac{\vec{F}_{02}}{q_0} + \cdots + \frac{\vec{F}_{0n}}{q_0}$$
$$= \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n. \tag{22-20}$$

Here $\vec{E}_i$ is the electric field that would be set up by point charge i acting alone. Equation 22-20 shows us that the principle of superposition applies to electric fields as well as to electrostatic forces.

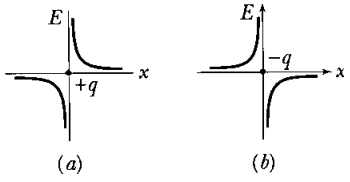
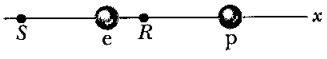


FIG. 22-22 (*a*) Electric field versus x for a positive point charge kept at the origin. Note that the electric field at positive x is positive, because it is in positive direction. At negative x it is negative, because it is in negative direction. (*b*) Electric field versus x for a negative point charge kept at the origin. Note that the electric field at positive x is negative, because it is in negative direction. At negative x it is positive, because it is in positive direction.

CHECKPOINT 6 The figure here shows a proton p and an electron e on an x axis. What is the direction of the electric field due to the electron at (a) point S and (b) point R? What is the direction of the net electric field at (c) point R and (d) point S?



Sample Problem 22-5

Figure 22-23*a* shows three particles with charges $q_1 = +2Q$, $q_2 = -2Q$, and $q_3 = -4Q$, each a distance d from the origin. What net electric field $\vec{E}$ is produced at the origin?

Conceptualize: Charges q_1, q_2, and q_3 produce electric field vectors $\vec{E}_1$, $\vec{E}_2$ and $\vec{E}_3$ respectively, at the origin, and the net electric field is the vector sum $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$. To find this sum, we first must find the magnitudes and orientations of the three field vectors.

Classify: To find the magnitude of $\vec{E}_1$ which is due to q_1, we use Eq. 22-19, substituting d for r and $2Q$ for q and obtaining

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}.$$

Similarly, we find the magnitudes of the fields $\vec{E}_2$ and $\vec{E}_3$ to be

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} \quad \text{and} \quad E_3 = \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}.$$

We next must find the orientations of the three electric field vectors at the origin. Because q_1 is a positive charge, the field vector it produces points directly *away* from it, and because q_2 and q_3 are both negative, the field vectors they produce point directly *toward* each of them. Thus, the three electric fields produced at the origin by the three charged particles are oriented as in Fig. 22-23*b*. (*Caution:* Note that we have placed the tails of the vectors at the point where the fields are to be evaluated; doing so decreases the chance of error.)

Compute: We can now add the fields vectorially as outlined for forces in Sample Problem 22-1c. However, here we can use symmetry to simplify the procedure. From Fig. 22-23*b*, we see that $\vec{E}_1$ and $\vec{E}_2$ have the same direction. Hence, their vector sum has that direction and has the magnitude

$$E_1 + E_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} + \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} = \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2},$$

which happens to equal the magnitude of field $\vec{E}_3$.

We must now combine two vectors, $\vec{E}_3$ and the vector sum $\vec{E}_1 + \vec{E}_2$ that have the same magnitude and that are oriented symmetrically about the x axis, as shown in Fig. 22-23*c*. From the symmetry of Fig. 22-23*c*, we realize that the equal y components of our two vectors cancel and the equal x components add. Thus, the net electric field $\vec{E}$ at the origin is in the positive direction of the x axis and has the magnitude

$$E = 2E_{3x} = 2E_3 \cos 30°$$
$$= (2)\frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}(0.866) = \frac{6.93Q}{4\pi\varepsilon_0 d^2}. \qquad \text{(Answer)}$$

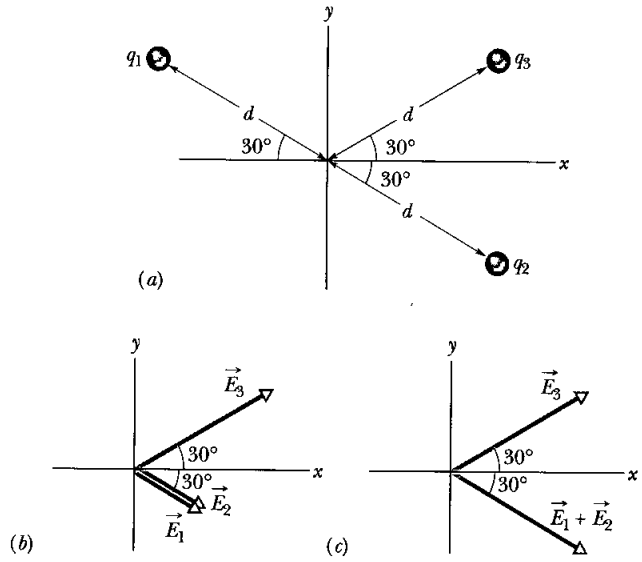


FIG. 22-23 (*a*) Three particles with charges q_1, q_2, and q_3 are at the same distance d from the origin. (*b*) The electric field vectors $\vec{E}_1$, $\vec{E}_2$ and $\vec{E}_3$ at the origin due to the three particles. (*c*) The electric field vector $\vec{E}_3$ and the vector sum $\vec{E}_1 + \vec{E}_2$ at the origin.

Sample Problem 22-6

Two charged particles lie along the x axis as shown in Fig. 22-24. The particle with charge $q_2 = +8\ \mu C$ is at $x = 6.00$ m, and the particle with charge $q_1 = +2\ \mu C$ is at the origin. Locate the point where the resultant electric field is zero.

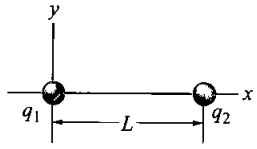


FIG. 22-24 Two charged particles kept on the x axis.

Conceptualize: Before calculating, let us physically see the location of the point where the electric field can be zero. At points other than the x axis, say above the x axis, both the charges will have a component of the electric field in the positive direction. This y component of the electric field does not cancel out. So the net electric field at that point will not be zero. The same statement also holds true for points which are not in xy plane.

On the x axis also we can see that on the points beyond $x = 6$ m and points on the negative x axis, both the electric fields will be in the same direction. So the net electric field cannot be zero. At some point between the two charges, the electric field due to both of them will be in opposite direction. So the electric field will be zero at a point between $x = 0$ and $x = 6$ m.

Classify: To find the electric field at any point, we can apply the superposition principle as in Eq. 22-20.

Compute:

$$E_x = \frac{k\times 2\times 10^{-6}}{x^2} - \frac{k\times 8\times 10^{-6}}{(6-x)^2} = 0.$$

Solving we get $x = 2$ m. We are ignoring the other trivial solution.

Conclude: To analyze this situation more deeply, let us draw a graph of the net electric field at different points on the x axis versus the position on the x axis (Fig. 22-25).

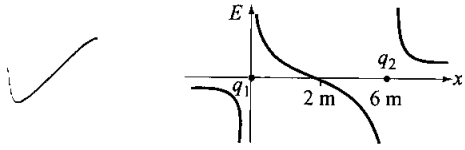


FIG. 22-25 Electric field at various points of x axis due to q_1 and q_2.

This point where the electric field is zero is called a neutral point. We can also say that if the electric field at this point is zero, a charged particle kept at this point will not experience any force.

$$\vec{E} = \frac{\vec{F}}{q}.$$

Then $\vec{F} = q\vec{E}$. Hence, at this point any charged particle will be in equilibrium.

From Sample Problem 22-5 it can be seen that if we want to find the graph of electric field along x axis when there are more than one charge present on the x axis, the graphs can be added to get the graph for the resultant electric field. To see this for yourself, compare Figs. 22-22*a* and 22-25.

Sample Problem 22-7

In the situation described above, find the condition that a charge kept on $x = 2$ m may be in a stable equilibrium. If the charge is of magnitude 1 μC and has a mass of 3 g, find the time period of small oscillations of the body from its equilibrium position.

Conceptualize: The equilibrium may be of three types: (1) Stable equilibrium is the position from which if the particle is displaced, it tends to come back to that position. Apart from the condition that the electric field at that point is zero, in this position potential energy is minimum and hence

$$\frac{d^2U}{dx^2} > 0.$$

Recall from Chapter 8, Section 8-11, Vol. I, that

$$F = -\frac{dU}{dx}.$$

This would mean that

$$\frac{dF}{dx} = -\frac{d^2U}{dx^2} < 0.$$

So if $dF/dx < 0$, then equilibrium is stable.
(2) Unstable equilibrium is the position from which if the particle is displaced, it tends to go away and does not come back. This situation is just the reverse of stable equilibrium. $dF/dx > 0$ implies unstable equilibrium.
(3) In neutral equilibrium, F is zero at all the positions.

Till now, we have been constructing E versus x graph. To construct F versus x graph, we should find $\vec{F} = q\vec{E}$. If the charge whose equilibrium is being considered is positive, the graph of F versus x retains the same shape as that of E versus x. However, if the charge is negative, the graph of F versus x becomes inverted in comparison to E versus x graph. For example, in this situation, if a negative charge is kept at different point on the x axis, its F–x graph is shown in Fig. 22-26.

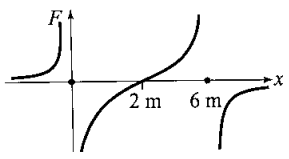


FIG. 22-26 Force on a test charge of $-q$ kept at different points on x axis.

As the slope of F–x graph is positive at the point $x = 2$ m, we can say that it will be in unstable equilibrium. This could have been stated without graph also. If we displace the charge slightly to the left from the position, it would be attracted more by the positive charge on the left and would not come back.

Classify: In electrostatics, if a body is displaced slightly from its stable equilibrium position, it may execute simple harmonic motion (SHM) if the net force is directed toward the equilibrium position and is proportional to x (i.e., displacement from equilibrium position). Recall that from Chapter 15 of Vol. I, we have

$$a = -\omega^2 x.$$

Here ω is the angular frequency of the SHM of the body. Often we need to use the binomial expansion $(1+x)^n \approx 1+nx$ to show that the motion is SHM. This binomial expansion can be used only if $|x| \ll 1$. n can be any real number, positive or negative, integer or noninteger.

Here, the slope of F–x graph should be negative at $x = 2$ m. We already saw that if the test charge kept at 2 m is negative, the equilibrium is unstable. So the charge which is in stable equilibrium for motion along x axis should be positive. Then, slope of F–x graph will be negative as can be seen from Fig. 22-27.

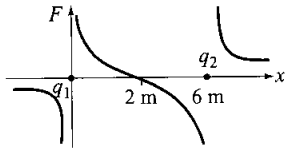


FIG. 22-27 The graph of force versus x for a positive charge kept at different points on the x axis.

Compute: Assume that the charge is displaced slightly to the right of point $x = 2$:

$$F = kq\left(\frac{2\times10^{-6}}{(2+x)^2} - \frac{8\times10^{-6}}{(4-x)^2}\right)$$

$$= kq\times10^{-6}\left(\frac{2}{4\times[1+(x/2)]^2} - \frac{8}{16\times[1-(x/4)]^2}\right)$$

$$= \frac{kq\times10^{-6}}{2}\left[\left(1+\frac{x}{2}\right)^{-2} - \left(1-\frac{x}{4}\right)^{-2}\right]$$

$$= \frac{9\times10^{9}\times1\times10^{-12}}{2}\left[\left(1-\frac{2x}{2}\right)-\left(1+\frac{2x}{4}\right)\right]$$

$$= -\frac{27\times10^{-3}}{4}x.$$

This force is proportional to x and toward mean position. So the motion is simple harmonic:

$$3\times10^{-3}a = -\frac{27\times10^{-3}}{4}x$$

$$a = -\frac{9x}{4}.$$

So $\omega = 3/2$ rad/s, therefore

$$T = \frac{2\pi}{\omega} = \frac{4\pi}{3}s.$$

Conclude: In this problem, we stated that the charge is in stable equilibrium for a longitudinal displacement. Actually, the same charge would be in unstable equilibrium if displaced along a perpendicular direction to the line joining the charges. On the contrary, a negative charge which would have been in unstable equilibrium for motion along the line joining the two, will be in a stable equilibrium for motion in a direction perpendicular to the line joining both the charges.

TABLE 22-3

Some Measures of Electric Charge

Name	Symbol	SI Unit
Charge	q	C
Linear charge density	λ	C/m
Surface charge density	σ	C/m^2
Volume charge density	ρ	C/m^3

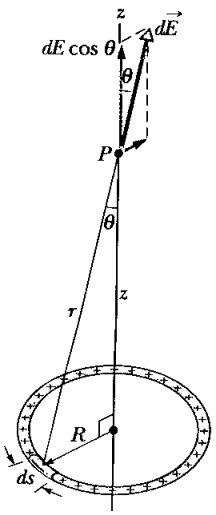


FIG. 22-28 A ring of uniform positive charge. A differential element of charge occupies a length ds (greatly exaggerated for clarity). This element sets up an electric field $d\vec{E}$ at point P. The component of $d\vec{E}$ along the central axis of the ring is $dE\cos\theta$.

22-9 | The Electric Field Due to Continuous Charge Distribution

So far we have considered the electric field that is produced by one or, at most, a few point charges. We now consider charge distributions that consist of a great many closely spaced point charges (perhaps billions) that are spread along a line, over a surface, or within a volume. Such distributions are said to be **continuous** rather than discrete. Since these distributions can include an enormous number of point charges, we find the electric fields that they produce by means of calculus rather than by considering the point charges one by one. In this section we discuss the electric field caused by a line of charge (a line can be straight or curved). We consider a charged surface in the next section. In the next chapter, we shall find the field inside a uniformly charged sphere.

When we deal with continuous charge distributions, it is most convenient to express the charge on an object as a *charge density* rather than as a total charge. For a line of charge, for example, we would report the *linear charge density* (or charge per unit length) λ, whose SI unit is the coulomb per meter. Table 22-3 shows the other charge densities we shall be using.

Figure 22-28 shows a thin ring of radius R with a uniform positive linear charge density λ around its circumference. We may imagine the ring to be made of plastic or some other insulator, so that the charges can be regarded as fixed in place. What is the electric field $\vec{E}$ at point P, a distance z from the plane of the ring along its central axis?

To answer, we cannot just apply Eq. 22-19, which gives the electric field set up by a point charge, because the ring is obviously not a point charge. However, we can mentally divide the ring into differential elements of charge that are so small that they are like point charges, and then we can apply Eq. 22-19 to each of them. Next, we can add the electric fields set up at P by all the differential elements. The vector sum of the fields gives us the field set up at P by the ring.

Let ds be the (arc) length of any differential element of the ring. Since λ is the charge per unit (arc) length, the element has a charge of magnitude

$$dq = \lambda\, ds. \tag{22-21}$$

This differential charge sets up a differential electric field $d\vec{E}$ at point P, which is a distance r from the element. Treating the element as a point charge and using Eq. 22-21, we can rewrite Eq. 22-19 to express the magnitude of $d\vec{E}$ as

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2}, \tag{22-22}$$

From Fig. 22-28, we can rewrite Eq. 22-22 as

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{(z^2+R^2)}. \tag{22-23}$$

Figure 22-28 shows that $d\vec{E}$ is at angle θ to the central axis (which we have taken to be a z axis) and has components perpendicular to and parallel to that axis.

Every charge element in the ring sets up a differential field $d\vec{E}$ at P, with magnitude given by Eq. 22-23. All the $d\vec{E}$ vectors have identical components parallel to the central axis, in both magnitude and direction. All these $d\vec{E}$ vectors have components perpendicular to the central axis as well; these perpendicular components are identical in magnitude but point in different directions. In fact, for any perpendicular component that points in a given direction, there is another one that points in the opposite direction. The sum of this pair of components, like the sum of all other pairs of oppositely directed components, is zero.

Thus, the perpendicular components cancel and we need not consider them further. This leaves the parallel components; they all have the same direction, so the net electric field at P is their sum.

The parallel component of $d\vec{E}$ shown in Fig. 22-28 has magnitude $dE\cos\theta$. The figure also shows us that

$$\cos\theta = \frac{z}{r} = \frac{z}{(z^2+R^2)^{1/2}}. \tag{22-24}$$

Then multiplying Eq. 22-23 by Eq. 22-24 gives us, for the parallel component of $d\vec{E}$,

$$dE\cos\theta = \frac{z\lambda}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\,ds. \tag{22-25}$$

To add the parallel components $dE\cos\theta$ produced by all the elements, we integrate Eq. 22-25 around the circumference of the ring, from $s=0$ to $s=2\pi R$. Since the only quantity in Eq. 22-14 that varies during the integration is s, the other quantities can be moved outside the integral sign. The integration then gives us

$$E = \int dE\cos\theta = \frac{z\lambda}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi R} ds$$

$$= \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}. \tag{22-26}$$

Since λ is the charge per length of the ring, the term $\lambda(2\pi R)$ in Eq. 22-26 is q, the total charge on the ring. We then can rewrite Eq. 22-26 as

$$E = \frac{qz}{4\pi\varepsilon_0(z^2+R^2)^{3/2}} \quad \text{(charged ring)}. \tag{22-27}$$

If the charge on the ring is negative, instead of positive as we have assumed, the magnitude of the field at P is still given by Eq. 22-27. However, the electric field vector then points toward the ring instead of away from it.

Let us check Eq. 22-27 for a point on the central axis that is so far away that $z \gg R$. For such a point, the expression $z^2 + R^2$ in Eq. 22-27 can be approximated as z^2, and Eq. 22-27 becomes

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2} \quad \text{(charged ring at large distance)}. \tag{22-28}$$

This is a reasonable result because from a large distance, the ring "looks like" a point charge. If we replace z with r in Eq. 22-28, we indeed do have Eq. 22-19, the magnitude of the electric field due to a point charge.

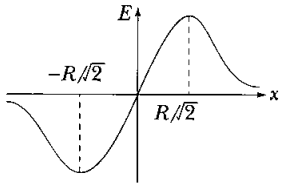


FIG. 22-29 Electric field plotted for different points of the axis of a uniformly charged circular ring.

Let us next check Eq. 22-27 for a point at the center of the ring—that is, for $z = 0$. At that point, Eq. 22-27 tells us that $E = 0$. This is a reasonable result because if we were to place a test charge at the center of the ring, there would be no net electrostatic force acting on it; the force due to any element of the ring would be canceled by the force due to the element on the opposite side of the ring. By Eq. 22-17, if the force at the center of the ring were zero, the electric field there would also have to be zero.

Next point that can be understood is that even if the charges are nonuniformly distributed, the electric field in x direction remains the same. But electric field in y direction will be nonzero. This electric field has a maximum value at $x = R/\sqrt{2}$ as can be seen by differentiating the expression of E (Fig. 22-29).

Sample Problem 22-8 Build your skill

Figure 22-30 shows a plastic rod having a uniformly distributed charge. The charge per unit length of the rod is λ. Find the electric field due to the rod at a point P which is at a distance r from the rod.

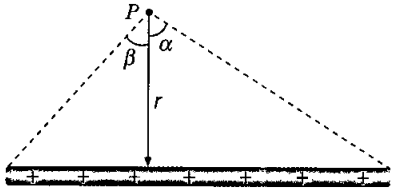


FIG. 22-30 An uniformly charged rod is kept along x axis we need to find the electric field at P.

Conceptualize/Classify: Because the rod has a continuous charge distribution, we must find an expression for the electric fields due to differential elements of the rod and then sum those fields via calculus.

Compute: An element: Consider a differential element having length dx. If the length of a line segment is dx, the charge on it is λdx. Here λ is the charge per unit length of the line (assumed to be constant).

Thus,

$$dq = \lambda dx.$$

The element's field: Our element produces a differential electric field. Treating this element as a point charge, the electric field due to this small charge at a distance r from the line is

$$dE = \frac{k\lambda dx}{(\sqrt{x^2 + r^2})^2}.$$

The electric field is away from the charge because the charge is positive (see Fig. 22-31).

Summing: Thus, to find the electric field set up by the rod, we need sum (via integration) the parallel as well as perpendicular components of the differential electric fields setup by all the differential elements of the rod. Let us resolve this electric field in a direction parallel and perpendicular to the line charge.

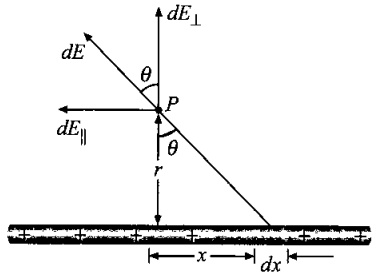


FIG. 22-31 A small element of length dx produces an electric field dE a point P. This electric field is resolved into two components $dE_\perp$ and $dE_\parallel$ as shown.

$$dE_\parallel = dE \sin\theta,$$

$$dE_\perp = dE \cos\theta,$$

$$dE_\parallel = \frac{k\lambda dx}{(x^2 + r^2)} \sin\theta.$$

From the geometry of Fig. 22-31, we can say that

$$\frac{x}{r} = \tan\theta,$$

$$x = r\tan\theta.$$

Since we need to replace x everywhere by the variable θ, we need to differentiate this expression with respect to θ.

$$\frac{dx}{d\theta} = r\sec^2\theta,$$

$$dx = r\sec^2\theta\, d\theta.$$

By replacing this in the expression for dE, we get

$$dE_{\parallel} = \frac{k\lambda \times r\sec^2\theta \times \sin\theta}{r^2 + r^2\tan^2\theta}d\theta,$$

$$dE_{\parallel} = \frac{k\lambda}{r}\sin\theta\, d\theta,$$

$$E_{\parallel} = \frac{k\lambda}{r}\int_{-\beta}^{\alpha}\sin\theta\, d\theta = -\frac{k\lambda}{r}\,[\cos\alpha - \cos(-\beta)$$

$$= +\frac{k\lambda}{r}\,[\cos\beta - \cos\alpha].$$

Similarly, resolving the electric field in the perpendicular direction.

$$E_{\perp} = \frac{k\lambda}{r}\int_{-\beta}^{\alpha}\cos\theta\, d\theta = \frac{k\lambda}{r}[\sin\alpha + \sin\beta] = \frac{k\lambda}{r}[\sin\alpha + \sin\beta]$$

Conclude: If the line is infinite which means that it extends from $-\infty$ to $+\infty$, we can assume $\alpha = 90°$ and $\beta = 90°$. Then

$$E_{\parallel} = \frac{k\lambda}{r}[\cos 90° - \cos 90°] = 0,$$

$$E_{\perp} = \frac{k\lambda}{r}[\sin 90° + \sin 90°] = \frac{2k\lambda}{r}.$$

If the line is semi-infinite, extending from $-\infty$ to the foot of the perpendicular, we can assume $\alpha = 90°$ and $\beta = 0°$. Then,

$$E_{\parallel} = \frac{k\lambda}{r}[\cos 0° - \cos 90°] = \frac{k\lambda}{r},$$

$$E_{\perp} = \frac{k\lambda}{r}[\sin 90° + \sin 0°] = \frac{k\lambda}{r}.$$

Sample Problem 22-9 Build your skill

Figure 22-32*a* shows a plastic rod having a uniformly distributed charge $-Q$. The rod has been bent in a 120° circular arc of radius r. We place coordinate axes such that the axis of symmetry of the rod lies along the x axis and the origin is at the center of curvature P of the rod. In terms of Q and r, what is the electric field $\vec{E}$ due to the rod at point P?

Conceptualize/Classify: Because the rod has a continuous charge distribution, we must find an expression for the electric fields due to differential elements of the rod and then sum those fields via calculus.

Compute: An element: Consider a differential element having arc length ds and located at an angle θ above the x axis (Fig. 22-32*b*). If we let λ represent the linear charge density of the rod, our element ds has a differential charge of magnitude

$$dq = \lambda\, ds. \qquad (22\text{-}29)$$

The element's field: Our element produces a differential electric field $d\vec{E}$ at point P, which is a distance r from the element. Treating the element as a point charge, we can rewrite Eq. 22-19 to express the magnitude of $d\vec{E}$ as

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2}. \qquad (22\text{-}30)$$

The direction of $d\vec{E}$ is toward ds because charge dq is negative.

Symmetric partner: Our element has a symmetrically located (mirror image) element ds' in the bottom half of the rod. The electric field $d\vec{E}'$ set up at P by ds' also has the magnitude given by Eq. 22-30, but the field vector points toward ds' as shown in Fig. 22-32*b*. If we resolve the electric field vectors of ds and ds' into x and y components as shown in Fig. 22-32*b*, we see that their y components cancel (because they have equal magnitudes and are in opposite directions). We also see that their x components have equal magnitudes and are in the same direction.

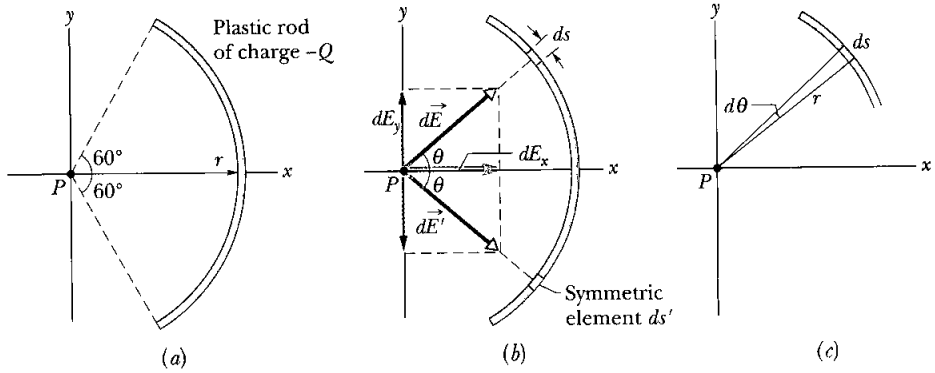


FIG. 22-32 (*a*) A plastic rod of charge $-Q$ is a circular section of radius r and central angle 120°; point P is the center of curvature of the rod. (*b*) A differential element in the top half of the rod, at an angle θ to the x axis and of arc length ds, sets up a differential electric field $d\vec{E}$ at P. An element ds', symmetric to ds about the x axis, sets up a field $d\vec{E}$ at P with the same magnitude. (*c*) Arc length ds makes an angle $d\theta$ about point P.

Summing: Thus, to find the electric field set up by the rod, we need sum (via integration) only the x components of the differential electric fields set up by all the differential elements of the rod. From Fig. 22-32*b* and Eq. 22-30, we can write the component dE_x set up by ds as

$$dE_x = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\cos\theta\, ds. \qquad (22\text{-}31)$$

Equation 22-31 has two variables, θ and s. Before we can integrate it, we must eliminate one variable. We do so by replacing ds, using the relation

$$ds = r\, d\theta,$$

in which $d\theta$ is the angle at P that includes arc length ds (Fig. 22-32*c*). With this replacement, we can integrate Eq. 22-31 over the angle made by the rod at P, from $\theta = -60°$ to $\theta = 60°$; that will give us the magnitude of the electric field at P due to the rod:

$$E = \int dE_x = \int_{-60°}^{60°} \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\cos\theta\, r\, d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 r}\int_{-60°}^{60°}\cos\theta\, d\theta = \frac{\lambda}{4\pi\varepsilon_0 r}[\sin\theta]_{-60°}^{60°}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 r}[\sin 60° - \sin(-60°)]$$

$$= \frac{\sqrt{3}\lambda}{4\pi\varepsilon_0 r}. \qquad (22\text{-}32)$$

Charge density: To evaluate λ, we note that the rod subtends an angle of 120° and so is one-third of a full circle. Its arc length is then $2\pi r/3$, and its linear charge density must be

$$\lambda = \frac{\text{charge}}{\text{length}} = \frac{Q}{2\pi r/3} = \frac{0.477Q}{r},$$

Substituting this into Eq. 22-21 and simplifying give us

$$E = \frac{(1.73)(0.477Q)}{4\pi\varepsilon_0 r^2}$$

$$= \frac{0.83Q}{4\pi\varepsilon_0 r^2}. \qquad \text{(Answer)}$$

The direction of $\vec{E}$ is toward the rod, along the axis of symmetry of the charge distribution. We can write $\vec{E}$ in unit-vector notation as

$$\vec{E} = \frac{0.83Q}{4\pi\varepsilon_0 r^2}\hat{i}.$$

PROBLEM-SOLVING TACTICS

Tactic 1: *A Field Guide for Lines of Charge* Here is a generic guide for finding the electric field $\vec{E}$ produced at a point P by a line of uniform charge, either circular or straight. The general strategy is to pick out an element dq of the charge, find $d\vec{E}$ due to that element, and integrate $d\vec{E}$ over the entire line of charge.

Step 1. If the line of charge is circular, let ds be the arc length of an element of the distribution. If the line is straight, run an x axis along it and let dx be the length of an element. Mark the element on a sketch.

Step 2. Relate the charge dq of the element to the length of the element with either $dq = \lambda\, ds$ or $dq = \lambda\, dx$. Consider dq and λ to be positive, even if the charge is actually negative. (The sign of the charge is used in the next step.)

Step 3. Express the field $d\vec{E}$ produced at P by dq with Eq. 22-19, replacing q in that equation with either $\lambda\, ds$ or $\lambda\, dx$. If the charge on the line is positive, then at P draw a vector $d\vec{E}$ that points directly away from dq. If the charge is negative, draw the vector pointing directly toward dq.

Step 4. Always look for any symmetry in the situation. If P is on an axis of symmetry of the charge distribution, resolve the field $d\vec{E}$ produced by dq into components that are perpendicular and parallel to the axis of symmetry. Then consider a second element dq' that is located symmetrically to dq about the line of symmetry. At P draw the vector $d\vec{E}'$ that this symmetrical element produces and resolve it into components. One of the components produced by dq is a *canceling component;* it is canceled by the corresponding component produced by dq' and needs no further attention. The other component produced by dq is an *adding component;* it adds to the corresponding component produced by dq'. Add the adding components of all the elements via integration.

Step 5. Here are four general types of uniform charge distributions, with strategies for simplifying the integral of step 4.

Ring, with point P on (central) axis of symmetry, as in Fig. 22-28. In the expression for dE, replace r^2 with $z^2 + R^2$, as in Eq. 22-23. Express the adding component of $d\vec{E}$ in terms of θ. That introduces $\cos\theta$, but θ is identical for all elements and thus is not a variable. Replace $\cos\theta$ as in Eq. 22-24. Integrate over s, around the circumference of the ring.

Circular arc, with point P at the center of curvature, as in Fig. 22-24. Express the adding component of $d\vec{E}$ in terms of θ. That introduces either $\sin\theta$ or $\cos\theta$. Reduce the resulting two variables s and θ to one, θ, by replacing ds with $r\, d\theta$. Integrate over θ, as in Sample Problem 22-9, from one end of the arc to the other end.

Straight line, with point P on an extension of the line, as in Fig. 22-33*a*. In the expression for dE, replace r with x. Integrate over x, from end to end of the line of charge.

Straight line, with point P at perpendicular distance y from the line of charge, as in Fig. 22-33*b*. In the

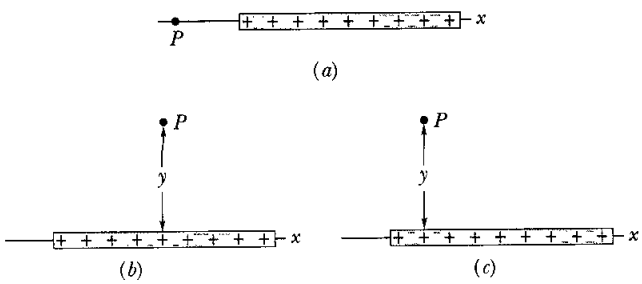
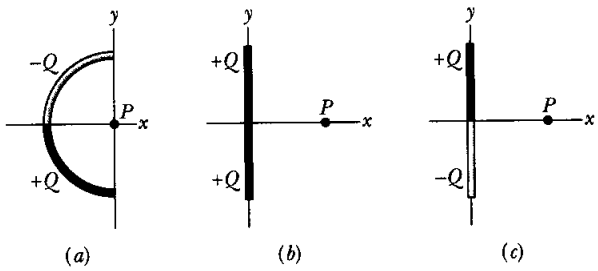


FIG. 22-33 (*a*) Point P is on an extension of the line of charge. (*b*) P is on a line of symmetry of the line of charge, at perpendicular distance y from that line. (*c*) Same as (*b*) except that P is not on a line of symmetry.

expression for dE, replace r with an expression involving x and y. If P is on the perpendicular bisector of the line of charge, find an expression for the adding component of $d\vec{E}$. That will introduce either $\sin\theta$ or $\cos\theta$. Reduce the resulting two variables x and θ to one, x, by replacing the trigonometric function with an expression (its definition) involving x and y. Integrate over x from end to end of the line of charge. If P is not on a line of symmetry, as in Fig. 22-33*c*, set up an integral to sum the components dE_x, and integrate over x to find E_x. Also set up an integral to sum the components dE_y, and integrate over x again to find E_y. Use the components E_x and E_y in the usual way to find the magnitude E and the orientation of $\vec{E}$.

CHECKPOINT 7 The figure here shows three nonconducting rods, one circular and two straight. Each has a uniform charge of magnitude Q along its top half and another along its bottom half. For each rod, what is the direction of the net electric field at point P?



22-10 | The Electric Field Due to a Charged Disk

Figure 22-34 shows a circular plastic disk of radius R that has a positive surface charge of uniform density σ on its upper surface (see Table 22-3). What is the electric field at point P, a distance z from the disk along its central axis?

Our plan is to divide the disk into concentric flat rings and then to calculate the electric field at point P by adding up (that is, by integrating) the contributions of all the rings. Figure 22-34 shows one such ring, with radius r and radial width dr. Since σ is the charge per unit area, the charge on the ring is

$$dq = \sigma\, dA = \sigma(2\pi r\, dr), \tag{22-33}$$

where dA is the differential area of the ring.

We have already solved the problem of the electric field due to a ring of charge. Substituting dq from Eq. 22-23 for q in Eq. 22-27, and replacing R in Eq. 22-27 with r, we obtain an expression for the electric field dE at P due to the arbitrarily chosen flat ring of charge shown in Fig. 22-34:

$$dE = \frac{z\sigma 2\pi r\, dr}{4\pi\varepsilon_0(z^2+r^2)^{3/2}},$$

which we may write as

$$dE = \frac{\sigma z}{4\varepsilon_0}\frac{2r\, dr}{(z^2+r^2)^{3/2}}. \tag{22-34}$$

We can now find E by integrating Eq. 22-34 over the surface of the disk—that is, by integrating with respect to the variable r from $r=0$ to $r=R$. Note that z remains constant during this process. We get

$$E = \int dE \frac{\sigma z}{4\varepsilon_0}\int_0^R (z^2+r^2)^{-3/2}(2r)dr. \tag{22-35}$$

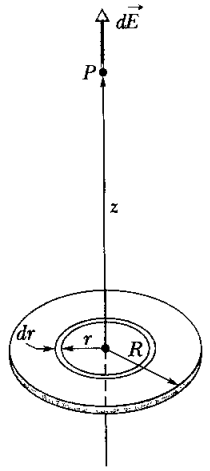


FIG. 22-34 A disk of radius R and uniform positive charge. The ring shown has radius r and radial width dr. It sets up a differential electric field $d\vec{E}$ at point P on its central axis.

To solve this integral, we cast it in the form $\int X^m dX$ by setting $X = (z^2 + r^2)$, $m = -\frac{3}{2}$, and $dX = (2r)\,dr$. For the recast integral we have

$$\int X^m dX = \frac{X^m + 1}{m + 1},$$

and so Eq. 22-35 becomes

$$E = \frac{\sigma z}{4\varepsilon_0}\left[\frac{(z^2 + r^2)^{-1/2}}{-1/2}\right]_0^R. \tag{22-36}$$

Taking the limits in Eq. 22-36 and rearranging, we find

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \quad \text{(charged disk)}. \tag{22-37}$$

as the magnitude of the electric field produced by a flat, circular, charged disk at points on its central axis. (In carrying out the integration, we assumed that $z \geq 0$.)

If we let $R \to \infty$ while keeping z finite, the second term in the parentheses in Eq. 22-37 approaches zero, and this equation reduces to

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(infinite sheet)}. \tag{22-38}$$

This is the electric field produced by an infinite sheet of uniform charge located on one side of a nonconductor such as plastic. The electric field lines for such a situation are shown in Fig. 22-18.

We also get Eq. 22-38 if we let $z \to 0$ in Eq. 22-37 while keeping R finite. This shows that at points very close to the disk, the electric field set up by the disk is the same as if the disk were infinite in extent.

Sample Problem 22-10 Build your skill

Find the electric field at point P as shown in Fig. 22-35. Assume that the rod is uniformly charged with a charge per unit length λ.

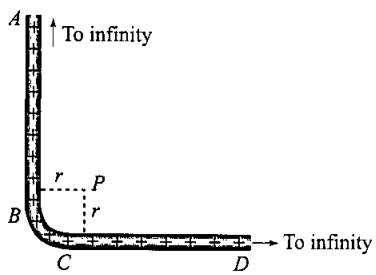


FIG. 22-35 (*a*) Electric field due to *L*-shaped wire at *P*.

Conceptualize/Classify: We will first find the electric field due to two straight semi-infinite wires.

Compute: The electric field due to the wire AB will be:
Parallel to the wire,

$$E_x = \frac{k\lambda}{r}[\cos 90° - \cos 0°] = -\frac{k\lambda}{r}.$$

Perpendicular to the wire,

$$E_y = \frac{k\lambda}{r}[\sin 90° + \sin 0°] = \frac{k\lambda}{r}.$$

Similarly, the electric field due to the wire CD will be:
Parallel to the wire,

$$E_y = \frac{k\lambda}{r}[\cos 90° - \cos 0°] = -\frac{k\lambda}{r}.$$

Perpendicular to the wire,

$$E_x = \frac{k\lambda}{r}[\sin 90° + \sin 0°] = \frac{k\lambda}{r}.$$

Electric field due to the circular arc BC will be:

$$E = \frac{2k\lambda}{r}\left(\sin\frac{90°}{2}\right) = \frac{\sqrt{2}k\lambda}{r}.$$

Since this electric field will be along the axis of symmetry of the arc, it will be at an angle of 45° to the x axis:

$$E_x = \frac{k\lambda}{r}; \qquad E_y = \frac{k\lambda}{r}$$

Therefore

$$\vec{E} = \frac{k\lambda}{r}[\hat{i} + \hat{j}].$$

22-11 | A Point Charge in an Electric Field

In the preceding four sections we worked at the first of our two tasks: given a charge distribution, to find the electric field it produces in the surrounding space. Here we begin the second task: to determine what happens to a charged particle when it is in an electric field set up by other stationary or slowly moving charges.

What happens is that an electrostatic force acts on the particle, as given by

$$\vec{F} = q\vec{E}, \tag{22-39}$$

in which q is the charge of the particle (including its sign) and $\vec{E}$ is the electric field that other charges have produced at the location of the particle. (The field is *not* the field set up by the particle itself; to distinguish the two fields, the field acting on the particle in Eq. 22-39 is often called the *external field.* A charged particle or object is not affected by its own electric field.) Equation 22-39 tells us

> The electrostatic force $\vec{F}$ acting on a charged particle located in an external electric field $\vec{E}$ has the direction of $\vec{E}$ if the charge q of the particle is positive and has the opposite direction if q is negative.

Sample Problem 22-11

An electron ($q = -e$, $m = 9.1 \times 10^{-31}$ kg) is projected out along the positive x axis with an initial speed of 3×10^6 m/s. It goes 45 cm and stops due to a uniform electric field in the region. Find the magnitude and direction of the field.

Conceptualize/Classify: The force, and hence acceleration, is constant, and must be in the negative x direction. This is necessary because if there were a y component of force, the velocity component in that direction would continually increase and never become zero for $t > 0$. Furthermore, the component in the x direction must act to slowdown the initial velocity of the ball.

Compute: From one-dimensional kinematics,

$$v_x^2 = v_{0x}^2 + 2a_x x.$$

For our case

$$v_{0x} = 3\times10^6 \text{ m/s},$$
$$v_x = 0,$$
$$x = 0.45 \text{ m}.$$

Then

$$a_x = \frac{-(3\times10^6)^2}{(2\times0.45)} = -1.0\times10^{13} \text{ m/s}^2.$$

The force causing this acceleration is

$$F_x = ma_x = (9.1\times10^{-31}\text{ kg})(-1.0\times10^{13}\text{ m/s}^2),$$

$$F_x = -9.1\times10^{-18} \text{ N}.$$

F_x is due to the electrostatic force on the electron caused by the electric field, so

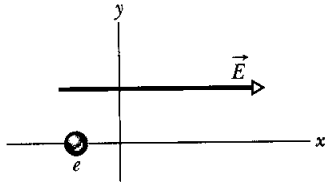
$$F_x = qE_x,$$

$$E_x = \frac{-9.1\times10^{-18}\text{ N}}{-1.6\times10^{-19}\text{ C}} = 57 \text{ N/C}.$$

Finally, as $F_y = 0$, we have the electric field = 57 N/C in the positive x direction.

Conclude: The force of gravity on the electron is completely negligible in comparison with the electric force and has therefore been ignored.

CHECKPOINT 8 (*a*) In the figure, what is the direction of the electrostatic force on the electron due to the external electric field shown? (*b*) In which direction will the electron accelerate if it is moving parallel to the y axis before it encounters the external field? (*c*) If, instead, the electron is initially moving rightward, will its speed increase, decrease, or remain constant?



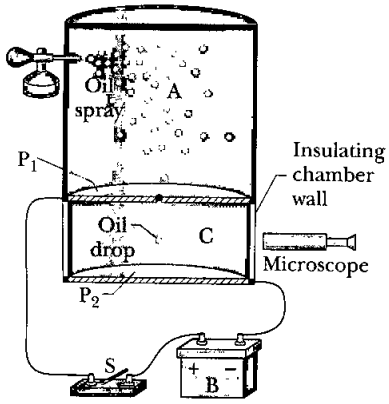


FIG. 22-36 The Millikan oil-drop apparatus for measuring the elementary charge e. When a charged oil drop drifted into chamber C through the hole in plate P_1, its motion could be controlled by closing and opening switch S and thereby setting up or eliminating an electric field in chamber C. The microscope was used to view the drop, to permit timing of its motion.

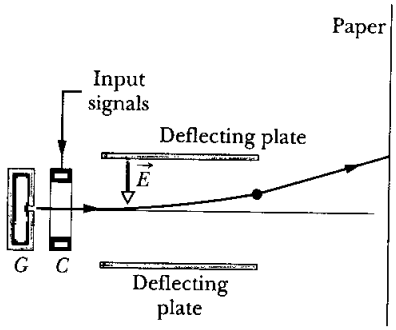
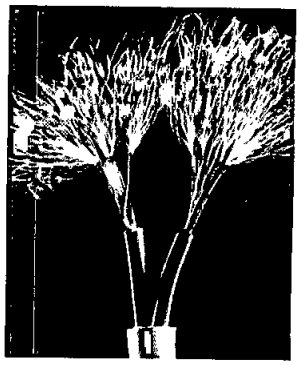


FIG. 22-37 The essential features of an ink-jet printer. Drops are shot out from generator G and receive a charge in charging unit C. An input signal from a computer controls the charge given to each drop and thus the effect of field $\vec{E}$ on the drop and the position on the paper at which the drop lands. About 100 tiny drops are needed to form a single character.



Measuring the Elementary Charge

Equation 22-39 played a role in the measurement of the elementary charge e by American physicist Robert A. Millikan in 1910–1913. Figure 22-36 is a representation of his apparatus. When tiny oil drops are sprayed into chamber A, some of them become charged, either positively or negatively, in the process. Consider a drop that drifts downward through the small hole in plate P_1 and into chamber C. Let us assume that this drop has a negative charge q.

If switch S in Fig. 22-36 is open as shown, battery B has no electrical effect on chamber C. If the switch is closed (the connection between chamber C and the positive terminal of the battery is then complete), the battery causes an excess positive charge on conducting plate P_1 and an excess negative charge on conducting plate P_2. The charged plates set up a downward-directed electric field $\vec{E}$ in chamber C. According to Eq. 22-39, this field exerts an electrostatic force on any charged drop that happens to be in the chamber and affects its motion. In particular, our negatively charged drop will tend to drift upward.

By timing the motion of oil drops with the switch opened and with it closed and thus determining the effect of the charge q, Millikan discovered that the values of q were always given by

$$q = ne, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{22-40}$$

in which e turned out to be the fundamental constant we call the *elementary charge,* 1.60×10^{-19} C. Millikan's experiment is convincing proof that charge is quantized, and he earned the 1923 Nobel Prize in physics in part for this work. Modern measurements of the elementary charge rely on a variety of interlocking experiments, all more precise than the pioneering experiment of Millikan.

Ink-Jet Printing

The need for high-quality, high-speed printing has caused a search for an alternative to impact printing, such as occurs in a standard typewriter. Building up letters by squirting tiny drops of ink at the paper is one such alternative.

Figure 22-37 shows a negatively charged drop moving between two conducting deflecting plates, between which a uniform, downward-directed electric field $\vec{E}$ has been set up. The drop is deflected upward according to Eq. 22-39 and then strikes the paper at a position that is determined by the magnitudes of $\vec{E}$ and the charge q of the drop.

In practice, E is held constant and the position of the drop is determined by the charge q delivered to the drop in the charging unit, through which the drop must pass before entering the deflecting system. The charging unit, in turn, is activated by electronic signals that encode the material to be printed.

Electrical Breakdown and Sparking

If the magnitude of an electric field in air exceeds a certain critical value E_c, the air undergoes *electrical breakdown,* a process whereby the field removes electrons from the atoms in the air. The air then begins to conduct electric current because the freed electrons are propelled into motion by the field. As they move, they collide with any atoms in their path, causing those atoms to emit light. We can see the paths, commonly called sparks, taken by the freed electrons because of that emitted light. Figure 22-38 shows sparks above charged metal wires where the electric fields due to the wires cause electrical breakdown of the air.

FIG. 22-38 The metal wires are so charged that the electric fields they produce in the surrounding space cause the air there to undergo electrical breakdown. *(Adam Hart-Davis/ Photo Researchers).*

Sample Problem 22-12

Figure 22-39 shows the deflecting plates of an ink-jet printer, with superimposed coordinate axes. An ink drop with a mass m of 1.3×10^{-10} kg and a negative charge of magnitude $Q = 1.5 \times 10^{-13}$ C enters the region between the plates, initially moving along the x axis with speed $v_x = 18$ m/s. The length L of each plate is 1.6 cm. The plates are charged and thus produce an electric field at all points between them. Assume that field $\vec{E}$ is downward directed, is uniform, and has a magnitude of 1.4×10^6 N/C. What is the vertical deflection of the drop at the far edge of the plates? (The gravitational force on the drop is small relative to the electrostatic force acting on the drop and can be neglected.)

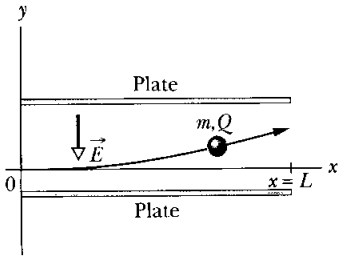


FIG. 22-39 An ink drop of mass m and charge magnitude Q is deflected in the electric field of an ink-jet printer.

Conceptualize: The drop is negatively charged and the electric field is directed *downward.* From Eq. 22-39, a constant electrostatic force of magnitude QE acts *upward* on the charged drop. Thus, as the drop travels parallel to the x axis at constant speed v_x, it accelerates upward with some constant acceleration a_y.

Compute: Applying Newton's second law ($F = ma$) for components along the y axis, we find that

$$a_y = \frac{F}{m} = \frac{QE}{m}, \tag{22-41}$$

Let t represent the time required for the drop to pass through the region between the plates. During t the vertical and horizontal displacements of the drop are

$$y = \frac{1}{2}a_y t^2 \quad \text{and} \quad L = v_x t, \tag{22-42}$$

respectively. Eliminating t between these two equations and substituting Eq. 22-41 for a_y, we find

$$y = \frac{QEL^2}{2mv_x^2}$$

$$= \frac{(1.5 \times 10^{13}\,\text{C})(1.4 \times 10^6\,\text{N/C})(1.6 \times 10^2\,\text{m})^2}{(2)(1.3 \times 10^{-10}\,\text{kg})(18\,\text{m/s})^2}$$

$$= 6.4 \times 10^{-4}\,\text{m}$$

$$= 0.64\,\text{mm}. \qquad \text{(Answer)}$$

Conclude: You can imagine how charged ink drops can be deflected to strike the appropriate point on the paper by manipulating the electric field.

Sample Problem 22-13

A semi-infinite-charged rod is arranged so that its one end coincides with the center of a charged ring as shown in Fig. 22-40. Both of them have a charge per unit length of λ. The radius of ring is R.

(a) What is the force on the ring due to the rod?

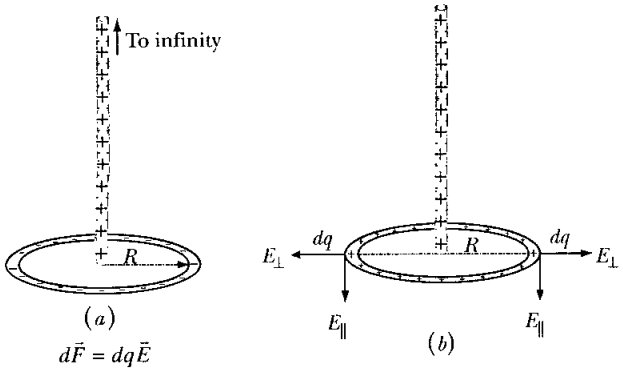


FIG. 22-40 (*a*) Force of interaction between a line charge and a ring. (*b*) The electric field of rod exerts force on an element of the ring in the direction shown.

Conceptualize/Classify: We can find the electric field due to rod, and find the force due to this electric field on an element of the ring. To find the net force of interaction between the two, we can integrate the two components of the force to obtain the net force on the ring.

Compute: As we studied in the Sample Problem 22-8, the electric field due to a semi-infinite rod at a point near the end has two components as shown in Fig. 22-40*b*. Then,

$$E_{\parallel} = \frac{k\lambda}{r}[\cos 0° - \cos 90°] = \frac{k\lambda}{r},$$

$$E_{\perp} = \frac{k\lambda}{r}[\sin 90° + \sin 0°] = \frac{k\lambda}{r}.$$

To find the force due to this electric field on an element of charge, let us consider a small part of the ring as shown in Fig. 22-41 .

The force due to electric field in the parallel direction will all add up as they are all in the same direction.

$$F = QE_{\parallel} = Q\frac{k\lambda}{R} = \lambda \times 2\pi R \times \frac{k\lambda}{R} = \frac{\lambda^2}{2\varepsilon_0}.$$

We can see that the direction of $E_{\perp}$ is different for every point on the ring. The net force due to this perpendicular component of the electric field will cancel out in pairs as seen in Fig. 22-40*b*.

Conclude: By Newton's third law, the force on the rod due to ring will also be $\lambda^2/2\varepsilon_0$. But its proof will be more tedious. Here, we can take a small element of the rod, find the force on it due to ring and integrate to find the net force on the rod. This is left for the reader to evaluate. But note carefully that the third law gives a relation between net forces. It does not give the relation for the force experienced by the individual elements.

(*b*) What is the increase in the tension of the ring due to the rod?

Conceptualize/Classify: Tension happens to be an internal force. So it will not appear in the free-body diagram of the ring. So to find its value, we need to take a small element of the ring and consider the forces exerted by the rest of the part of the ring on this element. The free-body diagram of this ring will show the tension exerted by the rest of the part of the ring on this element (Fig. 22-41). Now by applying Newton's laws, we can find the value of the tension in the ring.

Compute: The net force due to the perpendicular component of the electric field will cancel out in pairs as shown in Fig. 22-40*b*. But this force will produce a tension in the ring. To find this tension, consider a small element of the ring. The force experienced by it in the radial direction is

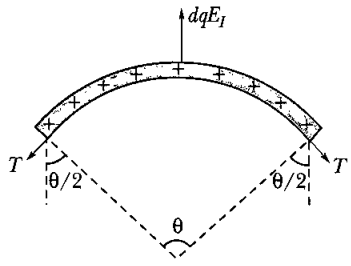


FIG. 22-41 The force exerted by semi-infinite rod on an element of the ring in the radial direction.

$$dF = dqE_{\perp} = dq \times \frac{k\lambda}{R}.$$

This force is balanced by the tension acting at the ends. From Fig. 22-41, we can say that

$$2T\sin\frac{\theta}{2} = dq \times \frac{k\lambda}{R} = \lambda \times R\theta \times \frac{k\lambda}{R}.$$

For small angle,

$$\sin\frac{\theta}{2} \approx \frac{\theta}{2}.$$

Therefore

$$T = \frac{\lambda^2}{4\pi\varepsilon_0}.$$

Conclude: Here, we took small element because for a larger arc, the force on different element will be in different directions, and resultant force has to be found by vector addition of force on the different elements.

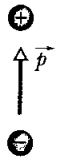


FIG. 22-42 The dipole moment $\vec{p}$ of the dipole points from the negative charge to the positive charge.

22-12 | A Dipole in an Electric Field

Figure 22-42 shows two charged particles of magnitude q but of opposite sign, separated by a distance d. As was noted in connection with Fig. 22-20, we call this configuration an *electric dipole*. The electric dipole moment $\vec{p}$ of an electric dipole is a vector that points from the negative to the positive end of the dipole. As you will see, the behaviour of a dipole in a uniform external electric field $\vec{E}$ can be described completely in terms of the two vectors $\vec{E}$ and $\vec{p}$, with no need of any details about the dipole's structure.

A molecule of water (H_2O) is an electric dipole; Fig. 22-43 shows why. There the black dots represent the oxygen nucleus (having eight protons) and the two hydrogen nuclei (having one proton each). The colored enclosed areas represent the regions in which electrons can be located around the nuclei.

In a water molecule, the two hydrogen atoms and the oxygen atom do not lie on a straight line but form an angle of about 105°, as shown in Fig. 22-43. As a result, the molecule has a definite "oxygen side" and "hydrogen side." Moreover, the 10 electrons of the molecule tend to remain closer to the oxygen nucleus than to the hydrogen nuclei. This makes the oxygen side of the molecule slightly more negative than the hydrogen side and creates an electric dipole moment $\vec{p}$ that points along the symmetry axis of the molecule as shown.

To examine this behavior, we now consider such an abstract dipole in a uniform external electric field $\vec{E}$, as shown in Fig. 22-44*a*. We assume that the dipole is a rigid structure that consists of two centers of opposite charge, each of magnitude q, separated by a distance d. The dipole moment $\vec{p}$ makes an angle θ with field $\vec{E}$.

Electrostatic forces act on the charged ends of the dipole. Because the electric field is uniform, those forces act in opposite directions (as shown in Fig. 22-44*a*) and with the same magnitude $F = qE$. Thus, *because the field is uniform,* the net force on the dipole from the field is zero and the center of mass of the dipole does not move. However, the forces on the charged ends do produce a net torque $\vec{\tau}$ on the dipole about its center of mass. The center of mass lies on the line connecting the charged ends, at some distance x from one end and thus a distance $d - x$ from the other end. Since $\tau = rF \sin \phi$, we can write the magnitude of the net torque $\vec{\tau}$ as

$$\tau = Fx \sin\theta + F(d - x) \sin\theta = Fd \sin\theta. \tag{22-43}$$

We can also write the magnitude of $\vec{\tau}$ in terms of the magnitudes of the electric field E and the dipole moment $p = qd$. To do so, we substitute qE for F and p/q for d in Eq. 22-43, finding that the magnitude of $\vec{\tau}$ is

$$\tau = pE \sin\theta. \tag{22-44}$$

We can generalize this equation to vector form as

$$\vec{\tau} = \vec{p} \times \vec{E} \quad \text{(torque on a dipole)}. \tag{22-45}$$

Vectors $\vec{p}$ and $\vec{E}$ are shown in Fig. 22-44*b*. The torque acting on a dipole tends to rotate $\vec{p}$ (hence the dipole) into the direction of field $\vec{E}$, thereby reducing θ. In Fig. 22-44, such rotation is clockwise. As we discussed in the chapter on Rigid Body Dynamics-I (Chapter 10, Vol. I), we can represent a torque that gives rise to a clockwise rotation by including a minus sign with the magnitude of the torque. With that notation, the torque of Fig. 22-44 is

$$\tau = -pE \sin\theta. \tag{22-46}$$

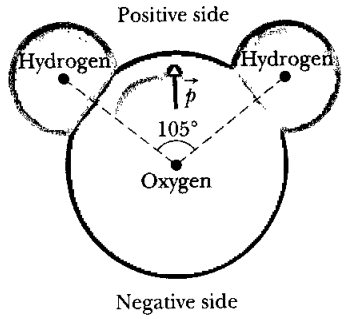


FIG. 22-43 A molecule of H_2O, showing the three nuclei (represented by dots) and the regions in which the electrons can be located. The electric dipole moment $\vec{p}$ points from the (negative) oxygen side to the (positive) hydrogen side of the molecule.

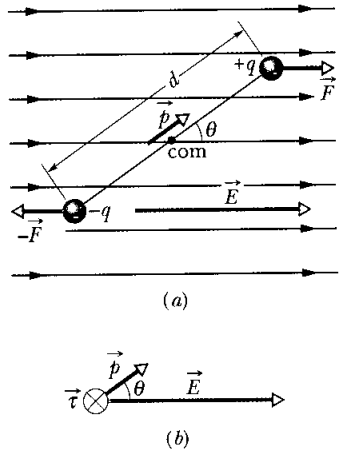


FIG. 22-44 (*a*) An electric dipole in a uniform external electric field $\vec{E}$. Two centers of equal but opposite charge are separated by distance d. The line between them represents their rigid connection. (*b*) Field $\vec{E}$ causes a torque $\vec{\tau}$ on the dipole. The direction of $\vec{\tau}$ is into the page, as represented by the symbol ⊗.

Potential Energy of an Electric Dipole

Potential energy can be associated with the orientation of an electric dipole in an electric field. The dipole has its least potential energy when it is in its equilibrium orientation, which is when its moment $\vec{p}$ is lined up with the field $\vec{E}$ (then $\vec{\tau} = \vec{p} \times \vec{E} = 0$). It has greater potential energy in all other orientations. Thus the dipole is like a pendulum, which has *its* least gravitational potential energy in *its* equilibrium orientation—at its lowest point. To rotate the dipole or the pendulum to any other orientation requires work by some external agent.

In any situation involving potential energy, we are free to define the zero-potential-energy configuration in a perfectly arbitrary way because only differences in potential energy have physical meaning. It turns out that the expression for the potential energy of an electric dipole in an external electric field is simplest if we choose the potential energy to be zero when the angle θ in Fig. 22-44 is 90°. We then can find the potential energy U of the dipole at any other value of θ with the definition of potential energy ($\Delta U = -W$) by calculating the work W done by the field on the dipole when the dipole is rotated to that value of θ from 90°. With the aid of the definition of work ($W = \int \tau\, d\theta$) and Eq. 22-46, we find that the potential energy U at any angle θ is

$$U = -W = -\int_{90°}^{\theta} \tau d\theta = \int_{90°}^{\theta} pE \sin\theta\, d\theta. \tag{22-47}$$

Evaluating the integral leads to

$$U = -pE \cos\theta. \tag{22-48}$$

We can generalize this equation to vector form as

$$U = -\vec{p} \cdot \vec{E} \quad \text{(potential energy of a dipole)}. \tag{22-49}$$

Equations 22-48 and 22-49 show us that the potential energy of the dipole is least ($U = -pE$) when $\theta = 0$ ($\vec{p}$ and $\vec{E}$ are in the same direction); the potential energy is greatest ($U = pE$) when $\theta = 180°$ ($\vec{p}$ and $\vec{E}$ are in opposite directions).

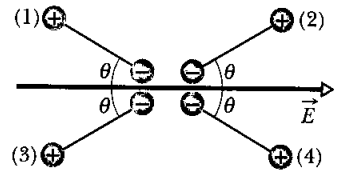
When a dipole rotates from an initial orientation θ_i to another orientation θ_f, the work W done on the dipole by the electric field is

$$W = -\Delta U = -(U_f - U_i), \tag{22-50}$$

where U_f and U_i are calculated with Eq. 22-49. If the change in orientation is caused by an applied torque (commonly said to be due to an external agent), then the work W_a done on the dipole by the applied torque is the negative of the work done on the dipole by the field; that is,

$$W_a = -W = (U_f - U_i). \tag{22-51}$$

CHECKPOINT 9 The figure shows four orientations of an electric dipole in an external electric field. Rank the orientations according to (a) the magnitude of the torque on the dipole and (b) the potential energy of the dipole, greatest first.



Microwave Cooking

Food can be warmed and cooked in a microwave oven if the food contains water because water molecules are electric dipoles. When you turn on the oven, the microwave source sets up a rapidly oscillating electric field $\vec{E}$ within the oven and thus also within the food. From Eq. 22-45, we see that any electric field $\vec{E}$ produces a torque on an electric dipole moment $\vec{p}$ to align $\vec{p}$ with $\vec{E}$. Because the oven's $\vec{E}$ oscillates, the water molecules continuously flip-flop in a frustrated attempt to align with $\vec{E}$.

Energy is transferred from the electric field to the thermal energy of the water (and thus of the food) where three water molecules happened to have bonded together to form a group. The flip-flop breaks some of the bonds. When the molecules reform the bonds, energy is transferred to the random motion of the group and then to the surrounding molecules. Soon, the thermal energy of the water is enough to cook the food. Sometimes the heating is surprising. If you heat a jelly donut, for example, the jelly (which holds a lot of water) heats far more than the donut material (which holds much less water). Although the exterior of the donut may not be hot, biting into the jelly can burn you. If water molecules were not electric dipoles, we would not have microwave ovens.

Sample Problem 22-14

A neutral water molecule (H_2O) in its vapor state has an electric dipole moment of magnitude 6.2×10^{-30} C · m.

(a) How far apart are the molecule's centers of positive and negative charge?

Conceptualize/Classify: A molecule's dipole moment depends on the magnitude q of the molecule's positive or negative charge and the charge separation d.

Compute: There are 10 electrons and 10 protons in a neutral water molecule; so the magnitude of its dipole moment is

$$p = qd = (10e)(d),$$

in which d is the separation we are seeking and e is the elementary charge. Thus,

$$d = \frac{p}{10e} = \frac{6.2\times10^{-30}\,\text{C}\cdot\text{m}}{(10)(1.60\times10^{-19}\,\text{C})}$$
$$= 3.9\times10^{-12}\,\text{m} = 3.9\ \text{pm}. \qquad \text{(Answer)}$$

Conclude: This distance is not only small, but it is also actually smaller than the radius of a hydrogen atom.

(b) If the molecule is placed in an electric field of 1.5×10^4 N/C, what maximum torque can the field exert on it? (Such a field can easily be set up in the laboratory.)

Conceptualize/Classify: The torque on a dipole is maximum when the angle θ between $\vec{p}$ and $\vec{E}$ is 90°.

Compute: Substituting $\theta = 90°$ in Eq. 22-44 yields

$$\tau = pE\sin\theta$$
$$= (6.2\times10^{-30}\ \text{C}\cdot\text{m})(1.5\times10^4\,\text{N/C})(\sin 90°)$$
$$= 9.3\times10^{-26}\,\text{N}\cdot\text{m}. \qquad \text{(Answer)}$$

Conclude: An interesting point about the torque on the dipole is that this torque is independent of the axis of rotation. In the language of rotational dynamics, we say that this type of force constitutes a couple. Here, the net force on the dipole is zero, but the net torque is not. So, if we want to find the angular acceleration of the dipole, we need not bother about the axis in finding the torque, but we need to bother about it in finding the moment of inertia of the dipole. After all, the moment of inertia does depend on the axis of rotation.

(c) How much work must an *external agent* do to rotate this molecule by 180° in this field, starting from its fully aligned position, for which $\theta = 0$?

Conceptualize/Classify: The work done by an external agent (by means of a torque applied to the molecule) is equal to the change in the molecule's potential energy due to the change in orientation.

Compute: From Eq. 22-51, we find

$$W_a = U_{180°} - U_0$$
$$= (-pE\cos 180°) - (-pE\cos 0)$$
$$= 2pE = (2)(6.2\times10^{-30}\,\text{C}\cdot\text{m})(1.5\times10^4\ \text{N/C})$$
$$= 1.9\times10^{-25}\,\text{J}. \qquad \text{(Answer)}$$

Conclude: Although, this is an equilibrium position as well, it is an unstable equilibrium.

Why is the Concept of Electric Field Necessary?

It may seem to you from the above discussion that the electric field is a convenient method to find out force on a charged body and it has no utility other than the mathematical convenience. However, it is not true. Electric field is an entity which has its existence along with the charge that produces it. We may not be able to see it; for that matter we may not be able to see even God. But by suitable experimentation, we can perceive His presence in our lives. Similarly, the electric field cannot be seen but its presence can be felt in experiments.

Suppose we consider the force between two distance charges q_1 and q_2 in accelerated motion. The greatest speed with which a signal or information can go from one point to another is c, the speed of light. Thus, the effect of any motion of q_1 on q_2 cannot arise instantaneously. There will be some time delay between the effect (force on q_2) and the cause (motion of q_1). It is precisely seen in Fig. 22-45 that the notion of electric field (strictly, electromagnetic field) is natural and very useful. The field picture is this: The accelerated motion of charge q_1 produces electromagnetic waves, which then propagate with the speed c, reach q_2, and cause a force on q_2. The notion of field elegantly accounts for the time delay. Thus, even though electric and magnetic fields can be detected only by their effects (forces) on charges, they are regarded as physical entities, not merely mathematical constructs. They have an independent dynamics of their own, that is, they evolve according to laws of their own. They can also transport energy. Thus, concept of field is now among the central concept in physics.

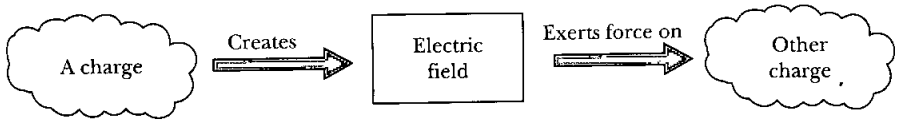


FIG. 22-45 The sequence in which a charge "feels" the presence of another charge.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: Two charged bodies placed near each other attract each other. They must be oppositely charged.

Statement 2: Like charges repel each other and unlike charges attract each other.

2. Statement 1: A point charge at rest is released in an electric field. It will move along the electric field line.

Statement 2: The force on a point charge is along the electric field line.

3. Statement 1: The law of superposition is a consequence of Newton's second law.

Statement 2: Newton's second law states that the net force on a body is equal to its mass × acceleration.

4. Statement 1: The atomic number of all the elements is an integer, but atomic mass of all the elements is not an integer.

Statement 2: Charge is quantized, but mass is not.

5. Statement 1: Electrons and protons do not decay.

Statement 2: The charge of an isolated system is conserved.

6. Statement 1: The number of field lines drawn from a charge is proportional to the magnitude of the charge.

Statement 2: The electric field at any point is proportional to the magnitude of the source charge.

7. Statement 1: When a body is positively charged by rubbing against a positively charged body, the mass of the initially charged body increases slightly.

Statement 2: Mass of proton is greater than the mass of the electron.

8. Statement 1: Two plastic rings, one full and another half, have the same radius and uniform linear charge density as shown in Fig. 22-46. The electric field at the center of full ring will be greater in magnitude.

Statement 2: In the situation described, full ring has greater charge than the half ring.

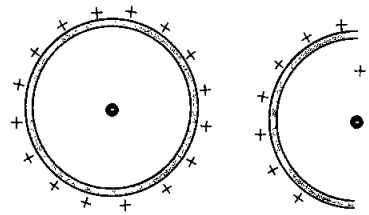


FIG. 22-46 Reasoning type question 8.

9. Statement 1: If a water molecule is kept near an infinite uniform sheet of charge, it may rotate, but it will not move toward or away from the sheet.

Statement 2: The net force on a dipole in a uniform electric field is zero.

MULTIPLE CHOICE QUESTIONS

sec. 22-2 Electric Charge

1. Two uncharged metal spheres, L and M, are in contact. A negatively charged rod is brought close to, but not touching L as shown in Fig. 22-47. The two spheres are slightly separated and the rod is then withdrawn. As a result, the spheres are charged.

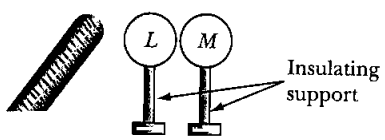


FIG. 22-47 Multiple choice question 1.

(a) Both positive

(b) L positive and M negative

(c) Both negative

(d) L negative and M positive

2. An electroscope is given a positive charge, causing its foil leaves to separate. When an object is brought near the top plate of the electroscope, the foils separate even further. We could conclude

(a) That the object is positively charged

(b) That the object is electrically neutral

(c) That the object is negatively charged

(d) Only that the object is charged

sec. 22-3 Coulomb's Law

3. A +2 C charge is located at the point $(x, y) = (5\text{ cm}, 0\text{ cm})$ and a +1 C charge is at the origin. The +2 C charge is moved to the point (3 cm, 4 cm). The force on the +1 C charge

(a) Has changed in magnitude only

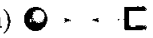
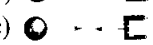
(b) Has changed in neither magnitude nor direction

(c) Has changed in direction only

(d) Has changed in both magnitude and direction.

4. The circular objects have an electrical charge of +1 mC; the square objects have an electrical charge of –2 mC. Which pair of arrows best represents the forces on the charges,

if the arrows' lengths are proportional to the magnitude of the forces?

(a) 
(b) 
(c) 
(d) 

5. Two small spheres with masses m_1 and m_2 hang on weightless, insulating threads with l_1 and l_2. The two spheres carry a charge of q_1 and q_2, respectively. The spheres hang such that they at are level with one another and the threads are inclined to the vertical at angles l_1 and l_2. Which of the following conditions is required for θ_1 and θ_2 to be equal.

(a) $m_1 = m_2$
(b) $|q_1| = |q_2|$
(c) $l_1 = l_2$
(d) None of these

6. Two particles, each of mass m and carrying charge Q, are separated by some distance. If they are in equilibrium under mutual gravitational and electrostatic forces, then Q/m (in C/kg) is of the order of

(a) 10^{-5}
(b) 10^{-10}
(c) 10^{-15}
(d) 10^{-20}

7. Figure 22-48 shows the arrangement of three charged hollow metal spheres A, B, and C. The arrows indicate the direction of the electric forces acting between the spheres. At least two of the spheres are positively charged. Which sphere, if any, could be negatively charged?

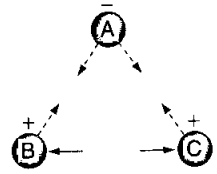


FIG. 22-48 Multiple choice question 7.

(a) Sphere A
(b) Sphere C
(c) Sphere B
(d) No sphere

sec. 22-4 Charge Is Quantized

8. What is the excess number of electrons that must be placed on each of two small sphere spaced 3 cm apart so that a force of repulsion that acts between them becomes 10^{-19} N?

(a) 25
(b) 225
(c) 625
(d) 125

sec. 22-7 Electric Field Lines

9. Assuming only electrostatic forces are present, electric field lines in the space surrounding a charge distribution show

(a) Only the directions in which static charges would accelerate when at points on those lines.
(b) Only the direction in which moving charges would accelerate when at points on those lines.
(c) Tangents to the directions in which either static or moving charges would accelerate when passing through points on those lines.
(d) The paths static or moving charges would take.

10. Two imaginary spherical surfaces or radius R and $2R$, respectively, surround a positive point charge Q located at the center of the concentric spheres. When compared to the number of field lines N_1 going through the sphere of radius R, the number of electric field lines N_2 going through the sphere of radius $2R$ is

(a) $N_2 = \frac{1}{4}N_1$
(b) $N_2 = \frac{1}{2}N_1$
(c) $N_2 = N_1$
(d) $N_2 = 2N_1$
(e) $N_2 = 4N_1$

11. In Fig. 22-49, the electric field lines on the left have twice the separation of those on the right. If the magnitude of the field at A is 40 N/C. What is the magnitude of the field at B?

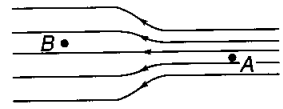


FIG. 22-49 Multiple choice question 11.

(a) 40 N/C
(b) 20 N/C
(c) 80 N/C
(d) 10 N/C

12. In space two point charges are placed as shown in Fig. 22-50. The electric field line at point P may be in direction

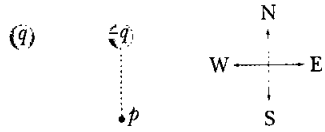


FIG. 22-50 Multiple choice question 12.

(a) South
(b) Southwest
(c) Northeast
(d) West

sec. 22-8 The Electric Field Due to a Point Charge

13. Charged beads are placed at the corners of a square in the various configurations as shown in Fig. 22-51. Black beads have charge $+Q$, whereas gray beads have charge $-Q$. Rank the configurations according to the magnitude of the electric field at the center of the square, least to greatest.

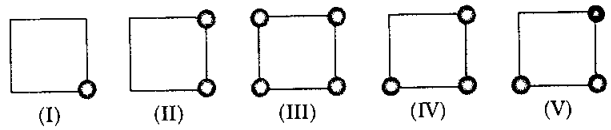


FIG. 22-51 Multiple choice question 13.

(a) I > V > III > II > IV
(b) III > V > I > IV > II
(c) I = III > V > II > IV
(d) V > II > I = IV > III

14. Which of the following graphs best represents the x components of the electric field on the x axis as a function of the x in Fig. 22-52?

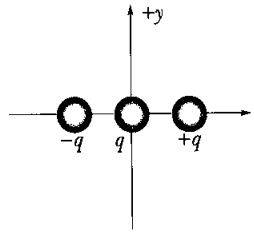
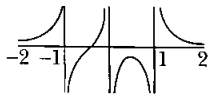
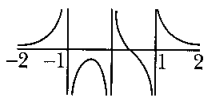
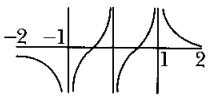
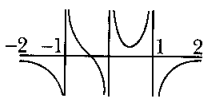


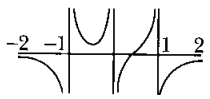
FIG. 22-52 Multiple choice question 14.

(a) 

(b) 

(c) 

(d) 

(e) 

15. Two point charges having the same magnitude but opposite sign are placed at a separation of l. P is a point on the line joining the charges, at a distance x from any one charge. The field at P is E. E is plotted against x for values of x from close to zero to slightly less than l (Fig. 22-53). Which of the following best represents the resulting curve?

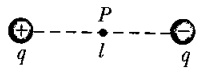
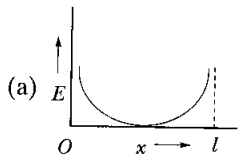
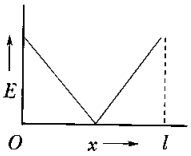
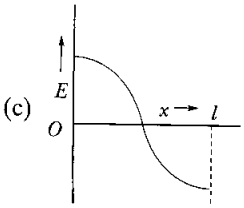
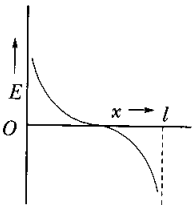


FIG. 22-53 Multiple choice question 15.

(a) 

(b) 

(c) 

(d) 

sec. 22-9 The Electric Field Due to Continuous Charge Distribution

16. A ring of radius R carries a charge Q, uniformly distributed along its circumference. What is the ratio of the electric field strength at a distance R to that at a distance $R/\sqrt{2}$ along the axis?

(a) $\dfrac{\sqrt{3}}{8}$

(b) $\dfrac{3\sqrt{3}}{8}$

(c) $\dfrac{3\sqrt{3}}{4\sqrt{2}}$

(d) $\dfrac{2\sqrt{2}}{3\sqrt{3}}$

17. Consider a charged filament (charge per unit length λ) of finite length l. Consider a point P at distance "a" from the filament. The electrostatic field at P due to this filament in the limit $a \ll l$ can correspond to that due to

(a) An infinitely long filament

(b) A semi-infinite filament

(c) A point charge

(d) A semi-circular charge (with P as the center)

(e) Either (a) or (b)

18. A charge of 80 nC is uniformly distributed along the x axis from $x = 0$ to $x = 2.0$ m. Determine the magnitude of the electric field at a point on the x axis with $x = 8.0$ m.

(a) 30 N/C

(b) 15 N/C

(c) 48 N/C

(d) 90 N/C

19. Charge Q is distributed uniformed along a semicircle of radius a. Which formula below gives the correct magnitude of the electric field at the center of the circle?

(a) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{\pi a}$

(b) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{\pi a^2}$

(c) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{\pi a}$

(d) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{\pi a^2}$

(e) $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a^2}$

20. Three non-conducting infinite planes of charge A, B, and C are vertical and parallel to one another. There is a uniform electric field $\vec{E}_\perp$ to the left of plane A and a uniform electric field $\vec{E}_\parallel$ to the right of plane C. The field $\vec{E}_\perp$ points to the left and the field $\vec{E}_\parallel$ points to the right. The signs of the charges on plates A, B, and C may be

(a) −, −, −

(b) +, −, −

(c) +, −, +

(d) +, +, +

sec. 22-11 A Point Charge in an Electric Field

21. A negatively charged particle enters a region where electric field is constant in direction, and perpendicular to the velocity of the particle. However, in the region entered, the electric field magnitude is not uniform. That particle will

(a) Continue to traverse region unaffected.

(b) Traverse the region with increasing velocity but in a straight line.

(c) Make a smooth turn to align itself with the electric field and traverse in the direction of the electric field.

(d) Make a smooth turn to align itself with in the direction opposite to that of the electric field and traverse the region.

22. A proton is at coordinates $(x; y) = (0; 0)$, and an electron at $(d; h)$, where $d = h$. At time $t = 0$, a uniform electric field E of unknown magnitude but pointing in the positive y direction is turned on. Assuming that d is large enough that the proton–electron interaction is negligible, the y coordinates of the two particles will be equal.

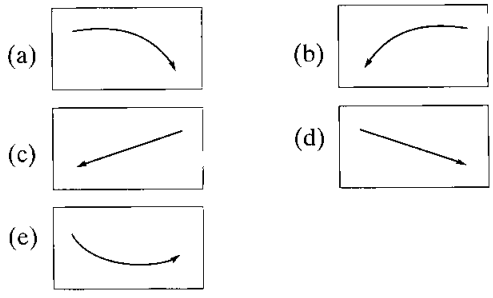
(a) At about $y = d/2000$

(b) At an undetermined value since E is unknown

(c) At about $y = d/43$

(d) Nowhere: they move in opposite directions

23. An electron is placed, initially at rest, in a constant uniform gravitational field and constant, uniform electric field as shown in Fig. 22-54. What qualitatively, is the shape of the trajectory of the electron?



24. Two equal charges $-q$ are fixed at a point $(0, \pm a)$ on y axis, one positive charge q is placed at $x = 2a$; this charge, when released, will

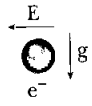


FIG. 22-54 Multiple choice question 23.

(a) Execute simple harmonic motion about the origin

(b) Oscillate but not execute simple harmonic motion

(c) Move toward origin and will become stationary

(d) Continues to move along negative x axis.

25. Two identical charges $+Q$ are kept fixed some distance apart. A small particle P with charge q is placed midway between them. If P is given a small displacement Δ, it will undergo simple harmonic motion if

(a) q is positive and the given displacement is along the line joining the charges.

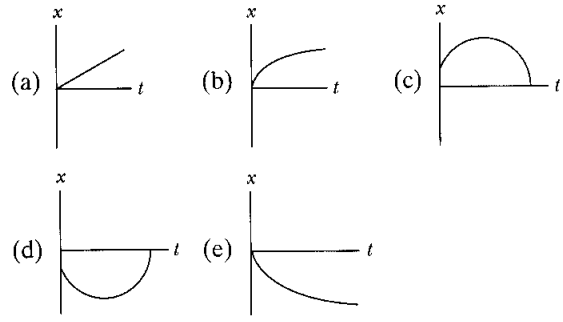
(b) q is positive and the given displacement is perpendicular to the line joining the charges

(c) q is negative and the given displacement is perpendicular to the line joining the charges.

(d) q is negative and the given displacement is along the line joining the charges.

(e) Both (a) and (c)

26. A negatively charged particle is moving in the positive x direction when it enters a region with a uniform electric field pointing in the positive x direction. Which graph gives its position as a function of time correctly? (Its initial position is $x = 0$ at $t = 0$.)



sec. 22-12 A Dipole in an Electric Field

27. An electric dipole is placed near a positive charge. It will experience

(a) A torque only

(b) A force only

(c) Both torque and force

(d) It will depend on the orientation of the dipole with respect to the charge.

28. An electric dipole in a uniform electric field has potential energy U. The magnitude of the torque acting on the dipole due to the field is τ.

(a) U is minimum and τ is zero when the dipole is perpendicular to the field

(b) U is zero and τ is a maximum when the dipole is perpendicular to the field

(c) U is a minimum and τ is a maximum when the dipole is perpendicular to the field

(d) U is a minimum and τ is zero when the dipole is antiparallel to the field

PROBLEMS

sec. 22-3 Coulomb's Law

1. A particle of charge $+3.00 \times 10^{-6}$ C is 12.0 cm distant from a second particle of charge -1.50×10^{-6} C. Calculate the magnitude of the electrostatic force between the particles.

2. Two objects have the same magnitude of charge, and the magnitude of force exerted by one on the other is 0.1 N when they are 30 mm apart. (a) What is the magnitude of charge on the two objects, assuming they are much smaller than 30 mm in diameter? (b) If the magnitude of the force increases when the charged objects are released, are the objects of the same charge or opposite charge?

3. Of the charge Q initially on a tiny sphere, a portion q is to be transferred to a second, nearby sphere. Both spheres can be treated as particles. For what value of q/Q will the electrostatic force between the two spheres be maximized?

4. Two small particles, each carrying 70 pC of charge, are released from rest on a frictionless, nonconducting surface. One particle accelerates initially at 7 m/s^2. Another particle whose mass is 0.49 mg accelerates initially at 9 m/s^2. (a) What is the mass of the first particle? (b) What is the separation between the particles when they are first released?

5. In the return stroke of a typical lightning bolt, a current of 2.5×10^4 A exists for 20 μs. How much charge is transferred in this event?

6. In Fig. 22-55, four particles form a square. The charges are $q_1 = q_4 = Q$ and $q_2 = q_3 = q$. (a) What is Q/q if the net electrostatic

force on particles 1 and 4 is zero? (b) Is there any value of q that makes the net electrostatic force on each of the four particles zero? Explain.

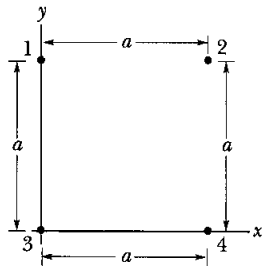


FIG. 22-55 Problem 6.

7. In Fig. 22-56, three charged particles lie on an x axis. Particles 1 and 2 are fixed in place. Particle 3 is free to move, but the net electrostatic force on it from particles 1 and 2 happens to be zero. If $L_{23} = L_{12}$, what is the ratio q_1/q_2?

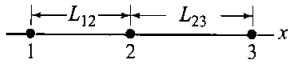


FIG. 22-56 Problem 7.

8. In Fig. 22-57, particles 1 and 2 of charge $q_1 = q_2 = +3.20 \times 10^{-19}$ C are on a y axis at distance d = 17.0 cm from the origin. Particle 3 of charge $q_3 = +6.40 \times 10^{-19}$ C is moved gradually along the x axis from $x = 0$ to $x = +5.0$ m. At what values of x will the magnitude of the electrostatic force on the third particle from the other two particles be (a) minimum and (b) maximum? What are the (c) minimum and (d) maximum magnitudes?

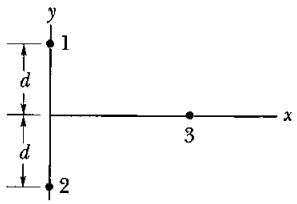


FIG. 22-57 Problem 8.

9. A charge A of +200 mC is placed at $(x, y) = (3, 4)$m. Another charge, B, of +30 μC is at the origin. What is the electrostatic force on B?

10. Two charged particles are on the y axis. One particle with charge $4Q$ is located at the origin and another particle with charge Q is located at y = 0.12 m. A third particle carrying a charge of 2.0 C can move along the y axis. It experiences a net force of zero when at $y = 0.08$ m and a net force of 126.4 j N when at $y = 0.04$ m. What is the value of Q? Neglect the weight of the charge.

11. A nonconducting spherical shell, with an inner radius of 4.0 cm and an outer radius of 6.0 cm, has charge spread nonuniformly through its volume between its inner and outer surfaces. The *volume charge density* ρ is the charge per unit volume, with the unit coulomb per cubic meter. For this shell $\rho = b/r$, where r is the distance in meters from the center of the shell and $b = 3.0$ μC/m^2. What is the net charge in the shell?

12. Two small spheres with equal unknown mass are suspended by lightweight strings of length 10 cm (Fig. 22-58). When a charge of +5 nC is placed on each sphere, each string makes an angle of 30° with the vertical in equilibrium. What is the mass of each ball?

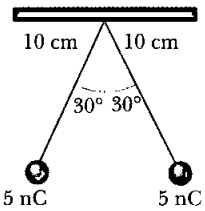


FIG. 22-58 Problem 12.

13. In the classical model of the hydrogen atom, a single electron orbits the single proton of the hydrogen's nucleus in a radius r. (a) How fast is the electron moving? (b) How long does it take to complete one orbit?

14. Assume that two charges are kept on x axis, Q at origin and $4Q$ at $(d, 0)$. What should be the location of another charge and its magnitude so that all the three charges are in equilibrium?

15. In crystals of the salt cesium chloride, cesium ions Cs^+ form the eight corners of a cube and a chlorine ion Cl^- is at the cube's center (Fig. 22-59). The edge length of the cube is 0.40 nm. The Cs^+ ions are each deficient by one electron (and thus each has a charge of $+e$), and the Cl^- ion has one excess electron (and thus has a charge of $-e$). (a) What is the magnitude of the net electrostatic force exerted on the Cl^- ion by the eight Cs^- ions at the corners of the cube? (b) If one of the Cs^+ ions is missing, the crystal is said to have a *defect*; what is the magnitude of the net electrostatic force exerted on the Cl^- ion by the seven remaining Cs^+ ions?

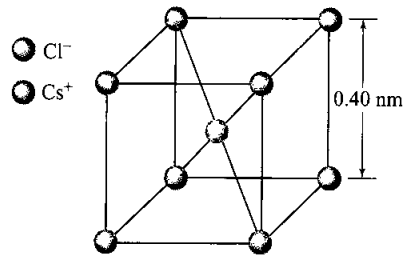


FIG. 22-59 Problem 15.

sec. 22-4 Charge Is Quantized

16. A conducting sphere has a net charge of -4.8×10^{-17} C. What is the approximate number of excess electrons on the sphere?

17. The magnitude of the electrostatic force between two identical ions that are separated by a distance of 5.0×10^{-10} m is 3.7×10^{-9} N. (a) What is the charge of each ion? (b) How many electrons are "missing" from each ion (thus giving the ion its charge imbalance)?

18. Earth's atmosphere is constantly bombarded by *cosmic ray protons* that originate somewhere in space. If the protons all passed through the atmosphere, each square meter of the Earth's surface would intercept protons at the average rate of 1500 protons per second. What would be the electric current intercepted by the total surface area of the planet?

19. Calculate the number of coulombs of positive charge in 250 cm^3 of (neutral) water. (*Hint:* A hydrogen atom contains one proton; an oxygen atom contains eight protons.)

sec. 22-5 Charge Is Conserved

20. Electrons and positrons are produced by the nuclear transformations of protons and neutrons known as *beta decay.* (a) If a proton transforms into a neutron, is an electron or a positron produced? (b) If a neutron transforms into a proton, is an electron or a positron produced?

21. Identify X in the following nuclear reactions: (a) $^1H + {}^9Be \rightarrow X + n$; (b) $^{12}C + {}^1H \rightarrow X$; (c) $^{15}N + {}^1H \rightarrow {}^4He + X$. (Appendix F will help.)

22. A metal sphere is coated with a thin layer of radioactive material and placed in vacuum. The material emits beta particles. It is seen that after 50 days the sphere acquires a charge of 4.32×10^{10} elementary charges. It is also reported that during the 50-day period, the sphere emits beta particles at an average rate of 1×10^4 per second. (a) What is the charge on a single beta particle? (b) What is it composed of?

sec. 22-8 The Electric Field Due to a Point Charge

23. What is the magnitude of a point charge whose electric field 50 cm away has the magnitude 2.0 N/C?

24. In Fig. 22-60, particle 1 of charge $q_1 = -5.00q$ and particle 2 of charge $q_2 = -2.00q$ are fixed to an x axis. As a multiple of distance L, at what coordinate on the axis is the net electric field of the particles zero?

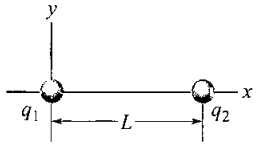


FIG. 22-60 Problem 24.

25. The electric field at some point beyond the region between two point charges vanishes. (a) What are the sign of the charges? (b) Can these charges be equal in magnitude? (c) Can the electric field due to two charges ever vanish except along a line connecting the two charges?

26. Figure 22-61 shows two charges q_1 and q_2 fixed to a circle. At the center of the circle they produce a net electric field that is directed along the vertical axis. What is the ratio of the charges?

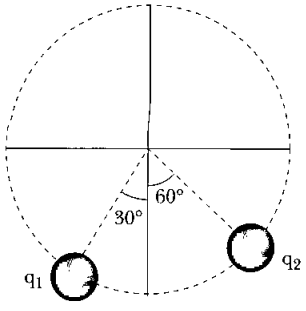


FIG. 22-61 Problem 26.

27. In Fig. 22-62, the four particles form a square of edge length $a = 5.00$ cm and have charges $q_1 = -10.0$ nC, $q_2 = -20.0$ nC, $q_3 = -20.0$ nC, and $q_4 = -10.0$ nC. In unit-vector notation, what net electric field do the particles produce at the square's center?

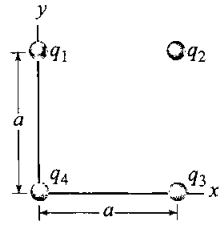


FIG. 22-62 Problem 27.

28. Figure 22-63*a* shows two charged particles fixed in place on a x axis with separation L. The ratio q_1/q_2 of their charge magnitudes is 4.00. Figure 22-63*b* shows the x component $E_{net,x}$ of their net electric field along the x axis just to the right of particle 2. The x axis scale is set by $x_s = 30.0$ cm. (a) At what value of $x > 0$ is $E_{net,x}$ maximum? (b) If particle 2 has charge $-q_2 = -3e$, what is the value of that maximum?

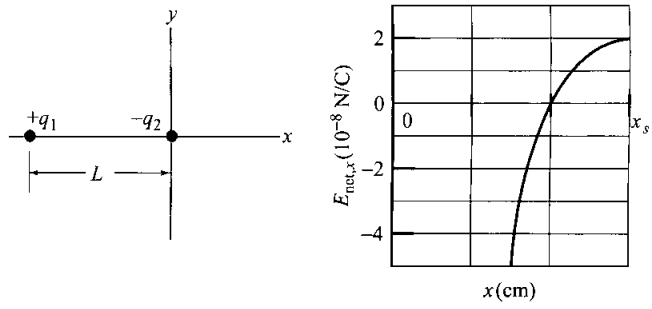


FIG. 22-63 Problem 28.

sec 22-9 The Electric Field Due to Continuous Charge Distribution

29. (a) A charge $-300e$ is uniformly distributed along a circular arc of radius 4.00 cm, which subtends an angle of 30° at its center. What is the linear charge density along the arc? (b) A charge $-300e$ is uniformly distributed over one face of a circular disk of radius 2.00 cm. What is the surface charge density over that face? (c) A charge $-300e$ is uniformly distributed over the surface of a sphere of radius 2.00 cm. What is the surface charge density over that surface? (d) A charge $-300e$ is uniformly spread through the volume of a sphere of radius 2.00 cm. What is the volume charge density in that sphere?

30. A circular plastic disk with radius $R = 2.00$ cm has a uniformly distributed charge $Q = +(2.00 \times 10^6)e$ on one face. A circular ring whose center is the same as the disk is cut out. The ring has a radius of 0.5 cm and a width of 30 μm. What is the charge on the ring?

31. A uniform charge lies on a string that is stretched along an x axis from $x = 0$ to $x = 3.0$ m (Fig. 22-64). Determine the magnitude of the electric field at $y = 4.0$ m on the y axis.

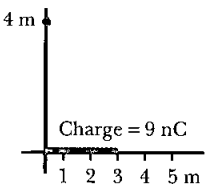


FIG. 22-64 Problem 31.

32. A semi-infinite line charge of uniform density λ *lies* along the negative x axis from $x = 0$ to $x = -\infty$. What is the magnitude of the

electric field at any point x on the positive x axis? What is the magnitude of electric field at any point y on the positive y axis?

33. Figure 22-65 shows two parallel nonconducting rings with their central axes along a common line. Ring 1 has uniform charge q_1 and radius R; ring 2 has uniform charge q_2 and the same radius R. The rings are separated by distance $d = 3.00R$. The net electric field at point P on the common line, at distance R from ring 1, is zero. What is the ratio q_1/q_2?

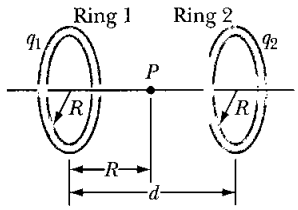


FIG. 22-65 Problem 33.

34. In Fig. 22-66, two curved plastic rods, one of charge $+q$ and the other of charge $-q$, form a circle of radius $R = 8.50$ cm in an xy plane. The x axis passes through both of the connecting points, and the charge is distributed uniformly on both rods. If $q = 15.0$ pC, what are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the electric field $\vec{E}$ produced at P, the center of the circle?

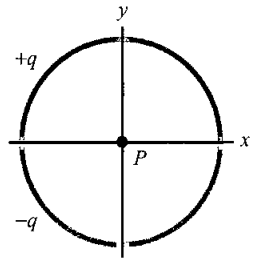


FIG. 22-66 Problem 34.

35. Figure 22-67 shows two concentric rings, of radii R and $R' = 3.00R$, that lie on the same plane. Point P lies on the central z axis, at distance $D = 2.00R$ from the center of the rings. The smaller ring has uniformly distributed charge $+Q$. In terms of Q, what is the uniformly distributed charge on the larger ring if the net electric field at P is zero?

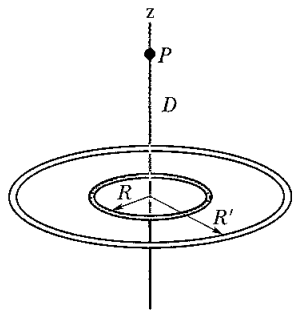


FIG. 22-67 Problem 35.

36. Three nonconducting infinite planar sheets are parallel to the y-z plane. Each sheet has a uniform surface charge density. The first sheet, with a negative surface charge density $-\sigma$, passes through the x axis at $x = 1$ m. The second sheet has an unknown surface charge density and passes through the x axis at $x = 2$ m. The third sheet has a negative surface charge density -3σ and passes through the x axis at $x = 4$ m. The net electric field due to the sheets is zero at $x = 1.5$ m. (a) What is the surface charge density on the second sheet? (b) If the electric field at $x = 0$ is $\vec{E}_0$, what is the electric field at (i) $x = -2$ m (ii) $x = 3$ m (iii) $x = 6$ m

37. Two infinite thin plates are perpendicular to each other. The charge densities are $+\sigma$ (along yz plane) and $-\sigma$ (along xz plane) on each plate, respectively. Calculate the E field in all the four quadrants formed by the plates.

sec. 22-10 The Electric Field Due to a Charged Disk

38. At what distance along the central perpendicular axis of a uniformly charged plastic disk of radius 0.600 m is the magnitude of the electric field equal to one-half the magnitude of the field at the center of the surface of the disk?

sec. 22-11 A Point Charge in an Electric Field

39. An electron is released from rest in a uniform electric field of magnitude 2.00×10^4 N/C. Calculate the acceleration of the electron. (Ignore gravitation.)

40. An electron with a speed of 5.00×10^8 cm/s enters an electric field of magnitude 1.00×10^3 N/C, traveling along a field line in the direction that retards its motion. (a) How far will the electron travel in the field before stopping momentarily, and (b) how much time will have elapsed? (c) If the region containing the electric field is 8.00 mm long (too short for the electron to stop within it), what fraction of the electron's initial kinetic energy will be lost in that region?

41. A region in space contains a uniform electric field of $E_x = 0$ and $E_y = 2$ V/m and no gravitational field. A particle of mass $M = 0.0015$ kg and charge $Q = -0.005$ C is at the origin (0, 0) at time $t = 0$, moving to the right (positive x direction) with a speed of 2 m/s. (a) Find the acceleration of the particle. (b) Find the x position of the particle at time $t = 4$ s. (c) Find the y position of the particle at time $t = 4$ s. (d) Calculate the kinetic energy of the particle when it reaches a point with $y = -0.5$ m.

42. At some instant, the velocity components of an electron moving between two charged parallel plates are $v_x = 1.5 \times 10^5$ m/s and $v_y = 3.0 \times 10^3$ m/s. Suppose the electric field between the plates is given by $\vec{E} = (120 \text{ N/C})\hat{j}$. In unit-vector notation, what are (a) the electron's acceleration in that field and (b) the electron's velocity when its x coordinate has changed by 2.0 cm?

43. Two large parallel copper plates are 5.0 cm apart and have a uniform electric field between them as depicted in Fig. 22-68. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other. (Does it surprise you that you need not know the electric field to solve this problem?)

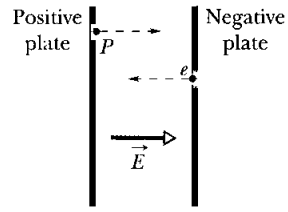


FIG. 22-68 Problem 43.

44. In Fig. 22-69, an electron (e) is to be released from rest on the central axis of a uniformly charged disk of radius R. The surface charge density on the disk is $+4.00\ \mu C/m^2$. What is the

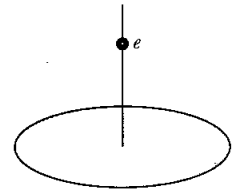


FIG. 22-69 Problem 44.

magnitude of the electron's initial acceleration if it is released at a distance (a) R, (b) $R/100$, and (c) $R/1000$ from the center of the disk? (d) Why does the acceleration magnitude increase only slightly as the release point is moved closer to the disk?

sec. 22-12 A Dipole in an Electric Field

45. Calculate the electric dipole moment of an electron and a proton 4.30 nm apart.

46. A uniform external electric field exerts a torque of magnitude 10.0×10^{-9} Nm on an electric dipole oriented at 30° from the field. The dipole moment of the dipole is 8.0×10^{-12} Cm. (a) What is the magnitude of the external field? (b) If the two particles making up the dipole are 2.5 mm apart, what is the magnitude of the charge on one of the particles?

47. An electric dipole consists of charges $+2e$ and $-2e$ separated by 0.78 nm. It is in an electric field of strength 3.4×10^6 N/C. Calculate the magnitude of the torque on the dipole when the dipole moment is (a) parallel to, (b) perpendicular to, and (c) antiparallel to the electric field.

48. An electric dipole consisting of charges of magnitude 1.50 nC separated by 6.20 μm is in an electric field of strength 1100 N/C. What are (a) the magnitude of the electric dipole moment and (b) the difference between the potential energies for dipole orientations parallel and antiparallel to $\vec{E}$?

49. Find an expression for the oscillation frequency of an electric dipole of dipole moment $\vec{p}$ and rotational inertia I for small amplitudes of oscillation about its equilibrium position in a uniform electric field of magnitude E.

ADDITIONAL PROBLEMS

50. A particle of charge Q is fixed at the origin of an xy coordinate system. At $t = 0$ a particle ($m = 0.800$ g, $q = 4.00$ μC) is located on the x axis at $x = 20.0$ cm, moving with a speed of 50.0 m/s in the positive y direction. For what value of Q will the moving particle execute circular motion? (Neglect the gravitational force on the particle.)

51. A charged nonconducting rod, with a length of 2.00 m and a cross-sectional area of 4.00 cm^2, lies along the positive side of an x axis with one end at the origin. The *volume charge density* ρ is charge per unit volume in coulombs per cubic meter. How many excess electrons are on the rod if ρ is (a) uniform, with a value of -4.00 μC/m^3, and (b) nonuniform, with a value given by $\rho = bx^2$, where $b = -2.00$ μC/m^5?

52. If you were able to isolate two identical charged particles of mass m from all external forces, (a) what would the charge of the particles have to be in terms of m and any necessary physical constants if the two particles remained at a fixed separation from each other? (b) Is it necessary that the charge on both the particles be equal? (c) What is the minimum value of m so that they can be held in such a configuration? (d) Comment on the nature of equilibrium.

53. A small sphere with charge Q and mass m is constrained to move inside a narrow vertical tube as shown in Fig. 22-70. Fixed at the bottom of the tube is another small sphere with charge q. (a) Find the equilibrium height h for the particle carrying charge Q (neglect friction). (b) Show that if the particle is displaced from this equilibrium position by a small amount and then released, it will exhibit simple harmonic motion with an angular frequency of $(2\,g/h)^{1/2}$.

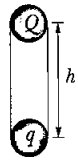


FIG. 22-70 Problem 53.

54. A thin, half-spherical insulating shell of radius R has a charge Q distributed uniformly over it. It is placed so that it is entirely in the $z = 0$ half-space with its base resting on the xy plane and centered about the z axis. (a) Write down the charge on an infinitesimal annular ring (a part of the shell) of width $d\theta$, where θ is the angle from the z axis to the annular ring. (b) What is the electric field due to the ring at origin? (c) What is the net electric field at the origin due to the entire shell?

55. A thin wire ring with a radius R carries an electric charge q. The center of the ring has a charge Q of the same sign as q. Find the extra tension to which the ring is subjected due to Q.

56. Two particles, each with a charge of magnitude 12 nC, are at two of the vertices of an equilateral triangle with edge length 2.0 m. What is the magnitude of the electric field at the third vertex if (a) both charges are positive and (b) one charge is positive and the other is negative?

57. An electron is constrained to the central axis of the ring of charge of radius R in Fig. 22-28, with $z \ll R$. Show that the electrostatic force on the electron can cause it to oscillate through the ring center with an angular frequency

$$\omega = \sqrt{\frac{eq}{4\pi\varepsilon_0 mR^3}},$$

where q is the ring's charge and m is the electron's mass.

58. An electric field $\vec{E}$ with an average magnitude of about 150 N/C points downward in the atmosphere near Earth's surface. We wish to "float" a sulfur sphere weighing 4.4 N in this field by charging the sphere. (a) What charge (both sign and magnitude) must be used? (b) Why is the experiment impractical?

59. The electric field in an xy plane produced by a positively charged particle is $7.2(4.0\hat{i} + 3.0\hat{j})$ N/C at the point (3.0, 3.0) cm and $100\,\hat{i}$ N/C at the point (2.0, 0) cm. What are the (a) x and (b) y coordinates of the particle? (c) What is the charge of the particle?

60. An electric dipole with dipole moment $\vec{p} = (3.00\hat{i} + 4.00\hat{j})(1.24 \times 10^{-30}\,\text{C} \cdot \text{m})$ is in an electric field $\vec{E} = (4000\,\text{N/C})\hat{i}$. (a) What is the potential energy of the electric dipole? (b) What is the torque acting on it? (c) If an external agent turns the dipole until its electric dipole moment is $\vec{p} = (-4.00\hat{i} + 3.00\hat{j})(1.24 \times 10^{-30}\,\text{C} \cdot \text{m})$ how much work is done by the agent?

61. Two particles, each of positive charge q, are fixed in place on a y axis, one at $y = d$ and the other at $y = -d$. (a) Write an expression that gives the magnitude E of the net electric field at points on the x axis given by $x = \alpha d$. (b) Graph E versus α for the range $0 < \alpha < 4$. (c) From the graph, determine the values of α that give the maximum value of E.

62. In Fig. 22-71, a uniform, upward electric field $\vec{E}$ of magnitude 2.00×10^3 N/C has been set up between two horizontal plates by charging the lower plate positively and the upper plate negatively. The plates have length $L = 10.0$ cm and separation $d = 2.00$ cm. An electron is then shot between the plates from the left edge of the lower plate. The initial velocity of the electron makes an angle $\theta = 45.0°$ with the lower plate and has a magnitude of 6.00×10^6 m/s. (a) Will the electron strike one of the plates? (b) If so, which plate and how far horizontally from the left edge will the electron strike?

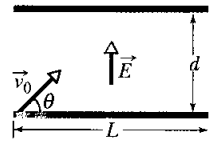


FIG. 22-71 Problem 62.

63. A ring-shaped wire of radius R is charged with charge Q. The ring is kept in a horizontal plane and a particle of equal charge Q and mass m is kept on its axis at a distance of R from the center. Find the mass m of the particle if it is in equilibrium under the action of electrostatic repulsion and its weight. Comment on the nature of the equilibrium.

ANSWERS

Checkpoints

1. C and D attract; B and D attract
2. (a) leftward; (b) leftward; (c) leftward
3. (a) a, c, b; (b) less than
4. Increases because

$$\left(\frac{q_1 + q_2}{2}\right)^2 \geq q_1 q_2$$

5. $-15e$ (net charge of $-30e$ is equally shared)
6. (a) rightward; (b) leftward; (c) leftward; (d) rightward (p and e have same charge magnitude, and p is farther)
7. (a) toward positive y; (b) toward positive x; (c) toward negative y
8. (a) leftward; (b) leftward; (c) decrease
9. (a) all tie; (b) 1 and 3 tie, then 2 and 4 tie

Reasoning Type Questions

1. (d)
2. (d)
3. (d)
4. (a)
5. (b)
6. (a)
7. (b)
8. (d)
9. (a)

Multiple Choice Questions

1. (b)
2. (a)
3. (c)
4. (c)
5. (a), (c)
6. (b)
7. (a)
8. (c)
9. (c)
10. (c)
11. (b)
12. (c)
13. (d)
14. (b)
15. (a)
16. (c)
17. (e)
18. (b)

19. (d)
20. (b), (c), (d)
21. (d)
22. (a)
23. (d)
24. (b)
25. (e)
26. (c)
27. (d)
28. (b)

Problems

1. 2.8125 N
2. (a) $q = 10^{-7}$ C = 100 nC; (b) opposite
3. 0.500 or 1/2
4. (a) 0.63 mg; (b) 0.1 mm
5. 0.50 C
6. (a) –2.83; (b) No
7. –1/4
8. (a) 0; (b) 12 cm; (c) 0; (d) 4.9×10^{-26} N
9. $\vec{F} = -1296\hat{i} - 1728\hat{j}$
10. $Q \approx 3$ pC
11. 3.8×10^{-8} C or 12π nC
12. 3.9×10^{-3} gm
13. (a) $e\sqrt{\frac{1}{4\pi\varepsilon_0 m_e r}}$; (b) $\frac{2\pi r}{e}\sqrt{4\pi\varepsilon_0 m_e r}$
14. $(d/3, 0)$; $q = -4Q/9$
15. (a) 0; (b) 1.92×10^{-9} N
16. 300
17. (a) 3.2×10^{-19} C; (b) 2
18. 122 mA
19. $\approx 1.3 \times 10^{7}$ C
20. (a) Since the proton is positively charged, the emitted particle must be a positron (as opposed to the negatively charged electron) in accordance with the law of charge conservation; (b) In this case, the initial state had zero charge (the neutron is neutral), so the sum of charges in the final state must be zero. Since there is a proton in the final state, there should also be an electron (as opposed to a positron) so that $\Sigma q = 0$.
21. ^{9}B; (b) ^{13}N; (c) ^{12}C
22. (a) 1.6×10^{-19} C; (b) electron or positron
23. 56 pC
24. $x = \frac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$, $y = 0$
25. (a) Opposite; (b) no; (c) no
26. $\sqrt{3}$
27. $7.2\sqrt{2} \times 10^{4}$ N/C
28. (a) 34 cm; (b) 2.2×10^{-8} N/C
29. (a) -2.29×10^{-15} C/m; (b) -3.82×10^{-14} C/m^2; (c) -9.56×10^{-15} C/m^2; (d) -1.43×10^{-12} C/m^3
30. 2.4×10^{-16} C
31. 4.27 N/C
32. (a) $\frac{\lambda}{4\pi\varepsilon_0 x}$; (b) $\frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 y}$
33. 0.506
34. (a) 23.8 N/C; (b) –90°
35. $-4.19Q$
36. (a) 2σ; (b) $\vec{E}_0$; (c) $2\vec{E}_0$; (d) $-\vec{E}_0$
37. I: $\frac{\sigma}{2\varepsilon_0}(\hat{i} - \hat{j})$; II: $\frac{\sigma}{2\varepsilon_0}(-\hat{i} - \hat{j})$; III: $\frac{\sigma}{2\varepsilon_0}(-\hat{i} + \hat{j})$; IV: $\frac{\sigma}{2\varepsilon_0}(\hat{i} + \hat{j})$
38. 0.346 m
39. 3.51×10^{15} m/s^2
40. (a) 7.12×10^{-2} m; (b) 2.85×10^{-8} s; (c) 0.112
41. (a) $a_y = -6.67$ m/s^2, $a_x = 0$; (b) 8 m; (c) –53.3 m; (d) 8×10^{-3} J
42. (a) $-(2.1 \times 10^{13}\ \text{m/s}^2)\,\hat{j}$; (b) $(1.5 \times 10^{5}\ \text{m/s})\hat{i} - (2.8 \times 10^{6}\ \text{m/s})\hat{j}$
43. 27 μm
44. (a) 1.16×10^{16} m/s^2; (b) 3.94×10^{16} m/s^2; (c) 3.97×10^{16} m/s^2
45. 6.88×10^{-28} C·m
46. (a) 2500 N/C; (b) 3.2×10^{-9} C
47. (a) 0; (b) 8.5×10^{-22} N·m; (c) 0
48. (a) 9.30×10^{-15} C·m; (b) 2.05×10^{-11} J
49. $(1/2\pi)(pE/I)^{0.5}$

Additional Problems

50. -1.11×10^{-5} C

51. (a) 2.00×10^{10} electrons; (b) 1.33×10^{10} electrons

52. (a) $q = m\sqrt{4\pi\varepsilon_0 G}$; (b) no; (c) $m = 1.86\times10^{-6}$ gm; (d) neutral

53. (a) $h = \sqrt{\dfrac{Qq}{4\pi\varepsilon_0 mg}}$

54. (a) $(Q\sin\theta)\,d\theta$; (b) $\dfrac{1}{4\pi\varepsilon_0}\dfrac{(Q\sin\theta)d\theta\cdot\cos\theta}{R^2}$; (c) $\dfrac{Q}{8\pi\varepsilon_0 R^2}$

55. $T = \dfrac{Qq}{8\pi^2\varepsilon_0 r^2}$

56. (a) 47 N/C; (b) 27 N/C

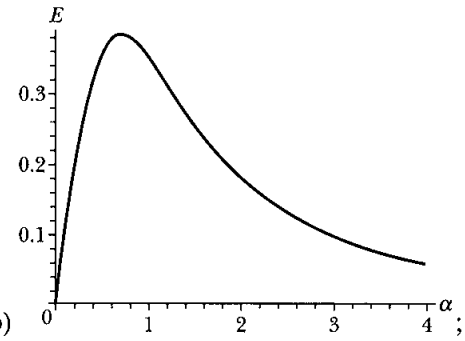
58. (a) –0.029 C; (b) repulsive forces would explode the sphere

59. (a) –1.0 cm; (b) 0; (c) 10 pC

60. (a) -1.49×10^{-26} J; (b) $(-1.98\times10^{-26}\ \text{N}\cdot\text{m})\hat{\text{k}}$;
(c) 3.47×10^{-26} J

61. (a) $E_x = \dfrac{q}{2p\varepsilon_0 d^2}\left(\dfrac{\alpha}{(\alpha^2+1)^{3/2}}\right)$;

(b)

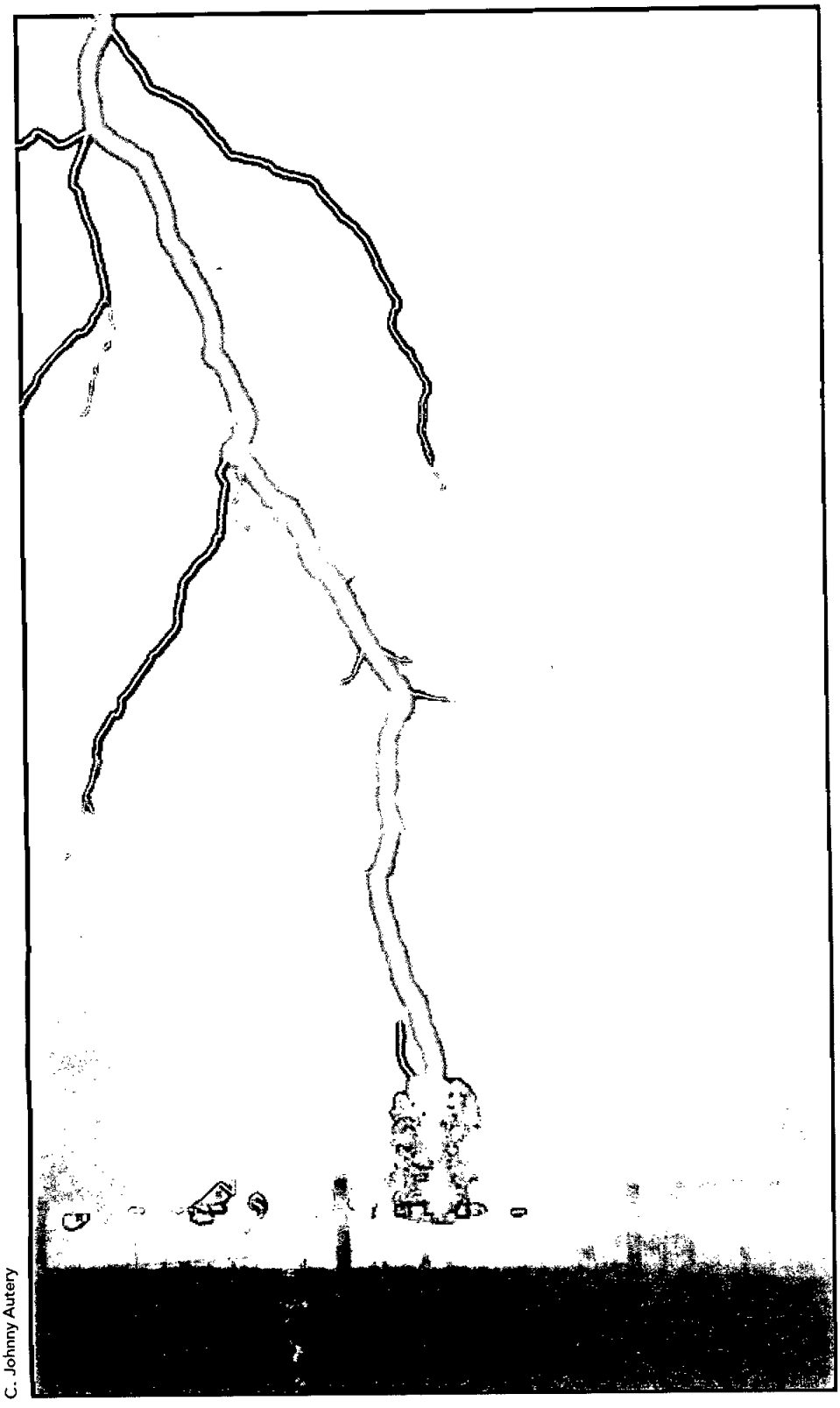


;

(c) $\alpha = 1/\sqrt{2}$

63. $m = \dfrac{Q}{8\sqrt{2}\pi\varepsilon_0 R^2 g}$, stable

Gauss' Law 23



C. Johnny Autery

Lightning storms are dangerous for several reasons: (1) If lightning strikes you or something you are touching, it will produce a fatal charge flow in your body. (2) If it strikes an object near you, a portion of the charge flow can jump through the air to you (an effect known as side flash*). (3) If lightning strikes the ground near you, part of the charge flow it produces along the ground can be diverted through your body. Recently a fourth danger of a lightning storm was recognized. The photograph here, in which lightning strikes a tree, contains a hint.*

What is this additional danger of a lightning storm?

The answer is in this chapter.

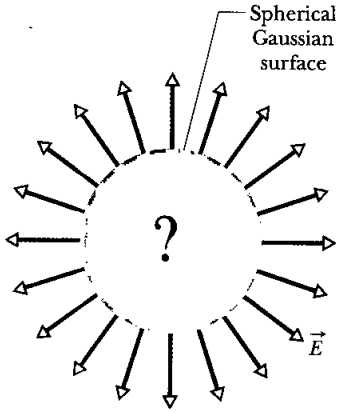


FIG. 23-1 A spherical Gaussian surface. If the electric field vectors are of uniform magnitude and point radially outward at all surface points, you can conclude that a net positive distribution of charge must lie within the surface and have spherical symmetry.

23-1 WHAT IS PHYSICS?

One of the primary goals of physics is to find simple ways of solving seemingly complex problems. One of the main tools of physics in attaining this goal is the use of symmetry. For example, in finding the electric field $\vec{E}$ of the charged ring of Fig. 22-7 and the charged rod of Fig. 22-13, we considered the fields $d\vec{E}(= k\,dq/r^2)$ of charge elements in the ring and rod. Then we simplified the calculation of $\vec{E}$ by using symmetry to discard the perpendicular components of the $d\vec{E}$ vectors. That saved us some work.

For certain charge distributions involving symmetry, we can save far more work by using a law called Gauss' law, developed by German mathematician and physicist Carl Friedrich Gauss (1777–1855). Instead of considering the fields $d\vec{E}$ of charge elements in a given charge distribution, Gauss' law considers a hypothetical (imaginary) closed surface enclosing the charge distribution. This **Gaussian surface**, as it is called, can have any shape, but the shape that minimizes our calculations of the electric field is one that mimics the symmetry of the charge distribution. For example, if the charge is spread uniformly over a sphere, we enclose the sphere with a spherical Gaussian surface, such as the one in Fig. 23-1, and then, as we discuss in this chapter, find the electric field on the surface by using the fact that

Gauss' law relates the electric fields at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

We can also use Gauss' law in reverse: If we know the electric field on a Gaussian surface, we can find the net charge enclosed by the surface. As a limited example, suppose that the electric field vectors in Fig. 23-1 all point radially outward from the center of the sphere and have equal magnitude. Gauss' law immediately tells us that the spherical surface must enclose a net positive charge that is either a particle or distributed spherically. However, to calculate how *much* charge is enclosed, we need a way of calculating how much electric field is intercepted by the Gaussian surface in Fig. 23-1. This measure of intercepted field is called *flux*, which we discuss next.

23-2 | Flux

The idea of flux was first discussed in fluid mechanics (although vaguely). Remember $R_v = Av$. Suppose that, as in Fig. 23-2*a*, you aim a wide airstream of uniform velocity $\vec{v}$ at a small square loop of area A. Let Φ represent the *volume flow rate* (volume per unit time) at which air flows through the loop. This rate depends on the angle between $\vec{v}$ and the plane of the loop. If $\vec{v}$ is perpendicular to the plane, the rate Φ is equal to vA.

If $\vec{v}$ is parallel to the plane of the loop, no air moves through the loop, so Φ is zero. For an intermediate angle θ, the rate Φ depends on the component of

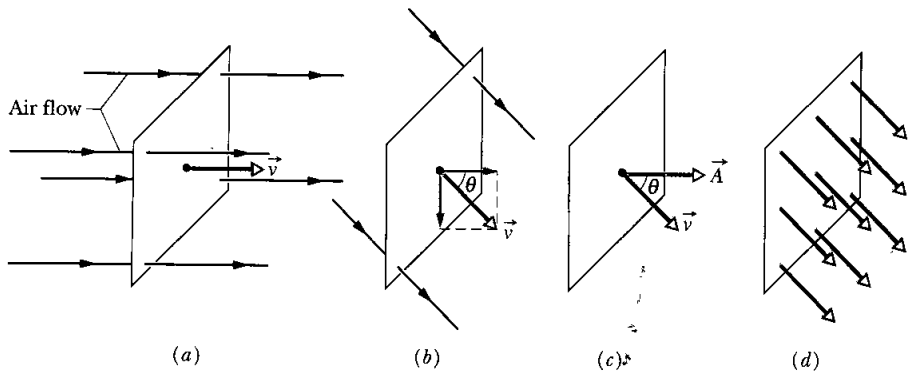


FIG. 23-2 (*a*) A uniform airstream of velocity $\vec{v}$ is perpendicular to the plane of a square loop of area A. (*b*) The component of $\vec{v}$ perpendicular to the plane of the loop is $v \cos\theta$, where θ is the angle between $\vec{v}$ and a normal to the plane. (*c*) The area vector $\vec{A}$ is perpendicular to the plane of the loop and makes an angle θ with $\vec{v}$. (*d*) The velocity field intercepted by the area of the loop.

$\vec{v}$ normal to the plane (Fig. 23-2*b*). Since that component is $v \cos \theta$, the rate of volume flow through the loop is

$$\Phi = (v \cos \theta)A. \tag{23-1}$$

This rate of flow through an area is an example of a **flux**—a *volume flux* in this situation.

Before we discuss a flux involved in electrostatics, we need to rewrite Eq. 23-1 in terms of vectors. To do this, we first define an *area vector* $\vec{A}$ as being a vector whose magnitude is equal to an area (here the area of the loop) and whose direction is normal to the plane of the area (Fig. 23-2*c*). We then rewrite Eq. 23-1 as the scalar (or dot) product of the velocity vector $\vec{v}$ of the airstream and the area vector $\vec{A}$ of the loop:

$$\Phi = vA \cos\theta = \vec{v} \cdot \vec{A}, \tag{23-2}$$

where θ is the angle between $\vec{v}$ and $\vec{A}$. Here you might be surprised that area is treated as a vector. This may sound strange but it is so. We take it to have a direction of its normal. This does make a lot of sense, because its direction does make a difference to the rate of volume flow. Recall that a quantity is a vector if it follows the vector law of addition. And, area does!

The word "flux" comes from the Latin word meaning "to flow." That meaning makes sense if we talk about the flow of air volume through the loop. However, Eq. 23-2 can be regarded in a more abstract way. To see this different way, note that we can assign a velocity vector to each point in the airstream passing through the loop (Fig. 23-2*d*). Because the composite of all those vectors is a *velocity field*, we can interpret Eq. 23-2 as giving the *flux of the velocity field through the loop.* With this interpretation, flux no longer means the actual flow of something through an area—rather it means the product of an area and the field across that area.

23-3 | Electric Flux

Similar to the flux of velocity which is measured by the product of a velocity of a fluid across a section and the area of cross-section, a concept associated with electrostatics is electric flux. Just as the flux of velocity can be appreciated and estimated by the number of streamlines crossing a given area (Chapter 14, Section *Fluid Dynamics*, Vol. I), electric flux is a measure of the number of electric field lines crossing a given area. Note the use of the word measure. It is not equal to the number of lines crossing a given area. Each person is free to draw as many number of lines he likes, the only restriction is that the number of lines should be proportional to the charge producing the electric field. We already know that the electric field can be estimated with help of the number of field lines per unit area. So number of field lines can be estimated by $E \times$ Area. To be more precise, electric flux ϕ is

$$\phi = \vec{E} \cdot \vec{A} = EA \cos\theta. \tag{23-3}$$

Its units are $\mathrm{N \cdot m^2/C}$ and it is a scalar.

Imagine an infinite charged sheet from which parallel electric field lines are emanating. If we take a cardboard sheet parallel to this charged sheet (Fig. 23-3*a*), the number of lines crossing the cardboard will be large. Note that the angle between area vector as we have defined and electric field vector is 0°. However, if we tilt this cardboard so that both surfaces are not parallel, the number of lines crossing it will keep on decreasing (Fig. 23-3*b*). When we orient it such that the cardboard becomes parallel to the electric field lines, the number of lines crossing it will be zero. In this position, according to our definition, angle between electric field vector and area vector is 90°. Therefore, electric flux will be zero (Fig. 23-3*c*).

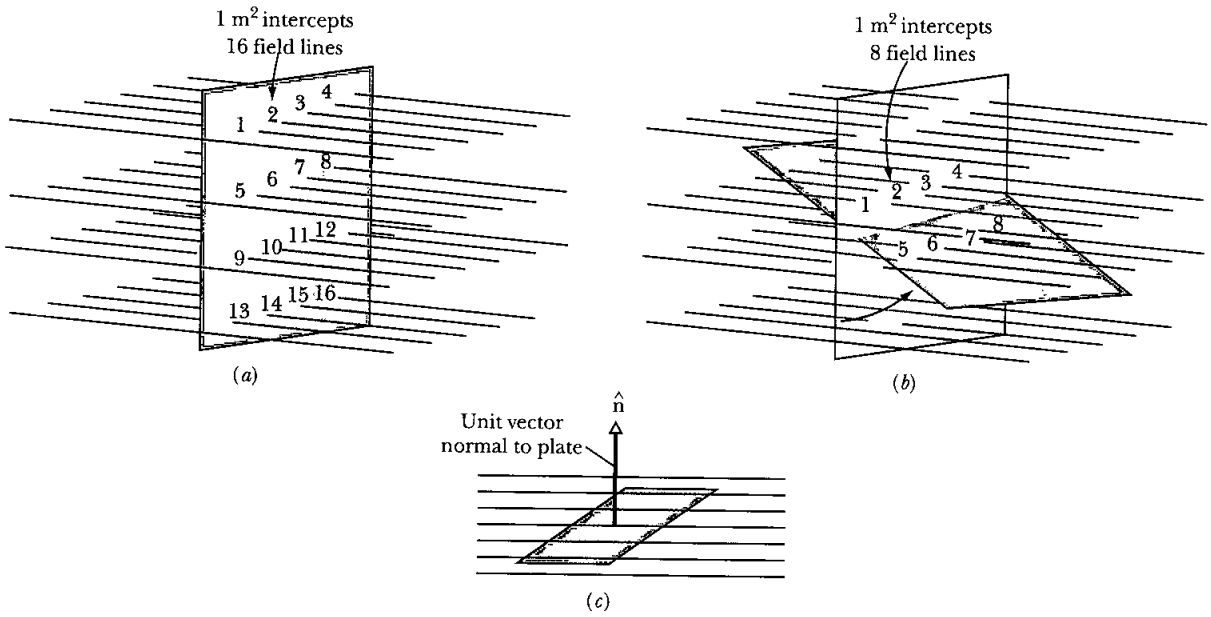


FIG. 23-3 The number of field lines crossing a surface of area 1 m^2 depends on the orientation of the surface relative to the field lines. In case (*a*), the flux is maximum. In case (*b*), the flux decreases and in case (*c*) it reduces to zero. Can you guess the angle at which the cardboard sheet is tilted in part (*b*)?

You might be surprised to see that the electric flux is a scalar, yet it has positive as well as negative values. The definition gives us the clue. If the angle between the electric field vector and the area is acute, the flux is positive. In other words, if the electric field lines are emanating from the surface, the flux is considered positive. However, if the electric field lines are entering into the surface, the angle between the electric field vector and the area would be obtuse and the flux is negative.

Normally, the electric field will not be constant. In such a case, the electric flux can be obtained by dividing the whole surface into small parts and integrating over the area.

$$\phi_E = \int \vec{E} \cdot d\vec{A}. \tag{23-4}$$

Here the value of $\vec{E}$ is at the surface and not at any point in space. This kind of integration carried out at different points of surface is called *surface integral.*

For a closed surface also, which is commonly called as Gaussian surface, we can apply a similar analysis. Here,

$$\phi_E = \oint \vec{E} \cdot d\vec{A}. \tag{23-5}$$

The circle drawn on the integral sign represents that the integration is carried over a closed surface. This equation instructs us to visit each square on the Gaussian surface, evaluate the scalar product $\vec{E} \cdot \Delta\vec{A}$ for the two vectors $\vec{E}$ and $\Delta\vec{A}$ we find there, and sum the results algebraically (that is, with signs included) for all the squares that make up the surface. The value of each scalar product (positive, negative, or zero) determines whether the flux through its square is positive, negative, or zero. Squares like square 1 in Fig. 23-4, in which $\vec{E}$ points inward, make a negative contribution to the sum of Eq. 23-5. Squares like square 2, in which $\vec{E}$ lies in the surface, make zero contribution. Squares like square 3, in which $\vec{E}$ points outward, make a positive contribution.

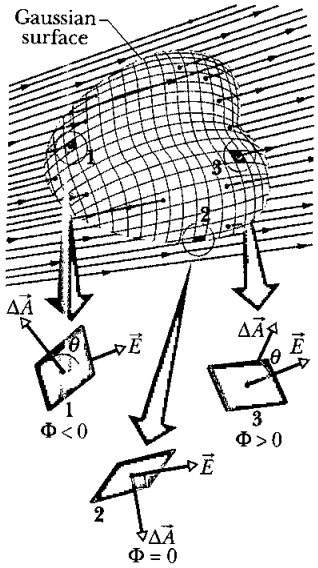
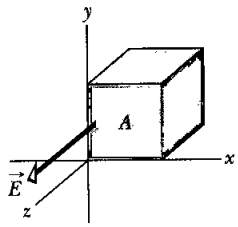


FIG. 23-4 A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area A. The electric field vectors $\vec{E}$ and the area vectors $\Delta\vec{A}$ for three representative squares, marked 1, 2, and 3, are shown.

CHECKPOINT 1 The figure here shows a Gaussian cube of face area A immersed in a uniform electric field $\vec{E}$ that has the positive direction of the z axis. In terms of E and A, what is the flux through (a) the front face (which is in the xy plane), (b) the rear face, (c) the top face, and (d) the whole cube?



Sample Problem 23-1

Figure 23-5 shows a Gaussian surface in the form of a cylinder of radius R immersed in a uniform electric field $\vec{E}$, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?

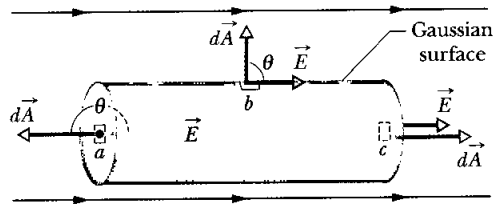


FIG. 23-5 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

Conceptualize/Classify: We can find the flux Φ through the Gaussian surface by integrating the scalar product $\vec{E} \cdot d\vec{A}$ over that surface.

Compute: We can do the integration by writing the flux as the sum of three terms: integrals over the left cylinder cap a, the cylindrical surface b, and the right cap c. Thus, from Eq. 23-4,

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

$$= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \qquad (23\text{-}6)$$

For all points on the left cap, the angle θ between $\vec{E}$ and $d\vec{A}$ is 180° and the magnitude E of the field is uniform. Thus,

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180°) dA = -E \int dA = -EA,$$

where $\int dA$ gives the cap's area A ($= \pi R^2$). Similarly, for the right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} \int E(\cos 0)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

Substituting these results into Eq. 23-6 leads us to

$$\Phi = -EA + 0 + EA = 0. \qquad \text{(Answer)}$$

Conclude: This result is perhaps not surprising because the field lines that represent the electric field all pass entirely through the Gaussian surface, entering through the left end cap, leaving through the right end cap, and giving a net flux of zero.

Sample Problem 23-2 Build your skill

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ pierces the Gaussian cube shown in Fig. 23-6. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in Sample Problem 23-5.)

Conceptualize/Classify: We can find the flux Φ through the surface by integrating the scalar product $\vec{E} \cdot d\vec{A}$ over each face.

Compute: Right face: An area vector $\vec{A}$ is always perpendicular to its surface and always points away from the interior of a Gaussian surface. Thus, the vector $d\vec{A}$ for the right face of the cube must point in the positive direction of the x axis. In unit-vector notation,

$$d\vec{A} = dA\hat{i}.$$

From Eq. 23-4, the flux Φ_r through the right face is then

$$\Phi_r = \int \vec{E}\cdot d\vec{A} = \int (3.0x\hat{i} + 4.0\hat{j})\cdot(dA\hat{i})$$
$$= \int [(3.0x)(dA)\hat{i}\cdot\hat{i} + (4.0)(dA)\hat{j}\cdot\hat{i}]$$
$$= \int (3.0x\,dA + 0) = 3.0\int x\,dA.$$

We are about to integrate over the right face, but we note that x has the same value everywhere on that face—namely, $x = 3.0$ m. This means we can substitute that constant value for x. Then

$$\Phi_r = 3.0\int (3.0)\,dA = 9.0\int dA.$$

The integral $\int dA$ merely gives us the area $A = 4.0\ \text{m}^2$ of the right face; so

$$\Phi_r = (9.0\ \text{N/C})(4.0\ \text{m}^2) = 36\ \text{N}\cdot\text{m}^2/\text{C}. \quad \text{(Answer)}$$

Left face: The procedure for finding the flux through the left face is the same as that for the right face. However, two factors change. (1) The differential area vector $d\vec{A}$ points in the negative direction of the x axis, and thus $d\vec{A} = -dA\hat{i}$. (2) The term x again appears in our integration, and it is again constant over the face being considered. However, on the left face, $x = 1.0$ m. With these two changes, we find that the flux Φ_l through the left face is

$$\Phi_l = -12\ \text{N}\cdot\text{m}^2/\text{C}. \quad \text{(Answer)}$$

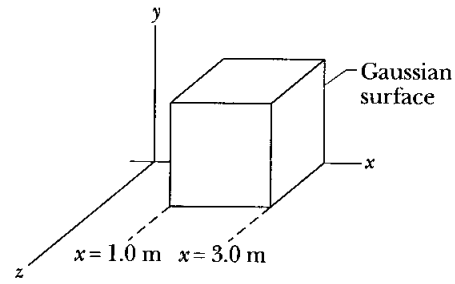


FIG. 23-6 A Gaussian cube with one edge on the x axis lies within a nonuniform electric field.

Top face: The differential area vector $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = +dA\hat{j}$. The flux Φ_t through the top face is then

$$\Phi_t = \int (3.0x\hat{i} + 4.0\hat{j})\cdot(dA\hat{j})$$
$$= \int [(3.0x)(dA)\hat{i}\cdot\hat{j} + (4.0)(dA)\hat{j}\cdot\hat{j}]$$
$$= \int (0 + 4.0\,dA) = 4.0\int dA$$
$$= 16\ \text{N}\cdot\text{m}^2/\text{C}. \quad \text{(Answer)}$$

Sample Problem 23-3

Calculate the flux through the flat disk held perpendicular to the end of a semi infinite wire of charge per unit length λ. The radius of the disk is R (Fig. 23-7).

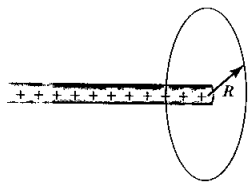


FIG. 23-7 Electric flux through a disk held perpendicular to a semi-infinite rod terminating at its center.

Conceptualize: Here we have a real situation where the electric field is not constant in space but variable. In fact, at different points of the disk, the electric field will not be constant in direction or magnitude. So obviously we need to resort to integration techniques.

Classify: As we are aware, there are two components of the electric field due to a semi-infinite rod. One is in the direction perpendicular to the rod and another is in the direction parallel to the rod. Both these components are drawn at a point on the disk as shown in Fig. 23-8.

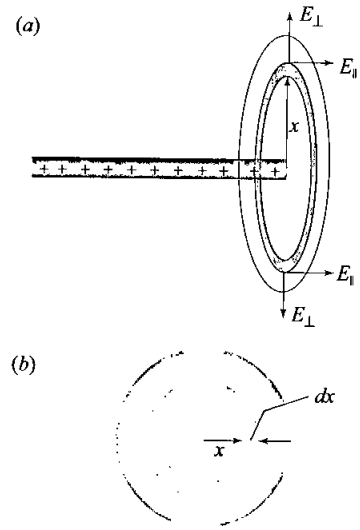


FIG. 23-8 (*a*) Pattern of electric field at various points of the disk. (*b*) A small element of the disk.

Compute: As can be easily seen, $E_\perp$ is parallel to the surface of the disk at all the points. So the angle between $d\vec{A}$ and $E_\perp$ is 90° at all the points. So the flux due to $E_\perp$ at all the points of the disk is

$$d\phi = \vec{E}_\perp\cdot d\vec{A} = E_\perp dA\cos 90° = 0.$$

However, $E_{\parallel}$ is perpendicular to the disk at all the points. So the angle between $E_{\parallel}$ and $d\vec{A}$ at any point is 0°. At a distance x from the end,

$$E_{\parallel} = \frac{k\lambda}{x}.$$

So considering a small ring of radius x on this disk whose width is dx, the flux through this small ring is

$$d\phi = \vec{E}_{\parallel} \cdot d\vec{A} = \frac{k\lambda}{x} \times 2\pi x dx \times \cos 0° = 2\pi k\lambda dx.$$

Integrating, we obtain,

$$\phi = 2\pi k\lambda R.$$

Since electric field obeys the law of superposition, electric flux obeys the law of superposition. Thus, the net flux through any area is the sum of the flux due to individual charges. To find the net flux, we should add the flux due to each charge without changing the sign of the flux.

Conclude: Suppose the disk was having uniform surface charge density σ. The force experienced by every part of the disk would be $dF = E\sigma dA$. But only the component of force perpendicular to the disk would contribute to the net force. Other component would cancel. So

$$F = \int \sigma E dA \cos\theta = \sigma\, \phi = 2\pi k\sigma\, \lambda R.$$

23-4 | Statement of Gauss' Law

Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is *enclosed* by that surface. It tells us that

$$\varepsilon_0 \Phi = q_{enc} \qquad \text{(Gauss' law).} \tag{23-7}$$

By substituting Eq. 23-4, the definition of flux, we can also write Gauss' law as

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \quad \text{(Gauss' law).} \tag{23-8}$$

Equations 23-7 and 23-8 hold only when the net charge is located in a vacuum or (what is the same for most practical purposes) in air. In Chapter 25, we modify Gauss' law to include situations in which a material such as mica, oil, or glass is present.

In Eqs. 23-7 and 23-8, the net charge q_{enc} is the algebraic sum of all the *enclosed* positive and negative charges, and it can be positive, negative, or zero. We include the sign, rather than just use the magnitude of the enclosed charge, because the sign tells us something about the net flux through the Gaussian surface: If q_{enc} is positive, the net flux is *outward*; if q_{enc} is negative, the net flux is *inward*.

Charge outside the surface, no matter how large or how close it may be, is not included in the term q_{enc} in Gauss' law. The exact form and location of the charges inside the Gaussian surface are also of no concern; the only things that matter on the right side of Eqs. 23-7 and 23-8 are the magnitude and sign of the net enclosed charge. The quantity $\vec{E}$ on the left side of Eq. 23-8, however, is the electric field resulting from *all* charges, both those inside and those outside the Gaussian surface. This may seem to be inconsistent, but keep in mind what we saw in Sample Problem 23-1: The electric field due to a charge outside the Gaussian surface contributes zero net flux *through* the surface, because as many field lines due to that charge enter the surface as leave it.

Let us apply these ideas to Fig. 23-9, which shows two point charges, equal in magnitude but opposite in sign, and the field lines describing the electric fields the charges set up in the surrounding space. Four Gaussian surfaces are also shown, in cross section. Let us consider each in turn.

Surface S_1. The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires. (That is, in Eq. 23-7, if Φ is positive, q_{enc} must be also.)

Surface S_2. The electric field is inward for all points on this surface. Thus, the flux of the electric field is negative and so is the enclosed charge, as Gauss' law requires.

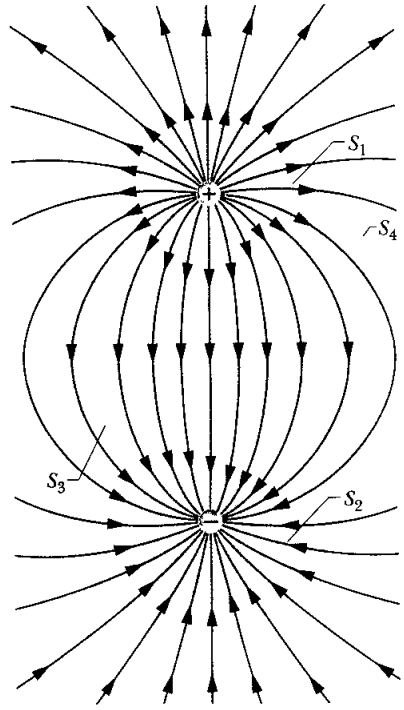


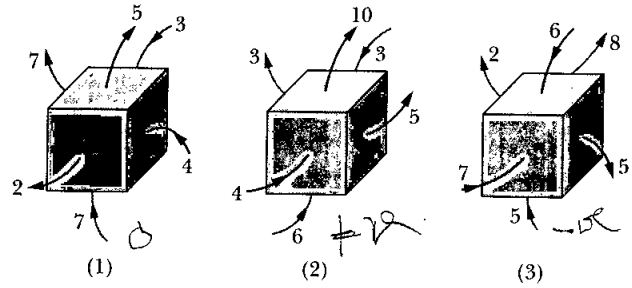
FIG. 23-9 Two point charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section. Surface S_1 encloses the positive charge. Surface S_2 encloses the negative charge. Surface S_3 encloses no charge. Surface S_4 encloses both charges and thus no net charge.

Surface S_3. This surface encloses no charge, and thus $q_{enc} = 0$. Gauss' law (Eq. 23-7) requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

Surface S_4. This surface encloses no *net* charge, because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface S_4 as entering it.

What would happen if we were to bring an enormous charge Q up close to surface S_4 in Fig. 23-9? The pattern of the field lines would certainly change, but the net flux for each of the four Gaussian surfaces would not change. We can understand this because the field lines associated with the added Q would pass entirely through each of the four Gaussian surfaces, making no contribution to the net flux through any of them. The value of Q would not enter Gauss' law in any way, because Q lies outside all four of the Gaussian surfaces that we are considering.

CHECKPOINT 2 The figure shows three situations in which a Gaussian cube sits in an electric field. The arrows and the values indicate the directions of the field lines and the magnitudes (in $N \cdot m^2/C$) of the flux through the six sides of each cube. (The lighter arrows are for the hidden faces.) In which situation does the cube enclose (a) a positive net charge, (b) a negative net charge, and (c) zero net charge?



Sample Problem 23-4

Figure 23-10 shows five charged lumps of plastic and an electrically neutral coin. The cross section of a Gaussian surface S is indicated. What is the net electric flux through the surface if $q_1 = q_4 = +3.1$ nC, $q_2 = q_5 = -5.9$ nC, and $q_3 = -3.1$ nC?

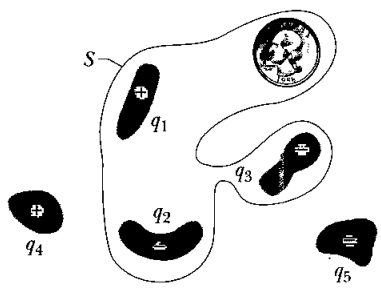


FIG. 23-10 Five plastic objects, each with an electric charge, and a coin, which has no net charge. A Gaussian surface, shown in cross section, encloses three of the plastic objects and the coin.

Conceptualize/Classify: The *net* flux Φ through the surface depends on the *net* charge q_{enc} enclosed by surface S.

Compute: The coin does not contribute to Φ because it is neutral and thus contains equal amounts of positive and negative charge. Charges q_4 and q_5 do not contribute because they are outside surface S. Thus, q_{enc} is $q_1 + q_2 + q_3$ and Eq. 23-7 gives us

$$\Phi = \frac{q_{enc}}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

$$= \frac{+3.1 \times 10^{-9}\,C - 5.9 \times 10^{-9}\,C - 3.1 \times 10^{-9}\,C}{8.85 \times 10^{-12}\,C^2/N \cdot m^2}$$

$$= -670\ N \cdot m^2/C. \qquad \text{(Answer)}$$

Conclude: The minus sign shows that the net flux through the surface is inward and thus that the net charge within the surface is negative.

Sample Problem 23-5 Build your skill

What is the net charge enclosed by the Gaussian cube of Sample Problem 23-2 and Fig. 23-6?

Conceptualize/Classify: The net charge enclosed by a (real or mathematical) closed surface is related to the total electric flux through the surface by Gauss' law as given by Eq. 23-6 ($\varepsilon_0\Phi = q_{enc}$).

Compute: To use Eq. 23-7, we need to know the flux through all six faces of the cube. We already know the flux through the right face ($\Phi_r = 36\ \text{N}\cdot\text{m}^2/\text{C}$), the left face ($\Phi_l = -12\ \text{N}\cdot\text{m}^2/\text{C}$), and the top face ($\Phi_t = 16\ \text{N}\cdot\text{m}^2/\text{C}$).

For the bottom face, our calculation is just like that for the top face *except* that the differential area vector $d\vec{A}$ is now directed downward along the y axis (recall, it must be *outward* from the Gaussian enclosure). Thus, we have $d\vec{A} = -dA\hat{j}$, and we find

$$\Phi_b = -16\ \text{N}\cdot\text{m}^2/\text{C}.$$

For the front face we have $d\vec{A} = dA\hat{k}$, and for the back face, $d\vec{A} = dA\hat{k}$. When we take the dot product of the given electric field $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ with either of these expressions for $d\vec{A}$, we get 0 and thus there is no flux through those faces. We can now find the total flux through the six sides of the cube:

$$\Phi = (36 - 12 + 16 - 16 + 0 + 0)\ \text{N}\cdot\text{m}^2/\text{C}$$
$$= 24\ \text{N}\cdot\text{m}^2/\text{C}.$$

Enclosed charge: Next, we use Gauss' law to find the charge q_{enc} enclosed by the cube:

$$q_{enc} = \varepsilon_0\Phi = (8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(24\ \text{N}\cdot\text{m}^2/\text{C})$$
$$= 2.1\times10^{-10}\ \text{C}. \qquad \text{(Answer)}$$

Conclude: Thus, the cube encloses a *net* positive charge.

23-5 | Gauss' Law and Coulomb's Law

Because Gauss' law and Coulomb's law are different ways of describing the relation between electric charge and electric field in static situations, we should be able to derive each from the other. Here we derive Coulomb's law from Gauss' law and some symmetry considerations.

Figure 23-11 shows a positive point charge q, around which we have drawn a concentric spherical Gaussian surface of radius r. Let us divide this surface into differential areas dA. By definition, the area vector $d\vec{A}$ at any point is perpendicular to the surface and directed outward from the interior. From the symmetry of the situation, we know that at any point the electric field $\vec{E}$ is also perpendicular to the surface and directed outward from the interior. Thus, since the angle θ between $\vec{E}$ and $d\vec{A}$ is zero, we can rewrite Eq. 23-8 for Gauss' law as

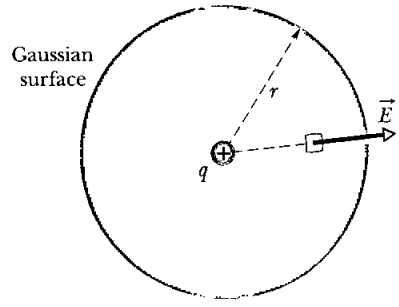


FIG. 23-11 A spherical Gaussian surface centered on a point charge q.

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = \varepsilon_0 \oint E\, dA = q_{enc}. \qquad (23\text{-}9)$$

Here $q_{enc} = q$. Although E varies radially with distance from q, it has the same value everywhere on the spherical surface. Since the integral in Eq. 23-9 is taken over that surface, E is a constant in the integration and can be brought out in front of the integral sign. That gives us

$$\varepsilon_0 E \oint dA = q. \qquad (23\text{-}10)$$

The integral is now merely the sum of all the differential areas dA on the sphere and thus is just the surface area, $4\pi r^2$. Substituting this, we have

$$\varepsilon_0 E(4\pi r^2) = q$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad (23\text{-}11)$$

This is exactly Eq. 22-19, which we found using Coulomb's law.

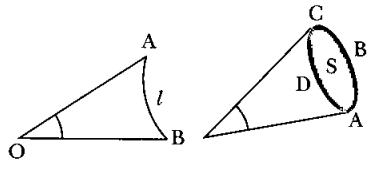


FIG. 23-12 Formation of (*a*) a plane angle and (*b*) a solid angle.

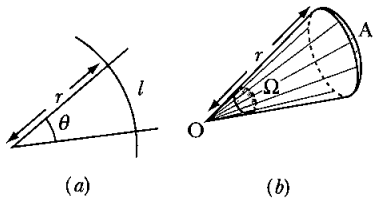


FIG. 23-13 Measurement of (*a*) a plane angle and (*b*) a solid angle.

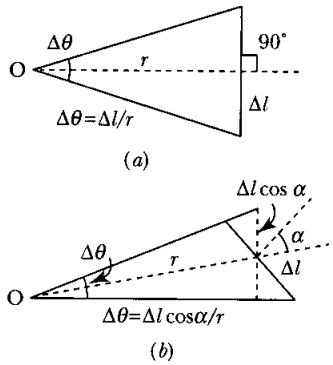


FIG. 23-14 Measurement of (*a*) a small angle and (*b*) angle subtended by an inclined line.

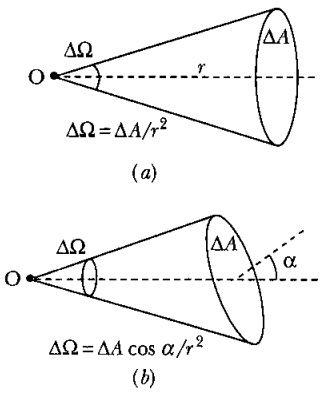


FIG. 23-15 Solid angle subtended by (*a*) a small plane area held normal to the radius and (*b*) by a small plane area held at an angle to the radius.

CHECKPOINT 3 There is a certain net flux Φ_i through a Gaussian sphere of radius r enclosing an isolated charged particle. Suppose the enclosing Gaussian surface is changed to (a) a larger Gaussian sphere, (b) a Gaussian cube with edge length equal to r, and (c) a Gaussian cube with edge length equal to $2r$. In each case, is the net flux through the new Gaussian surface greater than, less than, or equal to Φ_i?

*Deriving Gauss Law (Optional)**

One way of deriving Gauss's law involves solid angles. Solid angle is a generalization of the plane angle. In Fig. 23-12*a* we show a plane curve AB. The end point A and B are joined to point O. We say that the curve AB subtends an angle or a plane angle at O. An angle is formed at O by the two lines OA and OB passing through O.

To construct a solid angle, we start with a surface ABCD (Fig. 23-12*b*) and join all the points on the periphery such as A, B, C, D, etc. with the given point O. We then say that a solid angle is formed at O and that the surface ABCD has subtended the solid angle. The solid angle is formed by the lines joining the points on the periphery with O. The whole figure looks like a cone. As a typical example, think of the paper containers used by street vendors.

How do we measure a solid angle? Let us consider how do we measure a plane angle. See Fig. 23-13*a*. We draw a circle of any radius r with the center at O and measure the length l of the arc intercepted by the angle. The angle θ is then define as $\theta = l/r$. In order to measure a solid angle at the point O (Fig. 23-13*b*), we draw a sphere of any radius r with O as the center and measure the area A of the part of the sphere intercepted by the cone.

The solid angle Ω subtended at the center of the sphere by this element is defined to be

$$\Omega \equiv \frac{A}{r^2}.$$

From this equation, we see that Ω has no dimensions because A and r^2 both have dimensions L^2. The dimensionless unit of a solid angle is steradian. Because the surface area of a sphere is $4\pi r^2$, the total solid angle subtended by the sphere is

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradians}.$$

It is independent of the radius of the sphere drawn.

Next, consider a plane angle subtended at a point O by a small line segment Δl (Fig. 23-14*a*). Suppose, the line joining O to the middle point of Δl is perpendicular to Δl. As the segment is small, we can approximately write

$$\Delta\theta = \frac{\Delta l}{r}.$$

As Δl get smaller, the approximation becomes better. Now suppose, the line joining O to Δl is not perpendicular to Δl (Fig. 23-14*b*), but this line makes an angle α with the perpendicular to Δl. The angle subtended by Δl at O is

$$\Delta\theta = \frac{\Delta l \cos\alpha}{r}.$$

Similarly, if a small plane area ΔA (Fig. 23-15*a*) subtended a solid angle $\Delta\Omega$ at O in such a way that the line joining O to ΔA is normal to ΔA, we can write $\Delta\Omega = \Delta A/r^2$. But if the line joining O to ΔA makes an angle α with the normal to ΔA (Fig. 23-15*b*), we should write

*One may skip this section without losing continuity of text.

$$\Delta\Omega = \frac{\Delta A \cos\alpha}{r^2}.$$

A complete circle subtends an angle

$$\theta = \frac{l}{r} = \frac{2\pi r}{r} = 2\pi$$

at the center. In fact, any closed curve subtends an angle 2π at any of the internal points. Similarly, a complete sphere subtends a solid angle

$$\Omega = \frac{A}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi$$

at the center. Also, any closed surface subtends a solid angle 4π at any internal point. How much is the angle subtended by a closed plane curve at an external point? See Fig. 23-16. As we gradually close the curve, the angle finally diminishes to zero. A closed curve subtends zero angle at an external point. Similarly, a closed surface subtends zero solid angle at an external point.

We can also find the solid angle in terms of the half angle made by the cone at the center. We need to find the area of the part of the sphere which is used to make the solid angle. Here we should be careful. This area S is not the area of the base of the cone formed by the disk and the source (Fig. 23-17). It is the area of the bulgy portion of the sphere intercepted by the disk.

To find this area, we can divide this area in small parts, specifically rings. A typical ring is shown in the diagram. This ring has an angular position of θ with the x axis and its thickness makes an angle $d\theta$ at the center of the sphere. The radius of this ring x can be seen to be $R\sin\theta$. The width of the ring can be seen to be $Rd\theta$. Now

Length of the arc = Radius × Angle made by the arc at the center

If we straighten out the ring, it will make a rectangle of width $Rd\theta$ and length equal to the circumference of the ring. So the area of the ring will come out to be

$$dA = 2\pi R\sin\theta \times R\,d\theta = 2\pi R^2 \sin\theta\, d\theta.$$

To find the total area, we can integrate this expression as

$$A = \int_0^{\alpha} 2\pi R^2 \sin\theta\, d\theta = 2\pi R^2(1-\cos\alpha).$$

From the above discussion, we can say that

$$\Omega = 2\pi(1-\cos\theta). \tag{23-12}$$

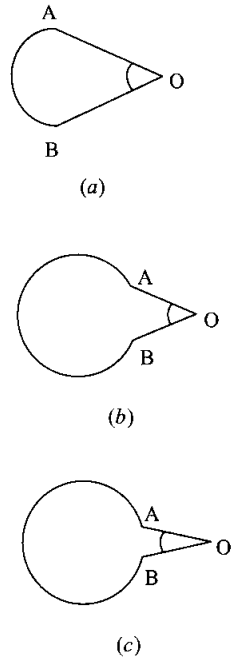


FIG. 23-16 Angle subtended by a plane curve at a point outside the curve is zero.

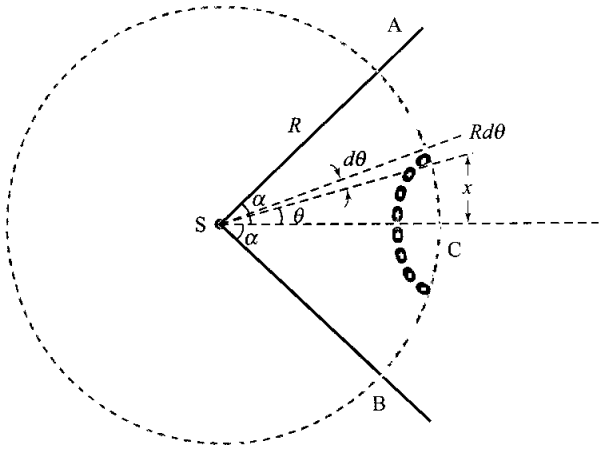


FIG. 23-17 Finding the area of an imaginary portion cut from an imaginary sphere of radius R by a cone ABS.

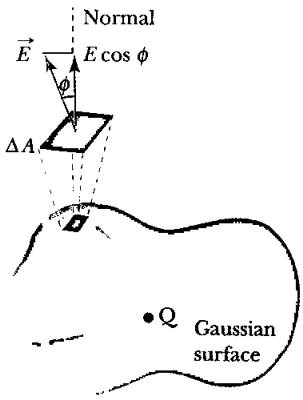


FIG. 23-18 An arbitrarily shaped Gaussian surface. The electric flux through any tiny segment of the surface is the product of $E\cos\phi$ and the area of the segment ΔA. The angle ϕ is the angle between the electric field and the normal to the surface.

Formal Derivation of Gauss' Law

Now consider a point charge q surrounded by a closed surface of arbitrary shape (Fig. 23-18). The total electric flux through this surface can be obtained by evaluating $\vec{E}\cdot\Delta\vec{A}$ for each small area element ΔA and summing over all elements. The flux through each element is

$$\Delta\Phi_E = \vec{E}\cdot\Delta\vec{A} = (E\cos\theta)\Delta A = k_e q\frac{\Delta A\cos\theta}{r^2}.$$

In Fig. 23-19, we see that the projection of the area element perpendicular to the radius vector is $\Delta A\cos\theta$. Thus, the quantity $(\Delta A\cos\theta)/r^2$ is equal to the solid angle $\Delta\Omega$ that the surface element ΔA subtends at the charge q. We also see that $\Delta\Omega$ is equal to the solid angle subtended by the area element of a spherical surface of radius r. Because the total solid angle at a point is 4π steradians, the total flux through the closed surface is

$$\Delta\Phi_E = kq\oint\frac{dA\cos\theta}{r^2} = kq\oint d\Omega = 4\pi kq = \frac{q}{\varepsilon_0}$$

If we consider a large number of charges within the Gaussian surface, each of them will produce a flux that will give the same expression. Also we have shown that the solid angle subtended by a charge outside the Gaussian surface will be zero. Since flux obeys the law of superposition, we can say that the net flux through the Gaussian surface is

$$\phi_E = \frac{\Sigma Q_{\text{enclosed}}}{\varepsilon_0},$$

where $\Sigma Q_{\text{enclosed}}$ refers to the total charge enclosed by the Gaussian surface. Thus we have derived Gauss's law. Note that this result is independent of the shape of the closed surface and independent of the position of the charge within the surface.

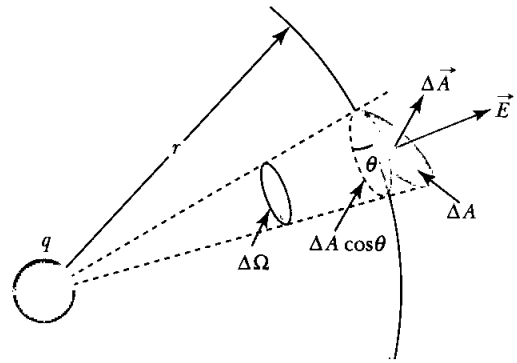


FIG. 23-19 The area element ΔA subtends a solid angle $\Delta\Omega = (\Delta A\cos\theta)/r^2$ at charge q, where r is the distance from the charge to the area element, θ is the angle between the electric field E and ΔA for the element, and $E = kq/r^2$ for a point charge.

Sample Problem 23-6

Find the flux through the disk shown in Fig. 23-20.

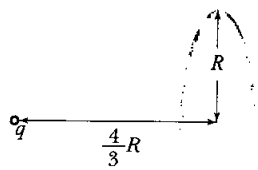


FIG. 23-20 Finding flux due to a point charge through a disk. The line joining the charge to the center of the disk is perpendicular to the disk.

Conceptualize: The electric flux through the disk cannot be found by the equation

$$\phi = \vec{E}\cdot\vec{A} = EA\cos\theta$$

because the electric field due to the point charge at different points of the disk is not the same. Hence we have to resort to integration methods enumerated earlier.

Classify: If we wish to use the basic formula, we can divide the disk into small rings as shown in Fig. 23-21 and find the electric field due to charge at all the rings:

$$\phi = \int\vec{E}\cdot\vec{ds}.$$

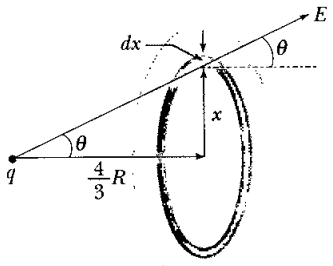


FIG. 23-21 Here we divide the entire disk into thin ring and find the flux due to the charge through the thin ring.

Compute: The electric field due to the point charge at the location of the ring shown is given by

$$E = \frac{kq}{(16/9)R^2 + x^2}.$$

As we discussed before, the area of the ring is $2\pi x dx$. But the electric field is not normal to the ring. The angle can be found as shown:

$$\cos\theta = \frac{4R/3}{\sqrt{(16R^2/9)+x^2}},$$

$$\tan\alpha = \tfrac{3}{4},$$

$$\phi = \int \vec{E}\cdot d\vec{s},$$

$$\phi = \int_0^{0.75R} \frac{kq \times (4R/3) \times 2\pi x dx}{[x^2 + (16R^2/9)] \times \sqrt{x^2 + (16R^2/9)}} = \frac{q}{10\varepsilon_0}.$$

Alternative method

Conceptualize/Classify: Let us visualize the electric field lines coming from the point charge. These electric field lines will emanate just as light rays emanate from a point source of light. A part of these electric field lines will pass through the disk (Fig. 23-22). As we discussed in the section on solid angle, the amount of flux can be calculated by using the concept of solid angle. The total flux due to the point charge through any sphere is given by

$$\phi_T = \frac{q}{\varepsilon_0}.$$

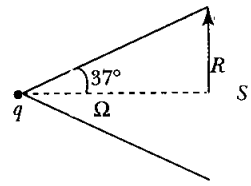


FIG. 23-22 The flux through the disk can be calculated by using an imaginary Gaussian surface centered at q. The flux through the disk and the curved portion of sphere will be in proportion to the solid angle Ω subtended by it at q.

The portion of flux passing through the disk will be a small fraction of it. In fact, since flux through any part of the spherical surface is proportional to the area of the spherical surface, it will also be proportional to the solid angle subtended by this part on the sphere. Since the total solid angle subtended by the sphere on the charge is 4π, the

$$\frac{\phi_{\text{disk}}}{\phi_T} = \frac{\Omega}{4\pi}.$$

Compute: We can imagine a Gaussian surface centered at the charge and intersecting the disk as shown. This disk makes a cone at the center whose solid angle can be calculated by Eq. 23-12:

$$\tan\alpha = \tfrac{3}{4},\ \cos\alpha = \tfrac{4}{5};$$

$$\phi_{\text{disk}} = \frac{q}{\varepsilon_0}\times\frac{\Omega}{4\pi} = \frac{q}{\varepsilon_0}\times\frac{2\pi}{4\pi}(1-\cos\theta)$$

$$= \frac{q}{\varepsilon_0}\times\frac{1}{10}.$$

Conclude: Gauss' law is definitely superior in finding the value of flux or the electric field.

PROBLEM-SOLVING TACTICS

Tactic 1: Choosing a Gaussian Surface Because the derivation of Eq. 23-11 using Gauss' law is a warm-up for derivations of electric fields produced by other charge configurations, let us go back over the steps involved. We started with a given positive point charge q; we know that electric field lines extend radially outward from q in a spherically symmetric pattern.

To use Gauss' law (Eq. 23-8) to find the magnitude E of the electric field at a distance r, we had to place a hypothetical

closed Gaussian surface around q, through a point that is a distance r from q. Then we had to sum via integration the values of $\vec{E} \cdot d\vec{A}$ over the full Gaussian surface. To make this integration as simple as possible, we chose a spherical Gaussian surface (to mimic the spherical symmetry of the electric field). That choice produced three simplifying features. (1) The dot product $\vec{E} \cdot d\vec{A}$ became simple, because at all points on the Gaussian surface the angle between $\vec{E}$ and $d\vec{A}$ is zero, and so at all points we have $\vec{E} \cdot d\vec{A} = E\,dA$. (2) The electric field magnitude E is the same at all points on the spherical Gaussian surface; so E was a constant in the integration and could be brought out in front of the integral sign. (3) The result was a very simple integration—a summation of the differential areas of the sphere, which we could immediately write as $4\pi r^2$.

Note that Gauss' law holds regardless of the shape of the Gaussian surface we choose to place around charge q_{enc}. However, if we had chosen, say, a cubical Gaussian surface, our three simplifying features would have disappeared and the integration of $\vec{E} \cdot d\vec{A}$ over the cubical surface would have been very difficult. The moral here is to choose the Gaussian surface that most simplifies the integration in Gauss' law.

23-6 | Applying Gauss' Law: Spherical Symmetry

Here we use Gauss' law to prove the two shell theorems.

☞ A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.

☞ If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell.

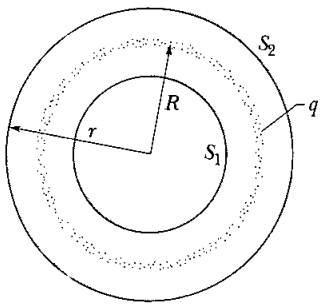


FIG. 23-23 A thin, uniformly charged, spherical shell with total charge q, in cross section. Two Gaussian surfaces S_1 and S_2 are also shown in cross section. Surface S_2 encloses the shell, and S_1 encloses only the empty interior of the shell.

Figure 23-23 shows a charged spherical shell of total charge q and radius R and two concentric spherical Gaussian surfaces, S_1 and S_2. If we followed the procedure of Section 23-5 as we applied Gauss' law to surface S_2, for which $r \geq R$, we would find that

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{(spherical shell, field at } r \geq R\text{)}. \tag{23-13}$$

This field is the same as one set up by a point charge q at the center of the shell of charge. Thus, the force produced by a shell of charge q on a charged particle placed outside the shell is the same as the force produced by a point charge q located at the center of the shell. This proves the first shell theorem.

Applying Gauss' law to surface S_1, for which $r < R$, leads directly to

$$E = 0 \qquad \text{(spherical shell, field at } r < R\text{)}, \tag{23-14}$$

because this Gaussian surface encloses no charge. Thus, if a charged particle were enclosed by the shell, the shell would exert no net electrostatic force on the particle. This proves the second shell theorem.

Any spherically symmetric charge distribution, such as that of Fig. 23-24, can be constructed with a nest of concentric spherical shells. For purposes of applying the two shell theorems, the volume charge density ρ should have a single value for each shell but need not be the same from shell to shell. Thus, for the charge distribution as a whole, ρ can vary, but only with r, the radial distance from the center. We can then examine the effect of the charge distribution "shell by shell."

In Fig. 23-24*a*, the entire charge lies within a Gaussian surface with $r > R$. The charge produces an electric field on the Gaussian surface as if the charge were a point charge located at the center, and Eq. 23-13 holds.

Figure 23-24*b* shows a Gaussian surface with $r < R$. To find the electric field at points on this Gaussian surface, we consider two sets of charged shells—one set inside the Gaussian surface and one set outside. Equation 23-14 says that the charge lying *outside* the Gaussian surface does not set up a net electric field on the Gaussian surface. Equation 23-13 says that the charge *enclosed* by the surface sets up an

electric field as if that enclosed charge were concentrated at the center. Letting q' represent that enclosed charge, we can then rewrite Eq. 23-13 as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2} \qquad \text{(spherical distribution, field at } r \le R\text{).} \tag{23-15}$$

If the full charge q enclosed within radius R is uniform, then q' enclosed within radius r in Fig. 23-24*b* is proportional to q:

$$\frac{\begin{pmatrix}\text{charge enclosed by}\\ \text{sphere of radius } r\end{pmatrix}}{\begin{pmatrix}\text{volume enclosed by}\\ \text{sphere of radius } r\end{pmatrix}} = \frac{\text{full charge}}{\text{full volume}}$$

or

$$\frac{q'}{\frac{4}{3}\pi r^3} = \frac{q}{\frac{4}{3}\pi R^3}. \tag{23-16}$$

This gives us

$$q' = q\frac{r^3}{R^3}. \tag{23-17}$$

Substituting this into Eq. 23-15 yields

$$E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r \qquad \text{(uniform charge, field at } r \le R\text{).} \tag{23-18}$$

We can also draw the graph of electric field due to a uniformly charged sphere and a uniformly charged shell with the distance from the center (Figs. 23-25 and 23-26).

The idea behind the first shell theorem can be explained without Gauss' law as well, in a manner that is quite interesting. Consider the homogeneous hollow shell in Fig. 23-27*a*. A particle of charge q is placed inside the hollow sphere.

To determine the force exerted by the hollow shell on the particle, consider first the force exerted by a very small patch q_1. The distance from q_1 to q is d_1 and so the magnitude of the electrostatic force exerted by this patch is kqq_1/d_1^2. Extending the cone defined by q and q_1 toward the opposite side of the hollow sphere, we find another very small patch q_2 at a distance d_2 from q. The force exerted by this patch on q is kqq_2/d_2^2. If q is closer to q_1 than to q_2 then patch q_2 has a larger area and so it contains more charge. A larger charge means a larger force, but we also have to account for the larger distance from q_2 to q which reduces the force. If the charge is uniformly distributed over the sphere, then the charge of each patch is proportional to its area. Because the area of the two patches are marked out by similar cones, their areas are proportional to the squares of their distances to q and so

$$\frac{q_1}{q_2} = \frac{d_1^2}{d_2^2}.$$

Rearranging terms we then find $q_1/d_1^2 = q_2/d_2^2$ regardless of the position of q, and so we see that the forces cancel—the two patches exert equal but opposite forces on q (see Fig. 23-27*b*).We can now apply the same arguments to other pairs of small patches on either side of q, each yielding equal and opposite forces on q (Fig. 23-27*c*). The vector sum of all of the forces exerted by all the small patches making up the hollow sphere on q is thus zero.

An interesting consequence of the shell theorem is that the interaction between two homogeneous spherical bodies having charge distributed uniformly

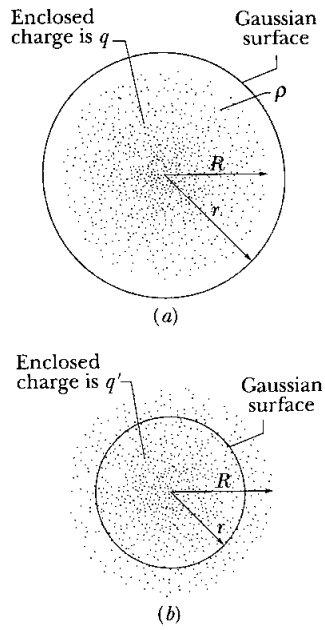


FIG. 23-24 The dots represent a spherically symmetric distribution of charge of radius R, whose volume charge density ρ is a function only of distance from the center. The charged object is not a conductor, and the refore the charge is assumed to be fixed in position. A concentric spherical Gaussian surface with $r > R$ is shown in (a). A similar Gaussian surface with $r < R$ is shown in (b).

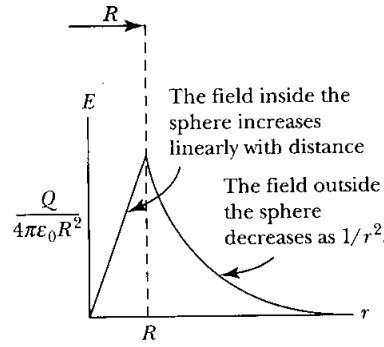


FIG. 23-25 The graph of electric field due to a uniformly charged sphere.

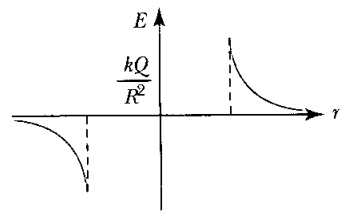


FIG. 23-26 Electric field due to a uniformly charged shell.

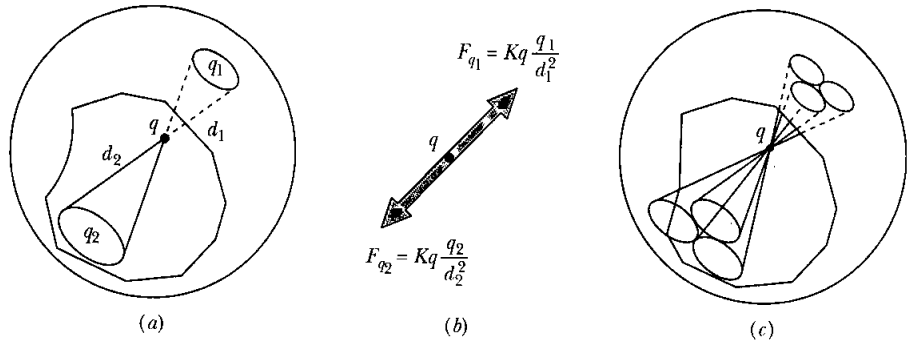


FIG. 23-27 (*a*) A uniformly charged shell is divided into small patches of charges q_1 and q_2 to find the force on a point charge q kept inside (not necessarily at the center). (*b*) The patches can be approximated by point charges and Coulomb's law can then be applied. The forces on q due to these patches are equal and opposite. (*c*) Similar to this we can find pair of patches which will cancel the force and the electric field at the location of q.

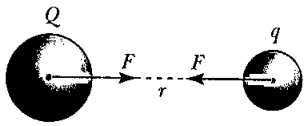


FIG. 23-28 The force of interaction between two uniformly charged spherical bodies.

over there surface depends only on the distance between their centers. Assume that we place a point charge q at a distance r from the center of a spherical charge distribution Q as in Fig. 23-28. The field that the point charge experiences is $E = KQ/r^2$, and the force is $F = qE = KQq/r^2$. By Newton's third law, q must exert an equal and opposite force on Q. This force can be understood as being due to the field created by point charge q in the region occupied by spherical charge q, the field around Q. Now, if we replace point charge q by a homogeneous spherical body of same charge q, the field around Q does not change, because of the result we have just proved, and therefore the force on Q remains the same. Again we use Newton's third law, and conclude that the force on the spherical charge q is still the same. Consequently, two homogeneous spherical charge bodies interact with each other according to Eq. 22-1, where r is the distance between their centers. If the charge distributions are neither spherical nor homogeneous, some geometrical factors, including angles defining their relative orientation, will appear in the expression for their interaction.

✓**CHECKPOINT 4** What if the smaller sphere of uniformly distributed charge q was placed inside the larger shell of charge Q. What will be the force on each of them? Consider two cases: (a) The two charge distributions are concentric and (b) the centers of charges do not coincide.

Sample Problem 23-7

If a cavity is made as shown in the uniformly charged sphere (Fig. 23-29), find the electric field at an arbitrary point inside the cavity.

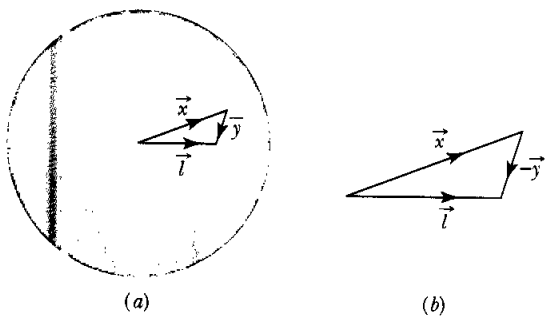


FIG. 23-29 Finding the electric field inside a spherical cavity made in a uniformly charged sphere.

Conceptualize: Here we can create a cavity by superimposing a charge density of $-\rho$ over a uniformly charged sphere of charge density ρ (Fig. 23-30).

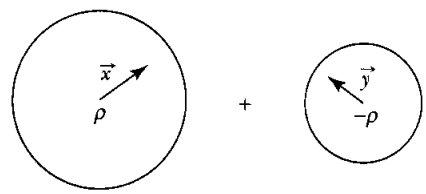


FIG. 23-30 Finding electric field by superposition.

Classify: The charge distribution present in Fig. 23-29 and the electric field inside the cavity can be represented by summing up the two charge distribution ρ and $-\rho$ which would neutralize each other. However, the electric filed does not cancel out.

Compute: Here we can see that the electric field in the cavity will be a superposition of the electric field due to ρ and $-\rho$. Due to each of the sphere, the electric field will be

$$\vec{E}_1 = \frac{\rho\vec{x}}{3\varepsilon_0}.$$

Note that we are using the vector sign for x as well as the electric field because it is in the radial direction. Hence the direction of electric field coincides with the direction of the radius x. Similarly,

$$\vec{E}_2 = -\frac{\rho\vec{y}}{3\varepsilon_0}.$$

The net electric field inside the cavity is given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho\vec{x}}{3\varepsilon_0} - \frac{\rho\vec{y}}{3\varepsilon_0}.$$

From Figure 23-29 it is clear that

$$\vec{x} - \vec{y} = \vec{l},$$

so the electric field E is expressed as

$$\vec{E} = \frac{\rho\vec{l}}{3\varepsilon_0}.$$

Conclude: Note that the electric field is constant in direction as well as magnitude. This is another example of uniform electric field.

Sample Problem 23-8

An electron is released from the surface of a uniformly and positively charged sphere (volume charge density ρ, Fig. 23-31). Show that the electron will execute SHM and find its time period.

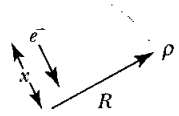


FIG. 23-31 An electron inside a uniformly charged sphere.

Conceptualize/Classify: Since the electron will be negatively charged, it will be pulled by the positively charged sphere. The condition for the simple harmonic motion (SHM) is that the force should be proportional to the displacement x from the mean position.

Compute: At a distance x from the center, the force on the electron is

$$F = -eE = -e\frac{\rho x}{3\varepsilon_0}.$$

Since this force is proportional to distance x from the mean position (center) and directed toward it, the motion should be simple harmonic. Solving this, we get

$$a = \frac{-e\rho}{3m\varepsilon_0}x.$$

But in an SHM,

$$a = -\omega^2 x; \quad \omega = \sqrt{\frac{e\rho}{3m\varepsilon_0}}; \quad T = 2\pi\sqrt{\frac{3m\varepsilon_0}{e\rho}}.$$

Conclude: Here the electron was moving along the diameter. Suppose that this electron was allowed to move only along a chord, starting from rest. In that case also the motion of the electron will be simple harmonic. A component of the force will be balanced by the walls of the tunnel in which electron moves. The net force will come out to be proportional to the distance from the center of the chord which will be the mean position in this case. Even the time period will come out to be the same (Fig. 23-32).

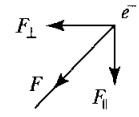


FIG. 23-32 An electron inside a smooth tunnel in a uniformly charged sphere.

23-7 | Applying Gauss' Law: Cylindrical Symmetry

Figure 23-33 shows a section of an infinitely long cylindrical plastic rod with a uniform positive linear charge density λ. Let us find an expression for the magnitude of the electric field $\vec{E}$ at a distance r from the axis of the rod.

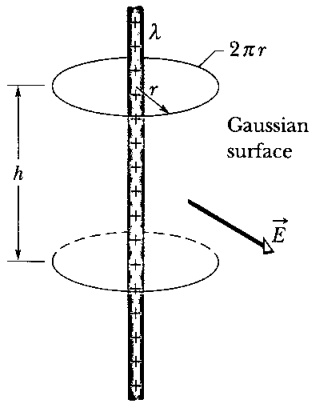


FIG. 23-33 A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

Our Gaussian surface should match the symmetry of the problem, which is cylindrical. We choose a circular cylinder of radius r and length h, coaxial with the rod. Because the Gaussian surface must be closed, we include two end caps as part of the surface.

Imagine now that, while you are not watching, someone rotates the plastic rod about its longitudinal axis or turns it end for end. When you look again at the rod, you will not be able to detect any change. We conclude from this symmetry that the only uniquely specified direction in this problem is along a radial line. Thus, at every point on the cylindrical part of the Gaussian surface, $\vec{E}$ must have the same magnitude E and (for a positively charged rod) must be directed radially outward.

Since $2\pi r$ is the cylinder's circumference and h is its height, the area A of the cylindrical surface is $2\pi rh$. The flux of $\vec{E}$ through this cylindrical surface is then

$$\Phi = EA \cos\theta = E(2\pi rh)\cos 0 = E(2\pi rh).$$

There is no flux through the end caps because $\vec{E}$, being radially directed, is parallel to the end caps at every point.

The charge enclosed by the surface is λh, which means Gauss' law,

$$\varepsilon_0\Phi = q_{enc},$$

reduces to

$$\varepsilon_0 E(2\pi rh) = \lambda h,$$

yielding

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge)}. \tag{23-19}$$

This is the electric field due to an infinitely long, straight line of charge, at a point that is a radial distance r from the line. The direction of $\vec{E}$ is radially outward from the line of charge if the charge is positive, and radially inward if it is negative. Equation 23-19 also approximates the field of a *finite* line of charge at points that are not too near the ends (compared with the distance from the line).

Sample Problem 23-9

Upward streamer in a lightning storm. The woman in Fig. 23-34 was standing on a lookout platform in the Sequoia National Park when a large storm cloud moved overhead. Some of the conduction electrons in her body were driven into the ground by the cloud's negatively charged base (Fig. 23-35*a*), leaving her positively charged. You can tell she was highly charged because her hair strands repelled one another and extended away from her along the electric field lines produced by the charge on her.

Lightning did not strike the woman, but she was in extreme danger because that electric field was on the verge of causing electrical breakdown in the surrounding



FIG. 23-34 This woman has become positively charged by an overhead storm cloud. *(Courtesy NOAA)*

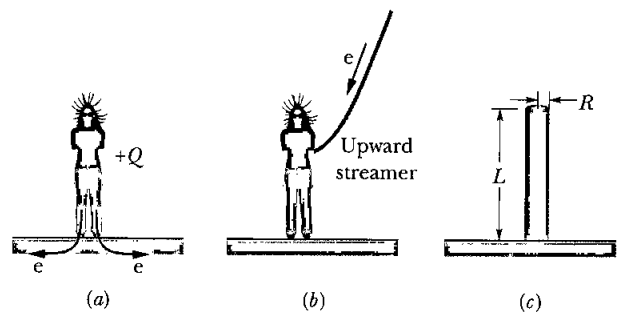


FIG. 23-35 (*a*) Some of the conduction electrons in the woman's body are driven into the ground, leaving her positively charged. (*b*) An upward streamer develops if the air undergoes electrical breakdown, which provides a path for electrons freed from molecules in the air to move to the woman. (*c*) A cylinder represents the woman.

air. Such a breakdown would have occurred along a path extending away from her in what is called an *upward streamer.* You can see a bright upward streamer near the top of the tree in the opening photograph for this chapter. An upward streamer is dangerous because the resulting ionization of molecules in the air suddenly frees a tremendous number of electrons from those molecules. Had the woman in Fig. 23-34 developed an upward streamer, the free electrons in the air would have moved to neutralize her (Fig. 23-35*b*), producing a large, perhaps fatal, charge flow through her body.

Let's model her body as a narrow vertical cylinder of height $L = 1.8$ m and radius $R = 0.10$ m (Fig. 23-35*c*). Assume that charge Q was uniformly distributed along the cylinder and that electrical breakdown would have occurred if the electric field magnitude along her body had exceeded the critical value $E_c = 2.4$ MN/C. What value of Q would have put the air along her body on the verge of breakdown?

Conceptualize/Classify: Because $R \ll L$, we can approximate the charge distribution as a long line of charge. Further, because we assume that the charge is uniformly distributed along this line, we can approximate the magnitude of the electric field along the side of her body with Eq. 23-19 ($E = \lambda/2\pi\varepsilon_0 r$).

Compute: Substituting the critical value E_c for E, the cylinder radius R for radial distance r, and the ratio Q/L for linear charge density λ, we have

$$E_c = \frac{Q/L}{2\pi\varepsilon_0 R},$$

or

$$Q = 2\pi\varepsilon_0 RLE_c.$$

Substituting given data then gives us

$$\begin{aligned} Q &= (2\pi)(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.10\ \text{m}) \\ &\quad \times (1.8\ \text{m})(2.4\times10^{6}\ \text{N/C}) \\ &= 2.402\times10^{-5}\ \text{C} \approx 24\ \mu\text{C}. \end{aligned} \qquad \text{(Answer)}$$

Conclude: This is the fourth danger of a lightning storm.

23-8 | Applying Gauss' Law: Planar Symmetry

Nonconducting Sheet

Figure 23-36 shows a portion of a thin, infinite, nonconducting sheet with a uniform (positive) surface charge density σ. A sheet of thin plastic wrap, uniformly charged on one side, can serve as a simple model. Let us find the electric field $\vec{E}$ a distance r in front of the sheet.

A useful Gaussian surface is a closed cylinder with end caps of area A, arranged to pierce the sheet perpendicularly as shown. From symmetry, $\vec{E}$ must be perpendicular to the sheet and hence to the end caps. Furthermore, since the charge is positive, $\vec{E}$ is directed *away* from the sheet, and thus the electric field lines pierce the two Gaussian end caps in an outward direction. Because the field lines do not pierce the curved surface, there is no flux through this portion of the Gaussian surface. Thus $\vec{E}\cdot d\vec{A}$ is simply $E\,dA$; then Gauss' law,

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\text{enc}},$$

becomes

$$\varepsilon_0(EA + EA) = \sigma A,$$

where σA is the charge enclosed by the Gaussian surface. This gives

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(sheet of charge)}. \qquad (23\text{-}20)$$

Since we are considering an infinite sheet with uniform charge density, this result holds for any point at a finite distance from the sheet. Equation 23-20 agrees with Eq. 22-38, which we found by integration of the electric field components produced by individual charges. (Look back to that time-consuming and challenging integration, and note how much more easily we obtain the result with Gauss' law. That is one reason for devoting a whole chapter to that law: for certain symmetric arrangements of charge, it is very much easier to use than integration of field components.)

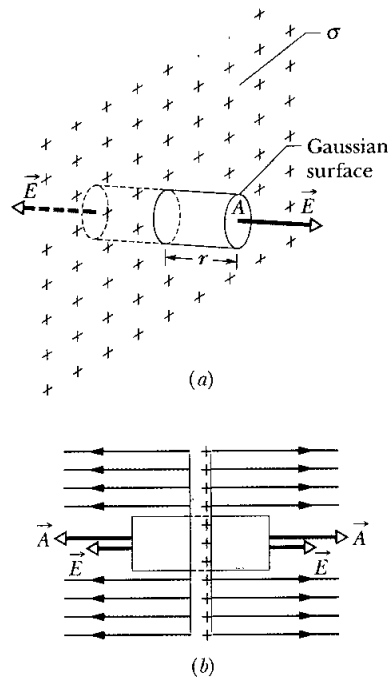


FIG. 23-36 (*a*) Perspective view and (*b*) side view of a portion of a very large, thin plastic sheet, uniformly charged on one side to surface charge density σ. A closed cylindrical Gaussian surface passes through the sheet and is perpendicular to it.

Sample Problem 23-10

Figure 23-37*a* shows portions of two large, parallel, nonconducting sheets, each with a fixed uniform charge on one side. The magnitudes of the surface charge densities are $\sigma_{(+)} = 6.8\ \mu C/m^2$ for the positively charged sheet and $\sigma_{(-)} = 4.3\ \mu C/m^2$ for the negatively charged sheet.

Find the electric field $\vec{E}$ (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

Conceptualize/Classify: With the charges fixed in place (they are on nonconductors), we can find the electric field of the sheets in Fig. 23-37*a* by (1) finding the field of each sheet as if that sheet were isolated and (2) algebraically adding the fields of the isolated sheets via the superposition principle. (We can add the fields algebraically because they are parallel to each other.)

Compute: At any point, the electric field $\vec{E}_{(+)}$ due to the positive sheet is directed *away* from the sheet and, from Eq. 23-20, has the magnitude

$$E_{(+)} = \frac{\sigma_{(+)}}{2\varepsilon_0} = \frac{6.8\times10^{-6}\ C/m^2}{(2)(8.85\times10^{-12}\ C^2/N\cdot m^2)} = 3.84\times10^5\ N/C.$$

Similarly, at any point, the electric field $\vec{E}_{(-)}$ due to the negative sheet is directed *toward* that sheet and has the magnitude

$$E_{(-)} = \frac{\sigma_{(-)}}{2\varepsilon_0} = \frac{4.3\times10^{-6}\ C/m^2}{(2)(8.85\times10^{-12}\ C^2/N\cdot m^2)} = 2.43\times10^5\ N/C.$$

Figure 23-37*b* shows the fields set up by the sheets to the left of the sheets (L), between them (B), and to their right (R).

The resultant fields in these three regions follow from the superposition principle. To the left, the field magnitude is

$$E_L = E_{(+)} - E_{(-)} = 3.84\times10^5\ N/C - 2.43\times10^5\ N/C = 1.4\times10^5\ N/C. \quad \text{(Answer)}$$

Because $E_{(+)}$ is larger than $E_{(-)}$, the net electric field $\vec{E}_L$ in this region is directed to the left, as Fig. 23-37*c* shows. To the right of the sheets, the electric field $\vec{E}_R$ has the same magnitude but is directed to the right, as Fig. 23-37*c* shows.

Between the sheets, the two fields add and we have

$$E_B = E_{(+)} + E_{(-)} = 3.84\times10^5\ N/C + 2.43\times10^5\ N/C = 6.3\times10^5\ N/C. \quad \text{(Answer)}$$

Conclude: The electric field $\vec{E}_B$ is directed to the right.

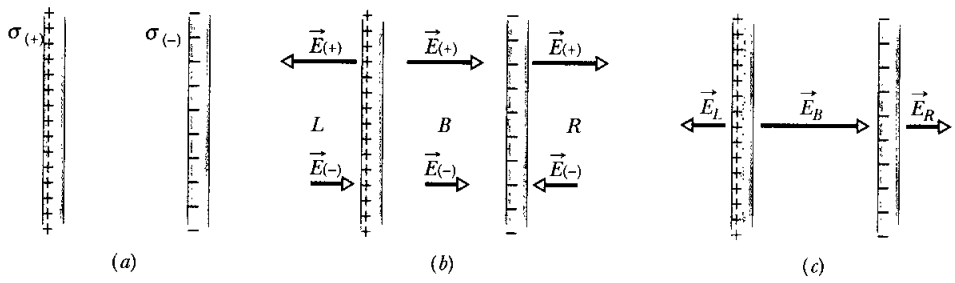
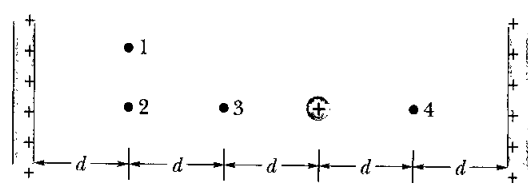


FIG. 23-37 (*a*) Two large, parallel sheets, uniformly charged on one side. (*b*) The individual electric fields resulting from the two charged sheets. (*c*) The net field due to both charged sheets, found by superposition.

CHECKPOINT 5 The figure shows two large, parallel, nonconducting sheets with identical (positive) uniform surface charge densities, and a sphere with a uniform (positive) volume charge density. Rank the four numbered points according to the magnitude of the net electric field there, greatest first.



REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: If there is no net charge inside a Gaussian surface, the net flux through the Gaussian surface is zero.

Statement 2: If there is no net charge inside a Gaussian surface, the electric field at every point of the Gaussian surface is zero.

2. Statement 1: We cannot find the electric field due to a finite line charge by using Gauss' law alone.

Statement 2: A finite line charge is not having any of the three symmetries.

3. Statement 1: The electric field inside a uniformly charged spherical shell due to itself is always zero.

Statement 2: At every point inside the shell, we can form a pair of charges on the surface of the shell which cancel the effect of each other.

4. Consider a case of finding electric field due to infinite uniform line charge.

Statement 1: While applying Gauss's law to this case, the axis of Gaussian cylinder should coincide with the line.

Statement 2: In this case, the Gaussian surface should be drawn in such a manner that it encloses the charge whose electric field is to be found.

MULTIPLE CHOICE QUESTIONS

sec. 23-3 Electric Flux

1. The flux of the electric field $\vec{E} = 24\hat{i} + 30\hat{j} + 28\hat{k}$ N/C through an area of 20 m^2 on the yz plane is

(a) 480 N · m^2/C

(b) 600 N · m^2/C

(c) 560 N · m^2/C

(d) 1640 N · m^2/C

2. A hemispherical surface (half of a spherical surface) of radius R is located in a uniform electric field E that is parallel to the axis of the hemisphere (Fig. 23-38). What is the magnitude of the electric flux through the hemisphere surface?

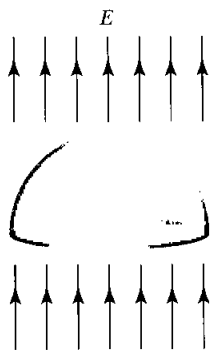


FIG. 23-38 Multiple choice question 2.

(a) 0

(b) $4\pi R^2 E/3$

(c) $2\pi R^2 E$

(d) $\pi R^2 E$

sec. 23-4 Statement of Gauss' Law

3. Choose the incorrect statement:

(a) Gauss' law can be derived from Coulomb's law

(b) Gauss' law states that the net number of lines crossing any closed surface in an outward direction is proportional to the net charge enclosed within the surface

(c) Gauss' law applies to a closed surface of any shape

(d) According to Gauss' law, if a closed surface encloses no charge, then the electric field must vanish everywhere on the surface

4. In Fig. 23-39, a full Gaussian surface encloses two of the four positively charged particles. Answer the following:

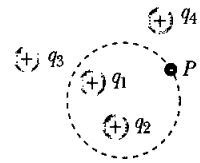


FIG. 23-39 Multiple choice question 4.

(i) Which of the four particles contribute to the electric field at point P on the surface?

(a) q_1

(b) q_2

(c) q_3

(d) q_4

(ii) Which of the four particles contribute to the net electric flux through the Gaussian surface?

(a) q_1

(b) q_2

(c) q_3

(d) q_4

5. Consider three different closed surfaces. Surface A is a sphere of radius R, surface B is a sphere of radius $2R$, and surface C is a cube with side length R. At the geometric center of each surface is a small ball carrying a positive charge $+Q$; each ball is completely enclosed by the surface. Assuming that there are no other charged objects inside any of the surfaces, rank the surfaces based on the field line flux through them, from largest to smallest.

(a) A > B > C

(b) A > C > B

(c) B > C > A

(d) All are equal

6. A charge Q is kept at the apex of the cone as shown in Fig. 23-40. Find the flux due to this charge through the curved face.



FIG. 23-40 Multiple choice question 6.

(a) $\frac{Q}{2\varepsilon_0}(1-\cos\theta)$

(b) $\frac{Q}{2\varepsilon_0}(1-\sin\theta)$

(c) Zero

(d) $\frac{Q}{2\varepsilon_0}$

7. A sphere of radius 2 m is kept in space such that its center is on y axis at $(0, a)$. A charge of 1 C is kept at $(1, 0)$. In which of following cases, flux through sphere is not zero.

(a) $a = 2$ m

(b) $a = -2$ m

(c) $a = 1$ m

(d) $a = -3$ m

8. A point particle with charge q is placed inside a cube but not at its center. The electric flux through any one side of the cube

(a) Is zero

(b) Is q/ε_0

(c) Is $q/6\varepsilon_0$

(d) Cannot be computed using Gauss' law

sec. 23-6 Applying Gauss' Law: Spherical Symmetry

9. In which of following cases, electric field is not constant

(a) Inside a uniformly charged spherical shell

(b) In a cavity inside a uniformly charged sphere

(c) In front of an infinite sheet of uniform surface charge density

(d) At a distance x from a point charge q

10. As shown in Fig. 23-41 a 1 μC charge is uniformly distributed on a spherical shell given by equation $x^2 + y^2 + z^2 = 25$. What will be intensity of electric field at a point $(1, 1, 2)$?

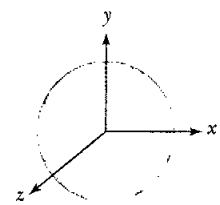


FIG. 23-41 Multiple choice question 10.

(a) 5 N/C

(b) 45 N/C

(c) 112.5 N/C

(d) Zero

sec. 23-7 Applying Gauss' Law: Cylindrical Symmetry

11. An electron is placed just in the middle between two long fixed line charges of charge density $+\lambda$ each. The wires are in the xy plane (Fig. 23-42). (Do not consider gravity.)

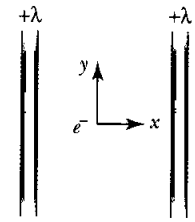


FIG. 23-42 Multiple choice question 11.

(a) The equilibrium of the electron will be unstable along x direction

(b) The equilibrium of the electron will be neutral along y direction

(c) The equilibrium of the electron will be stable along z direction

(d) The equilibrium of the electron will be stable along y direction

sec. 23-8 Applying Gauss' Law: Planar Symmetry

12. Figure 23-43 shows an edge-on view of three extremely large, parallel insulating sheets positively charged with uniform and identical surface charge densities. In which of the regions shown is the electric field a maximum?

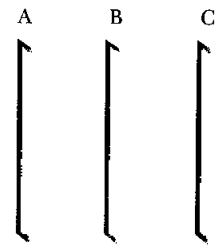


FIG. 23-43 Multiple choice question 12.

(a) B

(b) C

(c) A and D

(d) The electric field is the same in all four regions of space

PROBLEMS

sec. 23-3 Electric Flux

1. The square surface shown in Fig. 23-44 measures 5 mm on each side. It is immersed in a uniform electric field with magnitude E = 1800 N/C and with field lines at an angle of $\theta = 37°$ with a normal to the surface, as shown. Take that normal to be directed "outward," as though the surface were one face of a box. Calculate the electric flux through the surface.

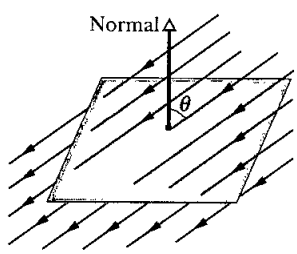


FIG. 23-44 Problem 1.

2. An electric field given by $\vec{E} = 4.0\hat{i} - 3.0(y^2 + 2.0)\hat{j}$ pierces a Gaussian cube of edge length 2.0 m and positioned as shown in Fig. 23-5. (The magnitude E is in newtons per coulomb and the position x is in meters.) What is the electric flux through the (a) top face, (b) bottom face, (c) left face, and (d) back face? (e) What is the net electric flux through the cube?

sec. 23-4 Statement of Gauss' Law

3. At each point on the surface of the cube shown in Fig. 23-45, the electric field is parallel to the z axis. The length of each edge of the cube is 3.0 m. On the top face of the cube $\vec{E} = -34\hat{k}$ N/C and on the bottom face $\vec{E} = +20\hat{k}$ N/C. Determine the net charge contained within the cube.

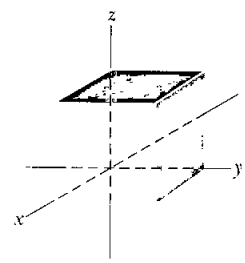


FIG. 23-45 Problems 3 and 8.

4. In Fig. 23-46, a butterfly net is in a uniform electric field of magnitude $E = 3.0$ m N/C. The rim, a circle of radius $a = 11$ cm, is aligned perpendicular to the field. The net contains no net charge. Find the electric flux through the netting.

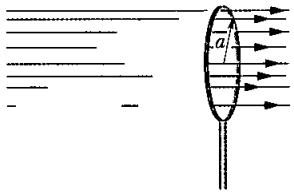


FIG. 23-46 Problem 4.

5. In Fig. 23-47, a proton is a distance $d/2$ directly above the center of a square of side d. What is the magnitude of the electric flux through the square? (*Hint:* Think of the square as one face of a cube with edge d.)

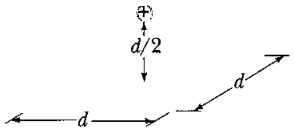


FIG. 23-47 Problem 5.

6. It is found experimentally that the electric field in a certain region of Earth's atmosphere is directed vertically down. At an altitude of 300 m the field has magnitude 60.0 N/C; at an altitude of 200 m, the magnitude is 100 N/C. Find the net amount of charge contained in a cube 100 m on edge, with horizontal faces at altitudes of 200 and 300 m.

7. When a shower is turned on in a closed bathroom, the splashing of the water on the bare tub can fill the room's air with negatively charged ions and produce an electric field in the air as great as 1000 N/C. Consider a bathroom with dimensions 2.5 m × 3.0 m × 2.0 m. Along the ceiling, floor, and four walls, approximate the electric field in the air as being directed perpendicular to the surface and as having a uniform magnitude of 600 N/C. Also, treat those surfaces as forming a closed Gaussian surface around the room's air. What are (a) the volume charge density ρ and (b) the number of excess elementary charges e per cubic meter in the room's air?

8. Figure 23-45 shows a Gaussian surface in the shape of a cube with edge length 1.40 m. What are (a) the net flux Φ through the surface and (b) the net charge q_{enc} enclosed by the surface if $\vec{E} = (3.00y\hat{j})$ N/C, with y in meters? What are (c) Φ and (d) q_{enc} if $\vec{E} = [-4.00\,\hat{i} + (6.00 + 3.00y)\,\hat{j}]$.

9. The net electric flux through each face of a die (singular of dice) has a magnitude in units of 10^3 N · m²/C that is exactly equal to the number of spots N on the face (1 through 6). The flux is inward for N odd and outward for N even. What is the net charge inside the die?

10. *Flux and nonconducting shells.* A charged particle is suspended at the center of two concentric spherical shells that are very thin and made of nonconducting material. Figure 23-48*a* shows a cross section. Figure 23-48*b* gives the net flux Φ through a Gaussian sphere centered on the particle, as a function of the radius r of the sphere. The scale of the vertical axis is set by $\Phi_s = 5.0 \times 10^5$ N · m²/C. (a) What is the charge of the central particle? What are the net charges of (b) shell A and (c) shell B?

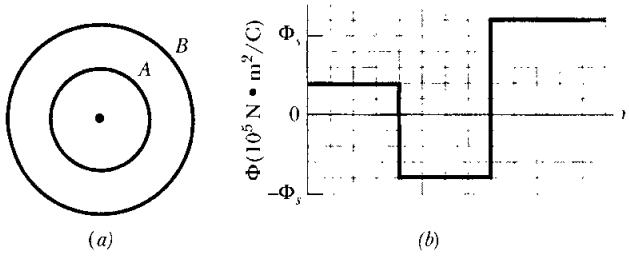


FIG. 23-48 Problem 10.

11. A particle of charge $+q$ is placed at one corner of a Gaussian cube. What multiple of q/ε_0 gives the flux through (a) each cube face forming that corner and (b) each of the other cube faces?

sec. 23-6 Applying Gauss' Law: Spherical Symmetry

12. A point charge causes an electric flux of −750 N · m²/C to pass through a spherical Gaussian surface of 10.0 cm radius centered on the charge. (a) If the radius of the Gaussian surface were doubled, how much flux would pass through the surface? (b) What is the value of the point charge?

13. Two charged concentric spherical shells have radii 10.0 cm and 15.0 cm. The charge on the inner shell is 4.00×10^{-8} C, and that on the outer shell is 2.00×10^{-8} C. Find the electric field (a) at $r = 12.0$ cm and (b) at $r = 20.0$ cm.

14. Figure 23-49 shows two nonconducting spherical shells fixed in place. Shell 1 has uniform surface charge density +6.0 μC/m² on its outer surface and radius 3.0 cm; shell 2 has uniform surface charge density +4.0 μC/m² on its outer surface and radius 2.0 cm; the shell centers are separated by L = 10 cm. In unit-vector notation, what is the net electric field at $x = 2.0$ cm?

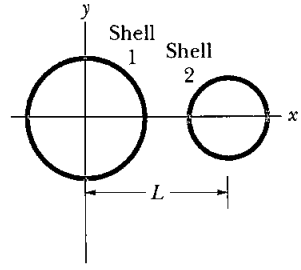


FIG. 23-49 Problem 14.

15. Figure 23-50 shows two nonconducting spherical shells fixed in place on an x axis. Shell 1 has uniform surface charge density +4.0 μC/m² on its outer surface and radius 0.50 cm, and shell 2 has uniform surface charge density −2.0 μC/m² on its outer surface and radius 2.0 cm; the centers are separated by L = 6.0 cm. Other than at $x = \infty$, where on the x axis is the net electric field equal to zero?

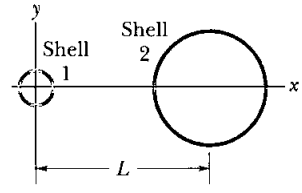


FIG. 23-50 Problem 15.

16. In Fig. 23-51, a solid sphere of radius $a = 2.00$ cm is concentric with a spherical conducting shell of inner radius $b = 2.00a$ and outer radius $c = 2.40a$. The sphere has a net uniform charge $q_1 = +5.00$ fC; the shell has a net charge $q_2 = -q_1$. What is the magnitude of the electric field at radial distances (a) $r = 0$, (b) $r = a/2.00$, (c) $r = a$, (d) $r = 1.50a$, (e) $r = 2.30a$, and (f) $r = 3.50a$? What is the net charge on the (g) inner and (h) outer surface of the shell?

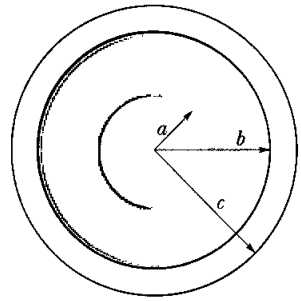


FIG. 23-51 Problem 16.

17. A charged particle is held at the center of a spherical shell. Figure 23-52 gives the magnitude E of the electric field versus radial distance r. The scale of the vertical axis is set by $E_s = 10.0 \times 10^7$ N/C. Approximately, what is the net charge on the shell?

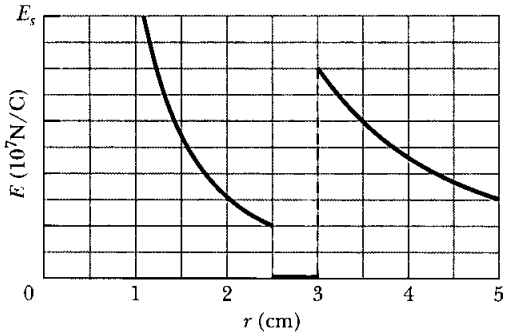


FIG. 23-52 Problem 17.

18. Figure 23-53 shows, in cross section, two solid spheres with uniformly distributed charge throughout their volumes. Each has radius R. Point P lies on a line connecting the centers of the spheres, at radial distance $R/2.00$ from the center of sphere 1. If the net electric field at point P is zero, what is the ratio q_2/q_1 of the total charge q_2 in sphere 2 to the total charge q_1 in sphere 1?

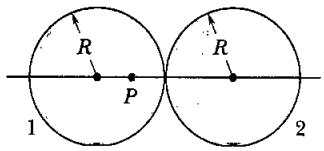


FIG. 23-53 Problem 18.

19. A solid nonconducting sphere of radius $R = 5.60$ cm has a nonuniform charge distribution of volume charge density $\rho = (14.1 \text{ pC/m}^3)r/R$, where r is radial distance from the sphere's center. (a) What is the sphere's total charge? What is the magnitude E of the electric field at (b) $r = 0$, (c) $r = R/2.00$, and (d) $r = R$? (e) Sketch a graph of E versus r.

sec. 23-7 Applying Gauss' Law: Cylindrical Symmetry

20. Figure 23-54 shows a section of a long, thin-walled metal tube of radius $R = 3.00$ cm, with a charge per unit length $\lambda = 2.00 \times 10^{-8}$ C/m. What is the magnitude E of the electric field at radial distance (a) $r = R/2.00$ and (b) $r = 2.00R$? (c) Graph E versus r for the range $r = 0$ to $2.00R$.

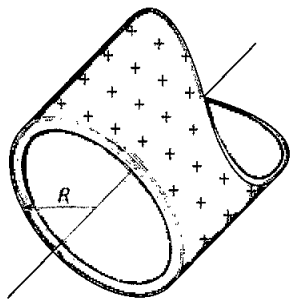


FIG. 23-54 Problem 20.

21. An infinite line of charge produces a field of magnitude 4.5×10^4 N/C at a distance of 2.0 m. Calculate the linear charge density.

22. In Fig. 23-55, short sections of two very long parallel lines of charge are shown, fixed in place, separated by $L = 8.0$ cm. The uniform linear charge densities are +6.0 μC/m for line 1 and −2.0 μC/m for line 2. Where along the x axis shown is the net electric field from the two lines zero?

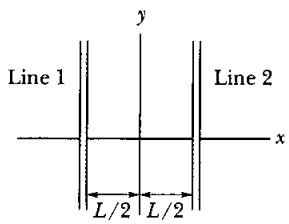


FIG. 23-55 Problem 22.

23. Figure 23-56 is a section of a conducting rod of radius $R_1 = 1.30$ mm and length $L = 11.00$ m inside a thin-walled coaxial conducting cylindrical shell of radius $R_2 = 10.0R_1$ and the (same) length L. The net charge on the rod is $Q_1 = +3.40 \times 10^{-12}$ C; that on the shell is $Q_2 = -2.00Q_1$. What are the (a) magnitude E and (b) direction (radially inward or outward) of the electric field at radial distance $r = 2.00R_2$? What are (c) E and (d) the direction at $r = 5.00R_1$? What is the charge on the (e) interior and (f) exterior surface of the shell?

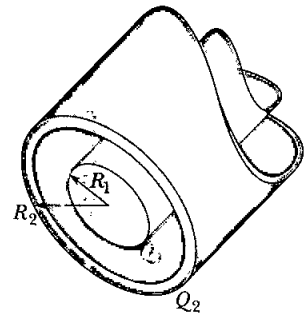


FIG. 23-56 Problem 23.

sec. 23-8 Applying Gauss' Law: Planar Symmetry

24. Figure 23-57 shows cross sections through two large, parallel, nonconducting sheets with identical distributions of positive charge with surface charge density $\sigma = 1.77 \times 10^{-22}$ C/m^2. In unit-vector notation, what is $\vec{E}$ at points (a) above the sheets, (b) between them, and (c) below them?

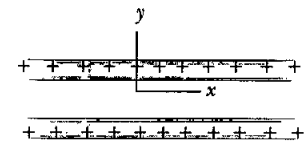


FIG. 23-57 Problem 24.

25. Figure 23-58*a* shows three plastic sheets that are large, parallel, and uniformly charged. Figure 23-58*b* gives the component of the net electric field along an x axis through the sheets. The scale of the vertical axis is set by $E_s = 6.0 \times 10^5$ N/C. What is the ratio of the charge density on sheet 3 to that on sheet 2?

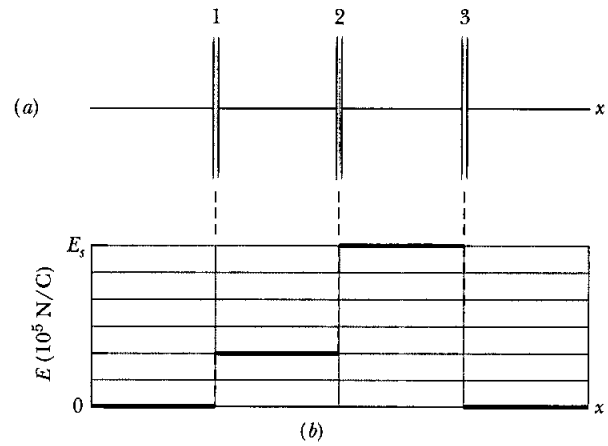


FIG. 23-58 Problem 25.

26. In Fig. 23-59, a small, nonconducting ball of mass $m = 1.0$ mg and charge $q = 2.0 \times 10^{-8}$ C (distributed uniformly through its volume) hangs from an insulating thread that

makes an angle $\theta = 30°$ with a vertical, uniformly charged nonconducting sheet (shown in cross section). Considering the gravitational force on the ball and assuming the sheet extends far vertically and into and out of the page, calculate the surface charge density σ of the sheet.

27. Figure 23-60 shows a cross section through a very large nonconducting slab of thickness $d = 9.40$ mm and uniform volume charge density $\rho = 5.80$ fC/m^3. The origin of an x axis is at the slab's center. What is the magnitude of the slab's electric field at an x coordinate of (a) 0, (b) 2.00 mm, (c) 4.70 mm, and (d) 26.0 mm?

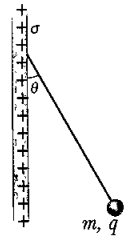


FIG. 23-59 Problem 26.

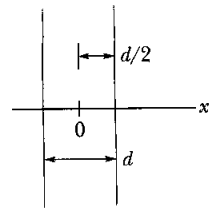


FIG. 23-60 Problem 27.

ADDITIONAL PROBLEMS

28. An infinitely long, straight string has a uniform linear charge density of 1 C/m. A sphere of radius $R = 1$ m has its center at a point on the string. What is the electric flux through the sphere? What is the flux through the sphere if the center of the sphere is at a distance of $R/2$ from the string?

29. The box-like Gaussian surface of Fig. 23-61 encloses a net charge of $+24.0\varepsilon_0$ C and lies in an electric field given by $\vec{E} = [(10.0 + 2.00x)\hat{i} - 3.00\hat{j} + bz\hat{k}]$ N/C, with x and z in meters and b a constant. The bottom face is in the xz plane; the top face is in the horizontal plane passing through $y_2 = 1.00$m. For $x_1 = 1.00$ m, $x_2 = 4.00$ m, $z_1 = 1.00$ m, and $z_2 = 3.00$ m, what is b?

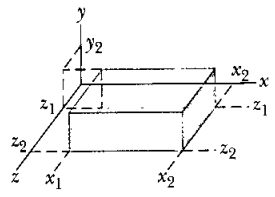


FIG. 23-61 Problem 29.

30. A long, nonconducting, solid cylinder of radius 4.0 cm has a nonuniform volume charge density ρ that is a function of radial distance r from the cylinder axis: $\rho = Ar^2$. For $A = 2.5$ μC/m^5, what is the magnitude of the electric field at (a) $r = 3.0$ cm and (b) $r = 5.0$ cm?

31. In Fig. 23-62, a small circular hole of radius $R = 1.80$ cm has been cut in the middle of an infinite, flat, nonconducting surface that has uniform charge density $\sigma = 4.50$ pC/m^2. A z axis, with its origin at the hole's center, is perpendicular to the surface. In unit-vector notation, what is the electric field at point P at $z = 2.56$ cm? (*Hint:* See Eq. 22-37 and use superposition.)

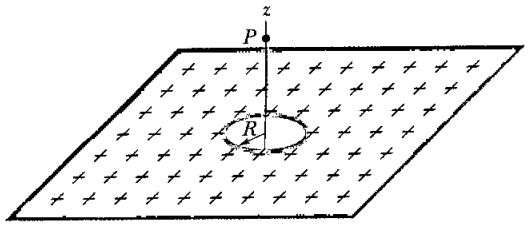


FIG. 23-62 Problem 31.

32. Figure 23-63 shows a very large nonconducting sheet that has a uniform surface charge density of $\sigma = -2.00$ μC/m^2; it also shows a particle of charge $Q = 6.00$ μC, at distance d from the sheet. Both are fixed in place. If $d = 0.200$ m, at what (a) positive and (b) negative coordinate on the x axis (other than infinity) is the net electric field $\vec{E}_{net}$ of the sheet and particle zero? (c) If $d = 0.800$ m, at what coordinate on the x axis is $\vec{E}_{net} = 0$?

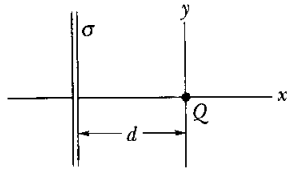


FIG. 23-63 Problem 32.

33. In Fig. 23-64, a nonconducting spherical shell of inner radius $a = 2.00$ cm and outer radius $b = 2.40$ cm has (within its thickness) a positive volume charge density $\rho = A/r$, where A is a constant and r is the distance from the center of the shell. In addition, a small ball of charge $q = 45.0$ fC is located at that center. What value should A have if the electric field in the shell ($a \le r \le b$) is to be uniform?

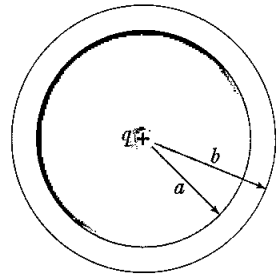


FIG. 23-64 Problem 33.

34. A charge distribution that is spherically symmetric but not uniform radially produces an electric field of magnitude $E = Kr^4$, directed radially outward from the center of the sphere. Here r is the radial distance from that center, and K is a constant. What is the volume density ρ of the charge distribution?

35. Charge Q is uniformly distributed in a sphere of radius R. (a) What fraction of the charge is contained within radius $r = R/2.00$? (b) What is the ratio of the electric field magnitude at $r = R/2.00$ to that on the surface of the sphere?

36. A uniform surface charge of density 8.0 nC/m^2 is distributed over the entire xy plane. What is the electric flux through a spherical Gaussian surface centered on the origin and having a radius of 5.0 cm?

37. Charge is distributed uniformly throughout the volume of an infinitely long solid cylinder of radius R. (a) Show that, at a distance $r \le R$ from the cylinder axis,

$$E = \frac{\rho r}{2\varepsilon_0},$$

where ρ is the volume charge density. (b) Write an expression for E when $r \ge R$.

38. A spherical ball of charged particles has a uniform charge density. In terms of the ball's radius R, at what radial distances (a) inside and (b) outside the ball is the magnitude of the ball's electric field equal to $\frac{1}{4}$ of the maximum magnitude of that field?

39. Charge of uniform surface density 8.00 nC/m^2 is distributed over an entire xy plane; charge of uniform surface density 3.00 nC/m^2 is distributed over the parallel plane defined by $z = 2.00$ m. Determine the magnitude of the electric field at any point having a z coordinate of (a) 1.00 m and (b) 3.00 m.

ANSWERS

Checkpoints

1. (a) $+EA$; (b) $-EA$; (c) 0; (d) 0
2. (a) 2; (b) 3; (c) 1
3. (a) equal; (b) equal; (c) equal
4. (a) zero; (b) zero
5. 3 and 4 tie, then 2,1

Reasoning Type Questions

1. (c)
2. (a)
3. (a)
4. (b)

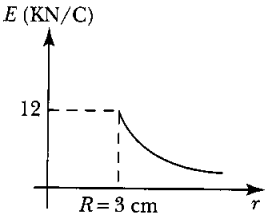
Multiple Choice Questions

1. (a)
2. (d)
3. (d)
4. (i) (a), (b), (c), (d); (ii) (a), (b)
5. (d)
6. (c)
7. (c)
8. (d)
9. (d)
10. (d)
11. (a), (b), (c)
12. (c)

Problems

1. $-0.027\ \text{N}\cdot\text{m}^2/\text{C}$
2. (a) $-72\ \text{N}\cdot\text{m}^2/\text{C}$; (b) $+24\ \text{N}\cdot\text{m}^2/\text{C}$; (c) $-16\ \text{N}\cdot\text{m}^2/\text{C}$; (d) 0; (e) $-48\ \text{N}\cdot\text{m}^2/\text{C}$
3. -4.3×10^{-9} C
4. $-1.1\times10^{-4}\ \text{N}\cdot\text{m}^2/\text{C}$
5. $3.01\ \text{nN}\cdot\text{m}^2/\text{C}$
6. 3.54 μC
7. (a) $1.3\times10^{-8}\ \text{C/m}^3$; (b) 8.2×10^{10} excess electrons per cubic meter
8. (a) $8.23\ \text{N}\cdot\text{m}^2/\text{C}$; (b) 72.9 pC; (c) $8.23\ \text{N}\cdot\text{m}^2/\text{C}$; (d) 72.9 pC
9. 2.66×10^{-8} C
10. (a) 1.8×10^{-6} C ; (b) -5.3×10^{-6} C; (c) +8.9 μC
11. (a) 0; (b) 0.0417
12. (a) $-750\ \text{N}\cdot\text{m}^2/\text{C}$; (b) -6.64×10^{-9} C
13. (a) 2.50×10^4 N/C; (b) 1.35×10^4 N/C
14. $-2.8\times10^4\ \text{N/C}\,\hat{\text{i}}$
15. −3.3 cm
16. (a) 0; (b) 56.2 mN/C; (c) 112 mN/C; (d) 49.9 mN/C; (e) 0; (f) 0; (g) −5.00 fC; (h) 0
17. 6.6×10^{-6} C
18. 1.125
19. (a) 7.78 fC; (b) 0; (c) 5.58 mN/C; (d) 22.3 mN/C
20. (a) 0; (b) 5.99×10^3 N/C;

(c)



21. 5.0 μC/m
22. 8.0 cm
23. (a) 0.214 N/C; (b) inward; (c) 0.855 N/C; (d) outward; (e) -3.40×10^{-12} C; (f) -3.40×10^{-12} C
24. (a) $(2.00\times10^{-11}\ \text{N/C})\hat{\text{j}}$; (b) 0; (c) $-(2.00\times10^{-11}\ \text{N/C})\hat{\text{j}}$
25. −1.5
26. $5.0\ \text{nC/m}^2$
27. (a) 0; (b) 1.31 μN/C; (c) 3.08 μN/C; (d) 3.08 μN/C

Additional Problems

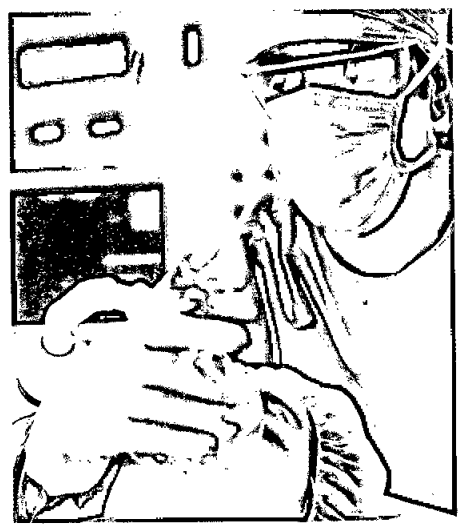
28. $\frac{2}{\varepsilon_0}, \frac{\sqrt{3}}{\varepsilon_0}\omega$
29. 2.00 N/C m
30. (a) 1.9 N/C; (b) 3.6 N/C
31. $(0.208\ \text{N/C})\,\hat{\text{k}}$
32. (a) $x = 0.691$ m; (b) $x = -0.691$ m; (c) $x = +0.691$ m
33. $1.79\times10^{-11}\ \text{C/m}^2$
34. $6K\varepsilon_0 r^3$
35. (a) 0.125; (b) 0.500
36. $7.1\ \text{N}\cdot\text{m}^2/\text{C}$
37. (a) $\frac{\rho r}{2\varepsilon_0}$; (b) $E_{\text{ext}} = \frac{R^2\rho}{2\varepsilon_0 r}$
38. (a) $0.25R$; (b) $2.0R$
39. (a) 2.82×10^2 N/C ; (b) 6.21×10^2 N/C

Finding the Electric Potential

24



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If a person peels off a jacket or sweater while working at a computer, the computer can be destroyed. If a child slides down a plastic slide and then reaches for another person, the child can be in for a painful surprise. If an anesthesiologist does not wear the right type of shoes during surgery, the patient can be fatally injured. If a customer at a self-serve gas station slides back into the car while the tank is being filled, a fire can erupt at the pump nozzle when the customer returns to pull the nozzle from the car.

What is the danger in these situations?

The answer is in this chapter.

24-1 WHAT IS PHYSICS?

One goal of physics is to identify basic forces in our world, such as the electric force we discussed in Chapter 22. A related goal is to determine whether a force is conservative—that is, whether a potential energy can be associated with it. The motivation for associating a potential energy with a force is that we can then apply the principle of the conservation of mechanical energy to closed systems involving the force. This extremely powerful principle allows us to calculate the results of experiments for which force calculations alone would be very difficult. Experimentally, physicists and engineers discovered that the electrostatic force is conservative and thus has an associated electric potential energy. In this chapter we first define this type of potential energy and then put it to use.

24-2 | Electric Potential Energy

Any charge in an electric field experiences the action of a force. Consequently, a certain work is done when a charge moves in the field. This work depends on the field strength at different points and on the charge displacement. However, *if a charge describes a closed curve, that is, returns to the original position, the work done by the field is equal to zero* irrespective of the field configuration and the shape of the path along which the charge has moved.

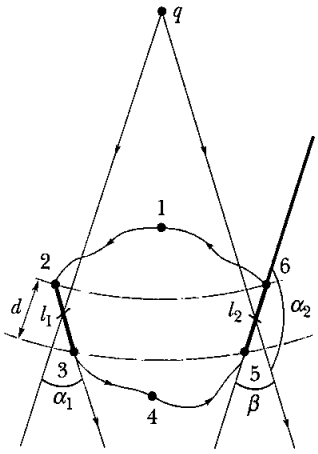


FIG. 24-1 To prove of the independence of the work done by electrostatic forces on the shape of path.

When a charge moves in this field, the work done by the forces of the field is positive on the segments of the path where the direction of the displacement forms an acute angle with direction of force (like 2–3 in Fig. 24-1). On the contrary, the work done by the force of electric field is negative in the region where the displacement forms an obtuse angle with the direction of force (like 5–6 in Fig. 24-1). When the charge, having traversed a closed path, returns to the initial point, the total work of electric forces over this path, which is equal to the algebraic sum of positive works done on some segments and negative works done on other segments, is zero.

A strict mathematical proof of the conservation nature of electric field is rather cumbersome in the general case, and for this reason we shall confine ourselves to the proof for a simple case of the field produced by a single point charge.

Let us suppose that a charge q_0 moves an arbitrary closed trajectory 1–2–3–4–5–6–1 (Fig. 24-1) in the electric field of a fixed point charge q and returns to the initial point 1 after following the curve. To calculate the work done in this case, we mentally draw a number of spheres with the center at the charge q, which divide the entire path of the charge q_0 into small segments. Let us consider two segments l_1 and l_2 contained between the same sphere (between points 2 and 3 and 5 and 6). If the segments l_1 and l_2 are small enough, we can assume that the force acting on the charge q_0 is constant at all points of these segments. As both segments are at equal distance from the charges, the electric field over the two segments are equal in magnitude, but have different directions, which form different angle α_1 and α_2 with the direction of displacement.

Finally, if l_1 and l_2 are sufficiently small, they can be regarded as segments of straight lines. Therefore, the work A_{23} done by electric forces over the path 2–3 is equal to the products of the force by the displacement and the cosine of the angle between the force and the displacement

$$A_{23} = Fl_1 \cos\alpha_1.$$

Similarly, the work A_{56} done over the segments 5–6 is

$$A_{56} = Fl_2 \cos\alpha_2.$$

However, $\cos\alpha_2 = \cos(180° - \beta) = -\cos\beta$, so that $A_{56} = Fl_2 \cos\beta$. Besides, it can be seen from Fig. 24-1 that

$$l_1 \cos\alpha_1 = l_2 \cos\beta = d,$$

where d is the separation between the spheres containing the segments l_1 and l_2. Therefore, we find that

$$A_{23} = -A_{56},$$

that is, the algebraic sum of the works over the segments of the path contained between the spheres is zero. Therefore, the total work done in circumventing the closed path, which is equal to the sum of the works on individual segments, will also be equal to zero.

We obtained this result for the electric field of a single point charge. It will be valid for any electrostatic field since the fields created by any charge distribution can be reduced to the field of a system of point charges.

Thus, *the work done in an electric field for displacing a charge over a closed path is always equal to zero.* When this condition is obeyed, the electric field can be said to be conservative in nature. So we can assign an **electric potential energy** U to the system. If the system changes its configuration from an initial state i to a different final state f, the electrostatic force does work W on the particles. From Eq. 8-42, we then know that the resulting change ΔU in the potential energy of the system is

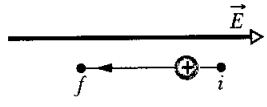
$$\Delta U = U_f - U_i = -W. \tag{24-1}$$

As with other conservative forces, the work done by the electrostatic force is *path independent.* Suppose a charged particle within the system moves from point i to point f while an electrostatic force between it and the rest of the system acts on it. Provided the rest of the system does not change, the work W done by the force on the particle is the same for *all* paths between points i and f.

For convenience, we usually take the *reference configuration* of a system of charged particles to be that in which the particles are all infinitely separated from one another. Also, we usually set the corresponding *reference potential energy* to be zero. Suppose that several charged particles come together from initially infinite separations (state i) to form a system of neighboring particles (state f). Let the initial potential energy U_i be zero, and let W_∞ represent the work done by the electrostatic forces between the particles during the move in from infinity. Then from Eq. 24-1, the final potential energy U of the system is

$$U = -W_\infty. \tag{24-2}$$

CHECKPOINT 1 In the figure, a proton moves from point i to point f in a uniform electric field directed as shown. (a) Does the electric field do positive or negative work on the proton? (b) Does the electric potential energy of the proton increase or decrease?



PROBLEM-SOLVING TACTICS

Tactic 1: *Electric Potential Energy; Work Done by a Field* An electric potential energy is associated with a system of particles as a whole. However, you will see statements (starting with Sample Problem 24-1) that associate it with only one particle within a system. For example, you might read, "An electron in an electric field has a potential energy of 10^{-7} J." Such statements are often acceptable, but you should always keep in mind that the potential energy is actually associated *with a system*—here, the electron plus the charged particles that set up the electric field. Also keep in mind that it makes sense to assign a particular potential energy value, such as 10^{-7} J here, to a particle or even a system *only* if the reference potential energy value is known.

If the potential energy is associated with only one particle within a system, you may read that work is done on the particle *by the electric field.* This means that work is done on the particle by the force due to the charges that set up the field.

Sample Problem 24-1

Electrons are continually being knocked out of air molecules in the atmosphere by cosmic-ray particles coming in from space. Once released, each electron experiences an electrostatic force $\vec{F}$ due to the electric field $\vec{E}$ that is produced in the atmosphere by charged particles already on Earth. Near Earth's surface the electric field has the magnitude $E = 150$ N/C and is directed downward. What is the change ΔU in the electric potential energy of a released electron when the electrostatic force causes it to move vertically upward through a distance $d = 520$ m (Fig. 24-2)?

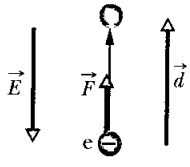


FIG. 24-2 An electron in the atmosphere is moved upward through displacement $\vec{d}$ by an electrostatic force $\vec{F}$ due to an electric field $\vec{E}$.

Conceptualize/Classify: (1) The change ΔU in the electric potential energy of the electron is related to the work W done on the electron by the electric field. Equation 24-1 ($\Delta U = -W$) gives the relation. (2) The work done by a constant force $\vec{F}$ on a particle undergoing a displacement $\vec{d}$ is

$$W = \vec{F} \cdot \vec{d}. \qquad (24\text{-}3)$$

(3) The electrostatic force and the electric field are related by $\vec{F} = q\vec{E}$, where here q is the charge of an electron ($= -1.6 \times 10^{-19}$ C).

Compute: Substituting for $\vec{F}$ in Eq. 24-3 and taking the dot product yield

$$W = q\vec{E} \cdot \vec{d} = qEd \cos\theta, \qquad (24\text{-}4)$$

where θ is the angle between the directions of $\vec{E}$ and $\vec{d}$. The field $\vec{E}$ is directed downward and the displacement $\vec{d}$ is directed upward; so $\theta = 180°$. Substituting this and other data into Eq. 24-4, we find

$$W = (-1.6 \times 10^{-19}\text{C})(150\text{ N/C})(520\text{ m})\cos 180°$$
$$= 1.2 \times 10^{-14}\text{ J}.$$

Equation 24-1 then yields

$$\Delta U = -W = -1.2 \times 10^{-14}\text{ J}. \qquad \text{(Answer)}$$

Conclude: This result tells us that during the 520 m ascent, the electric potential energy of the electron *decreases* by 1.2×10^{-14} J.

Suppose you push two bodies together that have charges of the same electrical sign. The work that you must do is stored as electric potential energy in the two-body system (provided the kinetic energy of the bodies does not change). If you later release the charges, you can recover this stored energy, in whole or in part, as kinetic energy of the charged bodies as they rush away from each other.

We define the electric potential energy *of a system of point charges*, held in fixed positions by forces not specified, as follows:

The electric potential energy of a system of fixed point charges is equal to the work that must be done by an external agent to assemble the system, bringing each charge in from reference configuration.

We assume that the charges are stationary both in their initial reference configuration and in their final assembled configuration. Figure 24-3 shows two point charges q_1 and q_2, separated by a distance r. To find the electric potential energy of this two-charge system, we must mentally build the system, starting with both charges infinitely far away and at rest. When we bring q_1 in from infinity and put it in place, we do no work because no electrostatic force acts on q_1. However, when we next bring q_2 in from infinity and put it in place, we must do work because q_1 exerts an electrostatic force on q_2 during the move.

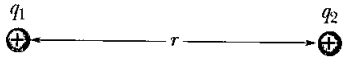


FIG. 24-3 Two charges held a fixed distance r apart.

As before, the work done by the electric field on the charge depends only on their configuration, it does not depend on the path taken in bringing the second charge from infinity to that point. It is emphasized here that the charge number 1 has been fixed and charge 2 has been brought from infinity. It would not make any difference if it is done the other way around. This potential energy is a property of the two charges. It does not belong to a single charge. Electrostatic potential energy is the energy of a system of two or more charges due to their electrostatic interaction. As the force exerted by the charge q_1 is variable, we need to integrate the expression for force.

$$W_{\text{ext}} = \int \vec{F} \cdot d\vec{x}.$$

If we bring the charge slowly from infinity, the force required to be exerted is the force required to balance the force due to electric field.

$$F_{ext} = -qE.$$

By definition, change in potential energy

$$\Delta U = W_{ext}.$$

So $$\Delta U = \int \vec{F} \cdot d\vec{x} = -q \int \vec{E} \cdot d\vec{x}.$$

For a two charge system,

$$\Delta U = -\int \frac{kq_1q_2}{x^2} dx.$$

We assume as stated earlier the reference configuration to be one where charges are at infinite separation from each other. Since in any situation, we are only concerned with the change in the potential energy, we could have taken any other point as of zero potential energy and it would not have made a difference to calculations.

$$U_f - U_i = \frac{kq_1q_2}{x}\Big|_{\infty}^{r}.$$

Taking U_i to be zero, we get

$$U_f = \frac{kq_1q_2}{r}. \tag{24-5}$$

To use this equation, we should remember that charges should be substituted with their sign. But substitute only the magnitude of the distance r between them. Thus, the potential energy is a scalar; however, it may be positive as well as negative depending on the sign of the charges.

What is the use of this new concept? After all we already have an expression for the electric field and from that force can be calculated. From that, we could have analyzed any situation. But actually, to find the new velocity when a charged body moves from one point to another becomes a formidable task in an electric field. In general, the electric field is not constant and so acceleration is not constant. Hence, we cannot apply the equations of motion for finding the velocity, etc.

But if we use the concept of potential energy and that the electric field is a conservative force, we can apply the law of conservation of mechanical energy:

$$U_i + K_i = U_f + K_f. \tag{24-6}$$

Sample Problem 24-2

An alpha particle (two protons and two neutrons) moves into a stationary gold atom (79 protons, and 118 neutrons), passing through the electron region that surrounds the gold nucleus like a shell and headed directly toward the nucleus (Fig. 24-4).The alpha particle slows until it momentarily stops when its center is at radial distance r = 9.23 fm from the nuclear center. Then it moves back along its incoming path. (Because the gold nucleus is much more massive than the alpha particle, we can assume the gold nucleus does not move.) What was the kinetic energy K_i of the alpha particle when

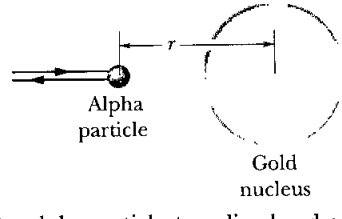


FIG. 24-4 An alpha particle, traveling head-on toward the center of a gold nucleus, comes to a momentary stop (at which time all its kinetic energy has been transferred to electric potential energy) and then reverses its path.

it was initially far away (hence external to the gold atom)? Assume that the only force acting between the alpha particle and the gold nucleus is the (electrostatic) Coulomb force.

Conceptualize: When the alpha particle is outside the atom, the system's initial electric potential energy U_i is zero because the atom has an equal number of electrons and protons, which produce a *net* electric field of zero.

However, once the alpha particle passes through the electron region surrounding the nucleus on its way to the nucleus, the electric field due to the electrons goes to zero. The reason is that the electrons act like a closed spherical shell of uniform negative charge and, as discussed in Section 23-6, such a shell produces zero electric field in the space it encloses. The alpha particle still experiences the electric field of the protons in the nucleus, which produces a repulsive force on the protons within the alpha particle.

As the incoming alpha particle is slowed by this repulsive force, its kinetic energy is transferred to electric potential energy of the system. The transfer is complete when the alpha particle momentarily stops and the kinetic energy is $K_f = 0$.

Classify: During the entire process, the mechanical energy of the *alpha particle and gold atom* system is conserved.

Compute: The principle of conservation of mechanical energy tells us that

$$K_i + U_i = K_f + U_f. \tag{24-7}$$

We know two values: $U_i = 0$ and $K_f = 0$. We also know that the potential energy U_f at the stopping point is given by the right side of Eq. 24-7, with $q_1 = 2e$, $q_2 = 79e$ (in which e is the elementary charge, 1.60×10^{-19} C), and $r = 9.23$ fm. Thus, we can rewrite Eq. 24-7 as

$$K_i = \frac{1}{4\pi\varepsilon_0}\frac{(2e)(79e)}{9.23 \text{ fm}}$$

$$= \frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)(158)(1.60 \times 10^{-19} \text{ C})^2}{9.23 \times 10^{-15} \text{ m}}$$

$$= 3.94 \times 10^{-12} \text{ J} = 24.6 \text{ MeV}. \qquad \text{(Answer)}$$

Conclude: As x increases, *the potential energy* will decrease. Thus, we can conclude that if a positive charge is released, it will move in the direction of the electric field. This also proves the fact that left to it, a system always tries to minimize its potential energy.

Apart from this law, we may also be required to use the law of conservation of angular momentum or the law of conservation of momentum to find the velocities.

Sample Problem 24-3

A block of mass 1 kg and charge 1 μC is directly projected from infinity toward another stationary block of mass 2 kg and same charge as the first. The initial velocity of projection is 10 m/s. Find the distance of closest separation and their velocities at the closest separation (Fig. 24-5).

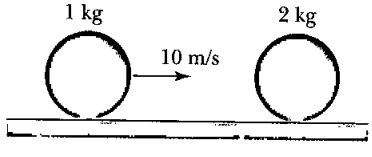


FIG. 24-5 A ball of mass 1 kg and charge 1 μC is projected from a large distance directly toward a ball of mass of 2 kg and same charge. The surface is horizontal and smooth.

Conceptualize/Classify: First let us analyze what will be the condition of the closest separation. At this, distance x between them will be minimum. Thus,

$$\frac{dx}{dt} = 0.$$

However, this dx/dt represents the relative velocity, or the rate of change of relative position. In other words, at the closest separation,

$$v_1 - v_2 = 0$$

$$v_1 = v_2.$$

In contrast to the previous example, here the system is free to move, so we cannot just say that the striking block will stop, rather both of them will move. Since no external force acts on the system, the momentum of the system should be conserved in addition to the conservation of energy.

Compute: $1 \times 10 = 1 \times v_1 + 2 \times v_2$. Solving, we get

$$v_1 = v_2 = \frac{10}{3} \text{ m/s}.$$

Note the force that moves the second charge was the electrostatic force but still we did not take that into account at all. All we are doing is applying the mechanics as we would do in a simple collision problem. This is

in fact true. The electrostatic force makes this a special case of collision, that's all.

However, to find the closest separation, we need to apply the concept of potential energy as well.

$$U_1 + K_1 = U_2 + K_2$$

$$0 + \frac{1}{2} \times 1 \times 10^2 = \frac{9 \times 10^9 \times (1 \times 10^{-6})^2}{r} + \frac{1}{2} \times 1 \times \left(\frac{10}{3}\right)^2 + \frac{1}{2} \times 2 \times \left(\frac{10}{3}\right)^2.$$

Solving, we get $r = 3.70 \times 10^3$ m.

Conclude: Let us extend this example further. The two blocks will still repel each other and the velocity of 2 kg block will increase even further under the influence of this repulsive force, and the velocity of the 1 kg block will decrease even further under the influence of this repulsive force. Finally, after a long time, they will be at a large distance from each other. The 2 kg block would be moving with a velocity of 6.67 m/s in the positive direction and the 1 kg block would be moving in the backward direction with a speed of 3.33 m/s. You can check that this turns out to be a special case of elastic collision!

Sample Problem 24-4

A ball of mass of 1 kg and charge 50 μC is directly projected from infinity toward another fixed ball of charge 1/6 mC (as shown in Fig. 24-6). The initial velocity of projection is 10 m/s. Find the distance and its velocity at the closest separation. Assume that the initial line of motion is at a distance of 1 m from the second fixed charge.

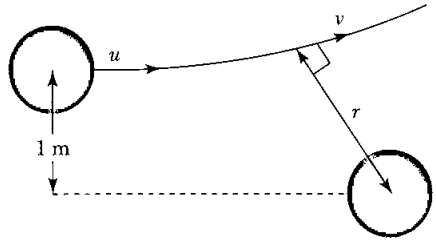


FIG. 24-6 A ball of mass of 1 kg and charge 50 μC is projected toward a fixed charge. The initial of motion is at a perpendicular distance of 1 m from fixed charge.

Conceptualize: Here, it is clear that the momentum of the system is not conserved because the second charge is fixed and the nail to which it is tied will exert some force to keep it at rest. But if we take the torque of this force by the nail about the nail, it will be zero ($r_\perp = 0$).

Classify: So the angular momentum of the system is conserved about the second charge. At the position of closest approach, its component of the velocity toward the second charge will be zero. But the velocity of the first charge will not be zero. Rather, it will take a path as shown in Fig. 24-6 due to repulsion. Its entire velocity will be in a direction perpendicular to the line joining the two charges at this time.

Compute: $L_i = mux = 1 \times 10 \times 1 = 10$ kg m^2/s,

$$L_f = mvr = 1 \times v \times r.$$

So we get,

$$v \times r = 10.$$

Also from the law of conservation of mechanical energy,

$$U_1 + K_1 = U_2 + K_2$$

$$0 + \frac{1}{2} \times 1 \times 10^2 = \frac{9 \times 10^9 \times 50 \times 10^{-6} \times 1/6 \times 10^{-3}}{r} + \frac{1}{2} \times 1 \times v^2.$$

Solving we get,

$$v_2 + 15v - 100 = 0.$$

Therefore

$v = 20$ m/s; $v = -5$ m/s (which is not possible)

$r = 0.5$ m.

Conclude: The charged particle will move with a velocity of 10 m/s after a long time, but it will be deflected from its path. This analysis was used by Rutherford in his famous gold foil experiment to postulate the existence of a massive positively charged nucleus.

24-3 | Electric Potential

As you can infer from Sample Problem 24-1, the potential energy of a charged particle in an electric field depends on the charge magnitude. However, the potential energy *per unit charge* has a unique value at any point in an electric field.

For an example of this, suppose we place a test particle of positive charge 1.60×10^{-19} C at a point in an electric field where the particle has an electric potential energy of 2.40×10^{-17} J. Then the potential energy per unit charge is

$$\frac{2.40 \times 10^{-17}\ \text{J}}{1.60 \times 10^{-19}\ \text{C}} = 150\ \text{J/C}.$$

Next, suppose we replace that test particle with one having twice as much positive charge, 3.20×10^{-19} C. We would find that the second particle has an electric potential energy of 4.80×10^{-17} J, twice that of the first particle. However, the potential energy per unit charge would be the same, still 150 J/C.

Thus, the potential energy per unit charge, which can be symbolized as U/q, is independent of the charge q of the particle we happen to use and is *characteristic only of the electric field* we are investigating. The potential energy per unit charge at a point in an electric field is called the **electric potential** V (or simply the **potential**) at that point. Thus,

$$V = \frac{U}{q}. \tag{24-8}$$

Note that electric potential is a scalar, not a vector.

The *electric potential difference* ΔV between any two points i and f in an electric field is equal to the difference in potential energy per unit charge between the two points:

$$\Delta V = V_f - V_i = \frac{U_f}{q} - \frac{U_i}{q} = \frac{\Delta U}{q}. \tag{24-9}$$

Using Eq. 24-1 to substitute $-W$ for ΔU in Eq. 24-9, we can define the potential difference between points i and f as

$$\Delta V = V_f - V_i = -\frac{W}{q} \quad \text{(potential difference defined)}. \tag{24-10}$$

The potential difference between two points is thus the negative of the work done by the electrostatic force to move a unit charge from one point to the other. A potential difference can be positive, negative, or zero, depending on the signs and magnitudes of q and W.

If we set $U_i = 0$ at infinity as our reference potential energy, then by Eq. 24-8, the electric potential V must also be zero there. Then from Eq. 24-10, we can define the electric potential at any point in an electric field to be

$$V = -\frac{W_\infty}{q} \quad \text{(potential defined)}, \tag{24-11}$$

where W_∞ is the work done by the electric field on a charged particle as that particle moves in from infinity to point f. A potential V can be positive, negative, or zero, depending on the signs and magnitudes of q and W_∞.

The SI unit for potential that follows from Eq. 24-11 is the joule per coulomb. This combination occurs so often that a special unit, the *volt* (abbreviated V), is used to represent it. Thus,

$$1\ \text{volt} = 1\ \text{joule per coulomb}. \tag{24-12}$$

This new unit allows us to adopt a more conventional unit for the electric field $\vec{E}$ which we have measured up to now in newtons per coulomb. With two unit conversions, we obtain

$$1\ \text{N/C} = \left(1\frac{\text{N}}{\text{C}}\right)\left(\frac{1\ \text{V}\cdot\text{C}}{1\ \text{J}}\right)\left(\frac{1\ \text{J}}{1\ \text{N}\cdot\text{m}}\right)$$
$$= 1\ \text{V/m}. \tag{24-13}$$

Finally, we can now define an energy unit that is a convenient one for energy measurements in the atomic and subatomic domain: One *electron-volt* (eV) is the energy equal to the work required to move a single elementary charge e, such as that of the electron or the proton, through a potential difference of exactly one volt. Equation 24-10 tells us that the magnitude of this work is $q\,\Delta V$; so

$$1\text{ eV} = e(1\text{ V})$$
$$= (1.60 \times 10^{-19}\text{ C})(1\text{ J/C}) = 1.60 \times 10^{-19}\text{ J}.$$

PROBLEM-SOLVING TACTICS

Tactic 2: *Electric Potential and Electric Potential Energy* Electric potential V and electric potential energy U are quite different quantities and should not be confused.

☞ *Electric potential* is a scalar property associated with an electric field, regardless of whether a charged object has been placed in that field; it is measured in joules per coulomb, or volts.

☞ *Electric potential energy* is an energy of a charged object in an external electric field (or more precisely, an energy of the system consisting of the object and the external electric field); it is measured in joules.

In essence, you should understand that the term potential is an abstract concept and although it is in vogue, yet it is only a means to calculate the potential energy which is the central concept. For example, if you touch a wire at 220 V with respect to Earth, you may suffer serious injuries because the amount of charge passing through your body and hence the potential energy associated with it is enormous. But if you touch a Van de Graff generator at even 22 000 V, it may not cause so much damage because the charge involved and hence the potential energy involved may not be so much.

✓**CHECKPOINT 2** In the figure of Checkpoint 1, we move the proton from point i to point f in a uniform electric field directed as shown. (a) Does our force do positive or negative work? (b) Does the proton move to a point of higher or lower potential?

24-4 | Equipotential Surfaces

Adjacent points that have the same electric potential form an **equipotential surface**, which can be either an imaginary surface or a real, physical surface. No net work W is done on a charged particle by an electric field when the particle moves between two points i and f on the same equipotential surface. This follows from Eq. 24-10, which tells us that W must be zero if $V_f = V_i$. Because of the path independence of work (and thus of potential energy and potential), $W = 0$ for *any* path connecting points i and f on a given equipotential surface regardless of whether that path lies entirely on that surface.

Figure 24-7 shows a *family* of equipotential surfaces associated with the electric field due to some distribution of charges. The work done by the electric field on a charged particle as the particle moves from one end to the other of paths I and II is zero because each of these paths begins and ends on the same equipotential surface. The work done as the charged particle moves from one end to the other of paths III and IV is not zero but has the same value for both these paths because the initial and final potentials are identical for the two paths; that is, paths III and IV connect the same pair of equipotential surfaces.

From symmetry, the equipotential surfaces produced by a point charge or a spherically symmetrical charge distribution are a family of concentric spheres. For a uniform electric field, the surfaces are a family of planes perpendicular to

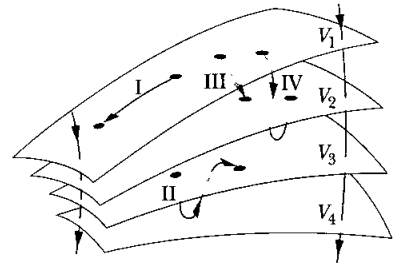


FIG. 24-7 Portions of four equipotential surfaces at electric potentials $V_1 = 100$ V, $V_2 = 80$ V, $V_3 = 60$ V, and $V_4 = 40$ V. Four paths along which a test charge may move are shown. Two electric field lines are also indicated.

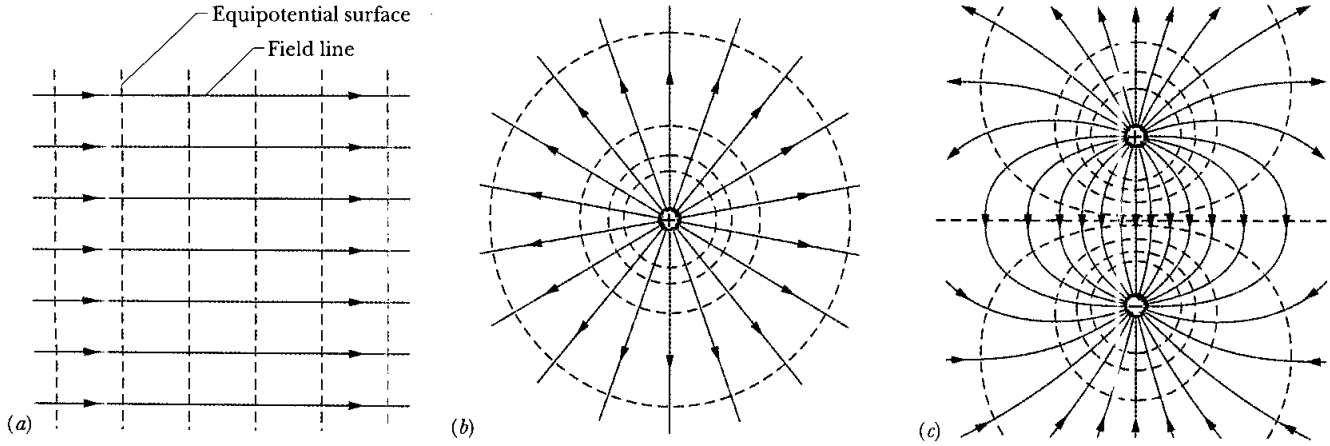


FIG. 24-8 Electric field lines (purple) and cross sections of equipotential surfaces (gold) for (*a*) a uniform electric field, (*b*) the field due to a point charge, and (*c*) the field due to an electric dipole.

the field lines. In fact, equipotential surfaces are always perpendicular to electric field lines and thus to $\vec{E}$, which is always tangent to these lines. If $\vec{E}$ were *not* perpendicular to an equipotential surface, it would have a component lying along that surface. This component would then do work on a charged particle as it moved along the surface. However, by Eq. 24-10 work cannot be done if the surface is truly an equipotential surface; the only possible conclusion is that $\vec{E}$ must be everywhere perpendicular to the surface. Figure 24-8 shows electric field lines and cross sections of the equipotential surfaces for a uniform electric field and for the field associated with a point charge and with an electric dipole.

24-5 | Calculating the Potential from the Field

We can calculate the potential difference between any two points i and f in an electric field if we know the electric field vector $\vec{E}$ all along any path connecting those points. To make the calculation, we find the work done on a positive test charge by the field as the charge moves from i to f, and then use Eq. 24-10.

Consider an arbitrary electric field, represented by the field lines in Fig. 24-9, and a positive test charge q_0 that moves along the path shown from point i to point f. At any point on the path, an electrostatic force $q_0\vec{E}$ acts on the charge as it moves through a differential displacement $d\vec{s}$. From Chapter 7, we know that the differential work dW done on a particle by a force $\vec{F}$ during a displacement $d\vec{s}$ is

$$dW = \vec{F} \cdot d\vec{s}. \tag{24-14}$$

For the situation of Fig. 24-9, $\vec{F} = q_0\vec{E}$ and Eq. 24-14 becomes

$$dW = q_0\vec{E} \cdot d\vec{s}. \tag{24-15}$$

To find the total work W done on the particle by the field as the particle moves from point i to point f, we sum—via integration—the differential works done on the charge as it moves through all the displacements $d\vec{s}$ along the path:

$$W = q_0 \int_i^f \vec{E} \cdot d\vec{s}. \tag{24-16}$$

If we substitute the total work W from Eq. 24-16 into Eq. 24-10, we find

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}. \tag{24-17}$$

Thus, the potential difference $V_f - V_i$ between any two points i and f in an electric field is equal to the negative of the *line integral* (meaning the integral along a

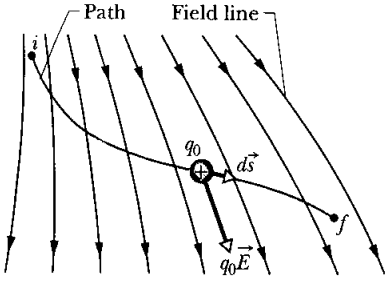


FIG. 24-9 A test charge q_0 moves from point i to point f along the path shown in a nonuniform electric field. During a displacement $d\vec{s}$, an electrostatic force $q_0\vec{E}$ acts on the test charge. This force points in the direction of the field line at the location of the test charge.

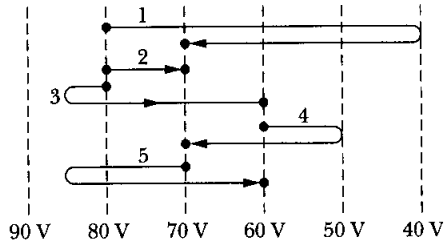
particular path) of $\vec{E}\cdot d\vec{s}$ from i to f. However, because the electrostatic force is conservative, all paths (whether easy or difficult to use) yield the same result.

Equation 24-17 allows us to calculate the difference in potential between any two points in the field. If we set potential $V_i = 0$, then Eq. 24-17 becomes

$$V = -\int_i^f \vec{E}\cdot d\vec{s}, \tag{24-18}$$

in which we have dropped the subscript f on V_f. Equation 24-18 gives us the potential V at any point f in the electric field *relative to the zero potential* at point i. If we let point i be at infinity, then Eq. 24-18 gives us the potential V at any point f relative to the zero potential at infinity.

CHECKPOINT 3 The figure here shows a family of parallel equipotential surfaces (in cross section) and five paths along which we shall move an electron from one surface to another. (a) What is the direction of the electric field associated with the surfaces? (b) For each path, is the work we do positive, negative, or zero? (c) Rank the paths according to the work we do, greatest first.



Sample Problem 24-5

(a) Figure 24-10*a* shows two points i and f in a uniform electric field $\vec{E}$. The points lie on the same electric field line (not shown) and are separated by a distance d. Find the potential difference $V_f - V_i$ by moving a positive test charge q_0 from i to f along the path shown, which is parallel to the field direction.

Conceptualize/Classify: We can find the potential difference between any two points in an electric field by integrating $\vec{E}\cdot d\vec{s}$ along a path connecting those two points according to Eq. 24-17.

Compute: We begin by mentally moving a test charge q_0 along that path, from initial point i to final point f. As we move such a test charge along the path in Fig. 24-10*a*, its differential displacement $d\vec{s}$ always has the same direction as $\vec{E}$. Thus, the angle θ between $\vec{E}$ and $d\vec{s}$ is zero and the dot product in Eq. 24-17 is

$$\vec{E}\cdot d\vec{s} = E\,ds\cos\theta = E\,ds. \tag{24-19}$$

Equations 24-17 and 24-19 then give us

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}k = -\int_i^f E\,ds. \tag{24-20}$$

Since the field is uniform, E is constant over the path and can be moved outside the integral, giving us

$$V_f - V_i = -E\int_i^f ds = -Ed, \qquad \text{(Answer)}$$

in which the integral is simply the length d of the path.

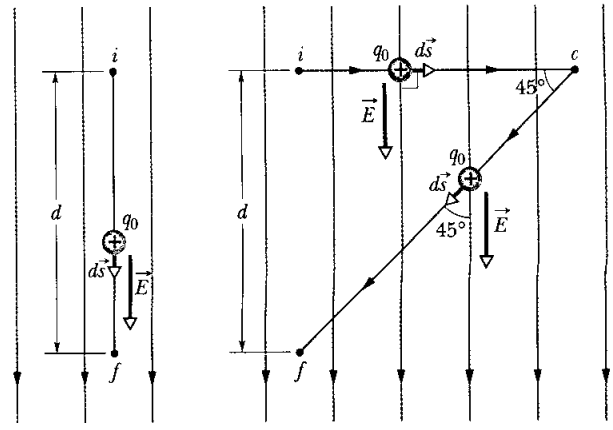


FIG. 24-10 (*a*) A test charge q_0 moves in a straight line from point i to point f, along the direction of a uniform external electric field. (*b*) Charge q_0 moves along path *icf* in the same electric field.

Conclude: The minus sign in the result shows that the potential at point f in Fig. 24-10*a* is lower than the potential at point i. This is a general result: The potential always decreases along a path that extends in the direction of the electric field lines.

(b) Now find the potential difference $V_f - V_i$ by moving the positive test charge q_0 from i to f along the path *icf* shown in Fig. 24-10*b*.

Compute: The Conceptualize/Classify of (a) applies here too, except now we move the test charge along a path that consists of two lines: *ic* and *cf*. At all points along line *ic*, the displacement $d\vec{s}$ of the test charge is perpendicular to $\vec{E}$. Thus, the angle θ between $\vec{E}$ and $d\vec{s}$ is 90°, and the dot product $\vec{E}\cdot d\vec{s}$ is 0. Equation 24-17 then tells us that points *i* and *c* are at the same potential: $V_c - V_i = 0$.

For line *cf* we have $\theta = 45°$ and, from Eq. 24-17,

$$V_f - V_i = -\int_c^f \vec{E}\cdot d\vec{s} = -\int_c^f E(\cos 45°)\, ds$$
$$= -E(\cos 45°)\int_c^f ds.$$

The integral in this equation is just the length of line *cf*; from Fig. 24-10*b*, that length is $d/\sin 45°$. Thus,

$$V_f - V_i = -E(\cos 45°)\frac{d}{\sin 45°} = -Ed. \qquad \text{(Answer)}$$

Conclude: This is the same result we obtained in (a), as it must be; the potential difference between two points does not depend on the path connecting them. Moral: When you want to find the potential difference between two points by moving a test charge between them, you can save time and work by choosing a path that simplifies the use of Eq. 24-17.

Sample Problem 24-6 Finding the Potential

Figure 24-11 is a graph of E_x, the x component of the electric field, versus position along the x axis. Find and graph $V(x)$. Assume $V = 0$; V at $x = 0$ m.

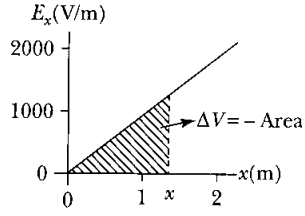


FIG. 24-11 Graph of E_x versus x.

Conceptualize: The potential difference is the negative of the area under the curve.

Classify: E_x is positive throughout this region of space, meaning that $\vec{E}$ points in the positive x direction.

Compute: If we integrate from $x - 0$, then $V_i = V(x - 0) - 0$. The potential for $x > 0$ is the negative of the triangular area under the E_x curve. We can see that $E_x = 1000x$ V/m, where x is in meters (m). Thus,

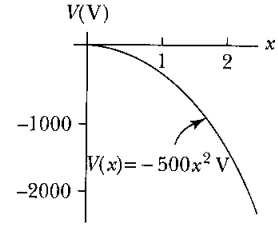


FIG. 24-12 Graph of V versus x.

$$V_i = V(x) = 0 - (\text{area under the } E_x \text{ curve})$$
$$= \frac{1}{2}\times \text{base} \times \text{height} = -\frac{1}{2}(x)(1000x) = -500x^2 \text{ V}.$$

Figure 24-12 shows that the electric potential in this region of space is parabolic, decreasing from 0 V at $x = 0$ m to –2000 V at $x = 2$ m.

Conclude: The electric field points in the direction in which V is decreasing. We'll soon see that as this is a general rule.

24-6 | Potential Due to a Point Charge

We now use Eq. 24-17 to derive, for the space around a charged particle, an expression for the electric potential V relative to the zero potential at infinity. Consider a point P at distance R from a fixed particle of positive charge q (Fig. 24-13). To use Eq. 24-17, we imagine that we move a positive test charge q_0 from point P to infinity. Because the path we take does not matter, let us choose the simplest one—a line that extends radially from the fixed particle through P to infinity.

To use Eq. 24-17, we must evaluate the dot product

$$\vec{E}\cdot d\vec{s} = E\cos\theta\, ds. \qquad (24\text{-}21)$$

The electric field $\vec{E}$ in Fig. 24-13 is directed radially outward from the fixed particle. Thus, the differential displacement $d\vec{s}$ of the test particle along its path has

the same direction as $\vec{E}$. That means that in Eq. 24-21, angle $\theta = 0$ and $\cos\theta = 1$. Because the path is radial, let us write ds as dr. Then, substituting the limits R and ∞, we can write Eq. 24-17 as

$$V_f - V_i = -\int_R^\infty E\,dr. \tag{24-22}$$

Next, we set $V_f = 0$ (at ∞) and $V_i = V$ (at R). Then, for the magnitude of the electric field at the site of the test charge, we substitute from Eq. 22-19:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \tag{24-23}$$

With these changes, Eq. 24-22 then gives us

$$0 - V = -\frac{q}{4\pi\varepsilon_0}\int_R^\infty \frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_R^\infty$$
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \tag{24-24}$$

Solving for V and switching R to r, we then have

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{24-25}$$

as the electric potential V due to a particle of charge q at any radial distance r from the particle.

Although we have derived Eq. 24-23 for a positively charged particle, the derivation holds also for a negatively charged particle, in which case, q is a negative quantity. Note that the sign of V is the same as the sign of q:

A positively charged particle produces a positive electric potential. A negatively charged particle produces a negative electric potential.

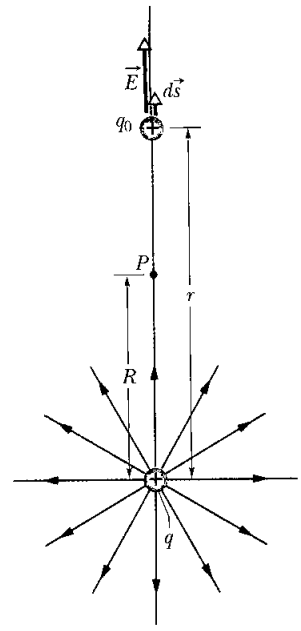


FIG. 24-13 The positive point charge q produces an electric field $\vec{E}$ and an electric potential V at point P. We find the potential by moving a test charge q_0 from P to infinity. The test charge is shown at distance r from the point charge, during differential displacement $d\vec{s}$.

Figure 24-14 shows a computer-generated plot of Eq. 24-25 for a positively charged particle; the magnitude of V is plotted vertically. Note that the magnitude increases as $r \to 0$. In fact, according to Eq. 24-25, V is infinite at $r = 0$, although Fig. 24-14 shows a finite, smoothed-off value there.

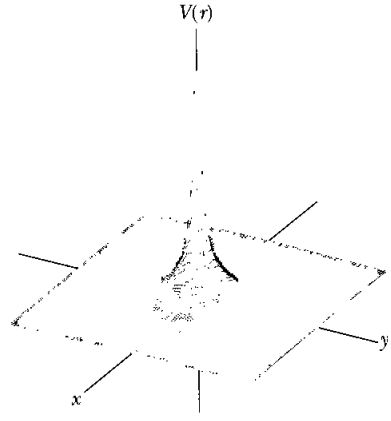


FIG. 24-14 A computer-generated plot of the electric potential $V(r)$ due to a positive point charge located at the origin of an xy plane. The potentials at points in the xy plane are plotted vertically. (Curved lines have been added to help you visualize the plot.) The infinite value of V predicted by Eq. 24-25 for $r = 0$ is not plotted.

PROBLEM-SOLVING TACTICS

Tactic 3: *Finding a Potential Difference* To find the potential difference ΔV between any two points in the field of an isolated point charge, evaluate Eq. 24-25 at each point and then subtract one result from the other. The value of ΔV will be the same for any choice of reference potential energy because that choice is eliminated by the subtraction.

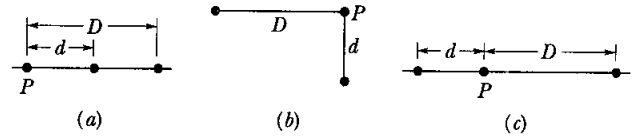
24-7 | Potential Due to a Group of Point Charges

We can find the net potential at a point due to a group of point charges with the help of the superposition principle. Using Eq. 24-25 with the sign of the charge included, we calculate separately the potential resulting from each charge at the given point. Then we sum the potentials. For n charges, the net potential is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} \qquad (n \text{ point charges}). \tag{24-26}$$

Here q_i is the value of the ith charge and r_i is the radial distance of the given point from the ith charge. The sum in Eq. 24-26 is an *algebraic sum,* not a vector sum like the sum that would be used to calculate the electric field resulting from a group of point charges. Herein lies an important computational advantage of potential over electric field: It is a lot easier to sum several scalar quantities than to sum several vector quantities whose directions and components must be considered.

CHECKPOINT 4 The figure here shows three arrangements of two protons. Rank the arrangements according to the net electric potential produced at point P by the protons, greatest first.



Sample Problem 24-7

What is the electric potential at point P, located at the center of the square of point charges shown in Fig. 24-15*a*? The distance d is 1.3 m, and the charges are

$$q_1 = +12\text{ nC}, \quad q_3 = +31\text{ nC},$$
$$q_2 = -24\text{ nC}, \quad q_4 = +17\text{ nC}.$$

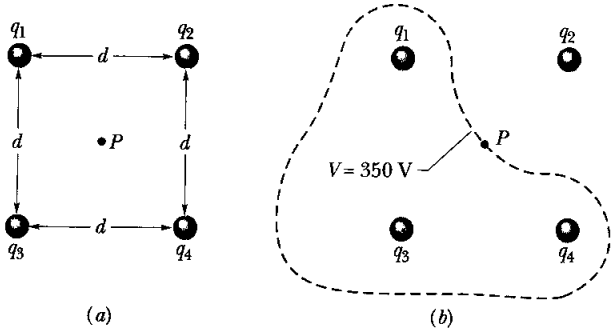


FIG. 24-15 (*a*) Four point charges are held fixed at the corners of a square. (*b*) The closed curve is a cross section, in the plane of the figure, of the equipotential surface that contains point P. (The curve is drawn only roughly.)

Conceptualize/Classify: The electric potential V at P is the algebraic sum of the electric potentials contributed by the four point charges. (Because electric potential is a scalar, the orientations of the point charges do not matter.)

Compute: From Eq. 24-26, we have

$$V = \sum_{i=1}^{4} V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r} + \frac{q_2}{r} + \frac{q_3}{r} + \frac{q_4}{r}\right).$$

The distance r is $d/\sqrt{2}$, which is 0.919 m, and the sum of the charges is

$$q_1 + q_2 + q_3 + q_4 = (12 - 24 + 31 + 17) \times 10^{-9}\text{ C}$$
$$= 36 \times 10^{-9}\text{ C}.$$

Thus,
$$V = \frac{(8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(36 \times 10^{-9}\text{ C})}{0.919\text{ m}}$$
$$\approx 350\text{ V}. \qquad \text{(Answer)}$$

Conclude: Close to any of the three positive charges in Fig. 24-15*a*, the potential has very large positive values. Close to the single negative charge, the potential has very large negative values. Therefore, there must be points within the square that have the same intermediate potential as that at point P. The curve in Fig. 24-15*b* shows the intersection of the plane of the figure with the equipotential surface that contains point P. Any point along that curve has the same potential as point P.

Sample Problem 24-8

(a) In Fig. 24-16*a*, 12 electrons (of charge $-e$) are equally spaced and fixed around a circle of radius R. Relative to $V = 0$ at infinity, what are the electric potential and electric field at the center C of the circle due to these electrons?

Conceptualize/Classify: (1) The electric potential V at C is the algebraic sum of the electric potentials contributed by all the electrons. (Because electric potential is a scalar, the orientations of the electrons do not matter.) (2) The electric field at C is a vector quantity and thus the orientation of the electrons *is* important.

Compute: Because the electrons all have the same negative charge $-e$ and are all the same distance R from C, Eq. 24-26 gives us

$$V = -12\frac{1}{4\pi\varepsilon_0}\frac{e}{R}. \qquad \text{(Answer)} \quad (24\text{-}27)$$

Because of the symmetry of the arrangement in Fig. 24-16*a*, the electric field vector at C due to any given electron is canceled by the field vector due to the electron that is diametrically opposite it. Thus, at C,

$$\vec{E} = 0. \qquad \text{(Answer)}$$

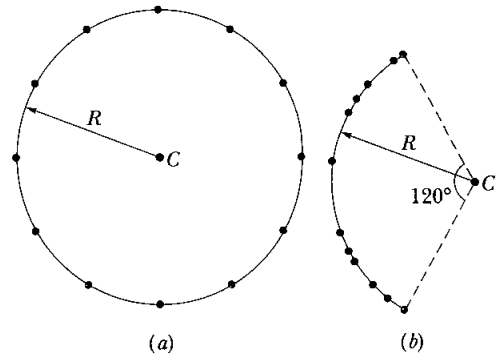


FIG. 24-16 (*a*) Twelve electrons uniformly spaced around a circle. (*b*) The electrons nonuniformly spaced along an arc of the original circle.

(b) If the electrons are moved along the circle until they are nonuniformly spaced over a 120° arc (Fig. 24-16*b*), what then is the potential at C? How does the electric field at C change (if at all)?

Compute: The potential is still given by Eq. 24-27, because the distance between C and each electron is unchanged and orientation is irrelevant. The electric field is no longer zero, however, because the arrangement is no longer symmetric. A net field is now directed toward the charge distribution.

Sample Problem 24-9

A point charge, $q_1 = +2\ \mu C$, is placed at the origin of coordinates. A second, $q_2 = -3\ \mu C$, is placed on the x axis at $x = 100$ cm (Fig. 24-17). At what point (or points) on the x axis will the potential be zero?

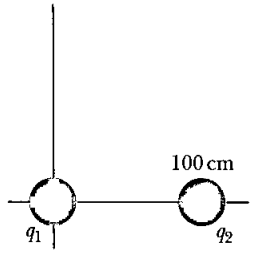


FIG. 24-17 Two charges are kept on x axis as shown. Find the points at which the potential will be zero.

Conceptualize/Classify: Since the potential obeys the law of superposition, we can find the potential due to each charge and add them. But here we have to be a little cautious, because although the point may be on the negative x axis, the distance from the charge should be taken as positive.

Compute: The two charges are as shown in Fig. 24-17. At a point x along the x axis, the potential is

$$V(x) = k\left(\frac{q_1}{|x|} + \frac{q_2}{|x-1|}\right).$$

Setting $V = 0$ yields

$$\frac{2}{|x|} - \frac{3}{|x-1|} = 0$$

or

$$2|x-1| = 3|x|.$$

We consider three cases $x > 1$, $0 < x < 1$, and $x < 0$.

For $x > 1$: $2(x-1) = 3x \Rightarrow x = -2$ (contradiction and no solution).

For $0 < x < 1$: $2(1-x) = 3x \Rightarrow x = 0.4$ or $x = 40$ cm.

For $x < 0$: $2(1-x) = -3x \Rightarrow x = -2$ or $x = -200$ cm.

Conclude: Thus, there are two points on the x axis where the potential is zero. In fact, we can make an equipotential surface in space on which potential is zero at every point. It will intersect x axis at two points.

24-8 | Potential Due to a Continuous Charge Distribution

When a charge distribution q is continuous (as on a uniformly charged thin rod or disk), we cannot use the summation of Eq. 24-26 to find the potential V at a point P. Instead, we must choose a differential element of charge dq, determine the potential dV at P due to dq, and then integrate over the entire charge distribution.

Let us again take the zero of potential to be at infinity. If we treat the element of charge dq as a point charge, then we can use Eq. 24-25 to express the potential dV at point P due to dq:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \quad \text{(positive or negative } dq\text{)}. \tag{24-28}$$

Here r is the distance between P and dq. To find the total potential V at P, we integrate to sum the potentials due to all the charge elements:

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}. \tag{24-29}$$

The integral must be taken over the entire charge distribution. Note that because the electric potential is a scalar, there are *no vector components* to consider in Eq. 24-29.

We now examine two continuous charge distributions, a disk and a spherical shell.

Charged Disk

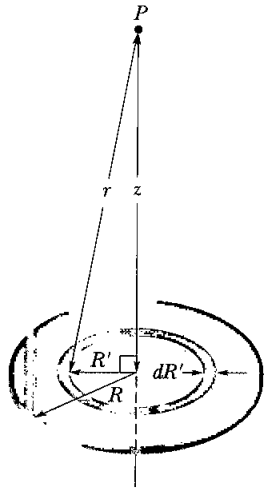


FIG. 24-18 A plastic disk of radius R, charged on its top surface to a uniform surface charge density σ. We wish to find the potential V at point P on the central axis of the disk.

In Section 22-10, we calculated the magnitude of the electric field at points on the central axis of a plastic disk of radius R that has a uniform charge density σ on one surface. Here we derive an expression for $V(z)$, the electric potential at any point on the central axis.

In Fig. 24-18, consider a differential element consisting of a flat ring of radius R' and radial width dR'. Its charge has magnitude

$$dq = \sigma(2\pi R')(dR'),$$

in which $(2\pi R')(dR')$ is the upper surface area of the ring. All parts of this charged element are the same distance r from point P on the disk's axis. With the aid of Fig. 24-18, we can use Eq. 24-28 to write the contribution of this ring to the electric potential at P as

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi R')(dR')}{\sqrt{z^2 + R'^2}}. \tag{24-30}$$

We find the net potential at P by adding (via integration) the contributions of all the rings from $R' = 0$ to $R' = R$:

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{R'\,dR'}{\sqrt{z^2 + R'^2}} = \frac{\sigma}{2\varepsilon_0}(\sqrt{z^2 + R^2} - z). \tag{24-31}$$

Note that the variable in the second integral of Eq. 24-31 is R' and not z, which remains constant while the integration over the surface of the disk is carried out. (Note also that, in evaluating the integral, we have assumed that $z \geq 0$.)

Sample Problem 24-10

Find the potential due to a ring at a point on its axis as shown in Fig. 24-19.

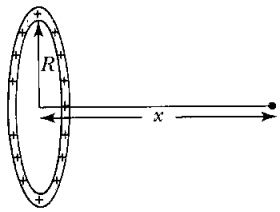


FIG. 24-19 Finding the electric potential due to a ring at a point on its axis.

Conceptualize/Classify: Consider a ring of total charge Q and radius a. If we wish to find the potential due to this ring at a point on its axis, we can divide the entire ring in small point charges. Each element of the ring is at the same distance from the point.

Compute: The potential due to this small element is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{a^2+x^2}}.$$

Integrating this expression, we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{a^2+x^2}}. \tag{24-32}$$

As in the case of a point charge, the potential is assumed to be zero at infinity.

Conclude: Note here that the ring need not be uniformly charged. Even if some point charges are located on the ring at specific points, the potential will have the same expression. In general, it is noted that the potential is a scalar, so its value can be obtained much easily (Fig. 24-20).

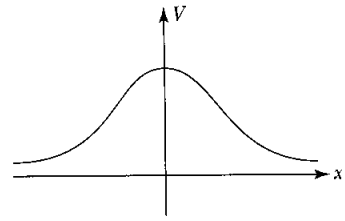


FIG. 24-20 Graph showing the potential due to ring at a point on its axis. Note that the graph is continuous, so the electric field is defined at all points on the axis.

Potential Due to Spherical Shell

Although here also we can find out the potential at different points in space by dividing the shell into rings and integrating, the integration turns out to be more tough than usual. So we will take a different approach: The potential at different points in space can be found by integrating the expression for the electric field. We will first find the potential outside the shell because the reference point happens to be outside the shell (at infinity).

$$V_{out} = -\int_{\infty}^{r}\vec{E}\cdot d\vec{l} = -\int_{\infty}^{r}\frac{q}{4\pi\varepsilon_0 r^2}dr = \frac{q}{4\pi\varepsilon_0 r}. \tag{24-33}$$

Although we are integrating from infinity to r, it is observed that $d\vec{l}$ is taken in outward direction by convention (Fig. 24-21). So angle between it and electric field is zero.

Inside the shell, electric field is zero. This does not mean that potential is zero too. Recall your integration lessons. If the electric field is zero inside, the potential will be constant. But how to figure out the value of the constant! Now we do not have the flexibility of assigning any potential to the inside points. That we can do only with the reference point and that is already done. There is a way out. We know the potential at the surface of the shell which happens to be common to both the regions.

$$\int_{V_R}^{V_{in}} dV = -\int_{R}^{r}\vec{E}\cdot d\vec{l} = -\int_{R}^{r} 0\,dr = 0.$$

$$V_{in} = V_R = \frac{q}{4\pi\varepsilon_0 R}. \tag{24-34}$$

The potential due to a uniform spherical shell is plotted in Fig. 24-22.

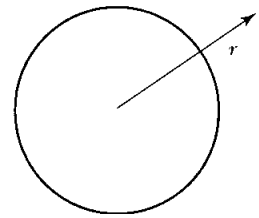


FIG. 24-21 Finding the potential due to a uniformly charged spherical shell at (a) point outside the shell and (b) at a point inside the shell.

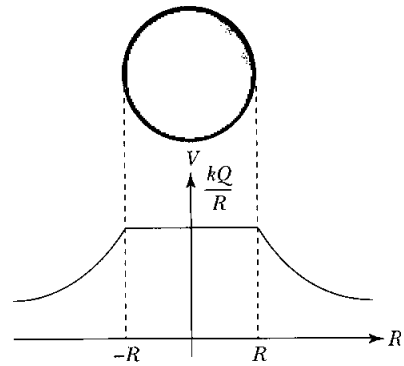


FIG. 24-22 A graph showing variation of potential due to a uniformly charged spherical shell at different points on its diameter extended. Note the discontinuity on the slope of V at R, this would mean that electric field is defined at all the points, but is discontinuous at the surface.

Sample Problem 24-11

Find the potential at the surface of the outer shell and at the center (Fig. 24-23).

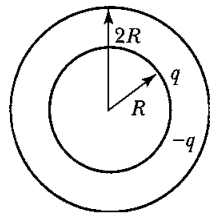


FIG. 24-23 Finding the potential due to two concentric spherical shells, one of charge q and radius R and another outer shell of charge $-q$ and radius $2R$.

Conceptualize: Remember that both the shells will contribute to the potential at any point. Center is a point which is inside both the shells.

Classify: It can be said that net potential at any point is

$$V = V_1 + V_2,$$

where V_1 and V_2 are potential due to inner and outer shells, respectively.

Compute: The potential at the center due to the shell is given by Eq. 24-34:

$$V_1 = \frac{kq}{R}, \quad V_2 = \frac{-kq}{2R}.$$

Therefore

$$V = \frac{kq}{R} + \frac{-kq}{2R} = \frac{kq}{2R}.$$

The surface of the outer shell is a point lying outside both the shells. So at that point the potential will not be constant but will be given by the equation

$$V = \frac{kq}{2R} + \frac{-kq}{2R} = 0.$$

Conclude: This is a result that may seem little surprising: the potential everywhere out of the system of shells is zero. Why? Recollect that according to the Gauss' law, the net charge in any Gaussian surface drawn outside the shells is zero. So the electric field outside is zero. Equation 24-16 says that this would mean that the potential outside is constant. But at an outside point, that is infinity, we have already defined the electric potential to be zero. So the potential at all the points outside must be zero.

Sample Problem 24-12 Build your skill

Find the potential due to a uniformly charged sphere of radius R and charge per unit volume ρ at different points in space.

Conceptualize/Classify: Equation 23-14 gave the electric field due to a sphere at a point outside and Eq. 23-17 gave the electric field due to a uniformly charged sphere at a point outside the sphere. We can use Eq. 24-18 to find the potential at a point, taking the reference to be at infinity.

Compute: For a point outside the sphere, we can integrate the electric field outside to obtain an expression for the electric field outside (Fig. 24-24).

$$V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = -\int_{\infty}^{r} \frac{\rho R^3}{3\varepsilon_0 x^2}\, dx = \frac{\rho R^3}{3\varepsilon_0 r}.$$

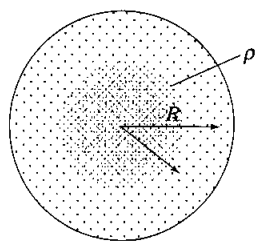


FIG. 24-24 The dots represent a spherically symmetric distribution of charge of radius R, whose volume charge density ρ is a constant.

For finding the potential at an inside point, we integrate as before. Keep in mind that the lower limit in this integration cannot be infinity, as infinity does not lie inside the sphere. But lower limit can be assumed to be R where potential is not zero but a known value.

$$\int_{V_R}^{V_{in}} dV = -\int_{R}^{r} \vec{E}\cdot d\vec{l} = -\int_{R}^{r} \frac{\rho x}{3\varepsilon_0} dx = \frac{\rho(R^2 - r^2)}{6\varepsilon_0}$$

$$V_R = \frac{\rho R^2}{3\varepsilon_0} \Rightarrow V_r = \frac{\rho(3R^2 - r^2)}{6\varepsilon_0}.$$

This potential is plotted in Fig. 24-25.

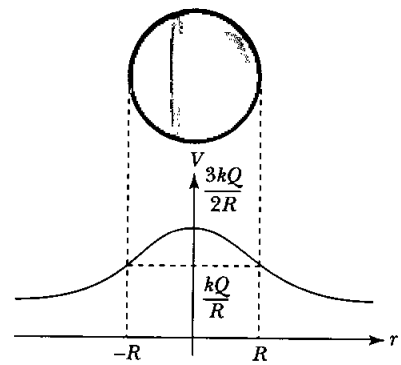


FIG. 24-25 The graph showing variation of potential due to a uniformly charged sphere at points on its diameter extended.

Conclude: The method looks simple, but it is deceptive. Remember that you can apply this method only when you know the electric field at all the points along the line that you are going to integrate. For example, you cannot find the electric potential due to a circular arc at its center by this formula. After all, we are not aware of its electric field at all the points. But if you divide the arc in small elements and integrate, you easily see that the potential at the center of the arc is

$$V = \frac{kq}{R}.$$

Sample Problem 24-13

Find the potential due to an infinite sheet of charge at a distance x from it. Assume that the potential is zero at a distance a from it.

Conceptualize/Classify: Although the infinite sheet can be treated as the special case of a disk, yet there are nuances to it that need be discussed. Let us do this by the formula we derived (Eq. 23-19), because the electric field of an infinite disk is known and is simple to handle.

Compute: Here we can use the formula we found before

$$\Delta V = -\int \vec{E}\cdot d\vec{x} = -\int_a^x \frac{\sigma}{2\varepsilon_0}\cdot dx.$$

Here, we have assumed the potential to be zero at a distance of a from the sheet.

$$V_x = \frac{\sigma(a-x)}{2\varepsilon_0}.$$

Conclude: We can note here that we cannot take the potential to be zero at infinity. If we do so, the potential will be infinite. This is because the electric field is also extending to infinity.

We can note that a is just an arbitrary number we chose according to our convenience. It can have any fixed value and still it will not affect our calculations because in any situation, we want to find the change in potential energy. For that, we need to find only the change in potential. It will be the same for every value of constant a.

Sample Problem 24-14

Assume that a point charge q is projected from a large distance toward the center of a fixed charged ring along its axis. What should be its initial velocity so that the charge may reach the other side of infinity?

Conceptualize: Here, we can say that the charge will be repelled till it reaches the center of the ring. After that, it will experience repulsive force again, but in the opposite direction. So if we are able to give the charge sufficient velocity so that it may reach the center of the ring, then it will cross the ring and reach the other side without any extra velocity.

Classify: Here, we can use the law of conservation of energy.

Compute: Assuming the velocity to be zero at the center of the ring,

$$U_i + K_i = U_f + K_f,$$

$$U = qV = \frac{kqQ}{\sqrt{a^2+x^2}},$$

$$0+\frac{1}{2}mv^2 = \frac{kqQ}{\sqrt{a^2+0^2}}+0$$

$$v = \sqrt{\frac{2kqQ}{ma}}.$$

Conclude: In practice, the initial velocity should be slightly greater than this velocity so that the charged particle may cross the center.

PROBLEM-SOLVING TACTICS

Tactic 4: Signs of Trouble with Electric Potential When you calculate the potential V at some point P due to a line of charge or any other continuous charge configuration, the signs can cause you trouble. Here is a generic guide to sort out the signs.

If the charge is negative, should the symbols dq and λ represent negative quantities, or should you explicitly show the signs, using $-dq$ and $-\lambda$? You can do either as long as you remember what your notation means, so that when you get to the final step, you can correctly interpret the sign of V.

Another approach, which can be used when the entire charge distribution is of a single sign, is to let the symbols dq and λ represent magnitudes only. The result of the calculation will give you the magnitude of V at P. Then add a sign to V based on the sign of the charge. (If the zero potential is at infinity, positive charge gives a positive potential and negative charge gives a negative potential.)

If you happen to reverse the limits on the integral used to calculate a potential, you will obtain a negative value for V. The magnitude will be correct, but discard the minus sign.

Then determine the proper sign for V from the sign of the charge. As an example, we would have obtained a minus sign in Eq. 24-25 if we had reversed the limits in the integral above that equation. We would then have discarded that minus sign and noted that the potential is positive because the charge producing it is positive.

24-9 | Calculating the Field from the Potential

In Section 24-5, you saw how to find the potential at a point f if you know the electric field along a path from a reference point to point f. In this section, we propose to go the other way—that is, to find the electric field when we know the potential. As Fig. 24-8 shows, solving this problem graphically is easy: If we know the potential V at all points near an assembly of charges, we can draw in a family of equipotential surfaces. The electric field lines, sketched perpendicular to those surfaces, reveal the variation of $\vec{E}$. What we are seeking here is the mathematical equivalent of this graphical procedure.

Figure 24-26 shows cross sections of a family of closely spaced equipotential surfaces, the potential difference between each pair of adjacent surfaces being dV. As the figure suggests, the field $\vec{E}$ at any point P is perpendicular to the equipotential surface through P.

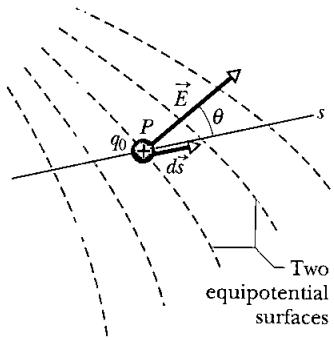


FIG. 24-26 A test charge q_0 moves a distance $d\vec{s}$ from one equipotential surface to another. (The separation between the surfaces has been exaggerated for clarity.) The displacement $d\vec{s}$ makes an angle θ with the direction of the electric field $\vec{E}_0$.

Suppose that a positive test charge q_0 moves through a displacement $d\vec{s}$ from one equipotential surface to the adjacent surface. From Eq. 24-10, we see that the work the electric field does on the test charge during the move is $-q_0\,dV$. From Eq. 24-15 and Fig. 24-26, we see that the work done by the electric field may also be written as the scalar product $(q_0\vec{E}) \cdot d\vec{s}$ or $q_0E(\cos\theta)\,ds$. Equating these two expressions for the work yields

$$-q_0\,dV = q_0E(\cos\theta)\,ds, \tag{24-35}$$

or

$$E\cos\theta = -\frac{dV}{ds}. \tag{24-36}$$

Since $E\cos\theta$ is the component of $\vec{E}$ in the direction of $d\vec{s}$ Eq. 24-36 becomes

$$E_s = -\frac{\partial V}{\partial s}. \tag{24-37}$$

We have added a subscript to E and switched to the partial derivative symbols to emphasize that Eq. 24-37 involves only the variation of V along a specified axis (here called the s axis) and only the component of $\vec{E}$ along that axis. In words, Eq. 24-37 (which is essentially the reverse operation of Eq. 24-17) states:

☞ The component of $\vec{E}$ in any direction is the negative of the rate at which the electric potential changes with distance in that direction.

If we take the s axis to be, in turn, the x, y, and z axes, we find that the x, y, and z components of $\vec{E}$ at any point are

$$E_x = -\frac{\partial V}{\partial x};\quad E_y = -\frac{\partial V}{\partial y};\quad E_z = -\frac{\partial V}{\partial z}. \tag{24-38}$$

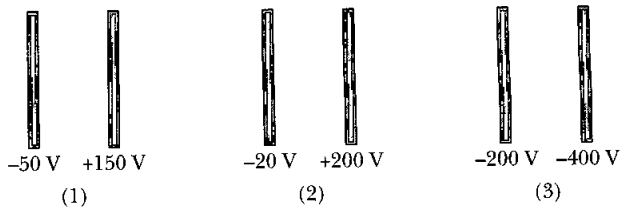
Thus, if we know V for all points in the region around a charge distribution—that is, if we know the function $V(x, y, z)$—we can find the components of $\vec{E}$, and thus $\vec{E}$ itself, at any point by taking partial derivatives.

For the simple situation in which the electric field $\vec{E}$ is uniform, Eq. 24-37 becomes

$$E = -\frac{\Delta V}{\Delta s}, \tag{24-39}$$

where s is perpendicular to the equipotential surfaces. The component of the electric field is zero in any direction parallel to the equipotential surfaces.

CHECKPOINT 5 The figure shows three pairs of parallel plates with the same separation, and the electric potential of each plate. The electric field between the plates is uniform and perpendicular to the plates. (a) Rank the pairs according to the magnitude of the electric field between the plates, greatest first. (b) For which pair is the electric field pointing rightward? (c) If an electron is released midway between the third pair of plates, does it remain there, move rightward at constant speed, move leftward at constant speed, accelerate rightward, or accelerate leftward?



Sample Problem 24-15

The electric potential at any point on the central axis of a uniformly charged disk is given by Eq. 24-31,

$$V = \frac{\sigma}{2\varepsilon_0}(\sqrt{z^2 + R^2} - z).$$

Starting with this expression, derive an expression for the electric field at any point on the axis of the disk.

Conceptualize/Classify: We want the electric field $\vec{E}$ as a function of distance z along the axis of the disk. For any value of z, the direction of $\vec{E}$ must be along that axis because the disk has circular symmetry about that axis. Thus, we want the component E_z of $\vec{E}$ in the direction of z. This component is the negative of the rate at which the electric potential changes with distance z.

Compute: Thus, from the last of Eqs. 24-38, we can write

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\frac{d}{dz}(\sqrt{z^2 + R^2} - z)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right). \qquad \text{(Answer)}$$

Conclude: This is the same expression that we derived in Section 22-7 by integration, using Coulomb's law.

Your body is a fairly good electrical conductor and can be easily charged if you move around or change clothing. Such an action produces a great many contact points between your clothing and your skin. For many types of fabrics, such contact allows some of the conduction electrons on one surface to move to the other surface. For example, you might gain conduction electrons when you remove a sweater. If the air humidity is high, these electrons are quickly drained from you by airborne water drops. If the humidity is low, however, you may have so much excess charge that the potential difference between your body and your surroundings is 5 kV or more. If you touch a computer keyboard while charged like this, the excess charge on your body can flow through the computer's circuit chips, overloading and ruining them.

There are countless examples in which contact between a person and some other type of material leaves the person so highly charged that the person might discharge with a spark. Children sliding down a plastic slide on a dry day have been measured to have a potential of about 60 kV. If such a child reaches for any conducting object (such as another person), the child probably will discharge to the object with a very painful spark. Such a spark discharge would be disastrous in a hospital operating room where a flammable gas (such as an anesthetic gas) is present. To drain the charges they collect as they move around, a surgical team wears conducting shoes and stands on a conducting floor. Spark discharges have also caused a number of fires at self-serve gasoline stations when a customer has slid back into a car seat to wait for the car's tank to be filled. Contact with the car seat can leave the customer with so much charge that a spark might jump between fingers and pump nozzle when the customer goes back to remove the nozzle from the car. The spark can ignite the gasoline vapor that surrounds the nozzle.

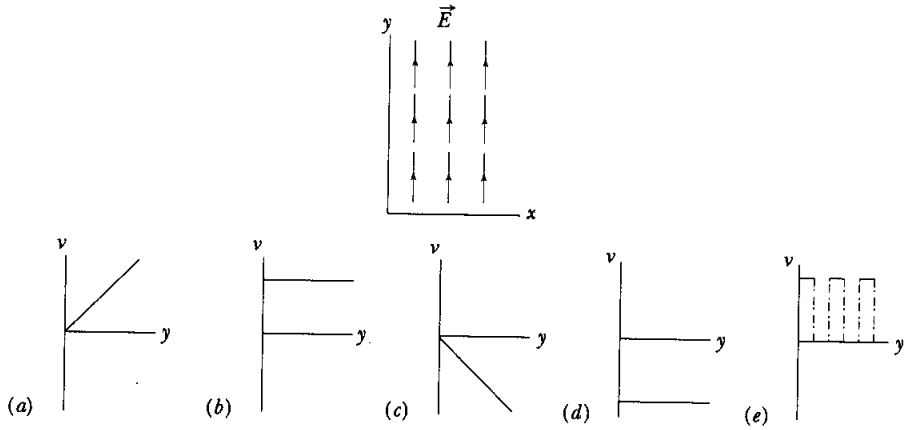
A very important conclusion can be drawn from this. The electric field will always point from high potential to low potential. We will show how in the following equation:

$$E_x = -\frac{dV}{dx}.$$

If potential increases in x direction, dV/dx will be positive and the electric field will be negative. In other words, electric field points in negative direction from higher potential to lower potential.

The electric field is away from the sheet of charge. But from the expression for the potential, we can say that the potential decreases as x increases. So the electric field points from higher potential to lower potential.

CHECKPOINT 6 Which potential graph describes the electric field shown here?



Sample Problem 24-16

The potential at a point (x, y) is expressed as $V = 4x + 3y$. A charge 1 μC and mass 1 mg is released from (1, 1) from rest. (a) Find the acceleration of the charge.

Conceptualize/Classify: The potential at a given point can be differentiated to obtain the electric field. This electric field can then be used to find the acceleration of the charge.

Compute:

$$E_x = -\frac{\partial V}{\partial x} \text{ and } E_y = -\frac{\partial V}{\partial y}.$$

$$\vec{E} = -4\hat{i} - 3\hat{j}.$$

$$\vec{a} = \frac{q\vec{E}}{m} = \frac{(-4\hat{i} - 3\hat{j}) \times 10^{-6}}{10^{-6}} = -4\hat{i} - 3\hat{j}.$$

(b) Find the time when it crosses x axis.

Conceptualize/Classify: Since the acceleration is constant, we can apply the equations of motion.

Compute: According to the equation of motion:

$$S = ut + \tfrac{1}{2}at^2.$$

When the charge crosses x axis, its y coordinate is 0. So displacement in y direction is $0 - 1 = -1$ m. Hence

$$-1 = \tfrac{-3}{2}t^2 \Rightarrow t = \sqrt{\tfrac{2}{3}}\,\text{s}.$$

Conclude: The charged particle would move in a straight line.

CHECKPOINT 7 Is the electrostatic field pattern shown in the figure possible? (*Hint*: Assume a rectangular loop in the field and find the work done by the electric field on a test charge in moving along that loop. Did you arrive at a contradiction?)



24-10 | Potential Due to an Electric Dipole

Before trying to understand the electric field of a dipole, we need to define some new mathematics which will come handy here.

Polar Coordinates

The polar coordinate (r, θ) of a point P are defined in Fig. 24-27; r is the distance from the origin (the magnitude of the position vector), (the angle down from the z axis) is called the **polar angle**. The relation to Cartesian coordinates (x, y, z) can be read from Fig. 24-27 as

$$x = r\sin\theta,\ z = r\cos\theta. \tag{24-40}$$

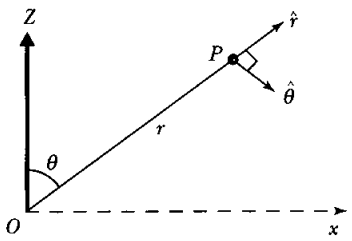


FIG. 24-27 Polar coordinates of a general point in xz plane. For simplicity, we ignore the y axis.

Figure 24-27 also shows two unit vector $\hat{r}$ and $\hat{\theta}$ pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{i}$ and $\hat{j}$, and any vector A can be expressed in terms of those in the usual way):

$$A = A_r\hat{r} + A_\theta\hat{\theta}. \tag{24-41}$$

A_r and A_θ are the radial and polar components of A.

However, there is something very important one must be aware of: $\hat{r}$ and $\hat{\theta}$ are associated with a particular point P, and they change direction as P moves around. For example, $\hat{r}$ always points radially outward, but "radially outward" can be the x direction, the y direction, or any other direction, depending on where you are. In Fig. 24-28, $\vec{A} = \hat{y}$ and $\vec{B} = -\hat{y}$, and yet both of them would be written as $\hat{r}$ in polar coordinates. In particular, do not naively combine the spherical components of vectors associated with different points (in Fig. 24-28, $\vec{A} + \vec{B} = 0$, not $2\hat{r}$, and $A \cdot B = -1$, not $+1$). Beware of differentiating a vector that is expressed in polar coordinates, as the unit vectors themselves are functions of position ($\partial\hat{r}/\partial\theta = \hat{\theta}$, for example). And do not take $\hat{r}$ and $\hat{\theta}$ outside an integral, as we did with $\hat{i}$ and $\hat{j}$.

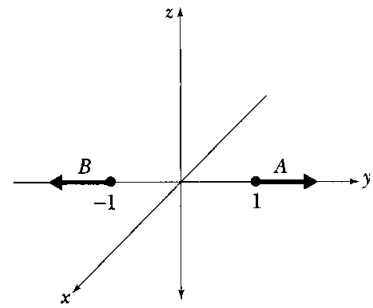


FIG. 24-28 Vector A and B in polar coordinates.

An infinitesimal displacement in the $\hat{r}$ direction is simply dr (Fig. 24-29*a*), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr. \tag{24-42}$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (Fig. 24-29*b*) is not just $d\theta$ (that's an angle—it doesn't even have the right units for a length), but rather $r\,d\theta$.

$$dl_\theta = rd\theta. \tag{24-43}$$

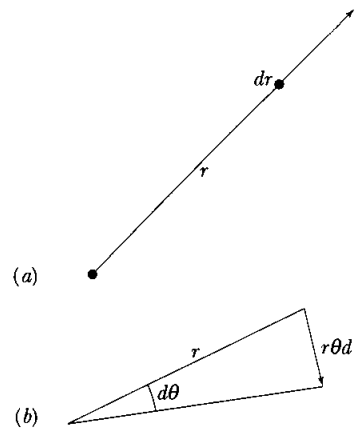


FIG. 24-29 (*a*) dr represents a change in polar coordinate r; (*b*) $d\theta$ represents a change in polar coordinate θ.

Thus, the general infinitesimal displacement dl is

$$dl = dr\,\hat{r} + rd\theta\hat{\theta}. \tag{24-44}$$

This plays the role (in line integrals, for example) that $dl = dx\,\hat{i} + dy\,\hat{j}$ played in Cartesian coordinates.

Now let us apply Eq. 24-24 to an electric dipole to find the potential at an arbitrary point P in Fig. 24-30*a*.

At P, the positive point charge (at distance $r_{(+)}$) sets up potential $V_{(+)}$ and the negative point charge (at distance $r_{(-)}$) sets up potential $V_{(-)}$. Then the net potential at P is given by Eq. 24-26 as

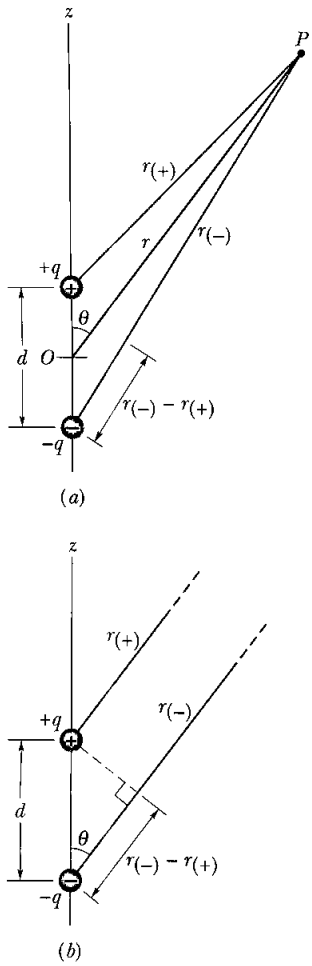


FIG. 24-30 (*a*) Point P is a distance r from the midpoint O of a dipole. The line OP makes an angle θ with the dipole axis. (*b*) If P is far from the dipole, the lines of lengths $r_{(+)}$ and $r_{(-)}$ are approximately parallel to the line of length r, and the dashed black line is approximately perpendicular to the line of length $r_{(-)}$.

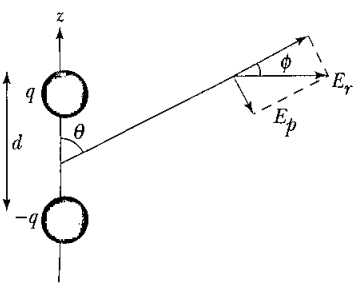


FIG. 24-31 Electric field of a dipole kept along z axis. Note that the resultant electric field makes an angle of $\theta + \phi$ with the z axis.

$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{r_{(-)} - r_{(+)}}{r_{(-)}\, r_{(+)}}. \tag{24-45}$$

Naturally occurring dipoles—such as those possessed by many molecules—are quite small; so we are usually interested only in points that are relatively far from the dipole, such that $r \gg d$, where d is the distance between the charges. Under those conditions, the approximations that follow from Fig. 24-30*b* are

$$r_{(-)} - r_{(+)} \approx d\cos\theta \quad \text{and} \quad r_{(-)}r_{(+)} \approx r^2.$$

If we substitute these quantities into Eq. 24-40, we can approximate V to be

$$V = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2},$$

where θ is measured from the dipole axis as shown in Fig. 24-30*a*. We can now write V as

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \quad \text{(electric dipole)}. \tag{24-46}$$

in which ($p = qd$) is the magnitude of the electric dipole moment $\vec{p}$ defined in Section 22-12. The vector $\vec{p}$ is directed along the dipole axis, from the negative to the positive charge. (Thus, θ is measured from the direction of $\vec{p}$.) The plane perpendicular to the dipole axis is called equator or more appropriately equatorial plane.

If the location of the points and the dipole is known in terms of their x, y, z coordinates, it is more convenient to use the formula of potential in vector form as follows:

$$V = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}. \tag{24-47}$$

CHECKPOINT 8 Suppose that three points are set at equal (large) distances r from the center of the dipole in Fig. 24-30: Point a is on the dipole axis above the positive charge, point b is on the axis below the negative charge, and point c is on a perpendicular bisector through the line connecting the two charges. Rank the points according to the electric potential of the dipole there, greatest (most positive) first.

Electric Field Due to a Dipole

As we have studied in Section 24-9, the electric field can be obtained if we know the potential. But the hitch is that we have derived the equation in terms of x and y coordinates, whereas here we have the expression of potential in terms of r and θ. So in this situation, we make use of polar coordinate system. In this system, the polar coordinates (r, θ) are as shown in Fig. 24-31.

To get the electric field from the potential, we take the derivative of the potential in radial and polar directions.

$$E_r = \frac{-\partial V}{\partial r}.$$

For displacement in the tangential direction, the radius will be the same but the displacement will be $r\,d\theta$:

$$E_p = \frac{-\partial V}{r\partial\theta},$$

$$V = \frac{kp\cos\theta}{r^2}.$$

By differentiating the above expressions, we get

$$E_r = \frac{2kp\cos\theta}{r^3}, \quad E_p = \frac{kp\sin\theta}{r^3}. \tag{24-48}$$

Therefore,

$$E_T = \sqrt{E_r^2 + E_p^2} = \frac{kp}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{kp}{r^3}\sqrt{1+3\cos^2\theta}. \tag{24-49}$$

The angle made by resultant with the radial direction is

$$\tan\phi = \frac{E_p}{E_r} = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2}. \tag{24-50}$$

Sample Problem 24-17

A dipole of dipole moment p is placed at origin along x axis. Another dipole of dipole moment also p is kept at $(0, 1, 0)$ along y axis. Find the resultant potential and electric field at $(1, 0, 0)$.

Conceptualize/Classify: We have studied that the electric field and potential both obey the law of superposition. We can find the electric field and the potential due to each of the dipole at the point concerned and then add them. The caution to be exercised is that while finding the electric field/potential due to a given dipole, we need to consider the center of that dipole as the origin.

Compute: Due to the first dipole at the origin, the potential is

$$V_1 = \frac{kp\cos 0^\circ}{r^2} = \frac{kp}{1^2} = kp.$$

Due to the second dipole at $y = 1$, the potential is

$$V_2 = \frac{kp\cos\theta}{r^2} = \frac{kp}{(\sqrt{2})^2}\cos 135^\circ = -\frac{kp}{2\sqrt{2}}.$$

Therefore

$$V = V_1 + V_2 = kp - \frac{kp\sqrt{2}}{4} = kp\left(\frac{2\sqrt{2}-1}{2\sqrt{2}}\right). \quad \text{(Answer)}$$

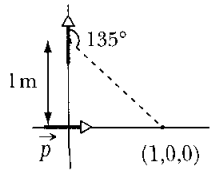


FIG. 24-32 Finding the electric field and potential due to two dipole (*a*) at origin and (*b*) at (0,1,0). Note that $r = 1$ m and $\theta = 0°$ for first dipole and $r = \sqrt{2}$ m and $\theta = 135°$ for the second.

Similarly, the electric field in the radial direction due to the first dipole is

$$\vec{E}_{r1} = \frac{2kp\cos 0}{r^3}\hat{\mathrm{i}} = 2kp\hat{\mathrm{i}}.$$

Here, the radial direction is also the x direction as can be seen from Fig. 24-32.

$$\vec{E}_{p1} = \frac{kp\sin 0}{r^3}\hat{\mathrm{j}} = 0\hat{\mathrm{j}}.$$

Here, the polar direction is the positive y direction. The magnitude of the electric field in the radial direction due to the second dipole is

$$E_{r2} = \frac{2kp}{2\sqrt{2}}\cos 135^\circ = -\frac{kp}{2}.$$

Here, the radial direction is shown in Fig. 24-33. A unit vector in the radial direction will be $\cos 45\hat{\mathrm{i}} - \sin 45\hat{\mathrm{j}}$. So

$$\vec{E}_{r2} = -\frac{kp}{2}\left(\frac{\hat{\mathrm{i}}}{\sqrt{2}} - \frac{\hat{\mathrm{j}}}{\sqrt{2}}\right).$$

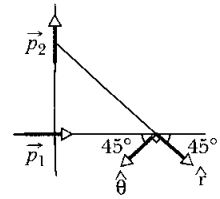


FIG. 24-33 Radial and polar direction for dipole $\vec{p}_2$.

Similarly, the electric field in the polar direction will be given by

$$\vec{E}_{p2} = \frac{kp\sin 135°}{(\sqrt{2})^3}\left(-\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}\right).$$

Because a unit vector in the polar direction will be $-\cos 45\hat{i} - \sin 45\hat{j}$, so we have

$$\vec{E} = 2kp\hat{i} - \frac{kp}{2\sqrt{2}}\hat{i} + \frac{kp}{2\sqrt{2}}\hat{j} - \frac{kp}{4\sqrt{2}}\hat{i} - \frac{kp}{4\sqrt{2}}\hat{j}$$

$$\vec{E} = \left(\frac{kp(8\sqrt{2}-3)\hat{i}}{4\sqrt{2}} + \frac{kp\hat{j}}{4\sqrt{2}}\right). \qquad \text{(Answer)}$$

Force on a Dipole Kept in a Nonuniform Electric Field

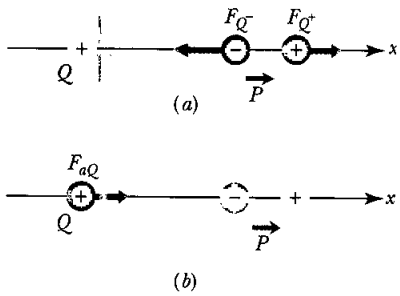


FIG. 24-34 (*a*) The vector sum of the electric forces exerted by a charged particle on a dipole is nonzero. (*b*) The force exerted by the charged particle on the dipole and that exerted by the dipole on the charged particle form an action–reaction pair.

The vector sum of the forces exerted on dipoles in *nonuniform* fields is not zero. Consider, for example, the situation illustrated in Fig. 24-34*a*. A dipole, with its dipole moment $\vec{P}$ aligned with the x axis is placed in the nonuniform electric field of a particle carrying a charge Q on the x axis. Because the negative end of the dipole is closer to Q than the positive end, the magnitude of the attractive force on the negative end is larger than the repulsive force on the positive end. Consequently, the vector sum of the forces exerted by the nonuniform field of Q on the dipole is nonzero and the dipole is attracted toward the charged particle.

How does this attraction vary with the distance d? To answer this question, we can write down an expression for the vector sum of the forces exerted on the two ends of the dipole and examine how this sum varies with the distance d. But a simpler method would be to calculate the force exerted by the electric field of the dipole on Q using the formula for the electric field of a dipole. This force and the vector sum of the forces exerted by Q on the dipole form an action–reaction pair and so their magnitudes are the same (Fig. 24-34*b*). Equation 24-49 tells us that the magnitude of the electric field of the dipole along the dipole axis is

$$E_r = \frac{2kp}{r^3},$$

and so the magnitude of the force on Q is

$$F_{dQ} = E_d Q = 2k\frac{p}{d^3}Q = k\frac{2pQ}{d^3}.$$

The force exerted by Q on the dipole, being equal in magnitude to, is thus

$$F_{dQ} = F_{Q^+} + F_{Q^-} = k\frac{2pQ}{d^3}.$$

Many times we come across situations where the total charge of a system is zero, but they are made up of more than two charges. In such cases, we may be required to make pairs of charges and find the dipole moment of each pair. Then we can add the dipole moments as vectors to obtain the total dipole moment.

Sample Problem 24-18

Three charges Q, Q, and $-2Q$ are located on the vertices of an equilateral triangle of side a. Find the dipole moment of the combination (Fig. 24-35).

Conceptualize/Classify: In the situation shown in Fig. 24-36, we can divide the charge $-2Q$ into two parts: $-Q$ and $-Q$. This makes two dipoles of moment Qa at an angle of 60° to each other.

Compute: As can be easily seen their resultant should be in the y direction.

$$\vec{p} = 2Qa\cos 30°\hat{j} = \sqrt{3}Qa\hat{j}.$$

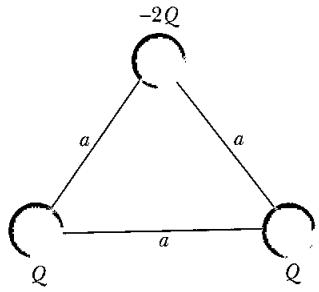


FIG. 24-35 Finding dipole moment of three charges.

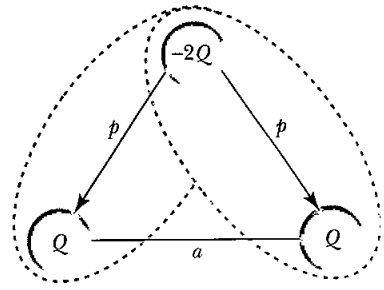


FIG. 24-36 Making pairs of equal and opposite charges.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: It is less dangerous to touch a faulty electrical appliance at 200 V than a Van de Graff generator at 20 000 V.

Statement 2: If we have two similar bodies at the different potential, the body with a higher potential will have more charge on it.

2. Statement 1: The potential difference between two points in space is independent of the choice of reference point where we have assumed the potential to be zero.

Statement 2: The work done by an electrostatic force is independent of the path chosen.

3. Statement 1: The electrostatic force on a charged particle located on an equipotential surface is zero.

Statement 2: x component of electric field is given by

$$E_x = -\frac{\partial V}{\partial x},$$

and on equipotential surface potential V is constant.

4. Statement 1: If the distance between two point charges increases, their potential energy decreases.

Statement 2: The potential energy of two point charges is inversely proportional to the distance between the two charges.

5. Statement 1: If the electric potential at all the points in space is zero, the electric field at all the points in space must be zero.

Statement 2: If the electric field at all the points in space is zero, the electric potential at all the points in space must be zero.

6. Statement 1: The potential of an electric dipole is constant on its perpendicular bisector.

Statement 2: The electric field on the perpendicular bisector of a dipole in the radial direction is zero.

MULTIPLE CHOICE QUESTIONS

sec. 24-2 Electric Potential Energy

1. A point charge Q is moved along a circular path around another fixed point charge. The work done by the electric field for one complete circle is zero

(a) If fixed charge is at the center of the circular path
(b) If the two charges have the same magnitude
(c) If the two charges have the same magnitude and opposite signs
(d) All above cases

2. Three charged particles are initially in a position 1. They are free to move and they come to another position 2 after some time. Let U_1 and U_2 be the electrostatic potential energies in position 1 and 2. Then

(a) $U_1 > U_2$
(b) $U_2 > U_1$
(c) $U_1 = U_2$

3. When a negative charge moves in a direction opposite to the direction of an electric field,

(a) The field does work on the charge
(b) The charge does work on the field
(c) The charge gains potential energy
(d) The charge loses potential energy

4. A point charge q moves from point P to point S along the path P, Q, R, and S in a uniform electric field pointing parallel to the positive direction of the x axis. The coordinates of the points P, Q, R, and S are $(a, b, 0)$, $(2a, 0, 0)$, $(a, -b, 0)$, and $(0, 0, 0)$, respectively (Fig. 24-37). The work done by the field in the above process is given by the expression

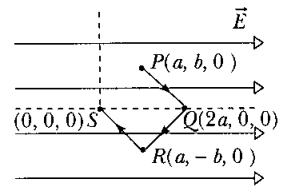


FIG. 24-37 Multiple choice question 4.

(a) qaE

(b) $-qaE$

(c) $q(\sqrt{a^2+b^2})E$

(d) $3qE\sqrt{a^2+b^2}$

5. There is a fixed positive charge Q at O and A and B are points equidistant from O. A positive charge +q is taken slowly by an external agent from A to B along the line AC and then along the line CB (Fig. 24-38).

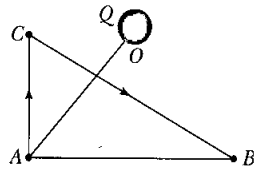


FIG. 24-38 Multiple choice question 5.

(a) The total work done on the charge is zero

(b) The work done by the electrostatic force from A to C is negative

(c) The work done by the electrostatic force from C to B is positive

(d) The work done by electrostatic force in taking the charge from A to B is dependent on the actual path followed

6. Two balls kept on a smooth horizontal ground are connected by a light inextensible thread as shown in Fig. 24-39. They are given equal charge as shown in the figure. Choose the correct statement.



FIG. 24-39 Multiple choice question 6.

(a) During the subsequent motion, the momentum of the system is always conserved, but the mechanical energy is not

(b) During the subsequent motion, the momentum as well as the mechanical energy of the system is conserved

(c) During the subsequent motion, the mechanical energy of the system is always conserved, but the momentum of the system is not

(d) During the subsequent motion, neither the momentum nor the mechanical energy of the system is conserved

7. Three balls A, B, and C of which two are charged, but the third is uncharged as shown in Fig. 24-40. They are connected by light inextensible thread and kept on smooth level ground. All the strings are taut. If the string connecting A and B breaks,

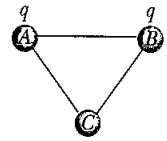


FIG. 24-40 Multiple choice question 7.

(a) During the subsequent motion, the momentum of the system is always conserved, but the mechanical energy is not

(b) During the subsequent motion, the momentum as well as the mechanical energy of the system is conserved

(c) During the subsequent motion, the mechanical energy of the system is always conserved, but the momentum of the system is not

(d) During the subsequent motion, neither the momentum nor the mechanical energy of the system is conserved

8. Two rigid insulated sphere A and B of mass m and $2m$, respectively, are given charge Q each and kept on rough horizontal ground as shown in Fig. 24-41. The balls always roll purely. When the balls are released,



FIG. 24-41 Multiple choice question 8.

(a) Mechanical energy of system is conserved

(b) Linear momentum of system is conserved

(c) Angular momentum of the system is conserved about a point on the ground

(d) None of these

9. Which statement about a system of point charges that are fixed in space is necessarily true? Assuming electrostatic potential energy at infinity to be zero

(a) If the electrostatic potential energy of the system is negative, net positive work by an external agent is required to take the charges in the system back to infinity

(b) If the electrostatic potential energy of the system is zero, all charges in the configuration cannot have same sign

(c) If the electrostatic potential energy of the system is negative, net positive work by an external agent was required to assemble the system of charges

(d) If the electrostatic potential energy of the system is negative, then there is no electric force anywhere in space on any other charged particle not part of the system

10. When the separation between two charges is increased, the electric potential energy of the charges

(a) Increases

(b) Decreases

(c) Remains the same

(d) May increase or decrease

sec. 24-3 Electric Potential

11. An electron at a potential of −10 kV moves to a point where its potential is −1 kV. Its potential energy has

(a) Decreased

(b) Increased

(c) Not changed

(d) One needs to know the distance between the points to say

12. A ball of mass 1 g and charge 10^{-4} C moves from point A, whose potential is 1500 V to the point B whose potential is zero volt. If the velocity of the ball at B is 20 m/s, what is its velocity at A?

(a) 5 m/s

(b) 7.5 m/s

(c) 10 m/s

(d) 15 m/s

sec. 24-4 Equipotential Surfaces

13. Figure 24-42 shows a group of equipotential lines of the electric potential, which result from a charge distribution in the plane of the paper. At which location will an electron ($q = -1.6 \times 10^{-19}$ C) experience a force directed toward the bottom of the paper?

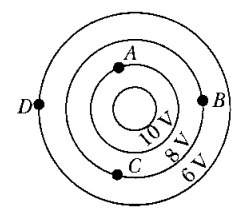
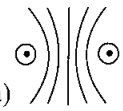
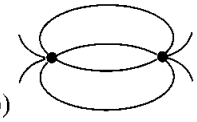
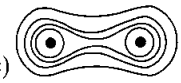


FIG. 24-42 Multiple choice question 13.

14. Which figure most closely represents the equipotential lines corresponding to two equal negative point charges?

(a) 

(b) 

(c) 

(d) None of these

15. Figure 24-43 shows three spherical and equipotential surfaces 1, 2, and 3 around a point charge q. The potential difference $V_1 - V_2 = V_2 - V_3$. If t_1 and t_2 be the distance between them, then

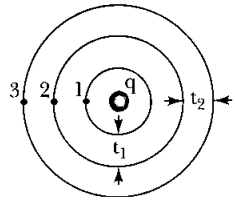


FIG. 24-43 Multiple choice question 15.

(a) Their radii are in AP
(b) Their radii are in GP
(c) The inverse of their radii are in AP
(d) The inverse of their radii are in GP

sec. 24-5 Calculating the Potential from the Field

16. In a certain region of space, the electric field is zero. From this, we can conclude that the electric potential in this region is

(a) Constant
(b) Zero
(c) Positive
(d) Negative

17. Suppose a region of space has a uniform electric field, directed toward the right, as shown in Fig. 24-44. Which statement is true?

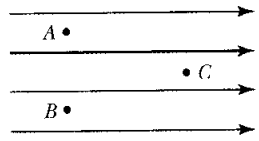


FIG. 24-44 Multiple choice question 17.

(a) The potential at all three locations is the same
(b) The potential at points A and B are equal, the potential at point C is higher than the potential at point A
(c) The potential at points A and B are equal, and the potential at point C is lower than the potential at point A
(d) The potential at point A is the highest, the potential at point B is the second highest, and the potential at point C is the lowest

18. A uniform electric field exists in the vertically downward direction. The magnitude is 10 N/C. What is the increase in electrostatic potential as one goes up through a height of 50 cm?

(a) 20 V
(b) 15 V
(c) 10 V
(d) 5 V

sec. 24-6 Potential Due to a Point Charge

19. At a certain distance from a point charge there is a fixed point A. At that point there is an electric field of 500 V/m and potential of 3 kV. Then, the distance of A from the point charge is

(a) 6 m
(b) 12 m
(c) 36 m
(d) 144m

20. A charge of 10 μC is placed at the origin of $(x-y)$ coordinate system. The potential difference between two points $(a/\sqrt{2}, a/\sqrt{2})$ and $(a, 0)$ in volts is

(a) $9\times10^4/a$
(b) $9\times10^4/a\sqrt{2}$
(c) $9\times10^4/2a$
(d) 0

sec. 24-7 Potential Due to a Group of Point Charges

21. In total, 10 charges (5 of them are $+q$ each and other 5 are $-q$ each) are placed randomly on the circumference of a circle. The radius of the circle is R. The electric potential at the center of this circle due to these charges will be

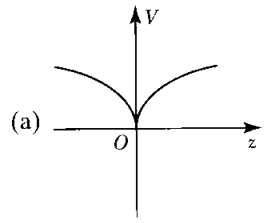
(a) 0
(b) $10\,Kq/R$
(c) $5\,Kq/R$
(d) Cannot be calculated unless the positions of the charges on the circle are specified

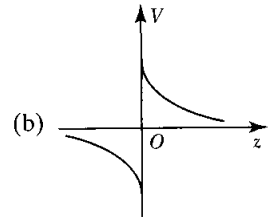
22. Four identical charges are placed at the points (1, 0, 0), (0, 1, 0), (−1, 0, 0), and (0, −1, 0), then

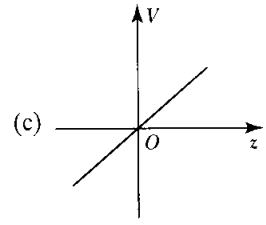
(a) The potential at the origin is zero
(b) The field at the origin is zero
(c) The potential at all points on the z axis, other than the origin, is zero
(d) The field at all points on the z axis, other than the origin, is along the z axis

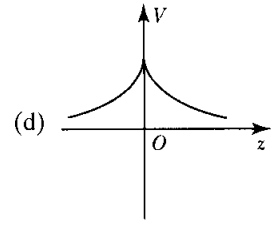
sec. 24-8 Potential Due to a Continuous Charge Distribution

23. A circular disk carries a uniformly distributed positive charge and lies in xy plane with center at origin of coordinate system. If at a point $(0, 0, z)$, electric potential is V, which of the following is correct?

(a) 

(b) 

(c) 

(d) 

24. Electric potential at a point P, r distance away due to a point charge q at point A is V. If this charge is distributed uniformly on the surface of a hollow sphere of radius $4r$ with center at point A, the potential at P now is

(a) V
(b) $V/2$
(c) $V/4$
(d) $V/8$

25. Two concentric shells of radii 1 cm and 2 cm are charged to 3 V and 5 V, respectively. The charge on the inner shell is

(a) $-4/9 \times 10^{-11}$ C
(b) $4/9 \times 10^{-11}$ C
(c) $-2/9 \times 10^{-11}$ C
(d) $-2/9 \times 10^{-11}$ C

26. The electrostatic potential is measured to be $v(x, y, z) = 4|x| + v_0$, where v_0 is a constant. The charge distribution responsible for this potential is

(a) A uniformly charged thread in the xy plane
(b) A point charge at origin
(c) A uniformly charged sheet in the yz plane
(d) A uniformly charged sphere of radius $1/\pi$ at the origin

27. Consider the two large oppositely charged plates as shown Fig. 24-45. At which of the marked points shown in the figure would a negatively charged particle have the greatest electrical potential energy?

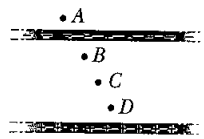


FIG. 24-45 Multiple choice question 27.

(a) A
(b) B
(c) C
(d) D

28. A solid sphere of radius R is charged uniformly. At what distance from its surface is the electrostatic potential half of the potential at the center?

(a) R
(b) $R/2$
(c) $R/3$
(d) $2R$

29. A large solid sphere with uniformly distributed positive charge has a smooth narrow tunnel through its center. A small particle with negative charge, initially at rest far from the sphere, approaches it along the line of the tunnel, reaches its surface with a speed v, and passes through the tunnel. What will be its speed at the center of the tunnel?

(a) Zero
(b) v
(c) $\sqrt{2}v$
(d) $\sqrt{1.5}v$

30. An infinitely long uniformly charged nonconducting rod is placed along x axis. A test charge is slowly taken from point A to B in the x–y plane along a semi-circular path $A \rightarrow C \rightarrow B$ as shown in Fig. 24-46, then which alternative (s) may be correct

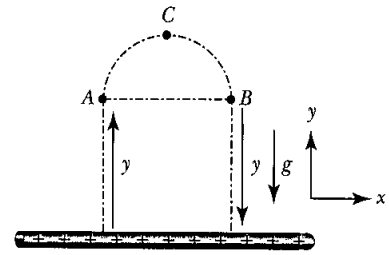


FIG. 24-46 Multiple choice question 30.

(a) Work done by external agent is zero
(b) The work done by external agent is path independent
(c) $W_{A \rightarrow C} = -W_{B \rightarrow C}$
(d) At point C, the force applied by external agent is maximum

sec. 24-9 Calculating the Field from the Potential

31. Two point charges are kept at a certain distance from one another. Figure 24-47 represents the variation of the potential along the straight line connecting the two charges. At what point is the electric field zero?

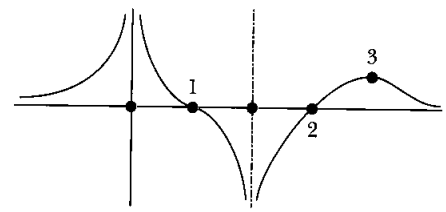


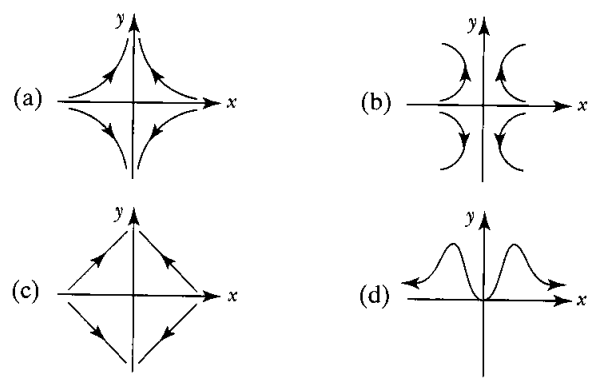
FIG. 24-47 Multiple choice question 31.

(a) 1
(b) 2
(c) 3
(d) None of these

32. The electric potential V at any point (x, y, z) (all in meters) in space is given by $V = 4x^2$ V. The electric field at the point (1 m, 0, 2 m) in V/m is

(a) 8 V/m along negative x axis
(b) 8 V/m along positive x axis
(c) 16 V/m along negative x axis
(d) 16 V/m along positive x axis

33. The potential field depends on x and y coordinates as $V = (x^2 - y^2)$. Corresponding electric field lines in xy plane are as



34. Which of the following equation cannot possibly represent the shape of an electric field line?

(a) $x^2 + y^2 = 1$

(b) $xy = 1$

(c) $y = ax - bx^2$

(d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

sec. 24-10 Potential Due to an Electric Dipole

35. An electric dipole is placed at the origin O such that its equator is y axis. At a point P far away from dipole, the electric field direction is along y direction. OP makes an angle α with the x axis such that

(a) $\tan\alpha = \sqrt{3}$

(b) $\tan\alpha = \sqrt{2}$

(c) $\tan\alpha = 1$

(d) $\tan\alpha = \frac{1}{\sqrt{2}}$

36. Electric field on the axis of a small electric dipole at a distance r is $\vec{E}_1$ and at a distance of $2r$ on a line of perpendicular bisector is $\vec{E}_2$. Then

(a) $\vec{E}_2 = -\frac{\vec{E}_1}{8}$

(b) $\vec{E}_2 = -\frac{\vec{E}_1}{16}$

(c) $\vec{E}_2 = -\frac{\vec{E}_1}{4}$

(d) $\vec{E}_2 = -\frac{\vec{E}_1}{8}$

PROBLEMS

sec. 24-2 Electric Potential Energy

1. Electrostatic work w is done on a charged particle, going from point A to point B, without any other external force acting on it. You next apply a force to move the particle back to point A, increasing its kinetic energy by an amount equal to $2w$. How much work did you do?

2. How much work is required to set up the arrangement of Fig. 24-48 if q = 2.0 μC, a = 9.0 cm, and the particles are initially infinitely far apart and at rest?

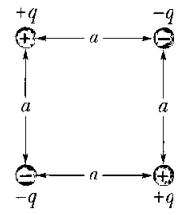


FIG. 24-48 Problem 2.

3. A particle of charge +7.5 μC is released from rest at the point x = 60 cm on an x axis. The particle begins to move due to the presence of a charge Q that remains fixed at the origin. What is the kinetic energy of the particle at the instant it has moved 40 cm if Q = +20 μC?

4. Two charged objects are held a distance r apart. The first object has a mass m and a charge $+2q$, while the second object has a mass $2m$ and a charge $+q$. The objects are released from rest. Assume that the only force acting on either charge is the electrostatic force from the other charge. (a) When the objects are in motion what is the ratio of the first object's kinetic energy to that of the second object? What is the speed of (b) the first object and (c) the second object when they are at a large disctance from each other?

5. Two electrons are fixed 2.0 cm apart. Another electron is shot from infinity and stops midway between the two. What is its initial speed?

sec. 24-3 Electric Potential

6. A particular 12 V car battery can send a total charge of 84 A-h (ampere-hours) through a circuit, from one terminal to the other. (a) How many coulombs of charge does this represent? (*Hint:* See Eq. 21-2.) (b) If this entire charge undergoes a change in electric potential of 12 V, how much energy is involved?

7. A positron (charge $+e$, mass equal to the electron mass) is moving at 1.0×10^7 m/s in the positive direction of an x axis when, at x = 0, it encounters an electric field directed along the x axis. The electric potential V associated with the field is given in Fig. 24-49. The scale of the vertical axis is set by V_s = 500.0 V. (a) Does the positron emerge from the field at x = 0 (which means its motion is reversed) or at x = 0.50 m (which means its motion is not reversed)? (b) What is its speed when it emerges?

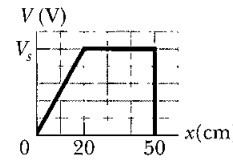


FIG. 24-49 Problem 7.

sec. 24-5 Calculating the Potential from the Field

8. When an electron moves from A to B along an electric field line in Fig. 24-50, the electric field does 3.84×10^{-19} J of work on it. What are the electric potential differences (a) $V_B - V_A$, (b) $V_C - V_A$, and (c) $V_C - V_B$?

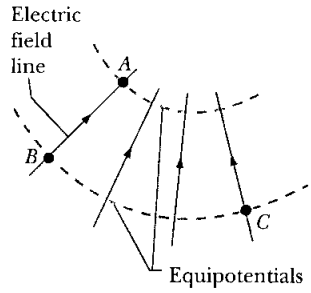


FIG. 24-50 Problem 8.

9. A graph of the x component of the electric field as a function of x in a region of space is shown in Fig. 24-51. The scale of the vertical axis is set by E_{xs} = 20.0 N/C. The y and z components of the electric field are zero in this

region. If the electric potential at the origin is 10 V, (a) what is the electric potential at x = 2.0 m, (b) what is the greatest positive value of the electric potential for points on the x axis for which $0 \le x \le 6.0$ m, and (c) for what value of x is the electric potential zero?

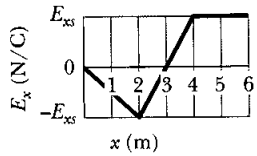


FIG. 24-51 Problem 9.

10. The electric field in a region of space has the components $E_y = E_z = 0$ and $E_x = (4.00\ \text{N/C})x$. Point A is on the y axis at $y = 3.00$ m, and point B is on the x axis at $x = 4.00$ m. What is the potential difference $V_B - V_A$?

11. The electric field at any point (x, y, z) in space is given by $E_x = 3x^2 + yz$, $E_y = 3y^2 + xz$, $E_z = 3z^2 + xy$. Find the potential at that point.

12. Find the potential $V(x, y, z)$ in an electrostatic field $\vec{E} = ay\hat{i} + (ax + bz)\hat{j} + by\hat{k}$, where a and b are constants.

sec. 24-6 Potential due to a Point Charge

13. In the rectangle of Fig. 24-52, the sides have lengths 5.0 cm and 15 cm, $q_1 = -5.0\ \mu$C, and $q_2 = +2.0\ \mu$C. With $V = 0$ at infinity, what is the electric potential at (a) corner A and (b) corner B? (c) How much work is required to move a charge $q_3 = +3.0\ \mu$C from B to A along a diagonal of the rectangle? (d) Does this work increase or decrease the electric potential energy of the three-charge system? Is more, less, or the same work required if q_3 is moved along a path that is (e) inside the rectangle but not on a diagonal and (f) outside the rectangle?

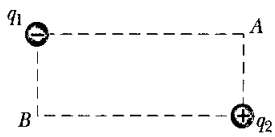


FIG. 24-52 Problem 13.

14. A particle carrying a charge +9 nC is at the origin. Taking the potential to be zero at infinity, locate the equipotential surfaces at 20 V intervals from 20 to 100 V, and sketch them to scale. Are these surfaces equally spaced?

15. Assume the electric potential to be zero at infinity. A charge q is located on the x axis at x = b. One additional charge is placed elsewhere on the x axis so that all points at a distance c from the origin have zero potential. What is that additional charge and where is it located? Express your answer in terms of q, c, and b (Fig. 24-53).

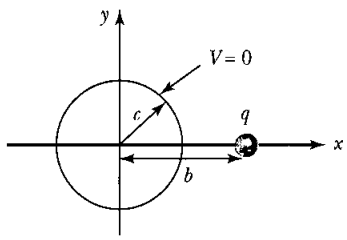


FIG. 24-53 Problem 15.

sec. 24-7 Potential Due to a Group of Point Charges

16. Consider a point charge $q = 1.0\ \mu$C, point A at distance $d_1 = 2.0$m from q, and point B at distance $d_2 = 1.0$ m. (a) If A and B are diametrically opposite each other, as in Fig. 24-54*a*, what is the electric potential difference $V_A - V_B$? (b) What is that electric potential difference if A and B are located as in Fig. 24-54*b*?

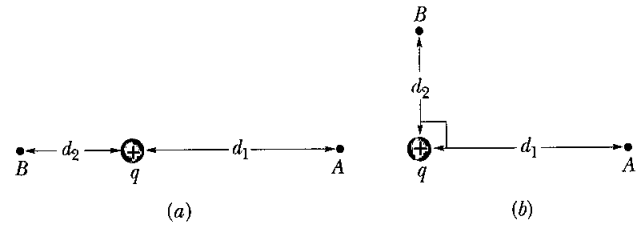


FIG. 24-54 Problem 16.

17. Two particles, of charges q_1 and q_2, are separated by distance d in Fig. 24-55. The net electric field due to the particles is zero at $x = d/4$. With $V = 0$ at infinity, locate (in terms of d) any point on the x axis (other than at infinity) at which the electric potential due to the two particles is zero.

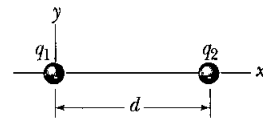


FIG. 24-55 Problem 17.

sec. 24-8 Potential Due to a Continuous Charge Distribution

18. In Fig. 24-56, a plastic rod having a uniformly distributed charge $Q = -25.6$ pC has been bent into a circular arc of radius $R = 3.71$ cm and central angle $\phi = 120°$. With $V = 0$ at infinity, what is the electric potential at P, the center of curvature of the rod?

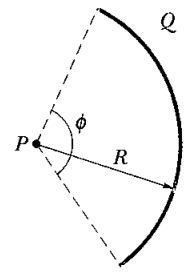


FIG. 24-56 Problem 18.

19. (a) Figure 24-57*a* shows a nonconducting rod of length $L = 6.00$ cm and uniform linear charge density $\lambda = +3.68$ pC/m. Take $V = 0$ at infinity. What is V at point P at distance $d = 8.00$ cm along the rod's perpendicular bisector? (b) Figure 24-57*b* shows an identical rod except that one half is now negatively charged. Both halves have a linear charge density of magnitude 3.68 pC/m. With $V = 0$ at infinity, what is V at P?

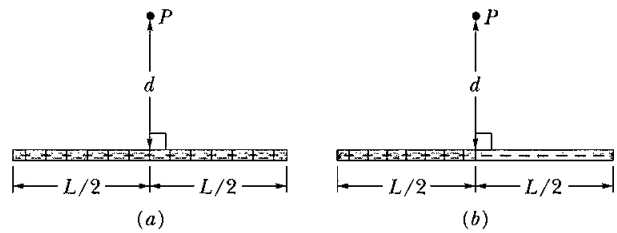


FIG. 24-57 Problem 19.

20. A wire having a uniform linear charge density λ is bent into the shape as shown in Fig. 24-58. Find (a) the electrical potential and (b) the electric field at point O.

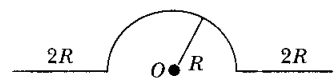


FIG. 24-58 Problem 20.

21. A plastic disk of radius $R = 32.0$ cm is charged on one side with a uniform surface charge density $\sigma = 35.4\ \text{fC/m}^2$, and then three quadrants of the disk are removed. The remaining quadrant is shown in Fig. 24-59. With $V = 0$ at infinity, what is the potential due to the remaining quadrant at point P, which

is on the central axis of the original disk at distance $D = 24.0$ cm from the original center?

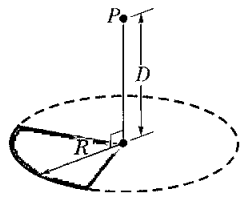


FIG. 24-59 Problem 21.

22. In Fig. 24-60, three thin plastic rods form quarter-circles with a common center of curvature at the origin. The uniform charges on the rods are $Q_1 = +30$ nC, $Q_2 = +3.0Q_1$, and $Q_3 = -8.0Q_1$. What is the net electric potential at the origin due to the rods?

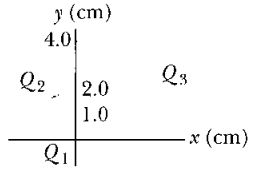


FIG. 24-60 Problem 22.

23. Find the electric field potential at the center of a hemisphere of radius R charged uniformly with the surface charge density σ.

24. A spherical drop of water carrying a charge of 30 pC has a potential of 500 V at its surface (with $V = 0$ at infinity). (a) What is the radius of the drop? (b) If two such drops of the same charge and radius combine to form a single spherical drop, what is the potential at the surface of the new drop?

25. Much of the material making up Saturn's rings is in the form of tiny dust grains having radii on the order of 10^{-6} m. These grains are located in a region containing a dilute ionized gas, and they pick up excess electrons. As an approximation, suppose each grain is spherical, with radius $R = 9.0 \times 10^{-6}$ m. How many electrons would one grain have to pick up to have a potential of -400 V on its surface (taking $V = 0$ at infinity)?

26. A charge of -9.0 nC is uniformly distributed around a thin plastic ring lying in a yz plane with the ring center at the origin. A -6.0 pC point charge is located on the x axis at $x = 3.0$ m. For a ring radius of 1.5 m, how much work must an external force do on the point charge to move it to the origin?

27. What is the *escape speed* for an electron initially at rest on the surface of a sphere with a radius of 1.0 cm and a uniformly distributed charge of 1.6×10^{-15} C? That is, what initial speed must the electron have in order to reach an infinite distance from the sphere and have zero kinetic energy when it gets there?

sec. 24-9 Calculating the Field from the Potential

28. The electric potential V in the space between two flat parallel plates 1 and 2 is given (in volts) by $V = 1500x^2$, where x (in meters) is the perpendicular distance from plate 1. At $x = 1.3$ cm, (a) what is the magnitude of the electric field and (b) is the field directed toward or away from plate 1?

29. The electric potential at points in an xy plane is given by $V = (2.0\ \text{V/m}^2)x^2 - (3.0\ \text{V/m}^2)y^2$. In unit-vector notation, what is the electric field at the point (3.0 m, 2.0 m)?

30. Two large parallel metal plates are 1.5 cm apart and have charges of equal magnitudes but opposite signs on their facing surfaces. Take the potential of the negative plate to be zero. If the potential halfway between the plates is then +5.0 V, what is the electric field in the region between the plates?

31. An electron is placed in an xy plane where the electric potential depends on x and y as shown in Fig. 24-61 (the potential does not depend on z).The scale of the vertical axis

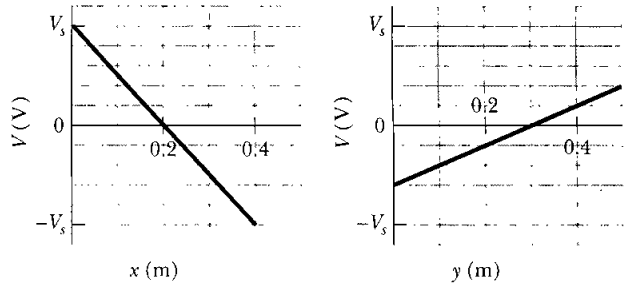


FIG. 24-61 Problem 31.

is set by $V_s = 500$ V. In unit-vector notation, what is the electric force on the electron?

32. What is the magnitude of the electric field at the point $(3.00\hat{i} - 2.00\hat{j} + 4.00\hat{k})$ m if the electric potential is given by $V = 2.00xyz^2$, where V is in volts and x, y, and z are in meters?

sec. 24-10 Potential Due to an Electric Dipole

33. The ammonia molecule NH_3 has a permanent electric dipole moment equal to 1.47 D, where 1 D = 1 debye unit $= 3.34 \times 10^{-30}$ C · m. Calculate the electric potential due to an ammonia molecule at a point 52.0 nm away along the axis of the dipole. (Set $V = 0$ at infinity.)

34. What is the equation of equipotential lines in x-y plane for a dipole kept at the origin and oriented along x axis?

35. An electron on the axis of an electric dipole is 25 nm from the center of the dipole. What is the magnitude of the electrostatic force on the electron if the dipole moment is 3.6×10^{-29} Cm? Assume that 25 nm is much larger than the dipole charge separation.

36. (a) Does the magnitude of dipole's electric field remain constant at a given distance r away from it when its orientation is changed? (b) If not, what is the ratio of the greatest to least magnitudes?

37. A small sphere carrying a uniform charge of -3.0 nC experiences a force of magnitude 200 nN in the positive y direction when it is along the perpendicular bisector of a permanent dipole. (a) If the charges making up the dipole have a magnitude of 10 nC each and a separation of 2.0 cm, how far away from the dipole is the sphere? (b) What is the orientation of the dipole?

38. If a dipole of moment $10^{-9}\ (\hat{i} + \hat{j})$ C · m is kept at the origin, what is the electric field and potential at (a) (1, 0, 0); (b) (0, 1, 0); and (c) (0, 0, 1)?

39. Four charges are arranged on the four corners of a square as shown in Fig. 24-62. Find the resultant dipole moment of the arrangement.

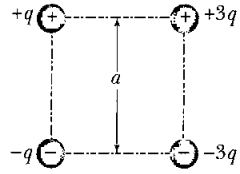


FIG. 24-62 Problem 39.

40. A thin nonconducting ring of radius R has a linear density $\lambda = \lambda_0 \cos\theta$, where λ_0 is a constant and θ is the polar angle (Fig. 24-63). (a) Show by integration that the total charge on the ring is 0. (b) Find the dipole moment of the ring. (c) Find the magnitude of electric field at a point on the axis of the ring at a distance x ($x \gg R$) from the center.

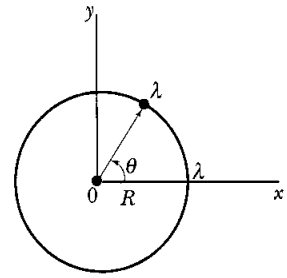


FIG. 24-63 Problem 40.

ADDITIONAL PROBLEMS

41. A nonconducting sphere has radius R = 2.31 cm and uniformly distributed charge q = +3.50 fC. Take the electric potential at the sphere's center to be $V_0 = 0$. What is V at radial distance (a) r = 1.45 cm and (b) $r = R$.

42. Two charged particles are shown in Fig. 24-64*a*. Particle 1, with charge q_1, is fixed in place at distance d. Particle 2, with charge q_2, can be moved along the x axis. Figure 24-64*b* gives the net electric potential V at the origin due to the two particles as a function of the x coordinate of particle 2. The scale of the x axis is set by x_s = 16.0 cm. The plot has an asymptote of $V = 5.76 \times 10^{-7}$ V as $x \to \infty$. What is q_2 in terms of e?

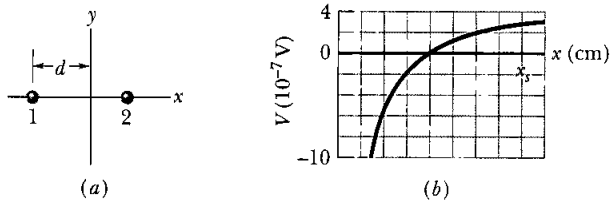


FIG. 24-64 Problem 42.

43. A nonuniform linear charge distribution given by $\lambda = bx$, where b is a constant, is located along an x axis from x = 0 to x = 0.20 m. If b = 20 nC/m^2 and V = 0 at infinity, what is the electric potential at (a) the origin and (b) the point y = 0.15 m on the y axis?

44. *Proton in a well.* Figure 24-65 shows electric potential V along an x axis. The scale of the vertical axis is set by V_s = 10.0 V. A proton is to be released at x = 3.5 cm with initial kinetic energy 4.00 eV. (a) If it is initially moving in the negative direction of the axis, does it reach a turning point (if so, what is the x coordinate of that point) or does it escape from the plotted region (if so, what is its speed at x = 0)? (b) If it is initially moving in the positive direction of the axis, does it reach a turning point (if so, what is the x coordinate of that point) or does it escape from the plotted region (if so, what is its speed at x = 6.0 cm)? What are the (c) magnitude F and (d) direction (positive or negative direction of the x axis) of the electric force on the proton if the proton moves just to the left of x = 3.0 cm? What are (e) F and (f) the direction if the proton moves just to the right of x = 5.0 cm?

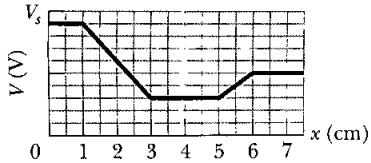


FIG. 24-65 Problem 44.

45. The smiling face of Fig. 24-66 consists of three items:

(a) a thin rod of charge −3.0 μC that forms a full circle of radius 6.0 cm;

(b) a second thin rod of charge 2.0 μC that forms a circular arc of radius 4.0 cm, subtending an angle of 90° about the center of the full circle;

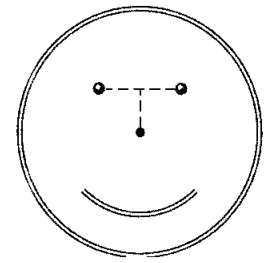


FIG. 24-66 Problem 45.

(c) an electric dipole with a dipole moment that is perpendicular to a radial line and has magnitude 1.28×10^{-21} C · m.

What is the net electric potential at the center?

46. In Fig. 24-67, we move a particle of charge +2e from infinity to the x axis. How much work do we do? Distance D is 4.00 m.

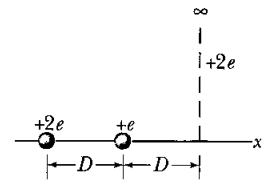


FIG. 24-67 Problem 46.

47. A particle of positive charge Q is fixed at point P. A second particle of mass m and negative charge $-q$ moves at constant speed in a circle of radius r_1, centered at P. Derive an expression for the work W that must be done by an external agent on the second particle to increase the radius of the circle of motion to r_2.

48. In a certain situation, the electric potential varies along an x axis as shown in 24-68. The scale of the vertical axis is set by V_s = 12.0 V. For the intervals (a) ab, (b) bc, (c) cd, (d) de, and (e) fg, determine the x component of the electric field, and then plot E_x versus x. (Ignore behavior at the interval end points.)

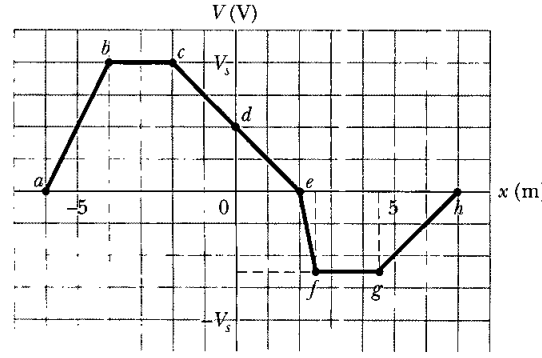


FIG. 24-68 Problem 48.

49. A disk has radius $R = 2.20$ cm. Its surface charge density is 1.50×10^{-6} C/m^2 from $r = 0$ to $r = R/2$ and 8.00×10^{-7} C/m^2 from $r = R/2$ to $r = R$. (a) What is the total charge on the disk? (b) With $V = 0$ at infinity, what is the electric potential at a point on the perpendicular central axis of the disk, at distance $z = R/2$ from the center of the disk?

50. Point charges of equal magnitudes (25 nC) and opposite signs are placed on diagonally opposite corners of a 60 cm × 80 cm rectangle. Point A is the unoccupied corner nearest the positive charge, and point B is the other unoccupied corner. Determine the potential difference $V_B - V_A$.

51. A particle having charge $q = +2.00\ \mu$C and mass $m = 0.010$ kg is connected to a string that is $L = 1.50$ m long and is tied to the pivot point P in Fig. 24-69. The particle, string and pivot point all lie on a frictionless horizontal table. The particle is released from rest when the string makes an angle $\theta = 60.0°$ with a uniform electric field of magnitude $E = 300$ V/m. Determine the speed of the particle when the string is parallel to the electric field.

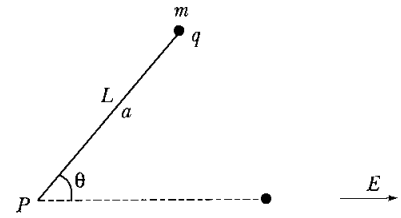


FIG. 24-69 Problem 51.

ANSWERS

Checkpoints

1. (a) negative; (b) increase

2. (a) positive; (b) higher

3. (a) rightward; (b) 1, 2, 3, 5: positive; 4, negative; (c) 3, then 1, 2, and 5 tie, then 4

4. all tie

5. (a) 2, then 1 and 3 tie; (b) 3; (c) accelerated leftward

6. (c)

7. Not possible

8. $V_a > V_c > V_b$

Reasoning Type Questions

1. (d)

2. (b)

3. (d)

4. (d)

5. (c)

6. (a)

Multiple Choice Questions

1. (d)

2. (a)

3. (a), (d)

4. (b)

5. (a), (b), (c)

6. (a)

7. (b)

8. (a), (c)

9. (a), (b)

10. (d)

11. (a)

12. (c)

13. (a)

14. (c)

15. (c)

16. (a)

17. (c)

18. (d)

19. (a)

20. (d)

21. (a)

22. (b), (d)

23. (d)

24. (c)

25. (b)

26. (c)

27. (a)

28. (c)

29. (d)

30. (a), (b)

31. (c)

32. (a)

33. (a)

34. (b)

35. (a), (d)

36. (b)

Problems

1. $3w$

2. $0.4(\sqrt{2}-4)$ J

3. 0.90 J

4. (a) 2:1; (b) $2\sqrt{\frac{Q^2}{6\pi\varepsilon_0 mr}}$; (c) $\sqrt{\frac{kQ^2}{6\pi\varepsilon_0 mr}}$

5. 0.32 km/s

6. (a) 3.0×10^5 C; (b) 3.6×10^6 J

7. (a) Yes; (b) 10^7 m/s

8. (a) 2.4 V; (b) 2.4 V; (c) 0

9. (a) 30 V; (b) 40 V; (c) 5.5 m

10. −32.0 V

11. $V=-x^3-y^3-z^3-xyz+C$

12. $V=-byz-axy+C$

13. (a) $+6.0\times10^4$ V; (b) -7.8×10^5 V; (c) 2.5 J; (d) increase; (e) same; (f) same

14. No

15. $\left(x=\frac{c^2}{b},\frac{-qc}{b}\right)$

16. (a) -4.5×10^3 V; (b) since $V(r)$ depends only on the magnitude of $\vec{r}$, the result is unchanged, that is -4.5×10^3 V

17. No such point exists

18. −6.20 V

19. (a) 24.3 mV; (b) 0

20. (a) $V=\frac{\pi\lambda+2\lambda\ln 3}{4\pi\varepsilon_0}$; (b) $E=\frac{\lambda}{2\pi\varepsilon_0 R}$

21. 4×10^{-15} V

22. 13 kV

23. $\frac{\sigma R}{2\varepsilon_0}$

24. (a) 0.54 mm; (b) 790 V

25. 2.5×10^6

26. 1.8×10^{-10} J

27. 22.5 km/s

28. (a) 39 V/m; (b) the direction of $\vec{E}$ is $-\hat{i}$, or toward plate 1

29. $(-12\text{ V/m})\hat{i}+(12\text{ V/m})\hat{j}$

30. 6.7×10^2 V/m

31. $-4\times10^{-16}(\hat{i})+1.6\times10^{-16}$ N $(\hat{j})$

32. 150 N/C

33. 16.3 μV

34. $\frac{x}{(x^2+y^2)^{3/2}}=C$

35. 6.6×10^{-15} N

36. (a) No; (b) 2:1

37. (a) 30 cm; (b) positive y direction

38. (a) $9\text{ V}, 18\hat{i}-9\hat{j}$ V/m; (b) $9\text{ V}, 18\hat{j}-9\hat{i}$ V/m; (c) $0\text{ V}, -(9\hat{i}+9\hat{j})$ V/m

39. $4qa\hat{j}$

40. (b) $\pi R^2\lambda_0\hat{i}$; (c) $-\frac{R^2\lambda}{4\varepsilon_0 x^3}(\hat{i})$

Additional Problems

41. (a) −0.268 mV; (b) −0.681 mV

42. $-5.13\times10^{-18}\text{ C}=-32e$

43. (a) 36 V; (b) 18 V

44. (a) 1.7 cm; (b) it escapes, 20 km/s; (c) $+4.8\times10^{-17}$ N; (d) the force $\vec{F}$ points in the $+x$ direction, as the electric field $\vec{E}$; (e) 3.2×10^{-17} N; (f) the force $\vec{F}$ points in the $-x$ direction, as the electric field $\vec{E}$

45. 0

46. 2.30×10^{-28} J

47. $W=\frac{Qq}{8\pi\varepsilon_0}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$

48. (a) −6.0 V/m; (b) 0; (c) 3.0 V/m; (d) 3.0 V/m; (e) 0

49. (a) 1.48 nC; (b) 795 V

50. −187.5 V

51. 0.3 m/s

Conductors 25



Reproduction in many flower species depends on insects carrying pollen from one flower to another. Honeybees commonly provide this service as they visit flowers to collect nectar. However, they do not merely brush against the flower, collecting the pollen as you would collect chalk from a dusty chalkboard by brushing against it. Instead, the pollen jumps from a flower to the bee, clings to the bee during the flight to a second flower, and then jumps to the second flower.

What causes the pollen to jump, first to the bee and then away from it?

The answer is in this chapter.

25-1 | WHAT IS PHYSICS?

Although we have already discussed about charges in last three chapters, but the only part in our daily life where we come across charges is those moving in the wires conducting electricity. Most of the time these charges are not in electrostatic equilibrium and the considerations we have discussed till now strictly do not apply, except Gauss' law. We need a bridge to go to the other side. Therefore, before we come across the conductors which do conduct electricity, let us lay down the foundations to that science.

In this chapter, we will discuss the behavior of conductors in the presence of the electric field. It is interesting to know that the basic principles remain the same, although we will need a lot of common sense to figure out things here.

25-2 | Conductors and Insulators

We can classify materials generally according to the ability of charge to move through them. **Conductors** are materials through which charge can move rather freely; examples include metals (such as copper in common lamp wire), the human body, and tap water. **Nonconductors**—also called **insulators**—are materials through which charge cannot move freely; examples include rubber (such as the insulation on common lamp wire), plastic, glass, and chemically pure water. **Semiconductors** are materials that are intermediate between conductors and insulators; examples include silicon and germanium in computer chips. **Superconductors** are materials that are *perfect* conductors, which allow charge to move without *any* hindrance.

The properties of conductors and insulators are due to the structure and electrical nature of atoms. When atoms of a conductor like copper come together to form the solid, some of their outermost (and so most loosely held) electrons become free to wander about within the solid, leaving behind positively charged atoms (*positive ions*). We call the mobile electrons *conduction electrons*. There are few (if any) free electrons in a nonconductor.

Thorough observation shows that electric charges can pass through glass as well as through any other insulator. However, the charge that can pass through bodies known as insulators during a certain time (other conditions being equal) is much smaller than the charge passing through a conductor of the same shape and size. When a substance is said to be an insulator this only means that in the given case the charges passing through it can be neglected.

Insulating properties of a substance are also determined by its state and may change significantly. Figure 25-1 represents an experiment showing that glass completely loses its insulating properties at high temperatures. Let us cut one

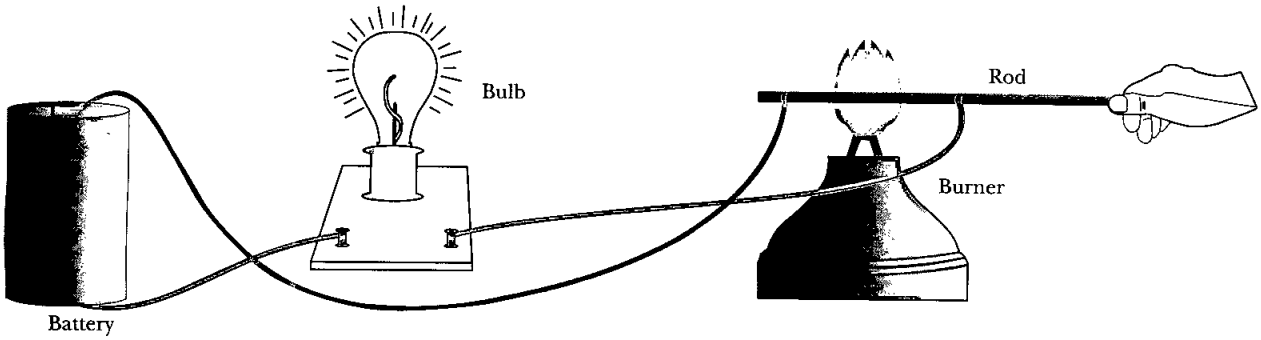


FIG. 25-1 Ordinarily glass is an insulator. However, when it is heated, it starts conducting.

of the wires leading to an electric bulb, remove the insulation and connect the terminals to a glass rod. If we close the circuit, the bulb will not glow since glass is a sufficiently good insulator at room temperature. If, however, we heat the glass rod with the help of burner, the bulb starts to glow. Thus, electric current can pass through the heated glass rod. Here, we can observe one more phenomenon. Passing through the glass rod, the electric current itself heats it, the heating being the stronger the larger the current. Therefore, if we take a sufficiently powerful bulb, that is, such that a strong electric current can pass through it, this current will heat the glass rod considerably. After that the burner can be removed, and the glass will remain hot and conduct well. The heating of the glass rod will continue to increase, and ultimately the glass will melt.

25-3 | Electric Field Inside a Conductor

In a conductor (metal), there are free electrons which are contributed by the valence shell of the individual atoms. Because of the metallic bonding between the individual atoms, these electrons are relatively free. Their number is also quite large. These electrons cannot come out of the surface of the metal easily because of a surface barrier. Suppose you put a neutral ideal conducting solid sphere in a region of space in which there is, initially, a uniform electric field. Its influence will be felt by both the free electrons and the bound charges. But only free electrons can move. So they will move in a direction opposite to the electric field. Soon charges will be redistributed on the conductor surface as shown in Fig. 25-2. This occurs in less than a microsecond:

$$F = qE.$$

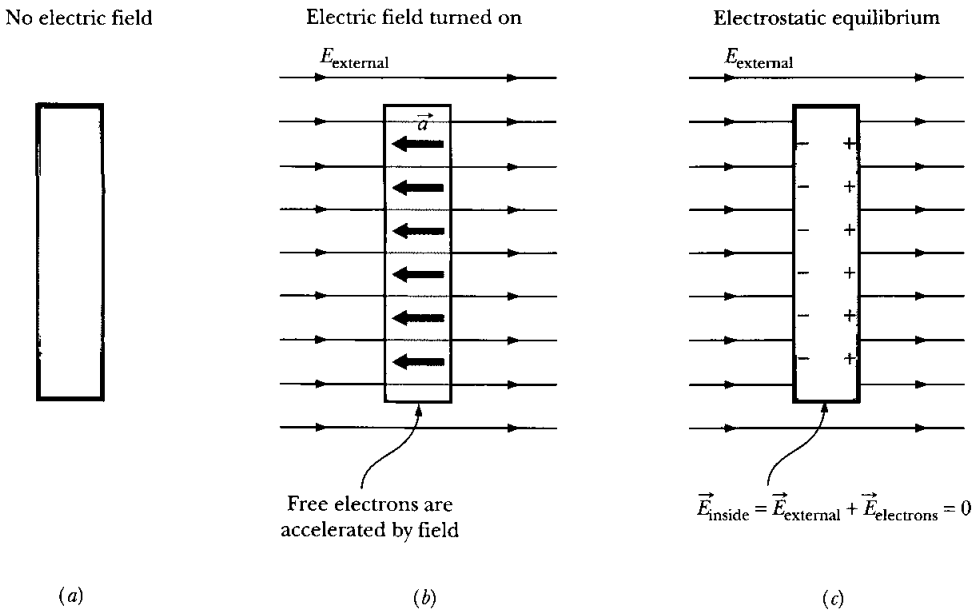


FIG. 25-2 A metallic object contains many so-called "free electrons"—electrons that are not bound to a particular location in the metal. (*a*) When the object is not subject to an electric field, there is no force on the electrons. (*b*) When an external electric field is applied, the field accelerates these free electrons in a direction opposite that of the field. (*c*) The electrons accumulate on the left side leaving positively charged ions on the right. The flow of electrons stops once the electric field set up by the accumulated charges cancels the external field. The number of free electrons in a metal is so large that practically any electric field can be canceled by this redistribution of charges.

Accumulation of small number of charges can also build up sizeable field; this field due to the induced charges is in opposite direction to the external electric field.

Since they are oppositely directed, the two contributions to the electric field inside the conductor tend to cancel each other. Now comes the profound part of the argument: the two contributions to the electric field at any point in the conductor exactly cancel. We know that they have to completely cancel because, if they don't, the free-to-move-charge in the conductor would move as a result of the force exerted on it by the electric field. The force on the charge is always in a direction that causes the charge to be redistributed to positions in which it will create its own electric field that tends to cancel the electric field that caused the charge to move. The point is that the charge will not stop responding to the electric field until the net electric field at every point in the conductor is zero.

☞ Thus, we can say that the electric field is zero at all points inside the conductor.

Remember that any conductor has practically infinite supply of electrons to nullify any practical electric field:

$$\vec{E}_{net} = \vec{E}_{ext} + \vec{E}_{in} = 0.$$

Sample Problem 25-1

Find the electric field at the center of the sphere due to the induced charges on the surface of the sphere, as shown in Fig. 25-3.

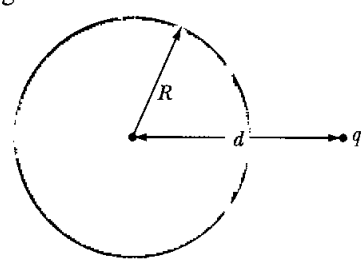


FIG. 25-3 A point charge is kept near a conducting sphere.

Conceptualize: Here, we note that the induced charges will form a certain pattern on the surface of the sphere which cannot be predicted. But we can find the electric field due to q at any point.

Classify: The net electric field inside the conductor is zero. This is produced by the induced charges and the external charge.

Compute: At the center this electric field due to q in the positive direction will be (Fig. 25-4):

$$E_Q = \frac{q}{4\pi\varepsilon_0 d^2}.$$

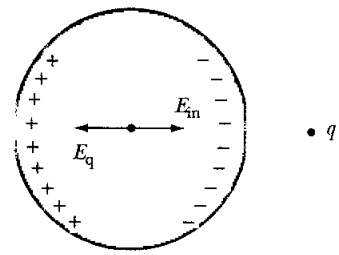


FIG. 25-4 Due to the point charge, charges redistribute over the surface of the conductor. These redistributed charges produce electric field E_{in} at the center, as shown.

Since the net electric field at the center has to be zero,

$$E_{induced} + E_Q = 0 \Rightarrow E_{induced} = -\frac{q}{4\pi\varepsilon_0 d^2}.$$

Conclude: Here, the charges will be induced on the surface of the sphere, but they will not be uniformly distributed. So we cannot apply the formula for the electric field due to a uniformly charged spherical shell.

When we say the electric field is zero, this implies that electric field should be zero inside the core of a solid metal. These are the points where the free electrons are present within the lattice. These free electrons should not experience any force.

25-4 | A Charged Isolated Conductor

Gauss' law permits us to prove an important theorem about conductors:

☞ If an excess charge is placed on an isolated conductor, then that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

This might seem reasonable, considering that charges with the same sign repel one another. You might imagine that, by moving to the surface, the added charges are getting as far away from one another as they can. We turn to Gauss' law for verification of this speculation.

Figure 25-5*a* shows, in cross-section, an isolated lump of copper hanging from an insulating thread and having an excess charge q. We place a Gaussian surface just inside the actual surface of the conductor. The electric field inside this conductor must be zero as we have discussed in previous section.

If it is zero everywhere inside our copper conductor, it must be zero for all points on the Gaussian surface because that surface, though close to the surface of the conductor, is definitely inside the conductor. This means that the flux through the Gaussian surface must be zero. Gauss' law then tells us that the net charge inside the Gaussian surface must also be zero. As the excess charge is not inside the Gaussian surface, it must be outside that surface, which means it must lie on the actual surface of the conductor.

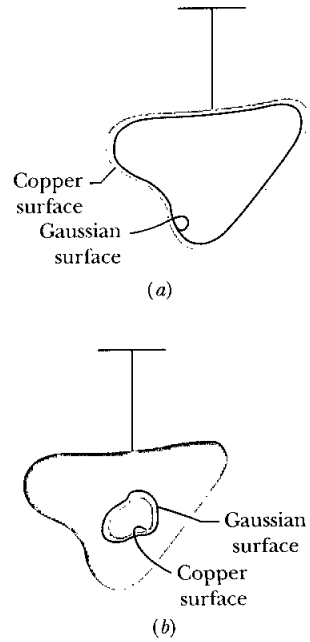


FIG. 25-5 (*a*) A lump of copper with a charge q hangs from an insulating thread. A Gaussian surface is placed within the metal, just inside the actual surface. (*b*) The lump of copper now has a cavity within it. A Gaussian surface lies within the metal, close to the cavity surface.

An Isolated Conductor with a Cavity

Figure 25-5*b* shows the same hanging conductor, but now with a cavity that is totally within the conductor. It is perhaps reasonable to suppose that when we scoop out the electrically neutral material to form the cavity, we do not change the distribution of charge or the pattern of the electric field that exists in Fig. 25-5*a*. Again, we must turn to Gauss' law for a quantitative proof. We draw a Gaussian surface surrounding the cavity, close to its surface but inside the conducting body. This is because inside the conductor, there can be no flux through this new Gaussian surface. Hence, we have

$$\phi_{\text{Gauss surface}} = 0 = \frac{q_{\text{in}}}{\varepsilon_0} \Rightarrow q_{\text{in}} = 0.$$

Therefore, from Gauss' law, we have that surface can enclose no net charge. We conclude that there is no net charge on the cavity walls; all the excess charge remains on the outer surface of the conductor, as in Fig. 25-5*a*.

But what prevents us from the charge being distributed as shown in the cavity in Fig. 25-6? In this situation also, the net charge in the cavity is zero. Gauss' law is obeyed, and hence the net flux through the cavity is zero.

Many arguments can be made against this charge configuration. One of the arguments is that the electric field in the lattice cannot be zero with this charge configuration—a necessary condition for electrostatic equilibrium. But a deeper argument lies in the conservative nature of the electric field. Imagine a loop in the region: partly inside the cavity and partly inside the metal. The net value of $\oint \vec{E} \cdot d\vec{l} = 0$ by this nature. But this is not followed by this electric field pattern. To appreciate this, follow the electric field line along the path shown in Fig. 25-6. You can see that in the cavity, $\oint \vec{E} \cdot d\vec{l}$ is positive, but inside the metal $\oint \vec{E} \cdot d\vec{l}$ is zero. However, if there is a charge suspended inside the cavity as in Fig. 25-7*a*, then what should happen? We can logically say that some charge will be induced on the inner side of the cavity (Fig. 25-7*b*). It seems that this principle is violated. It is not!

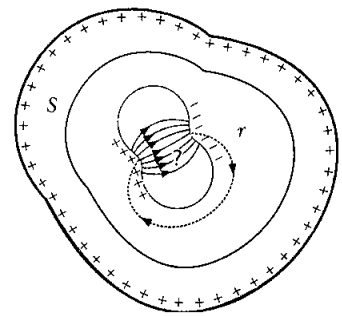


FIG. 25-6 A metal conductor having a cavity cannot have a charge distribution on the inner surface of the cavity, as shown.

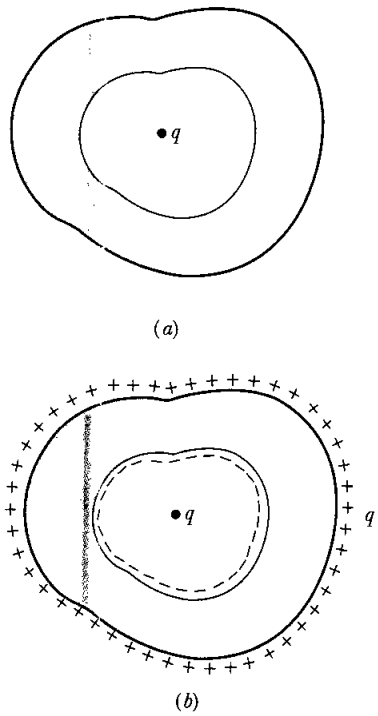


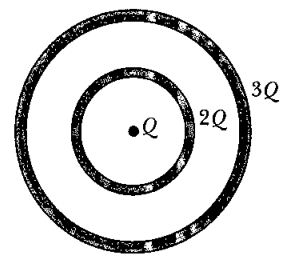
FIG. 25-7 (*a*) A charge is suspended inside a cavity in a conductor. (*b*) The pattern of charge distribution on the cavity and outside surface.

Let us look more closely. When a charge is suspended inside a cavity, charge is induced so that the electric field inside the conductor should become zero. From Gauss' law, the flux within the conductor should be zero, as the field is zero:

$$\phi_T = \frac{q_{\text{induced}} + Q}{\varepsilon_0} = 0.$$

So an equal and opposite charge is induced on the walls of the cavity. From the previous analysis, we can say that this induced charge will be borrowed from the outer surface of the conductor. So charge distribution will look like as shown in Fig. 25-7*b*. There will be an electric field inside the cavity in this case. This does not violate our original statement. We only said that the electric field is zero within the core of the *conductor*, not within the cavity inside the conductor.

✓**CHECKPOINT 1** As shown in the figure, a charge of Q is kept in the inner cavity and a charge of $2Q$ is imparted to the inner shell. A charge of $3Q$ is imparted to the outermost shell. (a) Find charge at each of the surfaces. (b) Find $\vec{E}$ in the first annular region and outside.



The Conductor Removed

Suppose that, by some magic, the excess charges could be "frozen" into position on the conductor's surface, perhaps by embedding them in a thin plastic coating, and then the conductor could be removed completely. This is equivalent to enlarging the cavity of Fig. 25-5*b* until it consumes the entire conductor, leaving only the charges. The electric field would not change at all; it would remain zero inside the thin shell of charge and would remain unchanged for all external points. This shows us that the electric field is set up by the charges and not by the conductor. The conductor simply provides an initial pathway for the charges to take up their positions.

25-5 | The Net Electric Field Near the Surface of a Conductor

You have seen that the excess charge on an isolated conductor moves entirely to the conductor's surface. However, unless the conductor is spherical, the charge does not distribute itself uniformly. Let us put it in another way, the surface charge density σ (charge per unit area) varies over the surface of any nonspherical conductor. Generally, this variation makes the determination of the electric field set up by the surface charges very difficult. However, the electric field just outside

the surface of a conductor is easy to determine using Gauss' law. To do this, we consider a section of the surface that is small enough to permit us to neglect any curvature and thus to take the section to be flat. We then imagine a tiny cylindrical Gaussian surface to be embedded in the section as in Fig. 25-8: One end cap is fully inside the conductor, the other is fully outside, and the cylinder is perpendicular to the conductor's surface. The electric field at and just outside the conductor's surface must also be perpendicular to that surface. If it were not, then it would have a component along the conductor's surface that would exert forces on the surface charges, causing them to move. However, such motion would violate our implicit assumption that we are dealing with electrostatic equilibrium.

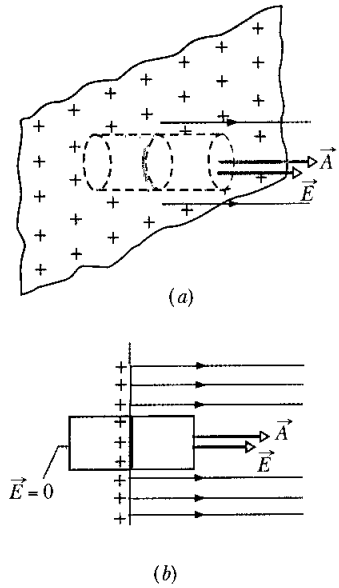


FIG. 25-8 (*a*) Perspective view and (*b*) side view of a tiny portion of a large, isolated conductor with excess positive charge on its surface. A (closed) cylindrical Gaussian surface, embedded perpendicularly in the conductor, encloses some of the charge. Electric field lines pierce the external end cap of the cylinder, but not the internal end cap. The external end cap has area A and area vector $\vec{A}$.

☞ Therefore, $\vec{E}$ is perpendicular to the conductor's surface.

We will show that,

☞ Net electric field at points near the conductor is σ/ε_0.

σ is the charge per unit area at that point on the conductor which is close to us. To show this, consider a small patch on the surface of the conductor. Assume that this patch has a charge per unit area of σ. Note that this charge per unit area may not be constant. Let us make a Gaussian surface which is pill box straddling the surface as shown in Fig. 25-8. There is no flux through the internal end cap, because the electric field within the conductor is zero. There is no flux through the curved surface of the cylinder, because internally (in the conductor) there is no electric field and externally the electric field is parallel to the curved portion of the Gaussian surface. The only flux through the Gaussian surface is that through the external end cap, which is perpendicular to the plane of the cap. We assume that the cap area A is small enough that the field magnitude E is constant over the cap. Then the flux through the cap is EA, and that is the net flux ϕ_{net} through the Gaussian surface.

The charge q enclosed by the Gaussian surface lies on the conductor's surface in an area A.

$$\phi_{net} = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

or

$$EA = \frac{\sigma A}{\varepsilon_0}$$

or

$$E = \frac{\sigma}{\varepsilon_0}. \qquad (25\text{-}1)$$

Thus, the magnitude of the electric field just outside a conductor is proportional to the surface charge density on the conductor. If the charge on the conductor is positive, the electric field is directed away from the conductor as in Fig. 25-8. It is directed toward the conductor if the charge is negative.

The field lines in Fig. 25-8 must terminate on negative charges somewhere in the environment. If we bring those charges near the conductor, the charge density at any given location on the conductor's surface changes, and so does the magnitude of the electric field. However, the relation between σ and E is still given by Eq. 25-1.

Do not be under the mistaken conception that this is the electric field due to the conductor. This is the ***net electric field*** in the vicinity of the conductor.

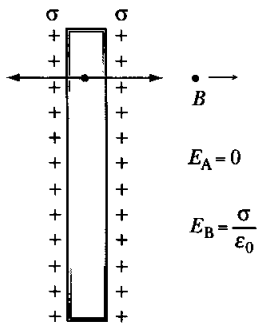


FIG. 25-9 The electric field near the surface of an infinite conducting sheet.

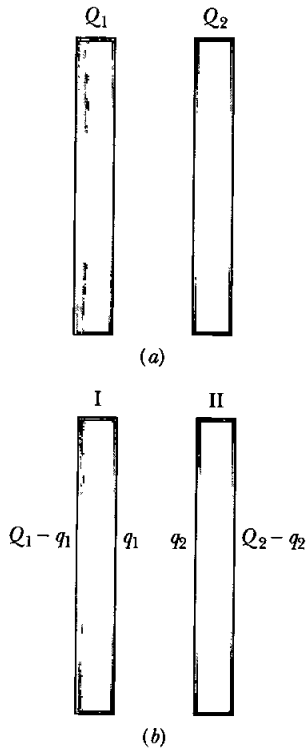


FIG. 25-10 (*a*) Two conducting sheets having a charge Q_1 and Q_2. (*b*) The distribution of charges on them.

The interesting fact of nature is that the charge gets redistributed on the surface of the conductor so that the electric field comes out to be the found value.

It is quite possible that the charge per unit area is not constant. In that case the electric field will not be having the same magnitude at all the points.

To understand this more deeply, let us take the case of a large conducting sheet on which we have kept a charge Q (see Fig. 25-9). By symmetry, this charge will spread on the surface such that the charge per unit area will be the same on both the surfaces say σ. These two surfaces will form two sheets of charge. They will superpose to produce zero electric field inside the conductor and σ/ε_0 outside the conductor. We can see that the charge on each surface will be $Q/2$. So σ for each surface is $Q/2A$. There will be two infinite sheets of charge. Each of them will produce an electric field of $\sigma/2\varepsilon_0$. (Recall what I said in Chapter 22.) Inside the conductor, these two fields will oppose each other, and outside the conductor, these two fields will add up. So the net electric field outside the conductor comes out to be σ/ε_0.

Let us consider another case where there are two large sheets of conductors rather than one as in Fig. 25-10*a*. One of the sheets is given a charge of Q_1 and another a charge of Q_2. We know that the charges will be redistributed on both the surfaces. Let us assume that the charge on facing surfaces be q_1 and q_2, respectively. Then the charge on the outer surface of the conductors will be $Q_1 - q_1$ and $Q_2 - q_2$, respectively, as can be seen in Fig. 25-10*b*.

To find the values of q_1 and q_2, we have two conditions: (a) The electric field in the first conductor is zero and (b) the electric field in the second conductor is zero. Since the sheets are large, we can assume the charge distribution is uniform. We can also use the fact that the electric field due to a sheet of charge (note: not a conducting sheet) is $\sigma/2\varepsilon_0$.

There are two conducting sheets having four surfaces as in Fig. 25-10*b*. So there are four sheets of charges. (Note, I am saying four sheets of charges, not the conductors.)

Inside the first conductor, the electric field is given by

$$E_I = \frac{Q_1 - q_1}{2A\varepsilon_0} - \frac{q_1}{2A\varepsilon_0} - \frac{q_2}{2A\varepsilon_0} - \frac{Q_2 - q_2}{2A\varepsilon_0}.$$

Inside the second conductor, the electric field is given by

$$E_{II} = \frac{Q_1 - q_1}{2A\varepsilon_0} + \frac{q_1}{2A\varepsilon_0} + \frac{q_2}{2A\varepsilon_0} - \frac{Q_2 - q_2}{2A\varepsilon_0}.$$

Both these electric fields are zero. So we obtain

$$q_1 = -q_2,$$

$$q_1 = \frac{Q_1 - Q_2}{2}.$$

$$Q_{out} = Q_1 - q_1 = Q_1 - \left(\frac{Q_1 - Q_2}{2}\right) = \left(\frac{Q_1 + Q_2}{2}\right)$$

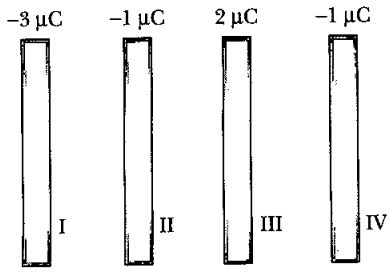
$$Q_{out} = Q_2 - \left(\frac{Q_1 - Q_2}{2}\right) = \left(\frac{Q_1 + Q_2}{2}\right)$$

This gives an interesting result that the charge on the outside surface is the same and is equal to total charge divided by 2. We can also say that the charge on the facing plates will always be equal and opposite.

If there are many sheets placed parallel to each other, the charge on outermost surfaces will be (total charge)/2 (i.e., $\sum_{i=1}^{n} Q_i/2$, here Q_i is the charge on ith sheet.

The charge on the inner faces can be found by the law of conservation of charge. The charge on the facing surface will be equal and opposite.

CHECKPOINT 2 Four large charged conducting sheets kept parallel to each other. Find the charge on each of the eight faces and electric field in region I, II, III, IV as shown in the figures below.



25-6 | Electrostatic Shielding

A conductor divides the whole space into two parts, (i) outside the conductor and (ii) inside the conductor (Fig. 25-11*a*). We know that the electric field at a point in region II due to charges outside the conductor is cancelled by the charges induced on the surface of the conductor. Suppose that we move the charges outside the conductor. The induced charge pattern will quickly change to nullify the electric field inside the conductor. So if a charge is kept inside the conductor (obviously in a cavity), it will not feel any net force due to charges outside (Fig. 25-11*b*). In fact, it will not be even able to detect the motion of charges outside. This is called ***electrostatic shielding***. This is why we are protected inside a car when lightning strikes as in Fig. 25-12.

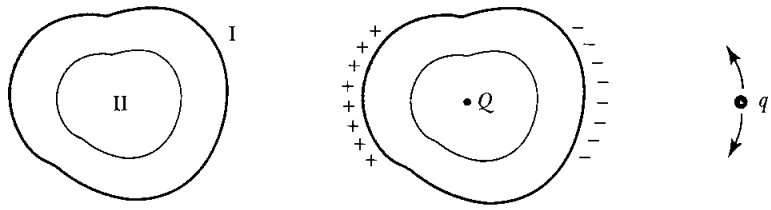


FIG. 25-11 (*a*) A conductor divides the entire space in two regions. (*b*) The motion of the charge outside the conductor changes the pattern of the induced charge on the surface of the conductor but does not influence the charges inside the cavity.

So we can say that

☞ The electric field lines cannot penetrate the surface of a conductor.

Similarly, I could say that the motion of charges inside the cavity in a conductor will not be detected by the charges outside the conductor. Some induced charges will appear on the inner surface of the cavity which will cancel the effect of these charges in region I.

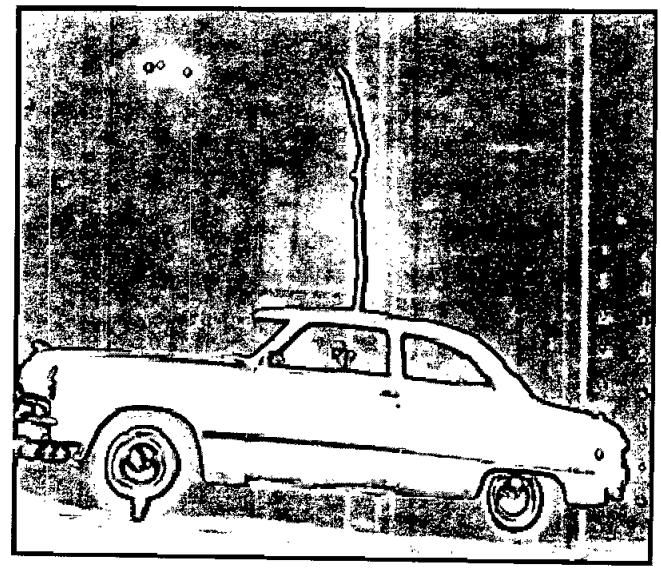


FIG. 25-12 A large spark jumps to a car's body and then exits by moving across the insulating left front tire (note the flash there), leaving the person inside unharmed. The reason the person is unharmed is due to electrostatic shielding. *(Courtesy Westinghouse Electric Corporation)*

Sample Problem 25-2

Find the electric field at A (a) due to q; (b) due to charges induced on the inner surface of the shell and (c) due to charges induced on the outer surface of the shell (Fig. 25-13).

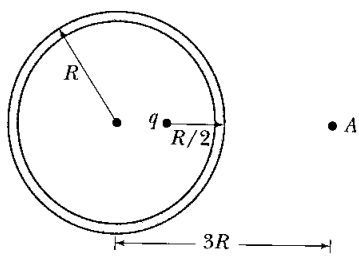


FIG. 25-13 A charge is kept inside a thin conducting shell.

Conceptualize/Classify: First we note that the charge q is not symmetrically located in the shell. Nevertheless, the electric field due to a point charge is easy to calculate.

$$E_q = \frac{q}{4\pi\varepsilon_0(5R/2)^2} = \frac{q}{25\pi\varepsilon_0 R^2}.$$

For finding the electric field due to the inner shell, we note that since the charge q is not symmetrically placed, the charge distribution on inner surface will not be symmetric. At the points close to q, charge density will be higher and lower at points farther away from the shell. If the charge was distributed uniformly, we could possibly use the formula for the electric field due to a uniformly charged shell at an external point. But this is not the case here. Not only that, we are not even aware of the pattern in which the induced charges will be distributed.

Compute: This dilemma can be easily resolved by concept of electrostatic shielding. We know that the effect of the charge q will be cancelled by the charges induced on the inner surface of the cavity:

$$\vec{E}_q + \vec{E}_{\text{induced}} = \vec{0}.$$

$$E_{\text{in}} = -\frac{q}{25\pi\varepsilon_0 R^2}.$$

To find the electric field due to the charge on the outer surface, we again do not know the pattern of induced charges on that surface. Without that, we cannot find the electric field at A. Let us use a neat trick. Suppose the charge q was at the center of the shell. What would the charge distribution on the outer surface look like? Yes, it would be uniform. From this position, when we move q, the charge on outer surface will not come to know about it because it has been shielded from cavity. So the charge on the outer surface will still be uniform. Therefore, we can use the formula of the electric field due to a uniform shell:

$$\vec{E}_{\text{outer}} = \frac{kq}{R^2}.$$

So the net electric field will be

$$\vec{E}_{\text{net}} = \frac{kq}{R^2}.$$

Sample Problem 25-3 Build your skill

Figure 25-14*a* shows a cross-section of a spherical metal shell of inner radius R. A point charge of 5.0 μC is located at a distance $R/2$ from the center of the shell. If the shell is electrically neutral, what are the (induced) charges on its inner and outer surfaces? Are those charges uniformly distributed? What is the field pattern inside and outside the shell?

Conceptualize/Classify: Figure 25-14*b* shows a cross-section of a spherical Gaussian surface within the metal, just outside the inner wall of the shell. The electric field must be zero inside the metal (and thus on the Gaussian surface inside the metal). This means that the electric flux through the Gaussian surface must also be zero. Gauss' law then tells us that the *net* charge enclosed by the Gaussian surface must be zero.

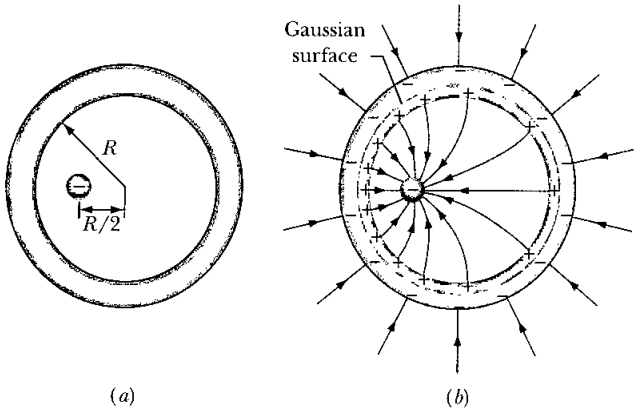


FIG. 25-14 (*a*) A negative point charge is located within a spherical metal shell that is electrically neutral. (*b*) As a result, positive charge is nonuniformly distributed on the inner wall of the shell, and an equal amount of negative charge is uniformly distributed on the outer wall.

Compute: *Reasoning*: With a point charge of 5.0 μC within the shell, a charge of –5.0 μC must lie on the inner wall of the shell in order that the net enclosed charge be zero. If the point charge were centered, this positive charge would be uniformly distributed along the inner wall. However, since the point charge is off-center, the distribution of positive charge is skewed, as suggested by Fig. 25-14*b*, because the positive charge tends to collect on the section of the inner wall nearest the (negative) point charge.

Because the shell is electrically neutral, its inner wall can have a charge of –5.0 μC only if electrons, with a total charge of 5.0 μC, leave the inner wall and move to the outer wall. There they spread out uniformly, as is also suggested by Fig. 25-14*b*. This distribution of negative charge is uniform because the shell is spherical and because the skewed distribution of positive charge on the inner wall cannot produce an electric field in the shell to affect the distribution of charge on the outer wall. The field lines inside and outside the shell are shown approximately in Fig. 25-14*b*. All the field lines intersect the shell and the point charge perpendicularly. Inside the shell, the pattern of field lines is skewed because of the skew of the positive charge distribution. Outside the shell the pattern is the same as if the point charge were centered and the shell were missing. In fact, this would be true no matter where inside the shell the point charge happened to be located.

25-7 | Potential of a Charged Isolated Conductor

In Section 25-6, we concluded that for all points inside an isolated conductor, the electric field is zero. Here, we use this fact to prove an extension of it:

☞ An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that all points of the conductor—whether on the surface or inside—come to the same potential.

This is true even if the conductor has an internal cavity and even if that cavity contains a net charge.

Our proof follows directly from

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$$

Since for all points within a conductor, it follows directly that $V_f - V_i$ for all possible pairs of points i and f in the conductor. Figure 25-15*a* is a plot of potential against radial distance r from the center or an isolated spherical conducting shell

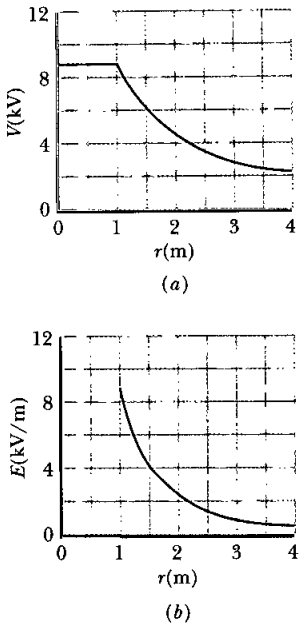


FIG. 25-15 (*a*) A plot of $V(r)$ both inside and outside a charged spherical shell of radius 1.0 m. (*b*) A plot of $E(r)$ for the same shell.

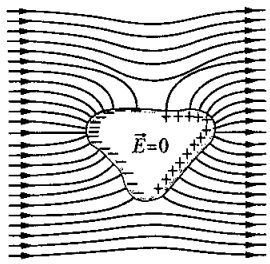


FIG. 25-16 An uncharged conductor is suspended in an external electric field. The free electrons in the conductor distribute themselves on the surface, as shown, so as to reduce the net electric field inside the conductor to zero and make the net field at the surface perpendicular to the surface.

of 1.0 m radius, having a charge of 1.0 μC. For points outside the shell, we can calculate $V(r)$ from

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

because the charge q behaves for such external points as if it were concentrated at the center of the shell. This equation holds right up to the surface of the shell. Now let us push a small test charge through the shell—assuming a small hole exists—to its center. No extra work is required to do this because no net electric force acts on the test charge once it is inside the shell. Thus, the potential at all points inside the shell has the same value as that on the surface, as shown in Fig. 25-15*a*. Figure 25-15*b* shows the variation of electric field with radial distance for the same shell. Note that $E = 0$ everywhere inside the shell. The curves in Fig. 25-15*b* can be derived from the curve in Fig. 25-15*a* by differentiating with respect to r, using

$$E_s = -\frac{\partial V}{\partial s}.$$

(Recall that the derivative of any constant is zero.) The curve in Fig. 25-15*a* can be derived from the curves in Fig. 25-15*b* by integrating with respect to r, using

$$V = -\int_i^f \vec{E}\cdot d\vec{s}.$$

Isolated Conductor in an External Electric Field

If an isolated conductor is placed in an *external electric field*, as in Fig. 25-16, all points of the conductor still come to a single potential regardless of whether the conductor has an excess charge. The free conduction electrons distribute themselves on the surface in such a way that the electric field they produce at interior points cancels the external electric field that would otherwise be there. Furthermore, the electron distribution causes the net electric field at all points on the surface to be perpendicular to the surface. If the conductor in Fig. 25-16 could be somehow removed, leaving the surface charges frozen in place, the pattern of the electric field would remain absolutely unchanged, for both exterior and interior points.

The surface of a conductor is an equipotential surface. Since a conductor has electric field lines perpendicular to its surface, a conductor should always be an equipotential surface. Not only is the surface of the conductor equipotential, but also all the points in the conductor should be equipotential because the electric field is zero. If, however, there is a cavity inside the conductor containing some charge, there will be some electric field within the cavity due to the charge. So the points within the cavity will not be at the same potential as that of the conductor; however, the surface of cavity will be at the same potential.

Sample Problem 25-4

Find the potential at point A due to (*a*) charge Q; (*b*) charge on the outer surface of the conducting shell; and (*c*) total potential at A (Fig. 25-17).

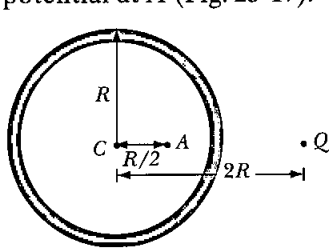


FIG. 25-17 A point charge is kept outside a spherical conducting shell.

Conceptualize/Classify: It is easy to find the potential due to a point charge. But the charge on the surface of the conducting shell is nonuniform and its potential can be tricky. However, we can use the fact that the entire conductor is an equipotential surface.

Compute: The first part, potential due to Q is easy. It can be seen that

$$V_Q = \frac{q}{6\pi\varepsilon_0 R}.$$

There are induced charges on outer surface of the shell. As discussed before, they are nonuniformly distributed. So we

cannot calculate the potential due to a nonuniform shell at A. But let us try to calculate the potential due to this shell at C, the center of the shell. We cannot use the formula for the uniformly charged shell, but the center is equidistant from all the induced charges at the surface. Since potential is a scalar, V at the center due to the induced charges will be

$$V_{C\text{in}} = \frac{\sum q_{\text{in}}}{4\pi\varepsilon_0 R}.$$

Since the shell is initially uncharged, the total charge on it will be zero. So the potential at C due to induced charges will be zero. So the total potential at C will be

$$V_{CQ} + V_{C\text{in}} = \frac{Q}{8\pi\varepsilon_0 R} + 0.$$

As we discussed before, the potential at all the points within the cavity should be the same. So $V_A = V_C$.

From this, we can say that

$$V_A = \frac{Q}{8\pi\varepsilon_0 R}$$

$$V_A = V_{AQ} + V_{A\text{in}} = \frac{Q}{8\pi\varepsilon_0 R}$$

$$V_{A\text{in}} = \frac{-Q}{24\pi\varepsilon_0 R}.$$

Conclude: We can see why the potential at A is negative. The induced charges near Q will be negative; and equal and opposite charges will be induced on the other side of the conductor. They will be farther off, producing a net negative potential at A.

A Corollary: Corona Discharge

If some charge is given to a conductor, the charge on the conductor surface is not evenly distributed. The charge per unit area is maximum at those points where the radius of curvature is minimum. To understand this, consider two conducting spheres of radii r and R kept at a large distance from each other. They are joined by a conducting wire as shown in Fig. 25-18.

If $l \gg r$ and R then effect of one sphere on the potential of another will be negligible. After we have joined the two with conducting wire, the charge will be redistributed so that both the bodies acquire the same potential. Since the bodies are at a large distance, we can safely assume that the charge will be distributed evenly on the surface. So to find the potential, we can use the formula of the potential due to a uniformly charged shell:

$$\frac{q'}{4\pi\varepsilon_0 r} = \frac{q''}{4\pi\varepsilon_0 R} = c \qquad \text{(say)},$$

$$q' = \frac{rq''}{R},$$

$$\sigma_1 = \frac{q'}{4\pi r^2} = \frac{c\varepsilon_0}{r},$$

$$\sigma_2 = \frac{q''}{4\pi R^2} = \frac{c\varepsilon_0}{R}.$$

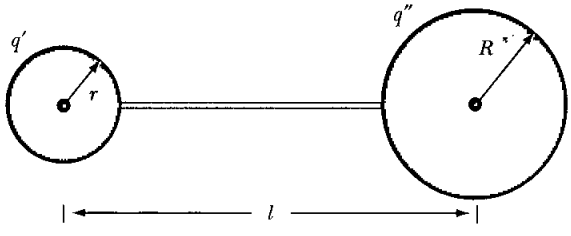


FIG. 25-18 Two conductors are joined by a thin wire to bring them to the same potential. Once they reach the same potential, the wire can even be removed and the spheres will remain at the same potential.

From this, we can note that:

☞ The charge per unit area is inversely proportional to the radius of curvature of the surface, that is,

$$\sigma \propto \frac{1}{r}.$$

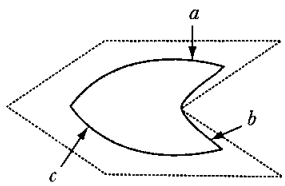


FIG. 25-19 The distribution of the surface charge density over a conductor of a complex shape. If we imagine that the conductor is enveloped by a layer whose thickness is proportional to the surface charge density, the layer will have the shape of the figure shown by the dashed line.

For conductors of a more intricate shape, the charge density distribution is much more complex. After charging the conductor shown in Fig. 25-19 and touching its lateral surface *a*, concave surface *b*, and the pointed region *c* with an electroscope, we find that the surface charge density for a conductor of an arbitrary shape is different at different regions of the surface. The minimum value corresponds to the concave surface, while the maximum value is observed on protruding tips. It should be recalled once again that although the surface of such a conductor is an equipotential surface, the density of the distributed charge can be quite nonuniform.

Spark Discharge from a Charged Conductor

On nonspherical conductors, a surface charge does not distribute itself uniformly over the surface of the conductor. At sharp points or sharp edges, the surface charge density—and thus the external electric field, which is proportional to it—may reach very high values. The air around such sharp points or edges may become ionized, producing the corona discharge that golfers and mountaineers see on the tips of bushes, golf clubs, and rock hammers when thunderstorms threaten. Such corona discharges, like hair that stands on end, are often the precursors of lightning strikes. In such circumstances, it is wise to enclose yourself in a cavity inside a conducting shell, where the electric field is guaranteed to be zero. A car (unless it is a convertible or made with a plastic body) is almost ideal as can be seen in Fig. 25-12. In Fig. 25-20, we see a lady posing for a photograph when she is in such a danger. Had she known a little physics, she would be running for her life

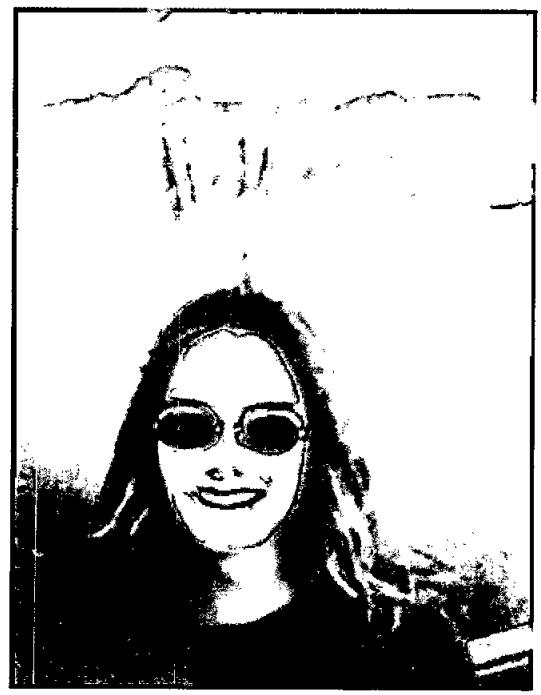


FIG. 25-20 This woman has become positively charged by an overhead storm cloud. *(Courtesy NOAA)*

rather than taking photographs. There are a lot of benefits of studying physics, after all.

If the distance between the charges is not small, the potential at any point will be due to both the conductors. We will also be required to take the effect of the induced charges into account.

Surface charge density on a conductor can also be different for the same radii of curvature when there is a charge in the vicinity. For example, consider the situation shown in Fig. 25-21. Here, it is clear that the surface charge density at different points of the sphere will not be the same even though R for all points is the same.

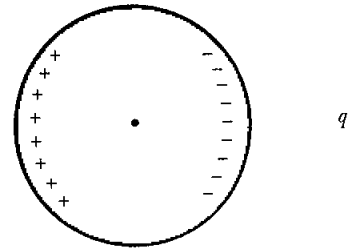


FIG. 25-21 The radius of curvature of the surface is constant, yet the electric field at all the points near the surface is not the same.

Sample Problem 25-5

Find the (a) potential of the shells in the situation shown in Fig. 25-22 and (b) the charge distribution on each of the shells after the inner shell and the outermost shells are connected by a conducting wire.

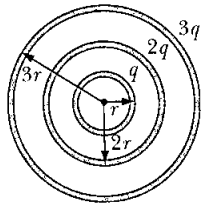


FIG. 25-22 Three concentric spherical conducting shells are given charges, as shown.

Conceptualize: Here, the problem is simplified because the shells are concentric. So the charge distribution on them will be uniform.

Classify: In Chapter 23, we have introduced the formula for the potential of a uniformly charged shell. We can use the same here.

Compute:

(a) $$V_A = \frac{q}{4\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q}{3\times 4\pi\varepsilon_0 r} = \frac{3q}{4\pi\varepsilon_0 r}.$$

$$V_B = \frac{q}{8\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q}{3\times 4\pi\varepsilon_0 r}$$

$$= \frac{2q}{4\pi\varepsilon_0 r} + \frac{q}{2\times 4\pi\varepsilon_0 r} = \frac{5q}{8\pi\varepsilon_0 r}.$$

$$V_C = \frac{6q}{3\times 4\pi\varepsilon_0 r} = \frac{q}{2\pi\varepsilon_0 r}.$$

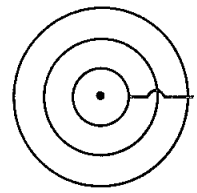


FIG. 25-23 The charge will be redistributed between the inner and outer shells so that they come to the same potential.

(b) Once the shells are connected by a wire, then

$$V_A = V_C.$$

Assume that the charges on inner shell and outer shell are q' and q'', respectively (Fig. 25-23):

$$V_A = \frac{q'}{4\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q''}{3\times 4\pi\varepsilon_0 r}.$$

$$V_C = \frac{q'}{3\times 4\pi\varepsilon_0 r} + \frac{2q}{3\times 4\pi\varepsilon_0 r} + \frac{3q''}{3\times 4\pi\varepsilon_0 r}.$$

By charge conservation, we have

$$q' + q'' = 4q.$$

$$q' = -\frac{q}{2}.$$

$$q'' = -\frac{9q}{2}.$$

Conclude: Charge on the middle shell will not change because it is not connected. Note that although there will be two pairs of charges on each shell, one on its inner surface and another on its outer surface, both of them have been clubbed together. Why?

25-8 | Earthing

Here is an example of how conduction can eliminate excess charge on an object. If you rub a copper rod with wool, charge is transferred from the wool to the rod. However, if you are holding the rod while also touching a water tap, you

cannot charge the rod in spite of the transfer. The reason is that you, the rod, and the water tap are all conductors connected, via the plumbing, to Earth's surface, which is a huge conductor. As the excess charges put on the rod by the wool repel one another, they move away from one another by moving first through the rod, then through you, and then through the water tap and plumbing to reach Earth's surface, where they can spread out.

Thus, in setting up a pathway of conductors between an object and Earth's surface, we are said to *ground* the object, and in neutralizing the object (by eliminating an unbalanced positive or negative charge), we are said to *discharge* the object. If instead of holding the copper rod in your hand, you hold it by an insulating handle, you eliminate the conducting path to Earth, and the rod can then be charged by rubbing (the charge remains on the rod), as long as you do not touch it directly with your hand.

Earth can be assumed to be a large spherical conductor. Since its radius is very large any change in amount of charge on it will not produce any substantial change in its potential:

$$V = \frac{Q}{6.4\times10^{6}\times 4\pi\varepsilon_0} \simeq 0.$$

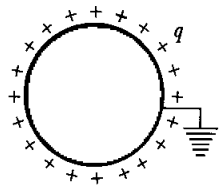


FIG. 25-24 Earthing is shown with this sign.

Thus, we can assume the potential of Earth to be zero always. If any conductor is connected to Earth, charges will flow from Earth or to the Earth so that the potential of that conductor will also become zero. Remember, it is said that the potential will become zero, not the charge! Actually the charge on Earth at any moment is very large. But we are talking about change in it. Potential is assumed to be 0, taking earth as reference point as shown in Fig. 25-24.

Let us analyze what happens when a conductor is earthed. We will take an example to illustrate the point.

Sample Problem 25-6

Assume that two spherical concentric conducting shells of radius r and R are kept as shown in Fig. 25-25. The inner shell is given a charge Q and the outer shell is earthed. Find the charge supplied by the Earth and the potential of the inner shell.

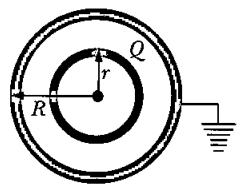


FIG. 25-25 A charged spherical conducting shell is enclosed by another conducting shell and then the outer shell is connected to the Earth.

Conceptualize/Classify: Let q be the charge given by the Earth. Note that when the body is connected to Earth, its charge will not be conserved. Some amount of charge can flow from the Earth or flow to the Earth to make its potential zero. The body which is not connected to the Earth will retain its original charge.

Compute: Potential at surface of outer shell:

$$\frac{Q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = 0.$$

So, $q = -Q$. So the Earth has to supply a charge of $-Q$ to make the potential of the outer shell zero. But the potential of inner shell need not be zero.

$$V_{\text{in}} = \frac{Q}{4\pi\varepsilon_0 r} - \frac{Q}{4\pi\varepsilon_0 R}.$$

This potential will be positive.

Conclude: Another interesting point that we can note here is that since the shells are conducting, the charge $-Q$ will reside on the inner surface of the outer shell. So the charge on the outer surface of the shell will be zero!

Can you guess why! Let us recall that the joining of the outer shell to the Earth brings both the Earth and the shell at the same potential. So the electric field between the Earth and the outer shell will be zero. This is possible only if the charge on the outer surface of shell is zero. The charge inside the cavity cannot produce any electric field outside because electric field lines cannot penetrate a conductor.

Electric Field of the Earth

Experiments show that an electroscope connected to a probe shows noticeable divergence even if there are no charged bodies near it. The larger the divergence of its leaves, the higher the point above the surface of the Earth. This means that there is a potential difference between points in the atmosphere at different altitudes, that is, there exists an electric field around the Earth's surface. The variation of potential with height is different in different seasons and for different regions. Near the surface, its approximate value is about 130 V/m. As we rise above the surface, this field rapidly attenuates, and at an altitude of 1 km its strength amounts to only 40 V/m, while at an altitude of 10 km it becomes negligibly weak. Thus, we always live and work in a considerable electric field.

Experimental investigation of this field and appropriate calculations show that the Earth as a whole has a negative charge whose mean value is estimated at 0.5 million coulombs. This charge remains practically unchanged due to some processes in the Earth's atmosphere and beyond it (in space), which have not been studied completely so far.

A natural question arises: if a constant negative charge exists on the surface of the Earth, then where are the corresponding positive charges? Where do the electric field lines terminating at the Earth's surface start? It can be easily seen that these positive charges cannot be located very far from the Earth, for example, on the Moon, stars, or planets. Otherwise, the field near the Earth would have the same configuration as the field of an isolated ball. The strength of this field would decrease in inverse proportion to the square of the distance, from the center of the Earth (and not from the Earth's surface). However, the Earth's radius is equal to about 6400 km, and therefore the change in the distance by several kilometers or tens of kilometers from the center of the Earth would have caused only a negligible change in the electric field strength. However, experiments show that, as was mentioned earlier, the strength of the Earth's electric field decreases very rapidly as we move away from it. This points to the fact that the positive charge corresponding to the negative charge of the Earth is located at a not very large altitude above its surface. Indeed, a layer of positively charged (ionized) molecules was observed at an altitude of a few tens of kilometers above the Earth. The positive bulk charge of this "cloud" compensates the negative charge of the Earth. The lines of the electric field of the Earth stretch from this layer to the surface of the Earth.

Pollination and Electrostatics

The ability of a bee to transport pollen from flower to flower depends on two features. (1) Bees become charged while flying through the air. (2) The anther of a flower (Fig. 25-26*a*) is electrically isolated from the ground but the stigma is electrically connected to the ground. In other words, it is earthed. When a bee hovers near an anther, the electric field due to the charge on the bee induces charge on a neutral pollen grain, making the closer side slightly more negative than the farther side (Fig. 25-26*b*). The charges on the two sides are equal but their distances from the bee are not, and the attractive force on the closer side is slightly larger than the repulsive force on the farther side. As a result, the pollen grain is pulled to the bee, where it clings to hairs during the bee's flight to the next flower.

When the bee happens to come close to a stigma in the next flower, the charge on the bee and induced charge on the grain bring conduction electrons up to the tip of the stigma (Fig. 25-26*c*) because the stigma is electrically connected to the ground. Those electrons attract the closer side of the grain and repel the farther side. If the grain is close enough to the stigma, the net force causes the grain to jump to the stigma, starting the fertilization of the flower.

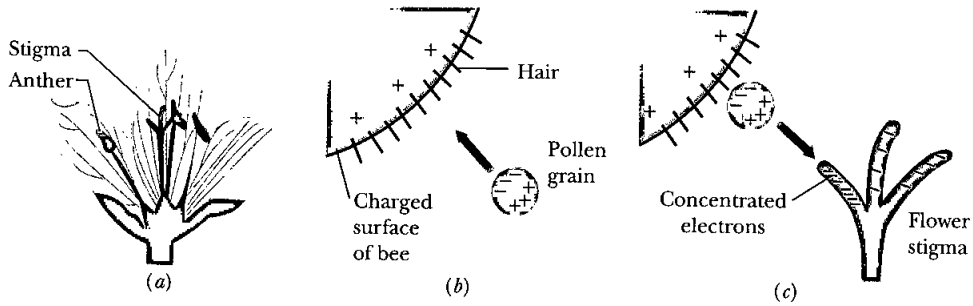


FIG. 25-26 (*a*) The anther and stigma portions of a flower. (*b*) A pollen grain near an anther has induced charge due to a bee. (*c*) Electrons collect at a stigma tip, attracting the grain.

Agricultural engineers now mimic this process by spraying plants with charged pollen grains, so that the grains collect on the stigma rather than fall uselessly to the ground.

25-9 | Electrostatic Self-Energy

Self-energy of a system of charge distribution is the work required to be done by an external agent in building the charge configuration to the present state from a state of infinite separation.

$$W_{\text{ext}} + W_E = 0$$
$$W_{\text{ext}} = -W_E = -(-\Delta U) = \Delta U.$$

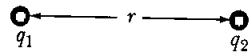


FIG. 25-27 Two charges at a distance r from each other have an electrostatic potential energy which gets converted into potential energy as soon as they are released to move apart.

In this situation (Fig. 25-27),

$$W_{\text{ext}} = U_f - U_i = \frac{q_1 q_2}{4\pi\varepsilon_0 r}.$$

In the case of a continuous charge distribution, this is the energy of interaction of all parts of the charge distribution with each other.

To find it, let us consider a small part of the charge distribution having charge dq. Find the potential due to all the charges at dq's location (since dq is a small charge this will be equal to the potential at that point):

$$dU = V dq.$$

dU is the potential energy of interaction of dq with rest of the charges:

$$U = \tfrac{1}{2}\int V dq.$$

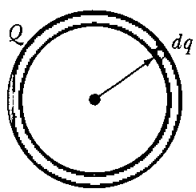


FIG. 25-28 A uniformly charged shell has some self-energy due to interaction of different parts of the charged shell. Here, we are considering the shell to be made of a large number of charges dq.

We divided the entire value by 2 because each of the pair will be covered twice. We now calculate the self-energy of a uniformly charged spherical shell shown in Fig. 25-28.

$$V = \frac{kQ}{R} \quad \text{(due to all other charges; } dq \text{ is not considered).}$$

$$U = \frac{1}{2}\int V dq = \frac{1}{2}\int_0^Q \frac{Q dq}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

Sample Problem 25-7

Find the self-energy of a uniformly charged sphere of radius R as shown in Fig. 25-29. Assume that the charge density is ρ.

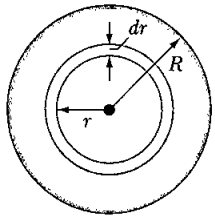


FIG. 25-29 A uniformly charged sphere.

Conceptualize: The charged sphere can be divided into small shells as shown in Fig. 25-29. We can find the potential due to rest of the charges at the location of the shell. This will give us interaction energy of the shell with the rest of the sphere. This energy can then be integrated to find the total energy.

Classify: Here, we have to find the energy of interaction of different parts of the shell with each other.

Compute: By referring to Sample Problem 24-12, we have

$$V = \frac{\rho}{6\varepsilon_0}(3R^2 - x^2).$$

The charge on a spherical shell can be found by multiplying the charge density with the volume of the shell.

$$dq = \rho \times \frac{4}{3}\pi \times 3r^2 dr.$$

$$U = \frac{1}{2}\int \frac{\rho}{6\varepsilon_0}(3R^2 - r^2) \times \rho \times 4\pi r^2 dr$$

$$= \frac{4\pi\rho^2}{6 \times 2\varepsilon_0}\int_0^R (3R^2 r^2 dr - r^4 dr)$$

$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}$$

$$= \frac{3}{20\pi\varepsilon_0}\frac{Q^2}{R}.$$

Conclude: Do not commit mistake. This is a uniformly charged sphere, it cannot be a conductor.

Sample Problem 25-8

(a) Find the total electrostatic energy of the configuration shown in Fig. 25-30.

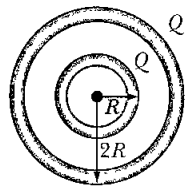


FIG. 25-30 Finding the total electrostatic energy of two charged conducting shells.

Conceptualize/Classify: Here, we find that there are three shells of charges. In the innermost shell, charges on the inner surface of the outer shell form one shell and charges on the outer shell also form a shell. Each of these shells will have a self-energy and an interaction energy with the other two. Since there are three shells, there will be three pairs of interaction energy as seen in Fig. 25-31. So in total, there will be six terms in the electrostatic potential energy expression.

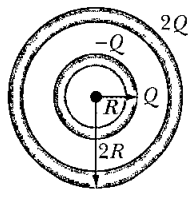


FIG. 25-31 Distribution of charges on the surface of a conductor.

Compute:

$$U_{\text{self}} = \frac{Q^2}{2\times 4\pi\varepsilon_0 R} + \frac{(-Q)^2}{2\times 2\times 4\pi\varepsilon_0 R} + \frac{4Q^2}{2\times 2\times 4\pi\varepsilon_0 R}.$$

$$U_{AB} = Q \times \frac{-Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U_{AC} = Q \times \frac{2Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U_{BC} = -Q \times \frac{2Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U = \frac{5Q^2}{16\pi\varepsilon_0 R}.$$

Conclude: Note that the interaction energies of outer shell with charge Q and with inner surface of the outer shell cancel each other. This is expected. Since there is no electric field due to the outer shell at a point inside, its interaction energy with inner shells will also be zero.

(b) If the inner shell is earthed, find the new charges on each surface and the total electrostatic energy of the configuration.

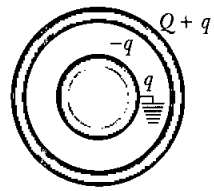


FIG. 25-32 The charge distribution after the inner shell is earthed.

Conceptualize/Classify: When the inner shell is earthed, its potential becomes zero. Let us assume that its charge becomes q as seen in Fig. 25-32.

Compute:

$$V = \frac{Q}{2\times 4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = 0.$$

Conclude: Here the potential due to the inner surface and the outer surface have been combined because the outer shell is thin and both the surfaces have the same radii. So charge on the inner shell will be $-Q/2$. The facing surface will get a charge of $Q/2$, and charge on outer surface will be $Q/2$.

(c) Find the energy lost in form of heat when the inner shell is earthed.

Compute: Self-energy is equal to

$$\frac{(Q/2)^2}{2\times 4\pi\varepsilon_0 R} + \frac{(Q/2)^2}{2\times 2\times 4\pi\varepsilon_0 R} + \frac{(Q/2)^2}{2\times 2\times 4\pi\varepsilon_0 R}.$$

Interaction energy of three shells is given by

$$-\frac{Q}{2}\times\frac{Q}{4\times 4\pi\varepsilon_0 R} = -\frac{Q^2}{32\pi\varepsilon_0 R}.$$

Total energy of the system will be

$$U_f = \frac{3Q^2}{32\pi\varepsilon_0 R}.$$

Heat energy will be dissipated because the potential energy of the system will change.

$$\Delta U = U_f - U_i$$

$$\Delta U = \frac{5Q^2}{16\pi\varepsilon_0 R} - \frac{3Q^2}{32\pi\varepsilon_0 R} = \frac{7Q^2}{32\pi\varepsilon_0 R}.$$

It is reasonable to assume that the loss in potential energy appears as thermal energy.

Conclude: We know that the heat energy is always dissipated. So if two charged metallic shells are connected, their energy will always decrease, it can never increase.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1

(c) If Statement 1 is true but Statement 2 is false

(d) If Statement 1 is false but Statement 2 is true

1. Statement 1: An uncharged conducting slab is placed normally in a uniform electric field. The resultant electric field inside the slab is zero.

Statement 2: The equal and opposite charges appearing on two surfaces of slab cancel the external field.

2. Statement 1: Any net charge of a conductor resides on its outer surface.

Statement 2: The electrostatic energy of a conductor is minimum when charge is spread over the surface.

3. Statement 1: If a metal block is placed near a large uniformly charged nonconducting sheet, it does not experience any net force.

Statement 2: A dipole does not experience any net force in a uniform electric field.

4. A large nonconducting sheet is replaced by a large conducting sheet carrying the same charge.
Statement 1: The electric field near the sheet increases.

Statement 2: The electric field near a large nonconducting sheet is $\sigma/2\varepsilon_0$ and near a conducting sheet is σ/ε_0.

MULTIPLE CHOICE QUESTIONS

sec. 25-3 Electric Field Inside a Conductor

1. A large negatively charged object was placed on an insulated table. A neutral metallic ball rolled straight toward the object but stopped before touching it. A second neutral metallic ball rolled along the same path as the first ball, struck the first ball driving it a bit closer to the negatively charged object and stopped. After all balls stopped rolling, the first ball was closer to the negatively charged object than the second ball. At no time did either ball touch the charged object. Which statement is correct concerning the final charge on each ball?

(a) The first ball is positive and the second negative

(b) The first ball is negative and the second positive

(c) Both balls remain neutral

(d) Both balls are positive

2. Figure 25-33 shows a closed surface which intersects a conducting sphere. If a positive charge is placed at the point P, the flux of the electric field through the closed surface

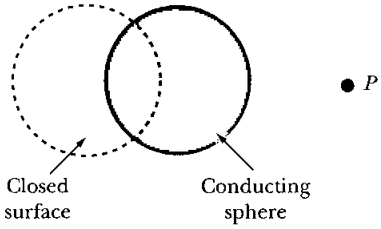


FIG. 25-33 Multiple choice question 2.

(a) Will remain zero

(b) Will become positive

(c) Will become negative

(d) Will become undefined

3. A conducting ball is charged and another similar point charge is brought closer to the ball

(a) The ball may attract the point charge

(b) The ball may repel the point charge

(c) The electric field inside the ball due to ball's charges is nonzero

(d) The net electric field inside the ball is zero

sec. 25-4 A Charged Isolated Conductor

4. A positive charge Q is brought near an isolated metal cube.

(a) The cube becomes negatively charged

(b) The cube becomes positively charged

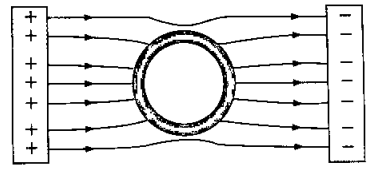
(c) The interior becomes positively charged and the surface becomes negatively charged

(d) The interior remains charge free and the surface gets nonuniform charge distribution

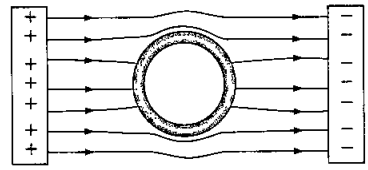
sec. 25-5 The Net Electric Field Near the Surface of a Conductor

5. There is a conducting sphere in the middle of two charged, parallel conducting plates as shown in the figure. Which electric field line distribution describes the effect of the conducting sphere better?

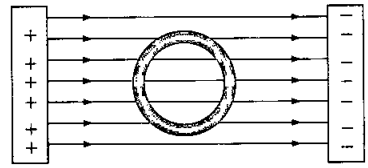
(a)



(b)



(c)



(d) None of these

sec. 25-6 Electrostatic Shielding

6. If we seal a pipe with two metal end caps around a point charge Q, the electric field outside the pipe will be

(a) Identical to the field of an isolated point charge

(b) Identically zero, because the metal shields the field

(c) Nonzero but dependent on where the charge is within the pipe

(d) Nonzero, but independent of where the charge is within the pipe

7. An electrically isolated, hollow, conducting sphere has a small positively charged ball suspended by an insulating rod from its inside surface as shown in Fig. 25-34. This causes the inner surface of the sphere to become negatively charged. When the ball is centered in the sphere the electric field outside, and far from the conducting sphere is approximately

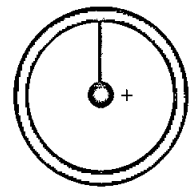


FIG. 25-34 Multiple choice question 7.

(a) Zero

(b) The same as if the ball wasn't in the center

(c) Twice of what it would be if the sphere wasn't in the center

(d) Half of what it would be if the sphere wasn't in the center

(e) Equal in magnitude but opposite in direction to what it would be if the sphere wasn't in the center

8. A negative charge $Q_1 = -q$ is placed at the center of a hollow metal sphere. A positive charge $Q_2 = +q$ is placed outside the sphere as shown in Fig. 25-35. The sphere is ungrounded. How do the magnitudes of the net forces on the charges compare?

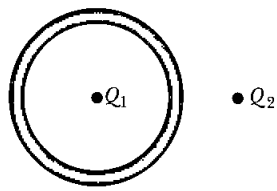


FIG. 25-35 Multiple choice question 8.

(a) Force on Q_1 > Force on Q_2

(b) Force on Q_1 = Force on Q

(c) Force on Q_1 < Force on Q_2

9. A charge Q_0 is placed at the center of spherical cavity of a conductor and another charge Q is placed outside the conductor (Fig. 25-36). The correct statement is

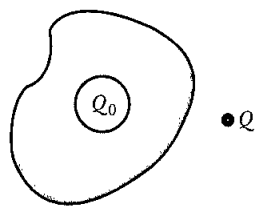


FIG. 25-36 Multiple choice question 9.

(a) Force on Q_0 and Q are equal and opposite

(b) Force on Q_0 is zero and force on Q is nonzero

(c) Force on Q and Q_0 is zero

(d) Force on Q_0 is nonzero and force on Q is zero

10. A positive point charge Q is kept (as shown in Fig. 25-37) inside a neutral conducting shell whose center is at C. An external uniform electric field E is applied. Then

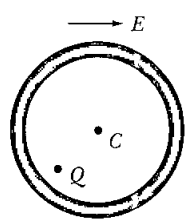


FIG. 25-37 Multiple choice question 10.

(a) Force on Q due to E is zero

(b) Net force on Q is zero

(c) Net force acting on Q and conducting shell considered as a system is zero

(d) Net force acting on the shell due to E is toward right

sec. 25-7 Potential of a Charged Isolated Conductor

11. A solid spherical conducting object has a nonzero charge on it. Which of the following statements is most appropriate?

(a) Electric potential is zero throughout the volume but the electric field is nonzero constant

(b) Electric field is zero throughout the volume but the electric potential is nonzero constant

(c) Electric potential as well as field are zero throughout the volume

(d) Electric potential as well as field are nonzero constant throughout the volume

12. Two conducting spheres, A of radius a and B of radius b, are shown in Fig. 25-38. Both are positively charged and isolated from their surroundings. They had been connected by a conducting wire but the wire has been removed. Assuming the potential an infinite distance away is zero, which of the following statements about the spheres are true:

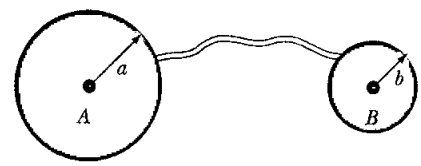


FIG. 25-38 Multiple choice question 12.

(a) Sphere A has the larger charge

(b) Sphere B has the larger charge

(c) Both spheres have the same potential

(d) Both spheres have the same charge

13. Two concentric spherical shells are given conducting positive charges

(a) Outer shell will be at higher potential

(b) Inner shell will be at higher potential

(c) Outer will always be at higher potential irrespective of the sign of charges given to two spheres

(d) No prediction can be made using the given data

14. Which of the following statements about solid conductors in electrostatics is (are) true?

(I) The electric field inside the conductor is always zero

(II) The electric potential inside the conductor is always zero

(III) Any net charge is on the surface

(a) I only

(b) III only

(c) II and III only

(d) I and III only

15. An ellipsoidal cavity is carved within a perfect conductor. A positive charge q is placed at the center of the cavity. The points A and B are on the cavity surface as shown in Fig. 25-39.

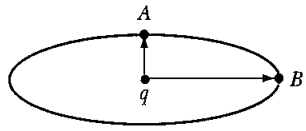


FIG. 25-39 Multiple choice question 15.

(a) Electric field near A in the cavity = electric field near B in the cavity

(b) Charge density at A = charge density at B

(c) Potential at A = potential at B

(d) None of these

sec. 25-8 Earthing

16. A conductor A is given a charge of amount $+Q$ and then placed inside an uncharged deep metal can B, without touching it.

(a) The potential of A does not change when it is placed inside B

(b) If B is earthed, $+Q$ amount of charge flows from it into the Earth

(c) If B is earthed, the potential of A is reduced

(d) If B is earthed, the potential of A and B both becomes zero

17. Consider the arrangement of two concentric conducting shells with radii a and b, respectively, as shown in Fig. 25-40. The inner shell carries a charge Q. The outer shell is grounded

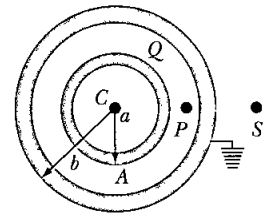


FIG. 25-40 Multiple choice question 17.

(a) Potential at C equals zero

(b) Potential at S equals zero

(c) Potential at P equals zero

(d) None

sec. 25-9 Electrostatic Self-Energy

18. A very large sphere having charge Q uniformly distributed on the surface is compressed uniformly till its radius reduces to R. The work done by electric forces in this process is

(a) $\dfrac{Q^2}{8\pi\varepsilon_0 R}$

(b) $\dfrac{-Q^2}{8\pi\varepsilon_0 R}$

(c) ∞

(d) 0

19. A sphere of radius R is charged uniformly with a volume charge density. It is disassembled and reassembled in two identical spheres kept far from each other.

(a) Positive work is done by external agent

(b) Potential energy of system decreases

(c) Potential energy of system increases

(d) Work is done by the system

PROBLEMS

sec. 25-2 Conductors and Insulators

1. Identical isolated conducting spheres 1 and 2 have equal charges and are separated by a distance that is large compared with their diameters (Fig. 25-41*a*). The electrostatic force acting on sphere 2 due to sphere 1 is $\vec{F}$. Suppose now that a third identical sphere 3, having an insulating handle and initially neutral, is touched first to sphere 1 (Fig. 25-41*b*), then to sphere 2 (Fig. 25-41*c*), and finally removed (Fig. 25-36*d*). The electrostatic force that now acts on sphere 2 has magnitude F'. What is the ratio F/F'?

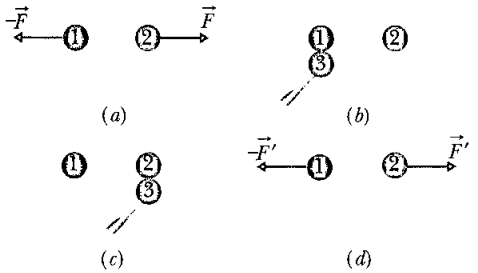


FIG. 25-41 Problem 1.

2. Two small identical conducting spheres, fixed in place, attract each other with an electrostatic force of 0.108 N when their center-to-center separation is 50.0 cm. The spheres are then connected by a thin conducting wire. When the wire is removed, the spheres repel each other with an electrostatic force of 0.0360 N. Of the initial charges on the spheres, with a positive net charge, what was (a) the negative charge on one of them and (b) the positive charge on the other?

sec. 25-3 Electric Field Inside a Conductor

3. A point charge $q = 1.0 \times 10^{-7}$ C is at the center of a spherical cavity of radius 3.0 cm in a chunk of metal. Use Gauss' law to find the electric field (a) 1.5 cm from the cavity center and (b) any place in the metal.

4. A positive charge q is placed in front of a conducting solid body at a distance d from its center. (a) Find the electric field at the center of the cube due to the charges appearing on its surface. (b) Does the body have to be spherical for the answer to be true?

sec. 25-4 A Charged Isolated Conductor

5. An isolated conductor of arbitrary shape has a net charge of $+10 \times 10^{-6}$ C. Inside the conductor is a cavity within which is a point charge $q = +3.0 \times 10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor?

6. The electric field at point P just outside the outer surface of a hollow spherical conductor of inner radius 10 cm and outer radius 20 cm has magnitude 450 N/C and is directed outward. When an unknown point charge Q is introduced into the center of the sphere, the electric field at P is still directed outward but is now 180 N/C. (a) What was the net charge enclosed by the outer surface before Q was introduced? (b) What is charge Q? After Q is introduced, what is the charge on the (c) inner and (d) outer surface of the conductor?

7. A metal sphere centered at the origin has a radius R and a net charge Q. The electric field at the point $x = 5R$ is E_0. The sphere is replaced by a different metal sphere centered at the origin with radius $2R$ and a net charge Q_0. The field at $x = 5R$ is still E_0. What is the ratio of Q and Q_0?

8. A uniformly charged conducting sphere of 1.2 m diameter has a surface charge density of 8.1 μC/m^2. (a) Find the net charge on the sphere. (b) What is the total electric flux leaving the surface of the sphere?

9. A square metal plate of edge length 8.0 cm and negligible thickness has a total charge of 6.0×10^{-6} C. (a) Estimate the magnitude E of the electric field just off the center of the plate (at, say, a distance of 0.50 mm from the center) by assuming that the charge is spread uniformly over the two faces of the plate. (b) Estimate E at a distance of 30 m (large relative to the plate size) by assuming that the plate is a point charge.

10. Two large metal plates of area 1.0 m^2 face each other. They are 5.0 cm apart and have equal but opposite charges on their inner surfaces. If the magnitude E of the electric field between the plates is 55 N/C, what is the magnitude of the charge on each plate? Neglect edge effects.

11. An electron is shot directly toward the center of a large metal plate that has surface charge density -2.0×10^{-6} C/m^2. If the initial kinetic energy of the electron is 1.60×10^{-17} J and if the electron is to stop (due to electrostatic repulsion from the plate) just as it reaches the plate, how far from the plate must the launch point be?

12. Two large parallel conducting plates are 10 cm apart and carry equal but opposite charges on their facing surfaces. An electron placed midway between the plates experiences a force of 3.2×10^{-17} N. What is the potential difference between the plates?

13. Three identical metal plates with large surface areas are kept parallel to each other. The leftmost plate is given a charge Q, the rightmost a charge $-2Q$, and the middle one remains neutral. Find the charge appearing on each of surfaces.

14. Three uncharged conducting large plates are placed parallel to each other in a uniform electric field E which is perpendicular to the plates. Find the induced charge density on each surface of each plate.

15. Two large metal plates of area A each with charges Q and $-Q$ are kept facing each other. Show that they attract each other with a force of $Q^2/2A\varepsilon_0$.

sec. 25-6 Electrostatic Shielding

16. A conducting spherical shell of outer radius R and inner radius $r = 3R/4$ has no net charge on it. At its center there is a point charge q, and at a distance $2R$ from its center there is a point charge Q (Fig. 25-42). (a) What is the magnitude of the electrostatic force on the charge q at the center due to the shell? (b) What is the magnitude of net force on it?

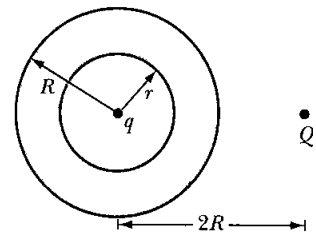


FIG. 25-42 Problem 16.

17. A pellet with charge $+Q$ is centered inside a spherical cavity in a conducting ball. The conducting ball has a charge $-2Q$ on it. The cavity in the ball is off-center, as shown in Fig. 25-43. (a) How much charge resides on the inner surface of the cavity? (b) Is the charge on the cavity surface distributed uniformly or nonuniformly? Why? (c) How much charge resides on the outer surface of the conducting ball? (d) Is the charge on the ball's outer surface distributed uniformly or nonuniformly?

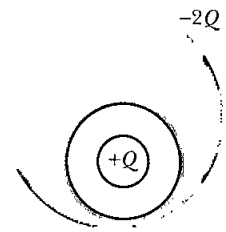


FIG. 25-43 Problem 17.

sec. 25-7 Potential of a Charged Isolated Conductor

18. What are (a) the charge and (b) the charge density on the surface of a conducting sphere of radius 0.15 m whose potential is 200 V (with $V = 0$ at infinity)?

19. What is the excess charge on a conducting sphere of radius $r = 0.15$ m if the potential of the sphere is 1500 V and $V = 0$ at infinity?

20. Sphere 1 with radius R_1 has positive charge q. Sphere 2 with radius $2.00R_1$ is far from sphere 1 and initially uncharged. After the separated spheres are connected with a wire thin enough to retain only negligible charge, (a) is potential V_1 of sphere 1 greater than, less than, or equal to potential V_2 of sphere 2? What fraction of q ends up on (b) sphere 1 and (c) sphere 2? (d) What is the ratio σ_1/σ_2 of the surface charge densities of the spheres?

21. Two metal spheres, each of radius 3.0 cm, have a center-to-center separation of 2.0 m. Sphere 1 has charge $+1.0 \times 10^{-8}$ C; sphere 2 has charge -3.0×10^{-8} C. Assume that the separation is large enough for us to assume that the charge on each sphere is uniformly distributed (the spheres do not affect each other).

With $V = 0$ at infinity, calculate (a) the potential at the point halfway between the centers and the potential on the surface of (b) sphere 1 and (c) sphere 2.

22. Two isolated, concentric, conducting spherical shells have radii $R_1 = 0.500$ m and $R_2 = 1.00$ m, uniform charges $q_1 = +2.00\ \mu C$ and $q_2 = +1.00\ \mu C$, and negligible thicknesses. What is the magnitude of the electric field E at radial distance (a) $r = 4.00$ m, (b) $r = 0.700$ m, and (c) $r = 0.200$ m? With $V = 0$ at infinity, what is V at (d) $r = 4.00$ m, (e) $r = 1.00$ m, (f) $r = 0.700$ m, (g) $r = 0.500$ m, (h) $r = 0.200$ m, and (i) $r = 0$? (j) Sketch $E(r)$ and $V(r)$.

23. Thousand small mercury drops each of the radius r and charge q coalesce together to form one spherical drop. What is the ratio of the potential of the bigger drop to that of the smaller one?

24. A particle with an unknown charge is placed at the center of a conducting spherical shell. The electric field outside of the shell is $3k|Q|/r^2$, directed toward the center. The total charge on the inner surface of the shell is $-2|Q|$.

(a) What is the net charge on the shell?

(b) What is the charge on the particle at the center of the shell?

(c) What is the potential of the shell?

25. A point object with a charge $+Q$ is placed at the center of a conducting shell of inner radius R, outer radius $2R$, and a charge of $-4Q$. A thin-walled conducting shell of radius $3R$ and a charge of $+4Q$ is concentric with the point object and the first shell. Defining $V = 0$ at infinity, find all the distances from the center at which the electric potential is zero.

26. Two concentric, spherical, metal shells are insulated from each other and the surroundings. The inner shell carries a total charge of $+2Q$, and the outer shell a total charge of $-Q$. In electrostatic equilibrium, what is the charge on

(a) the outer surface of the inner shell?

(b) the inner surface of the outer shell?

(c) the outer surface of the outer shell?

(d) the metal shells are now connected by a metal wire?

(e) answer part (a) for the situation obtained after electrostatic equilibrium is reached.

27. At a distance R from the center of the isolated metallic ball of radius r we put a point charge Q. Determine the potential of the ball.

sec. 25-8 Earthing

28. On the middle one of three thin concentric spherical metal shells of radius R there is electric charge of magnitude Q. The inner spherical shell of radius $R/2$ and the outer spherical shell of radius $3R/2$ are earthed. (a) What is the amount of the electric charge on the earthed spherical shells? (b) Plot the electric field strength versus the distance from the center.

sec. 25-9 Electrostatic Self-Energy

29. If a thin spherical shell of radius r and charge q is kept inside a thick spherical shell of inner radius $2r$, outer radius $3r$, and total charge $2q$. (a) Find the charge distribution on each surface. (b) Find the self-energy of each charge distribution. (c) Find the total electrostatic energy stored in the system.

30. Two isolated metallic solid spheres of radii R and $2R$ are charged such that both of these have same charge Q. The spheres are located far away from each other, and connected by a thin conducting wire. Find the heat dissipated in the wire.

ADDITIONAL PROBLEMS

31. A solid conducting sphere of radius 3.0 cm has a charge of 30 nC distributed uniformly over its surface. Let A be a point 1.0 cm from the center of the sphere, S be a point on the surface of the sphere, and B be a point 5.0 cm from the center of the sphere. What are the electric potential differences (a) $V_S - V_B$ and (b) $V_A - V_B$?

32. A solid copper sphere whose radius is 1.0 cm has a very thin surface coating of nickel. Some of the nickel atoms are radioactive, each atom emitting an electron as it decays. Half of these electrons enter the copper sphere, each depositing 100 keV of energy there. The other half of the electrons escape, each carrying away a charge $-e$. The nickel coating has an activity of 3.70×10^8 radioactive decays per second. The sphere is hung from a long, nonconducting string and isolated from its surroundings. (a) How long will it take for the potential of the sphere to increase by 1000 V? (b) How long will it take for the temperature of the sphere to increase by 5.0 K due to the energy deposited by the electrons? The heat capacity of the sphere is 14 J/K.

33. Four large parallel thin metallic plates are located, as shown in Fig. 25-44. Find the potential difference between the inner plates. Dimensions a, d, and the potential difference $\Delta\Phi_1$ and $\Delta\Phi_2$ are known.

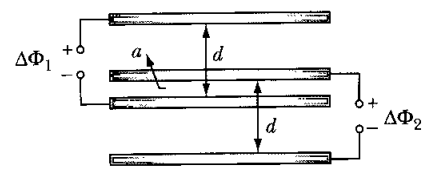


FIG. 25-44 Problem 33.

34. A charge $-Q$ is kept in an uncharged metallic shell as shown in Fig. 25-45. (a) Find the potential at a point outside the shell at a distance $2R$ from the center. (b) Find the potential due to $-Q$ at that point, if it is located at a distance x from the point. (c) Also find the potential at that point due to the charges induced on the inner surface.

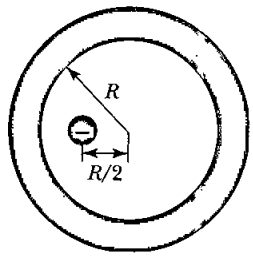


FIG. 25-45 Problem 34.

35. Three metal plates are positively charged and laid parallel to each other as shown in Fig. 25-46. The net charge on the left plate is q. The net force on the middle plate is P. If the right plate was removed, the force acting on the middle plate will be equal to $P/2$. Find charges on each of the plates. The area of each plate is S. The field created by the plates is considered to be uniform.



FIG. 25-46 Problem 35.

36. The electric field in space is produced by an external uniform electric field and the field of a charged metal plate (which can be considered uniform) as shown in Fig. 25-47. The electric field above the plate $E_2 = 2 \times 10^4$ V/m, and under it $E_1 = 5 \times 10^4$ V/m. What is the charge on the plate, if the force exerted on the plate by the external field is 0.7 N.

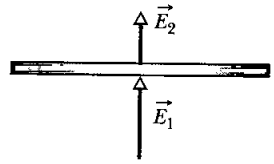


FIG. 25-47 Problem 36.

37. A thin ring of radius R has been uniformly charged with an amount of electric charge Q and placed in relation to a conducting sphere in such a way that the center of sphere O, lies on the axis at a distance of l from the plane of the ring (Fig. 25-48). (a) Determine the potential of the sphere. (b) When the sphere is grounded, what is the total charge on it?

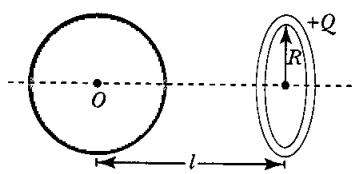


FIG. 25-48 Problem 37.

38. A point object with a positive charge of $+Q$ is located at the center of one or more concentric spherical shells. The shells are either conducting or nonconducting with uniform surface charge density. Figure 25-49 is the graph of electric flux passing through Gaussian spheres, centered on the object, as a function of the distance r from it.

(a) What is the minimum number of spherical shells required to produce the graph?

(b) Describe each shell completely, specifying its net charge, its inner and outer radius, and whether it is conducting or nonconducting.

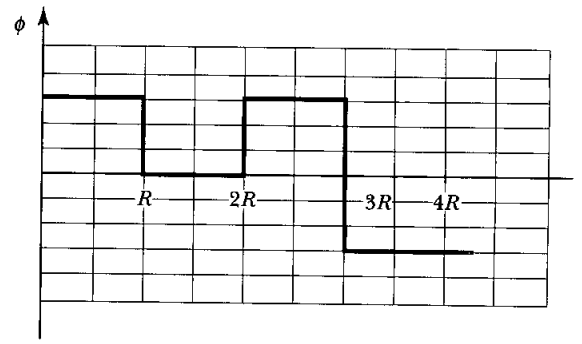


FIG. 25-49 Problem 38.

39. A charge q is kept at a distance $2R$ from the center of a metallic shell of radius R. The metallic shell is uncharged. (a) What is the net electric field at the center of the shell? What is the electric field due to the induced charges at the center of the shell? (b) What is the potential at the center of the shell? What is the potential at the center of the shell due to the induced charges? What is the potential due to the induced charges at a distance $R/2$ from the center on the line joining the center and q?

40. Mercury from a metal vessel maintained at a potential of 3 V falls in spherical drops of 2 mm diameter through a small hole into a thin-walled metal sphere of diameter 8 cm, isolated and placed in air until the sphere is filled with mercury. Ignoring the thickness of the metal, calculate the (a) final potential and (b) electrical energy of the metal sphere.

ANSWERS

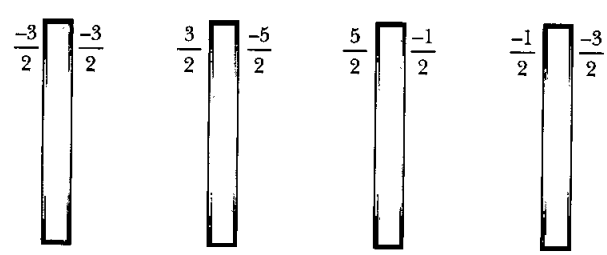
Checkpoints

1. (a) Q, $-Q$, $3Q$, $-3Q$, $6Q$;

(b) $E_I = \frac{kQ}{x^2}$, $E_{II} = \frac{3kQ}{x^2}$, $E_{III} = \frac{6kQ}{x^2}$

2. $E_I = \frac{\sigma}{\varepsilon_0} = \frac{-3\times10^{-6}}{2A\varepsilon_0}$; $E_{II} = \frac{\sigma}{\varepsilon_0} = \frac{-5\times10^{-6}}{2A\varepsilon_0}$;

$E_{III} = -\frac{1\times10^{-6}}{2A\varepsilon_0}$; $E_{IV} = -\frac{3\times10^{-6}}{2A\varepsilon_0}$



Reasoning Type Questions

1. (a)
2. (a)
3. (a)
4. (d)

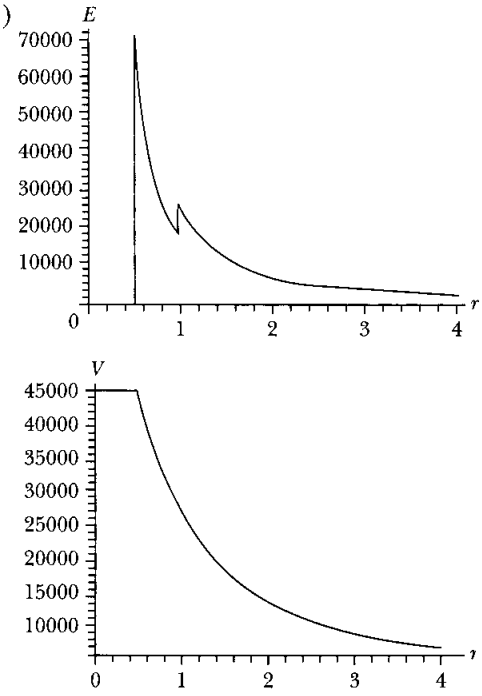
Multiple Choice Questions

1. (a)
2. (b)
3. (a), (b), (c), (d)
4. (d)
5. (a)
6. (d)
7. (b)
8. (c)
9. (b)
10. (d)
11. (b)
12. (a), (c)
13. (b)
14. (d)
15. (c)
16. (b), (c)
17. (b)
18. (b)
19. (b), (d)

Problems

1. 0.375
2. (a) $-1.00\ \mu C$; (b) $3.00\ \mu C$
3. (a) 4.0×10^{6} N/C; (b) 0, since P_2 is inside the metal
4. (a) $\frac{q}{4\pi\varepsilon_0 d^2}$; (b) no
5. (a) -3.0×10^{-6} C; (b) $+1.3\times10^{-5}$ C
6. (a) +2.0 nC; (b) −1.2 nC; (c) +1.2 nC; (d) +0.80 nC
7. 1
8. (a) 37 μC; (b) 4.1×10^{6} N·m^2/C
9. (a) 5.3×10^{7} N/C; (b) 60 N/C
10. 4.9×10^{-10} C
11. 0.44 mm
12. 60 V
13. $-Q/2$; $3Q/2$; $-3Q/2$; $3Q/2$; $-3Q/2$; $-Q/2$
14. $-\varepsilon_0 E, \varepsilon_0 E$, ... and so on
16. (a) $\frac{kQq}{4R^2}$ toward right; (b) 0
17. (a) $-Q$; (b) uniformly; (c) $-Q$; (d) uniformly
18. (a) 3.3 nC; (b) 12 nC/m^2
19. 2.5×10^{-8}C
20. (a) $q_1 = q/3, q_2 = 2q/3$; (b) 0.333; (c) 0.667; (d) 2.00
21. (a) -1.8×10^{2} V; (b) 2.9 kV; (c) −8.9 kV
22. (a) 1.69×10^{3} V/m; (b) 3.67×10^{4} V/m; (c) 0; (d) 6.74×10^{3} V; (e) 2.70×10^{4} V; (f) 3.47×10^{4} V; (g) 4.50×10^{4} V; (h) 4.50×10^{4} V; (i) $V = 4.50\times10^{4}$ V;

(j)



23. 100
24. (a) $-5Q$; (b) $2Q$; (c) $\frac{-3kQ}{R}$
25. $\frac{9R}{4}, \frac{6R}{7}$
26. (a) $2Q$; (b) $-2Q$; (c) Q; (d) 0, 0, Q
27. $\frac{kQ}{R}$
28. (a) $\frac{-Q}{4}$; (b) $\frac{-3Q}{4}$
29. (a) $0, q, -q, 3q$; (b) $0, \frac{kq^2}{2R}, \frac{kq^2}{4R}, \frac{3kq^2}{2R}$; (c) $\frac{7q^2}{16\pi\varepsilon_0 R}$
30. $\frac{kQ^2}{12R}$

Additional Problems

31. (a) 3.6×10^3 V; (b) 3.6×10^3 V

32. (a) 38 s; (b) 280 days

33. $\dfrac{(\Delta\phi_1 + \Delta\phi_2)a}{a+d}$

34. (a) $-\dfrac{kQ}{2R}$; (b) $-\dfrac{kQ}{x}$; (c) $\dfrac{kQ}{x}$

35. $q_1 = q, q_2 = \dfrac{PS\varepsilon_0}{q}, q_3 = 3q$

36. 20 μC

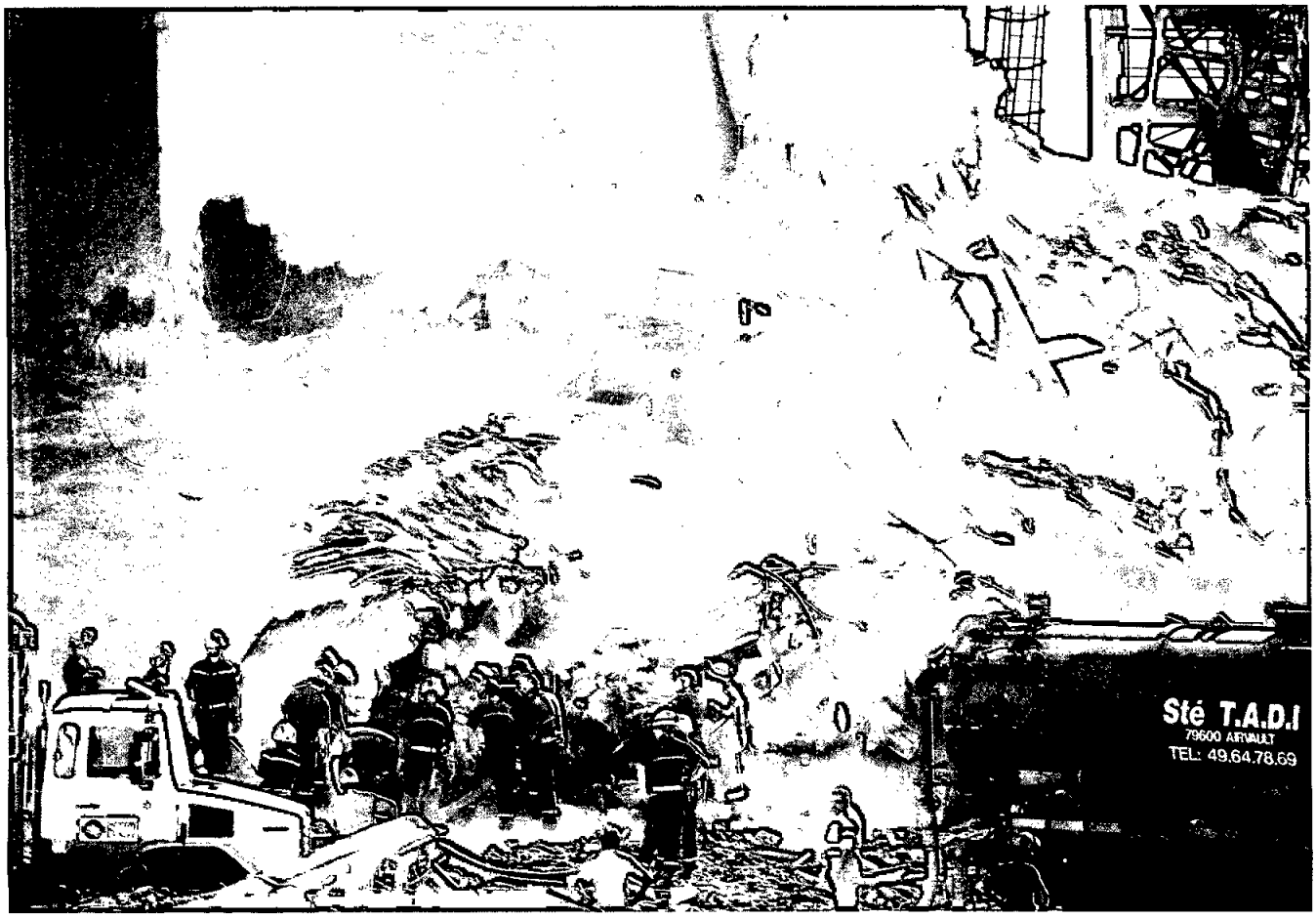
37. $\dfrac{kQ}{\sqrt{l^2+R^2}}, \dfrac{-QR}{\sqrt{l^2+R^2}}$

38. (a) Two; (b) first shell is conducting with no net charge, its inner radius is R and outer radius is $2R$. Second shell is nonconducting having a net charge of $-2Q$ and the radius of its charged surface is $3R$.

39. (a) $0, -\dfrac{kQ}{4R^2}$; (b) $\dfrac{kQ}{2R}, 0, -\dfrac{kQ}{6R}$

40. (a) 4800 V; (b) 5120 μJ

Capacitors and Capacitance 26



Tienda Brothers/Gamma-Presse, Inc.

Explosions of airborne dust in grain storage bins (as above), coal mines, flour mills, and many powder industries are a common occurrence, often with loss of life and much property damage. Usually the explosions are due to sparking between charged objects or between a charged object and a grounded connection. Engineers cannot eliminate the possibility of sparking, but they can take measures to reduce the chance that a spark will set off an explosion.

What determines whether sparking will cause an explosion of airborne dust?

The answer is in this chapter.

26-1 WHAT IS PHYSICS?

One goal of physics is to provide the basic science for practical devices designed by engineers. The focus of this chapter is on one extremely common example—the capacitor, a device in which electrical energy can be stored. For example, the batteries in a camera store energy in the photoflash unit by charging a capacitor. The batteries can supply energy at only a modest rate, too slowly for the photoflash unit to emit a flash of light. However, once the capacitor is charged, it can supply energy at a much greater rate when the photoflash unit is triggered—enough energy to allow the unit to emit a burst of bright light.

The physics of capacitors can be generalized to other devices and to any situation involving electric fields. For example, Earth's atmospheric electric field is modeled by meteorologists as being produced by a huge spherical capacitor that partially discharges via lightning. The charge that skis collect as they slide along snow can be modeled as being stored in a capacitor that frequently discharges as sparks (which can be seen by nighttime skiers on dry snow).

The first step in our discussion of capacitors is to determine how much charge can be stored. This "how much" is called capacitance.

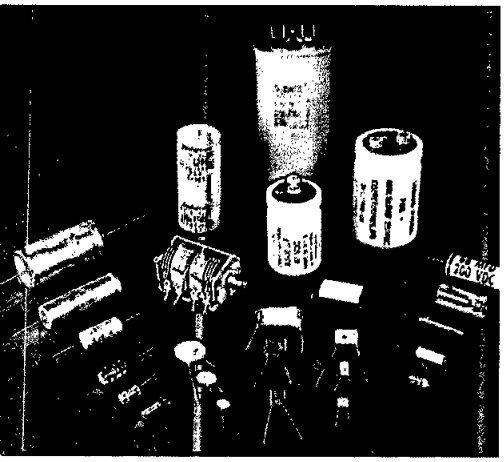


FIG. 26-1 An assortment of capacitors.

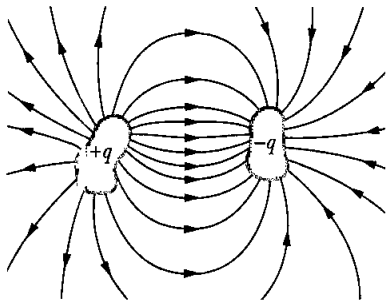


FIG. 26-2 Two conductors, isolated electrically from each other and from their surroundings, form a *capacitor.* When the capacitor is charged, the charges on the conductors, or *plates* as they are called, have the same magnitude q but opposite signs. *(Paul Silvermann/ Fundamental Photographs)*

26-2 | Capacitance

Figure 26-1 shows some of the many sizes and shapes of capacitors. Figure 26-2 shows the basic elements of *any* capacitor—two isolated conductors of any shape. No matter what their geometry, flat or not, we call these conductors *plates.*

Figure 26-3*a* shows a less general but more conventional arrangement, called a *parallel-plate capacitor,* consisting of two parallel conducting plates of area A separated by a distance d. The symbol we use to represent a capacitor (⊣⊢) is based on the structure of a parallel-plate capacitor but is used for capacitors of all geometries. We assume for the time being that no material medium (such as glass or plastic) is present in the region between the plates. In Section 26-6, we shall remove this restriction.

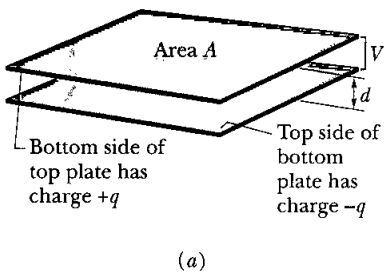
When a capacitor is *charged,* its plates have charges of equal magnitudes but opposite signs: $+q$ and $-q$. However, we refer to the *charge of a capacitor* as being q, the absolute value of these charges on the plates. (Note that q is not the net charge on the capacitor, which is zero.)

Because the plates are conductors, they are equipotential surfaces; all points on a plate are at the same electric potential. Moreover, there is a potential difference between the two plates. For historical reasons, we represent the absolute value of this potential difference with V rather than with the ΔV we used in previous notation.

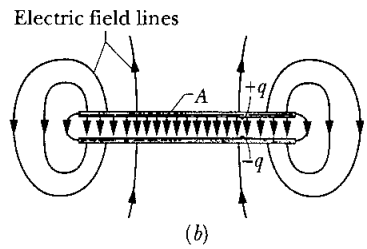
The charge q and the potential difference V for a capacitor are proportional to each other; that is,

$$q = CV. \tag{26-1}$$

The proportionality constant C is called the **capacitance** of the capacitor. Its value depends only on the geometry of the plates and *not* on their charge or potential



(*a*)



(*b*)

FIG. 26-3 (*a*) A parallel-plate capacitor, made up of two plates of area A separated by a distance d. The charges on the facing plate surfaces have the same magnitude q but opposite signs. (*b*) As the field lines show, the electric field due to the charged plates is uniform in the central region between the plates. The field is not uniform at the edges of the plates, as indicated by the "fringing" of the field lines there.

difference. The capacitance is a measure of how much charge must be put on the plates to produce a certain potential difference between them: The *greater the capacitance, the more charge is required.*

The SI unit of capacitance that follows from Eq. 26-1 is the coulomb per volt. This unit occurs so often that it is given a special name, the *farad* (F):

$$1 \text{ farad} = 1 \text{ F} = 1 \text{ coulomb per volt} = 1 \text{ C/V}. \tag{26-2}$$

As you will see, the farad is a very large unit. Submultiples of the farad, such as the microfarad ($1\ \mu\text{F} = 10^{-6}$ F) and the picofarad ($1\ \text{pF} = 10^{-12}$ F), are more convenient units in practice.

Charging a Capacitor

One way to charge a capacitor is to place it in an electric circuit with a battery. An *electric circuit* is a path through which charge can flow. A *battery* is a device that maintains a certain potential difference between its *terminals* (points at which charge can enter or leave the battery) by means of internal electrochemical reactions in which electric forces can move internal charge.

In Fig. 26-4*a*, a battery B, a switch S, an uncharged capacitor C, and interconnecting wires form a circuit. The same circuit is shown in the *schematic diagram* of Fig. 26-4*b*, in which the symbols for a battery, a switch, and a capacitor represent those devices. The battery maintains potential difference V between its terminals. The terminal of higher potential is labeled + and is often called the *positive* terminal; the terminal of lower potential is labeled – and is often called the *negative* terminal.

The circuit shown in Figs. 26-4*a* and *b* is said to be *incomplete* because switch S is *open;* that is, the switch does not electrically connect the wires attached to it. When the switch is *closed,* electrically connecting those wires, the circuit is complete and charge can then flow through the switch and the wires. As we discussed in Chapter 25, the charge that can flow through a conductor, such as a wire, is that of electrons. When the circuit of Fig. 26-4 is completed, electrons are driven through the wires by an electric field that the battery sets up in the wires. The field drives electrons from capacitor plate h to the positive terminal of the battery; thus, plate h, losing electrons, becomes positively charged. The field drives just as many electrons from the negative terminal of the battery to capacitor plate l; thus, plate l, gaining electrons, becomes negatively charged *just as much* as plate h, losing electrons, becomes positively charged.

Initially, when the plates are uncharged, the potential difference between them is zero. As the plates become oppositely charged, that potential difference increases until it equals the potential difference V between the terminals of the battery. Then plate h and the positive terminal of the battery are at the same potential, and there is no longer an electric field in the wire between them. Similarly, plate l and the negative terminal reach the same potential, and there is then no electric field in the wire between them. Thus, with the field zero, there is no further drive of electrons. The capacitor is then said to be *fully charged,* with a potential difference V and charge q that are related by Eq. 26-1.

In this book we assume that during the charging of a capacitor and afterward, charge cannot pass from one plate to the other across the gap separating them. Also, we assume that a capacitor can retain (or *store*) charge indefinitely, until it is put into a circuit where it can be *discharged.*

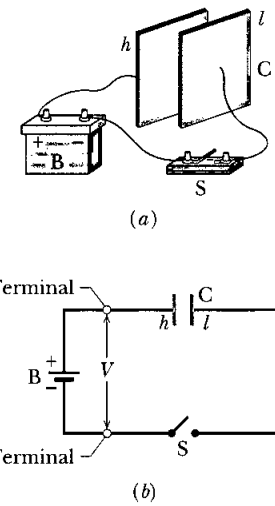


FIG. 26-4 (*a*) Battery B, switch S, and plates h and l of capacitor C, connected in a circuit. (*b*) A schematic diagram with the *circuit elements* represented by their symbols.

CHECKPOINT 1 Does the capacitance C of a capacitor increase, decrease, or remain the same (a) when the charge q on it is doubled and (b) when the potential difference V across it is tripled?

PROBLEM-SOLVING TACTICS

Tactic 1: *The Symbol V and Potential Difference* In previous chapters, the symbol V represents an electric potential at a point or along an equipotential surface. However, in matters concerning electrical devices, V often represents a *potential difference* between two points or two equipotential surfaces. Equation 26-1 is an example of this second use of the symbol. In Section 26-3, you will see a mixture of the two meanings of V. There and in later chapters, you need to be alert as to the intent of this symbol.

You will also be seeing, in this book and elsewhere, a variety of phrases regarding potential difference. A potential difference or a "potential" or a "voltage" may be *applied* to a device, or it may be *across* a device. A capacitor can be charged to a potential difference, as in "a capacitor is charged to 12 V." Also, a battery can be characterized by the potential difference across it, as in "a 12 V battery." Always keep in mind what is meant by such phrases: There is a potential difference between two points, such as two points in a circuit or at the terminals of a device such as a battery.

26-3 | Calculating the Capacitance

Our goal here is to calculate the capacitance of a capacitor once we know its geometry. Because we shall consider a number of different geometries, it seems wise to develop a general plan to simplify the work. In brief our plan is as follows: (1) Assume a charge q on the plates; (2) calculate the electric field $\vec{E}$ between the plates in terms of this charge, using Gauss' law; (3) knowing $\vec{E}$, calculate the potential difference V between the plates from Eq. 24-17; (4) calculate C from Eq. 26-1.

Before we start, we can simplify the calculation of both the electric field and the potential difference by making certain assumptions. We discuss each in turn.

Calculating the Electric Field

To relate the electric field $\vec{E}$ between the plates of a capacitor to the charge q on either plate, we shall use Gauss' law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q. \tag{26-3}$$

Here q is the charge enclosed by a Gaussian surface and $\oint \vec{E} \cdot d\vec{A}$ is the net electric flux through that surface. In all cases that we shall consider, the Gaussian surface will be such that whenever there is an electric flux through it, $\vec{E}$ will have a uniform magnitude E and the vectors $\vec{E}$ and $d\vec{A}$ will be parallel. Equation 26-3 then reduces to

$$q = \varepsilon_0 EA \qquad \text{(special case of Eq. 26-3)}, \tag{26-4}$$

in which A is the area of that part of the Gaussian surface through which there is a flux. For convenience, we shall always draw the Gaussian surface in such a way that it completely encloses the charge on the positive plate; see Fig. 26-5 for an example.

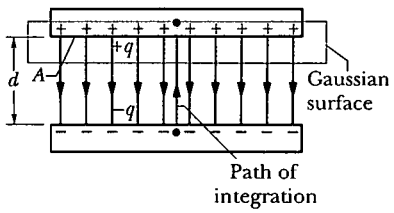


FIG. 26-5 A charged parallel-plate capacitor. A Gaussian surface encloses the charge on the positive plate. The integration of Eq. 26-6 is taken along a path extending directly from the negative plate to the positive plate.

Calculating the Potential Difference

In the notation of Chapter 24 (Eq. 24-17), the potential difference between the plates of a capacitor is related to the field $\vec{E}$ by

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}, \tag{26-5}$$

in which the integral is to be evaluated along any path that starts on one plate and ends on the other. We shall always choose a path that follows an electric field line, from the negative plate to the positive plate. For this path, the vectors $\vec{E}$ and $d\vec{s}$ will have opposite directions; so the dot product $\vec{E} \cdot d\vec{s}$ will be equal

to $-E\,ds$. Thus, the right side of Eq. 26-5 will then be positive. Letting V represent the difference $V_f - V_i$, we can then recast Eq. 26-5 as

$$V = \int_{-}^{+} E\,ds \qquad \text{(special case of Eq. 26-5)}, \tag{26-6}$$

in which the – and + remind us that our path of integration starts on the negative plate and ends on the positive plate.

We are now ready to apply Eqs. 26-4 and 26-6 to some particular cases.

A Parallel-Plate Capacitor

We assume, as Fig. 26-5 suggests, that the plates of our parallel-plate capacitor are so large and so close together that we can neglect the fringing of the electric field at the edges of the plates, taking $\vec{E}$ to be constant throughout the region between the plates.

We draw a Gaussian surface that encloses just the charge q on the positive plate, as in Fig. 26-5. From Eq. 26-4 we can then write

$$q = \varepsilon_0 EA, \tag{26-7}$$

where A is the area of the plate.

Equation 26-6 yields

$$V = \int_{-}^{+} E\,ds = E\int_{0}^{d} ds = Ed. \tag{26-8}$$

In Eq. 26-8, E can be placed outside the integral because it is a constant; the second integral then is simply the plate separation d.

If we now substitute q from Eq. 26-7 and V from Eq. 26-8 into the relation $q = CV$ (Eq. 26-1), we find

$$C = \frac{\varepsilon_0 A}{d} \qquad \text{(parallel-plate capacitor)}. \tag{26-9}$$

Thus, the capacitance does indeed depend only on geometrical factors—namely, the plate area A and the plate separation d. Note that C increases as we increase area A or decrease separation d.

As an aside, we point out that Eq. 26-9 suggests one of our reasons for writing the electrostatic constant in Coulomb's law in the form $1/4\pi\varepsilon_0$. If we had not done so, Eq. 26-9—which is used more often in engineering practice than Coulomb's law—would have been less simple in form. We note further that Eq. 26-9 permits us to express the permittivity constant ε_0 in a unit more appropriate for use in problems involving capacitors; namely,

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}. \tag{26-10}$$

We have previously expressed this constant as

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2. \tag{26-11}$$

A Cylindrical Capacitor

Figure 26-6 shows, in cross section, a cylindrical capacitor of length L formed by two coaxial cylinders of radii a and b. We assume that $L \gg b$ so that we can neglect the fringing of the electric field that occurs at the ends of the cylinders. Each plate contains a charge of magnitude q.

As a Gaussian surface, we choose a cylinder of length L and radius r, closed by end caps and placed as is shown in Fig. 26-6. Equation 26-4 then yields

$$q = \varepsilon_0 EA = \varepsilon_0 E(2\pi r L),$$

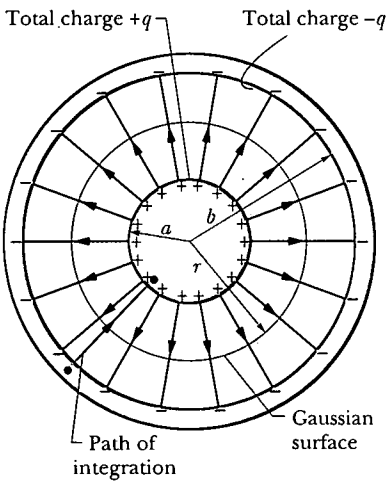


FIG. 26-6 A cross section of a long cylindrical capacitor, showing a cylindrical Gaussian surface of radius r (that encloses the positive plate) and the radial path of integration along which Eq. 26-6 is to be applied. This figure also serves to illustrate a spherical capacitor in a cross section through its center.

in which $2\pi rL$ is the area of the curved part of the Gaussian surface. There is no flux through the end caps. Solving for E yields

$$E = \frac{q}{2\pi\varepsilon_0 Lr}. \tag{26-12}$$

Substitution of this result into Eq. 26-6 yields

$$V = \int_-^+ E\,ds = -\frac{q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right), \tag{26-13}$$

where we have used the fact that here $ds = -dr$ (we integrated radially inward). From the relation $C = q/V$, we then have

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \quad \text{(cylindrical capacitor)}. \tag{26-14}$$

We see that the capacitance of a cylindrical capacitor, like that of a parallel-plate capacitor, depends only on geometrical factors, in this case L, b, and a.

A Spherical Capacitor

Figure 26-6 can also serve as a central cross section of a capacitor that consists of two concentric spherical shells, of radii a and b. As a Gaussian surface we draw a sphere of radius r concentric with the two shells; then Eq. 26-4 yields

$$q = \varepsilon_0 EA = \varepsilon_0 E(4\pi r^2),$$

in which $4\pi r^2$ is the area of the spherical Gaussian surface. We solve this equation for E, obtaining

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \tag{26-15}$$

which we recognize as the expression for the electric field due to a uniform spherical charge distribution (Eq. 23-13).

If we substitute this expression into Eq. 26-6, we find

$$V = \int_-^+ E\,ds = -\frac{q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\frac{b-a}{ab}, \tag{26-16}$$

where again we have substituted $-dr$ for ds. If we now substitute Eq. 26-16 into Eq. 26-1 and solve for C, we find

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \quad \text{(spherical capacitor)}. \tag{26-17}$$

An Isolated Sphere

We can assign a capacitance to a *single* isolated spherical conductor of radius R by assuming that the "missing plate" is a conducting sphere of infinite radius. After all, the field lines that leave the surface of a positively charged isolated conductor must end somewhere; the walls of the room in which the conductor is housed can serve effectively as our sphere of infinite radius.

To find the capacitance of the conductor, we first rewrite Eq. 26-17 as

$$C = 4\pi\varepsilon_0 \frac{a}{1 - a/b}.$$

If we then let $b \to \infty$ and substitute R for a, we find

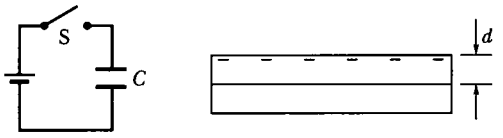
$$C = 4\pi\varepsilon_0 R \quad \text{(isolated sphere)}. \tag{26-18}$$

Note that this formula and the others we have derived for capacitance (Eqs. 26-9, 26-14, and 26-17) involve the constant ε_0 multiplied by a quantity that has the dimensions of a length.

CHECKPOINT 2 For capacitors charged by the same battery, does the charge stored by the capacitor increase, decrease, or remain the same in each of the following situations? (a) The plate separation of a parallel-plate capacitor is increased. (b) The radius of the inner cylinder of a cylindrical capacitor is increased. (c) The radius of the outer spherical shell of a spherical capacitor is increased.

Sample Problem 26-1

In Fig. 26-7*a*, switch S is closed to connect the uncharged capacitor of capacitance $C = 0.25\ \mu F$ to the battery of potential difference $V = 12$ V. The lower capacitor plate has thickness $L = 0.50$ cm and face area $A = 2.0 \times 10^{-4}\ m^2$, and it consists of copper, in which the density of conduction electrons is $n = 8.49 \times 10^{28}$ electrons/m^3. From what depth d within the plate (Fig. 26-7*b*) must electrons move to the plate face as the capacitor becomes charged?



(*a*) (*b*)

FIG. 26-7 (*a*) A battery and capacitor circuit. (*b*) The lower capacitor plate.

Conceptualize/Classify: The charge collected on the plate is related to the capacitance and the potential difference across the capacitor by Eq. 26-1 ($q = CV$).

Compute: Because the lower plate is connected to the negative terminal of the battery, conduction electrons move up to the face of the plate. From Eq. 26-1, the total charge magnitude that collects there is

$$q = CV = (0.25 \times 10^{-6}\ F)(12\ V)$$
$$= 3.0 \times 10^{-6}\ C.$$

Dividing this result by e gives us the number N of conduction electrons that come up to the face:

$$N = \frac{q}{e} = \frac{3.0 \times 10^{-6}\ C}{1.602 \times 10^{-19}\ C}$$
$$= 1.873 \times 10^{13}\ \text{electrons}.$$

These electrons come from a volume that is the product of the face area A and the depth d we seek. Thus, from the density of conduction electrons (number per volume), we can write

$$n = \frac{N}{Ad},$$

or

$$d = \frac{N}{An} = \frac{1.873 \times 10^{13}\ \text{electrons}}{(2.0 \times 10^{-4}\ m^2)(8.49 \times 10^{28}\ \text{electrons}/m^3)}$$
$$= 1.1 \times 10^{-12}\ m = 1.1\ \text{pm}. \quad \text{(Answer)}$$

Conclude: In common speech, we would say that the battery charges the capacitor by supplying the charged particles. But what the battery really does is set up an electric field in the wires and plate such that electrons very close to the plate face move up to the negative face.

26-4 | Combination of Capacitors

When there is a combination of capacitors in a circuit, we can sometimes replace that combination with an **equivalent capacitor**—that is, a single capacitor that has the same capacitance as the actual combination of capacitors. With such a replacement, we can simplify the circuit, affording easier solutions for unknown quantities of the circuit. Here we discuss two basic combinations of capacitors that allow such a replacement.

Capacitors in Parallel

Figure 26-8*a* shows an electric circuit in which three capacitors are connected *in parallel* to battery B. This description has little to do with how the capacitor plates are drawn. Rather, "in parallel" means that the capacitors are directly wired together at one plate and directly wired together at the other plate, and that the same potential difference V is applied across the two groups of wired-together plates. Thus, each capacitor has the same potential difference V, which produces charge on the capacitor. (In Fig. 26-8*a*, the applied potential V is maintained by the battery.) In general,

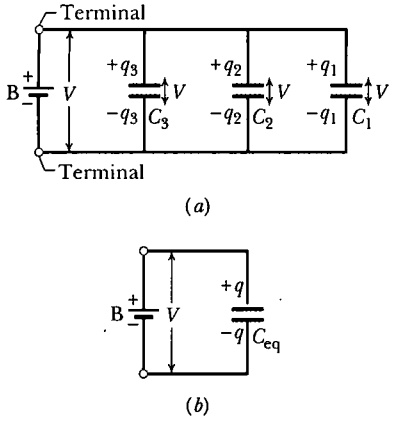


FIG. 26-8 (*a*) Three capacitors connected in parallel to battery B. The battery maintains potential difference V across its terminals and thus across *each* capacitor. (*b*) The equivalent capacitor, with capacitance C_{eq}, replaces the parallel combination.

> When a potential difference V is applied across several capacitors connected in parallel, that potential difference V is applied across each capacitor. The total charge q stored on the capacitors is the sum of the charges stored on all the capacitors.

When we analyze a circuit of capacitors in parallel, we can simplify it with this mental replacement:

> Capacitors connected in parallel can be replaced with an equivalent capacitor that has the same *total* charge q and the same potential difference V as the actual capacitors.

(You might remember this result with the nonsense word "par-V," which is close to "party," to mean "capacitors in parallel have the same V.") Figure 26-8*b* shows the equivalent capacitor (with equivalent capacitance C_{eq}) that has replaced the three capacitors (with actual capacitances C_1, C_2, and C_3) of Fig. 26-8*a*.

To derive an expression for C_{eq} in Fig. 26-8*b*, we first use Eq. 26-1 to find the charge on each actual capacitor:

$$q_1 = C_1V, \quad q_2 = C_2V, \quad \text{and} \quad q_3 = C_3V.$$

The total charge on the parallel combination of Fig. 26-8*a* is then

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V.$$

The equivalent capacitance, with the same total charge q and applied potential difference V as the combination, is then

$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3,$$

a result that we can easily extend to any number n of capacitors, as

$$C_{eq} = \sum_{j=1}^{n} C_j \qquad (n \text{ capacitors in parallel}). \tag{26-19}$$

Thus, to find the equivalent capacitance of a parallel combination, we simply add the individual capacitances.

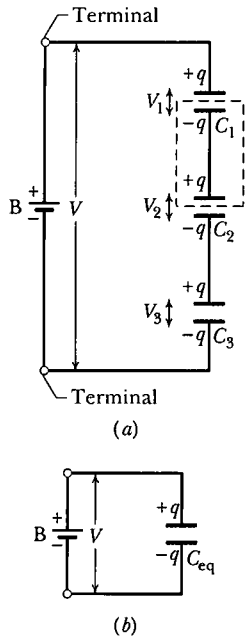


FIG. 26-9 (*a*) Three capacitors connected in series to battery B. The battery maintains potential difference V between the top and bottom plates of the series combination. (*b*) The equivalent capacitor, with capacitance C_{eq}, replaces the series combination.

Capacitors in Series

Figure 26-9*a* shows three capacitors connected *in series* to battery B. This description has little to do with how the capacitors are drawn. Rather, "in series" means that the capacitors are wired serially, one after the other, and that a potential difference V is applied across the two ends of the series. (In Fig. 26-9*a*, this potential difference V is maintained by battery B.) The potential differences that then exist across the capacitors in the series produce identical charges q on them.

> When a potential difference V is applied across several capacitors connected in series, the capacitors have identical charge q. The sum of the potential differences across all the capacitors is equal to the applied potential difference V.

We can explain how the capacitors end up with identical charge by following a *chain reaction* of events, in which the charging of each capacitor causes the charging of the next capacitor. We start with capacitor 3 and work upward to capacitor 1. When the battery is first connected to the series of capacitors, it produces charge $-q$ on the bottom plate of capacitor 3. That charge then repels negative charge from the top plate of capacitor 3 (leaving it with charge $+q$). The repelled negative charge moves to the bottom plate of capacitor 2 (giving it charge $-q$). That charge on the bottom plate of capacitor 2 then repels negative charge from the top plate of capacitor 2 (leaving it with charge $+q$) to the bottom

plate of capacitor 1 (giving it charge $-q$). Finally, the charge on the bottom plate of capacitor 1 helps move negative charge from the top plate of capacitor 1 to the battery, leaving that top plate with charge $+q$.

Here are two important points about capacitors in series:

1. When charge is shifted from one capacitor to another in a series of capacitors, it can move along only one route, such as from capacitor 3 to capacitor 2 in Fig. 26-9*a*. If there are additional routes, the capacitors are not in series. An example is given in Sample Problem 26-2.
2. The battery directly produces charges on only the two plates to which it is connected (the bottom plate of capacitor 3 and the top plate of capacitor 1 in Fig. 26-9*a*). Charges that are produced on the other plates are due merely to the shifting of charge already there. For example, in Fig. 26-9*a*, the part of the circuit enclosed by dashed lines is electrically isolated from the rest of the circuit. Thus, the net charge of that part cannot be changed by the battery—its charge can only be redistributed.

When we analyze a circuit of capacitors in series, we can simplify it with this mental replacement:

☞ Capacitors that are connected in series can be replaced with an equivalent capacitor that has the same charge q and the same *total* potential difference V as the actual series capacitors.

Obviously the charges on connected plates are equal and opposite if they are initially uncharged. (You might remember this with the nonsense word "seri-q" to mean "capacitors in series have the same q".) Figure 26-9*b* shows the equivalent capacitor (with equivalent capacitance C_{eq}) that has replaced the three actual capacitors (with actual capacitances C_1, C_2, and C_3) of Fig. 26-9*a*.

To derive an expression for C_{eq} in Fig. 26-9*b*, we first use Eq. 26-1 to find the potential difference of each actual capacitor:

$$V_1 = \frac{q}{C_1}, \quad V_2 = \frac{q}{C_2}, \quad \text{and} \quad V_3 = \frac{q}{C_3}.$$

The total potential difference V due to the battery is the sum of these three potential differences. Thus,

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

The equivalent capacitance is then

$$C_{eq} = \frac{q}{V} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3},$$

or

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

We can easily extend this to any number n of capacitors as

$$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j} \qquad (n \text{ capacitors in series}). \tag{26-20}$$

Using Eq. 26-20 you can show that the equivalent capacitance of a series of capacitances is always *less* than the least capacitance in the series.

✓**CHECKPOINT 3** A battery of potential V stores charge q on a combination of two identical capacitors. What are the potential difference across and the charge on either capacitor if the capacitors are (a) in parallel and (b) in series?

Sample Problem 26-2

(a) Find the equivalent capacitance for the combination of capacitances shown in Fig. 26-10*a*, across which potential difference V is applied. Assume

$$C_1 = 12.0\ \mu\text{F},\ C_2 = 5.30\ \mu\text{F},\ \text{and}\ C_3 = 4.50\ \mu\text{F}.$$

Conceptualize: Capacitors 1 and 3 are connected one after the other, but are they in series? No. The potential V that is applied to the capacitors produces charge on the bottom plate of capacitor 3. That charge causes charge to shift from the top plate of capacitor 3. However, note that the shifting charge can move to the bottom plates of both capacitor 1 and capacitor 2. Because there is more than one route for the shifting charge, capacitor 3 is not in series with capacitor 1 (or capacitor 2).

Are capacitor 1 and capacitor 2 in parallel? Yes. Their top plates are directly wired together and their bottom plates are directly wired together, and electric potential is applied between the top-plate pair and the bottom-plate pair. Thus, capacitor 1 and capacitor 2 are in parallel.

Classify: Any capacitors connected in series can be replaced with their equivalent capacitor, and any capacitors connected in parallel can be replaced with their equivalent capacitor. Therefore, we should first check whether any of the capacitors in Fig. 26-10*a* are in parallel or series.

Compute: Equation 26-19 tells us that their equivalent capacitance C_{12} is

$$C_{12} = C_1 + C_2 = 12.0\ \mu\text{F} + 5.30\ \mu\text{F} = 17.3\ \mu\text{F}.$$

In Fig. 26-10*b*, we have replaced capacitors 1 and 2 with their equivalent capacitor, called capacitor 12 (say "one two" and not "twelve"). (The connections at points A and B are exactly the same in Figs. 26-10*a* and *b*.)

Is capacitor 12 in series with capacitor 3? Again applying the test for series capacitances, we see that the charge that shifts from the top plate of capacitor 3 must entirely go to the bottom plate of capacitor 12. Thus, capacitor 12 and capacitor 3 are in series, and we can replace them with their equivalent C_{123} ("one two three"), as shown in Fig. 26-10*c*. From Eq. 26-20, we have

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{1}{17.3\ \mu\text{F}} + \frac{1}{4.50\ \mu\text{F}} = 0.280\ \mu\text{F}^{-1},$$

from which

$$C_{123} = \frac{1}{0.280\ \mu\text{F}^{-1}} = 3.57\ \mu\text{F}. \qquad \text{(Answer)}$$

(b) The potential difference applied to the input terminals in Fig. 26-10*a* is $V = 12.5$ V. What is the charge on C_1?

Conceptualize: To get the charge q_1 on capacitor 1, we work backwards to that capacitor, starting with the equivalent capacitor 123. Because the given potential difference V (= 12.5 V) is applied across the actual combination of three capacitors in Fig. 26-10*a*, it is also applied across C_{123} in Fig. 26-10*c*.

Classify: We now need to work backwards from the equivalent capacitance to get the charge on a particular capacitor. We have two techniques for such "backwards work": (1) Seri-q: Series capacitors have the same charge as their equivalent capacitor. (2) Par-V: Parallel capacitors have the same potential difference as their equivalent capacitor.

Compute: Thus, Eq. 26-1 ($q = CV$) gives us

$$q_{123} = C_{123}V = (3.57\ \mu\text{F})(12.5\ \text{V}) = 44.6\ \mu\text{C}.$$

The series capacitors 12 and 3 in Fig. 26-10*b* each have the same charge as their equivalent capacitor 123. Thus, capacitor 12 has charge $q_{12} = q_{123} = 44.6\ \mu\text{C}$. From Eq. 26-1, the potential difference across capacitor 12 must be

$$V_{12} = \frac{q_{12}}{C_{12}} = \frac{44.6\ \mu\text{C}}{17.3\ \mu\text{F}} = 2.58\ \text{V}.$$

The parallel capacitors 1 and 2 each have the same potential difference as their equivalent capacitor 12. Thus, capacitor 1 has potential difference $V_1 = V_{12} = 2.58$ V, and, from Eq. 26-1, the charge on capacitor 1 must be

$$q_1 = C_1V_1 = (12.0\ \mu\text{F})(2.58\ \text{V}) = 31.0\ \mu\text{C}. \qquad \text{(Answer)}$$

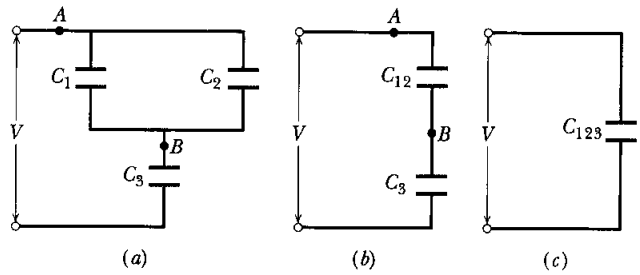


FIG. 26-10 (*a*) Three capacitors. (*b*) C_1 and C_2, a parallel combination, are replaced by the equivalent capacitance C_{12}. (*c*) C_{12} and C_3, a series combination, are replaced by the equivalent capacitance C_{123}.

Sample Problem 26-3 Build your skill

Capacitor 1, with $C_1 = 3.55\ \mu F$, is charged to a potential difference $V_0 = 6.30$ V, using a 6.30 V battery. The battery is then removed, and the capacitor is connected as in Fig. 26-11 to an uncharged capacitor 2, with $C_2 = 8.95\ \mu F$. When switch S is closed, charge flows between the capacitors. Find the charge on each capacitor when equilibrium is reached.

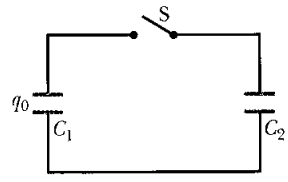


FIG. 26-11 A potential difference V_0 is applied to capacitor 1 and the charging battery is removed. Switch S is then closed so that the charge on capacitor 1 is shared with capacitor 2.

Conceptualize: As the electric potential across capacitor 1 decreases, that across capacitor 2 increases. Equilibrium is reached when the two potentials are equal because, with no potential difference between connected plates of the capacitors, there is no electric field within the connecting wires to move conduction electrons. The initial charge on capacitor 1 is then shared between the two capacitors.

Classify: The situation here differs from the previous example because here an applied electric potential is *not* maintained across a combination of capacitors by a battery or some other source. Here, just after switch S is closed, the only applied electric potential is that of capacitor 1 on capacitor 2, and that potential is decreasing. Thus, the capacitors in Fig. 26-11 are not connected *in series;* and although they are drawn parallel, in this situation they are not *in parallel.*

Compute: Initially, when capacitor 1 is connected to the battery, the charge it acquires is, from Eq. 26-1,

$$q_0 = C_1 V_0 = (3.55 \times 10^{-6}\ \text{F})\ (6.30\ \text{V})$$
$$= 22.365 \times 10^{-6}\ \text{C}.$$

When switch S in Fig. 26-11 is closed and capacitor 1 begins to charge capacitor 2, the electric potential and charge on capacitor 1 decrease and those on capacitor 2 increase until

$$V_1 = V_2 \quad \text{(equilibrium)}.$$

From Eq. 26-1, we can rewrite this as

$$\frac{q_1}{C_1} = \frac{q_2}{C_2} \quad \text{(equilibrium)}.$$

Because the total charge cannot magically change, the total after the transfer must be

$$q_1 + q_2 = q_0 \quad \text{(charge conservation)};$$

thus $q_2 = q_0 - q_1$.

We can now rewrite the second equilibrium equation as

$$\frac{q_1}{C_1} = \frac{q_0 - q_1}{C_2}.$$

Solving this for q_1 and substituting given data, we find

$$q_1 = 6.35\ \mu\text{C}. \qquad \text{(Answer)}$$

The rest of the initial charge ($q_0 = 22.365\ \mu$C) must be on capacitor 2:

$$q_2 = 16.0\ \mu\text{C}. \qquad \text{(Answer)}$$

PROBLEM-SOLVING TACTICS

Tactic 2: *Multiple-Capacitor Circuits* Let us review the procedure used in the solution of Sample Problem 26-2, in which several capacitors are connected to a battery. To find a single equivalent capacitance, we simplify the given arrangement of capacitances by replacing them, in steps, with equivalent capacitances, using Eq. 26-19 when we find capacitances in parallel and Eq. 26-20 when we find capacitances in series. Then, to find the charge stored by that single equivalent capacitance, we use Eq. 26-1 and the potential difference V applied by the battery.

That result tells us the net charge stored on the actual arrangement of capacitors. However, to find the charge on, or the potential difference across, any particular capacitor in the actual arrangement, we need to reverse our steps of simplification. With each reversed step, we use these two rules: When capacitances are in parallel, they have the same potential difference as their equivalent capacitance, and we use Eq. 26-1 to find the charge on each capacitance; when they are in series, they have the same charge as their equivalent capacitance, and we use Eq. 26-1 to find the potential difference across each capacitance.

Tactic 3: *Batteries and Capacitors* A battery maintains a certain potential difference across its terminals. Thus, when capacitor 1 of Sample Problem 26-3 is connected to the 6.30 V battery, charge flows between the capacitor and the battery until the capacitor has the same potential difference across it as the battery.

A capacitor differs from a battery in that a capacitor lacks the internal electrochemical reactions needed to release charged particles (electrons) from internal atoms and molecules. Thus, when the charged capacitor 1 of Sample Problem 26-3 is disconnected from the battery and then connected to the uncharged capacitor 2 with switch S closed, the potential difference across capacitor 1 is not maintained. The quantity that *is* maintained is the charge q_0 of the two-capacitor system; that is, charge obeys a conservation law, *not* electric potential.

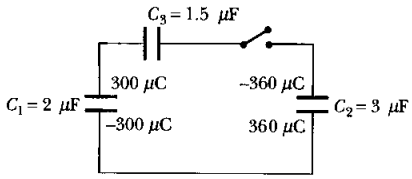


FIG. 26-12 Three capacitors connected to charge the capacitor C_3.

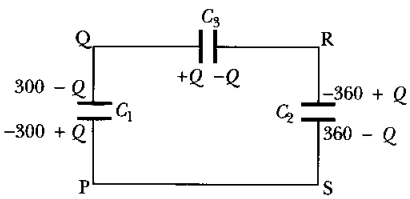


FIG. 26-13 Steady-state charges on C_1, C_2, and C_3.

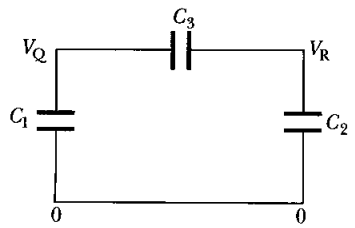


FIG. 26-14 Potentials for the circuit of Fig. 26-13.

For more complicated circuits, we are going to introduce two methods. You are free to choose any one of them as per your convenience. We will formalize them in Chapter 28.

Let us consider a circuit consisting of three capacitors as shown in Fig. 26-12. Two of them C_1 and C_2 are already charged as shown and the third one C_3 has just now been connected. This arrangement is a little deceptive. The capacitors seem to be in series, but they are not. The charges on them are not equal.

Method 1: Let us assume that the left plate of a C_3 gets a charge of $+Q$ in the steady state. Then its right plate will acquire a charge of $-Q$. By the law of conservation of charge, we can say that the upper plate of C_2 will get an additional charge of $+Q$. So its lower plate charge will reduce by Q. This plate acquires this charge from the lower plate of C_1. Thus, in steady state, the plate charges will look the way shown in Fig. 26-13.

Now we know that if we start from point P and traverse around the loop PQRS. By the conservative nature of electric field

$$\oint \vec{E} \cdot d\vec{l} = 0.$$

So if we denote V_1, V_2 and V_3 as the potential drops across C_1, C_2 and C_3, respectively, then conservative nature of electric field gives

$$V_1 + V_2 + V_3 = 0.$$

In this situation, it implies that

$$\frac{300 - Q}{C_1} - \frac{Q}{C_3} + \frac{360 - Q}{C_2} = 0.$$

Note that potential increases when we move from negative plate to positive plate and vice versa. Solving the above equation, we get

$$Q = 180\ \mu C.$$

Method 2: Here we will use the law of conservation of charges. At all the three nodes P, Q, and R, the charge before the connection should be equal to the charge after the connection. Thus, the charge at node P before connection was 360 – 300 = 60 μC, at Q it was 300 μC, at R it was –360 μC.

Let us assume that point P is at zero potential. (Recall what we said in Chapter 23: Any point can be taken to be reference at zero potential.) Assume that potential of Q in steady state is V_Q and that of R in steady state is V_R as seen in Fig. 26-14.

The total charge at Q is now

$$C_1(V_Q - 0) + C_3(V_Q - V_R) \quad \text{(at upper plate of } C_1 \text{ and left plate of } C_3\text{)}$$

$$2(V_Q - 0) + 1.5(V_Q - V_R) = 300.$$

The total charge at R is now

$$C_3(V_R - V_Q) + C_2(V_R - 0) \quad \text{(at right plate of } C_3 \text{ and upper plate of } C_2\text{)}$$

$$1.5(V_R - V_Q) + 3V_R = -360.$$

Solving these two equations, we get $V_Q = 60$ V and $V_R = -60$ V. So the charge on C_3 in steady state is $Q = 1.5\ [60 - (-60)] = 180\ \mu C$ as we found before.

26-5 | Energy Stored in an Electric Field

Work must be done by an external agent to charge a capacitor. Starting with an uncharged capacitor, for example, imagine that—using "magic tweezers"—you remove electrons from one plate and transfer them one at a time to the other plate. The electric field that builds up in the space between the plates has a direction that tends to oppose further transfer. Thus, as charge accumulates on the

capacitor plates, you have to do increasingly larger amounts of work to transfer additional electrons. In practice, this work is done by a battery, at the expense of its store of chemical energy.

We visualize the work required to charge a capacitor as being stored in the form of electric potential energy U in the electric field between the plates. You can recover this energy at will, by discharging the capacitor in a circuit, just as you can recover the potential energy stored in a stretched bow by releasing the bowstring to transfer the energy to the kinetic energy of an arrow.

Suppose that, at a given instant, a charge q' has been transferred from one plate of a capacitor to the other. The potential difference V' between the plates at that instant will be q'/C. If an extra increment of charge dq' is then transferred, the increment of work required will be, from Eq. 24-10,

$$dW = V'\,dq' = \frac{q'}{C}dq'.$$

The work required to bring the total capacitor charge up to a final value q is

$$W = \int dW = \frac{1}{C}\int_0^q q'dq' = \frac{q^2}{2C}.$$

This work is stored as potential energy U in the capacitor, so that

$$U = \frac{q^2}{2C} \qquad \text{(potential energy).} \tag{26-21}$$

From Eq. 26-1, we can also write this as

$$U = \tfrac{1}{2}CV^2 \qquad \text{(potential energy).} \tag{26-22}$$

Equations 26-21 and 26-22 hold no matter what the geometry of the capacitor is.

To gain some physical insight into energy storage, consider two parallel-plate capacitors that are identical except that capacitor 1 has twice the plate separation of capacitor 2. Then capacitor 1 has twice the volume between its plates and also, from Eq. 26-9, half the capacitance of capacitor 2. Equation 26-4 tells us that if both capacitors have the same charge q, the electric fields between their plates are identical. And Eq. 26-21 tells us that capacitor 1 has twice the stored potential energy of capacitor 2. Thus, of two otherwise identical capacitors with the same charge and same electric field, the one with twice the volume between its plates has twice the stored potential energy. Arguments like this tend to verify our earlier assumption:

☞ The potential energy of a charged capacitor may be viewed as being stored in the electric field between its plates.

Explosions in Airborne Dust

As we discussed in Section 24-10, making contact with certain materials, such as clothing, carpets, and even playground slides, can leave you with a significant electrical potential. You might become painfully aware of that potential if a spark leaps between you and a grounded object, such as a faucet. In many industries involving the production and transport of powder, such as in the cosmetic and food industries, such a spark can be disastrous. Although the powder in bulk may not burn at all, when individual powder grains are airborne and thus surrounded by oxygen, they can burn so fiercely that a cloud of the grains burns as an explosion. Safety engineers cannot eliminate all possible sources of sparks in the powder industries. Instead, they attempt to keep the amount of energy available in the sparks below the threshold value U_t ($\approx$ 150 mJ) typically required to ignite airborne grains.

Suppose a person becomes charged by contact with various surfaces as he walks through an airborne powder. We can roughly model the person as a spherical capacitor of radius $R = 1.8$ m. From Eq. 26-18 ($C = 4\pi\varepsilon_0 R$) and Eq. 26-22 ($U = \frac{1}{2}CV^2$), we see that the energy of the capacitor is

$$U = \tfrac{1}{2}(4\pi\varepsilon_0 R)V^2.$$

From this we see that the threshold energy corresponds to a potential of

$$V = \sqrt{\frac{2U_i}{4\pi\varepsilon_0 R}} = \sqrt{\frac{2(150\times10^{-3}\ \text{J})}{4\pi(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(1.8\ \text{m})}}$$
$$= 3.9\times10^4\ \text{V}.$$

Safety engineers attempt to keep the potential of the personnel below this level by "bleeding" off the charge through, say, a conducting floor.

Energy Density

In a parallel-plate capacitor, neglecting fringing, the electric field has the same value at all points between the plates. Thus, the **energy density** u—that is, the potential energy per unit volume between the plates—should also be uniform. We can find u by dividing the total potential energy by the volume Ad of the space between the plates. Using Eq. 26-22, we obtain

$$u = \frac{U}{Ad} = \frac{CV^2}{2Ad}.$$

With Eq. 26-9 ($C = \varepsilon_0 A/d$), this result becomes

$$u = \tfrac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2.$$

However, from Eq. 24-39 ($E = -\Delta V/\Delta s$), V/d equals the electric field magnitude E; so

$$u = \tfrac{1}{2}\varepsilon_0 E^2 \quad \text{(energy density)}. \tag{26-23}$$

Although we derived this result for the special case of a parallel-plate capacitor, it holds generally, whatever may be the source of the electric field. If an electric field $\vec{E}$ exists at any point in space, we can think of that point as a site of electric potential energy whose amount per unit volume is given by Eq. 26-23.

Sample Problem 26-4

Often, a burn victim is treated while lying on a gurney in an enclosed chamber filled with oxygen-enriched air (a hyperbaric chamber). Once a treatment session is over, a hospital worker pulls the gurney and patient from the chamber onto a trolley, to be rolled away. On at least two occasions, the gurney caught fire at the end that was last to leave the chamber. Obviously, a burning gurney holding a patient already suffering from burns is a dangerous situation, and obviously fires burn easily in air rich in oxygen, but the question remains: What caused the gurneys to catch fire?

Investigators realized that charge separation occurred between the patient's skin, the hospital gown on the patient, and the sheet on the gurney. They also found that the gurney and the part of the chamber's metal framework below the gurney formed a parallel-plate capacitor (Fig. 26-15) of capacitance $C_i = 250$ pF. If the gurney discharged its excess charge and the associated energy by sparking, could the spark ignite the gurney? Measurements revealed that a spark could occur only if the potential difference V on the gurney–framework capacitor exceeded 2000 V and that a fire could start only if the capacitor's potential energy U exceeded 0.20 mJ. However, the potential difference on the gurney–framework capacitor was only $V_i = 600$ V, not enough to produce a spark.

(a) As the gurney was withdrawn from the chamber, the area of the gurney–framework overlap decreased. Thus, the plate area of the capacitor decreased from its initial value A_i. What was the potential difference V_f when the overlap plate area was $A_f = 0.10A_i$?



FIG. 26-15 (*a*) A hyperbaric chamber. (*b*) A gurney and the chamber's metal framework form a capacitor that is charged by stray electrostatic charge. (*c*) As the gurney is pulled from the chamber, the charge crowds onto a smaller area. *(BARA-MED® Monoplace chamber designed and manufactured by BioMedical Systems Group, Environmental Tectonics Corp.)*

Conceptualize/Classify: (1) The potential difference V across a capacitor is related to the charge q and capacitance C according to Eq. 26-1 ($q = CV$). (2) As the gurney was withdrawn from the chamber, the charge q did not change. (3) The capacitance of a parallel-plate capacitor is related to the plate area according to Eq. 26-9 ($C = \varepsilon_0 A/d$).

Compute: From Eq. 26-1, the charge q was

$$q = C_f V_f = C_i V_i,$$

or

$$V_f = \frac{C_i}{C_f} V_i. \tag{26-24}$$

From Eq. 26-9, we can write

$$C_f = \frac{\varepsilon_0 A_f}{d} = \frac{\varepsilon_0 (0.10 A_i)}{d}$$

$$= 0.10 \frac{\varepsilon_0 A_i}{d} = 0.10 C_i. \tag{26-25}$$

Substituting this result into Eq. 26-24 brings us to

$$V_f = \frac{C_i}{0.10 C_i} V_i = 10 V_i = (10)(600\,\text{V})$$

$$= 6000\,\text{V}. \qquad \text{(Answer)}$$

Conclude: As the gurney was withdrawn, the potential difference increased because the charge on the capacitor was crowded into a smaller plate area. The potential difference $V_f = 6000$ V was more than enough to produce a spark.

(b) What was the energy U_f of the gurney—framework capacitor when the plate area was $0.10A_i$?

Conceptualize/Classify: The potential energy U stored in a capacitor is related to the capacitance C and potential difference V according to Eq. 26-22 ($U = \frac{1}{2}CV^2$) .

Compute: Using Eq. 26-25, we write

$$U_f = \tfrac{1}{2} C_f V_f^2 = \tfrac{1}{2} (0.10 C_i) V_f^2$$

$$= \tfrac{1}{2} (0.10)(250 \times 10^{-12}\,\text{F})(6000\,\text{V})^2$$

$$= 4.5 \times 10^{-4}\,\text{J} = 0.45\,\text{mJ}. \qquad \text{(Answer)}$$

This was more than enough energy to ignite the gurney.

Conclude: The investigators concluded that the gurney fire was due to a spark produced by the gurney—framework capacitor as the charge became crowded into a smaller area while the gurney was being withdrawn from the chamber.

Sample Problem 26-5

An isolated conducting sphere whose radius R is 6.85 cm has a charge $q = 1.25$ nC.

(a) How much potential energy is stored in the electric field of this charged conductor?

Conceptualize/Classify: (1) An isolated sphere has capacitance given by Eq. 26-18 ($C = 4\pi\varepsilon_0 R$). (2) The energy U stored in a capacitor depends on the capacitor's charge q and capacitance C according to Eq. 26-21 ($U = q^2/2C$).

Compute: Substituting $C = 4\pi\varepsilon_0 R$ into Eq. 26-21 gives us

$$U = \frac{q^2}{2C} = \frac{q^2}{8\pi\varepsilon_0 R}$$

$$= \frac{(1.25 \times 10^{-9}\,\text{C})^2}{(8\pi)(8.85 \times 10^{-12}\,\text{F/m})(0.0685\,\text{m})}$$

$$= 1.03 \times 10^{-7}\,\text{J} = 103\,\text{nJ}. \qquad \text{(Answer)}$$

(b) What is the energy density at the surface of the sphere?

Conceptualize/Classify: The density u of the energy stored in an electric field depends on the magnitude E of the field, according to Eq. 26-23 ($u = \frac{1}{2}\varepsilon_0 E^2$).

Compute: Here we must first find E at the surface of the sphere, as given by Eq. 23-13:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}.$$

The energy density is then

$$u = \tfrac{1}{2}\varepsilon_0 E^2 = \frac{q^2}{32\pi^2\varepsilon_0 R^4}$$

$$= \frac{(1.25\times10^{-9}\,\text{C})^2}{(32\pi^2)(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.0685\,\text{m})^4}$$

$$= 2.54\times10^{-5}\,\text{J/m}^3 = 25.4\,\mu\text{J/m}^3. \qquad \text{(Answer)}$$

Sample Problem 26-6

A capacitor of capacitance C is charged by connecting it to a battery of emf $\mathscr{E}$. The capacitor is now disconnected and reconnected to the battery with the polarity reversed. Calculate the heat dissipated in the connecting wires (Fig. 26-16).

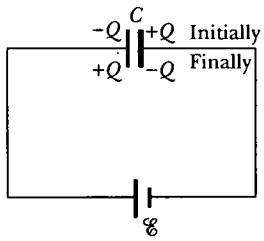


FIG. 26-16 A capacitor being charged by a battery.

Conceptualize: The work done by the external agent on the system goes into two parts: (a) in increasing the potential energy of the system and (b) in heat. Here the battery is the agent which drives the charges around in the circuit. Obviously it is doing work at the expense of its internal chemical energy. Although a part of it goes in the form of potential energy of the charges on the capacitor, a part of it goes as heat too.

Classify: To find the heat dissipated in the circuit, we need to invoke the first law of thermodynamics.

Compute: When the capacitor is connected to the battery, a charge $Q = C\mathscr{E}$ appears on one plate and $-Q$ on the other. When the polarity is reversed, a charge $-Q$ appears on the first plate and $+Q$ on the second. A charge $2Q$, therefore, passes through the battery from the negative terminal to the positive terminal. The battery does a work equal to

$$W = (2Q)\mathscr{E} = 2C\mathscr{E}^2$$

in the process. The potential energy stored in the capacitor is the same in the two cases. Thus, the work done by the battery appears as heat in the connecting wires. The heat produced is, therefore, $2C\mathscr{E}^2$.

Conclude: You might be wondering that where did this heat energy go. After all, if we had mentioned that the connecting wires had some resistance, then it makes sense that the wires get heated up as the charge passes through them. But since that was not at all mentioned, where does this energy go? The answer is that even if the wires were superconductors, some amount of energy will be definitely dissipated because the charges would get accelerated in the process and according to Maxwell's equations, accelerating charges would always radiate some energy. The surprising part is that no matter what the resistance of the circuit is, the energy dissipated would turn out to be the same; the only difference is that one process would be very quick and another rather slow.

26-6 | Capacitor with a Dielectric

If you fill the space between the plates of a capacitor with a *dielectric,* which is an insulating material such as mineral oil or plastic, what happens to the capacitance? Michael Faraday—to whom the whole concept of capacitance is largely due and for whom the SI unit of capacitance is named—first looked into this matter in 1837. Using simple equipment much like that shown in Fig. 26-17, he found that the capacitance *increased* by a numerical factor κ, which he called the **dielectric constant** of the insulating material. Table 26-1 shows some dielectric materials and their dielectric constants. The dielectric constant of a vacuum is unity by definition. Because air is mostly empty space, its measured dielectric constant is only slightly greater than unity.

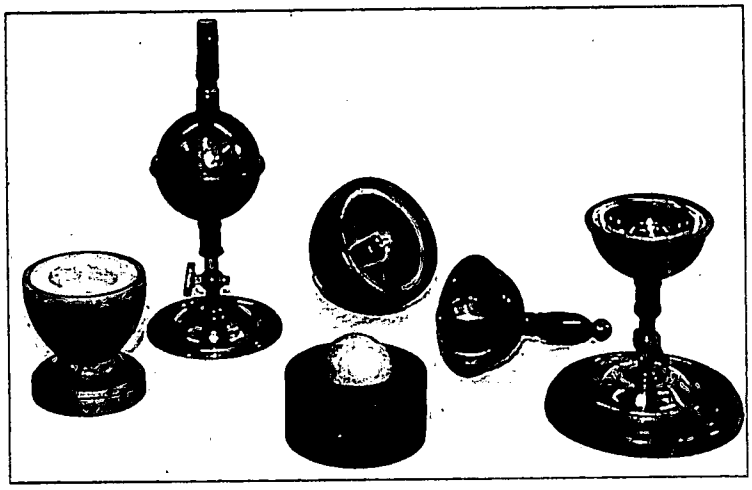


FIG. 26-17 The simple electrostatic apparatus used by Faraday. An assembled apparatus (second from left) forms a spherical capacitor consisting of a central brass ball and a concentric brass shell. Faraday placed dielectric materials in the space between the ball and the shell. *(The Royal Institute, England/ Bridgeman Art Library/NY)*

Another effect of the introduction of a dielectric is to limit the potential difference that can be applied between the plates to a certain value V_{max}, called the *breakdown potential.* If this value is substantially exceeded, the dielectric material will break down and form a conducting path between the plates. Every dielectric material has a characteristic *dielectric strength,* which is the maximum value of the electric field that it can tolerate without breakdown. A few such values are listed in Table 26-1.

As we discussed just after Eq. 26-18, the capacitance of any capacitor can be written in the form

$$C = \varepsilon_0 \mathcal{L}, \tag{26-26}$$

in which $\mathcal{L}$ has the dimension of length. For example, $\mathcal{L} = A/d$ for a parallel-plate capacitor. Faraday's discovery was that, with a dielectric *completely* filling the space between the plates, Eq. 26-26 becomes

$$C = \kappa \varepsilon_0 \mathcal{L} = \kappa C_{air}, \tag{26-27}$$

where C_{air} is the value of the capacitance with only air between the plates.

Figure 26-18 provides some insight into Faraday's experiments. In Fig. 26-18*a* the battery ensures that the potential difference V between the plates will remain constant. When a dielectric slab is inserted between the plates, the charge q on the plates increases by a factor of κ; the additional charge is delivered to the capacitor plates by the battery. In Fig. 26-18*b* there is no battery, and therefore the charge q must remain constant when the dielectric slab is inserted; then the potential difference V between the plates decreases by a factor of κ. Both these observations are consistent (through the relation $q = CV$) with the increase in capacitance caused by the dielectric.

Comparison of Eqs. 26-26 and 26-27 suggests that the effect of a dielectric can be summed up in more general terms:

> In a region completely filled by a dielectric material of dielectric constant κ, all electrostatic equations containing the permittivity constant ε_0 are to be modified by replacing ε_0 with $\kappa\varepsilon_0$.

Thus, the magnitude of the electric field produced by a point charge inside an infinite liquid dielectric is given by this modified form of Eq. 23-13:

$$E = \frac{1}{4\pi\kappa\varepsilon_0}\frac{q}{r^2}. \tag{26-28}$$

Also, the expression for the electric field just outside an isolated conductor immersed in a dielectric (see Eq. 25-1) becomes

$$E = \frac{\sigma}{\kappa\varepsilon_0}. \tag{26-29}$$

TABLE 26-1

Some Properties of Dielectrics[a]

Material	Dielectric Constant κ	Dielectric Strength (kV/mm)
Air (1 atm)	1.00054	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	
Pyrex	4.7	14
Ruby mica	5.4	
Porcelain	6.5	
Silicon	12	
Germanium	16	
Ethanol	25	
Water (20°C)	80.4	
Water (25°C)	78.5	
Titania ceramic	130	
Strontium titanate	310	8

For a vacuum, κ = unity.

[a]Measured at room temperature, except for the water.

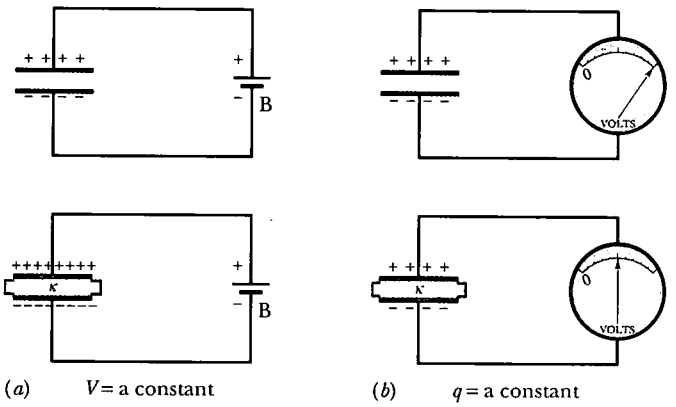


FIG. 26-18 (*a*) If the potential difference between the plates of a capacitor is maintained, as by battery B, the effect of a dielectric is to increase the charge on the plates. (*b*) If the charge on the capacitor plates is maintained, as in this case, the effect of a dielectric is to reduce the potential difference between the plates. The scale shown is that of a *potentiometer,* a device used to measure potential difference (here, between the plates). A capacitor cannot discharge through a potentiometer.

Because κ is always greater than unity, both these equations show that *for a fixed distribution of charges, the effect of a dielectric is to weaken the electric field* that would otherwise be present.

Sample Problem 26-7

A parallel-plate capacitor whose capacitance C is 13.5 pF is charged by a battery to a potential difference $V = 12.5$ V between its plates. The charging battery is now disconnected, and a porcelain slab ($\kappa = 6.50$) is slipped between the plates.

(a) What is the potential energy of the capacitor before the slab is inserted?

Conceptualize/Classify: We can relate the potential energy U_i of the capacitor to the capacitance C and either the potential V (with Eq. 26-22) or the charge q (with Eq. 26-21):

$$U_i = \frac{1}{2}CV^2 = \frac{q^2}{2C}.$$

Compute: Because we are given the initial potential V (= 12.5 V), we use Eq. 26-22 to find the initial stored energy:

$$U_i = \frac{1}{2}CV^2 = \frac{1}{2}(13.5 \times 10^{-12}\ \text{F})(12.5\ \text{V})^2$$
$$= 1.055 \times 10^{-9}\ \text{J} = 1055\ \text{pJ} \approx 1100\ \text{pJ}. \qquad \text{(Answer)}$$

(b) What is the potential energy of the capacitor–slab device after the slab is inserted?

Conceptualize/Classify: Because the battery has been disconnected, the charge on the capacitor cannot change when the dielectric is inserted. However, the potential *does* change.

Compute: Thus, we must now use Eq. 26-21 to write the final potential energy U_f, but now that the slab is within the capacitor, the capacitance is κC. We then have

$$U_f = \frac{q^2}{2\kappa C} = \frac{U_i}{\kappa} = \frac{1055\ \text{pJ}}{6.50}$$
$$= 162\ \text{pJ} \approx 160\ \text{pJ}. \qquad \text{(Answer)}$$

When the slab is introduced, the potential energy decreases by a factor of κ.

The "missing" energy, in principle, would be apparent to the person who introduced the slab. The capacitor would exert a tiny tug on the slab and would do work on it, in amount

$$W = U_i - U_f = (1055 - 162)\ \text{pJ} = 893\ \text{pJ}.$$

If the slab were allowed to slide between the plates with no restraint and if there were no friction, the slab would oscillate back and forth between the plates with a (constant) mechanical energy of 893 pJ, and this system energy would transfer back and forth between kinetic energy of the moving slab and potential energy stored in the electric field.

26-7 | Dielectrics: An Atomic View

What happens, in atomic and molecular terms, when we put a dielectric in an electric field? There are two possibilities, depending on the type of molecule:

1. *Polar dielectrics.* The molecules of some dielectrics, like water, have permanent electric dipole moments. In such materials (called *polar dielectrics*), the electric dipoles tend to line up with an external electric field as in Fig. 26-19. Because the molecules are continuously jostling each other as a result of their random thermal motion, this alignment is not complete, but it becomes more complete as the magnitude of the applied field is increased (or as the temperature, and thus the jostling, are decreased). The alignment of the electric dipoles produces an electric field that is directed opposite the applied field and is smaller in magnitude.
2. *Nonpolar dielectrics.* Regardless of whether they have permanent electric dipole moments, molecules acquire dipole moments by induction when placed in an external electric field. This occurs because the external field tends to "stretch" the molecules, slightly separating the centers of negative and positive charge.

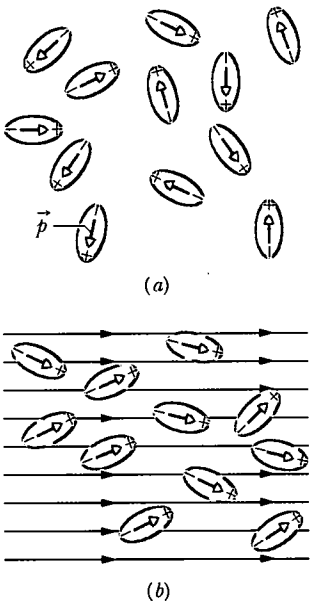


FIG. 26-19 (*a*) Molecules with a permanent electric dipole moment, showing their random orientation in the absence of an external electric field. (*b*) An electric field is applied, producing partial alignment of the dipoles. Thermal agitation prevents complete alignment.

Figure 26-20*a* shows a nonpolar dielectric slab with no external electric field applied. In Fig. 26-20*b*, an electric field $\vec{E}_0$ is applied via a capacitor, whose plates are charged as shown. The result is a slight separation of the centers of the positive and negative charge distributions within the slab, producing positive charge on one face of the slab (due to the positive ends of dipoles there) and negative charge on the opposite face (due to the negative ends of dipoles there). The slab as a whole remains electrically neutral and—within the slab—there is no excess charge in any volume element.

Figure 26-20*c* shows that the induced surface charges on the faces produce an electric field $\vec{E}'$ in the direction opposite that of the applied electric field $\vec{E}_0$. The resultant field $\vec{E}$ inside the dielectric (the vector sum of fields $\vec{E}_0$ and $\vec{E}'$) has the direction of $\vec{E}_0$ but is smaller in magnitude.

Both the field $\vec{E}'$ produced by the surface charges in Fig. 26-20*c* and the electric field produced by the permanent electric dipoles in Fig. 26-19 act in the same way—they oppose the applied field $\vec{E}$. Thus, the effect of both polar and nonpolar dielectrics is to weaken any applied field within them, as between the plates of a capacitor.

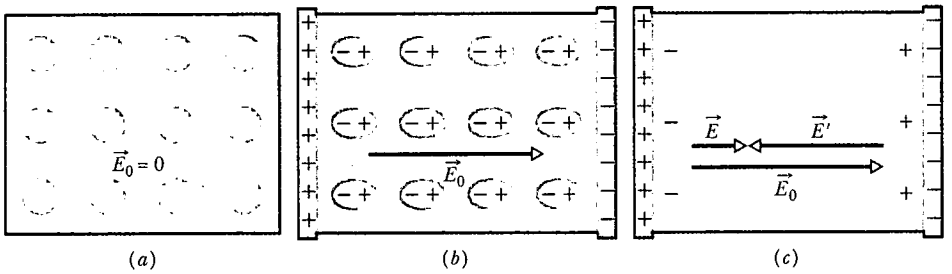


FIG. 26-20 (*a*) A nonpolar dielectric slab. The circles represent the electrically neutral atoms within the slab. (*b*) An electric field is applied via charged capacitor plates; the field slightly stretches the atoms, separating the centers of positive and negative charge. (*c*) The separation produces surface charges on the slab faces. These charges set up a field $\vec{E}'$, which opposes the applied field $\vec{E}_0$. The resultant field $\vec{E}$ inside the dielectric (the vector sum of $\vec{E}_0$ and $\vec{E}'$) has the same direction as $\vec{E}_0$ but a smaller magnitude.

Induced Dipole Moment

Many molecules, such as water, have *permanent* electric dipole moments. In other molecules (called *nonpolar molecules*) and in every isolated atom, the centers of the positive and negative charges coincide (Fig. 26-21*a*) and thus no dipole moment is set up. However, if we place an atom or a nonpolar molecule in an external electric field, the field distorts the electron orbits and separates the centers of positive and negative charge (Fig. 26-21*b*). Because the electrons are negatively charged, they tend to be shifted in a direction opposite the field. This shift sets up a dipole moment $\vec{P}$ that points in the direction of the field. This dipole

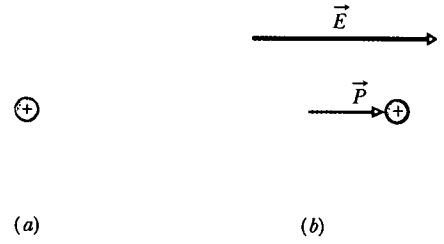


FIG. 26-21 (*a*) An atom showing the positively charged nucleus (green) and the negatively charged electrons (gold shading). The centers of positive and negative charge coincide. (*b*) If the atom is placed in an external electric field, the electron orbits are distorted so that the centers of positive and negative charge no longer coincide. An induced dipole moment $\vec{p}$ appears. The distortion is greatly exaggerated here.

moment is said to be *induced* by the field and the atom or molecule is then said to be *polarized* by the field (i.e., it has a positive side and a negative side). When the field is removed, the induced dipole moment and the polarization disappear.

CHECKPOINT 4 In a cylindrical capacitor, if the potential difference is too great, dielectric breakdown will occur. Will such breakdown begin near the inner or the outer conductor? Why?

26-8 | Dielectrics and Gauss' Law

In our discussion of Gauss' law in Chapter 23, we assumed that the charges existed in a vacuum. Here we shall see how to modify and generalize that law if dielectric materials, such as those listed in Table 26-1, are present. Figure 26-22 shows a parallel-plate capacitor of plate area A, both with and without a dielectric. We assume that the charge q on the plates is the same in both situations. Note that the field between the plates induces charges on the faces of the dielectric by one of the methods described in Section 26-7.

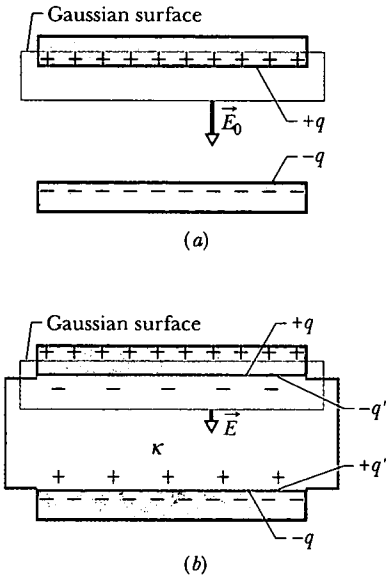


FIG. 26-22 A parallel-plate capacitor (*a*) without and (*b*) with a dielectric slab inserted. The charge q on the plates is assumed to be the same in both cases.

For the situation of Fig. 26-22*a*, without a dielectric, we can find the electric field $\vec{E}_0$ between the plates as we did in Fig. 26-5: We enclose the charge $+q$ on the top plate with a Gaussian surface and then apply Gauss' law. Letting E_0 represent the magnitude of the field, we find

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E_0 A = q, \tag{26-30}$$

or

$$E_0 = \frac{q}{\varepsilon_0 A}. \tag{26-31}$$

In Fig. 26-22*b*, with the dielectric in place, we can find the electric field between the plates (and within the dielectric) by using the same Gaussian surface. However, now the surface encloses two types of charge: It still encloses charge $+q$ on the top plate, but it now also encloses the induced charge $-q'$ on the top face of the dielectric. The charge on the conducting plate is said to be *free charge* because it can move if we change the electric potential of the plate; the induced charge on the surface of the dielectric is not free charge because it cannot move from that surface.

The net charge enclosed by the Gaussian surface in Fig. 26-22*b* is $q - q'$, so Gauss' law now gives

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = q - q', \tag{26-32}$$

or

$$E = \frac{q - q'}{\varepsilon_0 A}. \tag{26-33}$$

The effect of the dielectric is to weaken the original field E_0 by a factor of κ; so we may write

$$E = \frac{E_0}{\kappa} = \frac{q}{\kappa \varepsilon_0 A}. \tag{26-34}$$

Comparison of Eqs. 26-33 and 26-34 shows that

$$q - q' = \frac{q}{\kappa}. \tag{26-35}$$

Equation 26-35 shows correctly that the magnitude q' of the induced surface charge is less than that of the free charge q and is zero if no dielectric is present (because then $\kappa = 1$ in Eq. 26-35).

By substituting for $q - q'$ from Eq. 26-35 in Eq. 26-32, we can write Gauss' law in the form

$$\varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q \qquad \text{(Gauss' law with dielectric).} \tag{26-36}$$

This equation, although derived for a parallel-plate capacitor, is true generally and is the most general form in which Gauss' law can be written. Note:

1. The flux integral now involves $\kappa\vec{E}$ not just $\vec{E}$. (The vector $\varepsilon_0\kappa\vec{E}$ is sometimes called the *electric displacement* $\vec{D}$ so that Eq. 26-36 can be written in the form $\oint \vec{D} \cdot d\vec{A} = q$.)
2. The charge q enclosed by the Gaussian surface is now taken to be the *free charge only.* The induced surface charge is deliberately ignored on the right side of Eq. 26-36, having been taken fully into account by introducing the dielectric constant κ on the left side.
3. Equation 26-36 differs from Eq. 23-8, our original statement of Gauss' law, only in that ε_0 in the latter equation has been replaced by $\kappa\varepsilon_0$. We keep κ inside the integral of Eq. 26-36 to allow for cases in which κ is not constant over the entire Gaussian surface.

Sample Problem 26-8

Figure 26-23 shows a parallel-plate capacitor of plate area A and plate separation d. A potential difference V_0 is applied between the plates. The battery is then disconnected, and a dielectric slab of thickness b and dielectric constant κ is placed between the plates as shown. Assume $A = 115\ \text{cm}^2$, $d = 1.24\ \text{cm}$, $V_0 = 85.5\ \text{V}$, $b = 0.780\ \text{cm}$, and $\kappa = 2.61$.

(a) What is the capacitance C_0 before the dielectric slab is inserted?

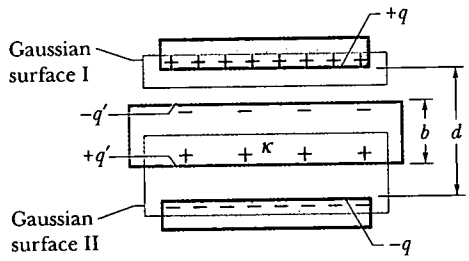


FIG. 26-23 A parallel-plate capacitor containing a dielectric slab that only partially fills the space between the plates.

Conceptualize/Classify: We use the formula for capacitance of a parallel-plate capacitor.

Compute: From Eq. 26-9 we have

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{(8.85\times10^{-12}\ \text{F/m})(115\times10^{-4}\ \text{m}^2)}{1.24\times10^{-2}\ \text{m}}$$
$$= 8.21\times10^{-12}\ \text{F} = 8.21\ \text{pF}. \qquad \text{(Answer)}$$

(b) What free charge appears on the plates?

Compute: From Eq. 26-1,

$$q = C_0 V_0 = (8.21\times10^{-12}\ \text{F})(85.5\ \text{V})$$
$$= 7.02\times10^{-10}\ \text{C} = 702\ \text{pC}. \qquad \text{(Answer)}$$

Conclude: Because the battery was disconnected before the slab was inserted, the free charge is unchanged.

(c) What is the electric field E_0 in the gaps between the plates and the dielectric slab?

Conceptualize/Classify: We need to apply Gauss' law, in the form of Eq. 26-36, to Gaussian surface I in Fig. 26-23.

Compute: That surface passes through the gap, and so it encloses *only* the free charge on the upper capacitor plate. Because the area vector $d\vec{A}$ and the field vector $\vec{E}_0$ are both directed downward, the dot product in Eq. 26-36 becomes

$$\vec{E}_0 \cdot d\vec{A} = E_0\, dA \cos 0° = E_0\, dA.$$

Equation 26-36 then becomes

$$\varepsilon_0 \kappa E_0 \oint dA = q.$$

The integration now simply gives the surface area A of the plate. Thus, we obtain

$$\varepsilon_0 \kappa E_0 A = q,$$

or

$$E_0 = \frac{q}{\varepsilon_0 \kappa A}.$$

We must put $\kappa = 1$ here because Gaussian surface I does not pass through the dielectric. Thus, we have

$$E_0 = \frac{q}{\varepsilon_0 \kappa A} = \frac{7.02\times10^{-10}\ \text{C}}{(8.85\times10^{-12}\ \text{F/m})(1)(115\times10^{-4}\ \text{m}^2)}$$
$$= 6900\ \text{V/m} = 6.90\ \text{kV/m}. \qquad \text{(Answer)}$$

Conclude: Note that the value of E_0 does not change when the slab is introduced because the amount of charge enclosed by Gaussian surface I in Fig. 26-23 does not change.

(d) What is the electric field E_1 in the dielectric slab?

Conceptualize/Classify: Now we apply Gauss' law in the form of Eq. 26-36 to Gaussian surface II in Fig. 26-23.

Compute: That surface encloses free charge $-q$ and induced charge $+q'$, but we ignore the latter when we use Eq. 26-36. We find

$$\varepsilon_0 \oint \kappa \vec{E}_1 \cdot d\vec{A} = -\varepsilon_0 \kappa E_1 A = -q. \qquad (26\text{-}37)$$

(The first minus sign in this equation comes from the dot product $\vec{E}_1 \cdot d\vec{A}$, because now the field vector $\vec{E}_1$ is directed downward and the area vector $d\vec{A}$ is directed upward.) Now $\kappa = 2.61$. Thus, Eq. 26-37 gives us

$$E_1 = \frac{q}{\varepsilon_0 \kappa A} = \frac{E_0}{\kappa} = \frac{6.90\ \text{kV/m}}{2.61}$$
$$= 2.64\ \text{kV/m}. \qquad \text{(Answer)}$$

(e) What is the potential difference V between the plates after the slab has been introduced?

Conceptualize/Classify: We find V by integrating along a straight line directly from the bottom plate to the top plate.

Compute: Within the dielectric, the path length is b and the electric field is E_1. Within the two gaps above and below the dielectric, the total path length is $d - b$ and the electric field is E_0. Equation 26-6 then yields

$$V = \int_-^+ E\, ds = E_0(d-b) + E_1 b$$
$$= (6900\ \text{V/m})(0.0124\ \text{m} - 0.00780\ \text{m})$$
$$+ (2640\ \text{V/m})(0.00780\ \text{m})$$
$$= 52.3\ \text{V}. \qquad \text{(Answer)}$$

Conclude: This is less than the original potential difference of 85.5 V.

(f) What is the capacitance with the slab in place between the plates of the capacitor?

Conceptualize/Classify: The capacitance C is related to the free charge q and the potential difference V via Eq. 26-1.

Compute: Taking q from (b) and V from (e), we have

$$C = \frac{q}{V} = \frac{7.02\times10^{-10}\ \text{C}}{52.3\ \text{V}}$$
$$= 1.34\times10^{-11}\ \text{F} = 13.4\ \text{pF}. \qquad \text{(Answer)}$$

Conclude: This is greater than the original capacitance of 8.21 pF.

Force on a Dielectric Slab Inside a Capacitor

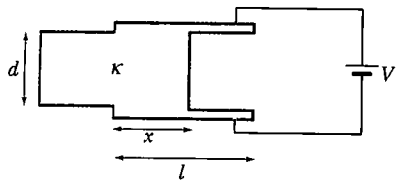


FIG. 26-24 A dielectric slab with same dimensions as the inner dimensions of the capacitor, is partially inserted inside the capacitor.

Figure 26-24 shows a parallel-plate capacitor with plates of width w and length l. The separation between the plates is d. The plates are rigidly clamped and connected to a battery of emf V. A dielectric slab of thickness d and dielectric constant K is slowly inserted between the plates. We wish to find the force exerted by the electric field on the dielectric slab.

Just as a conductor is attracted to an electric field, so it too is dielectric—and for essentially the same reason, the bound charge tends to accumulate near the free charge of the opposite sign. But the calculation of forces on dielectrics can be surprisingly tricky.

Consider, for example, the case of a slab of linear dielectric material, partially inserted between the places of a parallel-plate capacitor (Fig. 26-24). We have always pretended that the field is uniform inside a parallel-plate capacitor and

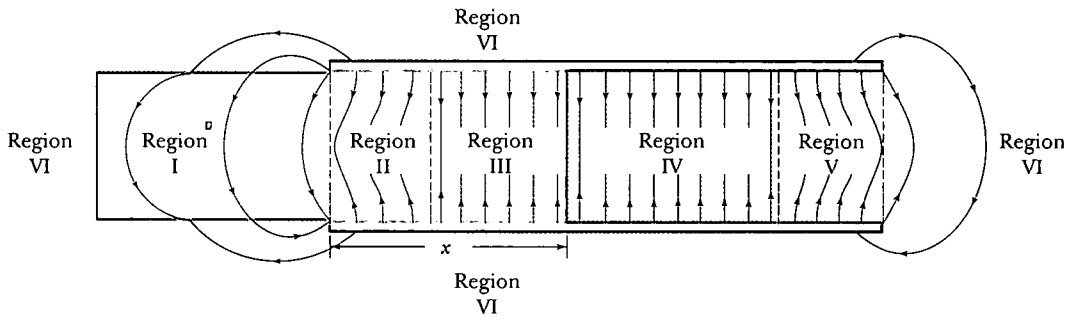


FIG. 26-25 Division of all space into six regions for the purpose of evaluating the total field energy. Region I, II, and III are within the dielectric and regions IV, V, and VI are in vacuum.

zero outside. If this were literally true, there would be no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates. However, there is in reality a fringing field around the edges, which for most purposes can be ignored but in this case is responsible for the whole effect. Indeed, the field could not terminate abruptly at the edge of the capacitor, for if it did the line integral of E around any closed loop in Fig. 26-24 would not be zero. It is this nonuniform fringing field that pulls the dielectric into the capacitor.

In order to use the principle of work correctly, the presence of fringing must be acknowledged. To this end, let all space be divided into six regions as shown in Fig. 26-25. Region I consists of the dielectric volume up to the left-hand edge of the plates, region II is the dielectric volume from this point up to where the field is uniform, and region III is the remainder of the dielectric volume. The other regions are in vacuum. Region IV consists of the volume of uniform field between the plates, region V continues from this point to the right-hand edge of the plates, and region VI is the rest of space.

Consider the special case where the left-hand edge of the dielectric extends beyond the end of the fringe field and its right-hand edge lies in the region of uniform field, as shown in Fig. 26-25. In this case, an infinitesimal displacement dx to the right does not change the field energy in regions I, II, V, and VI. Instead, the only effect is to increase the volume of region III by an amount $(wd)dx$ while decreasing that of region IV by the same amount. Since the field is given by $|E| = V/d$ in both regions, the resulting change in the total field energy is

$$dU = \frac{1}{2}K\varepsilon_0\left(\frac{V}{d}\right)^2(wd)dx - \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2(wd)dx$$
$$= \frac{(K-1)\varepsilon_0 wV^2}{2d}dx.$$

This can be in turn represented in terms of the capacitance: The plate area of the part with the dielectric is wx. Its capacitance is

$$C_1 = \frac{K\varepsilon_0 wx}{d}.$$

Similarly, the capacitance of the part without the dielectric is

$$C_2 = \frac{\varepsilon_0 w(l-x)}{d}.$$

These two parts are connected in parallel. The capacitance of the system is, therefore,

$$C = C_1 + C_2 = \frac{\varepsilon_0 w}{d}[l + x(K-1)].$$

The energy of the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{\varepsilon_0 wV^2}{2d}[l + x(K-1)].$$

So it follows that

$$dU = \frac{(K-1)\varepsilon_0 w V^2}{2d} dx.$$

Suppose, the electric field attracts the dielectric slab with a force F. An external force of magnitude equal to F should be applied in opposite direction so that the plate moves slowly (no acceleration). As the slab moves inside, the capacitance increases to $C + dC$. As the potential difference remains constant at V, the battery has to supply a further charge of

$$dQ = (dC)V$$

to the capacitor. The work done by the battery is, therefore,

$$dW_b = VdQ = (dC)V^2.$$

The external force F does a work

$$dW_e = (-Fdx)$$

during the displacement. The total work done on the capacitor is

$$dW_b + dW_e = (dC)V^2 - Fdx.$$

This should be equal to the increase dU in the stored energy. Thus,

$$\tfrac{1}{2}(dC)V^2 = (dC)V^2 - Fdx$$

$$F = \frac{1}{2}V^2\frac{dC}{dx}$$

$$F = \frac{\varepsilon_0 w V^2 (K-1)}{2d}.$$

Thus, the electric field attracts the dielectric into the capacitor with a force $\varepsilon_0 w V^2 (K-1)/2d$ and this much force should be applied in the opposite direction to move it slowly.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: A capacitor with vacuum between the plates can store any amount of charge.

Statement 2: The dielectric strength of vacuum is infinity.

2. Statement 1: Equation $C = \varepsilon_0 A/d$ is an underestimate for the capacitance of a real parallel-plate capacitor.

Statement 2: While deriving this equation, we ignored the effects of fringing electric field.

3. Statement 1: When a dielectric slab is inserted inside a capacitor connected to a voltage source, the electric field inside the dielectric is less than what was in the capacitor before the dielectric was inserted.

Statement 2: In the dielectric, the electric field is lesser than the electric field outside.

4. Statement 1: The capacitance of any capacitor is inversely proportional to the potential difference across it.

Statement 2: Capacitance of a capacitor is defined as $C = Q/V$.

5. Statement 1: The capacitance of a solid copper sphere is greater than the capacitance of a hollow copper sphere of the same radius.

Statement 2: The mass of a solid copper sphere is greater than the mass of a hollow copper sphere of the same radius.

MULTIPLE CHOICE QUESTIONS

sec. 26-3 Calculating the Capacitance

1. The plate areas and plate separations of five parallel-plate capacitors are

Capacitor 1: Area A_0, separation d_0

Capacitor 2: Area $2A_0$, separation $2d_0$

Capacitor 3: Area $2A_0$, separation $d_0/2$

Capacitor 4: Area $A_0/2$, separation $2d_0$

Capacitor 5: Area A_0, separation $d_0/2$

Rank these according to their capacitances, least to greatest.

(a) 1, 2, 3, 4, 5

(b) 5, 4, 3, 2, 1

(c) 5, 3, and 4 tie, then 1, 2

(d) 4, 1, and 2 tie, then 5, 3

2. Electric potential difference between the positive plate and halfway point between the two parallel plates of a fully charged capacitor is

(a) 0

(b) One-half of the total potential difference between the two plates

(c) One-quarter of the total potential difference between the two plates

(e) None of the above

sec. 26-4 Combination of Capacitors

3. What is the equivalent capacitance of the system of capacitors between A and B as shown in Fig. 26-26.

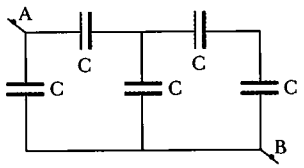


FIG. 26-26 Multiple choice question 3.

(a) 1 C

(b) 1.6 C

(c) C

(d) None of these

4. How many different capacitances can you produce using three capacitors if you have all three connected in the circuit each time?

(a) 5

(b) 6

(c) 7

(d) 8

5. Two capacitors with C_A greater than C_B and are connected in series with a battery. Which of the following is true?

(a) There is more charge stored on C_A

(b) There is more charge stored on C_B

(c) There is the same charge stored on each capacitor

(d) There is the same potential difference across both capacitors

(e) There is a larger potential difference across the capacitor with C_A

6. Four metallic plates each with a surface area of one side A are placed at a distance d from each other as shown in Fig. 26-27. The capacitance of the system is

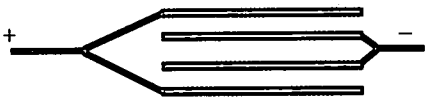


FIG. 26-27 Multiple choice question 6.

(a) $\frac{\varepsilon_0 A}{d}$

(b) $\frac{2\varepsilon_0 A}{d}$

(c) $\frac{3\varepsilon_0 A}{d}$

(d) $\frac{4\varepsilon_0 A}{d}$

7. Consider the circuit shown in Fig. 26-28.

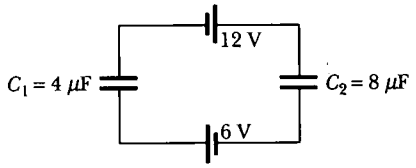


FIG. 26-28 Multiple choice question 7.

(a) The charge on C_2 is greater that on C_1

(b) The charge on C_1 and C_2 are the same

(c) The potential drops across C_1 and C_2 are the same

(d) The potential drops across C_1 is greater than that across C_2

8. A 2 μF and a 1 μF capacitor are connected in series and charged from a battery. They store charges P and Q, respectively. When disconnected and charged separately using the same battery, they have charges R and S, respectively. Then

(a) $R > S > Q = P$

(b) $P > Q > R = S$

(c) $R > P = Q > S$

(d) $R = P > S = Q$

(e) $R > P > S = Q$

sec. 26-5 Energy Stored in an Electric Field

9. You have five capacitors. To store maximum amount of energy when using a source with terminal voltage V

(a) You need to connect them all in series

(b) You need to connect them all in parallel

(c) You need to connect three capacitors in parallel and the rest in series with that combination

(d) You need to connect two capacitors in parallel and the rest in series with that combination

(e) It does not matter how you connect them as same amount of energy is stored for all combinations

10. You charge a parallel-plate capacitor, remove it from the battery, and prevent the wires connected to the plates from touching each other. When you pull the plates farther apart, which one of the following quantities increases?

(a) Electric field between the plates

(b) Charge

(c) Capacitance

(d) Voltage between the plates

(e) Energy stored in the capacitor

11. Two identical capacitors are charged individually to different potentials and are disconnected from the source. The capacitors are then connected in parallel to each other.

(a) Net charge is less than the sum of initial individual charges

(b) Net charge equals the sum of initial charge

(c) The net potential difference across them is different from the sum of the individual initial potential difference

(d) The net energy stored in the two capacitors is less than the sum of the initial individual energies

12. Which of following graph in Fig. 26-29 correctly represents the force between plates of an isolated charged parallel-plate capacitor with distance x between them.

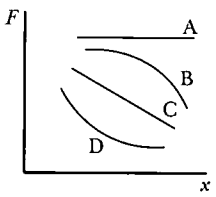


FIG. 26-29 Multiple choice question 12.

(a) A

(b) B

(c) C

(d) D

13. The top plate of a charged plane capacitor is fixed, whereas the bottom one is kept in equilibrium by the gravitational and electrostatic forces.

(a) When the capacitor is isolated, the plate is in stable equilibrium

(b) When the capacitor is connected to a battery, the plate is in unstable equilibrium

(c) When the capacitor is isolated, the plate is in neutral equilibrium

(d) When the capacitor is connected to a battery, the plate is in neutral equilibrium

sec. 26-6 Capacitor with a Dielectric

14. Consider an isolated capacitor made of two parallel metallic plates separated by a distance d. The top plate has a surface charge density of $-\sigma$. A slab of metal of thickness $l < d$ is inserted between the plates, not touching either plates. Upon insertion of the metal slab, the potential difference between the plates

(a) Increase

(b) Decreases

(c) Remains the same

15. A parallel-plate capacitor is charged by a battery. The battery is removed and a thick glass slab is inserted between the plates. Now,

(a) The capacity of the capacitor is increased

(b) The electrical energy stored in the capacitor is decreased

(c) The potential acorss the plates is decreased

(d) The electric field between the plates is dereased

16. A parallel-plate capacitor has parallel sheet of copper inserted between and parallel to the two plate, without touching the plates. The capacity of the capacitor after the introduction of the copper sheet is

(a) Minimum when the copper sheet touches one of the plates

(b) Maximum when the copper sheet is midway between the two plates

(c) Maximum when the copper sheet touches one of the plates

(d) Invariant for all positions of the sheet between the plates

(e) Greater than that before introducing the sheet

17. Consider a parallel-plate capacitor originally with a charge q_0, capacitance C_0, and potential difference ΔV_0. There is an electrostatic force of magnitude F_0 between the plates, and the capacitor has a stored energy U_0. The terminals of the capacitor are connected to another capacitor of same capacitance and charge.

(i) A dielectric slab with $k_e > 1$ is inserted between the plates of the first capacitor. Which quantities increase? (Choose all that apply).

(a) q

(b) C

(c) ΔV

(d) F

(ii) What is the direction of the electrostatic force on the dielectric slab while it is being inserted?

(a) The force pulls the slab into the capacitor

(b) The force pushes the slab out of the capacitor

(c) There is no electrostatic force on the slab

(d) Can't be said

(iii) Later the dielectric slab is removed. While the slab is being removed

(a) Charge on second capacitor increases

(b) Charge on second capacitor decreases

(c) Charge on second capacitor is constant

(d) Can't be said

18. Two identical capacitors are connected as shown in Fig. 26-30. If a dielectric slab is inserted in B, choose correct statement.

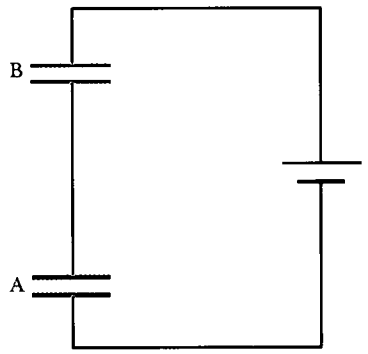


FIG. 26-30 Multiple choice question 18.

(a) Energy of both capacitors will increase
(b) Energy of both capacitors will decrease
(c) Energy of A will increase but B will decrease
(d) Energy of B will increase but A will decrease

19. Figure 26-31 shows a capacitor having three layers between its plates. Layer x is vacuum, y is conductor, and z is a dielectric. Which of the following change(s) will result in increase in capacitance?

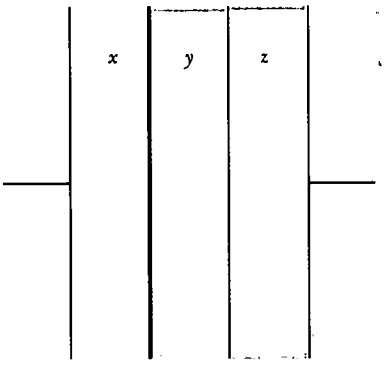


FIG. 26-31 Multiple choice question 19.

(a) Replace x by conductor
(b) Replace y by dielectric
(c) Replace z by conductor
(d) Replace x by dielectric

PROBLEMS

sec. 26-2 Capacitance

1. The two metal objects in Fig. 26-32 have net charges of +70 pC and −70 pC, which result in a 20 V potential difference between them. (a) What is the capacitance of the system? (b) If the charges are changed to +200 pC and −200 pC, what does the capacitance become? (c) What does the potential difference become?

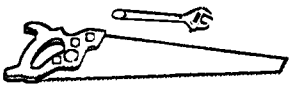


FIG. 26-32 Problem 1.

2. The capacitor in Fig. 26-33 has a capacitance of 25 μF and is initially uncharged. The battery provides a potential difference of 120 V. After switch S is closed, how much charge will pass through it?

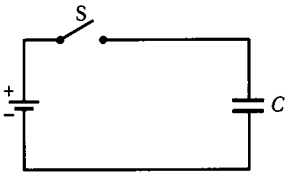


FIG. 26-33 Problem 2.

sec. 26-3 Calculating the Capacitance

3. You have two flat metal plates, each of area 1.00 m^2, with which to construct a parallel-plate capacitor. (a) If the capacitance of the device is to be 1.00 F, what must be the separation between the plates? (b) Could this capacitor actually be constructed?

4. The plates of a spherical capacitor have radii 38.0 mm and 40.0 mm. (a) Calculate the capacitance. (b) What must be the plate area of a parallel-plate capacitor with the same plate separation and capacitance?

5. What is the capacitance of a drop that results when two mercury spheres, each of radius R = 2.00 mm, merge?

sec. 26-4 Combination of Capacitors

6. In Fig. 26-34 find the equivalent capacitance of the combination. Assume that C_1 is 10.0 μF, C_2 is 5.00 μF, and C_3 is 4.00 μF.

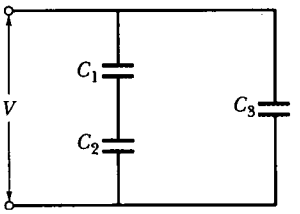


FIG. 26-34 Problem 6.

7. In Fig. 26-35, find the equivalent capacitance of the combination. Assume that C_1 = 10.0 μF, C_2 = 5.00 μF, and C_3 = 4.00 μF.

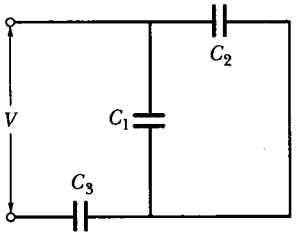


FIG. 26-35 Problems 7 and 9.

8. In Fig. 26-36, the battery has a potential difference of V = 10.0 V and the five capacitors each have a capacitance of 10.0 μF. What is the charge on (a) capacitor 1 and (b) capacitor 2?

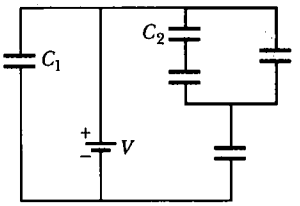


FIG. 26-36 Problem 8.

9. In Fig. 26-35, a potential difference of V = 100.0 V is applied across a capacitor arrangement with capacitances C_1 = 10.0 μF,

$C_2 = 5.00\ \mu F$, and $C_3 = 4.00\ \mu F$. If capacitor 3 undergoes electrical breakdown so that it becomes equivalent to conducting wire, what is the increase in (a) the charge on capacitor 1 and (b) the potential difference across capacitor 1?

10. Two parallel-plate capacitors, 6.0 μF each, are connected in parallel to a 10 V battery. One of the capacitors is then squeezed so that its plate separation is 50.0% of its initial value. Because of the squeezing, (a) how much additional charge is transferred to the capacitors by the battery and (b) what is the increase in the charge stored on the capacitors?

11. A 100 pF capacitor is charged to a potential difference of 50 V, and the charging battery is disconnected. The capacitor is then connected in parallel with a second (initially uncharged) capacitor. If the potential difference across the first capacitor drops to 35 V, what is the capacitance of this second capacitor?

12. Figure 26-37 shows a circuit section of four air-filled capacitors that is connected to a larger circuit. The graph below the section shows the electric potential $V(x)$ as a function of position x along the lower part of the section, through capacitor 4. Similarly, the graph above the section shows the electric potential $V(x)$ as a function of position x along the upper part of the section, through capacitors 1, 2, and 3. Capacitor 3 has a capacitance of 0.80 μF. What are the capacitances of (a) capacitor 1 and (b) capacitor 2?

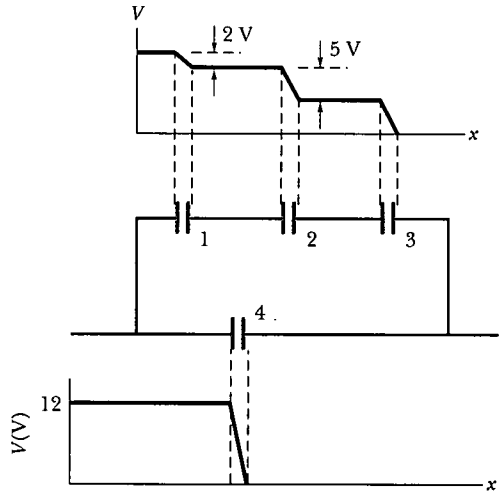


FIG. 26-37 Problem 12.

13. In Fig. 26-38, a 20.0 V battery is connected across capacitors of capacitances $C_1 = C_6 = 3.00\ \mu F$ and $C_3 = C_5 = 2.00$ $C_2 = 2.00$ $C_4 = 4.00\ \mu F$. What are (a) the equivalent capacitance C_{eq} of the capacitors and (b) the charge stored by C_{eq}? What are (c) V_1 and (d) q_1 of capacitor 1, (e) V_2 and (f) q_2 of capacitor 2, and (g) V_3 and (h) q_3 of capacitor 3?

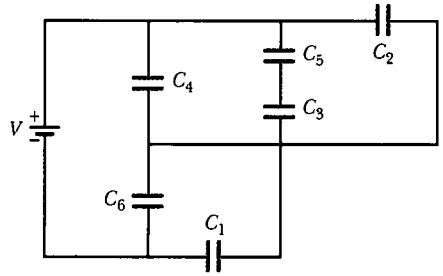


FIG. 26-38 Problem 13.

14. Plot 1 in Fig. 26-39*a* gives the charge q that can be stored on capacitor 1 versus the electric potential V set up across it. The vertical scale is set by $q_s = 16.0\ \mu C$, and the horizontal scale is set by $V_s = 2.0$ V. Plots 2 and 3 are similar plots for capacitors 2 and 3, respectively. Figure 26-39*b* shows a circuit with those three capacitors and a 6.0 V battery. What is the charge stored on capacitor 2 in that circuit?

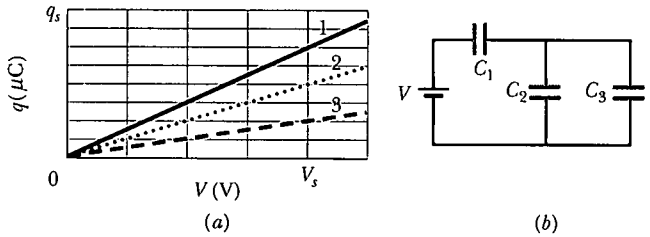


FIG. 26-39 Problem 14.

15. In Fig. 26-40, the capacitances are $C_1 = 1.0\ \mu F$ and $C_2 = 3.0\ \mu F$, and both capacitors are charged to a potential difference of $V = 100$ V but with opposite polarity as shown. Switches S_1 and S_2 are now closed. (a) What is now the potential difference between points a and b? What now is the charge on capacitor (b) 1 and (c) 2?

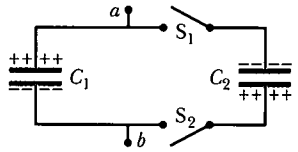


FIG. 26-40 Problem 15.

16. In Fig. 26-41, $V = 10$ V, $C_1 = 10\ \mu F$, and $C_2 = C_3 = 20\ \mu F$. Switch S is first thrown to the left side until capacitor 1 reaches equilibrium. Then the switch is thrown to the right. When equilibrium is again reached, how much charge is on capacitor 1?

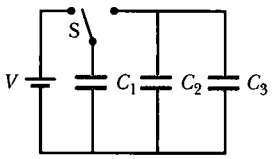


FIG. 26-41 Problem 16.

17. In Fig. 26-42, two parallel-plate capacitors (with air between the plates) are connected to a battery. Capacitor 1 has a plate area of 1.5 cm^2 and an electric field (between its plates) of magnitude 2000 V/m. Capacitor 2 has a plate area of 0.70 cm^2 and an electric field of magnitude 1500 V/m. What is the total charge on the two capacitors?

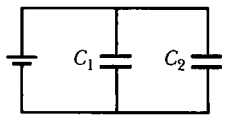


FIG. 26-42 Problem 17.

18. In Fig. 26-43, the battery has potential difference $V = 9.0$ V, $C_2 = 3.0\ \mu F$, $C_4 = 4.0\ \mu F$, and all the capacitors are initially uncharged. When switch S is closed, a total charge of 12 μC passes through point a and a total charge of 8.0 μC passes through point b. What are (a) C_1 and (b) C_3?

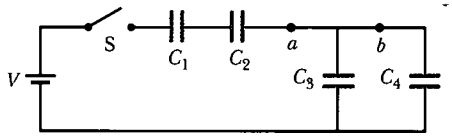


FIG. 26-43 Problem 18.

19. Figure 26-44 shows a variable "air gap" capacitor for manual tuning. Alternate plates are connected together; one group of plates is fixed in position, and the other group is capable of rotation. Consider a capacitor of $n = 8$ plates of alternating

polarity, each plate having area $A = 1.25\ \text{cm}^2$ and separated from adjacent plates by distance $d = 3.40$ mm. What is the maximum capacitance of the device?

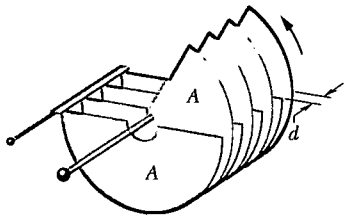


FIG. 26-44 Problem 19.

20. Figure 26-45 displays a 12.0 V battery and 3 uncharged capacitors of capacitances $C_1 = 4.00\ \mu\text{F}$, $C_2 = 6.00\ \mu\text{F}$, and $C_3 = 3.00\ \mu\text{F}$. The switch is thrown to the left side until capacitor 1 is fully charged. Then the switch is thrown to the right. What is the final charge on (a) capacitor 1, (b) capacitor 2, and (c) capacitor 3?

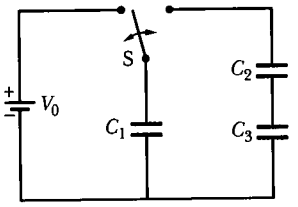


FIG. 26-45 Problem 20.

21. Figure 26-46 shows a 12.0 V battery and four uncharged capacitors of capacitances $C_1 = 1.00\ \mu\text{F}$, $C_2 = 2.00\ \mu\text{F}$, $C_3 = 3.00\ \mu\text{F}$, and $C_4 = 4.00\ \mu\text{F}$. If only switch S_1 is closed, what is the charge on (a) capacitor 1, (b) capacitor 2, (c) capacitor 3, and (d) capacitor 4? If both switches are closed, what is the charge on (e) capacitor 1, (f) capacitor 2, (g) capacitor 3, and (h) capacitor 4?

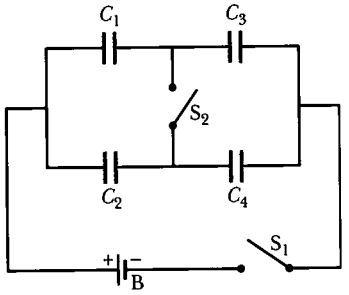


FIG. 26-46 Problem 21.

22. Capacitor 3 in Fig. 26-47*a* is a *variable capacitor* (its capacitance C_3 can be varied). Figure 26-47*b* gives the electric potential V_1 across capacitor 1 versus C_3. The horizontal scale is set by $C_{3s} = 12.0\ \mu\text{F}$. Electric potential V_1 approaches an asymptote of 10 V as $C_3 \to \infty$. What are (a) the electric potential V across the battery, (b) C_1, and (c) C_2?

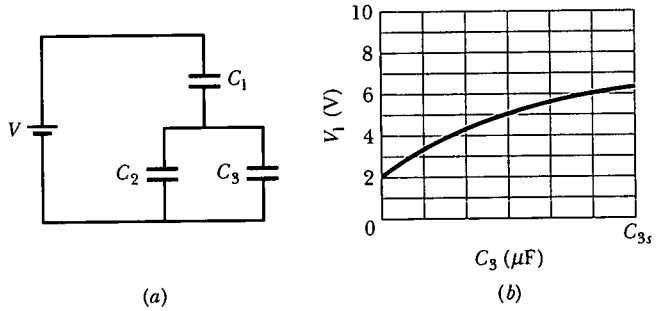


FIG. 26-47 Problem 22.

sec. 26-5 Energy Stored in an Electric Field

23. A 2.0 μF capacitor and a 4.0 μF capacitor are connected in parallel across a 300 V potential difference. Calculate the total energy stored in the capacitors.

24. A parallel-plate air-filled capacitor having area 40 cm^2 and plate spacing 1.0 mm is charged to a potential difference of 600 V. Find (a) the capacitance, (b) the magnitude of the charge on each plate, (c) the stored energy, (d) the electric field between the plates, and (e) the energy density between the plates.

25. How much energy is stored in 1.00 m^3 of air due to the "fair weather" electric field of magnitude 150 V/m?

26. Assume that a stationary electron is a point of charge. What is the energy density u of its electric field at radial distances (a) $r = 1.00$ mm, (b) $r = 1.00\ \mu$m, (c) $r = 1.00$ nm, and (d) $r = 1.00$ pm? (e) What is u in the limit as $r \to 0$?

27. The parallel plates in a capacitor, with a plate area of 8.50 cm^2 and an air-filled separation of 3.00 mm, are charged by a 6.00 V battery. They are then disconnected from the battery and pulled apart (without discharge) to a separation of 8.00 mm. Neglecting fringing, find (a) the potential difference between the plates, (b) the initial stored energy, (c) the final stored energy, and (d) the work required to separate the plates.

28. In Fig. 26-48, $C_1 = 10.0\ \mu\text{F}$, $C_2 = 20.0\ \mu\text{F}$, and $C_3 = 25.0\ \mu\text{F}$. If no capacitor can withstand a potential difference of more than 100 V without failure, what are (a) the magnitude of the maximum potential difference that can exist between points A and B and (b) the maximum energy that can be stored in the three-capacitor arrangement?

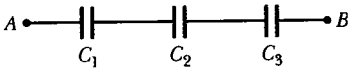


FIG. 26-48 Problem 28.

29. A charged isolated metal sphere of diameter 10 cm has a potential of 8000 V relative to $V = 0$ at infinity. Calculate the energy density in the electric field near the surface of the sphere.

sec. 26-6 Capacitor with a Dielectric

30. A parallel-plate air-filled capacitor has a capacitance of 50 pF. (a) If each of its plates has an area of 0.35 m^2, what is the separation? (b) If the region between the plates is now filled with material having $\kappa = 5.6$, what is the capacitance?

31. An air-filled parallel-plate capacitor has a capacitance of 1.3 pF. The separation of the plates is doubled, and wax is inserted between them. The new capacitance is 2.6 pF. Find the dielectric constant of the wax.

32. A coaxial cable used in a transmission line has an inner radius of 0.10 mm and an outer radius of 0.60 mm. Calculate the capacitance per meter for the cable. Assume that the space between the conductors is filled with polystyrene.

33. In Fig. 26-49, how much charge is stored on the parallel-plate capacitors by the 12.0 V battery? One is filled with air, and the other is filled with a dielectric for which $\kappa = 3.00$; both capacitors have a plate area of $5.00 \times 10^{-3}\ \text{m}^2$ and a plate separation of 2.00 mm.

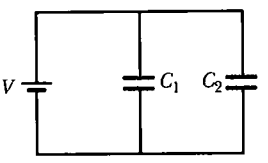


FIG. 26-49 Problem 33.

34. A certain substance has a dielectric constant of 2.8 and a dielectric strength of 18 MV/m. If it is used as the dielectric material in a parallel-plate capacitor, what minimum area should the plates of the capacitor have to obtain a capacitance of 7.0×10^{-2} μF and to ensure that the capacitor will be able to withstand a potential difference of 4.0 kV?

35. You are asked to construct a capacitor having a capacitance near 1 nF and a breakdown potential in excess of 10000 V. You think of using the sides of a tall Pyrex drinking glass as a dielectric, lining the inside and outside curved surfaces with aluminum foil to act as the plates. The glass is 15 cm tall with an inner radius of 3.6 cm and an outer radius of 3.8 cm. What are the (a) capacitance and (b) breakdown potential of this capacitor?

36. Figure 26-50 shows a parallel-plate capacitor with a plate area $A = 5.56$ cm^2 and separation $d = 5.56$ mm. The left half of the gap is filled with material of dielectric constant $\kappa_1 = 7.00$; the right half is filled with material of dielectric constant $\kappa_2 = 12.0$. What is the capacitance?

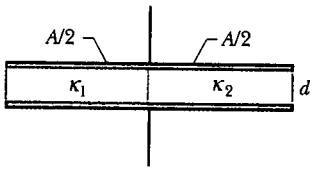


FIG. 26-50 Problem 36.

37. Figure 26-51 shows a parallel-plate capacitor with a plate area $A = 7.89$ cm^2 and plate separation $d = 4.62$ mm. The top half of the gap is filled with material of dielectric constant $\kappa_1 = 11.0$; the bottom half is filled with material of dielectric constant $\kappa_2 = 12.0$. What is the capacitance?

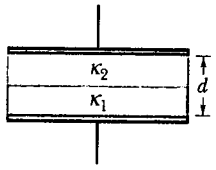


FIG. 26-51 Problem 37.

38. Figure 26-52 shows a parallel-plate capacitor of plate area $A = 10.5$ cm^2 and plate separation $2d = 7.12$ mm. The left half of the gap is filled with material of dielectric constant $\kappa_1 = 21.0$; the top of the right half is filled with material of dielectric constant $\kappa_2 = 42.0$; the bottom of the right half is filled with material of dielectric constant $\kappa_3 = 58.0$. What is the capacitance?

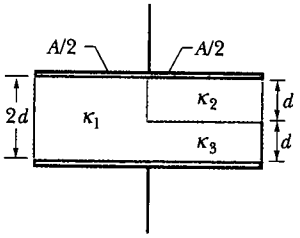


FIG. 26-52 Problem 38.

sec. 26-8 Dielectrics and Gauss' Law

39. A parallel-plate capacitor has a capacitance of 100 pF, a plate area of 100 cm^2, and a mica dielectric ($\kappa = 5.4$) completely filling the space between the plates. At 50 V potential difference, calculate (a) the electric field magnitude E in the mica, (b) the magnitude of the free charge on the plates, and (c) the magnitude of the induced surface charge on the mica.

40. In Sample Problem 26-8, suppose that the battery remains connected while the dielectric slab is being introduced. Calculate (a) the capacitance, (b) the charge on the capacitor plates, (c) the electric field in the gap, and (d) the electric field in the slab, after the slab is in place.

41. The space between two concentric conducting spherical shells of radii $b = 1.70$ cm and $a = 1.20$ cm is filled with a substance of dielectric constant $\kappa = 23.5$. A potential difference $V = 73.0$ V is applied across the inner and outer shells. Determine (a) the capacitance of the device, (b) the free charge q on the inner shell, and (c) the charge q' induced along the surface of the inner shell.

42. Two parallel plates of area 100 cm^2 are given charges of equal magnitudes 8.9×10^{-7} C but opposite signs. The electric field within the dielectric material filling the space between the plates is 1.4×10^6 V/m. (a) Calculate the dielectric constant of the material. (b) Determine the magnitude of the charge induced on each dielectric surface.

43. A parallel-plate capacitor has plates of area 0.12 m^2 and a separation of 1.2 cm. A battery charges the plates to a potential difference of 120 V and is then disconnected. A dielectric slab of thickness 4.0 mm and dielectric constant 4.8 is then placed symmetrically between the plates. (a) What is the capacitance before the slab is inserted? (b) What is the capacitance with the slab in place? What is the free charge q (c) before and (d) after the slab is inserted? What is the magnitude of the electric field (e) in the space between the plates and dielectric and (f) in the dielectric itself? (g) With the slab in place, what is the potential difference across the plates? (h) How much external work is involved in inserting the slab?

ADDITIONAL PROBLEMS

44. Figure 26-53 shows capacitor 1 ($C_1 = 8.00$ μF), capacitor 2 ($C_2 = 6.00$ μF), and capacitor 3 ($C_3 = 8.00$ μF) connected to a 12.0 V battery. When switch S is closed so as to connect uncharged capacitor 4 ($C_4 = 6.00$ μF), (a) how much charge passes through point P from the battery and (b) how much charge shows up on capacitor 4? (c) Explain the discrepancy in those two results.

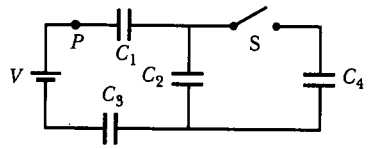


FIG. 26-53 Problem 44.

45. Figure 26-54 represents two air-filled cylindrical capacitors connected in series across a battery with potential $V = 10$ V. Capacitor 1 has an inner plate radius of 5.0 mm, an outer plate radius of 1.5 cm, and a length of 5.0 cm. Capacitor 2 has an inner plate radius of 2.5 mm, an outer plate radius of 1.0 cm, and a length of 9.0 cm. The outer plate of capacitor 2 is a conducting organic membrane that can be stretched, and the capacitor can be inflated to increase the plate separation. If the outer plate radius is increased to 2.5 cm by inflation, (a) how many electrons move through point P and (b) do they move toward or away from the battery?

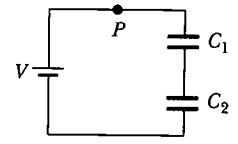


FIG. 26-54 Problem 45.

46. Two air-filled, parallel-plate capacitors are to be connected to a 10 V battery, first individually, then in series, and then in parallel. In those arrangements, the energy stored in the capacitors turns out to be, listed least to greatest: 75 μJ, 100 μJ, 300 μJ, and 400 μJ. Of the two capacitors, what is the (a) smaller and (b) greater capacitance?

47. The capacitances of the four capacitors shown in Fig. 26-55 are given in terms of a certain quantity C. (a) If $C = 50\ \mu$F, what is the equivalent capacitance between points A and B? (*Hint:* First imagine that a battery is connected between those two points; then reduce the circuit to an equivalent capacitance.) (b) Repeat for points A and D.

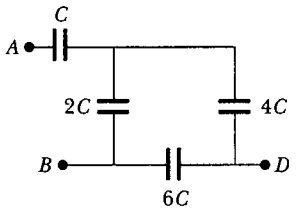


FIG. 26-55 Problem 47.

48. A capacitor of unknown capacitance C is charged to 100 V and connected across an initially uncharged 60 μF capacitor. If the final potential difference across the 60 μF capacitor is 40 V, what is C?

49. A cylindrical capacitor has radii a and b as in Fig. 26-6. Show that half the stored electric potential energy lies within a cylinder whose radius is $r = \sqrt{ab}$.

50. A certain capacitor is charged to a potential difference V. If you wish to increase its stored energy by 10%, by what percentage should you increase V?

51. You have two plates of copper, a sheet of mica (thickness = 0.10 mm, $\kappa = 5.4$), a sheet of glass (thickness = 2.0 mm, $\kappa = 7.0$), and a slab of paraffin (thickness = 1.0 cm, $\kappa = 2.0$). To make a parallel-plate capacitor with the largest C, which sheet should you place between the copper plates?

52. In Fig. 26-56, the parallel-plate capacitor of plate area 2.00×10^{-2} m^2 is filled with two dielectric slabs, each with thickness 2.00 mm. One slab has dielectric constant 3.00, and the other, 4.00. How much charge does the 7.00 V battery store on the capacitor?

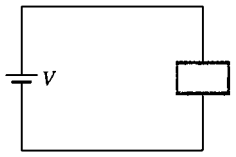


FIG. 26-56 Problem 52.

53. A slab of copper of thickness $b = 2.00$ mm is thrust into a parallel-plate capacitor of plate area $A = 2.40$ cm^2 and plate separation $d = 5.00$ mm, as shown in Fig. 26-57; the slab is exactly halfway between the plates. (a) What is the capacitance after the slab is introduced? (b) If a charge $q = 3.40\ \mu$C is maintained on the plates, what is the ratio of the stored energy before to that after the slab is inserted? (c) How much work is done on the slab as it is inserted? (d) Is the slab sucked in or must it be pushed in?

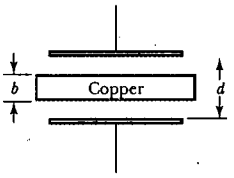


FIG. 26-57 Problems 53 and 54.

54. Repeat Problem 53, assuming that a potential difference $V = 85.0$ V, rather than the charge, is held constant.

55. You have many 2.0 μF capacitors, each capable of withstanding 200 V without undergoing electrical breakdown (in which they conduct charge instead of storing it). How would you assemble a combination having an equivalent capacitance of (a) 0.40 μF and (b) 1.2 μF, each combination capable of withstanding 1000 V?

56. A 10 V battery is connected to a series of n capacitors, each of capacitance 2.0 μF. If the total energy stored in the capacitors is 25 μJ, what is n?

57. A capacitor is made of two plane parallel plates of width "a" and length 'l' separated by a distance d ($d \ll a, l$) as shown in Fig. 26-58. The capacitor has a dielectric slab of relative permittivity κ fully occupying the space between the plates. The capacitor plates are charged to a potential difference V_0 using battery. The battery is then removed. Find the force on the dielectric when the dielectric slab is partially pulled out of the plates such that length x remains between the plates.

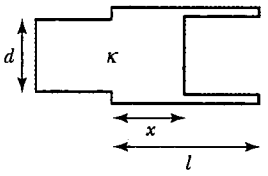


FIG. 26-58 Problem 57.

ANSWERS

Checkpoints

1. (a) same; (b) same

2. (a) decreases; (b) increases; (c) decreases

3. (a) $V, q/2$; (b) $V/2, q$

4. Inner conductor as electric filed strength will reach its breakdown value there first

Reasoning Type Questions

1. (a)

2. (a)

3. (d)

4. (d)

5. (d)

Multiple Choice Questions

1. (d)

2. (b)

3. (b)

4. (d)

5. (c)

6. (b)

7. (b), (d)
8. (a)
9. (b)
10. (d), (e)
11. (b), (c), (d)
12. (a)
13. (b), (c)
14. (b)
15. (a), (b), (c), (d)
16. (d), (e)
17. (i) (a), (b), (d); (ii) (a); (iii) (a)
18. (a)
19. (a), (c), (d)

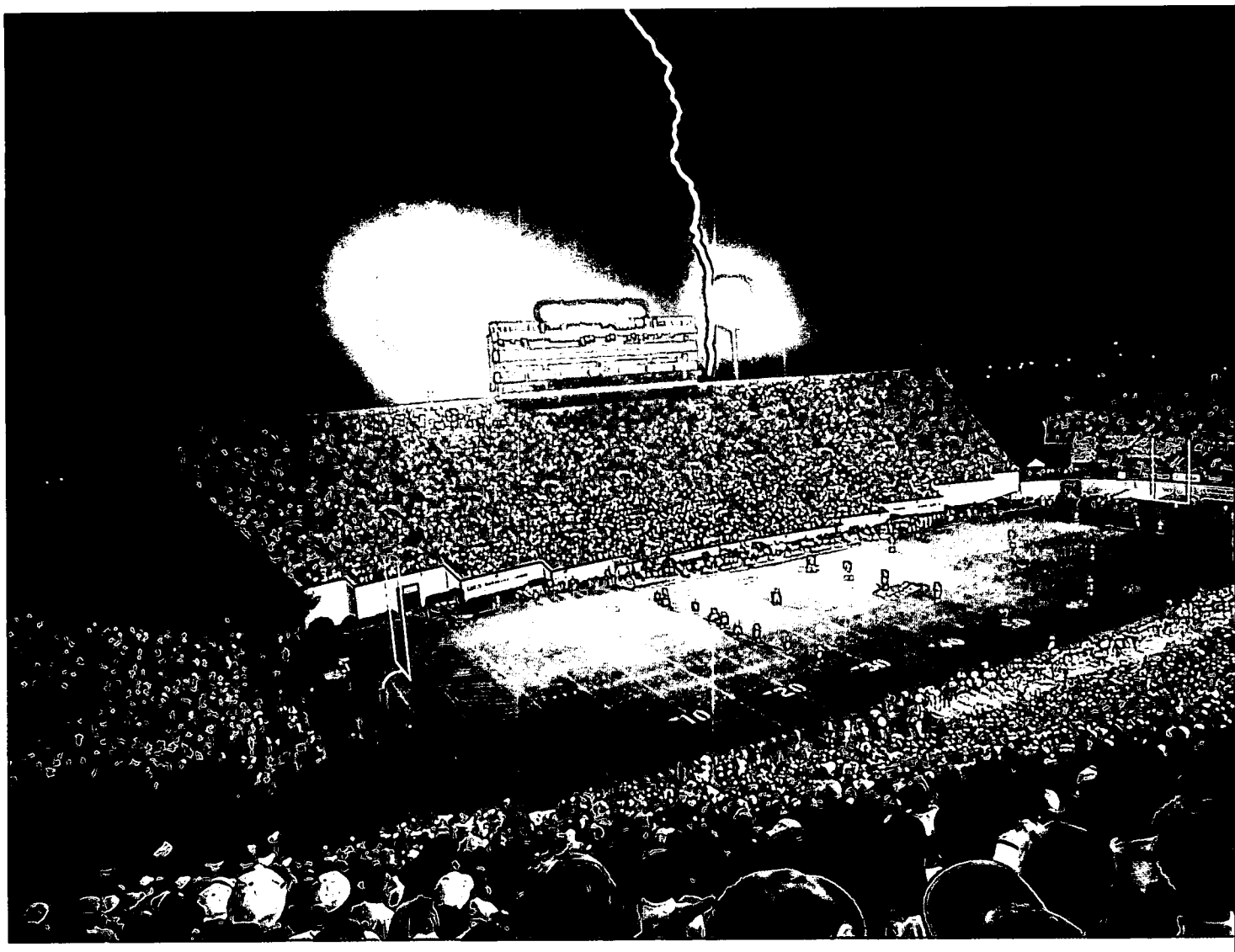
Problems

1. (a) 3.5 pF; (b) 3.5 pF; (c) 57 V
2. 3.0×10^{-3} C
3. (a) 8.85×10^{-12} m; (b) since d is much less than the size of an atom ($\sim 10^{-10}$ m), this capacitor cannot be constructed.
4. (a) 84 pF; (b) 191 cm^2
5. 0.280 pF
6. 7.33 μF
7. 3.16 μF
8. (a) 1.00×10^{-4} C; (b) 2.00×10^{-5} C
9. (a) 789 μC; (b) 78.9 V
10. (a) 60 μC; (b) 60 μC
11. 43 pF
12. (a) 2.0 μF; (b) 0.80 μF
13. (a) 3.00 μF; (b) 60.0 μC; (c) 10.0 V; (d) 30.0 μC; (e) 10.0 V; (f) 20.0 μC; (g) 5.00 V; (h) 20.0 μC
14. 12 μC
15. (a) 50 V; (b) 5.0×10^{-5} C; (c) 1.5×10^{-4} C
16. 20 μC
17. 3.6 pC
18. (a) 4.0 μF; (b) 2.0 μF
19. 2.28×10^{-12} F
20. (a) 32.0 μC; (b) 16.0 μC; (c) 16.0 μC
21. (a) 9.00 μC; (b) 16.0 μC; (c) 9.00 μC; (d) 16.0 μC; (e) 8.40 μC; (f) 16.8 μC; (g) 10.8 μC; (h) 14.4 μC
22. (a) 10 V; (b) 8.0 μF; (c) 2.0 μF
23. 0.27 J
24. (a) 35 pF; (b) 21 nC; (c) 6.3 μJ; (d) 6.0×10^5 V/m; (e) 1.6 J/m^3
25. 9.96×10^{-8} J
26. (a) 9.16×10^{-18} J/m^3; (b) 9.16×10^{-6} J/m^3; (c) 9.16×10^{6} J/m^3; (d) 9.16×10^{18} J/m^3; (e) ∞
27. (a) 16.0 V; (b) 45.1 pJ; (c) 120 pJ; (d) 75.2 pJ
28. (a) 190 V; (b) 95 mJ
29. 0.11 J/m^3
30. (a) 6.2×10^{-2} m; (b) 280 pF
31. 4.0
32. 81 pF/m
33. 1.06×10^{-9} C
34. 0.63 m^2
35. (a) 0.73 nF; (b) 28 kV
36. 8.41×10^{-12} F
37. 17.3 pF
38. 4.55×10^{-11} F
39. (a) 10 kV/m; (b) 5.0 nC; (c) 4.1 nC
40. (a) 13.4 pF; (b) 1.15 nC; (c) 1.13×10^4 N/C; (d) 4.33×10^3 N/C
41. (a) 0.107 nF; (b) 7.79 nC; (c) 7.45 nC
42. (a) 7.2; (b) 7.7×10^{-7} C
43. (a) 89 pF; (b) 0.12 nF; (c) 11 nC; (d) 11 nC; (e) 10 kV/m; (f) 2.1 kV/m; (g) 88 V; (h) -0.17 μJ

Additional Problems

44. (a) 7.20 μC; (b) 18.0 μC; (c) Battery supplies charges only to plates to which it is connected; charges on other plates are due to electron transfers between plates, in accord with new distribution of voltages across the capacitors. So battery does not directly supply charge on capacitor 4.
45. (a) 2.0×10^7; (b) In this problem some electrons have "returned" to the positive plates (making them less positive).
46. (a) 2.0 μF; (b) 6.0 μF
47. (a) 41 μF; (b) 42 μF
48. 40 μF
50. 4.9%
51. Mica
52. 1.06 nC
53. (a) 0.708 pF; (b) 1.67; (c) -5.44 J; (d) since $W < 0$ the slab is sucked in
54. (a) 0.708 pF; (b) 0.600; (c) 1.02×10^{-9} J; (d) sucked in
55. (a) Put five such capacitors in series. (b) As one possibility, you can take three identical arrays of capacitors, each array being a five-capacitor combination described in part (a) above, and hook up the arrays in parallel.
56. 4
57. $F = \dfrac{\kappa^2 \varepsilon_0^2 l^2 a^2 (\kappa - 1)}{2d^2 [l + x(\kappa - 1)]^2}$

Ohm's Law 27

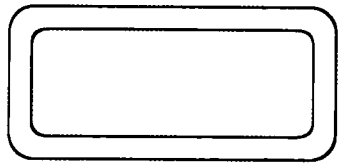


Courtesy of The Roanoke Times/©Eric Brady

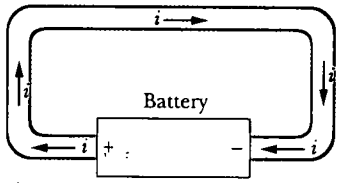
Lightning strikes the ground less than 1 km from a Virginia Tech football game. The chance of lightning striking a person directly is slim. The much greater danger lies in the ground current—the current that spreads out from the strike point. Everyone on the field or in the stands could have been knocked down, paralyzed, or killed by the ground current. If you are caught in the open during a lightning storm like this, there is a simple procedure for reducing your risk from ground current.

How can you reduce your risk from ground current?

The answer is in this chapter.



(*a*)



(*b*)

FIG. 27-1 (*a*) A loop of copper in electrostatic equilibrium. The entire loop is at a single potential, and the electric field is zero at all points inside the copper. (*b*) Adding a battery imposes an electric potential difference between the ends of the loop that are connected to the terminals of the battery. The battery thus produces an electric field within the loop, from terminal to terminal, and the field causes charges to move around the loop. This movement of charges is a current *i*.

27-1 | WHAT IS PHYSICS?

In the last five chapters we discussed electrostatics—the physics of stationary charges. In this and the next chapter, we discuss the physics of **electric currents**—that is, charges in motion.

Examples of electric currents abound and involve many professions. Meteorologists are concerned with lightning and with the less dramatic slow flow of charge through the atmosphere. Biologists, physiologists, and engineers working in medical technology are concerned with the nerve currents that control muscles and especially with how those currents can be reestablished after spinal cord injuries. Electrical engineers are concerned with countless electrical systems, such as power systems, lightning protection systems, information storage systems, and music systems. Space engineers monitor and study the flow of charged particles from our Sun because that flow can wipe out telecommunication systems in orbit and even power transmission systems on the ground.

In this chapter we discuss the basic physics of electric currents and why they can be established in some materials but not in others. We begin with the meaning of electric current.

27-2 | Electric Current

Although an electric current is a stream of moving charges, not all moving charges constitute an electric current. If there is to be an electric current through a given surface, there must be a net flow of charge through that surface. The free electrons (conduction electrons) in an isolated length of copper wire are in random motion at speeds of the order of 10^6 m/s. If you pass a hypothetical plane through such a wire, conduction electrons pass through it *in both directions* at the rate of many billions per second—but there is *no net transport* of charge and thus *no current* through the wire. However, if you connect the ends of the wire to a battery, you slightly bias the flow in one direction, with the result that there now is a net transport of charge and thus an electric current through the wire.

In this chapter we restrict ourselves largely to the study—within the framework of classical physics—of *steady* currents of *conduction electrons* moving through *metallic conductors* such as copper wires.

As Fig. 27-1*a* reminds us, any isolated conducting loop—regardless of whether it has an excess charge—is all at the same potential. No electric field can exist within it or along its surface. Although conduction electrons are available, no net electric force acts on them and thus there is no current.

If, as in Fig. 27-1*b*, we insert a battery in the loop, the conducting loop is no longer at a single potential. Electric fields act inside the material making up the loop, exerting forces on the conduction electrons, causing them to move and thus establishing a current. After a very short time, the electron flow reaches a constant value and the current is in its *steady state.* Steady current means that the amount of charge entering an end of the wire at any time is equal to the amount of charge leaving the wire at the other end. In other words, no charge is accumulating at any point.

The current may change with time, but its value is the same at every cross section. If the current out of any cross section is more than the incoming current, then some negative charge will build up at the cross section. First, this would mean that there is source or sink of charge which does not exist in nature. Even if there are such points where we hypothetically assume that the charges are stored, then this negative charge built up at the cross section will decrease the outgoing current (by decelerating the positive charge flow) and increase the incoming current (by accelerating the positive charge inflow). The net effect is that the current will be equal in a very short time. This mechanism of nature to

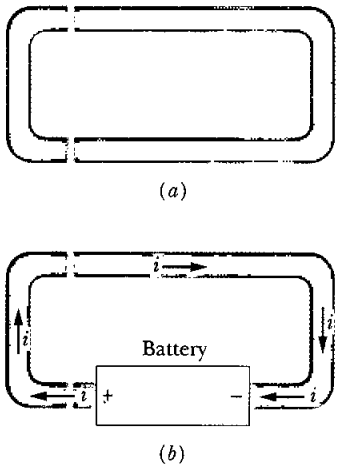


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keep the current constant is so refined that even if the current is changing the direction with time (as in AC), still the current at all the cross sections at any given time will have the same value.

Figure 27-2 shows a section of a conductor, part of a conducting loop in which current has been established. If charge dq passes through a hypothetical plane (such as aa') in time dt, then the current i through that plane is defined as

$$i = \frac{dq}{dt} \qquad \text{(definition of current).} \tag{27-1}$$

We can find the charge that passes through the plane in a time interval extending from 0 to t by integration:

$$q = \int dq = \int_0^t i\,dt, \tag{27-2}$$

in which the current i may vary with time.

Under steady-state conditions, the current is the same for planes aa', bb', and cc' and indeed for all planes that pass completely through the conductor, no matter what their location or orientation. This follows from the fact that charge is conserved. Under the steady-state conditions assumed here, an electron must pass through plane aa' for every electron that passes through plane cc'.

The SI unit for current is the coulomb per second, or the ampere (A), which is an SI base unit:

$$1 \text{ ampere} = 1 \text{ A} = 1 \text{ coulomb per second} = 1 \text{ C/s}.$$

The formal definition of the ampere is discussed in Chapter 30.

Current, as defined by Eq. 27-1, is a scalar because both charge and time in that equation are scalars. Yet, as in Fig. 27-1*b*, we often represent a current with an arrow to indicate that charge is moving. Such arrows are not vectors, however, and they do not require vector addition. Figure 27-3*a* shows a conductor with current i_0 splitting at a junction into two branches. Because charge is conserved, the magnitudes of the currents in the branches must add to yield the magnitude of the current in the original conductor, so that

$$i_0 = i_1 + i_2. \tag{27-3}$$

As Fig. 27-3*b* suggests, bending or reorienting the wires in space does not change the validity of Eq. 27-3. Current arrows show only a direction (or sense) of flow along a conductor, not a direction in space.

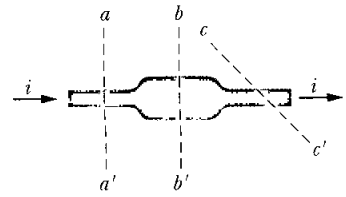


FIG. 27-2 The current i through the conductor has the same value at planes aa', bb', and cc'.

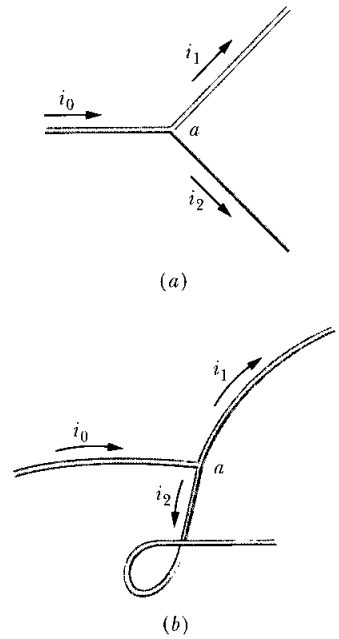


FIG. 27-3 The relation $i_0 = i_1 + i_2$ is true at junction a no matter what the orientation in space of the three wires. Currents are scalars, not vectors.

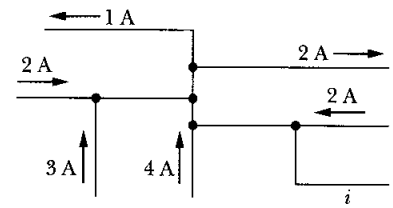
The Directions of Currents

In Fig. 27-1*b* we drew the current arrows in the direction in which positively charged particles would be forced to move through the loop by the electric field. Such positive *charge carriers*, as they are often called, would move away from the positive battery terminal and toward the negative terminal. Actually, the charge carriers in the copper loop of Fig. 27-1*b* are electrons and thus are negatively charged. The electric field forces them to move in the direction opposite the current arrows, from the negative terminal to the positive terminal. For historical reasons, however, we use the following convention:

☞ A current arrow is drawn in the direction in which positive charge carriers would move, even if the actual charge carriers are negative and move in the opposite direction.

We can use this convention because in *most* situations, the assumed motion of positive charge carriers in one direction has the same effect as the actual motion of negative charge carriers in the opposite direction. (When the effect is not the same, we shall drop the convention and describe the actual motion.)

CHECKPOINT 1 The figure here shows a portion of a circuit. What are the magnitude and direction of the current i in the lower right-hand wire?



Sample Problem 27-1

Water flows through a garden hose at a volume flow rate dV/dt of 450 cm^3/s. What is the current of negative charge?

Conceptualize/Classify: The current i of negative charge is due to the electrons in the water molecules moving through the hose. The current is the rate at which that negative charge passes through any plane that cuts completely across the hose.

Compute: We can write the current in terms of the number of molecules that pass through such a plane per second as

$$i = \begin{pmatrix}\text{charge}\\ \text{per}\\ \text{electron}\end{pmatrix}\begin{pmatrix}\text{electrons}\\ \text{per}\\ \text{molecule}\end{pmatrix}\begin{pmatrix}\text{molecules}\\ \text{per}\\ \text{second}\end{pmatrix}$$

or
$$i = (e)(10)\frac{dN}{dt}.$$

We substitute 10 electrons per molecule because a water (H_2O) molecule contains 8 electrons in the single oxygen atom and 1 electron in each of the two hydrogen atoms.

We can express the rate dN/dt in terms of the given volume flow rate dV/dt by first writing

$$\begin{pmatrix}\text{molecules}\\ \text{per}\\ \text{second}\end{pmatrix} = \begin{pmatrix}\text{molecules}\\ \text{per}\\ \text{mole}\end{pmatrix}\begin{pmatrix}\text{moles}\\ \text{per unit}\\ \text{mass}\end{pmatrix} \times \begin{pmatrix}\text{mass}\\ \text{per unit}\\ \text{volume}\end{pmatrix}\begin{pmatrix}\text{volume}\\ \text{per}\\ \text{second}\end{pmatrix}.$$

"Molecules per mole" is Avogadro's number N_A. "Moles per unit mass" is the inverse of the mass per mole, which is the molar mass M of water. "Mass per unit volume" is the (mass) density ρ_{mass} of water. The volume per second is the volume flow rate dV/dt. Thus, we have

$$\frac{dN}{dt} = N_A\left(\frac{1}{M}\right)\rho_{mass}\left(\frac{dV}{dt}\right) = \frac{N_A\rho_{mass}}{M}\frac{dV}{dt}.$$

Substituting this into the equation for i, we find

$$i = 10eN_AM^{-1}\rho_{mass}\frac{dV}{dt}.$$

N_A is 6.02×10^{23} molecules/mol, or $6.02\times10^{23}\ mol^{-1}$, and ρ_{mass} is 1000 kg/m^3. We can get the molar mass of water from the molar masses listed in Appendix F: We add the molar mass of oxygen (16 g/mol) to twice the molar mass of hydrogen (1 g/mol), obtaining 18 g/mol = 0.018 kg/mol. Then

$$\begin{aligned} i &= (10)(1.6\times10^{-19}\text{ C})(6.02\times10^{23}\ mol^{-1})\\ &\quad\times(0.018\text{ kg/mol})^{-1}(1000\text{ kg/m}^3)(450\times10^{-6}\text{ m}^3/\text{s})\\ &= 2.41\times10^7\text{ C/s} = 2.41\times10^7\text{ A}\\ &= 24.1\text{ MA}. \qquad \text{(Answer)}\end{aligned}$$

Conclude: Although this seems to be a large value of current compared to what we are used to, this current of negative charge is exactly compensated by a current of positive charge associated with the nuclei of the three atoms that make up the water molecule. Thus, there is no net flow of charge through the hose.

27-3 | Current Density

Sometimes we are interested in the current i in a particular conductor. At other times we take a localized view and study the flow of charge through a cross section of the conductor at a particular point. To describe this flow, we can use the **current density** $\vec{J}$, which has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative. For each element of the cross section, the magnitude J is equal to the current per unit area through that element. We can write the amount of current through the element as $\vec{J}\cdot d\vec{A}$,

where $d\vec{A}$ is the area vector of the element, perpendicular to the element. The total current through the surface is then

$$i = \int \vec{J} \cdot d\vec{A}. \tag{27-4}$$

If the current is uniform across the surface and parallel to $d\vec{A}$ then $\vec{J}$ is also uniform and parallel to $d\vec{A}$. Then Eq. 27-4 becomes

$$i = \int J\, dA = J \int dA = JA,$$

so

$$J = \frac{i}{A}, \tag{27-5}$$

where A is the total area of the surface. From Eq. 27-4 or 27-5 we see that the SI unit for current density is the ampere per square meter (A/m^2).

In Chapter 22 we saw that we can represent an electric field with electric field lines. Figure 27-4 shows how current density can be represented with a similar set of lines, which we can call *streamlines*. The current, which is toward the right in Fig. 27-4, makes a transition from the wider conductor at the left to the narrower conductor at the right. Because charge is conserved during the transition, the amount of charge and thus the amount of current cannot change. However, the current density does change—it is greater in the narrower conductor. The spacing of the streamlines suggests this increase in current density; streamlines that are closer together imply greater current density.

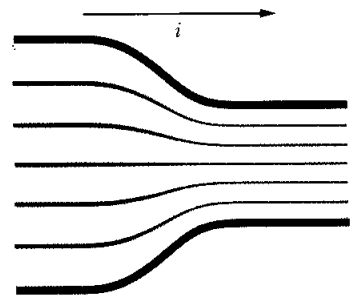


FIG. 27-4 Streamlines representing current density in the flow of charge through a constricted conductor.

Drift Speed

When a conductor does not have a current through it, its conduction electrons move randomly, with no net motion in any direction. When the conductor does have a current through it, these electrons actually still move randomly, but now they tend to *drift* with a **drift speed** v_d in the direction opposite that of the applied electric field that causes the current. The drift speed is tiny compared with the speeds in the random motion. For example, in the copper conductors of household wiring, electron drift speeds are perhaps 10^{-5} or 10^{-4} m/s, whereas the random-motion speeds are around 10^6 m/s.

We can use Fig. 27-5 to relate the drift speed v_d of the conduction electrons in a current through a wire to the magnitude J of the current density in the wire. For convenience, Fig. 27-5 shows the equivalent drift of *positive* charge carriers in the direction of the applied electric field $\vec{E}$. Let us assume that these charge carriers all move with the same drift speed v_d and that the current density J is uniform across the wire's cross-sectional area A. The number of charge carriers in a length L of the wire is nAL, where n is the number of carriers per unit volume. The total charge of the carriers in the length L, each with charge e, is then

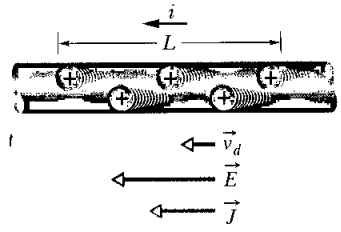


FIG. 27-5 Positive charge carriers drift at speed v_d in the direction of the applied electric field $\vec{E}$. By convention, the direction of the current density $\vec{J}$ and the sense of the current arrow are drawn in that same direction.

$$q = (nAL)e.$$

Because the carriers all move along the wire with speed v_d, this total charge moves through any cross section of the wire in the time interval

$$t = \frac{L}{v_d}.$$

Equation 27-1 tells us that the current i is the time rate of transfer of charge across a cross section, so here we have

$$i = \frac{q}{t} = \frac{nALe}{L/v_d} = nAev_d. \tag{27-6}$$

Solving for v_d and recalling Eq. 27-5 ($J = i/A$), we obtain

$$v_d = \frac{i}{nAe} = \frac{J}{ne},$$

or, extended to vector form,

$$\vec{J} = (ne)\vec{v}_d. \tag{27-7}$$

Here the product ne, whose SI unit is the coulomb per cubic meter (C/m^3), is the *carrier charge density*. For positive carriers, ne is positive and Eq. 27-7 predicts that $\vec{J}$ and $\vec{v}_d$ have the same direction. For negative carriers, ne is negative and $\vec{J}$ and $\vec{v}_d$ have opposite directions.

CHECKPOINT 2 The figure shows conduction electrons moving leftward in a wire. Are the following leftward or rightward: (a) the current i, (b) the current density $\vec{J}$, (c) the electric field $\vec{E}$ in the wire?



Sample Problem 27-2

(a) The current density in a cylindrical wire of radius $R = 2.0$ mm is uniform across a cross section of the wire and is $J = 2.0 \times 10^5\ A/m^2$. What is the current through the outer portion of the wire between radial distances $R/2$ and R (Fig. 27-6*a*)?

Conceptualize/Classify: Because the current density is uniform across the cross section, the current density J, the current i, and the cross-sectional area A are related by Eq. 27-5 ($J = i/A$).

Compute: We want only the current through a reduced cross-sectional area A' of the wire (rather than the entire area), where

$$A' = \pi R^2 - \pi\left(\frac{R}{2}\right)^2 = \pi\left(\frac{3R^2}{4}\right)$$
$$= \frac{3\pi}{4}(0.0020\,m)^2 = 9.424 \times 10^{-6}\,m^2.$$

So, we rewrite Eq. 27-5 as

$$i = JA'$$

and then substitute the data to find

$$i = (2.0 \times 10^5\ A/m^2)(9.424 \times 10^{-6}\,m^2)$$
$$= 1.9\ A. \qquad \text{(Answer)}$$

(b) Suppose, instead, that the current density through a cross section varies with radial distance r as $J = ar^2$, in which $a = 3.0 \times 10^{11}\ A/m^4$ and r is in meters. What now is the current through the same outer portion of the wire?

Conceptualize/Classify: Because the current density is not uniform across a cross section of the wire, we must resort to Eq. 27-4 ($i = \int \vec{J} \cdot d\vec{A}$) and integrate the current density over the portion of the wire from $r = R/2$ to $r = R$.

Compute: The current density vector $\vec{J}$ (along the wire's length) and the differential area vector $d\vec{A}$ (perpendicular to a cross section of the wire) have the same direction. Thus,

$$\vec{J} \cdot d\vec{A} = J\,dA \cos 0 = J\,dA.$$

We need to replace the differential area dA with something we can actually integrate between the limits $r = R/2$ and $r = R$. The simplest replacement (because J is given as a function of r) is the area $2\pi r\,dr$ of a thin ring of circumference $2\pi r$ and width dr (Fig. 27-6*b*). We can then integrate with r as the variable of integration. Equation 27-4 then gives us

$$i = \int \vec{J} \cdot d\vec{A} = \int J\,dA$$
$$= \int_{R/2}^{R} ar^2 2\pi r\,dr = 2\pi a \int_{R/2}^{R} r^3 dr$$
$$= 2\pi a\left[\frac{r^4}{4}\right]_{R/2}^{R} = \frac{\pi a}{2}\left[R^4 - \frac{R^4}{16}\right] = \frac{15}{32}\pi a R^4$$
$$= \frac{15}{32}\pi(3.0 \times 10^{11}\ A/m^4)(0.0020\,m)^4$$
$$= 7.1\ A. \qquad \text{(Answer)}$$

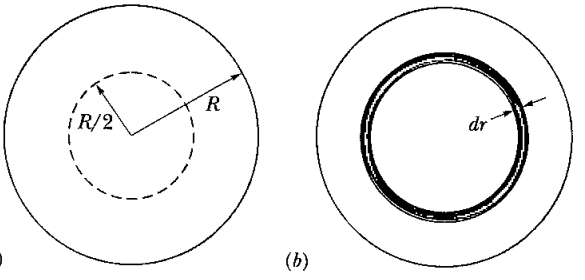


FIG. 27-6 (*a*) Cross section of a wire of radius R. (*b*) A thin ring has width dr and circumference $2\pi r$, and thus a differential area $dA = 2\pi r\,dr$.

Sample Problem 27-3

What is the drift speed of the conduction electrons in a copper wire with radius $r = 900\ \mu m$ when it has a uniform current $i = 17$ mA? Assume that each copper atom contributes one conduction electron to the current and that the current density is uniform across the wire's cross section.

Conceptualize/Classify:

1. The drift speed v_d is related to the current density $\vec{J}$ and the number n of conduction electrons per unit volume according to Eq. 27-7, which we can write as $J = nev_d$.
2. Because the current density is uniform, its magnitude J is related to the given current i and wire size by Eq. 27-5 ($J = i/A$, where A is the cross-sectional area of the wire).
3. Because we assume one conduction electron per atom, the number n of conduction electrons per unit volume is the same as the number of atoms per unit volume.

Compute: Let us start with the third idea by writing

$$n = \begin{pmatrix}\text{atoms}\\ \text{per unit}\\ \text{volume}\end{pmatrix} = \begin{pmatrix}\text{atoms}\\ \text{per}\\ \text{mole}\end{pmatrix}\begin{pmatrix}\text{moles}\\ \text{per unit}\\ \text{mass}\end{pmatrix}\begin{pmatrix}\text{mass}\\ \text{per unit}\\ \text{volume}\end{pmatrix}.$$

The number of atoms per mole is just Avogadro's number N_A ($= 6.02 \times 10^{23}\ \text{mol}^{-1}$). Moles per unit mass is the inverse of the mass per mole, which here is the molar mass M of copper. The mass per unit volume is the (mass) density ρ_{mass} of copper. Thus,

$$n = N_A\left(\frac{1}{M}\right)\rho_{mass} = \frac{N_A\,\rho_{mass}}{M}.$$

Taking copper's molar mass M and density ρ_{mass} from Appendix F, we then have (with some conversions of units)

$$n = \frac{(6.02\times10^{23}\ \text{mol}^{-1})(8.96\times10^{3}\ \text{kg/m}^3)}{63.54\times10^{-3}\ \text{kg/mol}}$$
$$= 8.49\times10^{28}\ \text{electrons/m}^3$$

or
$$n = 8.49\times10^{28}\ \text{m}^{-3}.$$

Next let us combine the first two key ideas by writing

$$\frac{i}{A} = nev_d.$$

Substituting for A with πr^2 ($= 2.54\times10^{-6}\ \text{m}^2$) and solving for v_d, we then find

$$v_d = \frac{i}{ne(\pi r^2)}$$
$$= \frac{17\times10^{-3}\ \text{A}}{(8.49\times10^{28}\ \text{m}^{-3})(1.6\times10^{-19}\ \text{C})(2.54\times10^{-6}\ \text{m}^2)}$$
$$= 4.9\times10^{-7}\ \text{m/s}, \qquad \text{(Answer)}$$

which is only 1.8 mm/h, slower than a sluggish snail.

Conclude: You may well ask: "If the electrons drift so slowly, why do the room lights turn on so quickly when I throw the switch?" Confusion on this point results from not distinguishing between the drift speed of the electrons and the speed at which *changes* in the electric field configuration travel along wires. This latter speed is nearly that of light; electrons everywhere in the wire begin drifting almost at once, including into the lightbulbs.

27-4 | Resistance and Resistivity

If we apply the same potential difference between the ends of geometrically similar rods of copper and of glass, very different currents result. The characteristic of the conductor that enters here is its electrical **resistance**. We determine the resistance between any two points of a conductor by applying a potential difference V between those points and measuring the current i that results. The resistance R is then

$$R = \frac{V}{i} \qquad \text{(definition of } R\text{)}. \qquad (27\text{-}8)$$

The SI unit for resistance that follows from Eq. 27-8 is the volt per ampere. This combination occurs so often that we give it a special name, the **ohm** (symbol Ω); that is,

$$1\ \text{ohm} = 1\ \Omega = 1\ \text{volt per amp}$$
$$= 1\ \text{V/A}. \qquad (27\text{-}9)$$

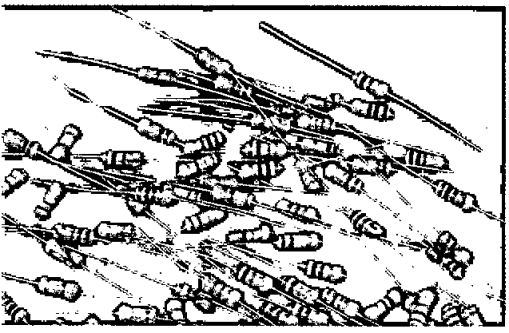


FIG. 27-7 An assortment of resistors. The circular bands are color-coding marks that identify the value of the resistance. *(The Image Works)*

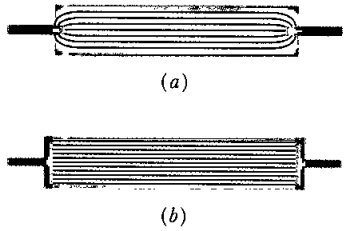


FIG. 27-8 Two ways of applying a potential difference to a conducting rod. The gray connectors are assumed to have negligible resistance. When they are arranged as in (*a*) in a small region at each rod end, the measured resistance is larger than when they are arranged as in (*b*) to cover the entire rod end.

A conductor whose function in a circuit is to provide a specified resistance is called a **resistor** (see Fig. 27-7). In a circuit diagram, we represent a resistor and a resistance with the symbol $\neg\!\!\wedge\!\!\vee\!\!\wedge\!\!\vee\!\!\wedge\!\!-$. If we write Eq. 27-8 as

$$i = \frac{V}{R},$$

we see that, for a given V, the greater the resistance, the smaller the current.

The resistance of a conductor depends on the manner in which the potential difference is applied to it. Figure 27-8, for example, shows a given potential difference applied in two different ways to the same conductor. As the current density streamlines suggest, the currents in the two cases—hence the measured resistances—will be different. Unless otherwise stated, we shall assume that any given potential difference is applied as in Fig. 27-8*b*.

As we have done several times in other connections, we often wish to take a general view and deal not with particular objects but with materials. Here we do so by focusing not on the potential difference V across a particular resistor but on the electric field $\vec{E}$ at a point in a resistive material. Instead of dealing with the current i through the resistor, we deal with the current density $\vec{J}$ at the point in question. Instead of the resistance R of an object, we deal with the **resistivity** ρ of the *material:*

$$\rho = \frac{E}{J} \qquad \text{(definition of } \rho\text{)}. \tag{27-10}$$

(Compare this equation with Eq. 27-8.)

If we combine the SI units of E and J according to Eq. 27-10, we get, for the unit of ρ, the ohm-meter ($\Omega \cdot \text{m}$):

$$\frac{\text{unit}(E)}{\text{unit}(J)} = \frac{\text{V/m}}{\text{A/m}^2} = \frac{\text{V}}{\text{A}}\text{m} = \Omega \cdot \text{m}.$$

(Do not confuse the *ohm-meter,* the unit of resistivity, with the *ohmmeter,* which is an instrument that measures resistance.) Table 27-1 lists the resistivities of some materials.

We can write Eq. 27-10 in vector form as

$$\vec{E} = \rho \vec{J}. \tag{27-11}$$

Equations 27-10 and 27-11 hold only for *isotropic* materials—materials whose electrical properties are the same in all directions.

We often speak of the **conductivity** σ of a material. This is simply the reciprocal of its resistivity, so

$$\sigma = \frac{1}{\rho} \qquad \text{(definition of } \sigma\text{)}. \tag{27-12}$$

The SI unit of conductivity is the reciprocal ohm-meter, $(\Omega \cdot \text{m})^{-1}$. The unit name mhos per meter is sometimes used (mho is ohm backwards). The definition of σ allows us to write Eq. 27-11 in the alternative form

$$\vec{J} = \sigma \vec{E}. \tag{27-13}$$

TABLE 27-3

Resistivities of Some Materials at Room Temperature (20°C)

Material	Resistivity, ρ ($\Omega \cdot m$)	Temperature Coefficient of Resistivity, α (K^{-1})
	Typical Metals	
Silver	1.62×10^{-8}	4.1×10^{-3}
Copper	1.69×10^{-8}	4.3×10^{-3}
Gold	2.35×10^{-8}	4.0×10^{-3}
Aluminum	2.75×10^{-8}	4.4×10^{-3}
Manganin[a]	4.82×10^{-8}	0.002×10^{-3}
Tungsten	5.25×10^{-8}	4.5×10^{-3}
Iron	9.68×10^{-8}	6.5×10^{-3}
Platinum	10.6×10^{-8}	3.9×10^{-3}
	Typical Semiconductors	
Silicon, pure	2.5×10^{3}	-70×10^{-3}
Silicon, *n*-type[b]	8.7×10^{-4}	
Silicon, *p*-type[c]	2.8×10^{-3}	
	Typical Insulators	
Glass	10^{10}–10^{14}	
Fused quartz	$\sim 10^{16}$	

[a]An alloy specifically designed to have a small value of α.
[b]Pure silicon doped with phosphorus impurities to a charge carrier density of 10^{23} m^{-3}.
[c]Pure silicon doped with aluminum impurities to a charge carrier density of 10^{23} m^{-3}.

Calculating Resistance from Resistivity

We have just made an important distinction:

☞ Resistance is a property of an object. Resistivity is a property of a material.

If we know the resistivity of a substance such as copper, we can calculate the resistance of a length of wire made of that substance. Let A be the cross-sectional area of the wire, let L be its length, and let a potential difference V exist between its ends (Fig. 27-9). If the streamlines representing the current density are uniform throughout the wire, the electric field and the current density will be constant for all points within the wire and, from Eqs. 24-39 and 27-5, will have the values

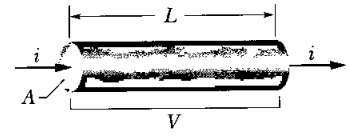


FIG. 27-9 A potential difference V is applied between the ends of a wire of length L and cross section A, establishing a current i.

$$E = V/L \text{ and } J = i/A. \tag{27-14}$$

We can then combine Eqs. 27-10 and 27-14 to write

$$\rho = \frac{E}{J} = \frac{V/L}{i/A}. \tag{27-15}$$

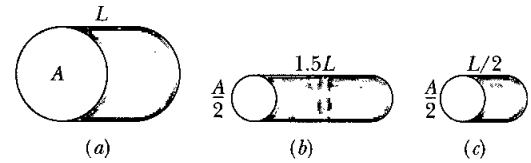
However, V/i is the resistance R, which allows us to recast Eq. 27-15 as

$$R = \rho \frac{L}{A}. \tag{27-16}$$

Equation 27-16 can be applied only to a homogeneous isotropic conductor of uniform cross section, with the potential difference applied as in Fig. 27-8*b*.

The macroscopic quantities V, i, and R are of greatest interest when we are making electrical measurements on specific conductors. They are the quantities that we read directly on meters. We turn to the microscopic quantities E, J, and ρ when we are interested in the fundamental electrical properties of materials.

CHECKPOINT 3 The figure here shows three cylindrical copper conductors along with their face areas and lengths. Rank them according to the current through them, greatest first, when the same potential difference V is placed across their lengths.



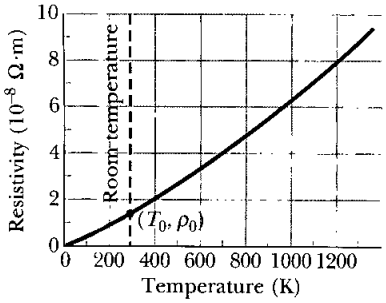


FIG. 27-10 The resistivity of copper as a function of temperature. The dot on the curve marks a convenient reference point at temperature $T_0 = 293$ K and resistivity $\rho_0 = 1.69 \times 10^{-8}\ \Omega \cdot m$.

Variation with Temperature

The values of most physical properties vary with temperature, and resistivity is no exception. Figure 27-10, for example, shows the variation of this property for copper over a wide temperature range. The relation between temperature and resistivity for copper—and for metals in general—is fairly linear over a rather broad temperature range. For such linear relations we can write an empirical approximation that is good enough for most engineering purposes:

$$\rho - \rho_0 = \rho_0 \alpha (T - T_0). \tag{27-17}$$

Here T_0 is a selected reference temperature and ρ_0 is the resistivity at that temperature. Usually $T_0 = 293$ K (room temperature), for which $\rho_0 = 1.69 \times 10^{-8}\ \Omega \cdot m$ for copper.

Because temperature enters Eq. 27-17 only as a difference, it does not matter whether you use the Celsius or Kelvin scale in that equation because the sizes of degrees on these scales are identical. The quantity α in Eq. 27-17, called the *temperature coefficient of resistivity*, is chosen so that the equation gives good agreement with experiment for temperatures in the chosen range. Some values of α for metals are listed in Table 27-1.

Sample Problem 27-4

A rectangular block of iron has dimensions 1.2 cm × 1.2 cm × 15 cm. A potential difference is to be applied to the block between parallel sides and in such a way that those sides are equipotential surfaces (as in Fig. 27-8*b*). What is the resistance of the block if the two parallel sides are (1) the square ends (with dimensions 1.2 cm × 1.2 cm) and (2) two rectangular sides (with dimensions 1.2 cm × 15 cm)?

Conceptualize/Classify: The resistance R of an object depends on how the electric potential is applied to the object. In particular, it depends on the ratio L/A, according to Eq. 27-16 ($R = \rho L/A$), where A is the area of the surfaces to which the potential difference is applied and L is the distance between those surfaces.

Compute: For arrangement 1, we have $L = 15$ cm = 0.15 m and

$$A = (1.2\ \text{cm})^2 = 1.44 \times 10^{-4}\ \text{m}^2.$$

Substituting into Eq. 27-16 with the resistivity ρ from Table 27-1, we then find that for arrangement 1,

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega \cdot \text{m})(0.15\ \text{m})}{1.44 \times 10^{-4}\ \text{m}^2}$$

$$= 1.0 \times 10^{-4}\ \Omega = 100\ \mu\Omega. \qquad \text{(Answer)}$$

Similarly, for arrangement 2, with distance $L = 1.2$ cm and area $A = (1.2\ \text{cm})(15\ \text{cm})$, we obtain

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega \cdot \text{m})(1.2 \times 10^{-2}\ \text{m})}{1.80 \times 10^{-3}\ \text{m}^2}$$

$$= 6.5 \times 10^{-7}\ \Omega = 0.65\ \mu\Omega. \qquad \text{(Answer)}$$

Sample Problem 27-5 Build your skill

Figure 27-11 shows a person and a cow, each a radial distance $D = 60.0$ m from the point where lightning of current $I = 100$ kA strikes the ground. The current spreads through the ground uniformly over a

hemisphere centered on the strike point. The person's feet are separated by radial distance $\Delta r_{per} = 0.50$ m; the cow's front and rear hooves are separated by radial distance $\Delta r_{cow} = 1.50$ m. The resistivity of the ground is $\rho_{gr} = 100\ \Omega \cdot m$. The resistance both across the person, between left and right feet, and across the cow, between front and rear hooves, is $R = 4.00\ k\Omega$.

(a) What is the current i_p through the person?

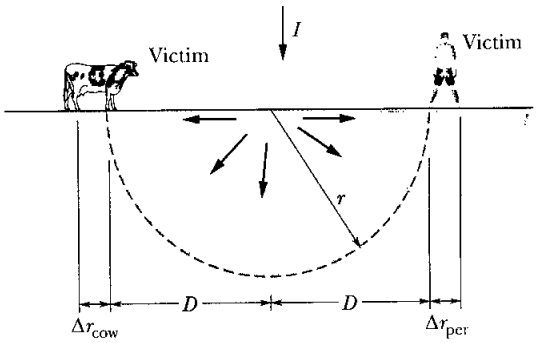


FIG. 27-11 Current from a lightning strike spreads hemispherically through the ground and reaches a cow and a person, each located a distance D from the strike point. The danger to them depends on the separation Δr.

Conceptualize/Classify:

1. The lightning strike sets up an electric field and an electric potential in the surrounding ground.
2. Because one foot is closer to the strike point than the other foot, a potential difference ΔV is set up across the person.
3. That ΔV drives a current i_p through the person.

Compute: Potential difference: Because the lightning's current I spreads uniformly over a hemisphere in the ground, the current density at any given radius r from the strike point is, from Eq. 27-5 ($J = i/A$),

$$J = \frac{I}{2\pi r^2}, \tag{27-18}$$

where $2\pi r^2$ is the area of the curved surface of a hemisphere. From Eq. 27-10 ($\rho = E/J$), the magnitude of the electric field is then

$$E = \rho_{gr} J = \frac{\rho_{gr} I}{2\pi r^2}. \tag{27-19}$$

From Eq. 24-17 ($\Delta V = -\int \vec{E} \cdot d\vec{s}$), the potential difference ΔV between a point at radial distance D and a point at radial distance $D + \Delta r$ is

$$\Delta V = -\int_D^{D+\Delta r} E\, dr. \tag{27-20}$$

Substituting Eq. 27-19 into this integral and then integrating give us the potential difference:

$$\Delta V = -\int_D^{D+\Delta r} \frac{\rho_{gr} I}{2\pi r^2} dr = -\frac{\rho_{gr} I}{2\pi}\left[-\frac{1}{r}\right]_D^{D+\Delta r}$$
$$= \frac{\rho_{gr} I}{2\pi}\left(\frac{1}{D+\Delta r} - \frac{1}{D}\right)$$
$$= -\frac{\rho_{gr} I}{2\pi}\frac{\Delta r}{D(D+\Delta r)}. \tag{27-21}$$

Current: If one of the person's feet is at radial distance D from the strike point and the other foot is at radial distance $D + \Delta r$, the potential difference ΔV between the feet is given by Eq. 27-21. This ΔV drives a current i_p through the person. To find that current, we use Eq. 27-8 ($R = V/i$), in which V represents the magnitude of the potential difference. Substituting the magnitude of ΔV from Eq. 27-21 for V in Eq. 27-8, we find the current to be

$$i = \frac{V}{R} = \frac{\rho_{gr} I}{2\pi}\frac{\Delta r}{D(D+\Delta r)}\frac{1}{R}. \tag{27-22}$$

Substituting known values, including the foot-to-foot separation $\Delta r_{per} = 0.50$ m, gives the current through the person:

$$i_p = \frac{(100\ \Omega \cdot m)(100\ kA)}{2\pi} \times \frac{0.50\ m}{(60\ m)(60.0\ m + 0.50\ m)}\frac{1}{4.00\ k\Omega}$$
$$= 0.0548\ A = 54.8\ mA. \quad \text{(Answer)}$$

Conclude: This amount of current causes involuntary muscle contraction; the person will collapse but probably soon recover. Note that the person could reduce the current an order of magnitude by standing with feet together so that Δr is only a few centimeters.

(b) What is the current i_c through the cow?

Compute: We again use Eq. 27-22, but now Δr is $\Delta r_{cow} = 1.50$ m. We now find that the current through the cow is

$$i_c = 0.162\ A = 162\ mA. \quad \text{(Answer)}$$

Conclude: This is fatal. The cow is in more danger from the ground current because of its greater value of Δr. The cow is, of course, unable to reduce its danger by standing with its hooves together (which would be a bizarre sight).

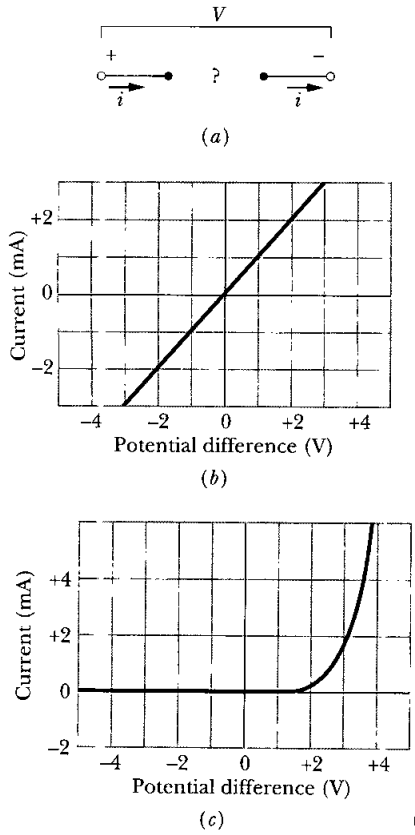


FIG. 27-12 (*a*) A potential difference V is applied to the terminals of a device, establishing a current i. (*b*) A plot of current i versus applied potential difference V when the device is a 1000 Ω resistor. (*c*) A plot when the device is a semiconducting *pn* junction diode.

27-5 | Ohm's Law

As we just discussed in Section 27-4, a resistor is a conductor with a specified resistance. It has that same resistance no matter what the magnitude and direction (*polarity*) of the applied potential difference are. Other conducting devices, however, might have resistances that change with the applied potential difference.

Figure 27-12*a* shows how to distinguish such devices. A potential difference V is applied across the device being tested, and the resulting current i through the device is measured as V is varied in both magnitude and polarity. The polarity of V is arbitrarily taken to be positive when the left terminal of the device is at a higher potential than the right terminal. The direction of the resulting current (from left to right) is arbitrarily assigned a plus sign. The reverse polarity of V (with the right terminal at a higher potential) is then negative; the current it causes is assigned a minus sign.

Figure 27-12*b* is a plot of i versus V for one device. This plot is a straight line passing through the origin, so the ratio i/V (which is the slope of the straight line) is the same for all values of V. This means that the resistance $R = V/i$ of the device is independent of the magnitude and polarity of the applied potential difference V.

Figure 27-12*c* is a plot for another conducting device. Current can exist in this device only when the polarity of V is positive and the applied potential difference is more than about 1.5 V. When current does exist, the relation between i and V is not linear; it depends on the value of the applied potential difference V.

We distinguish between the two types of device by saying that one obeys Ohm's law and the other does not.

☞ **Ohm's law** is an assertion that the current through a device is *always* directly proportional to the potential difference applied to the device.

(This assertion is correct only in certain situations; still, for historical reasons, the term "law" is used.) The device of Fig. 27-12*b*—which turns out to be a 1000 Ω resistor—obeys Ohm's law. The device of Fig. 27-12*c*—which is called a *pn* junction diode—does not.

☞ A conducting device obeys Ohm's law when the resistance of the device is independent of the magnitude and polarity of the applied potential difference.

Modern microelectronics depends almost totally on devices that do *not* obey Ohm's law. Your calculator, for example, is full of them.

It is often contended that $V = iR$ is a statement of Ohm's law. That is not true! This equation is the defining equation for resistance, and it applies to all conducting devices, whether they obey Ohm's law or not. If we measure the potential difference V across, and the current i through, any device, even a *pn* junction diode, we can find its resistance *at that value of* V as $R = V/i$. The essence of Ohm's law, however, is that a plot of i versus V is linear; that is, R is independent of V.

We can express Ohm's law in a more general way if we focus on conducting *materials* rather than on conducting *devices*. The relevant relation is then Eq. 27-11 ($\vec{E} = \rho\vec{J}$) which corresponds to $V = iR$.

☞ A conducting material obeys Ohm's law when the resistivity of the material is independent of the magnitude and direction of the applied electric field.

All homogeneous materials, whether they are conductors like copper or semiconductors like pure silicon or silicon containing special impurities, obey Ohm's law within some range of values of the electric field. If the field is too strong, however, there are departures from Ohm's law in all cases.

CHECKPOINT 4 The following table gives the current i (in amperes) through two devices for several values of potential difference V (in volts). From these data, determine which device does not obey Ohm's law.

Device 1		Device 2	
V	i	V	i
2.00	4.50	2.00	1.50
3.00	6.75	3.00	2.20
4.00	9.00	4.00	2.80

27-6 | A Microscopic View of Ohm's Law

To find out *why* particular materials obey Ohm's law, we must look into the details of the conduction process at the atomic level. Here we consider only conduction in metals, such as copper. We base our analysis on the *free-electron model,* in which we assume that the conduction electrons in the metal are free to move throughout the volume of a sample, like the molecules of a gas in a closed container. We also assume that the electrons collide not with one another but only with atoms of the metal.

According to classical physics, the electrons should have a Maxwellian speed distribution somewhat like that of the molecules in a gas, and thus the average electron speed should depend on the temperature. The motions of electrons are, however, governed not by the laws of classical physics but by those of quantum physics. As it turns out, an assumption that is much closer to the quantum reality is that conduction electrons in a metal move with a single effective speed v_{eff}, and this speed is essentially independent of the temperature. For copper, $v_{\text{eff}} \approx 1.6 \times 10^6$ m/s.

When we apply an electric field to a metal sample, the electrons modify their random motions slightly and drift very slowly—in a direction opposite that of the field—with an average drift speed v_d. As we saw in Sample Problem 27-3, the drift speed in a typical metallic conductor is about 5×10^{-7} m/s, less than the effective speed (1.6×10^6 m/s) by many orders of magnitude. Figure 27-13 suggests the relation between these two speeds. The gray lines show a possible random path for an electron in the absence of an applied field; the electron proceeds from A to B, making six collisions along the way. The green lines show how the same events *might* occur when an electric field $\vec{E}$ is applied. We see that the electron drifts steadily to the right, ending at B' rather than at B. Figure 27-13 was drawn with the assumption that $v_d \approx 0.02 v_{\text{eff}}$. However, because the actual value is more like $v_d \approx (10^{-13}) v_{\text{eff}}$, the drift displayed in the figure is greatly exaggerated.

The motion of conduction electrons in an electric field $\vec{E}$ is thus a combination of the motion due to random collisions and that due to $\vec{E}$. When we consider all the free electrons, their random motions average to zero and make no contribution to the drift speed. Thus, the drift speed is due only to the effect of the electric field on the electrons.

If an electron of mass m is placed in an electric field of magnitude E, the electron will experience an acceleration given by Newton's second law:

$$a = \frac{F}{m} = \frac{eE}{m}. \tag{27-23}$$

The nature of the collisions experienced by conduction electrons is such that, after a typical collision, each electron will—so to speak—completely lose its memory of its previous drift velocity. Each electron will then start off fresh after every encounter, moving off in a random direction. In the average time τ between collisions, the average electron will acquire a drift speed of $v_d = a\tau$. Moreover, if we measure the drift speeds of all the electrons at any instant, we will find that their

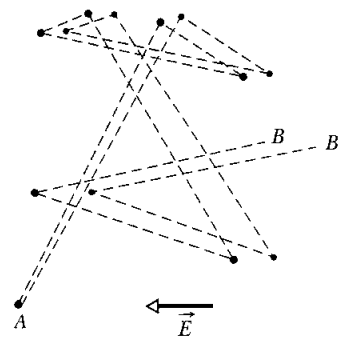


FIG. 27-13 The gray lines show an electron moving from A to B, making six collisions en route. The green lines show what the electron's path might be in the presence of an applied electric field $\vec{E}$. Note the steady drift in the direction of $-\vec{E}$. (Actually, the green lines should be slightly curved, to represent the parabolic paths followed by the electrons between collisions, under the influence of an electric field.)

average drift speed is also $a\tau$. Thus, at any instant, on average, the electrons will have drift speed $v_d = a\tau$. Then Eq. 27-23 gives us

$$v_d = a\tau = \frac{eE\tau}{m}. \tag{27-24}$$

Combining this result with Eq. 27-7 ($\vec{J} = ne\vec{v}_d$), in magnitude form, yields

$$v_d = \frac{J}{ne} = \frac{eE\tau}{m},$$

which we can write as

$$E = \left(\frac{m}{e^2 n\tau}\right) J.$$

Comparing this with Eq. 27-11 ($\vec{E} = \rho\vec{J}$), in magnitude form, leads to

$$\rho = \frac{m}{e^2 n\tau}. \tag{27-25}$$

Equation 27-25 may be taken as a statement that metals obey Ohm's law if we can show that, for metals, their resistivity ρ is a constant, independent of the strength of the applied electric field $\vec{E}$. Because n, m, and e are constant, this reduces to convincing ourselves that τ, the average time (or *mean free time*) between collisions, is a constant, independent of the strength of the applied electric field. Indeed, τ can be considered to be a constant because the drift speed v_d caused by the field is so much smaller than the effective speed v_{eff} that the electron speed—and thus τ—is hardly affected by the field.

The field inside circuit wires can be shown to be due not only to charges in and on the battery but also to those on the surfaces of the circuit elements.

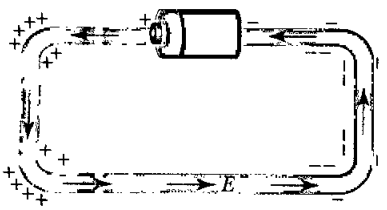


FIG. 27-14 The steady-state pattern of the electric field and a schematic surface charge distribution for a simple circuit in the steady state. The surface charge is shown at sample locations along the wire to depict a charge gradient.

Figure 27-14 shows the pattern of the electric field and a schematic charge distribution on a simple circuit consisting of a battery connected to a resistive wire of constant cross section and composition. You might be wondering what makes the electrons in the wire realize the electric field of the battery. The answer is simple. In the steady state there is a gradient of surface charge density on the surface of the resistive wire, which is a major contributor to the field inside the wire (charges in and on the battery also contribute). These charges produce an electric field which keeps on moving the electrons despite their being subjected to the resistive forces with the resistance. Thus, when there is a bend in a wire, charge accumulates at the bend at the right places to change the direction of the electric field. Note that the battery does not supply any charge. The net charge on the wires is still zero. It is just redistributed to produce the desired effect.

Consideration of surface charge can also help us understand the function of a resistor in a circuit. Let us consider a resistor made of carbon, and for simplicity we will give it the same diameter as the conducting wires in our circuit (Fig. 27-15). Carbon is a poor conductor, but not quite a good insulator—its resistivity is about $3 \times 10^{-5}\ \Omega\cdot\text{m}$ about 2000 times larger than that of copper. Because the resistor and the wires have the same cross-sectional area, the current density is the same in both. Using our previous current density for the 1 A current in the 1 mm radius wire, we can find the electric field in the resistor

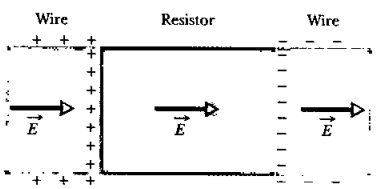


FIG. 27-15 Details of surface changes near a resistor. The build-up charge on the wire ends produces the large electric field in the resistor.

$$E = \rho J = \frac{3\times10^{-5}\ \Omega\cdot\text{m}}{3\times10^{-6}\ \text{A/m}^2} \approx 10\ \text{V/m},$$

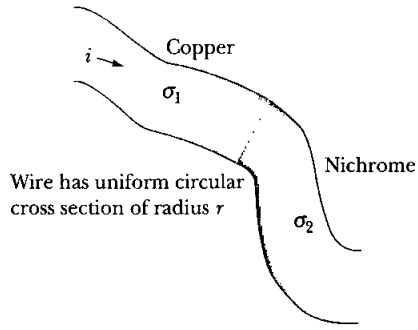
giving a field about 2000 times larger than the field in the copper wires. (Can you now see why the potential drop in the wires is negligible compared with the potential drop in the resistor?) This large electric field is necessary to force the electrons

through the resistor. As shown in Fig. 27-15, charges building up on the end of the wire (similar to the charges on a capacitor) are responsible for producing this large electric field. It does not take a whole lot; in fact this electric field can be produced by only about 1000 electrons at the interface.

What we have described here is a wonderful, self-regulating system. The battery provides the initial "burst" of current to the circuit, and almost instantly the charge finds its way to the locations where it guides the steady current and prevents further build-up of charge on the surface of the wires. This equilibrium is maintained as long as the battery continues to pump charge around the circuit.

Do you know that the Gauss' law is also applicable in the electric circuits?

✓CHECKPOINT 5 Can you see how the two wires having different conductivities connected in a circuit will have different values of the electric field inside them? What will be the sign of excess charge built up on the interface between two cylindrical conductors as shown in the figure?



Sample Problem 27-6

(a) What is the mean free time τ between collisions for the conduction electrons in copper?

Conceptualize/Classify: The mean free time τ of copper is approximately constant, and in particular does not depend on any electric field that might be applied to a sample of the copper. Thus, we need not consider any particular value of applied electric field. However, because the resistivity ρ displayed by copper under an electric field depends on τ, we can find the mean free time τ from Eq. 27-25 ($\rho = m/e^2 n\tau$).

Compute: That equation gives us

$$\tau = \frac{m}{ne^2\rho}.$$

We take the value of n, the number of conduction electrons per unit volume in copper, from Sample Problem 27-3. We take the value of ρ from Table 27-1. The denominator then becomes

$$(8.49\times10^{28}\ \mathrm{m^{-3}})(1.6\times10^{-19}\ \mathrm{C})^2(1.69\times10^{-8}\ \Omega\cdot\mathrm{m})$$
$$= 3.67\times10^{-17}\ \mathrm{C^2\cdot\Omega/m^2} = 3.67\times10^{-17}\ \mathrm{kg/s},$$

where we converted units as

$$\frac{\mathrm{C^2\cdot\Omega}}{\mathrm{m^2}} = \frac{\mathrm{C^2\cdot V}}{\mathrm{m^2\cdot A}} = \frac{\mathrm{C^2\cdot J/C}}{\mathrm{m^2\cdot C/s}} = \frac{\mathrm{kg\cdot m^2/s^2}}{\mathrm{m^2/s}} = \frac{\mathrm{kg}}{\mathrm{s}}.$$

Using these results and substituting for the electron mass m, we then have

$$\tau = \frac{9.1\times10^{-31}\ \mathrm{kg}}{3.67\times10^{-17}\ \mathrm{kg/s}} = 2.5\times10^{-14}\ \mathrm{s}. \quad \text{(Answer)}$$

(b) The mean free path λ of the conduction electrons in a conductor is the average distance traveled by an electron between collisions. What is λ for the conduction electrons in copper, assuming that their effective speed v_{eff} is 1.6×10^6 m/s?

Conceptualize/Classify: The distance d any particle travels in a certain time t at a constant speed v is $d = vt$.

Compute: For the electrons in copper, this gives us

$$\lambda = v_{\mathrm{eff}}\tau = (1.6\times10^6\ \mathrm{m/s})(2.5\times10^{-14}\ \mathrm{s})$$
$$= 4.0\times10^{-8}\ \mathrm{m} = 40\ \mathrm{nm}. \quad \text{(Answer)}$$

Conclude: This is about 150 times the distance between nearest-neighbor atoms in a copper lattice. Thus, on the average, each conduction electron passes many copper atoms before finally hitting one.

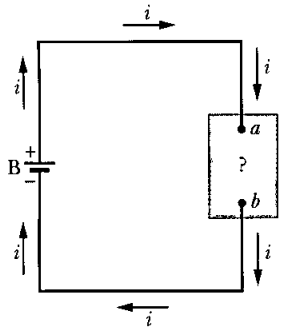


FIG. 27-16 A battery B sets up a current i in a circuit containing an unspecified conducting device.

27-7 | Power in Electric Circuits

Figure 27-16 shows a circuit consisting of a battery B that is connected by wires, which we assume have negligible resistance, to an unspecified conducting device. The device might be a resistor, a storage battery (a rechargeable battery), a motor, or some other electrical device. The battery maintains a potential difference of magnitude V across its own terminals and thus (because of the wires) across the terminals of the unspecified device, with a greater potential at terminal a of the device than at terminal b.

Because there is an external conducting path between the two terminals of the battery, and because the potential differences set up by the battery are maintained, a steady current i is produced in the circuit, directed from terminal a to terminal b. The amount of charge dq that moves between those terminals in time interval dt is equal to $i\,dt$. This charge dq moves through a decrease in potential of magnitude V, and thus its electric potential energy decreases in magnitude by the amount

$$dU = dq\,V = i\,dt\,V.$$

The principle of conservation of energy tells us that the decrease in electric potential energy from a to b is accompanied by a transfer of energy to some other form. The power P associated with that transfer is the rate of transfer dU/dt, which is

$$P = iV \qquad \text{(rate of electrical energy transfer).} \tag{27-26}$$

Moreover, this power P is also the rate at which energy is transferred from the battery to the unspecified device. If that device is a motor connected to a mechanical load, the energy is transferred as work done on the load. If the device is a storage battery that is being charged, the energy is transferred to stored chemical energy in the storage battery. If the device is a resistor, the energy is transferred to internal thermal energy, tending to increase the resistor's temperature.

The unit of power that follows from Eq. 27-26 is the volt-ampere (V · A). We can write it as

$$1\,\mathrm{V\cdot A} = \left(1\,\frac{\mathrm{J}}{\mathrm{C}}\right)\left(1\,\frac{\mathrm{C}}{\mathrm{s}}\right) = 1\,\frac{\mathrm{J}}{\mathrm{s}} = 1\,\mathrm{W}.$$

As an electron moves through a resistor at constant drift speed, its average kinetic energy remains constant and its lost electric potential energy appears as thermal energy in the resistor and the surroundings. On a microscopic scale this energy transfer is due to collisions between the electron and the molecules of the resistor, which leads to an increase in the temperature of the resistor lattice. The mechanical energy thus transferred to thermal energy is *dissipated* (lost) because the transfer cannot be reversed.

For a resistor or some other device with resistance R, we can combine Eqs. 27-8 ($R = V/i$) and 27-26 to obtain, for the rate of electrical energy dissipation due to a resistance, either

$$P = i^2R \qquad \text{(resistive dissipation)} \tag{27-27}$$

or

$$P = \frac{V^2}{R} \qquad \text{(resistive dissipation).} \tag{27-28}$$

Caution: We must be careful to distinguish these two equations from Eq. 27-26: $P = iV$ applies to electrical energy transfers of all kinds; $P = i^2R$ and $P = V^2/R$ apply only to the transfer of electric potential energy to thermal energy in a device with resistance.

CHECKPOINT 6 A potential difference V is connected across a device with resistance R, causing current i through the device. Rank the following variations according to the change in the rate at which electrical energy is converted to thermal energy due to the resistance, greatest change first: (a) V is doubled with R unchanged, (b) i is doubled with R unchanged, (c) R is doubled with V unchanged, (d) R is doubled with i unchanged.

Sample Problem 27-7

You are given a length of uniform heating wire made of a nickel–chromium–iron alloy called Nichrome; it has a resistance R of 72 Ω. At what rate is energy dissipated in each of the following situations? (1) A potential difference of 120 V is applied across the full length of the wire. (2) The wire is cut in half, and a potential difference of 120 V is applied across the length of each half.

Conceptualize/Classify: Current in a resistive material produces a transfer of mechanical energy to thermal energy; the rate of transfer (dissipation) is given by Eqs. 27-26 to 27-28.

Compute: Because we know the potential V and resistance R, we use Eq. 27-28, which yields, for situation 1,

$$P = \frac{V^2}{R} = \frac{(120\text{ V})^2}{72\ \Omega} = 200\text{ W}. \qquad \text{(Answer)}$$

In situation 2, the resistance of each half of the wire is (72 Ω)/2, or 36 Ω. Thus, the dissipation rate for each half is

$$P' = \frac{(120\text{V})^2}{36\ \Omega} = 400\text{ W},$$

and that for the two halves is

$$P = 2P' = 800\text{ W}. \qquad \text{(Answer)}$$

Conclude: This is four times the dissipation rate of the full length of wire. Thus, you might conclude that you could buy a heating coil, cut it in half, and reconnect it to obtain four times the heat output. Why is this unwise? (What would happen to the amount of current in the coil?)

27-8 | Superconductors

In 1911, Dutch physicist Kamerlingh Onnes discovered that the resistivity of mercury absolutely disappears at temperatures below about 4 K (Fig. 27-17). This phenomenon of **superconductivity** is of vast potential importance in technology because it means that charge can flow through a superconducting conductor without losing its energy to thermal energy. Currents created in a superconducting ring, for example, have persisted for several years without loss; the electrons making up the current require a force and a source of energy at start-up time but not thereafter.

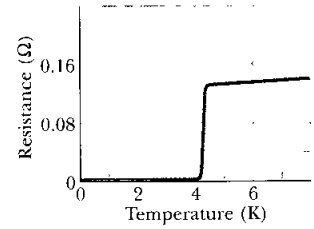
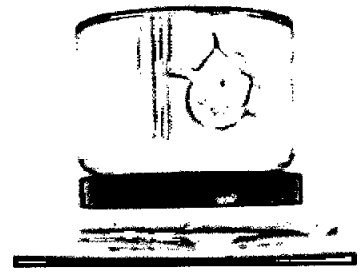


FIG. 27-17 The resistance of mercury drops to zero at a temperature of about 4 K.

Prior to 1986, the technological development of superconductivity was throttled by the cost of producing the extremely low temperatures required to achieve the effect. In 1986, however, new ceramic materials were discovered that become superconducting at considerably higher (and thus cheaper to produce) temperatures. Practical application of superconducting devices at room temperature may eventually become commonplace.



A disk-shaped magnet is levitated above a superconducting material that has been cooled by liquid nitrogen. The goldfish is along for the ride. *(Courtesy Shoji Tonaka/ International Superconductivity Technology Center, Tokyo, Japan)*

Superconductivity is a phenomenon much different from conductivity. In fact, the best of the normal conductors, such as silver and copper, cannot become superconducting at any temperature, and the new ceramic superconductors are actually good insulators when they are not at low enough temperatures to be in a superconducting state.

One explanation for superconductivity is that the electrons that make up the current move in coordinated pairs. One of the electrons in a pair may electrically distort the molecular structure of the superconducting material as it moves through, creating nearby a short-lived concentration of positive charge.

The other electron in the pair may then be attracted toward this positive charge. According to the theory, such coordination between electrons would prevent them from colliding with the molecules of the material and thus would eliminate electrical resistance. The theory worked well to explain the pre-1986, lower temperature superconductors, but new theories appear to be needed for the newer, higher temperature superconductors.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: Current density is a vector.

Statement 2: Current is a vector.

2. Statement 1: Although the average velocity of electrons in a current carrying wire is very low, still the bulb glows as soon as the switch is turned on.

Statement 2: The average speed of electrons in the wire is very high.

3. Statement 1: If a heater is connected to a power supply, the current decreases when the heater becomes hot.

Statement 2: For most of the resistors, resistance increases with increase in temperature.

4. A filament emits electrons at a constant rate as shown in Fig. 27-18. The electrons are then subjected to a constant electric field. The cross section area of the beam remains constant.

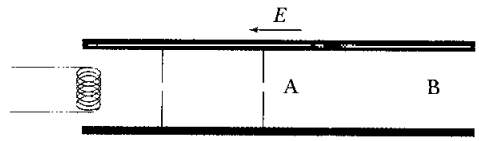


FIG. 27-18 Reasoning type question 4.

Statement 1: The current at cross section B is greater than the current at A.

Statement 2: The speed of the electrons is greater at B than at A.

5. A steady current I_{in} flows through the wire that connects to a resistor. A steady current I_{out} flows through the wire that is connected to the other end. The potential at the first end is V_{in} and the potential at the second end is V_{out}.

Statement 1: I_{in} is greater than I_{out}.

Statement 2: V_{in} is greater than V_{out}.

MULTIPLE CHOICE QUESTIONS

sec. 27-2 Electric Current

1. Which one of the following statements is false?

(a) The electrons in a wire carrying an electrical current normally move very slowly (< 1 m/s)

(b) A battery supplies the same current to any object to which it is connected

(c) The electric current leaving the positive terminal of a battery is the same as the current returning to the negative terminal

(d) A conductor in which an electric current is flowing has an electric field in its interior

(e) The electric current flows in a direction opposite to the motion of the electrons

2. In which one of the following situations does a conventional electric current flow due north?

(a) Protons in a beam are moving due south

(b) A water molecule is moving due north

(c) Electrons in a beam are moving due south

(d) Electrons in a wire connected to a battery are moving from south to north

3. A nonconducting ring of radius R has charge Q distributed over it. If it rotates with an angular velocity ω about its axis, the equivalent electric current will be

(a) Zero

(b) $Q\omega$

(c) $Q\omega/2\pi$

(d) $Q\omega/2\pi R$

4. A long conductor having charge q, with charge density λ is moving with a velocity $2v$ parallel to its own axis. The convectional current I due to motion of conductor is

(a) $I = \lambda v/2$

(b) $I = 2\lambda v$

(c) $I = \lambda v$

(d) $I = 3\lambda v$

5. In an ionic solution, sodium ions (Na^+) are moving to the right and chlorine ions (Cl^-) are moving to the left. In which direction is the current due to the motion of (1) the sodium ions and (2) the chlorine ions?

(a) Both are to the right

(b) Current due to Na^+ is to the left; current due to Cl^- is to the right

(c) Current due to Na^+ is to the right; current due to Cl^- is to the left

(d) Both are to the left

sec. 27-3 Current Density

6. When you flip a switch to turn on a light, the delay before the light turns on is determined by:

(a) The speed of the electric field moving in the wire

(b) The density of electrons in the wire

(c) The drift speed of the electrons in the wire

(d) The number of electron collisions per second in the wire

(e) None of these, since the light comes on instantly

7. Figure 27-19 shows the cross section of a wire. Which of the following statement(s) about current (I) and drift velocity (v) is true?

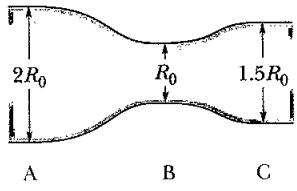


FIG. 27-19 Multiple choice question 7.

(a) $I_A = I_B = I_C$ and $v_A = v_C > v_B$

(b) $I_A = I_B = I_C$ and $v_A > v_C > v_B$

(c) $I_A = I_B < I_C$ and $v_A = v_C = v_B$

(d) $I_A = I_B > I_C$ and $v_A = v_C = v_B$

(e) $I_A = I_B = I_C$ and $v_A < v_C < v_B$

sec. 27-4 Resistance and Resistivity

8. A hollow copper tube of 5 m length has got external diameter equal to 10 cm and its walls are 5 mm thick. The specific resistance of copper is $1.7 \times 10^{-8}\ \Omega \cdot m$. The resistance of the copper tube approximately

(a) $5.4 \times 10^{-3}\ \Omega$

(b) $5.4 \times 10^{-9}\ \Omega$

(c) $5.4 \times 10^{-5}\ \Omega$

(d) $5.4 \times 10^{-7}\ \Omega$

9. A copper wire is fabricated that has a gradually increasing diameter along its length as shown in Fig. 27-20. If an electric current is moving through the wire, which quantities vary along the length of the wire?

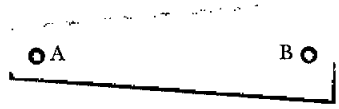


FIG. 27-20 Multiple choice question 9.

(a) Current only

(b) Current and current density only

(c) Current density and electric field only

(d) Resistivity and current only

(e) Current, resistivity, current density, and electric field

10. The magnitude of the electric field within a conducting wire depends on which of the following quantities?

(a) Potential difference, wire diameter, wire length, and wire conductivity

(b) Potential difference and wire conductivity only

(c) Potential difference, wire length, and wire conductivity only

(d) Wire diameter, wire length, and wire conductivity only

(e) The magnitude of the electric field inside the conductor is always zero

11. Which one of the following statements concerning the electric field inside a conductor is true if electrons are moving from right to left in a conducting wire?

(a) The electric field must be zero in this case

(b) The electric field is directed perpendicular to the direction the electrons are moving

(c) The electric field is directed toward the left

(d) The electric field is directed toward the right

sec. 27-5 Ohm's Law

12. Wires A and B are identical except that each is made from a different material. The one end of wire A is connected to one end of wire B. The two remaining ends are connected across a battery and current flows through the two wires. Which one of the following statements concerning this situation is true?

(a) The potential difference across the two wires is the same, but the current through each wire is different

(b) The potential difference across the two wires is the same; and the current through each wire is the same

(c) The potential difference across the two wires is different, but the current through each wire is the same

(d) Both the potential difference across the two wires and the current through each wire is different

sec. 27-6 A Microscopic View of Ohm's Law

13. A conductor of area of cross-section A having charge carries, each having a charge q is subjected to a potential difference V. The number density of charge carries in the conductor is n and the charge carries (along with their random) are moving with a velocity v. If σ is the conductivity of the conductor and τ is the average relaxation time, then

(a) $\tau = \dfrac{m}{nq^2\sigma}$

(b) $\tau = \dfrac{m\sigma}{nq^2}$

(c) $\tau = \dfrac{2m\sigma}{nq^2}$

(d) $\tau = \dfrac{m\sigma}{2nq^2}$

14. Which of the following statements are true for a metallic conductor?

(a) The electrical conductivity depends on the mass of atoms

(b) The electrical conductivity decreases with rise in temperature

(c) The current density depends upon the drift velocity of electrons

(d) The electrical conductivity increases with increase in voltage across it

15. How does the resistivity of a metal wire change if either the number of electrons per unit volume increases or the mean free time increases?

(a) In both cases, the resistivity will increase

(b) In both cases, the resistivity will decrease

(c) Increasing the number of electrons will increase the resistivity, but it will decrease if the mean free time increases

(d) Increasing the number of electrons will decrease the resistivity, but it will increase if the mean free time increases

16. Figure 27-21 represents a portion of a wire in a circuit. A current is flowing in the wire in the direction shown. Under the convention that it is positive charge that flows, the electric field points in the direction of the current. There will be some charge accumulation at the bend to change the direction of the electric field. What is the direction of the electric field due to charges at the bend?

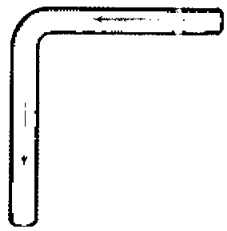
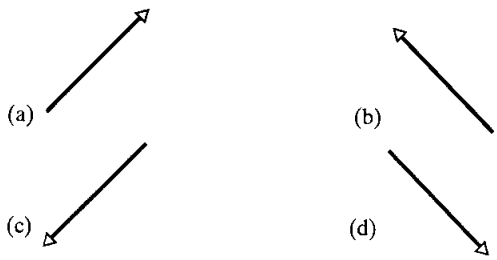


FIG. 27-21 Multiple choice question 16.



sec. 27-7 Power in Electric Circuits

17. Lengths and cross-sectional areas of four pieces of nichrome wire are (L, A), $(2L, A)$, $(L, 2A)$ and $(2L, 2A)$, respectively. If the same voltage difference V is applied across their lengths, which of the wires will get the hottest in steady state?

(a) Wire (L, A)

(b) Wire $(2L, A)$

(c) Wire $(L, 2A)$

(d) Wire $(2L, 2A)$

(e) All wires get hot to the same temperature

18. The insulated wiring in a house can safely carry a maximum current of 18 A. The electrical outlets in the house provide an alternating voltage of 120 V. A space heater when plugged into the outlet operates at an average power of 1500 W. How many space heaters can safely be plugged into a single electrical outlet and turned on for an extended period of time?

(a) Zero

(b) One

(c) Two

(d) Three

(e) Four

19. If the current in an electric bulb decreases by 0.5%, then the power in the bulb decreases approximately by

(a) 0.5%

(b) 1%

(c) 2%

(d) 0.2%

20. When a 500 W electric bulb and 500 W heater operate at their rated voltages, the filament of the bulb reaches a much higher temperature than the filament of the heater. The most important reason for this is that

(a) Their resistances are not equal

(b) They are made of different materials

(c) Their dimensions are very different

(d) The emissivity of their surface is different

PROBLEMS

sec. 27-2 Electric Current

1. During the 4.0 min a 5.0 A current is set up in a wire, how many (a) coulombs and (b) electrons pass through any cross section across the wire's width?

sec. 27-3 Current Density

2. A small but measurable current of 1.2×10^{-10} A exists in a copper wire whose diameter is 2.5 mm. The number of charge carriers per unit volume is 8.49×10^{28} m^{-3}. Assuming the current is uniform, calculate the (a) current density and (b) electron drift speed.

3. A certain cylindrical wire carries current. We draw a circle of radius r around its central axis in Fig. 27-22*a* to determine the current i within the circle. Figure 27-18*b* shows current i as a function of r^2. The vertical scale is set by $i_s = 4.0$ mA, and the horizontal scale is set by $r_s^2 = 4.0$ mm^2. (a) Is the current density uniform? (b) If so, what is its magnitude?

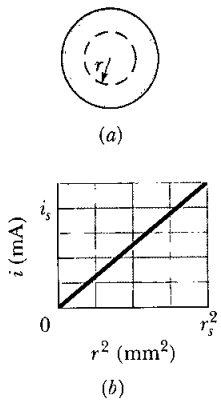


FIG. 27-22 Problem 3.

4. How long does it take electrons to get from a car battery to the starting motor? Assume the current is 300 A and the electrons travel through a copper wire with cross-sectional area 0.21 cm^2 and length 0.85 m. The number of charge carriers per unit volume is 8.49×10^{28} m^{-3}.

sec. 27-4 Resistance and Resistivity

5. A wire of Nichrome (a nickel–chromium–iron alloy commonly used in heating elements) is 1.0 m long and 1.0 mm^2 in cross-sectional area. It carries a current of 4.0 A when a 2.0 V potential difference is applied between its ends. Calculate the conductivity

6. A human being can be electrocuted if a current as small as 50 mA passes near the heart. An electrician working with sweaty hands makes good contact with the two conductors he is holding, one in each hand. If his resistance is 2000 Ω, what might the fatal voltage be?

7. A wire with a resistance of 6.0 Ω is drawn out through a die so that its new length is three times its original length. Find the resistance of the longer wire, assuming that the resistivity and density of the material are unchanged.

8. Figure 27-23*a* gives the magnitude $E(x)$ of the electric fields that have been set up by a battery along a resistive rod of length 9.00 mm (Fig. 27-23*b*). The vertical scale is set by $E_s = 4.00 \times 10^3$ V/m. The rod consists of three sections of the same material but with different radii. (The schematic diagram of Fig. 27-23*b* does not indicate the different radii.) The radius of section 3 is 2.00 mm. What is the radius of (a) section 1 and (b) section 2?

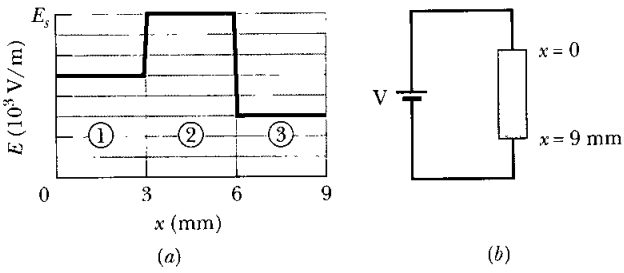


FIG. 27-23 Problem 8.

9. Figure 27-24 gives the electric potential $V(x)$ along a copper wire carrying uniform current, from a point of higher potential $V_s = 12.0$ μV at $x = 0$ to a point of zero potential at $x_s = 3.00$ m. The wire has a radius of 2.00 mm. What is the current in the wire?

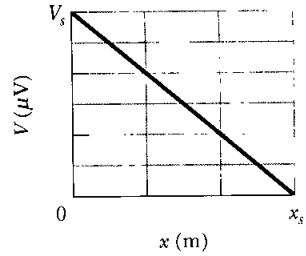


FIG. 27-24 Problem 9.

10. *Kiting during a storm.* The legend that Benjamin Franklin flew a kite as a storm approached is only a legend—he was neither stupid nor suicidal. Suppose a kite string of radius 2.00 mm extends directly upward by 0.800 km and is coated with a 0.500 mm layer of water having resistivity 150 Ω. If the potential difference between the two ends of the string is 160 MV, what is the current through the water layer? The danger is not this current but the chance that the string draws a lightning strike, which can have a current as large as 500,000 A (way beyond being just lethal).

11. In Fig. 27-25*a*, a 9.00 V battery is connected to a resistive strip that consists of three sections with the same cross-sectional areas but different conductivities. Figure 27-21*b* gives the electric potential $V(x)$ versus position x along the strip. The horizontal scale is set by $x_s = 8.00$ mm. Section 3 has conductivity 3.00×10^7 (Ω · m)$^{-1}$. What is the conductivity of section (a) 1 and (b) 2?

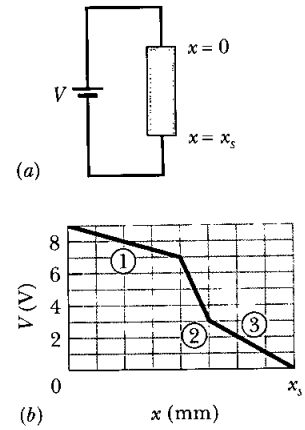


FIG. 27-25 Problem 11.

12. Earth's lower atmosphere contains negative and positive ions that are produced by radioactive elements in the soil and cosmic rays from space. In a certain region, the atmospheric electric field strength is 120 V/m and the field is directed vertically down. This field causes singly charged positive ions, at a density of 620 cm^{-3}, to drift downward and singly charged negative ions, at a density of 550 cm^{-3}, to drift upward (Fig. 27-26). The measured conductivity of the air in that region is 2.70×10^{-14} (Ω · m)$^{-1}$. Calculate (a) the magnitude of the current density and (b) the ion drift speed, assumed to be the same for positive and negative ions.

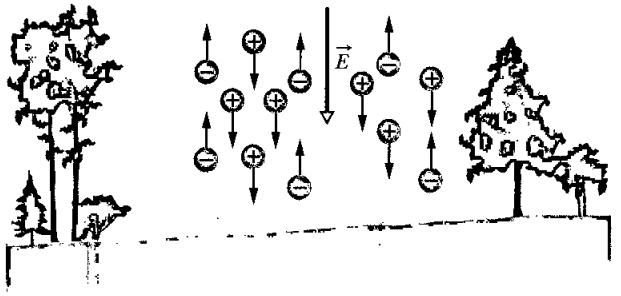


FIG. 27-26 Problem 12.

13. Figure 27-27 shows wire section 1 of diameter $D_1 = 4.00R$ and wire section 2 of diameter $D_2 = 2.00R$, connected by a tapered section. The wire is copper and carries a current. Assume that the current is uniformly distributed across any cross-sectional area through the wire's width. The electric potential change V along the length $L = 2.00$ m shown in section 2 is 10.0 μV. The number of charge carriers per unit volume is 8.49×10^{28} m^{-3}. What is the drift speed of the conduction electrons in section 1?

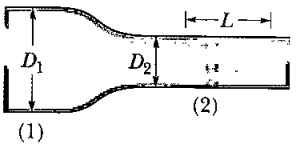


FIG. 27-27 Problem 13.

sec. 27-7 Power in Electric Circuits

14. In Fig. 27-28, a battery of potential difference $V = 12$ V is connected to a resistive strip of resistance $R = 6.0\ \Omega$. When an electron moves through the strip from one end to the other, (a) in which direction in the figure does the electron move, (b) how much work is done on the electron by the electric field in the strip, and (c) how much energy is transferred to the thermal energy of the strip by the electron?

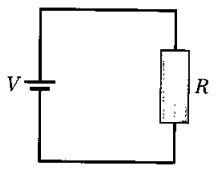


FIG. 27-28 Problem 14.

15. An unknown resistor is connected between the terminals of a 3.00 V battery. Energy is dissipated in the resistor at the rate of 0.540 W. The same resistor is then connected between the terminals of a 1.50 V battery. At what rate is energy now dissipated?

16. The current through the battery and resistors 1 and 2 in Fig. 27-29*a* is 2.00 A. Energy is transferred from the current to thermal energy E_{th} in both resistors. Curves 1 and 2 in Fig. 27-29*b* give that thermal energy E_{th} for resistors 1 and 2, respectively, as a function of time t. The vertical scale is set by $E_{th,s} = 40.0$ mJ, and the horizontal scale is set by $t_s = 5.00$ s. What is the power of the battery?

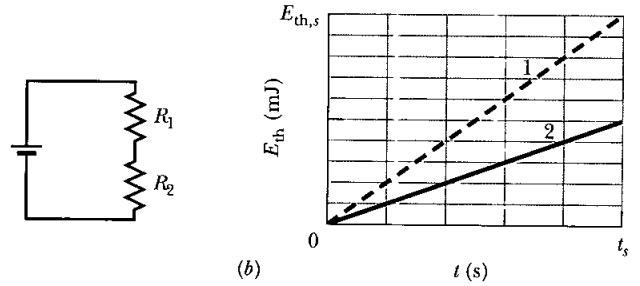


FIG. 27-29 Problem 16.

17. The current-density magnitude in a certain circular wire is $J = (2.75 \times 10^{10}\ \text{A/m}^4)r^2$, where r is the radial distance out to the wire's radius of 3.00 mm. The potential applied to the wire (end to end) is 60.0 V. How much energy is converted to thermal energy in 1.00 h?

18. Figure 27-30*a* shows a rod of resistive material. The resistance per unit length of the rod increases in the positive direction of the x axis. At any position x along the rod, the resistance dR of a narrow (differential) section of width dx is given by $dR = 5.00x\ dx$, where dR is in ohms and x is in meters. Figure 27-30*b* shows such a narrow section. You are to slice off a length of the rod between $x = 0$ and some position $x = L$ and then connect that length to a battery with potential difference $V = 5.0$ V (Fig. 27-30*c*). You want the current in the length to transfer energy to thermal energy at the rate of 200 W. At what position $x = L$ should you cut the rod?

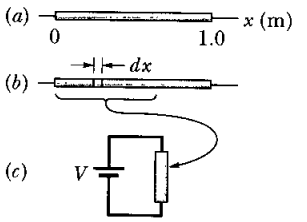


FIG. 27-30 Problem 18.

ADDITIONAL PROBLEMS

19. A steady beam of alpha particles ($q = +2e$) traveling with constant kinetic energy 20 MeV carries a current of 0.25 μA. (a) If the beam is directed perpendicular to a flat surface, how many alpha particles strike the surface in 3.0 s? (b) At any instant, how many alpha particles are there in a given 20 cm length of the beam? (c) Through what potential difference is it necessary to accelerate each alpha particle from rest to bring it to an energy of 20 MeV?

20. A 500 W heating unit is designed to operate with an applied potential difference of 115 V. (a) By what percentage will its heat output drop if the applied potential difference drops to 110 V? Assume no change in resistance. (b) If you took the variation of resistance with temperature into account, would the actual drop in heat output be larger or smaller than that calculated in (a)?

21. A Nichrome heater dissipates 500 W when the applied potential difference is 110 V and the wire temperature is 800°C. What would be the dissipation rate if the wire temperature were held at 200°C by immersing the wire in a bath of cooling oil? The applied potential difference remains the same, and α at 800°C is $4.0 \times 10^{-4}\ K^{-1}$.

22. The headlights of a moving car require about 10 A from the 12 V alternator, which is driven by the engine. Assume the alternator is 80% efficient (its output electrical power is 80% of its input mechanical power), and calculate the horsepower the engine must supply to run the lights.

23. An 18.0 W device has 9.00 V across it. How much charge goes through the device in 4.00 h?

24. A coil of current-carrying Nichrome wire is immersed in a liquid. (Nichrome is a nickel–chromium–iron alloy commonly used in heating elements.) When the potential difference across the coil is 12 V and the current through the coil is 5.2 A, the liquid evaporates at the steady rate of 21 mg/s. Calculate the latent heat of vaporization of the liquid.

25. A caterpillar of length 4.0 cm crawls in the direction of electron drift along a 5.2-mm-diameter bare copper wire that carries a uniform current of 12 A. (a) What is the potential difference between the two ends of the caterpillar? (b) Is its tail positive or negative relative to its head? (c) How much time does the caterpillar take to crawl 1.0 cm if it crawls at the drift speed of the electrons in the wire? (The number of charge carriers per unit volume is $8.49 \times 10^{28}\ m^{-3}$.)

ANSWERS

Checkpoints

1. 8 A, rightward

2. (a)–(c) rightward

3. *a* and *c* tie, then *b*

4. device 2

5. $E_{Cu} < E_{Ni}$, positive

6. (a) and (b) tie, then (d), then (c)

Reasoning Type Questions

1. (c)

2. (b)

3. (a)

4. (d)

5. (d)

Multiple Choice Questions

1. (a), (b)

2. (c)

3. (c)

4. (b)

5. (a)

6. (e)

7. (e)

8. (c)

9. (c)

10. (c)

11. (d)

12. (c)

13. (b)

14. (b), (c)

15. (b)

16. (d)

17. (c)

18. (b)

19. (b)

20. (c)

Problems

1. (a) 1.2 kC; (b) 7.5×10^{21}

2. (a) $2.4 \times 10^{-5}\ A/m^2$; (b) 1.8×10^{-15} m/s

3. (a) yes; (b) $4.0 \times 10^2\ A/m^2$

4. 13 min 28 sec

5. $2.0 \times 10^6\ (\Omega \cdot m)^{-1}$

6. 100 V

7. 54 Ω

8. (a) 1.55 mm; (b) 1.22 mm

9. 3.0 mA

10. 9.42×10^{-3} A

11. (a) $6.00 \times 10^{7}\ (\Omega \cdot \text{m})^{-1}$; (b) $7.50 \times 10^{6}\ (\Omega \cdot \text{m})^{-1}$

12. (a) 3.24×10^{-12} A/m^2; (b) 1.73 cm/s

13. 5.44×10^{-9} m/s

14. (a) upward; (b) 12 eV or 1.9×10^{-18} J; (c) 12 eV or 1.9×10^{-18} J

15. 0.135 W

16. 12 mW

17. 7.56×10^{5} J

18. 0.224 m

Additional Problems

19. (a) 2.3×10^{12}; (b) 5.0×10^{3}; (c) 10 MV

20. (a) −8.6%; (b) smaller

21. 660 W

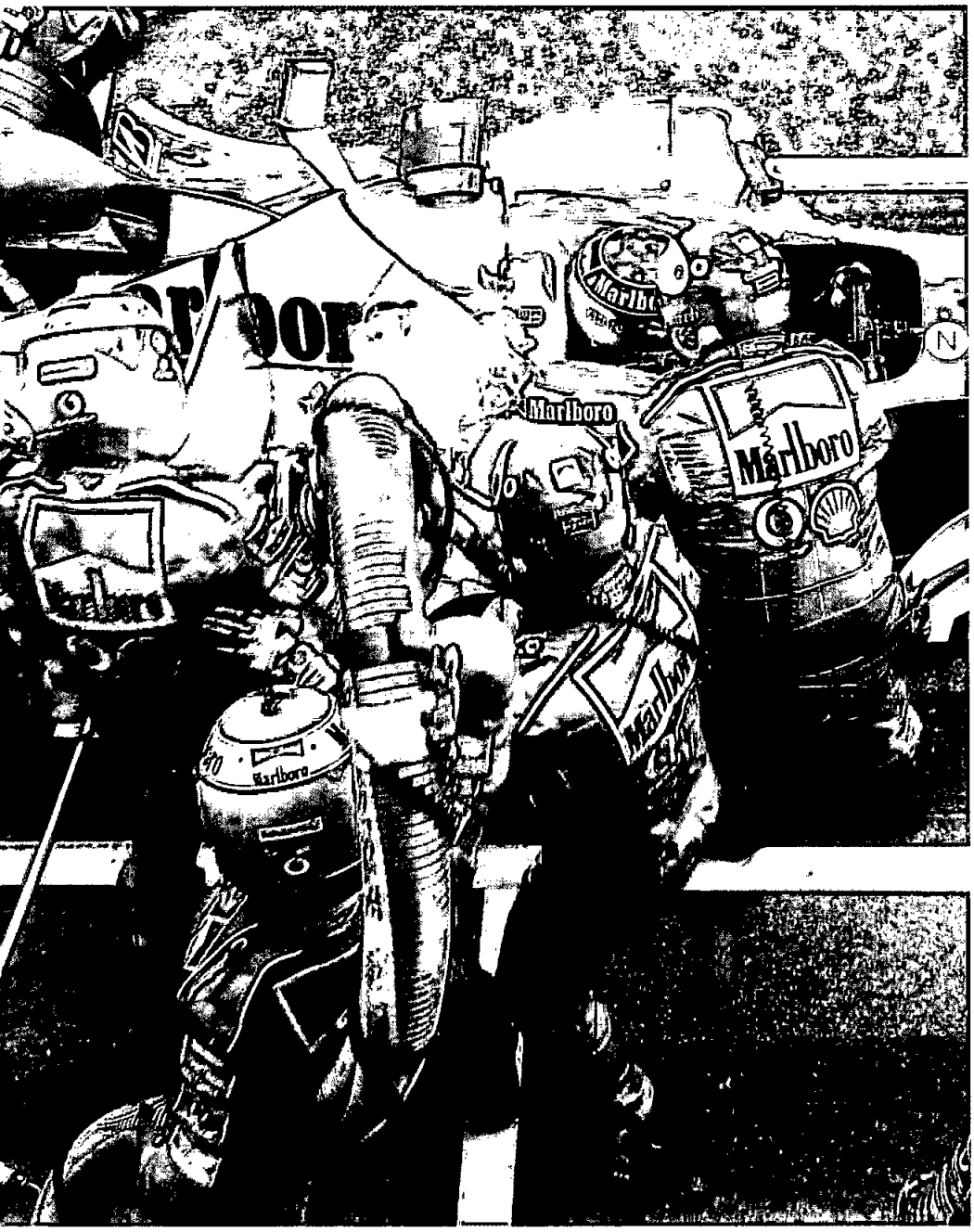
22. 0.20 hp

23. 28.8 kC

24. 3.0×10^{6} J/kg

25. (a) 3.8×10^{-4} V; (b) its tail is negative compared to its head; (c) 3 min 58 sec

Circuit Theory 28



© AP/Wide World Photos

When a race car pulls in for a pit stop, the pit crew scurries to make any needed car adjustments and to refuel the car. The crew members pride themselves on their speed because even a fraction of a second wasted during the pit stop can cost the driver the race. However, rapid application of the fuel dispenser seems dangerous because the car may be charged to a potential of −30 kV when it arrives in the pit. If the car sparks through any of the fuel vapor, a fire can break out.

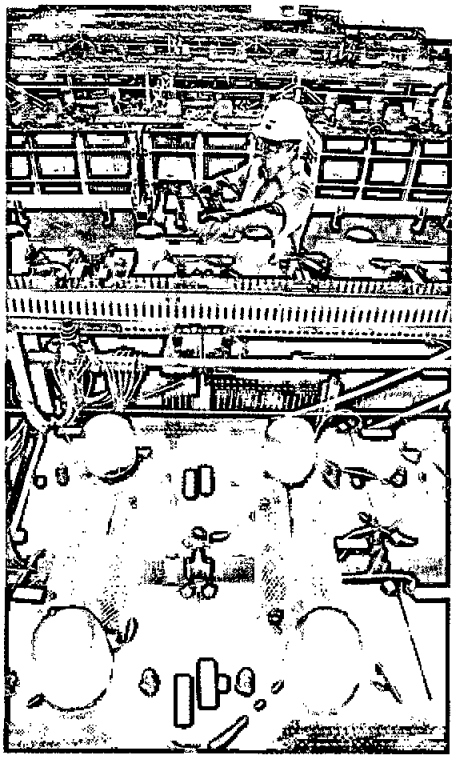
What precaution must be taken to prevent such a fire?

The answer is in this chapter.

28-1 WHAT IS PHYSICS?

You are surrounded by electric circuits. You might take pride in the number of electrical devices you own and might even carry a mental list of the devices you wish you owned. Every one of those devices, as well as the electrical grid that powers your home, depends on modern electrical engineering. We cannot easily estimate the current financial worth of electrical engineering and its products, but we can be certain that the financial worth continues to grow yearly as more and more tasks are handled electrically. Radios are now tuned electronically instead of manually. Messages are now sent by email instead of through the postal system. Research journals are now read on a computer instead of in a library building, and research papers are now copied and filed electronically instead of photocopied and tucked into a filing cabinet.

The basic science of electrical engineering is physics. In this chapter we cover the physics of electric circuits that are combinations of resistors and batteries (and, in Section 28-9, capacitors). We restrict our discussion to circuits through which charge flows in one direction, which are called either *direct-current circuits* or *DC circuits.* We begin with the question: How can you get charges to flow?



The world's largest battery energy storage plant (dismantled in 1996) connected over 8000 large lead-acid batteries in 8 strings at 1000 V each with a capability of 10 MW of power for 4 hours. Charged up at night, the batteries were then put to use during peak power demands on the electrical system. *(Courtesy Southern California Edison Company)*

28-2 | Work, Energy, and Emf

If you want to make charge carriers flow through a resistor, you must establish a potential difference between the ends of the device. One way to do this is to connect each end of the resistor to one plate of a charged capacitor. The trouble with this scheme is that the flow of charge acts to discharge the capacitor, quickly bringing the plates to the same potential. When that happens, there is no longer an electric field in the resistor, and thus the flow of charge stops.

To produce a steady flow of charge, you need a "charge pump," a device that—by doing work on the charge carriers—maintains a potential difference between a pair of terminals. We call such a device an **emf device**, and the device is said to provide an **emf** $\mathscr{E}$, which means that it does work on charge carriers. An emf device is sometimes called a *seat of emf.* The term *emf* comes from the outdated phrase *electromotive force,* which was adopted before scientists clearly understood the function of an emf device.

In Chapter 27, we discussed the motion of charge carriers through a circuit in terms of the electric field set up in the circuit—the field produces forces that move the charge carriers. In this chapter we take a different approach: We discuss the motion of the charge carriers in terms of the required energy—an emf device supplies the energy for the motion via the work it does.

A common emf device is the *battery,* used to power a wide variety of machines from wristwatches to submarines. The emf device that most influences our daily lives, however, is the *electric generator,* which, by means of electrical connections (wires) from a generating plant, creates a potential difference in our homes and workplaces. The emf devices known as *solar cells,* long familiar as the wing-like panels on spacecraft, also dot the countryside for domestic applications. Less familiar emf devices are the *fuel cells* that power the space shuttles and the *thermopiles* that provide onboard electrical power for some spacecraft and for remote stations in Antarctica and elsewhere. An emf device does not have to be an instrument—living systems, ranging from electric eels and human beings to plants, have physiological emf devices.

Although the devices we have listed differ widely in their modes of operation, they all perform the same basic function—they do work on charge carriers and thus maintain a potential difference between their terminals.

Figure 28-1 shows an emf device (consider it to be a battery) that is part of a simple circuit containing a single resistance R (the symbol for resistance and a resistor is —WW—). The emf device keeps one of its terminals (called the positive terminal and often labeled +) at a higher electric potential than the other terminal (called the negative terminal and labeled –). We can represent the emf of the device with an arrow that points from the negative terminal toward the positive terminal as in Fig. 28-1. A small circle on the tail of the emf arrow distinguishes it from the arrows that indicate current direction.

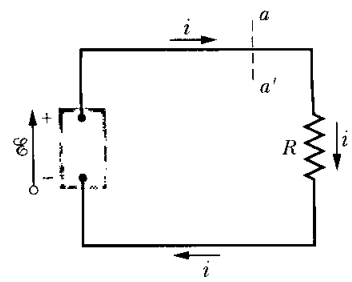


FIG. 28-1 A simple electric circuit, in which a device of emf $\mathscr{E}$ does work on the charge carriers and maintains a steady current i in a resistor of resistance R.

When an emf device is not connected to a circuit, the internal chemistry of the device does not cause any net flow of charge carriers within it. However, when it is connected to a circuit as in Fig. 28-1, its internal chemistry causes a net flow of positive charge carriers from the negative terminal to the positive terminal, in the direction of the emf arrow. This flow is part of the current that is set up around the circuit in that same direction (clockwise in Fig. 28-1).

Within the emf device, positive charge carriers move from a region of low electric potential and thus low electric potential energy (at the negative terminal) to a region of higher electric potential and higher electric potential energy (at the positive terminal). This motion is just the opposite of what the electric field between the terminals (which is directed from the positive terminal toward the negative terminal) would cause the charge carriers to do.

Thus, there must be some source of energy within the device, enabling it to do work on the charges by forcing them to move as they do. The energy source may be chemical, as in a battery or a fuel cell. It may involve mechanical forces, as in an electric generator. Temperature differences may supply the energy, as in a thermopile; or the Sun may supply it, as in a solar cell.

Let us now analyze the circuit of Fig. 28-1 from the point of view of work and energy transfers. In any time interval dt, a charge dq passes through any cross section of this circuit, such as aa'. This same amount of charge must enter the emf device at its low-potential end and leave at its high-potential end. The device must do an amount of work dW on the charge dq to force it to move in this way. We define the emf of the emf device in terms of this work:

$$\mathscr{E} = \frac{dW}{dq} \qquad \text{(definition of } \mathscr{E}\text{)}. \tag{28-1}$$

In words, the emf of an emf device is the work per unit charge that the device does in moving charge from its low-potential terminal to its high-potential terminal. The SI unit for emf is the joule per coulomb; in Chapter 24 we defined that unit as the *volt*.

An **ideal emf device** is one that lacks any internal resistance to the internal movement of charge from terminal to terminal. The potential difference between the terminals of an ideal emf device is equal to the emf of the device. For example, an ideal battery with an emf of 12.0 V always has a potential difference of 12.0 V between its terminals.

A **real emf device**, such as any real battery, has internal resistance to the internal movement of charge. When a real emf device is not connected to a circuit, and thus does not have current through it, the potential difference between its terminals is equal to its emf. However, when that device has current through it, the potential difference between its terminals differs from its emf. We shall discuss such real batteries in Section 28-3.

When an emf device is connected to a circuit, the device transfers energy to the charge carriers passing through it. This energy can then be transferred from the charge carriers to other devices in the circuit, for example, to light a bulb. Figure 28-2*a* shows a circuit containing two ideal rechargeable (*storage*) batteries A and B, a resistance R, and an electric motor M that can lift an object by using energy it obtains from charge carriers in the circuit. Note that the batteries are connected

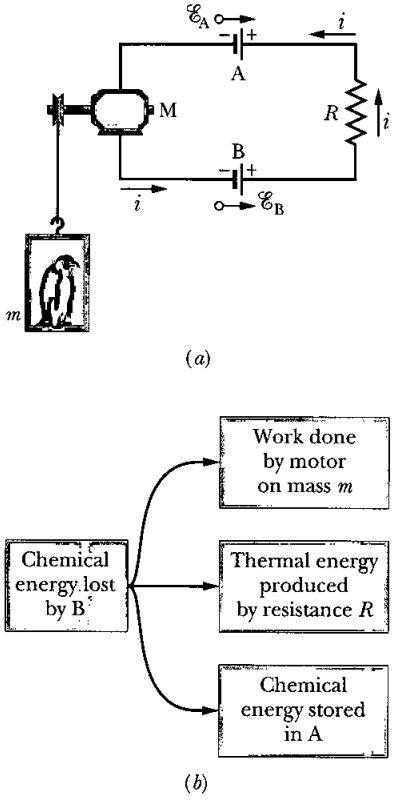


FIG. 28-2 (*a*) In the circuit, $\mathscr{E}_B > \mathscr{E}_A$; so battery B determines the direction of the current. (*b*) The energy transfers in the circuit.

so that they tend to send charges around the circuit in opposite directions. The actual direction of the current in the circuit is determined by the battery with the larger emf, which happens to be battery B, so the chemical energy within battery B is decreasing as energy is transferred to the charge carriers passing through it. However, the chemical energy within battery A is increasing because the current in it is directed from the positive terminal to the negative terminal. Thus, battery B is charging battery A. Battery B is also providing energy to motor M and energy that is being dissipated by resistance R. Figure 28-2*b* shows all three energy transfers from battery B; each decreases that battery's chemical energy.

28-3 | Calculating the Current in a Single-Loop Circuit

We discuss here two equivalent ways to calculate the current in the simple *single-loop* circuit of Fig. 28-3; one method is based on energy conservation considerations, and the other on the concept of potential. The circuit consists of an ideal battery B with emf $\mathscr{E}$, a resistor of resistance R, and two connecting wires. (Unless otherwise indicated, we assume that wires in circuits have negligible resistance. Their function, then, is merely to provide pathways along which charge carriers can move.)

Energy Method

Equation 27-27 ($P = i^2R$) tells us that in a time interval dt an amount of energy given by $i^2R\,dt$ will appear in the resistor of Fig. 28-3 as thermal energy. As noted in Section 27-7, this energy is said to be *dissipated*. (Because we assume the wires to have negligible resistance, no thermal energy will appear in them.) During the same interval, a charge $dq = i\,dt$ will have moved through battery B, and the work that the battery will have done on this charge, according to Eq. 28-1, is

$$dW = \mathscr{E}\,dq = \mathscr{E} i\,dt.$$

From the principle of conservation of energy, the work done by the (ideal) battery must equal the thermal energy that appears in the resistor:

$$\mathscr{E} i\,dt = i^2R\,dt.$$

This gives us

$$\mathscr{E} = iR.$$

The emf $\mathscr{E}$ is the energy per unit charge transferred to the moving charges by the battery. The quantity iR is the energy per unit charge transferred *from* the moving charges to thermal energy within the resistor. Therefore, this equation means that the energy per unit charge transferred to the moving charges is equal to the energy per unit charge transferred from them. Solving for i, we find

$$i = \frac{\mathscr{E}}{R}. \tag{28-2}$$

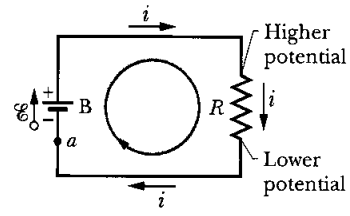


FIG. 28-3 A single-loop circuit in which a resistance R is connected across an ideal battery B with emf $\mathscr{E}$. The resulting current i is the same throughout the circuit.

Potential Method

Suppose we start at any point in the circuit of Fig. 28-3 and mentally proceed around the circuit in either direction, adding algebraically the potential differences that we encounter. Then when we return to our starting point, we must also have returned to our starting potential. Before actually doing so, we shall formalize this idea in a statement that holds not only for single-loop circuits such as that of Fig. 28-3 but also for any complete loop in a *multiloop* circuit, as we shall discuss in subsection "Kirchhoff's Junction Rule."

LOOP RULE: The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.

This is often referred to as *Kirchhoff's loop rule* (or *Kirchhoff's voltage law*), after German physicist Gustav Robert Kirchhoff. This rule is equivalent to saying that each point on a mountain has only one elevation above sea level. If you start from any point and return to it after walking around the mountain, the algebraic sum of the changes in elevation that you encounter must be zero.

In Fig. 28-3, let us start at point a, whose potential is V_a, and mentally walk clockwise around the circuit until we are back at a, keeping track of potential changes as we move. Our starting point is at the low-potential terminal of the battery. Because the battery is ideal, the potential difference between its terminals is equal to $\mathscr{E}$. When we pass through the battery to the high-potential terminal, the change in potential is $+\mathscr{E}$.

As we walk along the top wire to the top end of the resistor, there is no potential change because the wire has negligible resistance; it is at the same potential as the high-potential terminal of the battery. So too is the top end of the resistor. When we pass through the resistor, however, the potential changes according to Eq. 27-8 (which we can rewrite as $V = iR$). Moreover, the potential must decrease because we are moving from the higher potential side of the resistor. Thus, the change in potential is $-iR$.

We return to point a by moving along the bottom wire. Because this wire also has negligible resistance, we again find no potential change. Back at point a, the potential is again V_a. Because we traversed a complete loop, our initial potential, as modified for potential changes along the way, must be equal to our final potential; that is,

$$V_a + \mathscr{E} - iR = V_a.$$

The value of V_a cancels from this equation, which becomes

$$\mathscr{E} - iR = 0.$$

Solving this equation for i gives us the same result, $i = \mathscr{E}/R$, as the energy method (Eq. 28-2).

If we apply the loop rule to a complete *counterclockwise* walk around the circuit, the rule gives us

$$-\mathscr{E} + iR = 0$$

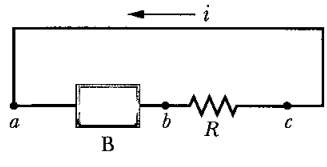
and we again find that $i = \mathscr{E}/R$. Thus, you may mentally circle a loop in either direction to apply the loop rule.

To prepare for circuits more complex than that of Fig. 28-3, let us set down two rules for finding potential differences as we move around a loop:

RESISTANCE RULE: For a move through a resistance in the direction of the current, the change in potential is $-iR$; in the opposite direction it is $+iR$.

EMF RULE: For a move through an ideal emf device in the direction of the emf arrow, the change in potential is $+\mathscr{E}$; in the opposite direction it is $-\mathscr{E}$.

CHECKPOINT 1 The figure shows the current i in a single-loop circuit with a battery B and a resistance R (and wires of negligible resistance). (a) Should the emf arrow at B be drawn pointing leftward or rightward? At points a, b, and c, rank (b) the magnitude of the current, (c) the electric potential, and (d) the electric potential energy of the charge carriers, greatest first.



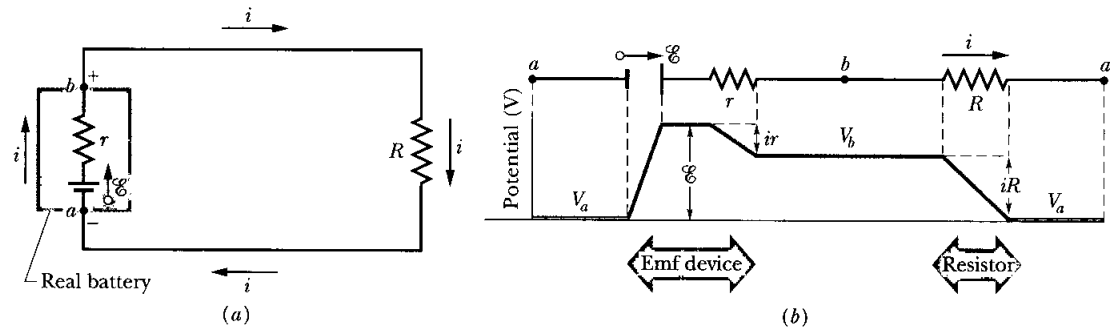


FIG. 28-4 (*a*) A single-loop circuit containing a real battery having internal resistance r and emf $\mathscr{E}$. (*b*) The same circuit, now spread out in a line. The potentials encountered in traversing the circuit clockwise from *a* are also shown. The potential V_a is arbitrarily assigned a value of zero, and other potentials in the circuit are graphed relative to V_a.

Internal Resistance

Figure 28-4*a* shows a real battery, with internal resistance r, wired to an external resistor of resistance R. The internal resistance of the battery is the electrical resistance of the conducting materials of the battery and thus is an unremovable feature of the battery. In Fig. 28-4*a*, however, the battery is drawn as if it could be separated into an ideal battery with emf $\mathscr{E}$ and a resistor of resistance r. The order in which the symbols for these separated parts are drawn does not matter.

If we apply the loop rule clockwise beginning at point *a*, the *changes* in potential give us

$$\mathscr{E} - ir - iR = 0. \tag{28-3}$$

Solving for the current, we find

$$i = \frac{\mathscr{E}}{R+r}. \tag{28-4}$$

Note that this equation reduces to Eq. 28-2 if the battery is ideal—that is, if $r = 0$.

Figure 28-4*b* shows graphically the changes in electric potential around the circuit. (To better link Fig. 28-4*b* with the *closed circuit* in Fig. 28-4*a*, imagine curling the graph into a cylinder with point *a* at the left overlapping point *a* at the right.) Note how traversing the circuit is like walking around a (potential) mountain back to your starting point—you return to the starting elevation.

In this book, when a battery is not described as real or if no internal resistance is indicated, you can generally assume that it is ideal—but, of course, in the real world batteries are always real and have internal resistance.

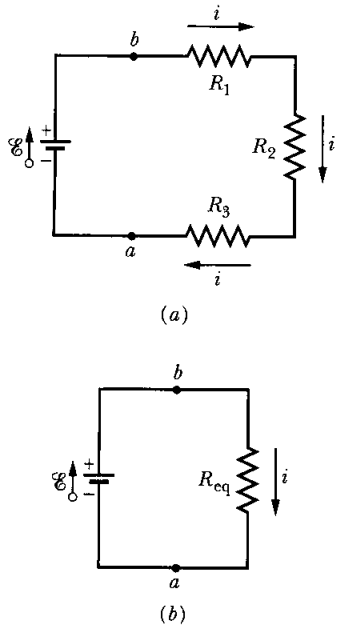


FIG. 28-5 (*a*) Three resistors are connected in series between points *a* and *b*. (*b*) An equivalent circuit, with the three resistors replaced with their equivalent resistance R_{eq}.

Resistances in Series

Figure 28-5*a* shows three resistances connected **in series** to an ideal battery with emf $\mathscr{E}$. This description has little to do with how the resistances are drawn. Rather, "in series" means that the resistances are wired one after another and that a potential difference V is applied across the two ends of the series. In Fig. 28-5*a*, the resistances are connected one after another between *a* and *b*, and a potential difference is maintained across *a* and *b* by the battery. The potential differences that then exist across the resistances in the series produce identical currents i in them. In general,

When a potential difference V is applied across resistances connected in series, the resistances have identical currents i. The sum of the potential differences across the resistances is equal to the applied potential difference V.

Note that charge moving through the series resistances can move along only a single route. If there are additional routes, so that the currents in different resistances are different, the resistances are not connected in series.

Resistances connected in series can be replaced with an equivalent resistance R_{eq} that has the same current i and the same *total* potential difference V as the actual resistances.

You might remember that R_{eq} and all the actual series resistances have the same current i with the nonsense word "ser-i." Figure 28-5*b* shows the equivalent resistance R_{eq} that can replace the three resistances of Fig. 28-5*a*.

To derive an expression for R_{eq} in Fig. 28-5*b*, we apply the loop rule to both circuits. For Fig. 28-5*a*, starting at a and going clockwise around the circuit, we find

$$\mathscr{E} - iR_1 - iR_2 - iR_3 = 0,$$

or

$$i = \frac{\mathscr{E}}{R_1 + R_2 + R_3}. \tag{28-5}$$

For Fig. 28-5*b*, with the three resistances replaced with a single equivalent resistance R_{eq}, we find

$$\mathscr{E} - iR_{eq} = 0,$$

or

$$i = \frac{\mathscr{E}}{R_{eq}}. \tag{28-6}$$

Comparison of Eqs. 28-5 and 28-6 shows that

$$R_{eq} = R_1 + R_2 + R_3.$$

The extension to n resistances is straightforward and is

$$R_{eq} = \sum_{j=1}^{n} R_j \qquad (n \text{ resistances in series}). \tag{28-7}$$

Note that when resistances are in series, their equivalent resistance is greater than any of the individual resistances.

CHECKPOINT 2 In Fig. 28-5*a*, if $R_1 > R_2 > R_3$, rank the three resistances according to (a) the current through them and (b) the potential difference across them, greatest first.

Potential Difference Between Two Points

We often want to find the potential difference between two points in a circuit. For example, in Fig. 28-6, what is the potential difference $V_b - V_a$ between points a and b? To find out, let's start at point a (at potential V_a) and move through the battery to point b (at potential V_b) while keeping track of the potential changes we encounter. When we pass through the battery's emf, the potential increases by $\mathscr{E}$. When we pass through the battery's internal resistance r, we move in the direction of the current and thus the potential decreases by ir. We are then at the potential of point b and we have

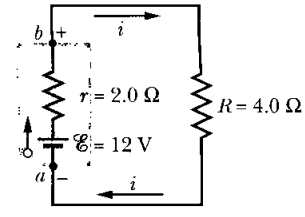


FIG. 28-6 Points a and b, which are at the terminals of a real battery, differ in potential.

$$V_a + \mathscr{E} - ir = V_b,$$

or

$$V_b - V_a = \mathscr{E} - ir. \tag{28-8}$$

To evaluate this expression, we need the current i. Note that the circuit is the same as in Fig. 28-4*a*, for which Eq. 28-4 gives the current as

$$i = \frac{\mathscr{E}}{R + r}. \tag{28-9}$$

Substituting this equation into Eq. 28-8 gives us

$$V_b - V_a = \mathscr{E} - \frac{\mathscr{E}}{R + r} r$$

$$= \frac{\mathscr{E}}{R + r} R. \tag{28-10}$$

Now substituting the data given in Fig. 28-6, we have

$$V_b - V_a = \frac{12\text{ V}}{4.0\ \Omega + 2.0\ \Omega} 4.0\ \Omega = 8.0\text{ V}. \tag{28-11}$$

Suppose, instead, we move from a to b counterclockwise, passing through resistor R rather than through the battery. Because we move opposite the current, the potential increases by iR. Thus,

$$V_a + iR = V_b$$

or

$$V_b - V_a = iR. \tag{28-12}$$

Substituting for i from Eq. 28-9, we again find Eq. 28-10. Hence, substitution of the data in Fig. 28-6 yields the same result, $V_b - V_a = 8.0$ V. In general,

☞ To find the potential between any two points in a circuit, start at one point and traverse the circuit to the other point, following any path, and add algebraically the changes in potential you encounter.

Potential Difference Across a Real Battery

In Fig. 28-6, points a and b are located at the terminals of the battery. Thus, the potential difference $V_b - V_a$ is the terminal-to-terminal potential difference V across the battery. From Eq. 28-8, we see that

$$V = \mathscr{E} - ir. \tag{28-13}$$

If the internal resistance r of the battery in Fig. 28-6 were zero, Eq. 28-13 tells us that V would be equal to the emf $\mathscr{E}$ of the battery—namely, 12 V. However, because $r = 2.0\ \Omega$, Eq. 28-13 tells us that V is less than $\mathscr{E}$. From Eq. 28-11, we know that V is only 8.0 V. Note that the result depends on the value of the current through the battery. If the same battery were in a different circuit and had a different current through it, V would have some other value.

Grounding a Circuit

Figure 28-7*a* shows the same circuit as Fig. 28-6 except that here point a is directly connected to *ground*, as indicated by the common symbol ⏚. *Grounding a circuit* usually means connecting the circuit to a conducting path to Earth's surface (actually to the electrically conducting moist dirt and rock below ground). Here, such a connection means only that the potential is defined to be zero at the grounding point in the circuit. Thus in Fig. 28-7*a*, the potential at a is defined to be $V_a = 0$. Equation 28-11 then tells us that the potential at b is $V_b = 8.0$ V.

Figure 28-7*b* is the same circuit except that point b is now directly connected to ground. Thus, the potential there is defined to be $V_b = 0$. Equation 28-11 now tells us that the potential at a is $V_a = -8.0$ V.

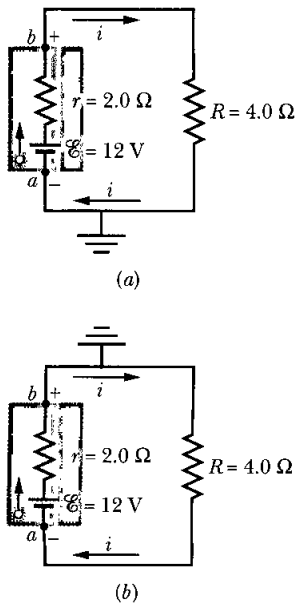


FIG. 28-7 (*a*) Point a is directly connected to ground. (*b*) Point b is directly connected to ground.

Power, Potential, and Emf

When a battery or some other type of emf device does work on the charge carriers to establish a current i, the device transfers energy from its source of energy (such as the chemical source in a battery) to the charge carriers. Because a real emf device has an internal resistance r, it also transfers energy to internal thermal energy via resistive dissipation (Section 26-7). Let us relate these transfers.

The net rate P of energy transfer from the emf device to the charge carriers is given by Eq. 27-26:

$$P = iV, \tag{28-14}$$

where V is the potential across the terminals of the emf device. From Eq. 28-13,

we can substitute $V = \mathscr{E} - ir$ into Eq. 28-14 to find

$$P = i(\mathscr{E} - ir) = i\mathscr{E} - i^2 r. \tag{28-15}$$

From Eq. 27-27, we recognize the term i^2r in Eq. 28-15 as the rate P_r of energy transfer to thermal energy within the emf device:

$$P_r = i^2 r \qquad \text{(internal dissipation rate)}. \tag{28-16}$$

Then the term $i\mathscr{E}$ in Eq. 28-15 must be the rate P_{emf} at which the emf device transfers energy *both* to the charge carriers and to internal thermal energy. Thus,

$$P_{emf} = i\mathscr{E} \qquad \text{(power of emf device)}. \tag{28-17}$$

If a battery is being *recharged*, with a "wrong way" current through it, the energy transfer is then *from* the charge carriers *to* the battery—both to the battery's chemical energy and to the energy dissipated in the internal resistance r. The rate of change of the chemical energy is given by Eq. 28-17, the rate of dissipation is given by Eq. 28-16, and the rate at which the carriers supply energy is given by Eq. 28-14.

CHECKPOINT 3 A battery has an emf of 12 V and an internal resistance of 2 Ω. Is the terminal-to-terminal potential difference greater than, less than, or equal to 12 V if the current in the battery is (a) from the negative to the positive terminal, (b) from the positive to the negative terminal, and (c) zero?

Sample Problem 28-1

The emfs and resistances in the circuit of Fig. 28-8*a* have the following values:

$$\mathscr{E}_1 = 4.4\text{ V}, \quad \mathscr{E}_2 = 2.1\text{ V},$$
$$r_1 = 2.3\ \Omega, \quad r_2 = 1.8\ \Omega, \quad R = 5.5\ \Omega.$$

(a) What is the current i in the circuit?

Conceptualize/Classify: We can get an expression involving the current i in this single-loop circuit by applying the loop rule.

Compute: Although knowing the direction of i is not necessary, we can easily determine it from the emfs of the two batteries. Because $\mathscr{E}_1$ is greater than $\mathscr{E}_2$, battery 1 controls the direction of i, so the direction is clockwise. Let us then apply the loop rule by going counterclockwise—against the current—and starting at point a. We find

$$-\mathscr{E}_1 + ir_1 + iR + ir_2 + \mathscr{E}_2 = 0.$$

Check that this equation also results if we apply the loop rule clockwise or start at some point other than a. Also, take the time to compare this equation term by term with Fig. 28-8*b*, which shows the potential changes graphically (with the potential at point a arbitrarily taken to be zero).

Solving the above loop equation for the current i, we obtain

$$i = \frac{\mathscr{E}_1 - \mathscr{E}_2}{R + r_1 + r_2} = \frac{4.4\text{ V} - 2.1\text{ V}}{5.5\ \Omega + 2.3\ \Omega + 1.8\ \Omega}$$
$$= 0.2396\text{ A} \approx 240\text{ mA}. \qquad \text{(Answer)}$$

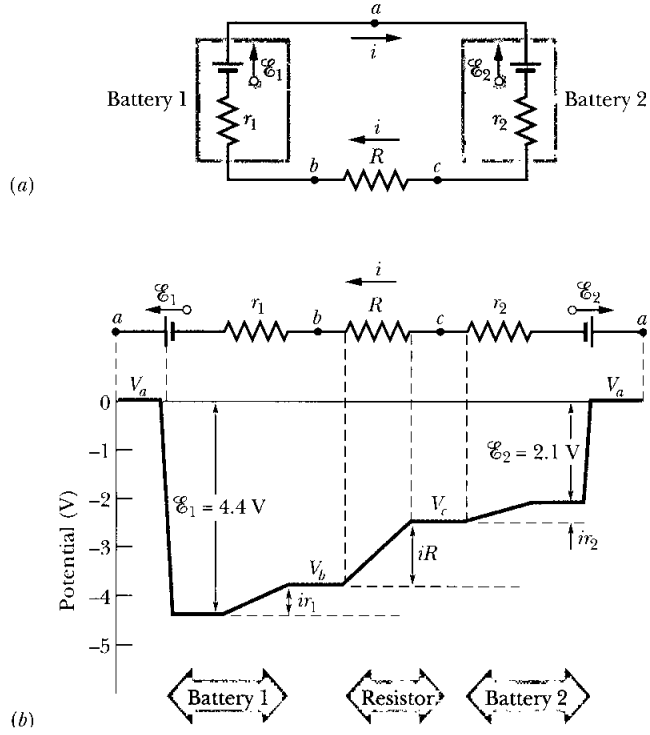


FIG. 28-8 (*a*) A single-loop circuit containing two real batteries and a resistor. The batteries oppose each other; that is, they tend to send current in opposite directions through the resistor. (*b*) A graph of the potentials, counterclockwise from point *a*, with the potential at *a* arbitrarily taken to be zero. (To better link the circuit with the graph, mentally cut the circuit at *a* and then unfold the left side of the circuit toward the left and the right side of the circuit toward the right.)

(b) What is the potential difference between the terminals of battery 1 in Fig. 28-8*a*?

Conceptualize/Classify: We need to sum the potential differences between points *a* and *b*.

Compute: Let us start at point *b* (effectively the negative terminal of battery 1) and travel clockwise through battery 1 to point *a* (effectively the positive terminal), keeping track of potential changes. We find that

$$V_b - ir_1 + \mathscr{E}_1 = V_a,$$

which gives us

$$\begin{aligned} V_a - V_b &= ir_1 + \mathscr{E}_1 \\ &= -(0.2396\ \text{A})(2.3\ \Omega) + 4.4\ \text{V} \\ &= +3.84\ \text{V} \approx 3.8\ \text{V}, \end{aligned} \qquad \text{(Answer)}$$

which is less than the emf of the battery.

Conclude: You can verify this result by starting at point *b* in Fig. 28-8*a* and traversing the circuit counterclockwise to point *a*.

PROBLEM-SOLVING TACTICS

Tactic 1: *Assuming the Direction of a Current Can Be an Arbitrary Decision* In solving circuit problems, you do not need to know the direction of a current in advance. Instead, you can just assume its direction, although that may take some physics courage. To show this, assume the current in Fig. 28-8*a* is counterclockwise; that is, reverse the direction of the current arrows shown. Applying the loop rule counterclockwise from point *a* now yields

$$-\mathscr{E}_1 - ir_1 - iR - ir_2 + \mathscr{E}_2 = 0,$$

or

$$i = -\frac{\mathscr{E}_1 - \mathscr{E}_2}{R + r_1 + r_2}.$$

Substituting the numerical values of Sample Problem 28-1 yields $i = -240$ mA for the current. The minus sign is a signal that the current is opposite the direction we initially assumed.

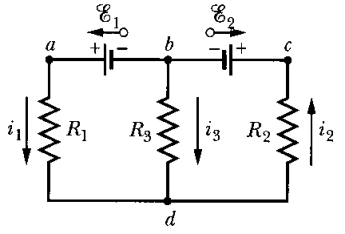


FIG. 28-9 A multiloop circuit consisting of three branches: left-hand branch *bad*, right-hand branch *bcd*, and central branch *bd*. The circuit also consists of three loops: left-hand loop *badb*, right-hand loop *bcdb*, and big loop *badcb*.

Kirchhoff's Junction Rule

Figure 28-9 shows a circuit containing more than one loop. For simplicity, we assume the batteries are ideal. There are two *junctions* in this circuit, at *b* and *d*, and there are three *branches* connecting these junctions. The branches are the left branch (*bad*), the right branch (*bcd*), and the central branch (*bd*). What are the currents in the three branches?

We arbitrarily label the currents, using a different subscript for each branch. Current i_1 has the same value everywhere in branch *bad*, i_2 has the same value everywhere in branch *bcd*, and i_3 is the current through branch *bd*. The directions of the currents are assumed arbitrarily.

Consider junction *d* for a moment: Charge comes into that junction via incoming currents i_1 and i_3, and it leaves via outgoing current i_2. Because there is no variation in the charge at the junction, the total incoming current must equal the total outgoing current:

$$i_1 + i_3 = i_2 \qquad (28\text{-}18)$$

You can easily check that applying this condition to junction *b* leads to exactly the same equation. Equation 28-18 thus suggests a general principle:

JUNCTION RULE: The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.

This rule is often called *Kirchhoff's junction rule* (or *Kirchhoff's current law*). It is simply a statement of the conservation of charge for a steady flow of charge—there is neither a buildup nor a depletion of charge at a junction. Thus, our basic tools for solving complex circuits are the *loop rule* (based on the conservation of energy) and the *junction rule* (based on the conservation of charge).

Equation 28-18 is a single equation involving three unknowns. To solve the circuit completely (that is, to find all three currents), we need two more equations involving those same unknowns. We obtain them by applying the loop rule twice. In the circuit of Fig. 28-9, we have three loops from which to choose: the left-hand loop (*badb*), the right-hand loop (*bcdb*), and the big loop (*badcb*). Which two loops we choose does not matter—let's choose the left-hand loop and the right-hand loop.

If we traverse the left-hand loop in a counterclockwise direction from point b, the loop rule gives us

$$\mathscr{E}_1 - i_1R_1 + i_3R_3 = 0. \tag{28-19}$$

If we traverse the right-hand loop in a counterclockwise direction from point b, the loop rule gives us

$$-i_3R_3 - i_2R_2 - \mathscr{E}_2 = 0. \tag{28-20}$$

We now have three equations (Eqs. 28-18, 28-19, and 28-20) in the three unknown currents, and they can be solved by a variety of techniques.

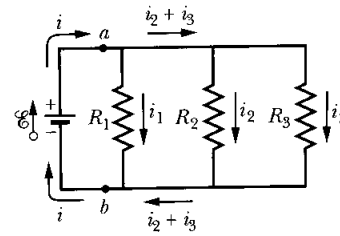
If we had applied the loop rule to the big loop, we would have obtained (moving counterclockwise from b) the equation

$$\mathscr{E}_1 - i_1R_1 - i_2R_2 - \mathscr{E}_2 = 0.$$

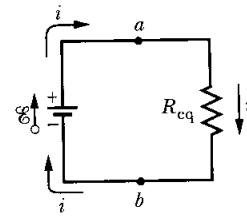
This equation may look like fresh information, but in fact it is only the sum of Eqs. 28-19 and 28-20. (It would, however, yield the proper results when used with Eq. 28-18 and either Eq. 28-19 or Eq. 28-20.)

Resistances in Parallel

Figure 28-10*a* shows three resistances connected *in parallel* to an ideal battery of emf $\mathscr{E}$. The term "in parallel" means that the resistances are directly wired together on one side and directly wired together on the other side, and that a potential difference V is applied across the pair of connected sides. Thus, all three resistances have the same potential difference V across them, producing a current through each. In general,



(*a*)



(*b*)

FIG. 28-10 (*a*) Three resistors connected in parallel across points *a* and *b*. (*b*) An equivalent circuit, with the three resistors replaced with their equivalent resistance R_{eq}.

☞ When a potential difference V is applied across resistances connected in parallel, the resistances all have that same potential difference V.

In Fig. 28-10*a*, the applied potential difference V is maintained by the battery. In Fig. 28-10*b*, the three parallel resistances have been replaced with an equivalent resistance R_{eq}.

☞ Resistances connected in parallel can be replaced with an equivalent resistance R_{eq} that has the same potential difference V and the same *total* current i as the actual resistances.

You might remember that R_{eq} and all the actual parallel resistances have the same potential difference V with the nonsense word "par-V."

To derive an expression for R_{eq} in Fig. 28-10*b*, we first write the current in each actual resistance in Fig. 28-10*a* as

$$i_1 = \frac{V}{R_1}, \quad i_2 = \frac{V}{R_2}, \quad \text{and} \quad i_3 = \frac{V}{R_3},$$

where V is the potential difference between a and b. If we apply the junction rule at point a in Fig. 28-10*a* and then substitute these values, we find

$$i = i_1 + i_2 + i_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right). \tag{28-21}$$

If we replaced the parallel combination with the equivalent resistance R_{eq} (Fig. 28-10*b*), we would have

$$i = \frac{V}{R_{eq}}. \tag{28-22}$$

Comparing Eqs. 28-21 and 28-22 leads to

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \tag{28-23}$$

TABLE 28-1

Series and Parallel Resistors and Capacitors

Series	Parallel	Series	Parallel
Resistors		Capacitors	
$R_{eq} = \sum_{j=1}^{n} R_j$ Eq. 28-7	$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j}$ Eq. 28-24	$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j}$ Eq. 26-20	$C_{eq} = \sum_{j=1}^{n} C_j$ Eq. 26-19
Same current through all resistors	Same potential difference across all resistors	Same charge on all capacitors	Same potential difference across all capacitors

Extending this result to the case of n resistances, we have

$$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j} \quad (n \text{ resistances in parallel}). \tag{28-24}$$

For the case of two resistances, the equivalent resistance is their product divided by their sum; that is,

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}. \tag{28-25}$$

If you accidentally took the equivalent resistance to be the sum divided by the product, you would notice that this result would be dimensionally incorrect.

Note that when two or more resistances are connected in parallel, the equivalent resistance is smaller than any of the combining resistances. Table 28-1 summarizes the equivalence relations for resistors and capacitors in series and in parallel.

✓**CHECKPOINT 4** A battery, with potential V across it, is connected to a combination of two identical resistors and then has current i through it. What are the potential difference across and the current through either resistor if the resistors are (a) in series and (b) in parallel?

Sample Problem 28-2 Build your skill!

Figure 28-11*a* shows a multiloop circuit containing one ideal battery and four resistances with the following values:

$$R_1 = 20\ \Omega, \quad R_2 = 20\ \Omega, \quad \mathscr{E}_1 = 12\ \text{V},$$
$$R_3 = 30\ \Omega, \quad R_4 = 8.0\ \Omega$$

(a) What is the current through the battery?

Conceptualize: Noting that the current through the battery must also be the current through R_1, we see that we might find the current by applying the loop rule to a loop that includes R_1 because the current would be included in the potential difference across R_1.

Classify: Incorrect method: Either the left-hand loop or the big loop should do. Noting that the emf arrow of the battery points upward, so the current the battery supplies is clockwise, we might apply the loop rule to the left-hand loop, clockwise from point a. With i being the current through the battery, we would get

$$+\mathscr{E} - iR_1 - iR_2 - iR_4 = 0 \quad \text{(incorrect)}.$$

However, this equation is incorrect because it assumes that R_1, R_2, and R_4 all have the same current i. Resistances R_1 and R_4 do have the same current, because the current passing through R_4 must pass through the battery and then through R_1 with no change in value. However, that current splits at junction point b—only part passes through R_2, the rest through R_3.

Dead-end method: To distinguish the several currents in the circuit, we must label them individually as in Fig. 28-11*b*. Then, circling clockwise from a, we can write the loop rule for the left-hand loop as

$$+\mathscr{E} - i_1 R_1 - i_2 R_2 - i_1 R_4 = 0.$$

Unfortunately, this equation contains two unknowns, i_1 and i_2; we would need at least one more equation to find them.

Compute: Successful method: A much easier option is to simplify the circuit of Fig. 28-11*b* by finding

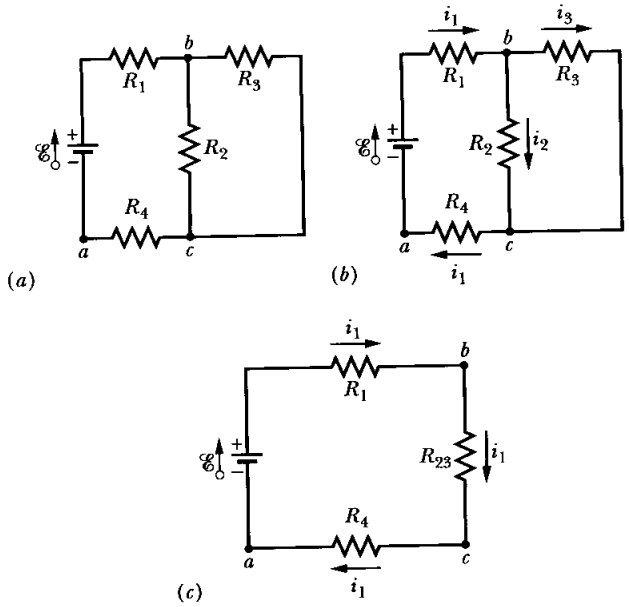


FIG. 28-11 (*a*) A multiloop circuit with an ideal battery of emf $\mathscr{E}$ and four resistances. (*b*) Assumed currents through the resistances. (*c*) A simplification of the circuit, with resistances R_2 and R_3 replaced with their equivalent resistance R_{23}. The current through R_{23} is equal to that through R_1 and R_4.

equivalent resistances. Note carefully that R_1 and R_2 are *not* in series and thus cannot be replaced with an equivalent resistance. However, R_2 and R_3 are in parallel, so we can use either Eq. 28-24 or Eq. 28-25 to find their equivalent resistance R_{23}. From the latter,

$$R_{23} = \frac{R_2 R_3}{R_2 + R_3} = \frac{(20\ \Omega)(30\ \Omega)}{50\ \Omega} = 12\ \Omega.$$

We can now redraw the circuit as in Fig. 28-11*c*; note that the current through R_{23} must be i_1 because charge that moves through R_1 and R_4 must also move through R_{23}. For this simple one-loop circuit, the loop rule (applied clockwise from point *a*) yields

$$+\mathscr{E} - i_1 R_1 - i_1 R_{23} - i_1 R_4 = 0.$$

Substituting the given data, we find

$$12\ \text{V} - i_1(20\ \Omega) - i_1(12\ \Omega) - i_1(8.0\ \Omega) = 0,$$

which gives us

$$i_1 = \frac{12\ \text{V}}{40\ \Omega} = 0.30\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the current i_2 through R_2?

Conceptualize/Classify: (1) We must now work backward from the equivalent circuit of Fig. 28-11*c*, where R_{23} has replaced R_2 and R_3. (2) Because R_2 and R_3 are in parallel, they both have the same potential difference across them as R_{23}.

Compute: Working backword: We know the current through R_{23} is $i_1 = 0.30$ A. Thus, we can use Eq. 27-8 ($R = V/i$) to find the potential difference V_{23} across R_{23}:

$$V_{23} = i_1 R_{23} = (0.30\ \text{A})(12\ \Omega) = 3.6\ \text{V}.$$

The potential difference across R_2 is thus 3.6 V, so the current i_2 in R_2 must be, by Eq. 27-8,

$$i_2 = \frac{V_2}{R_2} = \frac{3.6\ \text{V}}{20\ \Omega} = 0.18\ \text{A}. \qquad \text{(Answer)}$$

(c) What is the current i_3 through R_3?

Conceptualize/Classify: We can answer by using the same technique as in (b), or we can use this idea: The junction rule tells us that at point *b* in Fig. 28-11*b*, the incoming current i_1 and the outgoing currents i_2 and i_3 are related by

$$i_1 = i_2 + i_3.$$

Compute: Rearranging this junction-rule result yields

$$i_3 = i_1 - i_2 = 0.30\ \text{A} - 0.18\ \text{A} = 0.12\ \text{A}. \qquad \text{(Answer)}$$

28-4 | Multiloop Circuits

Let us now consider some of the complicated circuits and see how they can be solved with the help of principles that we enunciated just now.

Wheatstone Bridge

As shown in Fig. 28-12, the equivalent circuit diagram of a Wheatstone bridge consists of three known resistances P, Q and R and the unknown resistance X, connected to form the four sides of a quadrilateral. A battery is connected across the junctions of P and R and the junctions of Q and X, while a galvanometer connected across the junctions P–Q and R–X completes the circuit.

In general, P and Q, which are said to constitute the ratio arms, may be varied within the simple limits $P = Q$, $P = 10Q$ or $P = 100Q$, respectively. The relation is reciprocal, that is, Q may be made equal to $10P$ or $100P$; the basic value when

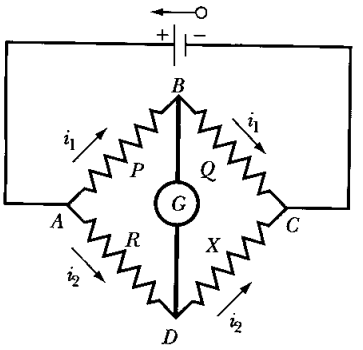


FIG. 28-12 Circuit diagram for multiloop circuit consisting of a network of resistors connected across a battery. The network of resistors includes a variable resistor R, the resistance of which can be adjusted. There is also a galvanometer connected in the middle of the network, which shows deflection if the value of the variable resistor is such that a nonzero current flows through the galvanometer. This type of circuit, called a Wheatstone bridge, is used to determine the value of the unknown resistor X by adjusting the variable resistor to such a value that no current flows through the galvanometer.

$P = Q$ is usually 10 Ω. The resistance R is variable over a wide range, usually from 1 Ω to several thousand ohms. Any sensitive galvanometer is suitable.

The action of the instrument depends on the adjustment of the value of R until there is no deflection in the galvanometer. This null condition is realized when no current flows in the galvanometer, that is, when the potential at B equals the potential of D. From Fig. 28-12 we have

$$V_{AB} = i_1 P, \quad V_{AD} = i_2 R.$$

Now the potential of A is common to both P and R, and if no current flows in the galvanometer, the potentials of B and D are equal. Hence,

$$V_{AB} = V_{AD};$$

and therefore,

$$i_1 P = i_2 R. \tag{28-26}$$

Similarly for the resistances Q and X,

$$V_{BC} = V_{DC}, \quad i_1 Q = i_2 X. \tag{28-27}$$

Hence, finally from Eqs. 28-26 and 28-27, we have

$$\frac{Q}{P} = \frac{X}{R} \tag{28-28}$$

from which X is easily determined if P, Q, and R are known.

Sample Problem 28-3

Figure 28-13 shows a circuit whose elements have the following values:

$$\mathscr{E}_1 = 3.0 \text{ V}, \quad \mathscr{E}_2 = 6.0 \text{ V},$$
$$R_1 = 2.0\ \Omega, \quad R_2 = 4.0\ \Omega.$$

The three batteries are ideal batteries. Find the magnitude and direction of the current in each of the three branches.

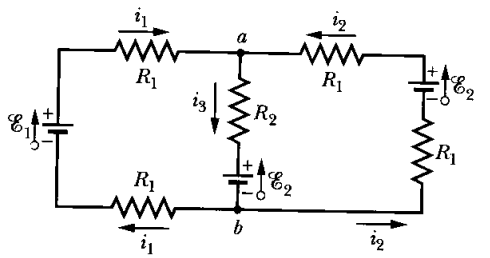


FIG. 28-13 A multiloop circuit with three ideal batteries and five resistances.

Conceptualize: It is not worthwhile to try to simplify this circuit, because no two resistors are in parallel, and the resistors that are in series (those in the right branch or those in the left branch) present no problem. So, our plan is to apply the junction and loop rules.

Classify: Junction rule: Using arbitrarily chosen directions for the currents as shown in Fig. 28-13, we apply the junction rule at point a by writing

$$i_3 = i_1 + i_2. \tag{28-29}$$

An application of the junction rule at junction b gives only the same equation, so we next apply the loop rule to any two of the three loops of the circuit.

Compute: Left-hand rule: We first arbitrarily choose the left-hand loop, arbitrarily start at point b, and arbitrarily traverse the loop in the clockwise direction, obtaining

$$-i_1 R_1 + \mathscr{E}_1 - i_1 R_1 - (i_1 + i_2) R_2 - \mathscr{E}_2 = 0,$$

where we have used $(i_1 + i_2)$ instead of i_3 in the middle branch. Substituting the given data and simplifying yield

$$i_1(8.0\ \Omega) + i_2(4.0\ \Omega) = -3.0 \text{ V}. \tag{28-30}$$

Right-hand loop: For our second application of the loop rule, we arbitrarily choose to traverse the right-hand loop counterclockwise from point b, finding

$$-i_2 R_1 + \mathscr{E}_2 - i_2 R_1 - (i_1 + i_2) R_2 - \mathscr{E}_2 = 0.$$

Substituting the given data and simplifying yield

$$i_1(4.0\ \Omega) + i_2(8.0\ \Omega) = 0. \qquad (28\text{-}31)$$

Combining equations: We now have a system of two equations (Eqs. 28-30 and 28-31) in two unknowns (i_1 and i_2) to solve either "by hand" (which is easy enough here) or with a "math package." (One solution technique is Cramer's rule, given in Appendix E.) We find

$$i_1 = -0.50\ \text{A}. \qquad (28\text{-}32)$$

(The minus sign signals that our arbitrary choice of direction for i_1 in Fig. 28-23 is wrong, but we must wait to correct it.) Substituting $i_1 = -0.50$ A into Eq. 28-31 and solving for i_2 then give us

$$i_2 = 0.25\ \text{A}. \qquad \text{(Answer)}$$

With Eq. 28-29 we then find that

$$i_3 = i_1 + i_2 = -0.50\ \text{A} + 0.25\ \text{A} = -0.25\ \text{A}.$$

The positive answer we obtained for i_2 signals that our choice of direction for that current is correct. However, the negative answers for i_1 and i_3 indicate that our choices for those currents are wrong. Thus, as a *last step* here, we correct the answers by reversing the arrows for i_1 and i_3 in Fig. 28-13 and then writing

$$i_1 = 0.50\ \text{A} \quad \text{and} \quad i_3 = 0.25\ \text{A}. \qquad \text{(Answer)}$$

Conclude: Caution: Always make any such correction as the last step and not before calculating *all* the currents.

Another useful method here is nodal analysis. A node is another name for a junction or a point in a circuit. For example, in Sample Problem 28-3, points *a* and *b* are the nodes. The first step in nodal analysis is to choose a reference node. Recall that in Chapter 24 we stated that any point in space can be taken to be at zero potential and all the other potentials measured with respect to it. Here we choose a reference node and assign it zero potential. So the node voltage is the potential difference between the node and the reference node. If we do not know the potential difference, we assign an arbitrary potential to the node. The current through all the resistances connected to the node can then be calculated and we can then use Kirchhoff's current law to evaluate the unknown potential.

Sample Problem 28-4

Let us illustrate this method by solving Sample Problem 28-3 by this method (Fig. 28-14).

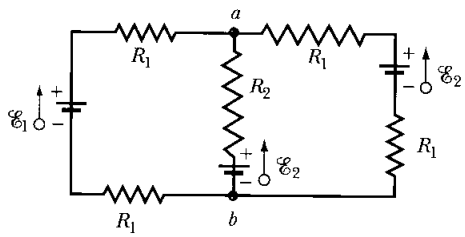


FIG. 28-14 Sample Problem 28-3 by nodal analysis.

Conceptualize: Here we note that the resistance R_1 at the lower left corner is like an internal resistance of the battery $\mathscr{E}_1$. There will be no change in the currents if we swap the battery and R_1 to simplify the circuit a little. Similarly, the circuit will not change if we do the same with the right-hand side of the circuit. This new figure is shown in Fig. 28-15.

Classify: In this new circuit, we will assign zero potential to point *b* and potential *V* to point *a*. We can see that there are two nodes in the circuit, and applying

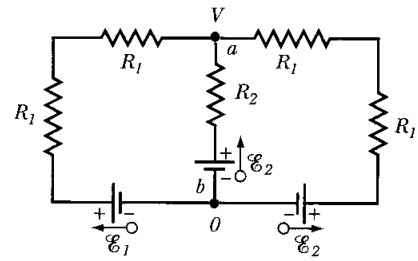


FIG. 28-15 Circuit 28-14 rearranged.

Kirchhoff's first law to the node *a* will yield a relation that can be solved to get the potential at point *a*. From that we can find the current through each branch.

Compute: The negative terminal of battery $\mathscr{E}_1$ is connected to *b*. So at the lower end of resistance R_2 the potential is 6 V. Similarly, we have marked out the potentials at different points of the circuit as shown in Fig. 28-16.

We do not know which point is at a higher potential, point *a* or any of the other three points connecting it to batteries. So we arbitrarily assume that *a* is at a higher potential so that current will be flowing away from node *a* through each of the three resistances.

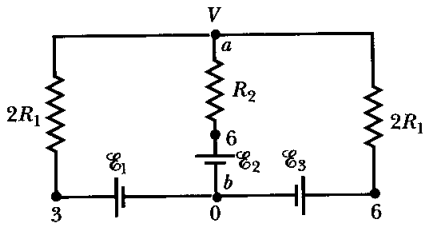


FIG. 28-16 Potential at the nodes.

Kirchhoff's junction rule yields $i_1 + i_2 + i_3 = 0$; therefore

$$\frac{V-3}{2R_1} + \frac{V-6}{R_2} + \frac{V-6}{2R_1} = 0.$$

Each of the terms represents the current in the left branch, middle branch and right branch, respectively. Solving this, we get

$$V = 5 \text{ volts.}$$

So the current in each of the branch is

$$i_1 = \frac{V-3}{2R_1} = 0.5 \text{ A},$$

$$i_2 = \frac{V-6}{R_2} = -0.25 \text{ A},$$

$$i_3 = \frac{V-6}{2R_1} = -0.25 \text{ A},$$

which is the same as we obtained in Sample Problem 28-3 using a lengthier method.

Conclude: Even if we had assumed point a to be at potential V_1 and point b to be at a potential V_2, the answer would be the same because in a circuit current flows through the resistance only on account of the potential difference. The negative current implies that we chose a wrong direction for the current in that branch. The current will flow in that branch toward node b, not away from it.

Grouping of Batteries

Series Connection

Suppose two batteries having emfs $\mathscr{E}_1$ and $\mathscr{E}_2$ and internal resistances r_1 and r_2 are connected in series as shown in Fig. 28-17. The points a and b act as the terminals of the combination. Suppose an external resistance R is connected across the combination. From Kirchhoff's loop law,

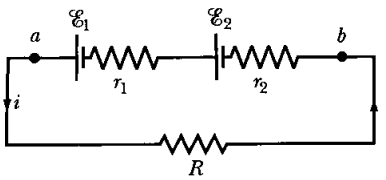


FIG. 28-17 Connecting two batteries in series.

$$Ri + r_2 i - \mathscr{E}_2 + r_1 i - \mathscr{E}_1 = 0,$$

or

$$i = \frac{\mathscr{E}_1 + \mathscr{E}_2}{R + (r_1 + r_2)} = \frac{\mathscr{E}}{R + r_0},$$

where i is the current through the resistance R.

We see that the combination acts as a battery of emf $\mathscr{E}_0 = \mathscr{E}_1 + \mathscr{E}_2$ having an internal resistance $r_0 = r_1 + r_2$. If the polarity of one of the batteries is reversed, the equivalent emf will be $|\mathscr{E}_1 - \mathscr{E}_2|$.

Parallel Connection

Now suppose the batteries are connected in parallel as shown in Fig. 28-18. The currents are also shown in the figure. Applying Kirchhoff's loop law in the loop containing $\mathscr{E}_1$, r_1, and R,

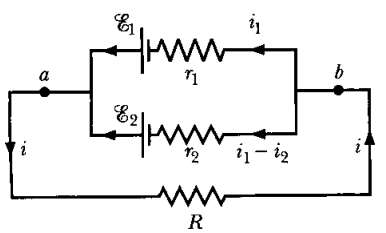


FIG. 28-18 Connecting two batteries in parallel.

$$iR + r_1 i_1 - \mathscr{E}_1 = 0. \tag{28-33}$$

Similarly, applying Kirchhoff's law in the loop containing $\mathscr{E}_2$, r_2, and R, we get

$$iR + r_2(i - i_1) - \mathscr{E}_2 = 0. \tag{28-34}$$

By multiplying Eq. 28-33 by r_2, Eq. 28-34 by r_1, and then adding, we get

$$iR(r_1 + r_2) + r_1 r_2 i - \mathscr{E}_1 r_2 - \mathscr{E}_2 r_1 = 0,$$

or $$i = \frac{\mathscr{E}_1 r_2 + \mathscr{E}_2 r_1}{R(r_1 + r_2) + r_1 r_2} = \frac{(\mathscr{E}_1 r_2 + \mathscr{E}_2 r_1)/(r_1 + r_2)}{R + (r_1 r_2)/(r_1 + r_2)} = \frac{\mathscr{E}_0}{R + r_0}. \qquad (28\text{-}35)$$

We see that the combination acts as a battery of emf

$$\mathscr{E}_0 = \frac{\mathscr{E}_1 r_2 + \mathscr{E}_2 r_1}{r_1 + r_2}$$

and internal resistance

$$r_0 = \frac{r_1 r_2}{r_1 + r_2}.$$

Sample Problem 28-5

Twelve wires, each having resistance r, are joined to form a cube as shown in Fig. 28-19. Find the equivalent resistance between the ends of a face diagonal such as a and c.

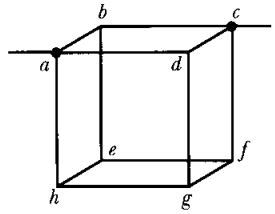


FIG. 28-19 Twelve equal resistors are arranged in the shape of a cube. The battery is connected between points a and c.

Conceptualize: This circuit is quite complicated and use of nodal analysis or loop law will yield a lot of equations which cannot be solved so easily. But the circuit is symmetric and symmetry considerations will help us in simplifying the circuit.

Classify: Suppose a potential difference V is applied between the points a and c so that a current i enters at a and the same current leaves at c. The current distribution is shown in Fig. 28-20*a*.

By symmetry, the paths ad and ab are equivalent and hence will carry the same current i_1. The path ah will carry the remaining current $i - 2i_1$ (using Kirchhoff's junction law). Similarly at junction c, currents coming from dc and bc will be i_1 each and from fc will be $i - 2i_1$. Kirchhoff's junction law at b and d shows that current through be and dg will be zero and hence may be ignored for further analysis. Omitting these two wires, the circuit is redrawn in Fig. 28-20*b*.

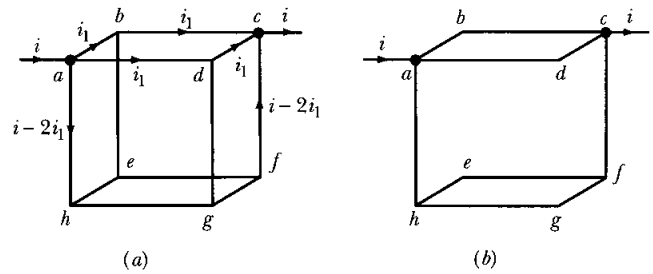


FIG. 28-20 (*a*) Current distribution in the cube and (*b*) equivalent circuit.

Compute: The wire hef and hgf are joined in parallel and have equivalent resistance $(2r)(2r)/(2r + 2r) = r$ between h and f. This equivalent is joined in series with ah and fc giving equivalent resistance $r + r + r = 3r$. This $3r$ is joined in parallel with adc $(2r)$ and abc $(2r)$ between a and c.

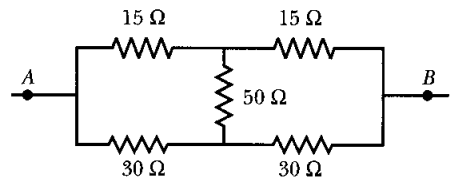
The equivalent resistance R between a and c is, therefore, given by

$$\frac{1}{R} = \frac{1}{3r} + \frac{1}{2r} + \frac{1}{2r},$$

or $$R = \frac{3}{4}r.$$

Conclude: In many of the situations, we can considerably simplify the circuit by guessing the branches in which the current is the same.

CHECKPOINT 5 In the adjoining circuit, find the branches in which the current is the same. Hence, redraw the circuit so that we can find the equivalent resistance of the circuit by simple series parallel analysis.



Sample Problem 28-6 Symmetry of a Different Kind

Find the equivalent resistance between points a and b of the circuit shown in Fig. 28-21.

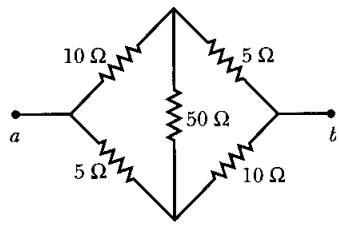


FIG. 28-21 A network of five resistances.

Conceptualize: Here we can imagine the circuit to be connected to a battery of 10 V (Chosen arbitrarily). Since resistances follow ohm's law, the current through the circuit is proportional to the applied voltage, the constant of proportionality will be $1/R_{eq}$.

Classify: Do not mistake this to be an example of Wheatstone bridge. Note here that

$$\frac{P}{Q} \neq \frac{R}{X}.$$

Instead we use nodal analysis to find the current through the source. Let's assume the right end of battery to be at zero potential. Other potentials are as indicated in Fig. 28-22.

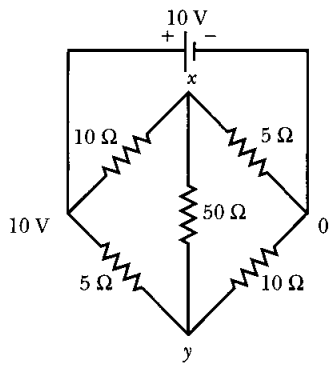


FIG. 28-22 Circuit for nodal analysis.

Compute: For applying Kirchhoff's junction rule at the node x, we assume that x is greater than 10 V as well as y.

$$\frac{x-10}{10} = i_{10\Omega},$$

$$\frac{x-y}{5} = i_{5\Omega},$$

$$\frac{x-0}{5} = i_{5\Omega}.$$

By summing up and equating to zero, we get

$$5x - 2y = 10. \tag{28-36}$$

Let us now apply the Kirchhoff's junction rule at the node y. We get

$$\frac{y-10}{5} = i_{5\Omega},$$

$$\frac{y-x}{5} = i_{5\Omega},$$

$$\frac{y-10}{10} = i_{10\Omega}.$$

By summing up and equating to zero, we get

$$5y - 2x = 20. \tag{28-37}$$

By multiplying Eq. 28-36 by 2 and Eq. 28-37 by 5 and then adding, we get

$$21y = 120 \Rightarrow y = \frac{40}{7}\text{ V}.$$

Substitute $x = (30/7)$ V. Thus, current through the battery is equal to

$$i_{10\Omega} + i_{5\Omega} = \frac{10-(30/7)}{10} + \frac{10-(40/7)}{5} = \frac{10}{7}\text{A}.$$

Thus,

$$R_{eq} = \frac{V}{i} = 7\,\Omega. \qquad \text{(Answer)}$$

Conclude: An interesting point that can be noted is that the potential difference across both the 10 Ω resistance is equal (10 – 30/7) and (40/7 – 0). This is no coincidence. Suppose we were to reverse the terminals of the battery. The current, instead of entering from a into left 10 Ω resistor and 5 Ω resistor would enter from right 10 Ω resistor and 5 Ω resistor. The magnitude of potential difference across each of the resistors would remain the same. Also the circuit has a certain symmetry: the resistors at the input are the same as resistors at the output. We can safely conclude that the current through these resistors would be the same. This kind of symmetry is known as "input–output symmetry." When such symmetry occurs, we need to write the potential at one node, say x and the potential at the other node would be $10 - x$. This indeed eases the problem at hand. So we can solve such a circuit using only one equation.

28-5 | Electricity-Related Instruments

Galvanometer shown in Fig. 28-23 is a device used for detecting currents. Normally, commercially available galvanometers are very sensitive instruments and can detect very small currents. As it is used to detect currents, it can detect currents flowing in either direction. When connected suitably to a high resistance or a low resistance, it can act as a voltmeter or an ammeter.

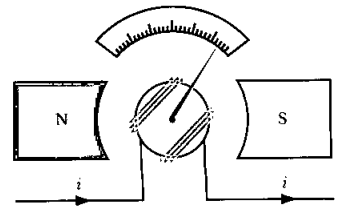


FIG. 28-23 A typical galvanometer.

Ammeter and Voltmeter

Ammeter is a device to measure an electric current and **voltmeter** is a device to measure a potential difference. In both the instruments there is a coil, suspended between the poles of a magnet as shown in Fig. 28-23. When a current is passed through the coil, it deflects. The angle of deflection is proportional to the current going through the coil. A needle is fixed to the coil. When the coil deflects, the needle moves on a graduated scale.

Ammeter

In an ammeter, a resistor having a small resistance is connected in parallel with the coil. This resistor is called the *shunt*. The current to be measured is passed through the ammeter by connecting it in series with the segment which carries the current. Plus and minus signs are marked near the terminals of the ammeter. The current should enter the ammeter through the terminal marked "plus." When no current passes through the ammeter, the needle stays at zero which is marked at the left end of the scale. The equivalent circuit is shown in Fig. 28-24.

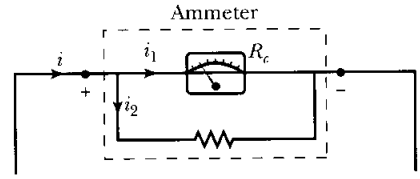


FIG. 28-24 The equivalent circuit of a ammeter. The dotted lines indicate the ammeter.

Suppose the coil has a resistance R_c and the small resistance connected in parallel (shunt) has a value r. When a current i is sent through the ammeter, the current gets divided into two parts. A part i_1 goes through the coil and the rest, $i - i_1$, through the shunt. As the potential difference across R_c is the same as that across r,

$$i_1 R_c = (i - i_1) r,$$

or

$$i_1 = \frac{r}{R_c + r} i. \tag{28-38}$$

The deflection is proportional to i_1 and hence to i. The scale is graduated to read the value of i directly.

The equivalent resistance of an ammeter is given by

$$R_{eq} = \frac{R_c r}{R_c + r}.$$

When the ammeter is connected in a segment of a circuit, the resistance of the segment increases by this amount R_{eq}. This reduces the main current which we wish to measure. To minimize this error, the equivalent resistant R_{eq} should be small. This is one reason why the shunt having a small resistance r is connected in parallel to the coil. This reduces the value of R_{eq}.

Voltmeter

In a voltmeter, a resistor having a high resistance R is connected in series with the coil. The end points (terminals) are connected to the points A and B between which the potential difference is to be measured. Plus and minus signs are marked

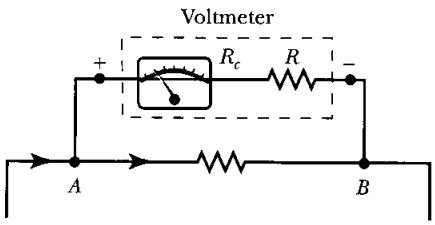


FIG. 28-25 The equivalent circuit of a voltmeter. The dotted lines indicate the voltmeter.

on the terminals. The terminal marked "plus" should be connected to the point at higher potential. When no potential difference is applied between the terminals, the needle stays at zero which is marked at the left extreme of the scale. When the potential difference is applied, a current passes through the coil and the high resistance. Its equivalent circuit is shown in Fig. 28-25. If R_c be the resistance of the coil and V be the potential difference applied to the voltmeter, the current in the coil is

$$i = \frac{V}{R_c + R}.$$

The deflection is proportional to the current i and hence to V. The scale is graduated to read to potential difference directly.

When the voltmeter is used in a circuit, its resistance $R_{eq} = R_c + R$ is connected in parallel to the element of the circuit across which we want to measure the potential difference. This changes the overall current in the circuit and hence, the potential difference to be measured is also changed. To minimize the error due to this, the equivalent resistance R_{eq} of the voltmeter should be large. (When a large resistance is connected in parallel to a small resistance, the equivalent resistance is only slightly less than the smaller one.) That is why, a large resistance R is added in series with the coil of a voltmeter.

CHECKPOINT 6 A galvanometer has a coil of resistance 100 Ω showing a full-scale deflection at 50 μA. What resistance should be added to use it as (a) a voltmeter of range 50 V and (b) an ammeter of range 10 mA?

Often a single meter is packaged so that, by means of a switch, it can be made to serve as either an ammeter or a voltmeter—and usually also as an *ohmmeter,* designed to measure the resistance of any element connected between its terminals. Such a versatile unit is called a *multimeter.*

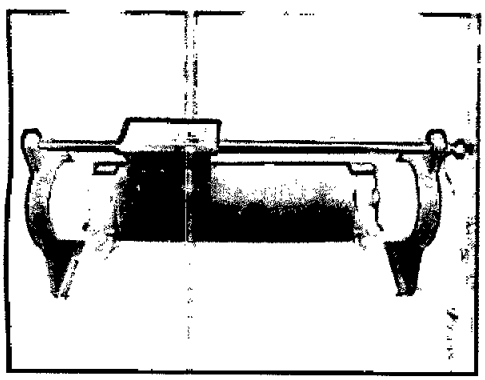


FIG. 28-26 A rheostat.

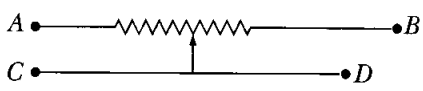


FIG. 28-27 Rheostat.

Current-Limiting Rheostats

In many cases it is necessary to vary and control a current. The nature of the variation required determines the type of resistance to choose for the purpose. Resistances designed for this purpose are called rheostats.

In general, we can divide rheostats into two classes: slide-wire types as shown in Fig. 28-26 and carbon compression types. The first kind is used when the range of variation is to be large and, for practical purposes, it is satisfactory if the resistance change occurs in very small jumps. Slide-wire rheostats are used nowadays almost exclusively. When large amounts of current have to be controlled and finer continuous adjustment is necessary, a carbon rheostat is used. The resistance change can be made practically infinitesimal by changing the pressure on the blocks slightly.

Most of the variable rheostats have connections as shown in Fig. 28-27. When the instrument is connected into a circuit at A and C, we have a variable rheostat. In this connection, the rheostat is acting as a current controller.

Another use of rheostat is in acting as a potential divider. To demonstrate the use of rheostat as a potential divider, we are showing here an experiment which is used to calibrate a voltmeter to read from 0 to 1.5 V.

Figure 28-28 shows the connections. Above the dotted line we have our homemade voltmeter, consisting of galvanometer and a variable high resistance R'. We shall calibrate this voltmeter by comparison with a standard voltmeter V. The 150 Ω rheostat serves as a potential divider. By varying the position of the slider, the

potential difference applied to the two voltmeters can be varied from 0 to 2 V. Adjust the Rheostat so that the standard voltmeter reads 1.5 V. Adjust R' so that G reads full scale. Now reduce the current in the main circuit and take readings on both instruments for about 10 points on the scale. In taking these readings set the standard voltmeter as exactly as possible on a division line and read the galvanometer (estimating to tenths of a division). Calibrate for various points on the scale, tabulating your results.

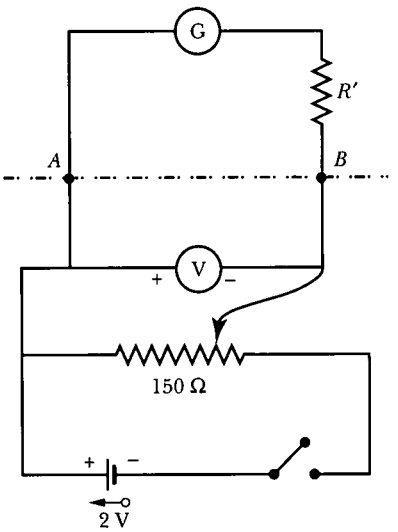


FIG. 28-28 Calibration of a voltmeter.

Verification of Ohm's law

Perhaps the simplest way of measuring the resistance of a wire is to connect it in series with an ammeter and a cell and place a voltmeter across the resistance as shown in Fig. 28-29. Then application of Ohm's law gives $R = \mathscr{E}/i$, where $\mathscr{E}$ is the reading on the voltmeter and i is the reading on the ammeter. The result can only be approximately correct, because when the voltmeter is placed in parallel with the resistance alone as in Fig. 28-29, the current i registered by the ammeter is only partially going through the resistance, and partly going through the voltmeter itself. Strictly Ohm's law would require $R = \mathscr{E}/i_R$, where i_R is the current in the resistance. If, however, i_R is determined by placing the voltmeter across the ammeter and resistance together as in Fig. 28-30, the voltage $\mathscr{E}$ is now no longer that across the resistance alone but across ammeter and resistance together. However, the lower the resistance of the ammeter and the higher the resistance of the voltmeter, the more accurate is the result, and if an ideal voltmeter is used, an accurate value for the resistance may be obtained, as the ideal voltmeter takes no current.

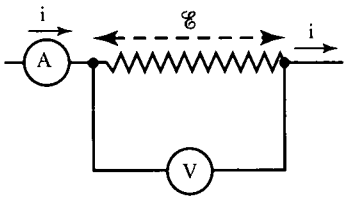


FIG. 28-29 Approximate measurement of resistance.

Use of Wheatstone Bridge to Find the Unknown Resistance

Very accurate values of resistance may be obtained with a special arrangement of resistances termed as the Wheatstone bridge that we discussed earlier in this chapter.

In using the bridge, the resistance R is changed until ideally the galvanometer shows no deflection. In practice, for values of R slightly greater than a certain amount, the galvanometer will generally show a small deflection in one direction, whereas for values of R slightly less than the same amount, the galvanometer will be deflected in the reverse direction. As we said in Section 28-4 (Eq. 28-28),

$$\frac{Q}{P} = \frac{X}{R}.$$

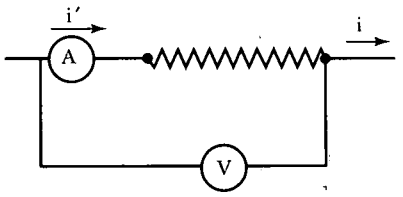


FIG. 28-30 Approximate measurement of resistance.

Post Office Box and Meter Bridge

The Wheatstone bridge is available in many different forms, two of which are **the post office box** and **the meter bridge**. The former, which is shown diagrammatically in Fig. 28-31, is fitted with two tapping keys which are situated in the battery and galvanometer circuits, respectively. Incidentally, the bridge is symmetrical in that the battery and galvanometer may be interchanged without affecting the balance point. Owing to the existence of electromagnetic induction (see Chapter 31), the battery circuit should be closed before the galvanometer circuit, so that steady current conditions are already established in the circuit.

The resistances in the box are usually non-inductively wound so as to reduce induction effects to a minimum. In any case, it is the final steady deflection of the galvanometer which is observed and not any temporary kick which may occur when the keys are depressed (see Chapter 31). Since resistance varies with temperature, only small currents are permissible and boxes are usually rated at

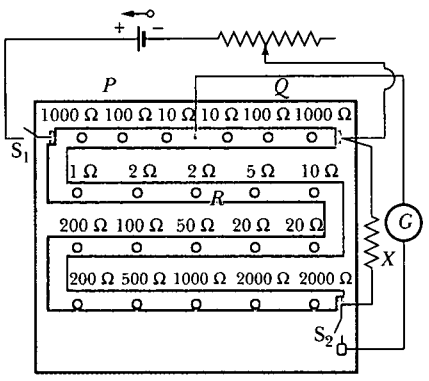


FIG. 28-31 Post office box.

a definite maximum current which can be carried with safety. This should on no account be exceeded or the resistances may be permanently damaged and acquire values differing from their normal amounts. Figure 28-31 is same as that represented by the circuit diagram in Fig. 28-12.

If the ratio arms are equal, so that $P = Q$, from Eq. 28-28, $R = X$ for a balance to be obtained, so X will usually lie somewhere between two integral values of R differing by 1 Ω.

When an approximate value for X has been obtained as above, the ratio arms are changed so that $P = 10Q$. The value of R, which now gives a balance, from Eq. 28-28 is equal to $10X$. Hence, the accuracy of determination of X is carried to one-tenth of an ohm. By further increasing the ratio to $P = 100Q$, the value of $R = 100X$, and X is determined to one-hundredth of an ohm. Of course, the value of X must be such that the resistances available for R in the given instrument are sufficient to cover the range $R = 100X$, or alternatively $R = X/100$ if the ratio arms are reversed. Theoretically, by having greater ratio arms the accuracy could be extended indefinitely. In practice, the introduction of very large resistances reduces the current in the circuit to such an extent that the galvanometer sensitivity becomes insufficient to respond to small changes in R. So a natural limit is set to the accuracy obtainable. In general, R can be varied by steps of 1 Ω or larger so that the value of X can be found only within certain accuracy.

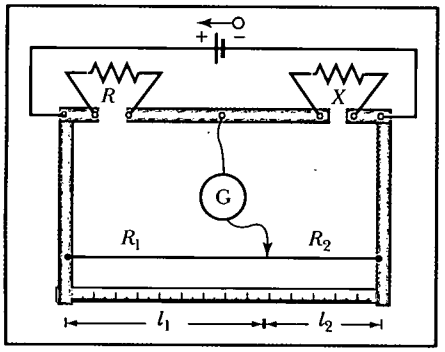


FIG. 28-32 Meter bridge. Here the switch in the galvanometer branch, the switch in series with the battery and the rheostat are omitted for simplicity.

The **meter bridge** or **slide-wire bridge** was devised to allow a continuous change in the value of the resistances instead of only integral values such as the post office box affords. As shown in Fig. 28-32, it consists of a rectangular board to which a meter rule is attached, stretched beside which is a wire of about 1 m long. The wire is made of some resistance material, such as Eureka alloy, so that the whole wire has a resistance of about 1 Ω. The ends of the wire are soldered to terminals which are in turn connected to stout copper strips of negligible resistance. These strips are provided with two gaps in which a known resistance R and the unknown resistance X are inserted, respectively. A third copper strip joins R and X, and a third terminal is attached to the center of this strip. This terminal is joined by a wire to a galvanometer, the other lead of which goes to a metal slider or jockey which slides along the bare wire of the meter bridge. The battery is joined to the extremities of the bridge wire, and the whole constitutes the usual Wheatstone bridge. A balance is obtained by sliding the jockey until no deflection is shown in the galvanometer. If the two resistances into which the bridge wire is then divided are R_1 and R_2, we have by Eq. 28-28

$$\frac{R_1}{R_2} = \frac{R}{X}.$$

The wire is assumed to be of uniform cross section; hence, the resistance of any portion of it is proportional to the length. Therefore $R_1 \propto l_1$ and $R_2 \propto l_2$. So it follows from the above that

$$X = R\frac{l_2}{l_1}.$$

If λ is the resistance of the wire of length 1 m, we can write by Eq. 28-28

$$\frac{X}{R} = \frac{\lambda(1-l)}{\lambda l}.$$

The value of distance "l" at which the current in the galvanometer is zero is known as null point. The error in the value of "X" will be minimum when the null point is obtained at the center of meter bridge. To show this, we assume that the resistance of the left and right copper strips is each equal to 'α'. If the null point is obtained at the center, then

$$\frac{X+\alpha}{R+\alpha} = \frac{1-0.5}{0.5} = 1.$$

Then the effect of "α" gets canceled. But if null point is not at the center, there will be some error because X will then not be equal to R. Another reason for this is related to the sensitivity of the instrument. Let us apply Eq. 28-28 to analyze for the indeterminate error in the unknown resistance X:

$$X = R\frac{(1-l)}{l}$$

$$\frac{dX}{X} = \frac{dR}{R} - \frac{dl}{1-l} - \frac{dl}{l} = \frac{dR}{R} - \frac{dl}{(1-l)l}.$$

The error in R and dR is zero because R is known accurately. The error in X and dX is minimum when denominator is maximum.

$$\frac{dX}{X} = -\frac{dl}{(1-l)l}.$$

We know that this is minimum when $l = 0.5$ m.

Potentiometer

In Section 28-3, we established that a real battery always has some internal resistance. So if we want to measure the emf of a real battery, we cannot use a voltmeter. The ordinary moving-coil voltmeter is fairly satisfactory if its resistance is high, but clearly it will cause great disturbance if it is introduced across a circuit in which resistances of several thousand ohms are present. The potentiometer is a device which allows the measurement of potential without taking any current from the system. So it can be used to measure the emf of a real battery also.

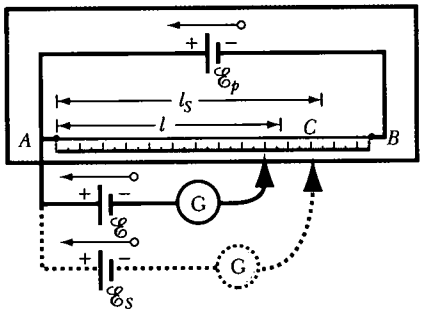


FIG. 28-33 Potentiometer.

In construction, the potentiometer is identical with the meter bridge, which is often used as a potentiometer, except that the resistances R and X are omitted, these spaces being left unoccupied. The instrument therefore consists, in its simplest form, of a wire stretched beside a meter scale and connected to a steady supply battery ($\mathscr{E}_p$) as shown in Fig. 28-33. The potentiometer will only compare voltages (potential differences, emf), that is, a standard voltage is first necessary to calibrate the potentiometer wire. Such a standard voltage is provided by the emf of a standard cell. We denote this by $\mathscr{E}_s$. The positive pole of the supply battery ($\mathscr{E}_p$) is connected to that end of the potentiometer wire to which the positive pole of the standard cell is attached. The negative pole of the standard cell is connected through a sensitive galvanometer to a jockey sliding on the potentiometer wire. Now at certain positions of the jockey, it will be found that the galvanometer shows no deflection. This means that the drop in potential between the common point A and the point of contact C of the jockey, due to the current supplied from the supply battery ($\mathscr{E}_p$) to the bridge wire, is exactly equal to the potential $\mathscr{E}_s$ between the poles of the standard cell. Remember, by definition the emf of a battery is the potential difference between its poles when no current is flowing through it.

To determine any other potential, therefore, the source to be measured simply replaces the standard cell and the new balance point l is obtained. Then from Eq. 28-39, its emf $\mathscr{E}$ can be measured. Since the wire is uniform, the drop $\mathscr{E}$ in potential across any length l is known by simple ratio,

$$\frac{\mathscr{E}}{\mathscr{E}_s} = \frac{l}{l_s}, \qquad (28\text{-}39)$$

where l_s is the distance AC.

The voltage to be measured must not exceed that of the supply battery ($\mathscr{E}_p$); otherwise no balance point will be found anywhere along the wire. Commercial potentiometers are available for measuring potentials over a very wide range.

28-6 | *RC* Circuits

In preceding sections we dealt only with circuits in which the currents did not vary with time. Here we begin a discussion of time-varying currents.

Charging a Capacitor

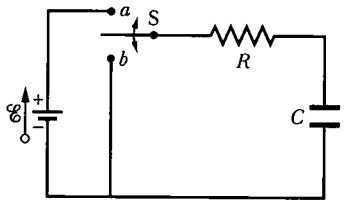


FIG. 28-34 When switch S is closed on *a*, the capacitor is *charged* through the resistor. When the switch is afterward closed on *b*, the capacitor *discharges* through the resistor.

The capacitor of capacitance C in Fig. 28-34 is initially uncharged. To charge it, we close switch S on point *a*. This completes an *RC series circuit* consisting of the capacitor, an ideal battery of emf $\mathscr{E}$, and a resistance R.

From Section 26-2, we already know that as soon as the circuit is complete, charge begins to flow (current exists) between a capacitor plate and a battery terminal on each side of the capacitor. This current increases the charge q on the plates and the potential difference V_C ($= q/C$) across the capacitor. When that potential difference equals the potential difference across the battery (which here is equal to the emf $\mathscr{E}$), the current is zero. From Eq. 26-1 ($q = CV$), the *equilibrium* (final) *charge* on the then fully charged capacitor is equal to $C\mathscr{E}$.

Here we want to examine the charging process. In particular we want to know how the charge $q(t)$ on the capacitor plates, the potential difference $V_C(t)$ across the capacitor, and the current $i(t)$ in the circuit vary with time during the charging process. We begin by applying the loop rule to the circuit, traversing it clockwise from the negative terminal of the battery. We find

$$\mathscr{E} - iR - \frac{q}{C} = 0. \tag{28-40}$$

The last term on the left side represents the potential difference across the capacitor. The term is negative because the capacitor's top plate, which is connected to the battery's positive terminal, is at a higher potential than the lower plate. Thus, there is a drop in potential as we move down through the capacitor.

We cannot immediately solve Eq. 28-40 because it contains two variables, i and q. However, those variables are not independent but are related by

$$i = \frac{dq}{dt}. \tag{28-41}$$

Substituting this for i in Eq. 28-40 and rearranging, we find

$$R\frac{dq}{dt} + \frac{q}{C} = \mathscr{E} \qquad \text{(charging equation)}. \tag{28-42}$$

This differential equation describes the time variation of the charge q on the capacitor in Fig. 28-34. To solve it, we need to find the function $q(t)$ that satisfies this equation and also satisfies the condition that the capacitor be initially uncharged; that is, $q = 0$ at $t = 0$.

We shall soon show that the solution to Eq. 28-42 is

$$q = C\mathscr{E}(1 - e^{-t/RC}) \qquad \text{(charging a capacitor)}. \tag{28-43}$$

(Here e is the exponential base, 2.718 . . . , and not the elementary charge.) Note that Eq. 28-43 does indeed satisfy our required initial condition, because at $t = 0$ the term $e^{-t/RC}$ is unity; so the equation gives $q = 0$. Note also that as t goes to infinity (that is, a long time later), the term $e^{-t/RC}$ goes to zero; so the equation gives the proper value for the full (equilibrium) charge on the capacitor—namely, $q = C\mathscr{E}$. A plot of $q(t)$ for the charging process is given in Fig. 28-35*a*.

The derivative of $q(t)$ is the current $i(t)$ charging the capacitor:

$$i = \frac{dq}{dt} = \left(\frac{\mathscr{E}}{R}\right)e^{-t/RC} \qquad \text{(charging a capacitor)}. \tag{28-44}$$

A plot of $i(t)$ for the charging process is given in Fig. 28-35*b*. Note that the current has the initial value $\mathscr{E}/R$ and that it decreases to zero as the capacitor becomes fully charged.

A capacitor that is being charged initially acts like ordinary connecting wire relative to the charging current. A long time later, it acts like a broken wire.

By combining Eq. 26-1 ($q = CV$) and Eq. 28-43, we find that the potential difference $V_C(t)$ across the capacitor during the charging process is

$$V_C = \frac{q}{C} = \mathscr{E}\,(1 - e^{-t/RC}) \qquad \text{(charging a capacitor).} \tag{28-45}$$

This tells us that $V_C = 0$ at $t = 0$ and that $V_C = \mathscr{E}$ when the capacitor becomes fully charged as $t \to \infty$.

The Time Constant

The product RC that appears in Eqs. 28-43, 28-44, and 28-45 has the dimensions of time (both because the argument of an exponential must be dimensionless and because, in fact, 1.0 Ω × 1.0 F = 1.0 s). The product RC is called the **capacitive time constant** of the circuit and is represented with the symbol τ:

$$\tau = RC \qquad \text{(time constant).} \tag{28-46}$$

From Eq. 28-43, we can now see that at time $t = \tau\,(= RC)$, the charge on the initially uncharged capacitor of Fig. 28-34 has increased from zero to

$$q = C\mathscr{E}(1 - e^{-1}) = 0.63C\mathscr{E}. \tag{28-47}$$

In words, during the first time constant τ the charge has increased from zero to 63% of its final value $C\mathscr{E}$. In Fig. 28-35, the small triangles along the time axes mark successive intervals of one time constant during the charging of the capacitor. The charging times for RC circuits are often stated in terms of τ; the greater τ is, the greater is the charging time.

Discharging a Capacitor

Assume now that the capacitor of Fig. 28-34 is fully charged to a potential V_0 equal to the emf $\mathscr{E}$ of the battery. At a new time $t = 0$, switch S is thrown from *a* to *b* so that the capacitor can *discharge* through resistance R. How do the charge $q(t)$ on the capacitor and the current $i(t)$ through the discharge loop of capacitor and resistance now vary with time?

The differential equation describing $q(t)$ is like Eq. 28-42 except that now, with no battery in the discharge loop, $\mathscr{E} = 0$. Thus,

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \text{(discharging equation).} \tag{28-48}$$

The solution to this differential equation is

$$q = q_0 e^{-t/RC} \qquad \text{(discharging a capacitor),} \tag{28-49}$$

where $q_0\,(= CV_0)$ is the initial charge on the capacitor. You can verify by substitution that Eq. 28-49 is indeed a solution of Eq. 28-48.

Equation 28-49 tells us that q decreases exponentially with time, at a rate that is set by the capacitive time constant $\tau = RC$. At time $t = \tau$, the capacitor's charge has been reduced to $q_0 e^{-1}$, or about 37% of the initial value. Note that a greater τ means a greater discharge time.

Differentiating Eq. 28-49 gives us the current $i(t)$:

$$i = \frac{dq}{dt} = -\left(\frac{q_0}{RC}\right)e^{-t/RC} \qquad \text{(discharging a capacitor).} \tag{28-50}$$

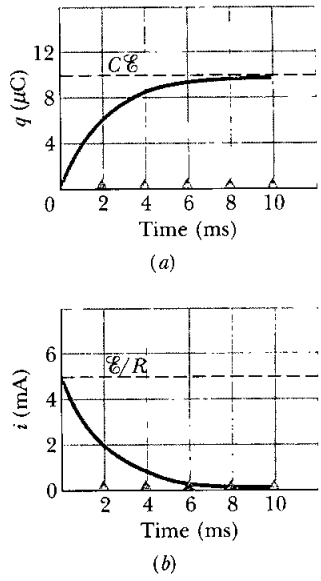


FIG. 28-35 (*a*) A plot of Eq. 28-43, which shows the buildup of charge on the capacitor of Fig. 28-34. (*b*) A plot of Eq. 28-44, which shows the decline of the charging current in the circuit of Fig. 28-34 The curves are plotted for $R = 2000\ \Omega$, $C = 1\ \mu\text{F}$, and $\mathscr{E} = 10$ V; the small triangles represent successive intervals of one time constant τ.

This tells us that the current also decreases exponentially with time, at a rate set by τ. The initial current i_0 is equal to q_0/RC. Note that you can find i_0 by simply applying the loop rule to the circuit at $t = 0$; just then the capacitor's initial potential V_0 is connected across the resistance R, so the current must be $i_0 = V_0/R = (q_0/C)/R = q_0/RC$. The minus sign in Eq. 28-50 can be ignored; it merely means that the capacitor's charge q is decreasing.

Derivation of Eq. 28-43

To solve Eq. 28-42, we first rewrite it as

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\mathscr{E}}{R}. \tag{28-51}$$

The general solution to this differential equation is of the form

$$q = q_p + Ke^{-at}, \tag{28-52}$$

where q_p is a *particular solution* of the differential equation, K is a constant to be evaluated from the initial conditions, and $a = 1/RC$ is the coefficient of q in Eq. 28-51. To find q_p, we set $dq/dt = 0$ in Eq. 28-51 (corresponding to the final condition of no further charging), let $q = q_p$, and solve, obtaining

$$q_p = C\mathscr{E}. \tag{28-53}$$

To evaluate K, we first substitute this into Eq. 28-52 to get

$$q = C\mathscr{E} + Ke^{-at}.$$

Then substituting the initial conditions $q = 0$ and $t = 0$ yields

$$0 = C\mathscr{E} + K,$$

or $K = -C\mathscr{E}$. Finally, with the values of q_p, a, and K inserted, Eq. 28-52 becomes

$$q = C\mathscr{E} - C\mathscr{E}e^{-t/RC},$$

which, with a slight modification, is Eq. 28-43.

CHECKPOINT 7 The table gives four sets of values for the circuit elements in Fig. 28-34. Rank the sets according to (a) the initial current (as the switch is closed on a) and (b) the time required for the current to decrease to half its initial value, greatest first.

	1	2	3	4
$\mathscr{E}$ (V)	12	12	10	10
R (Ω)	2	3	10	5
C (μF)	3	2	0.5	2

Sample Problem 28-7

As a car rolls along pavement, electrons move from the pavement first onto the tires and then onto the car body. The car stores this excess charge and the associated electric potential energy as if the car body were one plate of a capacitor and the pavement were the other plate (Fig. 28-36*a*). When the car stops, it discharges its excess charge and energy through the tires, just as a capacitor can discharge through a resistor. If a conducting object comes within a few centimeters of the car before the car is discharged, the remaining energy can be suddenly transferred to a spark between the car and the object. Suppose the conducting object is a fuel dispenser. The spark will not ignite the fuel and cause a fire if the spark energy is less than the critical value $U_{fire} = 50$ mJ.

When the car of Fig. 28-36*a* stops at time $t = 0$, the car–ground potential difference is $V_0 = 30$ kV. The car–ground capacitance is $C = 500$ pF, and the resistance of *each* tire is $R_{tire} = 100$ GΩ. How much time does the car take to discharge through the tires to drop below the critical value U_{fire}?

Conceptualize/Classify: (1) At any time t, a capacitor's stored electric potential energy U is related to its stored charge q according to Eq. 26-21 ($U = q^2/2C$). (2) While a capacitor is discharging, the charge decreases with time according to Eq. 28-49 ($q = q_0e^{-t/RC}$).

Compute: We can treat the tires as resistors that are connected to one another at their tops via the car body and at their bottoms via the pavement. Figure 28-36*b* shows how the four resistors are connected in parallel across the car's capacitance, and Fig. 28-36*c* shows their equivalent

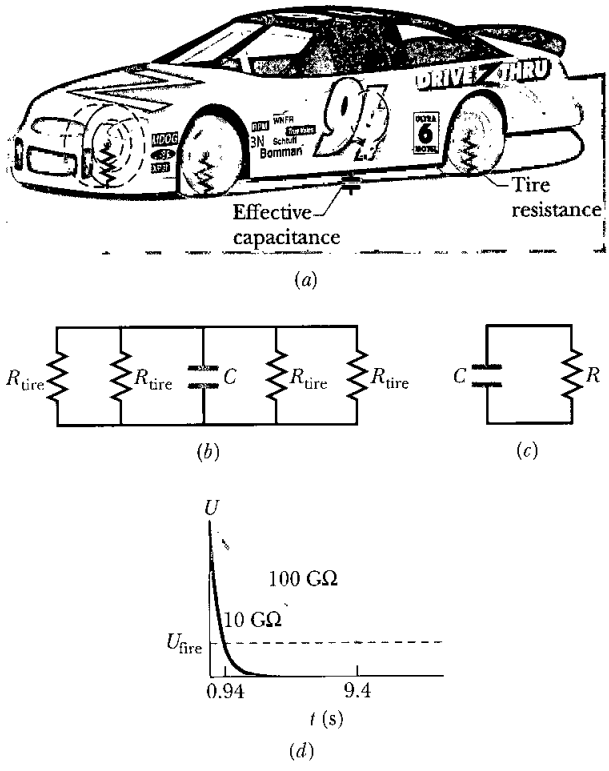


FIG. 28-36 (*a*) A charged car and the pavement acts like a capacitor that can discharge through the tires. (*b*) The effective circuit of the car–pavement capacitor, with four tire resistances R_{tire} connected in parallel. (*c*) The equivalent resistance R of the tires. (*d*) The electric potential energy U in the car–pavement capacitor decreases during discharge.

resistance R. From Eq. 28-24, R is given by

$$\frac{1}{R} = \frac{1}{R_{tire}} + \frac{1}{R_{tire}} + \frac{1}{R_{tire}} + \frac{1}{R_{tire}},$$

$$\text{or} \qquad R = \frac{R_{tire}}{4} = \frac{100\times10^9\ \Omega}{4} = 25\times10^9\ \Omega. \qquad (28\text{-}54)$$

When the car stops, it discharges its excess charge and energy through R.

We now use our two Key Ideas to analyze the discharge. Substituting Eq. 28-49 into Eq. 26-21 gives

$$U = \frac{q^2}{2C} = \frac{(q_0e^{-t/RC})^2}{2C} = \frac{q_0^2}{2C}e^{-2t/RC}. \qquad (28\text{-}55)$$

From Eq. 26-1 ($q = CV$), we can relate the initial charge q_0 on the car to the given initial potential difference V_0: $q_0 = CV_0$. Substituting this equation into Eq. 28-55 brings us to

$$U = \frac{(CV_0)^2}{2C}e^{-2t/RC} = \frac{CV_0^2}{2}e^{-2t/RC}$$

$$\text{or} \qquad e^{-2t/RC} = \frac{2U}{CV_0^2}. \qquad (28\text{-}56)$$

Taking the natural logarithms of both sides, we obtain

$$-\frac{2t}{RC} = \ln\left(\frac{2U}{CV_0^2}\right),$$

$$\text{or} \qquad t = \frac{RC}{2}\ln\left(\frac{2U}{CV_0^2}\right). \qquad (28\text{-}57)$$

Substituting the given data, we find that the time the car takes to discharge to the energy level $U_{fire} = 50$ mJ is

$$t = -\frac{(25\times10^9\ \Omega)(500\times10^{-12}\ \text{F})}{2} \times \ln\left(\frac{2(50\times10^{-3}\ \text{J})}{(500\times10^{-12}\ \text{F})(30\times10^3\ \text{V})^2}\right)$$

$$= 9.4\ \text{s}. \qquad \text{(Answer)}$$

Conclude: Fire or no fire: This car requires at least 9.4 s before fuel or a fuel dispenser can be brought safely near it. During a race, a pit crew cannot wait that long. Instead, tires for race cars include some type of conducting material (such as carbon black) to lower the tire resistance and thus increase the car's discharge rate. Figure 28-36*d* shows the stored energy U versus time t for tire resistances of $R = 100$ GΩ (the value we used in our calculations here) and $R = 10$ GΩ. Note how much more rapidly a car discharges to level U_{fire} with the lower R value.

Sample Problem 28-8

(a) Find the current i supplied by the battery in the network as shown in Fig. 28-37 in steady state. (b) Find the steady state charge on the capacitor. (c) Find the initial current through the battery.

Conceptualize: Here we do not have a simple RC circuit as seen in Section 28-6. So we cannot apply Eqs. 28-43 and 28-44 here.

Classify: We have studied in Section 28-6 that an uncharged capacitor will initially behave like short circuit and after a long time behave like a broken wire.

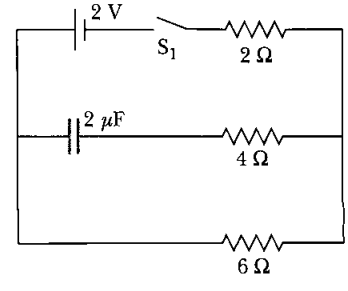


FIG. 28-37 A complicated RC circuit.

Compute: (a) Once the capacitor is charged, no current will go through it and the current through the middle branch of the circuit is zero in steady state. The 4 Ω resistor will have no current in it and may be omitted for current analysis. The 2 Ω and 6 Ω resistors are therefore connected in series and hence

$$i = \frac{2\text{ V}}{2\ \Omega + 6\ \Omega} = 0.25\text{ A}.$$

(b) The potential drop across the 6 Ω resistor is 6 Ω × 0.25 A = 1.5 V. As there is no current in the 4 Ω resistor, there is no potential drop across it. The potential difference across the capacitor is, therefore, 1.5 V. The charge on this capacitor is

$$Q = CV = 2\ \mu\text{F} \times 1.5\ \text{V} = 3\ \mu\text{C}.$$

(c) At the initial moment, an uncharged capacitor can be assumed to be short circuited. So the equivalent resistance of the circuit will be 4.4 Ω. The current through the battery will be

$$\frac{2}{4.4}\text{ A} = 0.45\text{ A}.$$

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: The voltage across an ideal battery is equal to its emf

Statement 2: An ideal battery has no internal resistance

2. In circuit shown in Fig. 28-38, identical bulbs are connected.

Statement 1: Bulb C glows brighter than bulb A.

Statement 2: The current through bulb C is more than the current through bulb A.

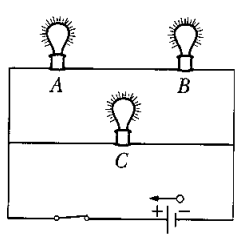


FIG. 28-38 Reasoning type question 2.

3. In circuit shown in Fig. 28-39, the battery is non-ideal.

Statement 1: The brightness of the bulb A remains the same after the switch is closed.

Statement 2: The voltage across elements in parallel is the same.

4. Statement 1: A voltmeter should draw very little current compared with that in the rest of the circuit.

Statement 2: A voltmeter should have a high resistance compared with the resistances of the other elements in the circuit.

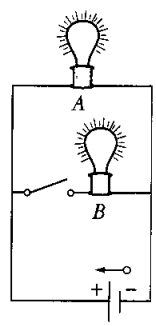


FIG. 28-39 Reasoning type question 3.

5. Statement 1: To measure the current in a circuit element, the ammeter should be connected in series with that element.

Statement 2: Connecting the ammeter in series with a circuit element causes at least a small reduction of the current in that element.

MULTIPLE CHOICE QUESTIONS

sec. 28-2 Work, Energy, and Emf

1. Somewhere in a circuit is a resistor. A constant current is flowing in the direction as indicated in Fig. 28-40. In going from A to B, we measure $\int_A^B \vec{E}\cdot d\vec{l}$. What do we find?

(a) A positive value

(b) A negative value

(c) Zero

(d) We do not have enough information to answer

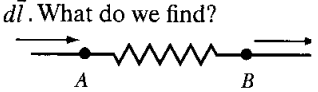


FIG. 28-40 Multiple choice question 1.

2. Does it make any difference whether we go from A to B through the wire to the resistor, or choose a random routing through free space (thus not through the resistor itself), starting at A and ending at B?

(a) No difference

(b) Yes, there is a difference

3. Ten coulombs of charge start from the negative terminal of a battery, flow through the battery and then leave the positive terminal through a wire, flow through a resistor and then return to the starting point on this closed circuit. In this complete process, the 10 C

(a) Have no net change in potential energy

(b) Receive heat energy from the resistor

(c) Have a net loss of potential energy

(d) Do positive work on the battery

(e) Are dissipated in the resistor, causing fewer than 10 C to return to the battery

sec. 28-3 Calculating the Current in a Single-Loop Circuit

4. A real battery has an emf $\mathscr{E}$ and an internal resistance r. A variable resistor R is connected across the terminals of the battery. A current i is drawn from the battery and the potential difference

across the terminals of the battery is V. If R is slowly decreased to zero, which of the following best describes i and V?

(a) i decreases to zero; V approaches $\mathscr{E}$

(b) i approaches an infinite value; V decreases to zero

(c) i approaches $\mathscr{E}r$; V approaches $\mathscr{E}$

(d) i approaches $\mathscr{E}/r$; V approaches zero

5. Figure 28-41 shows a network of wires carrying various currents. What is the current through A?

(a) 1 A

(b) 2 A

(c) 6 A

(d) 8 A

(e) There is not enough information given

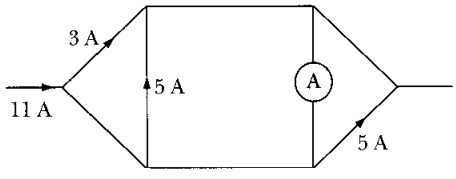


FIG. 28-41 Multiple choice question 5.

6. A current-carrying wire got pulled, stretching it and making it thinner in its middle. The cross-section of that wire is shown in Fig. 28-42. Which of the following statements about the current and the temperature of the wre is true?

(a) The current and temperature at the thin section of the wire is higher than at the thick section

(b) The current and temperature at the thick section of the wire is higher than at the thin section

(c) The current and temperature at the wire is the same everywhere

(d) The current in the wire is the same everywhere, but the temperature of the wire is higher in the thick section

(e) The current in the wire is the same everywhere, but the temperature of the wire is higher in the thin section

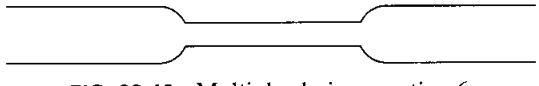


FIG. 28-42 Multiple choice question 6.

7. Four unequal resistors are connected in parallel. Which one of the following statements is correct?

(a) The total resistance is less than the smallest resistor

(b) The total resistance is equal to the average of the individual resistances

(c) The total resistance is more than the largest resistor

8. Three 100 W light bulbs are connected in series to a 220 V power source. If two of the light bulbs are replaced by 60 W bulbs, the brightness of the remaining 100 W bulb is

(a) Brighter than it was before

(b) Dimmer than it was before

(c) The same brightness as it was before

(d) Will not illuminate

9. An ideal battery is connected to a circuit of four resistors as shown in Fig. 28-43. If the resistance R_4 is increased, then

(a) The current through R_1 decreases

(b) The potential difference across R_1 increases

(c) The potential difference across R_4 increases

(d) The potential difference across R_2 decreases

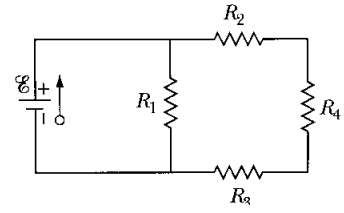


FIG. 28-43 Multiple choice question 9.

10. If voltage V is applied across terminals a and b of the circuit shown in Fig. 28-44, which of the statements below is true? (Notation: i_4 is current through 4.00-Ω resistor, ...)

(a) $i_7 < i_{10} < i_9$

(b) $i_4 = i_7 = i_9$

(c) $i_4 = i_9$, $i_4 > i_7$, and $i_7 = i_{10}$

(d) $i_4 > i_7 + i_{10} > i_9$

(e) $i_{10} < i_7$

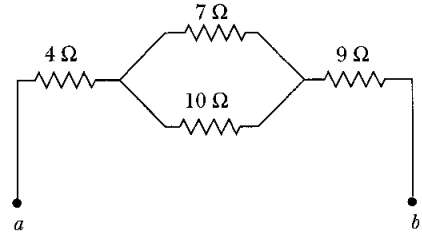


FIG. 28-44 Multiple choice question 10.

11. If the four light bulbs in Fig. 28-45 are identical, which circuit gives out more light?

(a) I

(b) The two emit the same amount of light

(c) II

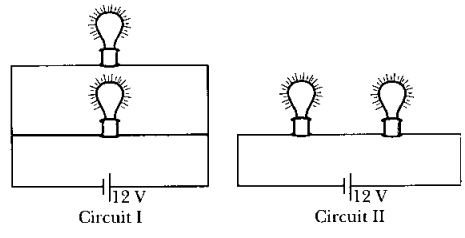


FIG. 28-45 Multiple choice question 11.

sec. 28-4 Multiloop Circuits

12. Five equal resistance each of value R are connected in a form shown in Fig. 28-46. The equivalent resistance of the network

(a) Between the points B and D is R

(b) Between the points B and D is $R/2$

(c) Between the points A and C is R

(d) Between the points A and C is $R/2$

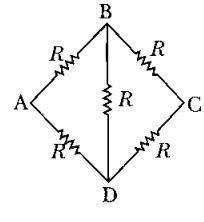
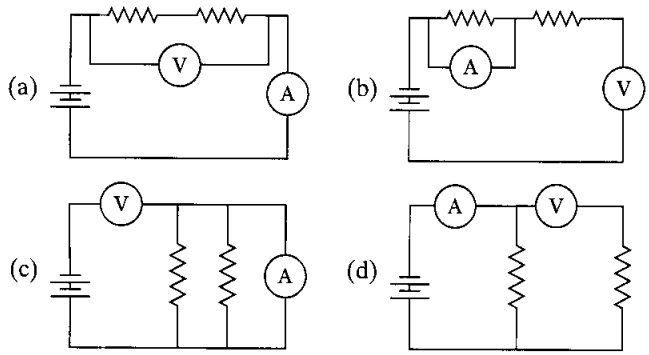


FIG. 28-46 Multiple choice question 12.

sec. 28-5 Electricity-Related Instruments

13. Which circuit diagram shows voltmeter V and ammeter A correctly positioned to measure the total potential difference of the circuit and the current through each resistor?



14. Two ends of an 0.1 MΩ potentiometer are connected in the way as shown in Fig. 28-47.

(a) $R_{AB} = 0.1$ MΩ for all values of x

(b) $R_{AB} > 0.1$ MΩ for all values of x

(c) $R_{AB} \leq 0.1$ MΩ for all values of x

(d) $R_{AB} \leq 0.05$ MΩ for all values of x

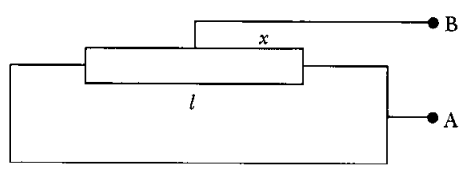


FIG. 28-47 Multiple choice question 14.

sec. 28-6 *RC* Circuits

15. When a capacitor is discharged through a resistor, which of the following is false?

(a) The electrical energy in the capacitor is converted into heat

(b) The current in the capacitor is equal to the current in the resistor

(c) The capacitor voltage decreases by half after a time equal to RC

(d) The voltage across the capacitor is equal to the voltage across the resistor at any instant

(e) The current in the circuit takes an infinite time to reach zero

16. Two identical resistors of magnitude R and two identical capacitors of magnitude C are used to form an RC circuit. In which case the time constant of that RC circuit is the highest?

(a) Two resistors and two capacitors connected in series with a power supply

(b) Two resistors and two capacitors connected in parallel with a power supply

(c) Two resistors connected in parallel, two capacitors connected in parallel, and those two combinations are connected in series with a power supply

(d) Two resistors connected in series, two capacitors connected in parallel, and those two combinations are connected in series with a power supply

17. Capacitor C_1 of capacitance 1 microfarad and capacitor C_2 of capacitance 2 μF are separately charged fully by a common battery. The two capacitors are then separately allowed to discharge through equal resistors at time $t = 0$.

(a) The current in each of the two discharging circuits is zero at $t = 0$

(b) The current in the two discharging circuits at $t = 0$ are equal but not zero

(c) The current in the two discharging circuits at $t = 0$ are unequal

(d) Capacitor C_1 loses 50% of its initial charge sooner than C_2 loses 50% of its initial charge

18. A capacitor of capacity C is charged to a steady potential difference V and connected in series with an open key and a pure resistor "R." At time $t = 0$, the key is closed if i = current at time t, a plot of log i against t is as shown in (1) in Fig. 28-48. Later one of the parameter, that is, V. R or C is changed keeping the other two constant, and the graph (2) is recorded. Then

(a) C is reduced

(b) C is increased

(c) R is reduced

(d) R is increased

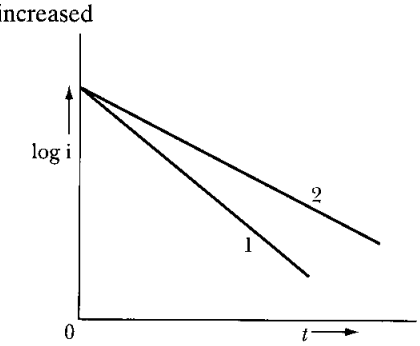


FIG. 28-48 Multiple choice question 18.

19. When the switch is closed in the circuit as shown in Fig. 28-49, the light bulb becomes bright in an imperceptibly short-time interval, and then gradually goes out. We can interpret this to mean that

(a) Electrons on the negative terminal of the battery reach a capacitor plate almost instantaneously

(b) Electrons from a capacitor plate travel to the positive terminal of the battery almost instantaneously

(c) The capacitor must already have had charge on its plates

(d) All the free electrons in the wire on that side travel to the capacitor plate while the light is still glowing

(e) The battery establishes an electric field in the circuit almost instantaneously

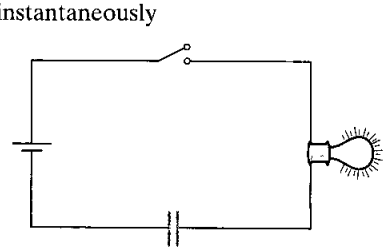


FIG. 28-49 Multiple choice question 19.

20. A resistor, capacitor, switch, and ideal battery are in series. Originally the capacitor is uncharged. The switch is then closed, allowing current of flow.

(i) While the current is flowing, the potential difference across the resistor is

(a) Increasing

(b) Decreasing

(c) Fixed

(ii) While the current is flowing, the potential difference across the capacitor is

(a) Increasing

(b) Decreasing

(c) Fixed

PROBLEMS

sec. 28-2 Work, Energy, and Emf

1. A car battery with a 12 V emf and an internal resistance of 0.040 Ω is being charged with a current of 50 A. What are (a) the potential difference V across the terminals, (b) the rate P_r of energy dissipation inside the battery, and (c) the rate P_{emf} of energy conversion to chemical form? When the battery is used to supply 50 A to the starter motor, what are (d) V and (e) P_r?

2. A 5.0 A current is set up in a circuit for 6.0 min by a rechargeable battery with a 6.0 V emf. By how much is the chemical energy of the battery reduced?

3. In Fig. 28-50, the ideal batteries have emfs $\mathscr{E}_1 = 150$ V and $\mathscr{E}_2 = 50$ V and the resistances are $R_1 = 3.0\ \Omega$ and $R_2 = 2.0\ \Omega$. If the potential at P is defined to be 100 V, what is the potential at Q?

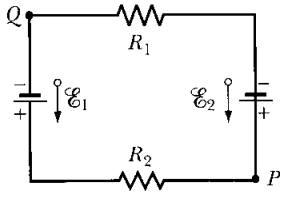


FIG. 28-50 Problem 3.

sec. 28-3 Calculating the Current in a Single-Loop Circuit

4. In Fig. 28-51, $R_1 = R_2 = 4.00\ \Omega$ and $R_3 = 2.50\ \Omega$. Find the equivalent resistance between points D and E. (*Hint:* Imagine that a battery is connected across those points.)

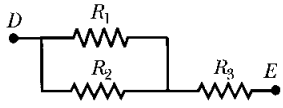


FIG. 28-51 Problem 4.

5. When resistors 1 and 2 are connected in series, the equivalent resistance is 16.0 Ω. When they are connected in parallel, the equivalent resistance is 3.0 Ω. What are (a) the smaller resistance and (b) the larger resistance of these two resistors?

6. A total resistance of 3.00 Ω is to be produced by connecting an unknown resistance to a 12.0 Ω resistance. (a) What must be the value of the unknown resistance, and (b) should it be connected in series or in parallel?

7. In Fig. 28-52, circuit section AB absorbs energy at a rate of 50 W when current $i = 1.0$ A through it is in the indicated direction. Resistance $R = 2.0\ \Omega$. (a) What is the potential difference between A and B? Emf device X lacks internal resistance. (b) What is its emf? (c) Is point B connected to the positive terminal of X or to the negative terminal?

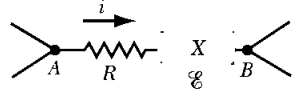


FIG. 28-52 Problem 7.

8. In Fig. 28-53, battery 1 has emf $\mathscr{E}_1 = 12.0$ V and internal resistance $r_1 = 0.016\ \Omega$ and battery 2 has emf $\mathscr{E}_2 = 12.0$ V and internal resistance $r_2 = 0.012\ \Omega$. The batteries are connected in series with an external resistance R. (a) What R value makes the terminal-to-terminal potential difference of one of the batteries zero? (b) Which battery is that?

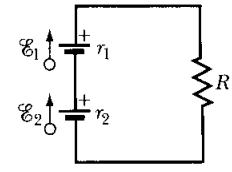


FIG. 28-53 Problem 8.

9. In Fig. 28-54, the current in resistance 6 is $i_6 = 1.40$ A and the resistances are $R_1 = R_2 = R_3 = 2.00\ \Omega$, $R_4 = 16.0\ \Omega$, $R_5 = 8.00\ \Omega$, and $R_6 = 4.00\ \Omega$. What is the emf of the ideal battery?

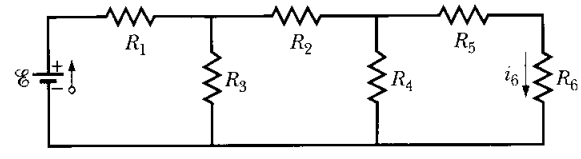
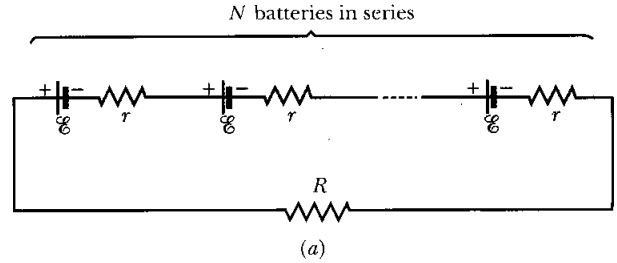


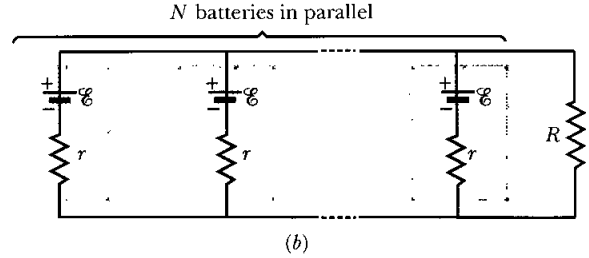
FIG. 28-54 Problem 9.

sec. 28-4 Multiloop Circuits

10. A group of N identical batteries of emf $\mathscr{E}$ and internal resistance r may be connected all in series (Fig. 28-55*a*) or all in parallel (Fig. 28-55*b*) and then across a resistor R. Show that both arrangements give the same current in R if $R = r$.



(*a*)



(*b*)

FIG. 28-55 Problem 10.

11. In Fig. 28-56, $R_1 = 100\ \Omega$, $R_2 = 50\ \Omega$, and the ideal batteries have emfs $\mathscr{E}_1 = 6.0$ V, $\mathscr{E}_2 = 5.0$ V, and $\mathscr{E}_3 = 4.0$ V. Find (a) the current in resistor 1, (b) the current in resistor 2, and (c) the potential difference between points a and b.

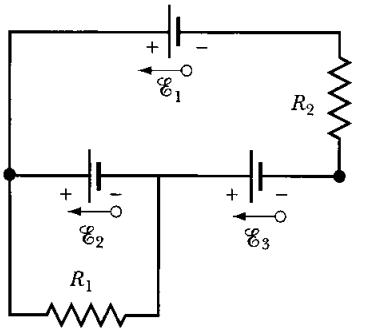


FIG. 28-56 Problem 11.

12. In Fig. 28-57, the ideal batteries have emfs $\mathscr{E}_1 = 10.0$ V and $\mathscr{E}_2 = 0.500\ \mathscr{E}_1$, and the resistances are each 4.00 Ω. What is the current in (a) resistance 2 and (b) resistance 3?

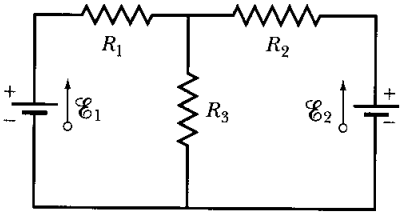


FIG. 28-57 Problem 12.

13. In Fig. 28-58, two batteries of emf $\mathscr{E} = 12.0$ V and internal resistance $r = 0.300\ \Omega$ are connected in parallel across a resistance R. (a) For what value of R is the dissipation rate in the resistor a maximum? (b) What is that maximum?

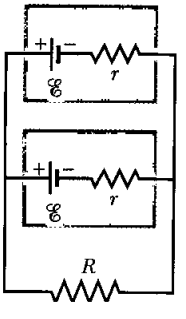


FIG. 28-58 Problem 13.

14. In the circuit shown in Fig. 28-59, there is a cube of resistances which are all having the same resistance of R. Find the equivalent resistance of the network between diagonally opposite points as shown in the figure.

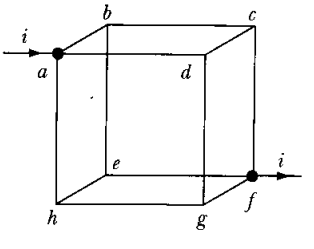


FIG. 28-59 Problem 14.

15. Find the equivalent resistance of the network as shown in Fig. 28-60 between points a and b.

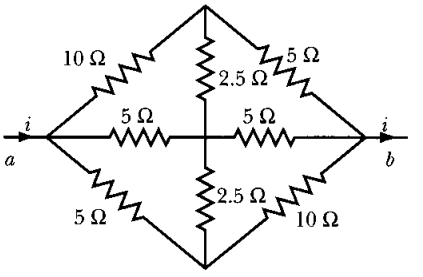


FIG. 28-60 Problem 15.

16. An infinite ladder is constructed with 1 Ω and 2 Ω resistors as shown in Fig. 28-61. Find the effective resistance between the points A and B.

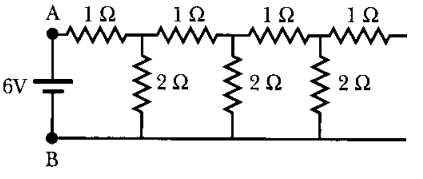


FIG. 28-61 Problem 16.

sec. 28-5 Electricity-Related Instruments

17. A voltmeter consists of a 25 Ω coil connected in series with a 575 Ω resistor. The coil takes 10 mA for full-scale deflection. What maximum potential difference can be measured on this voltmeter?

18. An ammeter is to be constructed which can read currents upto 2.0 A. If the coil has a resistance of 25 Ω and takes 1 mA for full-scale deflection, what should be the resistance of the shunt used?

19. A voltmeter coil has resistance 50.0 Ω and a resistor of 1.15 kΩ is connected in series. It can read potential differences up to 12 V. If this same coil is used to construct an ammeter which can measure currents up to 2.0 A, What should be the resistance of the shunt used?

20. In Fig. 28-62, assume that $\mathscr{E} = 3.0$ V, $r = 100\ \Omega$, $R_1 = 250\ \Omega$, and $R_2 = 300\ \Omega$. If the voltmeter resistance R_V is 5.0 kΩ, what percent error does it introduce into the measurement of the potential difference across R_1? Ignore the presence of the ammeter.

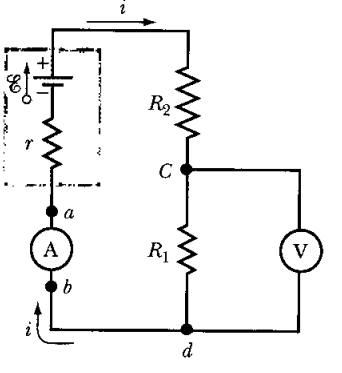


FIG. 28-62 Problem 20.

21. (a) In Fig. 28-63, what does the ammeter read if $\mathscr{E} = 5.0$ V (ideal battery), $R_1 = 2.0\ \Omega$, $R_2 = 4.0\ \Omega$, and $R_3 = 6.0\ \Omega$?

(b) The ammeter and battery are now interchanged. Show that the ammeter reading is unchanged.

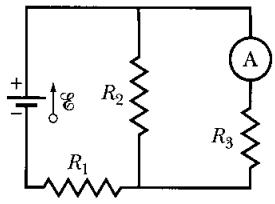


FIG. 28-63 Problem 21.

22. In Fig. 28-64, a voltmeter of resistance $R_V = 300\ \Omega$ and an ammeter of resistance $R_A = 3.00\ \Omega$ are being used to measure a resistance R in a circuit that also contains a resistance $R_0 = 100\ \Omega$ and an ideal battery of emf $\mathscr{E} = 12.0$ V. Resistance R is given by $R = V/i$, where V is the potential across R and i is the ammeter reading. The voltmeter reading is V', which is V plus the potential difference across the ammeter. Thus, the ratio of the two meter readings is not R but only an *apparent* resistance $R' = V'/i$. If $R = 85.0\ \Omega$, what are (a) the ammeter reading, (b) the voltmeter reading, and (c) R'? (d) If R_A is decreased, does the difference between R' and R increase, decrease, or remain the same?

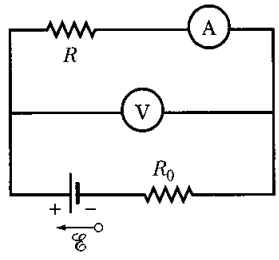


FIG. 28-64 Problem 22.

23. In Fig. 28-65, a voltmeter of resistance $R_V = 300\ \Omega$ and an ammeter of resistance $R_A = 3.00\ \Omega$ are being used to measure a resistance R in a circuit that also contains a resistance $R_0 = 100\ \Omega$ and an ideal battery of emf $\mathscr{E} = 12.0$ V. Resistance R is given by $R = V/i$, where V is the voltmeter reading and i is the current in resistance R. However, the ammeter reading is *not* i *but rather* i', which is i plus the current through the voltmeter. Thus, the ratio of the two meter readings is not R but only an *apparent* resistance $R' = V/i$. If $R = 85.0\ \Omega$, what are (a) the ammeter reading, (b) the voltmeter reading, and (c) R'? (d) If R_V is increased, does the difference between R' and R increase, decrease, or remain the same?

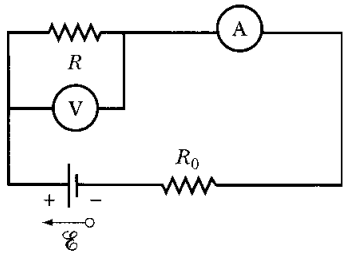


FIG. 28-65 Problem 23.

24. Figure 28-66 shows an arrangement to measure the emf $\mathscr{E}$ and internal resistance r of a battery. The voltmeter has resistance. The voltmeter reads 1.52 V when the switch S is open. When the switch is closed the voltmeter reading drops to 1.45 V and the ammeter reads 1.0 A. Find the (a) emf and (b) the internal resistance of the battery.

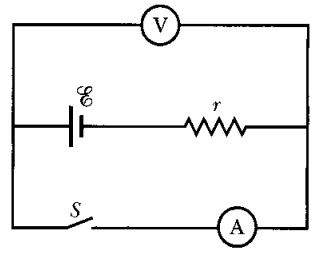


FIG. 28-66 Problem 24.

25. The wire AB of an arrangement like that of meter bridge as shown in Fig. 28-67 is 40 cm long. Where should the free end of the galvanometer be connected on AB so that the galvanometer may show zero deflection?

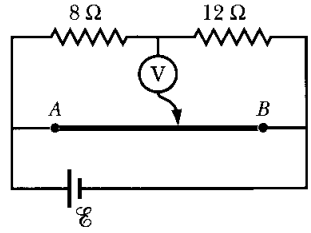


FIG. 28-67 Problem 25.

26. A 6 V battery of negligible internal resistance is connected across a uniform wire AB of length 100 cm. The positive terminal of another battery of emf 4 V and internal resistance 1 Ω is joined to the point A as shown in Fig. 28-68. Take the potential at B to be zero. (a) What are the potentials at the points A and C? (b) At which point D of the wire AB, the potential is equal to the potential at C? (c) If the points C and D are connected by a wire, what will be the current through it? (d) If the 4 V battery is replaced by 7.5 V battery, what would be the answers of part (a) and (b)?

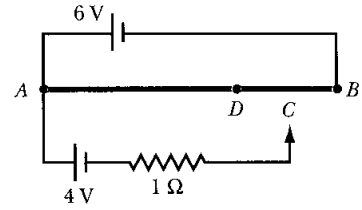


FIG. 28-68 Problem 26.

27. Consider the potentiometer circuit arranged as shown in Fig. 28-69. The potentiometer wire is 600 cm long. (a) At what distance from the point A should the jockey touch the wire to get zero deflection in the galvanometer? (b) If the jockey touches the wire at a distance of 560 cm from A, what will be the current in the galvanometer?

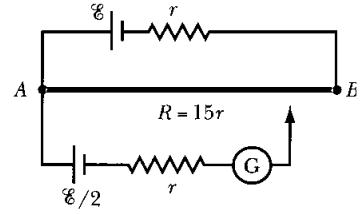


FIG. 28-69 Problem 27.

sec. 28-6 RC Circuits

28. A capacitor with initial charge q_0 is discharged through a resistor. What multiple of the time constant τ gives the time the capacitor takes to lose (a) the first one-third of its charge and (b) two-thirds of its charge?

29. In an RC series circuit, $\mathscr{E}$ = 12.0 V, R = 1.40 MΩ, and C = 1.80 μF. (a) Calculate the time constant. (b) Find the maximum charge that will appear on the capacitor during charging. (c) How long does it take for the charge to build up to 16.0 μC?

30. Switch S in Fig. 28-70 is closed at time $t = 0$, to begin charging an initially uncharged capacitor of capacitance C = 15.0 μF through a resistor of resistance R = 20.0 Ω. At what time is the potential difference across the capacitor equal to that across the resistor?

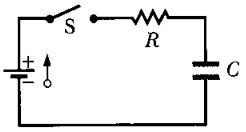


FIG. 28-70 Problem 30.

31. The potential difference between the plates of a leaky (meaning that charge leaks from one plate to the other) 2.0 μF capacitor drops to one-fourth its initial value in 2.0 s. What is the equivalent resistance between the capacitor plates?

32. In the circuit of Fig. 28-71, $\mathscr{E}$ = 1.2 kV, C = 6.5 μF, R_1 = R_2 = R_3 = 0.73 MΩ. With C completely uncharged, switch S is suddenly closed (at $t = 0$). At $t = 0$, what are (a) current i_1 in resistor 1, (b) current i_2 in resistor 2, and (c) current i_3 in resistor 3? At $t = \infty$ (that is, after many time constants), what are (d) i_1, (e) i_2, and (f) i_3? What is the potential difference V_2 across resistor 2 at (g) $t = 0$ and (h) $t = \infty$? (i) Sketch V_2 versus t between these two extreme times.

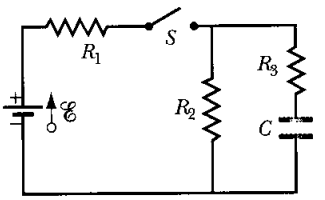


FIG. 28-71 Problem 32.

33. A 3.00 MΩ resistor and a 1.00 μF capacitor are connected in series with an ideal battery of emf $\mathscr{E}$ = 4.00 V. At 1.00 s after the connection is made, what is the rate at which (a) the charge of the capacitor is increasing, (b) energy is being stored in the capacitor, (c) thermal energy is appearing in the resistor, and (d) energy is being delivered by the battery?

ADDITIONAL PROBLEMS

34. In Fig. 28-72, R_1 = 7.00 Ω, R_2 = 12.0 Ω, R_3 = 4.00 Ω, and the ideal battery's emf is $\mathscr{E}$ = 24.0 V. For what value of R_4 will the rate at which the battery transfers energy to the resistors equal (a) 60.0 W, (b) the maximum possible rate P_{max}, and (c) the minimum possible rate P_{min}? What are (d) P_{max} and (e) P_{min}?

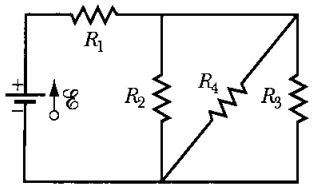


FIG. 28-72 Problem 34.

35. Figure 28-73 shows the circuit of a flashing lamp, like those attached to barrels at highway construction sites. The fluorescent lamp L (of negligible capacitance) is connected in parallel across the capacitor C of an RC circuit. There is a current through the lamp only when the potential difference across it reaches the breakdown voltage V_L; then the capacitor discharges completely through the lamp and the lamp flashes briefly. For a lamp with breakdown voltage V_L = 72.0 V, wired to a 95.0 V ideal battery and a 0.150 μF capacitor, what resistance R is needed for two flashes per second?

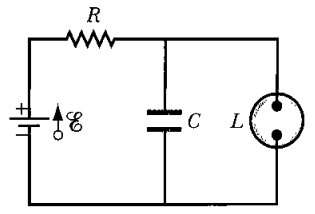


FIG. 28-73 Problem 35.

36. (a) In Fig. 28-4*a*, show that the rate at which energy is dissipated in R as thermal energy is a maximum when $R = r$. (b) Show that this maximum power is $P = \mathscr{E}^2/4r$.

37. In Fig. 28-74, the ideal batteries have emfs $\mathscr{E}_1$ = 20.0 V, $\mathscr{E}_2$ = 10.0 V, $\mathscr{E}_3$ = 5.00 V, and $\mathscr{E}_4$ = 5.00 V, and the resistances are each 2.00 Ω. What are the (a) size and (b) direction (left or right) of current i_1 and the (c) size and (d) direction of current i_2? (This can be answered with only mental calculation.) (e) At what rate is energy being transferred in battery 4, and (f) is the energy being supplied or absorbed by the battery?

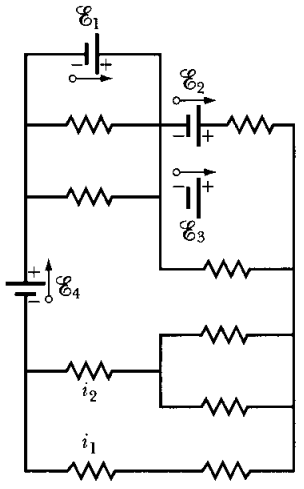


FIG. 28-74 Problem 37.

38. The circuit of Fig. 28-75 shows a capacitor, two ideal batteries, two resistors, and a switch S. Initially S has been open for a long time. If it is then closed for a long time, what is the change in the charge on the capacitor? Assume C = 10 μF, $\mathscr{E}_1$ = 1.0 V, $\mathscr{E}_2$ = 3.0 V, R_1 = 0.20 Ω, and R_2 = 0.40 Ω.

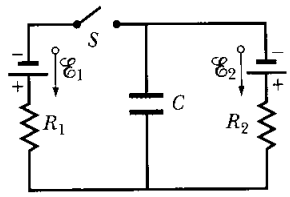


FIG. 28-75 Problem 38.

39. In Fig. 28-76, $R_1 = 5.00\ \Omega$, $R_2 = 10.0\ \Omega$, $R_3 = 15.0\ \Omega$, $C_1 = 5.00\ \mu F$, $C_2 = 10.0\ \mu F$, and the ideal battery has emf $\mathscr{E} = 20.0$ V. Assuming that the circuit is in the steady state, what is the total energy stored in the two capacitors?

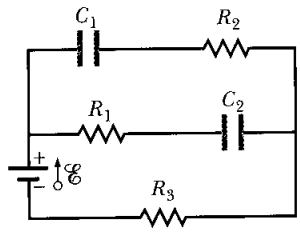


FIG. 28-76 Problem 39.

40. An initially uncharged capacitor C is fully charged by a device of constant emf $\mathscr{E}$ connected in series with a resistor R. (a) Show that the final energy stored in the capacitor is half the energy supplied by the emf device. (b) By direct integration of i^2R over the charging time, show that the thermal energy dissipated by the resistor is also half the energy supplied by the emf device.

41. A three-way 120 V lamp bulb that contains two filaments is rated for 100-200-300 W. One filament burns out. Afterward, the bulb operates at the same intensity (dissipates energy at the same rate) on its lowest as on its highest switch positions but does not operate at all on the middle position. (a) How are the two filaments wired to the three switch positions? What are the (b) smaller and (c) larger values of the filament resistances?

42. In Fig. 28-77 shows a conductor of length l having a circular cross section. The radius of cross section varies linearly from a to b. The resistivity of the material is ρ. Assuming that $b - a \ll l$, find the resistance of the conductor.

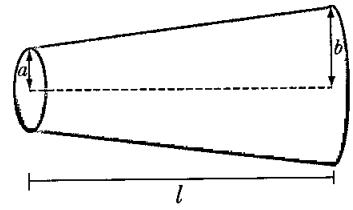


FIG. 28-77 Problem 42.

ANSWERS

Checkpoints

1. (a) rightward; (b) all tie; (c) b, then a and c tie; (d) b, then a and c tie

2. (a) all tie; (b) R_1, R_2, R_3

3. (a) less; (b) greater; (c) equal

4. (a) $V/2, i$; (b) $V, i/2$

5. $20\ \Omega$

6. (a) $R = 10^6\ \Omega - 100\ \Omega \approx 10^6\ \Omega$; (b) $r = 0.5\ \Omega$

7. (a) 1, 2, 4, 3; (b) 4, tie of 1 and 2; then 3

Reasoning Type Questions

1. (a)

2. (a)

3. (d)

4. (a)

5. (b)

Multiple Choice Questions

1. (a)

2. (a)

3. (a)

4. (d)

5. (b)

6. (e)

7. (a)

8. (b)

9. (c), (d)

10. (e)

11. (a)

12. (b), (c)

13. (a)

14. (c), (d)

15. (a), (d), (e)

16. (d)

17. (b), (d)

18. (b)

19. (e)

20. (i) (b); (ii) (a)

Problems

1. (a) 14 V; (b) 1.0×10^2 W; (c) 6.0×10^2 W; (d) 10 V; (e) 1.0×10^2 W

2. 11 kJ

3. −10 V

4. 4.50 Ω

5. (a) 4.0 Ω; (b) 12 Ω

6. (a) 4.00 Ω; (b) in parallel

7. (a) 50 V; (b) 48 V; (c) negative

8. (a) 0.004 Ω; (b) 1

9. 48.3 V

11. (a) 50 mA; (b) 60 mA; (c) 9.0 V

12. (a) 0; (b) 1.25 A

13. (a) 0.150 Ω; (b) 240 W

14. $\frac{5}{6}R$

15. 4.1 Ω

16. 2 Ω

17. 6 V

18. 1.25×10^{-2} Ω

19. 0.251 Ω

20. −3.0%

21. (a) 0.45 A

22. (a) 55.2 mA; (b) 4.86 V; (c) 88.0 Ω; (d) decrease

23. (a) 7.09×10^{-2} A; (b) 4.70 V; (c) 66.3 Ω; (d) decrease

24. (a) 1.52 V; (b) 0.07 Ω

25. 16 cm from A

26. (a) 6 V, 2 V; (b) $AD = 66.7$ cm; (c) zero; (d) 6 V, −1.5 V, no such point D exists

27. (a) 320 cm; (b) $3\mathscr{E}/22r$

28. (a) 0.41; (b) 1.1

29. (a) 2.52 s; (b) 21.6 μC; (c) 3.40 s

30. 0.208 ms

31. 0.72 MΩ

32. (a) 1.1 mA; (b) 0.55 mA; (c) 0.55 mA; (d) 0.82 mA; (e) 0.82 mA; (f) 0; (g) 4.0×10^2 V; (h) 6.0×10^2 V

33. (a) 0.955 μC/s; (b) 1.08 μW; (c) 2.74 μW; (d) 3.82 μW

Additional Problems

34. (a) 19.5 Ω; (b) 0; (c) ∞; (d) 82.3 W; (e) 57.6 W

35. 2.35×10^6 Ω

36. (a) 0

37. (a) 7.50 A; (b) leftward; (c) 10.0 A; (d) leftward; (e) 87.5 W; (f) the energy is being supplied to the circuit since the current is in the "forward" direction through the battery

38. −13 μC

39. 2.50×10^{-4} J

41. (a) Since $P = \varepsilon^2/R_{eq}$, the higher the power rating the smaller the value of R_{eq}. To achieve this, we can let the low position connect to the larger resistance (R_1), middle position connect to the smaller resistance (R_2), and the high position connect to both of them in parallel; (b) 72 Ω; (c) 144 Ω

42. $\rho l / \pi ab$

Magnetic Force 29



Tom Walker/Getty Images, Inc.

If you are outside on a dark night in the middle to high latitudes, you might be able to see an aurora, a ghostly "curtain" of light that hangs down from the sky. The curtain is not only local; it may be 200 km high and 4000 km long, stretching around Earth in an arc. However, it is only about 100 m thick.

What produces this huge display, and what makes it so thin?

The answers are in this chapter.

29-1 WHAT IS PHYSICS?

As we have discussed, one major goal of physics is the study of how an *electric field* can produce an *electric force* on a charged object. A closely related goal is the study of how a *magnetic field* can produce a *magnetic force* on a (moving) charged particle or on a magnetic object such as a magnet. You may already have a hint of what a magnetic field is if you have ever attached a note to a refrigerator door with a small magnet or accidentally erased a credit card by moving it near a magnet. The magnet acts on the door or credit card via its magnetic field.

The applications of magnetic fields and magnetic forces are countless and changing rapidly every year. Here are just a few examples. For decades, the entertainment industry depended on the magnetic recording of music and images on audiotape and videotape. Although digital technology has largely replaced magnetic recording, the industry still depends on the magnets that control CD and DVD players and computer hard drives; magnets also drive the speaker cones in headphones, TVs, computers, and telephones. A modern car comes equipped with dozens of magnets because they are required in the motors for engine ignition, automatic window control, sunroof control, and windshield wiper control. Most security alarm systems, doorbells, and automatic door latches employ magnets. In short, you are surrounded by magnets.

The science of magnetic fields is physics; the application of magnetic fields is engineering. Both the science and the application begin with the question "What produces a magnetic field?"



FIG. 29-1 Using an electromagnet to collect and transport scrap metal at a steel mill. *(Digital Vision/Getty Images)*

29-2 | What Produces a Magnetic Field?

Because an electric field $\vec{E}$ is produced by an electric charge, we might reasonably expect that a magnetic field $\vec{B}$ is produced by a magnetic charge. Although individual magnetic charges (called *magnetic monopoles*) are predicted by certain theories, their existence has not been confirmed. How then are magnetic fields produced? There are two ways.

One way is to use moving electrically charged particles, such as a current in a wire, to make an **electromagnet.** The current produces a magnetic field that can be used, for example, to control a computer hard drive or to sort scrap metal (Fig. 29-1). In Chapter 30, we discuss the magnetic field due to a current.

The other way to produce a magnetic field is by means of elementary particles such as electrons because these particles have an *intrinsic* magnetic field around them. That is, the magnetic field is a basic characteristic of each particle just as mass and electric charge (or lack of charge) are basic characteristics. The magnetic fields of the electrons in certain materials add together to give a net magnetic field around the material. Such addition is the reason why a **permanent magnet,** the type used to hang refrigerator notes, has a permanent magnetic field. In other materials, the magnetic fields of the electrons cancel out, giving no net magnetic field surrounding the material. Such cancellation is the reason you do not have a permanent field around your body, which is good because otherwise you might be slammed up against a refrigerator door every time you passed one.

If you touch the surface of a magnet with an object containing iron, such as a paper clip, you will discover that certain parts of the magnets, called magnetic poles, interact more strongly with the paper clip than other parts of the magnet. Disk-shaped refrigerator magnets usually have poles on either end; paper clips stick to the flat ends, but don't stick to the other surface.

Bar magnets (Fig. 29-2) also have poles at the ends; moreover, the sides of the bar don't interact very strongly with a paper clip. In fact, some magnets (like the magnetic strips) may have more than one pole. It just has as many north poles as south poles. Like charges, magnets can attract or repel without touching. This tells us that magnetic interactions are of long range and that there must be two types of magnetic poles.

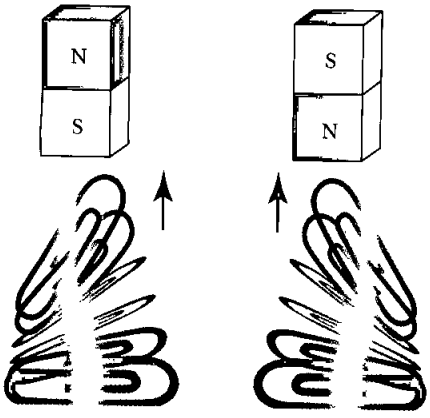


FIG. 29-2 Bar magnets.

A magnet interacts with an iron object via a magnetic field it produces in space. When a bar magnet with one pole on each end is suspended to rotate freely, it aligns itself so that its poles lie on a line roughly in south–north direction of Earth. Actually the Earth's magnetic field exerts a torque on this magnet to align it in the direction of stable equilibrium (and minimum energy). This north–south alignment of a freely rotating magnet is the basic operational principle of a compass needle, which is simply a freely rotating magnet (Fig. 29-3).

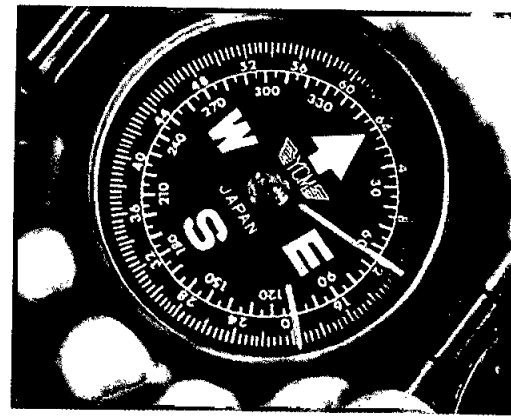


FIG. 29-3 A commercially available compass.

This alignment provides a way of distinguishing between the two types of poles. The end of the magnet which aligns toward the north pole is called the north pole (denoted by N), and the opposite end is called south pole (denoted by S). Commercially available magnets are painted red on the portion which acts as a north pole. To understand what causes this alignment, we must first examine how one magnet interacts with other. If you hold the poles of two magnets near one another, you find that like poles repel each other and opposite poles attract each other (see Fig. 29-4).

Materials that are attracted by both types of magnetic poles are iron, nickel, cobalt and certain alloys such as steel (but not stainless steel). These materials are called magnetic materials. Normally, two objects made from these materials do not interact with one another. For example, one paper clip containing iron does not attract another one. However, just like an object carrying charge induces a charge polarization on a neutral object, the magnetic field induces a magnetic polarization in a paper clip or any other object made from magnetic materials. The poles of the magnet then interact with the induced poles of the magnetized paper clip (see Fig. 29-2).

However, unlike induced polarizations, which vanish as soon as the charged object is removed, some magnetic materials retain their induced magnetic polarization. There is a fundamental difference between magnetic and electrostatic interactions: it is not possible to isolate the N pole or the S pole of a magnet the way one can separate positive and negative charges from each other. For example, if we cut a bar magnet in two, we find that two new poles have appeared and that the resulting pieces each have two opposite poles at the ends (see Fig. 29-5). The cutting has created an additional pair of opposite poles. Remarkably, if we carefully place the two pieces together again, the two new poles seem to vanish.

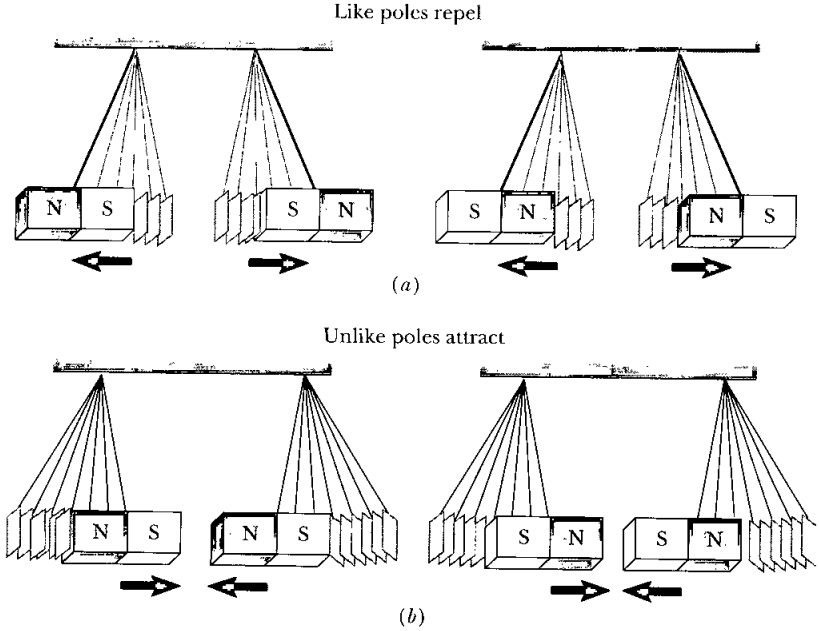


FIG. 29-4 (*a*) Like poles repel, (*b*) unlike poles attract.

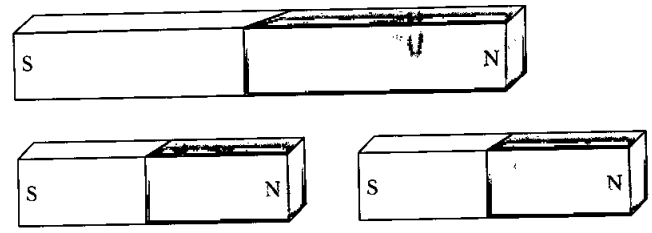


FIG. 29-5 If you break a magnet, each fragment becomes a separate magnet, with its own north and south poles.

CHECKPOINT 1 If we have two similar looking bars, one of which is a magnet and the other a bar of iron, how will we distinguish between the two by using these two bars? We cannot suspend them in the Earth's magnetic field.

Experiments show that the iron, nickel, or cobalt atoms which make up magnets are indeed "elementary magnets" each with an N pole and an S pole. A magnet thus simply consists of a large number of elementary magnets whose alignment is such that their combined effect is reinforced at the poles. Because we cannot cut the atoms that are the elementary magnets themselves, a pair of new poles appears every time we cut through the magnet. In an unmagnetized piece of magnetic material, the elementary magnets are randomly aligned—some push or pull in one direction while others push or pull in other directions—and so their effects cancel and the piece as a whole does not act like a magnet (Fig. 29-6*a*). When another magnet is brought nearby, however, the elementary magnets align themselves to the nearby magnet and the material becomes magnetized (Fig. 29-6*b*).

You can also use this model to visualize the demagnetization of a magnet. If a magnetized piece of material is heated or roughly handled, the elementary magnets are jarred around and they lose their alignment. Because there is no such thing as a "magnetic charge," it is not possible, as in electrostatics, to map out a magnetic field using a "test magnetic charge." We can, however, use a compass to map out the direction of the magnetic field. When a compass is placed near a bar magnet, as illustrated in Fig. 29-7, the S pole of the bar magnet exerts an attractive force on the N pole of the compass needle and a repulsive force on the S pole. These two forces cause a torque that tends to align the needle with its N pole pointing toward the magnet's S pole. Even in a general magnetic field, the N pole of a magnet experiences a force in the direction of the magnetic field and the S pole experience a force in the direction opposite to the magnetic field.

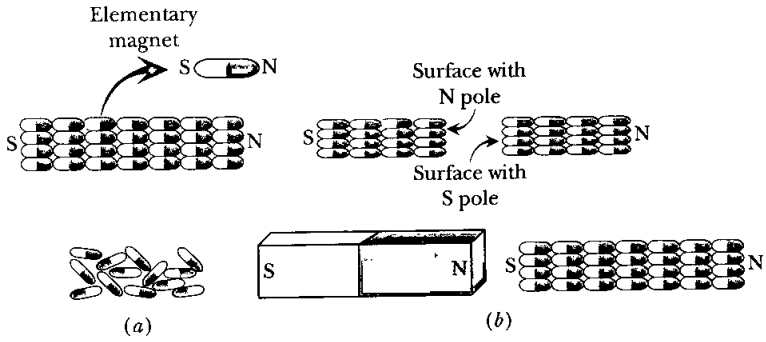


FIG. 29-6 (*a*) Unmagnetized and (*b*) magnetized piece of magnetic material.

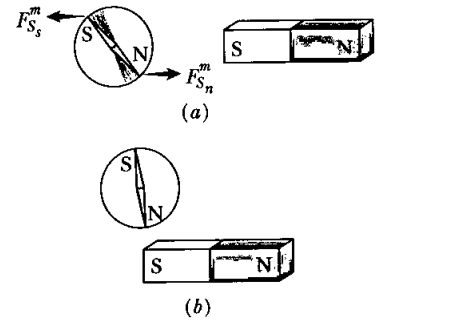


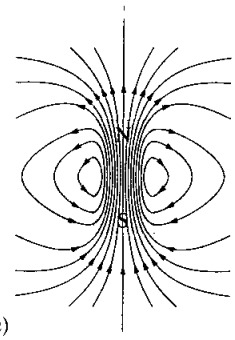
FIG. 29-7 (*a*) A magnet exerts a torque on a compass. We can safely assume that the magnetic field exerts a force on the north pole in the direction of the field and that on the south pole in the opposite direction. (*b*) In the equilibrium position, the compass will point in the direction of the magnetic field.

Magnetic Field Lines

We can represent magnetic fields with field lines, as we did for electric fields. Similar rules apply: (1) the direction of the tangent to a magnetic field line at any point gives the direction of magnetic field at that point, and (2) the spacing of the lines represents the magnitude of magnetic field—the magnetic field is stronger where the lines are closer together, and conversely.

Figure 29-8*a* shows how the magnetic field near a *bar magnet* (a permanent magnet in the shape of a bar) can be represented by magnetic field lines. The lines all pass through the magnet, and they all form closed loops (even those that are not shown closed in the figure). The external magnetic effects of a bar magnet are strongest near its ends, where the field lines are most closely spaced. Thus, the magnetic fields of the bar magnets in Figs. 29-8*b* and 29-9 collect the iron filings mainly near the two ends of the magnet. You might have done this experiment in your junior classes. The pattern which emerges is something similar to the electric field lines in space.

Two points can be noted here. At the points near the poles, the density of these magnetic field lines is high, and near the center of the magnet, the density of these lines is low. Another point is that if we keep a magnetic compass at some point, it will align itself in the same direction as the magnetic field lines.



(*a*)



(*b*)

FIG. 29-8 (*a*) The magnetic field lines for a bar magnet. (*b*) A "cow magnet"—a bar magnet that is intended to be slipped down into the rumen of a cow to prevent accidentally ingested bits of scrap iron from reaching the cow's intestines. The iron filings at its ends reveal the magnetic field lines. *(Courtesy Dr. Richard Cannon, Southeast Missouri State University, Cape Girardeau)*

The Magnetism of Earth

Earth is a huge magnet; for points near Earth's surface, its magnetic field can be approximated as the field of a huge bar magnet—a magnetic dipole—that straddles the center of the planet. Figure 29-10 is an idealized symmetric depiction of the magnetic field, without the distortion caused by passing charged particles from the Sun.

For the idealized field of Fig. 29-10, the magnet's axis makes an angle of 11.5° with the rotation axis (*RR*) of Earth. The *Magnet's axis* (*MM* in Fig. 29-10) lies along and intersects Earth's surface at the *geomagnetic north pole* in northwest Greenland and the *geomagnetic south pole* in Antarctica. The lines of the magnetic field generally emerge in the Southern Hemisphere and reenter Earth in the Northern Hemisphere. Thus, the magnetic pole that is in Earth's Northern Hemisphere and known as a "north magnetic pole" *is really the south pole of Earth's magnet.*

The direction of the magnetic field at any location on Earth's surface is commonly specified in terms of two angles. The **angle of declination** (ϕ) is the angle (left or right) between geographic north and the horizontal component of the field. The **angle of dip (or field inclination)** (θ) is the angle (up or down) between a horizontal plane and the field's direction. If we include the **horizontal component of the Earth's magnetic field** as well, these three are called the elements of the Earth's magnetic field. So if the magnitude of the magnetic field of Earth is B at a certain point, then its horizontal component is given by

$$B_h = B \cos \theta.$$

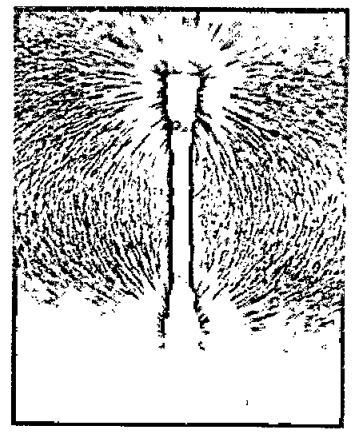


FIG. 29-9 Using iron fillings to map the magnetic field.

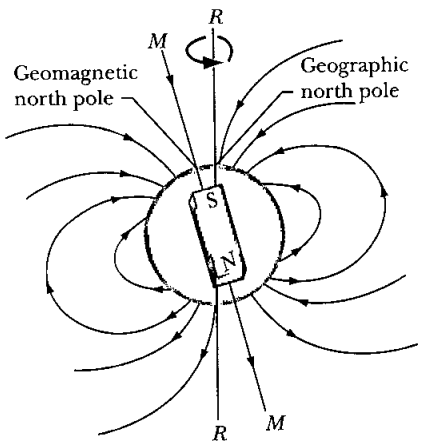
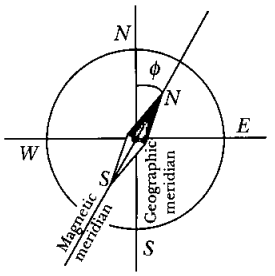
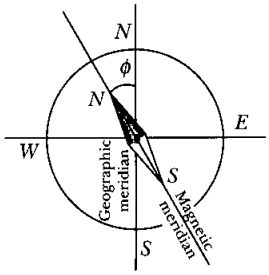


FIG. 29-10 Magnetic field lines due to Earth. The dipole axis *MM* makes an angle of 11.5° with Earth's rotational axis *RR*. The south pole of the magnet is in Earth's Northern Hemisphere.



(*a*)



(*b*)

FIG. 29-11 The arrangement of a compass: (*a*) in the region with eastern magnetic declination; (*b*) in the region with western magnetic declination. The eastern declination is often denoted by the plus sign and the western declination by the minus sign.

The vertical component of the magnetic field is given by

$$B_v = B \sin \theta.$$

We can imagine that at any point, a compass will not point in the true north direction. So if you are navigating in sea using only a compass, you are sure to land at a wrong place. You need to know the angle of declination at your location to find the true north direction (Fig. 29-11).

At any point on Earth's surface, the measured magnetic field may differ appreciably, in both magnitude and direction, from the idealized field of Fig. 29-10. In fact, the point where the field is actually perpendicular to Earth's surface and inward is not located at the geomagnetic North Pole in Greenland as we would expect; instead, this so-called *dip north pole* is located in the Queen Elizabeth Islands in northern Canada, far from Greenland. In addition, the field observed at any location on the surface of Earth varies with time, by measurable amounts over a period of a few years and by substantial amounts over, say, 100 years. In fact, the mechanism that produces Earth's magnetic field is only vaguely understood.

Our first job in this chapter is to define the magnetic field $\vec{B}$. We do so by using the experimental fact that when a charged particle moves through a magnetic field, a magnetic force $\vec{F}_B$ acts on the particle. As the magnetic field is a vector, its direction is given by the direction of the compass at that point.

29-3 | The Definition of $\vec{B}$

We determined the electric field $\vec{E}$ at a point by putting a test particle of charge q at rest at that point and measuring the electric force $\vec{F}_E$ acting on the particle. We then defined $\vec{E}$ as

$$\vec{E} = \frac{\vec{F}_E}{q}. \tag{29-1}$$

If a magnetic monopole were available, we could define $\vec{B}$ in a similar way. Because such particles have not been found, we must define $\vec{B}$ in another way, in terms of the magnetic force $\vec{F}_B$ exerted on a moving electrically charged test particle.

In principle, we do this by firing a charged particle through the point at which $\vec{B}$ is to be defined, using various directions and speeds for the particle and determining the force $\vec{F}_B$ that acts on the particle at that point. After many such trials we would find that when the particle's velocity $\vec{v}$ is along a particular axis through the point, force $\vec{F}_B$ is zero. For all other directions of $\vec{v}$, the magnitude of $\vec{F}_B$ is always proportional to $v \sin \phi$, where ϕ is the angle between the zero-force axis and the direction of $\vec{v}$. Furthermore, the direction of $\vec{F}_B$ is always perpendicular to the direction of $\vec{v}$. (These results suggest that a cross product is involved.)

We can then define a **magnetic field** $\vec{B}$ to be a vector quantity that is directed along the zero-force axis. We can next measure the magnitude of $\vec{F}_B$ when $\vec{v}$ is directed perpendicular to that axis and then define the magnitude of $\vec{B}$ in terms of that force magnitude:

$$B = \frac{F_B}{|q|v},$$

where q is the charge of the particle.

We can summarize all these results with the following vector equation:

$$\vec{F}_B = q\vec{v} \times \vec{B}; \tag{29-2}$$

that is, the force $\vec{F}_B$ on the particle is equal to the charge q times the cross product of its velocity $\vec{v}$ and the field $\vec{B}$ (all measured in the same reference frame). Using Eq. 3-27 (Vol. I) for the cross product, we can write the magnitude of $\vec{F}_B$ as

$$F_B = |q|vB \sin \phi, \tag{29-3}$$

where ϕ is the angle between the directions of velocity $\vec{v}$ and magnetic field $\vec{B}$.

Finding the Magnetic Force on a Particle

Equation 29-3 tells us that the magnitude of the force $\vec{F}_B$ acting on a particle in a magnetic field is proportional to the charge q and speed v of the particle. Thus, the force is equal to zero if the charge is zero or if the particle is stationary. Equation 29-3 also tells us that the magnitude of the force is zero if $\vec{v}$ and $\vec{B}$ are either parallel ($\phi = 0°$) or antiparallel ($\phi = 180°$), and the force is at its maximum when $\vec{v}$ and $\vec{B}$ are perpendicular to each other.

Equation 29-2 tells us all this plus the direction of $\vec{F}_B$. From Section 3-8 (Vol. I), we know that the cross product $\vec{v} \times \vec{B}$ in Eq. 29-2 is a vector that is perpendicular to the two vectors $\vec{v}$ and $\vec{B}$. The right-hand rule (Fig. 29-12*a*) tells us that the thumb of the right hand points in the direction of $\vec{v} \times \vec{B}$ when the fingers sweep $\vec{v}$ into $\vec{B}$. If q is positive, then (by Eq. 29-2) the force $\vec{F}_B$ has the same sign as $\vec{v} \times \vec{B}$ and thus must be in the same direction; that is, for positive q, $\vec{F}_B$ is directed along the thumb (Fig. 29-12*b*). If q is negative, then the force $\vec{F}_B$ and cross product $\vec{v} \times \vec{B}$ have opposite signs and thus must be in opposite directions. For negative q, $\vec{F}_B$ is directed opposite the thumb (Fig. 29-12*c*).

Regardless of the sign of the charge, however,

☞ The force $\vec{F}_B$ acting on a charged particle moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ is *always* perpendicular to $\vec{v}$ and $\vec{B}$.

Thus, $\vec{F}_B$ *never* has a component parallel to $\vec{v}$. This means that $\vec{F}_B$ cannot change the particle's speed v (and thus it cannot change the particle's kinetic energy). The force can change only the direction of $\vec{v}$ (and thus the direction of travel); only in this sense can $\vec{F}_B$ accelerate the particle.

To develop a feeling for Eq. 29-2, consider Fig. 29-13, which shows some tracks left by charged particles moving rapidly through a *bubble chamber*. The chamber, which is filled with liquid hydrogen, is immersed in a strong uniform magnetic field that is directed out of the plane of the figure. An incoming gamma ray particle—which leaves no track because it is uncharged—transforms into an electron (spiral track marked e^-) and a positron (track marked e^+) while it knocks an electron out of a hydrogen atom (long track marked e^-). Check with Eq. 29-2 and Fig. 29-12 that the three tracks made by these two negative particles and one positive particle curve in the proper directions.

The SI unit for $\vec{B}$ that follows from Eqs. 29-2 and 29-3 is the newton per coulomb-meter per second. For convenience, this is called the **tesla** (T):

$$1 \text{ tesla} = 1 \text{ T} = 1 \frac{\text{newton}}{(\text{coulomb})(\text{meter/second})}.$$

Recalling that a coulomb per second is an ampere, we have

$$1 \text{ T} = \frac{\text{newton}}{(\text{coulomb/second})(\text{meter})} = 1 \frac{\text{N}}{\text{A} \cdot \text{m}}. \tag{29-4}$$

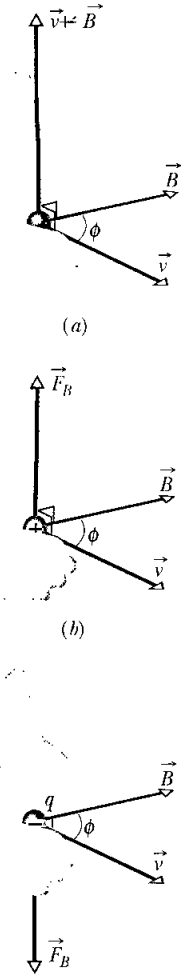


FIG. 29-12 (*a*) The right-hand rule (in which $\vec{v}$ is swept into $\vec{B}$ through the smaller angle ϕ between them) gives the direction of $\vec{v} \times \vec{B}$ as the direction of the thumb. (*b*) If q is positive, then the direction of $\vec{F}_B = q\vec{v} \times \vec{B}$ is in the direction of $\vec{v} \times \vec{B}$. (*c*) If q is negative, then the direction of $\vec{F}_B$ is opposite that of $\vec{v} \times \vec{B}$.

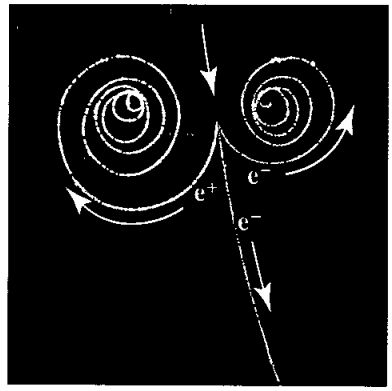


FIG. 29-13 The tracks of two electrons (e^-) and a positron (e^+) in a bubble chamber that is immersed in a uniform magnetic field that is directed out of the plane of the page. *(Lawrence Berkeley Laboratory/Photo Researchers)*

TABLE 29-1

Some Approximate Magnetic Fields

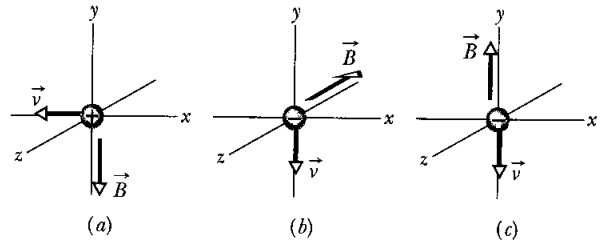
At surface of neutron star	10^8 T
Near big electromagnet	1.5 T
Near small bar magnet	10^{-2} T
At Earth's surface	10^{-4} T
In interstellar space	10^{-10} T
Smallest value in magnetically shielded room	10^{-14} T

An earlier (non-SI) unit for $\vec{B}$, still in common use, is the *gauss* (G), and

$$1 \text{ tesla} = 10^4 \text{ gauss}. \tag{29-5}$$

Table 29-1 lists the magnetic fields that occur in a few situations. Note that Earth's magnetic field near the planet's surface is about 10^{-4} T (= 100 μT or 1 G).

CHECKPOINT 2 The figure shows three situations in which a charged particle with velocity $\vec{v}$ travels through a uniform magnetic field $\vec{B}$. In each situation, what is the direction of the magnetic force $\vec{F}_B$ on the particle?



Sample Problem 29-1

A uniform magnetic field $\vec{B}$, with magnitude 1.2 mT, is directed vertically upward throughout the volume of a laboratory chamber. A proton with kinetic energy 5.3 MeV enters the chamber, moving horizontally from south to north. What magnetic deflecting force acts on the proton as it enters the chamber? The proton mass is 1.67×10^{-27} kg. (Neglect Earth's magnetic field.)

Conceptualize/Classify: Because the proton is charged and moving through a magnetic field, a magnetic force $\vec{F}_B$ can act on it. Because the initial direction of the proton's velocity is not along a magnetic field line, $\vec{F}_B$ is not simply zero.

Compute: Magnitude: To find the magnitude of $\vec{F}_B$, we can use Eq. 29-3 ($F_B = |q|vB \sin\phi$) provided we first find the proton's speed v. We can find v from the given kinetic energy because $K = \frac{1}{2}mv^2$. Solving for v, we obtain

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{(2)(5.3\text{ MeV})(1.60\times10^{-13}\text{ J/MeV})}{1.67\times10^{-27}\text{ kg}}}$$
$$= 3.2\times10^7 \text{ m/s}.$$

Equation 29-3 then yields

$$F_B = |q|vB\sin\phi$$
$$= (1.60\times10^{-19}\text{ C})(3.2\times10^7\text{ m/s})$$
$$\times (1.2\times10^{-3}\text{ T})(\sin 90°)$$
$$= 6.1\times10^{-15}\text{ N}. \qquad \text{(Answer)}$$

This may seem like a small force, but it acts on a particle of small mass, producing a large acceleration; namely,

$$a = \frac{F_B}{m} = \frac{6.1\times10^{-15}\text{ N}}{1.67\times10^{-27}\text{ kg}} = 3.7\times10^{12}\text{ m/s}^2.$$

Direction: To find the direction of $\vec{F}_B$, we use the fact that $\vec{F}_B$ has the direction of the cross product $q\vec{v}\times\vec{B}$. Because the charge q is positive, $\vec{F}_B$ must have the same direction as $\vec{v}\times\vec{B}$ which can be determined with the right-hand rule for cross products (as in Fig. 29-12*b*). We know that $\vec{v}$ is directed horizontally from south to north and $\vec{B}$ is directed vertically up. The right-hand rule shows us that the deflecting force $\vec{F}_B$ must be directed horizontally from west to east, as Fig. 29-14 shows. (The array of dots in the figure represents

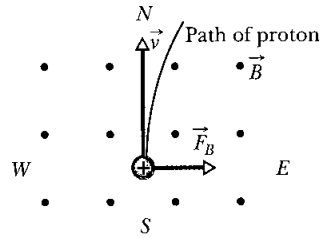


FIG. 29-14 An overhead view of a proton moving from south to north with velocity $\vec{v}$ in a chamber. A magnetic field is directed vertically upward in the chamber, as represented by the array of dots (which resemble the tips of arrows). The proton is deflected toward the east.

a magnetic field directed out of the plane of the figure. An array of **X**s would have represented a magnetic field directed into that plane.)

Conclude: If the charge of the particle were negative, the magnetic deflecting force would be directed in the opposite direction—that is, horizontally from east to west. This is predicted automatically by Eq. 29-2 if we substitute a negative value for q.

29-4 | A Circulating Charged Particle

If a particle moves in a circle at constant speed, we can be sure that the net force acting on the particle is constant in magnitude and points toward the center of the circle, always perpendicular to the particle's velocity. Think of a stone tied to a string and whirled in a circle on a smooth horizontal surface, or of a satellite moving in a circular orbit around Earth. In the first case, the tension in the string provides the necessary force and centripetal acceleration. In the second case, Earth's gravitational attraction provides the force and acceleration.

Figure 29-15 shows another example: A beam of electrons is projected into a chamber by an *electron gun* G. The electrons enter in the plane of the page with speed v and then move in a region of uniform magnetic field $\vec{B}$ directed out of that plane. As a result, a magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$ continuously deflects the electrons, and because $\vec{v}$ and $\vec{B}$ are always perpendicular to each other, this deflection causes the electrons to follow a circular path. The path is visible in the photo because atoms of gas in the chamber emit light when some of the circulating electrons collide with them.

We would like to determine the parameters that characterize the circular motion of these electrons, or of any particle of charge magnitude $|q|$ and mass m moving perpendicular to a uniform magnetic field $\vec{B}$ at speed v. From Eq. 29-3, the force acting on the particle has a magnitude of $|q|vB$. From Newton's second law ($\vec{F} = m\vec{a}$) applied to uniform circular motion (Eq. 6-18, Vol. I),

$$F = m\frac{v^2}{r}, \tag{29-6}$$

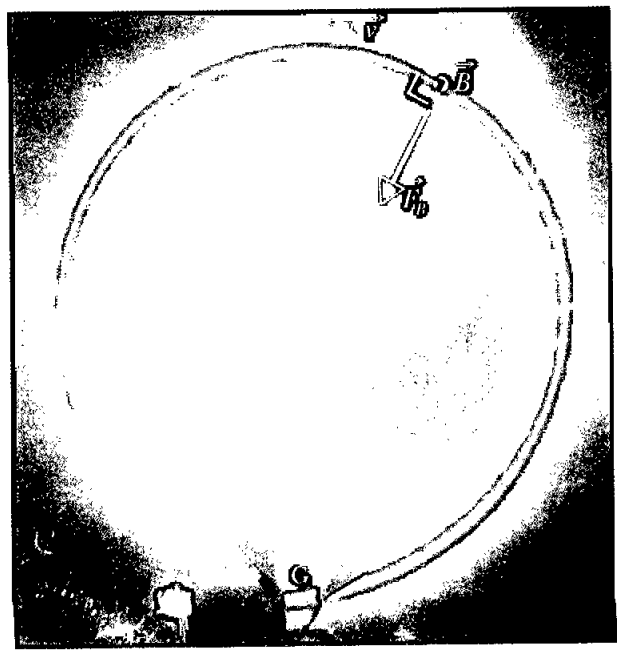


FIG. 29-15 Electrons circulating in a chamber containing gas at low pressure (their path is the glowing circle). A uniform magnetic field $\vec{B}$, pointing directly out of the plane of the page, fills the chamber. Note the radially directed magnetic force $\vec{F}_B$; for circular motion to occur, $\vec{F}_B$ *must* point toward the center of the circle. Use the right-hand rule for cross products to confirm that $\vec{F}_B = q\vec{v} \times \vec{B}$ gives $\vec{F}_B$ the proper direction. (Don't forget the sign of q.) *(Courtesy John Le P. Webb, Sussex University, England)*

we have

$$|q|vB = \frac{mv^2}{r}. \tag{29-7}$$

Solving for r, we find the radius of the circular path as

$$r = \frac{mv}{|q|B} \quad \text{(radius)}. \tag{29-8}$$

The period T (the time for one full revolution) is equal to the circumference divided by the speed:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{|q|B} = \frac{2\pi m}{|q|B} \quad \text{(period)}. \tag{29-9}$$

The frequency f (the number of revolutions per unit time) is

$$f = \frac{1}{T} = \frac{|q|B}{2\pi m} \quad \text{(frequency)}. \tag{29-10}$$

The angular frequency ω of the motion is then

$$\omega = 2\pi f = \frac{|q|B}{m} \quad \text{(angular frequency)}. \tag{29-11}$$

The quantities T, f, and ω do not depend on the speed of the particle (provided the speed is much less than the speed of light). Fast particles move in large circles and slow ones in small circles, but all particles with the same charge-to-mass ratio $|q|/m$ take the same time T (the period) to complete one round trip. Using Eq. 29-2, you can show that if you are looking in the direction of $\vec{B}$, the direction of rotation for a positive particle is always counterclockwise, and the direction for a negative particle is always clockwise.

Sample Problem 29-2

Figure 29-16 shows the essentials of a *mass spectrometer*, which can be used to measure the mass of an ion; an ion of mass m (to be measured) and charge q is produced in source S. The initially stationary ion is accelerated by the electric field due to a potential difference V. The ion leaves S and enters a separator chamber in which a uniform magnetic field $\vec{B}$ is perpendicular to the path of the ion. A wide detector lines the bottom wall of the chamber, and the $\vec{B}$ causes the ion to move in a semicircle and thus strike the detector. Suppose that $B = 80.000$ mT, $V = 1000.0$ V, and ions of charge $q = +1.6022 \times 10^{-19}$ C strike the detector at a point that lies at $x = 1.6254$ m. What is the mass m of the individual ions, in atomic mass units (Eq. 1-7, Vol. I: 1 u = 1.6605×10^{-27} kg)?

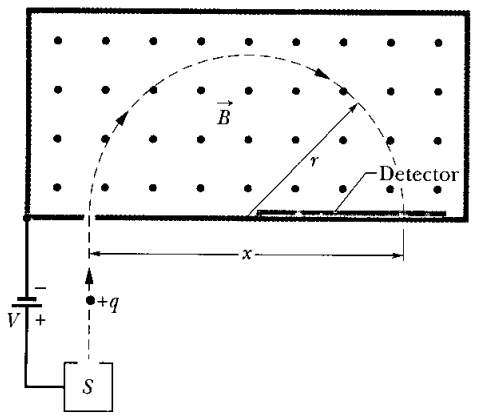


FIG. 29-16 Essentials of a mass spectrometer. A positive ion, after being accelerated from its source S by a potential difference V, enters a chamber of uniform magnetic field $\vec{B}$. There it travels through a semicircle of radius r and strikes a detector at a distance x from where it entered the chamber.

Conceptualize/Classify: (1) Because the (uniform) magnetic field causes the (charged) ion to follow a circular path, we can relate the ion's mass m to the path's radius r with Eq. 29-8 ($r = mv/|q|B$). From Fig. 29-16 we see that $r = x/2$, and we are given the magnitude B of the magnetic field. However, we lack the ion's speed v in the magnetic field after the ion has been accelerated due to the potential difference V. (2) To relate v and V, we use the fact that mechanical energy ($E_{mec} = K + U$) is conserved during the acceleration.

Compute: Finding speed: When the ion emerges from the source, its kinetic energy is approximately zero. At the end of the acceleration, its kinetic energy is $\frac{1}{2}mv^2$. Also, during the acceleration, the positive ion moves through a change in potential of $-V$. Thus, because the ion has positive charge q, its potential energy changes by $-qV$. If we now write the conservation of mechanical energy as

$$\Delta K + \Delta U = 0,$$

we get

$$\tfrac{1}{2}mv^2 - qV = 0$$

or
$$v = \sqrt{\frac{2qV}{m}}. \qquad (29\text{-}12)$$

Finding mass: Substituting this value for v into Eq. 29-8 gives us

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}.$$

Thus,
$$x = 2r = \frac{2}{B}\sqrt{\frac{2mV}{q}}.$$

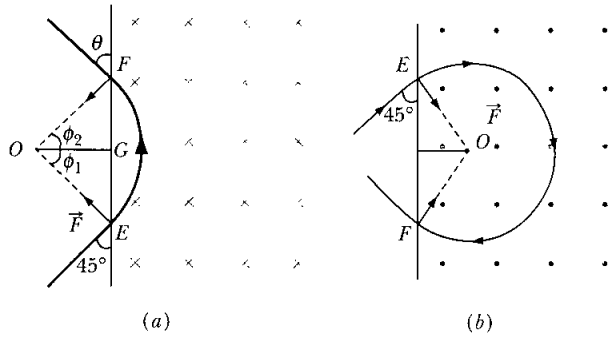
Solving this for m and substituting the given data yield

$$m = \frac{B^2qx^2}{8V}$$
$$= \frac{(0.080000\ \text{T})^2(1.6022\times10^{-19}\ \text{C})(1.6254\ \text{m})^2}{8(1000.0\ \text{V})}$$
$$= 3.3863\times10^{-25}\ \text{kg} = 203.93\ \text{u}. \qquad \text{(Answer)}$$

Conclude: We can easily see that the spectrometer can also be used to separate a given element into its isotopes. Any isotopes of the element will have the same electronic configuration and hence the same chemical properties. However, they will differ in their mass. The above analysis tells us that their radii of semicircular tracks will also be different. In fact, the detector can tell us the relative abundance of the isotopes too!

Sample Problem 29-3

A particle of mass $m = 1.6 \times 10^{-27}$ kg and charge $q = 1.6 \times 10^{-19}$ C enters a region of uniform magnetic field of strength 1 T along the direction shown in Fig. 29-17. The speed of the particle is 10^7 m/s. The magnetic field is directed along the inward normal to the plane of the paper. The particle leaves the region of the field at point F. Find the distance EF and the time spent by the charge in the magnetic field.



(a) (b)

FIG. 29-17 (*a*) A positively charged particle is projected in a magnetic field. (*b*) If the direction of the magnetic field is reversed, the particle follows a different path.

Conceptualize: First of all let us see how the path will look like. At least for the part when the particle is inside the magnetic field, the path will be part of a circle. To locate the center, let us find the direction of $q(\vec{v}\times\vec{B})$. As can be seen in the figure, it is directed perpendicular to velocity and is toward O. Since this force is centripetal in nature, the force is directed toward the center. The coordinates of center can be easily found out. The charge particle will move in anticlockwise direction and exit the magnetic field at the boundary.

Classify: When a charged particle moves perpendicular to a magnetic field the path is a circle with magnetic force qvB directed radially toward the center of the path. So if we draw normals at E and F they will meet at O which is the center of the circle. By geometry of Fig. 29-17*a*:

$$\theta = \phi_2 = \phi_1 = 45°.$$

Compute: As $\phi_1 = \phi_2 = \theta$, therefore

$$EF = 2EG = 2r\sin\theta.$$

Since

$$r = \frac{mv}{qB} = \frac{1.6\times10^{-27}\times10^7}{1.6\times10^{-19}\times1} = 0.1\ \text{m},$$

therefore

$$EF = (2\times0.1\times\sin45°)\ \text{m} = 10\sqrt{2}\ \text{cm}. \qquad \text{(Answer)}$$

Now, as in a magnetic field the speed of a particle remains constant, so if t is the time taken by the particle in the field, then

$$S = v \times t = r\theta$$

$$t = \frac{r\theta}{v}.$$

Hence,

$$t = \frac{0.1 \times \pi/2}{10^7} = 5\pi \times 10^{-9} = 1.57 \times 10^{-8} \text{ s.} \quad \text{(Answer)}$$

Conclude: Now if the direction of the field is out of the page the force will act as shown in Fig. 29-17*b* and so now O will be the center and the angle subtended by the path at O will be $[2\pi - (\pi/2)] = (3/2)\pi$. The time taken will be thrice. Can you predict what would happen if the charge would be negative instead of positive?

Helical Paths

If the velocity of a charged particle has a component parallel to the (uniform) magnetic field, the particle will move in a helical path about the direction of the field vector. Figure 29-18*a*, for example, shows the velocity vector $\vec{v}$ of such a particle resolved into two components, one parallel to $\vec{B}$ and one perpendicular to it:

$$v_{\parallel} = v \cos\phi \quad \text{and} \quad v_{\perp} = v \sin\phi. \tag{29-13}$$

The parallel component determines the *pitch* p of the helix—that is, the distance between adjacent turns (Fig. 29-18*b*). The perpendicular component determines the radius of the helix and is the quantity to be substituted for v in Eq. 29-8.

Figure 29-18*c* shows a charged particle spiraling in a nonuniform magnetic field. The more closely spaced field lines at the left and right sides indicate that the magnetic field is stronger there. When the field at an end is strong enough, the particle "reflects" from that end. If the particle reflects from both ends, it is said to be trapped in a *magnetic bottle.*

Electrons and protons are trapped in this way by the terrestrial magnetic field; the trapped particles form the *Van Allen radiation belts*, which loop well

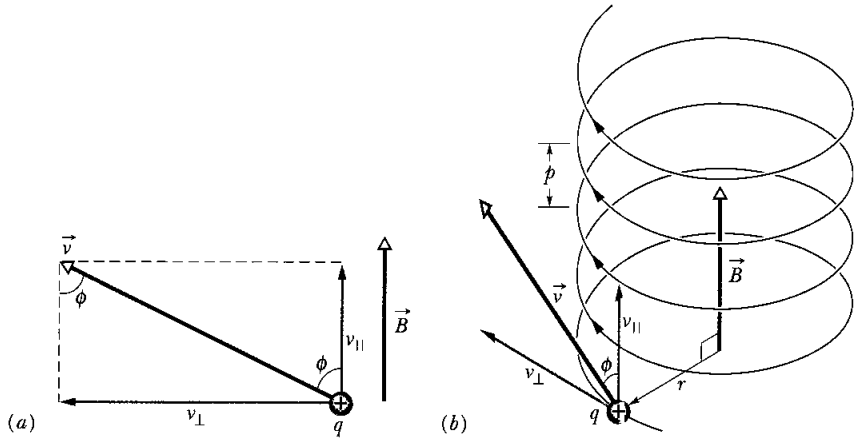


FIG. 29-18 (*a*) A charged particle moves in a uniform magnetic field $\vec{B}$, the particle's velocity $\vec{v}$ making an angle ϕ with the field direction. (*b*) The particle follows a helical path of radius r and pitch p. (*c*) A charged particle spiraling in a nonuniform magnetic field. (The particle can become trapped, spiraling back and forth between the strong field regions at either end.) Note that the magnetic force vectors at the left and right sides have a component pointing toward the center of the figure.

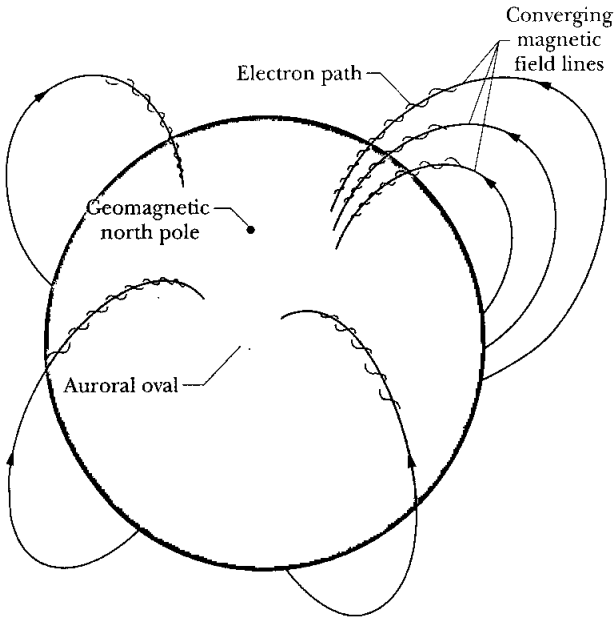


FIG. 29-19 The auroral oval surrounding Earth's geomagnetic north pole. Electrons moving toward Earth are "caught by" and spiral around these field lines, entering the terrestrial atmosphere at high latitudes and producing auroras within the oval.



FIG. 29-20 A false-color image of auroras inside the north auroral oval, recorded by the satellite *Dynamic Explorer*, using ultraviolet light emitted by oxygen atoms excited in the aurora. The sunlit portion of Earth is the crescent at the left. *(Courtesy Dr. L.A. Frank, University of Iowa)*

above Earth's atmosphere between Earth's north and south geomagnetic poles. These particles bounce back and forth, from one end of this magnetic bottle to the other, within a few seconds.

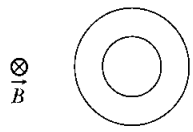
When a large solar flare shoots additional energetic electrons and protons into the radiation belts, an electric field is produced in the region where electrons normally reflect. This field eliminates the reflection and instead drives electrons down into the atmosphere, where they collide with atoms and molecules of air, causing that air to emit light. This light forms the aurora—a curtain of light that hangs down to an altitude of about 100 km. Green light is emitted by oxygen atoms, and pink light is emitted by nitrogen molecules, but often the light is so dim that we perceive only white light.

Auroras extend in arcs above Earth and can occur in a region called the *auroral oval* that is shown in Figs. 29-19 and 29-20 as seen from space. Although an aurora is long, it is only about 100 m thick (north to south) because the paths of the electrons producing it converge as the electrons spiral down the converging magnetic field lines (Fig. 29-19).

29-5 | Cyclotrons

Beams of high-energy particles, such as high-energy electrons and protons, have been enormously useful in probing atoms and nuclei to reveal the fundamental

CHECKPOINT 3 The figure here shows the circular paths of two particles that travel at the same speed in a uniform magnetic field $\vec{B}$, which is directed into the page. One particle is a proton; the other is an electron (which is less massive). (a) Which particle follows the smaller circle, and (b) does that particle travel clockwise or counterclockwise?



Sample Problem 29-4

An electron with a kinetic energy of 22.5 eV moves into a region of uniform magnetic field $\vec{B}$ of magnitude 4.55×10^{-4} T. The angle between the directions of $\vec{B}$ and the electron's velocity $\vec{v}$ is 65.5°. What is the pitch of the helical path taken by the electron?

Conceptualize/Classify: (1) The pitch p is the distance the electron travels parallel to the magnetic field $\vec{B}$ during one period T of circulation. (2) The period T is given by Eq. 29-9 regardless of the angle between the directions of $\vec{v}$ and $\vec{B}$ (provided the angle is not zero, for which there is no circulation of the electron).

Compute: Using Eqs. 29-13 and 29-9, we find

$$p = v_{\parallel} T = (v \cos \phi) \frac{2\pi m}{|q|B}. \qquad (29\text{-}14)$$

We can calculate the electron's speed v from its kinetic energy as we did for the proton in Sample Problem 29-1. We find that $v = 2.81 \times 10^6$ m/s. Substituting this and known data in Eq. 29-14 gives us

$$p = (2.81 \times 10^6 \text{ m/s})(\cos 65.5°) \times \frac{2\pi(9.11 \times 10^{-31} \text{ kg})}{(1.60 \times 10^{-19} \text{ C})(4.55 \times 10^{-4} \text{ T})}$$
$$= 9.16 \text{ cm}. \qquad \text{(Answer)}$$

Conclude: If we imagine the particle to be projected in xy plane from the origin with the magnetic field along $+y$ axis, the particle will move in a helix whose axis is parallel to the y axis. It will touch the y axis after regular time interval of T (after it has completed a revolution). Its maximum x coordinate will be R and the minimum will be $-R$. If the particle has a positive charge and is projected in first quadrant, its z coordinate will range from 0 to $2R$. Can you see this on the basis of Fig. 29-18*b*?

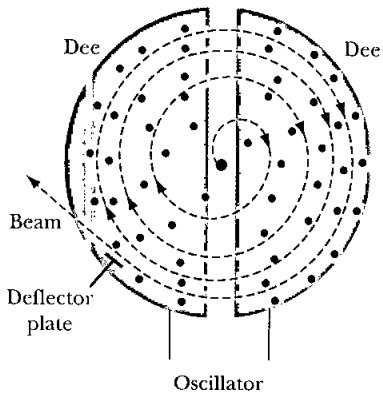


FIG. 29-21 The elements of a cyclotron, showing the particle source S and the dees. A uniform magnetic field is directed up from the plane of the page. Circulating protons spiral outward within the hollow dees, gaining energy every time they cross the gap between the dees.

structure of matter. Such beams were instrumental in the discovery that atomic nuclei consist of protons and neutrons and in the discovery that protons and neutrons consist of quarks and gluons. The challenge of such beams is how to make and control them. Because electrons and protons are charged, they can be accelerated to the required high energy if they move through large potential differences. Because electrons have low mass, accelerating them in this way can be done in a reasonable distance. However, because protons (and other charged particles) have greater mass, the distance required for the acceleration is too long.

A clever solution to this problem is first to let protons and other massive particles move through a modest potential difference (so that they gain a modest amount of energy) and then use a magnetic field to cause them to circle back and move through a modest potential difference again. If this procedure is repeated thousands of times, the particles end up with a very large energy.

Here we discuss an *accelerator* that employs a magnetic field to repeatedly bring particles back to an accelerating region, where they gain more and more energy until they finally emerge as a high-energy beam.

The Cyclotron

Figure 29-21 is a top view of the region of a *cyclotron* in which the particles (protons, say) circulate. The two hollow **D**-shaped objects (each open on its straight edge) are made of sheet copper. These *dees,* as they are called, are part of an electrical oscillator that alternates the electric potential difference across the gap between the dees. The electrical signs of the dees are alternated so that the electric field in the gap alternates in direction, first toward one dee and then toward the other dee, back and forth. The dees are immersed in a large magnetic field directed out of the plane of the page. The magnitude B of this field is set via a control on the electromagnet producing the field.

Suppose that a proton, injected by source S at the center of the cyclotron in Fig. 29-21, initially moves toward a negatively charged dee. It will accelerate

toward this dee and enter it. Once inside, it is shielded from electric fields by the copper walls of the dee; that is, the electric field does not enter the dee. The magnetic field, however, is not screened by the (nonmagnetic) copper dee, so the proton moves in a circular path whose radius, which depends on its speed, is given by Eq. 29-8 ($r = mv/|q|B$).

Let us assume that at the instant the proton emerges into the center gap from the first dee, the potential difference between the dees is reversed. Thus, the proton *again* faces a negatively charged dee and is *again* accelerated. This process continues, the circulating proton always being in step with the oscillations of the dee potential, until the proton has spiraled out to the edge of the dee system. There a deflector plate sends it out through a portal.

The key to the operation of the cyclotron is that the frequency f at which the proton circulates in the magnetic field (and that does *not* depend on its speed) must be equal to the fixed frequency f_{osc} of the electrical oscillator, or

$$f = f_{osc} \qquad \text{(resonance condition)}. \tag{29-15}$$

This *resonance condition* says that, if the energy of the circulating proton is to increase, energy must be fed to it at a frequency f_{osc} that is equal to the natural frequency f at which the proton circulates in the magnetic field.

Combining Eqs. 29-10 ($f = |q|B/2\pi m$) and 29-15 allows us to write the resonance condition as

$$|q|B = 2\pi m f_{osc}. \tag{29-16}$$

For the proton, q and m are fixed. The oscillator (we assume) is designed to work at a single fixed frequency f_{osc}. We then "tune" the cyclotron by varying B until Eq. 29-16 is satisfied, and then many protons circulate through the magnetic field, to emerge as a beam.

Other types of charged particles can be accelerated by a cyclotron to emerge as a high-energy beam. For example, *deuterons* (bundles of one proton and one neutron) can be accelerated because they each have the same charge as a proton. However, electrically neutral particles, such as neutrons, cannot be accelerated by a cyclotron. Still, a beam of high-energy neutrons can be produced by having a deuteron beam smash into a beryllium target. This is the arrangement used in hospitals providing fast-neutron cancer therapy. The deuteron beam emerging from a cyclotron at the hospital is sent into a beryllium target in front of the patient's cancerous region. As deuterons collide with beryllium nuclei, neutrons are knocked out of the beryllium nuclei and into the cancerous region, where they break bonds in the DNA of the cancer cells, "killing" the cells (Fig. 29-22). All this can be done within a hospital because the deuterons can be accelerated to high energy while circling in a cyclotron instead of along an extremely long straight path.

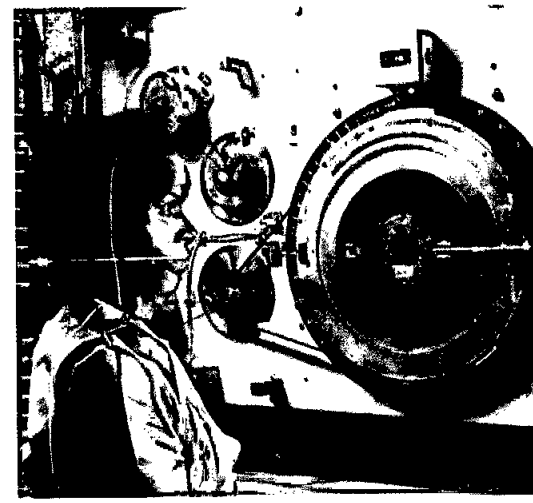


FIG. 29-22 Because it is invisible, a beam of neutrons from the portal at the right is aligned via laser crosshairs, seen superimposed on the patient. *(Fermilab/Science Photo Library/Photo Researchers)*

Sample Problem 29-5

Suppose a cyclotron is operated at an oscillator frequency of 12 MHz and has a dee radius $R = 53$ cm.

(a) What is the magnitude of the magnetic field needed for deuterons to be accelerated in the cyclotron? The deuteron mass is $m = 3.34 \times 10^{-27}$ kg.

Conceptualize/Classify: For a given oscillator frequency f_{osc}, the magnetic field magnitude B required to accelerate any particle in a cyclotron depends on the ratio $m/|q|$ of mass to charge for the particle, according to Eq. 29-16 ($|q|B = 2\pi m f_{osc}$).

Compute: For deuterons and the oscillator frequency $f_{osc} = 12$ MHz, we find

$$B = \frac{2\pi m f_{osc}}{|q|} = \frac{(2\pi)(3.34\times10^{-27}\,\text{kg})(12\times10^{6}\,\text{s}^{-1})}{1.60\times10^{-19}\,\text{C}}$$

$$= 1.57\,\text{T} \approx 1.6\,\text{T}. \qquad \text{(Answer)}$$

Conclude: Note that, to accelerate protons, B would have to be reduced by a factor of 2, provided the oscillator frequency remained fixed at 12 MHz.

(b) What is the resulting kinetic energy of the deuterons?

Conceptualize: The kinetic energy ($\frac{1}{2}mv^2$) of a deuteron exiting the cyclotron is equal to the kinetic energy it had just before exiting, when it was traveling in a circular path with a radius approximately equal to the radius R of the cyclotron dees.

Classify: We can find the speed v of the deuteron in that circular path with Eq. 29-8 ($r = mv/|q|B$).

Compute: Solving that equation for v, substituting R for r, and then substituting known data, we find

$$v = \frac{R|q|B}{m} = \frac{(0.53\text{ m})(1.60\times10^{-19}\text{ C})(1.57\text{ T})}{3.34\times10^{-27}\text{ kg}}$$
$$= 3.99\times10^{7}\text{ m/s}.$$

This speed corresponds to a kinetic energy of

$$K = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}(3.34\times10^{-27}\text{ kg})(3.99\times10^{7}\text{ m/s})^2$$
$$= 2.7\times10^{-12}\text{ J}, \qquad \text{(Answer)}$$

or about 17 MeV.

Conclude: As is obvious in Fig. 29-21, the radius of the deuteron does not increase uniformly throughout the motion. The change in the kinetic energy as it passes every time through the gap is the same; it passes through the same potential difference each time. But as its speed increases, the change in speed would be lesser. So the change in radius is also lesser.

29-6 | Crossed Fields: Discovery of the Electron

We have studied two kinds of forces that can be exerted on an electrically charged particle: the electric force given by Eq. 22-17 and the magnetic force given by Eq. 29-2. Taken together, they constitute all of the forces that can by exerted on a particle by virtue of its electric charge q. Their sum F is called Lorentz force and is given by

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}). \qquad (29\text{-}17)$$

We will now consider the Lorentz force in some important cases. When the two fields are perpendicular to each other, they are said to be *crossed fields.* Here we shall examine what happens to charged particles—namely, electrons—as they move through crossed fields. We use as our example the experiment that led to the discovery of the electron in 1897 by J. J. Thomson at Cambridge University.

Figure 29-23 shows a modern, simplified version of Thomson's experimental apparatus—a *cathode ray tube* (which is like the picture tube in a standard television set). Charged particles (which we now know as electrons) are emitted by a hot filament at the rear of the evacuated tube and are accelerated by an applied potential difference V. After they pass through a slit in screen C, they

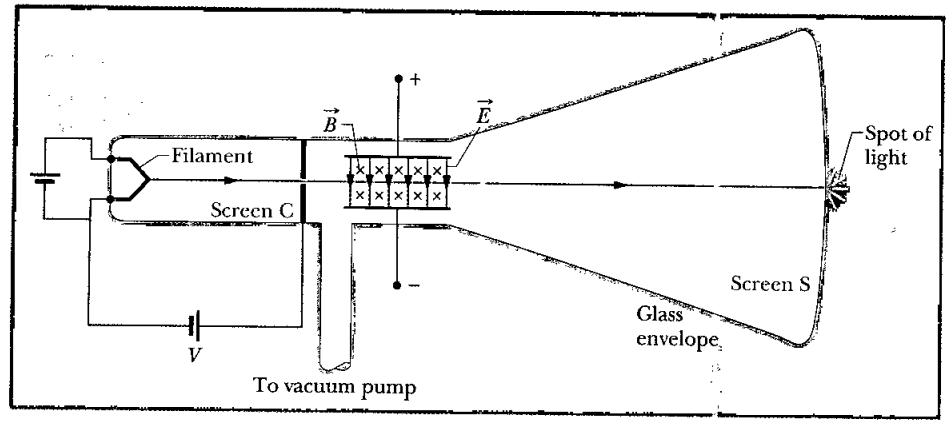


FIG. 29-23 A modern version of J. J. Thomson's apparatus for measuring the ratio of mass to charge for the electron. An electric field $\vec{E}$ is established by connecting a battery across the deflecting-plate terminals. The magnetic field $\vec{B}$ is set up by means of a current in a system of coils (not shown). The magnetic field shown is into the plane of the figure, as represented by the array of **X**s (which resemble the feathered ends of arrows).

form a narrow beam. They then pass through a region of crossed $\vec{E}$ and $\vec{B}$ fields, headed toward a fluorescent screen S, where they produce a spot of light (on a television screen the spot is part of the picture). The forces on the charged particles in the crossed-fields region can deflect them from the center of the screen. By controlling the magnitudes and directions of the fields, Thomson could thus control where the spot of light appeared on the screen. Recall that the force on a negatively charged particle due to an electric field is directed opposite the field. Thus, for the arrangement of Fig. 29-23, electrons are forced up the page by electric field $\vec{E}$ and down the page by magnetic field $\vec{B}$; that is, the forces are *in opposition*. Thomson's procedure was equivalent to the following series of steps.

1. Set $E = 0$ and $B = 0$ and note the position of the spot on screen S due to the undeflected beam.
2. Turn on $\vec{E}$ and measure the resulting beam deflection.
3. Maintaining $\vec{E}$, now turn on $\vec{B}$ and adjust its value until the beam returns to the undeflected position. (With the forces in opposition, they can be made to cancel.)

We discussed the deflection of a charged particle moving through an electric field $\vec{E}$ between two plates (step 2 here) in Sample Problem 22-12. We found that the deflection of the particle at the far end of the plates is

$$y = \frac{|q|EL^2}{2mv^2}, \tag{29-18}$$

where v is the particle's speed, m its mass, and q its charge, and L is the length of the plates. We can apply this same equation to the beam of electrons in Fig. 29-23; if need be, we can calculate the deflection by measuring the deflection of the beam on screen S and then working back to calculate the deflection y at the end of the plates. (Because the direction of the deflection is set by the sign of the particle's charge, Thomson was able to show that the particles that were lighting up his screen were negatively charged.)

When the two fields in Fig. 29-23 are adjusted so that the two deflecting forces cancel (step 3), we have from Eqs. 29-1 and 29-3

$$|q|E = |q|vB \sin(90°) = |q|vB$$

or

$$v = \frac{E}{B}. \tag{29-19}$$

Thus, the crossed fields allow us to measure the speed of the charged particles passing through them. Substituting Eq. 29-19 for v in Eq. 29-18 and rearranging yield

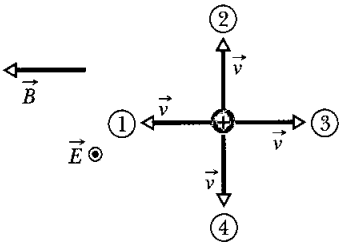
$$\frac{m}{|q|} = \frac{B^2L^2}{2yE}, \tag{29-20}$$

in which all quantities on the right can be measured. Thus, the crossed fields allow us to measure the ratio $m/|q|$ of the particles moving through Thomson's apparatus.

Thomson claimed that these particles are found in all matter. He also claimed that they are lighter than the lightest known atom (hydrogen) by a factor of more than 1000. (The exact ratio proved later to be 1836.15.) His $m/|q|$ measurement, coupled with the boldness of his two claims, is considered to be the "discovery of the electron."

Incidentally, this arrangement of adjusting electric field and magnetic field so that the net force on the charge particle becomes zero; this is called a ***velocity selector*** and is employed in mass spectrometer as well.

CHECKPOINT 4 The figure shows four directions for the velocity vector $\vec{v}$ of a positively charged particle moving through a uniform electric field $\vec{E}$ (directed out of the page and represented with an encircled dot) and a uniform magnetic field $\vec{B}$. (a) Rank directions 1, 2, and 3 according to the magnitude of the net force on the particle, greatest first. (b) Of all four directions, which might result in a net force of zero?

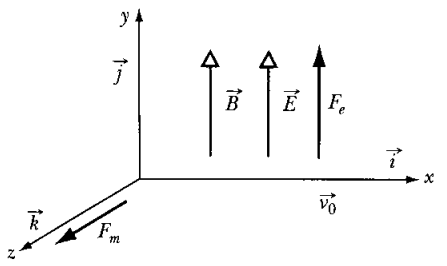


Sample Problem 29-6

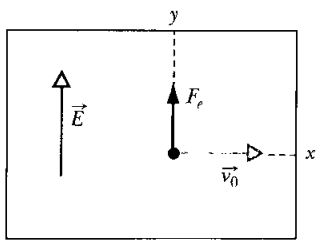
A particle of mass m and charge q is moving in a region where uniform, constant electric and magnetic fields $\vec{E}$ and $\vec{B}$ are present. $\vec{E}$ and $\vec{B}$ are parallel to each other. At time $t = 0$ the velocity $\vec{v}_0$ of the particle is perpendicular to $\vec{E}$. Find the velocity $\vec{v}$ of the particle at time t. You must express your answer in terms of t, q, m, $\vec{v}_0$, $\vec{E}$, and $\vec{B}$, and their magnitudes v_0, E and B.

Conceptualize: In this case, we cannot say that the path is circular or a helix with uniform pitch. The motion is rather complicated; here we have to use the Lorentz force (Eq. 29-17).

Classify: The forces due to parallel electric and magnetic fields on a charged particle moving perpendicular to the fields will be at right angles to each other (electric force being along the direction of $\vec{E}$ while magnetic force being perpendicular to the plane containing $\vec{v}$ and $\vec{B}$). Therefore the magnetic force will not affect the motion of charged particle in the direction of electric field and vice versa. Therefore the problem is equivalent to superposition of two independent motions as shown in Fig. 29-24.



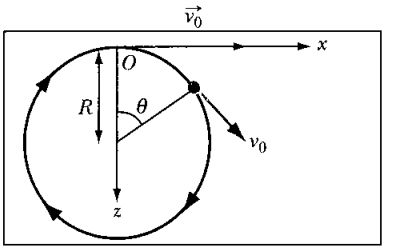
(a)



xy plane

(b)

FIG. 29-24 (a) Force on a charged particle in uniform electric and magnetic field. (b) Force on it due to electric field alone. Note that the charge is positive, so the force is in the direction of the electric field. This force is independent of the velocity.



xz plane

(c)

FIG. 29-24 (c) The force due to the magnetic field is always perpendicular to the velocity. So under its influence, the particle will move in a circular path.

Compute: For motion of particle under electric field alone,

$$a_y = \frac{qE}{m}, \quad \text{i.e.,} \quad \frac{dv_y}{dt} = \frac{qE}{m} \qquad \left(\text{as } a = \frac{F}{m} \text{ and } F = qE\right)$$

or

$$\int_0^{v_y} dv_y = \int_0^t \frac{qE}{m}\,dt, \quad \text{i.e.,} \quad v_y = \frac{qE}{m}t. \qquad (29\text{-}21)$$

While at the same time the charged particle under the action of magnetic field will describe a circle in the xz plane with

$$r = \frac{mv_0}{qB}, \qquad \text{i.e.,} \qquad \omega = \frac{v_0}{r}\frac{qB}{m}.$$

So the angular position of the particle at time t in the xz plane (or in a plane parallel to xz plane) will be given by

$$\theta = \omega t = \frac{qB}{m}t.$$

Hence, in accordance with Fig. 29-24c, we have

$$v_x = v_0\cos\theta = v_0\cos\omega t = v_0\cos\left(\frac{qB}{m}\right)t \qquad (29\text{-}22)$$

and

$$v_z = v_0\sin\theta = v_0\sin\omega t = v_0\sin\left(\frac{qB}{m}\right)t. \qquad (29\text{-}23)$$

Note that magnetic field does not do any work on the charged particle, so it can change the direction of the velocity v_0, but not its magnitude. Also the angular velocity of its circular motion is independent of its velocity.

So in the light of Eqs. 29-21 to 29-23 we have

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$
$$= v_0\cos\left(\frac{qB}{m}t\right)\hat{i} + \left(\frac{qE}{m}t\right)\hat{j} + v_0\sin\left(\frac{qB}{m}t\right)\hat{k}.$$

Since in this case

$$\hat{i} = \frac{\vec{v}_0}{v_0},\ \hat{j} = \frac{\vec{E}}{E} = \frac{\vec{B}}{B},\ \text{and}\ \hat{k} = \frac{\vec{v}_0 \times \vec{B}}{v_0 B};$$

therefore we have

$$\vec{v} = v_0\cos\left[\frac{qB}{m}t\right]\left[\frac{\vec{v}_0}{v_0}\right] + \frac{qE}{m}t\left[\frac{\vec{E}}{E}\right] + v_0\sin\left(\frac{qB}{m}t\right)\left[\frac{\vec{v}_0 \times \vec{B}}{v_0 B}\right].$$

Conclude: The answer seems to be quite complicated, but let us analyze the motion. In y direction, the motion is uniformly accelerated. The motion in xz plane is circular. So the resultant path is a helix with non-uniform pitch. The path looks like that shown in Fig. 29-25. As before, the particle repeatedly touches the y axis (if it was projected from the origin) after regular time intervals of T when it has completed one circle. But now the motion along the y axis is accelerated, so the distance between successive points on the y axis where it touches is not the same.

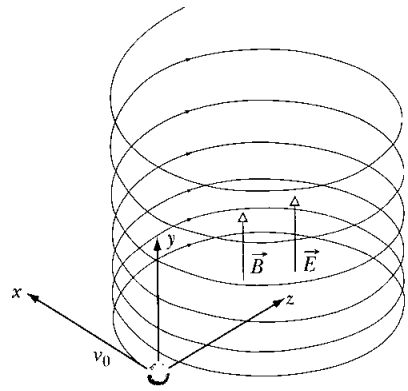


FIG. 29-25 In a uniform magnetic field which is parallel to a uniform electric field, the charged particle follows a helical path with variable pitch.

CHECKPOINT 5 Show that the successive points where the particle touches the y axis are in the ratio 1:3:5 ....

Sample Problem 29-7

If a uniform electric field is at right angles to the magnetic field and a charge is released from the origin as shown in Fig. 29-26. Suppose, for instance, that B points in the x direction and E in the z direction. What path will the charge follow?

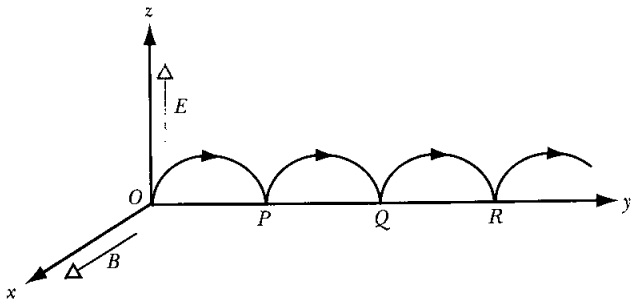


FIG. 29-26 A charged particle starts from rest under the influence of a uniform electric field and a uniform magnetic field. Note that its average displacement is not in the direction of the electric field but perpendicular to it.

Conceptualize: Let's think it through qualitatively. Initially, the particle is at rest, so the magnetic force on it is zero and the electric field accelerates the charge in the z direction. As it speeds up, a magnetic force develops which, according to Eq. 29-17, rotates its velocity vector. The faster it goes, the stronger the magnetic field force becomes; eventually, it curves the particle back around the y axis. At this point the charge is moving against the electrical force. So it begins to slow down. The magnetic force then decreases, and the electrical force takes over, bringing the charge to rest at point P (see Fig. 29-26). There the entire process commences anew, carrying the particle over to point Q, and so on.

Classify: This is a case where we need to apply Eq. 29-17 to decide on the path followed by the charge.

Compute: There being no force in the x direction, the velocity of the particle at any time t can be described by the vector

$$\vec{v} = v_y\hat{j} + v_z\hat{k}.$$

By applying the equation of Lorentz force, we get

$$\vec{F} = q[E\hat{k} + (v_y\hat{j} + v_z\hat{k}) \times B\hat{i}]$$
$$= q(E - v_y B)\hat{k} + qv_z B\hat{j}$$

Treating the y and z components separately we get

$$ma_y = qv_z B$$
$$ma_z = q(E - v_y B).$$

Let us take

$$\omega = \frac{qB}{m}.$$

Then the equation becomes

$$a_y = \omega v_z$$
$$a_z = \omega\left(\frac{E}{B} - v_y\right).$$

By differentiating the second and using the first to eliminate a_y, we get

$$\frac{d^2 v_z}{dt^2} = -\omega^2 v_z.$$

This is the familiar equation of simple harmonic motion. Its solution is given by

$$v_z = v_0 \sin(\omega t + \phi).$$

At $t = 0$, $v_z = 0$. This means that $\phi = 0$. Substituting this expression of v_z in the second equation, we get

$$v_y = \frac{E}{B} - v_0 \cos(\omega t).$$

At $t = 0$, v_y is also zero. This means that

$$v_0 = \frac{E}{B}.$$

Therefore we get

$$v_y = \frac{E}{B}[1 - \cos(\omega t)],$$
$$v_z = \frac{E}{B}\sin(\omega t).$$

By integrating these two expressions, we get

$$y = \frac{E}{\omega B}[\omega t - \sin(\omega t)],$$
$$z = \frac{E}{\omega B}[1 - \cos(\omega t)].$$

In this expression, if we let

$$R = \frac{E}{\omega B}$$

and eliminate sine and cosine, we find that

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

This is the equation of a circle of radius R whose center is at $(0, R\omega t, R)$. This means that the center travels in y direction at a constant speed of E/B.

Conclude: The particle moves as though it were a spot on the rim of a wheel, rolling down the y axis at a speed v. This path is called a cycloid.

CHECKPOINT 6 Of the three vectors in the equation $\vec{F} = q(\vec{v} \times \vec{B})$, which pairs are always at right angles? Which may have any angle between them?

29-7 | The Hall Effect and the Force on a Current-Carrying Wire

As we just discussed, a beam of electrons in a vacuum can be deflected by a magnetic field. Can the drifting conduction electrons in a copper wire also be deflected by a magnetic field? In 1879, Edwin H. Hall, then a 24-year-old graduate student at the Johns Hopkins University, showed that they can. This **Hall effect** allows us to find out whether the charge carriers in a conductor are positively or negatively charged. Beyond that, we can measure the number of such carriers per unit volume of the conductor.

Figure 29-27*a* shows a copper strip of width d, carrying a current i whose conventional direction is from the top of the figure to the bottom. The charge carriers are electrons and, as we know, they drift (with drift speed v_d) in the opposite direction, from bottom to top. At the instant shown in Fig. 29-27*a*, an external magnetic field $\vec{B}$, pointing into the plane of the figure, has just been turned on. From Eq. 29-2 we see that a magnetic deflecting force $\vec{F}_B$ will act on each drifting electron, pushing it toward the right edge of the strip.

As time goes on, electrons move to the right, mostly piling up on the right edge of the strip, leaving uncompensated positive charges in fixed positions at the left edge. The separation of positive and negative charges produces an electric field $\vec{E}$

within the strip, pointing from left to right in Fig. 29-27*b*. This field exerts an electric force $\vec{F}_E$ on each electron, tending to push it to the left.

An equilibrium quickly develops in which the electric force on each electron builds up until it just cancels the magnetic force. When this happens, as Fig. 29-27*b* shows, the force due to $\vec{B}$ and the force due to $\vec{E}$ are in balance. The drifting electrons then move along the strip toward the top of the page at velocity $\vec{v}_d$ with no further collection of electrons on the right edge of the strip and thus no further increase in the electric field $\vec{E}$.

A *Hall potential difference* V is associated with the electric field across strip width d. From Eq. 24-39, the magnitude of that potential difference is

$$V = Ed. \tag{29-24}$$

By connecting a voltmeter across the width, we can measure the potential difference between the two edges of the strip. Moreover, the voltmeter can tell us which edge is at higher potential. For the situation of Fig. 29-27*b*, we would find that the left edge is at higher potential, which is consistent with our assumption that the charge carriers are negatively charged.

For a moment, let us make the opposite assumption, that the charge carriers in current i are positively charged (Fig. 29-27*c*). Convince yourself that as these charge carriers move from top to bottom in the strip, they are pushed to the right edge by $\vec{F}_B$ and thus that the *right* edge is at higher potential. Because that last statement is contradicted by our voltmeter reading, the charge carriers must be negatively charged.

Now for the quantitative part. When the electric and magnetic forces are in balance (Fig. 29-27*b*), Eqs. 29-1 and 29-3 give us

$$eE = ev_dB. \tag{29-25}$$

From Eq. 27-7, the drift speed v_d is

$$v_d = \frac{J}{ne} = \frac{i}{neA}, \tag{29-26}$$

in which J (= i/A) is the current density in the strip, A is the cross-sectional area of the strip, and n is the *number density* of charge carriers (their number per unit volume).

In Eq. 29-25, substituting for E with Eq. 29-24 and substituting for v_d with Eq. 29-26, we obtain

$$n = \frac{Bi}{Vle}, \tag{29-27}$$

in which l (= A/d) is the thickness of the strip. With this equation we can find n from measurable quantities.

It is also possible to use the Hall effect to measure directly the drift speed v_d of the charge carriers, which you may recall is of the order of centimeters per hour. In this clever experiment, the metal strip is moved mechanically through the magnetic field in a direction opposite that of the drift velocity of the charge carriers. The speed of the moving strip is then adjusted until the Hall potential difference vanishes. At this condition, with no Hall effect, the velocity of the charge carriers *with respect to the laboratory frame* must be zero, so the velocity of the strip must be equal in magnitude but opposite the direction of the velocity of the negative charge carriers.

Hence a magnetic field exerts a sideways force on electrons moving in a wire. This force must then be transmitted to the wire itself, because the conduction electrons cannot escape sideways out of the wire.

In Fig. 29-28*a*, a vertical wire, carrying no current and fixed in place at both ends, extends through the gap between the vertical pole faces of a magnet. The magnetic field between the faces is directed outward from the page.

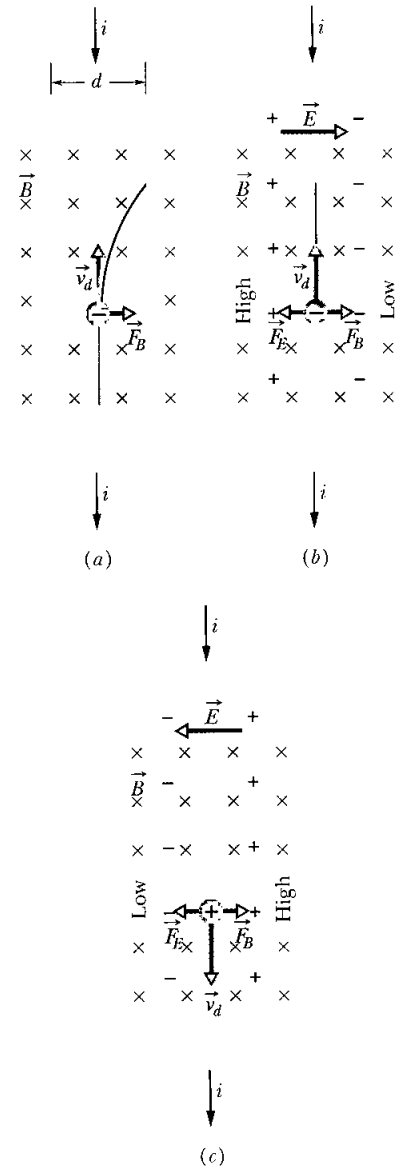


FIG. 29-27 A strip of copper carrying a current i is immersed in a magnetic field $\vec{B}$. (*a*) The situation immediately after the magnetic field is turned on. The curved path that will then be taken by an electron is shown. (*b*) The situation at equilibrium, which quickly follows. Note that negative charges pile up on the right side of the strip, leaving uncompensated positive charges on the left. Thus, the left side is at a higher potential than the right side. (*c*) For the same current direction, if the charge carriers were positively charged, *they* would pile up on the right side, and the right side would be at the higher potential.

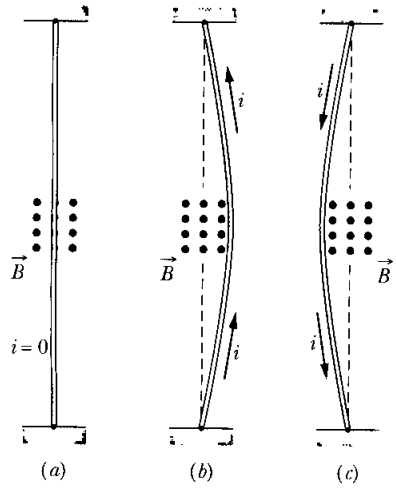


FIG. 29-28 A flexible wire passes between the pole faces of a magnet (only the farther pole face is shown). (*a*) Without current in the wire, the wire is straight. (*b*) With upward current, the wire is deflected rightward. (*c*) With downward current, the deflection is leftward. The connections for getting the current into the wire at one end and out of it at the other end are not shown.

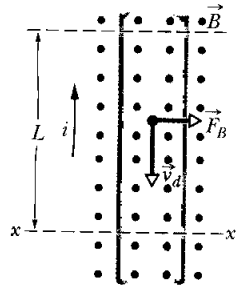


FIG. 29-29 A close-up view of a section of the wire of Fig. 29-28*b*. The current direction is upward, which means that electrons drift downward. A magnetic field that emerges from the plane of the page causes the electrons and the wire to be deflected to the right.

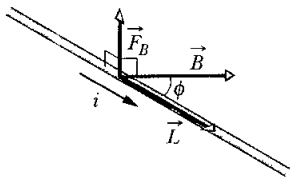


FIG. 29-30 A wire carrying current i makes an angle ϕ with magnetic field $\vec{B}$. The wire has length L in the field and length vector $\vec{L}$ (in the direction of the current). A magnetic force $\vec{F}_B = i\vec{L} \times \vec{B}$ acts on the wire.

In Fig. 29-28*b*, a current is sent upward through the wire; the wire deflects to the right. In Fig. 29-28*c*, we reverse the direction of the current and the wire deflects to the left.

Figure 29-29 shows what happens inside the wire of Fig. 29-28*b*. We see one of the conduction electrons, drifting downward with an assumed drift speed v_d. Equation 29-3, in which we must put $\phi = 90°$, tells us that a force $\vec{F}_B$ of magnitude ev_dB must act on each such electron. From Eq. 29-2 we see that this force must be directed to the right. We expect then that the wire as a whole will experience a force to the right, in agreement with Fig. 29-28*b*.

If, in Fig. 29-29, we were to reverse *either* the direction of the magnetic field *or* the direction of the current, the force on the wire would reverse, being directed now to the left. Note too that it does not matter whether we consider negative charges drifting downward in the wire (the actual case) or positive charges drifting upward. The direction of the deflecting force on the wire is the same. We are safe then in dealing with a current of positive charge.

Consider a length L of the wire in Fig. 29-29. All the conduction electrons in this section of wire will drift past plane xx in Fig. 29-29 in a time $t = L/v_d$. Thus, in that time a charge given by

$$q = it = i\frac{L}{v_d}$$

will pass through that plane. Substituting this into Eq. 29-3 yields

$$F_B = qv_dB\sin\phi = \frac{iL}{v_d}v_dB\sin 90°$$

or

$$F_B = iLB. \tag{29-28}$$

This equation gives the magnetic force that acts on a length L of straight wire carrying a current i and immersed in a magnetic field $\vec{B}$ that is perpendicular to the wire.

If the magnetic field is *not* perpendicular to the wire, as in Fig. 29-30, the magnetic force is given by a generalization of Eq. 29-28:

$$\vec{F}_B = i\vec{L} \times \vec{B} \qquad \text{(force on a current)}. \tag{29-29}$$

Here $\vec{L}$ is a *length vector* that has magnitude L and is directed along the wire segment in the direction of the (conventional) current.

Equation 29-29 is equivalent to Eq. 29-2 in that either can be taken as the defining equation for $\vec{B}$. In practice, we define $\vec{B}$ from Eq. 29-29 because it is much easier to measure the magnetic force acting on a wire than that on a single moving charge.

If a wire is not straight or the field is not uniform, we can imagine the wire broken up into small straight segments and apply Eq. 29-29 to each segment. The force on the wire as a whole is then the vector sum of all the forces on the segments that make it up. In the differential limit, we can write

$$d\vec{F}_B = i\,d\vec{L} \times \vec{B}, \tag{29-30}$$

and we can find the resultant force on any given arrangement of currents by integrating Eq. 29-30 over that arrangement.

In using Eq. 29-30, bear in mind that there is no such thing as an isolated current-carrying wire segment of length dL. There must always be a way to introduce the current into the segment at one end and take it out at the other end.

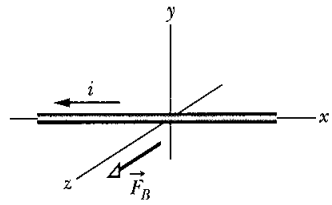
In a uniform field, the net force on a current loop is zero, that is

$$\vec{F} = i\oint(d\vec{L} \times \vec{B}) = i\oint d\vec{L} \times \vec{B} = 0.$$

The constant B comes outside the integral and $\oint d\vec{L}$ around a closed loop vanishes. However, in a nonuniform field, this is not the case. This analysis also implies that if a current-carrying wire of arbitrary shape is kept in a uniform magnetic field, the force experienced by it can be found by replacing the wire by a straight wire joining its ends:

$$\vec{F} = i\int(d\vec{L}\times\vec{B}) = i\int d\vec{L}\times\vec{B} = i\vec{L}\times\vec{B}.$$

CHECKPOINT 7 The figure shows a current i through a wire in a uniform magnetic field $\vec{B}$, as well as the magnetic force $\vec{F}_B$ acting on the wire. The field is oriented so that the force is maximum. In what direction is the field?



Sample Problem 29-8

A straight, horizontal length of copper wire has a current $i = 28$ A through it. What are the magnitude and direction of the minimum magnetic field $\vec{B}$ needed to suspend the wire—that is, to balance the gravitational force on it? The linear density (mass per unit length) of the wire is 46.6 g/m.

Conceptualize/Classify: (1) Because the wire carries a current, a magnetic force $\vec{F}_B$ can act on the wire if we place it in a magnetic field $\vec{B}$. To balance the downward gravitational force $\vec{F}_g$ on the wire, we want $\vec{F}_B$ to be directed upward (Fig. 29-31). (2) The direction of $\vec{F}_B$ is related to the directions of $\vec{B}$ and the wire's length vector $\vec{L}$ by Eq. 29-29 ($\vec{F}_B = i\vec{L}\times\vec{B}$).

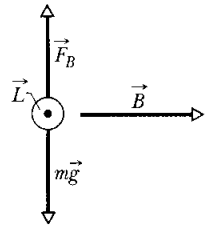


FIG. 29-31 A wire (shown in cross section) carrying current out of the page.

Compute: Because $\vec{L}$ is directed horizontally (and the current is taken to be positive), Eq. 29-29 and the right-hand rule for cross products tell us that $\vec{B}$ must be horizontal and rightward (in Fig. 29-31) to give the required upward $\vec{F}_B$.

The magnitude of $\vec{F}_B$ is $F_B = iLB\sin\phi$. Because we want $\vec{F}_B$ to balance $\vec{F}_g$, we want

$$iLB\sin\phi = mg, \tag{29-31}$$

where mg is the magnitude of $\vec{F}_g$ and m is the mass of the wire. We also want the minimal field magnitude B for $\vec{F}_B$ to balance $\vec{F}_g$. Thus, we need to maximize $\sin\phi$ in Eq. 29-31. To do so, we set $\phi = 90°$, thereby arranging for $\vec{B}$ to be perpendicular to the wire. We then have $\sin\phi = 1$, so Eq. 29-31 yields

$$B = \frac{mg}{iL\sin\phi} = \frac{(m/L)g}{i}. \tag{29-32}$$

We write the result this way because we know m/L, the linear density of the wire. Substituting known data then gives us

$$B = \frac{(46.6\times10^{-3}\ \text{kg/m})(9.8\ \text{m/s}^2)}{28\ \text{A}}$$
$$= 1.6\times10^{-2}\ \text{T}. \qquad \text{(Answer)}$$

Conclude: This is about 160 times the strength of Earth's magnetic field. Note that this equilibrium is neutral.

Sample Problem 29-9

A bar magnet with its north pole up is placed along the axis of symmetry of a horizontal conducting ring carrying current i, as shown in the Fig. 29-32. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

Conceptualize: Here the wire is in a non-uniform magnetic field. The magnitude of the magnetic field at all the points of the ring is the same, but the direction is different. So we cannot say that the net force on the wire is zero.

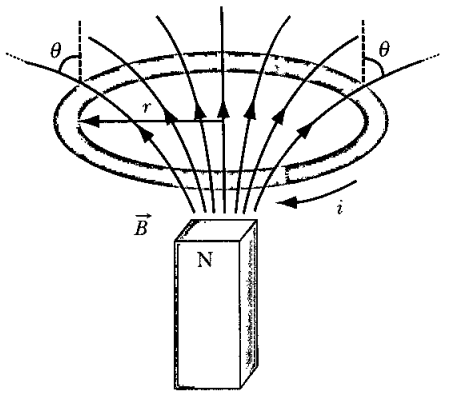


FIG. 29-32 A bar magnet kept below a conducting ring.

Classify: We have two options: we can either divide the entire ring into small elements and integrate to find the force on the ring, or divide the magnetic field into two components and then integrate the force due to each component of the field. We choose the second option.

Compute: The magnetic field at each point of the ring is having a vertical component of $B\cos\theta$. This field is uniform and hence the force due to this component of the magnetic field will be zero. The horizontal component of the magnetic field at each element of the ring will be radially outward. This will produce a force on the element in vertically upward direction as can be seen from Fig. 29-33.

The magnitude of this force is given by

$$dF = idLB\sin\theta.$$

The net force acting on the wire is given by

$$F = 2\pi irB\sin\theta.$$

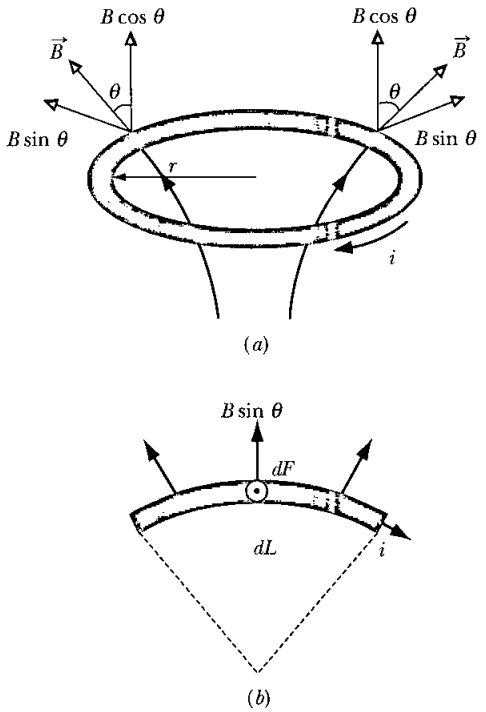


FIG. 29-33 Magnetic force acting (a) on a wire; (b) on the element.

Conclude: We can see that the vertical component of the magnetic field does not contribute to the net force on the wire; however, each element of the wire will be experiencing a force inward. This will tend to contract the wire. The current-carrying loop experiences a force in a nonuniform magnetic field. Does this remind you of an electric dipole?

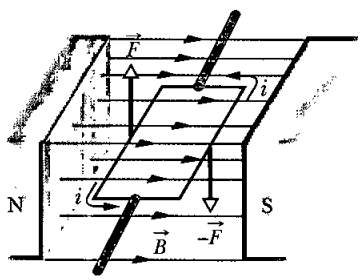


FIG. 29-34 The elements of an electric motor. A rectangular loop of wire, carrying a current and free to rotate about a fixed axis, is placed in a magnetic field. Magnetic forces on the wire produce a torque that rotates it. A commutator (not shown) reverses the direction of the current every half-revolution so that the torque always acts in the same direction.

29-8 | Torque on a Current Loop

Much of the world's work is done by electric motors. The forces behind this work are the magnetic forces that we studied in the preceding section—that is, the forces that a magnetic field exerts on a wire that carries a current.

Figure 29-34 shows a simple motor, consisting of a single current-carrying loop immersed in a magnetic field $\vec{B}$. The two magnetic forces $\vec{F}$ and $-\vec{F}$ produce a torque on the loop, tending to rotate it about its central axis. Although many essential details have been omitted, the figure does suggest how the action of a magnetic field on a current loop produces rotary motion. Let us analyze that action.

Figure 29-35a shows a rectangular loop of sides a and b, carrying current i through uniform magnetic field $\vec{B}$. We place the loop in the field so that its long sides, labeled 1 and 3, are perpendicular to the field direction (which is into the page), but its short sides, labeled 2 and 4, are not. Wires to lead the current into and out of the loop are needed but, for simplicity, are not shown.

To define the orientation of the loop in the magnetic field, we use a normal vector $\vec{n}$ that is perpendicular to the plane of the loop. Figure 29-35b shows a right-hand rule for finding the direction of $\vec{n}$. Point or curl the fingers of your right hand in the direction of the current at any point on the loop. Your extended thumb then points in the direction of the normal vector $\vec{n}$.

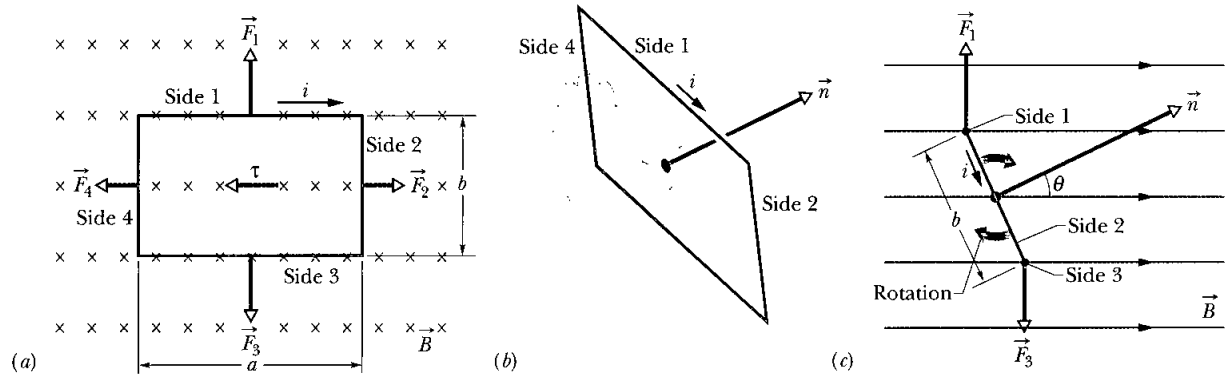


FIG. 29-35 A rectangular loop, of length a and width b and carrying a current i, is located in a uniform magnetic field. A torque τ acts to align the normal vector $\vec{n}$ with the direction of the field. (*a*) The loop as seen by looking in the direction of the magnetic field. (*b*) A perspective of the loop showing how the right-hand rule gives the direction of $\vec{n}$, which is perpendicular to the plane of the loop. (*c*) A side view of the loop, from side 2. The loop rotates as indicated.

In Fig. 29-35*c*, the normal vector of the loop is shown at an arbitrary angle θ to the direction of the magnetic field $\vec{B}$. We wish to find the net force and net torque acting on the loop in this orientation.

The net force on the loop is the vector sum of the forces acting on its four sides. For side 2 the vector $\vec{L}$ in Eq. 29-29 points in the direction of the current and has magnitude b. The angle between $\vec{L}$ and $\vec{B}$ for side 2 (see Fig. 29-35*c*) is $90° - \theta$. Thus, the magnitude of the force acting on this side is

$$F_2 = ibB \sin(90° - \theta) = ibB \cos \theta. \tag{29-33}$$

You can show that the force $\vec{F}_4$ acting on side 4 has the same magnitude as $\vec{F}_2$ but the opposite direction. Thus, $\vec{F}_2$ and $\vec{F}_4$ cancel out exactly. Their net force is zero and, because their common line of action is through the center of the loop, their net torque is also zero.

The situation is different for sides 1 and 3. For them, $\vec{L}$ is perpendicular to $\vec{B}$, so the forces $\vec{F}_1$ and $\vec{F}_3$ have the common magnitude iaB. Because these two forces have opposite directions, they do not tend to move the loop up or down. However, as Fig. 29-35*c* shows, these two forces do *not* share the same line of action; so they *do* produce a net torque. The torque tends to rotate the loop so as to align its normal vector $\vec{n}$ with the direction of the magnetic field $\vec{B}$. That torque has moment arm $(b/2) \sin \theta$ about the central axis of the loop. The magnitude τ' of the torque due to forces $\vec{F}_1$ and $\vec{F}_3$ is then (see Fig. 29-35*c*)

$$\tau' = \left(iaB\frac{b}{2}\sin\theta\right) + \left(iaB\frac{b}{2}\sin\theta\right) = iabB \sin\theta. \tag{29-34}$$

Suppose we replace the single loop of current with a *coil* of N loops, or *turns*. Further, suppose that the turns are wound tightly enough that they can be approximated as all having the same dimensions and lying in a plane. Then the turns form a *flat coil*, and a torque τ' with the magnitude given in Eq. 29-34 acts on each of them. The total torque on the coil then has magnitude

$$\tau = N\tau' = NiabB \sin\theta = (NiA)B \sin\theta, \tag{29-35}$$

in which A $(= ab)$ is the area enclosed by the coil. The quantities in parentheses (NiA) are grouped together because they are all properties of the coil: its number of turns, its area, and the current it carries. Equation 29-35 holds for all flat coils, no matter what their shape, provided the magnetic field is uniform.

Instead of focusing on the motion of the coil, it is simpler to keep track of the vector $\vec{n}$, which is normal to the plane of the coil. Equation 29-35 tells us that a current-carrying flat coil placed in a magnetic field will tend to rotate so that $\vec{n}$

has the same direction as the field. In a motor, the current in the coil is reversed as $\vec{n}$ begins to line up with the field direction, so that a torque continues to rotate the coil. This automatic reversal of the current is done via a commutator that electrically connects the rotating coil with the stationary contacts on the wires that supply the current from some source.

29-9 | The Magnetic Dipole Moment

As we have just discussed, a torque acts to rotate a current-carrying coil placed in a magnetic field. In that sense, the coil behaves like a magnetic compass placed in the magnetic field. Moreover, to account for the torque on the coil due to the magnetic field, we assign a **magnetic dipole moment** $\vec{\mu}$ to the coil. The direction of $\vec{\mu}$ is that of the normal vector $\vec{n}$ to the plane of the coil and thus is given by the same right-hand rule shown in Fig. 29-35. That is, grasp the coil with the fingers of your right hand in the direction of current i; the outstretched thumb of that hand gives the direction of $\vec{\mu}$. The magnitude of $\vec{\mu}$ is given by

$$\mu = NiA \qquad \text{(magnetic moment)}, \tag{29-36}$$

in which N is the number of turns in the coil, i is the current through the coil, and A is the area enclosed by each turn of the coil. From this equation, with i in amperes and A in square meters, we see that the unit of $\vec{\mu}$ is the ampere–square meter ($A \cdot m^2$).

Using $\vec{\mu}$, we can rewrite Eq. 29-35 for the torque on the coil due to a magnetic field as

$$\tau = \mu B \sin\theta, \tag{29-37}$$

in which θ is the angle between the vectors $\vec{\mu}$ and $\vec{B}$.

We can generalize this to the vector relation

$$\vec{\tau} = \vec{\mu} \times \vec{B}, \tag{29-38}$$

which reminds us very much of the corresponding equation for the torque exerted by an *electric* field on an *electric* dipole—namely, Eq. 22-45:

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

In each case the torque due to the field—either magnetic or electric—is equal to the vector product of the corresponding dipole moment and the field vector.

A magnetic dipole in an external magnetic field has a **magnetic potential energy** that depends on the dipole's orientation in the field. For electric dipoles we have shown (Eq. 22-49) that

$$U(\theta) = -\vec{p} \cdot \vec{E}.$$

In strict analogy, we can write for the magnetic case

$$U(\theta) = -\vec{\mu} \cdot \vec{B}. \tag{29-39}$$

A magnetic dipole has its lowest energy ($= -\mu B \cos 0 = -\mu B$) when its dipole moment $\vec{\mu}$ is lined up with the magnetic field (Fig. 29-36). It has its highest energy ($= -\mu B \cos 180° = +\mu B$) when $\vec{\mu}$ is directed opposite the field. From Eq. 29-36, with U in joules and B in teslas, we see that the unit of $\vec{\mu}$ can be the joule per tesla (J/T) instead of the ampere—square meter as suggested by Eq. 29-30.

Now you can perhaps appreciate why the force on the current-carrying coil in Sample Problem 29-9 was repulsive, directed away from the magnet. Any system left to itself tries to minimize its potential energy. In the given configuration, the magnetic moment of the loop is downward whereas the magnetic field is upward. So its magnetic potential energy is positive. If the coil moves outward,

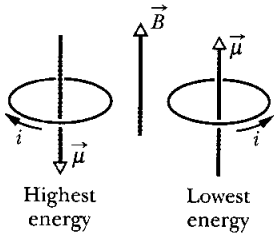


FIG. 29-36 The orientations of highest and lowest energy of a magnetic dipole (here a coil carrying current) in an external magnetic field $\vec{B}$. The direction of the current i gives the direction of the magnetic dipole moment $\vec{\mu}$ via the right-hand rule shown for $\vec{n}$ in Fig. 29-35*b*.

its potential energy will decrease because it will go to a region of lower magnetic field. Hence, the coil experiences a force in the upward direction.

If an applied torque (due to "an external agent") rotates a magnetic dipole from an initial orientation θ_i to another orientation θ_f, then work W_a is done on the dipole by the applied torque. *If the dipole is stationary* before and after the change in its orientation, then work W_a is

$$W_a = U_f - U_i, \tag{29-40}$$

where U_f and U_i are calculated with Eq. 29-39.

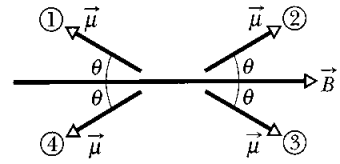
So far, we have identified only a current-carrying coil as a magnetic dipole. However, a simple bar magnet is also a magnetic dipole, as is a rotating sphere of charge. Earth itself is (approximately) a magnetic dipole. Finally, most subatomic particles, including the electron, the proton, and the neutron, have magnetic dipole moments. For comparison, some approximate magnetic dipole moments are shown in Table 29-2.

TABLE 29-2

Some Magnetic Dipole Moments

Small bar magnet	5 J/T
Earth	8.0×10^{22} J/T
Proton	1.4×10^{-26} J/T
Electron	9.3×10^{-24} J/T

CHECKPOINT 8 The figure shows four orientations, at angle θ, of a magnetic dipole moment $\vec{\mu}$ in a magnetic field. Rank the orientations according to (a) the magnitude of the torque on the dipole and (b) the potential energy of the dipole, greatest first.



Sample Problem 29-10

Figure 29-37 shows a circular coil with 250 turns, an area A of 2.52×10^{-4} m^2, and a current of 100 μA. The coil is at rest in a uniform magnetic field of magnitude $B = 0.85$ T, with its magnetic dipole moment $\vec{\mu}$ initially aligned with $\vec{B}$.

(a) In Fig. 29-37, what is the direction of the current in the coil?

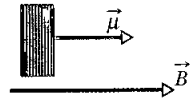


FIG. 29-37 A side view of a circular coil carrying a current and oriented so that its magnetic dipole moment $\vec{\mu}$ is aligned with magnetic field $\vec{B}$.

Right-hand rule: Imagine cupping the coil with your right hand so that your right thumb is outstretched in the direction of $\vec{\mu}$. The direction in which your fingers curl around the coil is the direction of the current in the coil. Thus, in the wires on the near side of the coil—those we see in Fig. 29-37—the current is from top to bottom.

(b) How much work would the torque applied by an external agent have to do on the coil to rotate it 90° from its initial orientation, so that $\vec{\mu}$ is perpendicular to $\vec{B}$ and the coil is again at rest?

Conceptualize/Classify: The work W_a done by the applied torque would be equal to the change in the coil's potential energy due to its change in orientation.

Compute: From Eq. 29-40 ($W_a = U_f - U_i$), we find

$$\begin{aligned} W_a &= U(90°) - U(0°) \\ &= -\mu B \cos 90° - (-\mu B \cos 0°) = 0 + \mu B \\ &= \mu B. \end{aligned}$$

Substituting for μ from Eq. 29-36 ($\mu = NiA$), we find that

$$\begin{aligned} W_a &= (NiA)B \\ &= (250)(100 \times 10^{-6}\,\text{A})(2.52 \times 10^{-4}\,\text{m}^2)(0.85\,\text{T}) \\ &= 5.355 \times 10^{-6}\,\text{J} \approx 5.4\,\mu\text{J}. \end{aligned}$$ (Answer)

Sample Problem 29-11

A uniform, constant magnetic field B is directed at an angle of 45° to the x-axis in the xy plane. $PQRS$ is a rigid square wire frame carrying a steady current i (clockwise), with its center at the origin O. At time $t = 0$, the frame is at rest in the position shown in Fig. 29-38, with its sides parallel to the x and y axes. Each side of the frame is of mass M and length L.

(a) What is the torque acting on the frame due to the magnetic field?

Conceptualize: Since this frame is a closed current-carrying loop, it will experience a torque in a constant magnetic field.

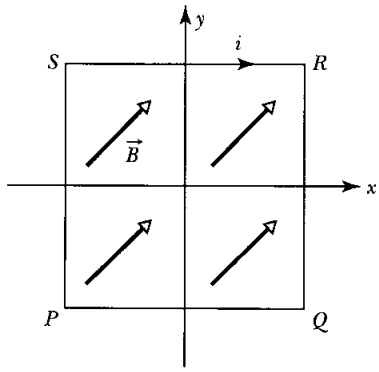


FIG. 29-38 A wire frame kept in a uniform magnetic field.

Classify: The torque due to the magnetic field can be found by Eq. 29-38.

Compute: The magnetic moment of the loop is given by $\mu = iL^2$ because the loop has only one turn. This is a vector whose direction is negative z direction by right hand rule.

The torque acting on the loop is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}.$$

Since $\vec{\mu}$ and $\vec{B}$ are perpendicular to each other, its magnitude is given by

$$\tau = \mu \times B \sin 90^\circ = iL^2 B.$$

This torque is directed in xy plane at an angle of 45° to the x axis as seen in Fig. 29-39. Use right-hand rule to convince yourself!

(b) Find the initial angular acceleration of the frame. Assume that the frame is in a gravity-free space free from any other influence.

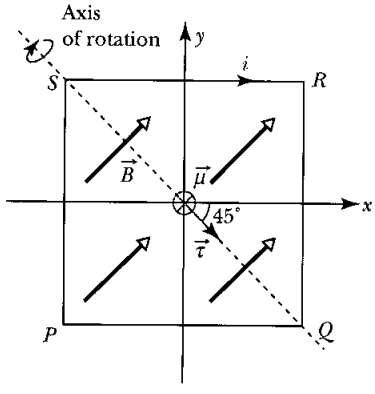


FIG. 29-39 The magnetic moment, torque, and the direction of axis of rotation of the wire frame.

Conceptualize: To find the angular acceleration of the frame, we need to know the moment of inertia of the frame for which we further need to know the axis about which the frame is rotating. Now, this can be a little tricky because as per the information given, the frame is not hinged at any point. We know that the force on a magnetic dipole in a uniform magnetic field is zero. So the torque that we have calculated is actually a couple. So whatever be the axis of rotation, the torque will turn out to be the same.

Since there is no net force, the center of mass of the body should be at rest. So the frame should rotate about its center of mass.

Classify: The torque vector is in the direction of the axis of rotation. So this means that the frame is rotating about an axis in the xy plane about its diagonal. We can use the perpendicular axis theorem to find the angular acceleration.

Compute: Using perpendicular axis theorem we can say that

$$I_1 + I_2 = I_z,$$

where I_1 and I_2 are the moments of inertia of the frame about the diagonals. Also by symmetry, $I_1 = I_2$.

By parallel axis theorem, the moment of inertia of each of the rods about the z axis is given by

$$\frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}.$$

So the moment of inertia of the entire frame about the z axis is given by

$$I_z = \frac{4ML^2}{3}.$$

But we can say that

$$I_1 = \frac{I_z}{2} = \frac{2ML^2}{3}.$$

By using the expression $\tau = I_1 \alpha$, we get

$$\alpha = \frac{\tau}{I_1} = \frac{iL^2 B}{2ML^2/3} = \frac{3iB}{2M}.$$

Conclude: We should note that this is the instantaneous angular acceleration. As the frame rotates, the angle between the magnetic moment vector and the magnetic field changes. So the angular acceleration will change with time.

Moving Coil Galvanometer

Analog voltmeters and ammeters work by measuring the torque exerted by a magnetic field on a current-carrying coil. The reading is displayed by means of

the deflection of a pointer over a scale. Figure 29-40 shows the essentials of a *galvanometer*, a device on which both analog ammeters and analog voltmeters are based. For any orientation of the coil, the net magnetic field through the coil is perpendicular to the normal vector of the coil (and thus parallel to the plane of the coil). A spring "Sp" provides a counter torque that balances the magnetic torque, so that a given steady current i in the coil results in a steady angular deflection ϕ. The greater the current is, the greater the deflection is, and thus the greater the torque required of the spring is.

With a constant current through the device, the resulting magnetic torque (Eq. 29-35) is balanced by the spring torque.

We write this balance as

$$NiAB = c\phi, \tag{29-41}$$

where c is the torsional constant of the spring Sp. Equation 29-41 then reduces to

$$i = \left(\frac{c}{NAB}\right)\phi. \tag{29-42}$$

Thus we can see that the deflection of the needle is directly proportional to the current passing through the coil.

The current sensitivity is the current necessary (usually given in amperes) to give a deflection of one scale. From Eq. 29-42 this current sensitivity is

$$\frac{\phi}{i} = \frac{NAB}{c}. \tag{29-43}$$

A convenient way to increase the sensitivity is to increase the number of turns N.

The moving coil galvanometer can also be used as a voltmeter to measure the voltage across a given section of the circuit. The scale of the voltmeter is calibrated and then graduated to read off the voltage value with ease. We define the voltage sensitivity as the deflection per unit voltage. From Eq. 29-42:

$$\frac{\phi}{V} = \left(\frac{NAB}{k}\right)\frac{i}{V} = \left(\frac{NAB}{k}\right)\frac{1}{R}. \tag{29-44}$$

An interesting point to note is that increasing the current sensitivity may not necessarily increase the voltage sensitivity. Let us take Eq. 29-43 which provides a measure of current sensitivity. If $N \rightarrow 2N$, that is, we double the number of turns; then

$$\frac{\phi}{i} \rightarrow 2\frac{\phi}{i}.$$

Thus, the current sensitivity doubles. However, the resistance of the galvanometer is also likely to double, since it is proportional to the length of the wire. In Eq. 29-44, $N \rightarrow 2N$ and $R \rightarrow 2R$; thus the voltage sensitivity remains unchanged.

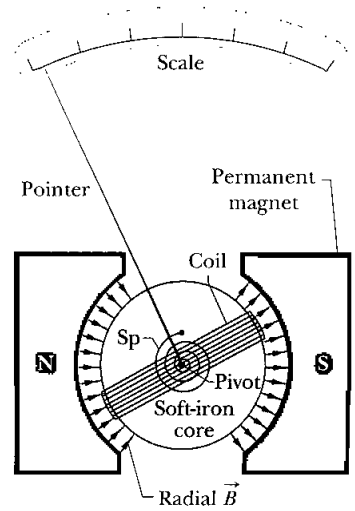


FIG. 29-40 The elements of a galvanometer. Depending on the external circuit, this device can be wired up as either a voltmeter or an ammeter.

Loop Model for Electron Orbits

We do not need a conducting wire loop to make a magnetic dipole. Rather, a circulating electron can also form a magnetic dipole. We imagine an electron moving at constant speed v in a circular path of radius r, counterclockwise as shown in Fig. 29-41.

The motion of the negative charge of the electron is equivalent to a conventional current i (of positive charge) that is clockwise, as also shown in Fig. 29-41. The magnitude of the orbital magnetic dipole moment of such a *current loop* is obtained from Eq. 29-36 with $N = 1$:

$$\mu_{orb} = iA, \tag{29-45}$$

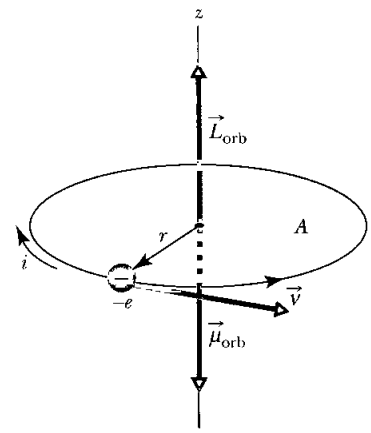


FIG. 29-41 An electron moving at constant speed v in a circular path of radius r that encloses an area A. The electron has an orbital angular momentum and an associated orbital magnetic dipole moment. A clockwise current i (of positive charge) is equivalent to the counterclockwise circulation of the negatively charged electron.

where A is the area enclosed by the loop. The direction of this magnetic dipole moment is, from the right-hand rule of Fig. 29-35, downward in Fig. 29-41.

To evaluate Eq. 29-45, we need the current i. Current is, generally, the rate at which charge passes some point in a circuit. Here, the charge of magnitude e takes a time $T = 2\pi r/v$ to circle from any point back through that point, so

$$i = \frac{e}{2\pi r/v}.$$

Substituting this and the area $A = \pi r^2$ of the loop into Eq. 29-45 gives us

$$\mu_{orb} = \frac{e}{2\pi r/v}\pi r^2 = \frac{evr}{2}. \tag{29-46}$$

To find the electron's orbital angular momentum $\vec{L}_{orb}$, we use the definition of angular momentum, $\vec{L} = m(\vec{r} \times \vec{v})$. Because $\vec{r}$ and $\vec{v}$ are perpendicular, $\vec{L}$ has the magnitude

$$L_{orb} = mrv. \tag{29-47}$$

The vector is directed upward in Fig. 29-41. Combining Eqs. 29-46 and 29-47, generalizing to a vector formulation, and indicating the opposite directions of the vectors with a minus sign yield

$$\vec{\mu}_{orb} = -\frac{e}{2m}\vec{L}_{orb}.$$

This relation can be readily seen to be valid for any rotating system of charges. The idea is that the calculation of angular momentum of a system is easy if we know its moment of inertia about the axis of rotation. This we saw in Section 10.8, Chapter 10, Vol. I. By this formula, we can easily calculate its magnetic moment. Note that the magnetic dipole moment is independent of the choice of origin. As we saw in Sample Problem 29-9, a magnetic dipole may feel a net force in a non-uniform magnetic field. In any atom, there are electrons revolving around the nucleus. This gives a magnetic dipole moment to the atom, and so the atom may get attracted or repelled in a magnetic field. The magnetic force experienced by so many atoms put together gives magnetic substances their magnetic properties.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.
(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.
(c) If Statement 1 is true but Statement 2 is false.
(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: Earth's magnetic field protects us from cosmic rays.
Statement 2: A charged particle deflects in presence of a magnetic field.

2. Statement 1: An electron cannot be accelerated by a uniform magnetic field.
Statement 2: Magnetic field does not do any work on charged particle.

3. Statement 1: Astronauts keep to altitudes beneath the Van Allen radiation belts while doing space walks.
Statement 2: Charge particles spiral in Van Allen belts from the one pole to another.

4. Statement 1: A current-carrying loop placed in a uniform magnetic field experiences no net force.
Statement 2: The net charge on a current-carrying loop is zero.

5. Statement 1: At most of the points in northern hemisphere, the magnetic field is inclined to an angle below horizontal, whereas in the southern hemisphere, the magnetic field is inclined to an angle above horizontal.
Statement 2: The magnetic south pole is near geographic north pole and vice versa.

MULTIPLE CHOICE QUESTIONS

sec. 29-2 What Produces a Magnetic Field?

1. Consider the two rectangular areas shown in Fig. 29-42 with a point P located at the midpoint between the two areas. The rectangular area on the left contains a bar magnet with the south pole near point P. The rectangle on the right is initially empty. How will the magnetic field at P change, if at all, when a second bar magnet is placed on the right rectangle with its south pole near point P?

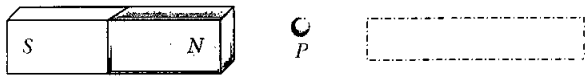


FIG. 29-42 Multiple choice question 1.

(a) The direction of the magnetic field will not change, but its magnitude will decrease

(b) The direction of the magnetic field will not change, but its magnitude will increase

(c) The magnetic field at P will be zero

(d) The direction of the magnetic field will change and its magnitude will increase

(e) The direction of the magnetic field will change and its magnitude will decrease

2. What is the direction of the magnetic field at the point P, directly below a point at the center of the magnet? The numbered arrows represent various directions. Direction "1" is to the right, "2" to the left, "3" is upward, "4" is downward, and "5" is toward you (Fig. 29-43).

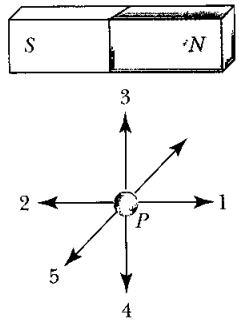


FIG. 29-43 Multiple choice question 2.

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

sec. 29-3 The Definition of $\vec{B}$

3. An observer turns on an electric field across a sealed glass tube and the electrons from one of the electrodes move toward the other, horizontally toward right. A magnetic field is applied pointing toward the observer. The force acting upon the electrons points

(a) To the right

(b) To the left

(c) Upward

(d) Downward

(e) The force will be zero

4. An electron enters a region that contains a magnetic field directed into the page as shown in Fig. 29-44. The velocity vector of the electron makes an angle of 30° with the $+y$ axis. What is the direction of the magnetic force on the electron when it enters the field?

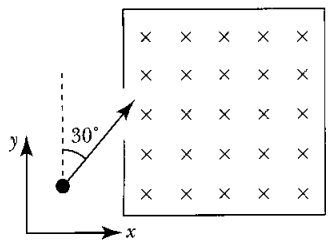


FIG. 29-44 Multiple choice question 4.

(a) Up, out of the page

(b) At an angle of 30° below the positive x axis

(c) At an angle of 30° above the positive x axis

(d) At an angle of 60° below the positive x axis

(e) At an angle of 60° above the positive x axis

5. Two particles move through a uniform magnetic field that is directed out of the plane of the page. Figure 29-45 shows the paths taken by the two paticles as they move through the field. The particles are not subject to any other forces or fields. Which one of the following statatements concerning these particles is true?

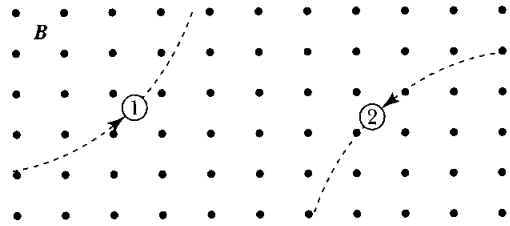


FIG. 29-45 Multiple choice question 5.

(a) The particles may both be neutral

(b) Particle 1 is positively charged; 2 is negative

(c) Particle 1 as well as 2 is positively charged

(d) Particle 1 as well as 2 is negatively charged

(e) Particle 1 is negatively charged; 2 is positive

6. A proton traveling at velocity v enters a region with magnetic field B pointing perpendicular to the original direction of the proton. Which of the following statements is true?

(a) Velocity v of the proton remains constant

(b) Speed v of the proton remains constant

(c) Both direction and speed of the proton change

(d) Direction of the proton remains constant although its speed increases

7. If an electron moves downward towards Earth surface near the Earth's equator, which direction is this particle deflected?

(a) Northward

(b) Southward

(c) Eastward

(d) Westward

8. Cosmic rays (atomic nuclei stripped bare of their electrons) would continuously bombard Earth's surface if most of them were not deflected by Earth's magnetic field. Given that Earth is, to an excellent approximation, a magnetic dipole, the intensity of cosmic rays bombarding its surface is greatest at the

(a) Poles

(b) Mid-latitudes

(c) Equator

sec. 29-4 A Circulating Charged Particle

9. A beam of electrons moving with a momentum p enters a uniform magnetic field B perpendicular to its motion. Which of the following statement(s) is(are) true?

(a) Energy gained is $p^2/2m$

(b) Centripetal force on the electron is $Be(p/m)$

(c) Radius of the electron's path is p/Be

(d) Work done on the electrons by the magnetic field is zero

10. The radius of curvature of the path of a charged particle moving in a static uniform magnetic field is

(a) Directly proportional to the magnitude of the charge on the particle

(b) Directly proportional to the magnitude of the liner momentum of the particle

(c) Directly proportional to the kinetic energy of the particle

(d) Inversely proportional to the magnitude of the magnetic field

11. A positively charged particle enters a region with magnetic field parallel to its direction of motion. The subsequent trajectory of that charged particle track will

(a) Curve to the right with radius, $R = mv/qB$

(b) Curve to the left with radius, $R = mv/qB$

(c) Curve upward with radius, $R = mv/qB$

(d) Curve downward with radius, $R = mv/qB$

(e) Not be affected at all

12. Two singly ionized isotopes of lead are accelerated through the same potential difference and enter perpendicular to a uniform magnetic field. Then

(a) Their kinetic energies before entering the magnetic field are equal and remain equal in the magnetic field.

(b) They trace circular orbits of the same radii in the magnetic field

(c) The radius of the circular orbit of the heavier isotope in the magnetic field is greater than that of the lighter isotope

(d) The radius of the circular orbit of the lighter isotope in the magnetic field is greater than that of the heavier isotope

(e) The radii of the circular orbits in the magnetic field are in the direct ratio of the magnitude of their momenta.

13. Figure 29-46 shows the respective paths taken by two particles, A and B, in a region containing a constant, uniform magnetic field directed into the plane of motion (into the page). There is no electric field. Both particles have charge $+e$. What can we correctly conclude about particles A and B?

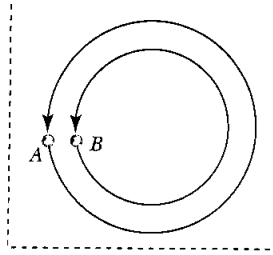


FIG. 29-46 Multiple choice question 13.

(a) The mass of A > the mass of B

(b) The speed of A > the speed of B

(c) The magnitude of momentum of A > the magnitude of momentum of B

(d) The kinetic energy of A > the kinetic energy of B

14. An electron is moving along the positive x axis. You want to apply a uniform magnetic field for a short time so that the electron may reverse its direction and move parallel to the negative x axis. This can be done by applying magnetic field along

(a) y axis

(b) z axis

(c) Any direction in yz plane

(d) z axis only

15. Figure 29-47 shows the path of an electron that passes through two regions containing uniform magnetic fields of magnitudes B_1 and B_2. Its path in each region is a half circle. The directions of the two fields are

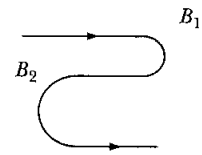


FIG. 29-47 Multiple choice question 15.

(a) Both B_1 and B_2 are directed into the page

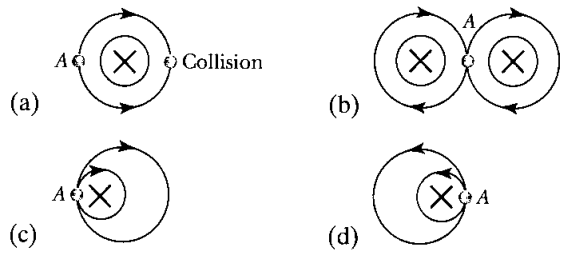
(b) Both B_1 and B_2 are directed out of the page

(c) B_1 is directed into the page and B_2 is directed out of the page

(d) B_1 is directed out of the page and B_2 is directed into the page

(e) There is no way to tell the directions of the fields

16. A neutral particle at rest at A decays into two charged particles of mass m each. Magnetic field is inside the paper. Which can be the correct path for two particles if we ignore electrostatic force between them?



17. A uniform magnetic field is directed into the page. A charged particle, moving in the plane of the page, follows a clockwise spiral of decreasing radius as shown in Fig. 29-48. A reasonable explanation is

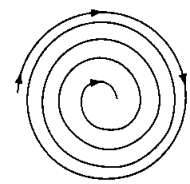


FIG. 29-48 Multiple choice question 17.

(a) The charge is positive and slowing down

(b) The charge is negative and slowing down

(c) The charge is positive and speeding up

(d) The charge is negative and speeding up

18. A charged particle of charge q and mass m is shot into a uniform magnetic field of induction B at an angle θ with the field. The frequency of revolution of the particle

(a) Depends on the angle θ
(b) Is independent of the angle θ
(c) Is proportional to the specific charge q/m of the particle
(d) Is inversly proportional to the value of B
(e) Is independent of the initial speed of the particle at the entry into the field

19. A particle of mass m and charge q moving with velocity $\vec{v}$ enters a region of uniform magnetic field $\vec{B}$. Then
(a) Its path in the region of the field is always circular
(b) Its path in the region of the field is circular if $\vec{v} \times \vec{B} = 0$
(c) Its path in the region of the field is a straight line if $\vec{v} \times \vec{B} = 0$
(d) Distance traveled by the particle in time T does not depend on the angle between $\vec{v}$ and $\vec{B}$.

20. If an electric dipole is rotating about its center with a uniform angular velocity in the anticlockwise direction in a uniform magnetic field which is in the direction of the angular velocity,
(a) Net magnetic force as well as torque on the dipole is zero
(b) Net magnetic force as well as torque on the dipole is nonzero
(c) Net magnetic force on the dipole is zero but the net torque on the dipole is nonzero
(d) Net magnetic force on the dipole is not zero but the net torque on the dipole is zero

sec. 29-6 Crossed Fields: Discovery of the Electron

21. A proton moves with constant velocity in a certain region of gravity-free space. Which one of the following conclusions is valid?
(a) There must be no electric or magnetic fields in this region
(b) There could be an electric field without any magnetic field
(c) There could be a magnetic field without any electric field
(d) There can be an electric field as well as magnetic field

22. A proton moves with constant speed in a certain region of space. Which one of the following conclusions is valid?
(a) There may be no electric or magnetic fields in this region
(b) There could be an electric field without any magnetic field
(c) There could be a magnetic field without any electric field
(d) There can be an electric field as well as magnetic field

23. A proton is moving in a uniform electric field, E, of unknown magnitude and direction. There is a uniform magnetic field of 0.01 T pointing in the y direction. The proton moves in the x direction at a constant velocity of 10 km/s. Which of the following is correct?
(a) $E = 100$ V/m, in the z direction
(b) $E = 100$ V/m, in the negative z direction
(c) $E = 100$ V/m, in the x direction
(d) $E = 100$ V/m, in the negative y direction

24. Which of the following configurations of electric and magnetic fields in a chamber can allow a positively charged particle to traverse it with no deflection?
(a) Chamber with electric field perpendicular and magnetic field parallel to the velocity of the charged particle
(b) Chamber with electric field parallel, and magnetic field perpendicular to the velocity of the charged particle
(c) Chamber with electric field parallel, and magnetic field anti-parallel to the velocity of the charged particle
(d) Chamber with electric field as well as magnetic field perpendicular to the velocity of the charged particle

25. If the acceleration of a charged particle, moving the velocity $\vec{v}$ through a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, is zero, then
(a) $\vec{E}$ must be perpendicular to $\vec{B}$
(b) $\vec{v}$ must be perpendicular to both $\vec{E}$ and $\vec{B}$
(c) $\vec{E}$ must be perpendicular to both $\vec{v}$ and $\vec{B}$
(d) $\vec{E}$ and $\vec{B}$ must be in same direction

26. A proton moves in the $+z$ direction after being accelerated from rest through a potential difference V. The proton then passes through a region with a uniform electric field E in the $+x$ direction and a uniform magnetic field B in the $+y$ direction, but the proton's trajectory is not affected. If the experiment were repeated using a potential difference of 2 V, the proton would then be
(a) Deflected in the $+x$ direction
(b) Deflected in the $-x$ direction
(c) Deflected in the $+y$ direction
(d) Deflected in the $-y$ direction
(e) Undeflected

sec. 29-7 The Hall Effect and the Force on a Wire Carrying Current

27. The current is from left to right in the conductor shown in Fig. 29-49. The magnetic field is into the page and point S is at a higher potential than point T. The charge carriers are
(a) Positive
(b) Negative
(c) Neutral
(d) Absent
(e) Moving near the speed of light

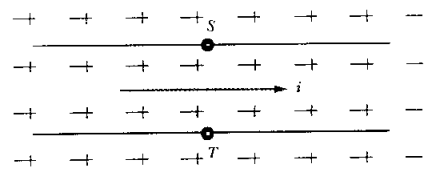


FIG. 29-49 Multiple choice question 27.

28. A portion of a loop of wire passes between the poles of a magnet as shown in Fig. 29-50. We are viewing the circuit from above. When the switch is closed and a current passes through the circuit, what is the movement, if any, of the wire between the poles of the magnet?

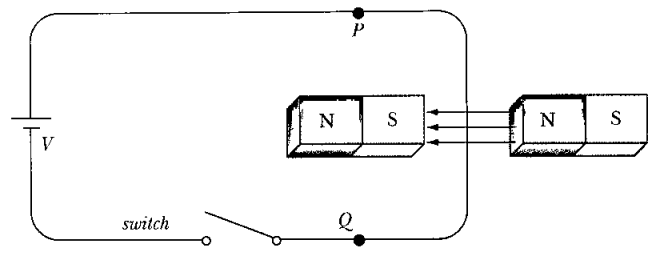


FIG. 29-50 Multiple choice question 28.

(a) The wire moves toward the north pole of the magnet
(b) The wire moves toward the south pole of the magnet
(c) The wire moves upward (toward us)
(d) The wire moves downward (away from us)
(e) The wire doesn't move

29. Three long, straight, identical wires are inserted one at a time into a magnetic field directed due east. Wire A carries a current of 2 A in the direction of 45° south of east. Wire B carries a current of 8 A, due north. Wire C carries a current of 10 A, due west. Rank the wires in terms of the magnitude of the magnetic force on each wire, with the largest force listed first and the smallest force listed last.

(a) A > B > C
(b) B > A > C
(c) C > B > A
(d) A > C > B
(e) B > C > A

sec. 29-8 Torque on a Current Loop

30. Which of the following setups can make a small conductor coil that is placed horizontally float on top of an object?

(a) The object is a magnet that provides a magnetic field pointing straight downward, whereas the coil carries an electric current flowing clockwise as viewed from top down
(b) The object is a magnet that provides a magnetic field pointing straight downward, while the coil carries an electric current flowing counterclockwise as viewed from top down
(c) The object carries positive electric charge while the coil carries negative charge
(d) The object as well as coil carries negative electric charge

31. An electric current runs counterclockwise in a rectangular loop around the outside edge of this page, which lies flat on your table. A uniform magnetic field is then turned on, directed parallel to the page from top to bottom. The magnetic force on the page will cause

(a) The left edge to lift up
(b) The right edge to lift up
(c) The top edge to lift up
(d) The bottom edge to lift up

32. The wire loop in Fig. 29-51 carries a clockwise current. There is a uniform magnetic field directed to the right. What is the direction of the net force on the current loop?

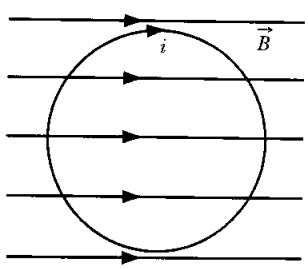


FIG. 29-51 Multiple choice question 32.

(a) Into the page
(b) Out of the page
(c) Up the page
(d) Down the page
(e) The net force is zero

33. For Q. 32, what is the direction of net torque on the current loop?

(a) Into the page
(b) Out of the page
(c) Up the page
(d) Down the page
(e) The net torque is zero

34. In Q. 32 assume that the field is not uniform; it is stronger at the top of the page than at the bottom. What is the direction of the net force on the current loop?

(a) Into the page
(b) Out of the page
(c) Up the page
(d) Down the page
(e) The net force is zero

sec. 29-9 The Magnetic Dipole Moment

35. Figure 29-52 shows a frame carrying a current in a uniform magnetic field.

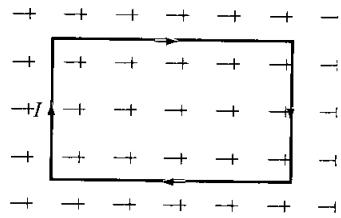


FIG. 29-52 Multiple choice question 35.

(a) The net force on the loop is zero, but the magnetic field compresses the frame
(b) The net torque on the loop is zero, but the magnetic field tries to extend the frame
(c) The net torque on the loop is zero, but the magnetic field compresses the frame
(d) The energy of interaction of the loop with the magnetic field is zero

36. The current sensitivity of a moving coil galvanometer can be increased by

(a) Increasing the magnetic field of the permanent magnet
(b) Increasing the area of the deflecting coil
(c) Increasing the number of turns in the coil
(d) Increasing the torsional constant of the coil

PROBLEMS

sec. 29-3 The Definition of $\vec{B}$

1. An electron that has velocity

$$\vec{v} = (2.0\times10^6 \text{ m/s})\hat{i} + (3.0\times10^6 \text{ m/s})\hat{j}$$

moves through the uniform magnetic field $\vec{B} = (0.030 \text{ T})\hat{i} - (0.15 \text{ T})\hat{j}$. (a) Find the force on the electron due to the magnetic field. (b) Repeat your calculation for a proton having the same velocity.

2. An alpha particle travels at a velocity of magnitude 550 m/s through a uniform magnetic field of magnitude 0.045 T. (An alpha particle has a charge of 3.2×10^{-19} C and a mass of 6.6×10^{-27} kg.) The angle between velocity and magnetic field is 53°. What is the magnitude of (a) the force acting on the particle due to the field and (b) the acceleration of the particle due to field? (c) Does the speed of the particle increase, decrease, or remain the same?

3. A proton traveling at 23° with respect to the direction of a magnetic field of strength 2.60 mT experiences a magnetic force of 6.50×10^{-17} N. Calculate (a) the proton's speed and (b) its kinetic energy in electron-volts. Take sin 23° = 5/13.

4. An electron moves through a uniform magnetic field given by $\vec{B} = B_x\hat{i} + (3.0B_x)\hat{j}$. At a particular instant, the electron has velocity $\vec{v} = (2.0\hat{i} + 4.0\hat{j})$ m/s and the magnetic force acting on it is $(6.4\times10^{-19} \text{ N})\hat{k}$. Find B_x.

5. A proton moves through a uniform magnetic field given by $\vec{B} = (10\hat{i} - 20\hat{j} + 30\hat{k})$ mT. At time t_1, the proton has a velocity given by $\vec{v} = v_x\hat{i} + v_y\hat{j} + (20 \text{ km/s})\hat{k}$ and the magnetic force on the proton is $\vec{F}_B = (4.0\times10^{-17} \text{ N})\hat{i} + (2.0\times10^{-17} \text{ N})\hat{j}$. At that instant, what are (a) v_x and (b) v_y?

sec. 29-4 A Circulating Charged Particle

6. An electron of kinetic energy 182 eV circles in a plane perpendicular to a uniform magnetic field. The orbit radius is 25.0 cm. Find (a) the electron's speed, (b) the magnetic field magnitude, and (c) the angular velocity.

7. What uniform magnetic field, applied perpendicular to a beam of electrons moving at 1.30×10^6 m/s, is required to make the electrons travel in a circular arc of radius 0.350 m?

8. A certain particle is sent into a uniform magnetic field, with the particle's velocity vector perpendicular to the direction of the field. Figure 29-53 gives the period T of the particle's motion versus the *inverse* of the field magnitude B. The vertical axis scale is set by $T_s = 40.0$ ns and the horizontal axis scale is set by $B_s^{-1} = 5.0 \text{ T}^{-1}$. What is the ratio m/q of the particle's mass to the magnitude of its charge?

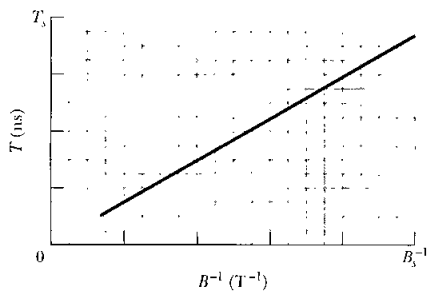


FIG. 29-53 Problem 8.

9. In Fig. 29-54, a charged particle moves into a region of uniform magnetic field $\vec{B}$, goes through half a circle, and then exits that region. The particle is either a proton or an electron (you must decide which). It spends 130 ns in the region. (a) What is the magnitude of $\vec{B}$? (b) If the particle is sent again through the magnetic field (along the same initial path) but with 2.00 times its previous kinetic energy, how much time does it spend in the field during this trip?

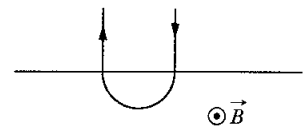


FIG. 29-54 Problem 9.

10. An electron follows a helical path in a uniform magnetic field of magnitude 0.300 T. The pitch of the path is 6.00 μm, and the magnitude of the magnetic force on the electron is 2.00×10^{-15} N. What is the electron's speed?

11. A positron (an elementary particle with the same mass as that of the electron but charge $+e$) with velocity 2.5×10^5 m/s is projected into a uniform magnetic field of magnitude 0.910 T, with its velocity vector making an angle of 53.0° with magnetic field. Find (a) the period, (b) the pitch p, and (c) the radius r of its helical path.

12. An electron follows a helical path in a uniform magnetic field given by $\vec{B} = (20\hat{i} - 50\hat{j} - 30\hat{k})$ mT. At time $t = 0$, the electron's velocity is given by $\vec{v} = (20\hat{i} - 30\hat{j} + 50\hat{k})$ m/s. (a) What is the angle ϕ between $\vec{v}$ and $\vec{B}$? The electron's velocity changes with time. Do (b) its speed and (c) the angle ϕ change with time.

13. Refer to Fig. 29-55, rectangle $PQRS$ represents the cross section of a uniform magnetic field region of 0.20 T. An electron is projected at a speed of $v = 2.0\times10^6$ m/s into the region at an angle of 30° to the direction of the magnetic field. The length of the magnetic field region is 0.01 m. Find the number of revolutions made by the electron before it leaves the magnetic field region.

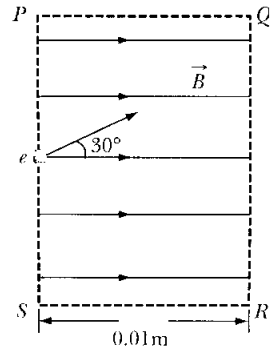


FIG. 29-55 Problem 13.

14. In Fig. 29-56, an electron with an initial kinetic energy of 4.0 keV enters region 1 at time $t = 0$. That region contains a uniform magnetic field directed into the page, with magnitude 0.010 T. The electron goes through a half-circle and then exits region 1, headed toward region 2 across a gap of 25.0 cm. There is an electric potential difference $\Delta V = 2000$ V across the gap, with a polarity such that the electron's speed increases uniformly as it traverses the gap. Region 2 contains a uniform magnetic field directed out of the page, with magnitude 0.020 T. The electron

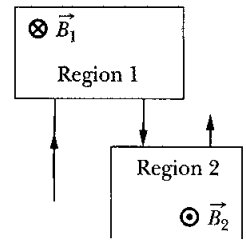


FIG. 29-56 Problem 14.

goes through a half-circle and then leaves region 2. At what time t does it leave?

15. A particular type of fundamental particle decays by transforming into an electron and a positron. Suppose the decaying particle is at rest in a uniform magnetic field of magnitude 4.55 mT and the electron and positron move away from the decay point in paths lying in a plane perpendicular to magnetic field. How long after the decay do the electron and positron collide? Ignore the electrostatic interaction between the two charges.

16. A source injects an electron of speed $v = 1.5 \times 10^7$ m/s into a uniform magnetic field of magnitude $B = 3.3$ mT. The velocity of the electron makes an angle of 67° with the direction of the magnetic field. Find the distance d from the point of injection at which the electron next crosses the field line that passes through the injection point.

sec. 29-5 Cyclotron

17. A cyclotron with dee radius 53.0 cm is operated at an oscillator frequency of 12.0 MHz to accelerate protons. (a) What magnitude B of magnetic field is required to achieve resonance? (b) At that field magnitude, what is the kinetic energy of a proton emerging from the cyclotron? Suppose, instead, that $B = 1.57$ T. (c) What oscillator frequency is required to achieve resonance now? (d) At that frequency, what is the kinetic energy of an emerging proton?

sec. 29-6 Crossed Fields: Discovery of the Electron

18. A particle of mass 10 g and charge 80 μC moves through a uniform magnetic field, in a region where the free-fall acceleration is $-9.8\hat{j}$ m/s^2. The velocity of the particle is a constant $20\hat{i}$ km/s, which is perpendicular to the magnetic field. What, then, is the magnetic field?

19. In Fig. 29-57, an electron accelerated from rest through potential difference $V_1 = 1.00$ kV enters the gap between two

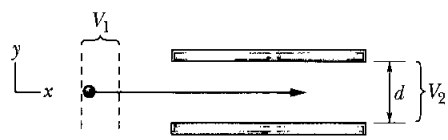


FIG. 29-57 Problem 19.

parallel plates having separation $d = 20.0$ mm and potential difference $V_2 = 100$ V. The lower plate is at the lower potential. Neglect fringing and assume that the electron's velocity vector is perpendicular to the electric field vector between the plates. In unit-vector notation, what uniform magnetic field allows the electron to travel in a straight line in the gap?

20. An electron has an initial velocity of $(12.0\hat{j} + 15.0\hat{k})$ km/s and a constant acceleration of $(2.00 \times 10^{12}$ m/s$^2)\hat{i}$ in a region in which uniform electric and magnetic fields are present. If $\vec{B} = (400\,\mu\text{T})\hat{i}$, find the electric field $\vec{E}$.

21. A proton travels through uniform magnetic and electric fields. The magnetic field is $\vec{B} = -2.50\hat{i}$ mT. At one instant the velocity of the proton is $\vec{v} = 2000\hat{j}$ m/s. At that instant and in unit-vector notation, what is the net force acting on the proton if the electric field is (a) $4.00\hat{k}$ V/m, (b) $-4.00\hat{k}$ V/m, and (c) $4.00\hat{i}$ V/m?

22. An electron with kinetic energy 2.5 keV moving along the positive direction of an x axis enters a region in which a uniform electric field of magnitude 10 kV/m is in the negative direction of the y axis. A uniform magnetic field $\vec{B}$ is to be set up to keep the electron moving along the x axis, and the direction of $\vec{B}$ is to be chosen to minimize the required magnitude of $\vec{B}$. In unit-vector notation, what $\vec{B}$ should be set up?

sec. 29-7 The Hall Effect and the Force on a Wire Carrying Current

23. A metal strip 8.50 cm long, 0.650 cm wide, and 0.760 mm thick moves with constant velocity v through a uniform magnetic field $B = 2.0$ mT directed perpendicular to the strip, as shown in Fig. 29-58. A potential difference of 3.90 μV is measured between points x and y across the strip. Calculate the speed v.

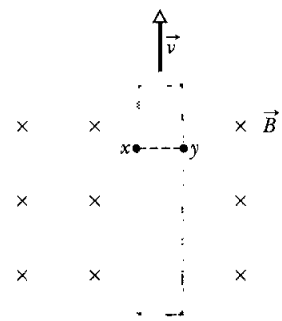


FIG. 29-58 Problem 23.

24. In Fig. 29-59, a conducting rectangular solid of dimensions $d_x = 5.00$ m, $d_y = 3.00$ m, and $d_z = 2.00$ m moves at constant velocity through a uniform magnetic field $\vec{B} = (30.0\text{ mT})\hat{j}$. What are the resulting (a) electric field within the solid, in unit-vector notation, and (b) potential difference across the solid?

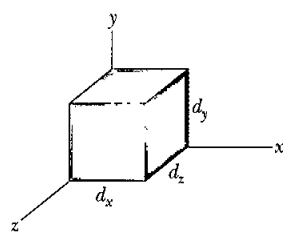


FIG. 29-59 Problem 24.

25. A 13.0 g wire of length $L = 70.0$ cm is suspended by a pair of flexible leads in a uniform magnetic field of magnitude 0.7 T (Fig. 29-60). What are the magnitude and direction (left or right) of the current required to remove the tension in the supporting leads?

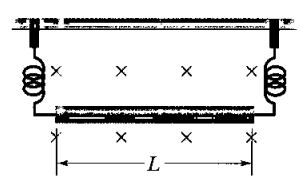


FIG. 29-60 Problem 25.

26. In Fig. 29-61, a metal wire of mass $m = 24$ mg can slide with negligible friction on two horizontal parallel rails separated by distance $d = 5$ cm. The track lies in a vertical uniform magnetic field of magnitude 56 mT. At time $t = 0$, device G is connected to the rails, producing a constant current $i = 9$ mA in

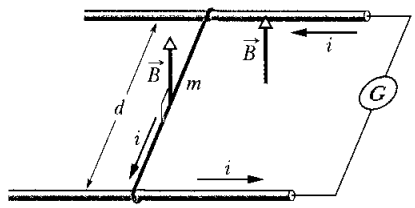


FIG. 29-61 Problem 26.

the wire and rails (even as the wire moves). At $t = 60$ ms, what are the wire's (a) speed and (b) direction of motion?

27. A 1.0 kg copper rod rests on two horizontal rails 1.0 m apart and carries a current of 50 A from one rail to the other. The coefficient of static friction between rod and rails is 0.60. What are the (a) magnitude and (b) angle (relative to the vertical) of the smallest magnetic field that puts the rod on the verge of sliding?

28. A single-turn current loop, carrying a current of 4.00 A, is in the shape of a right triangle with sides 50.0, 120, and 130 cm. The loop is in a uniform magnetic field of magnitude 75.0 mT whose direction is parallel to the current in the 130 cm side of the loop. What is the magnitude of the magnetic force on (a) the 130 cm side, (b) the 50.0 cm side, and (c) the 120 cm side? (d) What is the magnitude of the net force on the loop?

29. A straight wire 80 cm long is oriented at an angle of 37° to a uniform magnetic field (Fig. 29-62). (a) Which direction is the magnetic force? (b) If the bottom end of the wire is fixed but the rest of the wire is free to move, what is the torque on the wire? The field strength is 0.01 T and the current in the wire is 10 A.

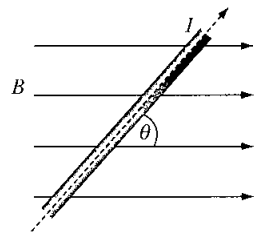


FIG. 29-62 Problem 29.

sec. 29-8 Torque on a Current Loop

30. Figure 29-63 shows a rectangular 20-turn coil of wire, of dimensions 10 cm by 5.0 cm. It carries a current of 0.10 A and is hinged along one long side. It is mounted in the xy plane, at angle θ 30° to the direction of a uniform magnetic field of magnitude 0.50 T. In unit-vector notation, what is the torque acting on the coil about the hinge line?

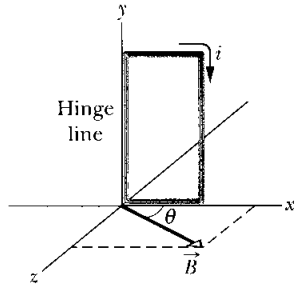


FIG. 29-63 Problem 30.

31. Figure 29-64 shows a wood cylinder of mass $m = 0.250$ kg and length $L = 0.100$ m, with $N = 10.0$ turns of wire wrapped around it longitudinally, so that the plane of the wire coil contains the long central axis of the cylinder. The cylinder is released on a plane inclined at an angle θ to the horizontal, with the plane of the coil parallel to the incline plane. If there is a vertical uniform magnetic field of magnitude 0.500 T, what is the least current i through the coil that keeps the cylinder from rolling down the plane?

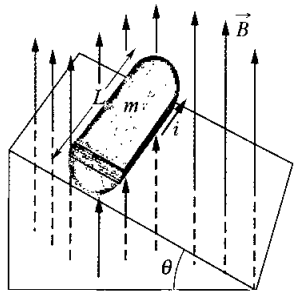


FIG. 29-64 Problem 31.

sec. 29-9 The Magnetic Dipole Moment

32. A circular wire loop of radius 5.0 cm carries a current of 2.80 A. It is placed so that the normal to its plane makes an angle of 37.0° with a uniform magnetic field of magnitude 12.0 T. (a) Calculate the magnitude of the magnetic dipole moment of the loop. (b) What is the magnitude of the torque acting on the loop?

33. A magnetic dipole with a dipole moment of magnitude 0.020 J/T is released from rest in a uniform magnetic field of magnitude 50 mT. The rotation of the dipole due to the magnetic force on it is unimpeded. When the dipole rotates through the orientation where its dipole moment is aligned with the magnetic field, its kinetic energy is 0.40 mJ. (a) What is the initial angle between the dipole moment and the magnetic field? (b) What is the angle when the dipole is next (momentarily) at rest?

34. Two concentric, circular wire loops, of radii $r_1 = 20.0$ cm and $r_2 = 30.0$ cm, are located in an xy plane; each carries a clockwise current of 7.00 A (Fig. 29-65). (a) Find the magnitude of the net magnetic dipole moment of the system. (b) Repeat for reversed current in the inner loop.

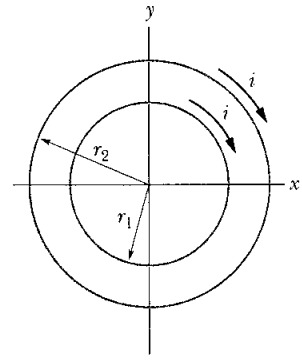


FIG. 29-65 Problem 34.

35. A circular loop of wire having a radius of 8.0 cm carries a current of 0.20 A. A vector of unit length and parallel to the dipole moment $\vec{u}$ of the loop is given by $0.60\hat{i} - 0.80\hat{j}$. If the loop is located in a uniform magnetic field given by $\vec{B} = (0.25\text{ T})\hat{i} + (0.30\text{ T})\hat{k}$, find (a) the torque on the loop (in unit-vector notation) and (b) the magnetic potential energy of the loop.

36. Figure 29-66 shows a current loop $ABCDEFA$ carrying a current $i = 5.00$ A. The sides of the loop are parallel to the coordinate axes shown, with $AB = 20.0$ cm, $BC = 30.0$ cm, and $FA = 10.0$ cm. In unit-vector notation, what is the magnetic dipole moment of this loop? (*Hint:* Imagine equal and opposite currents i in the line segment AD; then treat the two rectangular loops $ABCDA$ and $ADEFA$.)

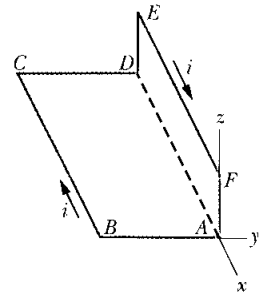


FIG. 29-66 Problem 36.

37. A spherical shell of radius "a" carries a uniform surface charge σ per unit area and spins with frequency f about an axis through its center. Show that the magnetic moment of the sphere is $(8/3)\pi^2\sigma a^4 f$.

ADDITIONAL PROBLEMS

38. A long, rigid conductor, lying along an x axis, carries a current of 5.0 A in the negative x direction. A magnetic field is present, given by with x in meters and in milliteslas. Find, in unit-vector notation, the force on the 2.0 m segment of the conductor that lies between $x = 1.0$ m and $x = 3.0$ m.

39. A stationary circular wall clock has a face with a radius of 15 cm. Six turns of wire are wound around its perimeter; the wire carries a current of 2.0 A in the clockwise direction. The clock is located where there is a constant, uniform external magnetic field of magnitude 70 mT (but the clock still keeps perfect time). At exactly 1:00 P. M., the hour hand of the clock points in the direction of the external magnetic field. (a) After how many minutes will the minute hand point in the direction of the torque on the winding due to the magnetic field? (b) Find the torque magnitude.

40. A proton of charge $+e$ and mass m enters a uniform magnetic field $\vec{B} = B\hat{i}$ with an initial velocity $\vec{v} = v_{0x}\hat{i} + v_{0y}\hat{j}$. Find an expression in unit-vector notation for its velocity $\vec{v}$ at any later time t.

41. At time $t = 0$, an electron with kinetic energy 12 keV moves through $x = 0$ in the positive direction of an x axis that is parallel to the horizontal component of Earth's magnetic field $\vec{B}$. The field's vertical component is downward and has magnitude 55.0 μT. (a) What is the magnitude of the electron's acceleration due to $\vec{B}$? (b) What is the electron's distance from the x axis when the electron reaches coordinate $x = 20$ cm?

42. The current loop of the radius b as shown in Fig. 29-67 is mounted rigidly on the axle, midway between the two supporting cords. In the absence of an external magnetic field the tensions in the cords are equal and are T_0. (a) What will be the tensions in the two cords when a vertical magnetic field $\vec{B}$ is present? (b) Repeat if the field is parallel to the axis.

43. A conducting rod of mass m and length l is placed in the east–west direction on a frictionless horizontal table. The horizontal component of earth's magnetic field is $\vec{B}$ from south to north. If a charge q is suddenly passed through the conductor, then what is the instantaneous velocity acquired by the rod?

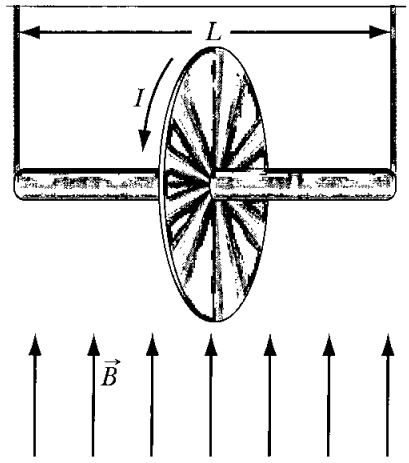


FIG. 29-67 Problem 42.

44. Let the first and second quadrant contain a constant magnetic field $\vec{B} = B\hat{k}$ (Fig. 29-68; $\vec{B}$ points out of the page). A particle with charge q and mass m travels along the y axis and enters the first quadrant with speed v_0. Assume that the particle is subject to a friction force, $\vec{F}_f$, proportional to its velocity, i.e., $\vec{F}_f = -\alpha\vec{v}$. (You may ignore gravity.) Assume this friction force is large enough so that the particle will remain in the first quadrant at all times. The particle will then spiral in toward the point P. What are the coordinates of the point P?

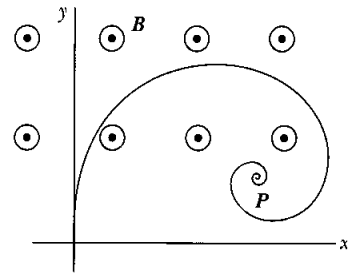


FIG. 29-68 Problem 44.

ANSWERS

Checkpoints

1. Touch a bar to the center of another, making a T. Iron bar will stick to the magnet if the end of the magnet touches the center of the bar, but not otherwise.

2. (a) z direction; (b) negative x direction; (c) zero.

3. (a) electron; (b) clockwise

4. (a) 2, then tie of 1 and 3 (zero); (b) 4

6. $\vec{v} \perp \vec{F}$, $\vec{B} \perp \vec{F}$, but $\vec{B}$ and $\vec{v}$ can be at an angle to each other.

7. $-Y$

8. (a) all tie; (b) 1 and 4 tie, then 2 and 3 tie

Reasoning Type Questions

1. (a)
2. (d)
3. (a)
4. (b)
5. (d)

Multiple Choice Questions

1. (b)
2. (b)
3. (c)
4. (b)
5. (d)
6. (b)
7. (d)
8. (a)
9. (b), (c), (d)
10. (b), (d)
11. (e)
12. (a), (c), (e)
13. (c)
14. (a), (b), (c)
15. (c)
16. (a)
17. (b)
18. (b), (c), (e)
19. (c), (d)
20. (d)
21. (c), (d)
22. (a), (c), (d)
23. (b)
24. (c), (d)
25. (a), (c)
26. (b)
27. (a)
28. (d)
29. (b)
30. (b), (d)
31. (d)
32. (e)
33. (d)
34. (e)
35. (b)
36. (a), (b), (c)

Problems

1. $(6.24\times10^{-14}\ \text{N})\,\hat{k}$; (b) $(6.24\times10^{-14}\ \text{N})\,\hat{k}$
2. (a) 6.336×10^{-18} N; (b) 9.6×10^{8} m/s^2; (c) remains the same
3. (a) 4.00×10^{5} m/s; (b) 835 eV
4. -2.0 T
5. (a) $v_x = 2.5\times10^{3}$ m/s; (b) $v_y = -5\times10^{3}$ m/s
6. (a) 8×10^{6} m/s; (b) 1.82×10^{-4} T; (c) 3.2×10^{7} rad/s
7. 21.1 μT
8. 1.2×10^{-9}
9. (a) $|\vec{B}| = 0.252$ T; (b) 130 ns
10. 65.4 km/s
11. (a) $5\pi/4\times10^{-11}$ s; (b) $15\pi/8\times10^{-6}$ m; (c) 1.25×10^{-6} m
12. (a) $\cos^{-1}(2/19)$; (b) no; (c) no
13. 32.31 revolutions
14. 8.7 ns
15. $\pi/8\times10^{-8}$ s
16. 6.25 cm
17. (a) 0.787 T; (b) 8.34×10^{6} eV; (c) 2.39×10^{7} Hz; (d) 3.32×10^{7} eV
18. $(-0.061\ \text{T})\,\hat{k}$
19. $-(2.67\times10^{-4}\ \text{T})\,\hat{k}$

20. $(-11.4\ \text{V/m})\hat{\text{i}} - (6.00\ \text{V/m})\hat{\text{j}} + (4.80\ \text{V/m})\hat{\text{k}}$

21. (a) $(1.44\times10^{-18}\ \text{N})\hat{\text{k}}$; (b) $(1.60\times10^{-19}\ \text{N})\hat{\text{k}}$; (c) $(6.4\times10^{-19}\ \text{N})\hat{\text{i}} + (8.0\times10^{-19}\ \text{N})\hat{\text{k}}$

22. $(-3.4\times10^{-4}\ \text{T})\hat{\text{k}}$

23. 0.3 m/s

24. (a) $(-600\ \text{mV/m})\hat{\text{k}}$; (b) 1.20 V

25. 0.26 A to right

26. (a) 6.3×10^{-2} m/s; (b) to the left

27. (a) 0.10 T; (b) 31°

28. (a) zero; (b) 0.138 N; (c) 0.138 N; (d) zero

29. (a) Inside the paper; (b) $1.92\times10^{-2}\ \text{N}\cdot\text{m}$

30. $\vec{\tau} = (-4.3\times10^{-3}\ \text{N}\cdot\text{m})\hat{\text{j}}$

31. 2.45 A

32. (a) $7\pi\times10^{-3}\ \text{A}\cdot\text{m}^2$; (b) $5.04\pi\times10^{-2}\ \text{N}\cdot\text{m}$

33. (a) 53°; (b) −53°

34. (a) $2.86\ \text{A}\cdot\text{m}^2$; (b) $1.10\ \text{A}\cdot\text{m}^2$

35. (a) $(-9.7\times10^{-4}\hat{\text{i}} - 7.2\times10^{-4}\hat{\text{j}} + 8.0\times10^{-4}\hat{\text{k}})\ \text{N}\cdot\text{m}$; (b) -6.0×10^{-4} J

36. $(0.150\hat{\text{j}} - 0.300\hat{\text{k}})\ \text{A}\cdot\text{m}^2$

Additional Problems

38. $(-0.35\ \text{N})\hat{\text{k}}$

39. (a) 20 min; (b) $5.9\times10^{-2}\ \text{N}\cdot\text{m}$

40. $\vec{v}(t) = v_{0x}\hat{\text{i}} + v_{0y}\cos(\omega t)\hat{\text{j}} - v_{0y}(\sin\omega t)\hat{\text{k}}$

41. (a) $6.3\times10^{14}\ \text{m/s}^2$; (b) 0.0030 m

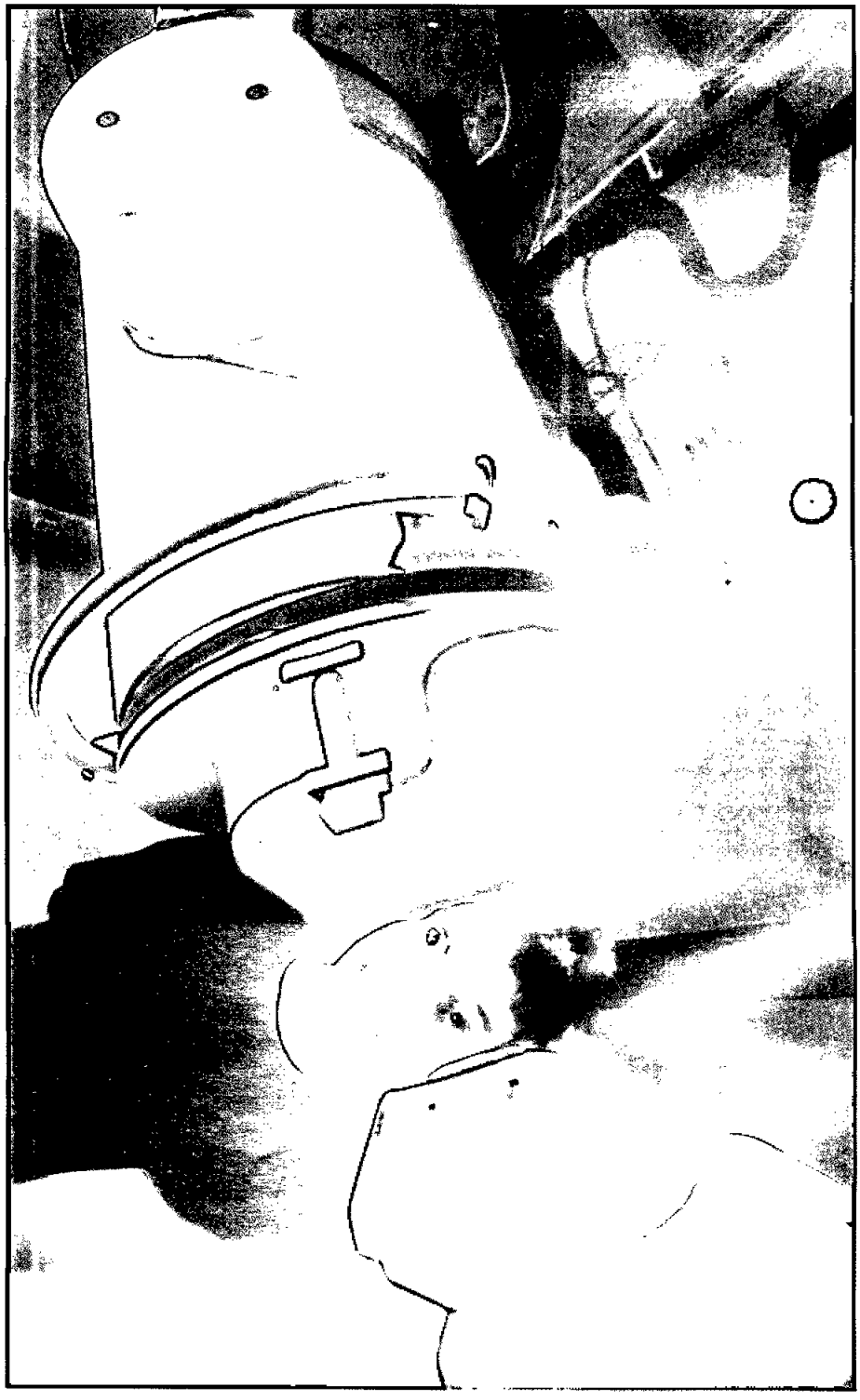
42. (a) $T_0 - \dfrac{(\pi b^2 IB)}{L}$ and $T_0 + \dfrac{(\pi b^2 IB)}{L}$; (b) T_0 each

43. $\dfrac{qBl}{m}$

44. $x = \dfrac{qBmv_0}{\alpha^2 + (qB)^2}$, $y = \dfrac{\alpha m v_0}{\alpha^2 + (qB)^2}$

Current-Produced Magnetic Fields

30



Jurgen Scriba/Photo Researchers

When you read this sentence, a certain region of your brain is activated. When you smell a rose or feel a pencil in your grip, other regions are activated. One of the best ways to determine which regions are activated is to detect the magnetic field produced by the activation. The apparatus in the photograph can detect the magnetic field set up by a person's brain so that a map of brain activity can be correlated with what the person does. However, there are no magnetic materials in the brain.

So, how can brain activation produce a magnetic field?

The answer is in this chapter.

30-1 WHAT IS PHYSICS?

One basic observation of physics is that a moving charged particle produces a magnetic field around itself. Thus a current of moving charged particles produces a magnetic field around the current. This feature of *electromagnetism,* which is the combined study of electric and magnetic effects, came as a surprise to the people who discovered it. Surprise or not, this feature has become enormously important in everyday life because it is the basis of countless electromagnetic devices. For example, a magnetic field produced by an electric current can be found in machines that record or read magnetically encoded tapes and disks such as old-fashioned computer floppies, cassette audio tapes, and videotapes. Such a field can also be found in magnetically lifted trains (maglev trains) and other devices used to lift heavy loads.

Our first step in this chapter is to determine the magnetic field due to the current in a very small section of current-carrying wire. Then we shall calculate the magnetic field due to the entire wire for several different arrangements of the wire. We will also study the magnetic field created by a moving charged particle.

30-2 | Calculating the Magnetic Field Due to a Current

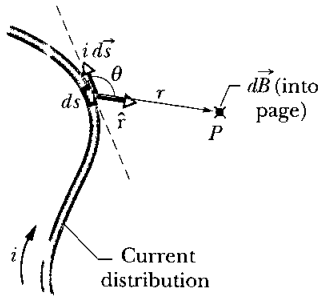


FIG. 30-1 A current-length element $i\,d\vec{s}$ produces a differential magnetic field $d\vec{B}$ at point P. The green × (the tail of an arrow) at the dot for point P indicates that $d\vec{B}$ is directed *into* the page there.

Figure 30-1 shows a wire of arbitrary shape carrying a current i. We want to find the magnetic field $\vec{B}$ at a nearby point P. We first mentally divide the wire into differential elements ds and then define for each element a length vector $d\vec{s}$ that has length ds and whose direction is the direction of the current in ds. We can then define a differential *current-length element* to be $i\,d\vec{s}$; we wish to calculate the field $d\vec{B}$ produced at P by a typical current-length element. From experiment we find that magnetic fields, like electric fields, can be superimposed to find a net field. Thus, we can calculate the net field $\vec{B}$ at P by summing, via integration, the contributions $d\vec{B}$ from all the current-length elements. However, this summation is more challenging than the process associated with electric fields because of a complexity; whereas a charge element dq producing an electric field is a scalar, a current-length element $i\,d\vec{s}$ producing a magnetic field is a vector, being the product of a scalar and a vector.

The magnitude of the field $d\vec{B}$ produced at point P at distance r by a current-length element $i\,d\vec{s}$ turns out to be

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}, \tag{30-1}$$

where θ is the angle between the directions of $d\vec{s}$ and $\hat{r}$, a unit vector that points from ds toward P. Symbol μ_0 is a constant, called the *permeability constant,* whose value is defined to be exactly

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\,\mathrm{T\cdot m/A}. \tag{30-2}$$

The direction of $d\vec{B}$, shown as being into the page in Fig. 30-1, is that of the cross product $d\vec{s} \times \hat{r}$. We can therefore write Eq. 30-1 in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2} \qquad \text{(Biot–Savart law).} \tag{30-3}$$

This vector equation and its scalar form, Eq. 30-1, are known as the **law of Biot and Savart** (rhymes with "Leo and bazaar"). The law, which is experimentally

deduced, is an inverse-square law. We shall use this law to calculate the net magnetic field $\vec{B}$ produced at a point by various distributions of current.

Magnetic Field Due to a Current in a Long Straight Wire

Shortly we shall use the law of Biot and Savart to prove that the magnitude of the magnetic field at a perpendicular distance R from a long (infinite) straight wire carrying a current i is given by

$$B = \frac{\mu_0 i}{2\pi R} \qquad \text{(long straight wire).} \tag{30-4}$$

The field magnitude B in Eq. 30-4 depends only on the current and the perpendicular distance R of the point from the wire. We shall show in our derivation that the field lines of $\vec{B}$ form concentric circles around the wire, as Fig. 30-2 shows and as the iron filings in Fig. 30-3 suggest. The increase in the spacing of the lines in Fig. 30-2 with increasing distance from the wire represents the $1/R$ decrease in the magnitude of $\vec{B}$ predicted by Eq. 30-4. The lengths of the two vectors $\vec{B}$ in the figure also show the $1/R$ decrease.

Here is a simple right-hand rule for finding the direction of the magnetic field set up by a current-length element, such as a section of a long wire:

Right-hand rule: Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

The result of applying this right-hand rule to the current in the straight wire of Fig. 30-2 is shown in a side view in Fig. 30-4*a*. To determine the direction of the magnetic field $\vec{B}$ set up at any particular point by this current, mentally wrap your right hand around the wire with your thumb in the direction of the current. Let your fingertips pass through the point; their direction is then the direction of the magnetic field at that point. In the view of Fig. 30-2, $\vec{B}$ at any point is *tangent to a magnetic field line;* in the view of Fig. 30-4, it is *perpendicular to a dashed radial line connecting the point and the current.*

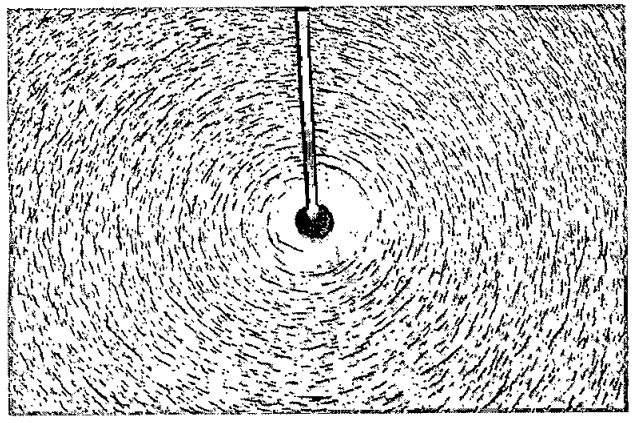


FIG. 30-3 Iron filings that have been sprinkled onto cardboard collect in concentric circles when current is sent through the central wire. The alignment, which is along magnetic field lines, is caused by the magnetic field produced by the current. *(Courtesy Education Development Center)*

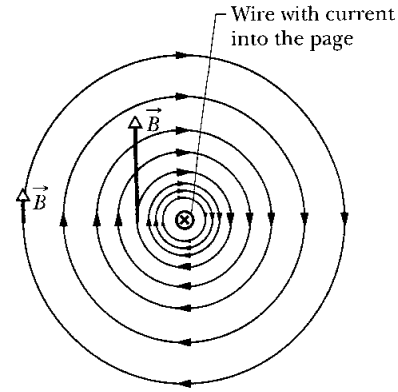
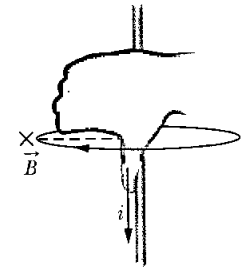
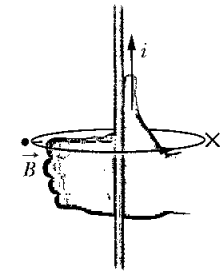


FIG. 30-2 The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire. Here the current is into the page, as indicated by the ×.



(*a*)



(*b*)

FIG. 30-4 A right-hand rule gives the direction of the magnetic field due to a current in a wire. (*a*) The situation of Fig. 30-2, seen from the side. The magnetic field $\vec{B}$ at any point to the left of the wire is perpendicular to the dashed radial line and directed into the page, in the direction of the fingertips, as indicated by the ×. (*b*) If the current is reversed, $\vec{B}$ at any point to the left is still perpendicular to the dashed radial line but now is directed out of the page, as indicated by the dot.

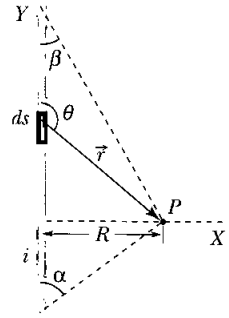


FIG. 30-5 The magnetic field due to a finite current-carrying wire. Note that according to the right-hand rule, the magnetic field at P is inside the page.

Proof of Equation 30-4

Figure 30-5, which is just like Fig. 30-1 except that now the wire is straight and of infinite length, illustrates the task at hand. We seek the field $\vec{B}$ at point P, a perpendicular distance R from the wire. The magnitude of the differential magnetic field produced at P by the current-length element $i\,d\vec{s}$ located a distance r from P is given by Eq. 30-1:

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}.$$

The direction of $d\vec{B}$ in Fig. 30-5 is that of the vector $d\vec{s}\times\hat{\mathrm{r}}$—namely, directly into the page.

Note that $d\vec{B}$ at point P has this same direction for all the current-length elements into which the wire can be divided. Thus, we can find the magnitude of the magnetic field produced at P by the current-carrying wire by integrating dB in Eq. 30-3.

Since the vector $d\vec{s}$ points in the direction of the current flow, we have

$$d\vec{s} = ds(\hat{\mathrm{j}}).$$

As can be seen from Fig. 30-5, we have

$$\vec{r} = r\cos(\pi-\theta)(-\hat{\mathrm{j}}) + r\sin(\pi-\theta)(\hat{\mathrm{i}}) = r\sin\theta(\hat{\mathrm{i}}) + r\cos\theta(\hat{\mathrm{j}}).$$

Note that the current element is chosen as the origin. Therefore

$$d\vec{s}\times\vec{r} = r\,ds\,\sin\theta(-\hat{\mathrm{k}}).$$

Also

$$\tan(\pi-\theta) = R/s, \quad \sin(\pi-\theta) = R/r \;\Rightarrow\; r = R\,\mathrm{cosec}\,\theta.$$

Therefore

$$s = -R\cot\theta \Rightarrow ds = R\,\mathrm{cosec}^2\theta\,d\theta$$

$$|d\vec{B}| = \frac{\mu_0}{4\pi}i\frac{ds\,r\sin\theta}{r^3} = \frac{\mu_0 i}{4\pi}\sin\theta\frac{R\,\mathrm{cosec}^2\theta}{R^2\,\mathrm{cosec}^2\theta}d\theta,$$

$$B = \int_{-\alpha}^{\pi-\beta}\frac{\mu_0 i}{4\pi R}\sin\theta\,d\theta = \frac{\mu_0 i}{4\pi R}(\cos\alpha+\cos\beta). \tag{30-5}$$

This can be easily extended to the case of an infinitely long wire, for which we can see that $\alpha=\beta=0°$. So the magnetic field due to an infinite wire at a distance R from it is

$$B = \frac{\mu_0 i}{2\pi R}, \tag{30-6}$$

which is the relation we set out to prove. Note that the magnetic field at P due to either the lower half or the upper half of the infinite wire in Fig. 30-5 is half this value; that is,

$$B = \frac{\mu_0 i}{4\pi R} \qquad \text{(semi-infinite straight wire).} \tag{30-7}$$

Magnetic Field Due to a Current in a Circular Arc of Wire

To find the magnetic field produced at a point by a current in a curved wire, we would again use Eq. 30-1 to write the magnitude of the field produced by a single current-length element, and we would again integrate to find the net field produced by all the current-length elements. That integration can be difficult, depending on the shape of the wire; it is fairly straightforward, however, when the wire is a circular arc and the point is the center of curvature.

Figure 30-6*a* shows such an arc-shaped wire with central angle ϕ, radius R, and center C, carrying current i. At C, each current-length element $i\,d\vec{s}$ of the wire produces a magnetic field of magnitude dB given by Eq. 30-1. Moreover, as Fig. 30-6*b* shows, no matter where the element is located on the wire, the angle θ between the vectors $d\vec{s}$ and $\hat{r}$ is 90°; also, $r = R$. Thus, by substituting R for r and 90° for θ in Eq. 30-1, we obtain

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90°}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}. \tag{30-8}$$

The field at C due to each current-length element in the arc has this magnitude.

An application of the right-hand rule anywhere along the wire (as in Fig. 30-6*c*) will show that all the differential fields $d\vec{B}$ have the same direction at C—directly out of the page. Thus, the total field at C is simply the sum (via integration) of all the differential fields $d\vec{B}$. We use the identity $ds = R\,d\phi$ to change the variable of integration from ds to $d\phi$ and obtain, from Eq. 30-8,

$$B = \int dB = \int_0^{\phi}\frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^{\phi} d\phi.$$

Integrating, we find that

$$B = \frac{\mu_0 i\phi}{4\pi R} \qquad \text{(at center of circular arc)}. \tag{30-9}$$

Note that this equation gives us the magnetic field *only* at the center of curvature of a circular arc of current. When you insert data into the equation, you must be careful to express ϕ in radians rather than degrees. For example, to find the magnitude of the magnetic field at the center of a full circle of current, you would substitute 2π rad for ϕ in Eq. 30-9, finding

$$B = \frac{\mu_0 i(2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \qquad \text{(at center of full circle)}. \tag{30-10}$$

Figure 30-7*a* shows a current loop, a circular loop of wire with radius R, that carries current I. Let us find the magnetic field of the current loop at distance z from the center on the axis of the loop.

Real coils need wires to bring the current in and out, but we'll model the coil as a current moving around the full circuit shown in Fig. 30-7*b*. You can easily see that if the wires are of small radius, their contribution to the magnetic field at any point in space will cancel.

Figure 30-8 shows a loop for which we've assumed that the current is circulating counterclockwise. We've chosen a coordinate system in which the loop lies in the xy plane, the center being the origin. Let k be the segment at the top of the

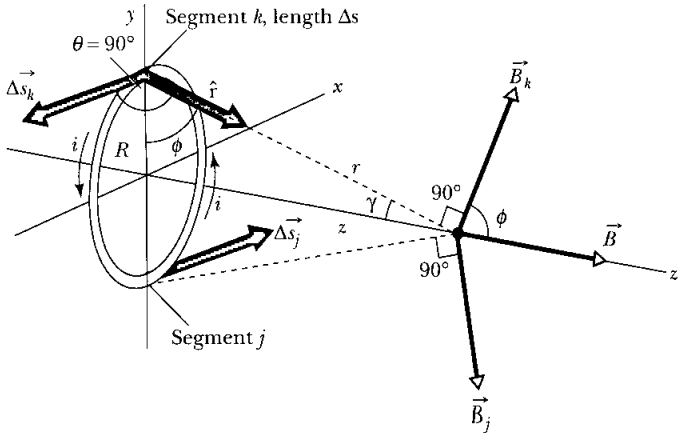


FIG. 30-8 Calculating the magnetic field of a current loop.

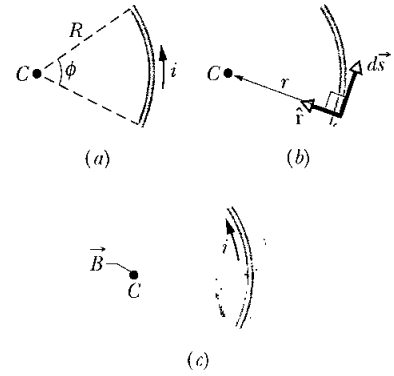


FIG. 30-6 (*a*) A wire in the shape of a circular arc with center C carries current i. (*b*) For any element of wire along the arc, the angle between the directions of $d\vec{s}$ and $\vec{r}$ is 90°. (*c*) Determining the direction of the magnetic field at the center C due to the current in the wire; the field is out of the page, in the direction of the fingertips, as indicated by the colored dot at C.

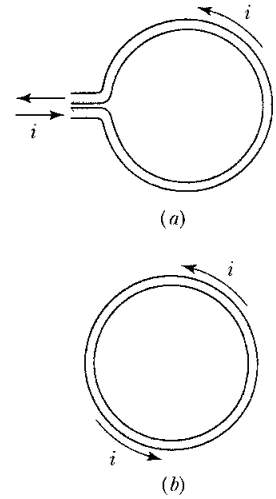


FIG. 30-7 A current loop. (*a*) A practical current loop; (*b*) an ideal current loop.

loop. Vector $\Delta\vec{s}_k$ is parallel to the x axis and unit vector $\hat{r}$ is in the yz plane; thus θ_k, the angle between $\Delta\vec{s}_k$ and $\hat{r}$, is 90°.

The direction of $\vec{B}_k$, the magnetic field due to the current in segment k, is given by the cross-product $\Delta\vec{s}_k \times \hat{r} \cdot \vec{B}_k$ must be perpendicular to both $\Delta\vec{s}_k$ and $\hat{r}$. You should convince yourself that $\vec{B}_k$ in Fig. 30-8 points in the correct direction. Notice that the y component of $\vec{B}_k$ is canceled by the y component of magnetic field $\vec{B}_j$ due to the current segment at the bottom of the loop, on the opposite side of the loop, such that the x and y components of $\vec{B}$ cancel and the components of $\vec{B}$ parallel the z axis add. In other words, the symmetry of the loop requires the on-axis magnetic field to point along the z axis. Knowing that we need to sum only the z components will simplify our calculation.

We can use the Biot–Savart law to find the z component $[(B_k)_z = B_k \cos\phi]$ of the magnetic field of segment k. The cross-product $\Delta\vec{s}_k \times \hat{r}$ has magnitude $(\Delta s)(1) \sin 90° = \Delta s$. Thus

$$(B_k)_z = \frac{\mu_0}{4\pi}\frac{i\Delta s}{r^2}\cos\phi = \frac{\mu_0 i\cos\phi}{4\pi(z^2+R^2)}\Delta s, \tag{30-11}$$

where we used $r = (z^2+R^2)^{1/2}$. You can see, because $\phi + \gamma = 90°$, that angle ϕ is also the angle between $\hat{r}$ and the radius of the loop. Hence $\cos\phi = R/r$, and

$$(B_k)_z = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}\Delta s. \tag{30-12}$$

The final step is to sum the magnetic fields due to all the segments:

$$B_{\text{loop}} = \sum_k (B_k)_z = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}\sum_k \Delta s. \tag{30-13}$$

In this case, unlike the straight wire, none of the terms multiplying with Δs depend on the position of segment k; therefore, all these terms can be factored out during the summation. We're left with summation that adds up the lengths of all the small segments. But this is just the circumference $2\pi R$. Thus the on-axis magnetic field of a current loop is

$$B_{\text{loop}} = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}2\pi R = \frac{\mu_0}{2}\frac{iR^2}{(z^2+R^2)^{3/2}}. \tag{30-14}$$

If instead of a single loop, we had N coils overlapped over each other, the magnetic field would have been

$$B_{\text{loop}} = \frac{\mu_0}{2}\frac{NiR^2}{(z^2+R^2)^{3/2}}. \tag{30-15}$$

Sample Problem 30-1

Find the magnetic field due to a solenoid having n turns per unit length at a point on its axis (Fig. 30-9).

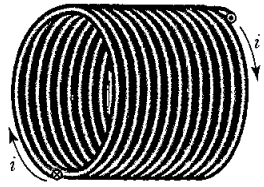


FIG. 30-9 A solenoid carrying current i.

Conceptualize: A solenoid consists of a large number of circular coils placed close to each other. Each of these coils is not in a single plane, but if the turns are close we can assume them to be in a single plane.

Classify: We can apply the formula for the magnetic field due to a single coil we derived just now (Eq. 30-15). Each of the coils has a different distance from the point, but we can find the magnetic field due to a short part of the solenoid, and then add up the contribution due to each.

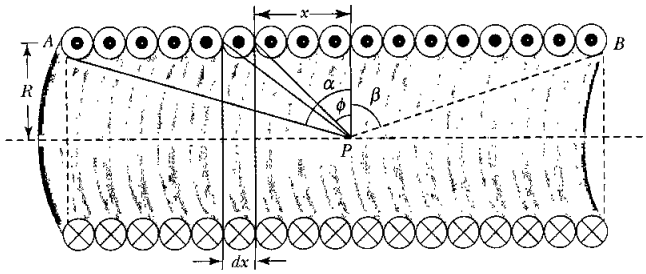


FIG. 30-10 The magnetic field at point P on the axis of the solenoid due to the different coils of the solenoid.

Compute: So if we consider a coil of width dx at a distance x from the point P on the axis of the solenoid as shown in Fig. 30-10, we have

$$dB = \frac{\mu_0}{4\pi}\frac{2\pi(dN)iR^2}{(R^2+x^2)^{3/2}}. \tag{30-16}$$

If n is the number of turns per unit length, $dN = n\,dx$. As in Fig. 30-10, $x = R\tan\phi$, that is, $dx = R\sec^2\phi\,d\phi$, so

$$dB = \frac{\mu_0}{4\pi}\frac{2\pi(n\,dx)iR^2}{(R^2+R^2\tan^2\phi)^{3/2}} = \frac{\mu_0}{4\pi}(2\pi ni)\cos\phi\,d\phi \tag{30-17}$$

and hence

$$B = \frac{\mu_0}{4\pi}(2\pi ni)\int_{-\alpha}^{\beta}\cos\phi\,d\phi. \tag{30-18}$$

That is

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(\sin\alpha+\sin\beta). \tag{30-19}$$

This is the desired result and from this it is clear that

1. If the solenoid is of infinite length and the point is well inside the solenoid,

$$\alpha = \beta = \frac{\pi}{2}.$$

So,

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(1+1) = \mu_0 ni. \tag{30-20}$$

2. If the solenoid is of infinite length and the point is near one end,

$$\alpha = 0;\quad \beta = \frac{\pi}{2}.$$

So,

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(1+0) = \frac{1}{2}(\mu_0 ni). \tag{30-21}$$

Conclude: The variation of the field with distance along the axis of a solenoid is shown in Fig. 30-11.

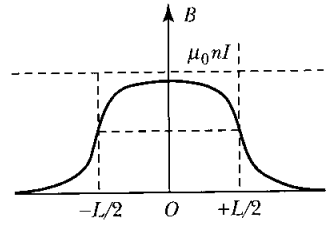


FIG. 30-11 Variation of the magnetic field along the axis of a finite solenoid.

A note about the current-carrying coil: You saw in Section 29-9 that such a coil behaves as a magnetic dipole in that, if we place it in an external magnetic field, a torque given by

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

acts on it. Here $\vec{\mu}$ is the magnetic dipole moment of the coil and has the magnitude NiA, where N is the number of turns, i is the current in each turn, and A is the area enclosed by each turn. (*Caution:* Don't confuse the magnetic dipole moment with the permeability constant μ_0.)

Recall that the direction of $\vec{\mu}$ is given by a curled-straight right-hand rule: Grasp the coil so that the fingers of your right hand curl around it in the direction of the current; your extended thumb then points in the direction of the dipole moment.

Magnetic Field of a Coil

We turn now to the other aspect of a current-carrying coil as a magnetic dipole. What magnetic field does *it* produce at a point in the surrounding space? The result is difficult to derive. We have derived it just now for the case when we want to find the magnetic field at the axis of the dipole. In terms of $\vec{\mu}$, we can rewrite the equation as

$$B = \frac{2\mu_0\mu}{4\pi(z^2+R^2)^{3/2}} \approx \frac{2\mu_0\mu}{4\pi z^3}, \tag{30-22}$$

when $z \gg R$.

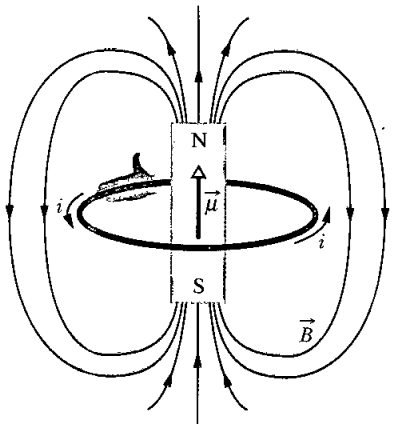
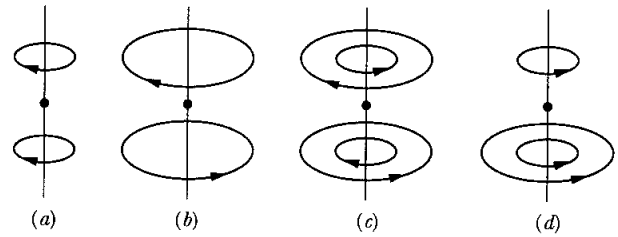


FIG. 30-12 A current loop produces a magnetic field like that of a bar magnet and thus has associated north and south poles. The magnetic dipole moment $\vec{\mu}$ of the loop, its direction given by a curled–straight right-hand rule, points from the south pole to the north pole, in the direction of the field $\vec{B}$ within the loop.

Thus, we have two ways in which we can regard a current-carrying coil as a magnetic dipole: (1) it experiences a torque when we place it in an external magnetic field; (2) it generates its own intrinsic magnetic field, given, for distant points along its axis, by Eq. 30-22. Figure 30-12 shows the magnetic field of a current loop; one side of the loop acts as a north pole (in the direction of $\vec{\mu}$) and the other side as a south pole, as suggested by the lightly drawn magnet in the figure.

CHECKPOINT 1 The figure here shows four arrangements of circular loops of radius r or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.



Magnetic Field Due to Brain Activity

Scientists, medical researchers, and physiologists would like to understand how the human brain works. One of their new tools is *magnetoencephalography* (MEG), a procedure in which magnetic fields of a person's brain are monitored as the person performs a task, such as reading a word. The task activates part of the brain, such as the part that processes reading, causing weak electrical pulses to be sent along conducting paths between brain cells. As with any other current, each pulse produces a magnetic field.

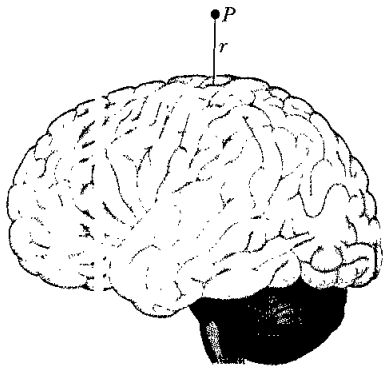


FIG. 30-13 A pulse along a fissure wall on the brain surface produces a magnetic field at point P at distance r.

The magnetic fields detected in MEG are probably produced by pulses along the walls of the fissures (crevices) on the brain surface (Fig. 30-13). Let's use Eq. 30-1 to estimate the magnitude of such a field at a point P located a distance $r = 2$ cm from the pulse. Let the path of the pulse be tangent to the brain surface so that the angle θ in Eq. 30-1 is 90°. In a typical pulse, the current is $i = 10\ \mu$A and the conducting path length is about 1 mm. Let's assume that this length is short enough to allow us to substitute 1 mm for the element length ds in Eq. 30-1. Then Eq. 30-1 gives us

$$B = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})}{4\pi} \frac{(10 \times 10^{-6}\ \text{A})(1 \times 10^{-3}\ \text{m})}{(2 \times 10^{-2}\ \text{m})^2} \sin 90°$$
$$= 2.5 \times 10^{-12}\ \text{T} \approx 3\ \text{pT}.$$

This is a *very* small magnetic field, more than a million times weaker than Earth's magnetic field at your location. Thus, if you want to detect the magnetic field of a brain, you cannot simply hold a compass near the brain and hope that brain activity will turn the compass needle. Instead, you need extremely sensitive instruments called SQUIDs (superconducting quantum interference devices) that can measure fields of magnitude less than 1 pT, and even then you must take care to eliminate other sources of varying magnetic fields in the detection area.

Sample Problem 30-2 Build your skill

The wire in Fig. 30-14*a* carries a current i and consists of a circular arc of radius R and central angle $\pi/2$ rad, and two straight sections whose extensions intersect the center C of the arc. What magnetic field $\vec{B}$ does the current produce at C?

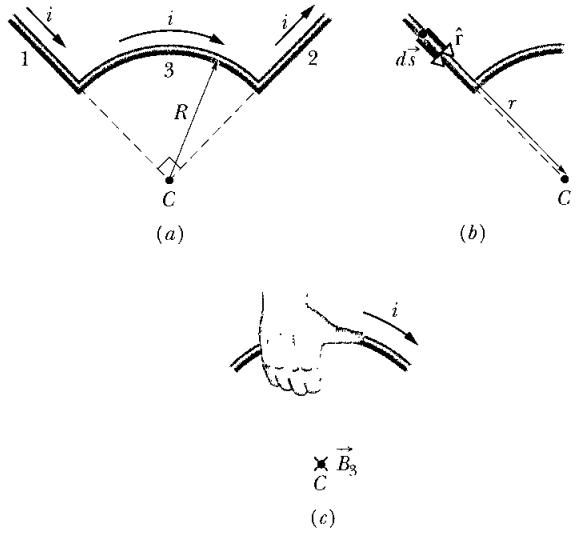


FIG. 30-14 (*a*) A wire consists of two straight sections (1 and 2) and a circular arc (3), and carries current i. (*b*) For a current-length element in section 1, the angle between $d\vec{s}$ and $\vec{r}$ is zero. (*c*) Determining the direction of magnetic field $\vec{B}_3$ at C due to the current in the circular arc; the field is into the page there.

Conceptualize/Classify: We can find the magnetic field $\vec{B}$ at point C by applying the Biot–Savart law of Eq. 30-3 to the wire. The application of Eq. 30-3 can be simplified by evaluating $\vec{B}$ separately for the three distinguishable sections of the wire—namely, (1) the straight section at the left, (2) the straight section at the right, and (3) the circular arc.

Compute Straight sections: For any current-length element in section 1, the angle θ between $d\vec{s}$ and $\hat{r}$ is zero (Fig. 30-14*b*); so Eq. 30-1 gives us

$$dB_1 = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 0}{r^2} = 0.$$

Thus, the current along the entire length of straight section 1 contributes no magnetic field at C:

$$B_1 = 0.$$

The same situation prevails in straight section 2, where the angle θ between $d\vec{s}$ and $\hat{r}$ for any current-length element is 180°. Thus,

$$B_2 = 0.$$

Circular arc: Application of the Biot–Savart law to evaluate the magnetic field at the center of a circular arc leads to Eq. 30-9 ($B = \mu_0 i\phi/4\pi R$). Here the central angle ϕ of the arc is $\pi/2$ rad. Thus from Eq. 30-9, the magnitude of the magnetic field $\vec{B}_3$ at the arc's center C is

$$B_3 = \frac{\mu_0 i(\pi/2)}{4\pi R} = \frac{\mu_0 i}{8R}.$$

To find the direction of $\vec{B}_3$, we apply the right-hand rule displayed in Fig. 30-4. Mentally grasp the circular arc with your right hand as in Fig. 30-14*c*, with your thumb in the direction of the current. The direction in which your fingers curl around the wire indicates the direction of the magnetic field lines around the wire. In the region of point C (inside the circular arc), your fingertips point *into the plane* of the page. Thus, $\vec{B}_3$ is directed into that plane.

Net field: Generally, when we must combine two or more magnetic fields to find the net magnetic field, we must combine the fields as vectors and not simply add their magnitudes. Here, however, only the circular arc produces a magnetic field at point C. Thus, we can write the magnitude of the net field $\vec{B}$ as

$$B = B_1 + B_2 + B_3 = 0 + 0 + \frac{\mu_0 i}{8R} = \frac{\mu_0 i}{8R}. \qquad \text{(Answer)}$$

The direction of $\vec{B}$ is the direction of $\vec{B}_3$—namely, into the plane of Fig. 30-14.

Sample Problem 30-3 Build your skill

Figure 30-15*a* shows two long parallel wires carrying currents i_1 and i_2 in opposite directions. What are the magnitude and direction of the net magnetic field at point P? Assume the following values: $i_1 = 15$ A, $i_2 = 32$ A, and $d = 5.3$ cm.

Conceptualize: The net magnetic field $\vec{B}$ at point P is the vector sum of the magnetic fields due to the currents in the two wires.

Classify: We can find the magnetic field due to any current by applying the Biot–Savart law to the current.

For points near the current in a long straight wire, that law leads to Eq. 30-4.

Finding the vectors: In Fig. 30-15*a*, point P is distance R from both currents i_1 and i_2. Thus, Eq. 30-4 tells us that at point P those currents produce magnetic fields $\vec{B}_1$ and $\vec{B}_2$ with magnitudes

$$B_1 = \frac{\mu_0 i_1}{2\pi R} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi R}.$$

In the right triangle of Fig. 30-15*a*, note that the base angles (between sides R and d) are both 45°. This allows us to write $\cos 45° = R/d$ and replace R with $d \cos 45°$. Then the field magnitudes B_1 and B_2 become

$$B_1 = \frac{\mu_0 i_1}{2\pi d \cos 45°} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi d \cos 45°}.$$

Compute: We want to combine $\vec{B}_1$ and $\vec{B}_2$ to find their vector sum, which is the net field $\vec{B}$ at P. To find the directions of $\vec{B}_1$ and $\vec{B}_2$, we apply the right-hand rule of Fig. 30-4 to each current in Fig. 30-15*a*. For wire 1, with current out of the page, we mentally grasp the wire with the right hand, with the thumb pointing out of the page. Then the curled fingers indicate that the field lines run counterclockwise. In particular, in the region of point P, they are directed upward to the left. Recall that the magnetic field at a point near a long, straight current-carrying wire must be directed perpendicular to a radial line between the point and the current. Thus, $\vec{B}_1$ must be directed upward to the left, as drawn in Fig. 30-15*b*. (Note carefully the perpendicular symbol between vector $\vec{B}_1$ and the line connecting point P and wire 1.)

Repeating this analysis for the current in wire 2, we find that $\vec{B}_2$ is directed upward to the right, as drawn in Fig. 30-15*b*. (Note the perpendicular symbol between vector $\vec{B}_2$ and the line connecting point P and wire 2.)

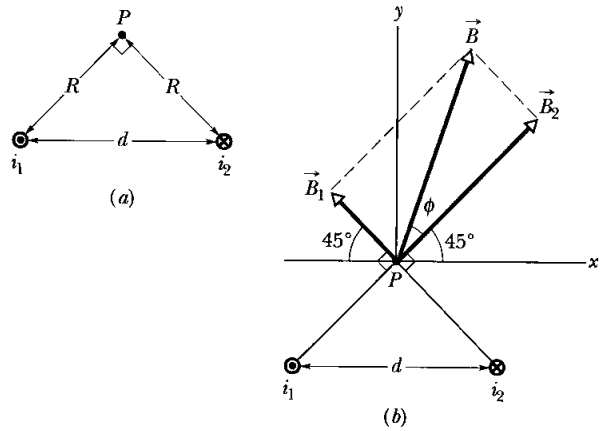


FIG. 30-15 (*a*) Two wires carry currents i_1 and i_2 in opposite directions (out of and into the page). Note the right angle at P. (*b*) The separate fields $\vec{B}_1$ and $\vec{B}_2$ are combined vectorially to yield the net field $\vec{B}$.

Adding the vectors: We can now vectorially add $\vec{B}_1$ and $\vec{B}_2$ to find the net magnetic field $\vec{B}$ at point P, either by using a vector-capable calculator or by resolving the vectors into components and then combining the components of $\vec{B}$. However, in Fig. 30-15*b*, there is a third method: Because $\vec{B}_1$ and $\vec{B}_2$ are perpendicular to each other, they form the legs of a right triangle, with $\vec{B}$ as the hypotenuse. The Pythagorean theorem then gives us

$$B = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi d(\cos 45°)}\sqrt{i_1^2 + i_2^2}$$

$$= \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})\sqrt{(15\ \text{A})^2 + (32\ \text{A})^2}}{(2\pi)(5.3 \times 10^{-2}\ \text{m})(\cos 45°)}$$

$$= 1.89 \times 10^{-4}\ \text{T} \approx 190\ \mu\text{T}. \qquad \text{(Answer)}$$

The angle ϕ between the directions of $\vec{B}$ and $\vec{B}_2$ in Fig. 30-15*b* follows from

$$\phi = \tan^{-1}\frac{B_1}{B_2},$$

which, with B_1 and B_2 as given above, yields

$$\phi = \tan^{-1}\frac{i_1}{i_2} = \tan^{-1}\frac{15\ \text{A}}{32\ \text{A}} = 25°.$$

The angle between the direction of $\vec{B}$ and the x axis shown in Fig. 30-15*b* is then

$$\phi + 45° = 25° + 45° = 70°. \qquad \text{(Answer)}$$

Sample Problem 30-4

A long, straight wire carries a current i. A particle having a positive charge q and mass m, kept at a distance x_0 from the wire, is projected toward it with a speed v. Find the minimum separation between the wire and the particle.

Conceptualize: Let the particle be initially at P (Fig. 30-16). Take the wire as the y axis and the foot of perpendicular from P to the wire as the origin. Take the line OP as the x axis. We have, $OP = x_0$. The magnetic field B at any point to the right of the wire is along the

negative z axis. The magnetic force on the particle is, therefore, in the xy plane. As there is no initial velocity along the z axis, the motion will be in the xy plane. We know that the magnetic field does not do any work. So the speed of the charged particle does not change as it moves through the magnetic field. Since the magnetic field is not uniform, the particle does not go along a circle but rather a curved path in the xy plane.

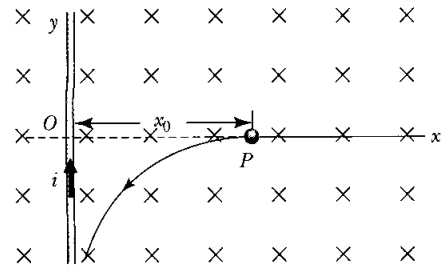


FIG. 30-16 A positively charged particle at P is projected toward a current-carrying wire.

Classify: The force at time t is $\vec{F} = q\vec{v} \times \vec{B}$. Here we have to use the basic definition of force and velocity.

Compute:

$$\vec{F} = q(v_x\hat{i} + v_y\hat{j}) \times \left(-\frac{\mu_0 i}{2\pi x}\hat{k}\right)$$

$$= qv_x\frac{\mu_0 i}{2\pi x}\hat{j} - qv_y\frac{\mu_0 i}{2\pi x}\hat{i}.$$

We can resolve this force along x and y directions. Thus

$$a_x = \frac{F_x}{m} = -\frac{\mu_0 qi}{2\pi m}\frac{v_y}{x} = -\lambda\frac{v_y}{x}, \tag{30-23}$$

where

$$\lambda = \frac{\mu_0 qi}{2\pi m}.$$

Also,

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{v_x dv_x}{dx}. \tag{30-24}$$

Since the speed remains constant,

$$v_x^2 + v_y^2 = v^2.$$

Differentiating this expression, we get

$$2v_x dv_x + 2v_y dv_y = 0$$

$$v_x dv_x = -v_y dv_y. \tag{30-25}$$

From Eqs. 30-24 and 30-25 we have

$$\frac{v_y dv_y}{dx} = \frac{\lambda v_y}{x}$$

$$\frac{dx}{x} = \frac{dv_y}{\lambda}.$$

Initially $x = x_0$ and $v_y = 0$. When x is minimum, by the concept of minima, $v_x = 0$ so that $v_y = -v$. Thus

$$\int_{x_0}^{x}\frac{dx}{x} = \int_0^{-v}\frac{dv_y}{\lambda}.$$

Solving we get

$$\ln\frac{x}{x_0} = -\frac{v}{\lambda}$$

$$x = x_0 e^{-v/\lambda} = x_0 e^{-2\pi mv/\mu_0 qi}.$$

Conclude: The charged particle will turn back without changing its speed. Even if a charged particle, say a stray electron, is emitted from the wire, it will move in a curved path and turn back at some point. However, in that case also, the path of the particle will not be circular.

PROBLEM-SOLVING TACTICS

Tactic 1: *Right-Hand Rules* To help you sort out the right-hand rules you have now seen (and the ones coming up), here is a review.

Right-Hand Rule for Cross Products. Introduced in Section 3-8, this is a way to determine the direction of the vector that results from a cross product. You point the fingers of your right hand so as to sweep the first vector expressed in the product into the second vector, through the smaller angle between the two vectors. Your outstretched thumb gives you the direction of the vector resulting from the cross product. In Chapter 11, we used this right-hand rule to find the directions of torque and angular momentum vectors; in Chapter 28, we used it to find the direction of the force on a current-carrying wire in a magnetic field.

Curled–Straight Right-Hand Rules for Magnetism. In many situations involving magnetism, you need to relate a "curled" element and a "straight" element. You can do so with the (curled) fingers and the (straight) thumb on your right hand. You have already seen an example in Section 28-9, where we related the current around a loop (curled element) to the normal vector $\vec{n}$ (straight element) of the loop: You curl the fingers of your right hand around in the direction of the current along the loop; your outstretched thumb then gives the direction of $\vec{n}$. This is also the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

In this section you were introduced to a second curled–straight right-hand rule for magnetism. To determine the direction of the magnetic field lines around a current-length element, you point the outstretched thumb of your right hand in the direction of the current. The fingers then naturally curl around the current-length element in the direction of the field lines around the element.

30-3 | Magnetic Field of Moving Charged Particles

Let us now use the Biot–Savart law to find an expression for the magnetic field caused by uniformly moving charged particles. Consider first a small wire carrying a current i, as illustrated in Fig. 30-1. The magnetic field due to a small segment of length dx is given by Eq. 30-3:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2}.$$

Suppose the segment contains an amount of charge dq; let this charge take an amount of time dt to move a distance dx due to the current i. According to the definition of the current we have

$$i = \frac{dq}{dt}.$$

Therefore,

$$i\,d\vec{s} = \frac{dq}{dt}d\vec{s} = dq\frac{d\vec{s}}{dt} = \vec{v}\,dq,$$

where v is the velocity of the charge carriers along the wire. In the limiting case that the segment contains just a single carrier carrying a charge q, dq becomes simply q and so

$$i\,d\vec{s} = \vec{v}\,q.$$

Thus, we obtain an expression for the magnetic field of a single moving charged particle:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}. \tag{30-26}$$

We would like to emphasize that the Biot–Savart law was valid only for steady currents; and obviously the motion of a single charge cannot constitute a steady current by any stretch of imagination. This equation is not strictly true; it is valid only when the charge moves with a velocity much smaller than the velocity of light.

Sample Problem 30-5 The Magnetic Field of a Proton

A proton moves along the x axis with velocity $v_x = 1.0\times10^7$ m/s. As it passes the origin, what is the magnetic field at the (x, y, z) position given by (1 mm, 0 mm, 0 mm), (0 mm, 1 mm, 0 mm), and (1 mm, 1 mm, 0 mm)?

Conceptualize: Figure 30-17 shows the geometry. The first point is on the x axis, directly in front of the proton, with $\theta_1 = 0°$. The second point is on the y axis, with $\theta_2 = 90°$, and the third is in the xy plane.

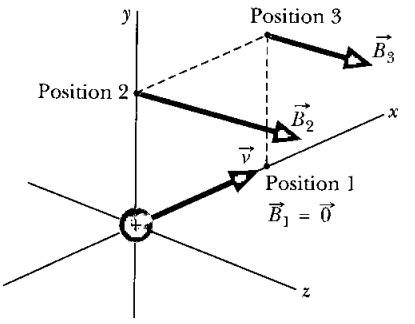


FIG. 30-17 The magnetic field of Problem 30-5.

Classify: The magnetic field is that of a moving charged particle.

Compute: Position 1, which is along the line of motion, has $\theta_1 = 0°$. Thus, $\vec{B}_1 = \vec{0}$.

Position 2 (at 0 mm, 1 mm, 0 mm) is at distance $r_2 = 1$ mm $= 0.001$ m. The magnetic field strength at this point is

$$B = \frac{\mu_0}{4\pi}\frac{qv\sin\theta_2}{r_2^2}$$

$$= \frac{4\pi\times10^{-7}\ \text{Tm/A}}{4\pi}\frac{(1.60\times10^{-19}\text{C})(1.0\times10^{7}\ \text{m/s})\sin 90°}{(0.0010\ \text{m})^2}$$

$$= 1.60\times10^{-13}\ \text{T}.$$

According to the right-hand rule, the field points in the positive z direction. Thus

$$\vec{B}_2 = 1.60\times10^{-13}\hat{\text{k}}\ \text{T},$$

where $\hat{\text{k}}$ is the unit vector in the positive z direction. The field at position 3, at (1 mm, 1 mm, 0 mm), also points in the z direction, but it is weaker than at position 2 both because r is larger and because q is smaller. From the geometry we know that $r_3 = \sqrt{2}$ mm and $\theta_3 = 45°$. So

$$B_3 = 0.57\times10^{-13}\hat{\text{k}}\ \text{T}.$$

Conclude: These magnetic fields are indeed very small. Although Faraday asserted in 1838 that a moving charge should produce the magnetic field in its vicinity as a current-carrying wire does, it took almost 37 years to demonstrate this.

The Magnetic Force and Conservation of Momentum

The magnetic force exerted by one moving charge on another is found by combining Eq. 29-2 for the force on a moving charge in a magnetic field and Eq. 30-26 for the magnetic field of a charge. The force exerted by a charge q_1 moving with velocity v_1 on a charge q_2 moving with velocity v_2 is given by

$$\vec{F}_{1,2} = q_2\vec{v}_2\times\vec{B}_1 = q_2\vec{v}_2\times\left(\frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1\times\hat{\text{r}}}{r_{1,2}^2}\right),$$

where $\vec{B}_1$ is the magnetic field at the position of charge q_2 due to charge q_1, and $\hat{\text{r}}$ is the unit vector pointing from q_1 to q_2. Similarly, the force exerted by a charge q_2 moving with velocity v_2 on a charge q_1 moving with velocity v_1 is given by

$$\vec{F}_{2,1} = q_1\vec{v}_1\times\vec{B}_2 = q_1\vec{v}_1\times\left(\frac{\mu_0}{4\pi}\frac{q_2\vec{v}_2\times(-\hat{\text{r}})}{r_{2,1}^2}\right).$$

Note that the position of 2 as seen by 1 is opposite to the position of 1 as seen by 2. These relations are remarkable in that the force exerted by charge q_1 on charge q_2 is generally not equal and opposite to that exerted by charge q_2 on charge q_1. That is, these forces do not obey Newton's third law. Let us see this in light of a situation illustrated in Fig. 30-18.

Here, the magnetic field due to charge q_1 at the location of the charge q_2 is in the negative z direction, and the force on q_2 is to the left in the negative x direction. However, the magnetic field due to q_2 at q_1 is zero because q_1 lies along the line of motion of q_2. Thus, there is no magnetic force exerted by q_2 on q_1. So the internal forces in this two-charge system will not cancel out. The system will accelerate in the direction of this force, and linear momentum will not be conserved. It seems that the law of conservation of momentum is violated.

However the true picture is this: The charges interact with each other with the help of the magnetic field and electrostatic field. These fields carry energy as well as momentum. We saw in Chapter 24 that there is energy associated with an electric field, and we will see later that there is also energy associated with a magnetic field. Similarly, these fields also carry momentum. When the charges move, as in Fig. 30-18, the linear momentum produced when the system accelerates to the left is balanced by momentum in the opposite direction carried by the fields.

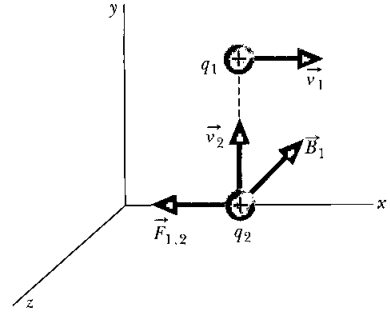


FIG. 30-18 The forces exerted by moving charges on each other are not equal and opposite. q_1 is moving in positive x direction and q_2 is moving in the y direction. At this instant q_2 moves toward q_1.

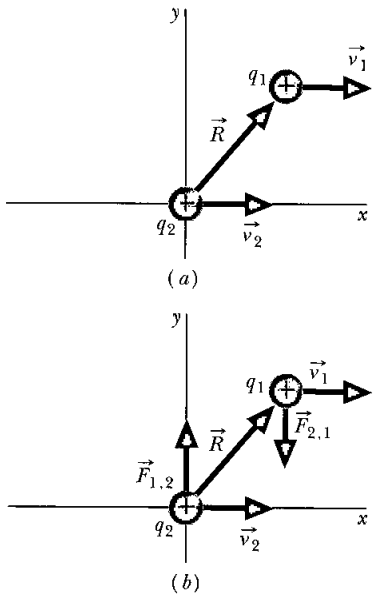


FIG. 30-19 (*a*) Two charges moving in parallel directions. (*b*) The magnetic forces exerted by the charges on each other are equal and opposite, but they are not along the line joining the charges.

Thus, when we include the momentum of the fields, the total momentum of the system is conserved.

Angular momentum is also carried by the electromagnetic fields produced by moving charges. Consider Fig. 30-19, which shows a point charge q_1 in the xy plane moving parallel to the x axis with velocity $\vec{v}_1 = v_1\hat{\mathrm{i}}$, and a second point charge q_2 at the origin and moving along the x axis with velocity $\vec{v}_2 = v_2\hat{\mathrm{i}}$. Let us calculate the magnetic force exerted by each charge on the other, assuming both charges to be positive.

We first find the force on charge q_1. We note that the vector $\vec{r}_{2,1}$ from charge q_2 to charge q_1 is just $\vec{R}$. Using $\vec{r}_{2,1} = \vec{R}/R$ we have

$$\frac{\vec{v}_2 \times \hat{\mathrm{r}}_{2,1}}{r_{2,1}^2} = \frac{\vec{v}_2 \times \vec{R}}{R^3} = \frac{\vec{v}_2\hat{\mathrm{i}} \times (x\hat{\mathrm{i}} + y\hat{\mathrm{j}})}{R^3} = \frac{yv_2}{R^3}\hat{\mathrm{k}}.$$

So the magnetic field produced by charge q_2 at the position of charge q_1 is

$$\vec{B}_2 = \frac{\mu_0}{4\pi}\frac{q_2\vec{v}_2 \times \hat{\mathrm{r}}_{2,1}}{r_{2,1}^2} = \frac{\mu_0}{4\pi}\frac{q_2 yv_2}{R^3}\hat{\mathrm{k}}.$$

The magnetic force exerted by charge q_2 on charge q_1 is then

$$\vec{F}_{2,1} = q_1\vec{v}_1 \times \vec{B}_2 = q_1 v_1\hat{\mathrm{i}} \times \frac{\mu_0}{4\pi}\frac{q_2 yv_2}{R^3}\hat{\mathrm{k}} = -\frac{\mu_0}{4\pi}\frac{q_1 q_2 v_1 v_2 y}{R^3}\hat{\mathrm{j}}.$$

This force is downward and parallel to the y axis, as shown.

To find the magnetic force exerted by charge q_1 on charge q_2, we note that the vector $\vec{r}_{1,2}$ from charge q_1 to charge q_2 is $-\vec{R}$. Then the magnetic field produced by charge q_1 at the position of charge q_2 is

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1 \times \vec{r}_{1,2}}{r_{1,2}^2} = \frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1 \times (-\vec{R})}{R^3} = -\frac{\mu_0}{4\pi}\frac{q_1 yv_1}{R^3}\hat{\mathrm{k}}.$$

The magnetic force exerted by charge q_1 on charge q_2 is then

$$\vec{F}_{1,2} = q_2\vec{v}_2 \times \vec{B}_1 = q_2 v_2\hat{\mathrm{i}} \times \left(-\frac{\mu_0}{4\pi}\frac{q_1 yv_1}{R^3}\hat{\mathrm{k}}\right) = +\frac{\mu_0}{4\pi}\frac{q_1 q_2 v_1 v_2 y}{R^3}\hat{\mathrm{j}}.$$

In this case, the forces are equal and opposite, as shown in Fig. 30-19*b*, but they are not along the line joining the two particles. The magnetic forces thus exert a torque on the two-particle system. Here also it seems that the angular momentum of the two particle system is not conserved. But in actuality the angular momentum is carried by the field.

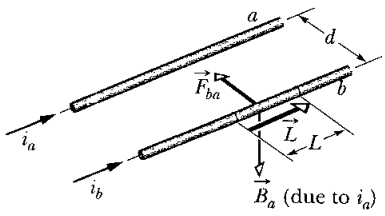


FIG. 30-20 Two parallel wires carrying currents in the same direction attract each other. $\vec{B}_a$ is the magnetic field at wire b produced by the current in wire a. $\vec{F}_{ba}$ is the resulting force acting on wire b because it carries current in $\vec{B}_a$.

30-4 | Force Between Two Parallel Currents

Two long parallel wires carrying currents exert forces on each other. Figure 30-20 shows two such wires, separated by a distance d and carrying currents i_a and i_b. Let us analyze the forces on these wires due to each other.

We seek first the force on wire b in Fig. 30-20 due to the current in wire a. That current produces a magnetic field $\vec{B}_a$ and it is this magnetic field that actually causes the force we seek. To find the force, then, we need the magnitude and direction of the field $\vec{B}_a$ *at the site of wire b*. The magnitude of $\vec{B}_a$ at every point of wire b is, from Eq. 30-4,

$$B_a = \frac{\mu_0 i_a}{2\pi d}. \qquad (30\text{-}27)$$

The curled–straight right-hand rule tells us that the direction of $\vec{B}_a$ at wire b is down, as Fig. 30-20 shows.

Now that we have the field, we can find the force it produces on wire b. Equation 29-29 tells us that the force $\vec{F}_{ba}$ on a length L of wire b due to the external magnetic field $\vec{B}_a$ is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a, \tag{30-28}$$

where $\vec{L}$ is the length vector of the wire. In Fig. 30-20, vectors $\vec{L}$ and $\vec{B}_a$ are perpendicular to each other, and so with Eq. 30-27, we can write

$$F_{ba} = i_b L B_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d}. \tag{30-29}$$

The direction of $\vec{F}_{ba}$ is the direction of the cross product $\vec{L} \times \vec{B}_a$. Applying the right-hand rule for cross products to $\vec{L}$ and $\vec{B}_a$ in Fig. 30-20, we see that $\vec{F}_{ba}$ is directly toward wire a, as shown.

The general procedure for finding the force on a current-carrying wire is this:

☞ To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

We could now use this procedure to compute the force on wire a due to the current in wire b. We would find that the force is directly toward wire b; hence, the two wires with parallel currents attract each other. Similarly, if the two currents were antiparallel, we could show that the two wires repel each other. Thus,

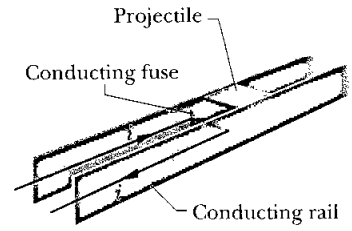
☞ Parallel currents attract each other, and antiparallel currents repel each other.

The force acting between currents in parallel wires is the basis for the definition of the ampere, which is one of the seven SI base units. The definition, adopted in 1946, is this: The ampere is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude 2×10^{-7} newton per meter of wire length.

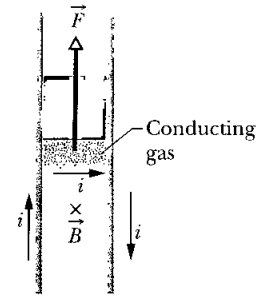
Rail Gun

One application of the physics of Eq. 30-29 is a rail gun. In this device, a magnetic force accelerates a projectile to a high speed in a short time. The basics of a rail gun are shown in Fig. 30-21*a*. A large current is sent out along one of two parallel conducting rails, across a conducting "fuse" (such as a narrow piece of copper) between the rails, and then back to the current source along the second rail. The projectile to be fired lies on the far side of the fuse and fits loosely between the rails. Immediately after the current begins, the fuse element melts and vaporizes, creating a conducting gas between the rails where the fuse had been.

The curled–straight right-hand rule of Fig. 30-4 reveals that the currents in the rails of Fig. 30-21*a* produce magnetic fields that are directed downward between the rails. The net magnetic field $\vec{B}$ exerts a force $\vec{F}$ on the gas due to the current i through the gas (Fig. 30-21*b*). With Eq. 30-28 and the right-hand rule for cross products, we find that $\vec{F}$ points outward along the rails. As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as $5 \times 10^6 g$, and then launches it with a speed of 10 km/s, all within 1 ms. Someday rail guns may be used to launch materials into space from mining operations on the Moon or an asteroid.



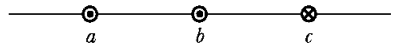
(*a*)



(*b*)

FIG. 30-21 (*a*) A rail gun, as a current i is set up in it. The current rapidly causes the conducting fuse to vaporize. (*b*) The current produces a magnetic field $\vec{B}$ between the rails, and the field causes a force $\vec{F}$ to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.

CHECKPOINT 2 The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



Sample Problem 30-6

Two long parallel wires carry currents of equal magnitude but in opposite directions. These wires are suspended from rod PQ by four chords of same length L as shown in Fig. 30-22*a*. The mass per unit length of wires is λ. Determine the value of θ at equilibrium (assuming it to be small).

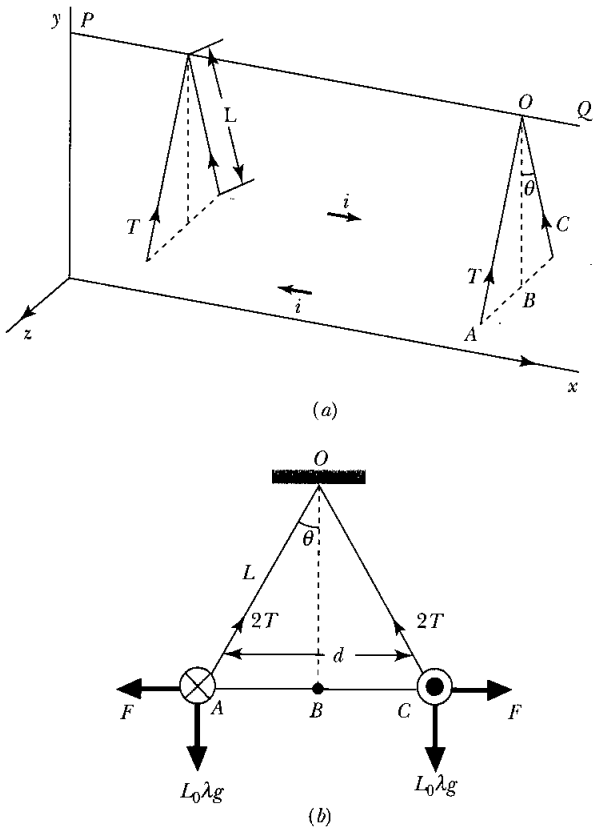


FIG. 30-22 (*a*) Two parallel wires carrying current in the opposite direction; (*b*) free body diagram of the wires.

Conceptualize: We know that current-carrying wires carrying antiparallel currents repel each other. Due to this repulsion, they acquire the configuration as shown in the figure.

Classify: The force between two parallel current-carrying wires is given by

$$F = \frac{\mu_0}{4\pi}\frac{2i_1 i_2 L}{d}.$$

The force is repulsive if the currents are in opposite direction. The *free body diagram* of the problem is as shown in Fig. 30-22*b*.

Compute: If L_0 is the length of each wire, for its vertical and horizontal equilibrium we have, respectively,

$$2T\cos\theta = L_0\lambda g,$$
$$2T\sin\theta = F.$$

Therefore,

$$\tan\theta = \frac{F}{L_0\lambda g}. \tag{30-30}$$

But as,

$$F = \frac{\mu_0}{4\pi}\left[\frac{2I\times I}{2L\sin\theta}L_0\right] \quad (\text{as } d = 2L\sin\theta).$$

So,

$$\tan\theta = \frac{\mu_0}{4\pi}\frac{I^2}{\lambda g L\sin\theta}.$$

Since as for small θ, $\tan\theta = \sin\theta = \theta$

$$\theta = I\sqrt{\frac{\mu_0}{4\pi\lambda g L}}.$$

Conclude: This force is quite small in magnitude and this is the reason why we do not observe its effect in our daily life.

30-5 | Ampere's Law

We can find the net electric field due to *any* distribution of charges by first writing the differential electric field $d\vec{E}$ due to a charge element and then summing the contributions of $d\vec{E}$ from all the elements. However, if the distribution

is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss' law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field due to *any* distribution of currents by first writing the differential magnetic field $d\vec{B}$ (Eq. 30-3) due to a current-length element and then summing the contributions of $d\vec{B}$ from all the elements. Again we may have to use a computer for a complicated distribution. However, if the distribution has some symmetry, we may be able to apply **Ampere's law** to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André-Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell.

Ampere's law is

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \qquad \text{(Ampere's law).} \tag{30-31}$$

The loop on the integral sign means that the scalar (dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around a *closed* loop, called an *Amperian loop.* The current i_{enc} is the *net* current encircled by that closed loop.

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampere's law to the general situation of Fig. 30-23. The figure shows cross sections of three long straight wires that carry currents i_1, i_2, and i_3 either directly into or directly out of the page. An arbitrary Amperian loop lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 30-31.

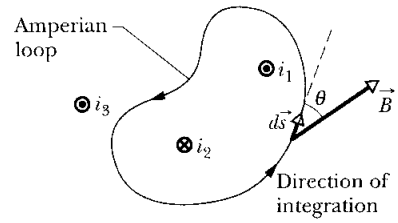


FIG. 30-23 Ampere's law applied to an arbitrary Amperian loop that encircles two long straight wires but excludes a third wire. Note the directions of the currents.

To apply Ampere's law, we mentally divide the loop into differential vector elements $d\vec{s}$ that are everywhere directed along the tangent to the loop in the direction of integration. Assume that at the location of the element $d\vec{s}$ shown in Fig. 30-23, the net magnetic field due to the three currents is $\vec{B}$. Because the wires are perpendicular to the page, we know that the magnetic field at $d\vec{s}$ due to each current is in the plane of Fig. 30-23; thus, their net magnetic field $\vec{B}$ at $d\vec{s}$ must also be in that plane. However, we do not know the orientation of $\vec{B}$ within the plane. In Fig. 30-23, $\vec{B}$ is arbitrarily drawn at an angle θ to the direction of $d\vec{s}$.

The scalar product $\vec{B} \cdot d\vec{s}$ on the left side of Eq. 30-31 is equal to $B \cos\theta\, ds$. Thus, Ampere's law can be written as

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos\theta\, ds = \mu_0 i_{enc}. \tag{30-32}$$

We can now interpret the scalar product $\vec{B} \cdot d\vec{s}$ as being the product of a length ds of the Amperian loop and the field component $B \cos\theta$ tangent to the loop. Then we can interpret the integration as being the summation of all such products around the entire loop.

When we can actually perform this integration, we do not need to know the direction of $\vec{B}$ before integrating. Instead, we arbitrarily assume $\vec{B}$ to be generally

☞ Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

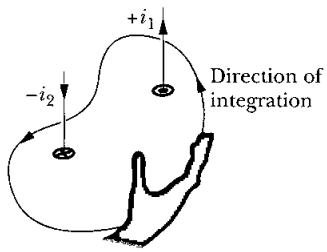


FIG. 30-24 A right-hand rule for Ampere's law, to determine the signs for currents encircled by an Amperian loop. The situation is that of Fig. 30-23.

in the direction of integration (as in Fig. 30-23). Then we use the following curled—straight right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current i_{enc}:

Finally, we solve Eq. 30-33 for the magnitude of $\vec{B}$. If B turns out positive, then the direction we assumed for $\vec{B}$ is correct. If it turns out negative, we neglect the minus sign and redraw $\vec{B}$ in the opposite direction.

In Fig. 30-24 we apply the curled–straight right-hand rule for Ampere's law to the situation of Fig. 30-23. With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{enc} = i_1 - i_2.$$

(Current i_3 is not encircled by the loop.) We can then rewrite Eq. 30-33 as

$$\oint B \cos\theta \, ds = \mu_0(i_1 - i_2). \tag{30-33}$$

You might wonder why, since current i_3 contributes to the magnetic-field magnitude B on the left side of Eq. 30-33, it is not needed on the right side. The answer is that the contributions of current i_3 to the magnetic field cancel out because the integration in Eq. 30-33 is made around the full loop. In contrast, the contributions of an encircled current to the magnetic field do not cancel out.

We need to explain more precisely just what is meant by the term "current encircled by the loop." The term means that the current pierces any surface that has the loop as a boundary (see Fig. 30-25). The loop need not be planar. Another way to think of the loop is as the perimeter of various hat-shaped surfaces; there is a wide variety of hat surfaces for a given hat size (perimeter), as a visit to a hat department in a department store will easily confirm! To thread the loop, the current must pierce the bubble surface or hat. The same path can bound many different hat-shaped surfaces.

Strictly speaking, the current must pierce the hat-shaped surface an odd number of times (see Fig. 30-26).

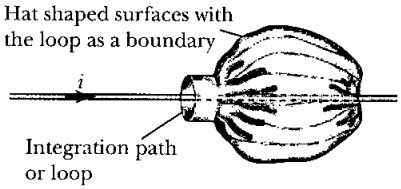


FIG. 30-25 Hat-shaped surfaces with the loop as boundary.

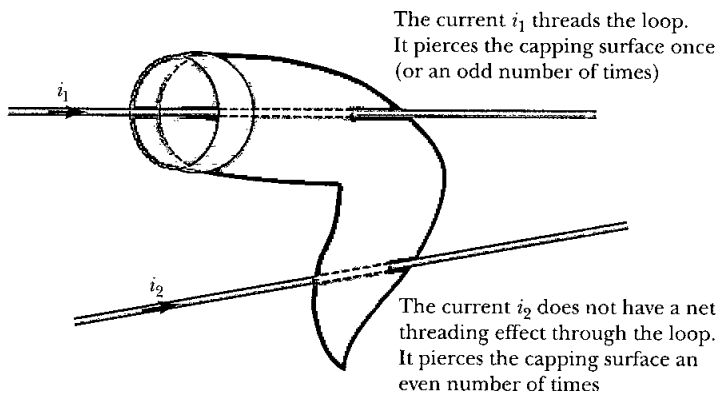


FIG. 30-26 The current must pierce the surface an odd number of times.

We cannot solve Eq. 30-33 for the magnitude B of the magnetic field because for the situation of Fig. 30-23 we do not have enough information to simplify and solve the integral. However, we do know the outcome of the integration; it must be equal to $\mu_0(i_1 - i_2)$, the value of which is set by the net current passing through the loop.

We shall now apply Ampere's law to two situations in which symmetry does allow us to simplify and solve the integral, hence to find the magnetic field.

Remember, just as in the Gauss' law, the electric field was due to all the charges, similarly, the magnetic field is due to all the currents in the vicinity and not just the current threading the loop.

Magnetic Field Outside a Long Straight Wire with Current

Figure 30-27 shows a long straight wire that carries current i directly out of the page. Equation 30-4 tells us that the magnetic field $\vec{B}$ produced by the current has the same magnitude at all points that are the same distance r from the wire; that is, the field $\vec{B}$ has cylindrical symmetry about the wire. We can take advantage of that symmetry to simplify the integral in Ampere's law (Eqs. 30-31 and 30-32) if we encircle the wire with a concentric circular Amperian loop of radius r, as in Fig. 30-27. The magnetic field $\vec{B}$ then has the same magnitude B at every point on the loop. We shall integrate counterclockwise, so that $d\vec{s}$ has the direction shown in Fig. 30-27.

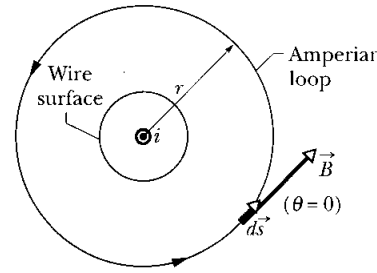


FIG. 30-27 Using Ampere's law to find the magnetic field that a current i produces outside a long straight wire of circular cross section. The Amperian loop is a concentric circle that lies outside the wire.

We can further simplify the quantity $B \cos\theta$ in Eq. 30-32 by noting that $\vec{B}$ is tangent to the loop at every point along the loop, as is $d\vec{s}$. Thus, $\vec{B}$ and $d\vec{s}$ are either parallel or antiparallel at each point of the loop, and we shall arbitrarily assume the former. Then at every point the angle θ between $d\vec{s}$ and $\vec{B}$ is 0°, so $\cos\theta = \cos 0° = 1$. The integral in Eq. 30-32 then becomes

$$\oint \vec{B}\cdot d\vec{s} = \oint B\cos\theta\, ds = B\oint ds = B(2\pi r).$$

Note that $\oint ds$ is the summation of all the line segment lengths ds around the circular loop; that is, it simply gives the circumference $2\pi r$ of the loop.

Our right-hand rule gives us a plus sign for the current of Fig. 30-27. The right side of Ampere's law becomes $+\mu_0 i$, and we then have

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(outside straight wire).} \tag{30-34}$$

With a slight change in notation, this is Eq. 30-4, which we derived earlier—with considerably more effort—using the law of Biot and Savart. In addition, because the magnitude B turned out positive, we know that the correct direction of $\vec{B}$ must be the one shown in Fig. 30-27.

Magnetic Field Inside a Long Straight Wire with Current

Figure 30-28 shows the cross section of a long straight wire of radius R that carries a uniformly distributed current i directly out of the page. Because the current is uniformly distributed over a cross section of the wire, the magnetic field $\vec{B}$ produced by the current must be cylindrically symmetrical. Thus, to find the magnetic field at points inside the wire, we can again use an Amperian loop of radius r, as shown in Fig. 30-28, where now $r < R$. Symmetry again suggests that $\vec{B}$ is tangent to the loop, as shown; so the left side of Ampere's law again yields

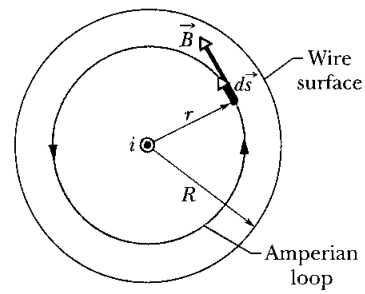


FIG. 30-28 Using Ampere's law to find the magnetic field that a current i produces inside a long straight wire of circular cross section. The current is uniformly distributed over the cross section of the wire and emerges from the page. An Amperian loop is drawn inside the wire.

$$\oint \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r). \tag{30-35}$$

To find the right side of Ampere's law, we note that because the current is uniformly distributed, the current i_{enc} encircled by the loop is proportional to the area encircled by the loop; that is,

$$i_{enc} = i\frac{\pi r^2}{\pi R^2}. \tag{30-36}$$

Our right-hand rule tells us that i_{enc} gets a plus sign. Then Ampere's law gives us

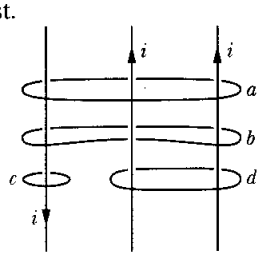
$$B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2}$$

or

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r \qquad \text{(inside straight wire).} \tag{30-37}$$

Thus, inside the wire, the magnitude B of the magnetic field is proportional to r; that magnitude is zero at the center and a maximum at the surface, where $r = R$. Note that Eqs. 30-34 and 30-37 give the same value for B at $r = R$; that is, the expressions for the magnetic field outside the wire and inside the wire yield the same result at the surface of the wire.

CHECKPOINT 3 The figure here shows three equal currents i (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ along each, greatest first.



Sample Problem 30-7

Figure 30-29*a* shows the cross section of a long conducting cylinder with inner radius $a = 2.0$ cm and outer radius $b = 4.0$ cm. The cylinder carries a current out of the page, and the magnitude of the current density in the cross section is given by $J = cr^2$, with $c = 3.0 \times 10^6$ A/m^4 and r in meters. What is the magnetic field $\vec{B}$ at a point that is 3.0 cm from the central axis of the cylinder?

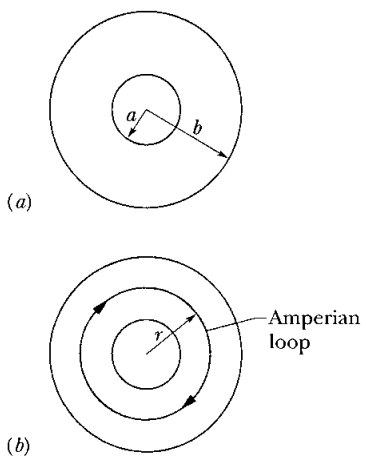


FIG. 30-29 (*a*) Cross section of a conducting cylinder of inner radius a and outer radius b. (*b*) An Amperian loop of radius r is added to compute the magnetic field at points that are a distance r from the central axis.

Conceptualize: The point at which we want to evaluate $\vec{B}$ is inside the material of the conducting cylinder, between its inner and outer radii. We note that the current distribution has cylindrical symmetry (it is the same all around the cross section for any given radius). Thus, the symmetry allows us to use Ampere's law to find $\vec{B}$ at the point. We first draw the Amperian loop shown in Fig. 30-29*b*. The loop is concentric with the cylinder and has radius $r = 3.0$ cm because we want to evaluate $\vec{B}$ at that distance from the cylinder's central axis.

Classify: Next, we must compute the current i_{enc} that is encircled by the Amperian loop. However, we *cannot* set up a proportionality as in Eq. 30-36, because here the current is not uniformly distributed. Instead, following the procedure of Sample Problem 27-2b, we must integrate the current density magnitude from the cylinder's inner radius a to the loop radius r.

Compute: We write the integral as

$$i_{enc} = \int J\, dA = \int_a^r cr^2 (2\pi r\, dr)$$

$$= 2\pi c \int_a^r r^3 dr = 2\pi c \left[\frac{r^4}{4}\right]_a^r$$

$$= \frac{\pi c(r^4 - a^4)}{2}.$$

The direction of integration indicated in Fig. 30-29*b* is (arbitrarily) clockwise. Applying the right-hand rule for Ampere's law to that loop, we find that we should take i_{enc} as negative because the current is directed out of the page but our thumb is directed into the page.

We next evaluate the left side of Ampere's law exactly as we did in Fig. 30-28, and we again obtain Eq. 30-35. Then Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc},$$

gives us

$$B(2\pi r) = -\frac{\mu_0 \pi c}{2}(r^4 - a^4).$$

Solving for B and substituting known data yield

$$B = -\frac{\mu_0 c}{4r}(r^4 - a^4)$$

$$= -\frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(3.0 \times 10^6\ \text{A/m}^4)}{4(0.030\ \text{m})} \times [(0.030\ \text{m})^4 - (0.020\ \text{m})^4]$$

$$= -2.0 \times 10^{-5}\ \text{T}.$$

Thus, the magnetic field $\vec{B}$ at a point 3.0 cm from the central axis has magnitude

$$B = 2.0 \times 10^{-5}\ \text{T} \qquad \text{(Answer)}$$

and forms magnetic field lines that are directed opposite our direction of integration, hence counterclockwise in Fig. 30-29*b*.

Conclude: If we carry out the integration over a circular loop that has radius greater than that of the cylinder, we will find that the magnetic field for a point outside the wire is effectively due to current seemingly concentrated at its axis. Similarly, the magnetic field inside the cavity is zero.

30-6 | Solenoids and Toroids

Magnetic Field of a Solenoid

We now turn our attention to another situation in which Ampere's law proves useful. It concerns the magnetic field produced by the current in a **solenoid** (Fig. 30-9). Recall Sample Problem 30-1. We found the magnetic field due to the solenoid at points on the axis. Now we wish to find the magnetic field at off-axis points as well. In a real solenoid, it will be difficult to find that, since we do not know the magnetic field at off-axis points of a loop. But it turns out that for a particular case which is closely followed by real-life situation, we can find the required magnetic field.

We assume that the length of the solenoid is much greater than the diameter. Figure 30-30 shows a section through a portion of a "stretched-out" solenoid. For points very close to a turn, the wire behaves magnetically, almost like a long

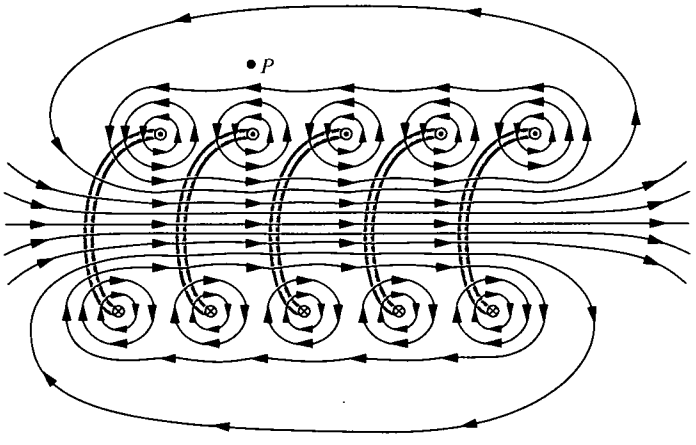


FIG. 30-30 A vertical cross section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near itself. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak.

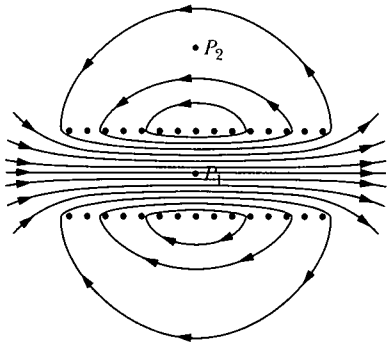


FIG. 30-31 Magnetic field lines for a real solenoid of finite length. The field is strong and uniform at interior points such as P_1 but relatively weak at external points such as P_2.

straight wire, and the lines of $\vec{B}$ there are almost concentric circles. Figure 30-30 suggests that the field tends to cancel between adjacent turns. It also suggests that, at points inside the solenoid and reasonably far from the wire, $\vec{B}$ is approximately parallel to the (central) solenoid axis. In the limiting case of an *ideal solenoid*, which is infinitely long and consists of tightly packed (*close-packed*) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis. At points above the solenoid, such as P in Fig. 30-30, the magnetic field set up by the upper parts of the solenoid turns (these upper turns are marked ⊙) is directed to the left (as drawn near P) and tends to cancel the field set up at P by the lower parts of the turns (these lower turns are marked ⊗), which is directed to the right (not drawn). In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is zero. The direction of the magnetic field along the solenoid axis is given by a curled–straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.

Figure 30-31 shows the lines of $\vec{B}$ for a real solenoid. The spacing of these lines in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field, however, is relatively weak.

As we stated in Sample Problem 30-1, the magnetic field at the axis of such an infinite solenoid is $B = \mu_0 ni$. To find the magnetic field at the off-axis points, we take a rectangular Amperian loop as shown in the Fig. 30-32*a* by loop *abefa*. We can apply Ampere's law to find the magnetic field:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}.$$

As we can see there are four parts of the loop. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment:

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^e \vec{B} \cdot d\vec{s} + \int_e^f \vec{B} \cdot d\vec{s} + \int_f^a \vec{B} \cdot d\vec{s}.$$

Along the line segment *ab*, $\vec{B}$ is parallel to $d\vec{s}$, whereas along the line segment *be* and *af*, $\vec{B}$ and $d\vec{s}$ are mutually perpendicular. Along the line segment *ef*, $\vec{B}$ is antiparallel to $d\vec{s}$.

So the first integral is $\mu_0 nih$, the second and fourth are zero, whereas the third is $-Bh$. Here, B is the magnetic field at the off-axis points (at point *e* or *f*). The current enclosed by the loop is zero. So we obtain:

$$\mu_0 nih + 0 + Bh + 0 = 0. \tag{30-38}$$

For a point outside the solenoid, we use the rectangular Amperian loop *abcda* as shown in Fig. 30-32*b*. We again apply Ampere's law to find the magnetic field:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}.$$

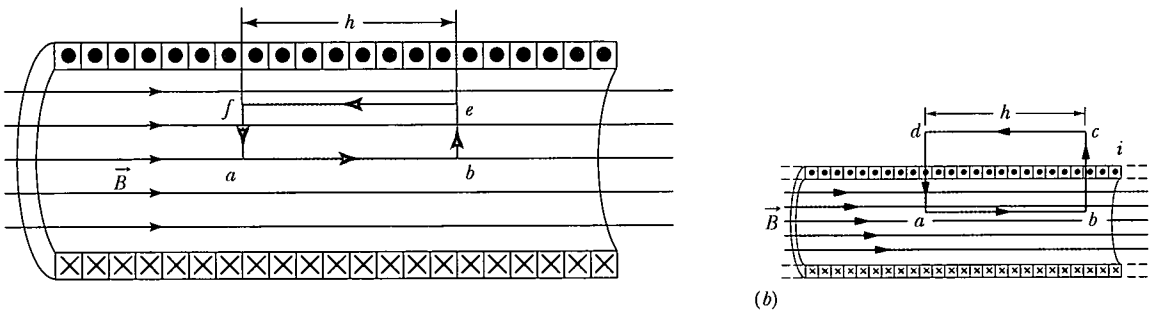


FIG. 30-32 Ampere's law applied to find the magnetic field (*a*) inside the solenoid at off-axis points; (*b*) outside the solenoid.

Again we can see that there are four parts of the loop. We write $\oint \vec{B}\cdot d\vec{s}$ as the sum of four integrals, one for each loop segment:

$$\oint \vec{B}\cdot d\vec{s} = \int_a^b \vec{B}\cdot d\vec{s} + \int_b^e \vec{B}\cdot d\vec{s} + \int_e^d \vec{B}\cdot d\vec{s} + \int_d^a \vec{B}\cdot d\vec{s}.$$

Along the line segment ab, $\vec{B}$ is parallel to $d\vec{s}$, whereas along the line segment bc and ad, $\vec{B}$ and $d\vec{s}$ are mutually perpendicular. So the first integral is $\mu_0 nih$ as before, the second and fourth are zero, whereas the third is $-Bh$. In this situation, the current enclosed by the loop is not zero. The net current i_{enc} enclosed by the rectangular Amperian loop in Fig. 30-32*b* is not the same as the current i in the solenoid windings because the windings pass more than once through this loop. Let n be the number of turns per unit length of the solenoid; then the loop encloses nh turns and $i_{enc} = i(nh)$. So we obtain

$$\begin{aligned} Bh + \mu_0 nih &= \mu_0 nih, \\ B &= 0. \end{aligned} \qquad (30\text{-}39)$$

So the magnetic field outside the loop is zero. Although we derived Eq. 30-20 for an infinitely long ideal solenoid, it holds quite well for actual solenoids if we apply it only at interior points and well away from the solenoid ends. Equation 30-20 is consistent with the experimental fact that the magnetic field magnitude B within a solenoid does not depend on the diameter or the length of the solenoid and that B is uniform over the solenoidal cross section. A solenoid thus provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

Magnetic Field of a Toroid

Figure 30-33*a* shows a **toroid**, which we may describe as a (hollow) solenoid that has been curved until its two ends meet, forming a sort of hollow bracelet. What magnetic field $\vec{B}$ is set up inside the toroid (inside the hollow of the bracelet)? We can find out from Ampere's law and the symmetry of the bracelet.

From the symmetry, we see that the lines of $\vec{B}$ form concentric circles inside the toroid, directed as shown in Fig. 30-33*b*. Let us choose a concentric circle of radius r as an Amperian loop and traverse it in the clockwise direction. Ampere's law (Eq. 30-31) yields

$$(B)(2\pi r) = \mu_0 iN,$$

where i is the current in the toroid windings (and is positive for those windings enclosed by the Amperian loop) and N is the total number of turns. This gives

$$B = \frac{\mu_0 iN}{2\pi}\frac{1}{r} \qquad \text{(toroid)}. \qquad (30\text{-}40)$$

In contrast to the situation for a solenoid, B is not constant over the cross section of a toroid.

It is easy to show, with Ampere's law, that $B = 0$ for points outside an ideal toroid (as if the toroid were made from an ideal solenoid). The direction of the magnetic field within a toroid follows from our curled-straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

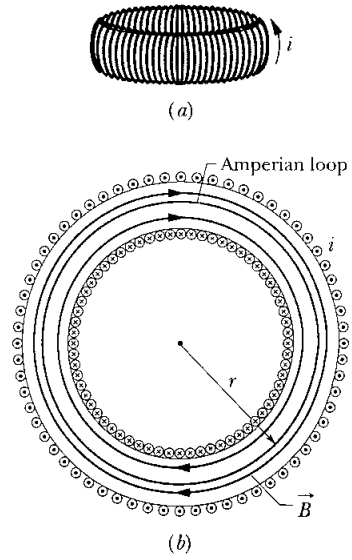


FIG. 30-33 (*a*) A toroid carrying a current i. (*b*) A horizontal cross section of the toroid. The interior magnetic field (inside the bracelet-shaped tube) can be found by applying Ampere's law with the Amperian loop shown.

Sample Problem 30-8

A solenoid has length $L = 1.23$ m and inner diameter $d = 3.55$ cm, and it carries a current $i = 5.57$ A. It consists of five close-packed layers, each with 850 turns along length L. What is B at its center?

Conceptualize/Classify: The magnitude B of the magnetic field along the solenoid's central axis is related to the solenoid's current i and number of turns per unit length n by Eq. 30-20.

Compute: Because B does not depend on the diameter of the windings, the value of n for five identical layers is simply five times the value for each layer. Equation 30-20 then tells us

$$B = \mu_0 in = (4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.57\ \text{A})\frac{5 \times 850\ \text{turns}}{1.23\ \text{m}}$$

$$= 2.42 \times 10^{-2}\ \text{T} = 24.2\ \text{mT}. \qquad \text{(Answer)}$$

Sample Problem 30-9 Magnetic Field of a Large Current-Carrying Sheet

A large flat current-carrying sheet carries a current K per unit length as shown in Figure 30-34*a*. (That is, a strip of width w of the sheet carries a current Kw.) What is the magnitude of the magnetic field at a distance d from the sheet?

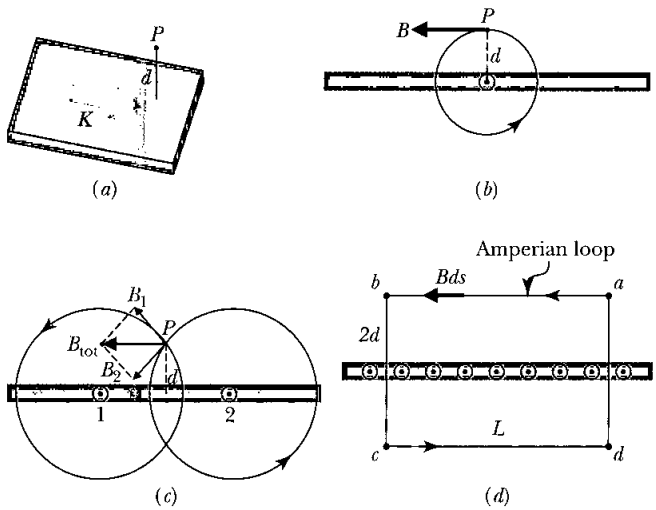


FIG. 30-34 (*a*) Current-carrying sheet. (*b*) Contributions from a strip right underneath point P. (*c*) Contributions from two strips to the magnetic field at point P. (*d*) Amperian loop.

Conceptualize: This situation has *planar symmetry*, so we expect that, in analogy to the electric field of a flat charged sheet, the field on either side of the plate is uniform. That is to say, the magnitude and direction of the magnetic field does not depend on the distance from the plane. Before we can use Ampere's law to verify this dependence, we must determine the direction of the magnetic field at an arbitrary point P above the charged sheet.

Classify: We divide the sheet into thin parallel strips in the direction of the current. We then treat each strip as a current-carrying wire. A cross-sectional view of the sheet, looking into a direction opposite that of the current through the sheet, is shown in Fig. 30-34*b*. Using the right-hand rule, we find that the strip right underneath point P contributes a magnetic field that points parallel to the plate and to the left. Next we look at the contributions from two strips, 1 and 2, equidistant on either side of P (see Fig. 30-34*c*). The two circles show the field lines from either strip that go through P. The magnitudes of the contributions from the two strips are equal and so we see that the sum of the contributions from these two strips also point parallel to the plate and to the left. Arguing this way, we can say that the entire magnetic field due to the sheet is parallel to the sheet and points to the left.

Compute: To use this fact in Ampere's law, we choose the rectangular loop *abcd* as shown in Fig. 30-34*d* as the Amperian loop. The width of this rectangular loop is L and its height is $2d$. The line integral around the loop can then be written as the sum of four line integrals over each side of the rectangle:

$$\oint \vec{B}\cdot d\vec{s} = \int_a^b \vec{B}\cdot d\vec{s} + \int_b^c \vec{B}\cdot d\vec{s} + \int_c^d \vec{B}\cdot d\vec{s} + \int_d^a \vec{B}\cdot d\vec{s}.$$

Along the vertical sides bc and da, the magnetic field is perpendicular to the path and so the dot product $\vec{B}\cdot d\vec{s}$ is zero. Over the other two sides the magnetic field is parallel to the path and so $\vec{B}\cdot d\vec{s} = B\,ds$.

Symmetry dictates that the magnitude of the magnetic field a distance d below the sheet is the same as that a distance d above it (because flipping the plane upside-down should not alter the physical situation) and so we can take this magnitude, B out of the integral, that is

$$\oint \vec{B}\cdot d\vec{s} = \int_{ab} B\,ds + \int_{cd} B\,ds = B\int_{ab} ds + B\int_{cd} ds.$$

The two remaining line integrals simply yield the lengths of the two sides and so the left-hand side of Ampere's law becomes

$$\oint \vec{B}\cdot d\vec{s} = B(2L) = 2BL. \qquad (30\text{-}41)$$

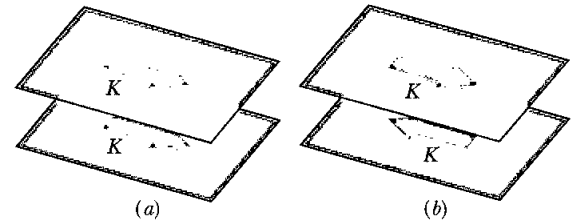
Substituting for the right-hand side of Eq. (30-41) into Ampere's law, we get

$$2BL = \mu_0 KL$$

$$B = \tfrac{1}{2}\mu_0 K.$$

Conclude: So we see that the magnetic field does not depend on the distance to the plane, just like the electric field of a large uniformly charged sheet.

CHECKPOINT 4 What are the directions of the magnetic field between and outside a pair of two parallel sheets carrying currents in (*a*) the same and (*b*) opposite directions, as illustrated in the adjoining figure?



Limitations of Ampere's Law

Ampere's law is useful for calculating the magnetic field only when there is a high degree of symmetry. Consider the current loop shown in Fig. 30-35. According to Ampere's law, the line integral around a loop such as curve C in the figure equals μ_0 times the current I in the loop. Although Ampere's law is valid for this curve, the magnetic field is not constant along any curve encircling the current, nor is it everywhere tangent to any such loop. Thus, there is not enough symmetry in this situation to allow us to calculate using Ampere's law.

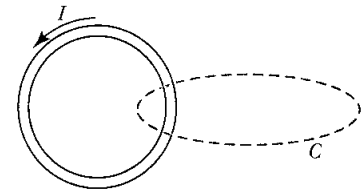


FIG. 30-35 Ampere's law holds for curve C encircling the current in the circular loop, but it is not useful for finding the magnetic field because it is neither constant along the curve nor tangent to it.

Figure 30-36 shows a finite current segment of length. We wish to find the magnetic field at point P, which is equidistant from the ends of the segment and at a distance r from the center of the segment. A direct application of Ampere's law gives

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}.$$

This result is the same as for an infinitely long wire, since the same symmetry arguments apply. It does not agree with the result obtained from the Biot–Savart law, which depends on the length of the current segment and also agrees with experiment. Actually, there can never be just a finite current-carrying wire as shown in Fig. 30-36. There must be some leading wires which connect this current-carrying wire to the battery. As can be seen in Fig. 30-37, Ampere's law for the curve C is valid, but there is not enough symmetry to use it to find the magnetic field at point P.

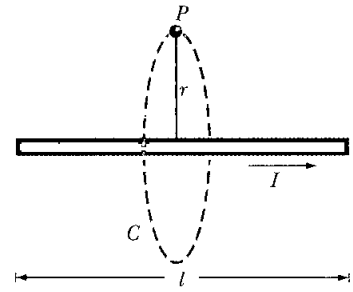


FIG. 30-36 The application of Ampere's law to find the magnetic field on the bisector of a finite current segment gives an incorrect result.

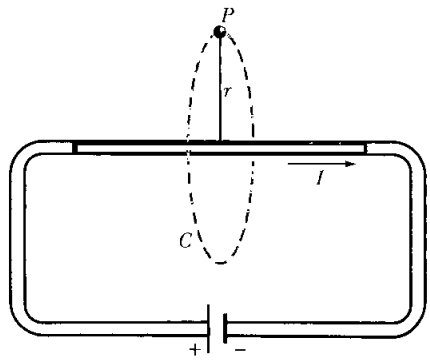


FIG. 30-37 The current segment in Fig. 30-35 is part of a complete circuit.

In Fig. 30-38, the current in the segment arises from a small spherical conductor with initial charge $+Q$ at the left of the segment and another one at the right with charge $-Q$. When they are connected, a current $I = -dQ/dt$ exists in the segment for a short time, until the spheres are discharged. For this case, we do have the symmetry needed to assume that magnetic field is tangential to the curve and constant in magnitude along the curve. For a situation like this, in which the electric field is varying with time, Ampere's law is not valid. Maxwell was able to modify Ampere's law so that it holds for all currents. When Maxwell's generalized form of Ampere's law is used to calculate the magnetic field for a current segment, such as that shown in Fig. 30-38, the result agrees with that found from the Biot–Savart law.

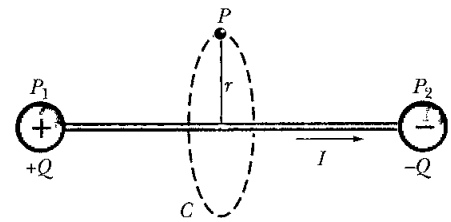


FIG. 30-38 If the current segment in Fig. 30-36 is due to a momentary flow of charge from a small conductor on the left to the one at the right, there is enough symmetry to use Ampere's law to compute the magnetic field at P, but Ampere's law is not valid because the Electric field in the space is varying with time.

A Note About the Frames of Reference

Now it seems clear that charges create an electric field $\vec{E}$ and that moving charges, or currents, create a magnetic field $\vec{B}$ in addition to it. Charges other than the source charges always respond to $\vec{E}$, but only moving charges respond to $\vec{B}$. There seems to be a problem, however.

Assume that there are two friends **A** and **B** as seen in Fig. 30-39. **A** is at rest and is carrying a magnet that produces a magnetic field $\vec{B}$ at the location of **B**. **B** is running with a velocity $\vec{v}$ as shown in the figure. **A** will claim that the charge carried by **B** will experience a force of $q\vec{v} \times \vec{B}$.

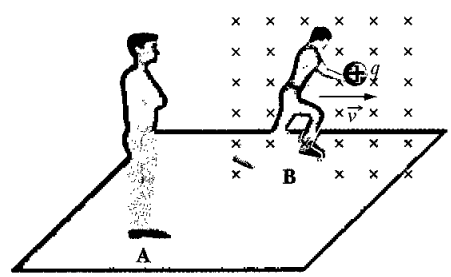


FIG. 30-39 **B** carries a charge past **A**. Charge q moves with a velocity v relative to **A**. The charge q moves through a magnetic field created by **A**.

Using the right-hand rule, **A** determines that the force points straight up. But for **B** the charge is still at rest. Stationary charges don't experience magnetic forces, so **B** claims that $\vec{F} = \vec{0}$.

Figure 30-39 shows **B** running past **A** with velocity $\vec{v}$ while carrying charge q. Also, A sees a moving charge, and he knows that this charge creates a magnetic field. But from **B**'s prespective, the charge is at rest. Stationary charges don't create magnetic fields, so **B** claims that the magnetic field is zero. Is there or is there not a magnetic field?

Now, we may be a bit uncertain about magnetic fields because they are an abstract concept, but surely there can be no disagreement over forces. After all,

either the charge is going to accelerate upward or it isn't, and **A** and **B** should be able to agree on the outcome.

Here we have a genuine paradox, not merely faulty reasoning. This paradox has arisen because we have fields and forces that depend on velocity. The difficulty is that we haven't looked at the issue of *velocity with respect to what or velocity as measured by whom.* A closer look at how electromagnetic fields are viewed by two experimenters moving relative to each other will lead us to conclude that $\vec{E}$ and $\vec{B}$ are not as we've been assuming: separate and independent entities. They are closely intertwined. We will explore this relationship in the chapter on relativity.

B and **A** may measure different positions and velocities for a particle, but they agree on its acceleration. This agreement is important because acceleration is directly related to force. An experimenter in frame S would find that a force $\vec{F} = m\vec{a}$ is acting on the particle. Similarly, the force measured in frame S' is $\vec{F}' = m\vec{a}$. But $\vec{a}' = \vec{a}$, hence

$$\vec{F}' = \vec{F}. \tag{30-42}$$

Experiments in all inertial reference frames agree about the force acting on a particle. This conclusion is the key to understanding how different experimenters see electric and magnetic fields. We are not going to study this in any more detail, but we need to understand that **the magnetic field and the electric field seen by observers in different inertial frames will be different**. So the conclusion is that do not change the frame of reference: it may seem to simplify the problem, but the electric field and magnetic field will not remain the same as before.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Consider two small elements of current-carrying wire as shown in Fig. 30-40.

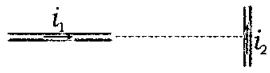


FIG. 30-40 Reasoning type question 1.

Statement 1: The force on 1 due to 2 is downwards, but the force on 2 due to 1 is zero.

Statement 2: Electromagnetic fields carry momentum with them.

2. Statement 1: A magnet exerts a force on current-carrying wire.

Statement 2: A current-carrying wire exerts a force on the magnet.

3. Statement 1: When a current is passed through a helically coiled spring, the spring contracts as if it is compressed.

Statement 2: Two parallel current-carrying wires attract each other.

4. Statement 1: A long wire carries current from south to north. At no point in space is the magnetic field directed from south to north.

Statement 2: If $\vec{C} = \vec{A} \times \vec{B}$, $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

5. Statement 1: In Ampere's law, $\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$, the current outside the curve is not included in right hand side.

Statement 2: The magnetic field $\vec{B}$ in left hand side is only due to currents enclosed by the curve.

6. Statement 1: Two current-carrying wires may repel each other.

Statement 2: Two charged particles stationary w.r.t. the observer may repel each other.

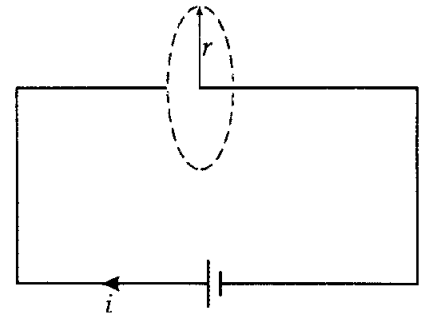


FIG. 30-41 Reasoning type question 6.

7. Statement 1: For the loop shown in Fig. 30-41 by Ampere's law, $B = \mu_0 i / 2\pi r$.

Statement 2: By Ampere's law $\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$.

MULTIPLE CHOICE QUESTIONS

sec. 30-2 Calculating the Magnetic Field Due to a Current

1. A wire carries current toward the north. In the region beneath this wire, what is the direction of the magnetic field due to the current?

(a) Northward
(b) Southward
(c) Eastward
(d) Westward

2. Two infinitely long perpendicular wires carry equal currents in the directions indicated in Fig. 30-42. At which of the labeled points is the magnitude of the magnetic field created by the currents the greatest? Note: All gridlines are equally spaced.

(a) A
(b) B
(c) C
(d) D
(e) Magnetic field at A, C, and D has the same magnitude, and is greater than the magnitude at B

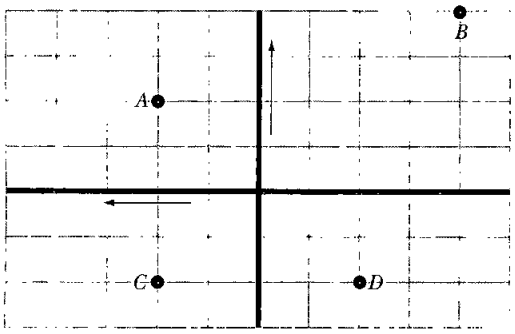


FIG. 30-42 Multiple choice question 2.

3. A circular loop is kept in a vertical plane with its axis in the East–West direction. It carries a current that is toward north at the top most point. Let A be the point of the axis of the circle to the east of it and B a point on the axis to the west of it. The magnetic field due to the loop

(a) Is toward east at A and toward west at B
(b) Is toward east at both A and B
(c) Is toward west at A and toward east at B
(d) Is toward west at both A and B

4. A long, straight wire carries a current along the z axis. One can find two points in the xy plane such that

(a) The magnetic field are equal
(b) The directions of the magnetic field are the same
(c) The magnitudes of the magnetic fields are equal
(d) The field at one point is opposite to that at the other point

5. Figure 30-43 shows a regular hexagon having uniform resistance per unit length. If a battery is connected across the ends A and B then the magnetic field at the center is B_0. If the same battery is connected across A and D then magnetic field at the center would be

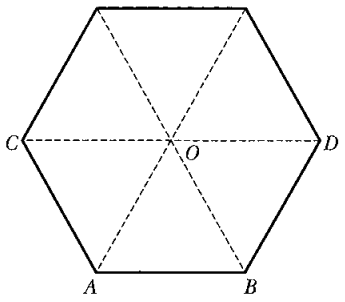


FIG. 30-43 Multiple choice question 5.

(a) Equal to B_0
(b) Greater than B_0
(c) Less than B_0
(d) Less than B_0

6. Figure 30-44 shows two long wires carrying equal currents i_1 and i_2 flowing in opposite direction. Which of the arrows labeled A–D correctly represents the direction of the magnetic field due to the wires at a point located at an equal distance d from each wire?

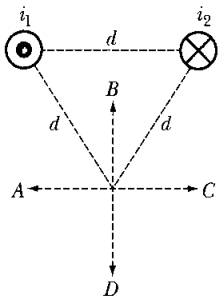


FIG. 30-44 Multiple choice question 6.

(a) A
(b) B
(c) C
(d) D
(e) The magnetic field is equal to zero at that point

sec. 30-3 Magnetic Field of Moving Charged Particles

7. Rowland (1876) placed some charges (in a fixed location) on a nonconducting disk, which he then rotated at high speed about its central axis near a delicate compass (Fig. 30-45). Rowland observed that

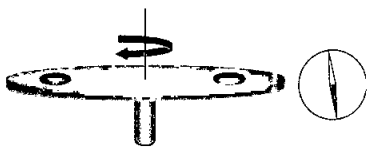


FIG. 30-45 Multiple choice question 7.

(a) There was no magnetic field, so nothing happened to the compass
(b) The current created a magnetic field that deflected the compass
(c) The charges repelled the compass by Coulomb's Law
(d) None of the above

8. Consider a long, straight wire of cross-sectional area A carrying a current i. Let there be n free electrons per unit volume. An observer is opposite to the current with a speed $v = i/nAe$ and separated from the wire by a distance r. The magnetic field seen by the observer is very nearly

(a) $\dfrac{\mu_0 i}{2\pi r}$

(b) Zero

(c) $\dfrac{\mu_0 i}{\pi r}$

(d) $\dfrac{\mu_0 i}{\pi r}$

sec. 30-4 Force Between Two Parallel Currents

9. Four very long straight wires carry equal electric currents in the $+z$ direction. They intersect the xy plane at $(x, y) = (-a, 0)$, $(0, a)$, $(a, 0)$, and $(0, -a)$. The magnetic force exerted on the wire at position $(-a, 0)$ is along

(a) $+y$

(b) $-y$

(c) $+x$

(d) $-x$

10. Cross section of three wires is shown in Fig. 30-46. If the current in the bottom two wires is opposite in direction to each other as indicated by ⊗ for into the page and ⊙ for out of the page, and the top wire has current flowing into the page ⊗, what is the direction of net force on the top wire?

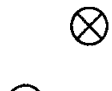


FIG. 30-46 Multiple choice question 10.

(a) Left

(b) Right

(c) Up

(d) Down

11. Consider the situation shown in Fig. 30-47. The straight wire is fixed but the loop can move under magnetic force. The loop will

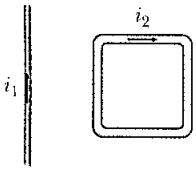


FIG. 30-47 Multiple choice question 11.

(a) Remain stationary

(b) Move toward the wire

(c) Move away from the wire

(d) Rotate about the wire

12. Two long straight thin wires carrying steady currents pass near each other at right angles to each other. As time passes

(a) They will move away from each other, parallel to their original positions

(b) They will move toward each other parallel to their original position.

(c) They will rotate about the line of the shortest distance between them and tend to be parallel to each other

(d) They will rotate about the line of the shortest distance between them and tend to be anti-parallel to each other

sec. 30-5 Ampere's Law

13. Which one of the following line integrals is correct? Note: The direction of the loops orientation is shown in Fig. 30-48.

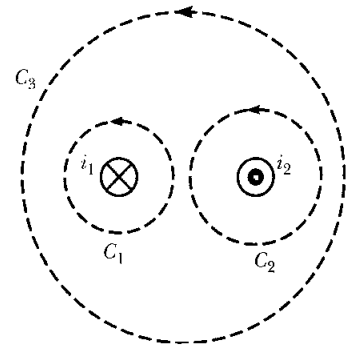


FIG. 30-48 Multiple choice question 13.

(a) $\oint_{C_1} \vec{B} \cdot \vec{dl} = \mu_0 i_1$

(b) $\oint_{C_2} \vec{B} \cdot \vec{dl} = \mu_0 i_2$

(c) $\oint_{C_1} \vec{B} \cdot \vec{dl} = -\mu_0 i_2$

(d) $\oint_{C_3} \vec{B} \cdot \vec{dl} = -\mu_0 (i_2 - i_1)$

(e) $\oint_{C_3} \vec{B} \cdot \vec{dl} = -\mu_0 (i_1 - i_2)$

14. In a coaxial, straight cable, the central conductor and the outer conductor carry equal currents in opposite directions. The magnetic field is zero

(a) Outside the cable

(b) Inside the inner conductor

(c) Inside the outer conductor

(d) In between the two conductor

15. A steady electric current is flowing through a cylindrical conductor.

(a) The electric field at the axis of the conductor is zero

(b) The magnetic field at the axis of the conductor is zero

(c) The electric field in the vicinity of the conductor is zero

(d) The magnetic field in the vicinity of the conductor is zero

sec. 30-6 Solenoids and Toroids

16. A solenoid exerts a force of attraction on an iron plate when a current flows through it. This force can be increased by

(a) Putting an iron core in the solenoid

(b) Putting more loops on the solenoid

(c) Cooling the solenoid to reduce the resistance of the coil

(d) Increasing the current flow through the coil

(e) All of the above

PROBLEMS

sec. 30-2 Calculating the Magnetic Field Due to a Current

1. Two long straight wires are parallel and 8.0 cm apart. They are to carry equal currents such that the magnetic field at a point halfway between them has magnitude 300 μT. (a) Should the currents be in the same or opposite directions? (b) How much current is needed?

2. A straight conductor carrying current $i = 5.0$ A splits identical semicircular arcs as shown in Fig. 30-49. What is the magnetic field at the center C of the resulting circular loop?

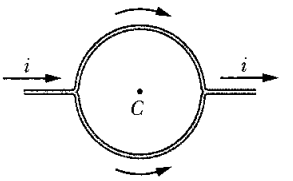


FIG. 30-49 Problem 2.

3. In Fig. 30-50, two semicircular arcs have radii $R_2 = 7.80$ cm and $R_1 = 3.15$ cm, carry current $i = 0.281$ A, and share the same center of curvature C. What are the (a) magnitude and (b) direction (into or out of the page) of the net magnetic field at C?

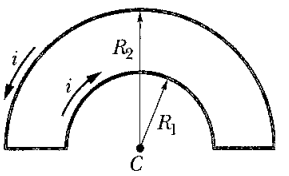


FIG. 30-50 Problem 3.

4. In Fig. 30-51, a current $i = 10$ A is set up in a long hairpin conductor formed by bending a wire into a semicircle of radius $R = 5.0$ mm. Point b is midway between the straight sections and so distant from the semicircle that each straight section can be approximated as being an infinite wire. What are the (a) magnitude and (b) direction (into or out of the page) of $\vec{B}$ at a and the (c) magnitude and (d) direction of $\vec{B}$ at b?

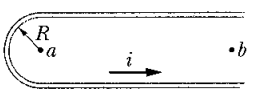


FIG. 30-51 Problem 4.

5. Figure 30-52*a* shows, in cross section, two long, parallel wires carrying current and separated by distance L. The ratio i_1/i_2 of their currents is 4.00; the directions of the currents are not indicated. Figure 30-52*b* shows the y component B_y of their net magnetic field along the x axis to the right of wire 2. The vertical scale is set by $B_{ys} = 4.0$ nT, and the horizontal scale is set by $x_s = 20.0$ cm. (a) At what value of $x > 0$ is B_y maximum? (b) If $i_2 = 3$ mA, what is the value of that maximum? What is the direction (into or out of the page) of (c) i_1 and (d) i_2?

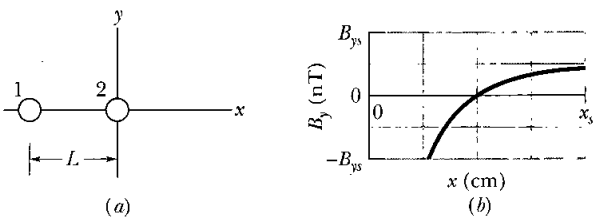


FIG. 30-52 Problem 5.

6. In Fig. 30-53, four long straight wires are perpendicular to the page, and their cross sections form a square of edge length $a = 20$ cm. The currents are out of the page in wires 1 and 4 and into the page in wires 2 and 3, and each wire carries 20 A. In unit-vector notation, what is the net magnetic field at the square's center?

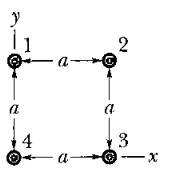


FIG. 30-53 Problems 6 and 19.

7. In Fig. 30-54, part of a long insulated wire carrying current $i = 5.78$ mA is bent into a circular section of radius $R = 1.89$ cm. In unit-vector notation, what is the magnetic field at the center of curvature C if the circular section (a) lies in the plane of the page as shown and (b) is perpendicular to the plane of the page after being rotated 90° counterclockwise as indicated?

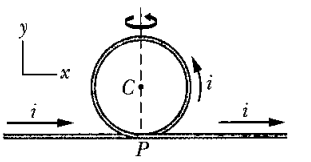


FIG. 30-54 Problem 7.

8. One long wire lies along an x axis and carries a current of 30 A in the positive x direction. A second long wire is perpendicular to the xy plane, passes through the point (0, 4.0 m, 0), and carries a current of 40 A in the positive z direction. What is the magnitude of the resulting magnetic field at the point (0, 2.0 m, 0)?

9. In Fig. 30-55, length a is 4.7 cm (short) and current i is 13 A. What are the (a) magnitude and (b) direction (into or out of the page) of the magnetic field at point P?

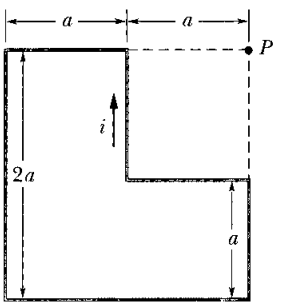
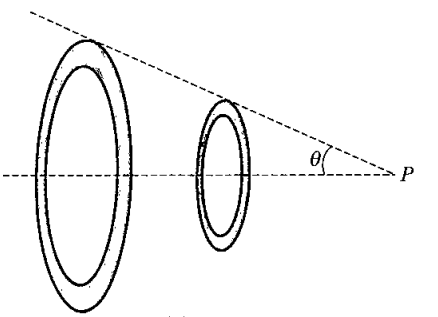


FIG. 30-55 Problem 9.

10. Find the intensity of a magnetic field on the axis of a circular ring at a distance of 3 cm from its plane. The ring radius is 4 cm and the current in the ring is 2 A.

11. Two circular coils shown in Fig. 30-56 have the same number of turns and carry equal current in the same sense. They have different diameters but subtend the same angle at P. (a) Which coil makes the larger contribution to the magnetic field at P? (b) If the smaller coil is midway between P and the larger one, and the smaller coil has radius R whereas the larger coil has a radius $2R$ and the distance of the smaller coil to the point P is also R, what is the ratio of the magnetic field due to larger coil at P to the magnetic field due to the smaller coil at P?



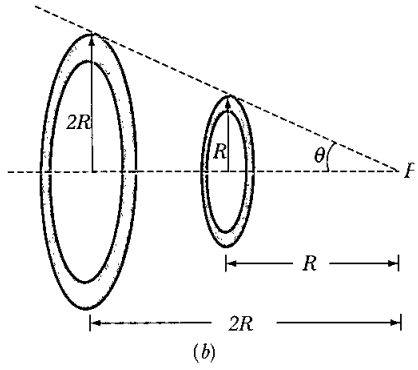


FIG. 30-56 Problem 11.

12. A circular coil connected to a cell of emf $\mathscr{E}$ with no internal resistance produces a magnetic induction B at its center. The coil is unwound, stretched to double its length and rewound into a coil of half the original radius. What must be the emf of the cell connected to the coil to produce the same induction as in the first case? Assume that the volume of the material of the wire and its resistivity remains the same.

sec. 30-3 Magnetic Field of Moving Charged Particles

13. An electron moves in a straight line at a speed of 6×10^7 m/s. Calculate the magnitude and direction of the magnetic field at (a) point 5 mm behind the electron and (b) 5 mm below its line of motion.

14. In a homogeneous magnetic field B there is an electron moving in a circular orbit with a speed v. Find the ratio of the magnetic field generated by the moving electron at the middle of the circle and the magnetic field making it revolve.

15. Let the position from q to $P(x, y, z)$ be $r = 30\hat{i}+20\hat{j}+60\hat{k}$ mm. Let $q = 400\ \mu$C and velocity of the charge q be $\vec{v} = 3\hat{i}-6\hat{j}+9\hat{k}\times 10^6$ m/s. Find the magnetic field at P.

16. A phonograph record of radius R, which carries a uniformly distributed charge Q, is rotating with constant angular speed ω (Fig. 30-57). Show that the magnetic field at the center of the disk is given by $B = (\mu_0 \omega Q)/(2\pi R)$.

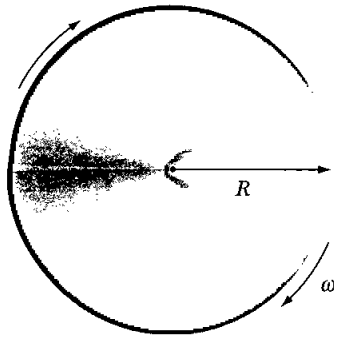


FIG. 30-57 Problem 16.

17. Two protons move parallel to each other with an equal velocity $v = 300$ km/s. Find the ratio of forces of magnetic and electrical interaction of the protons.

sec. 30-4 Force Between Two Parallel Currents

18. In Fig. 30-58, five long parallel wires in an xy plane are separated by distance $d = 50.0$ cm. The currents into the page are $i_1 = 2.00$ A, $i_3 = 0.250$ A, $i_4 = 4.00$ A, and $i_5 = 2.00$ A; the current out of the page is $i_2 = 4.00$ A. What is the magnitude of the net force *per unit length* acting on wire 3 due to the currents in the other wires?

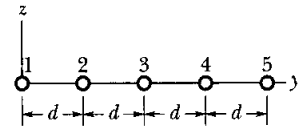


FIG. 30-58 Problem 18.

19. In Fig. 30-53, four long straight wires are perpendicular to the page, and their cross sections form a square of edge length $a = 8.50$ cm. Each wire carries 15.0 A, and all the currents are out of the page. In unit-vector notation, what is the net magnetic force *per meter of wire length* on wire 1?

sec. 30-5 Ampere's Law

20. Each of the eight conductors in Fig. 30-59 carries 2.0 A of current into or out of the page. Two paths are indicated for the line integral $\oint \vec{B}\cdot d\vec{s}$. What is the value of the integral for (a) path 1 and (b) path 2?

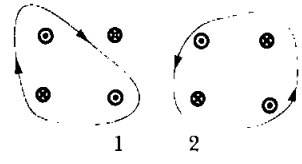


FIG. 30-59 Problem 20.

21. Figure 30-60 shows two closed paths wrapped around two conducting loops carrying currents $i_1 = 5.0$ A and $i_2 = 3.0$ A. What is the value of the integral $\oint \vec{B}\cdot d\vec{s}$. for (a) path 1 and (b) path 2?

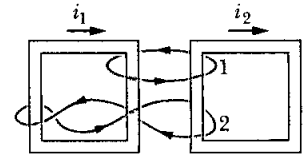


FIG. 30-60 Problem 21.

22. Eight wires cut the page perpendicularly at the points shown in Fig. 30-61. A wire labeled with the integer k ($k = 1, 2, \ldots, 8$) carries the current ki, where $i = 4.50$ mA. For those wires with odd k, the current is out of the page; for those with even k, it is into the page. Evaluate $\oint \vec{B}\cdot d\vec{s}$. along the closed path in the direction shown.

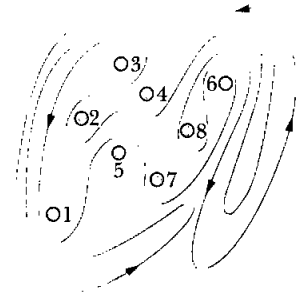


FIG. 30-61 Problem 22.

23. Figure 30-62 shows a cross section across a diameter of a long cylindrical conductor of radius $a = 2.00$ cm carrying uniform current 170 A. What is the magnitude of the current's

magnetic field at radial distance (a) 0, (b) 1.00 cm, (c) 2.00 cm (wire's surface), and (d) 4.00 cm?

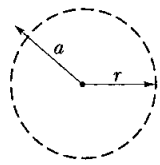


FIG. 30-62 Problem 23.

24. The current density $\vec{j}$ inside a long, solid, cylindrical wire of radius $a = 3.1$ mm is in the direction of the central axis, and its magnitude varies linearly with radial distance r from the axis according to $J = J_0 r/a$, where $J_0 = 310\ \text{A/m}^2$. Find the magnitude of the magnetic field at (a) $r = 0$, (b) $r = a/2$, and (c) $r = a$.

25. A cylindrical cable of radius 8.00 mm carries a current of 25.0 A, uniformly spread over its cross-sectional area. At what distance from the center of the wire is there a point within the wire where the magnetic field magnitude is 0.100 mT?

26. Show that a uniform magnetic field $\vec{B}$ cannot drop abruptly to zero (as is suggested by the lack of field lines to the right of point a in Fig. 30-63) as one moves perpendicular to $\vec{B}$, say along the horizontal arrow in the figure. (*Hint:* Apply Ampere's law to the rectangular path shown by the dashed lines.) In actual magnets, "fringing" of the magnetic field lines always occurs, which means that $\vec{B}$ approaches zero in a gradual manner. Modify the field lines in the figure to indicate a more realistic situation.

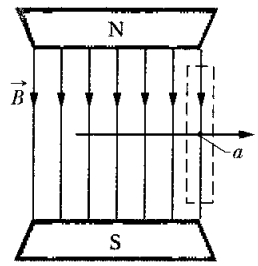


FIG. 30-63 Problem 26.

sec. 30-6 Solenoids and Toroids

27. A 200-turn solenoid having a length of 25 cm and a diameter of 10 cm carries a current of 0.29 A. Calculate the magnitude of the magnetic field $\vec{B}$ inside the solenoid.

28. A toroid having a square cross section, 5.00 cm on a side, and an inner radius of 15.0 cm has 500 turns and carries a current of 0.800 A. (It is made up of a square solenoid—instead of a round one as in Fig. 30-9—bent into a doughnut shape.) What is the magnetic field inside the toroid at (a) the inner radius and (b) the outer radius?

29. A long solenoid with 10.0 turns/cm and a radius of 7.00 cm carries a current of 20.0 mA. A current of 6.00 A exists in a straight conductor located along the central axis of the solenoid. (a) At what radial distance from the axis will the direction of the resulting magnetic field be at 45.0° to the axial direction? (b) What is the magnitude of the magnetic field there?

ADDITIONAL PROBLEMS

30. Figure 30-64 shows, in cross section, four thin wires that are parallel, straight, and very long. They carry identical currents in the directions indicated. Initially all four wires are at distance $d = 15.0$ cm from the origin of the coordinate system, where they create a net magnetic field $\vec{B}$. (a) To what value of x must you move wire 1 along the x axis in order to rotate $\vec{B}$ counterclockwise by 30°? (b) With wire 1 in that new position, to what value of x must you move wire 3 along the x axis to rotate $\vec{B}$ by 30° back to its initial orientation?

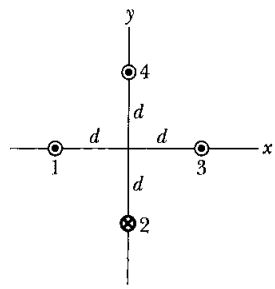


FIG. 30-64 Problem 30.

31. In Fig. 30-65, two concentric circular loops of wire carrying current in the same direction lie in the same plane. Loop 1 has radius 1.50 cm and carries 4.00 mA. Loop 2 has radius 2.50 cm and carries 6.00 mA. Loop 2 is to be rotated about a diameter while the net magnetic field $\vec{B}$ set up by the two loops at their common center is measured. Through what angle must loop 2 be rotated so that the magnitude of that net field is 100 nT?

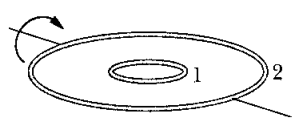


FIG. 30-65 Problem 31.

32. Figure 30-66 shows a cross section of a long thin ribbon of width $w = 4.91$ cm that is carrying a uniformly distributed total current $i = 4.61\ \mu$A into the page. In unit-vector notation, what is the magnetic field $\vec{B}$ at a point P in the plane of the ribbon at a distance $d = 2.16$ cm from its edge? (*Hint:* Imagine the ribbon as being constructed from many long, thin, parallel wires.)

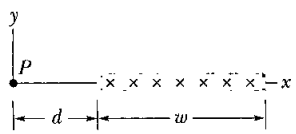


FIG. 30-66 Problem 32.

33. An electron is shot into one end of a solenoid. As it enters the uniform magnetic field within the solenoid, its speed is 800 m/s and its velocity vector makes an angle of 30° with the central axis of the solenoid. The solenoid carries 4.0 A and has 8000 turns along its length. How many revolutions does the electron make along its helical path within the solenoid by the time it emerges from the solenoid's opposite end? (In a real solenoid, where the field is not uniform at the two ends, the number of revolutions would be slightly less than the answer here.)

34. A long solenoid is fabricated by closely winding a wire of radius 0.5 mm over a cylindrical nonmagnetic frame so that the successive turns nearly touch each other. What would be the magnetic field B at the center of the solenoid if it carries a current of 5 A?

35. Figure 30-67 shows a cross section of a long cylindrical conductor of radius $a = 4.00$ cm containing a long cylindrical hole of radius $b = 1.50$ cm. The central axes of the cylinder and hole are parallel and are distance $d = 2.00$ cm apart; current $i = 5.25$ A is uniformly distributed over the tinted area. (a) What is the magnitude of the magnetic field at the center of the hole? (b) Discuss the two special cases $b = 0$ and $d = 0$.

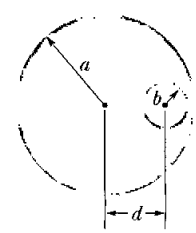


FIG. 30-67 Problem 35.

36. Figure 30-68 shows, in cross section, two long parallel wires spaced by distance $d = 10.0$ cm; each carries 100 A, out of the page in wire 1. Point P is on a perpendicular bisector of the line connecting the wires. In unit-vector notation, what is the net magnetic field at P if the current in wire 2 is (a) out of the page and (b) into the page?

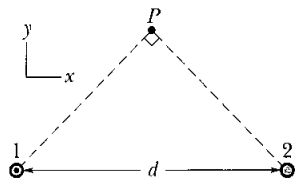


FIG. 30-68 Problem 36.

37. A long wire carrying 100 A is perpendicular to the magnetic field lines of a uniform magnetic field of magnitude 5.0 mT. At what distance from the wire is the net magnetic field equal to zero?

38. Two long wires lie in an xy plane, and each carries a current in the positive direction of the x axis. Wire 1 is at $y = 10.0$ cm and carries 6.00 A; wire 2 is at $y = 5.00$ cm and carries 10.0 A. (a) In unit-vector notation, what is the net magnetic field $\vec{B}$ at the origin? (b) At what value of y does $\vec{B}=0$? (c) If the current in wire 1 is reversed, at what value of y does $\vec{B}=0$?

39. Figure 30-69 is an idealized schematic drawing of a rail gun. Projectile P sits between two wide rails of circular cross section; a source of current sends current through the rails and through the (conducting) projectile (a fuse is not used). (a) Let w be the distance between the rails, R the radius of each rail, and i the current. Show that the force on the projectile is directed to the right along the rails and is given approximately by

$$F = \frac{i^2\mu_0}{2\pi} \ln \frac{w+R}{R}.$$

(b) If the projectile starts from the left end of the rails at rest, find the speed v at which it is expelled at the right. Assume that $i = 450$ kA, $w = 12$ mm, $R = 6.7$ cm, $L = 4.0$ m, and the projectile mass is 10 g.

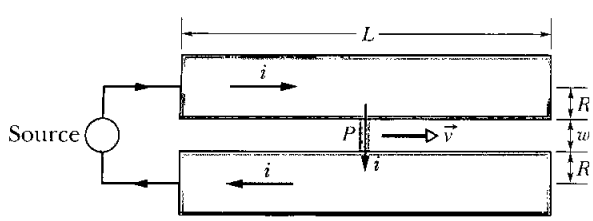


FIG. 30-69 Problem 39.

40. Two long parallel conductors are carrying currents in the opposite direction as shown in Fig. 30-70. The upper conductor (A) carries a current of 100 A is free to slide up and down. The lower conductor (B) carries a current of 50 A and and is held firmly in position. The linear mass density of the upper conductor is 0.01 kg m^{-1}. (a) What is the distance between the two conductor at which the upper conductor is in equilibrium? (b) Show that the equilibrium is stable. (c) Find the time period of its small oscillations about the equilibrium position.

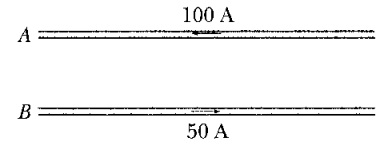


FIG. 30-70 Problem 40.

41. A thin insulated wire forms a plane spiral of N turns very close to each other carrying a current of i The radii of inside and outside turns (Fig. 30-71) are equal to a and b respectively. Find (a) the magnetic field at the center of the spiral and (b) the magnetic moment of the spiral with the given current.

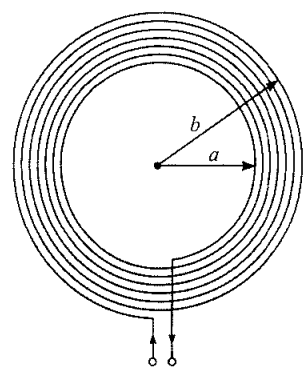


FIG. 30-71 Problem 41.

42. Find the magnetic induction at the point O if the wire carrying a current i has the shape shown in Fig. 30-72. The radius of the curved part of the wire is R, the linear parts of the wire are very long.

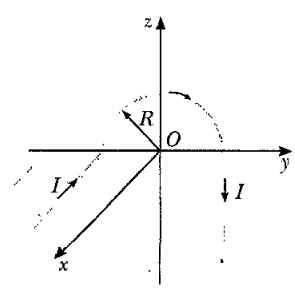


FIG. 30-72 Problem 42.

43. A current I flows in a long straight wire with cross-section having the form of a thin half-pipe of radius R (Fig. 30-73). (a) Find the magnetic field at a point on the axis of the pipe. (b) If an equal and opposite current flows in a long wire placed along the axis of the pipe, find the force per unit length on the central wire.

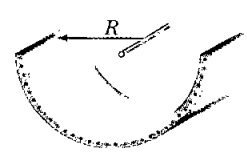


FIG. 30-73 Problem 43.

44. Consider the figure in Checkpoint 4. Find the force per unit area acting on each of the plates.

ANSWERS

Checkpoints

1. d, a, tie of b and c (zero)

2. b, c, a

3. d, tie of a and c, then b

4. (a) zero in between and μ_0 K outside; (b) μ_0 K in between and zero outside

Reasoning Type Questions

1. (b)
2. (a)
3. (a)
4. (a)
5. (c)
6. (b)
7. (d)

Multiple Choice Questions

1. (d)
2. (c)
3. (d)
4. (b), (c), (d)
5. (a)
6. (b)
7. (b)
8. (a)
9. (c)
10. (a)
11. (b)
12. (c)
13. (b), (e)
14. (a)
15. (b), (c)
16. (e)

Problems

1. (a) opposite; (b) 30 A
2. 0
3. (a) 1.67×10^{-6} T; (b) into the page
4. (a) 1.0×10^{-3} T; (b) out of the page; (c) 8.0×10^{-4} T; (d) out of the page
5. (a) 30 cm; (b) 2.0 nT; (c) out of page; (d) into the page
6. $(80\mu\text{T})\hat{\text{j}}$
7. (a) $(2.53\times10^{-7}\text{T})\hat{\text{k}}$; (b) $(1.92\times10^{-7}\text{ T})\hat{\text{i}}+(6.12\times10^{-8}\text{ T})\hat{\text{k}}$
8. 5.0 μT
9. (a) 20 μT; (b) into the page
10. $\dfrac{64\pi}{125}\times10^{-5}$ T
11. (a) the smaller coil; (b) $\frac{1}{2}$
12. $\mathscr{E}/2$
13. (a) 0; (b) 3.84×10^{-14} T
14. $\dfrac{\mu_0 e^3 B}{4\pi m^2 v}$
15. $-\dfrac{21.6}{343}\hat{\text{i}}+\dfrac{3.6}{343}\hat{\text{j}}+\dfrac{9.6}{343}\hat{\text{k}}$ T
17. $F_m/F_e=\mu_0\varepsilon_0 v^2=(v/c)^2=1.00\times10^{-6}$
18. 8.00×10^{-7} N/m
19. $(7.94\times10^{-4}\text{N/m})\hat{\text{i}}+(-7.94\times10^{-4}\text{N/m})\hat{\text{j}}$
20. (a) $-2.5\ \mu\text{T}\cdot\text{m}$; (b) 0
21. (a) $-2.5\times10^{-6}\text{ T}\cdot\text{m}$; (b) $-1.6\times10^{-5}\text{ T}\cdot\text{m}$
22. $2.83\times10^{-8}\text{ T}\cdot\text{m}$
23. (a) 0; (b) 0.850 mT; (c) 1.70 mT; (d) 0.850 mT
24. (a) 0; (b) 0.10 μT; (c) 0.40 μT
25. 1.28 mm
27. 0.30 mT
28. (a) 533 μT; (b) 400 μT
29. (a) 4.77 cm; (b) 35.5 μT

Additional Problems

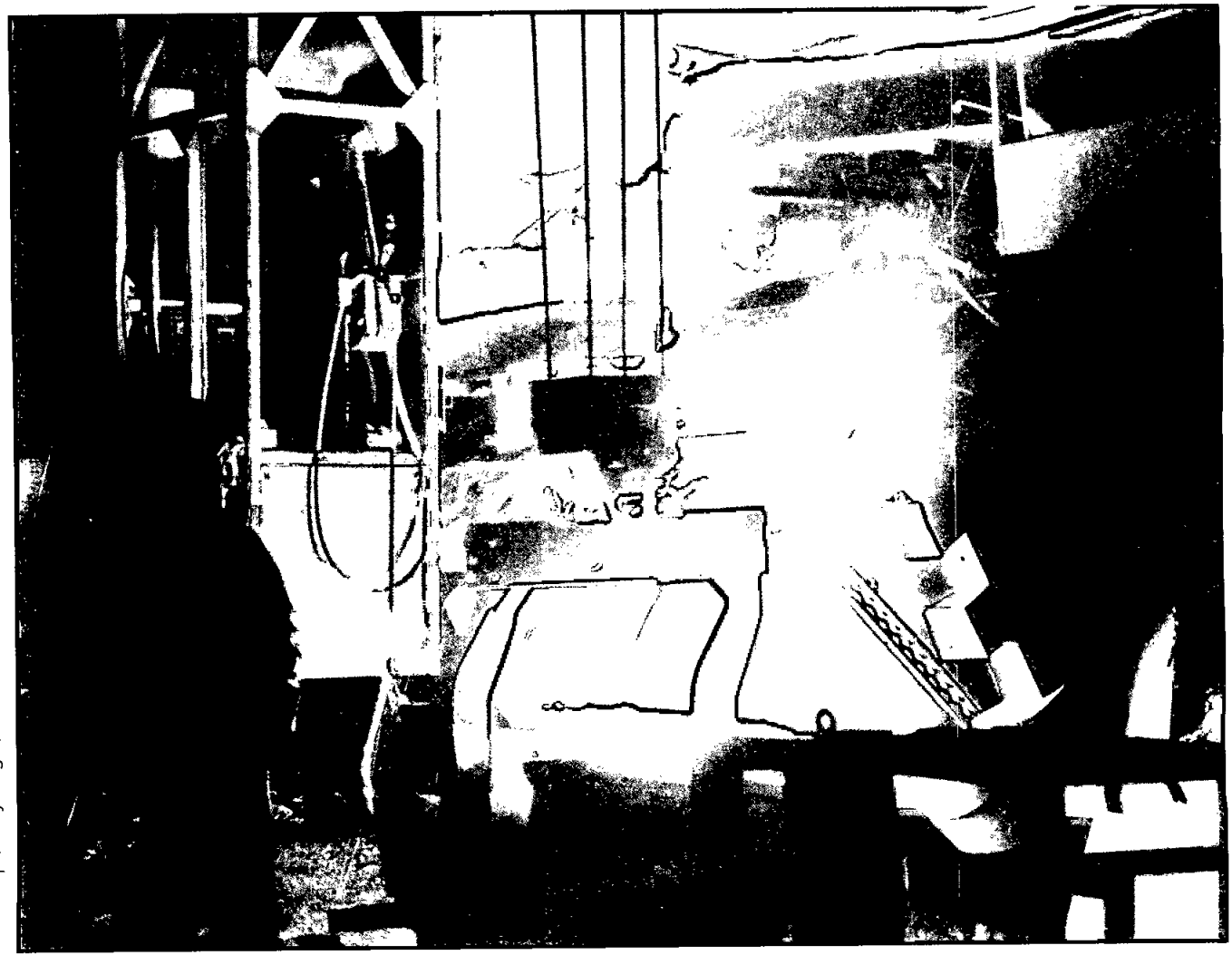
30. (a) −7.0 cm; (b) +7.0 cm
31. 144°
32. $(22.3\text{pT})\hat{\text{j}}$
33. $\approx1.62\times10^{6}$
34. $2\pi\times10^{-3}$ T
35. (a) 15.3 μT; (b) If $b=0$, then

$$B=\frac{\mu_0 id}{2\pi a^2}.$$

Now for $d=0$ the formula gives $B=0$

36. (a) $(-4.00\times10^{-4}\text{ T})\hat{\text{i}}$; (b) $(4.00\times10^{-4}\text{ T})\hat{\text{j}}$
37. 4.0×10^{-3} m
38. (a) $(-52.0\times10^{-6}\text{ T})\hat{\text{k}}$; (b) 0.0813 m; (c) 0.0175 m
39. (b) 2.3×10^{3} m/s
40. (a) 1 cm; (c) 0.2 s
41. (a) $\dfrac{\mu_0 Ni\ln(b/a)}{2(b-a)}$; (b) $1/3\ \pi N i\cdot(a^2+ab+b^2)$
42. $-\dfrac{\mu_0}{4\pi}\dfrac{i}{R}[\hat{\text{k}}+(\pi+1)\vec{i}\,]$
43. (a) $B=\mu_0 i/\pi^2 R$; (b) $\mu_0 i^2/\pi^2 R$
44. $\mu_0 K^2/2$ in both the cases. In part (a) it is attractive and in part (b) it is repulsive.

Electromagnetic Induction 31



Mark Joseph/Getty Images, Inc.

Traditionally, foundries used flame-heated furnaces to melt metals. However, many modern foundries avoid the resulting air pollution by using an induction furnace *in which the metal is heated by the current in insulated wires wrapped around the crucible that holds the metal. However, the wires themselves do not get hot enough to melt the metal (or they would also melt). Indeed, they are kept cool by a water bath.*

How then does the metal become hot enough to melt?

The answer is in this chapter.

31-1 WHAT IS PHYSICS?

In Chapter 30 we discussed the fact that a current produces a magnetic field. That fact came as a surprise to the scientists who discovered the effect. Perhaps even more surprising was the discovery of the reverse effect: A magnetic field can produce an electric field that can drive a current. This link between a magnetic field and the electric field it produces (*induces*) is now called *Faraday's law of induction.*

The observations by Michael Faraday and other scientists that led to this law were at first just basic science, interesting in that they revealed another facet of how our universe works. Today, however, applications of that basic science are almost everywhere. For example, induction is the basis of the electric guitars that revolutionized early rock and still drive heavy metal and punk today. It is also the basis of the electric generators that power cities and transportation lines. Although induction stoves for the kitchen have yet to catch on with either professional chefs or amateur cooks, huge induction furnaces are commonplace in foundries where large amounts of metal must be melted rapidly.

Before we get to applications like the electric guitar, we must examine two simple experiments about Faraday's law of induction. We will begin with the concept of magnetic flux and then go on to the experiments of Faraday.

31-2 | Gauss' Law for Magnetic Fields

Before we understand the process of electromagnetic induction, we would define an associated concept of magnetic flux. Actually, we need a way to calculate the *amount of magnetic field* that passes through a loop. In Chapter 23, in a similar situation, we needed to calculate the amount of electric field that passes through a surface. There we defined an electric flux as

$$\Phi_E = \int \vec{E} \cdot d\vec{A}.$$

Here we define *magnetic flux:* Suppose a loop enclosing an area A is placed in a magnetic field $\vec{B}$. Then the **magnetic flux** through the loop is

$$\Phi_B = \int \vec{B} \cdot d\vec{A}. \tag{31-1}$$

As in Chapter 23, $d\vec{A}$ is a vector of magnitude dA that is perpendicular to a differential area dA. As a special case of Eq. 31-1, suppose that the loop lies in a plane and that the magnetic field is perpendicular to the plane of the loop. Then we can write the dot product in Eq. 31-1 as $B\,dA\cos 0° = B\,dA$. If the magnetic field is also uniform, then B can be brought out in front of the integral sign. The remaining $\int dA$ then gives just the area A of the loop. Thus, Eq. 31-1 reduces to

$$\Phi_B = BA \qquad (\vec{B} \perp \text{area } A, \vec{B} \text{ uniform}). \tag{31-2}$$

From Eqs. 31-1 and 31-2, we see that the SI unit for magnetic flux is the tesla–square meter, which is called the *weber* (abbreviated Wb):

$$1 \text{ weber} = 1 \text{ Wb} = 1 \text{ Tm}^2.$$

Gauss' Law

In Chapter 29 we established that we cannot separate a magnet into its north and south poles. They are inseparable. We can conclude that the simplest magnetic structure that can exist is a magnetic dipole. Magnetic monopoles do not exist (as far as we know). Gauss' law for magnetic fields is a formal way of saying that

magnetic monopoles do not exist. The law asserts that the net magnetic flux ϕ through any closed Gaussian surface is zero:

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0 \quad \text{(Gauss' law for magnetic fields).}$$

Contrast this with Gauss' law for electric fields,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \quad \text{(Gauss' law for electric fields).}$$

In both equations, the integral is taken over a *closed* Gaussian surface. Gauss' law for electric fields says that this integral (the net electric flux through the surface) is proportional to the net electric charge q_{enc} enclosed by the surface. Gauss' law for magnetic fields says that there can be no net magnetic flux through the surface because there can be no net "magnetic charge" (individual magnetic poles) enclosed by the surface. The simplest magnetic structure that can exist, and thus be enclosed, by a Gaussian surface is a dipole, which consists of both a source and a sink for the field lines. Thus, there must always be as much magnetic flux into the surface as out of it, and the net magnetic flux must always be zero.

Gauss' law for magnetic fields holds for structures more complicated than a magnetic dipole, and it holds even if the Gaussian surface does not enclose the entire structure. Gaussian surface II near the bar magnet of Fig. 31-1 encloses no poles, and we can easily conclude that the net magnetic flux through it is zero. Gaussian surface I is more difficult. It may seem to enclose only the north pole of the magnet because it encloses the label N and not the label S. However, a south pole must be associated with the lower boundary of the surface because magnetic field lines enter the surface there. (The enclosed section is like one piece of the broken bar magnet.) Thus, Gaussian surface I encloses a magnetic dipole, and the net flux through the surface is zero.

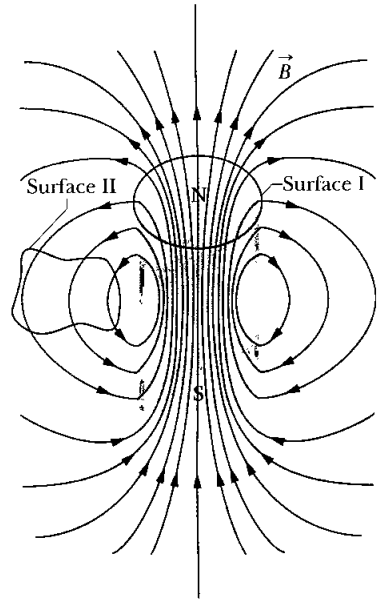


FIG. 31-1 The field lines for the magnetic field of a short bar magnet. The red curves represent cross sections of closed, three-dimensional Gaussian surfaces.

31-3 | Two Experiments

Let us examine two simple experiments to prepare for our discussion of Faraday's law of induction.

First Experiment. Figure 31-2 shows a conducting loop connected to a sensitive ammeter. Because there is no battery or other source of emf included, there is no current in the circuit. However, if we move a bar magnet toward the loop, a current suddenly appears in the circuit. The current disappears when the magnet stops. If we then move the magnet away, a current again suddenly appears, but now in the opposite direction. If we experimented for a while, we would discover the following:

1. A current appears only if there is relative motion between the loop and the magnet (one must move relative to the other); the current disappears when the relative motion between them ceases.
2. Faster motion produces a greater current.
3. If moving the magnet's north pole toward the loop causes, say, clockwise current, then moving the north pole away causes counterclockwise current. Moving the south pole toward or away from the loop also causes currents, but in the reversed directions.

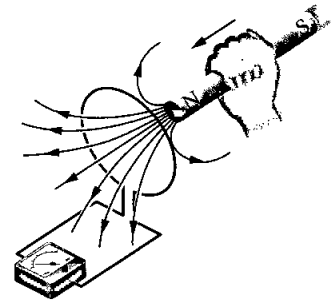


FIG. 31-2 An ammeter registers a current in the wire loop when the magnet is moving with respect to the loop.

The current produced in the loop is called an **induced current**; the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced emf**; and the process of producing the current and emf is called **induction**.

Second Experiment. For this experiment we use the apparatus of Fig. 31-3, with the two conducting loops close to each other but not touching. If we close switch S, to turn on a current in the right-hand loop, the meter suddenly and

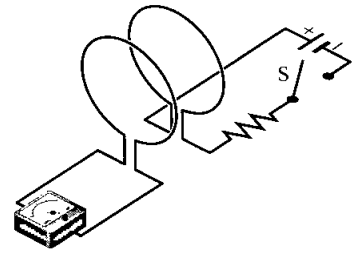


FIG. 31-3 An ammeter registers a current in the left-hand wire loop just as switch S is closed (to turn on the current in the right-hand wire loop) or opened (to turn off the current in the right-hand loop). No motion of the coils is involved.

briefly registers a current—an induced current—in the left-hand loop. If we then open the switch, another sudden and brief induced current appears in the left-hand loop, but in the opposite direction. We get an induced current (and thus an induced emf) only when the current in the right-hand loop is changing (either turning on or turning off) and not when it is constant (even if it is large).

The induced emf and induced current in these experiments are apparently caused when something changes—but what is that "something"? Faraday knew.

31-4 | Faraday's Law of Induction

Faraday realized that an emf and a current can be induced in a loop, as in our two experiments, by changing the *amount of magnetic field* passing through the loop. He further realized that the "amount of magnetic field" can be visualized in terms of the magnetic field lines passing through the loop. **Faraday's law of induction**, stated in terms of our experiments, is this:

☞ An emf is induced in the loop at the left in Figs. 31-2 and 31-3 when the number of magnetic field lines that pass through the loop is changing. In terms of what we stated in Section 31-2, the emf is induced when the magnetic flux linked with the loop is changing.

The actual number of field lines passing through the loop does not matter; the values of the induced emf and induced current are determined by the *rate* at which the magnetic flux changes.

In our first experiment (Fig. 31-2), the magnetic field lines spread out from the north pole of the magnet. Thus, as we move the north pole closer to the loop, the number of field lines passing through the loop increases. That increase apparently causes conduction electrons in the loop to move (the induced current) and provides energy (the induced emf) for their motion. When the magnet stops moving, the number of field lines through the loop no longer changes and the induced current and induced emf disappear.

In our second experiment (Fig. 31-3), when the switch is open (no current), there are no field lines. However, when we turn on the current in the right-hand loop, the increasing current builds up a magnetic field around that loop and at the left-hand loop. While the field builds, the number of magnetic field lines through the left-hand loop increases. As in the first experiment, the increase in field lines through that loop apparently induces a current and an emf there. When the current in the right-hand loop reaches a final, steady value, the number of field lines through the left-hand loop no longer changes, and the induced current and induced emf disappear.

Faraday's law does not explain *why* a current and an emf are induced in either experiment; it is just a statement that helps us visualize the induction.

A Quantitative Treatment

With the notion of magnetic flux, we can state Faraday's law in a more quantitative and useful way:

☞ The magnitude of the emf $\mathscr{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time.

As you will see in the next section, the induced emf $\mathscr{E}$ tends to oppose the flux change, so Faraday's law is formally written as

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law)}, \tag{31-3}$$

with the minus sign indicating that opposition. We often neglect the minus sign in Eq. 31-3, seeking only the magnitude of the induced emf.

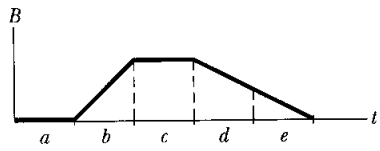
If we change the magnetic flux through a coil of N turns, an induced emf appears in every turn and the total emf induced in the coil is the sum of these individual induced emfs. If the coil is tightly wound (*closely packed*), so that the same magnetic flux Φ_B passes through all the turns, the total emf induced in the coil is

$$\mathscr{E} = -N\frac{d\Phi_B}{dt} \qquad \text{(coil of } N \text{ turns)}. \tag{31-4}$$

Here are the general means by which we can change the magnetic flux through a coil:

1. Change the magnitude B of the magnetic field within the coil.
2. Change either the total area of the coil or the portion of that area that lies within the magnetic field (for example, by expanding the coil or sliding it into or out of the field).
3. Change the angle between the direction of the magnetic field $\vec{B}$ and the plane of the coil (for example, by rotating the coil so that field $\vec{B}$ is first perpendicular to the plane of the coil and then is along that plane).

CHECKPOINT 1 The graph gives the magnitude $B(t)$ of a uniform magnetic field that exists throughout a conducting loop, with the direction of the field perpendicular to the plane of the loop. Rank the five regions of the graph according to the magnitude of the emf induced in the loop, greatest first.



Sample Problem 31-1 Build your skill

The long solenoid S shown (in cross section) in Fig. 31-4 has 220 turns/cm and carries a current $i = 1.5$ A; its diameter D is 3.2 cm. At its center we place a 130-turn closely packed coil C of diameter $d = 2.1$ cm. The current in the solenoid is reduced to zero at a steady rate in 25 ms. What is the magnitude of the emf that is induced in coil C while the current in the solenoid is changing?

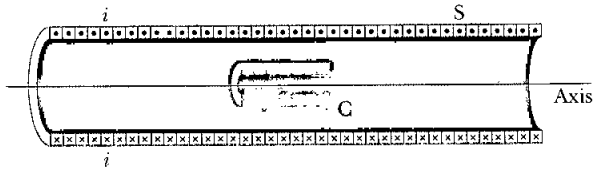


FIG. 31-4 A coil C is located inside a solenoid S, which carries current i.

Conceptualize/Classify:

1. Because it is located in the interior of the solenoid, coil C lies within the magnetic field produced by current i in the solenoid; thus, there is a magnetic flux Φ_B through coil C.
2. Because current i decreases, flux Φ_B also decreases.
3. As Φ_B decreases, emf $\mathscr{E}$ is induced in coil C, according to Faraday's law.
4. The flux through each turn of coil C depends on the area A and orientation of that turn in the solenoid's magnetic field $\vec{B}$. Because $\vec{B}$ is uniform and directed perpendicular to area A, the flux is given by $\Phi_B = BA$.
5. The magnitude B of the magnetic field in the interior of a solenoid depends on the solenoid's current i and its number n of turns per unit length, according to $B = \mu_0 in$.

Compute: Because coil C consists of more than one turn, we apply Faraday's law in the form of Eq. 31-4 ($\mathscr{E} = -N\, d\Phi_B/dt$), where the number of turns N is 130 and $d\Phi_B/dt$ is the rate at which the flux changes.

Because the current in the solenoid decreases at a steady rate, flux Φ_B also decreases at a steady rate, and so we can write $d\Phi_B/dt$ as $\Delta\Phi_B/\Delta t$. Then, to

evaluate $\Delta\Phi_B$, we need the final and initial flux values. The final flux $\Phi_{B,f}$ is zero because the final current in the solenoid is zero. To find the initial flux $\Phi_{B,i}$, we first note that the area A is $\frac{1}{4}\pi d^2$ ($= 3.464 \times 10^{-4}$ m^2) and the number n is 220 turns/cm, or 22 000 turns/m. Substituting $B = \mu_0 in$ into $\Phi_B = BA$ then leads to

$$\begin{aligned}\Phi_{B,i} &= BA = (\mu_0 in)A \\ &= (4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(1.5\,\text{A})(22\,000\,\text{turns/m}) \\ &\quad \times (3.464 \times 10^{-4}\,\text{m}^2) \\ &= 1.44 \times 10^{-5}\,\text{Wb}.\end{aligned}$$

Now we can write

$$\frac{d\Phi_B}{dt} = \frac{\Delta\Phi_B}{\Delta t} = \frac{\Phi_{B,f} - \Phi_{B,i}}{\Delta t}$$

$$= \frac{0 - 1.44 \times 10^{-5}\,\text{Wb}}{25 \times 10^{-3}\,\text{s}}$$

$$= -5.76 \times 10^{-4}\,\text{Wb/s} = -5.76 \times 10^{-4}\,\text{V}.$$

We are interested only in magnitudes; so we ignore the minus signs here and in Eq. 31-4, writing

$$\mathscr{E} = N\frac{d\Phi_B}{dt} = (130\,\text{turns})(5.76 \times 10^{-4}\,\text{V})$$

$$= 7.5 \times 10^{-2}\,\text{V} = 75\,\text{mV}. \qquad \text{(Answer)}$$

Conclude: Note that if the coil were positioned such that its plane contains the axis of the solenoid, no emf would be induced in the loop even though the magnetic field in space is changing. This is because no flux is linked with the coil.

Sample Problem 31-2

A small coil of N turns has its plane perpendicular to a uniform magnetic field as shown in Figure 31-5. The coil is connected to a Ballistic galvanometer, a device designed to measure the total charge passing through it. Find the charge passing through the coil if the coil is rotated through 180° about its diameter.

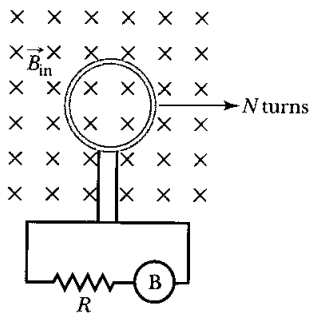


FIG. 31-5 A coil of N turns is rotated inside a magnetic field.

Conceptualize/Classify: When the coil in Fig. 31-5 is rotated, the magnetic flux through it changes, causing an induced emf, E. The emf in turn causes a current $I = E/R$, where R is the total resistance of the circuit. Since $I = dQ/dt$, we can find the charge passing through the coil by integrating I; that is,

$$Q = \int dQ = \int I dt. \qquad (31\text{-}5)$$

Compute: As we stated just now, the total charge flown is the integral of the current:

$$Q = \int dQ = \int I dt.$$

The current is related to the emf by Ohm's law as

$$I = \frac{E}{R},$$

and the emf is only due to the changing magnetic flux, that is

$$E = \frac{d\Phi_B}{dt}.$$

Thus, the induced current is given by

$$I = \frac{E}{R} = \frac{1}{R}\frac{d\Phi_B}{dt}.$$

Substituting it back in Eq. 31-5, we get

$$Q = \int I dt = \frac{\Delta\Phi_B}{R}.$$

The initial flux through the loop is given by

$$\Phi_i = NBA$$

and after rotating the loop through 180°, the flux would be

$$\Phi_f = -NBA.$$

Hence the charge flowing through the Ballistic galvanometer is given by

$$Q = \frac{2NBA}{R}.$$

Conclude: Note that the charge Q is independent of the time involved in rotating the coil—all that matters is the change in magnetic flux. A coil used in this way is called a flip coil. It is used to measure magnetic fields. For example, if the Ballistic galvanometer measures a total charge Q passing through the coil when it is flipped, the magnetic field can be found from the equation above.

31-5 | Lenz's Law

Soon after Faraday proposed his law of induction, Heinrich Friedrich Lenz devised a rule—now known as **Lenz's law**—for determining the direction of an induced current in a loop:

☞ An induced current has a direction such that the magnetic field due to *the current* opposes the change in the magnetic flux that induces the current.

Furthermore, the direction of an induced emf is that of the induced current. To get a feel for Lenz's law, let us apply it in two different but equivalent ways to Fig. 31-6, where the north pole of a magnet is being moved toward a conducting loop.

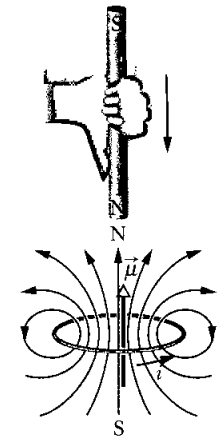


FIG. 31-6 Lenz's law at work. As the magnet is moved toward the loop, a current is induced in the loop. The current produces its own magnetic field, with magnetic dipole moment $\vec{\mu}$ oriented so as to oppose the motion of the magnet. Thus, the induced current must be counterclockwise as shown.

1. ***Opposition to Pole Movement.*** The approach of the magnet's north pole in Fig. 31-4 increases the magnetic flux through the loop and thereby induces a current in the loop. From Fig. 30-12, we know that the loop then acts as a magnetic dipole with a south pole and a north pole, and that its magnetic dipole moment $\vec{\mu}$ is directed from south to north. To *oppose* the magnetic flux increase being caused by the approaching magnet, the loop's north pole (and thus $\vec{\mu}$) must face *toward* the approaching north pole so as to repel it (Fig. 31-6). Then the curled–straight right-hand rule for $\vec{\mu}$ (Fig. 30-12) tells us that the current induced in the loop must be counterclockwise in Fig. 31-6.

   If we next pull the magnet away from the loop, a current will again be induced in the loop. Now, however, the loop will have a south pole facing the retreating north pole of the magnet, so as to oppose the retreat. Thus, the induced current will be clockwise.

2. ***Opposition to Flux Change.*** In Fig. 31-6, with the magnet initially distant, no magnetic flux passes through the loop. As the north pole of the magnet then nears the loop with its magnetic field $\vec{B}_{ind}$ directed *downward*, the flux through the loop increases. To oppose this increase in flux, the induced current i must set up its own field $\vec{B}_{ind}$ directed *upward* inside the loop, as shown in Fig. 31-7*a*; then the upward flux of field $\vec{B}_{ind}$ opposes the increasing downward flux of field $\vec{B}$. The curled–straight right-hand rule of Fig. 30-12 then tells us that i must be counterclockwise in Fig. 31-7*a*.

   Note carefully that the flux of $\vec{B}_{ind}$ always opposes the *change* in the flux of $\vec{B}$, but that does not always mean that $\vec{B}_{ind}$ points opposite $\vec{B}$. For example, if we next pull the magnet away from the loop in Fig. 31-6, the flux Φ_B from the magnet is still directed downward through the loop, but it is now decreasing. The flux of $\vec{B}_{ind}$ must now be downward inside the loop, to oppose the *decrease* in Φ_B, as shown in Fig. 31-7*b*. Thus, $\vec{B}_{ind}$ and $\vec{B}$ are now in the same direction.

   Figures 31-7*c* and *d* show the situations in which the south pole of the magnet approaches and retreats from the loop, respectively.

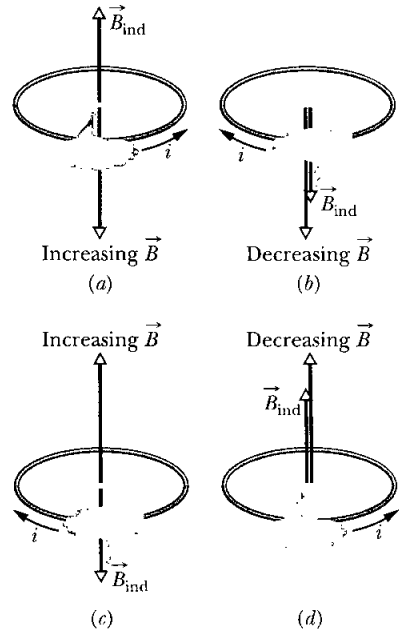


FIG. 31-7 The direction of the current i induced in a loop is such that the current's magnetic field $\vec{B}_{ind}$ opposes the *change* in the magnetic field $\vec{B}$ inducing i. The field $\vec{B}_{ind}$ is always directed opposite an increasing field $\vec{B}(a, c)$ and in the same direction as a decreasing field $\vec{B}(a, d)$. The curled–straight right-hand rule gives the direction of the induced current based on the direction of the induced field.

Lenz's law follows from the law of conservation of energy. Let us say that the current was induced in the opposite direction. In other words, for the time being, let us assume that the current is induced in the loop so as to attract the magnet. Then after giving a slight push, magnet would move in the forward direction being attracted by the induced current. We need not supply any energy to produce the current. Why even pay for the electricity bill, then? Just buy a powerful magnet and voila, you have zero electricity bill.

Sorry! But this is not the way God has designed the world. You need to pay for every joule of energy that you consume. So what we assumed just now would be converse of the real world.

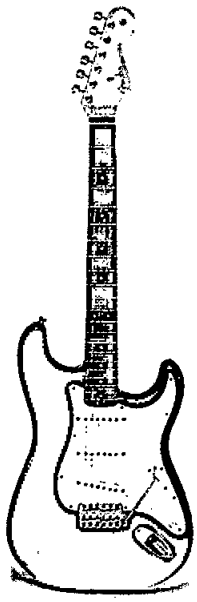


FIG. 31-8 A Fender Stratocaster guitar. *(Courtesy Fender Musical Instruments Corporation)*

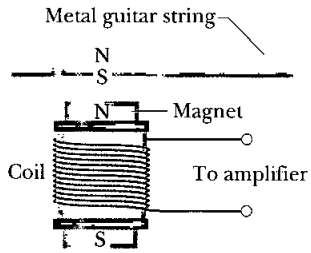


FIG. 31-9 A side view of an electric guitar pickup. When the metal string (which acts like a magnet) is made to oscillate, it causes a variation in magnetic flux that induces a current in the coil.

Electric Guitars

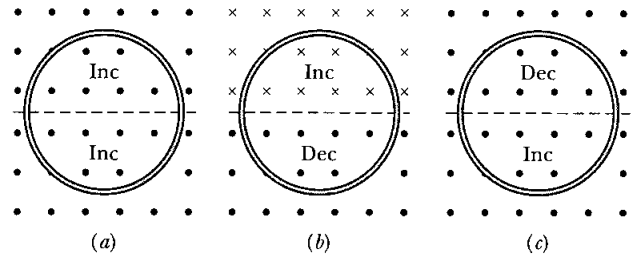
Figure 31-8 shows a Fender® Stratocaster®, one type of electric guitar. Whereas an acoustic guitar depends for its sound on the acoustic resonance produced in the hollow body of the instrument by the oscillations of the strings, an electric guitar is a solid instrument, so there is no body resonance. Instead, the oscillations of the metal strings are sensed by electric "pickups" that send signals to an amplifier and a set of speakers.

The basic construction of a pickup is shown in Fig. 31-9. Wire connecting the instrument to the amplifier is coiled around a small magnet. The magnetic field of the magnet produces a north and south pole in the section of the metal string just above the magnet. That section of string then has its own magnetic field. When the string is plucked and thus made to oscillate, its motion relative to the coil changes the flux of its magnetic field through the coil, inducing a current in the coil. As the string oscillates toward and away from the coil, the induced current changes direction at the same frequency as the string's oscillations, thus relaying the frequency of oscillation to the amplifier and speaker.

On a Stratocaster, there are three groups of pickups, placed at the near end of the strings (on the wide part of the body). The group closest to the near end better detects the high-frequency oscillations of the strings; the group farthest from the near end better detects the low-frequency oscillations. By throwing a toggle switch on the guitar, the musician can select which group or which pair of groups will send signals to the amplifier and speakers.

To gain further control over his music, the legendary Jimi Hendrix sometimes rewrapped the wire in the pickup coils of his guitar to change the number of turns. In this way, he altered the amount of emf induced in the coils and thus their relative sensitivity to string oscillations. Even without this additional measure, you can see that the electric guitar offers far more control over the sound that is produced than can be obtained with an acoustic guitar.

CHECKPOINT 2 The figure shows three situations in which identical circular conducting loops are in uniform magnetic fields that are either increasing (Inc) or decreasing (Dec) in magnitude at identical rates. In each, the dashed line coincides with a diameter. Rank the situations according to the magnitude of the current induced in the loops, greatest first.



Sample Problem 31-3

Figure 31-10 shows a conducting loop consisting of a half-circle of radius $r = 0.20$ m and three straight sections. The half-circle lies in a uniform magnetic field $\vec{B}$ that is directed out of the page; the field magnitude is given by $B = 4.0t^2 + 2.0t + 3.0$, with B in teslas and t in seconds. An ideal battery with emf $\mathscr{E}_{bat} = 2.0$ V is connected to the loop. The resistance of the loop is 2.0 Ω.

(a) What are the magnitude and direction of the emf $\mathscr{E}_{ind}$ induced around the loop by field $\vec{B}$ at $t = 10$ s?

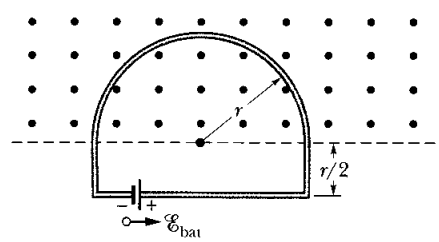


FIG. 31-10 A battery is connected to a conducting loop that includes a half-circle of radius r lying in a uniform magnetic field. The field is directed out of the page; its magnitude is changing.

Conceptualize/Classify:

1. According to Faraday's law, the magnitude of $\mathscr{E}_{ind}$ is equal to the rate $d\Phi_B/dt$ at which the magnetic flux through the loop changes.
2. The flux through the loop depends on how much of the loop's area lies within the flux and how the area is oriented in the magnetic field $\vec{B}$.
3. Because $\vec{B}$ is uniform and is perpendicular to the plane of the loop, the flux is given by $\Phi_B = BA$. (We don't need to integrate B over the area to get the flux.)
4. The induced field B_{ind} (due to the induced current) must always oppose the *change* in the magnetic flux.

Compute: Magnitude: Using $\Phi_B = BA$ and realizing that only the field magnitude B changes in time (not the area A), we rewrite Faraday's law, Eq. 31-3, as

$$\mathscr{E}_{ind} = \frac{d\Phi_B}{dt} = \frac{d(BA)}{dt} = A\frac{dB}{dt}.$$

Because the flux penetrates the loop only within the half-circle, the area A in this equation is $\frac{1}{2}\pi r^2$. Substituting this and the given expression for B yields

$$\mathscr{E}_{ind} = A\frac{dB}{dt} = \frac{\pi r^2}{2}\frac{d}{dt}(4.0t^2 + 2.0t + 3.0)$$
$$= \frac{\pi r^2}{2}(8.0t + 2.0).$$

At $t = 10$ s, then,

$$\mathscr{E}_{ind} = \frac{\pi(0.20\text{ m})^2}{2}[8.0(10) + 2.0]$$
$$= 5.152\text{ V} \approx 5.2\text{ V}. \qquad \text{(Answer)}$$

Direction: To find the direction of $\mathscr{E}_{ind}$, we first note that in Fig. 31-10 the flux through the loop is out of the page and increasing. Because the induced field B_{ind} (due to the induced current) must oppose that increase, it must be *into* the page. Using the curled-straight right-hand rule (Fig. 31-7*c*), we find that the induced current is clockwise around the loop, and thus so is the induced emf $\mathscr{E}_{ind}$.

(b) What is the current in the loop at $t = 10$ s?

Conceptualize/Classify: The point here is that *two* emfs tend to move charges around the loop.

Compute: The induced emf $\mathscr{E}_{ind}$ tends to drive a current clockwise around the loop; the battery's emf $\mathscr{E}_{bat}$ tends to drive a current counterclockwise. Because $\mathscr{E}_{ind}$ is greater than $\mathscr{E}_{bat}$, the net emf $\mathscr{E}_{net}$ is clockwise, and thus so is the current. To find the current at $t = 10$ s, we use $i = \mathscr{E}/R$:

$$i = \frac{\mathscr{E}_{net}}{R} = \frac{\mathscr{E}_{ind} - \mathscr{E}_{bat}}{R}$$
$$= \frac{5.152\text{ V} - 2.0\text{ V}}{2.0\,\Omega} = 1.58\text{ A} \approx 1.6\text{ A}. \qquad \text{(Answer)}$$

Sample Problem 31-4

Figure 31-11 shows a rectangular loop of wire immersed in a nonuniform and varying magnetic field $\vec{B}$ that is perpendicular to and directed into the page. The field's magnitude is given by $B = 4t^2x^2$, with B in teslas, t in seconds, and x in meters. The loop has width $W = 3.0$ m and height $H = 2.0$ m. What are the magnitude and direction of the induced emf $\mathscr{E}$ around the loop at $t = 0.10$ s?

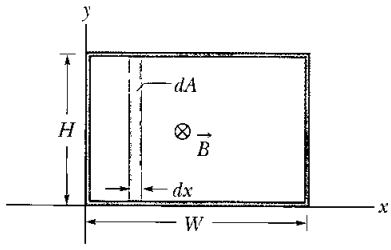


FIG. 31-11 A closed conducting loop, of width W and height H, lies in a nonuniform, varying magnetic field that points directly into the page. To apply Faraday's law, we use the vertical strip of height H, width dx, and area dA.

Conceptualize: Because the magnitude of the magnetic field $\vec{B}$ is changing with time, the magnetic flux Φ_B through the loop is also changing.

Classify: The changing flux induces an emf $\mathscr{E}$ in the loop according to Faraday's law, which we can write as $\mathscr{E} = d\Phi_B/dt$. To use that law, we need an expression for the flux Φ_B at any time t. However, because B is *not* uniform over the area enclosed by the loop, we *cannot* use $\Phi_B = BA$ to find that expression; instead we must use $\Phi_B = \int \vec{B}\cdot d\vec{A}$.

Compute: In Fig. 31-11, $\vec{B}$ is perpendicular to the plane of the loop (and hence parallel to the differential area vector $d\vec{A}$); so the dot product in $\Phi_B = \int \vec{B}\cdot d\vec{A}$ gives $B\,dA$. Because the magnetic field varies with the coordinate x but not with the coordinate y, we can take the differential area dA to be the area of a vertical strip of height H and width dx (as shown in Fig. 31-11). Then $dA = H\,dx$, and the flux through the loop is

$$\Phi_B = \int \vec{B}\cdot d\vec{A} = \int B\,dA = \int BH\,dx = \int 4t^2x^2H\,dx.$$

Treating t as a constant for this integration and inserting the integration limits $x = 0$ and $x = 3.0$ m, we obtain

$$\Phi_B = 4t^2 H \int_0^{3.0} x^2 dx = 4t^2 H \left[\frac{x^3}{3}\right]_0^{3.0} = 72t^2,$$

where we have substituted $H = 2.0$ m and Φ_B is in webers. Now we can use Faraday's law to find the magnitude of $\mathscr{E}$ at any time t:

$$\mathscr{E} = \frac{d\Phi_B}{dt} = \frac{d(72t^2)}{dt} = 144t,$$

in which $\mathscr{E}$ is in volts. At $t = 0.10$ s,

$$\mathscr{E} = (144 \text{ V/s})(0.10 \text{ s}) \approx 14 \text{ V}. \qquad \text{(Answer)}$$

Conclude: The flux of $\vec{B}$ through the loop is into the page in Fig. 31-11 and is increasing in magnitude because B is increasing in magnitude with time. By Lenz's law, the field B_i of the induced current opposes this increase and so is directed out of the page. The curled-straight right-hand rule in Fig. 31-7 then tells us that the induced current is counterclockwise around the loop, and thus so is the induced emf $\mathscr{E}$.

31-6 | Motional EMF

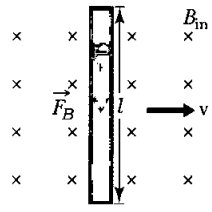


FIG. 31-12 A straight electrical conductor of length l moving with a velocity v through a uniform magnetic field B directed perpendicular to v. A potential difference $\Delta V = B\,l\,v$ appears between the ends of the conductor.

In Sample Problems 31-1, 31-3, and 31-4, we considered cases in which an emf is induced in a stationary circuit placed in a magnetic field when the field changes with time. In this section, we describe what is called motional emf, which is the emf induced in a conductor moving through a constant magnetic field. The straight conductor of length l shown in Fig. 31-12 is moving through a uniform magnetic field directed into the page.

For simplicity, we assume that the conductor is moving in a direction perpendicular to the field with constant velocity under the influence of some external agent. The electrons in the conductor experience a force

$$\vec{F} = q(\vec{v} \times \vec{B})$$

that is directed along the length l, perpendicular to both v and B (Eq. 29-2). Under the influence of this force, the electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. As a result of this charge separation, an electric field is produced inside the conductor. The charges accumulate at both ends until the downward magnetic force qvB is balanced by the upward electric force qE. At this point, electrons stop moving. The condition for equilibrium requires that

$$q[\vec{E} + (\vec{v} \times \vec{B})] = 0,$$

or in other words,

$$\vec{E} = -(\vec{v} \times \vec{B}).$$

The electric field produced in the conductor (once the electrons stop moving and E is constant) is related to the potential difference across the ends of the conductor according to the relationship

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}. \qquad (31\text{-}6)$$

Obviously, that end will be at a higher potential toward which $\vec{v} \times \vec{B}$ points. Thus, a potential difference is maintained between the ends of the conductor as long as the conductor continues to move through the magnetic field. This moving conductor can be deemed to be a battery of emf given by the Eq. 31-6. If the direction of the motion is reversed, the polarity of the equivalent battery is also reversed.

A more interesting situation occurs when the moving conductor is part of a closed conducting path. This situation is particularly useful for illustrating how a changing magnetic flux causes an induced current in a closed circuit. Consider a circuit consisting of a conducting bar of length l sliding along two fixed parallel conducting rails, as shown in Fig. 31-13*a*.

For simplicity, we assume that the bar has zero resistance and that the stationary part of the circuit has a resistance R. A uniform and constant magnetic field B is applied perpendicular to the plane of the circuit. As the bar is pulled to the right with a velocity v, free charges in the bar experience a magnetic force directed along the length of the bar. This force sets up an induced current because the charges are free to move in the closed conducting path.

Since the magnetic field is uniform here, and the velocity of the rod at any instant is perpendicular to the rod, we can simplify Eq. 31-6 as

$$\Delta V = Blv.$$

You can see for yourself that the flux through the loop is changing, and Faraday's law also predicts the same value of the induced emf. This rod will act as a source of emf, and the current will flow in the circuit. In Fig. 31-12, the rod was not the part of a closed loop, so there was no possibility of charge flowing continuously. The interesting part is that due to this flow of current, the rod will experience a force which will tend to retard its motion. Check for yourself that the force due to the magnetic field on the rod is indeed given by $\vec{F}_B$. We will explore this point more in the next section.

If the magnetic field is constant and the velocity of all the points of the rod is the same, the emf induced in the rod can be given by

$$\Delta V = (\vec{v} \times \vec{B}) \cdot \vec{l}.$$

If you recollect from your study in mathematics classes, this is scalar triple product, and all that matters in scalar triple product is the components of the vectors that are mutually perpendicular to each other. Thus, if we take the velocity of the rod as the reference direction, then

$$\Delta V = B_\perp l_\perp v,$$

where $B_\perp$ is the component of the magnetic field perpendicular to the velocity and $l_\perp$ is the component of the length perpendicular to the velocity.

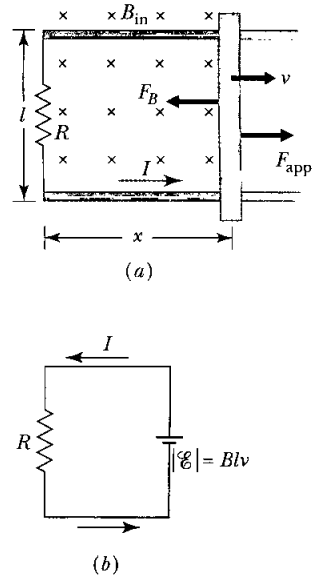


FIG. 31-13 (*a*) A conducting rod is moving on two rails under the influence of a force F_{app}. This induces an emf in the rod. Since the circuit is complete, it produces a current and the rod experiences a force F_B due to the magnetic field. (*b*) An equivalent circuit diagram of the situation. The rod is resistanceless, but if it had some resistance, it would form the internal resistance of the equivalent battery.

Sample Problem 31-5

A conducting rod of length l is rotated about its end with an angular velocity ω in a uniform magnetic field directed opposite to its axis of rotation. Find the emf induced in the rod.

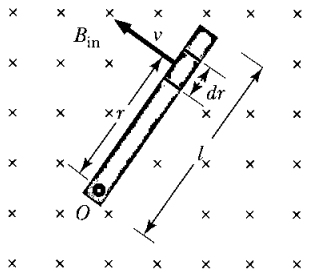


FIG. 31-14 A conducting rod rotating around one end in a uniform magnetic field that is perpendicular to the plane of rotation. A motional emf is induced across the ends of the rod.

Conceptualize: Although at a first glance, it seems that there should be no emf induced in the rod. After all, there is no flux associated with the rod. But when a conductor is moving across a magnetic field, a motional emf is induced.

Classify: We can use Eq. 31-6 here:

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}.$$

Compute: Consider a segment of the bar of length dr having a velocity v. According to Eq. 31-6, the magnitude of the emf induced in the entire rod is

$$\Delta V = \int (\vec{v} \times \vec{B}) \cdot d\vec{r} = \int B\omega r \, dr = \frac{B\omega l^2}{2}. \quad \text{(Answer)}$$

Conclude: Even in the situation shown in Fig. 31-15, there is a current in the external circuit. This device is popularly known as Faraday's disk. It is wrong to say that the flux through the disk is changing, but the conducting disk is moving in the magnetic field and hence there is a motional emf across the radius of the disk.

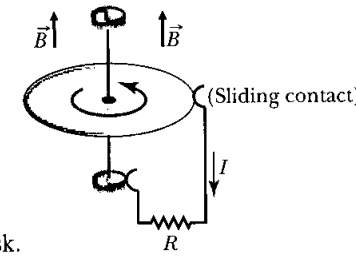


FIG. 31-15 Faraday's disk.

CHECKPOINT 3 If the rod were translating as well as rotating, what will be its emf? Assume that the center of mass has a velocity v and the rod is rotating with an angular velocity ω about its center of mass.

Sample Problem 31-6 Generator

A coil containing N conducting loops of area A is rotated at constant angular speed ω in a uniform magnetic field B, as shown in Fig. 31-16. What is the emf induced in the coil?

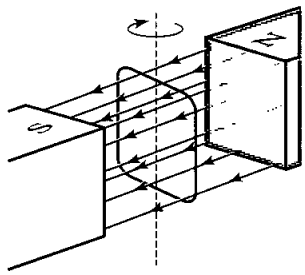


FIG. 31-16 A sinusoidal emf is induced in a conducting coil rotating in a magnetic field. The figure depicts the initial position of the coil in the magnetic field.

Conceptualize/Classify: Equation 31-3 tells us that the emf induced in the coil equals the rate of change of the magnetic flux through the coil with respect to time. So let's write an expression for the magnetic flux through the entire coil as a function of time, and then differentiate it to obtain the emf.

Compute: When the coil lies in the plane perpendicular to the magnetic field, the magnetic flux through the coil has its greatest possible value, NBA. As the coil rotates, the surface area vector associated with the coil is no longer perpendicular to the magnetic field, and thus changes with time. At any time, the angle between the surface area vector and the magnetic field is ωt, so the flux through the coil is

$$\phi = NBA\cos\omega t.$$

Therefore, the emf is

$$\text{emf} = -\frac{d}{dt}(NBA\cos\omega t) = NBA\omega\sin\omega t.$$

Conclude: Note that the emf starts at zero and then increases to a positive value as the enclosed flux decreases due to the rotation of the loop. The resulting induced emf oscillates sinusoidally in time and this emf can be used to establish an oscillating potential difference across two points. Oscillating sources of potential differences are usually referred to as "alternating" sources of potential difference. We will discuss these in more detail in Chapter 32.

31-7 | Induction and Energy Transfers

By Lenz's law, whether you move the magnet toward or away from the loop in Fig. 31-2, a magnetic force resists the motion, requiring your applied force to do positive work. At the same time, thermal energy is produced in the material of the loop because of the material's electrical resistance to the current that is induced by the motion. The energy you transfer to the closed *loop + magnet* system via your applied force ends up in this thermal energy. (For now, we neglect energy that is radiated away from the loop as electromagnetic waves during the induction.) The faster you move the magnet, the more rapidly your applied force does work and the greater the rate at which your energy is transferred to thermal energy in the loop; that is, the transfer of the power is greater.

Regardless of how current is induced in a loop, energy is always transferred to thermal energy during the process because of the electrical resistance of the loop (unless the loop is superconducting). For example, in Fig. 31-3, when switch S is closed and a current is briefly induced in the left-hand loop, energy is transferred from the battery to thermal energy in that loop.

Figure 31-17 shows another situation involving induced current. A rectangular loop of wire of width L has one end in a uniform external magnetic field that is directed perpendicularly into the plane of the loop. This field may be produced, for example, by a large electromagnet. The dashed lines in Fig. 31-17 show the assumed limits of the magnetic field; the fringing of the field at its edges is neglected. You are to pull this loop to the right at a constant velocity $\vec{v}$.

The situation of Fig. 31-17 does not differ in any essential way from that of Fig. 31-1. In each case a magnetic field and a conducting loop are in relative motion; in each case the flux of the field through the loop is changing with time. It is true that in Fig. 31-1 the flux is changing because $\vec{B}$ is changing and in Fig. 31-17 the flux is changing because the area of the loop still in the magnetic field is changing, but that difference is not important. The important difference between the two arrangements is that the arrangement of Fig. 31-17 makes calculations easier. Let us now calculate the rate at which you do mechanical work as you pull steadily on the loop in Fig. 31-17.

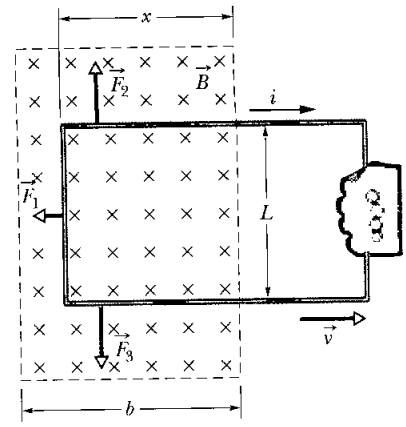


FIG. 31-17 You pull a closed conducting loop out of a magnetic field at constant velocity $\vec{v}$. While the loop is moving, a clockwise current i is induced in the loop, and the loop segments still within the magnetic field experience forces, $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$.

As you will see, to pull the loop at a constant velocity $\vec{v}$, you must apply a constant force $\vec{F}$ to the loop because a magnetic force of equal magnitude but opposite direction acts on the loop to oppose you. From $P = \vec{F} \times \vec{v}$, the rate at which you do work—that is, the power—is then

$$P = Fv, \tag{31-7}$$

where F is the magnitude of your force. We wish to find an expression for P in terms of the magnitude B of the magnetic field and the characteristics of the loop–namely, its resistance R to current and its dimension L.

As you move the loop to the right in Fig. 31-17, the portion of its area within the magnetic field decreases. Thus, the flux through the loop also decreases and, according to Faraday's law, a current is produced in the loop. It is the presence of this current that causes the force that opposes your pull.

To find the current, we first apply Faraday's law. When x is the length of the loop still in the magnetic field, the area of the loop still in the field is Lx. Then from $\Phi_B = BA$, the magnitude of the flux through the loop is

$$\Phi_B = BA = BLx. \tag{31-8}$$

As x decreases, the flux decreases. Faraday's law tells us that with this flux decrease, an emf is induced in the loop. Dropping the minus sign in Eq. 31-3 and using Eq. 31-8, we can write the magnitude of this emf as

$$\mathscr{E} = \frac{d\Phi_B}{dt} = \frac{d}{dt}BLx = BL\frac{dx}{dt} = BLv, \tag{31-9}$$

in which we have replaced dx/dt with v, the speed at which the loop moves.

Figure 31-18 shows the loop as a circuit: induced emf $\mathscr{E}$ is represented on the left, and the collective resistance R of the loop is represented on the right. The direction of the induced current i is obtained with a right-hand rule as in Fig. 31-7*b* for decreasing flux; applying the rule tells us that the current must be clockwise, and $\mathscr{E}$ must have the same direction.

To find the magnitude of the induced current, we cannot apply the loop rule for potential differences in a circuit because, as you will see in Section 31-8, we cannot define a potential difference for an induced emf. However, we can apply the equation $i = \mathscr{E}/R$, as we did in Sample Problem 31-2. With Eq. 31-9, this becomes

$$i = \frac{BLv}{R}. \tag{31-10}$$

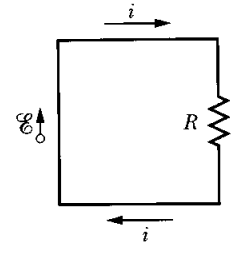


FIG. 31-18 A circuit diagram for the loop of Fig. 31-17 while the loop is moving.

Because three segments of the loop in Fig. 31-17 carry this current through the magnetic field, sideways deflecting forces act on those segments. From Eq. 29-29 we know that such a deflecting force is, in general notation,

$$\vec{F}_d = i\vec{L} \times \vec{B}. \tag{31-11}$$

In Fig. 31-17, the deflecting forces acting on the three segments of the loop are marked $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$. Note, however, that from the symmetry, forces $\vec{F}_2$ and $\vec{F}_3$ are equal in magnitude and cancel. This leaves only force $\vec{F}_1$, which is directed opposite your force $\vec{F}$ on the loop and thus is the force opposing you. So, $\vec{F} = -\vec{F}_1$.

Using Eq. 31-17 to obtain the magnitude of $\vec{F}_1$ and noting that the angle between $\vec{B}$ and the length vector $\vec{L}$ for the left segment is 90°, we write

$$F = F_1 = iLB \sin 90° = iLB. \tag{31-12}$$

Substituting Eq. 31-10 for i in Eq. 31-12 then gives us

$$F = \frac{B^2L^2v}{R}. \tag{31-13}$$

Because B, L, and R are constants, the speed v at which you move the loop is constant if the magnitude F of the force you apply to the loop is also constant.

By substituting Eq. 31-13 into Eq. 31-7, we find the rate at which you do work on the loop as you pull it from the magnetic field:

$$P = Fv = \frac{B^2L^2v^2}{R} \quad \text{(rate of doing work)}. \tag{31-14}$$

To complete our analysis, let us find the rate at which thermal energy appears in the loop as you pull it along at constant speed. We calculate it from Eq. 27-27,

$$P = i^2R. \tag{31-15}$$

Substituting for i from Eq. 31-10, we find

$$P = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v^2}{R} \quad \text{(thermal energy rate)}, \tag{31-16}$$

which is exactly equal to the rate at which you are doing work on the loop (Eq. 31-14). Thus, the work that you do in pulling the loop through the magnetic field appears as thermal energy in the loop.

This is a very elaborate topic and I can go on quoting examples after examples to show that the magnetic force does not do any work. If you recollect, we said in Chapter 28, Section 28-3, that the magnetic forces cannot change the speed of the charged particles and in that sense do not do any work on them. Here also we see that the magnetic forces do not do any work. All they do is transfer energy from mechanical to electrical one.

You might wonder that in Chapter 29, Section 29-4, we found that two current-carrying wires attract each other. Magnetic force is doing work after all. But if you look closely, note that the wires must be a part of a closed circuit to have a current. As the wires come close, the magnetic flux linked with the loops changes. So an induced emf is generated in the loop which opposes the change in flux. This kinetic energy of the wire is due to the work done by the battery maintaining the current in the wire.

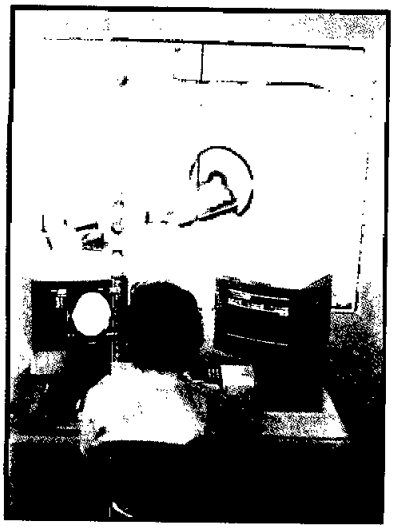


FIG. 31-19 A patient in an MRI apparatus. *(Michael Rosenfeld/Getty Images, Inc.)*

Burns During MRI Scans

A patient undergoing an MRI scan (Fig. 31-19) lies in an apparatus containing two magnetic fields: a large constant field $\vec{B}_{con}$ and a small sinusoidally varying

field $\vec{B}(t)$. Normally the scan requires the patient to lie motionless for a long time. Any patient unable to lie motionless, such as a child, say, is sedated. Because sedation, especially a general anesthetic, can be dangerous, a sedated patient must be carefully monitored, usually with a *pulse oximeter,* a device that measures the oxygen level in the patient's blood. This device includes a probe attached to one of the patient's fingers and a cable running from the probe to a monitor located outside the MRI apparatus.

MRI scans should be perfectly harmless to a patient. In a few cases, however, disregard of Faraday's law of induction led to a sedated patient receiving severe burns. In those cases, the oximeter cable was allowed to touch the patient's arm (Fig. 31-20). The cable and the lower part of the arm then formed a closed loop through which the varying magnetic field $\vec{B}(t)$ produced a varying flux. This flux variation induced an emf around the loop. Although the cable insulation and the skin both had high electrical resistance, the induced emf was large enough to drive a significant current around the loop. As with any other circuit in which there is resistance, the current transferred energy to thermal energy at the points of resistance. In this way, the finger and the skin where the cable touched the lower arm were burned. MRI staff are now trained to keep any monitor cable from touching a patient at more than one point.

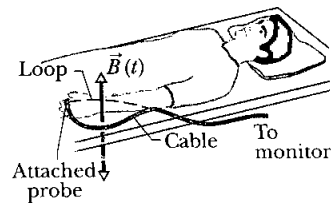


FIG. 31-20 A probe attached to a finger of a patient undergoing an MRI scan, in which a vertical magnetic field $\vec{B}(t)$ varies sinusoidally. The probe cable touches the patient's skin along the arm, and the cable and the lower part of the arm form a closed loop.

Eddy Currents

Suppose we replace the conducting loop of Fig. 31-17 with a solid conducting plate. If we then move the plate out of the magnetic field as we did the loop (Fig. 31-21*a*), the relative motion of the field and the conductor again induces a current in the conductor. Thus, we again encounter an opposing force and must do work because of the induced current. With the plate, however, the conduction electrons making up the induced current do not follow one path as they do with the loop. Instead, the electrons swirl about within the plate as if they were caught in an eddy (whirlpool) of water. Such a current is called an *eddy current* and can be represented, as it is in Fig. 31-21*a*, *as if* it followed a single path.

As with the conducting loop of Fig. 31-17, the current induced in the plate results in mechanical energy being dissipated as thermal energy. The dissipation is more apparent in the arrangement of Fig. 31-21*b*; a conducting plate, free to rotate about a pivot, is allowed to swing down through a magnetic field like a pendulum. Each time the plate enters and leaves the field, a portion of its mechanical energy is transferred to its thermal energy. After several swings, no mechanical energy remains and the warmed-up plate just hangs from its pivot.

Similar dissipation of energy is the basis of an induction furnace. Figure 31-22 shows the basic design: Metal is held within a crucible around which insulated wires are wrapped. The (AC) current in the wires alternates in direction and magnitude. Thus, the magnetic field due to the current continuously varies in direction and magnitude. This changing field $\vec{B}(t)$ creates eddy currents within the metal, and electrical energy is dissipated as thermal energy at the rate given by Eq. 31-15 ($P = i^2R$). The dissipation increases the temperature of the metal to the melting point, and then the molten metal can be poured.

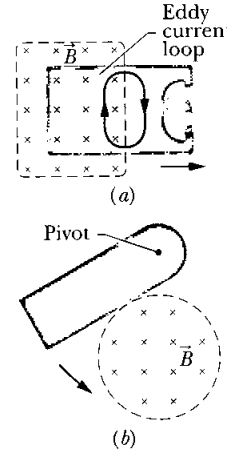


FIG. 31-21 (*a*) As you pull a solid conducting plate out of a magnetic field, *eddy currents* are induced in the plate. A typical loop of eddy current is shown. (*b*) A conducting plate is allowed to swing like a pendulum about a pivot and into a region of magnetic field. As it enters and leaves the field, eddy currents are induced in the plate.

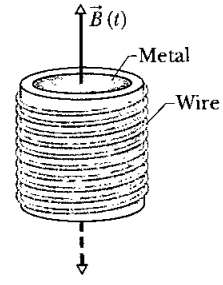


FIG. 31-22 An induction furnace.

Sample Problem 31-7

Assume that two conducting wires with negligible resistance are laid parallel to each other, as shown in Fig. 31-23, to make a smooth inclined plane of inclination θ with the horizontal. A conducting rod of mass m and negligible resistance is kept on the wires. A uniform magnetic field is directed in vertically upward direction. At the upper end, the wires are connected by a resistance R. Find the steady-state velocity of the rod.

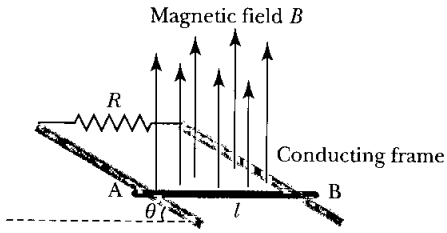


FIG. 31-23 A conducting rod is released from rest on a smooth conducting frame inclined at an angle θ to the horizontal.

Conceptualize: As the wire slides downward, its velocity increases under the influence of gravitational force. But the flux passing through the loop also increases. Hence, an emf is induced in the loop which opposes the change in flux, or in other words, opposes the motion of the rod. Looking from the perspective of motional emf, an emf is induced in the rod making the end A at higher potential. A current will flow in the loop in the clockwise direction as seen from above. Due to this current, the rod experiences a force due to the magnetic field. A component of this force opposes the motion of the rod. In steady state, the force due to the magnetic field will balance the force due to gravity.

Classify: We have two possible approaches to this problem.

(a) As stated just now, we can apply Newton's laws to find the acceleration of the rod. In steady state, its acceleration will be zero.

(b) We know that magnetic field does not do any work on the system. The gravitational force does work on the rod, increasing its kinetic energy, whereas the resistance of the resistor at the other end dissipates the energy. In steady state, when the kinetic energy becomes constant, the rate at which gravity is doing work will be balanced by the rate at which energy will be dissipated in the resistance.

Compute: **(a) First method:** Let us assume that the rod has acquired a velocity v. By applying the concept of motional emf (Eq. 31-6), the emf of the rod is

$$\Delta V = (\vec{v}\times\vec{B})\cdot l = Blv\cos\theta.$$

This emf will be such that the current is flowing toward the observer as in Fig. 31-24. The resulting current at that instant is

$$i = \frac{\Delta V}{R} = \frac{Blv\cos\theta}{R}.$$

Applying Eq. 29-29, we get

$$|\vec{F}| = |i\vec{l}\times\vec{B}| = \frac{B^2l^2v\cos\theta}{R}.$$

This force is directed horizontally. A component of this force will oppose the motion of the rod as can be seen in Fig. 31-23.

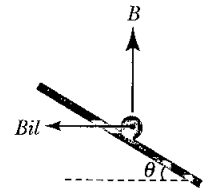


FIG. 31-24 A figure showing the direction of current flow and the force on the rod.

Hence the acceleration of the rod is given by

$$mg\sin\theta - \frac{B^2l^2v\cos^2\theta}{R} = ma.$$

It can be easily seen that the acceleration of the rod decreases as its velocity increases. After a long time, the acceleration of the rod will tend to zero. The velocity of the rod at this instant is known as the terminal velocity:

$$v_T = \frac{mgR\sin\theta}{B^2l^2\cos^2\theta}.$$

Alternative method:
Let us find the rate of work done by gravity when the velocity of the rod is v.

We know that the power developed by gravity is

$$P = \vec{F}\cdot\vec{v} = mgv\sin\theta.$$

The rate at which power will be dissipated is

$$P_d = \frac{(\Delta V)^2}{R} = \frac{(Blv\cos\theta)^2}{R}.$$

In steady state,

$$mgv\sin\theta = \frac{(Blv\cos\theta)^2}{R}$$

$$v_T = \frac{mgR\sin\theta}{B^2l^2\cos^2\theta},$$

which is the same as before.

Conclude: We can also find the velocity of the rod as a function of time. Put

$$a = \frac{dv}{dt}.$$

We get

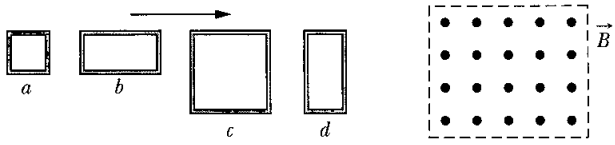
$$\frac{dv}{dt} = g\sin\theta\left(1-\frac{v}{v_T}\right).$$

By integrating, we get

$$v = v_T(1-e^{-g\sin\theta t/v_T})$$

The equation can be easily checked. By substituting $t = 0$, the velocity turns out to be zero, as expected. At $t = \infty$, the velocity turns out to be v_T.

CHECKPOINT 4 The figure shows four wire loops, with edge lengths of either L or $2L$. All four loops will move through a region of uniform magnetic field $\vec{B}$ (directed out of the page) at the same constant velocity. Rank the four loops according to the maximum magnitude of the emf induced as they move through the field, greatest first.



31-8 | Induced Electric Fields

Till now, we understood one reason of induced emf: a conducting body moving in a magnetic field. Now we consider another mechanism by which induced emf can be produced. To understand this, let us place a copper ring of radius r in a uniform external magnetic field, as in Fig. 31-25*a*. The field—neglecting fringing—fills a cylindrical volume of radius R. Suppose that we increase the strength of this field at a steady rate, perhaps by increasing—in an appropriate way—the current in the windings of the electromagnet that produces the field. The magnetic flux through the ring will then change at a steady rate and—by Faraday's law—an induced emf and thus an induced current will appear in the ring. From Lenz's law we can deduce that the direction of the induced current is counterclockwise in Fig. 31-25*a*.

If there is a current in the copper ring, an electric field must be present along the ring because an electric field is needed to do the work of moving the conduction electrons. Moreover, the electric field must have been produced by the changing magnetic flux. This **induced electric field** $\vec{E}$ is just as real as an electric field produced by static charges; either field will exert a force $q_0\vec{E}$ on a particle of charge q_0.

By this line of reasoning, we are led to a useful and informative restatement of Faraday's law of induction:

A changing magnetic field produces an electric field.

The striking feature of this statement is that the electric field is induced even if there is no copper ring.

To fix these ideas, consider Fig. 31-25*b*, which is just like Fig. 31-25*a* except the copper ring has been replaced by a hypothetical circular path of radius r. We assume, as previously, that the magnetic field $\vec{B}$ is increasing in magnitude at a constant rate dB/dt. The electric field induced at various points around the circular path must—from the symmetry—be tangent to the circle, as Fig. 31-25*b* shows.* Hence, the circular path is an electric field line. There is nothing special about the circle of radius r, so the electric field lines produced by the changing magnetic field must be a set of concentric circles, as in Fig. 31-25*c*.

As long as the magnetic field is *increasing* with time, the electric field represented by the circular field lines in Fig. 31-25*c* will be present. If the magnetic field remains *constant* with time, there will be no induced electric field and thus no electric field lines. If the magnetic field is *decreasing* with time (at a constant rate), the electric field lines will still be concentric circles as in Fig. 31-25*c*, but they will

*Arguments of symmetry show that there are three possible directions for the induced electric field: Radial, tangential and axial. But if the electric field were axial or radial, the emf $\mathcal{E} = -\oint \vec{E} \cdot d\vec{s}$ around the loop would be zero as the electric field at all the points would be perpendicular to the loop. The only possible direction of the induced electric field that would produce a nonzero emf around the loop is tangential.

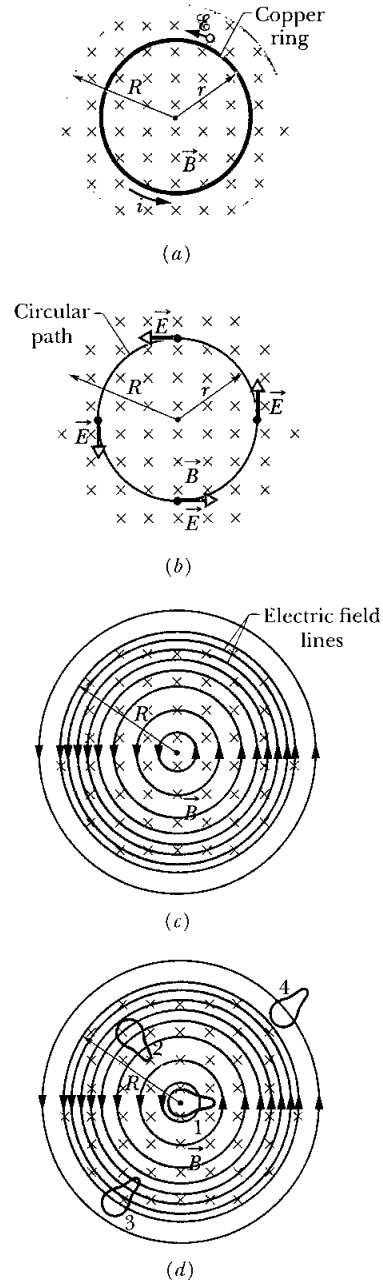


FIG. 31-25 (*a*) If the magnetic field increases at a steady rate, a constant induced current appears, as shown, in the copper ring of radius r. (*b*) An induced electric field exists even when the ring is removed; the electric field is shown at four points. (*c*) The complete picture of the induced electric field, displayed as field lines. (*d*) Four similar closed paths that enclose identical areas. Equal emfs are induced around paths 1 and 2, which lie entirely within the region of changing magnetic field. A smaller emf is induced around path 3, which only partially lies in that region. No net emf is induced around path 4, which lies entirely outside the magnetic field.

now have the opposite direction. All this is what we have in mind when we say "A changing magnetic field produces an electric field."

A Reformulation of Faraday's Law

Consider a particle of charge q_0 moving around the circular path of Fig. 31-25*b*. The work W done on it in one revolution by the induced electric field is $W = \mathscr{E}q_0$, where $\mathscr{E}$ is the induced emf—that is, the work done per unit charge in moving the test charge around the path. From another point of view, the work is

$$W = \int \vec{F} \cdot d\vec{s} = (q_0 E)(2\pi r), \tag{31-17}$$

where q_0E is the magnitude of the force acting on the test charge and $2\pi r$ is the distance over which that force acts. Setting these two expressions for W equal to each other and canceling q_0, we find that

$$\mathscr{E} = 2\pi r E. \tag{31-18}$$

Next we rewrite Eq. 31-17 to give a more general expression for the work done on a particle of charge q_0 moving along any closed path:

$$W = \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s}. \tag{31-19}$$

(The loop on each integral sign indicates that the integral is to be taken around the closed path.) Substituting $\mathscr{E}q_0$ for W, we find that

$$\mathscr{E} = \oint \vec{E} \cdot d\vec{s}. \tag{31-20}$$

This integral reduces at once to Eq. 31-18 if we evaluate it for the special case of Fig. 31-25*b*.

With Eq. 31-20, we can expand the meaning of induced emf. Up to this point, induced emf has meant the work per unit charge done in maintaining current due to a changing magnetic flux, or it has meant the work done per unit charge on a charged particle that moves around a closed path in a changing magnetic flux. However, with Fig. 31-25*b* and Eq. 31-20, an induced emf can exist without the need of a current or particle: An induced emf is the sum–via integration–of quantities $\vec{E} \cdot d\vec{s}$ around a closed path, where $\vec{E}$ is the electric field induced by a changing magnetic flux and $d\vec{s}$ is a differential length vector along the path.

If we combine Eq. 31-20 with Faraday's law in Eq. 31-3 ($\mathscr{E} = -d\Phi_B/dt$), we can rewrite Faraday's law as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law).} \tag{31-21}$$

This equation says simply that a changing magnetic field induces an electric field. The changing magnetic field appears on the right side of this equation, the electric field on the left.

Faraday's law in the form of Eq. 31-21 can be applied to *any* closed path that can be drawn in a changing magnetic field. Figure 31-25*d*, for example, shows four such paths, all having the same shape and area but located in different positions in the changing field. The induced emfs $\mathscr{E}(= \oint \vec{E} \cdot d\vec{s})$ for paths 1 and 2 are equal because these paths lie entirely in the magnetic field and thus have the same value of $d\Phi_B/dt$. This is true even though the electric field vectors at points along these paths are different, as indicated by the patterns of electric field lines in the figure. For path 3 the induced emf is smaller because the enclosed flux Φ_B (hence $d\Phi_B/dt$) is smaller, and for path 4 the induced emf is zero even though the electric field is not zero at any point on the path.

A New Look at Electric Potential

Induced electric fields are produced not by static charges but by a changing magnetic flux. Although electric fields produced in either way exert forces on charged particles, there is an important difference between them. The simplest evidence of this difference is that the field lines of induced electric fields form closed loops, as in Fig. 31-25*c*. Field lines produced by static charges never do so but must start on positive charges and end on negative charges.

In a more formal sense, we can state the difference between electric fields produced by induction and those produced by static charges in these words:

☞ Electric potential has meaning only for electric fields that are produced by static charges; it has no meaning for electric fields that are produced by induction.

You can understand this statement qualitatively by considering what happens to a charged particle that makes a single journey around the circular path in Fig. 31-25*b*. It starts at a certain point and, on its return to that same point, has experienced an emf $\mathscr{E}$ of, let us say, 5 V; that is, work of 5 J/C has been done on the particle, and thus the particle should then be at a point that is 5 V greater in potential. However, that is impossible because the particle is back at the same point, which cannot have two different values of potential. Thus, potential has no meaning for electric fields that are set up by changing magnetic fields.

We can take a more formal look by recalling Eq. 24-17, which defines the potential difference between two points i and f in an electric field $\vec{E}$:

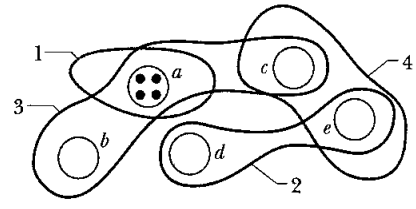
$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}. \tag{31-22}$$

In Chapter 24 we had not yet encountered Faraday's law of induction; so the electric fields involved in the derivation of Eq. 24-17 were those due to static charges. If i and f in Eq. 31-22 are the same point, the path connecting them is a closed loop, V_i and V_f are identical, and Eq. 31-22 reduces to

$$\oint \vec{E} \cdot d\vec{s} = 0. \tag{31-23}$$

However, when a changing magnetic flux is present, this integral is *not* zero but is $-d\Phi_B/dt$, as Eq. 31-22 asserts. Thus, assigning electric potential to an induced electric field leads us to a contradiction. We must conclude that electric potential has no meaning for electric fields associated with induction.

✓CHECKPOINT 5 The figure shows five lettered regions in which a uniform magnetic field extends either directly out of the page or into the page, with the direction indicated only for region *a*. The field is increasing in magnitude at the same steady rate in all five regions; the regions are identical in area. Also shown are four numbered paths along which $\oint \vec{E} \cdot d\vec{s}$ has the magnitudes given below in terms of a quantity "mag." Determine whether the magnetic field is directed into or out of the page for regions *b* through *e*.



Path	1	2	3	4
$\oint \vec{E} \cdot d\vec{s}$	mag	2(mag)	3(mag)	0

Sample Problem 31-8

In Fig. 31-25*b*, take $R = 8.5$ cm and $dB/dt = 0.13$ T/s.

(a) Find an expression for the magnitude E of the induced electric field at points within the magnetic field, at radius r from the center of the magnetic field. Evaluate the expression for $r = 5.2$ cm.

Conceptualize/Classify: An electric field is induced by the changing magnetic field, according to Faraday's law.

Compute: To calculate the field magnitude E, we apply Faraday's law in the form of Eq. 31-21. We use a circular

path of integration with radius $r \leq R$ because we want E for points within the magnetic field. We assume from the symmetry that $\vec{E}$ in Fig. 31-25*b* is tangent to the circular path at all points. The path vector $d\vec{s}$ is also always tangent to the circular path; so the dot product $\vec{E} \cdot d\vec{s}$ in Eq. 31-21 must have the magnitude $E\,ds$ at all points on the path. We can also assume from the symmetry that E has the same value at all points along the circular path. Then the left side of Eq. 31-21 becomes

$$\oint \vec{E} \cdot d\vec{s} = \oint E\,ds = E \oint ds = E(2\pi r). \qquad (31\text{-}24)$$

(The integral $\oint ds$ is the circumference $2\pi r$ of the circular path.)

Next, we need to evaluate the right side of Eq. 31-21. Because $\vec{B}$ is uniform over the area A encircled by the path of integration and is directed perpendicular to that area, the magnetic flux is given by $\Phi_B = BA$:

$$\Phi_B = BA = B(\pi r^2). \qquad (31\text{-}25)$$

Substituting this and Eq. 31-24 into Eq. 31-21 and dropping the minus sign, we find that

$$E(2\pi r) = (\pi r^2)\frac{dB}{dt}$$

or

$$E = \frac{r}{2}\frac{dB}{dt}. \qquad \text{(Answer)} \quad (31\text{-}26)$$

Equation 31-26 gives the magnitude of the electric field at any point for which $r \leq R$ (that is, within the magnetic field). Substituting given values yields, for the magnitude of $\vec{E}$ at $r = 5.2$ cm,

$$E = \frac{5.2 \times 10^{-2}\ \text{m}}{2}(0.13\ \text{T/s})$$
$$= 0.0034\ \text{V/m} = 3.4\ \text{mV/m}. \qquad \text{(Answer)}$$

(b) Find an expression for the magnitude E of the induced electric field at points that are outside the magnetic field, at radius r. Evaluate the expression for $r = 12.5$ cm.

Conceptualize/Classify: The idea of part (a) applies here also, except that we use a circular path of integration with radius $r \geq R$ because we want to evaluate E for points outside the magnetic field. Proceeding as in (a), we again obtain Eq. 31-24. However, we do not then obtain Eq. 31-25 because the new path of integration is now outside the magnetic field, and so the magnetic flux encircled by the new path is only that in the area πR^2 of the magnetic field region.

Compute: We can now write

$$\Phi_B = BA = B(\pi R^2). \qquad (31\text{-}27)$$

Substituting this and Eq. 31-24 into Eq. 31-21 (without the minus sign) and solving for E yield

$$E = \frac{R^2}{2r}\frac{dB}{dt}. \qquad \text{(Answer)} \quad (31\text{-}28)$$

Because E is not zero here, we know that an electric field is induced even at points that are outside the changing magnetic field, an important result that (as you will see in Section 31-12, subsection "Energy Density of a Magnetic Field") makes transformers possible.

With the given data, Eq. 31-28 yields the magnitude of $\vec{E}$ at $r = 12.5$ cm:

$$E = \frac{(8.5 \times 10^{-2}\ \text{m})^2}{(2)(12.5 \times 10^{-2}\ \text{m})}(0.13\ \text{T/S})$$
$$= 3.8 \times 10^{-3}\ \text{V/m} = 3.8\ \text{mV/m}. \qquad \text{(Answer)}$$

Conclude: Equations 31-26 and 31-27 give the same result for $r = R$. Figure 31-26 shows a plot of $E(r)$.

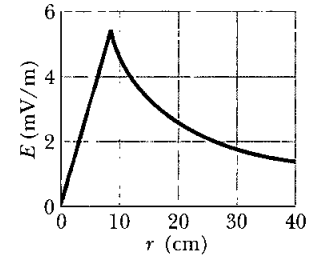


FIG. 31-26 A plot of the induced electric field $E(r)$ for the conditions of Sample Problem 31-8.

CHECKPOINT 6 A uniform copper ring is threaded by a steadily increasing magnetic field so that the magnetic flux through the ring is changing at a constant rate. Use a voltmeter to measure the voltage across the metal ring. What do you predict?

What I would like to emphasize here is that Faraday's law just sums up the two reasons for induced emf. It is an experimental law formulated by Faraday which was found to be true by him in all the situations that he experimented upon. But it is not a universal principle, nor is the changing flux the cause of induced emf. The cause is motion of a conductor, or induced electric field. We saw an example in conclusion to Sample Problem 31-5 where the Faraday's flux law is violated. Let us see another example which brings out the point clearly.

The circuit consists of a pair of spring clips and moves across the body of a magnet (see Fig. 31-27*a*). When the loop escapes the magnet, the clips rub over

the magnet and the body of the magnet becomes a part of the circuit. The use of Faraday's law in the form of Eq. 31-3 is misleading and predicts a nonzero emf, because the magnetic flux through the loop decreases. However, no emf is created. The magnetic field B does not change in time and no charge carrier moves through the magnetic field. The segment of the magnet closing the circuit, which changes at each instant, is at rest relative to the magnet. Note that if the loop were pulled out of the magnet between the magnetic poles (Fig. 31-27*b*), a motional emf would be created in the loop.

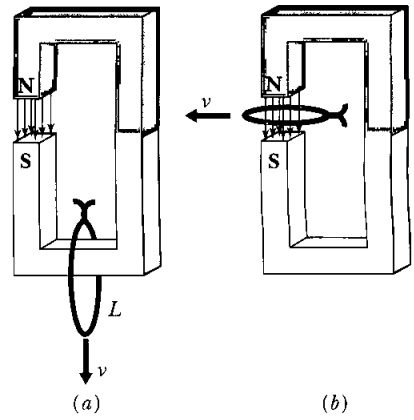


FIG. 31-27 (*a*) Clips allow a conducting loop L to escape the magnet. During the escape, the body of magnet (a conductor) becomes a segment of the closed circuit. Although the flux of the magnetic field through the circuit L decreases, no induction emf is created. (*b*) The conducting loop L escapes the magnet between the magnet's poles. Induced emf is created in the loop.

31-9 | Inductors and Inductance

We found in Chapter 26 that a capacitor can be used to produce a desired electric field. We considered the parallel-plate arrangement as a basic type of capacitor. Similarly, an **inductor** (symbol ⏦) can be used to produce a desired magnetic field. We shall consider a long solenoid (more specifically, a short length near the middle of a long solenoid) as our basic type of inductor.

If we establish a current i in the windings (turns) of the solenoid we are taking as our inductor, the current produces a magnetic flux Φ_B through the central region of the inductor. The **inductance** of the inductor is then

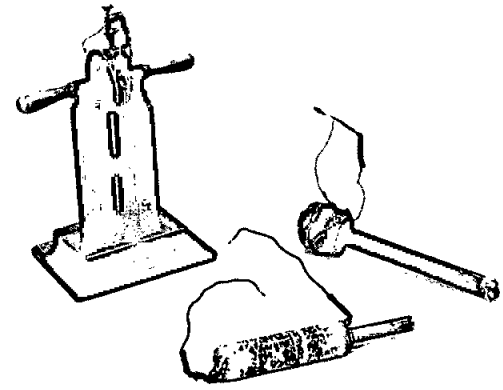
$$L = \frac{N\Phi_B}{i} \qquad \text{(inductance defined)}, \tag{31-29}$$

in which N is the number of turns. The windings of the inductor are said to be *linked* by the shared flux, and the product $N\Phi_B$ is called the *magnetic flux linkage*. The inductance L is thus a measure of the flux linkage produced by the inductor per unit of current.

Because the SI unit of magnetic flux is the tesla—square meter, the SI unit of inductance is the tesla—square meter per ampere ($T \cdot m^2/A$). We call this the **henry** (H), after American physicist Joseph Henry, the codiscoverer of the law of induction and a contemporary of Faraday. Thus,

$$1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}. \tag{31-30}$$

Through the rest of this chapter we assume that all inductors, no matter what their geometric arrangement, have no magnetic materials such as iron in their vicinity. Such materials would distort the magnetic field of an inductor.



The crude inductors with which Michael Faraday discovered the law of induction. In those days amenities such as insulated wire were not commercially available. It is said that Faraday insulated his wires by wrapping them with strips cut from one of his wife's petticoats. *(The Royal Institution/Bridgeman Art Library/NY)*

Inductance of a Solenoid

Consider a long solenoid of cross-sectional area A. What is the inductance per unit length near its middle?

To use the defining equation for inductance (Eq. 31-29), we must calculate the flux linkage set up by a given current in the solenoid windings. Consider a length l near the middle of this solenoid. The flux linkage for this section of the solenoid is

$$N\Phi_B = (nl)(BA),$$

in which n is the number of turns per unit length of the solenoid and B is the magnitude of the magnetic field within the solenoid.

The magnitude B is given by Eq. 30-38,

$$B = \mu_0 in,$$

and so from Eq. 31-29,

$$L = \frac{N\Phi_B}{i} = \frac{(nl)(BA)}{i} = \frac{(nl)(\mu_0 in)(A)}{i}$$
$$= \mu_0 n^2 lA. \tag{31-31}$$

Thus, the inductance per unit length near the center of a long solenoid is

$$\frac{L}{l} = \mu_0 n^2 A \qquad \text{(solenoid)}. \tag{31-32}$$

Inductance—like capacitance—depends only on the geometry of the device. The dependence on the square of the number of turns per unit length is to be expected. If you, say, triple n, you not only triple the number of turns (N) but you also triple the flux ($\Phi_B = BA = \mu_0 inA$) through each turn, multiplying the flux linkage $N\Phi_B$ and thus the inductance L by a factor of 9.

If the solenoid is very much longer than its radius, then Eq. 31-31 gives its inductance to a good approximation. This approximation neglects the spreading of the magnetic field lines near the ends of the solenoid, just as the parallel-plate capacitor formula ($C = \varepsilon_0 A/d$) neglects the fringing of the electric field lines near the edges of the capacitor plates.

From Eq. 31-31, and recalling that n is a number per unit length, we can see that an inductance can be written as a product of the permeability constant μ_0 and a quantity with the dimensions of a length. This means that μ_0 can be expressed in the unit henry per meter:

$$\begin{aligned}\mu_0 &= 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A} \\ &= 4\pi \times 10^{-7}\ \text{H/m}.\end{aligned} \tag{31-33}$$

This inductance can be increased substantially by inserting an iron rod inside the solenoid. For a given current, the magnetic flux is now much greater because of the increase in the field originating from the magnetization of the ferromagnetic material. For example, if the material has a magnetic permeability of $500\mu_0$, the inductance would increase by a factor of 500.

31-10 | Self-Induction

If two coils—which we can now call inductors—are near each other, a current i in one coil produces a magnetic flux Φ_B through the second coil. We have seen that if we change this flux by changing the current, an induced emf appears in the second coil according to Faraday's law. An induced emf appears in the first coil as well.

☞ An induced emf $\mathscr{E}_L$ appears in any coil in which the current is changing.

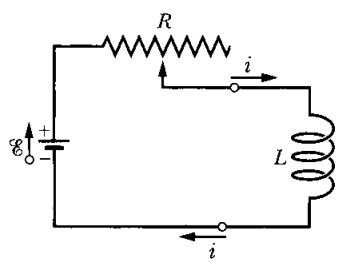


FIG. 31-28 If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced emf $\mathscr{E}_L$ will appear in the coil *while the current is changing.*

This process (see Fig. 31-28) is called **self-induction**, and the emf that appears is called a **self-induced emf**. It obeys Faraday's law of induction just as other induced emfs do.

For any inductor, Eq. 31-29 tells us that

$$N\Phi_B = Li. \tag{31-34}$$

Faraday's law tells us that

$$\mathscr{E}_L = -\frac{d(N\Phi_B)}{dt}. \tag{31-35}$$

By combining Eqs. 31-34 and 31-35 we can write

$$\mathscr{E}_L = -L\frac{di}{dt} \qquad \text{(self-induced emf)}. \tag{31-36}$$

Thus, in any inductor (such as a coil, a solenoid, or a toroid) a self-induced emf appears whenever the current changes with time. The magnitude of the current

has no influence on the magnitude of the induced emf; only the rate of change of the current counts.

You can find the *direction* of a self-induced emf from Lenz's law. The minus sign in Eq. 31-36 indicates that—as the law states—the self-induced emf $\mathscr{E}_L$ has the orientation such that it opposes the change in current i. We can drop the minus sign when we want only the magnitude of $\mathscr{E}_L$.

Suppose that, as in Fig. 31-29*a*, you set up a current i in a coil and arrange to have the current increase with time at a rate di/dt. In the language of Lenz's law, this increase in the current is the "change" that the self-induction must oppose. For such opposition to occur, a self-induced emf must appear in the coil, pointing—as the figure shows—so as to oppose the increase in the current. If you cause the current to decrease with time, as in Fig. 31-29*b*, the self-induced emf must point in a direction that tends to oppose the decrease in the current, as the figure shows.

In Section 31-8 we saw that we cannot define an electric potential for an electric field (and thus for an emf) that is induced by a changing magnetic flux. This means that when a self-induced emf is produced in the inductor of Fig. 31-28, we cannot define an electric potential within the inductor itself, where the flux is changing. However, potentials can still be defined at points of the circuit that are not within the inductor—points where the electric fields are due to charge distributions and their associated electric potentials.

Moreover, we can define a self-induced potential difference V_L *across an inductor* (between its terminals, which we assume to be outside the region of changing flux). For an *ideal inductor* (its wire has negligible resistance), the magnitude of V_L is equal to the magnitude of the self-induced emf $\mathscr{E}_L$.

If, instead, the wire in the inductor has resistance r, we mentally separate the inductor into a resistance r (which we take to be outside the region of changing flux) and an ideal inductor of self-induced emf $\mathscr{E}_L$. As with a real battery of emf $\mathscr{E}$ and internal resistance r, the potential difference across the terminals of a real inductor then differs from the emf. Unless otherwise indicated, we assume here that inductors are ideal.

Let us now find the equivalent inductance for two inductors when placed in (a) series as in Fig. 31-30*a* and (b) parallel as in Fig. 31-30*b*. Let us assume that the inductors are well separated so that their effect on each other can be ignored.

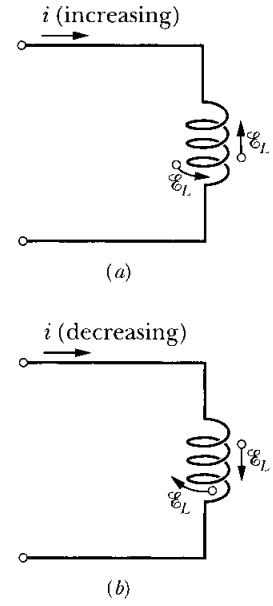


FIG. 31-29 (*a*) The current i is increasing, and the self-induced emf $\mathscr{E}_L$ appears along the coil in a direction such that it opposes the increase. The arrow representing $\mathscr{E}_L$ can be drawn along a turn of the coil or alongside the coil. Both are shown. (*b*) The current i is decreasing, and the self-induced emf appears in a direction such that it opposes the decrease.

1. From Fig. 31-30*a*, since the inductors are in series, the current through them will be the same. Hence the total induced emf across both of them will be a sum of the induced emf across the individual inductors. We have already stated that the inductors are well separated so that the current in one does not affect the magnetic field and hence the induced emf of the other:

$$E_{ab} = -L_1 \frac{di}{dt} - L_2 \frac{di}{dt}.$$

For equivalent inductance,

$$E_{ab} = -L \frac{di}{dt}.$$

Thus $L = L_1 + L_2$.

2. From Fig. 31-30*b*, the current in each of the inductors will not be the same. But since the inductors are placed in parallel, the induced emf across both of them will be equal:

$$E_{ab} = -L_1 \frac{di}{dt} = -L_2 \frac{di}{dt}.$$

By Kirchhoff's current law, we get

$$i = i_1 + i_2.$$

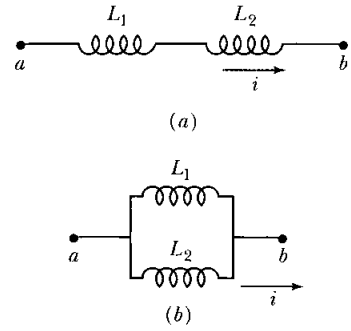


FIG. 31-30 (*a*) Inductance in series; (*b*) inductance in parallel.

By differentiating this expression, we get

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}.$$

For equivalent inductance:

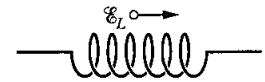
$$E_{ab} = -L\frac{di}{dt}.$$

Then

$$\frac{E_{ab}}{L} = \frac{E_{ab}}{L_1} + \frac{E_{ab}}{L_2} \quad \text{or} \quad \frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$

Thus, inductors in series and parallel add like resistance, as long as their mutual inductance can be ignored.

CHECKPOINT 7 The figure shows an emf $\mathscr{E}_L$ induced in a coil. Which of the following can describe the current through the coil: (a) constant and rightward, (b) constant and leftward, (c) increasing and rightward, (d) decreasing and rightward, (e) increasing and leftward, (f) decreasing and leftward?



31-11 | *RL* Circuits

In Section 28-6 we saw that if we suddenly introduce an emf $\mathscr{E}$ into a single-loop circuit containing a resistor R and a capacitor C, the charge on the capacitor does not build up immediately to its final equilibrium value $C\mathscr{E}$ but approaches it in an exponential fashion:

$$q = C\mathscr{E}(1 - e^{-t/\tau_C}). \tag{31-37}$$

The rate at which the charge builds up is determined by the capacitive time constant τ_C, defined in Eq. 28-46 as

$$\tau_C = RC. \tag{31-38}$$

If we suddenly remove the emf from this same circuit, the charge does not immediately fall to zero but approaches zero in an exponential fashion:

$$q = q_0 e^{-t/\tau_C}. \tag{31-39}$$

The time constant τ_C describes the fall of the charge as well as its rise.

An analogous slowing of the rise (or fall) of the current occurs if we introduce an emf $\mathscr{E}$ into (or remove it from) a single-loop circuit containing a resistor R and an inductor L. When the switch S in Fig. 31-31 is closed on a, for example, the current in the resistor starts to rise. If the inductor were not present, the current would rise rapidly to a steady value $\mathscr{E}/R$. Because of the inductor, however, a self-induced emf $\mathscr{E}_L$ appears in the circuit; from Lenz's law, this emf opposes the rise of the current, which means that it opposes the battery emf $\mathscr{E}$ in polarity. Thus, the current in the resistor responds to the difference between two emfs, a constant $\mathscr{E}$ due to the battery and a variable $\mathscr{E}_L$ ($= -L\,di/dt$) due to self-induction. As long as $\mathscr{E}_L$ is present, the current will be less than $\mathscr{E}/R$.

As time goes on, the rate at which the current increases becomes less rapid and the magnitude of the self-induced emf, which is proportional to di/dt, be-comes smaller. Thus, the current in the circuit approaches $\mathscr{E}/R$ asymptotically.

We can generalize these results as follows:

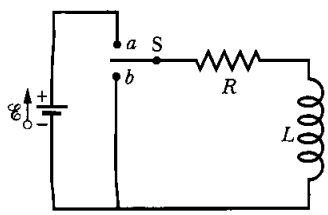


FIG. 31-31 An *RL* circuit. When switch S is closed on a, the current rises and approaches a limiting value $\mathscr{E}/R$.

Initially, an inductor acts to oppose changes in the current through it. A long time later, it acts like ordinary connecting wire.

Now let us analyze the situation quantitatively. With the switch S in Fig. 31-31 thrown to *a*, the circuit is equivalent to that of Fig. 31-32. Let us apply the loop rule, starting at point *x* in this figure and moving clockwise around the loop along with current *i*.

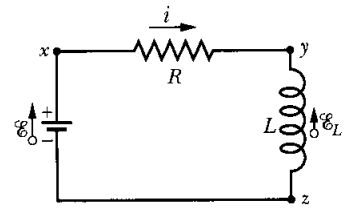


FIG. 31-32 The circuit of Fig. 31-31 with the switch closed on *a*. We apply the loop rule for circuits clockwise, starting at *x*.

1. *Resistor.* Because we move through the resistor in the direction of current *i*, the electric potential decreases by *iR*. Thus, as we move from point *x* to point *y*, we encounter a potential change of $-iR$.
2. *Inductor.* Because current *i* is changing, there is a self-induced emf $\mathscr{E}_L$ in the inductor. The magnitude of $\mathscr{E}_L$ is given by Eq. 31-36 as $L\,di/dt$. The direction of $\mathscr{E}_L$ is upward in Fig. 31-32 because current *i* is downward through the inductor *and* increasing. Thus, as we move from point *y* to point *z*, opposite the direction of $\mathscr{E}_L$, we encounter a potential change of $-L\,di/dt$.
3. *Battery.* As we move from point *z* back to starting point *x*, we encounter a potential change of $+\mathscr{E}$ due to the battery's emf.

Thus, the loop rule gives us

$$-iR - L\frac{di}{dt} + \mathscr{E} = 0$$

or

$$L\frac{di}{dt} + Ri = \mathscr{E} \qquad (RL \text{ circuit}). \tag{31-40}$$

Equation 31-40 is a differential equation involving the variable *i* and its first derivative *di/dt*. To solve it, we seek the function $i(t)$ such that when $i(t)$ and its first derivative are substituted in Eq. 31-40, the equation is satisfied and the initial condition $i(0) = 0$ is satisfied.

Equation 31-40 and its initial condition are of exactly the form of Eq. 28-42 for an *RC* circuit, with *i* replacing *q*, *L* replacing *R*, and *R* replacing $1/C$. The solution of Eq. 31-40 must then be of exactly the form of Eq. 28-43 with the same replacements. That solution is

$$i = \frac{\mathscr{E}}{R}(1 - e^{-Rt/L}), \tag{31-41}$$

which we can rewrite as

$$i = \frac{\mathscr{E}}{R}(1 - e^{-t/\tau_L}) \qquad \text{(rise of current)}. \tag{31-42}$$

Here τ_L, the **inductive time constant**, is given by

$$\tau_L = \frac{L}{R} \qquad \text{(time constant)}. \tag{31-43}$$

Let's examine Eq. 31-42 for when the switch is closed (at time $t = 0$) and for a time long after the switch is closed ($t \to \infty$). If we substitute $t = 0$ into Eq. 31-42, the exponential becomes $e^{-0} = 1$. Thus, Eq. 31-42 tells us that the current is initially $i = 0$, as we expected. Next, if we let *t* go to ∞, then the exponential goes to $e^{-\infty} = 0$. Thus, Eq. 31-42 tells us that the current goes to its equilibrium value of $\mathscr{E}/R$.

We can also examine the potential differences in the circuit. For example, Fig. 31-33 shows how the potential differences V_R ($= iR$) across the resistor and V_L ($= L\,di/dt$) across the inductor vary with time for particular values of $\mathscr{E}$, *L*, and *R*. Compare this figure carefully with the corresponding figure for an *RC* circuit (Fig. 28-35).

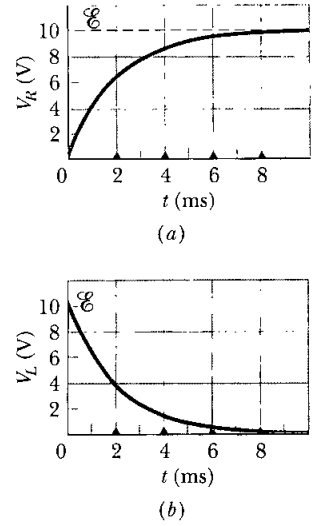


FIG. 31-33 The variation with time of (*a*) V_R, the potential difference across the resistor in the circuit of Fig. 31-32, and (*b*) V_L, the potential difference across the inductor in that circuit. The small triangles represent successive intervals of one inductive time constant $\tau_L = L/R$. The figure is plotted for $R = 2000\ \Omega$, $L = 4.0$ H, and $\mathscr{E} = 10$ V.

To show that the quantity τ_L (= L/R) has the dimension of time, we convert from henries per ohm as follows:

$$1\frac{\text{H}}{\Omega} = 1\frac{\text{H}}{\Omega}\left(\frac{1\,\text{V}\cdot\text{s}}{1\,\text{H}\cdot\text{A}}\right)\left(\frac{1\,\Omega\cdot\text{A}}{1\,\text{V}}\right) = 1\,\text{s}.$$

The first quantity in parentheses is a conversion factor based on Eq. 31-36, and the second one is a conversion factor based on the relation $V = iR$.

The physical significance of the time constant follows from Eq. 31-42. If we put $t = \tau_L = L/R$ in this equation, it reduces to

$$i = \frac{\mathscr{E}}{R}(1 - e^{-1}) = 0.63\frac{\mathscr{E}}{R}. \tag{31-44}$$

Thus, the time constant τ_L is the time it takes the current in the circuit to reach about 63% of its final equilibrium value $\mathscr{E}/R$. Since the potential difference V_R across the resistor is proportional to the current i, a graph of the increasing current versus time has the same shape as that of V_R in Fig. 31-33*a*.

If the switch S in Fig. 31-31 is closed on a long enough for the equilibrium current $\mathscr{E}/R$ to be established and then is thrown to b, the effect will be to remove the battery from the circuit. (The connection to b must actually be made an instant before the connection to a is broken. A switch that does this is called a *make-before-break* switch.) With the battery gone, the current through the resistor will decrease. However, it cannot drop immediately to zero but must decay to zero over time. The differential equation that governs the decay can be found by putting $\mathscr{E} = 0$ in Eq. 31-40:

$$L\frac{di}{dt} + iR = 0. \tag{31-45}$$

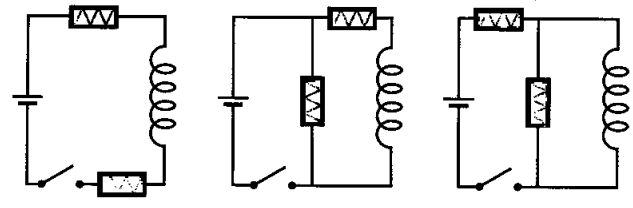
By analogy with Eqs. 28-48 and 28-49, the solution of this differential equation that satisfies the initial condition $i(0) = i_0 = \mathscr{E}/R$ is

$$i = \frac{\mathscr{E}}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \qquad \text{(decay of current)}. \tag{31-46}$$

We see that both current rise (Eq. 31-42) and current decay (Eq. 31-46) in an RL circuit are governed by the same inductive time constant, τ_L.

We have used i_0 in Eq. 31-46 to represent the current at time $t = 0$. In our case that happened to be $\mathscr{E}/R$, but it could be any other initial value.

CHECKPOINT 8 The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)



(1) (2) (3)

Sample Problem 31-9

Figure 31-34*a* shows a circuit that contains three identical resistors with resistance $R = 9.0\ \Omega$, two identical inductors with inductance $L = 2.0$ mH, and an ideal battery with emf $\mathscr{E} = 18$ V.

(a) What is the current i through the battery just after the switch is closed?

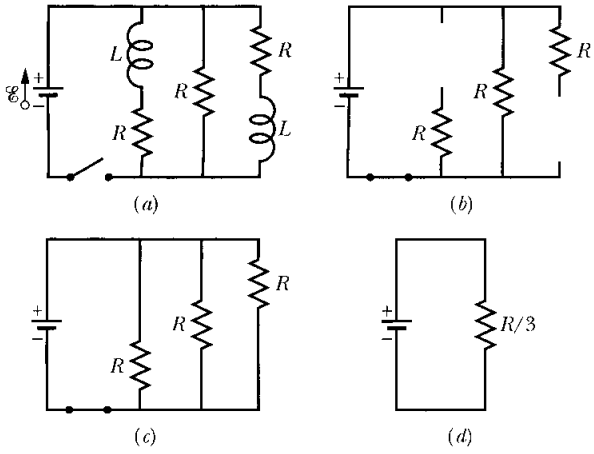


FIG. 31-34 (*a*) A multiloop RL circuit with an open switch. (*b*) The equivalent circuit just after the switch has been closed. (*c*) The equivalent circuit a long time later. (*d*) The single-loop circuit that is equivalent to circuit (*c*).

Conceptualize/Classify: Just after the switch is closed, the inductor acts to oppose a change in the current through it.

Compute: Because the current through each inductor is zero before the switch is closed, it will also be zero just afterward. Thus, immediately after the switch is closed, the inductors act as broken wires, as indicated in Fig. 31-34*b*. We then have a single-loop circuit for which the loop rule gives us

$$\mathscr{E} - iR = 0.$$

Substituting given data, we find that

$$i = \frac{\mathscr{E}}{R} = \frac{18\ \text{V}}{9.0\ \Omega} = 2.0\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the current i through the battery long after the switch has been closed?

Conceptualize/Classify: Long after the switch has been closed, the currents in the circuit have reached their equilibrium values, and the inductors act as simple connecting wires, as indicated in Fig. 31-34*c*.

Compute: We now have a circuit with three identical resistors in parallel; from Eq. 28-23, their equivalent resistance is $R_{eq} = R/3 = (9.0\ \Omega)/3 = 3.0\ \Omega$. The equivalent circuit shown in Fig. 31-34*d* then yields the loop equation $\mathscr{E} - iR_{eq} = 0$, or

$$i = \frac{\mathscr{E}}{R_{eq}} = \frac{18\ \text{V}}{3.0\ \Omega} = 6.0\ \text{A}. \qquad \text{(Answer)}$$

Sample Problem 31-10

A solenoid has an inductance of 53 mH and a resistance of 0.37 Ω. If the solenoid is connected to a battery, how long will the current take to reach half its final equilibrium value?

Conceptualize/Classify: We can mentally separate the solenoid into a resistance and an inductance that are wired in series with a battery, as in Fig. 31-32. Then application of the loop rule leads to Eq. 31-40, which has the solution of Eq. 31-42 for the current i in the circuit.

Compute: According to that solution, current i increases exponentially from zero to its final equilibrium value of $\mathscr{E}/R$. Let t_0 be the time that current i takes to reach half its equilibrium value. Then Eq. 31-42 gives us

$$\frac{1}{2}\frac{\mathscr{E}}{R} = \frac{\mathscr{E}}{R}(1 - e^{-t_0/\tau_L}).$$

We solve for t_0 by canceling $\mathscr{E}/R$, isolating the exponential, and taking the natural logarithm of each side. We find

$$t_0 = \tau_L \ln 2 = \frac{L}{R}\ln 2 = \frac{53\times10^{-3}\ \text{H}}{0.37\ \Omega}\ln 2$$
$$= 0.10\ \text{s}. \qquad \text{(Answer)}$$

Conclude: This is a reasonably small time interval. Actually, in our day-to-day experience, the inductances of the circuits are so small that the current builds up to approximate steady-state value within no time.

31-12 | Energy Stored in a Magnetic Field

When we pull two charged particles of opposite signs away from each other, we say that the resulting electric potential energy is stored in the electric field of the particles. We get it back from the field by letting the particles move closer together again. In the same way we say energy is stored in a magnetic field.

To derive a quantitative expression for that stored energy, consider again Fig. 31-32, which shows a source of emf $\mathscr{E}$ connected to a resistor R and an inductor L. Equation 31-40, restated here for convenience,

$$\mathscr{E} = L\frac{di}{dt} + iR, \tag{31-47}$$

is the differential equation that describes the growth of current in this circuit. Recall that this equation follows immediately from the loop rule and that the loop rule in turn is an expression of the principle of conservation of energy for single-loop circuits. If we multiply each side of Eq. 31-47 by i, we obtain

$$\mathscr{E}i = Li\frac{di}{dt} + i^2R, \tag{31-48}$$

which has the following physical interpretation in terms of work and energy:

1. If a differential amount of charge dq passes through the battery of emf $\mathscr{E}$ in Fig. 31-32 in time dt, the battery does work on it in the amount $\mathscr{E}\,dq$. The rate at which the battery does work is $(\mathscr{E}\,dq)/dt$, or $\mathscr{E}i$. Thus, the left side of Eq. 31-48 represents the rate at which the emf device delivers energy to the rest of the circuit.
2. The rightmost term in Eq. 31-48 represents the rate at which energy appears as thermal energy in the resistor.
3. Energy that is delivered to the circuit but does not appear as thermal energy must, by the conservation-of-energy hypothesis, be stored in the magnetic field of the inductor. Because Eq. 31-48 represents the principle of conservation of energy for RL circuits, the middle term must represent the rate dU_B/dt at which magnetic potential energy U_B is stored in the magnetic field.

Thus

$$\frac{dU_B}{dt} = Li\frac{di}{dt}. \tag{31-49}$$

We can write this as

$$dU_B = Li\,di.$$

Integrating yields

$$\int_0^{U_B} dU_B = \int_0^i Li\,di$$

or

$$U_B = \tfrac{1}{2}Li^2 \quad \text{(magnetic energy)}, \tag{31-50}$$

which represents the total energy stored by an inductor L carrying a current i. Note the similarity in form between this expression and the expression for the energy stored by a capacitor with capacitance C and charge q; namely,

$$U_E = \frac{q^2}{2C}. \tag{31-51}$$

(The variable i^2 corresponds to q^2, and the constant L corresponds to $1/C$.)

Sample Problem 31-11

A coil has an inductance of 53 mH and a resistance of 0.35 Ω.

(a) If a 12 V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

Conceptualize/Classify: The energy stored in the magnetic field of a coil at any time depends on the current through the coil at that time, according to Eq. 31-50 ($U_B = \frac{1}{2}Li^2$).

Compute: Thus, to find the energy $U_{B\infty}$ stored at equilibrium, we must first find the equilibrium current. From Eq. 31-42, the equilibrium current is

$$i_\infty = \frac{\mathscr{E}}{R} = \frac{12\text{ V}}{0.35\Omega} = 34.3\text{ A}. \qquad (31\text{-}52)$$

Then substitution yields

$$U_{B\infty} = \tfrac{1}{2}Li_\infty^2 = (\tfrac{1}{2})(53\times10^{-3}\text{ H})(34.3\text{ A})^2 = 31\text{ J}. \qquad \text{(Answer)}$$

(b) After how many time constants will half this equilibrium energy be stored in the magnetic field?

Compute: Now we are being asked: At what time t will the relation

$$U_B = \tfrac{1}{2}U_{B\infty}$$

be satisfied? Using Eq. 31-50 twice allows us to rewrite this energy condition as

$$\tfrac{1}{2}Li^2 = (\tfrac{1}{2})\tfrac{1}{2}Li_\infty^2$$

or

$$i = \left(\frac{1}{\sqrt{2}}\right)i_\infty \qquad (31\text{-}53)$$

However, i is given by Eq. 31-42 and i_∞ (see Eq. 31-52) is $\mathscr{E}/R$; so Eq. 31-53 becomes

$$\frac{\mathscr{E}}{R}(1-e^{-t/\tau_L}) = \frac{\mathscr{E}}{\sqrt{2}R}.$$

By canceling $\mathscr{E}/R$ and rearranging, we can write this as

$$e^{-t/\tau_L} = 1-\frac{1}{\sqrt{2}} = 0.293,$$

which yields

$$\frac{t}{\tau_L} = -\ln 0.293 = 1.23$$

or

$$t \approx 1.2\tau_L. \qquad \text{(Answer)}$$

Conclude: Thus, the energy stored in the magnetic field of the coil by the current will reach half its equilibrium value 1.2 time constants after the emf is applied.

Energy Density of a Magnetic Field

Consider a length l near the middle of a long solenoid of cross-sectional area A carrying current i; the volume associated with this length is Al. The energy U_B stored by the length l of the solenoid must lie entirely within this volume because the magnetic field outside such a solenoid is approximately zero. Moreover, the stored energy must be uniformly distributed within the solenoid because the magnetic field is (approximately) uniform everywhere inside.

Thus, the energy stored per unit volume of the field is

$$u_B = \frac{U_B}{Al}$$

or, since

$$U_B = \tfrac{1}{2}Li^2,$$

we have

$$u_B = \frac{Li^2}{2Al} = \frac{L}{l}\frac{i^2}{2A}.$$

Here L is the inductance of length l of the solenoid.

Substituting for L/l from Eq. 31-32, we find

$$u_B = \tfrac{1}{2}\mu_0 n^2 i^2, \tag{31-54}$$

where n is the number of turns per unit length. From Eq. 30-38 ($B = \mu_0 in$) we can write this *energy density* as

$$u_B = \frac{B^2}{2\mu_0} \quad \text{(magnetic energy density).} \tag{31-55}$$

This equation gives the density of stored energy at any point where the magnitude of the magnetic field is B. Even though we derived it by considering the special case of a solenoid, Eq. 31-55 holds for all magnetic fields, no matter how they are generated. The equation is comparable to Eq. 26-23,

$$u_E = \tfrac{1}{2}\varepsilon_0 E^2, \tag{31-56}$$

which gives the energy density (in a vacuum) at any point in an electric field. Note that both u_B and u_E are proportional to the square of the appropriate field magnitude, B or E.

CHECKPOINT 9 The table lists the number of turns per unit length, current, and cross-sectional area for three solenoids. Rank the solenoids according to the magnetic energy density within them, greatest first.

Solenoid	Turns per Unit Length	Current	Area
a	$2n_1$	i_1	$2A_1$
b	n_1	$2i_1$	A_1
c	n_1	i_1	$6A_1$

Sample Problem 31-12

A long coaxial cable (Fig. 31-35) consists of two thin-walled concentric conducting cylinders with radii a and b. The inner cylinder carries a steady current i, and the outer cylinder provides the return path for that current. The current sets up a magnetic field between the two cylinders.

(a) Calculate the energy stored in the magnetic field for a length l of the cable.

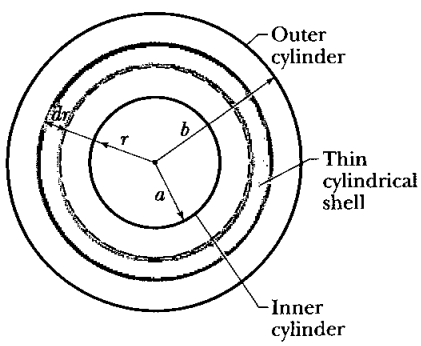


FIG. 31-35 A cross section of a long coaxial cable consisting of two thin-walled conducting cylinders, the inner cylinder of radius a and the outer cylinder of radius b.

Conceptualize: We can calculate the (total) energy U_B stored in the magnetic field from the energy density u_B of the field.

Classify: That energy density depends on the magnitude B of the magnetic field according to Eq. 31-55 ($u_B = B^2/2\mu_0$). Because of the circular symmetry of the cable, we can find B by using Ampere's law with the given current i.

Compute: Finding B: To apply these ideas, we begin with Ampere's law, using a circular path of integration with radius r such that $a < r < b$ (between the two cylinders, as indicated by the dashed line in Fig. 31-55). The only current enclosed by this path is current i on the inner cylinder. Thus, we can write Ampere's law as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i. \tag{31-57}$$

Next, we simplify the integral: Because of the circular symmetry, at all points along the circular path, $\vec{B}$ is tangent to the path and has the same magnitude B. Let us take the

direction of integration along the path as the direction of the magnetic field around the path. Then we can replace $\vec{B} \cdot d\vec{s}$ with $B\,ds \cos 0 = B\,ds$ and then move magnitude B in front of the integration symbol. The integral that remains is $\oint ds$, which just gives the circumference $2\pi r$ of the path. Thus, Eq. 31-57 simplifies to

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r}. \tag{31-58}$$

Finding u_B: Next, to obtain the energy density, we substitute Eq. 31-58 into Eq. 31-55:

$$u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 i^2}{8\pi^2 r^2}. \tag{31-59}$$

Finding U_B: Note that u_B is not uniform in the volume between the two cylinders, but instead depends on the radial distance r. Thus, to find the total energy U_B stored between the cylinders, we must integrate u_B over that volume.

Because the volume between the cylinders has circular symmetry about the cable's central axis, we consider the volume dV of a cylindrical shell located between the cylinders; the shell has inner radius r, outer radius $r + dr$ (Fig. 31-35), and length l. The shell's cross-sectional area is the product of its circumference $2\pi r$ and thickness dr. Thus, the shell's volume dV is $(2\pi r)(dr)(l)$; that is, $dV = 2\pi r l\,dr$.

Because points within this shell are all at approximately the same radial distance r, they all have approximately the same energy density u_B. Thus, the total energy dU_B contained in the shell of volume dV is given by

$$\text{energy} = \begin{pmatrix}\text{energy per}\\ \text{unit volume}\end{pmatrix}(\text{volume})$$

or

$$dU_B = u_B\,dV.$$

Substituting Eq. 31-59 for u_B and $2\pi r l\,dr$ for dV, we obtain

$$dU_B = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l)dr = \frac{\mu_0 i^2 l}{4\pi}\frac{dr}{r}.$$

To find the total energy contained between the two cylinders, we integrate this equation over the volume between the two cylinders:

$$U_B = \int dU_B = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r}$$

$$= \frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a}. \qquad \text{(Answer)} \tag{31-60}$$

No energy is stored outside the outer cylinder or inside the inner cylinder because the magnetic field is zero in both locations, as you can show with Ampere's law.

(b) What is the stored energy per unit length of the cable if $a = 1.2$ mm, $b = 3.5$ mm, and $i = 2.7$ A?

Compute: From Eq. 31-60 we have

$$\frac{U_B}{l} = \frac{\mu_0 i^2}{4\pi}\ln\frac{b}{a}$$

$$= \frac{(4\pi \times 10^{-7}\ \text{H/m})(2.7\ \text{A})^2}{4\pi}\ln\frac{3.5\ \text{mm}}{1.2\ \text{mm}}$$

$$= 7.8 \times 10^{-7}\ \text{J/m} = 780\ \text{J/m}. \qquad \text{(Answer)}$$

Conclude: By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 31-50, the energy can also be written as $(1/2)Li^2$. Comparing the two expressions, we get

$$L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right).$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume. In such cases different parts of the current may circle different amounts of flux, and it can be very tricky to get L directly from Eq. 31-29.

Magnetic Pressure

The idea of energy in a magnetic field gives a very important and useful result. Let us try to find the pressure due to magnetic field on a surface through which current flows. As we discussed in the previous chapter, a current-carrying wire experiences a force in a magnetic field. This force is perpendicular to the wire. Similarly, we can note that any current-carrying strip placed in a magnetic field will experience a force. If there is a component of the force perpendicular to the surface, there is a pressure due to magnetic field.

Consider the situation depicted in Fig. 31-36. The gray lines show the direction of the magnetic field above the strip and below the strip. The strip is a large one.

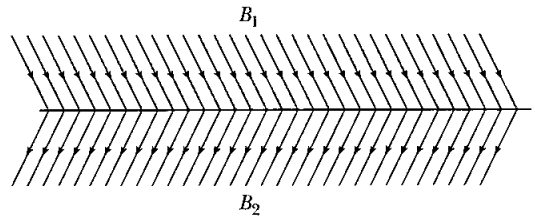


FIG. 31-36 Magnetic field pattern near an infinite current-carrying sheet. Here an external uniform magnetic field superimposes with the magnetic field of the sheet to produce a magnetic field B_1 in the space above the sheet and B_2 in a space below the sheet.

In space, the magnetic field will be due to the current flowing on the surface and the external magnetic field as well. So one way for finding the force is to find the external magnetic field, the current flowing on the surface and then use Amperian force formula to find the pressure. However, the situation will be very simplified if we use the concept of pressure and energy.

Imagine that the magnetic field above the strip under consideration is B_1 and below the strip is B_2. In general, the magnetic fields above and below will be different due to the contribution of the strip. Suppose that under the influence of this magnetic field, the strip moves in the upward direction by a distance ds.

The energy related to the magnetic field will change. In the region above, the energy will decrease by

$$\Delta U_1 = \frac{B_1^2}{2\mu_0} A\, ds.$$

In the region above, the energy will increase by

$$\Delta U_2 = \frac{B_2^2}{2\mu_0} A\, ds.$$

We are assuming that the magnetic fields above and below are remaining the same. Actually, the current-carrying strip should be connected to a source of emf which will maintain the current in the strip. That source will do the work. We also know that work done by the external agent is the change in the potential energy of the system:

$$dW = F\, ds = \Delta U_2 - \Delta U_1 = \frac{B_2^2 - B_1^2}{2\mu_0} A\, ds.$$

By definition of pressure,

$$p = \frac{F}{A} = \frac{B_2^2 - B_1^2}{2\mu_0}. \tag{31-61}$$

31-13 | Mutual Induction

In this section we return to the case of two interacting coils, which we first discussed in Section 31-3, and we treat it in a somewhat more formal manner. We saw earlier that if two coils are close together as in Fig. 31-3, a steady current i in one coil will set up a magnetic flux Φ through the other coil (*linking* the other coil). If we change i with time, an emf $\mathscr{E}$ given by Faraday's law appears in the second coil; we called this process *induction*. We could better have called it **mutual induction**, to suggest the mutual interaction of the two coils and to distinguish it from *self-induction*, in which only one coil is involved.

Let us look a little more quantitatively at mutual induction. Figure 31-37*a* shows two circular close-packed coils near each other and sharing a common central axis. With the variable resistor set at a particular resistance R, the battery produces a steady current i_1 in coil 1. This current creates a magnetic field represented by the lines of $\vec{B}_1$ in the figure. Coil 2 is connected to a sensitive meter but contains no battery; a magnetic flux Φ_{21} (the flux through coil 2 associated with the current in coil 1) links the N_2 turns of coil 2.

We define the mutual inductance M_{21} of coil 2 with respect to coil 1 as

$$M_{21} = \frac{N_2\Phi_{21}}{i_1}, \tag{31-62}$$

which has the same form as Eq. 31-29 ($L = N\Phi/i$), the definition of inductance. We can recast Eq. 31-62 as

$$M_{21}i_1 = N_2\Phi_{21}.$$

If we cause i_1 to vary with time by varying R, we have

$$M_{21}\frac{di_1}{dt} = N_2\frac{d\Phi_{21}}{dt}.$$

The right side of this equation is, according to Faraday's law, just the magnitude of the emf $\mathscr{E}_2$ appearing in coil 2 due to the changing current in coil 1. Thus, with a minus sign to indicate direction,

$$\mathscr{E}_2 = -M_{21}\frac{di_1}{dt}, \tag{31-63}$$

which you should compare with Eq. 31-36 for self-induction ($\mathscr{E} = -L\,di/dt$).

Let us now interchange the roles of coils 1 and 2, as in Fig. 31-37*b*; that is, we set up a current i_2 in coil 2 by means of a battery, and this produces a magnetic flux Φ_{12} that links coil 1. If we change i_2 with time by varying R, we then have, by the argument given above,

$$\mathscr{E}_1 = -M_{12}\frac{di_2}{dt}. \tag{31-64}$$

Thus, we see that the emf induced in either coil is proportional to the rate of change of current in the other coil. The proportionality constants M_{21} and M_{12} seem to be different. We assert, without proof, that they are in fact the same so that no subscripts are needed. (This conclusion is true but is in no way obvious.) Thus, we have

$$M_{21} = M_{12} = M, \tag{31-65}$$

and we can rewrite Eqs. 31-63 and 31-64 as

$$\mathscr{E}_2 = -M\frac{di_1}{dt} \tag{31-66}$$

and

$$\mathscr{E}_1 = -M\frac{di_2}{dt}. \tag{31-67}$$

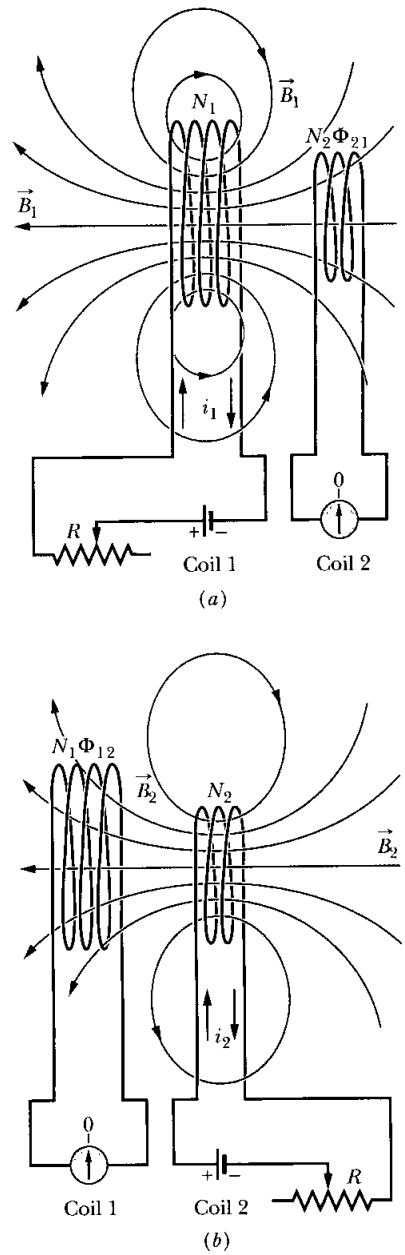


FIG. 31-37 Mutual induction. (*a*) The magnetic field $\vec{B}_1$ produced by current i_1 in coil 1 extends through coil 2. If i_1 is varied (by varying resistance R), an emf is induced in coil 2 and current registers on the meter connected to coil 2. (*b*) The roles of the coils interchanged.

Sample Problem 31-13

Figure 31-38 shows two circular close-packed coils, the smaller (radius R_2, with N_2 turns) being coaxial with the larger (radius R_1, with N_1 turns) and in the same plane.

(a) Derive an expression for the mutual inductance M for this arrangement of these two coils, assuming that $R_1 \gg R_2$.

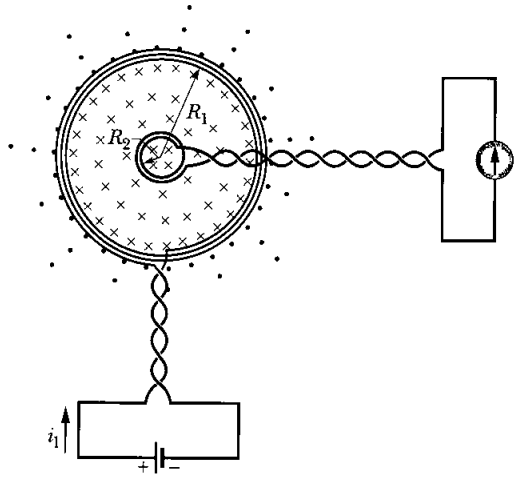


FIG. 31-38 A small coil is located at the center of a large coil. The mutual inductance of the coils can be determined by sending current i_1 through the large coil.

Conceptualize: The mutual inductance M for these coils is the ratio of the flux linkage ($N\Phi$) through one coil to the current i in the other coil, which produces that flux linkage.

Classify: We need to assume that currents exist in the coils; then we need to calculate the flux linkage in one of the coils.

Compute: The magnetic field through the larger coil due to the smaller coil is nonuniform in both magnitude and direction; so the flux through the larger coil due to the smaller coil is nonuniform and difficult to calculate. However, the smaller coil is small enough for us to assume that the magnetic field through it due to the larger coil is approximately uniform. Thus, the flux through it due to the larger coil is also approximately uniform. Hence, to find M we shall assume a current i_1 in the larger coil and calculate the flux linkage $N_2\Phi_{21}$ in the smaller coil:

$$M = \frac{N_2\Phi_{21}}{i_1}. \tag{31-68}$$

The flux Φ_{21} through each turn of the smaller coil is, from $\Phi_B = BA$,

$$\Phi_{21} = B_1 A_2,$$

where B_1 is the magnitude of the magnetic field at points within the small coil due to the larger coil and A_2 ($= \pi R_2^2$) is the area enclosed by the turn. Thus, the flux linkage in the smaller coil (with its N_2 turns) is

$$N_2\Phi_{21} = N_2 B_1 A_2. \tag{31-69}$$

To find B_1 at points within the smaller coil, we can use Eq. 30-14,

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}},$$

with z set to 0 because the smaller coil is in the plane of the larger coil. That equation tells us that each turn of the larger coil produces a magnetic field of magnitude $\mu_0 i_1/2R_1$ at points within the smaller coil. Thus, the larger coil (with its N_1 turns) produces a total magnetic field of magnitude

$$B_1 = N_1 \frac{\mu_0 i_1}{2R_1} \tag{31-70}$$

at points within the smaller coil.

Substituting Eq. 31-70 for B_1 and πR_2^2 for A_2 in Eq. 31-69 yields

$$N_2\Phi_{21} = \frac{\pi\mu_0 N_1 N_2 R_2^2 i_1}{2R_1}.$$

Substituting this result into Eq. 31-68, we find

$$M = \frac{N_2\Phi_{21}}{i_1} = \frac{\pi\mu_0 N_1 N_2 R_2^2}{2R_1}. \qquad \text{(Answer)} \tag{31-71}$$

(b) What is the value of M for $N_1 = N_2 = 1200$ turns, $R_2 = 1.1$ cm, and $R_1 = 15$ cm?

Compute: Equation 31-71 yields

$$M = \frac{(\pi)(4\pi \times 10^{-7}\,\text{H/m})(1200)(1200)(0.011\,\text{m})^2}{(2)(0.15\,\text{m})}$$
$$= 2.29 \times 10^{-3} \times 10^{-3}\,\text{H} \approx 2.3\,\text{mH}. \qquad \text{(Answer)}$$

Conclude: Consider the situation if we reverse the roles of the two coils—that is, if we produce a current i_2 in the smaller coil and try to calculate M from Eq. 31-62 in the form

$$M = \frac{N_1\Phi_{12}}{i_2}.$$

The calculation of Φ_{12} (the nonuniform flux of the smaller coil's magnetic field encompassed by the larger coil) is not simple. If we were to do the calculation numerically using a computer, we would find M to be 2.3 mH, as above! This emphasizes that Eq. 31-65 ($M_{21} = M_{12} = M$) is not obvious.

Sample Problem 31-14

In Fig. 31-39, coil AA′ (red colored) and coil BB′ (blue colored) are wound on a long plastic tube in the same sense. Ends A and B are joined together and a source of current I is connected to A′ and B′. Show that the induced emf is equal to $-(L_A + L_B - 2M_{AB})\,(di/dt)$, where L_A and L_B are self-inductances of coils AA′ and BB′, and M is mutual inductance.

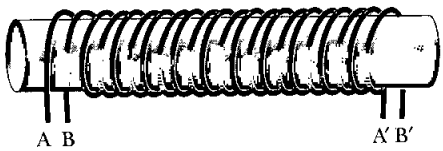


FIG. 31-39 Two coils of wire, red and blue, are wound over a plastic tube. The coils are wound in the same sense.

Conceptualize: The self-inductance of each coil is independent of the method of connection. If terminals A and B are connected, the sense of the current flow around coil AA′ is opposite to that around coil BB′. So we can imagine that inside the plastic tube, the magnetic field will be due to coils AA′ and BB′ both and both the magnetic fields will be in the opposite direction.

Classify: We can consider the induced emf as the sum of the induced emf due to the self-inductance and due to mutual inductance.

Compute: The emf induced across coil BB′ must be

$$V_B = -L_B \frac{di_B}{dt} + M \frac{di_A}{dt}.$$

Note that the magnetic field due to the coil itself and due to coil AA′ are in opposite directions. So their induced emf will also have opposite signs. Here M is the (magnitude of the) mutual inductance between the coils. Since the coils are connected in series $i_A = i_B = i$, so that

$$V_B = -(L_B - M)\frac{di}{dt}.$$

Similarly, we find that the emf induced across coil AA′ is

$$V_A = -(L_A - M)\frac{di}{dt}.$$

The emf across the entire series coil assembly is the sum of V_A and V_B:

$$V = V_A + V_B = -(L_A + L_B - 2M)\frac{di}{dt}. \qquad \text{(Answer)}$$

Conclude: The interesting point to note is that if both the coils have the same number of turns, $L_A = L_B$, and since no flux escapes from the sides, $M = \sqrt{L_A L_B}$, so the induced emf in this case turns out to be zero. This is the way we wind the resistances in a resistance box so that the effect of induced emf is cancelled. In the language of electrical engineers, the coil has non inductive winding.

CHECKPOINT 10 Suppose the situation in Fig. 31-39 is changed so that ends A and B′ are joined together and the current source is connected to A′ and B. Show that the induced emf is

$$V = V_A + V_B = -(L_A + L_B + 2M)\frac{di}{dt}.$$

Let us now find an expression for the total magnetic energy stored in two coils with inductances L_1 and L_2 and mutual inductance M, when the currents in the coils are I_1 and I_2, respectively. While the currents are building up we have for the emfs:

$$V_1 = -L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \quad \text{and} \quad V_2 = -L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt} \qquad (31\text{-}72)$$

where the ± appear consistently in both equations, and depend on the coil geometry and circuit sense. Then the external work done in time dt to push charge dq_1 and dq_2 through each circuit, respectively, is

$$\begin{aligned} dW &= -V_1\,dq_1 - V_2\,dq_2 \\ &= L_1\frac{di_1}{dt}dq_1 \mp M\frac{di_2}{dt}dq_1 + L_2\frac{di_2}{dt}dq_2 \mp M\frac{di_1}{dt}dq_2. \end{aligned}$$

Noting that $i_1 = dq_1/dt$ and $i_2 = dq_2/dt$, we have

$$\begin{aligned} dW &= L_1 i_1\,di_1 + L_2 i_2\,di_2 \mp (Mi_1\,di_1 + Mi_2\,di_2) \\ &= L_1 i_1\,di_1 + L_2 i_2\,di_2 \mp Md(i_1 i_2). \end{aligned}$$

Integrating from 0 to final current, we get

$$\begin{aligned} U = \int dW &= L_1\int_0^{I_1} i_1 di_1 + L_2\int_0^{I_2} i_2 di_2 \mp M\int_0^{I_1 I_2} d(i_1 i_2) \\ &= \tfrac{1}{2}L_1 I_1^{\,2} + \tfrac{1}{2}L_2 I_2^{\,2} \mp MI_1 I_2. \end{aligned} \qquad (31\text{-}73)$$

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. A metallic loop is placed in a nonuniform magnetic field.

Statement 1: No emf will be induced in the loop.

Statement 2: The magnetic flux through the loop does not change with time.

2. Two circular loops are placed coaxially but separated by a distance. A battery is connected to one of the loops establishing a current in it.

Statement 1: The two loops repel each other.

Statement 2: According to the Lenz's law, current is induced in a loop so as to oppose the change in magnetic flux through it.

3. The coil of a moving coil galvanometer is oscillating. We connect the two ends of the coil.

Statement 1: The oscillations will suddenly stop.

Statement 2: A current is induced in the coil which opposes the change in flux through the coil.

4. A length of wire is bent into a closed loop and a magnet is plunged into it. This induces an emf in the loop and hence a current in the loop. A second wire having twice the length is bent into two concentric loops of wire having the same radius as before. The same magnet is now plunged into it with the same speed as before.

Statement 1: The current induced in the second case is twice that of the first one.

Statement 2: The emf induced in the second case is twice that of the first one.

5. Statement 1: The induced current in the coil shown in Fig. 31-16 at this instant is maximum.

Statement 2: The magnetic flux through the coil shown in the figure at this instant is maximum.

6. An inductor is connected to a battery through a switch.

Statement 1: The emf induced in the inductor is much larger when the switch is opened as compared to the emf induced when the switch is closed.

Statement 2: When the switch is opened, the current suddenly tends to drop to zero. But when the switch is connected, due to the finite resistance of the circuit, the current does not suddenly build up to maximum value.

7. The separation between the two inductors as shown in Fig. 31-40 is decreased to the point at which it becomes small, and the inductors are wound in the same sense,

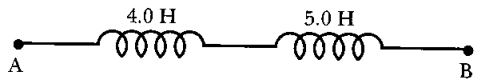


FIG. 31-40 Reasoning type question 7.

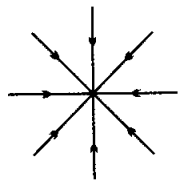
Statement 1: The inductance of the coils is greater than the inductance when they are farther apart.

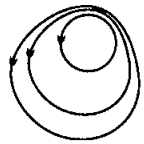
Statement 2: The flux due to one coil threads the other coil.

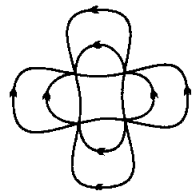
MULTIPLE CHOICE QUESTIONS

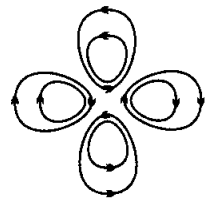
sec. 31-2 Gauss' Law for Magnetic Fields

1. Which of the following field line patterns could represent a magnetic field?

(a) 

(b) 

(c) 

(d) 

2. A spherical surface is split into two parts, S_1 and S_2, by a circular loop. The sphere is placed near a bar magnet, as shown in Fig. 31-41.

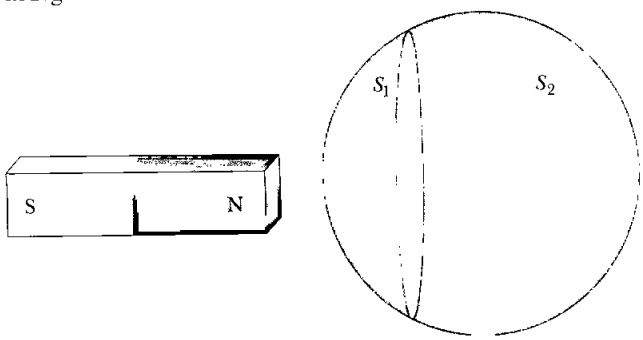


FIG. 31-41 Multiple choice question 2.

Through which of the two parts is the magnitude of the magnetic flux larger?

(a) S_1

(b) S_2

(c) The magnitude of the flux is the same for both

(d) Cannot tell without more information about the magnetic field

3. A sphere of radius R is placed near a long, straight wire that carries a steady current I. The magnetic field generated by the current is B. The total magnetic flux passing through the sphere is

(a) $\mu_0 I$

(b) $\mu_0 I/(4\pi R^2)$

(c) Zero

(d) Need more information

sec. 31-4 Faraday's Law of Induction

4. A coil of wire rotates about a horizontal north–south axis. Assume that at that point the magnetic field is along North direction. How does the Earth's magnetic field contribute to the emf induced in this coil?

(a) Both the horizontal and vertical components contribute to the induced emf

(b) Only the horizontal component induces an emf

(c) Only the vertical component induces an emf

(d) There is no emf induced in the coil

5. A magnet moves inside a coil. Which of the following factors can affect the emf induced in the coil?

I. The speed at which the magnet moves

II. The magnetic field of the magnet

III. The number of turns in the coil

(a) I only

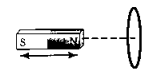
(b) I and II only

(c) II and III only

(d) I, II and III

6. A magnet is in form of a bar as shown in the figure. The line joining the center of the bar and the center of the coil (central axis) is perpendicular to the plane of the coil. Which of the following motions of the bar will not induce electric current in the coil?

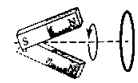
(a) Translation motion back and forth



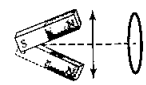
(b) Translation motion up and down



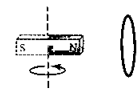
(c) The bar axis is at angle to the central axis and spins around it



(d) The bar axis is swings back and forth about the central axis perpendicular to paper plane



(e) The bar spins around an axis in the paper plane



7. A rectangular loop of wire is placed perpendicular to a uniform magnetic field and then spun around one of its sides at frequency f. The induced emf is a maximum when

(a) The flux is zero

(b) The flux is a maximum

(c) The flux is half its maximum value

(d) The derivative of the flux with respect to time is zero

sec. 31-5 Lenz's Law

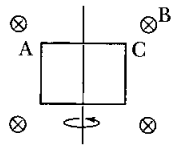
8. A loop of wire is turned round the vertical axis by 180° in a uniform magnetic field so that at the beginning of motion the point A moves toward the observer while the point C moves away from the observer. The direction of current, induced in the loop, is

(a) Permanently from A to C

(b) Permanently from C to A

(c) From A to C during the first half of the turn and then has the opposite direction

(d) From C to A during the first half of the turn and then has the opposite direction



9. A bar magnet with its north (N) and south (S) poles as shown in Fig. 31-42 is initially moving to the left, along the axis of, and away from a circular conducting loop. A current I is induced in the loop, with a being the acceleration of the magnet due to this current. As seen from the magnet looking in the direction of the loop,

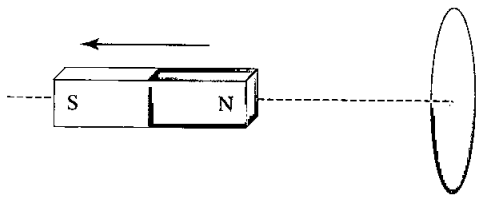


FIG. 31-42 Multiple choice question 9.

(a) I runs clockwise and a points to the left

(b) I runs counterclockwise and a points to the right

(c) I runs clockwise and a points to the right

(d) I runs counterclockwise and a points to the left

10. A coil is moving toward a straight long wire carrying a steady electric current. The wire and the motion are within the plane of the coil as shown in Fig. 31-43. The force exerted by the wire on the coil is in the direction

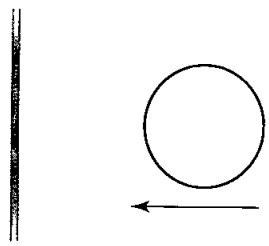


FIG. 31-43 Multiple choice question 10.

(a) Away from the wire

(b) Toward the wire

(c) Into the paper plane

(d) Out of the paper plane

(e) Upward

11. A circular loop of wire is positioned half in and half out of a square region of uniform magnetic field directed in the $+z$ direction, out of the paper, as shown in Fig. 31-44. To induce a counterclockwise current in this loop

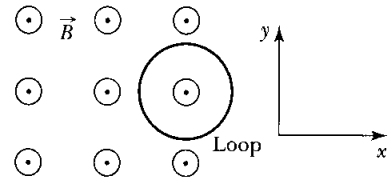


FIG. 31-44 Multiple choice question 11.

(a) Move it in the $+y$ direction

(b) Move it in the $-x$ direction

(c) Move it along the $+z$ axis, in the same direction as B

(d) Move it along the z axis, in the direction opposite to B

(e) Move it in the $+x$ direction

(f) Move it in the $-y$ direction

12. A rectangular loop, in edge view, as shown in Fig. 31-45, is rotating in a horizontal magnetic field. The axis of rotation is perpendicular to the page and is shown by a black dot in the middle of the figure. The parts of the loop perpendicular to the page are moving at a velocity v as shown in the figure. At the instant depicted, the normal to the loop makes an angle of 310° with the east direction. Which of the following statements, about the magnetic flux ϕ through the loop and the induced emf in the loop, are true, at the instant depicted?

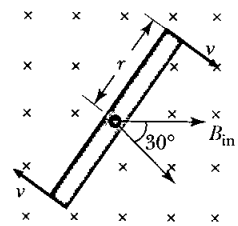


FIG. 31-45 Multiple choice question 12.

(a) ϕ is increasing and emf is increasing

(b) ϕ is increasing and emf is decreasing

(c) ϕ is decreasing and emf is increasing

(d) ϕ is decreasing and emf is decreasing

(e) ϕ and emf are both zero

13. A constant current of 3.0 A flows counterclockwise in the circular coil in Fig. 31-46. What is the direction of the induced current in the coil on the right?

(a) Counterclockwise

(b) Clockwise

(c) Out of the page

(d) There is no induced current

(e) Into the page

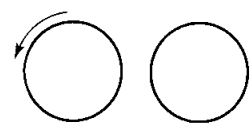


FIG. 31-46 Multiple choice question 13.

14. A long straight wire carries a current of 10.05 A to the right. A circular loop of wire lies below the long straight wire, as shown in Fig. 31-47. The current in the long straight wire decreases to zero in a time of 4.652 s. What direction does the induced current flow around the circular loop of wire?

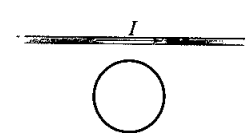


FIG. 31-47 Multiple choice question 14.

(a) Counterclockwise

(b) Clockwise

(c) No induced current flows around the loop

(d) The information given is insufficient for determining the direction of the induced current

15. Coils P and Q each have a large number of turns of insulated wire. When switch S is closed, the pointer of galvanometer G is deflected toward the left (Fig. 31-48). With S now closed, to make the pointer of G deflect toward the right one could

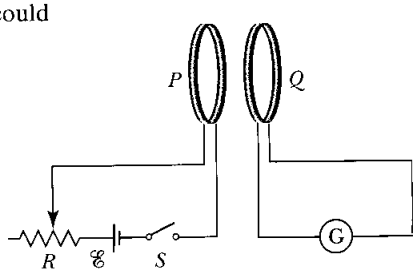


FIG. 31-48 Multiple choice question 15.

(a) Move the slide of the rheostat R quickly to the right

(b) Move coil P toward coil Q

(c) Move coil Q toward coil P

(d) Open S

16. Consider the situation as shown in Fig. 31-49. If the switch is closed and after some time it is opened again, the closed circular loop will show

(a) An anticlockwise current-pulse

(b) A clockwise current-pulse

(c) An anticlockwise current-pulse and then a clockwise current-pulse

(d) A clockwise current-pulse and then an anticlockwise current-pulse

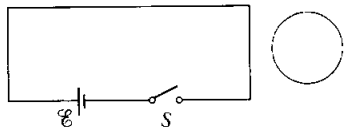


FIG. 31-49 Multiple choice question 16.

sec. 31-6 Motional Emf

17. A rod with resistance R lies across frictionless conducting rails in a constant uniform magnetic field $\vec{B}$, as shown in Fig. 31-50. Assume the rails have negligible resistance. The magnitude of the force that must be applied by a person to pull the rod to the right at constant speed v is

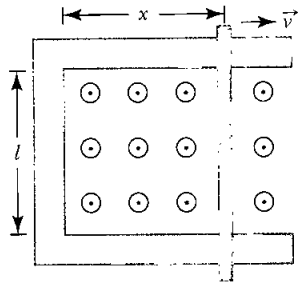


FIG. 31-50 Multiple choice question 17.

(a) 0

(b) Blv

(c) Blv/R

(d) B^2l^2v/R

(e) B^2lv/R

18. A rod of length l with resistance R lies in a constant uniform magnetic field $\vec{B}$ perpendicular to the rod. The surface is frictionless. The magnitude of the force that must be applied by a person to pull the rod at constant speed v is

(a) 0

(b) Blv

(c) Blv/R

(d) B^2l^2v/R

(e) B^2lv/R

19. A metal rod is falling toward the surface of the Earth near the equator. As it falls, one end of the rod become positively charged due to the motional emf of the rod through the Earth's magnetic field. The rod is oriented so that

(a) The rod is horizontal with the positive end toward the west

(b) The rod is vertical with the positive end toward the north

(c) The rod is vertical with the positive end higher

(d) The rod is horizontal with the positive end toward the east

20. A rectangular loop of wire in the plane of the paper starts outside a perpendicular uniform magnetic field directed into the paper as shown in Fig. 31-51. The loop is pulled with a constant velocity through the magnetic field and out the other side. Which of the following graphs would best represent the variation of the electric current in the wire with time?

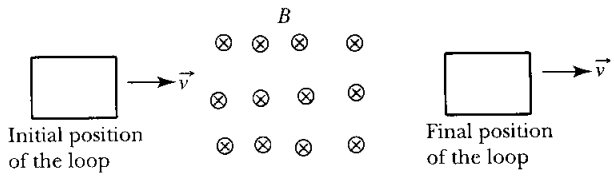
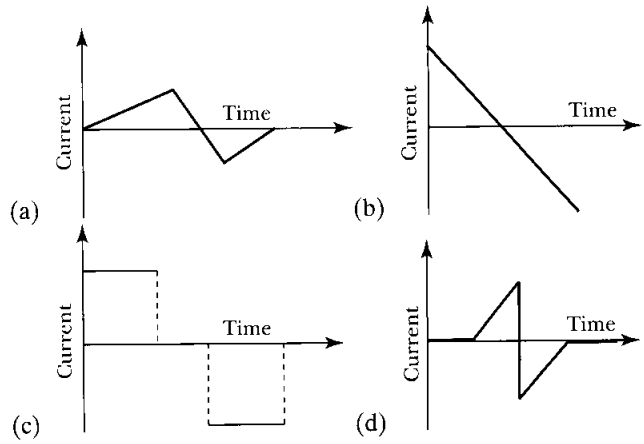


FIG. 31-51 Multiple choice question 20.



sec. 31-8 Induced Electric Fields

21. Which of the following is impossible in electromagnetism?

(a) Breaking a magnet in half, with one piece having only northpole and the other having only south pole

(b) Breaking a charged conductor in half with one piece carrying positive charge and the other carrying negative charge

(c) Detecting static electric charges with a compass

(d) Detecting steady electric currents with a compass

(e) Detecting static magnetic field with a voltmeter

22. A very long solenoid perpendicular to the page generates a downward magnetic field whose magnitude increases with time (Fig. 31-52). This induces an emf in a conducting wire loop inside the solenoid which lights two identical bulbs connected in series along the wire. Now two points diametrically opposed on the wire loop are shorted with another wire lying to the right of bulb B in the plane of the page. After the shorting wire is inserted,

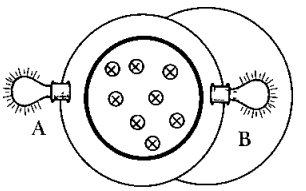


FIG. 31-52 Multiple choice question 22.

(a) Bulb A goes out, and bulb B dims

(b) Bulb A goes out, and bulb B gets brighter

(c) Bulb B goes out, and bulb A dims

(d) bulb B goes out, and bulb A gets brighter

23. A changing magnetic field pierces the interior of a circuit containing three identical resistors. Two ideal voltmeters are connected to the same points, as shown in Fig. 31-53. If V_1 reads 1 mV, then V_2 reads

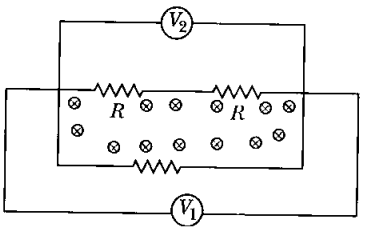


FIG. 31-53 Multiple choice question 23.

(a) 0

(b) $\frac{1}{3}$ mV

(c) $\frac{1}{2}$ mV

(d) 1 mV

(e) 2 mV

sec. 31-9 Inductors and Inductance

24. What will happen to the inductance of a solenoid if it is stretched to twice its length while maintaining the same number of turns and cross-sectional area?

(a) Its inductance will double

(b) Its inductance will be reduced to half

(c) Its inductance will remain unchanged

(d) Its inductance will be reduced to one fourth

sec. 31-10 Self-Induction

25. The potential at point A is higher than the potential at point B (Fig. 31-54). Which of the following statements about the inductor current i could be true?

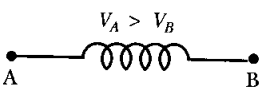


FIG. 31-54 Multiple choice question 25.

(a) i is from A to B and increasing

(b) i is from B to A and decreasing

(c) i is from B to A and increasing

(d) i is from A to B and decreasing

sec. 31-11 *RL* Circuits

26. An open switch in an RL circuit is closed at time $t = 0$, as shown in Fig. 31-55. The curve that best illustrates the variation of potential difference across the resistor as a function of time is

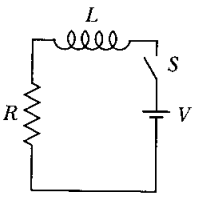
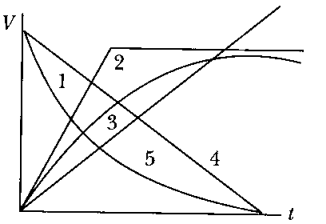


FIG. 31-55 Multiple choice question 26.



(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

27. A 2 H coil, a 10 μF resistor, and a 5 V battery are all connected in parallel. After a current is well established through the resistor and inductor, the battery is disconnected from the circuit by opening a switch as shown in Fig. 31-56. Immediately after the battery is disconnected, Lenz's law would predict

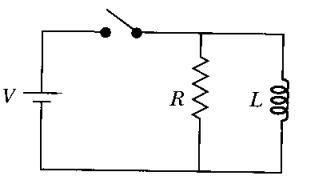


FIG. 31-56 Multiple choice question 27.

(a) The voltage across R is zero

(b) The current through R reverses direction

(c) The current through the inductor reverses direction, but its magnitude tends to remain the same

(d) The magnitude and direction of the voltage across the inductor tend to remain the same

(e) The magnitude and direction of the currents in the resistor and inductor tend to remain the same

28. The circuit shown in Fig. 31-57 is in a steady state with the switch open. When the switch is closed, which of the following will change immediately? (All of them eventually change, but three of them stay the same for an instant.)

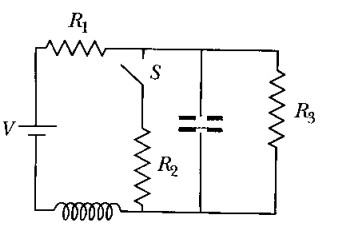


FIG. 31-57 Multiple choice question 28.

(a) The potential difference around the capacitor C

(b) The current through the inductor L

(c) The potential difference around the resistor R_1

(d) The current through the resistor R_3

29. In a circuit made up of inductor, resistance, ammeter, battery, and switch in series, at which of the following times after the switch is closed is the rate of increase of current greatest?

(a) The rate of current increase does not change in time

(b) One time constant

(c) Reciprocal of one time constant

(d) Zero

(e) Ten time constants

30. A battery is connected to an ideal solenoid and a light bulb in parallel (Fig. 31-58). When the switch is opened, the light bulb

(a) Remains off

(b) Instantly goes off

(c) Slowly dims out

(d) Keeps burning as brightly

(e) Flares up brightly, then dims and goes off

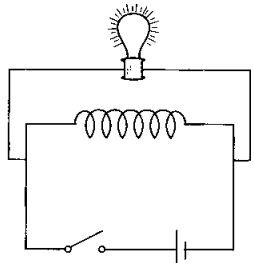


FIG. 31-58 Multiple choice question 30.

31. In the circuit shown in Fig. 31-59, the switch is moved to position 1 at $t = 0$. Which of the following most closely represents the current i as a function of time?

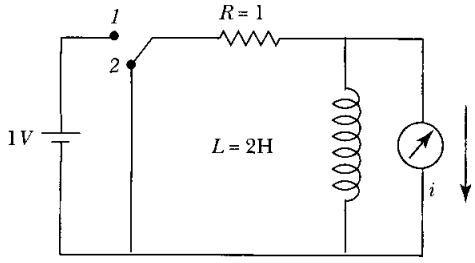
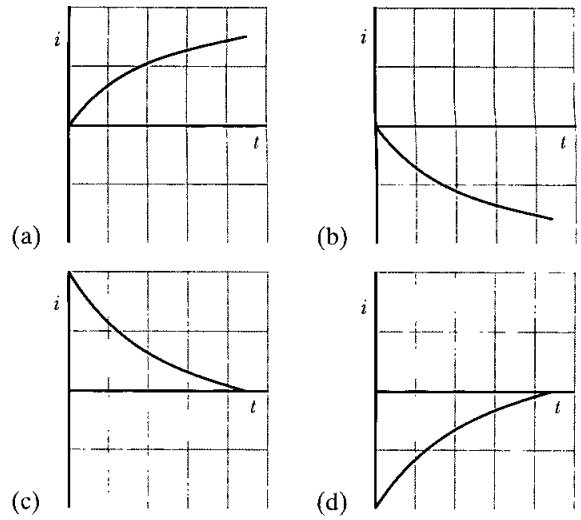


FIG. 31-59 Multiple choice question 31.



32. Figure 31-60 shows three circuits with identical batteries, identical inductors, and identical resistors. Rank them according to the current through the battery just after the switch is closed, from least to greatest.

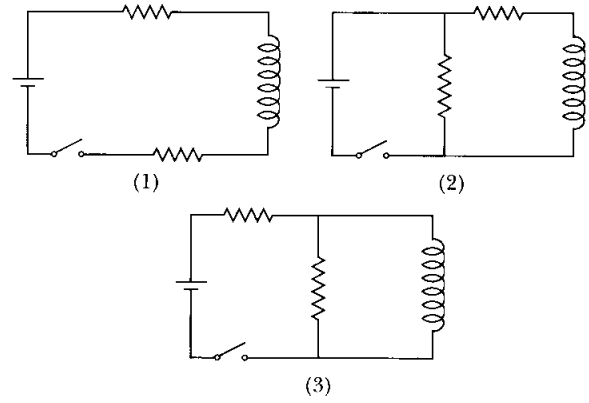


FIG. 31-60 Multiple choice question 32.

(a) 3,2,1

(b) 2 and 3 ties, then 1

(c) 1,3,2

(d) 1,2,3

(e) 3,1,2

PROBLEMS

sec. 31-2 Gauss' Law for Magnetic Fields

1. At the Earth's equator, B is horizontal and to the north, with a magnitude of approximately 7×10^{-5} T. (a) What is the magnetic flux through a loop of radius 0.1 m, lying flat on the ground? (b) If the same loop is balanced on its edge such that its axis points northwest, what is the magnetic flux through the loop? (c) If the loop is now rotated such that its axis points north, what is the magnetic flux through the loop? (d) If the loop is now rotated such that its axis points west, what is the magnetic flux through the loop?

2. A square loop 10 cm to a side is placed on a wooden table in a uniform magnetic field of magnitude 0.25 T. It is found that the biggest magnetic flux through the loop is measured when the loop is flat on the table. What is the flux through the loop when it is tilted such that the plane of the loop makes a 60° angle with the table?

3. Figure 31-61 shows a closed surface. Along the flat top face, which has a radius of 2.0 cm, a perpendicular magnetic field B of magnitude 0.31 T is directed outward. Along the flat bottom face, a magnetic flux of 0.70 mWb is directed outward. What

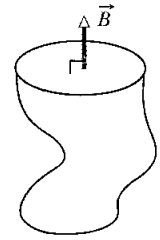


FIG. 31-61 Problem 3.

are the (a) magnitude and (b) direction (inward or outward) of the magnetic flux through the curved part of the surface?

4. The magnetic flux through each of five faces of a die (singular of "dice") is given by $\Phi_B = \pm N$ Wb, where N (= 1 to 5) is the number of spots on the face. The flux is positive (outward) for N even and negative (inward) for N odd. What is the flux through the sixth face of the die?

5. Consider the situation as shown in Fig. 31-62. The loop is a square of side a and its closest edge is at a distance a from the straight long wire. Find the magnetic flux through the loop.

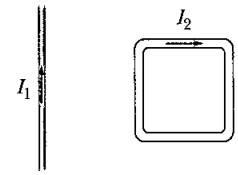


FIG. 31-62 Problem 5.

sec. 31-4 Faraday's Law of Induction

6. The magnetic flux through a conducting loop of area 100 cm^2 is increasing at a rate of 3.0 Tm^2/s. What is the magnitude of the induced emf in the loop?

7. Initially there is no magnetic flux through a certain conducting loop. A magnetic field near the loop is then suddenly turned on, and 5 s later, the magnetic flux through the loop is 1.0 Tm^2. What is the average magnitude of the induced emf in the loop during those 5 s?

8. A gold necklace has a resistance of 0.0021 Ω and forms three concentric loops with an area of 450 cm^2 each. There is a magnetic field of 2.0 T perpendicular to the plane of the necklace, and the field is changing at the rate of 0.028 T/s. What is the current induced in the necklace?

9. An elastic conducting material is stretched into a circular loop of 12.0 cm radius. It is placed with its plane perpendicular to a uniform 0.800 T magnetic field. When released, the radius of the loop starts to shrink at an instantaneous rate of 75.0 cm/s. What emf is induced in the loop at that instant?

10. A small loop of area 6.8 mm^2 is placed inside a long solenoid that has 854 turns/cm and carries a sinusoidally varying current i of amplitude 1.28 A and angular frequency 212 rad/s. The central axes of the loop and solenoid coincide. What is the amplitude of the emf induced in the loop?

11. In Fig. 31-63, a wire forms a closed circular loop, with radius R = 2.0 m and resistance 4.0 Ω. The circle is centered on a long straight wire; at time t = 0, the current in the long straight wire is 5.0 A rightward. Thereafter, the current changes according to i = 5.0 A − (2.0 A/s^2)t^2. (The straight wire is insulated; so there is no electrical contact between it and the wire of the loop.) What is the magnitude of the current induced in the loop at times $t > 0$?

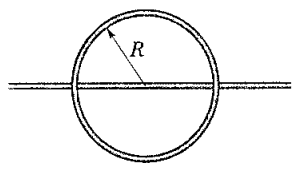


FIG. 31-63 Problem 11.

12. In Fig. 31-64, a stiff wire bent into a semicircle of radius a = 2.0 cm is rotated at constant angular speed 40 rev/s in a uniform 20 mT magnetic field. What are the (a) frequency and (b) amplitude of the emf induced in the loop?

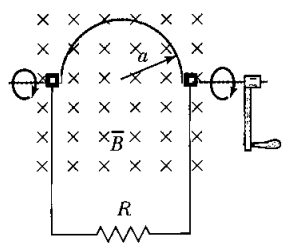


FIG. 31-64 Problem 12.

13. At a certain place, Earth's magnetic field has magnitude B = 0.590 gauss and is inclined downward at an angle of 70.0° to the horizontal. A flat horizontal circular coil of wire with a radius of 10.0 cm has 1000 turns and a total resistance of 85.0 Ω. It is connected in series to a meter with 140 Ω resistance. The coil is flipped through a half-revolution about a diameter, so that it is again horizontal. How much charge flows through the meter during the flip?

14. One hundred turns of (insulated) copper wire are wrapped around a wooden cylindrical core of cross-sectional area 1.20 × 10^{-3} m^2. The two ends of the wire are connected to a resistor. The total resistance in the circuit is 13.0 Ω. If an externally applied uniform longitudinal magnetic field in the core changes from 1.60 T in one direction to 1.60 T in the opposite direction, how much charge flows through a point in the circuit during the change?

15. In Fig. 31-65, a circular loop of wire 10 cm in diameter (seen edge-on) is placed with its normal $\vec{N}$ at an angle θ = 31° with the direction of a uniform magnetic field of magnitude 0.50 T. The loop is then rotated such that $\vec{N}$ rotates in a cone about the field direction at the rate 100 rev/min; angle θ remains unchanged during the process. What is the emf induced in the loop?

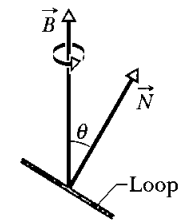


FIG. 31-65 Problem 15.

16. A small circular loop of area 2.00 cm^2 is placed in the plane of, and concentric with, a large circular loop of radius 1.00 m. The current in the large loop is changed at a constant rate from 200 A to −200 A (a change in direction) in a time of 1.00 s, beginning at t = 0. What is the magnitude of the magnetic field at the center of the small loop due to the current in the large loop at (a) t = 0, (b) t = 0.500 s, and (c) t = 1.00 s? (d) From t = 0 to t = 1.00 s, is $\vec{B}$ reversed? Because the inner loop is small, assume $\vec{B}$ is uniform over its area. (e) What emf is induced in the small loop at t = 0.500 s?

17. A single conducting loop with an area of 2.0 m^2 rotates in a uniform magnetic field so that the induced emf has a sinusoidal time dependence as shown in Fig. 31-66. Determine the strength of the magnetic field in which the loop rotates.

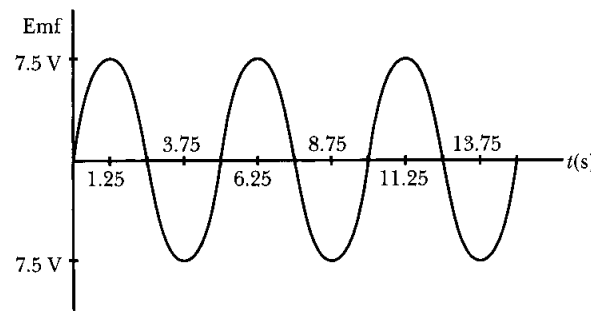


FIG. 31-66 Problem 17.

18. A uniform magnetic field $\vec{B}$ is perpendicular to the plane of a circular wire loop of radius r. The magnitude of the field varies with time according to $B = B_0 e^{-t/\tau}$, where B_0 and τ are constants. Find an expression for the emf in the loop as a function of time.

sec. 31-5 Lenz's Law

19. In Fig. 31-67, the magnetic flux through the loop increases according to the relation $\Phi_B = 6.0t^2 + 7.0t$, where Φ_B is in milliwebers and t is in seconds. (a) What is the magnitude of the emf induced in the loop when $t = 2.0$ s? (b) Is the direction of the current through R to the right or left?

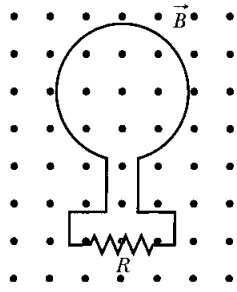


FIG. 31-67 Problem 19.

20. A wire is bent into three circular segments, each of radius $r = 10$ cm, as shown in Fig. 31-68. Each segment is a quadrant of a circle, *ab* lying in the *xy* plane, *bc* lying in the *yz* plane, and *ca* lying in the *zx* plane. (a) If a uniform magnetic field $\vec{B}$ points in the positive x direction, what is the magnitude of the emf developed in the wire when B increases at the rate of 3.0 mT/s? (b) What is the direction of the current in segment *bc*?

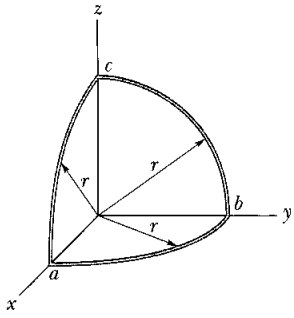


FIG. 31-68 Problem 20.

sec. 31-6 *Motional Emf*

21. A 1 m rod is falling vertically with a velocity of 50 m/s. The rod is always horizontal and in the east–west direction. If the horizontal component of Earth's magnetic field (the angle of declination is zero) is 3×10^{-5} T, find the motional emf between the ends of the rod.

22. An airliner with a wingspan of 33.5 m travels toward the east at a speed of 280 m/s in a region where the Earth's magnetic field is 4.2×10^{-5} T directed toward the north and 53° below horizontal. (a) What is the induced emf between the plane's wingtips? (b) Will a bulb glow if we connect the tip of wings by a conducting wire with a bulb in between?

23. In Fig. 31-69, a long rectangular conducting loop, of width L, resistance R, and mass m, is hung in a horizontal, uniform magnetic field $\vec{B}$ that is directed into the page and that exists only above line *aa*. The loop is then dropped; during its fall, it accelerates until it reaches a certain terminal speed v_t. Ignoring air drag, find an expression for v_t.

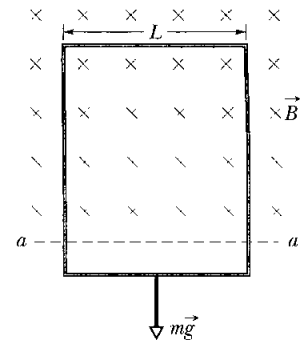


FIG. 31-69 Problem 23.

24. As shown in Fig. 31-70, a perfectly conducting slope frame with inclining angle θ is placed in a uniform magnetic field $\vec{B}$ pointing upward. A conducting rod of length L, mass M, resistance R, is moving down the frame at steady speed v. (a) Find v in terms of L, M, R, B, and any other physics constants you think appropriate. (b) If the direction of B is reversed, what will be the direction of the steady velocity when the rod is released from rest? (c) If R = infinity, describe the motion of the rod after it is released from rest.

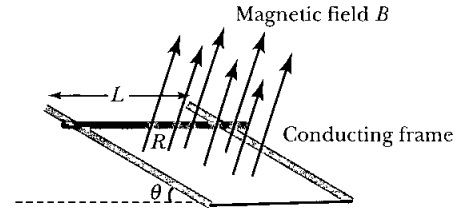


FIG. 31-70 Problem 24.

25. The long, horizontal pair of rails as shown in Fig. 31-71 is connected using resistance R. The distance between the rails is l, the electrical resistance of the rails is negligible. A conducting rod of mass m and length l can slide without friction on the pair of rails, in a vertical, homogeneous magnetic field of induction B. A force of magnitude F_0 is exerted for a long time onto the conducting rod, therefore the speed of the rod is constant. The external force stops at a certain point P. What distance does the conducting rod cover from point P before stopping?

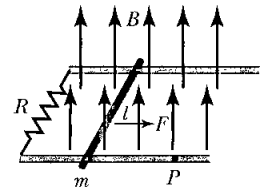


FIG. 31-71 Problem 25.

26. A wire of mass m and length l can slide freely on a pair of smooth, vertical rails in Fig. 31-72. A magnetic field $\vec{B}$ exists in the region in the direction perpendicular to the plane of the rails. The rails are connected at the top end by a capacitor of capacitance C. Find the acceleration of the wire neglecting any electric resistance.

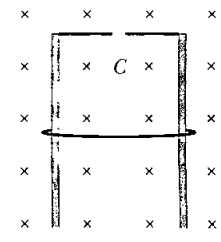


FIG. 31-72 Problem 26.

27. In Fig. 31-73, two straight conducting rails form a right angle. A conducting bar in contact with the rails starts at the

vertex at time $t = 0$ and moves with a constant velocity of 5.20 m/s along them. A magnetic field with $B = 0.350$ T is directed out of the page. Calculate (a) the flux through the triangle formed by the rails and bar at $t = 3.00$ s and (b) the emf around the triangle at that time. (c) If the emf is $E = at^n$, where a and n are constants, what is the value of n?

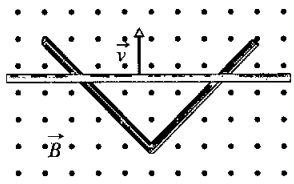


FIG. 31-73 Problem 27.

28. In Fig. 31-74, a rectangular loop of wire with length $a = 2.2$ cm, width $b = 0.80$ cm, and resistance $R = 0.40$ mΩ is placed near an infinitely long wire carrying current $i = 4.7$ A. The loop is then moved away from the wire at constant speed $v = 3.2$ mm/s. When the center of the loop is at distance $r = 1.5b$, what are (a) the magnitude of the magnetic flux through the loop and (b) the current induced in the loop?

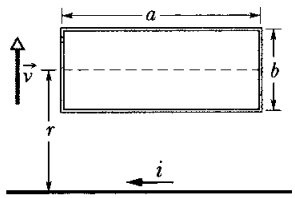


FIG. 31-74 Problem 28.

29. A straight rod of length l is rotating about an axis passing through O as shown in Fig. 31-75. A uniform magnetic field $\vec{B}$ exists parallel to the axis of rotation. What is the emf induced between P and Q?

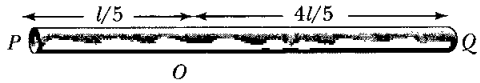


FIG. 31-75 Problem 29.

sec. 31-7 Induction and Energy Transfers

30. If 50.0 cm of copper wire (diameter = 1.00 mm) is formed into a circular loop and placed perpendicular to a uniform magnetic field that is increasing at the constant rate of 10.0 mT/s, at what rate is thermal energy generated in the loop?

31. A loop antenna of area 2.00 cm^2 and resistance 5.21 μΩ is perpendicular to a uniform magnetic field of magnitude 17.0 μT. The field magnitude drops to zero in 2.96 ms. How much thermal energy is produced in the loop by the change in field?

32. In Fig. 31-76, a metal rod is forced to move with constant velocity $\vec{v}$ along two parallel metal rails, connected with a strip of metal at one end. A magnetic field of magnitude $B = 0.350$ T points out of the page. (a) If the rails are separated by $L = 25.0$ cm and the speed of the rod is 55.0 cm/s, what emf is generated? (b) If the rod has a resistance of 18.0 Ω and the rails and connector have negligible resistance, what is the current in the rod? (c) At what rate is energy being transferred to thermal energy?

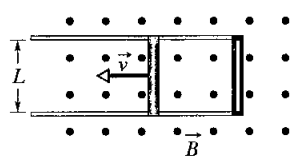


FIG. 31-76 Problems 32 and 33.

33. The conducting rod shown in Fig. 31-76 has length L and is being pulled along horizontal, frictionless conducting rails at a constant velocity $\vec{v}$. The rails are connected at one end with a metal strip. A uniform magnetic field $\vec{B}$, directed out of the page, fills the region in which the rod moves. Assume that $L = 10$ cm, $v = 5.0$ m/s, and $B = 1.2$ T. What are the (a) magnitude and (b) direction (up or down the page) of the emf induced in the rod? What are the (c) size and (d) direction of the current in the conducting loop? Assume that the resistance of the rod is 0.40 Ω and that the resistance of the rails and metal strip is negligibly small. (e) At what rate is thermal energy being generated in the rod? (f) What external force on the rod is needed to maintain $\vec{v}$? (g) At what rate does this force do work on the rod?

34. Figure 31-77 shows a rod of length $L = 10.0$ cm that is forced to move at constant speed $v = 5.00$ m/s along horizontal rails. The rod, rails, and connecting strip at the right form a conducting loop. The rod has resistance 0.400 Ω; the rest of the loop has negligible resistance. A current $I = 100$ A through the long straight wire at distance $a = 10.0$ mm from the loop sets up a (nonuniform) magnetic field through the loop. Find the (a) emf and (b) current induced in the loop. (c) At what rate is thermal energy generated in the rod? (d) What is the magnitude of the force that must be applied to the rod to make it move at constant speed? (e) At what rate does this force do work on the rod?

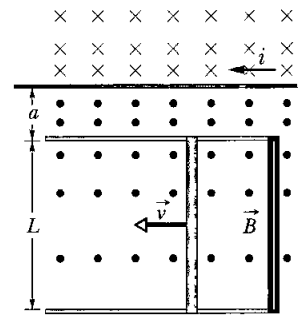


FIG. 31-77 Problem 34.

sec. 31-8 Induced Electric Fields

35. A long solenoid has a diameter of 12.0 cm. When a current i exists in its windings, a uniform magnetic field of magnitude $B = 31.0$ mT is produced in its interior. By decreasing i, the field is caused to decrease at the rate of 6.50 mT/s. Calculate the magnitude of the induced electric field (a) 2.20 cm and (b) 8.20 cm from the axis of the solenoid.

36. A circular region in an xy plane is penetrated by a uniform magnetic field in the positive direction of the z axis. The field's magnitude B (in teslas) increases with time t (in seconds) according to $B = at$, where a is a constant. The magnitude E of the electric field set up by that increase in the magnetic field is given by Fig. 31-78 versus radial distance r; the vertical axis scale is set by $E_s = 310$ μN/C, and the horizontal axis scale is set by $r_s = 4.00$ cm. Find a.

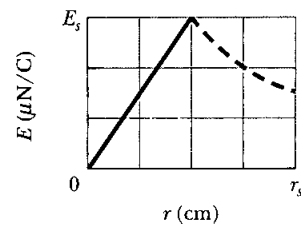


FIG. 31-78 Problem 36.

37. Figure 31-79 shows two circular regions R_1 and R_2 with radii $r_1 = 20.0$ cm and $r_2 = 31.0$ cm. In R_1 there is a uniform

magnetic field of magnitude $B_1 = 50.0$ mT directed into the page, and in R_2 there is a uniform magnetic field of magnitude $B_2 = 75.0$ mT directed out of the page (ignore fringing). Both fields are decreasing at the rate of 8.50 mT/s. Calculate $\oint \vec{E} \cdot d\vec{s}$ for (a) path 1, (b) path 2, and (c) path 3.

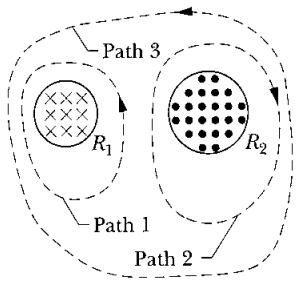


FIG. 31-79 Problem 37.

38. A uniform magnetic field $\vec{B}$ exists in a cylindrical region, shown dotted in Fig. 31-80. The magnetic field increase at a constant rate dB/dt. Consider a circle of radius r coaxial with the cylindrical region. (a) Find the magnitude of the electric field at a point on the circumference of the circle. (b) Consider a point P on the side of the square circumscribing the circle. Show that the component of the induced electric field at P along ba is the same as the magnitude found in part (a).

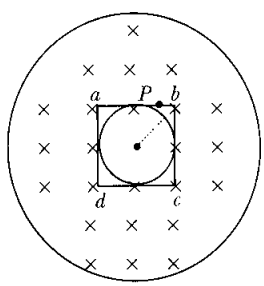


FIG. 31-80 Problem 38.

39. The magnetic field in the cylindrical region as shown in Fig. 31-81 increases at a constant rate of 20.0 mT/s. Each side of the square loop *abcd* and *defa* has a length of 1.00 cm and a resistance of 4.00 Ω. Find the current (magnitude and sense) in the wire *ad* if (a) the switch S_1 is closed by S_2 is open, (b) S_1 is open but S_2 is closed, (c) both S_1 and S_2 are open and (d) both S_1 and S_2 are closed.

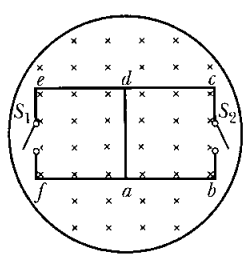


FIG. 31-81 Problem 39.

sec. 31-9 Inductors and Inductance

40. Calculate the inductance of a cylindrical solenoid with length 0.2 m, radius 0.03 m, and a total of 2000 coils.

41. The inductance of a closely packed coil of 400 turns is 8.0 mH. Calculate the magnetic flux through the coil when the current is 5.0 mA.

42. A circular coil has a 10.0 cm radius and consists of 31.0 closely wound turns of wire. An externally produced magnetic field of magnitude 2.60 mT is perpendicular to the coil. (a) If no current is in the coil, what magnetic flux links its turns? (b) When the current in the coil is 3.80 A in a certain direction, the net flux through the coil is found to vanish. What is the inductance of the coil?

43. Two identical long wires of radius $a = 1.53$ mm are parallel and carry identical currents in opposite directions. Their center-to-center separation is $d = 14.2$ cm. Neglect the flux within the wires but consider the flux in the region between the wires. What is the inductance per unit length of the wires?

sec. 31-10 Self-Induction

44. A 12 H inductor carries a current of 2.0 A. At what rate must the current be changed to produce a 60 V emf in the inductor?

45. At a given instant the current and self-induced emf in an inductor are directed as indicated in Fig. 31-82. (a) Is the current increasing or decreasing? (b) The induced emf is 17 V, and the rate of change of the current is 25 kA/s; find the inductance.

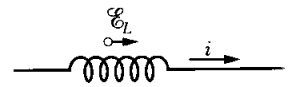


FIG. 31-82 Problem 45.

46. The inductor arrangement of Fig. 31-83, with $L_1 = 31.0$ mH, $L_2 = 50.0$ mH, $L_3 = 20.0$ mH, and $L_4 = 15.0$ mH, is to be connected to a varying current source. What is the equivalent inductance of the arrangement?

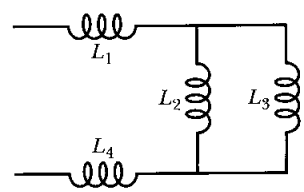


FIG. 31-83 Problem 46.

47. The inductance of a closely wound coil is such that an emf of 3.00 mV is induced when the current changes at the rate of 5.00 A/s. A steady current of 8.00 A produces a magnetic flux of 40.0 μWb through each turn. (a) Calculate the inductance of the coil. (b) How many turns does the coil have?

48. The induced emf in an inductor of inductance L varies with time according to $e = -2Ct$, where C is a positive constant. (a) If no current flows through the inductor at time $t = 0$, calculate the current flowing through the inductor as a function of time for $t > 0$. (b) Is the current increasing or decreasing for times $t > 0$?

sec. 31-11 *RL* Circuits

49. The current in an *RL* circuit builds up to one-third of its steady-state value in 5.00 s. Find the inductive time constant.

50. In Fig. 31-84, $\mathscr{E} = 100$ V, $R_1 = 10.0$ Ω, $R_2 = 20.0$ Ω, $R_3 = 31.0$ Ω, and $L = 2.00$ H. Immediately after switch S is closed, what are (a) i_1 and (b) i_2? (Let currents in the indicated directions have positive values and currents in the opposite directions have negative values.) A long time later, what are (c) i_1 and (d) i_2? The switch is then reopened. Just then, what are (e) i_1 and (f) i_2? A long time later, what are (g) i_1 and (h) i_2?

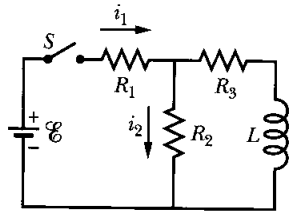


FIG. 31-84 Problem 50.

51. In Fig. 31-85, R = 15 Ω, L = 5.0 H, the ideal battery has $\mathscr{E}$ = 10 V, and the fuse in the upper branch is an ideal 3.0 A fuse. It has zero resistance as long as the current through it remains less than 3.0 A. If the current reaches 3.0 A, the fuse "blows" and thereafter has infinite resistance. Switch S is closed at time t = 0. (a) When does the fuse blow? (*Hint:* Equation 31-42 does not apply. Rethink Eq. 31-40.) (b) Sketch a graph of the current i through the inductor as a function of time. Mark the time at which the fuse blows.

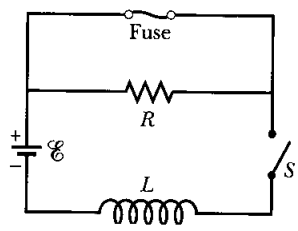


FIG. 31-85 Problem 51.

52. A wooden toroidal core with a square cross section has an inner radius of 10 cm and an outer radius of 12 cm. It is wound with one layer of wire (of diameter 1.0 mm and resistance per meter 0.020/m). What are (a) the inductance and (b) the inductive time constant of the resulting toroid? Ignore the thickness of the insulation on the wire.

53. Switch S in Fig. 31-86 is closed at time t = 0, initiating the buildup of current in the 15.0 mH inductor and the 20.0 resistor. At what time is the emf across the inductor equal to the potential difference across the resistor?

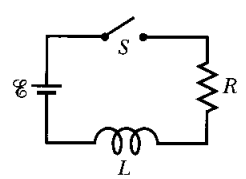


FIG. 31-86 Problem 53 and 77.

54. A 12 V battery is connected in series with a switch, resistor and inductor. If the circuit's time constant is 2.0×10^{-4} and the final steady current after the switch is closed becomes 1.0 A, what is the value of the inductance?

sec. 31-12 Energy Stored in a Magnetic Field

55. A coil is connected in series with a 10.0 kΩ resistor. An ideal 50.0 V battery is applied across the two devices, and the current reaches a value of 2.00 mA after 5.00 ms. (a) Find the inductance of the coil. (b) How much energy is stored in the coil at this same moment?

56. A coil with an inductance of 2.0 H and a resistance of 10 Ω is suddenly connected to an ideal battery with $\mathscr{E}$ = 100 V. (a) What is the equilibrium current? (b) How much energy is stored in the magnetic field when this current exists in the coil?

57. A coil with an inductance of 2.0 H and a resistance of 10 Ω is suddenly connected to an ideal battery with $\mathscr{E}$ = 100 V. At 0.10 s after the connection is made, what is the rate at which (a) energy is being stored in the magnetic field, (b) thermal energy is appearing in the resistance, and (c) energy is being delivered by the battery?

58. At t = 0, a battery is connected to a series arrangement of a resistor and an inductor. If the inductive time constant is 37.0 ms, at what time is the rate at which energy is dissipated in the resistor equal to the rate at which energy is stored in the inductor's magnetic field?

59. A solenoid that is 85.0 cm long has a cross-sectional area of 17.0 cm^2. There are 950 turns of wire carrying a current of 6.60 A. (a) Calculate the energy density of the magnetic field inside the solenoid. (b) Find the total energy stored in the magnetic field there (neglect end effects).

60. A toroidal inductor with an inductance of 90.0 mH encloses a volume of 0.0200 m^3. If the average energy density in the toroid is 70.0 J/m^3, what is the current through the inductor?

61. A length of copper wire carries a current of 10 A uniformly distributed through its cross section. Calculate the energy density of (a) the magnetic field and (b) the electric field at the surface of the wire. The wire diameter is 2.5 mm, and its resistance per unit length is 3.3 Ω/km.

62. A cubical region of space with side lengths 0.05 m contains a uniform magnetic field. If the total magnetic energy stored in this region is $\pi \cdot$ J, what is the magnitude of the magnetic field?

63. What pressure does the lateral surface of a long straight solenoid with n turns per unit length experience when a current I flows through it?

sec. 31-13 Mutual Induction

64. Two solenoids are part of the spark coil of an automobile. When the current in one solenoid falls from 6.0 A to zero in 2.5 ms, an emf of 31 kV is induced in the other solenoid. What is the mutual inductance M of the solenoids?

65. Two coils are at fixed locations. When coil 1 has no current and the current in coil 2 increases at the rate 15.0 A/s, the emf in coil 1 is 25.0 mV. (a) What is their mutual inductance? (b) When coil 2 has no current and coil 1 has a current of 3.60 A, what is the flux linkage in coil 2?

66. Two coils connected as shown in Fig. 31-87 separately have inductances L_1 and L_2. Their mutual inductance is M. (a) Show that this combination can be replaced by a single coil of equivalent inductance given by

$$L_{eq} = L_1 + L_2 + 2M.$$

(b) How could the coils in Fig. 31-65 be reconnected to yield an equivalent inductance of

$$L_{eq} = L_1 + L_2 - 2M?$$

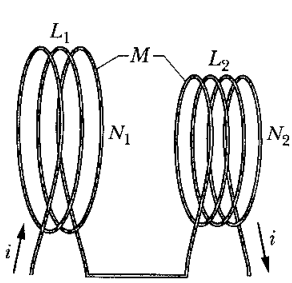


FIG. 31-87 Problem 66.

67. A coil C of N turns is placed around a long solenoid S of radius R and n turns per unit length, as in Fig. 31-88. (a) Show that the mutual inductance for the coil–solenoid combination

is given by $M = \mu_0 \pi R^2 nN$. (b) Explain why M does not depend on the shape, size, or possible lack of close packing of the coil.

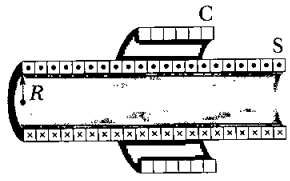


FIG. 31-88 Problem 67.

68. In Problem 67, show that $M_{12} = M_{21} = M$.

69. Two coils are close to each other (Fig. 31-89), with one connected to an alternating current source which produces a current of $I_1 = (5.0\text{ A}) \sin (240t)$, while the other is connected to a resister of 12 Ω. Due to the mutual inductance, current I_2 flows through the resistor. If the current I_2 has a maximum of 0.40 A, what is the mutual inductance of the two coils?

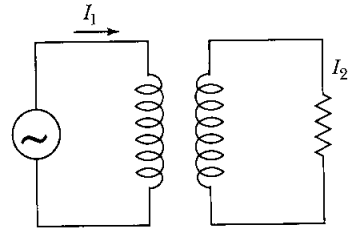


FIG. 31-89 Problem 69.

ADDITIONAL PROBLEMS

70. Two wires, parallel to a z axis and a distance $4r$ apart, carry equal currents i in opposite directions, as shown in Fig. 31-90. A circular cylinder of radius r and length L has its axis on the z axis, midway between the wires. Use Gauss' law for magnetism to derive an expression for the net outward magnetic flux through the half of the cylindrical surface above the x axis. (*Hint:* Find the flux through the portion of the xz plane that lies within the cylinder.)

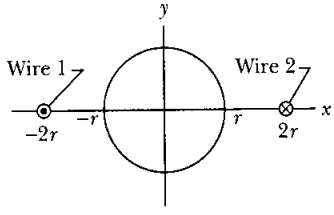


FIG. 31-90 Problem 70.

71. Figure 31-91 shows two parallel loops of wire having a common axis. The smaller loop (radius r) is above the larger loop (radius R) by a distance $x \gg R$ Consequently, the magnetic field due to the counterclockwise current i in the larger loop is nearly uniform throughout the smaller loop. Suppose that x is increasing at the constant rate $dx/dt = v$. (a) Find an expression for the magnetic flux through the area of the smaller loop as a function of x. In the smaller loop, find (b) an expression for the induced emf and (c) the direction of the induced current.

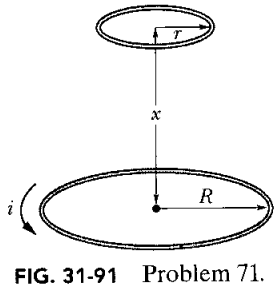


FIG. 31-91 Problem 71.

72. Figure 31-92 shows a metallic square frame of edge a in a vertical plane. A uniform magnetic field $\vec{B}$ exists in the space in a direction perpendicular to the plane of the figure. Two boys pull the opposite corners of the square to deform it into a rhombus. They start pulling the corners at $t = 0$ and displace the corners at a uniform speed u. (a) Find the induced emf in the frame at the instant when the angles at these corners reduce to 60°. (b) Find the induced current in the frame at this instant if the total resistance of the frame is R. (c) Find the total charge which flows through a side of the frame by the time the square is deformed into a straight line.

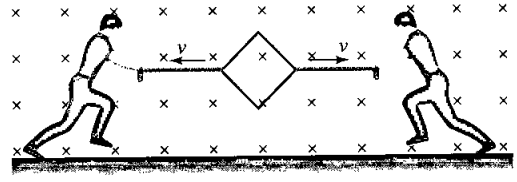


FIG. 31-92 Problem 72.

73. Figure 31-93 shows a uniform magnetic field $\vec{B}$ confined to a cylindrical volume of radius R. The magnitude of $\vec{B}$ is decreasing at a constant rate of 10 mT/s. In unit-vector notation, what is the initial acceleration of an electron released at (a) point a (radial distance $r = 5.0$ cm), (b) point b ($r = 0$), and (c) point c ($r = 5.0$ cm)?

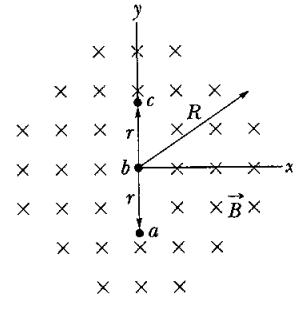


FIG. 31-93 Problem 73.

74. In Fig. 31-94, the battery is ideal, $\mathscr{E} = 10$ V, $R_1 = 5.0$ Ω, $R_2 = 10$ Ω, and $L = 5.0$ H. Switch S is closed at time $t = 0$. Just afterwards, what are (a) i_1, (b) i_2, (c) the current i_S through the switch, (d) the potential difference V_2 across resistor 2, (e) the potential difference V_L across the inductor, and (f) the rate of change di_2/dt? A long time later, what are (g) i_1, (h) i_2, (i) i_S, (j) V_2, (k) V_L, and (l) di_2/dt?

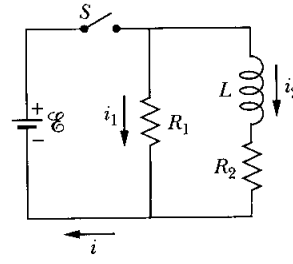


FIG. 31-94 Problem 74.

75. In Fig. 31-95*a*, switch S has been closed on A long enough to establish a steady current in the inductor of inductance $L_1 = 5.00$ mH and the resistor of resistance $R_1 = 25.0$ Ω. Similarly, in Fig. 31-73*b*, switch S has been closed on A long enough to establish a steady current in the inductor of inductance $L_2 = 3.00$ mH and the resistor of resistance $R_2 = 31.0$ Ω. The ratio Φ_{02} / Φ_{01} of the magnetic flux through a turn in inductor 2 to that in inductor 1 is 1.50. At time $t = 0$, the two switches are closed on B. At what time t is the flux through a turn in the two inductors equal?

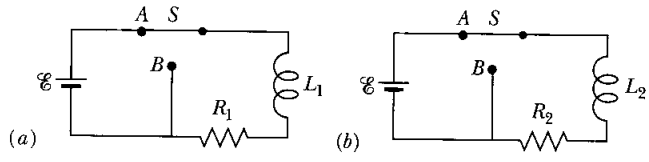


FIG. 31-95 Problem 75.

76. Figure 31-96*a* shows a rectangular conducting loop of resistance $R = 0.020\ \Omega$, height $H = 1.5$ cm, and length $D = 2.5$ cm being pulled at constant speed $v = 40$ cm/s through two regions of uniform magnetic field. Figure 31-96*b* gives the current i induced in the loop as a function of the position x of the right side of the loop. The vertical axis scale is set by $i_s = 3.0\ \mu$A. For example, a current equal to i_s is induced clockwise as the loop enters region 1. What are the (a) magnitude and (b) direction (into or out of the page) of the magnetic field in region 1? What are the (c) magnitude and (d) direction of the magnetic field in region 2?

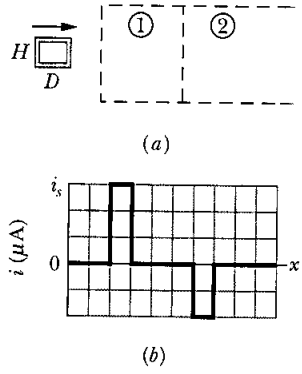


FIG. 31-96 Problem 76.

77. Once the switch S is closed in Fig. 31-86, the time required for the current to reach any obtainable value depends, in part, on the value of resistance R. Suppose the emf E of the ideal battery is 12 V and the inductance of the ideal (resistanceless) inductor is 18 mH. How much time is needed for the current to reach 2.00 A if R is (a) 1.00 Ω (b) 5.00 Ω, and (c) 6.00 Ω? (d) For what value of R is the time required for the current to reach 2.00 A least? (e) What is that least time? (*Hint:* Rethink Eq. 31-40.)

78. Figure 31-97 shows a coil of $N2$ turns wound as shown around part of a toroid of $N1$ turns. The toroid's inner radius is a, its outer radius is b, and its height is h. Show that the mutual inductance M for the toroid–coil combination is given by

$$M = \frac{\mu_0 N_1 N_2 h}{2\pi} \ln \frac{b}{a}.$$

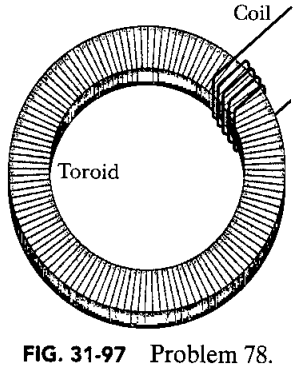


FIG. 31-97 Problem 78.

79. Figure 31-98 shows a conducting circular loop of radius a placed in a uniform, perpendicular magnetic field $\vec{B}$. A thick metal rod OA is pivoted at the center O. The other end of the rod touches the loop at A. The center O and a fixed point C on the loop are connected by a wire OC of resistance R. A force is applied at the middle point of the rod OA perpendicularly, s that the rod rotates clockwise at uniform angular velocity ω. Find the force.

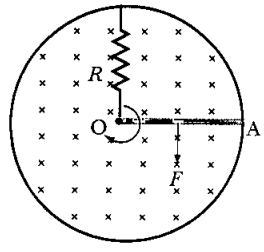


FIG. 31-98 Problem 79.

80. Suppose the emf of the battery in the circuit shown in Fig. 31-32 varies with time t so that the current is given by $i(t) = 3.0 + 5.0t$, where i is in amperes and t is in seconds. Take $R = 4.0\ \Omega$ and $L = 6.0$ H, and find an expression for the battery emf as a function of t. (*Hint:* Apply the loop rule.)

81. In Fig. 31-99, after switch S is closed at time $t = 0$, the emf of the source is automatically adjusted to maintain a constant current i through S. (a) Find the current through the inductor as a function of time. (b) At what time is the current through the resistor equal to the current through the inductor?

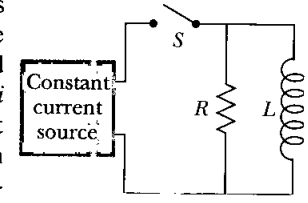


FIG. 31-99 Problem 81.

82. A loop as shown in Fig. 31-100 is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field with induction B. The distance between the conductors is equal to l. At the moment $t = 0$, the rod is imparted an initial velocity v_0 directed to the right. Find the displacement of the rod as a function of time if the electric resistance of the loop is negligible.

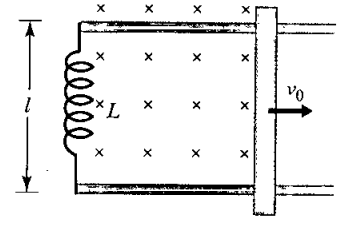


FIG. 31-100 Problem 82.

83. A superconducting round ring of radius a and inductance L was located in a uniform magnetic field of induction $\vec{B}$. The ring plane was parallel to the vector B, and the current in the ring was equal to zero. Then the ring was turned through 90° so that its plane became perpendicular to the field. Find (a) the current induced in the ring after the turn and (b) the work performed during the turn.

84. Show that the mutual inductance of two coils must be smaller than the geometric mean of their self-inductances.

85. Two identical conducting bars rest on two horizontal parallel conducting rails. The bars are perpendicular to the rails and parallel to each other as shown in Fig. 31-101. The distance between the bars is l. At a certain moment, a uniform vertical upward magnetic field is turned on. The field quickly reaches its maximum magnitude and then remains constant. Neglecting friction, find the final distance between the bars. Assume that the resistance of each bar is much greater than the resistance of the rails. Also assume that the force due to one bar on another is negligible.

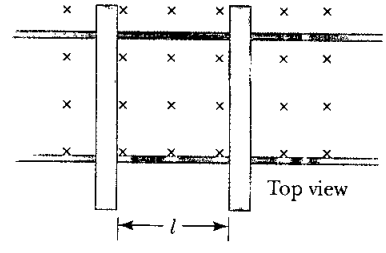


FIG. 31-101 Problem 85.

ANSWERS

Checkpoints

1. b, then d and e tie, and then a and c tie (zero)
2. a and b tie, then c (zero)
3. (a) We can break the motion in two parts. (a) Pure rotation about any point on the rod and (b) pure translation with the velocity of that point. The motional emf due to both kinds of motion can be added to give the total induced emf; (b) $\frac{B\omega l^2}{2} \pm Blv$
4. c and d tie, then a and b tie
5. b, out; c, out; d, into; e, into
6. The voltmeter will read zero because although there is induced electric field in space, but the loop forming the voltmeter and the metal ring has no induced emf across it
7. d and e
8. (a) 2, 3, 1 (zero); (b) 2, 3, 1
9. a and b tie, then c

Reasoning Type Questions

1. (a)
2. (a)
3. (a)
4. (d)
5. (d)
6. (a)
7. (a)

Multiple Choice Questions

1. (b), (d)
2. (c)
3. (c)
4. (d)
5. (d)
6. (c)
7. (a)
8. (c)
9. (c)
10. (a)
11. (e)
12. (e)
13. (d)
14. (b)
15. (d)
16. (d)
17. (d)
18. (a)
19. (d)
20. (c)
21. (a)
22. (d)
23. (e)
24. (b)
25. (a), (b)
26. (c)
27. (b)
28. (d)
29. (d)
30. (e)
31. (c)
32. (c)

Problems

1. (a) zero; (b) $11\sqrt{2} \times 10^{-7}$ Tm; (c) 22×10^{-7} Tm^2; (d) zero
2. 1.25×10^{-3} Tm^2
3. (a) 1.1 mWb; (b) inward
4. +3 Wb
5. $(\mu_0 Ia/2\pi)\ln 2$
6. 0.03 V
7. 0.2 V
8. 1.8 A
9. 0.452 V
10. 0.198 mV
11. 0
12. (a) 40 Hz; (b) 3.2 mV
13. 1.55×10^{-5} C
14. 29.5 mC
15. 0
16. (a) 1.26×10^{-4} T; (b) 0; (c) 1.26×10^{-4} T; (d) yes; (e) 5.04×10^{-8} V
17. 2.98 T

18. $\mathcal{E} = \pi r^2 \frac{B_0}{\tau} e^{-t/\tau}$

19. (a) 31 mV; (b) left

20. (a) 2.4×10^{-5} V; (b) the current in segment *bc* flows from *c* to *b* (following Lenz's law)

21. 1.5 mV

22. (a) 0.315 V; (b) no

23. $v_t = mgR/B^2L^2$

24. (a) $MgR\sin\theta / B^2L^2$; (b) downward; (c) moves down with constant acceleration of $g\sin\theta$

25. F_0mR^2/B^4l^4

26. $mg/m+CB^2l^2$

27. (a) 85.2 Wb; (b) 56.8 V; (c) 1

28. (a) 1.4×10^{-8} Wb; (b) 1.0×10^{-5} A

29. $\frac{3}{10} B\omega l^2$

30. 3.68 μW

31. 7.50×10^{-10} J

32. (a) 48.1 mV; (b) 2.67 mA; (c) 0.129 mW

33. (a) 0.60 V; (b) up; (c) 1.5 A; (d) clockwise; (e) 0.90 W; (f) 0.18 N; (g) 0.90 W

34. (a) 240 μV; (b) 0.600 mA; (c) 0.144 μW; (d) 2.87×10^{-8} N; (e) 0.144 μW

35. (a) 71.5 μV/m; (b) 143 μV/m

36. 0.031 T/s

37. (a) -1.07×10^{-3} V; (b) -2.40×10^{-3} V; (c) 1.33×10^{-3} V

38. (a) $\frac{r}{2}\frac{dB}{dt}$

39. (a) 1.25×10^{-27} A, *a* to *d*; (b) 1.25×10^{-27} A, *d* to *a*; (c) zero; (d) zero

40. 0.071 H

41. 1.0×10^{-7} Wb

42. (a) 2.45 mWb; (b) 0.645 mH

43. 1.81 μH/m

44. 5.0 A/s

45. (a) decreasing; (b) 0.68 mH

46. 59.3 mH

47. (a) 0.600 mH; (b) 120

48. (a) Cr^2/L; (b) increasing with time

49. 12.3 s

50. (a) 3.33 A; (b) 3.33 A; (c) 4.55 A; (d) 2.73 A; (e) 0; (f) –1.82 A; (g) 0; (h) 0

51. (a) 1.5 s; (b) we do not show the graph here; qualitatively, it would be similar to Fig. 31-22

52. (a) 2.9×10^{-4} H ; (b) 2.9×10^{-4}

53. 0.520 ms

54. 2.4 mH

55. (a) 97.9 H; (b) 0.196 mJ

56. (a) 10 A; (b) 1.0×10^2 J

57. (a) 0.20 s; (b) 1.5×10^2 W; (c) 3.9×10^2 W

58. 25.6 ms

59. (a) 34.2 J/m^3; (b) 49.4 mJ

60. 5.58 A

61. (a) 1.0 J/m^3; (b) 4.8×10^{-15} J/m^3

62. 80π mT

63. $p = 1/2\ \mu_0 n^2 I^2$

64. 13 H

65. (a) 1.67 mH; (b) 6.00 mWb

66. (b) Have the turns of the two solenoids wrapped in opposite directions

69. 4 mH

Additional Problems

70. $\Phi_B(S_1) = \Phi_B(S_2) = \int_{-r}^{r} B(x)L\,dx = 2\int_{-r}^{r} B_{\text{left}}(x)L\,dx$

$= 2\int_{-r}^{r} \frac{\mu_0 i}{2\pi}\frac{1}{2r-x} L\,dx = \frac{\mu_0 iL}{\pi}\ln 3$

71. (a) $\mu_0 iR^2\pi r^2 / 2x^3$; (b) $3\mu_0 i\pi R^2 r^2 v / 2x^4$; (c) counterclockwise

72. (a) $2Bav$; (b) $2Bav/R$; (c) a^2B/R

73. (a) $(4.4 \times 10^7\ \text{m/s}^2)$; (b) 0; (c) $-(4.4 \times 10^7\ \text{m/s}^2)\hat{\text{i}}$

74. (a) 2.0 A; (b) 0; (c) 2.0 A; (d) 0; (e) 10 V; (f) 2.0 A/s; (g) 2.0 A; (h) 1.0 A; (i) 3.0 A; (j) 10 V; (k) 0; (l) 0

75. 81.1 μs

76. (a) 10 μT; (b) out; (c) 3.3 μT; (d) out

77. (a) 3.28 ms; (b) 6.45 ms; (c) infinite time; (d) 0; (e) 3.00 ms

79. $\omega a^3 B^2/2R$ to the right of OA in the figure

80. $\varepsilon = 42 + 20t$

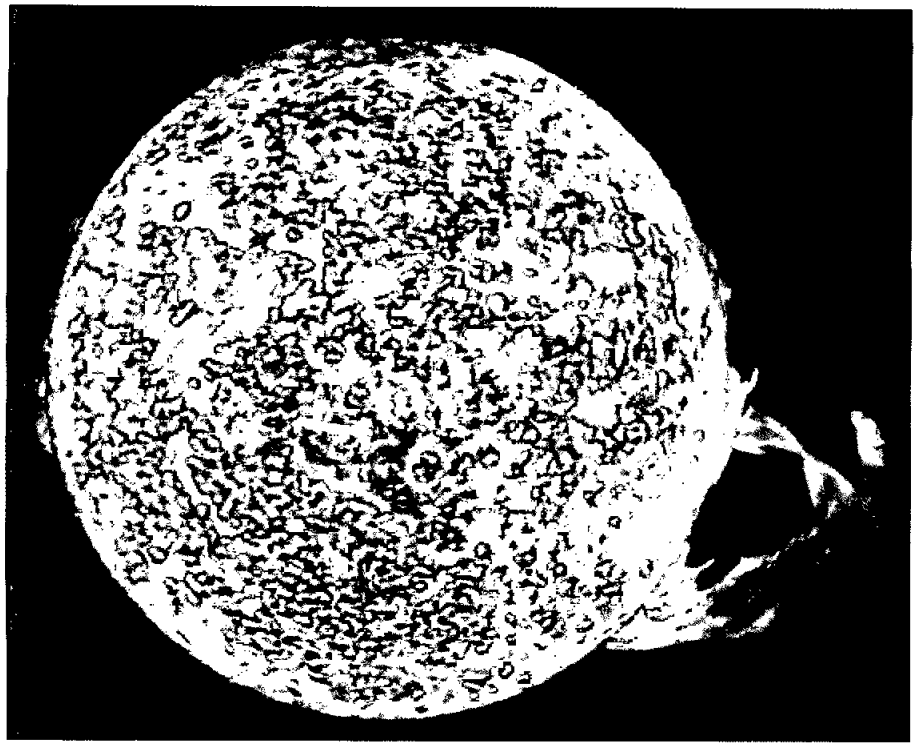
81. (a) $i[1 - \exp(-Rt/L)]$; (b) $(L/R)\ln 2$

82. $x = (n_0/\omega)\sin\omega t$, where $\omega = lB/\sqrt{mL}$

83. (a) $i = \pi a^2 B/L$; (b) $W = 1/2\ \pi^2 a^4 B^2/L$

85. $l/2$

Alternating Fields and Current 32



Courtesy NASA/JSC

At 2:45 A.M. on March 13, 1989, the entire power-grid system for the Canadian province of Quebec failed, leaving millions of people without power on that cold night. In fact, many power-grid systems in the Northern Hemisphere malfunctioned that night, creating a nightmare situation for the engineers who maintained the systems. The cause was not a sudden overtaxing demand for power or a failure of aging equipment. Rather, the cause was an explosion that had occurred on the Sun's surface three days earlier.

How can a solar explosion shut down a power-grid system?

The answer is in this chapter.

32-1 WHAT IS PHYSICS?

We have explored the basic physics of electric and magnetic fields and how energy can be stored in capacitors and inductors. We next turn to the associated applied physics, in which the energy stored in one location can be transferred to another location so that it can be put to use. For example, energy produced at a power plant can show up at your home to run a computer. The total worth of this applied physics is now so high that its estimation is almost impossible. Indeed, modern civilization would be impossible without this applied physics.

In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current, or ac). The challenge to both physicists and engineers is to design ac systems that transfer energy efficiently and to build appliances that make use of that energy.

In our discussion of electrically oscillating systems in this chapter, our first step is to examine oscillations in a simple circuit consisting of inductance L and capacitance C.

32-2 | *LC* Oscillations, Qualitatively

Of the three circuit elements resistance R, capacitance C, and inductance L, we have so far discussed the series combinations RC (in Section 28-9) and RL (in Section 31-11). In these two kinds of circuit we found that the charge, current, and potential difference grow and decay exponentially. The time scale of the growth or decay is given by a *time constant* τ, which is either capacitive or inductive.

We now examine the remaining two-element circuit combination LC. You will see that in this case the charge, current, and potential difference do not decay exponentially with time but vary sinusoidally (with period T and angular frequency ω). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**. Such a circuit is said to oscillate.

Parts *a* through *h* of Fig. 32-1 show succeeding stages of the oscillations in a simple LC circuit. From Eq. 26-21, the energy stored in the electric field of the capacitor at any time is

$$U_E = \frac{q^2}{2C}, \tag{32-1}$$

where q is the charge on the capacitor at that time. From Eq. 31-50, the energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}, \tag{32-2}$$

where i is the current through the inductor at that time.

We now adopt the convention of representing *instantaneous values* of the electrical quantities of a sinusoidally oscillating circuit with small letters, such as q, and the *amplitudes* of those quantities with capital letters, such as Q. With this convention in mind, let us assume that initially the charge q on the capacitor in Fig. 32-1 is at its maximum value Q and that the current i through the inductor is zero. This initial state of the circuit is shown in Fig. 32-1*a*. The bar graphs for energy included there indicate that at this instant, with zero current through the inductor and maximum charge on the capacitor, the energy U_B of the magnetic field is zero and the energy U_E of the electric field is a maximum.

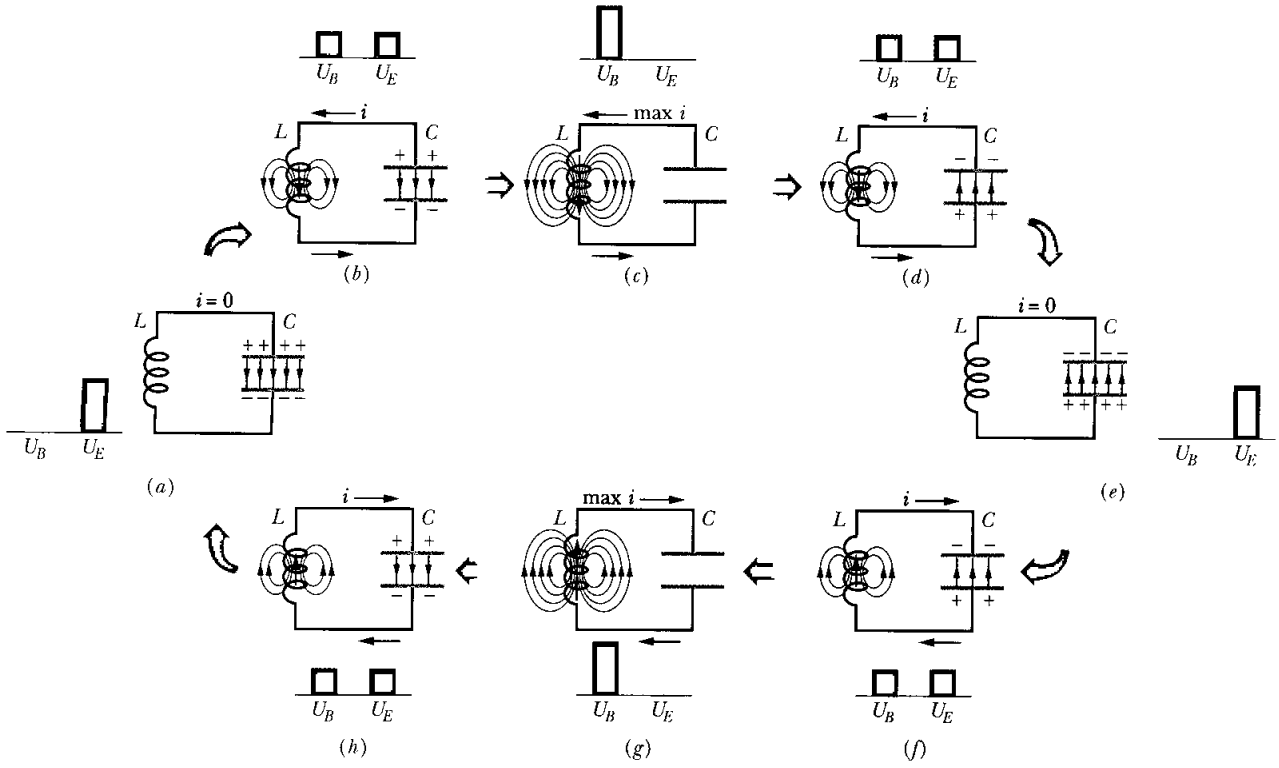


FIG. 32-1 Eight stages in a single cycle of oscillation of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electrical energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (*a*) Capacitor with maximum charge, no current. (*b*) Capacitor discharging, current increasing. (*c*) Capacitor fully discharged, current maximum. (*d*) Capacitor charging but with polarity opposite that in (*a*), current decreasing. (*e*) Capacitor with maximum charge having polarity opposite that in (*a*), no current. (*f*) Capacitor discharging, current increasing with direction opposite that in (*b*). (*g*) Capacitor fully discharged, current maximum. (*h*) Capacitor charging, current decreasing.

The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Fig. 32-1*b*. This means that a current i, given by dq/dt and pointing down in the inductor, is established. As the capacitor's charge decreases, the energy stored in the electric field within the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus, the electric field decreases and the magnetic field builds up as energy is transferred from the electric field to the magnetic field.

The capacitor eventually loses all its charge (Fig. 32-1*c*) and thus also loses its electric field and the energy stored in that field. The energy has then been fully transferred to the magnetic field of the inductor. The magnetic field is then at its maximum magnitude, and the current through the inductor is then at its maximum value I.

Although the charge on the capacitor is now zero, the counterclockwise current must continue because the inductor does not allow it to change suddenly to zero. The current continues to transfer positive charge from the top plate to the bottom plate through the circuit (Fig. 32-1*d*). Energy now flows from the inductor back to the capacitor as the electric field within the capacitor builds up again. The current gradually decreases during this energy transfer. When, eventually, the energy has been transferred completely back to the capacitor (Fig. 32-1*e*), the current has decreased to zero (momentarily). The situation of Fig. 32-1*e* is like the initial situation, except that the capacitor is now charged oppositely.

The capacitor then starts to discharge again but now with a clockwise current (Fig. 32-1*f*). Reasoning as before, we see that the clockwise current builds to a maximum (Fig. 32-1*g*) and then decreases (Fig. 32-1*h*), until the circuit eventually returns to its initial situation (Fig. 32-1*a*). The process then repeats at some frequency f and thus at an angular frequency $\omega = 2\pi f$. In the ideal LC circuit with no resistance, there are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor. Because of the conservation of energy, the oscillations continue indefinitely. The oscillations need not begin with the energy all in the electric field; the initial situation could be any other stage of the oscillation.

To determine the charge q on the capacitor as a function of time, we can put in a voltmeter to measure the time-varying potential difference (or *voltage*) v_C that exists across the capacitor C. From Eq. 26-1 we can write

$$v_C = \left(\frac{1}{C}\right) q,$$

which allows us to find q. To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference v_R across it; v_R is proportional to i through the relation

$$v_R = iR.$$

We assume here that R is so small that its effect on the behavior of the circuit is negligible. The variations in time of v_C and v_R, and thus of q and i, are shown in Fig. 32-2. All four quantities vary sinusoidally.

In an actual LC circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as thermal energy (the circuit may become warmer). The oscillations, once started, will die away as Fig. 32-3 suggests.

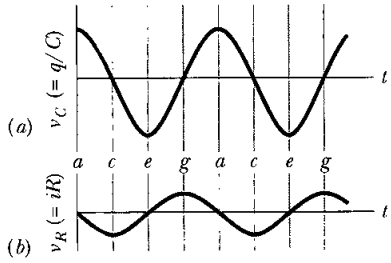


FIG. 32-2 (*a*) The potential difference across the capacitor of the circuit of Fig. 32-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 32-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 32-1.

CHECKPOINT 1 A charged capacitor and an inductor are connected in series at time $t = 0$. In terms of the period T of the resulting oscillations, determine how much later the following reach their maximum value: (a) the charge on the capacitor; (b) the voltage across the capacitor, with its original polarity; (c) the energy stored in the electric field; and (d) the current.

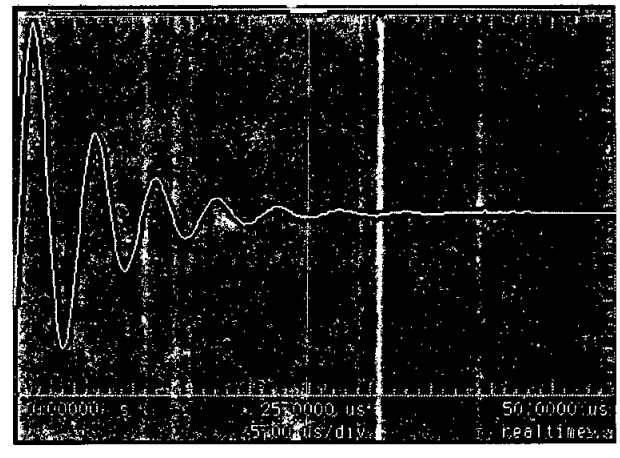


FIG. 32-3 An oscilloscope trace showing how the oscillations in an RLC circuit actually die away because energy is dissipated in the resistor as thermal energy. (*Courtesy Agilent Technologies*)

Sample Problem 32-1

A 1.5 μF capacitor is charged to 57 V. The charging battery is then disconnected, and a 12 mH coil is connected in series with the capacitor so that LC oscillations occur. What is the maximum current in the coil? Assume that the circuit contains no resistance, which would cause the energy to be dissipated.

Conceptualize: Because the circuit contains no resistance, the electromagnetic energy of the circuit is conserved as the energy is transferred back and forth between the electric field of the capacitor and the magnetic field of the coil (inductor).

Classify: At any time t, the energy $U_B(t)$ of the magnetic field is related to the current $i(t)$ through the coil by Eq. 32-2 ($U_B = Li^2/2$). When all the energy is stored as magnetic energy, the current is at its maximum value I and that energy is $U_{B,\max} = LI^2/2$. At any time t, the energy $U_E(t)$ of the electric field is related to the charge $q(t)$ on the capacitor by Eq. 32-1 ($U_E = q^2/2C$). When all the energy is stored as electrical energy, the charge is at its maximum value Q and that energy is $U_{E,\max} = Q^2/2C$.

Compute: With these ideas, we can now write the conservation of energy as

$$U_{B,\max} = U_{E,\max}$$

or

$$\frac{LI^2}{2} = \frac{Q^2}{2C}.$$

Solving for I gives us

$$I = \sqrt{\frac{Q^2}{LC}}.$$

We know L and C, but not Q. However, with Eq. 26-1 ($q = CV$) we can relate Q to the maximum potential difference V across the capacitor, which is the initial potential difference of 57 V. Thus, substituting $Q = CV$ leads to

$$I = V\sqrt{\frac{C}{L}} = (57\text{ V})\sqrt{\frac{1.5\times10^{-6}\text{ F}}{12\times10^{-3}\text{ H}}}$$

$$= 0.637\text{ A} \approx 640\text{ mA}. \qquad \text{(Answer)}$$

Conclude: As will be shown later in the chapter, the charge on the plates follows the equation of SHM.

32-3 | The Electrical–Mechanical Analogy

Let us look a little closer at the analogy between the oscillating LC system of Fig. 32-1 and an oscillating block–spring system. Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block. These two energies are given by the formulas in the left energy column in Table 32-1.

The table also shows, in the right energy column, the two kinds of energy involved in LC oscillations. By looking across the table, we can see an analogy between the forms of the two pairs of energies–the mechanical energies of the block–spring system and the electromagnetic energies of the LC oscillator. The equations for v and i at the bottom of the table help us see the details of the analogy. They tell us that q corresponds to x and i corresponds to v (in both equations, the former is differentiated to obtain the latter). These correspondences then suggest that, in the energy expressions, $1/C$ corresponds to k and L corresponds to m. Thus,

$$q \text{ corresponds to } x, \qquad 1/C \text{ corresponds to } k,$$

$$i \text{ corresponds to } v, \quad \text{and} \quad L \text{ corresponds to } m.$$

These correspondences suggest that in an LC oscillator, the capacitor is mathematically like the spring in a block–spring system and the inductor is like the block.

TABLE 32-1

Comparison of the Energy in Two Oscillating Systems

Block–Spring System		LC Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

In Section 15-3, Chapter 15, Vol. I, we saw that the angular frequency of oscillation of a (frictionless) block–spring system is

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(block–spring system)}. \tag{32-3}$$

The correspondences listed above suggest that to find the angular frequency of oscillation for an ideal (resistanceless) LC circuit, k should be replaced by $1/C$ and m by L, yielding

$$\omega = \frac{1}{\sqrt{LC}} \qquad (LC \text{ circuit}). \tag{32-4}$$

We derive this result in the next section.

32-4 | *LC* Oscillations, Quantitatively

Here we want to show explicitly that Eq. 32-4 for the angular frequency of LC oscillations is correct. At the same time, we want to examine even more closely the analogy between LC oscillations and block–spring oscillations. We start by extending somewhat our earlier treatment of the mechanical block–spring oscillator.

The Block–Spring Oscillator

We analyzed block–spring oscillations in Chapter 15 in terms of energy transfers and did not—at that early stage—derive the fundamental differential equation that governs those oscillations. We do so now.

We can write, for the total energy U of a block–spring oscillator at any instant,

$$U = U_b + U_s = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2, \tag{32-5}$$

where U_b and U_s are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction—which we assume—the total energy U remains constant with time, even though v and x vary. In more formal language, $dU/dt = 0$. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0. \tag{32-6}$$

However, $v = dx/dt$ and $dv/dt = d^2x/dt^2$. With these substitutions, Eq. 32-6 becomes

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{(block–spring oscillations)}. \tag{32-7}$$

Equation 32-7 is the fundamental *differential equation* that governs the frictionless block–spring oscillations.

The general solution to Eq. 32-7—that is, the function $x(t)$ that describes the block–spring oscillations—is

$$x = X\cos(\omega t + \phi) \qquad \text{(displacement)}, \tag{32-8}$$

in which X is the amplitude of the mechanical oscillations, ω is the angular frequency of the oscillations, and ϕ is a phase constant.

The LC Oscillator

Now let us analyze the oscillations of a resistanceless LC circuit, proceeding exactly as we just did for the block–spring oscillator. The total energy U present at any instant in an oscillating LC circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}, \tag{32-9}$$

in which U_B is the energy stored in the magnetic field of the inductor and U_E is the energy stored in the electric field of the capacitor. Since we have assumed the circuit resistance to be zero, no energy is transferred to thermal energy and U remains constant with time. In more formal language, dU/dt must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0. \tag{32-10}$$

However, $i = dq/dt$ and $di/dt = d^2q/dt^2$. With these substitutions, Eq. 32-10 becomes

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad (LC \text{ oscillations}). \tag{32-11}$$

This is the *differential equation* that describes the oscillations of a resistanceless LC circuit. Equations 32-11 and 32-7 are exactly of the same mathematical form.

Charge and Current Oscillations

Since the differential equations are mathematically identical, their solutions must also be mathematically identical. Because q corresponds to x, we can write the general solution of Eq. 32-11, by analogy to Eq. 32-8, as

$$q = Q\cos(\omega t + \phi) \qquad \text{(charge)}, \tag{32-12}$$

where Q is the amplitude of the charge variations, ω is the angular frequency of the electromagnetic oscillations, and ϕ is the phase constant.

Taking the first derivative of Eq. 32-12 with respect to time gives us the current of the LC oscillator:

$$i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi) \qquad \text{(current)}. \tag{32-13}$$

The amplitude I of this sinusoidally varying current is

$$I = \omega Q, \tag{32-14}$$

and so we can rewrite Eq. 32-13 as

$$i = -I\sin(\omega t + \phi). \tag{32-15}$$

Angular Frequencies

We can test whether Eq. 32-12 is a solution of Eq. 32-11 by substituting Eq. 32-12 and its second derivative with respect to time into Eq. 32-11. The first derivative of Eq. 32-12 is Eq. 32-13. The second derivative is then

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t + \phi).$$

Substituting for q and d^2q/dt^2 in Eq. 32-11, we obtain

$$-L\omega^2 Q\cos(\omega t + \phi) + \frac{1}{C}Q\cos(\omega t + \phi) = 0.$$

Canceling $Q\cos(\omega t + \phi)$ and rearranging lead to

$$\omega = \frac{1}{\sqrt{LC}}.$$

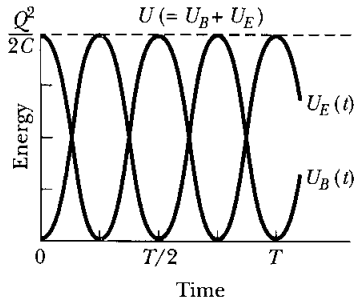


FIG. 32-4 The stored magnetic energy and electrical energy in the circuit of Fig. 32-1 as a function of time. Note that their sum remains constant. T is the period of oscillation.

Thus, Eq. 32-12 is indeed a solution of Eq. 32-11 if ω has the constant value $1/\sqrt{LC}$. Note that this expression for ω is exactly that given by Eq. 32-4, which we arrived at by examining correspondences.

The phase constant ϕ in Eq. 32-12 is determined by the conditions that exist at any certain time—say, $t = 0$. If the conditions yield $\phi = 0$ at $t = 0$, Eq. 32-12 requires that $q = Q$ and Eq. 32-13 requires that $i = 0$; these are the initial conditions represented by Fig. 32-1*a*.

Electrical and Magnetic Energy Oscillations

The electrical energy stored in the LC circuit at time t is, from Eqs. 32-1 and 32-12,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi). \tag{32-16}$$

The magnetic energy is, from Eqs. 32-2 and 32-13,

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 Q^2 \sin^2(\omega t + \phi).$$

Substituting for ω from Eq. 32-4 then gives us

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi). \tag{32-17}$$

Figure 32-4 shows plots of $U_E(t)$ and $U_B(t)$ for the case of $\phi = 0$. Note that

1. The maximum values of U_E and U_B are both $Q^2/2C$.
2. At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
3. When U_E is maximum, U_B is zero, and conversely.

✓**CHECKPOINT 2** A capacitor in an LC oscillator has a maximum potential difference of 17 V and a maximum energy of 160 μJ. When the capacitor has a potential difference of 5 V and an energy of 10 μJ, what are (a) the emf across the inductor and (b) the energy stored in the magnetic field?

Sample Problem 32-2

For the situation described in Sample Problem 32-1, let the coil (inductor) be connected to the charged capacitor at time $t = 0$. The result is an LC circuit like that in Fig. 32-1.

(a) What is the potential difference $v_L(t)$ across the inductor as a function of time?

Conceptualize: The current and potential differences of the circuit undergo sinusoidal oscillations.

Classify: We can still apply the loop rule to this oscillating circuit—just as we did for the nonoscillating circuits of Chapter 28.

Compute: At any time t during the oscillations, the loop rule and Fig. 32-1 give us

$$v_L(t) = v_C(t); \tag{32-18}$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with Eq. 26-1 ($q = CV$).

Because the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t = 0$, the charge q on the capacitor must also be maximum then. Thus, phase constant ϕ must be zero; so Eq. 32-12 gives us

$$q = Q\cos\omega t. \tag{32-19}$$

(Note that this cosine function does indeed yield maximum q ($= Q$) when $t = 0$.) To get the potential difference $v_C(t)$, we divide both sides of Eq. 32-19 by C to write

$$\frac{q}{C} = \frac{Q}{C}\cos\omega t,$$

and then use Eq. 26-1 to write

$$v_C = V_C\cos\omega t. \tag{32-20}$$

Here, V_C is the amplitude of the oscillations in the potential difference v_C across the capacitor.

Next, substituting $v_C = v_L$ from Eq. 32-18, we find

$$v_L = V_C \cos \omega t. \tag{32-21}$$

We can evaluate the right side of this equation by first noting that the amplitude V_C is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then, using the values of L and C from Sample Problem 32-1, we find ω with Eq. 32-4:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012\,\text{H})(1.5\times10^{-6}\,\text{F})]^{0.5}}$$
$$= 7454\ \text{rad/s} \approx 7500\ \text{rad/s}.$$

Thus, Eq. 32-21 becomes

$$v_L = (57\ \text{V})\cos(7500\ \text{rad/s})t. \qquad \text{(Answer)}$$

(b) What is the maximum rate $(di/dt)_{\text{max}}$ at which the current i changes in the circuit?

Conceptualize/Classify: With the charge on the capacitor oscillating as in Eq. 32-12, the current is in the form of Eq. 32-13. Because $\phi = 0$, that equation gives us

$$i = -\omega Q \sin \omega t.$$

Conepute: Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q) and $1/\sqrt{LC}$ for ω according to Eq. 32-4. We get

$$\frac{di}{dt} = -\frac{1}{LC} CV_C \cos \omega t = -\frac{V_C}{L} \cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57\ \text{V}}{0.012\ \text{H}} = 4750\ \text{A/s} \approx 4800\ \text{A/s}. \qquad \text{(Answer)}$$

32-5 | Alternating Current

The oscillations in an RLC circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance R. Circuits in homes, offices, and factories, including countless RLC circuits, receive such energy from local power companies. In most countries the energy is supplied via oscillating emfs and currents—the current is said to be an **alternating current**, or **ac** for short. (The nonoscillating current from a battery is said to be a **direct current**, or **dc**.) These oscillating emfs and currents vary sinusoidally with time, reversing direction (in India) 100 times per second and thus having frequency $f = 50$ Hz.

At first sight this may seem to be a strange arrangement. We have seen that the drift speed of the conduction electrons in household wiring may typically be 4×10^{-5} m/s. If we now reverse their direction every $\frac{1}{100}$, such electrons can move only about 4×10^{-7} m in a half-cycle. At this rate, a typical electron can drift past no more than about 10 atoms in the wiring before it is required to reverse its direction. How, you may wonder, can the electron ever get anywhere?

Although this question may be worrisome, it is a needless concern. The conduction electrons do not have to "get anywhere." When we say that the current in a wire is one ampere, we mean that charge passes through any plane cutting across that wire at the rate of one coulomb per second. The speed at which the charge carriers cross that plane does not matter directly; one ampere may correspond to many charge carriers moving very slowly or to a few moving very rapidly. Furthermore, the signal to the electrons to reverse directions—which originates in the alternating emf provided by the power company's generator—is propagated along the conductor at a speed close to that of light. All electrons, no matter where they are located, get their reversal instructions at about the same instant. Finally, we note that for many devices, such as lightbulbs and toasters, the direction of motion is unimportant as long as the electrons do move so as to transfer energy to the device via collisions with atoms in the device.

The basic advantage of alternating current is this: *As the current alternates, so does the magnetic field that surrounds the conductor.* This makes possible the use of Faraday's law of induction, which, among other things, means that we can step

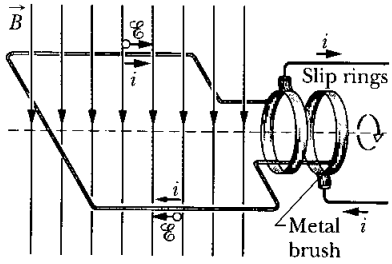


FIG. 32-5 The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

up (increase) or step down (decrease) the magnitude of an alternating potential difference at will, using a device called a transformer, as we shall discuss later. Moreover, alternating current is more readily adaptable to rotating machinery such as generators and motors than is (nonalternating) direct current.

Figure 32-5 shows a simple model of an ac generator. As the conducting loop is forced to rotate through the external magnetic field $\vec{B}$, a sinusoidally oscillating emf $\mathscr{E}$ is induced in the loop:

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t. \tag{32-22}$$

The *angular frequency* ω_d of the emf is equal to the angular speed with which the loop rotates in the magnetic field, the *phase* of the emf is $\omega_d t$, and the *amplitude* of the emf is $\mathscr{E}_m$ (where the subscript stands for maximum). When the rotating loop is part of a closed conducting path, this emf produces (*drives*) a sinusoidal (alternating) current along the path with the same angular frequency ω_d, which then is called the **driving angular frequency**. We can write the current as

$$i = I \sin(\omega_d t - \phi), \tag{32-23}$$

in which I is the amplitude of the driven current. (The phase $\omega_d t - \phi$ of the current is traditionally written with a minus sign instead of as $\omega_d t + \phi$.) We include a phase constant ϕ in Eq. 32-23 because the current i may not be in phase with the emf $\mathscr{E}$. (As you will see, the phase constant depends on the circuit to which the generator is connected.) We can also write the current i in terms of the **driving frequency** f_d of the emf, by substituting $2\pi f_d$ for ω_d in Eq. 32-33.

32-6 | Forced Oscillations

We have seen that once started, the charge, potential difference, and current in undamped LC circuits oscillate at angular frequency $\omega = 1/\sqrt{LC}$. Such oscillations are said to be *free oscillations* (free of any external emf), and the angular frequency ω is said to be the circuit's **natural angular frequency**.

When the external alternating emf of Eq. 32-22 is connected to an RLC circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations*. These oscillations always occur at the driving angular frequency ω_d:

☞ Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

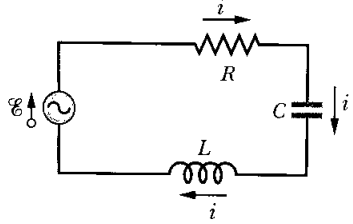


FIG. 32-6 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

However, as you will see in Section 32-8, the amplitudes of the oscillations very much depend on how close ω_d is to ω. When the two angular frequencies match—a condition known as **resonance**—the amplitude I of the current in the circuit is maximum.

32-7 | Three Simple Circuits

Later in this chapter, we shall connect an external alternating emf device to a series RLC circuit as in Fig. 32-6. We shall then find expressions for the amplitude I and phase constant ϕ of the sinusoidally oscillating current in terms of the amplitude $\mathscr{E}_m$ and angular frequency ω_d of the external emf. First, let's consider three simpler circuits, each having an external emf and only one other circuit element: R, C, or L. We start with a resistive element (a purely *resistive load*).

A Resistive Load

Figure 32-7 shows a circuit containing a resistance element of value R and an ac generator with the alternating emf of Eq. 32-22. By the loop rule, we have

$$\mathscr{E} - v_R = 0.$$

With Eq. 32-22, this gives us

$$v_R = \mathscr{E}_m \sin \omega_d t.$$

Because the amplitude V_R of the alternating potential difference (or voltage) across the resistance is equal to the amplitude $\mathscr{E}_m$ of the alternating emf, we can write this as

$$v_R = V_R \sin \omega_d t. \tag{32-24}$$

From the definition of resistance ($R = V/i$), we can now write the current i_R in the resistance as

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t. \tag{32-25}$$

From Eq. 32-23, we can also write this current as

$$i_R = I_R \sin(\omega_d t - \phi), \tag{32-26}$$

where I_R is the amplitude of the current i_R in the resistance. Comparing Eqs. 32-25 and 32-26, we see that for a purely resistive load the phase constant $\phi = 0°$. We also see that the voltage amplitude and current amplitude are related by

$$V_R = I_R R \qquad \text{(resistor)}. \tag{32-27}$$

Although we found this relation for the circuit of Fig. 32-7, it applies to any resistance in any ac circuit.

By comparing Eqs. 32-24 and 32-25, we see that the time-varying quantities v_R and i_R are both functions of sin $\omega_d t$ with $\phi = 0°$. Thus, these two quantities are *in phase,* which means that their corresponding maxima (and minima) occur at the same times. Figure 32-8*a*, which is a plot of $v_R(t)$ and $i_R(t)$, illustrates this fact. Note that v_R and i_R do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in R.

The time-varying quantities v_R and i_R can also be represented geometrically by *phasors.* Recall from Section 16-11, Vol. I, that phasors are vectors that rotate around an origin. Those that represent the voltage across and current in the resistor of Fig. 32-7 are shown in Fig. 32-8 at an arbitrary time t. Such phasors have the following properties:

Angular speed: Both phasors rotate counterclockwise about the origin with an angular speed equal to the angular frequency ω_d of v_R and i_R.

Length: The length of each phasor represents the amplitude of the alternating quantity: V_R for the voltage and I_R for the current.

Projection: The projection of each phasor on the *vertical* axis represents the value of the alternating quantity at time t: v_R for the voltage and i_R for the current.

Rotation angle: The rotation angle of each phasor is equal to the phase of the alternating quantity at time t. In Fig. 32-8*b*, the voltage and current are in phase; so their phasors always have the same phase $\omega_d t$ and the same rotation angle, and thus they rotate together.

Mentally follow the rotation. Can you see that when the phasors have rotated so that $\omega_d t = 90°$ (they point vertically upward), they indicate that just then $v_R = V_R$ and $i_R = I_R$? Equations 32-24 and 32-26 give the same results.

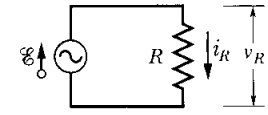


FIG. 32-7 A resistor is connected across an alternating-current generator.

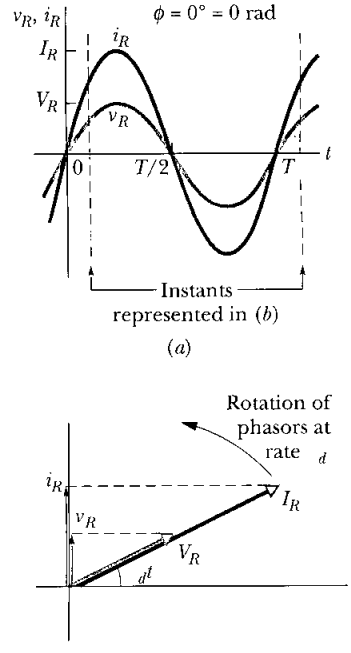


FIG. 32-8 (*a*) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T. (*b*) A phasor diagram shows the same thing as (*a*).

CHECKPOINT 3 If we increase the driving frequency in a circuit with a purely resistive load, do (a) amplitude V_R and (b) amplitude I_R increase, decrease, or remain the same?

Sample Problem 32-3

Purely resistive load. In Fig. 32-7, resistance R is 200 Ω and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What is the potential difference $v_R(t)$ across the resistance as a function of time t, and what is the amplitude V_R of $v_R(t)$?

Conceptualize/Classify: In a circuit with a purely resistive load, the potential difference $v_R(t)$ across the resistance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Compute: Here we have $v_R(t) = \mathscr{E}(t)$ and $V_R = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we can write

$$V_R = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_R(t)$, we use Eq. 32-22 to write

$$v_R(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t \qquad (32\text{-}27)$$

and then substitute $\mathscr{E}_m = 36.0$ V and

$$\omega_d = 2\pi f_d = 2\pi(60 \text{ Hz}) = 120\pi$$

to obtain

$$v_R = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

We can leave the argument of the sine in this form for convenience, or we can write it as (377 rad/s)t or as (377 s^{-1})t.

(b) What are the current $i_R(t)$ in the resistance and the amplitude I_R of $i_R(t)$?

Conceptualize/Classify: In an ac circuit with a purely resistive load, the alternating current $i_R(t)$ in the resistance is *in phase* with the alternating potential difference $v_R(t)$ across the resistance; that is, the phase constant ϕ for the current is zero.

Compute: Here we can write Eq. 32-23 as

$$i_R = I_R \sin(\omega_d t - \phi) = I_R \sin \omega_d t. \qquad (32\text{-}28)$$

From Eq. 32-27, the amplitude I_R is

$$I_R = \frac{V_R}{R} = \frac{36.0 \text{ V}}{200 \Omega} = 0.180 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 32-28, we have

$$i_R = (0.180 \text{ A}) \sin(120\pi t). \qquad \text{(Answer)}$$

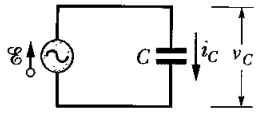


FIG. 32-9 A capacitor is connected across an alternating-current generator.

A Capacitive Load

Figure 32-9 shows a circuit containing a capacitance and a generator with the alternating emf of Eq. 32-22. Using the loop rule and proceeding as we did when we obtained Eq. 32-24, we find that the potential difference across the capacitor is

$$v_C = V_C \sin \omega_d t, \qquad (32\text{-}29)$$

where V_C is the amplitude of the alternating voltage across the capacitor. From the definition of capacitance we can also write

$$q_C = Cv_C = CV_C \sin \omega_d t. \qquad (32\text{-}30)$$

Our concern, however, is with the current rather than the charge. Thus, we differentiate Eq. 32-30 to find

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t. \qquad (32\text{-}31)$$

We now modify Eq. 32-31 in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_C, called the **capacitive reactance** of a capacitor, defined as

$$X_C = \frac{1}{\omega_d C} \quad \text{(capacitive reactance)}. \qquad (32\text{-}32)$$

Its value depends not only on the capacitance but also on the driving angular frequency ω_d. We know from the definition of the capacitive time constant ($\tau = RC$) that the SI unit for C can be expressed as seconds per ohm. Applying this to Eq. 32-32 shows that the SI unit of X_C is the *ohm*, just as for resistance R.

Second, we replace $\cos \omega_d t$ in Eq. 32-31 with a phase-shifted sine:

$$\cos \omega_d t = \sin(\omega_d t + 90°).$$

You can verify this identity by shifting a sine curve 90° in the negative direction.

With these two modifications, Eq. 32-31 becomes

$$i_C = \left(\frac{V_C}{X_C}\right)\sin(\omega_d t + 90°). \tag{32-33}$$

From Eq. 32-23, we can also write the current i_C in the capacitor of Fig. 32-9 as

$$i_C = I_C \sin(\omega_d t - \phi), \tag{32-34}$$

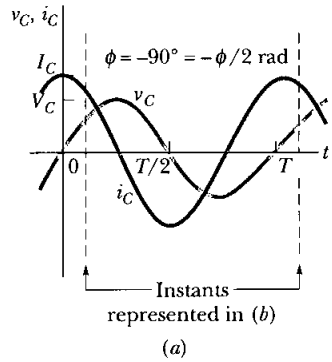
where I_C is the amplitude of i_C. Comparing Eqs. 32-33 and 32-34, we see that for a purely capacitive load the phase constant ϕ for the current is –90°. We also see that the voltage amplitude and current amplitude are related by

$$V_C = I_C X_C \quad \text{(capacitor)}. \tag{32-35}$$

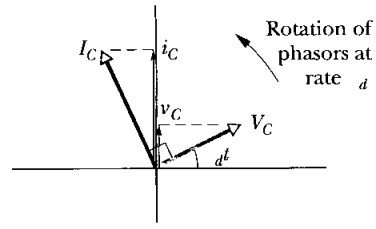
Although we found this relation for the circuit of Fig. 32-9, it applies to any capacitance in any ac circuit.

Comparison of Eqs. 32-29 and 32-33, or inspection of Fig. 32-10*a*, shows that the quantities v_C and i_C are 90°, π/2 rad, or one-quarter cycle, out of phase. Furthermore, we see that i_C *leads* v_C, which means that, if you monitored the current i_C and the potential difference v_C in the circuit of Fig. 32-9, you would find that i_C reaches its maximum *before* v_C does, by one-quarter cycle.

This relation between i_C and v_C is illustrated by the phasor diagram of Fig. 32-10*b*. As the phasors representing these two quantities rotate counterclockwise together, the phasor labeled I_C does indeed lead that labeled V_C, and by an angle of 90°; that is, the phasor I_C coincides with the vertical axis one-quarter cycle before the phasor V_C does. Be sure to convince yourself that the phasor diagram of Fig. 32-10*b* is consistent with Eqs. 32-29 and 32-33.



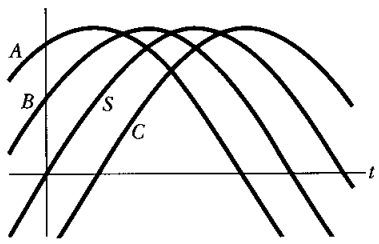
(*a*)



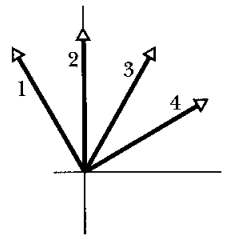
(*b*)

FIG. 32-10 (*a*) The current in the capacitor leads the voltage by 90° (= π/2 rad). (*b*) A phasor diagram shows the same thing.

CHECKPOINT 4 The figure shows, in (*a*), a sine curve $S(t) = \sin(\omega_d t)$ and three other sinusoidal curves $A(t)$, $B(t)$, and $C(t)$, each of the form $\sin(\omega_d t - \phi)$. (a) Rank the three other curves according to the value of ϕ, most positive first and most negative last. (b) Which curve corresponds to which phasor in (*b*) of the figure? (c) Which curve leads the others?



(*a*)



(*b*)

Sample Problem 32-4

Purely capacitive load. In Fig. 32-9, capacitance C is 15.0 μF and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_C(t)$ across the capacitance and the amplitude V_C of $v_C(t)$?

Conceptualize/Classify: In a circuit with a purely capacitive load, the potential difference $v_C(t)$ across the capacitance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Compute: Here we have $v_C(t) = \mathscr{E}(t)$ and $V_C = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we have

$$V_C = \mathscr{E}_m = 36.0\text{ V}. \qquad \text{(Answer)}$$

To find $v_C(t)$, we use Eq. 32-22 to write

$$v_C(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (32\text{-}36)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 32-36, we have

$$v_C = (36.0\text{ V})\sin(120\,\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_C(t)$ in the circuit as a function of time and the amplitude I_C of $i_C(t)$?

Conceptualize/Classify: In an ac circuit with a purely capacitive load, the alternating current $i_C(t)$ in the capacitance leads the alternating potential difference $v_C(t)$ by 90°; that is, the phase constant ϕ for the current is −90°, or $-\pi/2$ rad.

Compute: Thus, we can write Eq. 32-23 as

$$i_C = I_C \sin(\omega_d t - \phi) = I_C \sin(\omega_d t + \pi/2). \qquad (32\text{-}37)$$

We can find the amplitude I_C from Eq. 32-35 ($V_C = I_C X_C$) if we first find the capacitive reactance X_C. From Eq. 32-32 ($X_C = 1/\omega_d C$), with $\omega_d = 2\pi f_d$, we can write

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{(2\pi)(60.0\text{ Hz})(15.0\times10^{-6}\text{ F})} = 177\ \Omega.$$

Then Eq. 32-35 tells us that the current amplitude is

$$I_C = \frac{V_C}{X_C} = \frac{36.0\text{ V}}{177\ \Omega} = 0.203\text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 32-37, we have

$$i_C = (0.203\text{ A})\sin(120\pi t + \pi/2). \qquad \text{(Answer)}$$

Conclude: What is interesting to note is that if we change capacitive reactance, we can change the current in the circuit. As the capacitor does not dissipate any heat, such a circuit is used in fan regulators for increasing or decreasing fan speed.

An Inductive Load

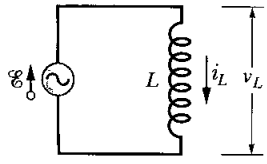


FIG. 32-11 An inductor is connected across an alternating-current generator.

Figure 32-11 shows a circuit containing an inductance and a generator with the alternating emf of Eq. 32-22. Using the loop rule and proceeding as we did to obtain Eq. 32-24, we find that the potential difference across the inductance is

$$v_L = V_L \sin \omega_d t, \qquad (32\text{-}38)$$

where V_L is the amplitude of v_L. From Eq. 31-35 ($\mathscr{E}_L = -L\,di/dt$), we can write the potential difference across an inductance L in which the current is changing at the rate di_L/dt as

$$v_L = L\frac{di_L}{dt}. \qquad (32\text{-}39)$$

If we combine Eqs. 32-38 and 32-39, we have

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin \omega_d t. \qquad (32\text{-}40)$$

Our concern, however, is with the current rather than with its time derivative. We find the former by integrating Eq. 32-40, obtaining

$$i_L = \int di_L = \frac{V_L}{L}\int \sin \omega_d t\,dt = -\left(\frac{V_L}{\omega_d L}\right)\cos \omega_d t. \qquad (32\text{-}41)$$

Note that Eq. 32-41 could have an integration constant i_0. An integration constant would represent a constant dc current throught the inductor, but there is no dc source in an ac circuit. Hence we set $i_0 = 0$ for an ac circuit.

We now modify this equation in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_L, called the **inductive reactance** of an inductor, which is defined as

$$X_L = \omega_d L \quad \text{(inductive reactance)}. \qquad (32\text{-}42)$$

The value of X_L depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the *ohm*, just as it is for X_C and for R.

Second, we replace $-\cos \omega_d t$ in Eq. 32-41 with a phase-shifted sine:

$$-\cos \omega_d t = \sin(\omega_d t - 90°).$$

You can verify this identity by shifting a sine curve 90° in the positive direction.

With these two changes, Eq. 32-41 becomes

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°). \tag{32-43}$$

From Eq. 32-23, we can also write this current in the inductance as

$$i_L = I_L \sin(\omega_d t - \phi), \tag{32-44}$$

where I_L is the amplitude of the current i_L. Comparing Eqs. 32-43 and 32-44, we see that for a purely inductive load the phase constant ϕ for the current is +90°. We also see that the voltage amplitude and current amplitude are related by

$$V_L = I_L X_L \quad \text{(inductor)}. \tag{32-45}$$

Although we found this relation for the circuit of Fig. 32-11, it applies to any inductance in any ac circuit.

Comparison of Eqs. 32-38 and 32-43, or inspection of Fig. 32-12*a*, shows that the quantities i_L and v_L are 90° out of phase. In this case, however, i_L *lags* v_L; that is, monitoring the current i_L and the potential difference v_L in the circuit of Fig. 32-11 shows that i_L reaches its maximum value *after* v_L does, by one-quarter cycle.

The phasor diagram of Fig. 32-12*b* also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled I_L does indeed lag that labeled V_L, and by an angle of 90°. Be sure to convince yourself that Fig. 32-12*b* represents Eqs. 32-38 and 32-43.

CHECKPOINT 5 If we increase the driving frequency in a circuit with a purely capacitive load, do (a) amplitude V_C and (b) amplitude I_C increase, decrease, or remain the same? If, instead, the circuit has a purely inductive load, do (c) amplitude V_L and (d) amplitude I_L increase, decrease, or remain the same?

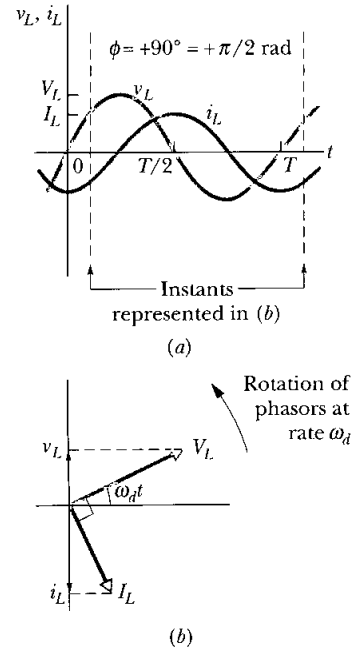


FIG. 32-12 (*a*) The current in the inductor lags the voltage by 90° (= $\pi/2$ rad). (*b*) A phasor diagram shows the same thing.

PROBLEM-SOLVING TACTICS

Tactic 1: *Leading and Lagging in AC Circuits* Table 32-2 summarizes the relations between the current i and the voltage v for each of the three kinds of circuit elements we have considered. When an applied alternating voltage produces an alternating current in these elements, the current is always in phase with the voltage across a resistor, always leads the voltage across a capacitor, and always lags the voltage across an inductor.

Many students remember these results with the mnemonic "*ELI* the *ICE* man." *ELI* contains the letter *L* (for inductor), and in it the letter *I* (for current) comes *after* the letter *E* (for emf or voltage). Thus, for an inductor, the current *lags* (comes after) the voltage. Similarly *ICE* (which contains a *C* for capacitor) means that the current *leads* (comes before) the voltage. You might also use the modified mnemonic "*ELI positively* is the *ICE* man" to remember that the phase constant ϕ is positive for an inductor.

If you have difficulty in remembering whether X_C is equal to $\omega_d C$ (wrong) or $1/\omega_d C$ (right), try remembering that C is in the "cellar"—that is, in the denominator.

TABLE 32-2

Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	0° (= 0 rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by 90° (= $\pi/2$ rad)	−90° (= $-\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by 90° (= $\pi/2$ rad)	+90° (= $+\pi/2$ rad)	$V_L = I_L X_L$

Sample Problem 32-5

Purely inductive load. In Fig. 32-11, inductance L is 230 mH and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0\,\text{V}$ and frequency $f_d = 60.0\,\text{Hz}$.

(a) What are the potential difference $v_L(t)$ across the inductance and the amplitude V_L of $v_L(t)$?

Conceptualize/Classify: In a circuit with a purely inductive load, the potential difference $v_L(t)$ across the inductance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Compute: Here we have $v_L(t) = \mathscr{E}(t)$ and $V_L = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we know that

$$V_L = \mathscr{E}_m = 36.0\,\text{V}. \qquad \text{(Answer)}$$

To find $v_L(t)$, we use Eq. 32-22 to write

$$v_L(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (32\text{-}46)$$

Then, substituting $\mathscr{E}_m = 36.0\,\text{V}$ and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 32-46, we have

$$v_L = (36.0\,\text{V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_L(t)$ in the circuit as a function of time and the amplitude I_L of $i_L(t)$?

Conceptualize/Classify: In an ac circuit with a purely inductive load, the alternating current $i_L(t)$ in the inductance lags the alternating potential difference $v_L(t)$ by 90°. (In the mnemonic of Problem-Solving Tactic 1, this circuit is "positively an *ELI* circuit," which tells us that the emf E leads the current I and that ϕ is *positive*.)

Compute: Because the phase constant ϕ for the current is +90°, or $+\pi/2$ rad, we can write Eq. 32-23 as

$$i_L = I_L \sin(\omega_d t - \phi) = I_L \sin(\omega_d t - \pi/2). \qquad (32\text{-}47)$$

We can find the amplitude I_L from Eq. 32-45 ($V_L = I_L X_L$) if we first find the inductive reactance X_L. From Eq. 32-42 ($X_L = \omega_d L$), with $\omega_d = 2\pi f_d$, we can write

$$X_L = 2\pi f_d L = (2\pi)(60.0\,\text{Hz})(230 \times 10^{-3}\,\text{H}) = 86.7\,\Omega$$

Then Eq. 32-45 tells us that the current amplitude is

$$I_L = \frac{V_L}{X_L} = \frac{36.0\,\text{V}}{86.7\,\Omega} = 0.415\,\text{A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 32-47, we have

$$i_L = (0.415\,\text{A}) \sin(120\pi t - \pi/2). \qquad \text{(Answer)}$$

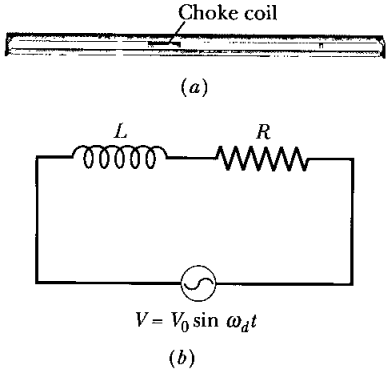


FIG. 32-13 (*a*) A tubelight. (*b*) The equivalent circuit of a tubelight. The tube effectively acts as a resistance R and the choke coil has inductance L.

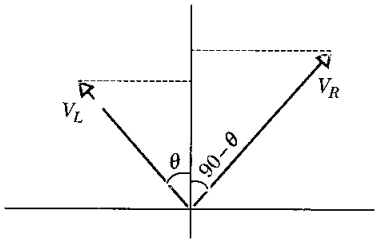


FIG. 32-14 The phasor of the voltages across the inductance and the resistance.

One familiar example of an ac circuit element is the tube light we use at our homes. It consists of a choke coil and a tube as shown in Fig. 32-13. (Note choke coil is an inductor.)

Since the inductor and the resistor are in series, the currents through them are the same. However at any instant, the voltage across them will not be necessarily the same. The instantaneous value of the voltage across the combination can be found by adding the voltage across the two, as in Kirchoff's law. As we can see in the phase diagram (Fig. 32-14), the instantaneous voltages will have a phase difference of 90° and so the total voltage across the combination will be

$$V = V_L \cos\theta + V_R \sin\theta.$$

As we know from our mathematics classes, the maximum value of V is

$$V_{\max} = \sqrt{V_L^2 + V_R^2} = I_{\max}\sqrt{(\omega_d L)^2 + R^2}.$$

Interestingly, as the equation suggests, the value of V is maximum when neither V_L nor V_R are maximum. If you remember, we encountered a similar situation before, while studying Simple Harmonic Motion, Chapter 15, Vol. I. There, in order to find out the resultant amplitude when two or more SHM are added, we added the amplitudes vectorially, taking into account the phase difference between them.

Mathematically, the situation that we have here is the same. So in the same spirit, we can add the amplitude of the voltage across the inductor and the resistor vectorially to obtain the amplitude of the applied voltage. Actually, we are already aware of the amplitude of the applied voltage. We are more interested in the amplitude of current. The current amplitude is

$$I_{max} = \frac{V_{max}}{\sqrt{(\omega_d L)^2 + R^2}}.$$

This gives that the impedance of the circuit as

$$Z = \frac{V_{max}}{I_{max}} = \sqrt{(\omega_d L)^2 + R^2}.$$

We can also see from the phase diagram that the current and the voltage across the resistor (tube in this case) are in the same phase. But here V_R will not be in the same phase as that of the applied voltage. Since the equation for the applied voltage is given by

$$V = V_0 \sin \omega_d t;$$

thus, the equation for current at any instant is

$$I = i_0 \sin(\omega_d t - \phi),$$

where ϕ is given by

$$\tan\phi = \frac{V_L}{V_R} = \frac{\omega_d L}{R}.$$

CHECKPOINT 6 A long wire of finite resistance is connected to an ac generator. This wire is then wound into a coil of many loops and reconnected to the generator. Is the current in the circuit with the coil greater than, less than, or the same as the current in the circuit with the uncoiled wire?

32-8 | The Series *RLC* Circuit

We are now ready to apply the alternating emf of Eq. 32-22,

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t \quad \text{(applied emf)}, \tag{32-48}$$

to the full *RLC* circuit of Fig. 32-6. Because R, L, and C are in series, the same current

$$i = I \sin(\omega_d t - \phi) \tag{32-49}$$

is driven in all three of them. We wish to find the current amplitude I and the phase constant ϕ. The solution is simplified by the use of phasor diagrams.

The Current Amplitude

We start with Fig. 32-15*a*, which shows the phasor representing the current of Eq. 32-49 at an arbitrary time t. The length of the phasor is the current amplitude I, the projection of the phasor on the vertical axis is the current i at time t, and the angle of rotation of the phasor is the phase $\omega_d t - \phi$ of the current at time t.

Figure 32-15*b* shows the phasors representing the voltages across R, L, and C at the same time t. Each phasor is oriented relative to the angle of rotation of current phasor I in Fig. 32-15*a*, based on the information in Table 32-2:

Resistor: Here current and voltage are in phase; so the angle of rotation of voltage phasor V_R is the same as that of phasor I.

Capacitor: Here current leads voltage by 90°; so the angle of rotation of voltage phasor V_C is 90° less than that of phasor I.

Inductor: Here current lags voltage by 90°; so the angle of rotation of voltage phasor v_L is 90° greater than that of phasor I.

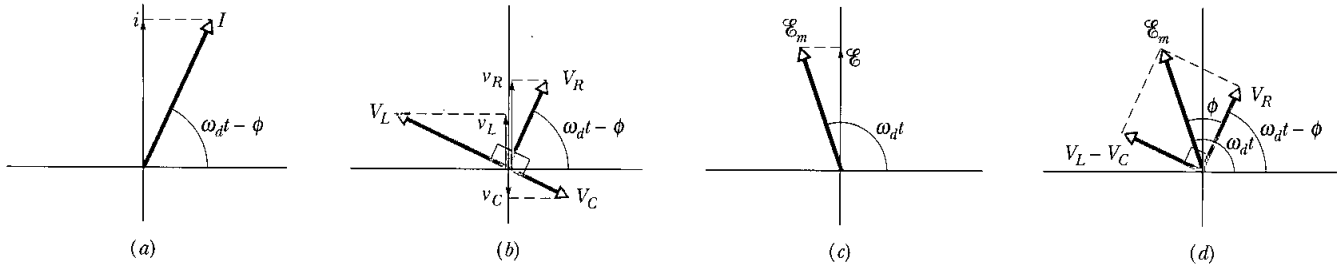


FIG. 32-15 (*a*) A phasor representing the alternating current in the driven *RLC* circuit of Fig. 32-6 at time *t*. The amplitude *I*, the instantaneous value *i*, and the phase $(\omega_d t - \phi)$ are shown. (*b*) Phasors representing the voltages across the inductor, resistor, and capacitor, oriented with respect to the current phasor in (*a*). (*c*) A phasor representing the alternating emf that drives the current of (*a*). (*d*) The emf phasor is equal to the vector sum of the three voltage phasors of (*b*). Here, voltage phasors V_L and V_C have been added vectorially to yield their net phasor $(V_L - V_C)$.

Figure 32-15*b* also shows the instantaneous voltages v_R, v_C, and v_L across *R*, *C*, and *L* at time *t*; those voltages are the projections of the corresponding phasors on the vertical axis of the figure.

Figure 32-15*c* shows the phasor representing the applied emf of Eq. 32-48. The length of the phasor is the emf amplitude $\mathscr{E}_m$, the projection of the phasor on the vertical axis is the emf $\mathscr{E}$ at time *t*, and the angle of rotation of the phasor is the phase $\omega_d t$ of the emf at time *t*.

From the loop rule we know that at any instant the sum of the voltages v_R, v_C, and v_L is equal to the applied emf $\mathscr{E}$:

$$\mathscr{E} = v_R + v_C + v_L. \tag{32-50}$$

Thus, at time *t* the projection $\mathscr{E}$ in Fig. 32-15*c* is equal to the algebraic sum of the projections v_R, v_C, and v_L in Fig. 32-15*b*. In fact, as the phasors rotate together, this equality always holds. This means that phasor $\mathscr{E}_m$ in Fig. 32-15*c* must be equal to the vector sum of the three voltage phasors V_R, V_C, and V_L in Fig. 32-15*b* (as we saw in the previous chapter).

That requirement is indicated in Fig. 32-15*d*, where phasor $\mathscr{E}_m$ is drawn as the sum of phasors V_R, V_L, and V_C. Because phasors V_L and V_C have opposite directions in the figure, we simplify the vector sum by first combining V_L and V_C to form the single phasor $V_L - V_C$. Then we combine that single phasor with V_R to find the net phasor. Again, the net phasor must coincide with phasor $\mathscr{E}_m$, as shown.

Both triangles in Fig. 32-15*d* are right triangles. Applying the Pythagorean theorem to either one yields

$$\mathscr{E}_m^2 = V_R^2 + (V_L - V_C)^2. \tag{32-51}$$

From the amplitude information displayed in Table 32-2 we can rewrite this as

$$\mathscr{E}_m^2 = (IR)^2 + (IX_L - IX_C)^2, \tag{32-52}$$

and then rearrange it to the form

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{32-53}$$

The denominator in Eq. 32-53 is called the **impedance** *Z* of the circuit for the driving angular frequency ω_d:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{(impedance defined)}. \tag{32-54}$$

We can then write Eq. 32-53 as

$$I = \frac{\mathscr{E}_m}{Z}. \tag{32-55}$$

If we substitute for X_C and X_L from Eqs. 32-32 and 32-42, we can write Eq. 32-53 more explicitly as

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \qquad \text{(current amplitude).} \qquad (32\text{-}56)$$

We have now accomplished half our goal: We have obtained an expression for the current amplitude I in terms of the sinusoidal driving emf and the circuit elements in a series RLC circuit.

The value of I depends on the difference between $\omega_d L$ and $1/\omega_d C$ in Eq. 32-56 or, equivalently, the difference between X_L and X_C in Eq. 32-53. In either equation, it does not matter which of the two quantities is greater because the difference is always squared.

The current that we have been describing in this section is the *steady-state current* that occurs after the alternating emf has been applied for some time. When the emf is first applied to a circuit, a brief *transient current* occurs. Its duration (before settling down into the steady-state current) is determined by the time constants $\tau_L = L/R$ and $\tau_C = RC$ as the inductive and capacitive elements "turn on." This transient current can, for example, destroy a motor on start-up if it is not properly taken into account in the motor's circuit design.

The Phase Constant

From the right-hand phasor triangle in Fig. 32-15*d* and from Table 32-2 we can write

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}, \qquad (32\text{-}57)$$

which gives us

$$\tan\phi = \frac{X_L - X_C}{R} \qquad \text{(phase constant).} \qquad (32\text{-}58)$$

This is the other half of our goal: an equation for the phase constant ϕ in a sinusoidally driven series RLC circuit. In essence, it gives us three different results for the phase constant, depending on the relative values of X_L and X_C:

$X_L > X_C$: The circuit is said to be *more inductive than capacitive.* Equation 32-58 tells us that ϕ is positive for such a circuit, which means that phasor I rotates behind phasor $\mathscr{E}_m$ (Fig. 32-16*a*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 32-16*b*. (Figures 32-15*c* and *d* were drawn assuming $X_L > X_C$.)

$X_C > X_L$: The circuit is said to be *more capacitive than inductive.* Equation 32-58 tells us that ϕ is negative for such a circuit, which means that phasor I rotates ahead of phasor $\mathscr{E}_m$ (Fig. 32-16*c*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 32-16*d*.

$X_C = X_L$: The circuit is said to be in *resonance,* a state that is discussed next. Equation 32-58 tells us that $\phi = 0°$ for such a circuit, which means that phasors $\mathscr{E}_m$ and I rotate together (Fig. 32-16*e*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 32-16*f*.

As illustration, let us reconsider two extreme circuits: In the *purely inductive circuit* of Fig. 32-11, where X_L is nonzero and $X_C = R = 0$, Eq. 32-58 tells us that

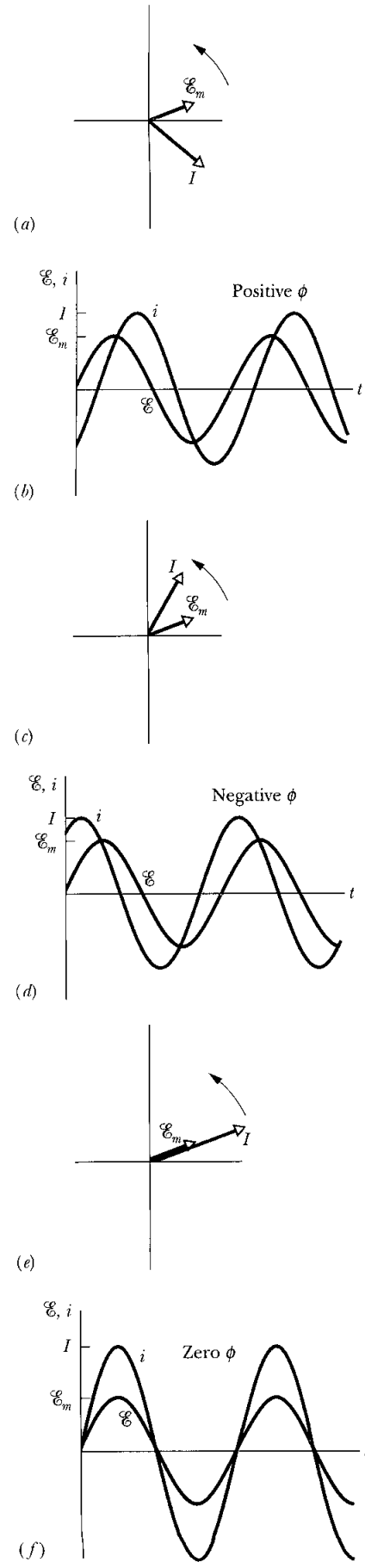


FIG. 32-16 Phasor diagrams and graphs of the alternating emf $\mathscr{E}$ and current i for the driven RLC circuit of Fig. 32-6. In the phasor diagram of (*a*) and the graph of (*b*), the current i lags the driving emf $\mathscr{E}$ and the current's phase constant ϕ is positive. In (*c*) and (*d*), the current i leads the driving emf $\mathscr{E}$ and its phase constant ϕ is negative. In (*e*) and (*f*), the current i is in phase with the driving emf $\mathscr{E}$ and its phase constant ϕ is zero.

$\phi = +90°$ (the greatest value of ϕ), consistent with Fig. 32-12*b*. In the *purely capacitive circuit* of Fig. 32-9, where X_C is nonzero and $X_L = R = 0$, Eq. 32-58 tells us that $\phi = -90°$ (the least value of ϕ), consistent with Fig. 32-10*b*.

Resonance

Equation 32-56 gives the current amplitude I in an RLC circuit as a function of the driving angular frequency ω_d of the external alternating emf. For a given resistance R, that amplitude is a maximum when the quantity $\omega_d L - 1/\omega_d C$ in the denominator is zero—that is, when

$$\omega_d L = \frac{1}{\omega_d C}$$

or

$$\omega_d = \frac{1}{\sqrt{LC}} \quad \text{(maximum } I\text{)}. \tag{32-59}$$

Because the natural angular frequency ω of the RLC circuit is also equal to $1/\sqrt{LC}$, the maximum value of I occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance. Thus, in an RLC circuit, resonance and maximum current amplitude I occur when

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad \text{(resonance)}. \tag{32-60}$$

Figure 32-17 shows three *resonance curves* for sinusoidally driven oscillations in three series RLC circuits differing only in R. Each curve peaks at its maximum current amplitude I when the ratio ω_d/ω is 1.00, but the maximum value of I decreases with increasing R. (The maximum I is always $\mathscr{E}_m/R$; to see why, combine Eqs. 32-54 and 32-55.) In addition, the curves increase in width (measured in Fig. 32-17 at half the maximum value of I) with increasing R.

To make physical sense of Fig. 32-17, consider how the reactances X_L and X_C change as we increase the driving angular frequency ω_d, starting with a value much less than the natural frequency ω. For small ω_d, reactance X_L $(= \omega_d L)$ is small and reactance X_C $(= 1/\omega_d C)$ is large. Thus, the circuit is mainly capacitive and the impedance is dominated by the large X_C, which keeps the current low.

As we increase ω_d, reactance X_C remains dominant but decreases while reactance X_L increases. The decrease in X_C decreases the impedance, allowing the

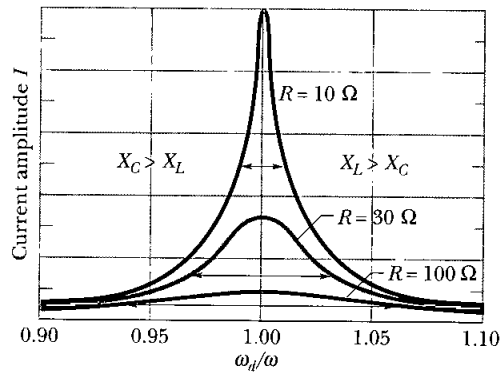


FIG. 32-17 *Resonance curves* for the driven RLC circuit of Fig. 32-6 with $L = 100\ \mu$H, $C = 100$ pF, and three values of R. The current amplitude I of the alternating current depends on how close the driving angular frequency ω_d is to the natural angular frequency ω. The horizontal arrow on each curve measures the curve's *half-width,* which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of $\omega_d/\omega = 1.00$, the circuit is mainly capacitive, with $X_C > X_L$; to the right, it is mainly inductive, with $X_L > X_C$.

current to increase, as we see on the left side of any resonance curve in Fig. 32-17. When the increasing X_L and the decreasing X_C reach equal values, the current is greatest and the circuit is in resonance, with $\omega_d = \omega$.

As we continue to increase ω_d, the increasing reactance X_L becomes progressively more dominant over the decreasing reactance X_C. The impedance increases because of X_L and the current decreases, as on the right side of any resonance curve in Fig. 32-17. In summary, then: The low-angular-frequency side of a resonance curve is dominated by the capacitor's reactance, the high-angular-frequency side is dominated by the inductor's reactance, and resonance occurs in the middle.

CHECKPOINT 7 Here are the capacitive reactance and inductive reactance, respectively, for three sinusoidally driven series *RLC* circuits: (1) 50 Ω, 100 Ω; (2) 100 Ω, 50 Ω; (3) 50 Ω, 50 Ω. (a) For each, does the current lead or lag the applied emf, or are the two in phase? (b) Which circuit is in resonance?

Sample Problem 32-6

In Fig. 32-6, let $R = 200\ \Omega$, $C = 15.0\ \mu\text{F}$, $L = 230\ \text{mH}$, $f_d = 60.0\ \text{Hz}$, and $\mathscr{E}_m = 36.0\ \text{V}$. (These parameters are those used in Sample Problems 32-3, 32-4, and 32-5.)

(a) What is the current amplitude I?

Conceptualize/Classify: The current amplitude I depends on the amplitude $\mathscr{E}_m$ of the driving emf and on the impedance Z of the circuit, according to Eq. 32-55 ($I = \mathscr{E}_m/Z$).

Compute: So, we need to find Z, which depends on resistance R, capacitive reactance X_C, and inductive reactance X_L. The circuit's only resistance is the given resistance R. Its only capacitive reactance is due to the given capacitance, and, from Sample Problem 32-4, $X_C = 177\ \Omega$. Its only inductive reactance is due to the given inductance, and, from Sample Problem 32-5, $X_L = 86.7\ \Omega$. Thus, the circuit's impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\,\Omega)^2 + (86.7\,\Omega - 177\,\Omega)^2}$$
$$= 219\ \Omega.$$

We then find

$$I = \frac{\mathscr{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the phase constant ϕ of the current in the circuit relative to the driving emf?

Conceptualize/Classify: The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 32-58.

Compute: Solving Eq. 32-58 for ϕ leads to

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega}$$
$$= -24.3^\circ = -0.424\ \text{rad}. \qquad \text{(Answer)}$$

Conclude: The negative phase constant is consistent with the fact that the load is mainly capacitive; that is, $X_C > X_L$. In the mnemonic of Problem-Solving Tactic 1, this circuit is an *ICE* circuit—the current *leads* the driving emf.

32-9 | Power in Alternating-Current Circuits

In the *RLC* circuit of Fig. 32-6, the source of energy is the alternating-current generator. Some of the energy that it provides is stored in the electric field in the capacitor, some is stored in the magnetic field in the inductor, and some is dissipated as thermal energy in the resistor. In steady-state operation—which we assume—the average energy stored in the capacitor and inductor together remains constant. The net transfer of energy is thus from the generator to the resistor, where electromagnetic energy is dissipated as thermal energy.

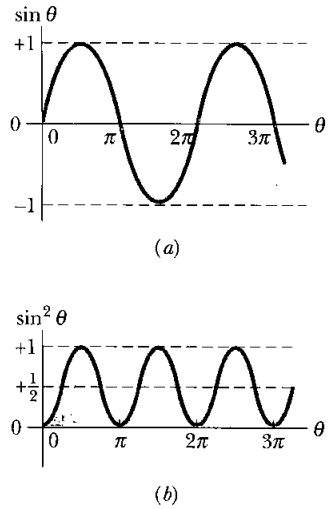


FIG. 32-18 (*a*) A plot of $\sin\theta$ versus θ. The average value over one cycle is zero. (*b*) A plot of $\sin^2\theta$ versus θ. The average value over one cycle is $\frac{1}{2}$.

The instantaneous rate at which energy is dissipated in the resistor can be written, with the help of Eqs. 27-27 and 32-23, as

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2 R = I^2R\sin^2(\omega_d t - \phi). \tag{32-61}$$

The *average* rate at which energy is dissipated in the resistor, however, is the average of Eq. 32-61 over time. Over one complete cycle, the average value of $\sin\theta$, where θ is any variable, is zero (Fig. 32-18*a*) but the average value of $\sin^2\theta$ is $\frac{1}{2}$ (Fig. 32-18*b*). (Note in Fig. 32-18*b* how the shaded areas under the curve but above the horizontal line marked $+\frac{1}{2}$ exactly fill in the unshaded spaces below that line.) Thus, we can write, from Eq. 32-61,

$$P_{\text{avg}} = \frac{I^2R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R. \tag{32-62}$$

The quantity $I/\sqrt{2}$ is called the **root-mean-square**, or **rms**, value of the current i:

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} \qquad \text{(rms current)}. \tag{32-63}$$

We can now rewrite Eq. 32-62 as

$$P_{\text{avg}} = I_{\text{rms}}^2 R \qquad \text{(average power)}. \tag{32-64}$$

Equation 32-64 looks much like Eq. 27-27 ($P = i^2R$); the message is that if we switch to the rms current, we can compute the average rate of energy dissipation for alternating-current circuits just as for direct-current circuits.

We can also define rms values of voltages and emfs for alternating-current circuits:

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad \text{and} \quad \mathscr{E}_{\text{rms}} = \frac{\mathscr{E}_m}{\sqrt{2}} \qquad \text{(rms voltage; rms emf)}. \tag{32-65}$$

Alternating-current instruments, such as ammeters and voltmeters, are usually calibrated to read I_{rms}, V_{rms}, and $\mathscr{E}_{\text{rms}}$. Thus, if you plug an alternating-current voltmeter into a household electrical outlet and it reads 220 V, that is an rms voltage. The *maximum* value of the potential difference at the outlet is $\sqrt{2}\times(220\text{ V})$ or 311 V.

Because the proportionality factor $1/\sqrt{2}$ in Eqs. 32-63 and 32-65 is the same for all three variables, we can write Eqs. 32-55 and 32-53 as

$$I_{\text{rms}} = \frac{\mathscr{E}_{\text{rms}}}{Z} = \frac{\mathscr{E}_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}, \tag{32-66}$$

and, indeed, this is the form that we almost always use.

We can use the relationship $I_{\text{rms}} = \mathscr{E}_{\text{rms}}/Z$ to recast Eq. 32-64 in a useful equivalent way. We write

$$P_{\text{avg}} = \frac{\mathscr{E}_{\text{rms}}}{Z} I_{\text{rms}} R = \mathscr{E}_{\text{rms}} I_{\text{rms}} \frac{R}{Z}. \tag{32-67}$$

From Fig. 32-15*d*, Table 32-2, and Eq. 32-55, however, we see that R/Z is just the cosine of the phase constant ϕ:

$$\cos\phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}. \tag{32-68}$$

Equation 32-67 then becomes

$$P_{\text{avg}} = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi \qquad \text{(average power)}, \tag{32-69}$$

in which the term cos ϕ is called the **power factor**. Because cos $\phi = \cos(-\phi)$, Eq. 32-69 is independent of the sign of the phase constant ϕ.

To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor cos ϕ as close to unity as possible. This is equivalent to keeping the phase constant ϕ in Eq. 32-23 as close to zero as possible. If, for example, the circuit is highly inductive, it can be made less so by putting more capacitance in the circuit, connected in series. (Recall that putting an additional capacitance into a series of capacitances decreases the equivalent capacitance C_{eq} of the series.) Thus, the resulting decrease in C_{eq} in the circuit reduces the phase constant and increases the power factor in Eq. 32-69. Power companies place series-connected capacitors throughout their transmission systems to get these results.

CHECKPOINT 8 (a) If the current in a sinusoidally driven series *RLC* circuit leads the emf, would we increase or decrease the capacitance to increase the rate at which energy is supplied to the resistance? (b) Would this change bring the resonant angular frequency of the circuit closer to the angular frequency of the emf or put it farther away?

Sample Problem 32-7

A series *RLC* circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with $X_L = 80.0\ \Omega$, and a capacitance with $X_C = 150\ \Omega$.

(a) What are the power factor cos ϕ and phase constant ϕ of the circuit?

Conceptualize/Classify: The power factor cos ϕ can be found from the resistance R and impedance Z via Eq. 32-68 (cos $\phi = R/Z$).

Compute: To calculate Z, we use Eq. 32-54:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\ \Omega)^2 + (80.0\ \Omega - 150\ \Omega)^2} = 211.90\ \Omega.$$

Equation 32-68 then gives us

$$\cos\phi = \frac{R}{Z} = \frac{200\ \Omega}{211.90\ \Omega} = 0.9438 \approx 0.944. \qquad \text{(Answer)}$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3°.$$

Both +19.3° and −19.3° have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because $X_C > X_L$, this circuit is mainly capacitive, with the current leading the emf. Thus, ϕ must be negative:

$$\phi = -19.3°. \qquad \text{(Answer)}$$

We could, instead, have found ϕ with Eq. 32-58. A calculator would then have given us the answer with the minus sign.

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

Conceptualize/Classify: There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 32-69 ($P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$). (2) The rate at which energy is dissipated in a resistance R depends on the square of the rms current I_{rms} through it, according to Eq. 32-64 ($P_{avg} = I^2_{rms} R$).

Compute: First way: We are given the rms driving emf $\mathscr{E}_{rms}$ and we already know cos ϕ from part (a). The rms current I_{rms} is determined by the rms value of the driving emf and the circuit's impedance Z (which we know), according to Eq. 32-66:

$$I_{rms} = \frac{\mathscr{E}_{rms}}{Z}.$$

Substituting this into Eq. 32-69 then leads to

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi = \frac{\mathscr{E}^2_{rms}}{Z} \cos\phi$$
$$= \frac{(120\ \text{V})^2}{211.90\ \Omega}(0.9438) = 64.1\ \text{W}. \qquad \text{(Answer)}$$

Second way: We then find

$$P_{avg} = I^2_{rms} R = \frac{\mathscr{E}^2_{rms}}{Z^2} R$$
$$= \frac{(120\ \text{V})^2}{(211.90\ \Omega)^2}(200\ \Omega) = 64.1\ \text{W}. \qquad \text{(Answer)}$$

(c) What new capacitance C_{new} is needed to maximize P_{avg} if the other parameters of the circuit are not changed?

Conceptualize/Classify: (1) The average rate P_{avg} at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when $X_C = X_L$.

Compute: From the given data, we have $X_C > X_L$. Thus, we must decrease X_C to reach resonance. From Eq. 32-32 ($X_C = 1/\omega_d C$), we see that this means we must increase C to the new value C_{new}.

Using Eq. 32-32, we can write the resonance condition $X_C = X_L$ as

$$\frac{1}{\omega_d C_{new}} = X_L.$$

Substituting $2\pi f_d$ for ω_d (because we are given f_d and not ω_d) and then solving for C_{new}, we find

$$C_{new} = \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60\text{ Hz})(80.0\ \Omega)}$$
$$= 3.32\times10^{-5}\text{ F} = 33.2\ \mu\text{F}. \qquad \text{(Answer)}$$

Conclude: Following the procedure of part (b), you can show that with C_{new}, P_{avg} would then be at its maximum value of 72.0 W.

Sample Problem 32-8

For the circuit of Fig. 32-19, find the instantaneous current through each element. Also find (a) the total instantaneous current through the source and (b) the expression for the phase angle of this current and the impedance of the circuit. Draw a phasor diagram to illustrate the relationship of the instantaneous current to the instantaneous emf.

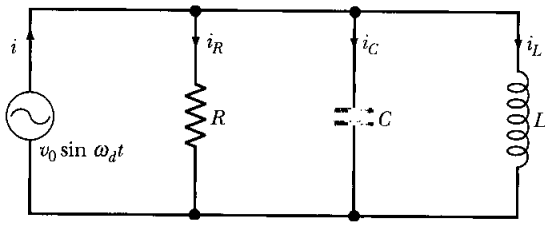


FIG 32-19 Three circuit elements are in parallel. The voltages across them will be the same, but not the current.

Conceptualize: Till now we have considered only the series circuits. In them the voltage across each element was added to the voltage across the other elements. The current was taken as reference phasor because the current was common through all of them. However, here we should take the voltage as the reference phasor as the voltage across each of the element is the same.

Classify: Here we can see that the circuit elements are in parallel. So the voltage across them will be the same but the current will be different.

Compute: The three current equations are

$$Ri_R = V_e; \quad L\frac{di_L}{dt} = V_e; \quad \frac{1}{C}i_C = \frac{dV_e}{dt}.$$

The steady-state solutions of these equations are

$$i_R = \frac{V_e}{R}\sin\omega_d t = I_R \sin\omega_d t;$$

$$i_L = -\frac{V_e}{\omega_d L}\cos\omega_d t = -\frac{V_e}{X_L}\cos\omega_d t = -I_L\cos\omega_d t;$$

$$i_C = -V_e\omega_d\cos\omega_d t = -\frac{V_e}{X_C}\cos\omega_d t = I_C\cos\omega_d t.$$

For the total current, we have

$$i = i_R + i_L + i_C = V_e\left[\frac{1}{R}\sin\omega_d t + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)\cos\omega_d t\right]$$
$$\equiv I\sin(\omega_d t + \phi),$$

where $I = V_e/Z$. The impedance and phase angle are then given by

$$\frac{1}{Z} \equiv \left[\left(\frac{1}{R}\right)^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2\right]^{1/2}.$$

$$\tan\phi \equiv \frac{(1/X_C) - (1/X_L)}{1/R}.$$

Figure 32-20 shows five phasors v_0, I_R, I_L, I_C, I, where I_R is in-phase with V_e, I_C leads V_e by 90°, I_L lags V_e by 90°, and I leads or lags V_e by $|\phi|$. The phasor equality

$$I = I_R + I_L + I_C$$

reflects conservation of charge in the circuit.

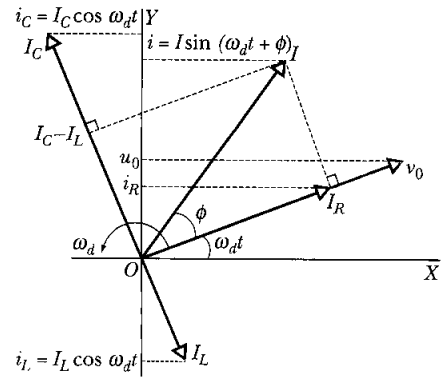


FIG. 32-20 Phasors representing the current through each of the circuit element.

Conclude: In addition to the series RCL and the parallel circuit, there are many different ways to connect resistors, capacitors, and inductors. The analysis of these circuits may be complex, but it helps to keep in mind the behavior of capacitors and inductors at the extreme limits of the frequency. When the frequency approaches zero (i.e., dc conditions), the reactance of a capacitor becomes so large that no charge can flow through the capacitor. It is as if the capacitor were cut out of circuit, leaving an open gap in the connecting wire. In the limit of zero frequency the reactance of an inductor is vanishingly small. The inductor offers no opposition to a dc current. It is as if the inductor were replaced with a wire of zero resistance. In the limit of very large frequency, the behavior of a capacitor and an inductor are reversed. The capacitor has a very small reactance and offers little opposition to the current, as if it were replaced by a wire with zero resistance. In contrast, the inductor has a very large reactance at a high frequency. The inductor offers so much opposition to the current that it might as well be cut out of the circuit, leaving an open gap in the connecting wire.

Sample Problem 32-9 Build your skill

Figure 32-21 shows a circuit. The rms voltage of the generator is V_0. The frequency of the ac generator is very near zero. What is the rms current through the circuit?

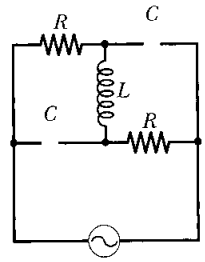


FIG. 32-21 Circuit covered in Sample Problem 32-9.

Conceptualize: As we just now discussed, at small frequencies, the circuit is like a dc circuit. So the capacitor can be replaced by a open circuit and inductor by a short circuit. Figure 32-22 shows the circuits as they would appear according to these changes.

Classify: It is obvious from the circuit that it is not a circuit of series elements, nor elements in parallel.

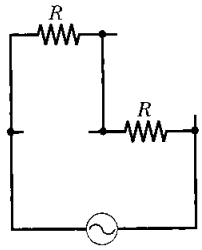


FIG. 32-22 The equivalent circuit of Fig. 31-21 at low frequency.

Compute: According to Eq. 32-66, the rms current is given by

$$I_{rms} = \frac{V_{rms}}{Z}.$$

However, the impedance Z is not given by Eq. 32-54 because the circuit in Fig. 32-21 is not series RCL circuit. The circuit behaves as if it contained only two identical resistors in series, with a total impedance of $Z = 2R$. Thus circuit has

$$I_{rms} = \frac{V_{rms}}{2R}.$$

Conclude: If the source had a very high frequency (much higher than the resonance frequency), the capacitor would be treated as short-circuited and inductor as open-circuited. Figure 32-23 shows the circuits as they would appear according to these changes. So at this frequency, the impedance of the circuit would be $R/2$. Hence

$$I_{rms} = \frac{2V_{rms}}{R}.$$

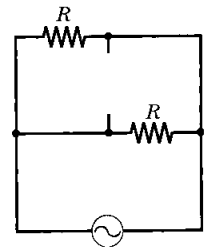


FIG. 32-23 Equivalent circuit at high frequency.

32-10 | Transformers

Energy Transmission Requirements

When an ac circuit has only a resistive load, the power factor in Eq. 32-69 is $\cos 0° = 1$ and the applied rms emf $\mathscr{E}_{rms}$ is equal to the rms voltage V_{rms} across the load. Thus, with an rms current I_{rms} in the load, energy is supplied and dissipated at the average rate of

$$P_{avg} = \mathscr{E} I = IV. \qquad (32\text{-}70)$$

(In Eq. 32-70 and the rest of this section, we follow conventional practice and drop the subscripts identifying rms quantities. Engineers and scientists assume that all time-varying currents and voltages are reported as rms values; that is what the meters read.) Equation 32-70 tells us that, to satisfy a given power requirement, we have a range of choices, from a relatively large current I and a relatively small voltage V to just the reverse, provided only that the product IV is as required.

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory). Nobody wants an electric toaster or a child's electric train to operate at, say, 10 kV. On the other hand, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called *ohmic losses*) in the transmission line.

As an example, consider the 735 kV line used to transmit electrical energy from the La Grande 2 hydroelectric plant in Quebec to Montreal, 1000 km away. Suppose that the current is 500 A and the power factor is close to unity. Then from Eq. 32-70, energy is supplied at the average rate

$$P_{avg} = \mathscr{E}I = (7.35 \times 10^5 \text{ V})(500 \text{ A}) = 368 \text{ MW}.$$

The resistance of the transmission line is about 0.220 Ω/km; thus, there is a total resistance of about 220 Ω for the 1000 km stretch. Energy is dissipated due to that resistance at a rate of about

$$P_{avg} = I^2R = (500 \text{ A})^2(220 \ \Omega) = 55.0 \text{ MW},$$

which is nearly 15% of the supply rate.

Imagine what would happen if we doubled the current and halved the voltage. Energy would be supplied by the plant at the same average rate of 368 MW as previously, but now energy would be dissipated at the rate of about

$$P_{avg} = I^2R = (1000 \text{ A})^2(220 \ \Omega) = 220 \text{ MW},$$

which is *almost 60% of the supply rate.* Hence the general energy transmission rule: Transmit at the highest possible voltage and the lowest possible current.

The Ideal Transformer

The transmission rule leads to a fundamental mismatch between the requirement for efficient high-voltage transmission and the need for safe low-voltage generation and consumption. We need a device with which we can raise (for transmission) and lower (for use) the ac voltage in a circuit, keeping the product current × voltage essentially constant. The **transformer** is such a device. It has no moving parts, operates by Faraday's law of induction, and has no simple direct-current counterpart.

The *ideal transformer* in Fig. 32-24 consists of two coils, with different numbers of turns, wound around an iron core. (The coils are insulated from the core.) In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf $\mathscr{E}$ at any time t is given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega t. \tag{32-71}$$

The secondary winding, of N_s turns, is connected to load resistance R, but its circuit is an open circuit as long as switch S is open (which we assume for the present). Thus, there can be no current through the secondary coil. We assume further for this ideal transformer that the resistances of the primary and secondary windings are negligible. Well-designed, high-capacity transformers can have energy losses as low as 1%; so our assumptions are reasonable.

For the assumed conditions, the primary winding (or *primary*) is a pure inductance and the primary circuit is like that in Fig. 32-11. Thus, the (very small) primary current, also called the *magnetizing current* I_{mag}, lags the primary voltage V_p by 90°; the primary's power factor (= cos ϕ in Eq. 32-69) is zero; so no power is delivered from the generator to the transformer.

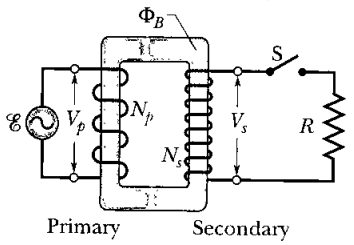


FIG. 32-24 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

However, the small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux Φ_B in the iron core. The core acts to strengthen the flux and to bring it through the secondary winding (or *secondary*). Because Φ_B varies, it induces an emf $\mathscr{E}_{turn}$ ($= d\Phi_B/dt$) in each turn of the secondary. In fact, this emf per turn $\mathscr{E}_{turn}$ is the same in the primary and the secondary. Across the primary, the voltage V_p is the product of $\mathscr{E}_{turn}$ and the number of turns N_p; that is, $V_p = \mathscr{E}_{turn}N_p$. Similarly, across the secondary the voltage is $V_s = \mathscr{E}_{turn}N_s$. Thus, we can write

$$\mathscr{E}_{turn} = \frac{V_p}{N_p} = \frac{V_s}{N_s},$$

or

$$V_s = V_p \frac{N_s}{N_p} \qquad \text{(transformation of voltage).} \tag{32-72}$$

If $N_s > N_p$, the transformer is called a *step-up transformer* because it steps the primary's voltage V_p *up* to a higher voltage V_s. Similarly, if $N_s < N_p$, the device is a *step-down transformer.*

So far, with switch S open, no energy is transferred from the generator to the rest of the circuit. Now let us close S to connect the secondary to the resistive load R. (In general, the load would also contain inductive and capacitive elements, but here we consider just resistance R.) We find that now energy *is* transferred from the generator. Let us see why.

Several things happen when we close switch S.

1. An alternating current I_s appears in the secondary circuit, with corresponding energy dissipation rate $I_s^2R(= V_s^2/R)$ in the resistive load.
2. This current produces its own alternating magnetic flux in the iron core, and this flux induces (from Faraday's law and Lenz's law) an opposing emf in the primary windings.
3. The voltage V_p of the primary, however, cannot change in response to this opposing emf because it must always be equal to the emf $\mathscr{E}$ that is provided by the generator; closing switch S cannot change this fact.
4. To maintain V_p, the generator now produces (in addition to I_{mag}) an alternating current I_p in the primary circuit; the magnitude and phase constant of I_p are just those required for the emf induced by I_p in the primary to exactly cancel the emf induced there by I_s. Because the phase constant of I_p is not 90° like that of I_{mag}, this current I_p can transfer energy to the primary.

We want to relate I_s to I_p. However, rather than analyze the foregoing complex process in detail, let us just apply the principle of conservation of energy. The rate at which the generator transfers energy to the primary is equal to I_pV_p. The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is I_sV_s. Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_pV_p = I_sV_s.$$

Substituting for V_s from Eq. 32-72, we find that

$$I_s = I_p \frac{N_p}{N_s} \qquad \text{(transformation of currents).} \tag{32-73}$$

This equation tells us that the current I_s in the secondary can differ from the current I_p in the primary, depending on the *turns ratio* N_p/N_s.

Current I_p appears in the primary circuit because of the resistive load R in the secondary circuit. To find I_p, we substitute $I_s = V_s/R$ into Eq. 32-73 and then we substitute for V_s from Eq. 32-72. We find

$$I_p = \frac{1}{R}\left(\frac{N_s}{N_p}\right)^2 V_p. \tag{32-74}$$

This equation has the form $I_p = V_p/R_{eq}$, where equivalent resistance R_{eq} is

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R. \tag{32-75}$$

This R_{eq} is the value of the load resistance as "seen" by the generator; the generator produces the current I_p and voltage V_p as if the generator were connected to a resistance R_{eq}.

Impedance Matching

Equation 32-75 suggests still another function for the transformer. For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. The same relation holds for ac circuits except that the *impedance* (rather than just the resistance) of the generator must equal that of the load. Often this condition is not met. For example, in a music-playing system, the amplifier has high impedance and the speaker set has low impedance. We can match the impedances of the two devices by coupling them through a transformer that has a suitable turns ratio N_p/N_s.

Solar Activity and Power-Grid Systems

In a *solar flare,* a huge loop of electrons and protons extends outward from the surface of the Sun, as shown in the photograph that opens this chapter. Some solar flares explode, shooting those charged particles into space. On March 10, 1989, a gigantic solar flare exploded toward Earth. When the particles arrived three days later, they produced a 10^6 A current, called an *electrojet,* in the high-altitude atmosphere above the Northern Hemisphere.

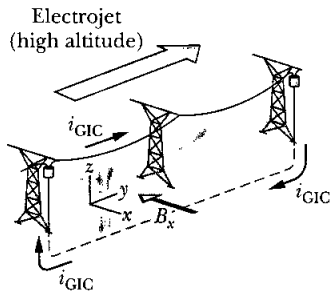


FIG. 32-25 An electrojet (current) in the ionosphere produces a magnetic field B_x through a vertical loop formed by a transmission line, the ground, and the wires grounding the transformers (located inside the cylinders at the ends of the transmission lines). Variations in B_x induce current i_{GIC} around the loop.

Because it is a current, an electrojet sets up a magnetic field $\vec{B}$ around itself, including along Earth's surface. Using the right-hand rule of Fig. 30-4, we see that the electrojet in Fig. 32-25 sets up a magnetic field component B_x along Earth's surface, directed perpendicular to the long power transmission line shown there. Grounded step-up or step-down transformers are attached at each end of the transmission line. Note that the transmission line, the ground, and the wires grounding the transformers form a conducting loop. A magnetic flux Φ due to B_x penetrates that loop.

An electrojet varies in both size and location, and the resulting variations in Φ induce emf and current in the loop. The current i_{GIC}, called the geomagnetically induced current (GIC), is directed along the transmission line and (more important) through the transformers.

Transmission of power by a power-grid system depends on the proper sinusoidal variations in current and voltage throughout the system. The presence of i_{GIC} through a transformer ruins the ability of the transformer's core to transfer the sinusoidal variations in the primary to the secondary. The reason is that the added flux in the core due to the i_{GIC} *saturates* the core, making it unable to respond properly to sinusoidal variations in the primary. The result is that the current and voltage in the secondary are highly distorted and no longer sinu-soidal, and this distortion disrupts the power transmission.

On March 13, 1989, this type of disruption caused the power-grid system of Quebec province to shut down. Today, whenever a solar flare explodes toward Earth, astronomers immediately warn power-grid engineers so that the engineers can brace for grid disruptions.

CHECKPOINT 9 An alternating-current emf device in a certain circuit has a smaller resistance than that of the resistive load in the circuit; to increase the transfer of energy from the device to the load, a transformer will be connected between the two. (a) Should N_s be greater than or less than N_p? (b) Will that make it a step-up or step-down transformer?

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: In an LC series circuit, current can flow in the circuit even though there is a capacitor in the circuit.

Statement 2: When there is a high voltage across the capacitor, its dielectric can break down and start conducting.

2. A series LCR circuit is operated at resonance.

Statement 1: The power transferred to the resistor is maximum.

Statement 2: The current in the circuit is maximum.

MULTIPLE CHOICE QUESTIONS

sec. 32-2 LC Oscillations, Qualitatively

1. A charged capacitor and an inductor are connected in series. At time $t = 0$, the current is zero, but the capacitor is charged. If T is the period of the resulting oscillations, the next time after $t = 0$, the maximum voltage across the inductor is

(a) T
(b) $T/4$
(c) $T/2$
(d) T
(e) $2T$

2. In the LC circuit shown in Fig. 32-26, the current is in the direction shown in the figure and the charges on the capacitor plates have the sign shown in the figure. At this time.

(a) i is increasing and Q is increasing
(b) i is increasing and Q is decreasing
(c) i is decreasing and Q is increasing
(d) i is decreasing and Q is decreasing

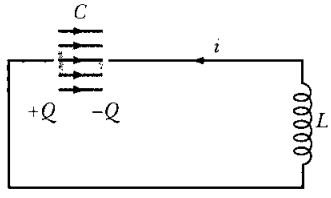


FIG. 32-26 Multiple choice question 2.

sec. 32-3 The Electrical—Mechanical Analogy

3. The electrical analog of a spring constant k is

(a) L
(b) $1/l$
(c) C
(d) $1/C$
(e) R

4. Consider the mechanical system consisting of two springs and a block, as shown in Fig. 32-27. Which one of the five electrical circuits is the analog of the mechanical system?

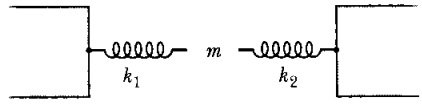
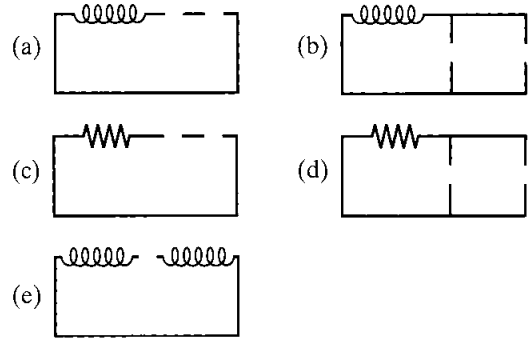


FIG. 32-27 Multiple choice question 4.



sec. 32-4 LC Oscillations, Quantitatively

5. An LC series circuit with an inductance L and a capacitance C has an oscillation frequency f. Two isolated inductors, each with inductance L, and two capacitors, each with capacitance C, are all wired in series and the circuit is completed. The oscillation frequency is

(a) $f/4$
(b) $f/2$
(c) f
(d) $2f$
(e) $4f$

6. In an oscillating LC circuit, the total stored energy is U and the maximum charge on the capacitor is Q. When the charge on the capacitor is $Q/2$, the energy stored in the inductor is

(a) $U/2$
(b) $U/4$
(c) $(4/3)U$
(d) $3U/4$

sec. 32-6 Forced Oscillations

7. The resonance frequency of a certain RLC series circuit is ω_0. A source of sinusoidal emf, with angular frequency $2\omega_0$,

is inserted into the circuit. After transients die out the angular frequency of the current oscillations is

(a) $\omega_0/2$
(b) ω_0
(c) $2\omega_0$
(d) $1.5\omega_0$
(e) $3\omega_0$

sec. 32-7 Three Simple Circuits

8. A variable inductor is connected to an ac source. What effect does increasing the inductance have on the reactance and current in this circuit?

	Reactance	Current
(a)	No change	No change
(b)	Decreases	No change
(c)	Increases	Increases
(d)	Decreases	Increases
(e)	Increases	Decreases

9. A variable capacitor is connected to an ac source. What effect does *decreasing* the capacitance have on the reactance and current in this circuit?

	Reactance	Current
(a)	No change	No change
(b)	Decreases	No change
(c)	Increases	Increases
(d)	Decreases	Increases
(e)	Increases	Decreases

sec. 32-8 The Series *RLC* Circuit

10. For the circuit shown in Fig. 32-28, the voltage of the source at any instant is equal to

(a) The sum of the maximum voltages across the elements
(b) The voltage drop across the resistor
(c) The sum of the instantaneous voltages across the elements
(d) The sum of rms voltages across the elements

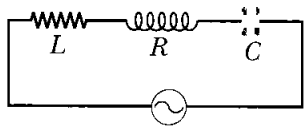


FIG. 32-28 Multiple choice question 10.

11. Resonance in the circuit shown in Fig. 32-28, will occur

(a) When the angular frequency of the source is equal to $1/\sqrt{LC}$
(b) When the capacitive and inductive reactances of the circuit are equal
(c) Independently of the value of R
(d) Independently of the maximum value of the voltage output of the source

12. In an *RLC* series circuit, if the ac frequency is increased to a very large value, what value does the phase angle between the current and voltage approach?

(a) 90°
(b) 0°
(c) 30°
(d) 45°

13. A series *LCR* circuit powered by a frequency generator has equal resistance, capacitive reactance, and inductive reactance. What is the phase angle between the voltages across the *generator* and the *resistor*?

(a) 0°
(b) 45°
(c) 60°
(d) 90°

14. In an *RL* series circuit driven by an ac voltage source with fixed amplitude,

(a) The source current leads the source voltage in phase
(b) The source current lags the source voltage in phase
(c) The source current amplitude increases as source frequency increase
(d) The inductor voltage lags the inductor current in phase
(e) The resistor voltage leads the resistor current in phase

15. An inductor, a capacitor, and a resistor are connected in series with a frequency generator to complete a circuit. The frequency generator can supply alternating current at frequencies ranging from about 0 Hz to 100 MHz. As the frequency is increased from the lowest to the highest value of the generator there is a particular value for which the voltage across the resistor is maximum. At that frequency

(a) The voltage across the inductor is zero at all times
(b) The voltage across the capacitor is zero at all times
(c) The voltage across the inductor and the capacitor are zero at all times
(d) The voltage across the inductor and the capacitor are opposite in phase at all times

16. In Fig. 32-29, the function $y(t) = y_m \sin(\omega t)$ is plotted as a solid curve. The other three curves have the form $y(t) = y_m \sin(\omega t + \phi)$, where ϕ is between $-\pi/2$ and $+\pi/2$. Rank the curves according to the value of ϕ from the most negative to the most positive.

(a) 1, 2, 3
(b) 2, 3, 1
(c) 3, 2, 1
(d) 1, 3, 2

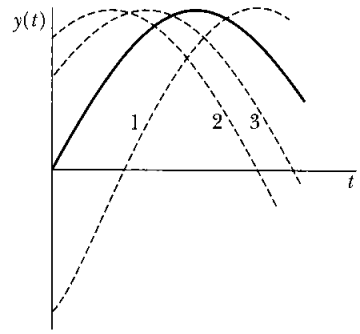


FIG. 32-29 Multiple choice question 16.

sec. 32-9 Power in Alternating-Current Circuits

17. Consider an ac circuit where an inductor is in series with an incandescent light bulb. If the frequency of the generator is increased, what will happen to the brightness of the bulb?

(a) The bulb will have the same brightness
(b) The bulb will shine brighter
(c) The bulb will become dimmer
(d) Depends on the value of L and R

18. Which one of the following is true for an ideal capacitor connected to a sinusoidal voltage source?

(a) Neither the average power nor the average current is zero
(b) The average current is zero but the average power is not zero
(c) The average power is zero but the average current is not zero
(d) Both the average power and the average current are zero
(e) The average power and the root mean square current are zero

19. A power outlet puts out 60 Hz AC. Which of the following statements is true?

(a) Voltage goes to zero only 120 times per second
(b) Current goes to zero only 120 times per second
(c) Power output goes to zero only 60 times per second
(d) Voltage, current, and power output stay constant
(e) Voltage, current, and power go to zero 120 times per second

20. The rms value of a sinusoidal voltage is $V_0/\sqrt{2}$, where V_0 is the amplitude (Fig. 32-30). What is the rms value of its fully rectified wave? Recall that $V_{rect}(t) = |V(t)|$.

(a) $\frac{V_0}{\sqrt{2}}$

(b) $V_0\sqrt{2}$

(c) V_0

(d) $\frac{V_0}{2}$

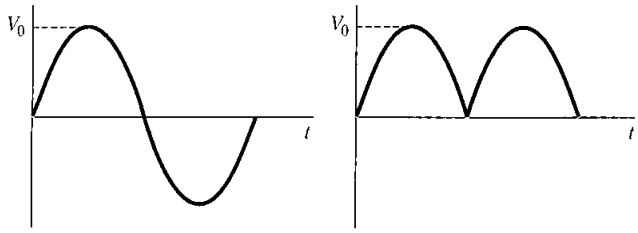


FIG. 32-30 Multiple choice question 20.

21. The ideal meters shown in Fig. 32-31 read rms current and voltage. The average power delivered to the load is

(a) Definitely equal to $V \times I$
(b) Definitely more than $V \times I$
(c) Possibly equal to $V \times I$ even if the load contains an inductor and a capacitor
(d) Definitely less than $V \times I$
(e) Zero, as is the average of any sine wave

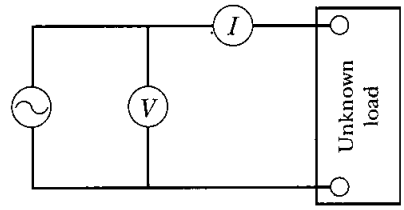


FIG. 32-31 Multiple choice question 21.

22. The average power supplied to the circuit shown in Fig. 32-32 passes through a maximum when which one of the following is increased continuously from a very low to a very high value?

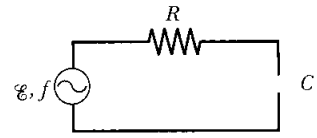


FIG. 32-32 Multiple choice question 21.

(a) Source emf $\mathscr{E}$
(b) R
(c) C
(d) Source frequency f

23. In a sinusoidally driven series RLC circuit the current lags the applied emf. The rate at which energy is dissipated in the resistor can be increased by

(a) Decreasing the capacitance and making no other changes
(b) Increasing the capacitance and making no other changes
(c) Increasing the inductance and making no other changes
(d) Increasing the driving frequency and making no other changes
(e) Decreasing the amplitude of the driving emf and making no other changes

sec. 32-10 Transformers

24. The core of a transformer is made in a laminated form to

(a) Facilitate easy assembly
(b) Reduce i^2R losses in the coils
(c) Increase the magnetic flux
(d) Prevent Eddy currents

25. The primary coil of an ideal transformer has 100 turns and the secondary coil has 600 turns. Then

(a) The power in the primary circuit is less than that in the secondary circuit
(b) The currents in the two circuits are the same
(c) The voltages in the two circuits are the same
(d) The primary current is six times the secondary current
(e) The frequency in the secondary circuit is six times that in the primary circuit

PROBLEMS

sec. 32-2 LC Oscillations, Qualitatively

1. In a certain oscillating LC circuit, the total energy is converted from electrical energy in the capacitor to magnetic energy in the inductor in 1.50 μs. What are (a) the period of oscillation and (b) the frequency of oscillation? (c) How long after the magnetic energy is a maximum will it be a maximum again?

2. In an oscillating LC circuit, $L = 1.10$ mH and $C = 4.00$ μF. The maximum charge on the capacitor is 3.00 μC. Find the maximum current.

3. The frequency of oscillation of a certain LC circuit is 200 kHz. At time $t = 0$, plate A of the capacitor has maximum positive charge. At what earliest time $t > 0$ will (a) plate A again

have maximum positive charge, (b) the other plate of the capacitor have maximum positive charge, and (c) the inductor have maximum magnetic field?

4. What is the capacitance of an oscillating LC circuit if the maximum charge on the capacitor is 1.60 μC and the total energy is 140 μJ?

sec. 32-3 The Electrical—Mechanical Analogy

5. A 0.50 kg body oscillates in SHM on a spring that, when extended 2.0 mm from its equilibrium position, has an 8.0 N restoring force. What are (a) the angular frequency of oscillation, (b) the period of oscillation, and (c) the capacitance of an LC circuit with the same period if L is 5.0 H?

sec. 32-4 LC Oscillations, Quantitatively

6. Using the loop rule, derive the differential equation for an LC circuit (Eq. 32-11).

7. In Fig. 32-33, $R = 14.0\ \Omega$, $C = 6.20\ \mu$F, and $L = 54.0$ mH, and the ideal battery has emf $\mathscr{E} = 34.0$ V. The switch is kept in position a for a long time and then thrown to position b. What are the (a) frequency and (b) current amplitude of the resulting oscillations?

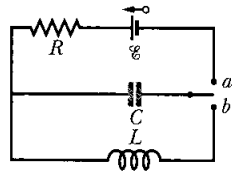


FIG. 32-33 Problem 7.

8. An inductor is connected across a capacitor whose capacitance can be varied by turning a knob. We wish to make the frequency of oscillation of this LC circuit vary linearly with the angle of rotation of the knob, going from 2×10^5 to 4×10^5 Hz as the knob turns through 180°. If $L = 1.0$ mH, plot the required capacitance C as a function of the angle of rotation of the knob.

9. A series circuit containing inductance L_1 and capacitance C_1 oscillates at angular frequency ω. A second series circuit, containing inductance L_2 and capacitance C_2, oscillates at the same angular frequency. In terms of ω, what is the angular frequency of oscillation of a series circuit containing all four of these elements? Neglect resistance.

10. In an oscillating LC circuit with $C = 64.0\ \mu$F, the current is given by $i = (1.60)\sin(2500t + 0.680)$, where t is in seconds, i in amperes, and the phase constant in radians. (a) How soon after $t = 0$ will the current reach its maximum value? What are (b) the inductance L and (c) the total energy?

11. In an oscillating LC circuit, $L = 25.0$ mH and $C = 7.80\ \mu$F. At time $t = 0$ the current is 9.20 mA, the charge on the capacitor is 3.80 μC, and the capacitor is charging. What are (a) the total energy in the circuit, (b) the maximum charge on the capacitor, and (c) the maximum current? (d) If the charge on the capacitor is given by $q = Q\cos(\omega t + \phi)$, what is the phase angle ϕ? (e) Suppose the data are the same, except that the capacitor is discharging at $t = 0$. What then is ϕ?

12. An oscillating LC circuit has a current amplitude of 7.50 mA, a potential amplitude of 250 mV, and a capacitance of 220 nF. What are (a) the period of oscillation, (b) the maximum energy stored in the capacitor, (c) the maximum energy stored in the inductor, (d) the maximum rate at which the current changes, and (e) the maximum rate at which the inductor gains energy?

sec. 32-7 Three Simple Circuits

13. (a) At what frequency would a 6.0 mH inductor and a 10 μF capacitor have the same reactance? (b) What would the reactance be? (c) Show that this frequency would be the natural frequency of an oscillating circuit with the same L and C.

14. An ac generator with emf $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, where $\mathscr{E}_m = 25.0$ V and $\omega_d = 377$ rad/s, is connected to a 4.15 μF capacitor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is −12.5 V and increasing in magnitude, what is the current?

sec. 32-8 The Series RLC Circuit

15. Remove the inductor from the circuit in Fig. 32-6 and set $R = 200\ \Omega$, $C = 15.0\ \mu$F, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

16. The current amplitude I versus driving angular frequency ω_d for a driven RLC circuit is given in Fig. 32-34, where the vertical axis scale is set by $I_s = 4.00$ A. The inductance is 200 μH, and the emf amplitude is 8.0 V. What are (a) C and (b) R?

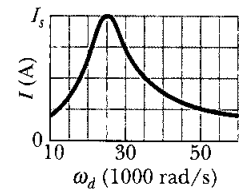


FIG. 32-34 Problem 16.

17. An alternating source drives a series RLC circuit with an emf amplitude of 6.00 V, at a phase angle of +30.0°. When the potential difference across the capacitor reaches its maximum positive value of +5.00 V, what is the potential difference across the inductor (sign included)?

18. (a) In an RLC circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator emf? (b) Consider an RLC circuit with $\mathscr{E}_m = 10$ V, $R = 10\ \Omega$, $L = 1.0$ H, and $C = 1.0\ \mu$F. Find the amplitude of the voltage across the inductor at resonance.

19. A series RLC circuit is driven by a generator at frequency 1050 Hz. The inductance is 90.0 mH; the capacitance is 0.500 μF; and the phase constant has a magnitude of 60.0° (you should supply the appropriate sign for the angle). (a) What is the resistance? To increase the current amplitude in the circuit, should we increase or decrease (b) the driving frequency, (c) the inductance, and (d) the capacitance?

sec. 32-9 Power in Alternating-Current Circuits

20. Figure 32-35 shows an ac generator connected to a "black box" through a pair of terminals. The box contains an RLC circuit, possibly even a multiloop circuit, whose elements and connections we do not know. Measurements outside the box reveal that

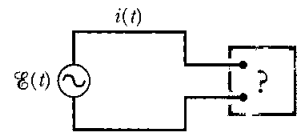


FIG. 32-35 Problem 20.

$$\mathscr{E}(t) = (75.0\text{ V})\sin\omega_d t$$

and

$$i(t) = (1.20\text{ A})\sin(\omega_d t + 42.0°).$$

(a) What is the power factor? (b) Does the current lead or lag the emf? (c) Is the circuit in the box largely inductive or largely capacitive? (d) Is the circuit in the box in resonance? (e) Must there be a capacitor in the box? (f) An inductor? (g) A resistor? (h) At what average rate is energy delivered to the box by the generator? (i) Why don't you need to know ω_d to answer all these questions?

21. An ac voltmeter with large impedance is connected in turn across the inductor, the capacitor, and the resistor in a series circuit having an alternating emf of 100 V (rms); the meter gives the same reading in volts in each case. What is this reading?

22. What direct current will produce the same amount of thermal energy, in a particular resistor, as an alternating current that has a maximum value of 2.60 A?

sec. 32-10 Transformers

23. A generator supplies 100 V to a transformer's primary coil, which has 50 turns. If the secondary coil has 500 turns, what is the secondary voltage?

24. Figure 32-36 shows an "autotransformer." It consists of a single coil (with an iron core). Three taps T_i are provided. Between taps T_1 and T_2 there are 200 turns, and between taps T_2 and T_3 there are 800 turns. Any two taps can be chosen as the primary terminals, and any two taps can be chosen as the secondary terminals. For choices producing a step-up transformer, what are the (a) smallest, (b) second smallest, and (c) largest values of the ratio V_s/V_p? For a step-down transformer, what are the (d) smallest, (e) second smallest, and (f) largest values of V_s/V_p?

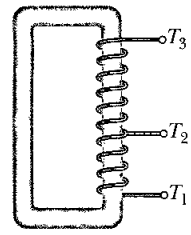


FIG. 32-36 Problem 24.

ADDITIONAL PROBLEMS

25. (a) In an oscillating LC circuit, in terms of the maximum charge Q on the capacitor, what is the charge there when the energy in the electric field is 50.0% of that in the magnetic field? (b) What fraction of a period must elapse following the time the capacitor is fully charged for this condition to occur?

26. In an oscillating LC circuit, $L = 8.00$ mH and $C = 1.40\ \mu$F. At time $t = 0$, the current is maximum at 12.0 mA. (a) What is the maximum charge on the capacitor during the oscillations? (b) At what earliest time $t > 0$ is the rate of change of energy in the capacitor maximum? (c) What is that maximum rate of change?

27. Figure 32-37 shows a driven RLC circuit that contains two identical capacitors and two switches. The emf amplitude is set at 12.0 V, and the driving frequency is set at 60.0 Hz. With both switches open, the current leads the emf by 30°. With switch S_1 closed and switch S_2 still open, the emf leads the current by 30°. With both switches closed, the current amplitude is 447 mA. What are (a) R, (b) C, and (c) L?

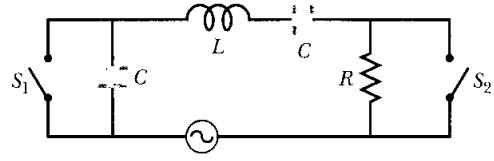


FIG. 32-37 Problem 27.

28. A typical light dimmer used to dim the stage lights in a theater consists of a variable inductor L (whose inductance is adjustable between zero and L_{max}) connected in series with a lightbulb B, as shown in Fig. 32-38. The electrical supply is 120 V (rms) at 60.0 Hz; the lightbulb is rated at 120 V, 1000 W. (a) What L_{max} is required if the rate of energy dissipation in the lightbulb is to be varied by a factor of 5 from its upper limit of 1000 W? Assume that the resistance of the lightbulb is independent of its temperature. (b) Could one use a variable resistor (adjustable between zero and R_{max}) instead of an inductor? (c) If so, what R_{max} is required? (d) Why isn't this done?

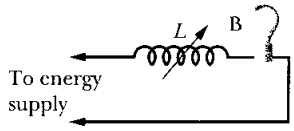


FIG. 32-38 Problem 28.

29. A series RLC circuit has a resonant frequency of 6.00 kHz. When it is driven at 8.00 kHz, it has an impedance of 1.00 kΩ and a phase constant of 45°. What are (a) R, (b) L, and (c) C for this circuit?

30. In the circuit shown in Fig. 32-39, the driving frequency $\omega = 1/\sqrt{LC}$. (a) What is the current through R? (b) What is the current through L?

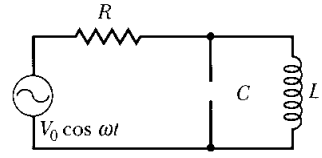


FIG. 32-39 Problem 30.

31. The generator delivers four times as much current at very low frequencies than it does at very high frequencies as shown in Fig. 32-40. Find the ratio R_2/R_1 of the resistance.

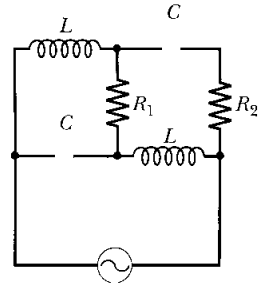


FIG. 32-40 Problem 31.

ANSWERS

Checkpoints

1. (a) $T/2$; (b) T; (c) $T/2$; (d) $T/4$

2. (a) 5 V; (b) 150 μJ

3. (a) remains the same; (b) remains the same

4. (a) C, B, A; (b) 1, A; 2, B; 3, S; 4, C; (c) A

5. (a) same; (b) increases; (c) same; (d) decreases

6. less than

7. (a) 1, lags; 2 leads; 3, in phase; (b) 3 ($\omega_d = \omega$ when $X_L = X_C$)

8. (a) increase (circuit is mainly capacitive; increase C to decrease X_C to be closer to resonance for maximum P_{avg}); (b) closer

9. (a) greater; (b) step-up

Reasoning Type Questions

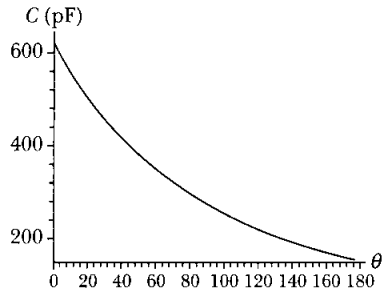
1. (b)

2. (b)

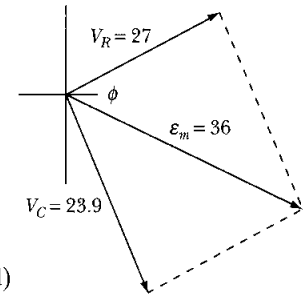
Multiple Choice Questions

1. (c)
2. (b)
3. (d)
4. (a)
5. (c)
6. (d)
7. (c)
8. (e)
9. (e)
10. (c)
11. (a), (b), (c), (d)
12. (a)
13. (a)
14. (b)
15. (d)
16. (d)
17. (c)
18. (d)
19. (a), (b)
20. (a)
21. (c)
22. (b)
23. (a)
24. (d)
25. (d)

Problems

1. (a) 6.00 μs; (b) 167 kHz; (c) 3.00 μs
2. 45.2 mA
3. (a) 5.00 μs; (b) 2.50 μs; (c) 1.25 μs
4. 9.14×10^{-9} F
5. (a) 89 rad/s; (b) 7.0×10^{-2} s; (c) 2.5×10^{-5} F
7. (a) 275 Hz; (b) 0.365 A
8. 

9. ω
10. (a) 356 μs; (b) 2.50 mH; (c) 3.20 mJ
11. (a) 1.98×10^{-6} J; (b) 5.56×10^{-6} C; (c) 1.26×10^{-2} A; (d) $-46.9°$; (e) $+46.9°$
12. (a) 46.1 μs; (b) 6.88 nJ; (c) 6.88 nJ; (d) 1.02×10^{3} A/s; (e) 0.938 mW
13. (a) 6.5×10^{2} Hz; (b) 24 Ω
14. (a) 39.1 mA; (b) zero; (c) 33.8 mA
15. (a) 267 Ω; (b) $-41.5°$; (c) 135 mA;



(d)

16. (a) 8.0 μF; (b) 2.0 Ω
17. –8.00 V
18. (a) Yes; (b) 1.0×10^{3} V
19. (a) 168 Ω; (b) decrease; (c) decrease; (d) decrease
20. (a) 0.743; (b) the current leads the emf; (c) largely capacitive; (d) the circuit is not in resonance; (e) the box must contain a capacitor and a resistor; (f) the inductive reactance may be zero, so there need not be an inductor; (g) yes, there is a resistor; (h) 33.4 W; (i) the answers above depend on the frequency only through the phase constant ϕ, which is given. If values were given for R, L and C then the value of the frequency would also be needed to compute the power factor.
21. 100 V in all three cases
22. 1.84 A
23. 1.00×10^{3} V
24. (a) 1.25; (b) 4.00; (c) 5.00; (d) 0.200; (e) 0.250; (f) 0.800

Additional Problems

25. (a) $0.577Q$; (b) 0.152
26. (a) 1.27×10^{-6} C; (b) 8.31×10^{-5} s; (c) 5.44×10^{-3} J/s
27. (a) $R = 100\ \Omega$; (b) $C = 30.6\ \mu$F; (c) $L = 301$ mH
28. (a) 7.64×10^{-2} H; (b) yes; (c) 17.8 Ω; (d) this is not done because the resistors would consume power, rather than temporarily store electromagnetic energy
29. (a) 707 Ω; (b) 32.2 mH; (c) 21.9 nF
30. (a) 0; (b) $\dfrac{V}{\omega L}\sin \omega t$
31. 3:1

Nature of Electromagnetic Waves



Norbert Rosing/National Geographic/Getty Images, Inc.

On bright days the sky can display a variety of bright spots and arcs (halos). Probably the most common display is a bright spot called a sun dog that appears to the left or right of the Sun and sometimes on both sides. Because sun dogs often show colors, they can be mistaken for rainbows. However, sun dogs are not produced by water drops as rainbows are.

What produces sun dogs?

The answer is in this chapter.

33-1 WHAT IS PHYSICS?

The information age in which we live is based almost entirely on the physics of electromagnetic waves. Like it or not, we are now globally connected by television, telephones, and the Web. And like it or not, we are constantly immersed in those signals because of television, radio, and telephone transmitters.

Much of this global interconnection of information processors was not imagined by even the most visionary engineers of 20 years ago. The challenge for today's engineers is trying to envision what the global interconnection will be like 20 years from now. The starting point in meeting that challenge is understanding the basic physics of electromagnetic waves, which come in so many different types that they are poetically said to form *Maxwell's rainbow.*

33-2 | Maxwell's Rainbow

The crowning achievement of James Clerk Maxwell was to show that a beam of light is a traveling wave of electric and magnetic fields—an **electromagnetic wave**—and thus that optics, the study of visible light, is a branch of electromagnetism. In this chapter we move from one to the other: we conclude our discussion of strictly electrical and magnetic phenomena, and we build a foundation for optics.

In Maxwell's time (the mid 1800s), the visible, infrared, and ultraviolet forms of light were the only electromagnetic waves known. Spurred on by Maxwell's work, however, Heinrich Hertz discovered what we now call radio waves and verified that they move through the laboratory at the same speed as visible light.

As Fig. 33-1 shows, we now know a wide *spectrum* (or range) of electromagnetic waves: Maxwell's rainbow. Consider the extent to which we are immersed in electromagnetic waves throughout this spectrum. The Sun, whose radiations define the environment in which we as a species have evolved and adapted, is the dominant source. We are also crisscrossed by radio and television signals. Microwaves from radar systems and from telephone relay systems may reach us. There are electromagnetic waves from lightbulbs, from the heated engine blocks of automobiles, from x-ray machines, from lightning flashes, and from buried radioactive

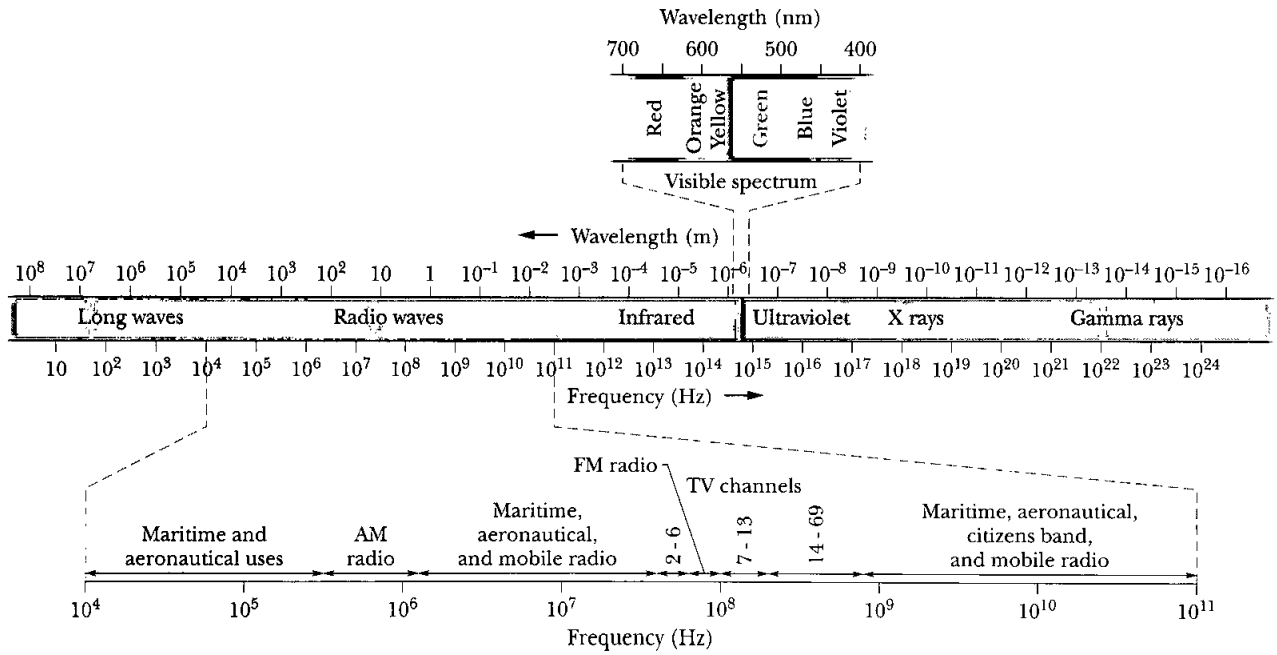


FIG. 33-1 The electromagnetic spectrum.

materials. Beyond this, radiation reaches us from stars and other objects in our galaxy and from other galaxies. Electromagnetic waves also travel in the other direction. Television signals, transmitted from Earth since about 1950, have now taken news about us (along with episodes of *I Love Lucy,* albeit *very* faintly) to whatever technically sophisticated inhabitants there may be on whatever planets may encircle the nearest 400 or so stars.

In the wavelength scale in Fig. 33-1 (and similarly the corresponding frequency scale), each scale marker represents a change in wavelength (and correspondingly in frequency) by a factor of 10. The scale is open-ended; the wavelengths of electromagnetic waves have no inherent upper or lower bound.

Certain regions of the electromagnetic spectrum in Fig. 33-1 are identified by familiar labels, such as *x rays* and *radio waves.* These labels denote roughly defined wavelength ranges within which certain kinds of sources and detectors of electromagnetic waves are in common use. Other regions of Fig. 33-1, such as those labeled TV channels and AM radio, represent specific wavelength bands assigned by law for certain commercial or other purposes. There are no gaps in the electromagnetic spectrum—and all electromagnetic waves, no matter where they lie in the spectrum, travel through *free space* (vacuum) with the same speed c.

The visible region of the spectrum is of course of particular interest to us. Figure 33-2 shows the relative sensitivity of the human eye to light of various wavelengths. The center of the visible region is about 555 nm, which produces the sensation that we call yellow-green.

The limits of this visible spectrum are not well defined because the eye-sensitivity curve approaches the zero-sensitivity line asymptotically at both long and short wavelengths. If we take the limits, arbitrarily, as the wavelengths at which eye sensitivity has dropped to 1% of its maximum value, these limits are about 430 and 690 nm; however, the eye can detect electromagnetic waves somewhat beyond these limits if they are intense enough.

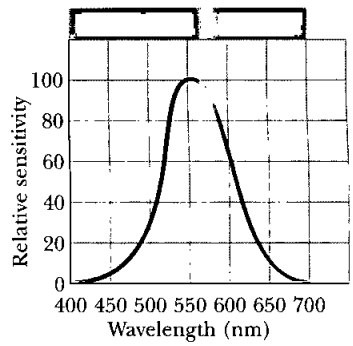


FIG. 33-2 The relative sensitivity of the average human eye to electromagnetic waves at different wavelengths. This portion of the electromagnetic spectrum to which the eye is sensitive is called *visible light.*

33-3 | The Traveling Electromagnetic Wave, Qualitatively

Some electromagnetic waves, including x rays, gamma rays, and visible light, are *radiated* (emitted) from sources that are of atomic or nuclear size, where quantum physics rules. Here we discuss how other electromagnetic waves are generated. To simplify matters, we restrict ourselves to that region of the spectrum (wavelength $\lambda \approx 1$ m) in which the source of the *radiation* (the emitted waves) is both macroscopic and of manageable dimensions. A charge at rest or moving at constant velocity does not generate electromagnetic waves. Electromagnetic waves are produced only by charges that accelerate. Electromagnetic waves consist of oscillating electric and magnetic fields that travel away from the accelerating charges.

Figure 33-3 shows, in broad outline, the generation of such waves. At its heart is an *LC oscillator,* which establishes an angular frequency $\omega(= 1/\sqrt{LC})$. Charges and currents in this circuit vary sinusoidally at this frequency, as depicted in Fig. 31-1. An external source—possibly an ac generator—must be included to supply energy to compensate both for thermal losses in the circuit and for energy carried away by the radiated electromagnetic wave.

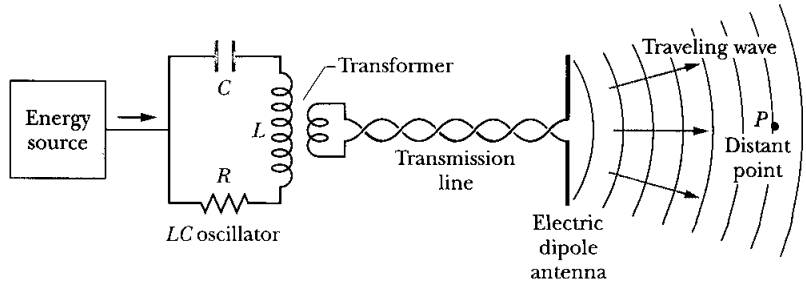


FIG. 33-3 An arrangement for generating a traveling electromagnetic wave in the shortwave radio region of the spectrum: an LC oscillator produces a sinusoidal current in the antenna, which generates the wave. P is a distant point at which a detector can monitor the wave traveling past it.

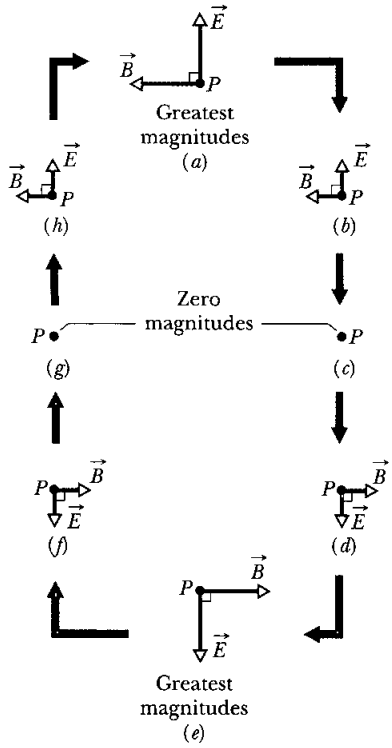


FIG. 33-4 (*a*)–(*h*) The variation in the electric field $\vec{E}$ and the magnetic field $\vec{B}$ at the distant point P of Fig. 33-3 as one wavelength of the electromagnetic wave travels past it. In this perspective, the wave is traveling directly out of the page. The two fields vary sinusoidally in magnitude and direction. Note that they are always perpendicular to each other and to the wave's direction of travel.

The LC oscillator of Fig. 33-3 is coupled by a transformer and a transmission line to an *antenna,* which consists essentially of two thin, solid, conducting rods. Through this coupling, the sinusoidally varying current in the oscillator causes charge to oscillate sinusoidally along the rods of the antenna at the angular frequency ω of the LC oscillator. The current in the rods associated with this movement of charge also varies sinusoidally, in magnitude and direction, at angular frequency ω. The antenna has the effect of an electric dipole whose electric dipole moment varies sinusoidally in magnitude and direction along the antenna.

Because the dipole moment varies in magnitude and direction, the electric field produced by the dipole varies in magnitude and direction. Also, because the current varies, the magnetic field produced by the current varies in magnitude and direction. However, the changes in the electric and magnetic fields do not happen everywhere instantaneously; rather, the changes travel outward from the antenna at the speed of light c. Together the changing fields form an electromagnetic wave that travels away from the antenna at speed c. The angular frequency of this wave is ω, the same as that of the LC oscillator.

Figure 33-4 shows how the electric field $\vec{E}$ and the magnetic field $\vec{B}$ change with time as one wavelength of the wave sweeps past the distant point P of Fig. 33-3; in each part of Fig. 33-4, the wave is traveling directly out of the page. (We choose a distant point so that the curvature of the waves suggested in Fig. 33-3 is small enough to neglect. At such points, the wave is said to be a *plane wave,* and discussion of the wave is much simplified.) Note several key features in Fig. 33-4; they are present regardless of how the wave is created:

1. The electric and magnetic fields $\vec{E}$ and $\vec{B}$ are always perpendicular to the direction in which the wave is traveling. Thus, the wave is a *transverse wave.*
2. The electric field is always perpendicular to the magnetic field.
3. The cross product $\vec{E} \times \vec{B}$ always gives the direction in which the wave travels.
4. The fields always vary sinusoidally, just like the transverse waves discussed in Chapter 16, Vol. I. Moreover, the fields vary with the same frequency and *in phase* (in step) with each other.

In keeping with these features, we can assume that the electromagnetic wave is traveling toward P in the positive direction of an x axis, that the electric field in Fig. 33-4 is oscillating parallel to the y axis, and that the magnetic field is then oscillating parallel to the z axis (using a right-handed coordinate system, of course). Then we can write the electric and magnetic fields as sinusoidal functions of position x (along the path of the wave) and time t:

$$E = E_m \sin(kx - \omega t), \tag{33-1}$$

$$B = B_m \sin(kx - \omega t), \tag{33-2}$$

in which E_m and B_m are the amplitudes of the fields and ω and k are the angular frequency and angular wave number of the wave, respectively. From these equations, we note that not only do the two fields form the electromagnetic wave but each also forms its own wave. Equation 33-1 gives the *electric wave component* of the electromagnetic wave, and Eq. 33-2 gives the *magnetic wave component.* As we shall discuss below, these two wave components cannot exist independently.

From Eq. 16-13, we know that the speed of the wave is ω/k. However, because this is an electromagnetic wave, its speed (in vacuum) is given the symbol c rather than v. In the next section you will see that c has the value

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{(wave speed)}, \tag{33-3}$$

which is about 3.0×10^8 m/s. In other words,

☞ All electromagnetic waves, including visible light, have the same speed c in vacuum.

You will also see that the wave speed c and the amplitudes of the electric and magnetic fields are related by

$$\frac{E_m}{B_m} = c \qquad \text{(amplitude ratio).} \tag{33-4}$$

If we divide Eq. 33-1 by Eq. 33-2 and then substitute with Eq. 33-4, we find that the magnitudes of the fields at every instant and at any point are related by

$$\frac{E}{B} = c \qquad \text{(magnitude ratio).} \tag{33-5}$$

We can represent the electromagnetic wave as in Fig. 33-5*a*, with a *ray* (a directed line showing the wave's direction of travel) or with *wavefronts* (imaginary surfaces over which the wave has the same magnitude of electric field), or both. The two wavefronts shown in Fig. 33-5*a* are separated by one wavelength λ ($= 2\pi/k$) of the wave. (Waves traveling in approximately the same direction form a *beam,* such as a laser beam, which can also be represented with a ray.)

We can also represent the wave as in Fig. 33-5*b*, which shows the electric and magnetic field vectors in a "snapshot" of the wave at a certain instant. The curves through the tips of the vectors represent the sinusoidal oscillations given by Eqs. 33-1 and 33-2; the wave components $\vec{E}$ and $\vec{B}$ are in phase, perpendicular to each other, and perpendicular to the wave's direction of travel.

Interpretation of Fig. 33-5*b* requires some care. The similar drawings for a transverse wave on a taut string that we discussed in Chapter 16, Vol. I, represented the up and down displacement of sections of the string as the wave passed (*something actually moved*). Figure 33-5*b* is more abstract. At the instant shown, the electric and magnetic fields each have a certain magnitude and direction (but always perpendicular to the x axis) at each point along the x axis. We choose to represent these vector quantities with a pair of arrows for each point, and so we must draw arrows of different lengths for different points, all directed away from the x axis, like thorns on a rose stem. However, the arrows represent field values only at points that are on the x axis. Neither the arrows nor the sinusoidal curves represent a sideways motion of anything, nor do the arrows connect points on the x axis with points off the axis.

Drawings like Fig. 33-5 help us visualize what is actually a very complicated situation. First consider the magnetic field. Because it varies sinusoidally, it induces (via Faraday's law of induction) a perpendicular electric field that also varies sinusoidally. However, because that electric field is varying sinusoidally, it induces (via Maxwell's law of induction) a perpendicular magnetic field that also

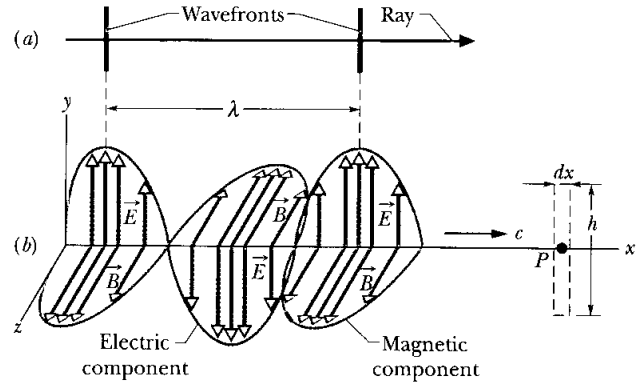


FIG. 33-5 (*a*) An electromagnetic wave represented with a ray and two wavefronts; the wavefronts are separated by one wavelength λ. (*b*) The same wave represented in a "snapshot" of its electric field $\vec{E}$ and magnetic field $\vec{B}$ at points on the x axis, along which the wave travels at speed c. As it travels past point P, the fields vary as shown in Fig. 33-4. The electric component of the wave consists of only the electric fields; the magnetic component consists of only the magnetic fields. The dashed rectangle at P is used in Fig. 33-6.

varies sinusoidally. And so on. The two fields continuously create each other via induction, and the resulting sinusoidal variations in the fields travel as a wave—the electromagnetic wave. Without this amazing result, we could not see; indeed, because we need electromagnetic waves from the Sun to maintain Earth's temperature, without this result we could not even exist.

A Most Curious Wave

The waves we discussed in Chapters 16 and 17, Vol. I, require a *medium* (some material) through which or along which to travel. We had waves traveling along a string, through Earth, and through the air. However, an electromagnetic wave (let's use the term *light wave* or *light*) is curiously different in that it requires no medium for its travel. It can, indeed, travel through a medium such as air or glass, but it can also travel through the vacuum of space between a star and us.

Once the special theory of relativity became accepted, long after Einstein published it in 1905, the speed of light waves was realized to be special. One reason is that light has the same speed regardless of the frame of reference from which it is measured. If you send a beam of light along an axis and ask several observers to measure its speed while they move at different speeds along that axis, either in the direction of the light or opposite it, they will all measure the *same speed* for the light. This result is an amazing one and quite different from what would have been found if those observers had measured the speed of any other type of wave; for other waves, the speed of the observers relative to the wave would have affected their measurements.

The meter has now been defined so that the speed of light (any electromagnetic wave) in vacuum has the exact value

$$c = 299\,792\,458 \text{ m/s},$$

which can be used as a standard. In fact, if you now measure the travel time of a pulse of light from one point to another, you are not really measuring the speed of the light but rather the distance between those two points.

33-4 | The Traveling Electromagnetic Wave, Quantitatively

We shall now derive Eqs. 33-3 and 33-4 and, even more important, explore the dual induction of electric and magnetic fields that gives us light.

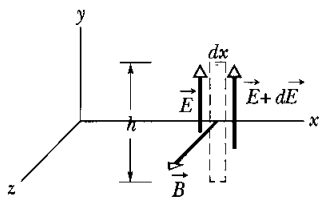


FIG. 33-6 As the electromagnetic wave travels rightward past point P in Fig. 33-5*b*, the sinusoidal variation of the magnetic field $\vec{B}$ through a rectangle centered at P induces electric fields along the rectangle. At the instant shown, $\vec{B}$ is decreasing in magnitude and the induced electric field is therefore greater in magnitude on the right side of the rectangle than on the left.

Equation 33-4 and the Induced Electric Field

The dashed rectangle of dimensions dx and h in Fig. 33-6 is fixed at point P on the x axis and in the xy plane (it is shown on the right in Fig. 33-5*b*). As the electromagnetic wave moves rightward past the rectangle, the magnetic flux Φ_B through the rectangle changes and—according to Faraday's law of induction—induced electric fields appear throughout the region of the rectangle. We take $\vec{E}$ and $\vec{E} + d\vec{E}$ to be the induced fields along the two long sides of the rectangle. These induced electric fields are, in fact, the electrical component of the electromagnetic wave.

Note the small red portion of the magnetic field component curve far from the y axis in Fig. 33-5*b*. Let's consider the induced electric fields at the instant when this red portion of the magnetic component is passing through the rectangle. Just then, the magnetic field through the rectangle points in the positive z direction and is decreasing in magnitude (the magnitude was greater just before the red section arrived). Because the magnetic field is decreasing, the magnetic flux Φ_B through the rectangle is also decreasing. According to Faraday's law, this

change in flux is opposed by induced electric fields, which produce a magnetic field $\vec{B}$ in the positive z direction.

According to Lenz's law, this in turn means that if we imagine the boundary of the rectangle to be a conducting loop, a counterclockwise induced current would have to appear in it. There is, of course, no conducting loop; but this analysis shows that the induced electric field vectors $\vec{E}$ and $\vec{E} + d\vec{E}$ are indeed oriented as shown in Fig. 33-6, with the magnitude of $\vec{E} + d\vec{E}$ greater than that of $\vec{E}$. Otherwise, the net induced electric field would not act counterclockwise around the rectangle.

Let us now apply Faraday's law of induction,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}, \tag{33-6}$$

counterclockwise around the rectangle of Fig. 33-6. There is no contribution to the integral from the top or bottom of the rectangle because $\vec{E}$ and $d\vec{s}$ are perpendicular to each other there. The integral then has the value

$$\oint \vec{E} \cdot d\vec{s} = (E + dE)h - Eh = h\,dE. \tag{33-7}$$

The flux Φ_B through this rectangle is

$$\Phi_B = (B)(h\,dx), \tag{33-8}$$

where B is the average magnitude of $\vec{B}$ within the rectangle and $h\,dx$ is the area of the rectangle. Differentiating Eq. 33-8 with respect to t gives

$$\frac{d\Phi_B}{dt} = h\,dx\frac{dB}{dt}. \tag{33-9}$$

If we substitute Eqs. 33-7 and 33-9 into Eq. 33-6, we find

$$h\,dE = -h\,dx\frac{dB}{dt}$$

or

$$\frac{dE}{dx} = -\frac{dB}{dt}. \tag{33-10}$$

Actually, both B and E are functions of *two* variables, x and t, as Eqs. 33-1 and 33-2 show. However, in evaluating dE/dx, we must assume that t is constant because Fig. 33-6 is an "instantaneous snapshot." Also, in evaluating dB/dt we must assume that x is constant because we are dealing with the time rate of change of B at a particular place, the point P in Fig. 33-5*b*. The derivatives under these circumstances are *partial derivatives,* and Eq. 33-10 must be written

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}. \tag{33-11}$$

The minus sign in this equation is appropriate and necessary because, although E is increasing with x at the site of the rectangle in Fig. 33-6, B is decreasing with t.

From Eq. 33-1 we have

$$\frac{\partial E}{\partial x} = kE_m \cos(kx - \omega t)$$

and from Eq. 33-2

$$\frac{\partial B}{\partial t} = -\omega B_m \cos(kx - \omega t).$$

Then Eq. 33-11 reduces to

$$kE_m \cos(kx - \omega t) = \omega B_m \cos(kx - \omega t). \tag{33-12}$$

The ratio ω/k for a traveling wave is its speed, which we are calling c. Equation 33-12 then becomes

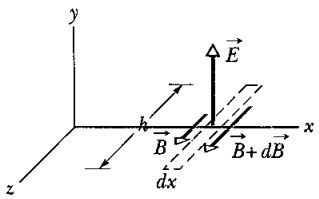


FIG. 33-7 The sinusoidal variation of the electric field through this rectangle, located (but not shown) at point P in Fig. 33-5*b*, induces magnetic fields along the rectangle. The instant shown is that of Fig. 33-6: $\vec{E}$ is decreasing in magnitude, and the magnitude of the induced magnetic field is greater on the right side of the rectangle than on the left.

$$\frac{E_m}{B_m} = c \qquad \text{(amplitude ratio)}, \tag{33-13}$$

which is just Eq. 33-4.

Equation 33-3 and the Induced Magnetic Field

Figure 33-7 shows another dashed rectangle at point P of Fig. 33-5*b*; this one is in the xz plane. As the electromagnetic wave moves rightward past this new rectangle, the electric flux Φ_E through the rectangle changes and—according to Maxwell's law of induction—induced magnetic fields appear throughout the region of the rectangle. These induced magnetic fields are, in fact, the magnetic component of the electromagnetic wave.

We see from Fig. 33-5*b* that at the instant chosen for the magnetic field represented in Fig. 33-6, marked in red on the magnetic component curve, the electric field through the rectangle of Fig. 33-7 is directed as shown. Recall that at the chosen instant, the magnetic field in Fig. 33-6 is decreasing. Because the two fields are in phase, the electric field in Fig. 33-7 must also be decreasing, and so must the electric flux Φ_E through the rectangle. By applying the same reasoning we applied to Fig. 33-6, we see that the changing flux Φ_E will induce a magnetic field with vectors $\vec{B}$ and $\vec{B} + d\vec{B}$ oriented as shown in Fig. 33-7, where field $\vec{B} + d\vec{B}$ is greater than field $\vec{B}$.

Let us apply Maxwell's law of induction,

$$\oint \vec{B}\cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}, \tag{33-14}$$

by proceeding counterclockwise around the dashed rectangle of Fig. 33-7. Only the long sides of the rectangle contribute to the integral, whose value is

$$\oint \vec{B}\cdot d\vec{s} = -(B+dB)h + Bh = -h\,dB. \tag{33-15}$$

The flux Φ_E through the rectangle is

$$\Phi_E = (E)(h\,dx), \tag{33-16}$$

where E is the average magnitude of $\vec{E}$ within the rectangle. Differentiating Eq. 33-16 with respect to t gives

$$\frac{d\Phi_E}{dt} = h\,dx\frac{dE}{dt}.$$

If we substitute this and Eq. 33-15 into Eq. 33-14, we find

$$-h\,dB = \mu_0\varepsilon_0\left(h\,dx\frac{dE}{dt}\right)$$

or, changing to partial-derivative notation as we did for Eq. 33-11,

$$-\frac{\partial B}{\partial x} = \mu_0\varepsilon_0\frac{\partial E}{\partial t}. \tag{33-17}$$

Again, the minus sign in this equation is necessary because, although B is increasing with x at point P in the rectangle in Fig. 33-7, E is decreasing with t.

Evaluating Eq. 33-17 by using Eqs. 33-1 and 33-2 leads to

$$-kB_m\cos(kx-\omega t) = -\mu_0\varepsilon_0\omega E_m\cos(kx-\omega t),$$

which we can write as

$$\frac{E_m}{B_m} = \frac{1}{\mu_0\varepsilon_0(\omega/k)} = \frac{1}{\mu_0\varepsilon_0 c}.$$

Combining this with Eq. 33-13 leads at once to

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{(wave speed)}, \tag{33-18}$$

which is exactly Eq. 33-3. Let us now explore another question: What is the speed of propagation of an electromagnetic pulse in a dielectric with dielectric constant k?

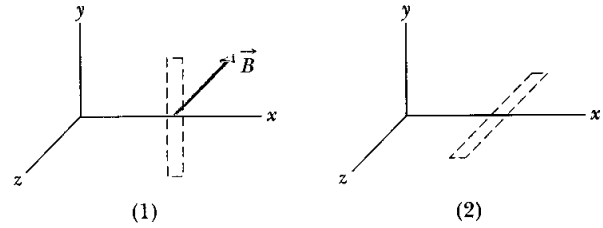
To work out the speed of light in a dielectric, we begin with the modification of the displacement current (Eq. 33-14) in the presence of a dielectric. In a dielectric, as in space, conventional current is zero. So this equation reduces to

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 k \frac{d\Phi_E}{dt}. \tag{33-19}$$

If we compare this with the analysis before, it is clear that this will yield the same answer for the speed of light except for the multiplying factor of k. We find the following for the speed of light in a dielectric:

$$c' = \frac{1}{\sqrt{k \mu_0 \varepsilon_0}} = \frac{c}{\sqrt{k}}. \tag{33-20}$$

CHECKPOINT 1 The magnetic field $\vec{B}$ through the rectangle of Fig. 33-6 is shown at a different instant in part 1 of the figure here; $\vec{B}$ is directed in the xz plane, parallel to the z axis, and its magnitude is increasing. (a) Complete part 1 by drawing the induced electric fields, indicating both directions and relative magnitudes (as in Fig. 33-6). (b) For the same instant, complete part 2 of the figure by drawing the electric field of the electromagnetic wave. Also draw the induced magnetic fields, indicating both directions and relative magnitudes (as in Fig. 33-7).



33-5 | Energy Transport and the Poynting Vector

All sunbathers know that an electromagnetic wave can transport energy and deliver it to a body on which the wave falls. The rate of energy transport per unit area in such a wave is described by a vector $\vec{S}$, called the **Poynting vector** after physicist John Henry Poynting (1852–1914), who first discussed its properties. This vector is defined as

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \text{(Poynting vector)}. \tag{33-21}$$

Its magnitude S is related to the rate at which energy is transported by a wave across a unit area at any instant (inst):

$$S = \left(\frac{\text{energy/time}}{\text{area}} \right)_{\text{inst}} = \left(\frac{\text{power}}{\text{area}} \right)_{\text{inst}}. \tag{33-22}$$

From this we can see that the SI unit for $\vec{S}$ is the watt per square meter (W/m^2).

The direction of the Poynting vector $\vec{S}$ of an electromagnetic wave at any point gives the wave's direction of travel and the direction of energy transport at that point.

Because $\vec{E}$ and $\vec{B}$ are perpendicular to each other in an electromagnetic wave, the magnitude of $\vec{E} \times \vec{B}$ is EB. Then the magnitude of $\vec{S}$ is

$$S = \frac{1}{\mu_0} EB, \tag{33-23}$$

in which S, E, and B are instantaneous values. The magnitudes E and B are so closely coupled to each other that we need to deal with only one of them; we choose E, largely because most instruments for detecting electromagnetic waves deal with the electric component of the wave rather than the magnetic component. Using $B = E/c$ from Eq. 33-5, we can rewrite Eq. 33-23 as

$$S = \frac{1}{c\mu_0} E^2 \qquad \text{(instantaneous energy flow rate).} \tag{33-24}$$

By substituting $E = E_m \sin(kx - \omega t)$ into Eq. 33-24, we could obtain an equation for the energy transport rate as a function of time. More useful in practice, however, is the average energy transported over time; for that, we need to find the time-averaged value of S, written S_{avg} and also called the **intensity** I of the wave. Thus from Eq. 33-22, the intensity I is

$$I = S_{avg} = \left(\frac{\text{energy/time}}{\text{area}}\right)_{avg} = \left(\frac{\text{power}}{\text{area}}\right)_{avg}. \tag{33-25}$$

From Eq. 33-24, we find

$$I = S_{avg} = \frac{1}{c\mu_0}[E^2]_{avg} = \frac{1}{c\mu_0}[E_m^2 \sin^2(kx - \omega t)]_{avg}. \tag{33-26}$$

Over a full cycle, the average value of $\sin^2\theta$, for any angular variable θ, is $\frac{1}{2}$ (see Fig. 32-18). In addition, we define a new quantity E_{rms}, the *root-mean-square* value of the electric field, as

$$E_{rms} = \frac{E_m}{\sqrt{2}}. \tag{33-27}$$

We can then rewrite Eq. 33-26 as

$$I = \frac{1}{c\mu_0} E_{rms}^2. \tag{33-28}$$

Because $E = cB$ and c is such a very large number, you might conclude that the energy associated with the electric field is much greater than that associated with the magnetic field. That conclusion is incorrect; the two energies are exactly equal. To show this, we start with Eq. 26-23, which gives the energy density $u(= \frac{1}{2}\varepsilon_0 E^2)$ within an electric field, and substitute cB for E; then we can write

$$u_E = \tfrac{1}{2}\varepsilon_0 E^2 = \tfrac{1}{2}\varepsilon_0 (cB)^2.$$

If we now substitute for c with Eq. 33-3, we get

$$u_E = \frac{1}{2}\varepsilon_0 \frac{1}{\mu_0 \varepsilon_0} B^2 = \frac{B^2}{2\mu_0}.$$

However, Eq. 31-55 tells us that $B^2/2\mu_0$ is the energy density u_B of a magnetic field $\vec{B}$; so we see that $u_E = u_B$ everywhere along an electromagnetic wave.

Variation of Intensity with Distance

How intensity varies with distance from a real source of electromagnetic radiation is often complex—especially when the source (like a searchlight at a movie premier) beams the radiation in a particular direction. However, in some situations we can assume that the source is a *point source* that emits the light *isotropically*—that is, with equal intensity in all directions. The spherical wavefronts spreading from such an isotropic point source S at a particular instant are shown in cross section in Fig. 33-8.

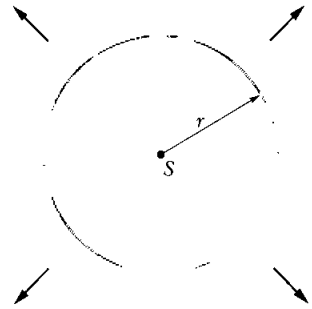


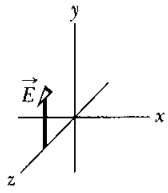
FIG. 33-8 A point source S emits electromagnetic waves uniformly in all directions. The spherical wavefronts pass through an imaginary sphere of radius r that is centered on S.

Let us assume that the energy of the waves is conserved as they spread from this source. Let us also center an imaginary sphere of radius r on the source, as shown in Fig. 33-8. All the energy emitted by the source must pass through the sphere. Thus, the rate at which energy passes through the sphere via the radiation must equal the rate at which energy is emitted by the source—that is, the source power P_s. The intensity I at the sphere must then be, from Eq. 33-25,

$$I = \frac{\text{power}}{\text{area}} = \frac{P_s}{4\pi r^2}, \tag{33-29}$$

where $4\pi r^2$ is the area of the sphere. Equation 33-29 tells us that the intensity of the electromagnetic radiation from an isotropic point source decreases with the square of the distance r from the source.

CHECKPOINT 2 The figure here gives the electric field of an electromagnetic wave at a certain point and a certain instant. The wave is transporting energy in the negative z direction. What is the direction of the magnetic field of the wave at that point and instant?



Sample Problem 33-1

When you look at the North Star (Polaris), you intercept light from a star at a distance of 431 ly and emitting energy at a rate of 2.2×10^3 times that of our Sun ($P_{sun} = 3.90 \times 10^{26}$ W). Neglecting any atmospheric absorption, find the rms values of the electric and magnetic fields when the starlight reaches you.

Conceptualize / Classify:

1. The rms value E_{rms} of the electric field in light is related to the intensity I of the light via Eq. 33-28 ($I = E_{rms}^2/c\mu_0$).
2. Because the source is so far away and emits light with equal intensity in all directions, the intensity I at any distance r from the source is related to the source's power P_s via Eq. 33-29 ($I = P_s/4\pi r^2$).
3. The magnitudes of the electric field and magnetic field of an electromagnetic wave at any instant and at any point in the wave are related by the speed of light c according to Eq. 33-5 ($E/B = c$). Thus, the rms values of those fields are also related by Eq. 33-5.

Compute: Electric field: Putting the first two ideas together gives us

$$I = \frac{P_s}{4\pi r^2} = \frac{E_{rms}^2}{c\mu_0}$$

and

$$E_{rms} = \sqrt{\frac{P_s c\mu_0}{4\pi r^2}}.$$

Substituting $P_s = (2.2 \times 10^3)(3.90 \times 10^{26}\ \text{W})$, $r = 431\ \text{ly} = 4.08 \times 10^{18}$ m, and values for the constants, we find

$$E_{rms} = 1.24 \times 10^{-3}\ \text{V/m} \approx 1.2\ \text{mV/m}. \quad \text{(Answer)}$$

Magnetic field: From Eq. 33-5, we write

$$B_{rms} = \frac{E_{rms}}{c} = \frac{1.24 \times 10^{-3}\ \text{V/m}}{3.00 \times 10^8\ \text{m/s}} = 4.1 \times 10^{-12}\ \text{T} = 4.1\ \text{pT}.$$

Conclude:

Cannot compare the fields: Note that E_{rms} (= 1.2 mV/m) is small as judged by ordinary laboratory standards, but B_{rms} (= 4.1 pT) is quite small. This difference helps to

explain why most instruments used for the detection and measurement of electromagnetic waves are designed to respond to the electric component of the wave. It is wrong, however, to say that the electric component of an electromagnetic wave is "stronger" than the magnetic component. You cannot compare quantities that are measured in different units. As we have seen, the electric and magnetic components are on an equal basis as far as the propagation of the wave is concerned because their average energies, which *can* be compared, are equal.

33-6 | Radiation Pressure

Electromagnetic waves have linear momentum as well as energy. This means that we can exert a pressure—a **radiation pressure**—on an object by shining light on it. However, the pressure must be very small because, for example, you do not feel a camera flash when it is used to take your photograph (which is good because otherwise every photographic flash could be like a punch).

To find an expression for the pressure, let us shine a beam of electromagnetic radiation—light, for example—on an object for a time interval Δt. Further, let us assume that the object is free to move and that the radiation is entirely **absorbed** (taken up) by the object. This means that during the interval Δt, the object gains an energy ΔU from the radiation. Maxwell showed that the object also gains linear momentum. The magnitude Δp of the momentum change of the object is related to the energy change ΔU by

$$\Delta p = \frac{\Delta U}{c} \qquad \text{(total absorption)}, \tag{33-30}$$

where c is the speed of light. The direction of the momentum change of the object is the direction of the *incident* (incoming) beam that the object absorbs.

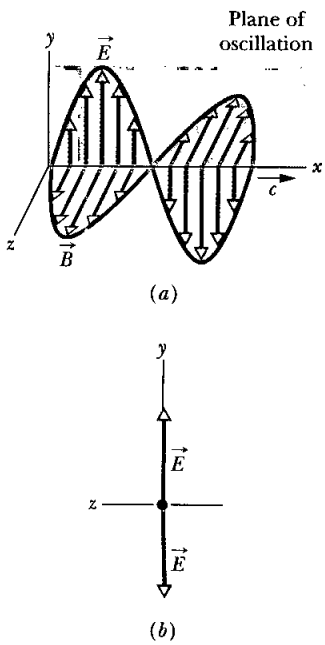


FIG. 33-9 (*a*) The plane of oscillation of a polarized electromagnetic wave. (*b*) To represent the polarization, we view the plane of oscillation head-on and indicate the directions of the oscillating electric field with a double arrow.

33-7 | Polarization

VHF (very high frequency) television antennas in England are oriented vertically, but those in North America are horizontal. The difference is due to the direction of oscillation of the electromagnetic waves carrying the TV signal. In England, the transmitting equipment is designed to produce waves that are **polarized** vertically; that is, their electric field oscillates vertically. Thus, for the electric field of the incident television waves to drive a current along an antenna (and provide a signal to a television set), the antenna must be vertical. In North America, the waves are polarized horizontally.

Figure 33-9*a* shows an electromagnetic wave with its electric field oscillating parallel to the vertical y axis. The plane containing the $\vec{E}$ vectors is called the **plane of oscillation** of the wave (hence, the wave is said to be *plane-polarized* in the y direction). We can represent the wave's *polarization* (state of being polarized) by showing the directions of the electric field oscillations in a head-on view of the plane of oscillation, as in Fig. 33-9*b*. The vertical double arrow in that figure indicates that as the wave travels past us, its electric field oscillates vertically—it continuously changes between being directed up and down the y axis.

Polarized Light

The electromagnetic waves emitted by a television station all have the same polarization, but the electromagnetic waves emitted by any common source of light (such as the Sun or a bulb) are **polarized randomly**, or **unpolarized** (the two terms mean the same thing). That is, the electric field at any given point is always perpendicular to the direction of travel of the waves but changes directions

randomly. Thus, if we try to represent a head-on view of the oscillations over some time period, we do not have a simple drawing with a single double arrow like that of Fig. 33-9*b*; instead we have a mess of double arrows like that in Fig. 33-10*a*.

In principle, we can simplify the mess by resolving each electric field of Fig. 33-10*a* into y and z components. Then as the wave travels past us, the net y component oscillates parallel to the y axis and the net z component oscillates parallel to the z axis. We can then represent the unpolarized light with a pair of double arrows as shown in Fig. 33-10*b*. The double arrow along the y axis represents the oscillations of the net y component of the electric field. The double arrow along the z axis represents the oscillations of the net z component of the electric field. In doing all this, we effectively change unpolarized light into the superposition of two polarized waves whose planes of oscillation are perpendicular to each other—one plane contains the y axis and the other contains the z axis. One reason to make this change is that drawing Fig. 33-10*b* is a lot easier than drawing Fig. 33-10*a*.

We can draw similar figures to represent light that is **partially polarized** (its field oscillations are not completely random as in Fig. 33-10*a*, nor are they parallel to a single axis as in Fig. 33-9*b*). For this situation, we draw one of the double arrows in a perpendicular pair of double arrows longer than the other one.

We can transform unpolarized visible light into polarized light by sending it through a *polarizing sheet,* as is shown in Fig. 33-11. Such sheets, commercially known as Polaroids or Polaroid filters, were invented in 1932 by Edwin Land while he was an undergraduate student. A polarizing sheet consists of certain long molecules embedded in plastic. When the sheet is manufactured, it is stretched to align the molecules in parallel rows, like rows in a plowed field. When light is then sent through the sheet, electric field components along one direction pass through the sheet, while components perpendicular to that direction are absorbed by the molecules and disappear.

We shall not dwell on the molecules but, instead, shall assign to the sheet a *polarizing direction,* along which electric field components are passed:

☞ An electric field component parallel to the polarizing direction is passed (*transmitted*) by a polarizing sheet; a component perpendicular to it is absorbed.

Thus, the electric field of the light emerging from the sheet consists of only the components that are parallel to the polarizing direction of the sheet; hence the light is polarized in that direction. In Fig. 33-11, the vertical electric field components are transmitted by the sheet; the horizontal components are absorbed. The transmitted waves are then vertically polarized.

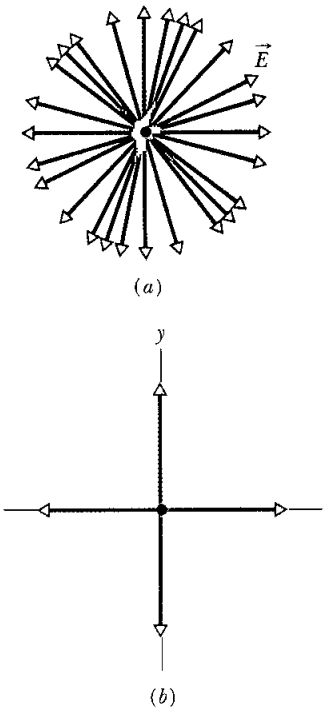


FIG. 33-10 (*a*) Unpolarized light consists of waves with randomly directed electric fields. Here the waves are all traveling along the same axis, directly out of the page, and all have the same amplitude E. (*b*) A second way of representing unpolarized light—the light is the superposition of two polarized waves whose planes of oscillation are perpendicular to each other.

Intensity of Transmitted Polarized Light

We now consider the intensity of light transmitted by a polarizing sheet. We start with unpolarized light, whose electric field oscillations we can resolve into y and z components as represented in Fig. 33-10*b*. Further, we can arrange for the y axis to be parallel to the polarizing direction of the sheet. Then only the y components of the light's electric field are passed by the sheet; the z components are absorbed. As suggested by Fig. 33-10*b*, if the original waves are randomly oriented, the sum of the y components and the sum of the z components are equal. When the z components are absorbed, half the intensity I_0 of the original light is lost. The intensity I of the emerging polarized light is then

$$I = \tfrac{1}{2} I_0. \tag{33-31}$$

Let us call this the *one-half rule*; we can use it *only* when the light reaching a polarizing sheet is unpolarized.

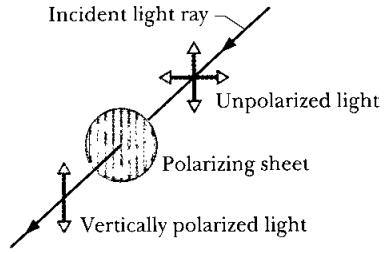


FIG. 33-11 Unpolarized light becomes polarized when it is sent through a polarizing sheet. Its direction of polarization is then parallel to the polarizing direction of the sheet, which is represented here by the vertical lines drawn in the sheet.

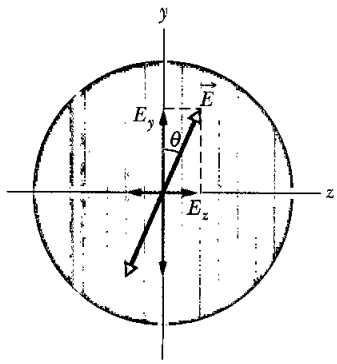


FIG. 33-12 Polarized light approaching a polarizing sheet. The electric field $\vec{E}$ of the light can be resolved into components E_y (parallel to the polarizing direction of the sheet) and E_z (perpendicular to that direction). Component E_y will be transmitted by the sheet; component E_z will be absorbed.

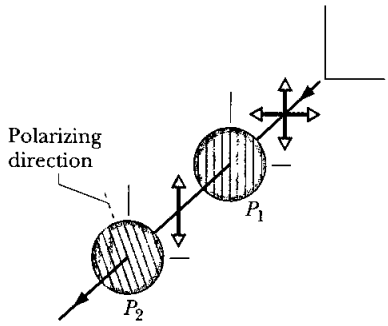


FIG. 33-13 The light transmitted by polarizing sheet P_1 is vertically polarized, as represented by the vertical double arrow. The amount of that light that is then transmitted by polarizing sheet P_2 depends on the angle between the polarization direction of that light and the polarizing direction of P_2 (indicated by the lines drawn in the sheet and by the dashed line).

Suppose now that the light reaching a polarizing sheet is already polarized. Figure 33-12 shows a polarizing sheet in the plane of the page and the electric field $\vec{E}$ of such a polarized light wave traveling toward the sheet (and thus prior to any absorption). We can resolve $\vec{E}$ into two components relative to the polarizing direction of the sheet: parallel component E_y is transmitted by the sheet, and perpendicular component E_z is absorbed. Since θ is the angle between $\vec{E}$ and the polarizing direction of the sheet, the transmitted parallel component is

$$E_y = E \cos \theta. \tag{33-32}$$

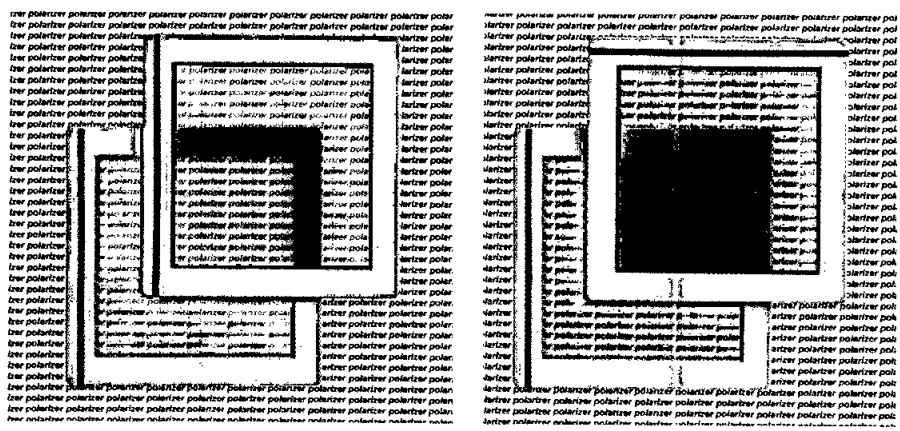
Recall that the intensity of an electromagnetic wave (such as our light wave) is proportional to the square of the electric field's magnitude (Eq. 33-28, $I = E_{rms}^2/c\mu_0$). In our present case then, the intensity I of the emerging wave is proportional to E_y^2 and the intensity I_0 of the original wave is proportional to E^2. Hence, from Eq. 33-32 we can write $I/I_0 = \cos^2 \theta$, or

$$I = I_0 \cos^2 \theta. \tag{33-33}$$

Let us call this the *cosine-squared rule;* we can use it *only* when the light reaching a polarizing sheet is already polarized. Then the transmitted intensity I is a maximum and is equal to the original intensity I_0 when the original wave is polarized parallel to the polarizing direction of the sheet (when θ in Eq. 33-33 is 0° or 180°). The transmitted intensity is zero when the original wave is polarized perpendicular to the polarizing direction of the sheet (when θ is 90°).

Figure 33-13 shows an arrangement in which initially unpolarized light is sent through two polarizing sheets P_1 and P_2. (Often, the first sheet is called the *polarizer,* and the second the *analyzer.*) Because the polarizing direction of P_1 is vertical, the light transmitted by P_1 to P_2 is polarized vertically. If the polarizing direction of P_2 is also vertical, then all the light transmitted by P_1 is transmitted by P_2. If the polarizing direction of P_2 is horizontal, none of the light transmitted by P_1 is transmitted by P_2. We reach the same conclusions by considering only the *relative* orientations of the two sheets: If their polarizing directions are parallel, all the light passed by the first sheet is passed by the second sheet (Fig. 33-14*a*). If those directions are perpendicular (the sheets are said to be *crossed*), no light is passed by the second sheet (Fig. 33-14*b*). Finally, if the two polarizing directions of Fig. 33-13 make an angle between 0° and 90°, some of the light transmitted by P_1 will be transmitted by P_2, as set by Eq. 33-33.

Light can be polarized by means other than polarizing sheets, such as by reflection (discussed in Section 33-9) and by scattering from atoms or molecules (discussed in Section 33-9). In *scattering,* light that is intercepted by an object, such as a molecule, is sent off in many, perhaps random, directions.

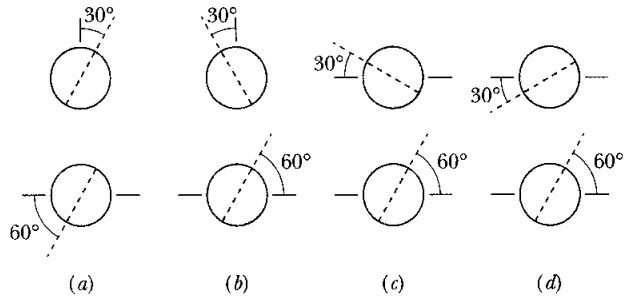


(*a*) (*b*)

FIG. 33-14 (*a*) Overlapping polarizing sheets transmit light fairly well when their polarizing directions have the same orientation, but (*b*) they block most of the light when they are crossed. *(Richard Megna/Fundamental Photographs)*

Although direct sunlight is unpolarized, light from much of the sky is at least partially polarized by such scattering. Bees use the polarization of sky light in navigating to and from their hives. Similarly, the Vikings used it to navigate across the North Sea when the daytime Sun was below the horizon (because of the high latitude of the North Sea). These early seafarers had discovered certain crystals (now called cordierite) that changed color when rotated in polarized light. By looking at the sky through such a crystal while rotating it about their line of sight, they could locate the hidden Sun and thus determine which way was south.

CHECKPOINT 3 The figure shows four pairs of polarizing sheets, seen face-on. Each pair is mounted in the path of initially unpolarized light. The polarizing direction of each sheet (indicated by the dashed line) is referenced to either a horizontal x axis or a vertical y axis. Rank the pairs according to the fraction of the initial intensity that they pass, greatest first.



Sample Problem 33-2 Build your skill

Figure 33-15*a*, drawn in perspective, shows a system of three polarizing sheets in the path of initially unpolarized light. The polarizing direction of the first sheet is parallel to the y axis, that of the second sheet is at an angle of 60° counterclockwise from the y axis, and that of the third sheet is parallel to the x axis. What fraction of the initial intensity I_0 of the light emerges from the three-sheet system, and in which direction is that emerging light polarized?

Conceptualize/Classify:

1. We work through the system sheet by sheet, from the first one encountered by the light to the last one.
2. To find the intensity transmitted by any sheet, we apply either the one-half rule or the cosine-squared rule, depending on whether the light reaching the sheet is unpolarized or already polarized.
3. The light that is transmitted by a polarizing sheet is always polarized parallel to the polarizing direction of the sheet.

Compute: First sheet: The original light wave is represented in Fig. 33-15*b*, using the head-on, double-arrow representation of Fig. 33-10*b*. Because the light is initially unpolarized, the intensity I_1 of the light transmitted by the first sheet is given by the one-half rule (Eq. 33-31):

$$I_1 = \tfrac{1}{2} I_0.$$

Because the polarizing direction of the first sheet is parallel to the y axis, the polarization of the light transmitted by it is also, as shown in the head-on view of Fig. 33-15*c*.

Second sheet: Because the light reaching the second sheet is polarized, the intensity I_2 of the light transmitted by that sheet is given by the cosine-squared rule (Eq. 33-33). The angle θ in the rule is the angle between the polarization direction of the entering light (parallel to the y axis) and the polarizing direction of the second sheet (60° counterclockwise from the y axis), and so θ is 60°. Then

$$I_2 = I_1 \cos^2 60°.$$

The polarization of this transmitted light is parallel to the polarizing direction of the sheet transmitting it—that is, 60° counterclockwise from the y axis, as shown in the head-on view of Fig. 33-15*d*.

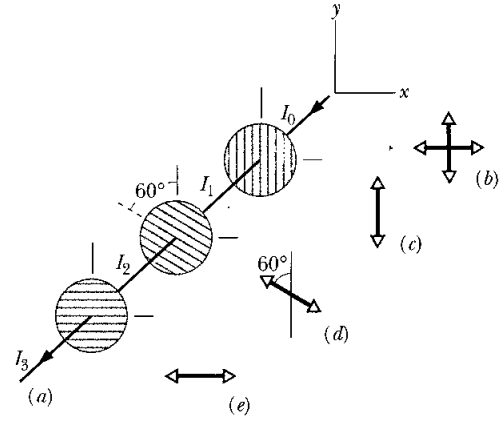


FIG. 33-15 (*a*) Initially unpolarized light of intensity I_0 is sent into a system of three polarizing sheets. The intensities I_1, I_2, and I_3 of the light transmitted by the sheets are labeled. Shown also are the polarizations, from head-on views, of (*b*) the initial light and the light transmitted by (*c*) the first sheet, (*d*) the second sheet, and (*e*) the third sheet.

Third sheet: Because the light reaching the third sheet is polarized, the intensity I_3 of the light transmitted by that sheet is given by the cosine-squared rule. The angle θ is now the angle between the polarization direction of the entering light (Fig. 33-15*d*) and the polarizing direction of the third sheet (parallel to the x axis), and so $\theta = 30°$. Thus,

$$I_3 = I_2 \cos^2 30°.$$

This final transmitted light is polarized parallel to the x axis (Fig. 33-15*e*). We find its intensity by substituting first for I_2 and then for I_1 in the equation above:

$$I_3 = I_2 \cos^2 30° = (I_1 \cos^2 60°)\cos^2 30°$$
$$= (\tfrac{1}{2} I_0)\cos^2 60° \cos^2 30° = 0.094 I_0.$$

Thus,
$$\frac{I_3}{I_0} = 0.094. \qquad \text{(Answer)}$$

Conclude: That is to say, 9.4% of the initial intensity emerges from the three-sheet system. (If we now remove the second sheet, what fraction of the initial intensity emerges from the system?)

33-8 | Reflection and Refraction

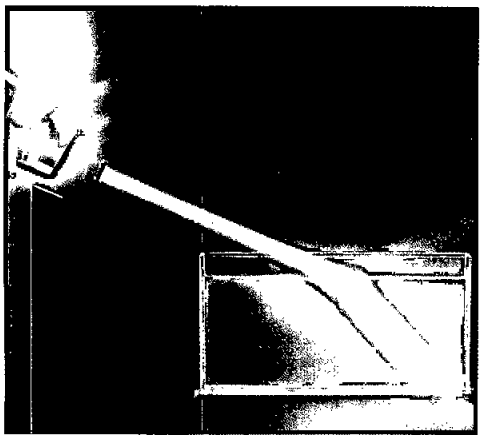
Although a light wave spreads as it moves away from its source, we can often approximate its travel as being in a straight line; we did so for the light wave in Fig. 33-5*a*. The study of the properties of light waves under that approximation is called *geometrical optics.* For the rest of this chapter and all of Chapters 34 and 35, we shall discuss the geometrical optics of visible light.

The photograph in Fig. 33-16*a* shows an example of light waves traveling in approximately straight lines. A narrow beam of light (the *incident* beam), angled downward from the left and traveling through air, encounters a *plane* (flat) water surface. Part of the light is **reflected** by the surface, forming a beam directed upward toward the right, traveling as if the original beam had bounced from the surface. The rest of the light travels through the surface and into the water, forming a beam directed downward to the right. Because light can travel through it, the water is said to be *transparent;* that is, we can see through it. (In this chapter we shall consider only transparent materials and not opaque materials, through which light cannot travel.)

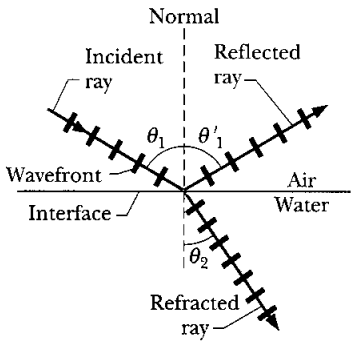
The travel of light through a surface (or *interface*) that separates two media is called **refraction**, and the light is said to be *refracted.* Unless an incident beam of light is perpendicular to the surface, refraction changes the light's direction of travel. For this reason, the beam is said to be "bent" by the refraction. Note in Fig. 33-16*a* that the bending occurs only at the surface; within the water, the light travels in a straight line. We are sure that you are aware that whenever a light ray moves from one medium to another, it changes its direction of motion. But why?

Here is a very simple and instructive illustration to show how light refracts. Fold your tablecloth and lay it on the table as shown in Fig. 33-17. Incline the table-top slightly. Then set a couple of wheels on one axle from a broken toy steam engine or some other toy rolling down it. When its path is set at right angles to the tablecloth fold there is no refraction, illustrating the law of refraction, according to which light falling perpendicularly on the boundary between two different media does not bend. However, when its path is set at an angle to the tablecloth fold, the direction of the couple of wheels changes at the boundary.

When passing from that part of the table where velocity is greater (the uncovered part) to that part where velocity is less (the covered part), the direction of the motion of wheels ("the ray") is nearer to the "normal incidence." When rolling the other way, the direction is farther away from the normal. This shows that refraction is due to the change in light velocity in the new medium. The greater this change is, the greater is the angle of refraction. The "refractive index," which shows how greatly the direction changes, is nothing but the ratio of the two velocities. If the refractive index in passing from air to water is 4/3, it means that light travels through the air roughly 1.33 times faster than through the water.



(*a*)



(*b*)

FIG. 33-16 (*a*) A photograph showing an incident beam of light reflected and refracted by a horizontal water surface. (*b*) A ray representation of (*a*). The angles of incidence (θ_1), reflection (θ'_1), and refraction (θ_2) are marked. (© *1974 FP/Fundamentals Photography*)

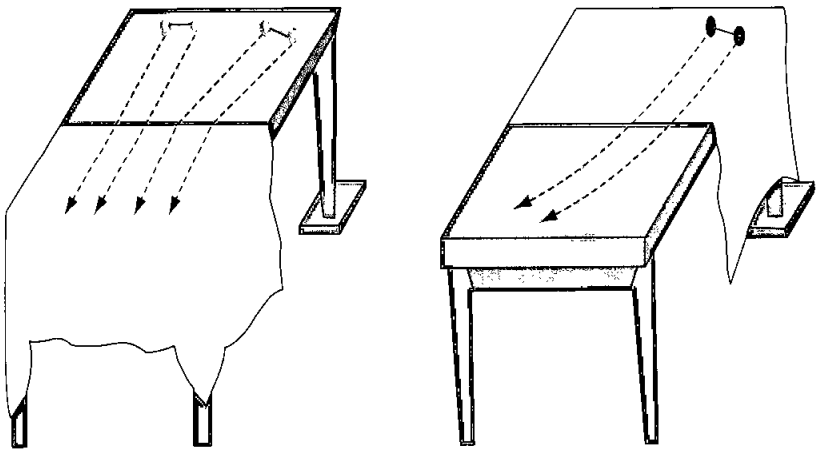


FIG. 33-17 Refraction of light explained.

In Figure 33-16*b*, the beams of light in the photograph are represented with an *incident ray,* a *reflected ray*, and a *refracted ray* (and wavefronts). Each ray is oriented with respect to a line, called the *normal*, that is perpendicular to the surface at the point of reflection and refraction. In Fig. 33-16*b*, the **angle of incidence** is θ_1, the **angle of reflection** is (θ_1'), and the **angle of refraction** is θ_2, all measured *relative to the normal.* The plane containing the incident ray and the normal is the *plane of incidence,* which is in the plane of the page in Fig. 33-16*b*.

Experiment shows that reflection and refraction are governed by two laws:

Law of reflection: A reflected ray lies in the plane of incidence and has an angle of reflection equal to the angle of incidence. In Fig. 33-16*b*, this means that

$$\theta_1' = \theta_1 \qquad \text{(reflection)}. \tag{33-34}$$

(We shall now usually drop the prime on the angle of reflection.)

Law of refraction: A refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by

$$n_2 \sin\theta_2 = n_1 \sin\theta_1 \qquad \text{(refraction)}. \tag{33-35}$$

Here each of the symbols n_1 and n_2 is a dimensionless constant, called the **index of refraction**, that is associated with a medium involved in the refraction. We derive this equation, called **Snell's law**, in Chapter 36. As we shall discuss there, the index of refraction of a medium is equal to c/v, where v is the speed of light in that medium and c is its speed in vacuum.

Table 33-1 gives the indexes of refraction of vacuum and some common substances. For vacuum, n is defined to be exactly 1; for air, n is very close to 1.0 (an approximation we shall often make). Nothing has an index of refraction below 1.

We can rearrange Eq. 33-35 as

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \tag{33-36}$$

to compare the angle of refraction θ_2 with the angle of incidence θ_1. We can then see that the relative value of θ_2 depends on the relative values of n_2 and n_1. In fact, we can have three basic results:

1. If n_2 is equal to n_1, then θ_2 is equal to θ_1 and refraction does not bend the light beam, which continues in the *undeflected direction,* as in Fig. 33-18*a*.
2. If n_2 is greater than n_1, then θ_2 is less than θ_1. In this case, refraction bends the light beam away from the undeflected direction and toward the normal, as in Fig. 33-18*b*.

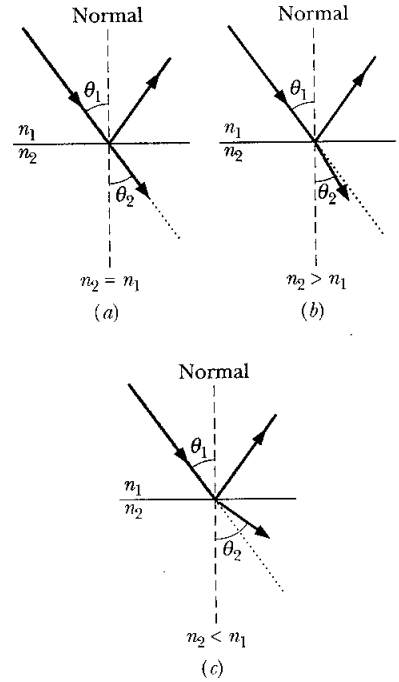


FIG. 33-18 Refraction of light traveling from a medium with an index of refraction n_1 into a medium with an index of refraction n_2. (*a*) The beam does not bend when $n_2 = n_1$; the refracted light then travels in the *undeflected direction* (the dotted line), which is the same as the direction of the incident beam. The beam bends (*b*) toward the normal when $n_2 > n_1$ and (*c*) away from the normal when $n_2 < n_1$.

TABLE 33-1

Some Indexes of Refraction[a]

Medium	Index	Medium	Index
Vacuum	Exactly 1	Typical crown glass	1.52
Air (STP)[b]	1.00029	Sodium chloride	1.54
Water (20°C)	1.33	Polystyrene	1.55
Acetone	1.36	Carbon disulfide	1.63
Ethyl alcohol	1.36	Heavy flint glass	1.65
Sugar solution (30%)	1.38	Sapphire	1.77
Fused quartz	1.46	Heaviest flint glass	1.89
Sugar solution (80%)	1.49	Diamond	2.42

[a]For a wavelength of 589 nm (yellow sodium light).

[b]STP means "standard temperature (0°C) and pressure (1 atm)."

3. If n_2 is less than n_1, then θ_2 is greater than θ_1. In this case, refraction bends the light beam away from the undeflected direction and away from the normal, as in Fig. 33-18*c*.

Refraction *cannot* bend a beam so much that the refracted ray is on the same side of the normal as the incident ray.

Sample Problem 33-3

(a) In Fig. 33-19*a*, a beam of monochromatic light reflects and refracts at point A on the interface between material 1 with index of refraction $n_1 = 1.33$ and material 2 with index of refraction $n_2 = 1.77$. The incident beam makes an angle of 50° with the interface. What is the angle of reflection at point A? What is the angle of refraction there?

Conceptualize/Classify: (1) The angle of reflection is equal to the angle of incidence, and both angles are measured relative to the normal to the surface at the point of reflection. (2) When light reaches the interface between two materials with different indexes of refraction (call them n_1 and n_2), part of the light can be refracted by the interface according to Snell's law, Eq. 33-35:

$$n_2 \sin\theta_2 = n_1 \sin\theta_1, \qquad (33\text{-}37)$$

where both angles are measured relative to the normal at the point of refraction.

Compute: In Fig. 33-19*a*, the normal at point A is drawn as a dashed line through the point. Note that the angle of incidence θ_1 is not the given 50° but rather is $90° - 50° = 40°$. Thus, the angle of reflection is

$$\theta_1' = \theta_1 = 40°. \qquad \text{(Answer)}$$

The light that passes from material 1 into material 2 undergoes refraction at point A on the interface between the two materials. Again we measure angles between light rays and a normal, here at the point of refraction. Thus, in Fig. 33-19*a*, the angle of refraction is the angle marked θ_2. Solving Eq. 33-37 for θ_2 gives us

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right) = \sin^{-1}\left(\frac{1.33}{1.77}\sin 40°\right)$$
$$= 28.88° \approx 29°. \qquad \text{(Answer)}$$

Conclude: This result means that the beam swings toward the normal (it was at 40° to the normal and is

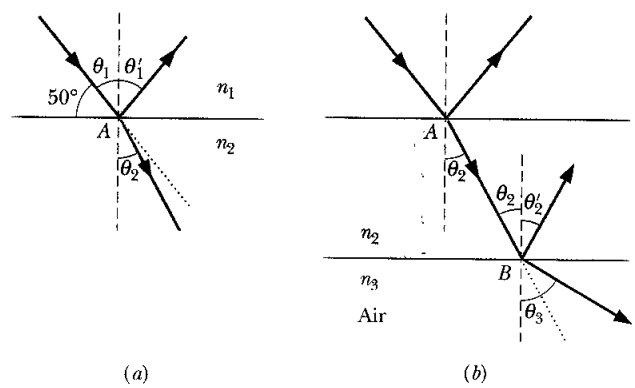


FIG. 33-19 (*a*) Light reflects and refracts at point A on the interface between materials 1 and 2. (*b*) The light that passes through material 2 reflects and refracts at point B on the interface between materials 2 and 3 (air). Each dashed line is a normal. Each dotted line gives the incident direction of travel.

now at 29°). The reason is that when the light travels across the interface, it moves into a material with a greater index of refraction.

(b) The light that enters material 2 at point A then reaches point B on the interface between material 2 and material 3, which is air, as shown in Fig. 33-22*b*. The interface through B is parallel to that through A. At B, some of the light reflects and the rest enters the air. What is the angle of reflection? What is the angle of refraction into the air?

Compute: We first need to relate one of the angles at point B with a known angle at point A. Because the interface through point B is parallel to that through point A, the incident angle at B must be equal to the angle of refraction θ_2, as shown in Fig. 33-19*b*. Then for reflection, we again use the law of reflection. Thus, the angle of reflection at B is

$$\theta_2' = \theta_2 = 28.88° \approx 29°. \qquad \text{(Answer)}$$

Next, the light that passes from material 2 into the air undergoes refraction at point B, with refraction angle θ_3. Thus, we again apply Snell's law of refraction, but this time we write Eq. 33-35 as

$$n_3 \sin\theta_3 = n_2 \sin\theta_2.$$

Solving for θ_3 then leads to

$$\theta_3 = \sin^{-1}\left(\frac{n_2}{n_3}\sin\theta_2\right) = \sin^{-1}\left(\frac{1.77}{1.00}\sin 28.88°\right)$$
$$= 58.75° \approx 59°. \qquad \text{(Answer)}$$

Conclude: This result means that the beam swings away from the normal (it was at 29° to the normal and is now at 59°). The reason is that when the light travels across the interface, it moves into a material (air) with a lower index of refraction.

Sun dogs occur when sunlight passes through atmospheric ice crystals (Fig. 33-20*a*) that have their hexagonal cross sections horizontal. Such a crystal can turn a ray (Fig. 33-20*b*) from its original direction of travel by an angle that depends on the crystal's orientation. However, there is a minimum turning angle θ_{dog} at which rays bunch up to form an especially bright spot in the sky—a sun dog. The situation is similar to how white wine in a wine glass can bunch up candlelight to form bright lines on a tablecloth. A sunray undergoing such minimum turning follows a path through the crystal that is parallel to one of the hexagonal sides, and its entrance and exit paths are symmetric. The minimum turning angle θ_{dog} is the angle to the left or right of the Sun at which you can see a sun dog.

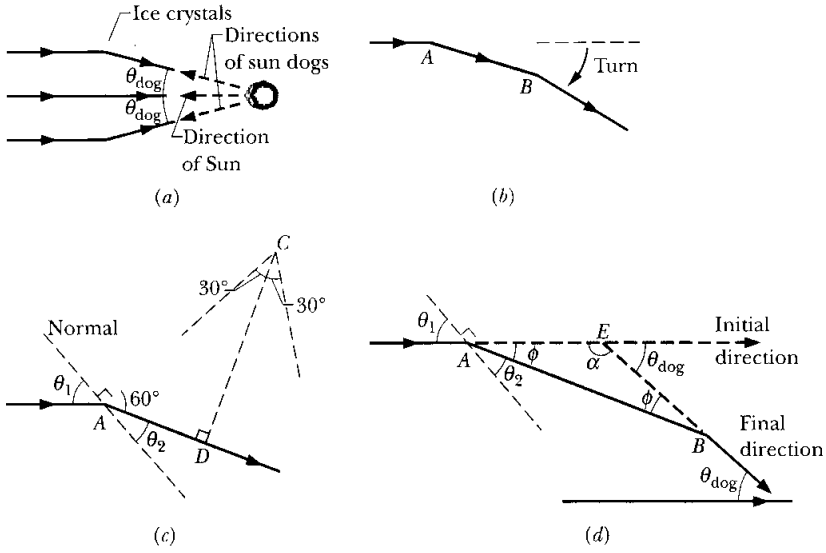


FIG. 33-20 (*a*) Overhead view of sunrays redirected by atmospheric ice crystals. (*b*) Overhead view of hexagonal cross section of a crystal. Path of ray (*c*) into and (*d*) out of the cross section.

33-9 | Polarization by Reflection

You can vary the glare you see in sunlight that has been reflected from, say, water by looking through a polarizing sheet (such as a polarizing sunglass lens) and then rotating the sheet's polarizing axis around your line of sight. You can do so because any light that is reflected from a surface is either fully or partially polarized by the reflection.

Figure 33-21 shows a ray of unpolarized light incident on a glass surface. Let us resolve the electric field vectors of the light into two components. The *perpendicular components* are perpendicular to the plane of incidence and thus also to the page in Fig. 33-21; these components are represented with dots (as if we see the tips of the vectors). The *parallel components* are parallel to the plane of incidence and the page; they are represented with double-headed arrows. Because the light is unpolarized, these two components are of equal magnitude.

In general, the reflected light also has both components but with unequal magnitudes. This means that the reflected light is partially polarized—the electric fields oscillating along one direction have greater amplitudes than those oscillating along other directions. However, when the light is incident at a particular incident angle, called the *Brewster angle* θ_B, the reflected light has only perpendicular components, as shown in Fig. 33-21. The reflected light is then fully polarized perpendicular to the plane of incidence. The parallel components of the incident light do not disappear but (along with perpendicular components) refract into the glass.

Glass, water, and the other dielectric materials discussed in Section 26-6 can partially and fully polarize light by reflection. When you intercept sunlight reflected from such a surface, you see a bright spot (the glare) on the surface where the reflection takes place. If the surface is horizontal as in Fig. 33-21, the reflected light is partially or fully polarized horizontally. To eliminate such glare from horizontal surfaces, the lenses in polarizing sunglasses are mounted with their polarizing direction vertical.

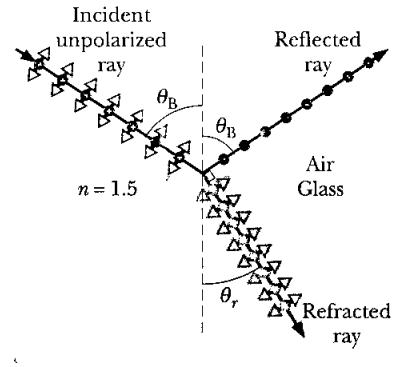


FIG. 33-21 A ray of unpolarized light in air is incident on a glass surface at the Brewster angle θ_B. The electric fields along that ray have been resolved into components perpendicular to the page (the plane of incidence, reflection, and refraction) and components parallel to the page. The reflected light consists only of components perpendicular to the page and is thus polarized in that direction. The refracted light consists of the original components parallel to the page and weaker components perpendicular to the page; this light is partially polarized.

Brewster's Law

For light incident at the Brewster angle θ_B, we find experimentally that the reflected and refracted rays are perpendicular to each other. Because the reflected ray is reflected at the angle θ_B in Fig. 33-21 and the refracted ray is at an angle θ_r, we have

$$\theta_B + \theta_r = 90°. \tag{33-38}$$

These two angles can also be related with Eq. 33-35. Arbitrarily assigning subscript 1 in Eq. 33-35 to the material through which the incident and reflected rays travel, we have, from that equation,

$$n_1 \sin \theta_B = n_2 \sin \theta_r.$$

Combining these equations leads to

$$n_1 \sin \theta_B = n_2 \sin(90° - \theta_B) = n_2 \cos \theta_B,$$

which gives us

$$\theta_B = \tan^{-1} \frac{n_2}{n_1} \qquad \text{(Brewster angle)}. \tag{33-39}$$

(Note carefully that the subscripts in Eq. 33-39 are *not* arbitrary because of our decision as to their meanings.) If the incident and reflected rays travel *in air*, we can approximate n_1 as unity and let n represent n_2 in order to write Eq. 33-39 as

$$\theta_B = \tan^{-1} n \qquad \text{(Brewster's law)}. \tag{33-40}$$

This simplified version of Eq. 33-39 is known as **Brewster's law**. Like θ_B, it is named after Sir David Brewster, who found both experimentally in 1812. It is not difficult to understand the physical reason why the light vibrating in the plane of incidence is not reflected at Brewster's angle. The incident light sets the electrons in the atoms of the material into oscillation, and it is the re-radiation from these that generates the reflected beam. When the latter is observed at 90° to the refracted beam, only the vibrations that are perpendicular to the plane of incidence can contribute. Those in the plane of incidence have no component transverse to the 90° direction and hence cannot radiate in that direction.

Polarization by Scattering

If a polarizing plate, such as Polaroid, is used to test the blue sky, the light is *partially plane-polarized*. A little exploration will show that maximum polarization occurs at an angle of 90° with the direction of the incoming sunlight. At dusk on a clear day, when the sun has just disappeared over the horizon, one can locate the direction of zero polarization and from it determine the sun's position.

Consider the scattering of light from a single air molecule P, as shown in Fig. 33-22. Suppose ordinary unpolarized light is incident from the left. We assume that it is composed of two plane-polarized components: one in xy plane and another in xz plane (Fig. 33-22). If the incident component is vibrating in the xy plane and is absorbed, it sets the particle vibrating in the y direction. In giving up this energy, the same wave can be emitted in any direction except along the y axis. This is because the light waves are transverse, not longitudinal.

Assume that the incident light component is vibrating in the xz plane. The particle at P will be set vibrating along the z axis. Reemission is now allowed in all directions except along the z axis. It can be seen from Fig. 33-22; therefore, an observer at O looking at the blue sky in a direction making 90° with the sun's rays will find the blue light *plane-polarized* with its direction of vibration parallel to the z axis. No particle at P can be set vibrating along the x axis, since this would violate the principle that light has no longitudinal component.

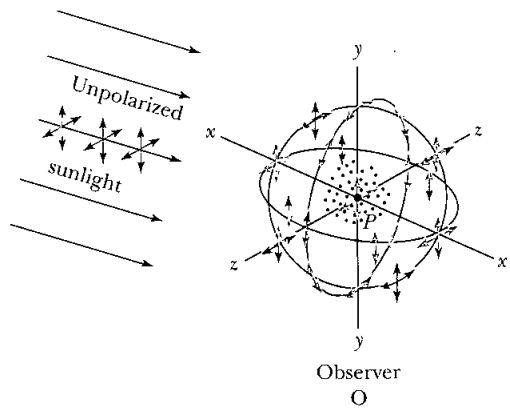


FIG. 33-22 The polarization of light by scattering from the particles.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: The velocity of electromagnetic waves in space is constant.

Statement 2: The law of conservation of energy holds true in case of electromagnetic waves also.

2. Statement 1: A radio wave and an x ray are essentially of the same nature.

Statement 2: A radio wave and x ray have same frequency.

3. Statement 1: If a beam of polarized light basses through a Polaroid with to plane to polarization at an angle θ to the axis of polarization of the sheet, the intensity of transmitted light is

$$I = I_0 \cos^2\theta.$$

Statement 2: In the situation described above, electric field amplitude is given by $E = E_0 \cos\theta$.

4. Statement 1: Polarizing filter passes almost none of the unpolarized light.

Statement 2: Average value of $\cos\theta$ is 0.

MULTIPLE CHOICE QUESTIONS

sec. 33-2 Maxwell's Rainbow

1. Select the correct statement

(a) Ultraviolet light has a longer wavelength than infrared

(b) Blue light has a higher frequency than x rays

(c) Radio waves have higher frequency than gamma rays

(d) Gamma rays have higher frequency than infrared waves

2. The order of increasing wavelength for blue (b), green (g), red (r), and yellow (y) light is

(a) r, y, g, b
(b) r, g, y, b
(c) g, y, b, r
(d) b, g, y, r

3. Of the following, human eyes are most sensitive to

(a) Red light
(b) Violet light
(c) Blue light
(d) Green light

4. If the wavelength of an electromagnetic wave is about the diameter of an apple, what type of radiation is it?

(a) X ray
(b) UV
(c) Infrared
(d) Microwave

sec. 33-3 The Traveling Electromagnetic Wave, Qualitatively

5. Figure 33-23 shows the electric and magnetic fields of an electromagnetic wave at a certain instant. The wave is traveling

(a) Into the page
(b) Out of it
(c) In the east direction
(d) In the west direction

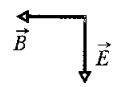


FIG. 33-23 Multiple choice question 5.

6. Which of the following types of electromagnetic radiation travels at the greatest speed in vacuum?

(a) Radio waves
(b) Visible light
(c) X rays
(d) All of these travel at the same speed

7. An electromagnetic wave is generated by

(a) Any moving charge
(b) Any accelerating charge
(c) Only a charge with changing acceleration
(d) Only a charge moving in a circle

8. Which of these statements correctly describes the orientation of the electric field ($\vec{E}$), the magnetic field ($\vec{B}$), and the velocity of propagation ($\vec{v}$) of an electromagnetic wave?

(a) $\vec{E}$ is perpendicular to $\vec{B}$; $\vec{v}$ may have any orientation relative to $\vec{E}$
(b) $\vec{E}$ is perpendicular to $\vec{B}$; $\vec{v}$ may have any orientation perpendicular to $\vec{E}$
(c) $\vec{E}$ is perpendicular to $\vec{B}$; $\vec{B}$ is parallel to $\vec{v}$
(d) Each of the three vectors is perpendicular to the other two

9. An electromagnetic wave going through vacuum is described by $E = E_0 \sin(kx - \omega t)$; $B = B_0 \sin(kx - \omega t)$. Then,

(a) $E_0 k = B_0 \omega$
(b) $E_0 B_0 = \omega k$
(c) $E_0 \omega = B_0 k$
(d) None of these

10. A free electron is placed in the path of a plane electromagnetic wave. The electron will start moving

(a) Along the electric field
(b) Along the magnetic field
(c) Along the direction of propagation of the wave
(d) In a plane containing the magnetic field and the direction of propagation

11. A beam of light traveling along x axis is described by the electric field as

$$E_y = (600 \text{ V/m}) \sin \omega(t - x/c).$$

Then maximum magnetic force on a charge $q = 2e$, moving along y axis with a speed of 3.0×10^7 m/s is ($e = 1.6 \times 10^{-19}$ C):

(a) 19.2×10^{-17} N
(b) 1.92×10^{-17} N
(c) 0.192 N
(d) None of these

12. A plane electromagnetic wave travels in free space along x axis. At a particular point in space, the electric field along y axis is 9.3 V/m. The magnetic induction is

(a) 3.1×10^{-8} T
(b) 3×10^{-5} T
(c) 3×10^{-6} T
(d) 9.3×10^{-6} T

13. A plane electromagnetic wave traveling along the x direction has a wavelength of 3 mm. The variation in the electric field occurs in the y direction with amplitude 66 V/m. The equation for the electric and magnetic fields as a function of x and t, respectively, is

(a) $E_y = 33\cos\pi \times 10^{11}\left(t - \frac{x}{c}\right)$;
$B_z = 1.1 \times 10^{-7} \cos\pi \times 10^{11}\left(t - \frac{x}{c}\right)$

(b) $E_y = 11\cos 2\pi \times 10^{11}\left(t - \frac{x}{c}\right)$;
$B_z = 11 \times 10^{-7} \cos 2\pi \times 10^{11}\left(t - \frac{x}{c}\right)$

(c) $E_x = 33\cos\pi \times 10^{11}\left(t - \frac{x}{c}\right)$;
$B_x = 11 \times 10^{-7} \cos\pi \times 10^{11}\left(t - \frac{x}{c}\right)$

(d) $E_y = 66\cos 2\pi \times 10^{11}\left(t - \frac{x}{c}\right)$;
$B_z = 2.2 \times 10^{-7} \cos 2\pi \times 10^{11}\left(t - \frac{x}{c}\right)$

sec. 33-4 The Traveling Electromagnetic Wave, Quantitatively

14. A plane electromagnetic wave is traveling in the positive x direction. At the instant shown in Fig. 33-24 the electric field at the extremely narrow dashed rectangle is in the negative z direction and its magnitude is decreasing. Which diagram correctly shows the direction and relative magnitudes of the magnetic field at the edges of the rectangle?

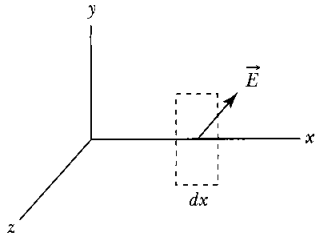
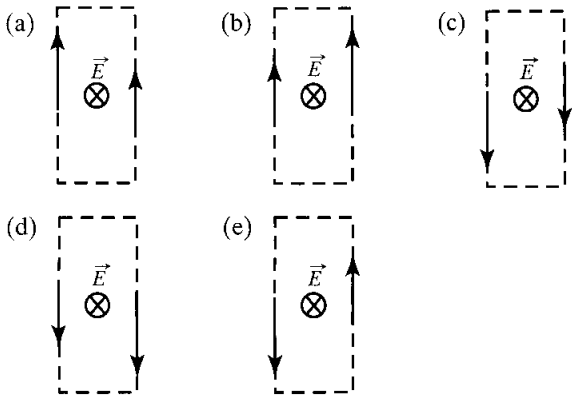


FIG. 33-24 Multiple choice question 14.



15. An electromagnetic wave is travelling in the positive x direction with its electric filed along the z axis and its magnetic field along the y axis. The fields are related by

(a) $\frac{\partial E}{\partial x} = \frac{\mu_0 \varepsilon_0 \partial B}{\partial x}$

(b) $\frac{\partial E}{\partial x} = \frac{\mu_0 \varepsilon_0 \partial B}{\partial t}$

(c) $\frac{\partial B}{\partial x} = \frac{\mu_0 \varepsilon_0 \partial E}{\partial x}$

(d) $\frac{\partial B}{\partial x} = \frac{\mu_0 \varepsilon_0 \partial E}{\partial t}$

(e) $\frac{\partial B}{\partial x} = -\frac{\mu_0 \varepsilon_0 \partial E}{\partial t}$

16. An electromagnetic wave of frequency $\nu = 3.0$ MHz passes from vacuum into a dielectric medium with permittivity $\varepsilon = 4.0$. Then,

(a) Wavelength is doubled and frequency unchanged

(b) Wavelength is doubled and frequency becomes half

(c) Wavelength is halved and frequency remains unchanged

(d) Wavelength and frequency both remains unchanged

17. A plane electromagnetic wave, $E_z = 100 \cos(6 \times 10^8 t + 4x)$ V/m, propagating in a medium of dielectric constant is

(a) 1.5

(b) 2.0

(c) 2.4

(d) 4.0

sec. 33-5 Energy Transport and the Poynting Vector

18. The dimensions of $\vec{S} = (1/\mu_0)\vec{E} \times \vec{B}$ are

(a) J/m^2

(b) J/s

(c) W/s

(d) J/m^3

19. The time-averaged energy in a sinusoidal electromagnetic wave is

(a) Overwhelmingly electrical

(b) Slightly more electrical than magnetic

(c) Equally divided between the electric and magnetic fields

(d) Overwhelmingly magnetic

20. An electromagnetic wave is transposing energy in the negative y direction. At one point and one instant the magnetic field is in the positive x direction. The electric field at that point and instant is

(a) Positive y direction

(b) Negative y direction

(c) Positive z direction

(d) Negative z direction

(e) Negative x direction

21. The light intensity 10 m from a point source is 1000 W/m^2. The intensity 100 m from the same source is

(a) 1000 W/m^2

(b) 100 W/m^2

(c) 10 W/m^2

(d) 1 W/m^2

(e) 0.1 W/m^2

22. The amplitude of electric field in a parallel light beam of intensity 4 W/m^2 is

(a) 35.5 N/C

(b) 45.5 N/C

(c) 49.5 N/C

(d) 55.5 N/C

23. Consider the following two statements regarding a linearly polarized, plane electromagnetic wave:

(A) The electric field and magnetic field have equal average values.

(B) The electric energy and the magnetic energy have equal average values.

(a) Both A and B are true

(b) A is false but B is true

(c) B is false but A is true

(d) Both A and B are false

24. The rms value of the electric field of the light coming from the sun is 720 N/C. The average total energy density of the electromagnetic wave is

(a) 4.58×10^{-6} J/m^3

(b) 6.37×10^{-9} J/m^3

(c) 81.35×10^{-12} J/m^3

(d) 3.3×10^{-3} J/m^3

sec. 33-6 Radiation Pressure

25. Evidence that electromagnetic waves carry momentum when

(a) The tail of a comet points away from the Sun

(b) Electron flow through a wire generates heat

(c) A charged particle in a magnetic field moves in a circular orbit

(d) Heat can be generated by rubbing two sticks together

26. Light of uniform intensity shines perpendicularly on a totally absorbing surface, fully illuminating the surface. If the area of the surface is decreased

(a) The radiation pressure increases and the radiation force increases

(b) The radiation pressure increases and the radiation force decreases

(c) The radiation pressure stays the same and the radiation force increases

(d) The radiation pressure stays the same and the radiation force decreases

sec. 33-7 Polarization

27. A linearly polarized transverse wave is propagating in z direction through a fixed point P in space. At time t_0, the x component E_x and the y component E_y of the displacement at P are 3 and 4 units, respectively. At a later time t_1, if E_x at P is 2 units, the value of E_y will be

(a) 5 units

(b) 8/3 units

(c) 3/8 units

(d) 1/3 units

28. A vertical automobile radio antenna is sensitive to electric fields that are polarized

(a) Horizontally

(b) In circles around the antenna

(c) Vertically

(d) Normal to the antenna in the forward direction

29. Which of the following are the properties of a plane polarized light?

(a) Electric field vectors lie in one plane

(b) Magnetic field vectors lie in one plane

(c) Both and vectors lie in the same plane

(d) Vector is in the direction of propagation

30. Polarized light

(a) Is a different form of light

(b) Can show interference pattern

(c) Travels in any medium with a velocity slightly more than that for unpolarized light

(d) Has direction of vibration restricted in some way

31. For linearly polarized light the plane of polarization is

(a) Perpendicular to both the direction of polarization and the direction of propagation

(b) Perpendicular to the direction of polarization and parallel to the direction of propagation

(c) Parallel to the direction of polarization and perpendicular to the direction of propagation

(d) Parallel to both the direction of polarization and the direction of propagation

32. An unpolarized beam of light has intensity I_0. It is incident on to ideal polarizing sheets. The angle between the axes of polarization of these sheets is θ. Find θ if the emerging light has intensity $I_0/4$

(a) $\sin^{-1}(1/2)$

(b) $\sin^{-1}(1/\sqrt{5})$

(c) $\cos^{-1}(1/2)$

(d) $\cos^{-1}(1/\sqrt{2})$

(e) $\tan^{-1}(1/4)$

33. Three polarizing sheets are placed in a stack with the polarizing directions of the first and third perpendicular to each other. What angle should the polarizing direction of the middle sheet make with the polarizing direction of the first sheet to obtain maximum transmitted intensity when unpolarized light is incident on the stack?

(a) 0

(b) 30°

(c) 45°

(d) 60°

(e) 90°

34. A light bulb shines light along the x axis and through two parallel ideal polarizing filters, one with a fixed polarizing axis and the other with an axis that rotates about the x axis and in the yz plane. Looking toward the light bulb through the combined filtering system, you see

(a) A bulb almost disappearing twice per revolution, and reaching a maximum intensity twice per revolution. The maximum brightness is not as bright as looking at an unfiltered bulb

(b) As in (a) but once per revolution

(c) As in (a) but with a maximum brightness as bright as an unfiltered bulb

(d) As in (a) but with a bulb that disappears completely

35. Two polaroids are kept crossed to each other. Now one of them is rotated through an angle of 45°. The percentage of incident light now transmitted through the system is

(a) 15%

(b) 25%

(c) 50%

(d) 60%

36. A beam of light is linearly polarized. You wish to rotate its direction of polarization by 90° using one or more ideal polarizing sheets. To get maximum transmitted intensity, how many sheets should you use?

(a) 1

(b) 2

(c) 3

(d) As many as possible

(e) There is no way to rotate the direction of polarization 90° using polarizing sheets

37. Electromagnetic waves transverse in nature are evident by
(a) Polarization
(b) Interference
(c) Reflection
(d) Diffraction

sec. 33-9 Polarization by Reflection

38. Linear polarized light is incident at Brewster angle on the surface of a medium. If the incident beam is polarized parallel to the plane of incidence, then the parallel component of light is
(a) Completely refracted
(b) Partly refracted
(c) Completely reflected
(d) Completely absorbed

39. When light is incident from air to glass at an angle 57°, the reflected beam is completely polarized. If the same beam is incident from water to glass, the angle of incidence at which reflected beam is completely polarized will be
(a) $\theta = 57°$
(b) $\theta > 57°$
(c) $\theta < 57°$
(d) Cannot be determined

40. Consider the following two statements regarding a linearly polarized, plane electromagnetic wave:
(A) The electric field and the magnetic field have equal average values.
(B) The electric energy and the magnetic energy have equal average values.
(a) Both A and B are true
(b) A is false but B is true
(c) B is false but A is true
(d) Both A and B are false

PROBLEMS

sec. 33-2 Maxwell's Rainbow

1. From Fig. 33-2, approximate the (a) smaller and (b) larger wavelength at which the eye of a standard observer has half the eye's maximum sensitivity. What are the (c) wavelength, (d) frequency, and (e) period of the light at which the eye is the most sensitive?

2. A certain helium—neon laser emits red light in a narrow band of wavelengths centered at 632.8 nm and with a "wavelength width" (such as on the scale of Fig. 33-1) of 0.0100 nm. What is the corresponding "frequency width" for the emission?

sec. 33-3 The Traveling Electromagnetic Wave, Qualitatively

3. What inductance must be connected to a 17 pF capacitor in an oscillator capable of generating 550 nm (i.e., visible) electromagnetic waves? Comment on your answer.

4. What is the wavelength of the electromagnetic wave emitted by the oscillator—antenna system of Fig. 33-3 if $L = 0.253\ \mu H$ and $C = 25.0$ pF?

sec. 33-5 Energy Transport and the Poynting Vector

5. Some neodymium—glass lasers can provide 100 TW of power in 1.0 ns pulses at a wavelength of 0.26 μm. How much energy is contained in a single pulse?

6. A plane electromagnetic wave has a maximum electric field magnitude of 3.20×10^{-4} V/m. Find the magnetic field amplitude.

7. A plane electromagnetic wave traveling in the positive direction of an x axis in vacuum has components $E_x = E_y = 0$ and $E_z = (2.0\ \text{V/m})\ \cos[(\pi \times 10^{15}\ \text{s}^{-1})(t - x/c)]$. (a) What is the amplitude of the magnetic field component? (b) Parallel to which axis does the magnetic field oscillate? (c) When the electric field component is in the positive direction of the z axis at a certain point P, what is the direction of the magnetic field component there?

8. In a plane radio wave the maximum value of the electric field component is 5.00 V/m. Calculate (a) the maximum value of the magnetic field component and (b) the wave intensity.

9. What is the intensity of a traveling plane electromagnetic wave if B_m is 1.0×10^{-4} T?

10. The maximum electric field 10 m from an isotropic point source of light is 2.0 V/m. What are (a) the maximum value of the magnetic field and (b) the average intensity of the light there? (c) What is the power of the source?

11. An electromagnetic wave with frequency 4.00×10^{14} Hz travels through vacuum in the positive direction of an x axis. The wave has its electric field directed parallel to the y axis, with amplitude E_m. At time $t = 0$, the electric field at point P on the x axis has a value of $+E_m/4$ and is decreasing with time. What is the distance along the x axis from point P to the first point with $E = 0$ if we search in (a) the negative direction and (b) the positive direction of the x axis?

12. An isotropic point source emits light at wavelength 500 nm, at the rate of 200 W. A light detector is positioned 400 m from the source. What is the maximum rate $\partial B/\partial t$ at which the magnetic component of the light changes with time at the detector's location?

sec. 33-6 Radiation Pressure

13. What is the radiation pressure 1.5 m away from a 500 W lightbulb? Assume that the surface on which the pressure is exerted faces the bulb and is perfectly absorbing and that the bulb radiates uniformly in all directions.

14. A black, totally absorbing piece of cardboard of area $A = 2.0\ \text{cm}^2$ intercepts light with an intensity of $10\ \text{W/m}^2$ from a camera strobe light. What radiation pressure is produced on the cardboard by the light?

sec. 33-7 Polarization

15. In Fig. 33-25, a beam of unpolarized light, with intensity $43\ \text{W/m}^2$, is sent into a system of two polarizing sheets with polarizing directions at angles $\theta_1 = 70°$ and $\theta_2 = 90°$ to the y axis. What is the intensity of the light transmitted by the system?

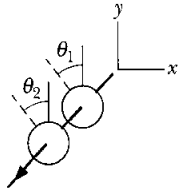


FIG. 33-25 Problem 15.

16. In Fig. 33-26, initially unpolarized light is sent into a system of three polarizing sheets whose polarizing directions make angles of $\theta_1 = 40°$, $\theta_2 = 20°$, and $\theta_3 = 40°$ with the direction of the y axis. What percentage of the light's initial intensity is transmitted by the system? (*Hint:* Be careful with the angles.)

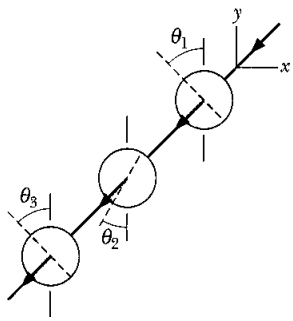


FIG. 33-26 Problem 16.

17. Unpolarized light of intensity 10 mW/m^2 is sent into a polarizing sheet as in Fig. 33-11. What is the amplitude of the electric field component of the transmitted light?

18. At a beach the light is generally partially polarized due to reflections off sand and water. At a particular beach on a particular day near sundown, the horizontal component of the electric field vector is 2.3 times the vertical component. A standing sunbather puts on polarizing sunglasses; the glasses eliminate the horizontal field component. (a) What fraction of the light intensity received before the glasses were put on now reaches the sunbather's eyes? (b) The sunbather, still wearing the glasses, lies on his side. What fraction of the light intensity received before the glasses were put on now reaches his eyes?

19. A beam of polarized light is sent into a system of two polarizing sheets. Relative to the polarization direction of that incident light, the polarizing directions of the sheets are at angles θ for the first sheet and 90° for the second sheet. If 0.10 of the incident intensity is transmitted by the two sheets, what is θ?

20. In Fig. 33-27, unpolarized light is sent into a system of three polarizing sheets, which transmits 0.0500 of the initial light intensity. The polarizing directions of the first and third sheets are at angles $\theta_1 = 0°$ and $\theta_3 = 90°$. What are the (a) smaller and (b) larger possible values of angle θ_2 (<90°) for the polarizing direction of sheet 2?

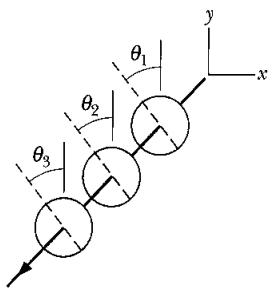


FIG. 33-27 Problem 20.

21. We want to rotate the direction of polarization of a beam of polarized light through 90° by sending the beam through one or more polarizing sheets. (a) What is the minimum number of sheets required? (b) What is the minimum number of sheets required if the transmitted intensity is to be more than 60% of the original intensity?

sec. 33-8 Reflection and Refraction

22. Light in vacuum is incident on the surface of a glass slab. In the vacuum the beam makes an angle of 32.0° with the normal to the surface, while in the glass it makes an angle of 21.0° with the normal. What is the index of refraction of the glass?

23. In Fig. 33-28*a*, a light ray in water is incident at angle θ_1 on a boundary with an underlying material, into which some of the light refracts. There are two choices of underlying material. For each, the angle of refraction θ_2 versus the incident angle θ_1 is given in Fig. 33-28*b*. The vertical axis scale is set by $\theta_{2s} = 90°$. Without calculation, determine whether the index of refraction of (a) material 1 and (b) material 2 is greater or less than the index of water ($n = 1.33$). What is the index of refraction of (c) material 1 and (d) material 2?

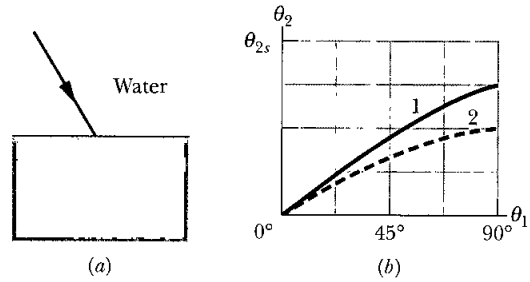


FIG. 33-28 Problem 23.

24. In Fig. 33-29, a 2.00-m-long vertical pole extends from the bottom of a swimming pool to a point 50.0 cm above the water. Sunlight is incident at angle $\theta = 55.0°$. What is the length of the shadow of the pole on the level bottomof the pool?

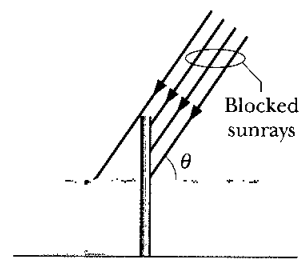


FIG. 33-29 Problem 24.

sec. 33-9 Polarization by Reflection

25. (a) At what angle of incidence will the light reflected from water be completely polarized? (b) Does this angle depend on the wavelength of the light?

26. In Fig. 33-30, a light ray in air is incident on a flat layer of material 2 that has an index of refraction $n_2 = 1.5$. Beneath material 2 is material 3 with an index of refraction n_3. The ray is incident on the air—material 2 interface at the Brewster angle for that interface. The ray of light refracted into material 3 happens to be incident on the material 2—material 3 interface at the Brewster angle for that interface. What is the value of n_3?

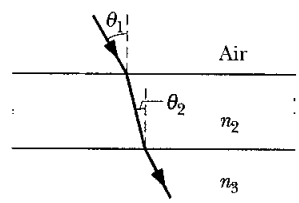


FIG. 33-30 Problem 26.

ADDITIONAL PROBLEMS

27. (a) How long does it take a radio signal to travel 150 km from a transmitter to a receiving antenna? (b) We see a full Moon by reflected sunlight. How much earlier did the light that enters our eye leave the Sun? The Earth–Moon and Earth–Sun distances are 3.8×10^5 km and 1.5×10^8 km, respectively. (c) What is the round-trip travel time for light between Earth and a spaceship orbiting Saturn, 1.3×10^9 km distant? (d) The Crab nebula, which is about 6500 light-years (ly) distant, is

thought to be the result of a supernova explosion recorded by Chinese astronomers in A.D. 1054. In approximately what year did the explosion actually occur?

28. The electric component of a beam of polarized light is

$$E_y = (5.00\ \text{V/m})\ \sin[(1.00 \times 10^6\ \text{m}^{-1})\ z + \omega t].$$

(a) Write an expression for the magnetic field component of the wave, including a value for ω. What are the (b) wavelength, (c) period, and (d) intensity of this light? (e) Parallel to which axis does the magnetic field oscillate? (f) In which region of the electromagnetic spectrum is this wave?

29. The magnetic component of an electromagnetic wave in vacuum has an amplitude of 85.8 nT and an angular wave number of 4.00 m^{-1}. What are (a) the frequency of the wave, (b) the rms value of the electric component, and (c) the intensity of the light?

30. A helium–neon laser, radiating at 632.8 nm, has a power output of 3.0 mW. The beam diverges (spreads) at angle θ = 0.17 mrad (Fig. 33-31). (a) What is the intensity of the beam 40 m from the laser? (b) What is the power of a point source providing that intensity at that distance?

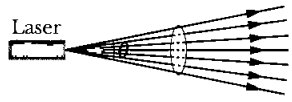


FIG. 33-31 Problem 30.

31. During a test, a NATO surveillance radar system, operating at 12 GHz at 180 kW of power, attempts to detect an incoming stealth aircraft at 90 km. Assume that the radar beam is emitted uniformly over a hemisphere. (a) What is the intensity of the beam when the beam reaches the aircraft's location? The aircraft reflects radar waves as though it has a cross-sectional area of only 0.22 m^2. (b) What is the power of the aircraft's reflection? Assume that the beam is reflected uniformly over a hemisphere. Back at the radar site, what are (c) the intensity, (d) the maximum value of the electric field vector, and (e) the rms value of the magnetic field of the reflected radar beam?

32. An unpolarized beam of light is sent into a stack of four polarizing sheets, oriented so that the angle between the polarizing directions of adjacent sheets is 30°. What fraction of the incident intensity is transmitted by the system?

33. A beam of initially unpolarized light is sent through two polarizing sheets placed one on top of the other. What must be the angle between the polarizing directions of the sheets if the intensity of the transmitted light is to be one-third the incident intensity?

34. An electromagnetic wave is traveling in the negative direction of a y axis. At a particular position and time, the electric field is directed along the positive direction of the z axis and has a magnitude of 100 V/m. What are the (a) magnitude and (b) direction of the corresponding magnetic field?

35. Calculate the (a) upper and (b) lower limit of the Brewster angle for white light incident on fused quartz. Assume that the wavelength limits of the light are 400 and 700 nm.

36. The magnetic component of a polarized wave of light is

$$B_x = (4.0 \times 10^{-6}\ \text{T})\ \sin[(1.57 \times 10^7\ \text{m}^{-1})y + \omega t].$$

(a) Parallel to which axis is the light polarized? What are the (b) frequency and (c) intensity of the light?

37. The rms value of the electric field in a certain light wave is 0.200 V/m. What is the amplitude of the associated magnetic field?

38. The magnetic component of a polarized wave of light is given by $B_x = (4.00\ \mu\text{T})\ \sin[ky + (2.00 \times 10^{15}\ \text{s}^{-1})t]$. (a) In which direction does the wave travel, (b) parallel to which axis is it polarized, and (c) what is its intensity? (d) Write an expression for the electric field of the wave, including a value for the angular wave number. (e) What is the wavelength? (f) In which region of the electromagnetic spectrum is this electromagnetic wave?

39. Start from Eqs. 33-11 and 33-17 and show that $E(x, t)$ and $B(x, t)$, the electric and magnetic field components of a plane traveling electromagnetic wave, must satisfy the "wave equations"

$$\frac{\partial^2 E}{\partial t^2} = c^2 \frac{\partial^2 E}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 B}{\partial t^2} = c^2 \frac{\partial^2 B}{\partial x^2}.$$

40. (a) Show that Eqs. 33-1 and 33-2 satisfy the wave equations displayed in Problem 102. (b) Show that any expressions of the form $E = E_m f(kx \pm \omega t)$ and $B = B_m f(kx \pm \omega t)$, where $f(kx \pm \omega t)$ denotes an arbitrary function, also satisfy these wave equations.

ANSWERS

Checkpoints

1. (a) (Use Fig. 33-5.) On right side of rectangle, $\vec{E}$ is in negative y direction; on left side, $\vec{E} + d\vec{E}$ is greater and in same direction; (b) $\vec{E}$ is downward. On right side, $\vec{B}$ is in negative z direction; on left side, $\vec{B} + d\vec{B}$ is greater and in same direction.

2. positive direction of x

3. a, d, b, c (zero)

Reasoning Type Questions

1. (a)

2. (c)

3. (a)

4. (d)

Multiple Choice Questions

1. (d)

2. (d)

3. (d)

4. (d)

5. (a)
6. (d)
7. (b)
8. (d)
9. (a)
10. (a)
11. (b)
12. (a)
13. (d)
14. (b)
15. (e)
16. (c)
17. (d)
18. (d)
19. (c)
20. (d)
21. (c)
22. (d)
23. (a)
24. (a)
25. (a)
26. (d)
27. (b)
28. (c)
29. (a), (b), (d)
30. (b) (d)
31. (d)
32. (d)
33. (c)
34. (a)
35. (c)
36. (d)
37. (a)
38. (a)
39. (c)
40. (a)

Problems

1. (a) 515 nm; (b) 610 nm; (c) 555 nm; (d) 5.41×10^{14} Hz; (e) 1.85×10^{-15} s
2. 7.49 GHz
3. 5.0×10^{-21} H
4. 4.74 m
5. 0.10 MJ
6. 1.07×10^{-12} T
7. (a) 6.7 nT; (b) y; (c) negative direction of y
8. (a) 1.67×10^{-8} T; (b) 3.31×10^{-2} W/m^2
9. 1.2 MW/m^2
10. (a) 6.7 nT; (b) 5.3 mW/m^2; (c) 6.7 W
11. (a) 30.1 nm; (b) 345 nm to the right of P
12. 3.44×10^6 T/s
13. 5.9×10^{-8} Pa
14. 3.3×10^{-8} Pa
15. 19 W/m^2
16. 3.1%
17. 1.9 V/m
18. (a) 0.16; (b) 0.84
19. 20° or 70°
20. (a) 19.6°; (b) 70.4°
21. (a) 2 sheets; (b) 5 sheets
22. 1.48
23. (a) $n_2 > n_1$; (b) the value of n_2 for material 2 is also greater than that of water (n_1); (c) $n_2 = 1.4$ for material 1; (d) $n_2 = 1.9$
24. 1.07 m
25. (a) 53.1°; (b) yes
26. $n_3 = 1.0$
27. (a) 0.50 ms; (b) 8.4 min; (c) 2.4 h; (d) 5446 B.C.
28. (a) (1.67 nT) $\sin[(1.00 \times 10^6\ \text{m}^{-1})z + (3.00 \times 10^{14}\ \text{s}^{-1})t]$; (b) 6.28 μm; (c) 20.9 fs; (d) 33.2 mW/m^2; (e) x; (f) infrared
29. (a) 1.91×10^8 Hz; (b) 18.2 V/m; (c) 0.878 W/m^2
30. (a) 83 W/m^2; (b) 1.7 MW
31. (a) 3.5 μW/m^2; (b) 0.78 μW; (c) 1.5×10^{-17} W/m^2; (d) 1.1×10^{-7} V/m; (e) 0.25 fT
32. 0.21
33. 35°
34. (a) 3.3×10^{-7} T; (b) $-x$ direction
35. (a) 55.8°; (b) 55.5°
36. (a) z axis; (b) 7.5×10^{14} Hz; (c) 1.9 kW/m^2
37. 9.43×10^{-10} T
38. (a) $-y$; (b) z; (c) 1.91 kW/m^2; (d) $E_z = (1.20\ \text{kV/m}) \sin[(6.67 \times 10^6\ \text{m}^{-1})y + (2.00 \times 10^{15}\ \text{s}^{-1})t]$; (e) 942 nm; (f) infrared

Geometrical Optics: Reflection 34



Edouard Manet's A Bar at the Folies-Bergère has enchanted viewers ever since it was painted in 1882. Part of its appeal lies in the contrast between the audience ready for entertainment and a bartender whose eyes betray her fatigue. Its appeal also depends on subtle distortions of reality that Manet hid in the painting—distortions that give an eerie feel to the scene even before you recognize what is "wrong."

Can you find those subtle distortions of reality?

The answer is in this chapter.

34-1 WHAT IS PHYSICS?

One goal of physics is to discover the basic laws governing light, such as the law of reflection. The application of the reflection is ubiquitous. Each one of us uses mirrors everyday, whether it is for dressing, for shaving, or so on. Every vehicle is fitted with a mirror, with the text "Objects in the mirror are closer than they appear." Apart from daily uses, mirrors are used widely in varying applications such as lasers, dentistry, and so on. In this chapter we are going to study the basics of image formation by the mirrors.

Our first step in this chapter is to define and classify images. Then we examine several basic ways in which they can be produced. In Chapter 33, Section 33-8, we laid down the basic laws of reflection and its laws. We will discuss those laws in this chapter.

34-2 | Principle of Rectilinear Propagation of Light

There are many principles on which this chapter is based. One of them is **law of rectilinear propagation of light**. The light ray travels in a straight line in a homogenous isotropic medium. When a ray of light encounters the surface of a mirror, or a transparent boundary such as a glass, the ray makes an abrupt change in direction, after which it travels along a new straight line path. This motion of light ray produces shadows, images, and so on.

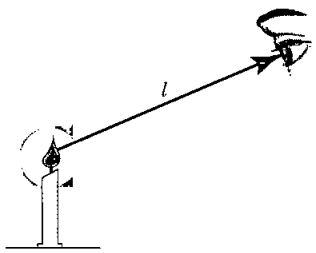


FIG. 34-1 Seeing a candle light with a single eye.

To understand this business of geometrical optics more clearly, let us try to understand how we see a candle in a room.

From every part of the candle, a ray is coming, rather an infinite number of rays are coming (Fig. 34-1). However, only a small fraction of these enters our eye. Our brain can deduce from this single ray that the flame lies somewhere on this line. Since we have two eyes, similarly the second eye will see another ray coming from the same point of the flame. The second eye will deduce that the flame lies somewhere on this second line. But the flame has to be at one point only. Therefore, we can say that this flame should be at the intersection of these two lines (Fig. 34-2).

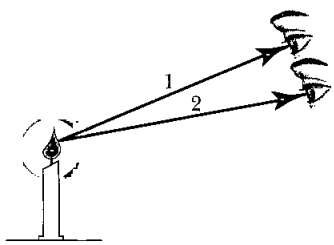


FIG. 34-2 Seeing a candle light with both the eyes.

The entire field of geometrical optics deals with tracing back the rays to see their origin. This will obviously involve geometrical considerations. Now you can see why we call this topic as ray optics or geometrical optics.

A very important and useful concept in this context is the **parallactic displacement** or simply **parallax**. The word may sound a little frightening, but as a matter of fact the parallax is a very simple as well as useful thing.

We may start our acquaintance with parallax by trying to put a thread into a needle's eye (Fig. 34-3). Try to do it with one eye closed, and you will find very quickly that it does not work; you will be bringing the end of the thread either too far behind the needle or stopping short in front of it. With only one eye, you are unable to judge the distance between the needle and the thread very clearly. (By the act of focusing, the eye gets only a vague idea of the distance.) However with two eyes open, you can do it very easily, or at least learn easily how to do it. When you look at the object with

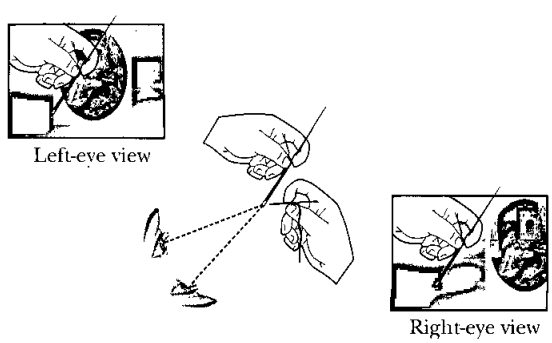


FIG. 34-3 Parallactic displacement of the object. Notice how the left eye sees a different background for the needle than the right eye. The brain superimposes these two images to get a clear idea of the distance.

two eyes, you automatically focus them both on the object. The closer the object, the more you have to turn your eyes toward each other, and the muscular feeling arising from such adjustment gives you a pretty good idea about the distance.

Now, if instead of looking with both eyes, you first close one and then the other, you will notice that the position of the object (the needle in this case) relative to the distant background (say, the Picture of Lord Krishna) has changed. This effect is known as parallactic displacement and is certainly familiar to everybody; if you have never heard about it, just try it out or look at Fig. 34-3 showing the needle and Lord Krishna as seen by the right eye and the left eye. The farther away the object, the smaller will be its parallactic displacement. Therefore, we can use it for measuring distances. Since parallactic displacement can be measured exactly in the degrees of the arc, this method is more precise than a simple judgment of the distance based on the muscular feeling in the eyeballs. But as the two eyes are set in our head, only about three inches apart, they are not good for the estimate of distances beyond a few feet. If two objects are placed such that their parallactic displacement is same, then we say that they have no parallax.

Formation of Shadow

For understanding the concept of rectilinear propagation of light, let us consider the formation of shadow. We should understand that the shadow is a region and not an area.

Sample Problem 34-1

Assume that sun rises at 6:00 A.M. and sets at 6:00 P.M. Find the angle made by the sunrays with the horizontal at 10:00 A.M. Also find the length of the shadow of a man 6 ft high at this time.

Conceptualize: We can assume that the sun rays are almost parallel to each other. A part of these rays will be blocked by a man. This blocked region will be the shadow (Fig. 34-4).

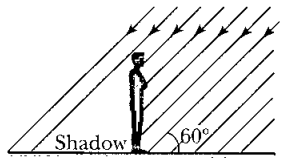


FIG. 34-4 The shadow of a man at 10:00 A.M.

Classify: As the time passes, the sun revolves around the Earth. We know that the sunrays are horizontal at 6:00 A.M. and at 6:00 P.M. At 10:00 A.M. the sun would have gone up in the sky.

Compute: As the sun covers 180° in sky in 12 hours from 6:00 A.M. to 12:00 P.M., it covers 15° in 1 hour. (Actually it is the Earth that is rotating, but the effect is same as seen from the Earth.) So till 10:00 A.M. the sun has covered

$$15° \times 4 = 60°.$$

Therefore the length of the man's shadow will be

$$L = 6 \cot 60° = 2\sqrt{3} \text{ ft.} \qquad \text{(Answer)}$$

Conclude: Now we can see how the length of the shadow will change as the day progresses. It will be the minimum at the midday. Actually, since the Earth is a sphere, at most of the points on the Earth, even at 12:00 A.M., the shadow length is not zero on any day of the year. Here, we have assumed that the man is standing at such a point that the sun is at its zenith at noon. Incidentally, this was the way that Eratosthenes first found the radius of the Earth. Can you guess how?

It is easy to explain why sharp shadows are cast by opaque obstacles if there is a point source of light. If there are two sources of light, the shadow cast by an opaque obstacle placed in the field of light of one source, S_1, is illuminated in certain regions by the light from the other source, S_2. Some portions of space will receive no light at all, while still others will be illuminated by both sources. This is shown in Fig. 34-5, where the black portion, which receives no light at all, is called the "umbra." The shaded portion that receives light from only one source is called the "penumbra."

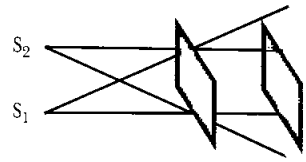


FIG. 34-5 If two sources are used to form the shadow; some regions are completely dark while some are partially dark.

CHECKPOINT 1 Keep a small cardboard between the wall and the bulb in your room. Observe the shadow pattern formed for different positions of the cardboard. Explain why and when is the shadow very sharp.

34-3 | Laws of Reflection

A **mirror** is a surface that can reflect a parallel beam of light in one direction instead of either scattering it widely in many directions or absorbing it. A shiny metal surface acts as a mirror whereas a concrete wall does not. In this section, we examine the images that a **plane mirror** (a flat reflecting surface) can produce.

First, let us briefly recollect the laws of reflection we introduced in Section 33-8.

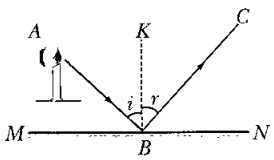


FIG. 34-6 The angle of reflection r is equal to the angle of incidence i.

The laws of reflection state that

1. The angle of incidence (i) equals the angle of reflection (r) (Fig. 34-6).
2. Incident ray, the normal ray, and the reflected ray lie in the same plane.

Fermat's Principle of Stationary Time (Optional)

It states that the light ray, while going from one point to another, follows a path which takes either minimum time or maximum time or constant time. This law forms the theoretical basis for geometrical optics. Even the law of rectilinear propagation of light can be deduced from it. We know that the smallest time will be taken by the light in going from A to B in the same medium, if it follows a straight line. Let us deduce the laws of reflection from this principle.

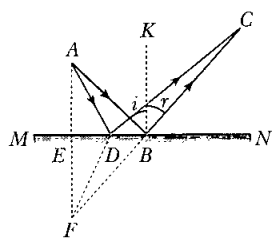


FIG. 34-7 Reflecting light chooses the shortest path.

The second law of reflection is easily seen from this principle. If we draw a set of rays going from A to C after reflection from a mirror as shown in Fig. 34-7, the set of rays in the plane of paper will constitute a smaller path and hence time required will be less. If the point of reflection is out of the plane of paper, both incident and reflected rays will have to travel correspondingly longer distance and hence that path is not acceptable.

For proving the first part of the law, let us assume a point F behind the mirror at the same perpendicular distance from the mirror as A. The ray coming from A can be incident on the mirror at any arbitrary point D. It can be easily seen that the time taken by ray in going from A to D will be the same as the time taken by a hypothetical ray in going from F to D. So all we need to do is choose a point D such that the time taken by the ray in going from C to F is minimum. It is clear that this will be when D is on the straight line joining C and F. Thus, the point of reflection for the ray reaching C is B. A little geometrical insight will then reveal that the angle of incidence is equal to the angle of reflection. Here, it should be noted that the angles are measured from the normal to the surface.

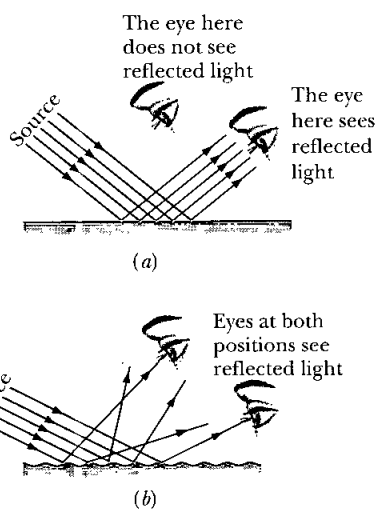


FIG. 34-8 (a) Specular reflection of a parallel beam of light and (b) diffuse reflection of a parallel beam of light.

Specular and Diffuse Reflection

Smooth surfaces reflect light **specularly**—as shown in (Fig. 34-8a). At the smooth surface, the normal to the surface does not change its direction abruptly. Therefore, the reflected rays do not change their directions abruptly. If the light is reflected specularly, the eye has to be in a particular position to receive light.

Surfaces that are not smooth reflect light in many directions, as shown in Fig. 34-8b. For such **diffuse reflection**, each individual ray still obeys the laws of reflection, but the direction of the normal to the surface varies over the surface and hence the angle of reflection also varies. How smooth is smooth surface? If the height and separation of irregularities on the surface are small compared to the wavelength of the incident light, the surface acts like a smooth surface and

most light is reflected specularly. In the case of diffuse reflection, the eye can be positioned at any point and still receive the light. You are reading the pages of this book in this manner only. Do you know that using this argument of diffuse reflection, Galileo showed that the surface of moon should be rough and not smooth as many people of his times thought. Can you see his line of logic?

Many times we are required to find the angle of the deviation of the light ray on reflection. Recollect that the angle between two vectors is the angle between their positive directions. So when we talk of the deviation of a light ray, it refers to the angle deviated by a light ray from its original path, and not the angle between the reflected ray and the incident ray. This deviation is shown in Fig. 34-9.

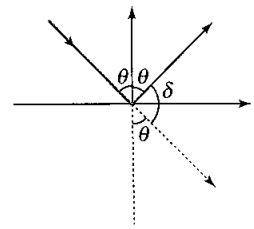


FIG. 34-9 Deviation (denoted by δ) of a light ray on reflection.

☞ The reflected ray deviates by an angle $\delta = \pi - 2\theta$.

Sample Problem 34-2

Find the deviation of the ray after two reflections as shown in Fig. 34-10.

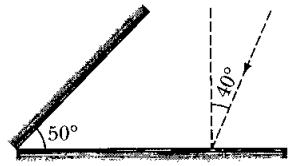


FIG. 34-10 Two plane mirrors are inclined at an angle to each other. A ray is incident on one of the mirrors.

Conceptualize/Classify: To solve this problem, we have to find the angle of incidence on each mirror. Then we can use the formula for deviation on each surface. If the deviation is in the same direction, it will be added, otherwise it will be subtracted.

Compute: Here, the angle of incidence on the first mirror is 40° as seen in Fig. 34-10. So the angle of reflection is also 40°. However, the angle of incidence on the second mirror is unknown. Consider the triangle formed by the two mirrors and the ray reflected from the first mirror. In this triangle, the angle between the first mirror and the reflected ray is 50°. The angle of reflection is measured from the normal. As the sum of all the angles in a triangle is 180°, the angle made by this ray with the second mirror is 180 – 50 – 50 = 80°. Therefore, the angle of incidence on the second mirror is 90° – 80° = 10°. The angle of reflection will also be 10°. From Fig. 34-11, it is clear that the angle of deviation is the angle between the reflected ray and the initial incident ray. This angle is the sum of the two interior angles made by the incident ray and the two reflected rays. So the deviation $\delta = 40° + 40° + 10° + 10° = 100°$ in the anticlockwise direction.

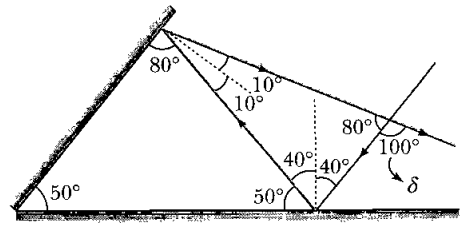


FIG. 34-11 The path of incident and the reflected rays.

Here, we should remember that the deviation is an angle between the directions in which the rays are traveling.

We could have also used the formula for the deviation of a ray on reflection. The ray reflected from the first surface undergoes a deviation

$$\delta_1 = 180° - 2 \times 40° \quad \text{(clockwise)}.$$

The ray reflected from the second surface undergoes a deviation

$$\delta_2 = 180° - 2 \times 10° \quad \text{(clockwise)}.$$

Total deviation will be $\delta_1 + \delta_2 = 100° + 160° = 260°$ clockwise which is same as 100° anticlockwise.

Conclude: Deviation after two reflections is 2θ, where θ is the angle between the two mirrors. In fact, a corner reflector is designed so that the light ray goes back in the direction from which it came. If we have three plane mirrors inclined at 90° to each other, the light ray reflects back to the direction from which it came. Such an arrangement is known as a **retro reflector**, a point we will discuss in the next chapter.

Sample Problem 34-3

A light ray is incident on a mirror rotating with a constant angular velocity of $\pi/30$ rad/s in the anticlockwise direction. This light ray is reflected from the mirror and falls on a wall nearby as shown in Fig. 34-12. Find the location of the spot on the wall and its velocity after 5 s.

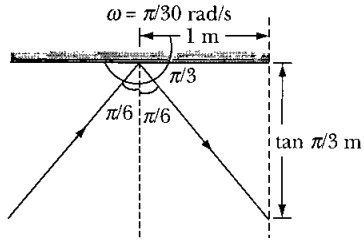


FIG. 34-12 Ray incident on a rotating mirror.

Conceptualize: If the mirror is rotating, the reflected ray will also rotate. This rotation produces a spot on the screen which moves due to the rotating reflected light.

Classify: If mirror is rotated by angle α, the normal to the mirror will also rotate by the same angle. Therefore, the angle of incidence increases by α. This would mean that the angle of reflection also increases by α. But remember that the angle of reflection is measured from the new normal. As seen from its original direction, the reflected ray is rotated by 2α. This is shown in Fig. 34-13. Even if the mirror is not rotated about the point of incidence, the reflected ray still rotates by 2α. If the mirror rotates in the clockwise direction, the reflected ray also rotates in the same direction. Thus, if the mirror rotates with an angular velocity ω, the reflected ray rotates with the same velocity 2ω.

Compute: In 5 s, the angle rotated by the mirror

$$\phi = \frac{\pi}{30} \times 5 = \frac{\pi}{6}.$$

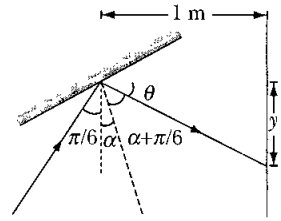


FIG. 34-13 The incident ray remains at the same position, when the mirror rotates. The angle made by the reflected ray with the new normal is $\alpha + \pi/6$. Therefore, the angle made by the reflected ray with the original normal would be $2\alpha + \pi/6$. The figure depicts the reflected ray at an intermediate position. At $t = 5$ s, angle θ would be zero.

Therefore, the reflected ray rotates by $= \pi/3$. The total angle is given by

$$= \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}.$$

As can be seen from Fig. 34-13, y is the position of the spot at angle $\theta = \tan(90 - \theta) = \cot^\circ \theta$. Therefore

$$y = \frac{dy}{dt} = \text{cosec}^2\theta \frac{d\theta}{dt} = -\text{cosec}^2\theta \cdot 2\omega,$$

$$-\text{cosec}^2(90^\circ) \cdot 2 \times \frac{\pi}{30} = \frac{-\pi}{15} \text{ m/s}.$$

Conclude: Those who are familiar with the Cavendish's experiment of measuring G can relate to this. In that, the twist of the wire is measured with the help of a plane mirror attached to the wire. A small twist of the wire can displace the light spot on a screen by quite a bit of distance.

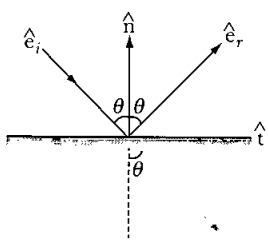


FIG. 34-14 The incident ray, the reflected ray, and the normal to the reflecting surface.

Laws of Reflection in Vector Form

Suppose that we are given a unit vector in the direction of the incident ray. We wish to find a unit vector in the direction of the reflected ray. For this, we will first represent the incident ray in terms of unit vector notation.

As can be seen from Fig. 34-14, the component of the incident ray in the direction of tangent to the reflecting surface would be

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

The component of the incident ray in the direction of normal to the reflecting surface would be

$$\cos(\pi - \theta) = -\cos\theta.$$

So the incident ray vector can be written as

$$\hat{e}_i = \sin\theta\,\hat{t} - \cos\theta\,\hat{n}.$$

The component of the reflected ray in the direction of tangent to the reflecting surface would be

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

The component of the reflected ray in the direction of normal to the reflecting surface would be cos θ. So the reflected ray can be written as

$$\hat{e}_r = \sin\theta\,\hat{t} + \cos\theta\,\hat{n}.$$

So we can say that

$$\hat{e}_r = \hat{e}_i + 2\cos\theta\,\hat{n}.$$

But

$$\cos(\pi - \theta) = \frac{\hat{e}_i \cdot \hat{n}}{|\hat{e}_i| \cdot |\hat{n}|} = -\cos\theta.$$

Therefore

$$\hat{e}_r = \hat{e}_i - 2(\hat{e}_i \cdot \hat{n})\hat{n}.$$

Note that the component of the ray parallel to the mirror surface does not change. But the component of the ray perpendicular to the mirror reverses in direction.

✓**CHECKPOINT 2** How many minimum mirrors arranged perpendicular to each other will make a light ray reverse its direction completely?

34-4 | Formation of Image by a Mirror

Before we go ahead in geometrical optics, we need to define what a point object is and what an image of a point object is.

In optics:

☞ ***Object*** is a point of intersection of the incident ray.

Many times the rays do not actually meet. In such cases, we have to trace the rays backward to make them meet.

☞ ***Image*** is a point at which all the reflected rays or refracted rays intersect after passing through an optical system.

Now the object and image can be of two types each. Besides the two mentioned above, we need to understand the following:

☞ An object is said to be **real** if the light rays incident on the surface are diverging in nature. An object is said to be **virtual** if the incident rays are converging in nature.

☞ If the light rays after reflection or refraction are converging in nature, then the image is a ***real image***.

A real image can be photographed directly by placing a photographic plate at the location of image. Since the light rays are actually converging at that point, the point of convergence will be a point of large intensity. So the photographic plate will record such an image. Real image can be obtained on a screen.

☞ An image is said to be ***virtual*** when the light rays after reflection or refraction are diverging in nature.

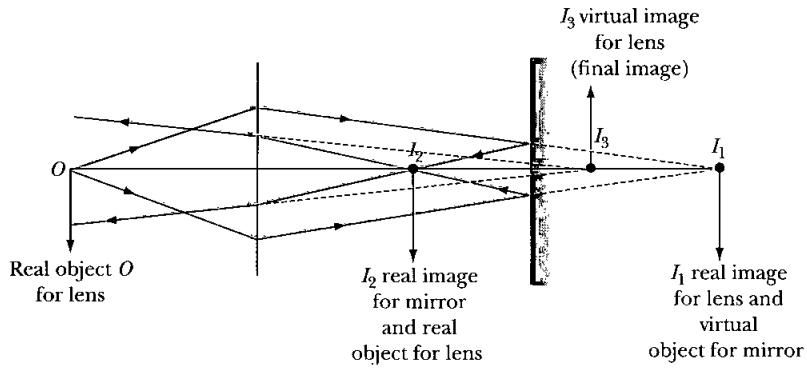


FIG. 34-15 A point object O is on the axis of a convex lens. After refraction from the lens, reflection from the mirror, and again reflection from the lens, a final image is formed.

A virtual image cannot be directly photographed by placing a photographic plate at that point. Since the rays are not meeting at that point, there is no need to keep photographic plate at that point. The image cannot be obtained on a screen.

In optics, we often encounter situation where the image is formed after multiple reflections or refractions. In order to form the final image, we work event by event. An object forms an image due to reflection or refraction from a surface. This image will act as an object for next surface and so on.

For example, in Fig. 34-15, the rays from O strike the lens first. As the rays incident on the lens are diverging, O is a real object for the lens. This object has no meaning for the mirror because rays from O do not directly reach the mirror. The rays after refraction from the lens converge to point I_1. As the refracted rays are converging in nature, I_1 is a real image formed by the lens. But before converging, the rays strike the mirror. These rays incident on the mirror meet at I_1. So I_1 will act as an object for the mirror. As the incident rays on the mirror are converging in nature, I_1 is a virtual object. After reflection from the mirror, the rays meet at I_2. So I_2 is an image formed by the mirror. As the reflected rays are converging, I_2 is a real image formed by the mirror. Again, these rays strike the lens. The rays incident on the lens intersect at I_2. So I_2 is an object for the lens. As the rays incident on the lens are diverging, I_2 is a real object for the lens. After refraction from the lens, the rays diverge. These diverging rays can be traced back to meet at I_3. As the refracted rays are diverging, I_3 is a virtual image formed by the lens. After this event, no further event will occur; rays will not return back. So I_3 is the final image formed by the system of lens. If someone asks us about the image formed by this system, the answer will be I_3.

Principle of Optical Reversibility

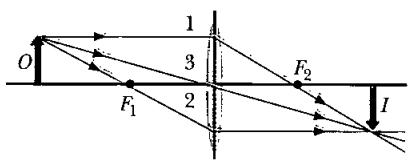


FIG. 34-16 Principle of optical reversibility is illustrated.

It states that if a light ray is reversed in its direction it will retrace its path. This essentially implies that the object and image can be interchanged. In other words, if a mirror or a lens forms an image at I when the object is at O, then due to this principle the image is formed at O if the object is instead kept at I (Fig. 34-16).

Image is formed with a large number of rays, practically infinite. But for ray tracing, only two or three rays are sufficient.

Image of a Point Object in a Plane Mirror

Suppose that a point object is kept in front of the mirror (Fig. 34-17*a*). The incident rays are diverging (the object is real). The reflected rays from the mirror are also seen to be diverging in nature (the image is virtual). So we can say that if the object is real, a plane mirror forms its virtual image. On the other hand, if converging rays fall on the mirror as shown in Fig. 34-17*b* (the object is then virtual), they converge after reflection (the image is then real). So the virtual object forms a real image in a plane mirror.

As can be seen in Fig. 34-17*a*, if the object is at a distance x from the mirror, the image is also at a distance x from the mirror. Here, we should be careful. The distance of the object from the mirror is always the perpendicular distance. Many times the object is in front of the plane of the mirror but not directly in front of the mirror. Still the image will be formed because some rays will be incident on the mirror whose reflection will form the image.

Only rays that are fairly close together can enter the eye after reflection from a mirror. For the eye position shown in Fig. 34-18, only a small portion of the mirror near point *a* (a portion smaller than the pupil of the eye) is useful in forming the image. To find this portion, close one eye and look at the mirror image of a small object such as the tip of a pencil. Then move your fingertip over the mirror surface until you cannot see the image. This implies that only the small portion of the mirror under your fingertip produced the image for you.

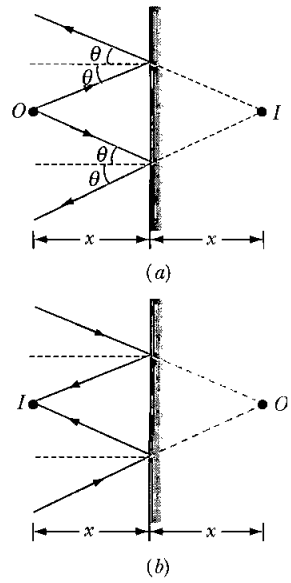


FIG. 34-17 (*a*) Formation of virtual image of a real object. (*b*) Formation of real image of a virtual object.

Velocity of the Image

If the object moves, its image also moves. To find the velocity of the image, we resolve the velocity of the object into two parts: one is parallel to the mirror and second is perpendicular to the mirror (Fig. 34-19).

The velocity of the object in the direction parallel to the mirror will be equal to the velocity of image in a direction parallel to the mirror. If the mirror is also moving parallel to itself, its velocity does not affect the velocity of the image in the parallel direction.

The speed of the object with respect to the mirror in a direction perpendicular to mirror surface is equal to the velocity of the image with respect to the mirror in the perpendicular direction. The only difference is that the object is in front of the mirror, and the image is behind the mirror. Let us show this.

As shown in Fig. 34-17*a*, the perpendicular distance of the image from the mirror is equal to the perpendicular distance of the object from the mirror. Putting it mathematically,

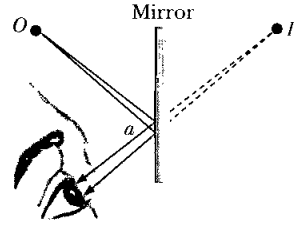


FIG. 34-18 A "pencil" of rays from O enters the eye after reflection at the mirror. Only a small portion of the mirror near a is involved in this reflection. The light appears to originate at point I behind the mirror.

$$X_{OM} = -X_{IM}.$$

By differentiating this position of the object and the image relative to the mirror, we obtain the velocity of the object and its image relative to the mirror:

$$\frac{d}{dt}(X_{OM}) = \frac{d}{dt}(-X_{IM}).$$

Thus, we can say that

$$V_{OM} = -V_{IM}.$$

Note that we are referring to the velocities perpendicular to the mirror surface here. By the definition of the relative velocity, we have

$$V_{IM_\perp} = V_{I_\perp} - V_{M_\perp} = -(V_{O_\perp} - V_{M_\perp}). \qquad (34\text{-}1)$$

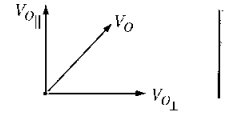


FIG. 34-19 A point object moves in front of a mirror. Its velocity is resolved in two directions: parallel and perpendicular to the mirror.

Sample Problem 34-4

Find the velocity of image and the object in the vector form when the ball is just about to hit the inclined plane as shown in Fig. 34-20*a*. The inclined plane is at an angle of 37° to the horizontal. The surface of the inclined plane acts like a mirror.

Conceptualize: As indicated in Eq. 34-1, the velocity of the object can be resolved in directions parallel and perpendicular to the mirror. Since the mirror is stationary, we need not bother about the motion of the mirror.

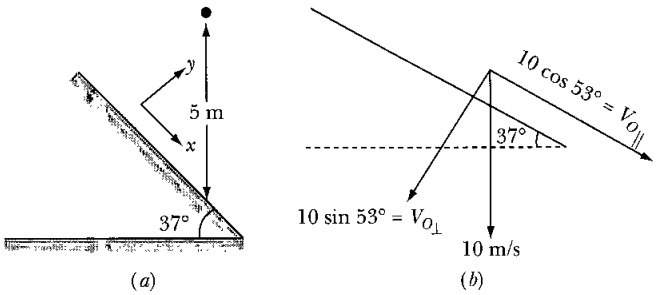


FIG. 34-20 (*a*) An object is dropped from a height on an inclined mirror. (*b*) Resolving the velocity of the object in a direction parallel to and perpendicular to the mirror.

Classify: In this situation, the velocity of the striking object is in the vertical direction. Since the object moves, image moves such that the distance of the image perpendicular to the mirror remains the same as the distance of the object perpendicular to the mirror.

Compute: The velocity of the object before striking the mirror can be found by equation of kinematics.

$$v^2 = u^2 + 2gs$$

$$v^2 = 0 + 2 \times 10 \times 5$$

$$v = 10 \text{ m/s}$$

$$v = +10 \text{ m/s}.$$

Resolving this in a direction perpendicular and parallel to the mirror as shown in Fig. 34-20*b*, we have

$$V_{O_\parallel} = 10 \cos 53° = 6 \text{ m/s},$$

$$V_{O_\perp} = -10 \cos 37° = -10 \times \frac{4}{5} = -8 \text{ m/s}.$$

Since the inclined plane is not moving,

$$V_{I_\perp} = -V_{O_\perp} = 8\hat{j}.$$

In the parallel direction,

$$V_{I_\parallel} = V_{O_\parallel} = 10 \cos 53° = 6 \text{ m/s}$$

$$v = v_x\hat{i} + v_y\hat{j} = 6\hat{i} + 8\hat{j}. \qquad \text{(Answer)}$$

Conclude: We can see that the velocity of the image makes the same angle with the mirror as the velocity of the object with the mirror.

Field of View

It is the region in which an observer can see the image of the object. For finding the field of view of a mirror, we should consider the mirror as a window through which we are viewing the image of the object. The region in which the observer is present to see this image is the field of view.

From Fig. 34-21*a* we can see that the reflected rays are present in a certain region only. These reflected rays can be traced back to see the image. We conclude this with the help of Fig. 34-21*b*. To see *I*, the observer can be at *A* but not at *B*, because no reflected rays reach *B*. The same point can be analyzed by treating the mirror as a window. If the mirror is a window, we cannot see *I* through the window by standing at any point.

We should also note that a human eye can form a sharp image on the retina only when diverging rays are incident on it. So even if an eye receives converging rays, it is not in the field of view as shown in Fig. 34-22.

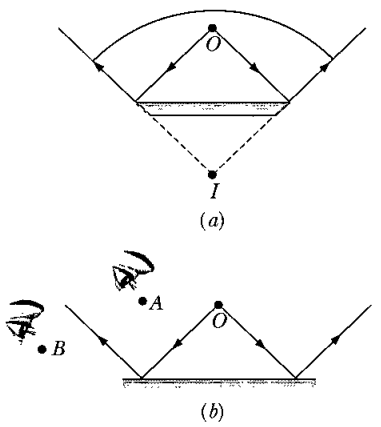


FIG. 34-21 (*a*) Field of view. (*b*) The observer can be at *A* to view the image but not at *B*.

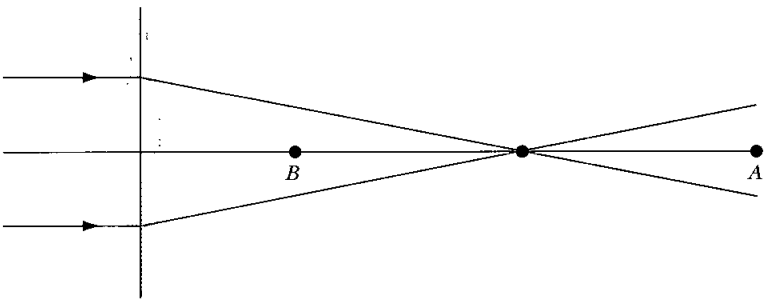


FIG. 34-22 The observer at *A* is in the field of view, but not at *B*.

Sample Problem 34-5

A mirror of length $2d$ is kept in front of a road as in Fig. 34-23. A point object is kept at a distance d from the mirror. An observer is able to see the image till distance $12d$ while moving along the road. What is the distance x between the road and the mirror?

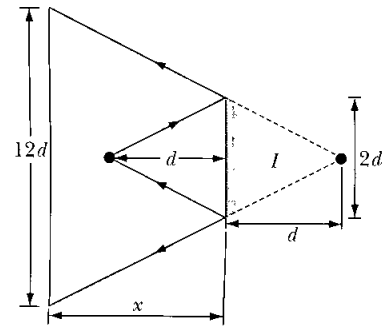


FIG. 34-23 An observer is walking on a road parallel to a mirror.

Conceptualize: We can first form the image of O, and then treat the mirror as a window through which we can view the image.

Classify: Let us see that the region in which the image is formed. This will define the region in which the observer can see the image.

Compute: Here we can see that the image, the mirror, and the length of the road when the image is visible from two triangles as shown in Fig. 34-23. We can say that the mirror acts like a window through which we can see the image of the object. This makes two triangles which are similar to each other.

By the property of similar triangles, we can say that

$$\frac{d}{d+x} = \frac{2d}{12d}$$

$$x = 5d.$$

Conclude: When we stand in front of a mirror, we can see only a part of the image depending on the length of the mirror. It is not that the image of the rest of the body is not formed by the mirror, only these reflected rays do not reach our eyes.

Image of an Extended Object

To form the image of an extended object, we should divide the object into many points and form the image of each point in the mirror. Then we should join all these point images to make the image of the object. If the object is a straight line, the image in a plane mirror will also be a straight line as can be seen in Fig. 34-24. To see the field of view of an extended object, we can make the image of the object as described above. Then we can see the field of view of the left end of the image by drawing rays reflected from the left end and the rays reflected from the right end. Similarly, we can draw the field of view of the right end of the image. The overlapping region for both the fields of view will give us the field of view of the complete image. This is clear from Fig. 34-24.

To see the image of oneself in the mirror, the mirror should be half of the height of person and the top of the mirror should be between the eye and head level. Similarly, the bottom of the mirror should be between the eye and the foot. This is clear from Fig. 34-25.

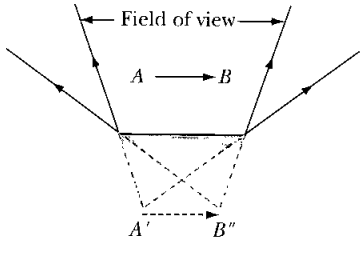


FIG. 34-24 The field of view of an extended object.

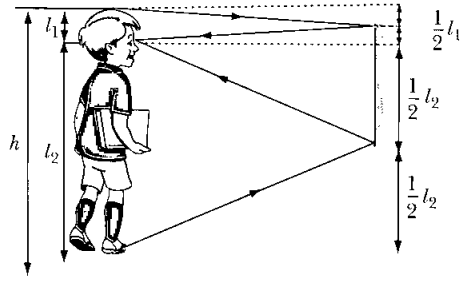


FIG. 34-25 The height and the position of the mirror to view one's full image.

Sample Problem 34-6

Find the minimum length of the mirror required so that the boy can see the image of himself as well as his father standing behind. The father is of height H and the boy is of height $H/2$ (Fig. 34-26). Assume that the eyes of the boy are at his head.

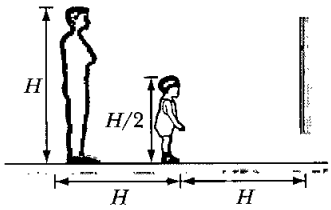


FIG. 34-26 A boy standing in front of a mirror such that he can see his own image as well as his father's image.

Conceptualize: The boy is of height $H/2$. The image of both the father and the boy is shown in Fig. 34-27. It is clear that the lowest point of the mirror is decided by the location of the boy's feet. If he can see his own feet, he can also see his father's feet. He can see his own head if the mirror is at the level of his head. But to see his father's head, he has to tilt his head to some angle above. So if he can see his father's head, he can see his own head as well.

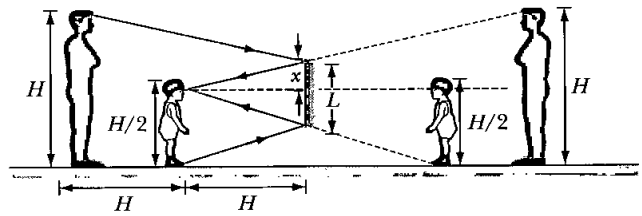


FIG. 34-27 Image of the father and the boy.

Classify: Here we can assume the mirror to act like a window by which the boy can see the image of the father as well as himself.

Compute: Let us assume that the mirror is at a height x above his own head. From the concept of similar triangles, we can say that

$$\frac{x}{H/2} = \frac{H}{3H}.$$

Therefore, $x = H/6$.

From Fig. 34-27, it can be seen that the boy can see his own feet if the bottom of the mirror is midway between the eye and the foot.

Therefore, the total length of mirror required is

$$\frac{H}{2} - \frac{H}{4} + \frac{H}{6} = \frac{5H}{12}.$$

Manet's "Folies-Bergère"

In *A Bar at Folies-Bergère* you see the bathroom via reflection by a large mirror on the wall behind the woman tending bar, but the reflection is subtly wrong in three ways. First note the bottles at the left. Manet painted their reflections on the mirror but misplaced them, painting them farther toward the front of the bar than they should be.

Now note the reflection of the woman. Since your view is from directly in front of the woman, her reflection should be behind her, with only a little of it (if any) visible to you; yet Manet painted her reflection well off to the right. Finally, note the reflection of the man facing her. He must be you, because the reflection shows that he is directly in front of the woman, and thus he must be the viewer of the painting. You are looking into Manet's work and seeing your reflection well off to your right. The effect is eerie because it is not what we expect from a painting or from a mirror.

Formation of Multiple Images

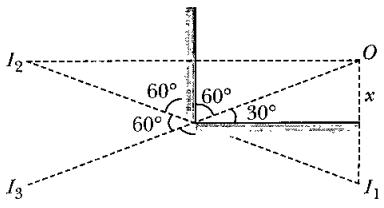


FIG. 34-28 Formation of three images by two mirrors held perpendicular to each other.

If an object is placed between two inclined mirrors at an angle to each other, multiple images will be formed. To consider the formation of images, let us take an example where there are two mirrors at 90° to each other. The object is at position O as shown Fig. 34-28. Each of the mirrors will form one image (I_1 and I_2 in Fig. 34-28). These images will act as an object for the next mirror.

We know that if the object is at perpendicular distance x from the mirror, the image will also be at a distance x behind the mirror. In other words, the angular position of image will be the same as the angular position of the object. So I_1 will be 30° behind the mirror. Similarly, I_2 will be 60° behind the mirror. This image will

also be in the front of plane of vertical mirror. Therefore, rays will reflect from the vertical mirror to form image I_3. For the horizontal mirror, I_2 acts as an object. I_2 is 150° ahead of the mirror. Therefore, its image will be 150° behind the mirror. Interestingly, these two images will coincide. No further images will be formed because reflected rays will not strike the mirrors again. Also we can see that to form I_3, left half of the horizontal mirror and bottom half of vertical mirror only will be useful.

From Fig. 34-27, we can understand that

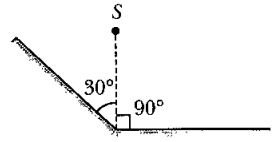
1. If the object is at angle θ in front of the mirrors, the image will also be at an angle θ behind mirrors.
2. All these images lie on a circle whose center is the point of intersection of the mirrors. The radius of the circle is equal to the distance between the point of intersection of the mirrors and the object.
3. The reflection will continue to take place till the image lies in front of the plane of the mirror. In other words, if reflected rays do not reach the next mirror again, further images will not form.

General rule for finding the number of image when the mirrors are inclined at an angle θ.

Let $m = 360/\theta$. Then

1. If m is an even integer, the number of images will be $(m - 1)$ irrespective of the position of the object.
2. If m is an odd integer, we have the following cases:
   Case-I: If the object is on angle bisector of the mirror, the number of images will be $(m - 1)$.
   Case-II: If the object is at any other place, the number of images will be m. We cannot apply this rule when $360/\theta$ is not an integer.

CHECKPOINT 3 If an object is at point S as shown in the figure and the mirrors are at an angle of 120° to each other, find the total number of images and their angular position from the x axis.



Specifically, if the mirrors are parallel to each other, the number of images will be infinite. But they will be of successively lesser intensity because some reflected rays will escape from the sides and some will be absorbed by the mirror.

34-5 | Formation of Image by a Spherical Mirror

You must have seen the image of an object on the back of a spoon. It seems funny, inverted, and distorted. The surface of the spoon acts as a mirror and not plane. In fact, it acts like a curved mirror. We will specifically discuss spherical mirrors here. Spherical mirror is a mirror whose reflecting surface is the part of a sphere. If the reflection occurs from the outer surface of the sphere, the mirror is called a **convex mirror** (Fig. 34-29*a*). If the reflection occurs from the inner surface of the sphere, then mirror is called a **concave mirror** (Fig. 34-29*b*).

There are some terminologies that we should be familiar with: **Aperture** is the diameter of the mirror disk (d in Figs. 34-29*a* and *b*). Center of the spherical surface is called the **center of curvature** (point C in Figs. 34-29*a* and *b*). The line joining center of curvature and the pole P is called the **principal axis**. r is the **radius of curvature** of the mirror surface. It is the distance between the pole

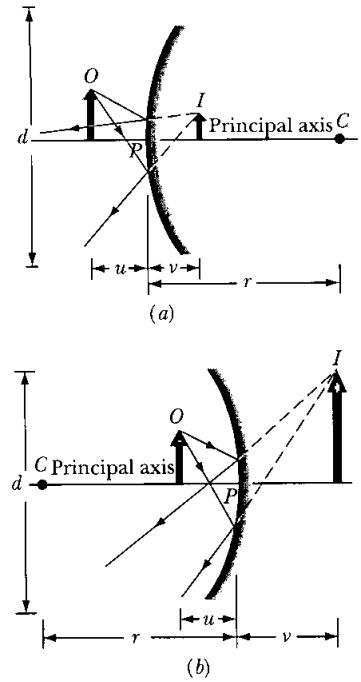


FIG. 34-29 Image formation by (*a*) a convex mirror and (*b*) a concave mirror. Note that both the images are virtual.

and the center of curvature. O is an object whose image is I. The distance of the object from the pole is denoted by u and the distance of the image from the pole is denoted by v.

Focus is a point on the principal axis where a beam parallel to the principal axis converges after reflection from the mirror. To find the location of focus, let us assume that a beam of light rays parallel to the principal axis gets reflected from the mirror and crosses the principal axis after reflection (Fig. 34-30a). We know that the line from center of curvature to the mirror will always be along the normal. So the angle of incidence will be θ. From the laws of reflection, we can say that angle of reflection will also be θ. Let us say that the reflected ray crosses the principal axis at a distance x from the pole. Since the incident ray is parallel to the principal axis, the angle made by the normal at the point of incidence with the principal axis will also be θ.

In the triangle formed by C, focus and the point of incidence (Fig. 34-30b), on applying sine rule we get

$$\frac{r-x}{\sin\theta} = \frac{r}{\sin(\pi-2\theta)}$$

$$x = r - \frac{r}{2}\sec\theta.$$

If θ is small, $\sec\theta = 1$, and so

$$x = \frac{r}{2}.$$

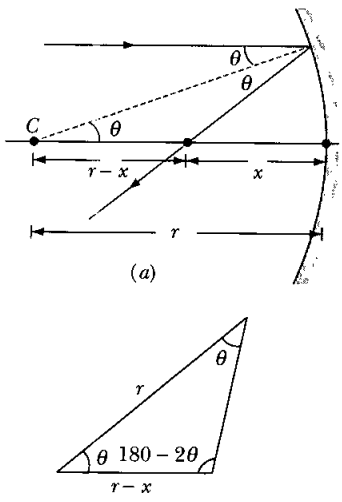


FIG. 34-30 (a) Reflection of a ray parallel to the principal axis from a spherical mirror. Note that the laws of reflection are valid for spherical as well as plane mirrors. (b) Tracing the reflected ray with help of sine rule.

Let us now define the **focal length of a mirror**. It is the distance of a point on the principal axis from the pole where a beam of light converges after reflection from the mirror. We can say that the focal length of a spherical mirror is

$$f = \frac{r}{2}.$$

The concept of focus is defined only for the angle of incidence close to 0°. Such rays are called **paraxial rays**. Paraxial rays approximation means that the angle of incidence should be very small ($\theta < 10°$). In such a case, parallel beam of light either converges at the focus after reflection from the mirror or diverges from it (Figs. 34-31a and b).

If the rays are not paraxial, the image is not sharp. This defect of image formation is known as **spherical aberration**. You know why? This defect is related to the spherical nature of the surface. If the same surface were parabolic in shape, all the rays parallel to the principal axis will converge at the focal point. Any standard text on mathematics will tell you how.

When two parallel rays, not parallel to principal axis, get reflected from the mirror, they meet at some point above the focus or below the focus. That point is also called **secondary focus**.

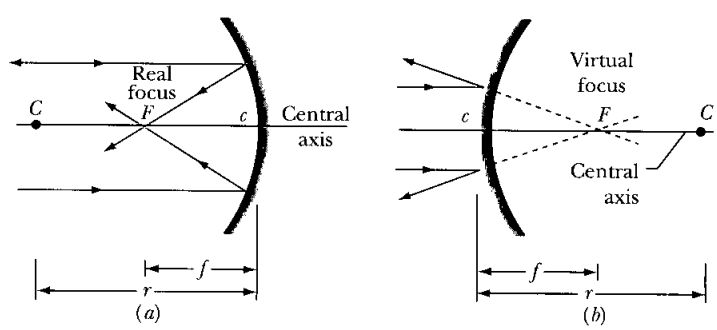


FIG. 34-31 (a) In a concave mirror, incident parallel light rays are brought to a real focus at F, on the same side of the mirror as the incident light rays. (b) In a convex mirror, incident parallel light rays seem to diverge from a virtual focus at F, on the side of the mirror opposite the light rays.

Focal plane is a plane passing through focus and perpendicular to the principal axis. A parallel beam of light meets in the focal plane after reflection.

Let us return to our original quest: finding the location of the image if we know the location of the object on the principal axis. To find the location of the image, we have to define a sign convention. In this chapter, we are going to follow **Cartesian sign convention**, in which

1. The principal axis of the mirror is taken as x axis and the pole as origin.
2. The direction of the incident light is taken as positive direction of x axis.
3. The upward direction is taken as positive and downward direction as negative.

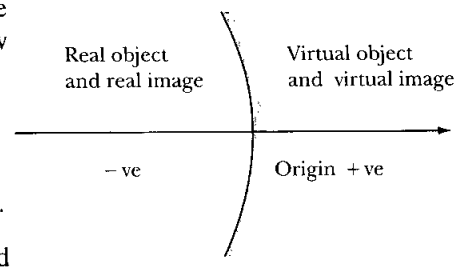


FIG. 34-32 Cartesian sign convention.

All the axial distances are always measured from the pole. This is represented in Fig. 34-32. Therefore, we can see that in the case of mirrors, real objects, u is always negative and for virtual objects, u is positive. For real image v is negative and for virtual image v is positive. Another useful concept is of **lateral magnification**. It is defined as the ratio of the height of the image and the height of the object from the principal axis and is denoted by m.

In Fig. 34-33, ab is the height of the object h_O and de is the height of the image h_I. Therefore, by our definition,

$$m = \frac{h_I}{h_O}.$$

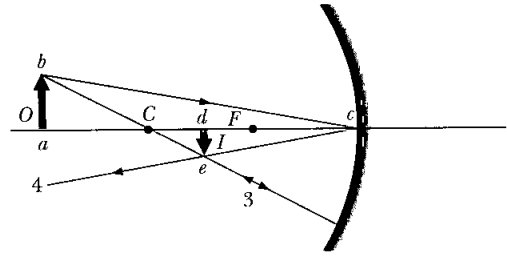


FIG. 34-33 Lateral magnification.

Let us try to calculate its value. From the figure, it can be seen that h_I is negative and h_O is positive according to our sign convention. Also abc and cde form two similar triangles. From the property of the similar triangles, we get

$$\frac{-h_I}{h_O} = \frac{cd}{ac},$$

where negative sign is introduced to take the modulus of h_I. Now according to the sign convention, $-ac = v$ and $-cd = u$. Therefore, we can say that

$$m = \frac{h_I}{h_O} = -\frac{v}{u}. \tag{34-2}$$

Be careful to substitute v and u with proper signs. This magnification has interesting implications. If the magnification is positive, it means that the image is formed on the same side of the principal axis as the object and vice versa. If the magnification is greater than 1 in magnitude, this would mean that the image is magnified; otherwise the image is diminished in size.

Case-I: If object and image both are real, according to Eq. 34-2, we get

$$m = \frac{-v}{u} < 0.$$

Therefore, v and u are both negative. The image is inverted. This means that if the object is above the principal axis (h_O is positive), the image will be below the principal axis (h_I is negative).

Case-II: If the object is real and the image virtual, according to Eq. 34-2, we get

$$m = -\left(\frac{v}{u}\right) > 0.$$

Therefore, v is positive and u is negative. This implies that the image is erect. This also means that if the object is above the principal axis (h_O is positive), the image will also be above the principal axis (h_I is positive).

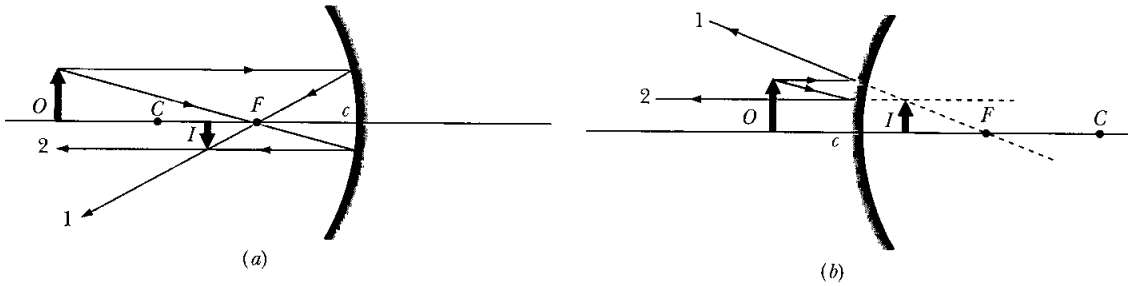


FIG. 34-34 (*a*) and (*b*) The real image of a real object is inverted but the virtual image of a real object is erect.

Case-III: If the object is virtual and the image is real, according to Eq. 34-2, we get

$$m = -\left(\frac{v}{u}\right) > 0.$$

Therefore, v is positive and u is positive. This implies that the image is erect.

Case-IV: If both the object and the image are virtual, according to Eq. 34-2, we get

$$m = -\left(\frac{v}{u}\right) < 0.$$

This implies that the image is inverted.

Although here we are discussing the case of image formation in a mirror, this happens in case of all the optical systems. This will be shown in each case: the combination of real object and real image (and virtual object and virtual image) is always inverted; and the combination of real object and virtual image (and virtual object and real image) is always erect. The same is shown in Figs. 34-34*a* and *b*.

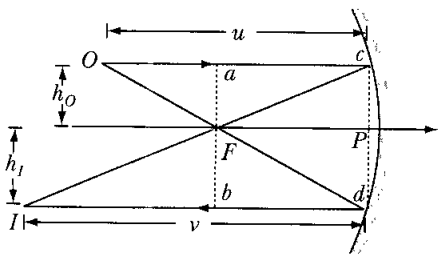


FIG. 34-35 Formation of image by a concave mirror. Can you guess the value of u and v from the ray diagram?

Mirror Formula

The Spherical Mirror Formula (Eq. 34-4)

Figure 34-35 shows a point object O placed above the central axis of a concave spherical mirror, between its center of curvature C and focus F. A large number of rays starting from O converge at point I to form an image. Here, we have shown only two of the rays for simplicity. Thus, I is the image of O; it is a *real* image. Let us find the image distance v.

There are two pairs of similar triangles in Fig. 34-35. The first pair is OaF and FPd. From the property of the similar triangles, we have

$$\frac{h_O}{f-u} = \frac{+h_I}{+f}$$

$$\frac{h_I}{h_O} = \frac{f}{f-u}. \tag{34-3}$$

Also from the triangles made by the IbF and FcP, we get

$$\frac{-h_I}{f-v} = \frac{h_O}{-f}$$

$$\frac{h_I}{h_O} = \frac{f-v}{f}. \tag{34-4}$$

Comparing Eqs. 34-3 and 34-4 we get

$$\frac{f}{f-u} = \frac{f-v}{f} \tag{34-5}$$

$$\frac{1}{1-(u/f)} = 1-\frac{v}{f}$$

$$f^2 = f^2 + uv - uf - vf$$
$$uv = uf + vf$$
$$\frac{uv}{f} = u + v$$
$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}. \tag{34-6}$$

This is commonly known as the mirror formula.

From Eq. 34-5, we have

$$f^2 = (f - v)(f - u).$$

If we write x_i = distance of image from focus and and x_o = distance of object from focus, then

$$x_o x_i = f^2. \tag{34-7}$$

This is what is known as the **Newton's formula**.

CHECKPOINT 4 If the object is on the right side of the focus, on what side of the focus will the image be? If the object is on the left side of the focus, on what side of the focus will the image be?

Finding the Details of the Image from 1/v versus 1/u Graph

Let us consider a concave mirror for which according to our sign convention, the focal length is negative. The mirror equation yields

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

Substituting $1/v = y$ and $1/u = x$, we obtain the equation of a straight line:

$$x + y = \frac{1}{f}.$$

This straight line has been plotted in Fig. 34-36. We note that for a real object $u < 0$, which means that $1/u$ is negative. Similarly, for a real image $v < 0$ and hence $1/v$ is negative. On the contrary, virtual object means $1/u$ is positive and virtual image means $1/v$ is positive.

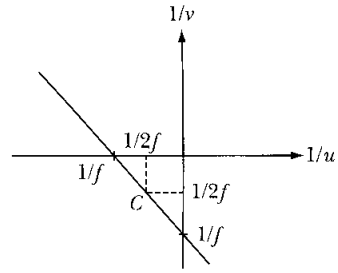


FIG. 34-36 $1/v$ versus $1/u$ graph for concave mirror.

Virtual image of virtual object would imply both x and y coordinates are positive. But no portion of the graph lies in that region. Hence, we can conclude that in a concave mirror, virtual object and virtual image combinations are not possible. Arguing along similar lines, we can say that in a convex mirror, real object cannot form a real image.

To see the magnification, we note that

$$m = -\frac{v}{u} = -\left(\frac{1/u}{1/v}\right) = -\frac{x}{y}.$$

In the second quadrant, where $1/u < 1/f$ or $|u| < |f|$ (focal length is negative), the image is virtual. Here, y coordinate is less than x coordinate. Therefore, $|m| > 1$.

Also when the object lies between pole and focus, the image is virtual, magnified, and erect. Similarly, we can deduce the information about other location of the object. Let us tabulate this finding. We divide the entire principal axis into four parts as shown in Fig. 34-37 and tabulate our findings in Table 34-1. Similarly,

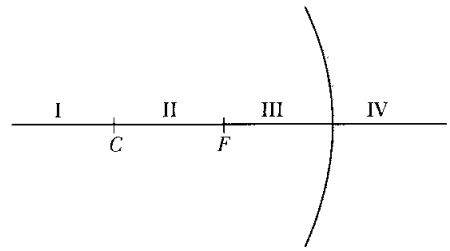


FIG. 34-37 The entire principal axis can be divided into four regions as shown in the figure. The region I, II, and III correspond to real side, and region IV refers to the virtual side.

TABLE 34-1

Concave Mirror: Nature of Object and Image

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Real Object & Real Image	Diminished	Inverted
II	I	Real Object & Real Image	Enlarged	Inverted
III	IV	Real Object & Virtual Image	Enlarged	Erect
IV	III	Virtual Object & Real Image	Diminished	Erect

TABLE 34-2

Convex Mirror: Nature of Object and Image

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Virtual Object & Virtual Image	Diminished	Inverted
II	I	Virtual Object & Virtual Image	Enlarged	Inverted
III	IV	Virtual Object & Real Image	Enlarged	Erect
IV	III	Real Object & Virtual Image	Diminished	Erect

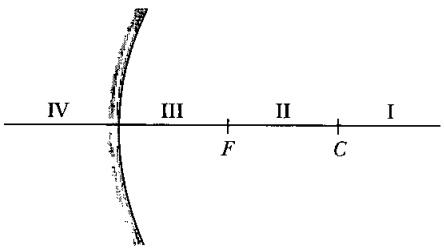


FIG. 34-38 The entire principal axis for the convex mirror can also be divided into four regions as shown in the figure. The region I, II, and III correspond to virtual side, and region IV refers to the real side.

for a convex mirror, the entire principal axis is divided into four parts as shown in Fig. 34-38 and the findings have been given in Table 34-2.

Some conclusions can be easily drawn from Tables 34-1 and 34-2. If the object moves along the principal axis, the image also moves along the principal axis. The object and image will cross each other at the center of curvature and the pole. At the point of crossing, the size of the object and the image are the same. At the center of curvature, the image is inverted ($m = -1$) and at the pole, the image is erect ($m = +1$). This can be understood in the following manner: At the pole, the spherical mirror will look like a plane mirror. Therefore, it will give erect image of the same size. Also $m = -v/u$. When u and v both approach 0, we should apply the limits to find that $m = +1$. This condition, where the object and the image coincide with each other is known as ***autocollimation***. It occurs when the object is at the center of curvature of the mirror or at its pole.

Sample Problem 34-7

A tarantula of height h sits cautiously before a spherical mirror whose focal length has absolute value $|f| = 40$ cm. The image of the tarantula produced by the mirror has the same orientation as the tarantula and has height $h' = 0.20h$.

(a) Is the image real or virtual, and is it on the same side of the mirror as the tarantula or the opposite side?

Because the image has the same orientation as the tarantula (the object), it must be virtual and on the opposite side of the mirror.

(b) Is the mirror concave or convex, and what is its focal length f, sign included?

Conceptualize/Classify: We *cannot* tell the type of mirror from the type of image because both types of mirror can produce virtual images. Similarly, we cannot tell the type of mirror from the sign of the focal length f, as obtained from Eq. 34-5 or Eq. 34-6, because we lack enough information to use either equation. However, we can make use of the magnification information.

Compute: We know that the ratio of image height h' to object height h is 0.20. Thus, from Eq. 34-7 we have

$$|m| = \frac{h'}{h} = 0.20.$$

Because the object and the image have the same orientation, we know that m must be positive: $m = +0.20$. Substituting this into Eq. 34-7 and solving for, say, v we get,

$$v = -0.20\,u,$$

which does not appear to be of help in finding f. However, it is helpful if we substitute it into Eq. 34-4 which yields

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}=\frac{1}{-0.20u}+\frac{1}{u}=\frac{1}{u}(-5+1),$$

from which we find

$$f=\frac{-u}{4}.$$

Now as u is negative, f must be positive, which means that the mirror is convex with

$$f=40\text{ cm}.$$

Conclude: In a concave mirror, an erect image will always be magnified. So if an erect image is formed which is diminished, it must be a convex mirror. The same thing can be understood by using a graph of u versus v (Figs. 34-39 and 34-40).

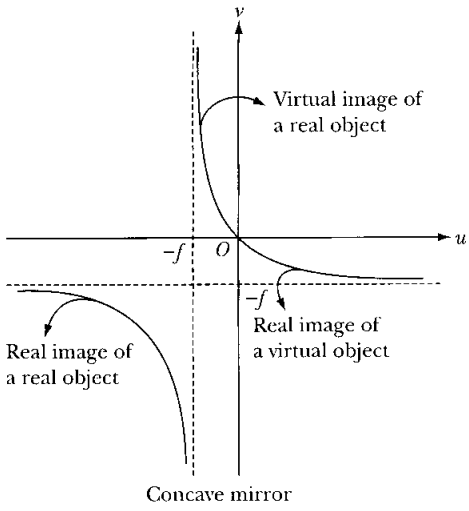


FIG. 34-39 u–v graph for concave mirror.

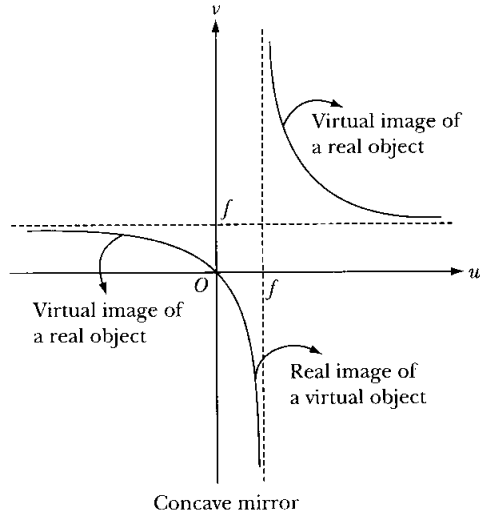


FIG. 34-40 u–v graph for convex mirror.

CHECKPOINT 5 A Central American vampire bat, dozing on the central axis of a spherical mirror, is magnified by $m=-4$. Is its image (a) real or virtual, (b) inverted or of the same orientation as the bat, (c) on the same side of the mirror as the bat or on the opposite side?

Locating Images by Drawing Rays

Figures 34-41*a* and *b* show an object O in front of a concave mirror. We can graphically locate the image of any off-axis point of the object by drawing a *ray diagram* with any two of the four special rays through the point:

1. A ray that is initially parallel to the central axis reflects through the focus F (ray 1 in Fig. 34-41*a*).
2. A ray that reflects from the mirror after passing through the focus emerges parallel to the principal axis (ray 2 in Fig. 34-41*a*).
3. A ray that reflects from the mirror after passing through the center of curvature C returns along itself (ray 3 in Fig. 34-11*b*). This essentially means that the line joining the object and the image passes through the center of curvature. The line joining the object and the image is always perpendicular to the mirror.
4. A ray that reflects from the mirror at point c is reflected symmetrically about that axis (ray 4 in Fig. 34-41*b*). From this, we can say that a parallel beam of light incident on the mirror will be forming an image at a height of $f\theta$ from the focus (in the focal plane) as shown in Fig. 34-42.

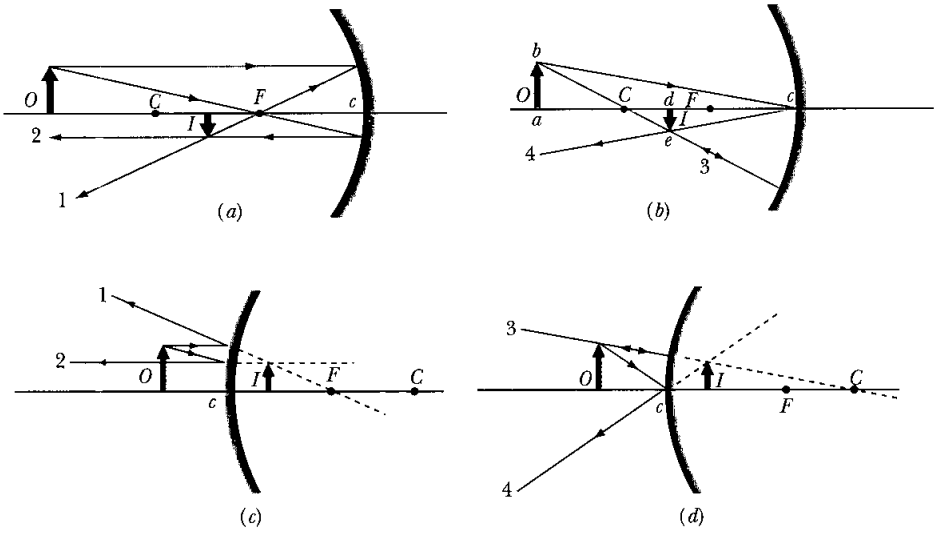


FIG. 34-41 (a, b) Four rays that may be drawn to find the image formed by a concave mirror. For the object position shown, the image is real, inverted, smaller than the object. (c, d) Four similar rays for the case of a convex mirror. For a convex mirror, the image is always virtual, oriented like the object, and smaller than the object. [In (c), ray 2 is initially directed toward focal point F. In (d), ray 3 is initially directed toward center of curvature C.]

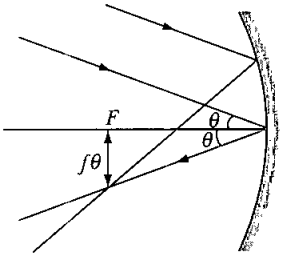


FIG. 34-42 Formation of an image in focal plane.

The image of the point is at the intersection of the two special rays you choose. The image of the object can then be found by locating the images of two or more of its off axis points. You need to modify the descriptions of the rays slightly to apply them to convex mirrors, as as shown in Figs. 34-41c and d.

Sample Problem 34-8

A light ray is incident on a concave mirror at an angle θ at a height h above the principal axis as shown in Fig. 34-43. Find the angle of deviation of the ray after reflection.

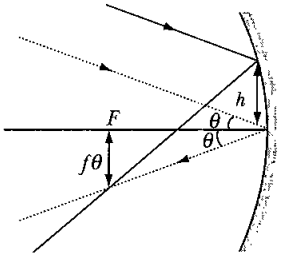


FIG. 34-43 Finding the path of a ray incident on a concave mirror. We draw an imaginary ray parallel to it incident at the pole for visualizing the image.

Conceptualize: Your mind must be throwing up a number of ways in which a ray diagram can be made. Here, we are showing only one method. Other methods should yield the same answer. We know that ray is neither passing through the focus nor the center of curvature. Therefore, we assume a ray parallel to it incident at the pole. This will make a set of rays parallel to each other. As discussed above, the reflected rays will meet at a point in the focal plane at a height $f\theta$ as shown as shown in Fig. 34-43.

Classify: Since here we are required to draw a ray which is not one of the principal rays, we have to make an imaginary ray to draw it.

Compute: So the angle made by the reflected ray with the horizontal is

$$\tan\phi = \frac{h + f\theta}{f} \approx \phi \qquad (\theta \text{ is small angle}).$$

So the angle of deviation is $\pi - (\phi + \theta)$.

Conclude: Many times to trace the image of an object, we draw imaginary rays to find the path followed by actual rays. This especially happens when we want to trace the image of a point object lying on the principal axis.

CHECKPOINT 6 Of the three types of mirrors: plane, concave, and convex, (a) which will have larger field of view, if aperture of all the three mirrors is the same? (b) Can you think of a practical application of this?

We have already stated that the lateral magnification or transverse magnification is the ratio of the height of image and the height of object. **Longitudinal magnification** is considered for an object lying along the principal axis or parallel to it.

We know in optics that to find the image of an extended object, we find the image of its end points. The image of the object will be the line joining the end points of the image. Suppose that the object is lying along the principal axis and its ends are at distances u_1 and u_2 from the pole, as seen in Fig. 34-44. Let us assume that the image of these end points is at v_1 and v_2, respectively. So the length of the object $l_O = (u_2 - u_1)$ and the length of the image $l_I = (v_2 - v_1)$.

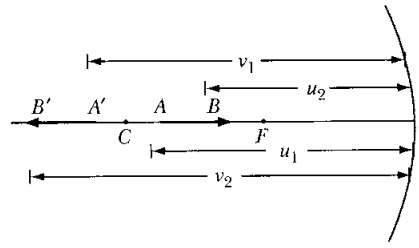


FIG. 34-44 The image of an object kept along the principal axis is laterally inverted.

Longitudinal magnification is defined as the ratio of the length of the object and the length of its image. Hence

$$\frac{l_I}{l_O} = m_L = \frac{v_2 - v_1}{u_2 - u_1}.$$

Note that in the case of mirror, longitudinal magnification will always come out to be negative because of **lateral inversion**. If we point our finger toward a mirror, the image will always point a finger at us. According to our sign convention, this means that if the object is oriented along the positive direction, the image will be oriented along the negative direction.

If the object is very small, we can replace $u_2 - u_1$ with du and $v_2 - v_1$ with dv, then m_L will be

$$m_L = \frac{dv}{du} = \frac{dv}{du} = \frac{d}{du}\left(\frac{uf}{u-f}\right),$$

$$\frac{du}{dv} = f\frac{[(u-f)-u]}{(u-f)^2} = \frac{-f^2}{(u-f)^2} = -m^2.$$

Therefore,

$$\frac{l_I}{l_O} = \text{Longitudinal magnification} = -m^2.$$

Remember that this is valid only for small objects.

Sample Problem 34-9

An L-shaped object with dimensions 1 cm each is kept on the principal axis of a concave mirror of focal length 10 cm as shown in Fig. 34-45. Its edge is 15 cm away from the pole. Find the location, shape, and size of the image.

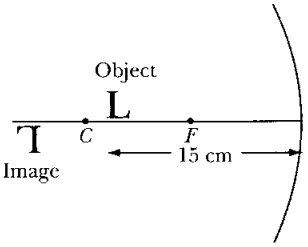


FIG. 34-45 An object kept on the axis of a mirror.

Conceptualize/Classify: Here, we can see that the object can be divided into two parts. For the part which is perpendicular to the principal axis, we can find the image by mirror formula and its magnification. The part of the image which lies on the principal axis has to be found with the help of concept of longitudinal magnification.

Compute: This problem can be solved by two methods:

Approximate method:

In the situation, we can take the object to be small.

$$\frac{h_I}{h_O} = -\frac{v}{u} = -\frac{f}{u-f} = \frac{10}{-15+10}.$$

So $h_I = -2$ cm. Also

$$\frac{l_I}{l_O} = -m^2 = -\left(\frac{f}{u-f}\right)^2 = -4$$

So $l_I = -4$ cm.

Exact method:

Here one end of L is at $u_1 = -15$ cm and another end at $u_2 = -14$ cm as seen in Fig. 34-45.

Solving by mirror formula for the image of these two points: $v_1 = -30$ cm and $v_2 = -35$ cm. So length of the image is 5 cm. Here, we can see that the length of the object is comparable to the focal length. So there is substantial error in length of the image.

Conclude: If we want to find the length of an image placed along the principal axis, we should not resort to the approximate method for longitudinal magnification unless the length of the object is very small in comparison to the focal length.

Velocity of Image with Respect to Mirror

Suppose the object and the mirror are both moving. Then naturally, the image might move. If we wish to find out the velocity of the image, we can apply the concepts that we have learnt till now. As usual, we take the x direction to be the principal axis.

In x direction, the location of the image is given by

$$v = \frac{uf}{u-f}.$$

We can say that v is the distance between the image and the mirror. If we differentiate both sides to find dv/dt, this derivative refers to velocity of image relative to the mirror in the x direction. Similarly, du/dt is the velocity of the object relative to mirror in the x direction.

$$v_{IM} = \frac{dv}{dt} = \frac{-f^2}{(u-f)^2} \times \frac{du}{dt}$$

$$v_{IM} = \frac{-f^2}{(u-f)^2} \times v_{OM}$$

But

$$-\frac{v}{u} = \frac{-f}{u-f} = m,$$

$$v_{IM} = -m^2 \times v_{OM}. \tag{34-8}$$

To find the velocity of the image in a direction perpendicular to the principal axis, we calculate

$$\frac{h_I}{h_O} = m = \frac{-f}{u-f}.$$

In y direction

$$h_I = h_O \times \frac{-f}{u-f}.$$

$$v_{Iy} = \frac{d}{dt}\left(\frac{-f h_O}{u-f}\right). \tag{34-9}$$

In kinematics, we have studied that the velocity of the objects in x and y direction are independent of each other. But here, the velocity of the image in y direction will depend on its x coordinate and velocity in x direction also. Remember that these velocities are with sign convention. So the velocity along positive x direction will be positive and so on.

Sample Problem 34-10

An object is traveling toward a concave mirror of focal length 10 cm. It moves with a velocity of 0.1 m/s parallel to the principal axis (Fig. 34-46). When it is at a distance of 15 cm from the mirror at a height of 1 cm from the principal axis, find the velocity of image in x and y direction?

Conceptualize/Classify: Here we note that the object is moving parallel to the principal axis. So although its height from the principal axis is constant, the image will be moving in x as well as y direction because the magnification is changing.

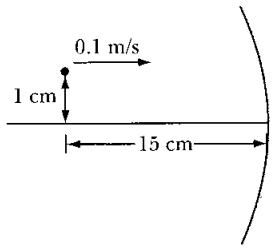


FIG. 34-46 An object is moving parallel to the principal axis with a speed of 10 cm/s.

Compute: $v_O = 0.1$ m/s $= 10$ cm/s. As we saw in Eq. 34-8

$$v_{Ix} = \frac{-f^2}{(u-f)} \times 10$$

$$= \frac{(10)^2}{(-15-10)^2} \times 0.10$$

$$= \frac{-100 \times 0.10}{25} = -0.40 \text{ m/s}.$$

For finding the velocity in y direction, we use Eq. 34-9:

$$v_{Iy} = \frac{d}{dt}\left(\frac{-fh_O}{u-f}\right) = \frac{-[f(u-f)(dh_O/dt) - fh_O(du/dt)]}{(u-f)^2}$$

$$= -f\frac{-[-(u-f)(dh_O/dt) + h_O(du/dt)]}{(u-f)^2}$$

$$= -\frac{(-10) \times 1 \times 0.1}{(-15-10)^2} = -\frac{1}{5^2} = -0.04 \text{ m/s}.$$

Conclude: The negative sign for the velocity of the image means that the image is moving away from the principal axis and farther from the pole. This is to be expected because the image is approaching the focus where its magnification is going to be infinite and its location is also infinity.

Sample Problem 34-11

A mirror is broken into two parts and these parts are separated by a distance of 1 cm as shown in Fig. 34-47. Find the location of the images formed by the two parts of mirror. The object is midway between the two principal axes (Fig. 34-47).

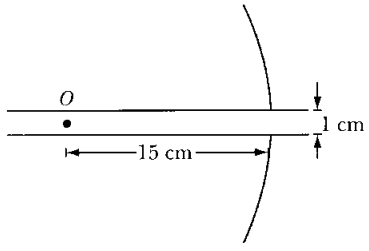


FIG. 34-47 A point object is kept midway between the principal axes of the two parts of the mirror. For the upper mirror, the object is below the principal axis and for the lower mirror, the object is above the principal axis.

Conceptualize/Classify: Here, we can see that for both the mirrors, the object is away from their respective principal axis. We can treat each individual mirrors to form the image separately.

Compute: Here for both the parts of the mirror, $u = -15$ cm. Focal length of both the parts of the mirror $= -10$ cm. So

$$v = \frac{uf}{u-f} = -30 \text{ cm}.$$

For the upper part, the object is 0.5 cm below the principal axis. The height of image can be found by the formula for magnification as

$$\frac{h_I}{h_O} = -\frac{v}{u} = -2.$$

So $$h_I = -2 \times -0.5 = 1 \text{ cm}.$$

Similarly the second image will be 1 cm below the principal axis.

Conclude: Dividing a mirror into two parts does not mean that the image is also halved. Remember that each point of the mirror reflects rays to form an image. Actually we use only principal rays to trace the image because they are easy to handle. It is obvious to see that if we use only two rays to form an image, the intensity of the image will be too little for us to perceive the image. Actually, halving the mirror area will lead to the power of the light rays forming the image being halved.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: It is difficult to thread a needle with one eye closed in comparison with keeping both eyes open.

Statement 2: A human eye cannot see an object clearly for an object placed at a distance smaller than a certain minimum distance.

2. Statement 1: It is not possible to photograph a virtual image.

Statement 2: An image is said to be virtual if the reflected or refracted rays are diverging in nature.

3. Statement 1: Pinhole camera forms an inverted image of an object.

Statement 2: Light travels in a straight line.

4. Statement 1: A convex mirror always forms a virtual image.

Statement 2: A diverging beam incident on a convex mirror is reflected as a diverging beam.

5. Statement 1: The formula connecting u, v, and f for a spherical mirror is valid only for mirrors whose sizes are very small compared to their radii of curvature.

Statement 2: Laws of reflection are strictly valid for plane surfaces, but not for large spherical surfaces.

MULTIPLE CHOICE QUESTIONS

sec. 34-3 Laws of Reflection

1. Two plane mirrors are arranged at right angles to each other as shown in Fig. 34-48. A ray of light is incident on the horizontal mirror at angle θ to the normal. For what value of θ the ray emerges parallel to the incoming ray after reflection from the vertical mirror.

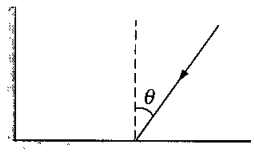


FIG. 34-48 Multiple choice question 1.

(a) 60°

(b) 30°

(c) 45°

(d) 70°

2. Two plane mirrors are inclined to each other at an angle 60°. If a ray of light incident on the first mirror is parallel to the second mirror, it is reflected from the second mirror

(a) Perpendicular to the first mirror

(b) Parallel to the first mirror

(c) Parallel to the second mirror

(d) Perpendicular to the second mirror

sec. 34-4 Formation of Image by a Mirror

3. A plane mirror is placed at origin parallel to y axis, facing the positive x axis. An object starts from (2 m, 0, 0) with a velocity of $(2\hat{i}+2\hat{j})$ m/s. The relative velocity of image with respect to object is along the

(a) Positive x axis

(b) Negative x axis

(c) Positive y axis

(d) Negative y axis

4. The person in Fig. 34-49 below is 2 m tall. A mirror is located 20 m away and is 0.5 m tall. The top of the mirror is at the same height as the top of the person's head. Assume for simplicity that the person's eyes are at the height of the top of his head. What part of his body does the person see in the mirror?

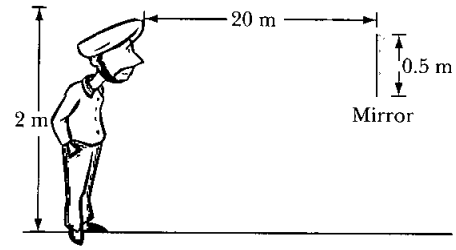


FIG. 34-49 Multiple choice question 4.

(a) Top 0.25 m

(b) Top 0.5 m

(c) Top 1m

(d) Entire 2 m

5. The number of roman capital letters such as A, B, C, D, ... which are not laterally inverted by a plane mirror is

(a) 6

(b) 7

(c) 11

(d) 13

6. From which point indicated in Fig. 34-50, if any, should you position your eyesight if you wish to see an image of an arrow in the mirror?

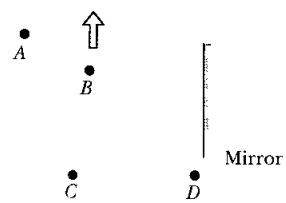


FIG. 34-50 Multiple choice question 6.

(a) Point A

(b) Point B

(c) Point C

(d) Point D

7. You are standing upright in a room in front of a vertical mirror. In this mirror, you can see from your position only the upper two-third of your body. You wish to see the entire length of your body reflected in the mirror. Which combination of the following three courses of action will achieve this?

(I) Move away from the mirror

(II) Move toward the mirror

(III) Use a mirror whose height will allow you to see your whole image

(a) (I) Only

(b) (II) Only

(c) (III) Only

(d) Either (I) or (III)

8. Two plane mirrors are placed parallel to each other at a distance L apart. A point object O placed between them, at a distance $L/3$ from one mirror. Both mirrors form multiple images. The distance between any two images cannot be

(a) $3L/2$

(b) $2L/3$

(c) $2L$

(d) None of these

9. The light reflected by a plane mirror may form a real image,

(a) If rays incident on the mirror are converging

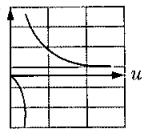
(b) If the rays incident on the mirror are diverging

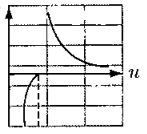
(c) If the object is placed very close to the mirror

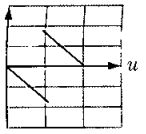
(d) Under no circumstances

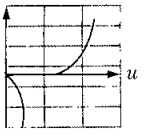
sec. 34-5 Formation of Image by a Spherical Mirror

10. As the position of an object (u) measured from a convex mirror varies, the position of the image (v) also varies. If u changes from 0 to $+\infty$, the graph between v and u will be

(a) 

(b) 

(c) 

(d) 

11. You walk slowly toward a large concave mirror. At first, you see your inverted image moving toward you. After you pass a certain point, you no longer see your image clearly. Moving still closer, you see a clear, enlarged and erect image of yourself behind the mirror. During the time when you cannot see a clear image

(a) You are closer to the mirror than the focal point and the image is now virtual and invisible

(b) You are between the center of curvature and the focal point, and the image is now behind you

(c) You are between the center of curvature and the focal point, and the image is now virtual and invisible

(d) You are closer to the mirror than the focal point and the image is now behind you

12. What can be the largest distance of an image of a real object from a convex mirror of radius of curvature 20 cm

(a) 10 cm

(b) Infinity

(c) 20 cm

(d) None

13. Two concave mirrors of the same focal length f are kept facing each other. A point source is kept symmetrically between them. What should be the distance between them so that only one image is formed at the source?

(a) f

(b) $2f$

(c) $3f$

(d) $4f$

14. Which pairs of rays (Fig. 34-51) from the object in the drawing are used to construct the image location produced by the convex spherical mirror of focal length f and radius R?

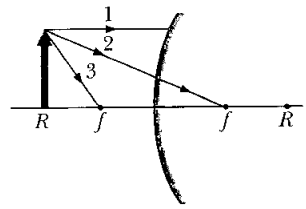


FIG. 34-51 Multiple choice question 14.

(a) 1 and 3

(b) 1 and 2

(c) 2 and 3

(d) Any pair of rays shown will work

15. A particle moves toward a concave mirror of focal length 30 cm along its axis and with a constant speed of 4 cm/s. What is the speed of its image when the particle is at 90 cm from the mirror ?

(a) 2 cm/s

(b) 8 cm/s

(c) 1 cm/s

(d) 4 cm/s

16. If a converging beam of light of incident on a concave mirror whose position can be adjusted then

(a) May form a real image

(b) Must form a real image

(c) May form a virtual image

(d) May be a parallel beam

17. A point objects P moves toward a convex mirror with a constant speed V, along its optics axis. The speed of the image

(a) Is always $\leq V$

(b) May be greater than, equal to, or smaller than V depending on the position of P

(c) Increases as P comes closer to the mirror

(d) Decreases as P comes closer to the mirror

PROBLEMS

sec. 34-3 Laws of Reflection

1. A ray of light makes an angle of 10° with the horizontal and strikes the mirror. Find the angle of inclination of this mirror to the horizontal if the ray of light is to be reflected vertically?

2. Prove that if a plane mirror is rotated through an angle α, the reflected beam is rotated through an angle 2α. Show that this result is reasonable for $\alpha = 45°$.

sec. 34-4 Formation of Image by a Mirror

3. A moth at about eye level is 10 cm in front of a plane mirror; you are behind the moth, 30 cm from the mirror. What is the distance between your eyes and the apparent position of the moth's image in the mirror.

4. A car moving at a speed of 110 km/h approaches a bus traveling at a speed of 80 km/h. What is the velocity of the image of the car in the plane rear view mirror as seen by a passenger on the bus.

5. Figure 34-52 shows an overhead view of a corridor with a plane mirror M mounted at one end. A burglar B sneaks along the corridor directly toward the center of the mirror. If $d = 3.0$ m, how far from the mirror will she be when the security guard S can first see her in the mirror.

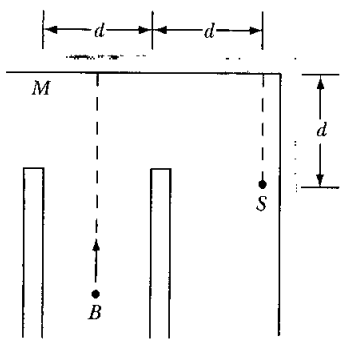


FIG. 34-52 Problem 5.

6. The plane mirrors are parallel to each other and spaced 20 cm apart. A luminous point is placed in between and 5 cm from one mirror. Determine the distance from each mirror of three nearest images in each.

7. Two plane mirrors are inclined at an angle of 100° to each other. If a point object is kept on their angle bisector, (a) find the number of images and (b) their angular coordinates with respect to one of the mirrors.

8. A fluorescent lamp of length 1 m is placed horizontally at a depth of 1.2 m below a ceiling. A plane mirror of length 0.60 m is placed below the lamp parallel to and symmetric to the lamp at a distance 2.4 m from it. Find the length of the reflected patch of light on the ceiling.

9. Two bodies A and B are moving toward a plane mirror with speeds v_A and v_B, respectively, as shown in Fig. 34-53. What is the speed of image of A with respect to the body B?

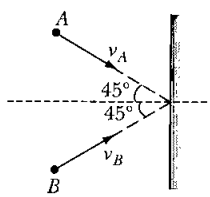


FIG. 34-53 Problem 9.

sec. 34-5 Formation of Image by a Spherical Mirror

10. A concave shaving mirror has a radius of curvature of 35.0 cm. It is positioned so that the (upright) image of a man's face is 2.50 times the size of the face. How far is the mirror from the face?

11. An object is 30.0 cm from a spherical mirror, along the mirror's central axis. The mirror produces an inverted image with a lateral magnification of absolute value 0.500. What is the focal length of the mirror?

12. A millipede sits 1.0 m in front of the nearest part of the surface of a shiny sphere of diameter 0.70 m. (a) How far from the surface does the millipede's image appear? (b) If the millipede's height is 2.0 mm, what is the image height? (c) Is the image inverted?

13. Converging rays fall on a convex spherical mirror with radius of curvature 56 cm so that the reflected rays converge on the principal axis of the mirror. The distance from the intersection of these reflected rays to the mirror is 20 cm. Find the intersection point of the rays without the mirror.

14. An object is placed before a concave mirror normal to its principal axis so that the magnification produced is 6/5. After the object has moved away from the mirror by 25 cm, the magnification becomes 2/5. Calculate the focal length of the mirror.

15. What will be the radius of the curvature of a concave mirror to form an image of the Sun 2 cm in diameter? The diameter of the sun may be assumed to be 1/100 of its distance from the Earth.

16. A small piece of wire bent into an L shape with upright and horizontal portions of equal lengths is placed with the horizontal portion along the axis of the concave mirror whose radius of curvature is 10 cm. If the bend is 20 cm from the pole of the mirror, then find the ratio of the lengths of the images of the upright and horizontal portions of the wire.

17. Two spherical mirrors (convex and concave) having the same focal length of 36 cm are arranged so that their optical axes coincide. The separation between the mirrors is 1 m (Fig. 34-54). At what distance from the concave mirror should an object be placed so that its images formed by the concave and convex mirror independently are identical in size?

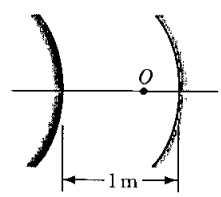


FIG. 34-54 Problem 17.

18. An object is placed exactly midway between a concave mirror of radius of curvature 40 cm and a convex mirror of radius of curvature 30 cm. The mirrors face each other and are 50 cm apart. Determine the nature and position of the image formed by the successive reflections (a) first at the concave mirror and (b) then at the convex mirror.

19. A police inspector is chasing a thief who is running away in a car with a speed of 3 m/s. The speed of police jeep is 12 m/s, then what is the speed of image of police jeep as seen by thief in the rear view mirror when police jeep is at a distance of 30 m. (Given that focal length of rear view mirror is 15 m.)

ADDITIONAL PROBLEMS

20. A point object is 10 cm away from a plane mirror, and the eye of an observer (with pupil diameter 5.0 mm) is 20 cm away. Assuming the eye and the object to be on the same line perpendicular to the mirror surface, find the area of the mirror used in observing the reflection of the point.

21. A point source of light is 60 cm from a screen and is kept at the focus of a concave mirror which reflects light on the screen. The focal length of the mirror is 20 cm. What is the ratio of average intensities of the illumination on the screen when the mirror is present and when the mirror is removed?

22. A plane mirror having a mass m is tied to the free end of a mass less spring constant k (Fig. 34-55). The other end of the spring is attached to a wall. The spring with the mirror held vertically to the floor can slide along it smoothly. When the spring is at its natural length, the mirror is found to be moving at a speed of v cm/s. What is the separation between the images of a man standing before the mirror, when the mirror is in its extreme position?

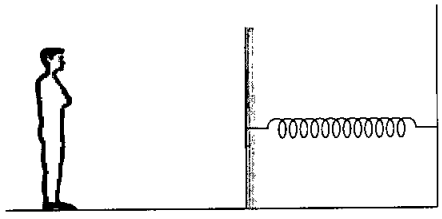


FIG. 34-55 Problem 22.

23. A bright source S is kept on the principal axis of a concave mirror of focal length $f = 20$ cm at a distance $d = 30$ cm from its pole. Find the distance in front of the concave mirror where a plane mirror should be placed so that the ray after being reflected first at the concave mirror and then by the plane mirror return to the point S. What would be your answer if the ray is first reflected at (a) the plane mirror and (b) then at the concave mirror?

24. Two rays are incident on a spherical mirror of radius of $R = 5$ cm parallel to its principal axis at the distance $h_1 = 0.5$ cm and $h_2 = 3$ cm. Determine the approximate distance Δx between the points at which these rays intersect the optical axis after being reflected at the mirror.

25. A small block of mass m and a concave mirror of radius 30 cm fitted with a stand lie on a smooth horizontal table with a separation 20 cm between them. The mirror together with its stand has mass m. The block is pushed at $t = 0$ toward the mirror at a constant speed 10 cm/s and collides with it. Then collision is perfectly elastic. Find the velocity of the image (a) at $t = 1$ s and (b) at $t = 3$ s.

ANSWERS

Checkpoints

1. When the cardboard is held close to the wall, the shadow formed is sharp because the light coming from different directions does not spread much behind the cardboard. Think about it: At a large distance, you will see an extended source as a point object only
2. 3
3. Three images at 150°, −90°, and at −150° to the x axis.
4. The object and image are always on the same side of the focus
5. (a) real; (b) inverted; (c) same
6. (a) Convex mirrors; (b) this is why they are used in rear view mirrors

Reasoning Type Questions

1. (a)
2. (d)
3. (a)
4. (d)
5. (c)

Multiple Choice Questions

1. (a), (b), (c), (d)
2. (b)
3. (b)
4. (c)
5. (a)
6. (c)
7. (c)
8. (a)
9. (a)
10. (a)
11. (b)
12. (a)
13. (b), (d)
14. (b)
15. (c)
16. (a), (c), (d)
17. (a), (c)

Problems

1. 40°

3. 40 cm

4. 50 km/h

5. 1.5 m

6. 5 cm, 35 cm, 45 cm and 15 cm, 25 cm, 55 cm

7. (a) Four images; (b) 150°, 210°, 250°, 310°

8. 3 m

9. $V_A + V_B$

10. 10.5 cm

11. 10.0 cm

12. (a) 0.15 m; (b) 0.30 m; (c) since $m > 0$, the image is erect

13. 11.67 cm

14. $f = 15$ cm

15. 400 cm

16. 3 : 1

17. 86 cm in front of concave mirror

18. (a) 100 cm; (b) 21.43 cm

19. 1 m/s

Additional Problems

20. 2.2 mm^2

21. 10 : 1

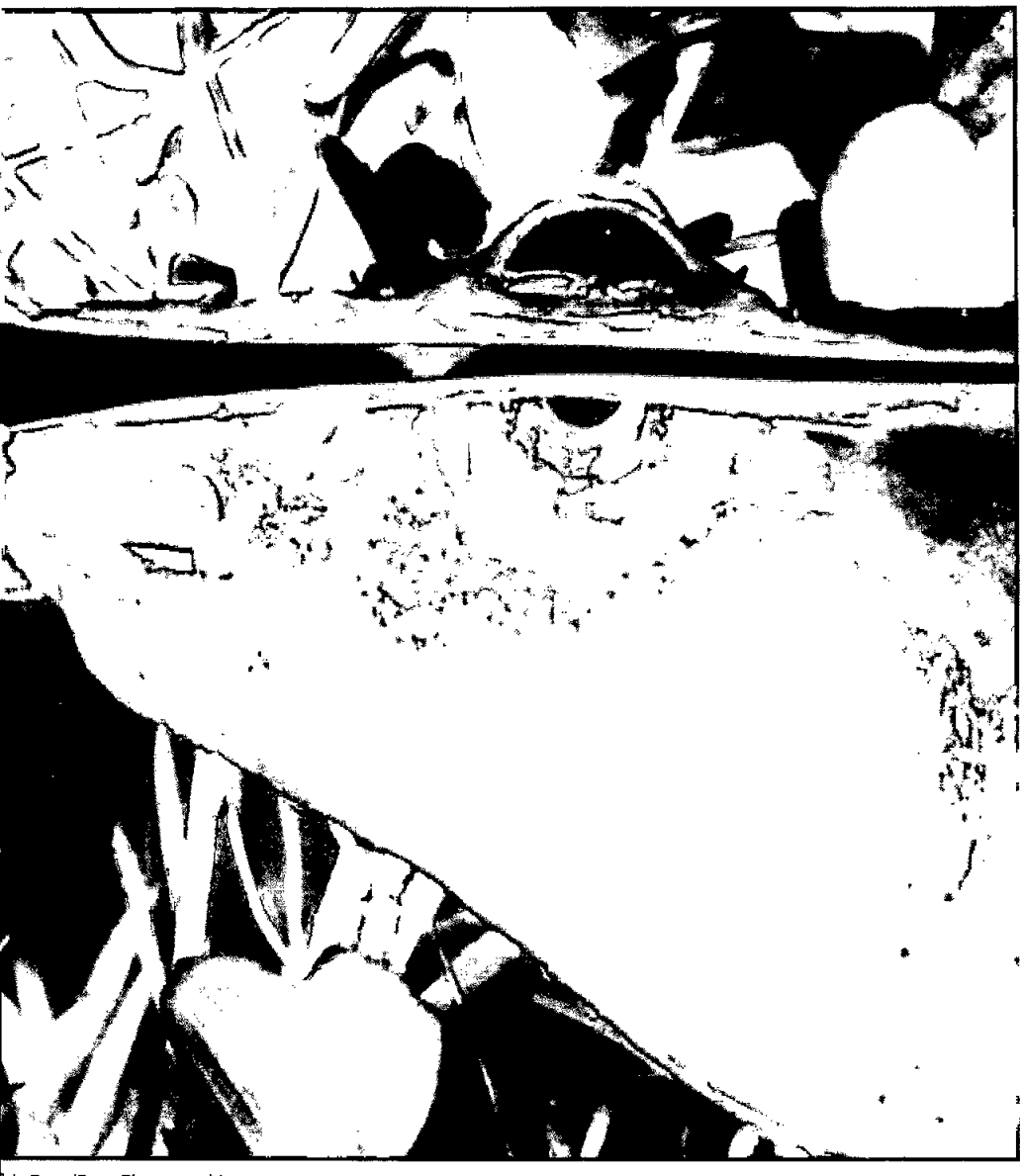
21. $4v\sqrt{m/k}$

23. (a) 45 cm, (b) same

24. 5/8 = 0.625 cm

25. (a) 90 cm/s, (b) 100 cm/s

Geometrical Optics: Refraction 35



dy Ryan/Ryan Photographic

Underwater vision is usually difficult even if you have perfect vision above water. The reason has to do with how water affects the refraction of light entering the eye. The refraction may be appropriate in the air but quite wrong in the water. However, the peculiar fish Anableps anableps swims with its eyes partially extending above the water surface so that it can see simultaneously above and below water.

How can its eyes function in both air and water?

The answer is in this chapter.

35-1 WHAT IS PHYSICS?

Refraction applications are ubiquitous. From the day we open our eyes, we see the world by refraction through our eye lens. When we are viewing very small objects, or very far ones, we make use of optical instruments which are based on laws of refraction. Even measuring the apparent depth of glass of water one is holding requires knowledge of laws of refraction laws. In this chapter, we are going to study the application of laws of refraction we had introduced in Chapter 33.

35-2 | Snell's Laws of Refraction

In Chapter 33, Section 33-8, we introduced the idea of refraction of light. To briefly recapitulate the idea, as light ray travels from one transparent medium to another transparent medium, it bends away from or toward the normal. Snell's law predicts a relationship between the angle of incidence and the angle of refraction.

Law of refraction: A refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{35-1}$$

Here, each of the symbols n_1 and n_2 is a dimensionless constant, known as the **index of refraction (or the refractive index)**, which is associated with a medium involved in the refraction.

If a light ray passes through many successive boundaries that are parallel to each other as shown in Fig. 35-1, then

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3. \tag{35-2}$$

This relationship is valid only if all the boundaries are parallel to each other. This is easy to see. The angle of refraction into one medium will be the angle of incidence into the next medium. So applying Snell's law successively from one medium to another, we arrive at Eq. 35-2.

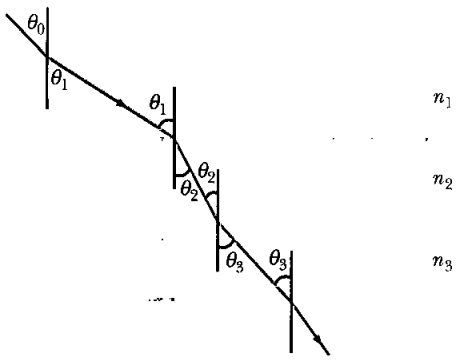


FIG. 35-1 Refraction of light through multiple slabs.

Sample Problem 35-1

A ray of light is incident on a parallel sided glass slab of refractive index n and thickness t as shown in Fig. 35-2. Find the lateral displacement of the ray as it passes through the slab.

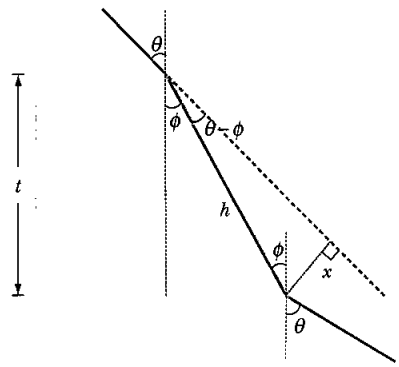


FIG. 35-2 A ray of light passes through a glass slab whose sides are parallel to each other. We wish to find the lateral displacement of the ray in passing through the slab.

Conceptualize: Once we apply Snell's law, all we need to do is apply geometry to the problem.

Classify: Let us start by drawing two triangles. From the second point from where the refracted ray enters back into air, we drop a perpendicular on the incident ray extended.

Compute:

$$1 \sin\theta = n \sin\phi \qquad \text{(from Snell's law).}$$

Here, we try to find the common hypotenuse (h) of both the triangle. From the definition of cosine and sine,

$$\cos\phi = \frac{t}{h} \Rightarrow h = \frac{t}{\cos\theta}.$$

Also,

$$\sin(\theta - \phi) = \frac{x}{h}.$$

Here, x is the lateral displacement as is clear from the figure. Therefore,

$$x = \frac{t\sin(\theta-\phi)}{\cos\phi}.$$

Conclude: If you look at the stars in the sky, they appear to be twinkling. This is due to slight variation in refractive index which changes the lateral displacement of rays coming from them.

Sample Problem 35-2

An observer is viewing along the line shown in Fig. 35-3. When there is no water filled in the tank, he can see only wall AB and no other part of the base of the tank. Then water is filled in till height h and he is just able to see the point E as shown in Fig. 35-3. Find the angle of refraction into air (θ) and angle of incidence from water (ϕ). Also find the depth to which water has been filled.

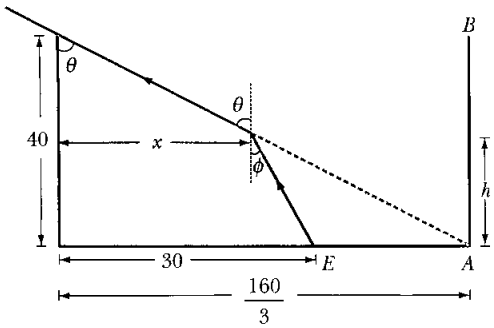


FIG. 35-3 When the water is filled in a tank, we can see the hidden part of the tank as well with the help of refracted rays.

Conceptualize/Classify: Here, we can see that refraction of ray enables the observer to see E. So we can apply Snell's law and geometry.

Compute: From Fig. 35-3, it is clear that the normal and wall AB are parallel to each other. So angle of refraction into air is

$$\tan\theta = \frac{160/3}{40} \Rightarrow \theta = 53°.$$

From Snell's law, we can say that

$$1\times\sin\theta = \frac{4}{3}\sin\phi \Rightarrow \phi = 37°.$$

From Fig. 35-3, we can say that

$$\tan 53° = \frac{x}{40-h} \quad \text{and} \quad \tan 37° = \frac{30-x}{h}.$$

Solving, we get

$$h = 40 \text{ cm}.$$

Sample Problem 35-3

A ray given by $-\hat{i}-2\hat{j}$ is incident on xz plane as shown in Fig. 35-4. Above the xz plane, refractive index is $n_1 = 2$ and below the xz plane refractive index is $n_2 = \sqrt{5/2}$. Find a unit vector in the direction of the refracted ray.

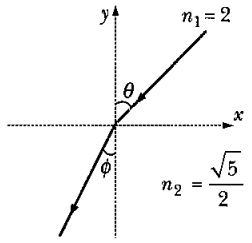


FIG. 35-4 A ray travelling in x–y plane is incident on a boundary which separates the two media. The boundary of the media is x–z plane.

Conceptualize: Note that although the incident ray is drawn in the first quadrant, its direction is in the third quadrant. This is because the direction of a vector is given by the direction in which its arrow points. To represent this vector in a vector form, we have to redraw it in such a way that its tail is at the origin. Note that refraction deviates a ray from its path; it does not change the quadrant. For the complete direction of the refracted ray, we can use Snell's law.

Classify: First we convert the incident ray vector into a unit vector by dividing it by its magnitude. Then we can apply Snell's law. As seen from Fig. 35-4, we can say that the tangential component of the incident ray is $1\sin\theta$ because the magnitude of the incident ray vector has been made unity.

Similarly, the tangential component of the refracted ray is given by $1\sin\phi$. By Snell's law,

$$n_1\sin\theta = n_2\sin\phi.$$

So we can say that

$n_1 \times$ tangential component of incident ray = $n_2 \times$ tangential component of refracted ray.

Once we get the tangential component of the refracted ray, we get the normal component by using the fact that the refracted ray is also a unit vector. So its magnitude is 1.

Compute: Converting the incident ray vector to a unit vector:

$$\hat{e}_i = \frac{(-\hat{i}-2\hat{j})}{\sqrt{(-1)^2+(-2)^2}} = -\frac{\hat{i}}{\sqrt{5}} - \frac{2}{\sqrt{5}}\hat{j}.$$

From Fig. 35-4, it can be seen that the normal is in the y direction.

The component of this vector in the tangential direction (x direction in this case) is $-\hat{i}/\sqrt{5}$. So by Snell's law, we can say that

$$-\frac{\hat{i}}{\sqrt{5}} \times 2 = \frac{\sqrt{5}}{2} \times \text{tangential component of refracted ray.}$$

$$\text{Tangential component of refracted ray} = -\frac{4\hat{i}}{5}.$$

Let the normal component of the refracted ray be $-c\hat{j}$. (The refracted ray should also travel in the third quadrant.) Therefore,

$$\hat{e}_r = -\frac{4\hat{i}}{5} - c\hat{j}.$$

So

$$1 = \sqrt{\left(-\frac{4}{5}\right)^2 + (-c)^2}$$

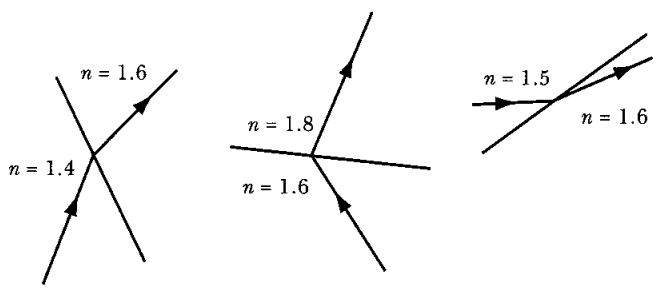
$$c = \frac{3}{5}.$$

Therefore,

$$\hat{e}_r = -\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}.$$

Conclude: In more involved problems, we can also involve the fact that incident ray, refracted ray, and normal, all three lie in the same plane. Hence, their scalar triple product is zero. If one finds it encouraging then one must try this out: $\hat{e}_i(\hat{e}_r \times \hat{n}) = 0$.

CHECKPOINT 1 Which of the three drawings here (if any) show physically possible refraction?



(*a*) (*b*) (*c*)

35-3 | Apparent Depth and Normal Shift

It is a common experience that when we view an object kept in water from above, the object seems to be shifted up. This happens due to refraction at the water surface. Thus, the real depth is different from apparent depth. This is clearly shown in Figs. 35-5*a–d*.

To understand this more clearly, consider Fig. 35-6. In it, rays are diverging from the real object O. For simplicity, we consider only two rays as shown in the figure. One of the rays is incident normally on the surface and therefore, does not deviate ($\theta = 0$; $\phi = 0$). The second ray goes from denser to rarer medium, so it bends away from the normal. These refracted rays meet at a point I. So I is the image. Since refracted rays are diverging, the image is virtual in nature. Let us find the value of the apparent depth.

In this figure, we have assumed that the rays go from denser to rarer medium. But the formula is equally valid for the other combinations also. The angle of

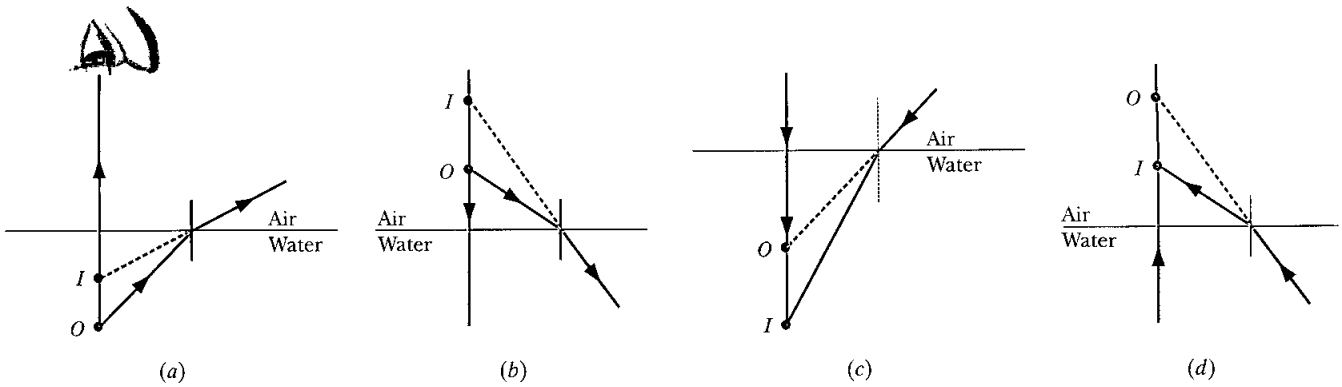


FIG. 35-5 (*a*) Real object at *O*, virtual image is formed at *I*. The observer is in air. (*b*) Real object is at *O*, virtual image is formed at *I*. The observer is in water. (*c*) Virtual object at *O*, real image formed at *I*. The observer is in water. (*d*) Virtual object at *O*, real image is formed at *I*. The observer is in air.

incidence is θ and angle of refraction is ϕ. The point at which refracted rays meet is called the image and the point at which incident rays meet is called the object.

Since the object is viewed normally and the pupil diameter is very small, only those rays enter our eyes which are very close to normal. So we can safely assume the angles involved (θ and ϕ) to be very small.

For small angles, we can say that

$$\theta = \frac{AB}{h_O} \text{ and } \phi = \frac{AB}{h_I}.$$

According to Snell's law, we can say that

$$n_r \sin \phi = n_i \sin \theta,$$

where n_i is the refractive index for the incident rays and n_r is the refractive index for the refracted rays.

For small angles, we can say that

$$n_r \phi = n_i \theta.$$

Solving we have

$$\frac{h_I}{n_r} = \frac{h_O}{n_i}. \tag{35-3}$$

If the object is viewed normally, there is no lateral shift. The shift is only in the normal direction. Here, we should note that the object and the image are always on the same side of the surface. So we should not take n_i as the refractive index corresponding to the object. Rather, it is the refractive index for the incident rays and n_r is the refractive index for the refracted rays. Be careful of the fact that the distances are always measured from surface. Here, we can note that a real object forms a virtual image and a virtual object forms a real image. Therefore, Eq. 35-3 is valid only when the object is viewed normally.

We have discussed in Chapter 34 that real–virtual combination will always be erect. Here, one can again see that this comes true.

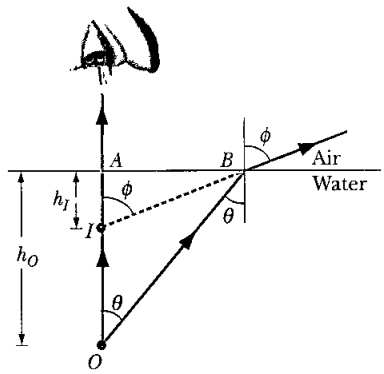


FIG. 35-6 An observer in air views an object *O* in water. The rays form an image at point *I*. In the figure, we have shown only two rays for simplicity. In reality, infinite rays can be back-tracked to form an image at *I*.

Sample Problem 35-4

A fish is floating in a tank filled with water to depth h. The bottom is silvered. A boy looks at the fish from above (Fig. 35-7).

(a) Find the apparent distance at which the boy sees the fish.

(a) Conceptualize: Notice that the boy is viewing the fish normally. The rays emanating from fish do not directly reach the boy; they undergo refraction at the water–air interface. It seems that the rays should not deviate at all when we are viewing normally. However,

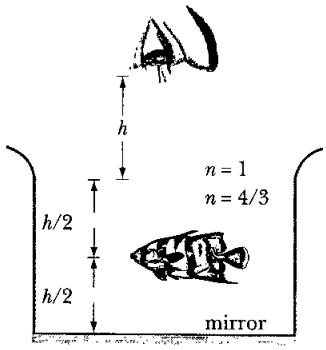


FIG. 35-7 A boy is viewing a fish kept in an aquarium whose bottom is made up of a plane mirror.

it is not one single ray that we are going to use for the formation of image. There is a narrow pencil of rays which refract to form an image.

Classify: Here, we can use the formula for the apparent depth (Eq. 35-3) that we derived just before.

Compute: Let us get the details straight: The rays are incident from water to air. So n_i is 4/3 and n_r is 1. The depth is measured from the water surface. So the depth is taken to be $h/2$. Now

$$\frac{d_I}{\mu_r} = \frac{d_O}{\mu_i}$$

$$\frac{d_{app}}{1} = \frac{h/2}{4/3}.$$

As we remarked before, this answer is the distance from the water surface.

Distance from the observer is

$$\frac{3h}{8} + h = \frac{11h}{8}.$$

(b) Find the apparent distance at which the boy sees the image of the fish (reflected in the mirror at the bottom).

Conceptualize: Here, we can see that first the rays from fish are reflected from the mirror. These reflected rays will again strike the water surface and after refraction from there, final image will be formed.

Classify: Here two events occur one after another.

Reflection from the mirror: For this, the object is at a distance $h/2$ from the mirror. So the image will be formed at a distance $h/2$ behind the mirror. Note that although this image will be formed behind the mirror, no refraction will occur to form this image. Therefore, we need not consider the apparent depth while considering this reflection. This image will act as an object for refraction from water surface. For this refraction, the distance is measured from the water surface.

Compute: The distance of this object from the water surface will be $h + (h/2) = 3h/2$. Therefore

$$\frac{d_{app}}{1} = \frac{3h/2}{4/3} = \frac{9h}{8}.$$

So distance from observer is

$$\frac{9h}{8} + h = \frac{17h}{8}.$$

Although the object here is in air (image behind the mirror), we need not consider the refractive index of air because in the formula for the apparent depth, we consider the refractive index for the incident rays and that for the refracted rays.

Conclude: If you are still having a doubt as to what refers to n_i and what refers to n_r, draw a ray diagram to see the formation of image.

✓**CHECKPOINT 2** Suppose now that the fish sees the boy. Find (a) apparent distance of eye as seen by fish. (b) apparent distance of image of eye as seen by fish.
(a) $11h/6$ from fish itself; (b) $17h/6$

Sample Problem 35-5

Find the location of the final image of O formed by the system (Fig. 35-8). Assume that O is on the principal axis of the concave mirror.

Conceptualize: Here, the image is formed after three events: refraction from water; reflection from mirror, and again refraction from water.

Classify: As we discussed in Chapter 34, image of one event will act as an object for the next event. Thus, first event we consider is refraction from water for which we can apply formula for apparent depth. Then next event will be refraction from concave mirror. The next event would be again refraction from water surface which will be the last event.

Compute: When the mirror sees the object O, it does not appear to be at 10 cm from the water surface. Rather, its apparent distance is

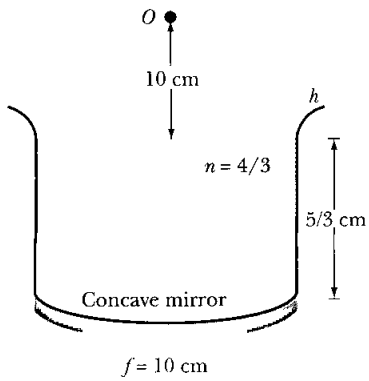


FIG. 35-8 Forming an image of object O by a concave mirror dipped in water.

$$\frac{d_{app}}{n_{refracted}} = \frac{d_{act}}{n_{incident}}$$

$$d_{app} = 10 \times \frac{4}{3} = \frac{40}{3} \text{ cm from water surface.}$$

The image formed by this refraction will act as an object for the next surface (mirror):

$$u = -\left(\frac{40}{3} + \frac{5}{3}\right) = -15 \text{ cm.}$$

Note that focal length of mirror ($r/2$) does not depend on the refractive index of surrounding medium.

The image is given by

$$v = \frac{uf}{u-f} = \frac{-15 \times -10}{-15-10} = -30 \text{ cm.}$$

These reflected rays will not actually be able to meet at 30 cm in front of the mirror. However, this image will act as an object for the next event: refraction from the water surface.

For this event, we have to measure distance of the object from the water surface:

$$\frac{d_{app}}{n_{refracted}} = \frac{d_{act}}{n_{incident}}.$$

In this event, the rays are incident from water and refracted into air:

$$\frac{d_{app}}{1} = \frac{85/3}{4/3}$$

$$d_{app} = \frac{85}{4} \text{ cm from water surface.}$$

Conclude: In all these examples, we can see that, because of refraction the depth of the object will seem to change but its transverse dimensions will not change.

So if an object is kept normal to the surface, its length will appear to be different from the actual length. Let us consider that an observer is observing an object of length y dipped in water (Fig. 35-9). Let us assume that water has refractive index n and the person is viewing it normally. The image of the upper end of the object is viewed by the observer to be at

$$\frac{d_{app}}{1} = \frac{d}{n}.$$

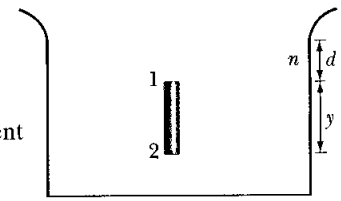


FIG 35-9 Finding the apparent length of an object dipped in water.

The image of the lower end of the object is viewed by the observer to be at

$$\frac{d_{app}}{1} = \frac{d+y}{n}.$$

Hence, the apparent length of the object will seem to be

$$\frac{d+y}{n} - \frac{d}{n} = \frac{y}{n}.$$

Sample Problem 35-6

A tank is filled with water up to a height of 15 cm and a fish is floating inside at a depth of 10 cm. The tank is now being drained so that the surface moves down with a velocity of 1 cm/s. An observer is viewing the fish from a height of 10 cm.

(a) Find the apparent velocity of the fish as seen by the observer.

(b) Find the apparent velocity of the observer as seen by the fish.

Conceptualize/Classify: If the object moves with a velocity V inside the medium, its apparent velocity can be found by rate of change of apparent depth.

Let us assume that x_{os} = depth of x with respect to surface. Then

$$\frac{d_{app}}{n_{refracted}} = \frac{d_{actual}}{n_{incident}}.$$

Here, the depth of image is changing because the depth of the object is changing with respect to surface.

Differentiating both sides, we get

$$\frac{V_{IS}}{n_r} = \frac{V_{OS}}{n_i} \Rightarrow \frac{V_I - V_S}{n_r} = \frac{V_O - V_S}{n_i}, \qquad (35\text{-}4)$$

where V_{IS} is the velocity of image with respect to the surface and V_{OS} is the velocity of image with respect to the surface. These refer to the velocities in the normal direction. In a

direction parallel to the surface, the velocity of the image will be the same as the velocity of the object.

Compute: Assuming downward direction to be positive, the velocity of the object (fish) is zero (fish is floating). Now $V_S = +1$ cm/s. Therefore

$$\frac{V_{IS}}{n_r} = \frac{V_{OS}}{n_i}$$

$$V_{IS} = \frac{n_r}{n_i} V_{OS} = \frac{1}{4/3} \times (V_O - V_S) - \frac{3}{4} \text{ m/s}$$

$$V_{IS} = V_I - V_S.$$

$$-\frac{3}{4} + 1 = V_I \Rightarrow V_I = \frac{1}{4} \text{ m/s.} \qquad (35\text{-}5)$$

Again

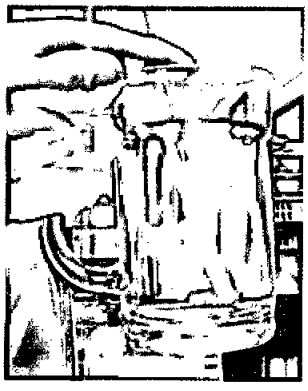
$$\frac{V_{IS}}{4/3} = \frac{V_{OS}}{1} = V_S \times \frac{4}{3} = \frac{-4}{3} \text{ m/s.}$$

$$V_I - V_S = \frac{4}{3}$$

$$V_I = 1 - \frac{-4}{3} = \frac{-1}{3} \text{ cm/s.} \qquad (35\text{-}6)$$

Conclude: Let us try to check this. The fish seems to be at a lesser depth than it actually is. So when depth of water decreases, the image of fish will approach its actual position. So V_I should be downward in Eq. 35-5. A similar argument can be applied to Eq. 35-6.

CHECKPOINT 3 We know that the light ray bends toward the normal when it goes from a rarer to a denser medium. This adjacent photograph seems to be wrong! The straw dipped in water seems to be bent away from the normal. Explain.



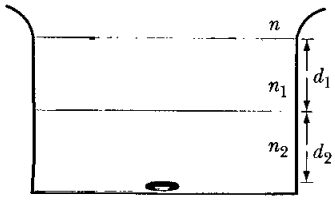


FIG. 35-10 Finding the apparent depth in the case where there are more than one medium.

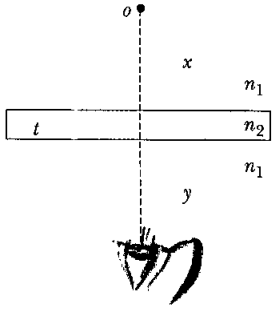


FIG. 35-11 Viewing an object through a glass slab makes the object look closer.

Normal Shift

If there are more than one medium separating the object and the observer (Fig. 35-10), then

$$\frac{d_{app}}{n} = \frac{d_1}{n_1} + \frac{d_2}{n_2}. \qquad (35\text{-}7)$$

Here n is the medium for final refracted rays as shown in Fig. 35-10.

Let us now try to visualize another situation related to refraction at plane surfaces. You might have observed that when you wear goggles, the whole world seems closer than it was before. This shift is due to refraction at the two plane surfaces of glass.

The shift of the object due to a thin parallel-sided slab placed on the path of light rays is called **normal shift**. The concept of normal shift is applicable only if the surface of the slab is perpendicular to the line joining the observer and the object. To understand this, let us assume that an object O is at a distance x from a slab of glass (Fig. 35-11). It is separated from the observer by a slab of refractive index n_2 and thickness t. Actual distance of the object from the observer is $x + t + y$.

Looking from medium II, the object seems to be closer than actual due to refraction at the two surfaces. From Eq. 35-7,

$$\frac{d_{app}}{n_1} = \frac{t}{n_2} + \frac{x}{n_1}$$

or

$$d_{app} = \frac{n_1 t}{n_2} + x.$$

This will be the depth of the object as seen from the surface of slab. According to the definition of normal shift,

$$\text{Normal shift} = d_{act} - d_{app} = x + t - \left(x + \frac{n_1 t}{n_2} \right)$$

$$= t\left(1 - \frac{n_1}{n_2} \right). \qquad (35\text{-}8)$$

Here t is the thickness of the slab, n_1 is the refractive index of the surroundings, and n_2 is the refractive index of the slab.

Let us try to understand Eq. 35-8 here. The equation says that the object seems to be shifted from its actual position by a distance called the normal shift. This formula is valid for any position of the slab between the observer and the object. (x does not appear in the equation.) Another interesting point is that the normal shift is positive and hence in the direction of propagation of the light if refractive index of slab is greater than that of the surrounding medium. Recall the Cartesian sign convention we introduced in Chapter 34. The direction of propagation of light is assumed to be the positive direction (Figs. 35-12*a* and *b*). On the contrary, the normal shift is opposite to direction of propagation of light if the slab is having lesser refractive index than the medium because normal shift will become negative.

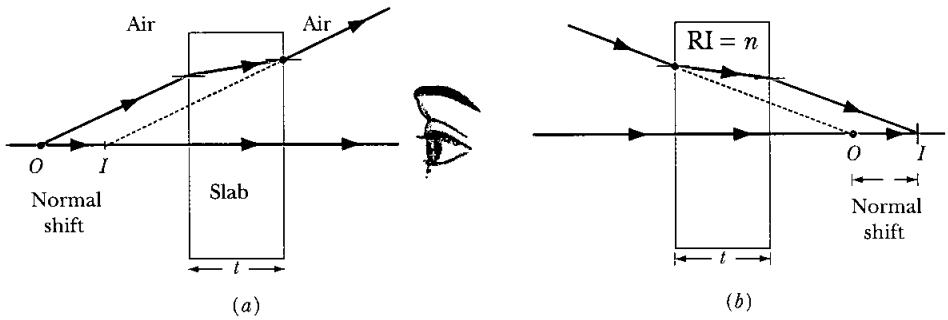


FIG. 35-12 (*a*) A real object at O is being viewed by an observer through a slab as shown. Due to refraction, the object is perceived by the eye to be at I. (*b*) Converging rays from a convex lens or a concave mirror are incident on the slab as shown. The incident rays would converge at O. Therefore, O is a virtual object for the glass slab. After refraction, the rays converge at I. Therefore, I is a real image.

Sample Problem 35-7

Find the location of final image as shown in Fig. 35-13. Take the refractive index of the slab as 4/3.

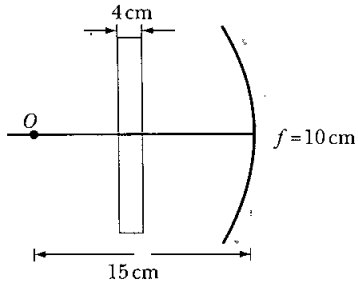


FIG. 35-13 An object O is located on the principal axis of a concave mirror. We wish to find the image of this optical system.

Conceptualize: This is again an example of multiple events before final image is formed. The rays coming from O will pass through the slab, get refracted by it and then reflected by the mirror, and again refracted by the slab. This refraction will form the final image. Whenever we refer to the image of the system, it refers to this image.

Classify: The mirror will not see the object at 15 cm but shifted toward it. After reflection, image will not be directly formed at the point where the mirror focuses the light but will be shifted in the direction of light incident on the slab.

Compute: The very first event is the refraction from the glass slab. It causes a shift in the direction of the incident rays. This normal shift is given by Eq. 35-8.

$$x = t\left(1-\frac{1}{n}\right)$$

$$= 4\times\left(1-\frac{3}{4}\right) = 1 \text{ cm}.$$

Thus, the object will seem to be shifted toward the mirror (in the direction of the incident rays). So the mirror will see the object at 15 – 1 = 14 cm. So for the mirror u = –14 cm. After reflection from the mirror,

$$v = \frac{14\times 10}{14-10} = -35 \text{ cm}.$$

One more time, refraction will occur as light after reflecting from mirror will pass through the slab.

Again normal shift will be 1 cm, but in the direction of propagation of light. Now the light rays are propagating in the left direction. Therefore, final image will be at a distance of 36 cm from the mirror.

Conclude: If there are more than one slabs, the total normal shift is

$$\text{Normal shift} = t_1\left[1-\frac{n}{n_1}\right]+t_2\left[1-\frac{n}{n_2}\right]+\cdots$$

CHECKPOINT 4 All along we have been considering the apparent depth of an object viewed normally. Suppose that the object is being viewed at an angle θ to the surface. If the actual depth is d and angle of ray inside water is ϕ, what is the apparent depth?

35-4 | Total Internal Reflection

We mentioned at the beginning of the chapter that when light is incident on a transparent surface, a part of the energy is reflected. Let us now consider the amount of reflected energy. The image of the face of a person in a good mirror is known to be brighter than that on the water surface in a lake or in a well. This is due to the fact that entire energy of light incident on the interface between two media is not reflected from it: a part of light penetrates through the interface into the second medium and passes through it or is partially absorbed by it. The fraction of the reflected energy depends on the optical properties of media and on the angle of incidence. If, for example, light is incident on a glass plate at right angles to its surface (the angle of incidence is zero), only about 5% of the energy is reflected, while 95% passes through the interface. As the angle of incidence increases, the fraction of the reflected energy increases.

At a particular angle of incidence the fraction of reflected energy may be even 100% as can be seen in Figs. 35-14*a* and *b*. Let us study this in some greater detail.

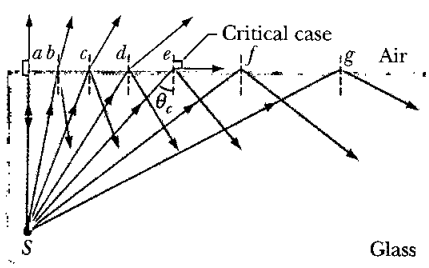
When a light ray travels from denser to a rarer medium, the light ray will bend away from normal, thus ray will bend at an angle $\phi > \theta$ (angle of incidence) to the normal 35-15*a*. If we keep on increasing the angle of incidence, the angle of refraction will also increase. At a particular value of angle θ, angle of refraction may even go to 90° (something we call as grazing emergence). This is depicted in Fig. 35-15.

In this situation,

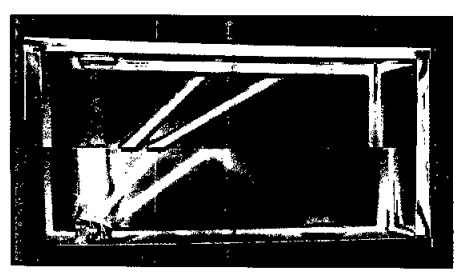
$$n_1 \sin\theta = n_2 \sin 90°. \tag{35-9}$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{35-10}$$

This phenomenon is called total internal reflection.



(*a*)



(*b*)

FIG. 35-14 (*a*) Total internal reflection of light from a point source S in glass occurs for all angles of incidence greater than the critical angle θ_c. At the critical angle, the refracted ray points along the air–glass interface. (*b*) A source in a tank of water. *(Ken Kay/ Fundamental Photographs)*

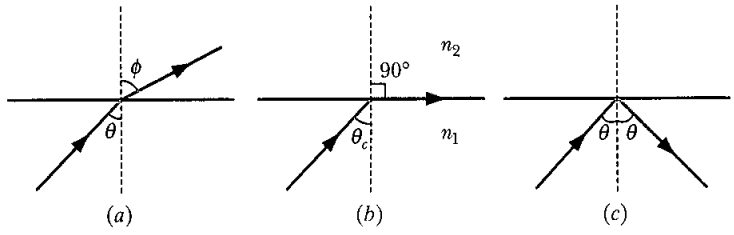


FIG. 35-15 (*a*) Refraction of a ray incident from denser to rarer medium. Note that the ray bends away from the normal. (*b*) The situation of total internal reflection of a ray incident from a medium of refractive index n_1 into a medium of refractive index n_2. Here, n_1 is a denser medium and n_2 is a rarer medium. (*c*) No light energy enters the rarer medium if the angle of incidence is greater than the critical angle θ_c.

Because the sine of an angle cannot exceed unity, n_2 cannot exceed n_1 in this equation. This restriction tells us that total internal reflection cannot occur when the incident light is in the medium of lower index of refraction. If source S were in the air as shown in Fig. 35-14*a*, all its rays that are incident on the air–glass interface (including f and g) would be both reflected and refracted at the interface. In this situation (ray incident from denser to rarer medium), let us plot the graph of angle of deviation versus angle of incidence. The graph is shown in Fig. 35-16.

Here, we can say that when the ray is incident at 0°, it will be refracted at 0°. So the angle of deviation is 0°. However, as we increase the angle of incidence, the angle of refraction also increases:

$$n \sin\theta = 1 \sin\phi.$$

So the angle of deviation is $\phi - \theta$. At the critical angle, angle of deviation becomes $(\pi/2) - \phi$. Beyond the critical angle, the ray is totally reflected so the angle of deviation follows the formula for reflection:

$$\delta = \pi - 2\theta.$$

This is the equation of a straight line. The maximum angle of incidence can be $\pi/2$. At that point, the angle of deviation becomes 0. Here, in Fig. 35-16, we can see that the deviation from 0 to $(\pi/2) - \theta_c$ can be obtained at two angles of incidence. The smaller angle corresponds to refraction and the larger angle corresponds to total internal reflection.

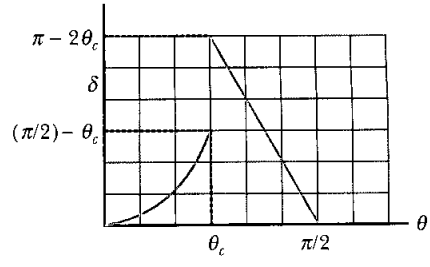


FIG. 35-16 The angle of deviation versus the angle of incidence graph of a ray incident from a denser medium to a rarer medium. Beyond the critical angle, the deviation F is due to reflection.

Sample Problem 35-8

The purpose of a diamond on a ring is, of course, to sparkle. Part of the art of cutting a diamond is to ensure that all the light entering through the top face or side facets leaves through those surfaces, to participate in the sparkle. Figure 35-17 shows part of a cross-sectional slice through a brilliant-cut diamond, with a ray entering at point A on the top face. In this type of cut, the top and bottom surfaces have normal lines that intersect at the indicated 48.84°. At point B, at least part of the light reflects and leaves the diamond properly, but part could refract and thus leak out of the diamond. Consider a light ray incident at angle $\theta_1 = 40°$ at A. Does light leak at B if air ($n_4 = 1.00$) lies next to the bottom surface? Does light leak if greasy grime ($n_4 = 1.63$) coats the surface? The index of refraction of diamond is $n_{dia} = 2.419$.

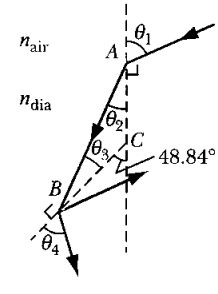


FIG. 35-17 Light entering the top of a brilliant-cut diamond. Can the light leak from the bottom surface at point B?

Conceptualize: When light reaches the interface between two materials with different indexes of refraction (call them n_1 and n_2), part or all of the light is reflected. The reflection is total if (1) the incident light is in the material with the *higher* index of refraction

($n_1 > n_2$) and (2) the incident angle exceeds a critical value given by Eq. 35-10:

$$\theta = \sin^{-1}\left(\frac{n_1}{n_2}\right).$$

If those two conditions are not met, part of the light is refracted across the interface according to Snell's law.

Classify: Clean diamond: We need to follow the light from point A to point B to see if it can leak at point B. The light incident at point A is in the material (air, $n_1 = 1.00$) with a *lower* index of refraction than the material on the other side of the interface (diamond, $n_{dia} = 2.419$). Thus, some of the light is refracted across the interface (the reflected portion is not shown in Fig. 35-17), and we find the angle of refraction from Snell's law.

Compute:

$$\theta_2 = \sin^{-1}\left(\frac{\sin 40°}{2.419}\right) = 15.41°.$$

Now note that the given angle of 48.84° at point C is an *exterior angle* of triangle ABC and thus (from Appendix E), we can write

$$\theta_3 + \theta_2 = 48.84°$$

or

$$\theta_3 = 48.84° - \theta_2 = 33.43°.$$

This is the angle of incidence of the ray at point B. Now the incident light is in the material (diamond) with the *greater* index of refraction than the material (air) on the other side of the interface. However, if we apply Snell's law at point B, we find no answer. The reason is that the light is incident at an angle greater than the critical value of

$$\theta_c = \sin^{-1}\left(\frac{n_4}{n_{dia}}\right) = \sin^{-1}\left(\frac{1.00}{2.419}\right) = 24.4°.$$

Thus, all the light reaching B reflects and none leaks out in the air.

Grimy diamond: We again follow the light from A to B, with the only difference lying in the last two calculations.

Now at B, the material on the other side of the interface is grime with an index $n_4 = 1.63$. The incident light is again in the material with the greater index, but now the critical angle is

$$\theta_c = \sin^{-1}\left(\frac{n_4}{n_{dia}}\right) = \sin^{-1}\left(\frac{1.63}{2.419}\right) = 42.4°.$$

Therefore, the incident angle $n_3 = 33.43°$ is *less* than the critical angle and light leaks through the bottom of the diamond.

From Snell's law, the angle of refraction is

$$\theta_4 = \sin^{-1}\left(\frac{2.419}{1.63}\sin 33.43°\right) = 54.8°.$$

Conclude: Thus, to keep a diamond sparkling, clean both the top and bottom surfaces.

Sample Problem 35-9

An optical fiber consists of a cylindrical core of unknown refractive index kept in air. A ray of light is incident on the flat face of the core so that it gets totally reflected at the curved surface (Fig. 35-18). What is the possible refractive index if the ray is totally reflected at any angle of incidence ϕ from air?

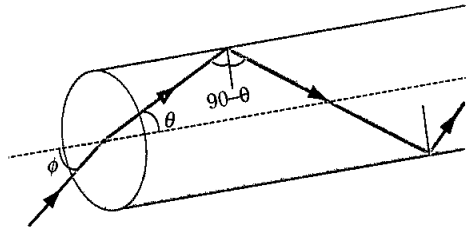


FIG. 35-18 A light ray is incident from air on a cylindrical optical fiber.

Conceptualize/Classify: When a ray refracts from air to glass, total internal reflection cannot occur. However, when this ray strikes curved surface of the fiber, the incidence is from denser to rarer medium. If n is such that total internal reflection occurs at this point, it will continue to occur as the ray strikes the curved surface repeatedly at the same angle.

Let us assume that the ray is incident at an angle ϕ from air and is refracted into the cylinder at an angle θ. It is then again incident on the air cylinder boundary at the curved surface as shown in Fig. 35-18. The condition is that the ray should undergo total internal reflection at the face.

For this, the angle of incidence here should be greater than critical angle.

$$90° - \theta > \theta_c.$$

Since sine is an increasing function, taking sine of both sides will not change the inequality.

$$\sin(90° - \theta) > \sin\theta_c = \frac{1}{n} \quad (n\sin\theta_c = 1 \times \sin 90°).$$

$$\cos\theta \geq \frac{1}{n}.$$

Again, applying Snell's law at flat face, we get

$$1 \times \sin\phi = n \times \sin\theta.$$

Squaring and adding, we get,

$$n^2 \sin^2\theta + n^2 \cos^2\theta \geq 1 + \sin^2\phi$$
$$n^2 \geq 1 + \sin^2\phi.$$

But the maximum value of $\sin\phi$ is 1 when the angle of incidence is 90°. For this condition,

$$n^2 \geq 2$$

or

$$n \geq \sqrt{2}.$$

Conclude: Interestingly, this yield $\mu \geq 1.41$. Thus, even ordinary glass can be used in optical fiber. But for practical reasons, this core is coated with a cylindrical shell called **cladding**. If cladding has refractive index of μ_1, the condition on μ_2 is

$$\mu_2 \geq \sqrt{\mu_1^2 + 1}.$$

Total internal reflection has found many applications in medical technology. For example, a physician can view the interior of an artery of a patient by running two thin bundles of *optical fibers* through the chest wall and into an artery (Fig. 35-19). Light introduced at the outer end of one bundle undergoes repeated total internal reflection within the fibers so that, even though the bundle provides a curved path, most of the light ends up exiting the other end and illuminating the interior of the artery. Some of the light reflected from the interior then comes back up, the second bundle in a similar way, to be detected and converted to an image on a monitor's screen for the physician to view.



FIG. 35-19 An endoscope used to inspect an artery. *(©Laurent/Phototake)*

Sample Problem 35-10

A point source of light is kept in a vessel which is filled with liquid (of refractive index 1.25) up to a depth of 1m (Fig. 35-20*a*). The light rays come out from the surface of water and make a circular patch on the surface (Fig. 35-20*b*).

(a) The radius of the circular patch through which the light can escape.

Conceptualize/Classify: Just vertically above the point source, the angle of incidence is 0°. As we move sideways, the angle of incidence will increase. So the angle of refraction will also increase. Since the rays are going from denser to the rarer medium, the rays will bend away from the normal. At the position where the angle of refraction becomes 90°, the angle of incidence will be critical angle. A little visualization says that this condition will be achieved at all the points on a circle just above the source.

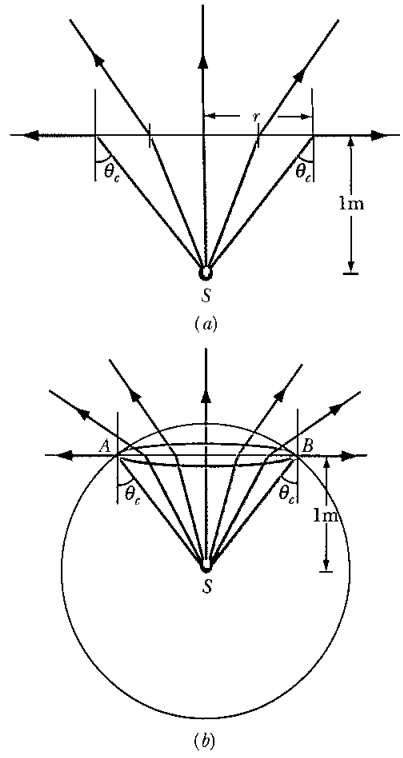


FIG. 35-20 (*a*) A source of light *S* is kept under water. Light is incident from it on water–air interface and gets refracted and totally reflected. (*b*) Light beam comes out of the water surface from a circular path known as the circle of illuminance.

Compute: From Eq. 35-10 $[\theta_c = \sin^{-1}(n_1/n_2)]$, we get

$$1.25 \sin\theta_c = 1 \sin 90°.$$

Therefore,

$$\theta_c = 53°.$$

From the triangle made by the radius of the circle and the vertical line, we can say that

$$\tan\theta = \frac{r}{1}$$

or

$$r = 1 \tan\theta = \frac{4}{3}.$$

(b) If the power of source is "P" find the power coming out of surface ($n_{water} = 4/3$).

Conceptualize/Classify: Here, we are required to find the part of the power which is incident on the circular patch whose radius we found just now. An associated concept here is the concept of intensity. If a source of power P is kept at a certain point, the light energy emitted by it will be transmitted equally in all the directions. So we can say that after some time, this light energy will be distributed on the surface of a sphere of radius $R = ct$.

Intensity at that point is defined as

$$\frac{\text{power}}{\text{area over which light is distributed}} = \frac{P}{4\pi R^2},$$

where R is the distance from source at which we are measuring the intensity.

It is clear here that if a disk is kept in the path of the light rays (this disk may be a mirror, a slab, etc.), a part of the light energy will be incident on the disk. If we wish to find the amount of light energy incident on the disk, we should note that disk is obstructing that part of the light energy which would have been distributed on the part of the sphere denoted by AB (see Fig. 35-20*b*).

Here we can see that since the light energy is equally distributed on all parts of the sphere, the amount of light energy incident on the disk is proportional to area A of the spherical portion above the water surface:

$$\frac{P_s}{P} = \frac{A}{4\pi R^2}.$$

Recall the concept of solid angle discussed in Chapter 23. There we showed how solid angle

$$\Omega = \frac{A}{R^2} = 2\pi(1 - \cos\theta).$$

To find the power passing through this circular patch, we should find the power emitted by the source in the cone of half angle $\theta_c = 53°$.

From the concept discussed above, the power contained in this cone is in proportion of solid angle subtended by the cone at the source. Therefore,

$$\Omega \text{ (solid angle)} = 2\pi(1 - \cos\theta) = \frac{A}{r^2}.$$

The solid angle is measured in steredians. Now

$$P_S = P \times \frac{\Omega}{4\pi}$$

$$= \frac{P}{4\pi} 2\pi(1 - \cos\theta) = \frac{3P}{10}.$$

Conclude: The actual power will be somewhat less than this value because we have neglected the partial reflection at the water surface.

A Common Mirage

A common example of a virtual image is a pool of water that appears to lie on the road some distance ahead of you on a sunny day that you can never reach. The pool is a *mirage* (a type of illusion) formed by light rays coming from the low section of the sky in front of you (Fig. 35-21*a*). As the rays approach the road, they travel through progressively warmer air that has been heated by the road, which is usually relatively warm. With an increase in air temperature, the index of refraction of the air decreases slightly. Thus, as the rays descend, encountering progressively smaller indexes of refraction, they continuously bend toward the horizontal (Fig. 35-21*b*).

Once a ray is horizontal, somewhat above the road's surface, it still bends because the lower portion of each associated wavefront is in slightly warmer air and is moving slightly faster than the upper portion of the wavefront (Fig. 35-21*c*). This nonuniform motion of the wavefronts bends the ray upward. As the ray then ascends, it continues to bend upward through progressively greater indexes of refraction (Fig. 35-21*d*).

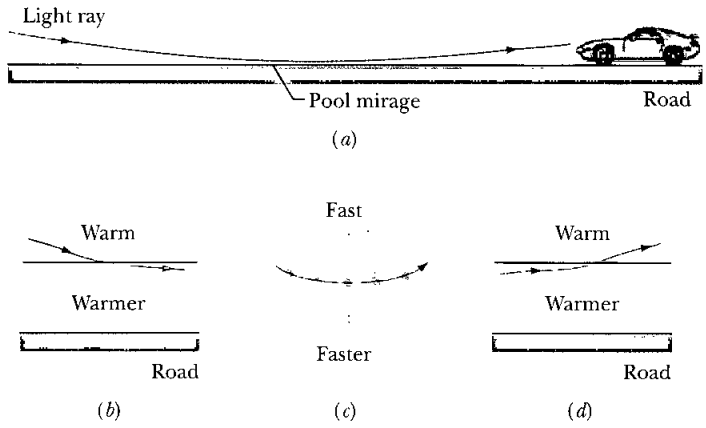
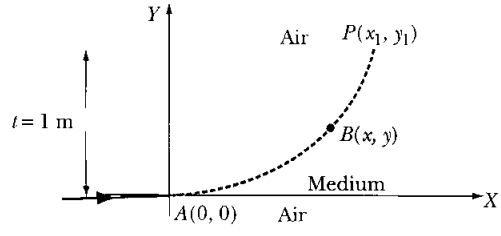


FIG. 35-21 (*a*) A ray from a low section of the sky refracts through air that is heated by a road (without reaching the road). An observer who intercepts the light perceives it to be from a pool of water on the road. (*b*) Bending (exaggerated) of a light ray descending across an imaginary boundary from warm air to warmer air. (*c*) Shifting of wavefronts and associated bending of a ray, which occur because the lower ends of wavefronts move faster in warmer air. (*d*) Bending of a ray ascending across an imaginary boundary to warm air from warmer air.

If you intercept some of this light, your visual system automatically infers that it originated along a backward extension of the rays you have intercepted and, to make sense of the light, assumes that it came from the road surface. If the light happens to be bluish from blue sky, the mirage appears bluish, like water. Because the air is probably turbulent due to the heating, the mirage shimmers, as if water waves were present. The bluish coloring and the shimmer enhance the illusion of a pool of water. However, one is actually seeing a virtual image of a low section of the sky.

Sample Problem 35-11

A ray of light traveling in air is incident at grazing angle (incident angle = 90°) on a long rectangular slab of a transparent medium of thickness $t = 1.0$ m (Fig. 35-22*a*). The point of incidence is the origin A (0, 0). The medium has a variable index of refraction given by $n = (ky^{3/2} + 1)^{1/2}$, where $k = 1.0\ (\text{m})^{-3/2}$. The refractive index of air is 1.0.



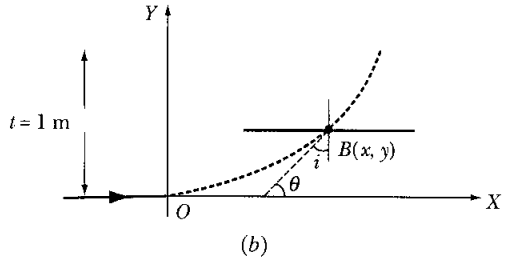


FIG. 35-22 (*a*) A light ray enters a medium at grazing incidence. Note that at the xz plane, the refractive index of the medium is 1.00, so the ray does not bend there. As the ray proceeds upward, the refractive index of the medium increases and hence the ray progressively bends toward the normal. (*b*) A ray being considered at an arbitrary point B (x, y) in the medium.

(a) Obtain a relation between the slope of the trajectory of the ray at a point $B(x, y)$ in the medium and the incident angle at that point.

(b) Obtain an equation for the trajectory $y(x)$ of the ray in the medium.

(c) Determine the coordinates (x_1, y_1) of the point P, where the ray intersects the upper surface of the slab–air boundary.

(d) Indicate the path of the ray subsequently.

Conceptualize: In situation where refractive index changes only in one direction, medium can be considered as a collection of large number of the thin layers. Let refractive index be a function of y, that is, $n = f(y)$ then medium can be considered as to made up of a large number of thin slabs placed parallel to x axis and normal at any surface is parallel to y axis.

Classify: Although the refractive index is not constant, we cannot apply Snell's laws here, but still we can divide the whole medium into thin layers as shown in Fig. 35-22. Within a layer, refractive index is constant. So here we have a case of many parallel slabs. We can apply Snell's law here.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3 = \ldots$$

$$n \sin\theta = \text{constant}.$$

Compute:

(a) From Fig. 35-22*b*,

$$i + \theta = 90°$$

$$\text{Slope of the tangent} = \tan\theta = \tan(90 - i) = \cot i.$$

(b) We know that at any point in the medium with variable refractive index $n \sin i$ = constant, where n = refractive index at any point y and i = angle of incidence at any point y. Therefore,

$$n \sin i = 1 \sin 90°$$

$$\sin i = \frac{1}{\operatorname{cosec} i} = \frac{1}{\sqrt{1+\cot^2 i}}.$$

Substituting the values $(ky^{3/2} + 1)^{1/2}$, we get

$$\frac{1}{\sqrt{1+m^2}} = 1$$

$$ky^{3/2} + 1 = 1 + m^2.$$

$$\text{Slope } m = \frac{dy}{dx} = \sqrt{k}\, y^{3/4}$$

$$y^{-3/4} dy = \sqrt{k}\, dx.$$

On integrating both sides, we get

$$\int_0^y y^{-3/4} dy = \int_0^x \sqrt{k}\, dx$$

$$4y^{1/4} = x \quad (\text{since } k = 1)$$

$$y = \frac{x^4}{256}.$$

(c) For $y = 1$ m, $x = 4$ m.

(d) Let α_e be the angle of emergence in the air, then by Snell's law,

$$n_e \sin\alpha_e = n_1 \sin\alpha_1 = 1 \quad (\text{refractive index of air, } n_e = 1).$$

This gives

$$\sin\alpha_e = 1 \Rightarrow \alpha_e = 90°.$$

Conclude: Thus, it emerges parallel to the incident ray; this could have been interpreted by Snell's law too.

35-5 | Refraction at Spherical Surfaces

Till now we have been considering the refracting surface to be a plane. A more frequent case is a curved refracting surface, specifically spherical. In such a situation, the concept of apparent depth or normal shift cannot be applied.

Let us consider an object O kept along the principal axis of the spherical refracting surface as shown in Fig. 35-23. The principal axis is defined by the line along the radius of the surface. In the figure, one of the rays is assumed to be incident along the principal axis. This ray will pass undeviated because the angle of incidence is 0°. Another ray is required to trace the image. Applying Snell's law for this ray, we obtain

$$n_1 \sin\theta = n_2 \sin\phi.$$

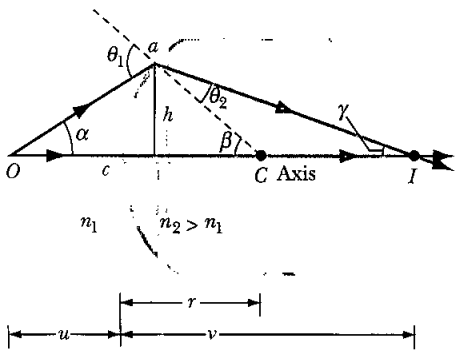


FIG. 35-23 A real point object O kept on the principal axis forms a real image I on the principal axis. A large number of paraxial rays form the image, but only two are shown for simplicity.

We know that if a surface is spherical, the normal will always pass through the center of curvature C. Let the angle made by the normal with the principal axis be β. This angle will be the exterior angle for the triangle formed by I, C, and a. So the angle at I is $\beta - \theta_2$. From the triangle formed by the image and the vertical height h, we can say, according to small angle approximation, that

$$\gamma = \beta - \theta_2 \approx \tan(\beta - \theta_2) = \frac{h}{v}.$$

Similarly, the angle between the horizontal and normal will also be β. The angle made at the object turns out to be $\alpha = \theta_1 - \beta$. According to small angle of approximation,

$$\theta_1 - \beta \approx \tan(\theta_1 - \beta) = \frac{h}{-u}.$$

Here, u happens to be negative according to our sign convention. Since tangent of an angle is always positive, we take modulus of u. Similarly

$$\beta \approx \tan(\beta) = \frac{h}{R}.$$

Also for the small angles, we can say that

$$\sin\theta_1 = \theta_1 \text{ and } \sin\theta_2 = \theta_2$$
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$
$$n_1\theta_1 = n_2\theta_2.$$

By substituting the values of θ_1 and θ_2, we get

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}. \tag{35-11}$$

Here, n_1 refers to the refractive index of the incident rays and n_2 refers to the refractive index of the refracted rays.

To apply this formula, we should substitute the values of u, v, and R with their proper sign. The sign convention remains the same as that of mirror. Seeing the ray diagram, we can note one interesting point. If the object is real and the radius of curvature of the surface is positive as shown the image cannot be formed between the center of curvature and optical center for any value of the refractive index. A similar conclusion can be drawn for other cases.

Cartesian Sign Convention

1. Principal axis of the spherical surface is taken as x axis and optical center as origin. Here, the principal axis is the diameter extended.
2. The direction of the incident light is taken as positive direction of x axis.
3. Upward direction is taken as positive and the downward direction as negative.

All the axial distances are always measured from the optical center. Object distance is taken to be u and image distance is taken to be v. For real objects, u is always negative and for virtual object u is positive as seen in Figs. 35-24*a–e*.

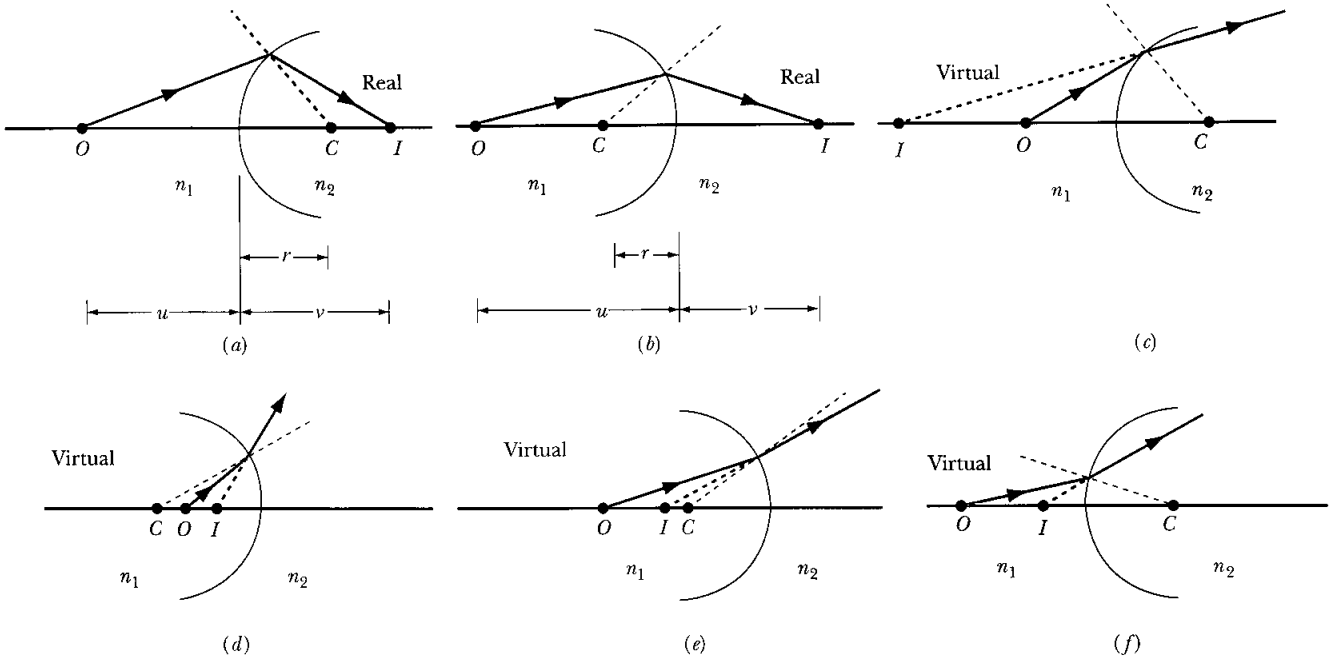
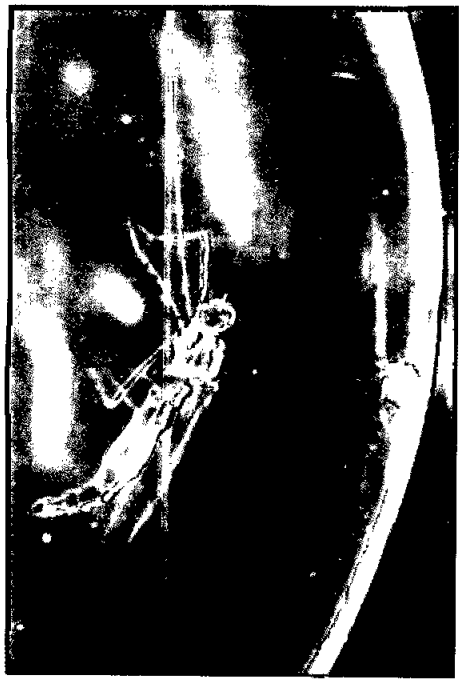


FIG. 35-24 (*a*)–(*e*) Here, we show the different situations where the image can be real and the situations where the image can be virtual. (*a*, *b*) Object as well as image are real. (*c–f*) Object is real but the image is virtual.



This insect has been entombed in amber for about 25 million years. Because we view the insect through a curved refracting surface, the location of the image we see does not coincide with the location of the insect. (*Dr. Paul A. Zahl/Photo Researchers*)

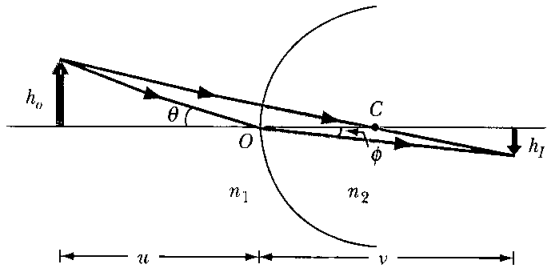


FIG. 35-25 If the image is kept perpendicular to the principal axis, the image in general is magnified or diminished.

For real image v is positive and for virtual image v is negative. Note here the difference between the mirror and the spherical surface. Since in the case of mirror, light is reflected, real image is formed on the negative side whereas in this case, the light is refracted. So a real image is formed on the positive side.

To find the transverse magnification, consider a ray incident from the extended object at the optical center (Fig. 35-25). Here, we can see that the angle of incidence is θ and angle of refraction is ϕ. Applying Snell's law, we get

$$n_1 \sin\theta = n_2 \sin\phi.$$

For small angles, we can say that

$$n_1\theta = n_2\phi$$

$$\frac{n_2 \times -h_I}{v} = \frac{n_1 \times h_O}{-u}$$

$$m = \frac{h_I}{h_O} = \frac{n_1 \times v}{n_2 \times u}. \quad \leftarrow \qquad (35\text{–}12)$$

Thus, we can see that for a real object forming a real image, magnification is negative (inverted image). We promised you that we will show this in each case, didn't we?

Sample Problem 35-12

Assume that a boy views a fish in a spherical aquarium of radius 10 cm kept at a distance of 2 cm from the center toward the left of center. The boy is viewing from the left (Fig. 35-26). Where will the image of fish be formed?

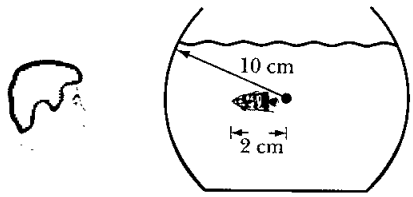


FIG. 35-26 A boy views a fish through a spherical aquarium.

Conceptualize/Classify: The positive direction is decided by the incident rays. Since the rays incident on the boundary of the aquarium is incident are from right to left so that they reach the eyes of the boy sitting to the left of the center, the positive direction will be from right to left. Here, the boundary of the aquarium will be the origin. All the distances to the left of this boundary will be positive and the distances to the right will be negative. So here we can see that the center of the surface is to the negative side and so the radius R will be taken as negative. Similarly, the distance of the object from the optical center is to the right side and hence it will be negative. Obviously, the refraction is from a spherical surface, so Eq. 35-11 will be used.

Compute: Rays are incident from water to air. So in Eq. 35-11, $n_2 = 1$; $n_1 = 4/3$; $R = -10$ cm; $u = -8$ cm. Substituting values in the formula, we get

$$\frac{1}{v} - \frac{4/3}{-8} = \frac{1-4/3}{-10}.$$

So, $v = -7.5$ cm.

Conclude: Here, the image formed is virtual. Students often tend to mistake that since the image is formed behind the water, refraction will occur again. Here, we should understand that if image is at −7.5 cm it means the refracted rays have to be traced to meet at that point. It does not mean that rays have reversed and met at that point.

CHECKPOINT 5 A bee is hovering in front of the concave spherical refracting surface of glass sculpture. (*a*) Which of the general situation of Fig. 35-24 is like this situation? (*b*) Is the image produced by the surface real or virtual, and is it on the same side as the bee or the opposite side?

Sample Problem 35-13

Figure 35-27 shows a vertical cross section through an eye of the *Anableps anableps* fish that swims with each eye half in and half out of the water, with a pigment band separating the two halves at the water surface. The front of the eye (the cornea) is a spherically convex refracting surface of radius $r = 1.95$ mm and index of refraction $n_2 = 1.335$. The refraction at the cornea is the first step in the eye's focusing of a *real* image onto the back of the eye (the retina), where visual processing begins. If the cornea faces an insect (lunch) at object distance $p = 0.20$ m, what is the image distance i of that refraction for the cornea in air ($n_1 = 1.000$) and in water ($n_1 = 1.333$)?

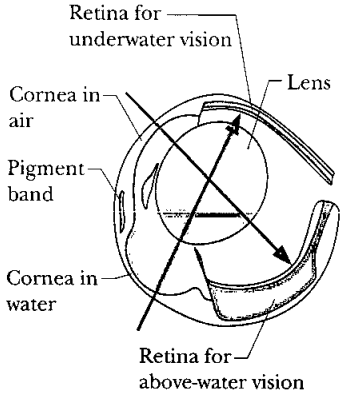


FIG. 35-27 Cross section of the eye of *Anableps anableps*.

Conceptualize/Classify: (1) Because the object and its image are on opposite sides of the refracting surface with its convex side facing the object, the situation is like either Fig. 34-24*a* (for a real image) or Fig. 34-24*c* (for a virtual image). (2) Image distance and object distance are related by

$$\frac{n_1}{p} + \frac{n_1}{i} = \frac{n_2 - n_1}{r}.$$

Compute: Solving this equation for i gives us

$$i = \frac{n_2}{\dfrac{n_2 - n_1}{r} - \dfrac{n_1}{p}}.$$

Substituting the given values and using the index of refraction $n_1 = 1.000$ for air, we find

$$i = \frac{1.335}{\dfrac{1.335 - 1.000}{0.00195} - \dfrac{1.000}{0.20}}$$

$$= 8.00 \text{ mm.} \qquad \text{(Answer)}$$

Repeating the calculation but using the index of refraction $n_1 = 1.333$ for water, we find

$$i = \frac{1.335}{\dfrac{1.335 - 1.333}{0.00195} - \dfrac{1.333}{0.20}}$$

$$= -0.237 \text{ m.} \qquad \text{(Answer)}$$

Conclude: After light is refracted by the cornea, it is further refracted by a lens to give a final real image on the retina. (The function of a lens is discussed in the next section.) Our first answer is positive (indicating a real image, as needed) and about twice the diameter of the eye. Thus, the role of the lens in focusing the light onto the retina is moderate because so much refraction occurs at the cornea. Our second answer is quite different: It is negative (indicating a virtual image) and much larger. So, far more focusing is required by the lens to put a real image onto the retina. To provide moderate focusing of light from the air and much stronger focusing of light from the water, the eye lens in *Anableps anableps* is egg shaped, with much greater curvature in the bottom half than in the top.

Origin Shifting

Many times we are required to consider that multiple events occur before final image is formed. In such situations, we take the image of one event as an object for another. The caution is that one should not worry about whether the image is

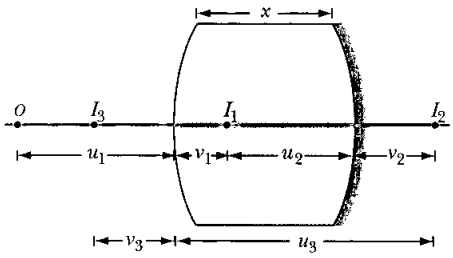


FIG. 35-28 Formation of the image of an object placed at O by a thick lens.

real or virtual while applying the sign convention. Instead, one should measure the distance from the optical center or pole and then apply the appropriate formula to form the image.

Suppose that a thick lens is silvered at the back to form an optical system (Fig. 35-28). Here, there will be three events before the final image is formed:

1. Refraction at the front surface of the lens.
2. Reflection at the back surface of the lens.
3. Refraction at the front surface of the lens again.

As we stated earlier, image for first event will act as an object for the second event. But for the second event, the pole is ahead of the optical center of first surface.

So to measure the value of u_2 (object distance for the mirror), the measurement has to be done from pole of mirror. This is similar to origin shifting we study in coordinate geometry.

From Fig. 35-28, we can say that

$$u_2 = v_1 - x.$$

Note that this includes the sign convention. For example, if image was formed at 5 cm and the thickness of the lens is 10 cm, this formula gives $u_2 = -5$ cm. This can be easily seen from Fig. 35-28 also.

The situation becomes a little tricky if we consider the event after the reflection. Here, the direction of light ray reverses. So what was positive before reflection becomes negative and vice versa. So here

$$u_3 = v_2 - x.$$

This formula again includes the sign convention. We can see this clearly from Fig. 35-28. If we have $v_2 = +10$ cm and $x = 10$ cm, u_3 will be -20 cm. To understand this clearly, we take an example.

Sample Problem 35-14

A sphere of radius 1 m and $n = 1.5$ is silvered at its back. A point object is kept at a distance of 1 m from the front face (Fig. 35-29). Where will the final image be formed?

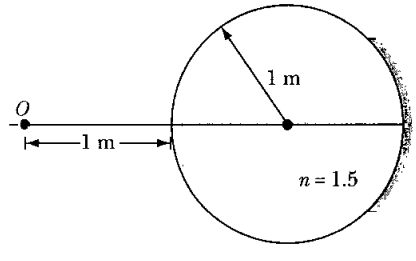


FIG. 35-29 A point object O is kept on the principal axis of a transparent sphere. The sphere is silvered at its rear surface.

Conceptualize/Classify: Here, we can clearly see that there will be three events before the final image will be formed. Refraction, reflection, and refraction back into the air.

Compute: For the first event O is the optical center, $n_1 = 1; n_2 = 1.5; u = -1$ m; $R = +1$ m. Therefore we get

$$\frac{1.5}{v_1} - \frac{1}{-1} = \frac{1.5-1}{1}$$

$$v_1 = 3 \text{ m}.$$

This image will act as an object for the rear silvered surface. However, for taking the object distance, we have to measure it from the pole of the mirror. As we have discussed before,

$$u_2 = v_1 - x$$

$$u_2 = -3 - 2 = -5 \text{ m}.$$

Applying mirror formula, we get (radius of curvature is -1 m, so focal length of the mirror is $-1/2$ m)

$$\frac{1}{v_2} + \frac{1}{-5} = \frac{1}{-0.5}$$

$$v_2 = -\frac{5}{9} \text{ m}.$$

These reflected rays will again be refracted from the spherical surface. Since now the direction of the light rays has reversed, therefore

$$u_3 = -v_2 - x$$
$$u_3 = -(-5/9) - 2 = -13/9 \text{ m}.$$

This can be seen from Fig. 35-28 also. Now

$$\frac{1}{v_3} - \frac{1.5}{-13/9} = \frac{1-1.5}{-1} = 0.5.$$

Note here that the radius of this surface is negative because the direction of the incident rays has reversed. Solving we get $v_3 = -13/7$ m.

Conclude: Negative means that the image is virtual formed inside the spherical surface. This is distance from the left edge of the spherical surface.

35-6 | Thin Spherical Lens

Lenses are common place. Probably half of you who are reading thisare doing so with the help of same glasses fitted with lens. Even those of us who don't need to use man-made glasses are endowed with a lens fitted within our eyes by Mother Nature. Before going further, let's try to study a simple spherical lens.

Lens are of two types: (1) convex lens (Figs. 35-30*a–c*) and (2) concave lens (Figs. 35-31*a–c*). We will consider only thin spherical lens. This means that the surfaces of lens are spherical in nature and the radius of curvature of the surface is very large in comparison to the thickness of the lens.

Before going ahead, let us define some common terminologies related to thin lens. ***First focus or object focus*** is point at which if an object is kept, the rays become parallel to the principal axis after refraction from the lens (Figs. 35-32*a* and *b*).

Second focus or image focus is a point at which rays parallel to principal axis will converge after refraction from the lens (Figs. 35-33*a* and *b*).



A fire is being started by focusing sunlight onto newspaper by means of a converging lens made of clear ice. The lens was made by melting both sides of a flat piece of ice into a convex shape in the shallow vessel (which has a curved bottom). *(Courtesy Matthew G. Wheeler)*

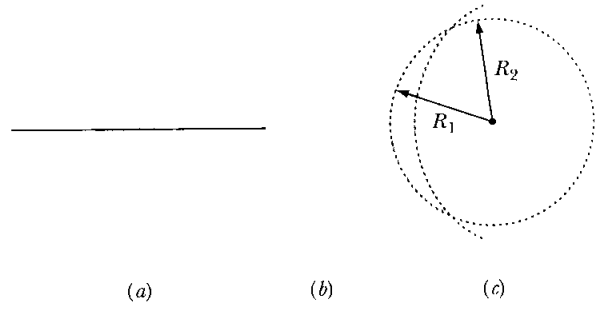


FIG. 35-30 (*a*) A typical convex lens. If the radii of curvature of both the refracting surfaces are equal in magnitude, the lens is said to be equiconvex, else it is a biconvex lens. (*b*) If one of the surfaces is plane, the lens is said to be plano-convex lens. (*c*) If the center of curvature of both the surfaces of the lens are on the same side, and the lens is thicker at the center than at the periphery, the lens is said to be concavo-convex lens.

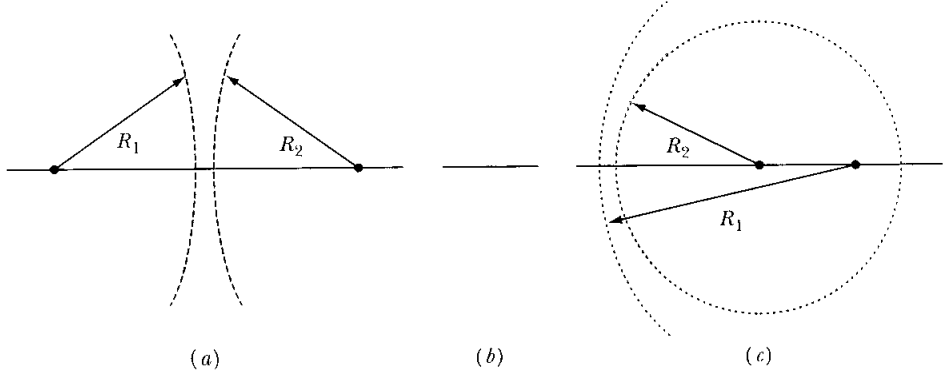


FIG. 35-31 (*a*) A typical concave lens. If the radii of curvature of both the refracting surfaces are equal, the lens is said to be equiconcave, else it is a biconcave lens. (*b*) If one of the surface is plane, the lens is said to be plano-concave lens. (*c*) If the center of curvature of both the surfaces of the lens are on the same side, and the lens is thinner at the center than at the periphery, the lens is said to be convexo-concave lens.

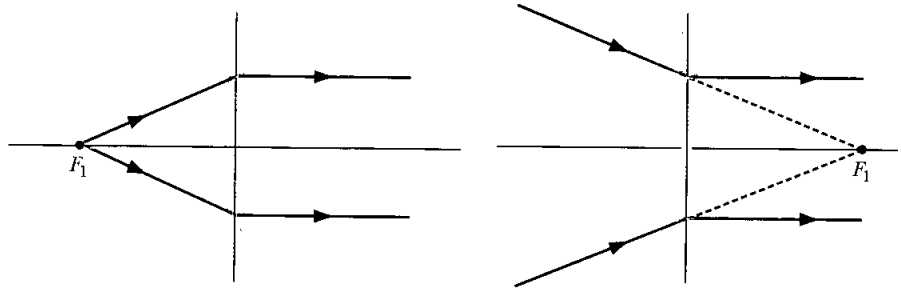


FIG. 35-32 First focus of a (a) convex lens and (b) concave lens.

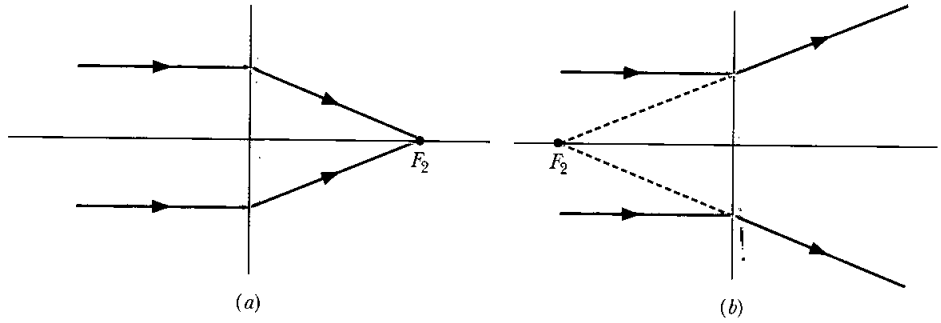


FIG. 35-33 Second focus of a (a) convex lens and (b) concave lens.

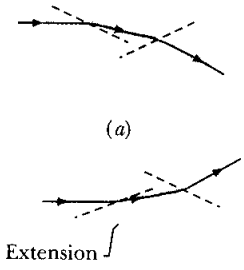


FIG. 35-34 (a) and (b) Although we are drawing the ray diagram as if the rays are refracted at the central line, the refraction occurs at both the surfaces. But to draw such a ray diagram is tedious. So we avoid it in general.

Both these focal lengths are equal in magnitude if the medium on both the sides of lens is same. Refraction in a lens occurs at both the surfaces (Figs. 35-34a and b).

Normally, focal length refers to the image focus.

Derivation of Lens Formula and Lens Maker's Formula

Consider a point object O kept on the principal axis of the lens (Fig. 35-35). The lens has a refractive index n_2 and the surroundings have a refractive index n_1. As we did before in the case of refraction from spherical surface, here there will be two events: refraction from the first face of radius of curvature R_1 and from the second face of radius of curvature R_2. For the first surface,

$$\frac{n_2}{v_1} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1}. \tag{35-13}$$

For the second surface, image of the first surface acts as an object. So $v_1 \approx u_2$ because the distance between the first surface and the second surface is small:

$$\frac{n_1}{v_2} - \frac{n_2}{u_2} = \frac{n_1 - n_2}{R_2} \quad (\text{Since } u_2 = v_1 - t \approx v_1)$$

$$\frac{n_1}{v_2} - \frac{n_2}{v_1} = \frac{n_1 - n_2}{R_2}. \tag{35-14}$$

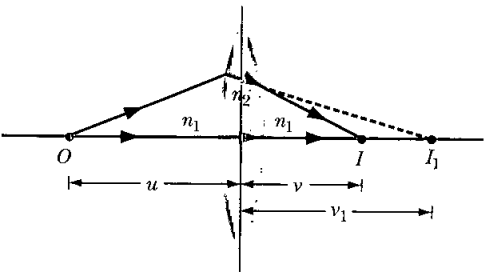


FIG. 35-35 Formation of image by a lens. Here, only two rays are drawn for simplicity.

Adding Eqs. 35-13 and 35-14, we get

$$\frac{n_1}{v} - \frac{n_1}{u} = \frac{(n_2 - n_1)}{R_1} + \frac{(n_1 - n_2)}{R_2}.$$

Since final image is at v_2, we can take $v_2 = v$. Therefore

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

According to the definition of second focus, if the object is at infinity, the image is formed at the focus, that is, $u = \infty \Rightarrow v = f$. We now have

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

This is known as **lens maker's formula** for obvious reasons.

Moreover

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

This is the lens formula.

Note that this f refers to second focus.

By using thin lens formula, for different positions of the object, we can find the location of images. To analyze this, we will divide space into two parts. For the object real side is left and virtual side is to the right of the lens. But for the image, real side is to the right of the lens and virtual side is to the left of the lens. As before (in the case of mirror), we will divide the entire space into four parts (Fig. 35-36). By plugging in different values of object distance in lens formula, we see that the image distance can be represented as in Table 35-1. From Tables 35-1 and 35-2, we can draw many important conclusions.

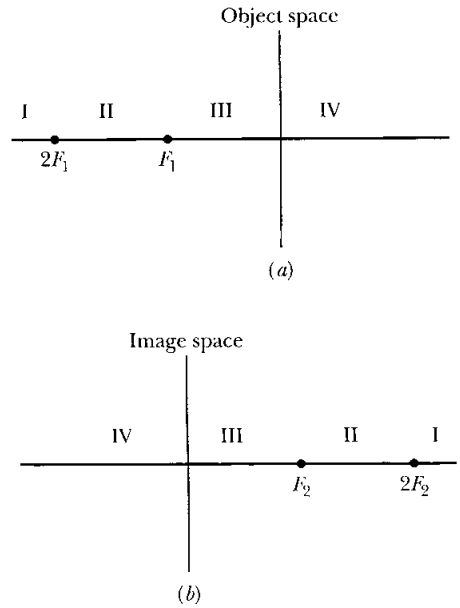


FIG. 35-36 (*a*) Object space and (*b*) image space for a convex lens.

I. In convex lens virtual object cannot form virtual image and in concave lens real object cannot form real image.

II. If an object moves from ∞ to $2f$, image moves from f to $2f$. So velocity of object is greater than velocity of image, when the object is in region I. When the object moves from $2f$ to f, the image moves from $2f$ to ∞ in the same time. So velocity of image is greater than that of object when the object is in

TABLE 35-1

Location, Nature, and Magnification of Image for Convex Lens

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Real object and real image	Diminished	Inverted
II	I	Real object and real image	Enlarged	Inverted
III	IV	Real object and virtual image	Enlarged	Erect
IV	III	Virtual object and real image	Diminished	Erect

TABLE 35-2

Location, Nature, and Magnification of Image for a Concave Lens

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Virtual object and virtual image	Diminished	Inverted
II	I	Virtual object and virtual image	Enlarged	Inverted
III	IV	Virtual object and real image	Enlarged	Erect
IV	III	Real object and virtual image	Diminished	Erect

region II. The velocity of the object is same as that of the image when the object is at $2f$. Note here that $2f$ does not mean R. In a lens, $2f$ is not equal to the radius of curvature in general.

III. When the object is at $2f$, the relative velocity of object and image is 0. So at this position, the distance between the real object and real image is minimum ($4f$).

IV. When the object is on the left side of the object focus, the image is on the right side of image focus and vice versa. So the object and its image are always on the opposite side of their respective focus. Yes, there is an analogous Newton's law for the lens also.

✓**CHECKPOINT 6** Show that for a lens, $x_o x_i = -f^2$, where x_o and x_i are the distance of the object from object focus and the distance of the image from the image focus, respectively.

Other useful terms are of converging lens and diverging lens.

Converging Lens

A **converging lens** is one which converges a parallel beam of light on the other side of the lens. This beam will converge at the second focus or image focus. For a converging lens this focal length (f) is positive. The object focus is located on the negative side.

As discussed before, for a converging lens, focal length should be positive. From the lens maker's formula, f is positive if $n_2 > n_1$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is positive. This is the situation with a convex lens kept in air. But in the case of concave lens also, f is positive if $n_1 > n_2$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is negative. This is the situation when a concave lens is dipped in a medium of refractive index greater than that of the lens.

Diverging Lens

A **diverging lens** is one which diverges a parallel beam of light passing through it. For such a lens, focal length is negative. So its image focus is on negative side and object focus is on the positive side.

As stated before, for a diverging lens, focal length should be negative. From the lens maker's formula, f is negative if $n_2 > n_1$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is negative. This is the situation with a concave lens. However, in the case of convex lens also, f is negative if $n_1 > n_2$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is positive. This is the situation when a convex lens is dipped in a medium of refractive index greater than that of the lens.

Thus, we can say that in the case of lens, the focal length depends on the properties of the medium. It may seem strange, but actually the focal length does not depend on which side is facing the incident light. For the application of the lens maker's formula, we take R_1 to be the radius of the surface on which light is first incident, but actually even if we invert the lens and make the second surface face the light; the answer for the focal length will be the same.

Sample Problem 35-15

A concavo-convex lens has radii of its faces 20 cm and 60 cm. If the refractive index of the material of the lens is 1.5, find its focal length.

Conceptualize: Here, we should be careful to see what will be the values of R_1 and R_2.

Classify: In Fig. 35-30c, $R_1 = +20$ cm and $R_2 = 60$ cm.

Compute: Surroundings have a refractive index of 1. So the lens maker's formula yields

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \left(\frac{1.5}{1} - 1\right)\left(\frac{1}{20} - \frac{1}{60}\right).$$

Therefore, $f = 60$ cm. Here note that sign convention is very important. As can be seen in Fig. 35-30, R_1 and R_2 are both positive because light is incident from the left and both the surfaces have center of curvature to the right of the optical center.

Conclude: Let us now assume that light is now incident from the right so that light is first incident on the surface of radius of curvature 60 cm. As can be seen from Fig. 35-30*c*, the side to the left of the optical center becomes the positive side. So now both the surfaces have their center of curvature on the negative sides. So now $R_1 = -60$ (light is first incident on it) and $R_2 = -20$ cm. Substituting the values, we get

$$\frac{1}{f} = \left(\frac{1.5}{1} - 1\right)\left(\frac{1}{-60} - \frac{1}{-20}\right) = \frac{1}{60}.$$

We can see that the focal length remains the same in both the cases.

CHECKPOINT 7 A thin symmetric lens provides an image of a fingerprint with a magnifications of +0.2 when the fingerprint is 1.0 cm farther from the lens than the focal point of the lens. What are the type and orientation of the image, and what is the type of lens?

Principles for Drawing Ray Diagram

Here, we illustrate the principles used for drawing ray diagrams (Figs. 35-37*a–e*).

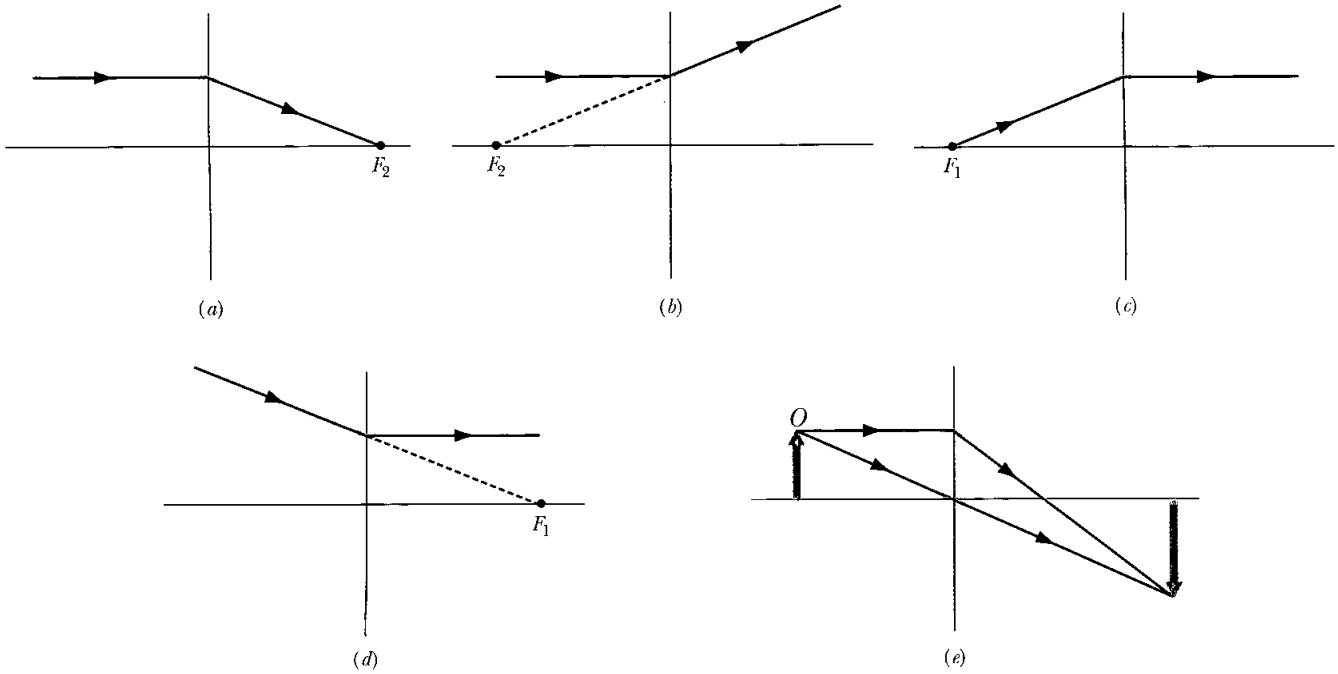


FIG. 35-37 (*a*, *b*) A ray parallel to the principal axis will pass through the second focus after refraction from the lens. (*c*, *d*) A ray passing through F_1 will become parallel to the principal axis after refraction. (*e*) A ray passing through the optical center will pass undeviated. Since on the line of incident ray, object will lie and on the line of the refracted ray, image will lie, this line is common to both the object and the image. So we can also say that the line joining the object and its image will always pass through the optical center.

Sample Problem 35-16

Find the point at which the first ray crosses the principal axis after refraction (Fig. 35-38). Also find its deviation.

Conceptualize/Classify: We can see that the two refracted rays will meet at a point in the focal plane. Since the ray passing through the optical center will pass

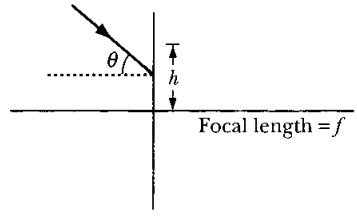


FIG. 35-38 Deviation of a ray incident on a converging lens.

undeviated, we can say that this meeting occurs at an angle of θ and hence at a distance $f\theta$ below the focus (Fig. 35-39).

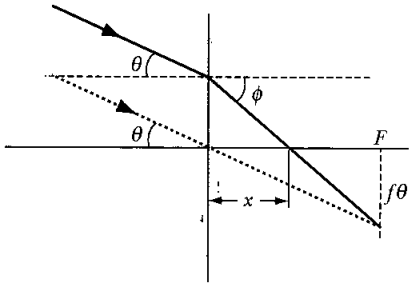


FIG. 35-39 We draw an imaginary ray passing through the optical center to find the path of the refracted rays. We know that a parallel beam of light will converge at a point in the focal plane. The ray passing through the optical center will pass undeviated.

Compute: By the property of the similar triangles, we can say that

$$\frac{h}{x}=\frac{f\theta}{f-x}=\phi.$$

Solving, we get

$$x=\frac{hf}{h+f\theta} \quad \text{and} \quad \phi=\frac{h+f\theta}{f}.$$

Therefore, the deviation is

$$\theta-\phi=\frac{h}{f}.$$

Sample Problem 35-17

Find the location of the final image after all the refractions in Fig. 35-40. The radius of both the plano-convex lens and plano-concave lens is 10 cm and refractive index is 1.5.

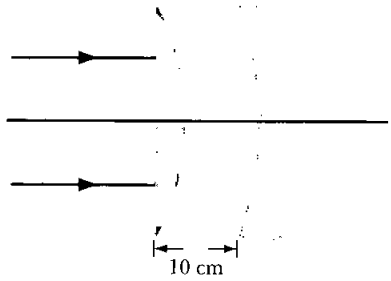


FIG. 35-40 Formation of image by a combination of converging and a diverging lens.

Conceptualize/Classify: Here again we are seeing an example of multiple events. The image of plano-convex lens will act as an object for the plano-concave lens. But to get there, we need to find the focal length of the lens first.

Compute: Focal length of the plano-convex lens is

$$\frac{1}{f}=(1.5-1)\left(\frac{1}{+10}-\frac{1}{\infty}\right)=\frac{1}{20}.$$

Focal length of plano-concave lens is

$$\frac{1}{f}=(1.5-1)\left(\frac{1}{\infty}-\frac{1}{+10}\right)=-\frac{1}{20}.$$

Since parallel beam is incident on the lens, its image from plano-convex lens will be formed at +20 cm from it (at the focus) and will act as an object for the plano-concave lens. Since the two lens are at a distance of 10 cm from each other, therefore, for the next lens u = +10 cm. Hence

$$v=\frac{uf}{u+f}=\frac{10\times-20}{10-20}=20 \text{ cm}.$$

Conclude: We can see that a diverging lens is forming a real image. This is because the object for it is virtual and is between pole and object focus.

Sample Problem 35-18

Find the distance between the convex lens and convex mirror, both of focal length 20 cm (Figs. 35-41*a* and *b*), so that for an object kept at 30 cm from the lens, the final image is at the object itself. (This is commonly known as ***condition for autocollimation*** that we discussed in Section 34-5.)

Conceptualize/Classify: This condition occurs when the object and image coincide. In the case of mirror, this happens when the object for the mirror is at its pole or at the center of curvature (Fig. 35-41*a* and *b*). In the second case, the angle of incidence is 0° and hence the angle of reflection will also be 0°.

In the case of single lens, image can never coincide with the object because the rays are refracted into the next medium. But if the lens is kept before the mirror, after refraction from the lens, rays will fall on the mirror. If the image of the lens is at the pole or at the center of the curvature of the mirror, the rays will retrace their path and hence by the principle of optical reversibility, we can say that image will again form on the object.

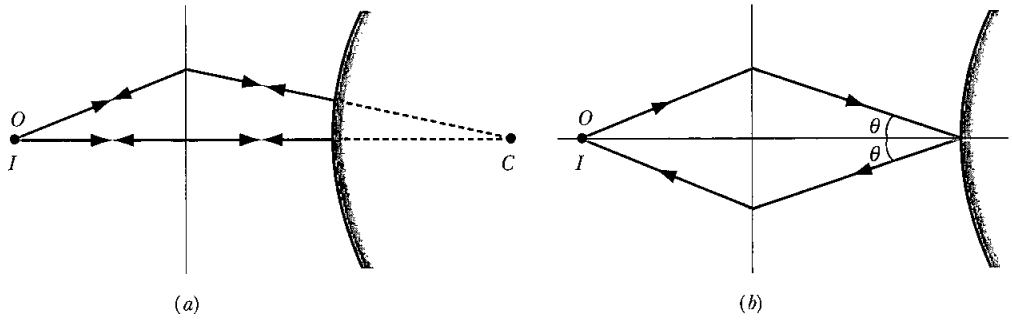


FIG. 35-41 (*a*) The principle of optical reversibility is illustrated. Here, the image of the object *O* formed by the lens acts as an object for the mirror. The final image will coincide with the object if the rays retrace their path. The rays retrace their path if they are incident normally on the mirror. (*b*) Another situation where the object and the image coincide with each other. The image of the lens is formed at the pole of the mirror. So the rays are reflected symmetrically.

Compute: For the lens,

$$v = \frac{uf}{u+f} = \frac{20\times -30}{-30+20} = 60 \text{ cm}.$$

For the mirror, this image should act as an object. The auto collimation will occur if this image is formed at pole. In this situation, the distance between the lens and the pole will be 60 cm.

Auto collimation will also occur if this image is formed at the center of curvature of the mirror. But the center of curvature is 40 cm away from the pole as can be seen in the figure. So the lens can also be at 60 – 40 = 20 cm from the mirror.

Conclude: In fact a beam of light can be made to retrace its path by using a transparent sphere silvered at its back. For this the condition of auto collimation requires that the intermediate image which acts as an object for the mirror at the back is at the center of the sphere (Fig. 35-42*b*) or at its pole (Fig. 35-42*a*). It is obvious that due to refraction from a sphere, image cannot be formed at its center (angle of refraction will become zero for a finite angle of incidence as can be seen in Fig. 35-42*b*). But the image due to refraction at the curved surface can be definitely formed at the pole of the mirror.

This would happen if the refractive index of the sphere is exactly two times the refractive index of the medium from which the radiation is incident. This is conventionally known as a *cat's eye* retroreflector.

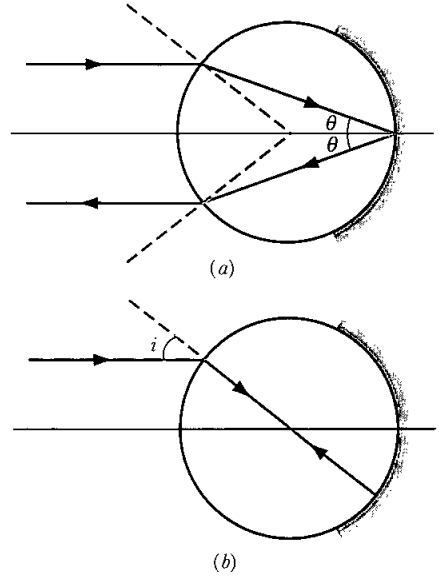


FIG. 35-42 A parallel beam of light is incident on a transparent sphere such that the image is formed at (*a*) pole of the mirror, (*b*) the center of curvature of the spherical surface.



The term *cat's eye* derives from the resemblance of the cat's eye retroreflector to the optical system that produces the well-known phenomenon of "glowing eyes" or eye shine in cats and other vertebrates (which are only reflecting light, rather than actually glowing). The combination of the eye's lens and the aqueous humor form the refractive converging system, while the tapetum lucidum behind the retina forms the spherical concave mirror. Because the function of the eye is to form an image on the retina, a beam of light naturally focuses at the retina. The rays get reflected from the back which gives cat's eyes the glow we see in the picture.

Sample Problem 35-19

Find the radius of the light spot on the screen as shown in Fig. 35-43. Consider only the light rays refracted from the lens. Find the average intensity on the screen. The source is point source of power 100 W. The lens has an aperture of 6 cm.

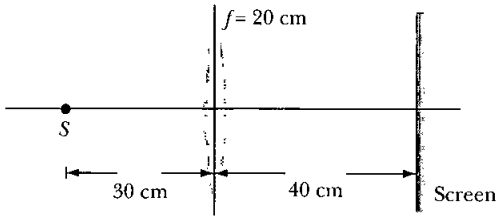


FIG. 35-43 A lens is used to form a light spot on the screen.

Conceptualize: Only a real image can be focused on a screen. A virtual image cannot be focused on a screen but can be seen with naked eyes. To see a real image with a naked eye, we should place the eye behind the image.

Classify: Here, we will try to locate the image formed by the lens. This image will be formed by all the rays collected by the lens. So if we consider the extreme rays which are being focused at the aperture (Fig. 35-44), we will get the field of view for the image. This will also show us the region in which the refracted light will be present.

Compute: If the screen was not present, the image would have formed at

$$v = \frac{uf}{u+f} = \frac{-30\times 20}{-30+20} = 60 \text{ cm}$$

from the lens.

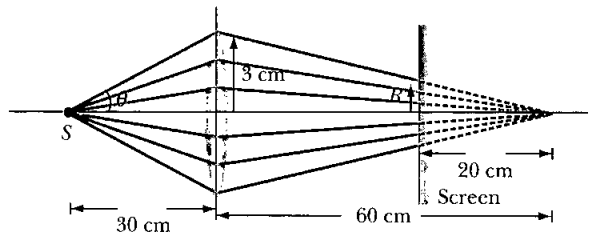


FIG. 35-44 The formation of a refracted spot on a screen.

From similar triangles, we can estimate the radius of the image formed:

$$\frac{R}{3} = \frac{20}{60}$$
$$R = 1 \text{ cm}.$$

The amount of light used to form the image can be estimated with the help of solid angle subtended by the lens at the location of the source:

$$\frac{\text{power of image}}{100 \text{ W}} = \frac{1-\cos\theta}{2}.$$

Here, $\tan\theta = 3/30 = 0.1$, for a small angle and $\cos\theta = (1-\theta^2/2)$ The power of the image $50\times(0.1)^2/2 = 0.25$ W.

$$\text{Intensity} = \frac{\text{power}}{\text{area}}$$
$$= \frac{0.25}{\pi\times(0.01)^2} = \frac{2500}{\pi} \text{ W/m}^2.$$

Conclude: Here, we can see how the intensity can be quite high even though the power transmitted by the lens is very small.

Sample Problem 35-20 Beam Expander

Assume that a parallel beam is incident on two convex lens of focal length 20 cm and 40 cm. The beam has a diameter of 2 cm. What should be the distance between them so that the final beam emerges parallel to the principal axis (Fig. 35-45)?

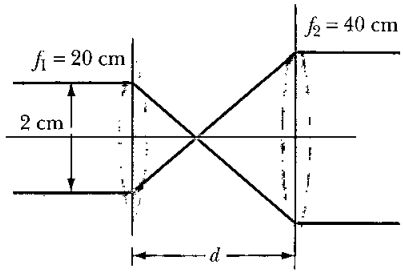


FIG. 35-45 Using two lens to expand or contract a parallel beam of light.

Conceptualize/Classify: Here, again we can see that the image of one will act as an image of another. However, we can also see that for the final beam to be parallel, the object for it should be at its first focus. The image of first lens is formed at its second focus. So we can say that the distance between them should be equal to sum of their focal length.

Compute: From Fig. 35-45, it is clear that

$$d = f_1 + f_2 = 60 \text{ cm}.$$

Conclude: If a parallel beam passes through a series of lens, under suitable arrangement, the width of the beam can be expanded. As can be seen from Fig. 35-45, the radius of the beam is changed. This new radius can be found with the help of similar triangles.

Here, we can say that

$$\frac{R_i}{R_f} = \frac{f_i}{f_f}.$$

Displacement Method for Finding the Focal Length

In this situation, we fix the object and its image is obtained on a fixed screen as shown in Fig. 35-46. We know that only a real image can be obtained on a screen. We can say that real object and real image distance can be

$$d > 4f,$$

where d is the distance between screen and object. So for this experiment, the object should be kept at a distance of not less than $4f$ from the screen.

Let us assume that the object is at a distance x from the lens. So the image will be formed at a distance of $d - x$ from the lens. By lens formula, we have

$$\frac{1}{d-x} - \frac{1}{-x} = \frac{1}{f}$$

or

$$\frac{1}{d-x} + \frac{1}{x} = \frac{1}{f}. \tag{35-15}$$

We know by the principle of optical reversibility that the object and its image can be interchanged. Here, although we cannot move the object and screen, we can move the lens. So if object distance is x, image is at $d - x$. If instead the object is kept at distance of $d - x$, the image is formed at a distance of x. So there is an alternative position of the lens for which an image can be formed on the screen. We can also apply the lens formula for this case and obtain the same formula as Eq. 35-15. The magnification in the first case is

$$m_1 = \frac{d-x}{-x}.$$

The magnification in the second case is

$$m_2 = \frac{x}{-(d-x)}.$$

Here, we note that both the magnifications are negative as real image of real object is always inverted. Also

$$m_1 \times m_2 = 1.$$

Therefore,

$$\frac{h_{I_1}}{h_O} \times \frac{h_{I_2}}{h_O} = 1. \tag{35-16}$$

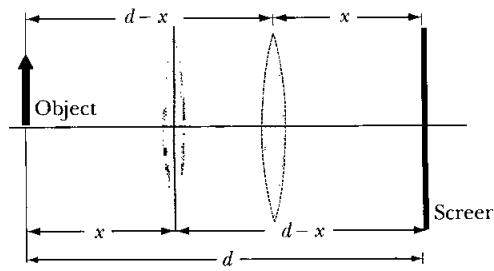


FIG. 35-46 Displacement method for finding the focal length of the lens. Here, the same lens is initially located at a distance of x from the object. Then the lens is moved to a distance of $d - x$ from the object so that a sharp image is again obtained on the screen.

Sample Problem 35-21

In an experiment for finding focal length of a convex lens, the object and screen are kept fixed. There are two position of lens between the object and screen for which an image is obtained on the screen. These positions are separated by 20 cm. Also the magnification in first situation is –2. Find the distance between the object and the screen, that is the focal length of the lens.

Conceptualize/Classify: As we can see from the analysis presented before, this refers to the situation of displacement method of finding the focal length of lens (Fig. 35-46).

Compute:

$$\frac{1}{d-x} + \frac{1}{x} = \frac{1}{f}.$$

From Fig. 35-46, it is clear that in the first situation, object is at a distance x from the lens and in second situation screen is at a distance x from the lens. Therefore,

$$d - 2x = 20 \text{ cm}.$$

Also

$$m_1 = \frac{d-x}{-x} = -2$$

$$d = 3x.$$

So $x = 20$ cm and $d = 60$ cm. Therefore,

$$\frac{1}{40} + \frac{1}{20} = \frac{1}{f}$$

$$f = 40/3 \text{ cm}.$$

Conclude: We stated previously that d must be greater than $4f$. This can be easily seen from the answer.

Equivalent Focal Length

When two lenses are joined in such away that the distance between them is negligible, the equivalent focal length is

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}.$$

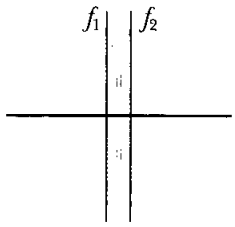


FIG. 35-47 Two lens are joined together. Here, we have taken the example of two converging lens. But the analysis will be equally well applied to the case of other lens combination.

Proof: Suppose that we have two lens of focal length f_1 and f_2 joined together (Fig. 35-47). As we have discussed numerous times, the image of one lens will act as an object for the other. Let us assume that a real object is situated to the left, say at a distance u. The final image is formed at v from the combination.

For the first lens, we can apply the lens formula:

$$\frac{1}{v_1}-\frac{1}{u}=\frac{1}{f_1}.$$

This image at v_1 from the first lens will act as an object for the next lens. The distance of the object from the second lens would be

$$u_2=v_1-t\approx u_1,$$

because the thickness t of both the lens is small.

For the second lens, we can apply the lens formula:

$$\frac{1}{v}-\frac{1}{u_2}=\frac{1}{f_2}$$

and

$$\frac{1}{v}-\frac{1}{v_1}=\frac{1}{f_2}.$$

Adding the two equations, we get

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f_{eq}}.$$

To solve such situations, we should first find the focal length of individual lens and then apply the above formula. Note that the focal lengths should be substituted with their sign.

Sample Problem 35-22

Two equiconcave lens of focal length –20 cm each having a refractive index of 1.50 are kept together and the space between them is filled with water as shown in Fig. 35-48. Find the equivalent focal length of the arrangement thus formed.

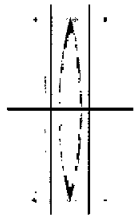


FIG. 35-48 Two equiconcave lens with their intermediate space filled with water.

Conceptualize: Here we can imagine a thin layer of air in between lens and water at both the surfaces (Fig. 35-49). This will give us two concave lens combined with a convex lens of water. This thin layer of air will not affect the formation of image but rather will simplify the calculations a lot.

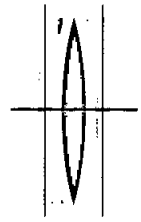


FIG. 35-49 Assuming a thin air gap between the lens.

Classify: First let us find the radius of curvature of the lens because this will be needed to find the focal length of the convex lens formed by water in between.

Compute:

$$\frac{1}{-20}=(1.5-1)\left(\frac{1}{-R}-\frac{1}{R}\right)$$

$$R=20\text{ cm}.$$

So the focal length of water convex lens (in Fig. 35-49) is

$$\frac{1}{f}=\left(\frac{4}{3}-1\right)\left(\frac{1}{20}-\frac{1}{-20}\right)$$

$$f = 30 \text{ cm}.$$

Note here carefully that we are taking the focal length of the water lens in air because when we are using the focal length of the concave lens, we are assuming that there is air on both the sides.

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_1}$$

$$\frac{1}{f}=\frac{1}{-20}+\frac{1}{30}+\frac{1}{-20}$$

$$f = -15 \text{ cm}.$$

Conclude: Similarly, if a lens is cut into two parts, the focal length of the two parts are given by

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2},$$

where f is the focal length of the original lens and f_1 and f_2 are focal lengths of the two cut parts. The interesting situation is that if we join these lenses back, we will get back the original lens. We have studied previously that the focal length of the lens does not depend on the side facing the light. So the focal length of all the following four arrangements (Figs. 35-50*a–d*) will be the same.

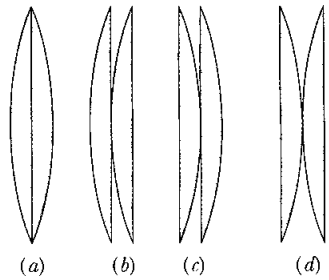


FIG. 35-50 (*a*) A convex lens is cut into two parts by a section done along the central line. (*b*)–(*d*) The two parts are recombined. It is found that the focal length in all these three cases is the same as that in (*a*).

Focal Length of a Lens if the Medium on Both the Sides of Lens is Different

Here we have a spherical lens, each of whose surfaces is spherical (Fig. 35-51). So instead of applying thin lens formula which we derived when the medium on both the sides of the lens is the same, we apply the formula for refraction from the spherical surface one by one. As usual, the image of one surface acts as an object for the next. Let us assume that an object is located at a distance of u from the left side of the lens in medium 1.

For the first surface,

$$\frac{n_2}{v_1}-\frac{n_1}{u}=\frac{n_2-n_1}{R_1}, \tag{35-17}$$

where v_1 is the distance of image from the left surface of the lens. For second surface, image of the first surface acts as an object. So $v_1 \approx u_2$ because the distance between the first surface and the second surface is small (it is a thin lens).

$$\frac{n_3}{v}-\frac{n_2}{u_2}=\frac{n_3-n_2}{R_2}. \tag{35-18}$$

Adding Eqs. 35-17 and 35-18, we get

and

$$\frac{n_3}{v}-\frac{n_1}{u}=\frac{(n_2-n_1)}{R_1}+\frac{(n_3-n_2)}{R_2}.$$

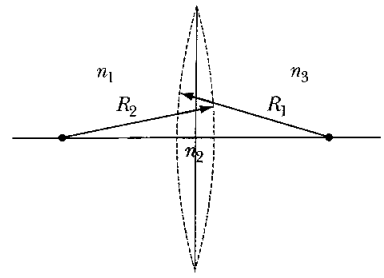


FIG. 35-51 A lens of refractive index n_2 is surrounded by a medium of refractive index n_1 on the left and n_3 on the right.

35-7 | Lens Mirror Combination

Many times we come across situations where the rear surface of the lens is silvered (Fig. 35-52) or it may be that a lens and a mirror are kept in close contact. As we stated before, the image of one system acts as an object for the next. However, it turns out that in this situation, there is a simple formula to find out the image.

Since the light rays will be reflected back, the system is behaving like a mirror. We are interested in replacing this with an equivalent mirror and we will work to find the focal length of a mirror with which this system can be replaced.

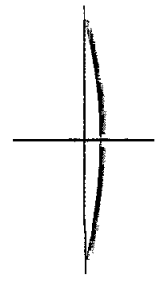


FIG. 35-52 A lens silvered at its back to form an equivalent mirror.

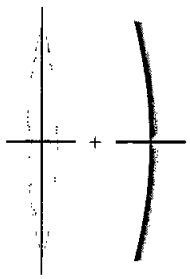


FIG. 35-53 A lens silvered at its back is equivalent to a combination of lens and a mirror.

As before, we will divide the system into three parts. We will assume that a thin layer of air is in between rear silvered surface and the lens (Fig. 35-53). This will not make any difference to the answer but will simplify the calculations a lot.

Let us assume that an object is located at a distance of u from the left side of the lens.

For the lens, using the lens formula will yield

$$\frac{1}{v_1}-\frac{1}{u}=\frac{1}{f_L}, \tag{35-19}$$

where v_1 is the distance of image from the lens. For the mirror, image of the lens acts as an object. So $v_1 \approx u_2$ because the distance between the lens and mirror is zero. Applying mirror formula, we get

$$\frac{1}{v_2}-\frac{1}{u_2}=\frac{1}{f_M}$$

or

$$\frac{1}{v_2}-\frac{1}{v_1}=\frac{1}{f_M}. \tag{35-20}$$

Again this image at a distance of v_2 from the mirror will act as an object for the lens. We can again apply lens formula to find the location of the final image.

$$\frac{1}{v_3}-\frac{1}{u_3}=\frac{1}{f_L}.$$

But now we have to be careful. Light rays have reversed their direction after striking the mirror. So although the lens is of the same nature and f_L remains the same, the sign convention for the object and the image reverses. Recall that in Cartesian sign convention, the direction of incident rays is assumed to be in positive direction. In fact, what is positive for the mirror will be negative for the lens. Therefore,

$$u_3 - v_2.$$

Since in the equivalent system, we are replacing with a mirror whose positive direction is same as that of the first lens, therefore, location of final image $v = -v_3$ and so

$$\frac{1}{-v}-\frac{1}{-v_2}=\frac{1}{f_L}. \tag{35-21}$$

By adding Eqs. 35-19, 35-20, and 35-21 according to the formula –(35-19) – (35-21) + (35-20), we get

$$\frac{1}{v}+\frac{1}{u}=-\frac{2}{f_L}+\frac{1}{f_M}.$$

So we can say that this system behaves like a mirror whose focal length is given by

$$\frac{1}{f_{eq}}=-\frac{2}{f_L}+\frac{1}{f_M}. \tag{35-22}$$

In this the focal length of a convex lens is considered positive. Similarly, the focal lengths of the mirror should be substituted with the respective sign. If the focal length of the mirror comes out to be positive, the system would be treated as an equivalent convex mirror. If the focal length of the mirror comes out to be negative, the system would be treated as an equivalent concave mirror.

Sample Problem 35-23

A convex lens of radii 20 cm and 30 cm is silvered at the surface which has smaller radius of curvature. Find the equivalent focal length ($n = 1.5$).

Conceptualize/Classify: Here, the focal length of equivalent mirror can be found by Eq. 35-22.

Compute: We know that

$$\frac{1}{f_L} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
$$= (1.5 - 1)\left(\frac{1}{20} + \frac{1}{30}\right) = 0.5 \times \frac{5}{60}.$$

Therefore

$$f_L = \frac{120}{5} = 24 \text{ cm}.$$

Also

$$f_M = \frac{-20}{2} = -10 \text{ cm}.$$

From Eq. 35-22 we get

$$\frac{1}{f_{eq}} = -\frac{2}{f_L} + \frac{1}{f_M}$$
$$= -\frac{2}{24} + \frac{1}{-10} = \frac{-60}{11} \text{cm}.$$

Conclude: So this system behaves like a concave mirror of focal length 60/11 cm.

35-8 | Prism

Figure 35-54 shows a ray of monochromatic light passing through a glass prism surrounded by air. When the light enters the prism at the left face, the refracted ray is bent toward the normal, because the refractive index of glass is greater than that of air. When the light leaves the prism at the right face, it is refracted away from the normal. Thus, the net effect of the prism is to change the direction of the ray, causing it to bend downward upon entering the prism, and downward again upon leaving. So the ray is deviated from its path. This is one of the primary uses of the prism: deviation of a ray from its path. The edge at which the refracting faces meet is called the refracting edge. The angle between the refracting faces is called refracting angle of the prism or angle of the prism (angle A in Fig. 35-54).

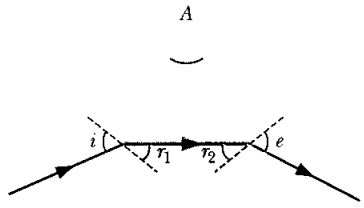


FIG. 35-54 A glass prism deviating a monochromatic beam of light.

From Fig. 35-54, it can be seen that

$$A + 90° - r_1 + 90° - r_2 = 180°$$

or
$$A = r_1 + r_2. \tag{35-23}$$

The incident ray is deviated from its path at each surface by the process of refraction. At the first surface, the angle of incidence is i and the angle of refraction is r_1. So the deviation at that face is

$$\delta_1 = i - r_1 \quad \text{(clockwise)}.$$

At the second surface, the angle of incidence is r_2 and angle of refraction is e. So the deviation at that face is

$$\delta_2 = e - r_2 \quad \text{(clockwise)}.$$

So the total deviation produced is

$$\delta = \delta_1 - \delta_2 = i + e - (r_1 + r_2)$$

or
$$\delta = i + e - A. \tag{35-24}$$

Here, to find the deviation when we know the angle of incidence, we can find the angle r_1 by applying Snell's law:

$$1 \sin i = n \sin r_1.$$

By applying Snell's law at the next surface, we get

$$1 \sin e = n \sin r_2$$

This value of deviation is obtained for a prism and the graph of deviation δ against angle of incidence i has been plotted as shown in Fig. 35-55. Take a realistic prism and you will obtain a curve as shown in Fig. 35-55.

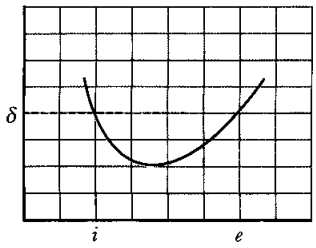


FIG. 35-55 A plot of angle of deviation *v/s* the angle of incidence. At an angle of incidence i as well as e, the deviation is the same.

Some conclusions can be easily drawn by observing the graph. This graph is not symmetric about the minimum. It seems from the graph that the same

deviation occurs for two different angles of incidence. One of these angles is the angle of incidence and another is the angle of emergence. But if the ray is incident at the angle of emergence on the prism, by the principle of optical reversibility, the angle of emergence will be the previous angle of incidence. So the angle of deviation remains the same.

Sample Problem 35-24

A ray is incident on an equilateral prism such that the angle of deviation is 30°. It is seen that if the angle of incidence is increased by 30°, then the deviation is again 30°. Find the refractive index of the prism.

Conceptualize/Classify: We can easily see that in one situation the angle of incidence will be i and in the next situation angle of incidence will be $i + 30°$. From the above discussion, we can say that in the first situation, the angle of emergence should be $i + 30°$ and in the second situation, the angle of emergence should be i.

Compute: In both cases

$$\delta = i + e - A$$
$$30° = i + i + 30° - 60°$$
$$i = 30°.$$

The angle of emergence thus is

$$i + 30° = 60°.$$

By Snell's law,

$$1 \sin 30 = n \sin r_1.$$

By Snell's law at the second surface, we have

$$1 \sin 60 = n \sin r_2$$
$$1 \sin 60 = n \sin(60 - r_1).$$

Solving for n, we get

$$n = \sqrt{\frac{4+\sqrt{3}}{3}}.$$

Angle of Minimum Deviation

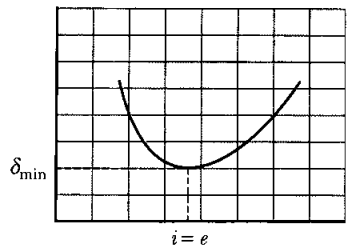


FIG. 35-56 At the minimum deviation, angle of incidence is equal to the angle of emergence. So in the situation of minimum deviation, the ray passes through the prism symmetrically.

From Fig. 35-56, we can say that at the minimum deviation, there is only one angle of incidence. So at the minimum deviation we can say that $i = e$. This gives

$$1 \times \sin i = n \sin r_1$$
$$n \sin r_2 = 1 \times \sin i.$$

Since $i = e$, we can also say that

$$r_1 = r_2 \text{ and } r_1 + r_2 = A. \tag{35-25}$$

Therefore, $r_1 = A/2$. From $\sin i = n \sin r_1$, we get

$$\sin i = n \sin \frac{A}{2}.$$

Now $\delta = i + e - A$. Therefore

$$\delta_{min} = 2i - A$$
$$i = \frac{\delta_{min} + A}{2}$$

or

$$n = \frac{\sin[(\delta_{min} + A)/2]}{\sin(A/2)}. \tag{35-26}$$

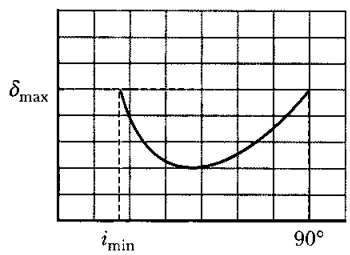


FIG. 35-57 At the maximum deviation, angle of incidence is equal to 90°.

Angle of Maximum Deviation

From Fig. 35-57, it can be easily seen that the deviation is maximum when the angle of incidence is maximum. But we can also note that this same maximum deviation can be obtained at another angle of incidence which we can call as i_{min}. It is clear that the maximum angle of incidence can be 90°. At this angle of

incidence, as we discussed earlier, the angle of emergence will be i_{min}. If we take the angle of incidence as less than i_{min} then r_2 will be greater than the critical angle for prism–air interface and hence total internal reflection will occur at the second surface. At an angle less than i_{min} the ray will not emerge from the prism. Normally, i_{min} can be any value depending on the angle of prism and the refractive index of the prism.

Sample Problem 35-25

A 60° prism has a refractive index of 1.5. (a) Calculate the angle of incidence for minimum deviation.

Conceptualize/Classify: As we have discussed in the section "Angle of Minimum Deviation," the angle of incidence will be equal to the angle of emergence.

Compute: As the prism is in the position of minimum deviation, $r = A/2 = 60°/2 = 30°$, so that at either face

$$\sin i = 1.5 \sin 30° = 0.75$$

$$i = \sin^{-1}(0.75) = 49°.$$

(b) Calculate the angle of emergence of light at maximum deviation.

Conceptualize/Classify: As we have discussed in the section "Angle of Maximum Deviation," the angle of incidence will be equal to 90°.

Compute: For maximum deviation, $i_1 = 90°$ so that $r_1 = \theta_C = \sin^{-1}(2/3) = 42°$. But as in a prism,

$$r_1 + r_2 = A.$$

Therefore,

$$r_2 = A - r_1 = 60° - 42° = 18°.$$

Now applying Snell's law at the second face,

$$n \sin r_2 = \sin i_2, \ \tfrac{3}{2} \sin 18° = \sin i_2.$$

that is,

$$i_2 = \sin^{-1}(1.5 \times 0.31) = \sin^{-1}(0.465) = 28°.$$

Sample Problem 35-26

What is the relationship between A and n, so that no rays come out of second face (Fig. 35-58)?

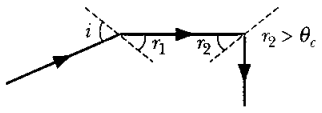


FIG. 35-58 The prism has such a refractive index and an angle of prism combination that all the incident rays undergo total internal reflection.

Conceptualize: Here, we want to choose the refractive index of the prism such that a ray will always undergo total internal reflection. Note that this total internal reflection can occur only when the light ray goes from denser to rarer medium. In other words, this will occur only when the ray is incident on the second surface.

The situation implies that for the smallest angle r_2 also, the total internal reflection should occur.

Classify: $r_1 + r_2 = A$. So when r_2 is minimum, r_1 is maximum. The angle of incidence i is also maximum by Snell's law. But the maximum angle of incidence can be 90°. This problem implies that if a total internal reflection occurs when angle of incidence is 90°, then total internal reflection will occur at all the angles.

Compute: Applying Snell's law at the first surface:

$$1 \times \sin 90° = n \sin r_1$$

$$r_1 = \sin^{-1}\left(\frac{1}{n}\right)$$

$$r_2 = A - \sin^{-1}\left(\frac{1}{n}\right).$$

But for total internal reflection at the second surface:

$$r_2 > \theta_c$$

$$\sin r_2 > 1 \sin \theta_c \left(\frac{1}{n}\right)$$

$$A - \sin^{-1}\left(\frac{1}{n}\right) > \sin^{-1}\left(\frac{1}{n}\right).$$

Thus, the condition becomes

$$n > \operatorname{cosec}\left(\frac{A}{2}\right).$$

Conclude: Many optical instruments, such as binoculars, periscopes, and telescopes, use glass prisms and total internal reflection to turn a beam of light through 90° or 180°.

Thin Prism

A very common situation in the laboratory is a thin prism. This refers to a prism whose angle is very small. As you might have studied in your mathematics classes, this would mean that the small angle approximation can be applied in the case of trigonometric ratios. Interestingly, in this situation, the deviation of the light ray comes out to be independent of the angle of incidence, if the angle of incidence is small too. The angle of deviation is given by

$$\delta = i + e - A.$$

From Eq. 35-25

$$\sin i = n \sin r_1.$$

For small angle of incidence

$$i = nr_1.$$

Since i is a small angle, r_1 turns out to be a small angle too. In turn,

$$r_1 + r_2 = A.$$

So r_2 is also a small angle. This would imply that e is also a small angle.

$$nr_2 = e.$$

Now

$$i + e = n(r_1 + r_2) = nA$$

Therefore,

$$\delta = nA - A$$

or

$$\delta = (n-1)A.$$

Interestingly, all the rays undergo the same deviation.

Sample Problem 35-27

A beam parallel to the principal axis is incident on a thin prism (Fig. 35-59). Find the location of final image.

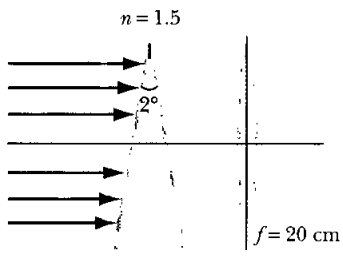


FIG. 35-59 A parallel beam of light is incident on a thin prism. After refraction from the prism, the light is incident on a converging lens.

Conceptualize: Here, two events occur before the formation of image, the first is refraction through the thin prism which will only deviate the parallel beam of rays. This parallel beam will be incident on the lens which would form the image.

Classify: When the rays pass through the prism, all the rays undergo the same deviation. A parallel beam incident on the lens will form an image somewhere on the focal plane.

Compute: The deviation of the rays while passing through the prism is given by:

$$\delta = (n-1)A = 0.5 \times 2 = 1°.$$

The parallel beam of light will pass through the lens. So the image will be formed in focal plane at 20 cm from the lens. The image will be located at an angle θ to the principal axis.

$$\theta = \frac{\pi}{180} \times 1°.$$

The distance of the image from the focal plane is given by

$$\frac{h}{20}=\frac{\pi}{180}$$

$$h=\frac{\pi}{9}\text{ cm}.$$

Conclude: If an object is viewed through a thin prism, all the rays coming from the object are deviated by the same angle. So the image is formed at some point above the object as shown in Fig. 35-60. Note that this is a virtual image.

Shift of image due to prism $= x\delta = x\,(n-1)A$.

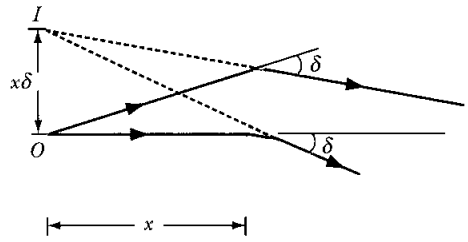


FIG. 35-60 Formation of a virtual image by a prism.

35-9 | Chromatic Dispersion

The index of refraction n encountered by light in any medium except vacuum depends on the wavelength of the light. The dependence of n on wavelength implies that when a light beam consists of rays of different wavelengths, the rays will be refracted at different angles by a surface; that is, the light will be spread out by refraction. This spreading of light is called **chromatic dispersion**, in which "chromatic" refers to the colors associated with the individual wavelengths and "dispersion" refers to the spreading of the light according to its wavelengths or colors. The refractions discussed till now do not show chromatic dispersion because the beams are *monochromatic* (of a single wavelength or color).

Generally, the index of refraction of a given medium is *greater* for a shorter wavelength (corresponding to, say, blue light) than for a longer wavelength (say, red light). As an example, Fig. 35-61 shows how the index of refraction of fused quartz depends on the wavelength of light.

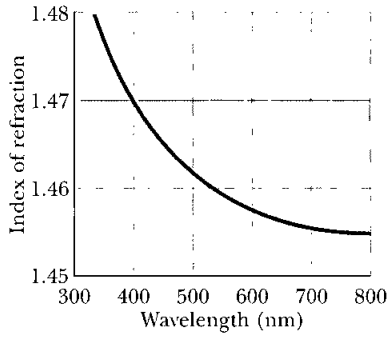


FIG. 35-61 The index of refraction as a function of wavelength for fused quartz. The graph indicates that a beam of short-wavelength light, for which the index of refraction is higher, is bent more upon entering or leaving quartz than a beam of long-wavelength light.

The relationship between n, of refraction of a given medium and the wavelength of light λ is given by

$$n=A+\frac{B}{\lambda^2}.$$

This relation is called **Cauchy's relation**. So if a ray of white light is incident on a boundary at an angle as shown in Fig. 35-62, the angle of refraction of different

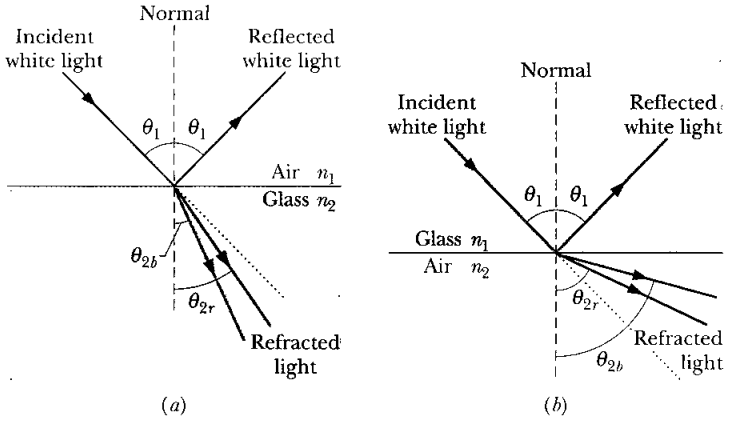


FIG. 35-62 Chromatic dispersion of white light. The blue component is bent more than the red component. (*a*) Passing from air to glass, the blue component ends up with the smaller angle of refraction. (*b*) Passing from glass to air, the blue component ends up with the greater angle of refraction. Each dotted line represents the direction in which the light would continue to travel if it were not bent by the refraction.

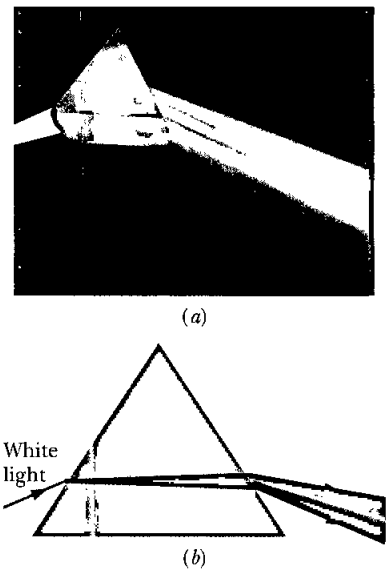


FIG. 35-63 (*a*) A triangular prism separating white light into its component colors. (*b*) Chromatic dispersion occurs at the first surface and is increased at the second surface. *(Courtesy Bausch & Lomb)*

colors will be different. Such dependence means that when a beam made up of waves of both blue and red light is refracted through a surface, such as from air into quartz or vice versa, the blue *component* (the ray corresponding to the wave of blue light) bends more than the red component.

A beam of *white light* consists of components of all (or nearly all) the colors in the visible spectrum with approximately uniform intensities. When you see such a beam, you perceive white rather than the individual colors. In Fig. 35-62*a*, a beam of white light in air is incident on a glass surface. (Because the pages of this book are white, a beam of white light is represented with a gray ray here. Also, a beam of monochromatic light is generally represented with a red ray.)

Of the refracted light in Fig. 35-62*a*, only the red and blue components are shown. Because the blue component is bent more than the red component, the angle of refraction θ_{2b} for the blue component is *smaller* than the angle of refraction θ_{2r} for the red component. (Remember, angles are measured relative to the normal.)

In Fig. 35-62*b*, a ray of white light in glass is incident on a glass—air interface. Again, the blue component is bent more than the red component, but now θ_{2b} is greater than θ_{2r}. To increase the color separation, we can use a solid glass prism with a triangular cross section, as in Fig. 35-63*a*.The dispersion at the first surface (on the left in Figs. 35-63*a*, *b*) is then enhanced by the dispersion at the second surface.

Rainbows

The most charming example of chromatic dispersion is a rainbow. When sunlight (which consists of all visible colors) is intercepted by a falling raindrop, some of the light refracts into the drop, reflects once from the drop's inner surface, and then refracts out of the drop. Figure 35-64*a* shows the situation when the Sun is on the horizon at the left (and thus when the rays of sunlight are horizontal).The first refraction separates the sunlight into its component colors, and the second refraction increases the separation. (Only the red and blue rays are shown in the figure.) If many falling drops are brightly illuminated, you can see the separated colors they produce when the drops are at an angle of 42° from the direction of the *antisolar point A*, the point directly opposite the Sun in your view. To locate the drops, face away from the Sun and point both arms directly away from the Sun, toward the shadow of your head. Then move your right arm directly up, directly rightward, or in any intermediate direction until the angle between your arms is 42°. If illuminated drops happen to be in the direction of your right arm, you see color in that direction.

Because any drop at an angle of 42° in any direction from *A* can contribute to the rainbow, the rainbow is always a 42° circular arc around *A* (Fig. 35-64*b*) and the top of a rainbow is never more than 42° above the horizon. When the Sun is above the horizon, the direction of *A* is below the horizon, and only a shorter, lower rainbow arc is possible (Fig. 35-64*c*).

Since rainbows formed in this way involve one reflection of light inside each drop, they are often called *primary rainbows.* A *secondary rainbow* involves two reflections inside a drop, as shown in Fig. 35-64*d*. Colors appear in the secondary rainbow at an angle of 52° from the direction of *A*. A secondary rainbow is wider and dimmer than a primary rainbow and thus is more difficult to see. Also, the

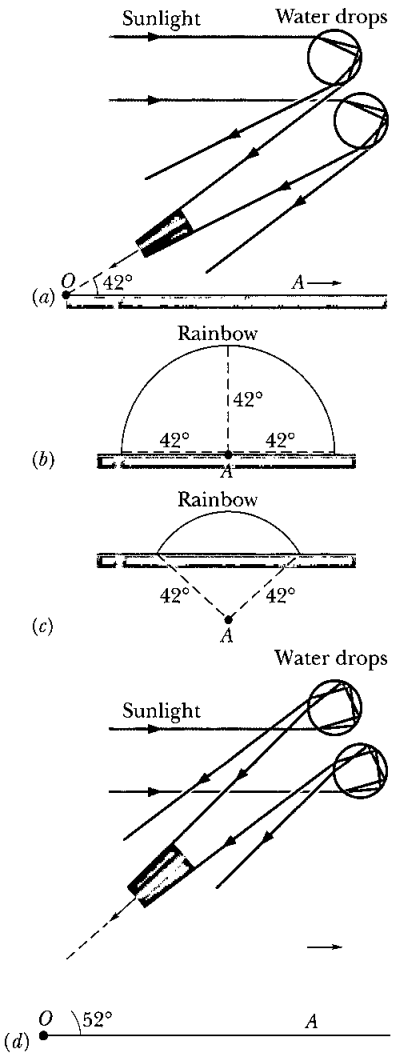


FIG. 35-64 (*a*) The separation of colors when sunlight refracts into and out of falling raindrops leads to a primary rainbow. The antisolar point *A* is on the horizon at the right. The rainbow colors appear at an angle of 42° from the direction of *A*. (*b*) Drops at 42° from *A* in any direction can contribute to the rainbow. (*c*) The rainbow arc when the Sun is higher (and thus *A* is lower). (*d*) The separation of colors leading to a secondary rainbow.

order of colors in a secondary rainbow is reversed from the order in a primary rainbow, as you can see by comparing parts *a* and *d* of Fig. 35-64.

Rainbows involving three or four reflections occur in the direction of the Sun and cannot be seen against the glare of sunshine in that part of the sky. Rainbows involving even more reflections inside the drops can occur in other parts of the sky but are always too dim to see.

Deviation and Dispersion through a Prism

In white light comprising of all colors, as wavelength increases, refractive index decreases. For a thin prism deviation is given by

$$\delta = (n-1)A.$$

From the analysis just presented, we can comment that the deviation decreases as we go from violet to red end of the spectrum. In other words, we can say that the deviation of violet is greater than the deviation of red color as seen in Fig. 35-63. This is why we say ultraviolet and infrared rays and not ultra-red and infra-violet. Let us now try to find difference in the deviation of these rays.

Angle of Dispersion

As shown in Fig. 35-65, we can say that the red ray and violet ray are separated by an angle. This angle is called angle of dispersion θ. It is the angle between emerging red ray and violet ray after a ray of white light is incident on the prism.

So we can see from Fig. 35-65, the deviation or red and violet rays are different. This difference will cause dispersion.

$$\begin{aligned}\theta &= \delta_V - \delta_R \\ &= (n_V - 1)A - (n_R - 1)A \\ &= (n_V - n_R)A.\end{aligned}$$

For any transparent material, dispersive power ω is defined as

$$\omega = \left(\frac{n_V - n_R}{n_Y - 1}\right).$$

It is the property of a given material. By rationalizing it by A, we get,

$$\omega = \frac{(n_V - n_R)A}{(n_Y - 1)A} = \frac{n_V - n_R}{n_Y - 1}$$

$$\omega = \frac{\theta}{\delta} \qquad \text{(for a thin prism).}$$

Normally, ω is much less than 1.

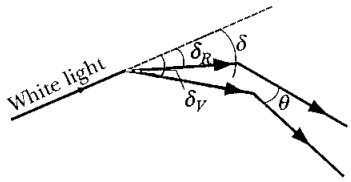


FIG. 35-65 The deviation of violet color is greater than the deviation of red color. The deviation of yellow color is the average of red and violet because the refractive index of yellow is the average of red and violet. The prism is thin but the figure has been exaggerated for clarity.

Sample Problem 35-28

What should be the condition on a combination of prism so that (a) angle of dispersion is zero but angle of deviation is not zero?

Deviation Without Dispersion

Conceptualize: Deviation without dispersion is not possible for a single prism but we can combine two or more prisms so that dispersion is zero. Such a combination is known as achromatic combination of prisms.

Classify: Mathematically we want that

$$\delta_1 + \delta_2 \neq 0,$$

but in this case

$$\theta_1 + \theta_2 = 0.$$

Compute: We have

$$(n_{V_1} - n_{R_1})A_1 + (n_{V_2} - n_{R_2})A_2 = 0.$$

Since $n_V - n_R$ is always greater than 0, this is possible only where A_1 and A_2 are of the opposite signs. That is

$$A_2 = -\left(\frac{n_{V_1} - n_{R_1}}{n_{V_2} - n_{R_2}}\right)A_1.$$

Now

$$\begin{aligned}\delta &= \delta_1 + \delta_2 \\ &= (n_{Y_1} - 1)A_1 + (n_{Y_2} - 1)A_2 \\ &= (n_{Y_1} - 1)A_1 - (n_{Y_2} - 1)\left(\frac{n_{V_1} - n_{R_1}}{n_{V_2} - n_{R_2}}\right)A_1 \\ &= (n_{Y_1} - 1)A_1\left[1 - \left(\frac{n_{V_1} - n_{R_1}}{n_{Y_1} - 1}\right)\left(\frac{n_{Y_2} - 1}{n_{V_2} - n_{R_2}}\right)\right] \\ &= \delta_1\left[1 - \frac{\omega_1}{\omega_2}\right].\end{aligned}$$

(b) Angle of deviation is zero but angle of dispersion is not zero.

Dispersion Without Deviation

Conceptualize: Dispersion without deviation is not possible for a single prism but we can combine two or more prisms so that deviation is zero. Such a combination is known as direct vision prism.

Classify: Mathematically we want that

$$\delta_1 + \delta_2 = 0$$

but in this case

$$\theta_1 + \theta_2 \neq 0.$$

Compute: We know that

$$\delta = \delta_1 + \delta_2 = 0,$$

where δ is average deviation for each prism, that is,

$$(n_{Y_1} - 1)A_1 + (n_{Y_2} - 1)A_2 = 0$$

$$A_2 = -\frac{(n_{Y_1} - 1)A_1}{(n_{Y_2} - 1)}.$$

This would mean that in this case also both angles of the prism are of opposite signs.

$$\theta = \theta_1 + \theta_2.$$

Now

$$\theta_1 = (n_{V_1} - n_{R_1})A_1$$

$$\theta_2 = (n_{V_2} - n_{R_2})A_2$$

$$\theta = (n_{V_1} - n_{R_1})A_1 - (n_{V_2} - n_{R_2})\frac{(n_{Y_1} - 1)A_1}{(n_{Y_2} - 1)}.$$

Substituting the value of A_2, we get

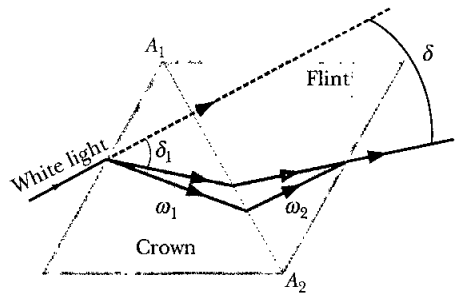
$$(n_{V_1} - n_{R_1})\left[1 - \left[\left(\frac{n_{V_2} - n_{R_2}}{n_{Y_1} - 1}\right)\left(\frac{n_{Y_2} - 1}{n_{V_1} - n_{R_2}}\right)\right]\right].$$

Finally, we have

$$\theta = \theta_1 + \theta_2 = \theta_1\left[1 - \frac{\omega_2}{\omega_1}\right].$$

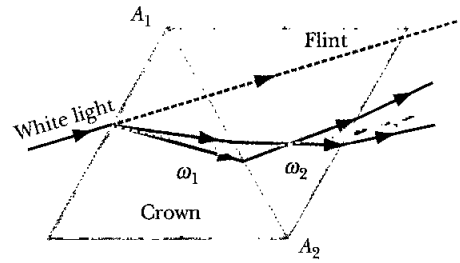
(Only yellow color ray will not deviate, all other colored rays will deviate.)

Conclude: This kind of combination of prisms is possible only if the prisms are kept inverted with respect to each other as shown in the Fig. 35-66.



Deviation without dispersion

(*a*)



Dispersion without deviation

(*b*)

FIG. 35-66 (*a*) Deviation without dispersion: When the crown and flint glass prism are combined as shown, the crown glass prism disperses white light into constituent colors but flint glass recombines them. The resultant beam is deviated from its path by an angle *d*. (*b*) Dispersion without deviation: When the crown and flint glass prism are combined as shown, the crown glass prism as well as flint glass prism disperse the light. The deviation produced by crown glass prism is positive but the deviation produced by the flint glass prism is negative. The net deviation (that of yellow color) is zero.

Chromatic Aberration in Lens

Just as in case of a prism, where deviation depends on the refractive index in the case of a lens the focal length depends on the refractive index of the lens. So obviously the previous discussion implies that the focal length of the lens will depend on the color of light used. Considering the lens to be kept in air, we have

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Since $n_V > n_R$, $f_V < f_R$ for a convex lens.

As is clear from Fig. 35-67, the violet beam will focus closer to the lens and the red beam will focus farther away. This distance between the violet focus and the red focus is known as longitudinal chromatic aberration. To find the distance between the red focus and violet focus, we can differentiate this expression

$$\frac{d}{dn}\left[\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]$$

$$-\frac{1}{f^2}\times\frac{df}{dn}=1\times\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$-\frac{df}{f^2}=dn\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\left(\frac{du}{f(n-1)}\right)$$

$$\frac{df}{f}=-\frac{dn}{n-1}.$$

Now

$$\frac{df}{f^2}=\frac{f_V-f_R}{f_{av}}=\frac{f_V-f_R}{f_Y}=\frac{n_V-n_R}{n_Y-1}=-\omega$$

$$f_R-f_V=-\omega f=\text{ longitudinal chromatic aberration.}$$

So it is clear that for a single lens, there will be some longitudinal chromatic aberration. Lens are commonly used in so many daily apparatus. Imagine taking a photograph with a defective lens. If a man is wearing a red T-shirt and blue jeans, at least one of them will be out of focus due to this aberration. This defect has to be removed. As you can easily guess, this is also done by combining two lens of different nature.

For a lens to be free of chromatic aberration, we should use two lenses in conjugation. For such a combination, we know that the equivalent focal length is given by:

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}.$$

Differentiating this equation, we get

$$\frac{-df_{eq}}{f^2}=\frac{-df_1}{f_1^2}+\frac{-df_2}{f_2^2}.$$

For the aberration of the combination to be 0, df_{eq} should be 0. So

$$\frac{df_1}{f_1^2}+\frac{df_2}{f_2^2}=0 \Rightarrow \quad \frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}=0.$$

Since ω_1 and ω_2 are always positive; at least one of f_1 and f_2 should be negative. Thus, the two lenses should be of opposite nature and should be kept close to each other.

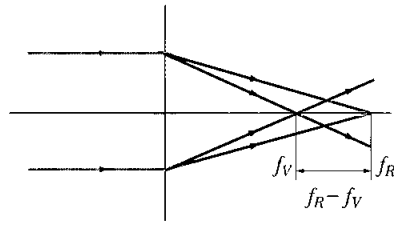


FIG. 35-67 A parallel beam of white light represented by gray colored line is incident on a lens. The lens splits the incident ray into its constituent colors of which only red and violet are shown. The red rays will converge farther than the violet rays.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: The Sun seems to rise before it actually rises above horizon and it seems to set after some time it actually sets below horizon.

Statement 2: A light ray bends toward the normal when it passes from a rarer to denser medium.

2. Statement 1: It is possible for a given lens to act as a converging lens in one medium, and as a diverging lens in another.

Statement 2: The focal length of the lens depends on the refractive index of the medium.

3. Statement 1: In vacuum, light of red color travels faster than light of violet color.

Statement 2: The refractive index depends on the wavelength and its value decreases with increase in wavelength.

4. Statement 1: On the Earth, summer occurs when the Earth is closest to the Sun and winter when the Earth is farthest from the Sun.

Statement 2: Intensity due to the Sun at a distance x from its center is inversely proportional to x^2.

5. Statement 1: The object and image both lie on the opposite side of focus for a spherical lens.

Statement 2: By Newton's formula, $x_o x_i = -f^2$.

6. Statement 1: The line joining object and its image formed by a thin lens passes through the optical center.

Statement 2: The ray passing through the optical center goes undeviated.

7. Statement 1: The image focus and the object focus are on the opposite side of the lens.

Statement 2: The radii of curvature of a biconvex lens and biconcave lens are on the opposite side of the lens.

8. Statement 1: Human eye cannot form image of a converging beam on the retina.

Statement 2: The light rays coming after refraction from a converging lens can never be diverging in nature.

9. Statement 1: A convex lens suffers from chromatic aberration.

Statement 2: The lens formula is strictly valid only for paraxial rays.

10. Statement 1: For a thin lens in air, there are two foci which are situated in different positions with respect to lens.

Statement 2: The center of curvature of the two surfaces is different.

MULTIPLE CHOICE QUESTIONS

sec. 35-2 Snell's Laws of Refraction

1. A laser beam and a sound wave from directional sources both enter a liquid at an angle of 60° from the horizontal surface of the liquid. The speed of sound in the liquid is 5 times than that of the speed of the sound in air. For light, the refractive index of the liquid is 1.8. What happens?

(a) Both the light and the sound refract at an angle of 74° from the horizontal

(b) The light refracts to an angle of 26° from the horizontal and the sound refracts to an angle of 74° from the horizontal.

(c) The light refracts to an angle of 26° from the horizontal and the sound reflection completely off the surface.

(d) The light refracts to an angle of 74° from the horizontal and the sound refracts to an angle of 26° from the horizontal.

sec. 35-3 Apparent Depth and Normal Shift

2. A tiny air bubble in a glass slab ($\mu = 1.5$) appears from one side to be 6 cm from the glass surface and from other side, 4 cm. The thickness of the glass slab is

(a) 10 cm

(b) 6.67 cm

(c) 15 cm

(d) None of these

3. A bird is flying 3 m above the surface of water. To a fish underwater, the height of the bird from the water surface appears to be

(a) 2.25 m

(b) 3.33 m

(c) 4 m

(d) 4/3 m

4. In the previous question, if the bird is diving vertically down with speed = 6 m/s, his apparent speed as seen by a stationary fish underwater is

(a) 8 m/s

(b) 6 m/s

(c) 12 m/s

(d) 4 m/s

5. A bucket of total height 60 cm is half filled with a liquid of index 1.5 and half with another liquid of index 2. The apparent depth of the bucket for an observer directly above the bucket

(a) 45 cm

(b) 30 cm

(c) 35 cm
(d) 45 cm

6. A flat glass slab of thickness 6 cm and index 1.5 is placed in front of a plane mirror. An observer is standing behind the glass slab and looking at the mirror. The actual distance of the observer from the mirror is 50 cm. The distance of his image from himself, as seen by the observer is

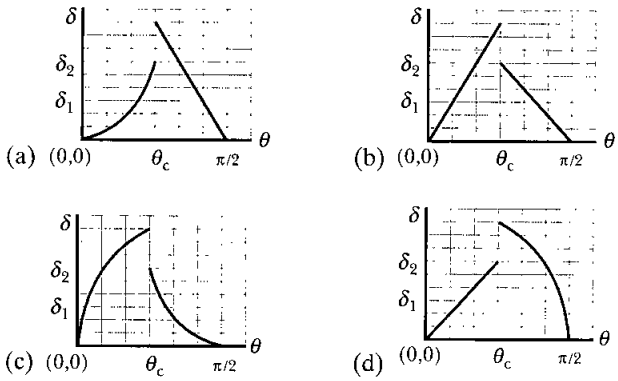
(a) 94 cm
(b) 96 cm
(c) 98 cm
(d) 100 cm

sec. 35-4 Total Internal Reflection

7. Critical angle of glass is θ_1 and that of water is θ_2. The critical angle for water and glass surface would be ($\mu_g = 3/2$, $\mu_w = 4/3$)

(a) Less than θ_2
(b) Between θ_1 and θ_2
(c) Greater than θ_2
(d) Greater than θ_1

8. A ray of light travels from a medium of refractive index μ into air. If the angle of incidence at the plane surface of separation is θ and the corresponding angle of deviation is δ, the variation δ with θ is shown correctly in the figure



sec. 35-5 Refraction at Spherical Surfaces

9. A real object is kept in air and refraction occurs at the convex boundary of a spherical glass surface. For the image to be real, the real object distance ($\mu g = 3/2$)

(a) Should be greater than three times the radius of curvature of the refracting surface
(b) Should be lesser than two times the radius of curvature of the refracting surface
(c) Should be greater than the radius of curvature of the refracting surface
(d) Is independent of the radius of curvature of the refracting surface

10. Half the surface of a transparent sphere of refractive index 2 is silvered. A narrow, parallel beam of light is incident on the unsilvered surface, symmetrically with respect to the silvered part. The light finally emerging from the sphere will be a

(a) Parallel beam
(b) Converging beam
(c) Slightly divergent beam
(d) Widely divergent beam

11. A solid, transparent sphere has a small, opaque dot at its center. When observed from outside, the apparent position of the dot will be

(a) Closer to the eye than its actual position
(b) The same as its actual position
(c) Farther away from the eye than its actual position
(d) Independent of the refractive index of the sphere

sec. 35-6 Thin Spherical Lens

12. An object was placed 25 cm in front of a converging lens with a focal length of 20 cm. A concave mirror with a focal length of 15 cm was placed 120 cm behind the lens. Which of these describes the final image?

(a) Real, erect
(b) Virtual, erect
(c) Real, inverted
(d) Virtual, inverted

13. A lens behaves as a converging lens in air and as a diverging lens in water. If refractive index of water is 1.33, the refractive index of the material of the lens will be

(a) Equal to unity
(b) Equal to 1.33
(c) Between unity and 1.33
(d) Greater than 1.33

14. A convergent lens of focal length 20 cm and made of a material with refractive index 1.1 is immersed in water. The lens will behave as a

(a) Converging lens of focal length 20 cm
(b) Converging lens of focal length less than 20 cm
(c) Converging lens of focal length more than 20 cm
(d) Divergent lens

15. A parallel beam of light is incident on a converging lens parallel to its principal axis. As one moves away from the lens on the other side on its principal axis, the intensity of light

(a) Remains constant
(b) Continuously increases
(c) Continuously decreases
(d) First increases then decreases

16. Which one of the rays of the converging lens shown in Fig. 35-68 is not correct?

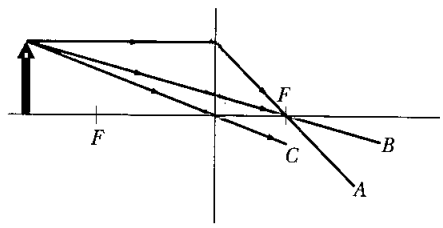


FIG. 35-68 Multiple choice question 16.

(a) *A*
(b) *B*

(c) C

(d) None of these

17. Which of the rays in Fig. 35-69 for the diverging lens shown is not correct?

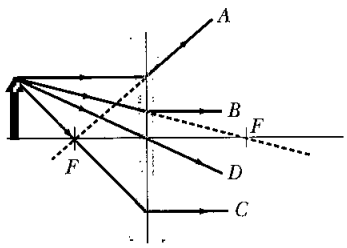


FIG. 35-69 Multiple choice question 17.

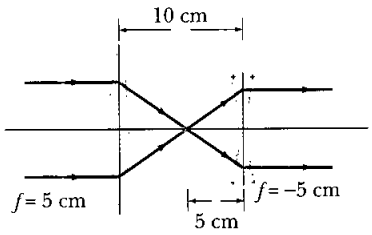
(a) A

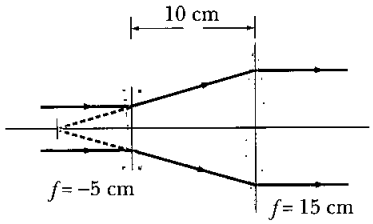
(b) B

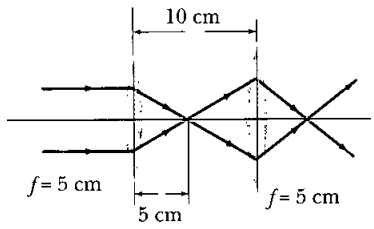
(c) C

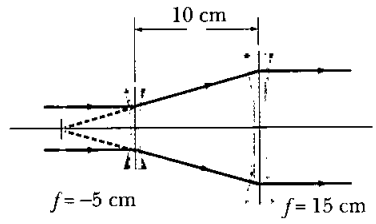
(d) D

18. Which one of these ray diagrams is most accurate?

(a)


(b)


(c)


(d)


19. A point source of light is placed at a distance of $2f$ from a converging lens of focal length f. The intensity on the other side of the lens is maximum at a distance

(a) f

(b) Between f and $2f$

(c) $2f$

(d) More than $2f$

20. A converging lens can from a virtual image if the object is placed

(a) Between the lens and its focus

(b) At the focus of the lens

(c) Between f and $2f$

(d) At infinity

21. A concave mirror and a convex lens are of the same focal length in air. When they are immersed in water

(a) The concave mirror will have its focal length increased

(b) The convex lens will have its focal length increased

(c) They will have equal focal lengths, different from those in air

(d) They will have equal focal lengths, same as those in the air

22. As shown in Fig. 35-70, an object is placed on the left side of a lens with its focal points labeled as "F." A flat mirror is placed at the image focus, with its reflecting surface facing the lens. The image of the object is at

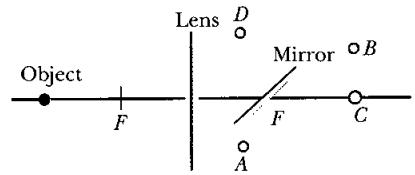


FIG. 35-70 Multiple choice question 22.

(a) Point A

(b) Point B

(c) Point C

(d) Point D

23. A section of a spherical shell of outer radius R_0 and inner radius R_1 is cut, and used as a lens. The light first strikes what was originally the outer surface of the glass shell. Which of the following statements about the resulting lens is true?

(a) The lens is converging

(b) The lens is diverging

(c) The lens has zero focal length

(d) The lens has infinite focal length

24. A slab of high quality flat glass, with parallel faces, is placed in the path of a parallel light beam before it is focused to a spot by a lens. The glass is rotated slightly back and forth from the vertical orientation, about an axis coming out of the page, as shown in Fig. 35-71. According to ray optics the effect on the focused spot is

(a) There is no effect on the spot

(b) The spot moves toward then away from the lens

(c) The spot moves up and down parallel to the lens

(d) The spot blurs out of focus

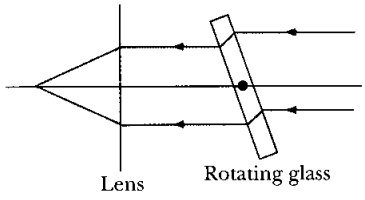


FIG. 35-71 Multiple choice question 24.

25. A biconcave glass lens having refractive index 1.5 has both surfaces of same radius of curvature R. On immersion in a medium of refractive index 1.75, it will behave as a

(a) Convergent lens of focal length 3.5 R

(b) Convergent lens of focal length 3.0 R

(c) Divergent lens of focal length 3.5 R

(d) Divergent lens of focal length 3.0 R

26. When an object is at a distance u_1 and u_2 from the optical center of a lens, a real and virtual image are formed, respectively, with the same magnification. The focal length of lens is

(a) $u_1 + u_2$

(b) $(u_1 + u_2)/2$

(c) $u_1 + u_2/2$

(d) $\sqrt{u_1 \cdot u_2}$

27. A virtual image larger than the object can be produced by

(a) Convex mirror

(b) Concave mirror

(c) Diverging lens

(d) Converging lens

28. If a convergent beam of light passes through a diverging lens, the result

(a) May be a convergent beam

(b) May be a divergent beam

(c) May be a parallel beam

(d) Must be a parallel beam

29. Which of the following form virtual and erect image for all positions of a self luminous subject

(a) Convex lens

(b) Concave lens

(c) Convex mirror

(d) Concave mirror

30. When the object is moved slightly closer to a converging lens, the image may

(a) Increase in size and move closer to the lens

(b) Increase in size and move farther away from the lens

(c) Decrease in size and move closer to the lens

(d) Decrease in size and move farther away from the lens

31. A convex lens forms an image of an object on a screen. The height of the image is 9 cm. The lens is now displaced until an image is again obtained on the screen. The height of this image is 4 cm. The distance between the object and the screen is 90 cm

(a) The distance between the two positions of the lens is 30 cm

(b) The distance of the object from the lens in its first position is 36 cm

(c) The height of the object is 6 cm

(d) The focal length of the lens is 21.6 cm

sec. 35-7 Lens Mirror Combination

32. A concavo-convex glass (index = 1.5) lens has radii of curvatures 60 cm and 40 cm, respectively. Its convex surface is silvered and it is placed on a horizontal table with concave surface up. The concave surface is then filled with a liquid of index 2.0. The combination behaves like

(a) Concave mirror

(b) Convex mirror

(c) Flat mirror

(d) Convex lens

sec. 35-8 Prism

33. A beam of light enters side 1 of a triangular piece of plastic (a prism) of index of refraction 1.5 as shown in Fig. 35-72. The three sides of the prism are of equal length. Which of the following statements is correct?

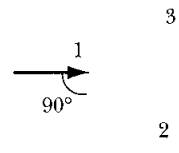


FIG. 35-72 Multiple choice question 33.

(a) The beam will emerge from side 1

(b) The beam will emerge from side 2

(c) The beam will emerge from side 3

(d) The beam will not emerge

34. Angle of minimum deviation is equal to the angle of prism A of an equilateral glass prism. The angle of incidence at which minimum deviation will be obtained is

(a) 60°

(b) 45°

(c) 30°

(d) $\sin^{-1}(2/3)$

35. A ray of light passes through an equilateral prism such that the angle of incidence is equal of emergence and later is equal to $3/4^{th}$ the angle of prism. The angle of deviation is

(a) 45°

(b) 20°

(c) 39°

(d) 30°

36. Light is incident normally on face AB of a prism as shown in Fig. 35-73. A liquid of refractive index n is placed on face AC of the prism. The prism is made of glass of refractive index 3/2. The limits of n for which total internal reflection takes place on face AC is

(a) $n > \dfrac{3\sqrt{3}}{2}$

(b) $n > \sqrt{3}$

(c) $n < \frac{3\sqrt{3}}{4}$

(d) $n < \frac{3\sqrt{3}}{8}$

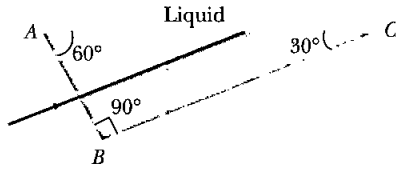


FIG. 35-73 Multiple choice question 36.

sec. 35-9 Chromatic Dispersion

37. The critical angle for glass to air refraction is least for which color?

(a) Orange

(b) Blue

(c) Violet

(d) Red

38. A ray of red light passes through a series of media with different refractive indices, n_1, n_2, and n_3. It then enters a medium with an index of refraction n_1 as shown in Fig. 35-74. If the incident ray of light in the figure is blue (instead of red), it will emerge from the interface between n_3 and n_1

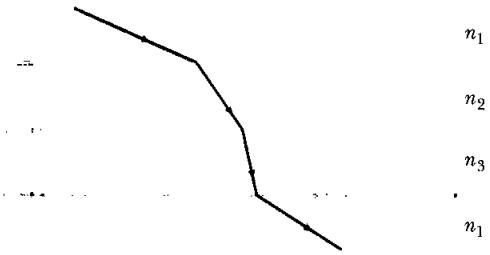


FIG. 35-74 Multiple choice question 38.

(a) To the right of where the red ray emerged

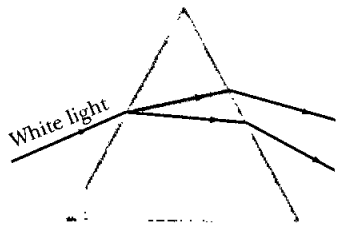
(b) At the same point as where the red ray emerged

(c) To the left of where the red ray emerged

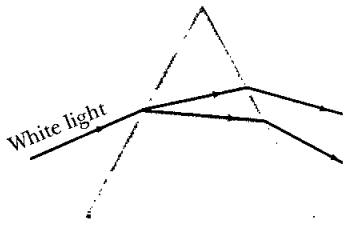
(d) Any of the above is possible

39. A prism is made of glass which has a higher index of refraction for violet light than for red light. Which diagram best indicates the paths of red and violet light rays through the prism?

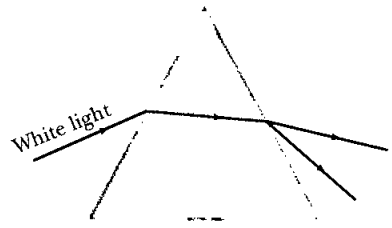
(a)



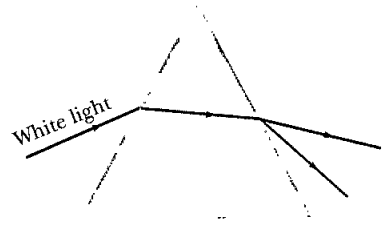
(b)



(c)



(d)



40. The focal lengths of a convex lens for blue and red colors of light are f_B and f_R, respectively, and those of a concave lens are F_B and F_R then

(a) $f_B > f_R$ and $F_B < F_R$

(b) $f_B < f_R$ and $F_B > F_R$

(c) $f_B > f_R$ and $F_B > F_R$

(d) $f_B < f_R$ and $F_B < F_R$

41. A secondary rainbow is formed when light rays coming from the Sun undergo the following through spherical water droplets

(a) A refraction, internal reflection, and then refraction

(b) Two refractions only

(c) A refraction, internal reflection, again internal reflection, and then refraction

(d) A refraction, internal reflection, and again internal reflection

42. The dispersive powers of two lenses are 0.01 and 0.02. If focal length of first lens is +10 cm, then what should be the focal length of the second lens, so that they form an achromatic combination?

(a) Diverging lens having focal length 20 cm

(b) Converging lens having focal length 20 cm

(c) Diverging lens having focal length 10 cm

(d) Converging lens having focal length 10 cm

43. Focal lengths of two lenses are f and f' and dispersive powers of their materials are ω and ω'. To form achromatic combination from these, which relation is correct?

(a) $\omega = \omega_0, \omega' = 2\omega_0, f' = 2f$

(b) $\omega = \omega_0, \omega' = 2\omega_0, f' = f/2$

(c) $\omega = \omega_0, \omega' = 2\omega_0, f' = -2f$

(d) $\omega = \omega_0, \omega' = 2\omega_0, f' = -f/2$

PROBLEMS

sec. 35-2 Refraction of Light

1. Light in vacuum is incident on the surface of a glass slab. In the vacuum the beam makes an angle of 37° with the normal to the surface, while in the glass it makes an angle of 23.5° with the normal. What is the index of refraction of the glass?

2. When the rectangular metal tank in Fig. 35-75 is filled to the top with an unknown liquid, observer O, with eyes level with the top of the tank, can just see corner E. A ray that refracts toward O at the top surface of the liquid is shown. If $D = 85.0$ cm and $L = 85$ cm, what is the index of refraction of the liquid?

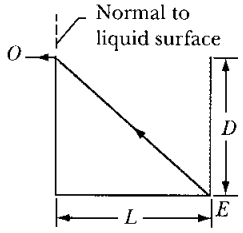


FIG. 35-75 Problem 2.

3. In Fig. 35-76, a 2.00-m long vertical pole extends from the bottom of a swimming pool to a point 50.0 cm above the water. Sunlight is incident at angle $\theta = 37.0°$. What is the length of the shadow of the pole on the level bottom of the pool?

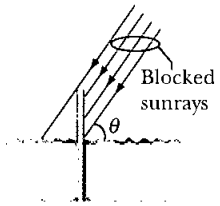


FIG. 35-76 Problem 3.

4. A small piece of wood is floating in the water of depth 1 m. The refractive index of water is 4/3 and the sun rays are incident at an angle of 53° to the vertical. Find the location of the shadow of the wood piece on the bottom of the ocean.

5. A person looking through the telescope just sees the point on the rim at the bottom of an empty cylindrical vessel. When the vessel is completely filled with a liquid of refractive index 1.5, he observes a mark at the center of the bottom without moving the telescope or the vessel. What is the height of the vessel if the diameter of the cross section is 10 cm (Fig. 35-77)?

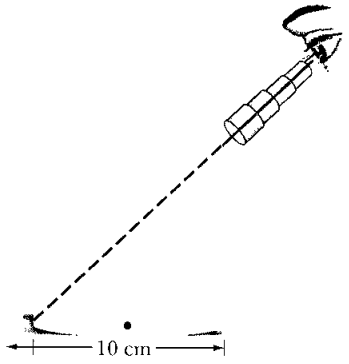


FIG. 35-77 Problem 5.

6. A ray of light falls on a transparent sphere with center at C as shown in Fig. 35-78. The ray gets refracted from the sphere and emerges from the sphere parallel to the line AB. With the help of ray diagram, (a) find out the angle of refraction. (b)What is the refractive index of the sphere?

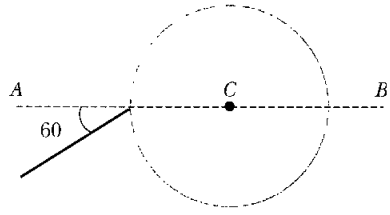


FIG. 35-78 Problem 6.

7. Light rays travel from liquid (refractive index = $\sqrt{2}$) to air. (a) What is the maximum angle of deviation? (b) What is the angle of incidence corresponding to this maximum angle of deviation? (c) At what angle(s) of incidence can the angle of deviation be 30°?

sec. 35-3 Apparent Depth and Normal Shift

8. A 20-mm-thick layer of water ($n = 1.33$) floats on a 40-mm-thick layer of carbon tetrachloride ($n = 1.46$) in a tank. A coin lies at the bottom of the tank. At what depth below the top water surface do you perceive the coin?

9. In Fig. 35-79, a fish watcher at point P watches a fish through a glass wall of a fish tank. The watcher is level with the fish; the index of refraction of the glass is 8/5, and that of the water is 4/3. The distances are $d_1 = 8.0$ cm, $d_2 = 3.0$ cm, and $d_3 = 6.8$ cm. To the fish, how far away does the watcher appear to be?

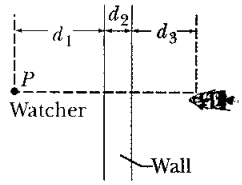


FIG. 35-79 Problem 9.

10. A rectangular slab of refractive index μ is placed over another slab of refractive index 3. Both slabs have identical dimensions. If a small coin is placed below the lower slab, for what value of μ will the coin appear at the interface between the two slabs, when viewed from the top?

11. A concave mirror of radius of curvature 1 m, placed at the bottom of a water tank, forms an image of the Sun when it is directly overhead. Determine the distance of the image from the mirror for a depth of 80 cm of water in the tank. Refractive index of water = 4/3.

12. A slab of glass of surface index 1.5 and thickness of 3 cm is placed with the faces perpendicular to the principal axis of the concave mirror. If the radius of the curvature of the mirror is 10 cm, find where an object must be placed so that the image coincides with the object?

13. A fish is rising up vertically inside a pond with velocity of 4 cm/s, and notices a bird, which is diving vertically downward and its velocity appears to be 16 cm/s (to the fish). What is the real velocity of the diving bird, if refractive index of water is 4/3?

sec. 35-4 Total Internal Reflection

14. The index of refraction of benzene is 1.8. What is the critical angle for a light ray traveling in benzene toward a flat layer of air above the benzene?

15. A point source of light is 80.0 cm below the surface of a body of water. Find the diameter of the circle at the surface through which light emerges from the water.

16. In the ray diagram of Fig. 35-80, where the angles are not drawn to scale, the ray is incident at the critical angle on the interface between materials 2 and 3. Angle $\phi = 60.0°$, and two of the indexes of refraction are $n_1 = 1.70$ and $n_2 = 1.60$. Find (a) index of refraction n_3 and (b) angle θ. (c) If θ is decreased, does light refract into material 3?

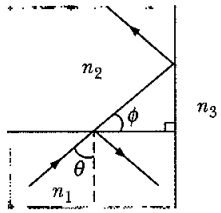


FIG. 35-80 Problem 16.

17. In Fig. 35-81, a ray of light is perpendicular to the face *ab* of a glass prism ($n = 1.50$). Find the largest value for the angle ϕ so that the ray is totally reflected at face *ac* if the prism is immersed (a) in air and (b) in water.



FIG. 35-81 Problem 65.

sec. 35-5 Refraction at Spherical Surfaces

18. A glass sphere has radius R = 5.0 cm and index of refraction 1.6. A paperweight is constructed by slicing through the sphere along a plane that is 2.0 cm from the center of the sphere, leaving height h = 3.0 cm. The paperweight is placed on a table and viewed from directly above by an observer who is at distance d = 8.0 cm from the tabletop (Fig. 35-82). When viewed through the paperweight, how far away does the tabletop appear to be to the observer?

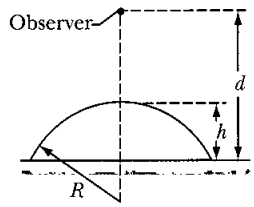


FIG. 35-82 Problem 18.

19. In Fig. 35-83, a beam of parallel light rays from a laser is incident on a solid transparent sphere of index of refraction n. (a) If a point image is produced at the back of the sphere, what is the index of refraction of the sphere? Assume paraxial rays. (b) What index of refraction, if any, will produce a point image at the center of the sphere?

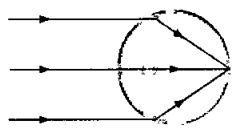


FIG. 35-83 Problem 19.

20. The boundary surface of the media of refractive indices n_1 and n_2 is spherical with radius R (Fig. 35-84). Find the location of a point on axis in medium I where we can keep a point object such that after refraction, the rays become parallel to the axis. (use only paraxial rays).

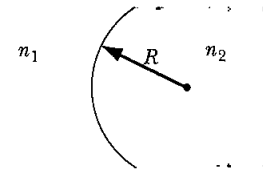


FIG. 35-84 Problem 20.

21. One end of a glass rod 8 cm in diameter has a hemispherical surface of 4 cm radius. Determine the position of the image of an object placed on the axis at following distance from the hemispherical end (a) at infinity, (b) 16 cm, (c) 4 cm, refractive index of the glass is 1.5

22. A goldfish in a spherical fish bowl of radius R is at the level of the center C of the bowl and at distance $R/2$ from the glass (Fig. 35-85). What magnification of the fish is produced by the water in the bowl for a viewer looking along a line that includes the fish and the center, with the fish on the near side of the center? The index of refraction of the water is 1.33. Neglect the glass wall of the bowl. Assume the viewer looks with one eye.

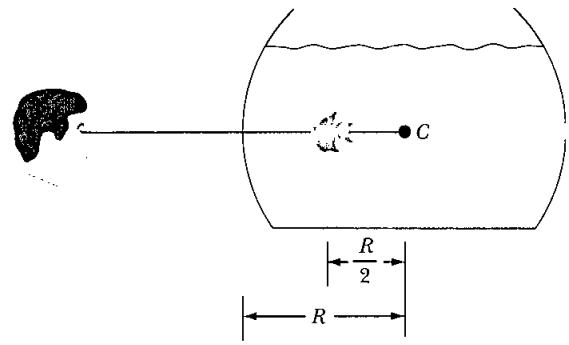


FIG. 35-85 Problem 22.

23. Two spherical-shaped equal-sized glass fishbowls filled with water are standing on the table. In the middle of each fishbowl a small fish is swimming. What will be the distance of the image as seen by another? The refraction index of water is 4/3. Take radius = R and distance between the centers of fishbowls as $3R$.

24. When viewed normally through the flat surface the apparent thickness of a plano-convex lens appears to be 2 mm. If the radius of curvature of the spherical surface is 10 cm then find thickness of the lens when viewed from the curved surface. Refractive index of the glass is 1.5.

sec. 35-6 Thin Spherical Lens

25. There are two spherical surface of radii R_1 = 30 cm and R_2 = 60 cm. (a) In how many ways these surfaces may be arranged to get different lenses. (b) If all the lenses are made of glass ($\mu = 1.5$), find the focal length of each lens.

26. A double-convex lens is to be made of glass with an index of refraction of 1.5. One surface is to have twice the radius of curvature of the other and the focal length is to be 60 mm. What is the (a) smaller and (b) larger radius?

27. Two watch glasses of radii of curvature 10 cm and 30 cm are cemented at the edges to form an air convex lens. (a) What is its focal length in air and water? (b) Is it convergent or divergent in water?

28. A movie camera with a (single) lens of focal length 75 mm takes a picture of a person standing 27 m away. If the person is 180 cm tall, what is the height of the image on the film?

29. An illuminated slide is held 44 cm from a screen. How far from the slide must a lens of focal length 11 cm be placed (between the slide and the screen) to form an image of the slide's picture on the screen?

30. What area of the ground below can be photographed at one time by a camera of focal length 45 cm and plate size 3 cm × 3 cm kept in a satellite at height of 1500 km?

31. A beam of light converges to a point on a screen. If a lens is placed in the path of the beam at 10 cm away from the screen, the screen has to be moved through 8 cm to obtain a sharp image. (a) Find the focal length of the lens and (b) determine whether it is concave or convex.

32. A convex lens focuses an object 40 cm from it on a screen placed 10 cm away from it. A glass plate ($\mu = 1.5$) and of thickness 3 cm is inserted between the lens and the screen. Where should the object be placed so that its image is again focused on the screen?

33. Figure 35-86 shows a *beam expander* made with two coaxial converging lenses of focal lengths f_1 and f_2 and separation $d = f_1 + f_2$. The device can expand a laser beam while keeping the light rays in the beam parallel to the central axis through the lenses. Suppose a uniform laser beam of width W_i = 2.5 mm and intensity I_i = 9.0 kW/m^2 enters a beam expander for which f_1 = 12.5 cm and f_2 = 30.0 cm. What are (a) W_f and (b) I_f of the beam leaving the expander? (c) What value of d is needed for the beam expander if lens 1 is replaced with a diverging lens of focal length $f_1 = -26.0$ cm?

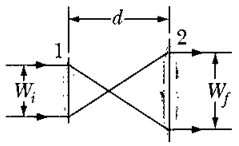


FIG. 35-86 Problem 33.

34. In Fig. 35-87, an object is placed in front of a converging lens at a distance equal to twice the focal length f_1 of the lens. On the other side of the lens is a concave mirror of focal length f_2 separated from the lens by a distance $2(f_1 + f_2)$. Light from the object passes rightward through the lens, reflects from the mirror, passes leftward through the lens, and forms a final image of the object. What are (a) the distance between the lens and that final image and (b) the overall lateral magnification M of the object? Is the image (c) real or virtual (if it is virtual, it requires someone looking through the lens toward the mirror), (d) to the left or right of the lens, and (e) inverted or noninverted relative to the object?

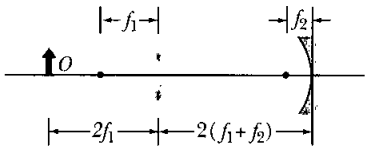


FIG. 35-87 Problem 98.

35. A small fish, 0.4 m below the surface of a lake, is viewed through a simple converging lens of focal length 3m. The lens is kept at 0.2m above the water surface such that the fish lies on the optical axis of the lens. Find the distance of the image of the fish from the lens as seen by the observer. The refractive index of water is 4/3.

36. A beam of light diverges from P on the axis of a convex lens and the after passing through the lens is reflected from the surface of a convex mirror. The reflected beam is brought to a focus by the lens at P itself. (a) Find the focal length of the lens. Given that the distance of the lens and the mirror is 10 cm. The distance of P from the mirror is 30 cm and the focal length of the mirror is 10 cm.

37. Find the nature and focal length of a lens which must be placed in contact with a concave lens of focal length 25 cm in order that the lens combination may produce a real image of an object placed at infinity.

38. Two thin symmetrical lenses of different nature and of different material have equal radii of curvature R = 15 cm. The lenses are put close together and immersed in water ($\mu_w = 4/3$). The focal length of the system in water is 30 cm. What is the difference between refractive indices of the two lenses?

39. An object is placed at a certain distance from a screen. A convex lens of focal length 20 cm is placed between the screen and the object. A real image is formed on the screen for two positions of the lens, which differ by a distance of 10 cm. What is the distance of the object from the screen?

40. A convex lens A of focal length 20 cm and a concave lens B of focal length 5 cm are kept along the same axis with a distance d between them. If a parallel beam of light falling on A leaves B as a parallel beam then what will be the distance d?

sec. 35-7 Lens Mirror Combination

41. A plano-convex lens of focal length 40 cm and refractive index = 1.5 is silvered at plane surface. (a) What will be its focal length in air? (b) What will be the equivalent mirror behave like? (c) What will be its focal length when it is silvered at curved surface instead of being silvered at the plane surface?

42. Two identical thin watch glasses are joined together with air inside as shown in Fig. 35-88. The surface B is silvered. A sharp image of object placed on optic axis at 20 cm from the lens is formed at distance 10 cm from lens. Find the position of the sharp image when the space between the glasses is filled with water ($\mu_w = 4/3$).

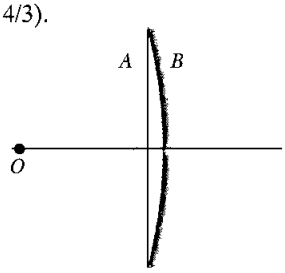


FIG. 35-88 Problem 42.

sec. 35-8 Prism

43. A prism of refractive index $\sqrt{2}$ has refracting angle = 60°. What should be the angle of incidence so that a light ray suffers minimum deviation?

44. The cross section of a glass prism $\mu = 2$ has the form of an isosceles right angled triangle. It is submerged in a liquid of refractive index 1.2. A ray is incident on to one of the equal faces perpendicularly. What is the angle between the direction of incident ray and the ray that emerges from the prism?

45. A ray of light is incident at angle i on a surface of a prism of small angle A and emerges normally from the opposite

surface. If the refractive index of the material of the prism is μ, what is the approximate value of the angle of incidence?

46. A prism having refractive index $\sqrt{2}$ and refracting angle 30° has one of the refracting surfaces polished. A beam of light incident on the other refracting surface is seen to trace its path. What is the angle of incidence?

47. Find the (a) angle of minimum and (b) the maximum deviation for an equilateral prism having refractive index $n = \sqrt{3}$.

48. Suppose the prism of Fig. 35-89 has apex angle $\phi = 60.0°$ and index of refraction $n = 1.60$. (a) What is the smallest angle of incidence θ for which a ray can enter the left face of the prism and exit the right face? (b) What angle of incidence θ is required for the ray to exit the prism with an identical angle θ for its refraction, as it does in Fig. 35-89

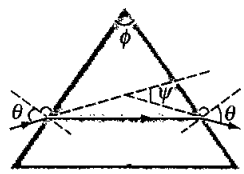


FIG. 35-89 Problem 48.

49. In Fig. 35-90, a light ray enters a glass slab at point A at incident angle $\theta_1 = 45.0°$ and then undergoes total internal reflection at point B. What minimum value for the index of refraction of the glass can be inferred from this information?

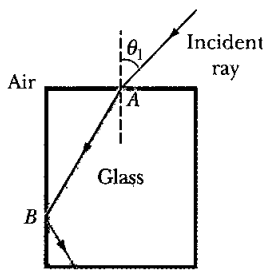


FIG. 35-90 Problem 73.

sec. 35-9 Chromatic Dispersion

50. White light falls on one face of crown glass prism of refracting angle 50°. The angle of incidence is 30°. Find the (a) mean deviation produced by the prism and (b) the angle of dispersion ($\mu_v = 1.538, \mu_r = 1.52$).

51. A flint glass prism and a crown glass prism are to be combined in such a way that the deviation of the mean ray is zero. The refractive index of the flint glass for the mean ray is 1.62 and for the crown glass it is 1.518. If the angle of the flint glass prism is 6°, what would be refracting angle of the crown prism.

52. The dispersive powers of crown and flint glasses are 0.03 and 0.05, respectively. The refractive index for yellow light for these glasses are 1.518 and 1.62, respectively. If we want to make an achromatic combination of these prisms that produces a mean deviation of 1° what are the angles of the two prisms needed?

53. A quartz glass prism of refracting angle 4° is to be combined with a crown glass prism to create a direct-vision prism combination. (a) Find the refracting angle of the crown prism. (b) Find the angular width of the spectrum found by the combination. (For crown glass $\mu_R = 1.52, \mu_v = 1.56$; and for quartz glass $\mu_R = 1.70, \mu_v = 1.74$.)

54. Focal lengths of two lenses are 10 cm and −20 cm and dispersive powers of their materials are 0.03 and ω. To form achromatic combination from these, (a) what is the value of ω? (b) What is the resulting focal length of the combination?

ADDITIONAL PROBLEMS

55. One end of a long glass rod ($n = 1.5$) is a convex surface of radius 6.0 cm. An object is located in air along the axis of the rod, at a distance of 10 cm from the convex end. (a) How far apart are the object and the image formed by the glass rod? (b) Within what range of distances from the end of the rod must the object be located in order to produce a virtual image?

56. An object is placed 1 cm above the principal axis of a convex lens of focal length of 40 cm. Object distance is 60 cm. If the object now starts moving perpendicularly away from optical axis with a speed of 10 cm/s, then what is the speed of the image?

57. Optic axis of a thin equiconvex lens is the x axis. The coordinates of point object and its image are (−40 cm, 1 cm) and (50 cm, −2 cm), respectively. (a) What is the location of the lens? (b) What is the focal length of the lens? (c) Where are their image focus and object focus located?

58. A point object is placed on the optic axis of a convex lens of focal length f at a distance of $2f$ to the left of it. The diameter of the lens is 'd.' An eye is placed at a distance of $3f$ to the right of the lens and a distance h below the optic axis. What is the maximum value of h to see the image?

59. A small air bubble O (as shown in Fig. 35-91) is formed inside the solid glass sphere of radius 20 cm. The refractive index of glass is $n = 1.5$. The position of air bubble is at a distance 5 cm from center. If one half of sphere is polished then find the distance between the images seen by observer on the left side of sphere. Assume paraxial rays conditions are satisfied.

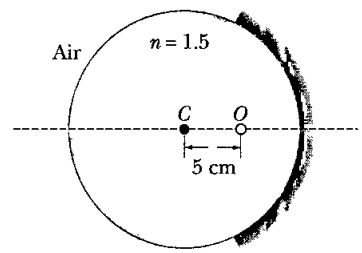


FIG. 35-91 Problem 59.

60. The cross section of a prism is shown in Fig. 35-92. One of its refracting surfaces (AC) is given by $y = x^2$. A ray of light traveling parallel to x axis is incident normally on face AB and refracted. Find the minimum distance of the incident ray from point A to get refracted ray from surface AC.

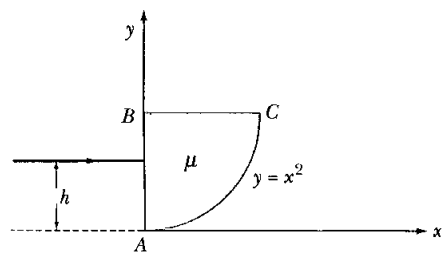


FIG. 35-92 Problem 60.

61. A thin equiconvex lens of refractive index 3/2 and radius of curvature 50 cm is placed on a reflecting convex surface of radius of curvature 100 cm. A point object is placed on the principle axis of the system such that its final image coincides

with itself. Now few drops of a transparent liquid is placed between the mirror and lens such that final image of the object is at infinity. Find (a) the refractive index of the liquid used and (b) the position of the object.

62. A concave mirror of radius R is kept on a horizontal table (Fig. 35-93). Water (refractive index = μ) is poured into it up to a height h. Where should an object be placed so that its image is formed on itself?

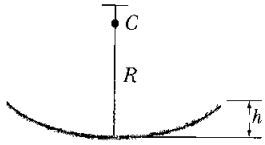


FIG. 35-93 Problem 62.

63. A light ray, going through a prism with the angle of prism 60° is found to deviate by 30°. What limit on the refractive index can be put from these data?

64. The apparatus shown in Fig. 35-94 consists of a liquid with a suitable refractive index poured into a cylindrical vessel and of a small laser. What should be the refractive index n of the liquid so that the light refracted by the liquid prism leaves the apparatus practically vertically when $\alpha < 10°$?

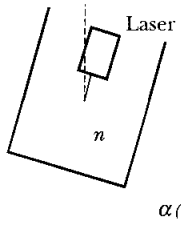


FIG. 35-94 Problem 64.

65. A fish hides under a floating plank that is circular in shape with radius R (Fig. 35-95). If the index of refraction of water is $\sqrt{2}$. Find the volume of water in which it can swim without being detected from above.

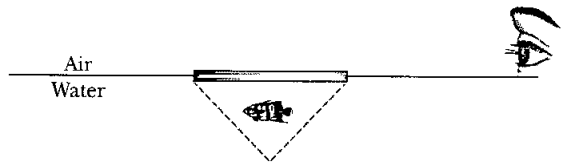


FIG. 35-95 Problem 65.

66. Figure 35-96 shows a transparent hemisphere of radius 3.0 cm made of a material of refractive index 2.0. (a) A narrow beam of parallel rays is incident on the hemisphere as shown in the figure. (a) Are the rays totally reflected at the plane surface? (b) Find the image formed by the refraction at the first surface. (c) Find the image formed by the reflection or by the refraction at the plane surface. (d) Trace qualitatively the final rays as they come out of the hemisphere.

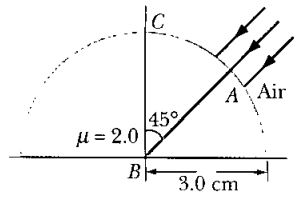


FIG. 35-96 Problem 66.

ANSWERS

Checkpoints

1. a
2. (a) $11h/6$ from fish itself; (b) $17\ h/6$
3. The straw is not a ray! Note that the straw has a lateral shift as well as a bend. Actually, the rays from the straw reach our eyes after refraction from the water and the apparent depth of different parts of the straw is different from its actual value. So the straw appears to be bent.
4. $d_a = \dfrac{d\cos^3\theta}{n\cos^3\phi}$
5. (a) e; (b) virtual, same
7. Virtual, same as object, diverging

Reasoning Type Questions

1. (a)
2. (a)
3. (d)
4. (d)
5. (a)
6. (a)
7. (b)
8. (c)
9. (b)
10. (b)

Multiple Choice Questions

1. (d)
2. (b)
3. (c)
4. (a)
5. (c)
6. (b)
7. (c), (d)
8. (a)
9. (b)
10. (a)
11. (b)
12. (c)
13. (c)
14. (d)
15. (d)
16. (b)
17. (c)
18. (b)
19. (c)
20. (a)
21. (b)
22. (d)

23. (b)
24. (a)
25. (a)
26. (b)
27. (b), (d)
28. (a), (b), (c)
29. (b), (c)
30. (b), (d)
31. (b), (c), (d)
32. (a)
33. (c)
34. (a)
35. (d)
36. (c)
37. (c)
38. (c)
39. (a)
40. (b)
41. (c)
42. (a)
43. (c)

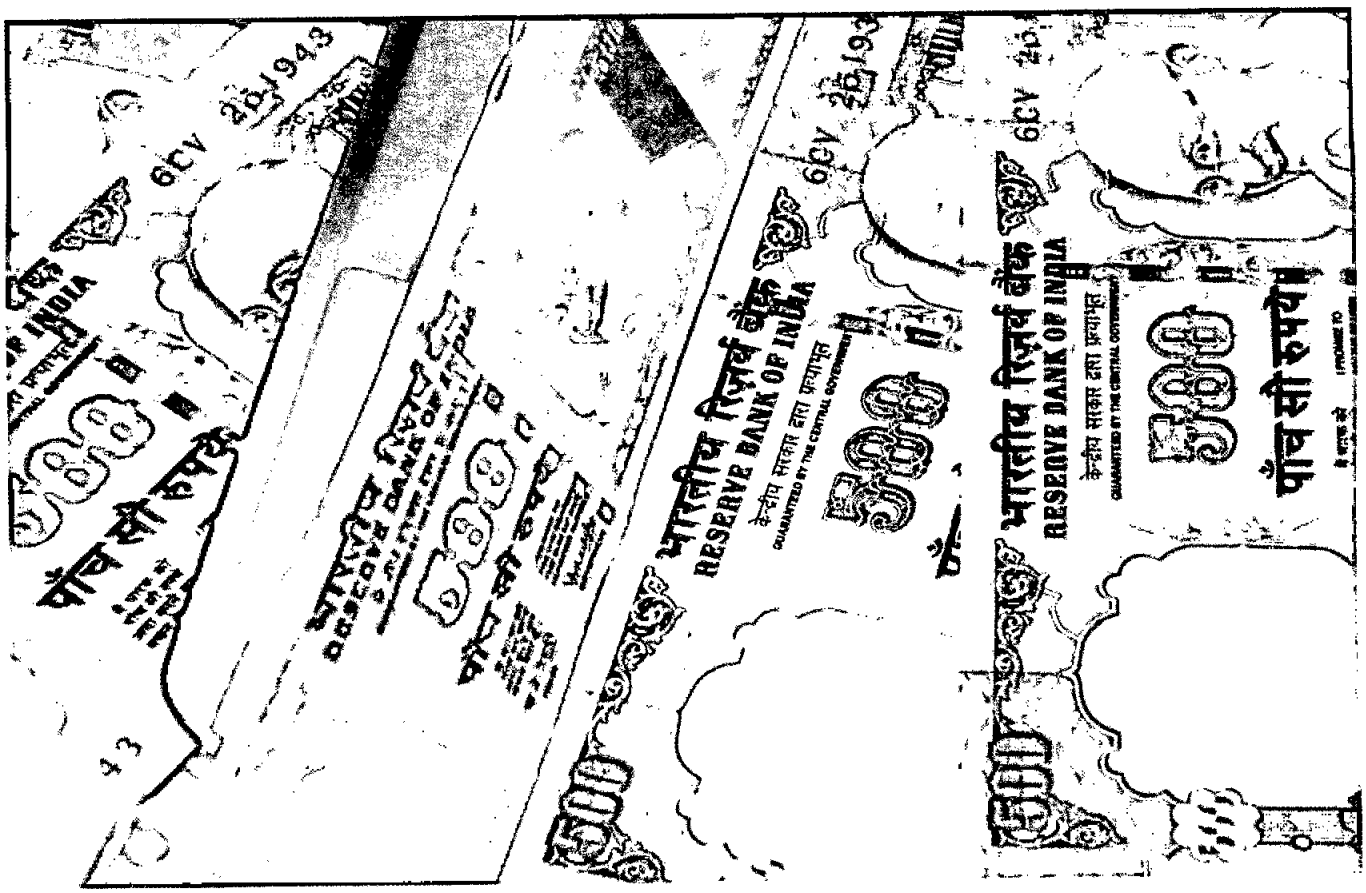
Problems

1. 1.56
2. $\sqrt{2}$
3. 43/24 m
4. 0.75 m away from the point vertically below the wood piece
5. $10\sqrt{5/7}$ cm
6. (a) 30°; (b) $\sqrt{3}$
7. (a) 90°; (b) 45°; (c) 42.4° and 75°
8. 42 mm
9. 20 cm
10. 1.5
11. 22.5 cm below water surface
12. 11 cm away from the mirror
13. 9 cm/s
14. 34°
15. 182 cm
16. (a) 60°; (b) 28.1°; (c) no
17. (a) 48.9°; (b) 29.0°
18. 75/31 cm
19. (a) $n = 2$; (b) not possible
20. $\frac{n_1 R}{n_2 - n_1}$ from the optical center
21. (a) 12 cm; (b) 24 cm; (c) 12 cm
22. 1.14
23. $9R$ from itself
24. 2.22 mm
25. (a) Four ways; (b) 40 cm, 120 cm, –40 cm, –120 cm
26. (a) 4.5 cm; (b) 9 cm
27. (a) infinite, 30 cm; (b) divergent lens
28. 0.5 cm
29. 22 cm
30. 100×100 km
31. (a) $f = 22.5$ cm; (b) concave lens
32. 72 cm away from the lens
33. (a) 6.0 mm; (b) 1.6 kW/m^2; (c) 4.0 cm
34. (a) $2f_1$; (b) –1.0; (c) real; (d) it is at a distance i_1' to the left of the lens; (e) inverted
35. 0.6 m
36. 12 cm and 20/3 cm
37. Convex lens of focal length less than 25 cm
38. 1/3
39. 0.9 m
40. 15 cm
41. (a) 20 cm; (b) concave mirror; (c) 20/3 cm
42. –5 cm
43. 45°
44. 90°
45. μA
46. 45°
47. (a) Minimum deviation = 60°; (b) maximum deviation = 76.5°
48. (a) 35.6°; (b) 53.1°
49. 1.22
50. (a) 31°45′; (b) 1°18′
51. 7.2°
52. 4.8° and 2.4°
53. (a) $\alpha = 3°$; (b) $\theta = 0.04°$
54. (a) $\omega = 0.06$; (b) $f = 20$ cm (convex lens)

Additional Problems

55. (a) 80 cm; (b) 12 cm
56. 20 cm/s
57. (a) $x = -10$ cm; (b) $f = 20$ cm; (c) $F_1 = +10$ cm and $F_0 = -30$ cm
58. $d/4$
59. 144/7 cm
60. $\frac{\mu^2 - 1}{4}$
61. (a) $\mu = 7/6$; (b) 100 cm from the lens
62. $(R - h)/\mu$ above the water surface
63. μ lies between $2/\sqrt{3}$ and $\sqrt{2}$
64. $n = 2$
65. $\pi R^3/3$
66. (a) They are reflected; (b) if the sphere is completed, the image is formed at a point diametrically opposite to A; (c) at a point on the sphere at an angle of 45° to line BC on the opposite side of A

Optical Interference 36



Governments worldwide scurry to stay ahead of counterfeiters who are quick to use the latest technology to duplicate paper currencies. Some of the security measures now used to thwart counterfeiters are security threads and special watermarks (both of which can be seen if the currency is held up against a light) and microprinting (which consists of dots too small to be picked up by a scanner). The feature that is probably the most difficult for a counterfeiter to duplicate is the variable tint that results from color-shifting inks. For example, the "500" in the center of the front face of a 500 note contains color-shifting ink. If you look directly down on the number, it is green. If you then tilt the bill and look at it obliquely, the color shifts to blue. A copy machine can duplicate color from only one perspective and therefore cannot duplicate this shift in color you see when you change your perspective.

How do color-shifting inks shift colors?

The answer is in this chapter.

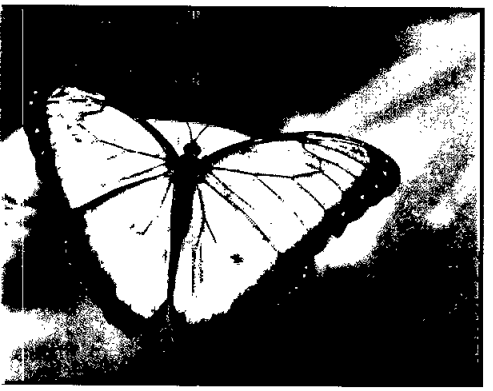


FIG. 36-1 The blue of the top surface of a *Morpho* butterfly wing is due to optical interference and shifts in color as your viewing perspective changes. *(Philippe Colombi/PhotoDisc//Getty Images)*

36-1 | WHAT IS PHYSICS?

One of the major goals of physics is to understand the nature of light. This goal has been difficult to achieve (and has not yet fully been achieved) because light is complicated. However, this complication means that light offers many opportunities for applications, and some of the richest opportunities involve the interference of light waves—**optical interference**.

Nature has long used optical interference for coloring. For example, the wings of a *Morpho* butterfly are a dull, uninspiring brown, as can be seen on the bottom wing surface, but the brown is hidden on the top surface by an arresting blue due to the interference of light reflecting from that surface (Fig. 36-1). Moreover, the top surface is color-shifting; if you change your perspective or if the wing moves, the tint of the color changes. Similar color shifting is used in the inks on many currencies to thwart counterfeiters, whose copy machines can duplicate color from only one perspective and therefore cannot duplicate any shift in color caused by a change in perspective.

Before we get to color shifting in butterfly wings and currency inks, we first must discuss the basic physics of optical interference. That means we must largely abandon the simplicity of geometrical optics (in which we describe light as rays) and return to the wave nature of light.

36-2 | Light as a Wave

The first person to advance a convincing wave theory for light was Dutch physicist Christian Huygens, in 1678. Although much less comprehensive than the later electromagnetic theory of Maxwell, Huygens' theory was simpler mathematically and remains useful today. Its great advantages are that it accounts for the laws of reflection and refraction in terms of waves and gives physical meaning to the index of refraction.

Huygens' wave theory is based on a geometrical construction that allows us to tell where a given wavefront will be at any time in the future if we know its present position. This construction is based on **Huygens' principle**, which is:

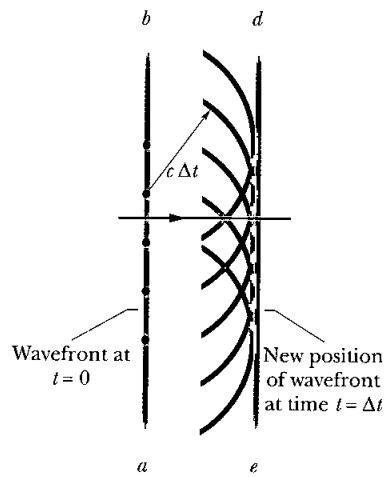


FIG. 36-2 The propagation of a plane wave in vacuum, as portrayed by Huygens' principle.

☞ All points on a wavefront serve as point sources of spherical secondary wavelets. After a time t, the new position of the wavefront will be that of a surface tangent to these secondary wavelets.

Here is a simple example. In Fig. 36-2, the present location of a wavefront of a plane wave traveling to the right in vacuum is represented by plane ab, perpendicular to the page. Where will the wavefront be at time Δt later? We let several points on plane ab (the dots) serve as sources of spherical secondary wavelets that are emitted at $t = 0$. At time Δt, the radius of all these spherical wavelets will have grown to $c\,\Delta t$, where c is the speed of light in vacuum. We draw plane de tangent to these wavelets at time Δt. This plane represents the wavefront of the plane wave at time Δt; it is parallel to plane ab and a perpendicular distance $c\,\Delta t$ from it.

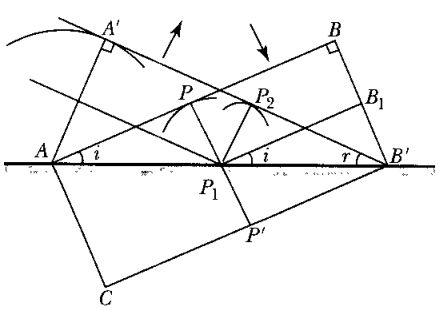


FIG. 36-3 Reflection of a plane wavefront AB incident on a plane mirror. $A'B'$ is the reflected wavefront; i and r correspond to angles of incidence and reflection, respectively.

Laws of Reflection

We now use Huygen's principle to prove the laws of reflection.

Let us consider a plane wave AB incident at an angle i on a plane mirror as shown in Fig. 36-3. We consider the reflection of the plane wave and try to obtain the shape of the reflected wavefront. Let AB be the position of the wavefront at $t = 0$. If the mirror was not present, then at a late time τ the position of the wavefront would have been CB', where $BB' = PP' = AC = v\tau$ and v is the

speed of propagation of the wave. In order to determine the shape of the reflected wavefront at the instant $t = \tau$, we consider an arbitrary point P on the wavefront AB and let τ_1 be the time taken by a disturbance to reach the point P_1 from P. According to Huygen's principle, the point P_1 will act as a source of secondary spherical wavefronts. From point P_1, we draw a sphere of radius $v(\tau - \tau_1)$. We draw a tangent plane on this sphere from the point B'. Since $BB_1 = PP_1 = v\tau_1$, the distance B_1B' will be equal to $P_1P_2 = [v(\tau - \tau_1)]$. If we consider triangles P_2P_1B' and B_1P_1B', then the side P_1B' is common to both and since $P_1P_2 = B'B_1$ and also since both the triangles are right-angled triangles, we have $\angle P_2B'P_1 = \angle PAP_1$. The former is the angle of reflection and the latter is the angle of incidence. Thus, we have the law of reflection, when a plane wavefront gets reflected from a plane surface, the angle of reflection is equal to the angle of incidence and the reflected wave is a plane wave.

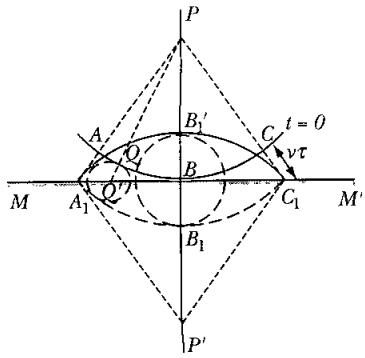


FIG. 36-4 P is a point source placed in front of a plane mirror MM'. ABC is the incident wavefront (which is spherical centered at P) and $A_1B_1'C_1$ is the corresponding reflected wavefront (which is spherical and centered at P'). P' is the virtual image of P.

Reflection of Light from a Point Source near a Mirror

As a second example, let us consider spherical waves (emanating from a point source P) incident on a plane mirror MM', as shown in Fig. 36-4. We know that a point source produces a point image behind the mirror. Let us try to see this by Huygen's principle. Let ABC denote the shape of the wavefront at time $t = 0$. In the absence of the mirror, the shape of the wavefront at a later time t would have been $A_1B_1C_1$ where $AA_1 = BB_1 = CC_1 = v\tau$, Q being an arbitrary point on the wavefront. If the time taken for the disturbance to traverse the distance QQ' be τ_1 then, in order to determine the shape of the reflected wavefront, we draw a sphere of radius $v(\tau - \tau_1)$ whose center is at the point Q'. In a similar manner we can draw the secondary wavelets emanating from other points on the mirror and, in particular, from the point B we have to draw a sphere of radius $v\tau$. The shape of the reflected wavefront is obtained by drawing a common tangent plane to all these spheres, which is shown as $A_1B_1'C_1$ in the figure. It can immediately be seen that $A_1B_1'C_1$ will have an exactly similar shape as $A_1B_1C_1$ except that $A_1B_1'C_1$ will have its center of curvature at the point P' where $PB = BP'$. Thus the reflected waves will appear to emanate from the point P' which will be the virtual image of the point P.

The Law of Refraction

We now use Huygens' principle to derive the law of refraction, Eq. 33-35 (Snell's law). Figure 36-5 shows three stages in the refraction of several wavefronts at a flat interface between air (medium 1) and glass (medium 2). We arbitrarily choose the wavefronts in the incident light beam to be separated by λ_1, the wavelength in medium 1. Let the speed of light in air be v_1 and that in glass be v_2. We assume that $v_2 < v_1$, which happens to be true.

Angle θ_1 in Fig. 36-5*a* is the angle between the wavefront and the interface; it has the same value as the angle between the *normal* to the wavefront (that is, the incident ray) and the *normal* to the interface. Thus, θ_1 is the angle of incidence.

As the wave moves into the glass, a Huygens wavelet at point e in Fig. 36-5*b* will expand to pass through point c, at a distance of λ_1 from point e. The time interval required for this expansion is that distance divided by the speed of the wavelet, or λ_1/v_1. Now note that in this same time interval, a Huygens wavelet at point h will expand to pass through point g, at the reduced speed v_2 and with wavelength λ_2. Thus, this time interval must also be equal to λ_2/v_2. By equating these times of travel, we obtain the relation

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}, \tag{36-1}$$

which shows that the wavelengths of light in two media are proportional to the speeds of light in those media.

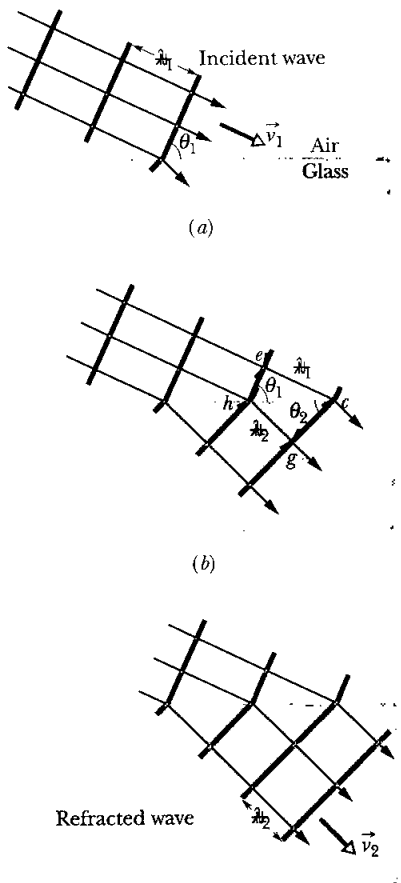


FIG. 36-5 The refraction of a plane wave at an air—glass interface, as portrayed by Huygens' principle. The wavelength in glass is smaller than that in air. For simplicity, the reflected wave is not shown. Parts (*a*) through (*c*) represent three successive stages of the refraction.

By Huygens' principle, the refracted wavefront must be tangent to an arc of radius λ_2 centered on h, say at point g. The refracted wavefront must also be tangent to an arc of radius λ_1 centered on e, say at c. Then the refracted wavefront must be oriented as shown. Note that θ_2, the angle between the refracted wavefront and the interface, is actually the angle of refraction.

For the right triangles *hce* and *hcg* in Fig. 36-5*b* we may write

$$\sin\theta_1 = \frac{\lambda_1}{hc} \qquad \text{(for triangle } hce\text{)}$$

and

$$\sin\theta_2 = \frac{\lambda_2}{hc} \qquad \text{(for triangle } hcg\text{)}.$$

Dividing the first of these two equations by the second and using Eq. 36-1, we find

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}. \tag{36-2}$$

We can define the **index of refraction** n for each medium as the ratio of the speed of light in vacuum to the speed of light v in the medium. Thus,

$$n = \frac{c}{v} \qquad \text{(index of refraction)}. \tag{36-3}$$

In particular, for our two media, we have

$$n_1 = \frac{c}{v_1} \quad \text{and} \quad n_2 = \frac{c}{v_2}. \tag{36-4}$$

If we combine Eqs. 36-2 and 36-4, we find

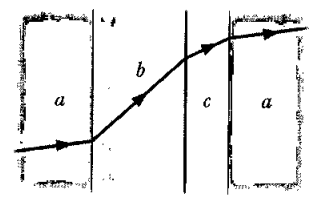
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1} \tag{36-5}$$

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \text{(law of refraction)}, \tag{36-6}$$

as introduced in Chapter 35.

CHECKPOINT 1 The figure shows a monochromatic ray of light traveling across parallel interfaces, from an original material *a*, through layers of materials *b* and *c*, and then back into material *a*. Rank the materials according to the speed of light in them, greatest first.



Wavelength and Index of Refraction

We have now seen that the wavelength of light changes when the speed of the light changes, as happens when light crosses an interface from one medium into another. Further, the speed of light in any medium depends on the index of refraction of the medium, according to Eq. 36-3. Thus, the wavelength of light in any medium depends on the index of refraction of the medium. Let a certain monochromatic light have wavelength λ and speed c in vacuum and wavelength λ_n and speed v in a medium with an index of refraction n. Now we can rewrite Eq. 36-1 as

$$\lambda_n = \lambda\frac{v}{c}. \tag{36-7}$$

Using Eq. 36-3 to substitute $1/n$ for v/c then yields

$$\lambda_n = \frac{\lambda}{n}. \tag{36-8}$$

This equation relates the wavelength of light in any medium to its wavelength in vacuum. It tells us that the greater the index of refraction of a medium, the smaller the wavelength of light in that medium.

What about the frequency of the light? Let f_n represent the frequency of the light in a medium with index of refraction n. Then from the general relation of $v = \lambda f$, we can write

$$f_n = \frac{v}{\lambda_n}.$$

Substituting Eqs. 36-3 and 36-8 then gives us

$$f_n = \frac{c/n}{\lambda/n} = \frac{c}{\lambda} = f,$$

where f is the frequency of the light in vacuum. Thus, although the speed and wavelength of light in the medium are different from what they are in vacuum, *the frequency of the light in the medium is the same as it is in vacuum.*

CHECKPOINT 2 The light waves of the rays in Fig. 36-6 have the same wavelength and amplitude and are initially in phase. (a) If 7.60 wavelengths fit within the length of the top material and 5.50 wavelengths fit within that of the bottom material, which material has the greater index of refraction? (b) If the rays are angled slightly so that they meet at the same point on a distant screen, will the interference there result in the brightest possible illumination, bright intermediate illumination, dark intermediate illumination, or darkness?

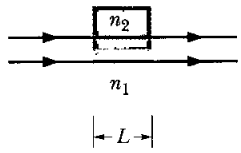


FIG. 36-6 Two light rays travel through two media having different indexes of refraction.

Sample Problem 36-1

In Fig. 36-6, the two light waves that are represented by the rays have wavelength 550.0 nm before entering media 1 and 2. They also have equal amplitudes and are in phase. Medium 1 is now just air, and medium 2 is a transparent plastic layer of index of refraction 1.600 and thickness 2.600 μm.

(a) What is the phase difference of the emerging waves in wavelengths, radians, and degrees? What is their effective phase difference (in wavelengths)?

Conceptualize/Classify: The phase difference of two light waves can change if they travel through different media, with different indexes of refraction. The reason is that their wavelengths are different in the different media. We can calculate the change in phase difference by counting the number of wavelengths that fits into each medium and then subtracting those numbers.

Compute: When the path lengths of the waves in the two media are identical, Eq. 36-11 gives the result of the subtraction. Here we have $n_1 = 1.000$ (for the air), $n_2 = 1.600$, $L = 2.600\ \mu$m, and $\lambda = 550.0$ nm. Thus, Eq. 36-11 yields

$$N_2 - N_1 = \frac{L}{\lambda}(n_2 - n_1)$$
$$= \frac{2.600\times10^{-6}\ \text{m}}{5.500\times10^{-7}\ \text{m}}(1.600 - 1.000)$$
$$= 2.84. \qquad \text{(Answer)}$$

Thus, the phase difference of the emerging waves is 2.84 wavelengths. Because 1.0 wavelength is equivalent to 2π rad and 360°, you can show that this phase difference is equivalent to

$$\text{phase difference} = 17.8\ \text{rad} \approx 1020°. \qquad \text{(Answer)}$$

The effective phase difference is the decimal part of the actual phase difference *expressed in wavelengths.* Thus, we have

$$\text{effective phase difference} = 0.84\ \text{wavelength}. \qquad \text{(Answer)}$$

You can show that this is equivalent to 5.3 rad and about 300°. *Caution:* We do *not* find the effective phase difference by taking the decimal part of the actual phase difference as expressed in radians or degrees. For example, we do *not* take 0.8 rad from the actual phase difference of 17.8 rad.

(b) If the waves reached the same point on a distant screen, what type of interference would they produce?

Reasoning: We need to compare the effective phase difference of the waves with the phase differences that give the extreme types of interference. Here the effective phase difference of 0.84 wavelength is between 0.5 wavelength (for fully destructive interference, or the darkest possible result) and 1.0 wavelength (for fully constructive interference, or the brightest possible result), but closer to 1.0 wavelength. Thus, the waves would produce intermediate interference that is closer to fully constructive interference—they would produce a relatively bright spot.

36-3 | Diffraction

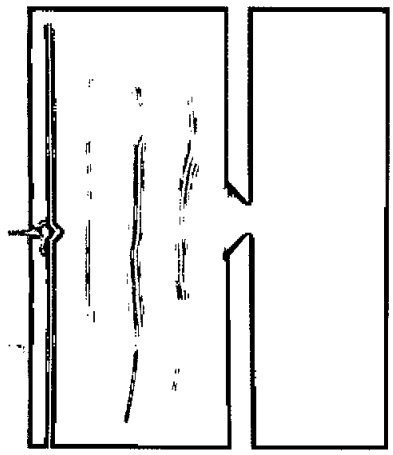


FIG. 36-7 Waves produced by an oscillating paddle at the left flare out through an opening in a barrier along the water surface. *(Runk Schoen- berger/Grant Heilman Photography)*

In the next section we shall discuss the experiment that first proved that light is a wave. To prepare for that discussion, we must introduce the idea of **diffraction** of waves, a phenomenon that we explore much more fully later in this chapter. Its essence is this: If a wave encounters a barrier that has an opening of dimensions similar to the wavelength, the part of the wave that passes through the opening will flare (spread) out—will *diffract*—into the region beyond the barrier. The flaring is consistent with the spreading of wavelets in the Huygens construction of Fig. 36-2. Diffraction occurs for waves of all types, not just light waves; Fig. 36-7 shows the diffraction of water waves traveling across the surface of water in a shallow tank.

Figure 36-8*a* shows the situation schematically for an incident plane wave of wavelength λ encountering a slit that has width $a = 6.0\lambda$ and extends into and out of the page. The part of the wave that passes through the slit flares out on the far side. Figures 36-8*b* (with $a = 3.0\lambda$) and 36-8*c* ($a = 1.5\lambda$) illustrate the main feature of diffraction: the narrower the slit, the greater the diffraction.

Diffraction limits geometrical optics, in which we represent an electromagnetic wave with a ray. If we actually try to form a ray by sending light through a narrow slit, or through a series of narrow slits, diffraction will always defeat our effort because it always causes the light to spread. Indeed, the narrower we make the slits (in the hope of producing a narrower beam), the greater the spreading is. Thus, geometrical optics holds only when slits or other apertures that might be located in the path of light do not have dimensions comparable to or smaller than the wavelength of the light.

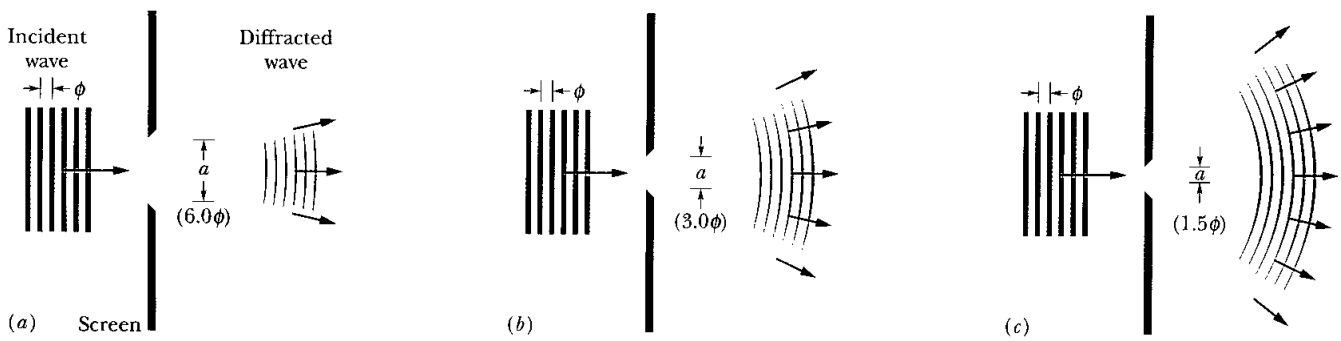


FIG. 36-8 Diffraction represented schematically. For a given wavelength λ, the diffraction is more pronounced the smaller the slit width a. The figures show the cases for (*a*) slit width $a = 6.0\lambda$, (*b*) slit width $a = 3.0\lambda$, and (*c*) slit width $a = 1.5\lambda$. In all three cases, the screen and the length of the slit extend well into and out of the page, perpendicular to it.

36-4 | Young's Interference Experiment

In 1801, Thomas Young experimentally proved that light is a wave, contrary to what most other scientists then thought. He did so by demonstrating that light undergoes

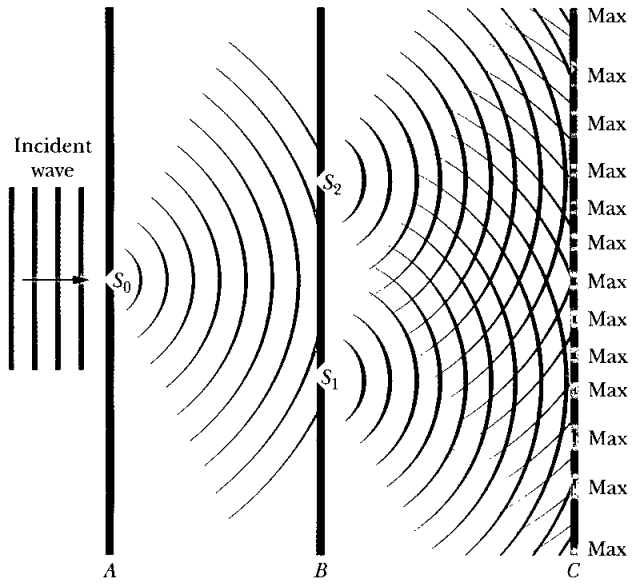


FIG. 36-9 In Young's interference experiment, incident monochromatic light is diffracted by slit S_0, which then acts as a point source of light that emits semicircular wavefronts. As that light reaches screen B, it is diffracted by slits S_1 and S_2, which then act as two point sources of light. The light waves traveling from slits S_1 and S_2 overlap and undergo interference, forming an interference pattern of maxima and minima on viewing screen C. This figure is a cross section; the screens, slits, and interference pattern extend into and out of the page. Between screens B and C, the semicircular wavefronts centered on S_2 depict the waves that would be there if only S_2 were open. Similarly, those centered on S_1 depict waves that would be there if only S_1 were open.

interference, as do water waves, sound waves, and waves of all other types. In addition, he was able to measure the average wavelength of sunlight; his value, 570 nm, is impressively close to the modern accepted value of 555 nm. We shall here examine Young's experiment as an example of the interference of light waves.

Figure 36-9 gives the basic arrangement of Young's experiment. Light from a distant monochromatic source illuminates slit S_0 in screen A. The emerging light then spreads via diffraction to illuminate two slits S_1 and S_2 in screen B. Diffraction of the light by these two slits sends overlapping circular waves into the region beyond screen B, where the waves from one slit interfere with the waves from the other slit.

The "snapshot" of Fig. 36-9 depicts the interference of the overlapping waves. However, we cannot see evidence for the interference except where a viewing screen C intercepts the light. Where it does so, points of interference maxima form visible bright rows—called *bright bands, bright fringes*, or (loosely speaking) *maxima*—that extend across the screen (into and out of the page in Fig. 36-9). Dark regions—called *dark bands, dark fringes*, or (loosely speaking) *minima*—result from fully destructive interference and are visible between adjacent pairs of bright fringes. (*Maxima* and *minima* more properly refer to the center of a band.) The pattern of bright and dark fringes on the screen is called an **interference pattern**. Figure 36-10 is a photograph of part of the interference pattern that would be seen by an observer standing to the left of screen C in the arrangement of Fig. 36-9.

Let us try to understand the phenomena of interference of waves. Any two waves reaching at a point may have a path difference between them, as we discussed in chapter on Longitudinal Waves, Vol. I. A path difference of 0.5 wavelength puts two waves exactly out of phase. If the waves had equal amplitudes and were to reach some common point, they would then undergo fully destructive interference, producing darkness at that point. With a path difference of 0.0 or 1.0 wavelength, they would, instead, undergo fully constructive interference, resulting in brightness at the common point. Our path difference of 0.6 wavelength is an intermediate situation but closer to fully destructive interference, and the waves would produce a dimly illuminated common point.



FIG. 36-10 A photograph of the interference pattern produced by the arrangement shown in Fig. 36-9. (The photograph is a front view of part of screen C.) The alternating maxima and minima are called *interference fringes* (because they resemble the decorative fringe sometimes used on clothing and rugs). *(From Michel Cagnet, Maurice Franzon, and Jean Claude Thierr, Atlas of Optical Phenomena. Springer-Verlag, New York, 1962. Reproduced with permission.)*

Locating the Fringes

Light waves produce fringes in a *Young's double-slit interference experiment*, as it is called, but what exactly determines the locations of the fringes? To answer,

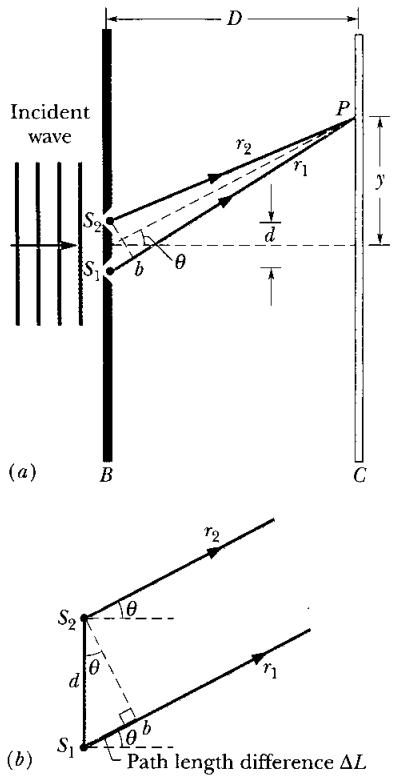


FIG. 36-11 (*a*) Waves from slits S_1 and S_2 (which extend into and out of the page) combine at P, an arbitrary point on screen C at distance y from the central axis. The angle θ serves as a convenient locator for P. (*b*) For $D \gg d$, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis.

we shall use the arrangement in Fig. 36-11*a*. There, a plane wave of monochromatic light is incident on two slits S_1 and S_2 in screen B; the light diffracts through the slits and produces an interference pattern on screen C. We draw a central axis from the point halfway between the slits to screen C as a reference. We then pick, for discussion, an arbitrary point P on the screen, at angle θ to the central axis. This point intercepts the wave of ray r_1 from the bottom slit and the wave of ray r_2 from the top slit.

These waves are in phase when they pass through the two slits because there they are just portions of the same incident wave. However, once they have passed the slits, the two waves must travel different distances to reach P. We saw a similar situation in Section 17-7, Vol. I, with sound waves and concluded that

☞ The phase difference between two waves can change if the waves travel paths of different lengths.

The change in phase difference is due to the *path length difference* ΔL in the paths taken by the waves. Consider two waves initially exactly in phase, traveling along paths with a path length difference ΔL, and then passing through some common point. When ΔL is zero or an integer number of wavelengths, the waves arrive at the common point exactly in phase and they interfere fully constructively there. If that is true for the waves of rays r_1 and r_2 in Fig. 36-11, then point P is part of a bright fringe. When, instead, ΔL is an odd multiple of half a wavelength, the waves arrive at the common point exactly out of phase and they interfere fully destructively there. If that is true for the waves of rays r_1 and r_2, then point P is part of a dark fringe. (And, of course, we can have intermediate situations of interference and thus intermediate illumination at P.) Thus,

☞ What appears at each point on the viewing screen in a Young's double-slit interference experiment is determined by the path length difference ΔL of the rays reaching that point.

We can specify where each bright fringe and each dark fringe is located on the screen by giving the angle θ from the central axis to that fringe. To find θ, we must relate it to ΔL. We start with Fig. 36-11*a* by finding a point b along ray r_1 such that the path length from b to P equals the path length from S_2 to P. Then the path length difference ΔL between the two rays is the distance from S_1 to b.

The relation between this S_1-to-b distance and θ is complicated, but we can simplify it considerably if we arrange for the distance D from the slits to the screen to be much greater than the slit separation d. In the experimental arrangement, the screen is far removed from the slits. This is necessary because the superposition of electromagnetic waves involves the superposition of electric field. At a large distance from the slits, the superposing waves are almost parallel to each other. This would mean that the electric fields are also parallel to each other. So the superposition can produce constructive and destructive interference. Then we can approximate rays r_1 and r_2 as being parallel to each other and at angle θ to the central axis (Fig. 36-11*b*). We can also approximate the triangle formed by S_1, S_2, and b as being a right triangle, and approximate the angle inside that triangle at S_2 as being θ. Then, for that triangle, $\sin\theta = \Delta L/d$ and thus

$$\Delta L = d\sin\theta \quad \text{(path length difference)}. \tag{36-9}$$

For a bright fringe, we saw that ΔL must be either zero or an integer number of wavelengths. Using Eq. 36-9, we can write this requirement as

$$\Delta L = d\sin\theta = (\text{integer})(\lambda), \tag{36-10}$$

or as

$$d\sin\theta = m\lambda, \qquad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright fringes)}. \tag{36-11}$$

For a dark fringe, ΔL must be an odd multiple of half a wavelength. Again using Eq. 36-9, we can write this requirement as

$$\Delta L = d \sin\theta = (\text{odd number})(\tfrac{1}{2}\lambda), \tag{36-12}$$

or as

$$d \sin\theta = (m + \tfrac{1}{2})\lambda, \quad \text{for } m = 0, 1, 2, \ldots \qquad \text{(minima—dark fringes).} \tag{36-13}$$

With Eqs. 36-11 and 36-12, we can find the angle θ to any fringe and thus locate that fringe; further, we can use the values of m to label the fringes. For the value and label $m = 0$, Eq. 36-11 tells us that a bright fringe is at $\theta = 0$ and thus on the central axis. This *central maximum* is the point at which waves arriving from the two slits have a path length difference $\Delta L = 0$, hence zero phase difference.

For, say, $m = 2$, Eq. 36-11 tells us that *bright* fringes are at the angle

$$\theta = \sin^{-1}\left(\frac{2\lambda}{d}\right)$$

above and below the central axis. Waves from the two slits arrive at these two fringes with $\Delta L = 2\lambda$ and with a phase difference of two wavelengths. These fringes are said to be the *second-order bright fringes* (meaning $m = 2$) or the *second side maxima* (the second maxima to the side of the central maximum), or they are described as being the second bright fringes from the central maximum.

For $m = 1$, Eq. 36-13 tells us that *dark* fringes are at the angle

$$\theta = \sin^{-1}\left(\frac{1.5\lambda}{d}\right)$$

above and below the central axis. Waves from the two slits arrive at these two fringes with $\Delta L = 1.5\lambda$ and with a phase difference, in wavelengths, of 1.5. These fringes are called the *second-order dark fringes* or *second minima* because they are the second dark fringes to the side of the central axis. (The first dark fringes, or first minima, are at locations for which $m = 0$ in Eq. 36-13.)

We derived Eqs. 36-11 and 36-13 for the situation $D \gg d$. However, they also apply if we place a converging lens between the slits and the viewing screen and then move the viewing screen closer to the slits, to the focal point of the lens. (The screen is then said to be in the *focal plane* of the lens; that is, it is in the plane perpendicular to the central axis at the focal point.) One property of a converging lens is that it focuses all rays that are parallel to one another to the same point on its focal plane. Thus, the rays that now arrive at any point on the screen (in the focal plane) were exactly parallel (rather than approximately) when they left the slits.

CHECKPOINT 3 In Fig. 36-11*a*, what are ΔL (as a multiple of the wavelength) and the phase difference (in wavelengths) for the two rays if point P is (a) a third side maximum and (b) a third minimum?

Let us now answer the basic question: What is the distance on screen C in Fig. 36-11*a* between adjacent maxima near the center of the interference pattern? Assume that θ in Fig. 36-11 is small enough to permit the use of the approximations $\sin\theta \approx \tan\theta \approx \theta$ in which θ is expressed in radian measure. This normally occurs in the case when we use visible light for performing Young's experiment. For this, first, let us pick a maximum with a low value of m to ensure that it is near the center of the pattern. Then, from the geometry of Fig. 36-11*a*, the maximum's vertical distance y_m from the center of the pattern is related to its angle θ from the central axis by

$$\tan\theta \approx \theta \approx \frac{y_m}{D}.$$

From Eq. 36-11, this angle θ for the mth maximum is given by

$$\sin\theta \approx \theta = \frac{m\lambda}{d}.$$

If we equate our two expressions for angle θ and then solve for y_m, we find

$$y_m = \frac{m\lambda D}{d}. \tag{36-14}$$

For the next maximum as we move away from the pattern center, we have

$$y_{m+1} = \frac{(m+1)\lambda D}{d}. \tag{36-15}$$

We find the distance between these adjacent maxima by subtracting Eq. 36-14 from Eq. 36-15:

$$\Delta y = y_{m+1} - y_m = \frac{\lambda D}{d}. \tag{36-16}$$

As long as d and λ in Fig. 36-11*a* are small, the separation of the interference fringes is independent of m, that is, the fringes are evenly spaced. In general, the equation is valid only if the wavelength of light is small in comparison to the slit separation d.

Sample Problem 36-2

Young's double-slit experiment is performed inside water (μ = 4/3) with light of frequency 6×10^{14} Hz. If the slits are separated by 0.2 mm and the screen kept 1 m from the slits, find the fringe width. Using the same setup, what will be the fringe width if the experiment is performed in air?

Conceptualize/Classify: Here the wavelength is much less than separation of the slits. Hence we can use the formula for fringe width.

Compute: The wavelength of given light in air can be obtained from the relation

$$c = f\lambda_{air}$$

$$\lambda_{air} = \frac{c}{f} = \frac{3\times10^8}{6\times10^{14}} = 5\times10^{-7}\,\text{m}.$$

The wavelength of light of same frequency (6×10^{14} Hz) in water will be different and

$$\lambda_{water} = \frac{\lambda_{air}}{\mu_{water}} = \frac{5 \times 10^{-7}}{4/3} = 3.75 \times 10^{-7}\ \text{m}.$$

As the experiment is performed in water, the fringe width is given by

$$\beta = \frac{\lambda_{water} D}{d}.$$

Here $D = 1$ m, $d = 0.2$ mm $= 0.2 \times 10^{-3}$ m; therefore

$$\beta = \frac{3.75\times10^{-7}\times1}{0.2\times10^{-3}}\ \text{m} = 1.87\ \text{mm}.$$

If the experiment is performed in air, the fringe width is given by

$$\beta = \frac{\lambda_{air} D}{d} = \frac{5\times10^{-7}\times1}{0.2\times10^{-3}}\ \text{m} = 2.5\ \text{mm}. \qquad \text{(Answer)}$$

Conclude: The fringe width in water reduces by the factor of refractive index because the wavelength reduces by the same factor.

Sample Problem 36-3

A coherent parallel beam of microwaves of wavelength $\lambda = 0.5$ mm falls on a Young's double-slit apparatus. The separation between the slits is 1.0 mm. The intensity of microwaves is measured on a screen placed parallel to the plane of the slits at a distance of 1.0 m from it as shown in Fig. 36-12.

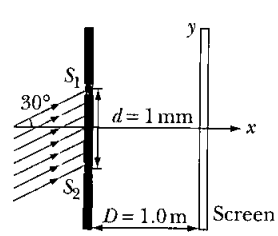


FIG. 36-12 A coherent parallel beam of microwaves falling on an YDSE apparatus.

(a) If the incident beam falls normally on the double-slit apparatus, find the y coordinates of all the interference minima on the screen.

(b) If the incident beam makes an angle of 30° with the x axis (as shown in the figure), find the y coordinates of the first minima on either side of the central maximum.

Conceptualize: Refer to Fig. 36-13. When the incident beam falls normally on the slits S_1 and S_2, the path difference at the central point P_0 of the screen is zero. Hence we have the central maximum at P_0. At the minima, the path difference between the waves is $\Delta = [m + (1/2)]\,\lambda$.

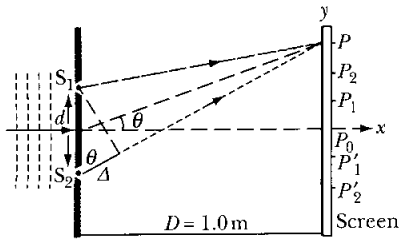


FIG. 36-13 Interference pattern when the waves are incident normally.

Classify: This is a classic case when the wavelength of the light is comparable to the slit separation d. So we cannot apply the equation for fringe width as in Eq. 36-16.

Compute: (a) Let the minima appear along directions θ with respect to the incident direction. Coherent waves from S_1 and S_2 along this direction are brought to a focus at P. It is clear that the path difference between the waves from S_1 and S_2 on reaching P is

$$\Delta = d\sin\theta.$$

The interference minima will appear on the screen if

$$\Delta = (m+\tfrac{1}{2})\lambda$$

$$d\sin\theta = (m+\tfrac{1}{2})\lambda,$$

where m is an integer. Thus the direction of minima are given by

$$\sin\theta = (m+\tfrac{1}{2})\times\left(\frac{\lambda}{d}\right).$$

It is given that $d = 1.0$ mm and $\lambda = 0.5$ mm. Therefore

$$\sin\theta = \left(m+\frac{1}{2}\right)\times\left(\frac{0.5}{1.0}\right) = \frac{1}{2}\left(m+\frac{1}{2}\right). \qquad (36\text{-}17)$$

The allowed values of m are those integers for which $\sin\theta$ is not more than +1 or less than –1. These values are m = 1, 0, –1, and –2. Hence, four minima will be observed. If follows from the figure that

$$\tan\theta = \frac{y}{D} \Rightarrow y = D\tan\theta$$

$$y = \frac{D\sin\theta}{(1-\sin^2\theta)^{1/2}}. \qquad (36\text{-}18)$$

The first minima above the central maximum (P_0) corresponds to $m = 0$. Using Eq. 36-17, we have $\sin\theta = 1/4$ for this minimum. Using this value in Eq. 36-18 the distance of the first minimum P_1 from P_0 is

$$y_1 = \frac{(1.0\text{ m})\times(1/4)}{[1-(1/16)]^{1/2}} = \frac{1}{\sqrt{15}}\text{ m}.$$

This is the y coordinate of the first minimum P_1 above P_0.

The second minima above P_0 corresponds to $m = 1$, for which, we have from Eq. 36-17, $\sin\theta = 3/4$. Using this value in Eq. 36-18, the y coordinate of the second minimum P_2 above P_0 is

$$y_2 = \frac{(1.0\text{ m})\times(3/4)}{[1-(9/16)]^{1/2}} = \frac{3}{\sqrt{7}}\text{ m}.$$

The first minimum P_1' below P_0 corresponds to $m = -1$, for which $\sin\theta = -1/4$. The y coordinate of this minimum P_1' is

$$y_1' = -\frac{1}{\sqrt{15}}\text{ m}.$$

The second minimum P_2' below P_0 corresponds to $m = -2$ for which $\sin\theta = -3/4$. The y coordinate of P_2' is (see Fig. 36-12)

$$P_2' = -\frac{3}{\sqrt{7}}\text{ m}.$$

(b) When the incident wave makes an angle $\alpha = 30°$ with the x axis, the position of zero path difference, that is the central maximum, shifts from P_0 to P_0' (see Fig. 36-14).

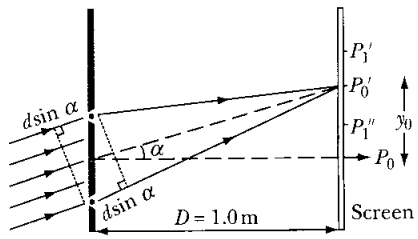


FIG. 36-14 Interference pattern when the waves are incident at an angle of 30° to the normal.

Here we can easily see that the waves arriving at slits are having an initial path difference of $d\sin\alpha$. To compensate for this, the central maxima should shift at a point where the path of the wave coming through the lower slit is longer by the same amount. Clearly, this will occur at an angle α to the central line.

We have

$$y_0 = D\tan\alpha = (1.0\text{ m})\times\tan 30° = \tfrac{1}{\sqrt{3}}\text{ m}.$$

Therefore, the y coordinate to the first minimum P_1' is the point where the path difference is $\lambda/2$. Therefore

$$d\sin\theta - d\sin\alpha = -\frac{\lambda}{2}.$$

Solving we get $\sin\theta = 3/4$. Therefore

$$y_1 = \frac{(1.0\text{ m})\times(3/4)}{[1-(9/16)]^{1/2}} = \frac{3}{\sqrt{7}}\text{ m}.$$

Similarly, the y coordinate of the first minimum P_1'' is where the path difference is $-\lambda/2$.

$$d\sin\theta - d\sin\alpha = -\frac{\lambda}{2}.$$

Solving, we get $\sin\theta = 1/4$. Therefore

$$y_1' = \frac{(1.0\text{m})\times(1/4)}{[1-(1/16)]^{1/2}} = \frac{1}{\sqrt{15}}\text{ m}.$$

Conclude: How do we distinguish between the central maxima and other maxima? For this, we note that the central maximum is the point where the path difference is zero for all the wavelengths. The location of other maxima are at a path difference of $n\lambda$, which depends on the wavelength. So if we use white light to obtain the interference pattern, the central maxima will be common to all the colors. Other fringes will be colored.

Geometrical Path and Optical Path

The fact that the wavelength of light depends on the index of refraction via Eq. 36-8 is important in certain situations involving the interference of light waves. For example, in Fig. 36-6, the *waves of the rays* (that is, the waves represented by the rays) have identical wavelengths and are initially in-phase in air ($n \approx 1$). One of the waves travels through medium 1 of index of refraction n_1 and length L. The other wave travels through medium 2 of index of refraction n_2 and the same length L. When the waves leave the two media, they will have the same wavelength λ: their wavelength λ in air. However, because their wavelengths differed in the two media, the two waves may no longer be in-phase.

☞ The phase difference between two light waves can change if the waves travel through different materials having different indexes of refraction.

As we will discuss soon, this change in the phase difference can determine how the light waves will interfere if they reach some common point.

The actual distance travelled by light in a medium is called geometrical path (Δx). Consider a light wave given by the equation:

$$E = E_0 \sin(\omega t - kx + \phi).$$

If the light travels by Δx, its phase changes by $k\Delta x = (\omega/v)\Delta x$, where ω which is the frequency of light does not depend on the medium, but v which is the speed of light depends on the medium as $v = c/n$.

Consequently, change in phase is given by

$$\Delta\phi = \frac{\omega}{C} n\Delta x.$$

It is clear that a wave travelling a distance Δx in a medium of refractive index n suffers the same phase change as when it travels a distance $n\Delta x$ in vacuum. That is a path length of Δx in medium of refractive index n is equivalent to a path length of $n\Delta x$ in vacuum.

The quantity $n\Delta x$ is called the optical path length of light, Δx_{opt}. In terms of optical path length, phase difference would be given by

$$\Delta\phi = \frac{2\pi}{\lambda_0}\Delta x_{opt},$$

where λ_0 is the wavelength of light in vacuum. However, in terms of the geometrical path length Δx,

$$\Delta\phi = \frac{2\pi}{\lambda_0} n\Delta x = \frac{2\pi}{\lambda}\Delta x,$$

where λ is the wavelength of light in the medium ($\lambda = \lambda_0/n$).

Figure 36-15 clearly shows that a thickness t of the medium has as many wavelengths as there are in a length nt of vacuum. Therefore, in terms of wavelengths, a thickness t in a medium of refractive index n is equivalent to a path length nt in vacuum. The quantity nt is called equivalent optical path length.

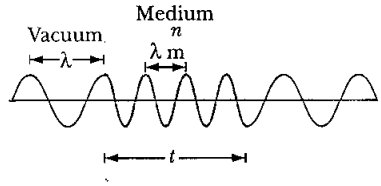


FIG. 36-15 The shortening of wavelength in a medium.

Displacement of Fringe on Introduction of a Glass Slab in the Path of the Light Coming out of the Slits

In Young's double-slit experiment (YDSE), we are concerned with the phase difference of the waves that arrive at the screen. It can be easily seen that a thin film in front of the upper slit will introduce additional path in comparison to the imaginary film of thickness t in air in front of the lower slit (Fig. 36-16).

On introduction of the thin glass-slab of thickness t and refractive index n, the optical path of the ray S_1P increases by $t(n-1)$ in comparison to the similar film of air in front of the lower slit. Now the path difference between waves coming from S_1 and S_2 at any point P is

$$\Delta x = S_2P - [S_1P + t(n-1)]$$
$$= (S_2P - S_1P) - t(n-1).$$

This implies

$$\Delta x = d\sin\theta - t(n-1) \text{ if } d \ll D.$$

$$\Delta x = \frac{yd}{D} - t(n-1) \text{ if } y \ll D \text{ as well.}$$

For central bright fringe,

$$\Delta\phi = 0 \Rightarrow \frac{yd}{D} = t(n-1).$$

The whole fringe pattern gets shifted by the same distance

$$y_0 = (n-1)t\frac{D}{d}.$$

The shift of the fringe pattern has been shown in Fig. 36-17.

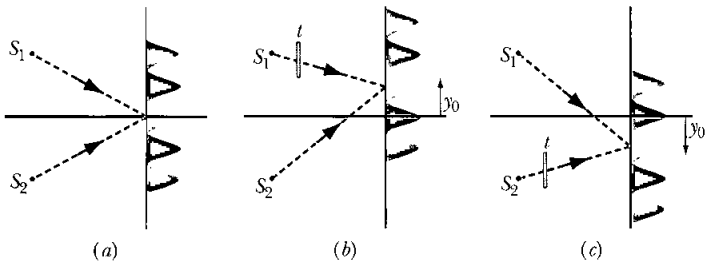


FIG. 36-17 Shift of the interference pattern in the presence of a thin glass slab. The intensity at all the maxima is the same, and to emphasize the central maxima, it has been shown white. Other fringes are colored. Notice that this shift is in the direction of the slit before which the glass slab is placed. If the glass slab is placed before the upper slit, the fringe pattern gets shifted upward and if the glass slab is placed before the lower slit the fringe pattern gets shifted downward.

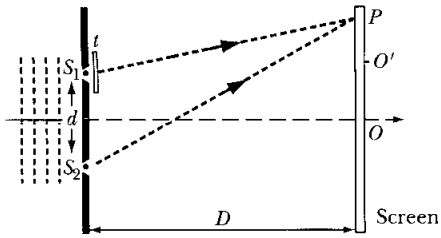


FIG. 36-16 In a standard YDSE apparatus, a thin film is introduced in front of one of the slits. The effect is to shift the entire interference pattern in the direction of that slit to point P (central maxima).

Sample Problem 36-4

A double-slit apparatus is immersed in a transparent liquid of refractive index 1.33. The slit separation is 1 mm and the distance between the plate of slits and screen is 1.33 m. The slits are illuminated by a parallel beam of light whose wavelength in air is 6300 Å.

(a) Find the fringe width.

(b) One of the slits of the apparatus is covered by a thin glass sheet of refractive index 1.53. Find the smallest thickness of the sheet to bring the adjacent minimum of the axis.

Conceptualize/Classify: In this situation, the wavelength of the light in the medium changes. So the fringe width will change. Also the refractive index has to be taken relative to the medium in which interference has to be done.

Compute: (a) The wavelength in liquid is

$$\lambda_1 = \frac{\lambda_a \mu_a}{\mu_1} = \frac{6300\ \text{Å} \times 1}{1.33} = \frac{6.3 \times 10^{-7}}{1.33}\ \text{m}.$$

Therefore, fringe width in liquid is

$$\beta_1 = \frac{\lambda_1 D}{d} = \frac{6.3 \times 10^{-7} \times 1.33}{1.33 \times 1 \times 10^{-3}} = 6.3 \times 10^{-4}\ \text{m} = 0.63\ \text{mm}.$$

(b) If t is the thickness of the glass plate, the fringes are displaced by an amount given by

$$\Delta y = \left(\frac{\mu_g}{\mu_l} - 1\right)\frac{tD}{d}. \qquad (36\text{-}19a)$$

In order to bring the adjacent minimum on the axis (where we had the maximum before the sheet was introduced), the fringes must be displaced by half the fringe width, that is

$$\Delta y = \frac{\beta_1}{2}. \qquad (36\text{-}19b)$$

Equating Eqs. 36-19a and 36-19b we have

$$\frac{\beta_1}{2} = \left(\frac{\mu_g}{\mu_l} - 1\right)\frac{tD}{d}.$$

Substituting $\beta_1 = 6.3 \times 10^{-4}$ m, $\mu_g = 1.53$, $\mu_1 = 1.33$, $D = 1.33$ m and $d = 1 \times 10^{-3}$ m and solving we get

$$t = 1.575 \times 10^{-6}\ \text{m}.$$

Conclude: By measuring this fringe shift, we can measure the thickness of very thin plates.

36-5 | Coherence

For the interference pattern to appear on viewing screen C in Fig. 36-9, the light waves reaching any point P on the screen must have a phase difference that does not vary in time. That is the case in Fig. 36-9 because the waves passing through slits S_1 and S_2 are portions of the single light wave that illuminates the slits. Because the phase difference remains constant, the light from slits S_1 and S_2 is said to be completely **coherent**.

Direct sunlight is partially coherent; that is, sunlight waves intercepted at two points have a constant phase difference only if the points are very close. The slits in a double-slit experiment, however, are not close enough, and in direct sunlight, the light at the slits would be **incoherent**. To get coherent light, we would have to send the sunlight through a single slit as in Fig. 36-9; because that single slit is small, light that passes through it is coherent. In addition, the smallness of the slit causes the coherent light to spread via diffraction to illuminate both slits in the double-slit experiment.

If we replace the double slits with two similar but independent monochromatic light sources, such as two fine incandescent wires, the phase difference between the waves emitted by the sources varies rapidly and randomly. (This occurs because the light is emitted by vast numbers of atoms in the wires, acting randomly and independently for extremely short times—of the order of nanoseconds.) As a result, at any given point on the viewing screen, the interference between the waves from the two sources varies rapidly and randomly between fully constructive and fully destructive. The eye (and most common optical detectors) cannot follow such changes, and no interference pattern can be seen. The fringes disappear, and the screen is seen as being uniformly illuminated.

A *laser* differs from common light sources in that its atoms emit light in a cooperative manner, thereby making the light coherent. Moreover, the light is almost monochromatic, is emitted in a thin beam with little spreading, and can be focused to a width that almost matches the wavelength of the light.

36-6 | Intensity in Double-Slit Interference

Equations 36-11 and 36-13 tell us how to locate the maxima and minima of the double-slit interference pattern on screen C of Fig. 36-11 as a function of the angle θ in that figure. Here we wish to derive an expression for the intensity I of the fringes as a function of θ.

The light leaving the slits is in phase. However, let us assume that the light waves from the two slits are not in phase when they arrive at point P. Instead, the electric field components of those waves at point P are not in phase and vary with time as

$$E_1 = E_0 \sin \omega t \tag{36-20}$$

and

$$E_2 = E_0 \sin(\omega t + \phi), \tag{36-21}$$

where ω is the angular frequency of the waves and ϕ is the phase constant of wave E_2. Note that the two waves have the same amplitude E_0 and a phase difference of ϕ. Because that phase difference does not vary, the waves are coherent. We shall show that these two waves will combine at P to produce an intensity I given by

$$I = 4I_0 \cos^2 \tfrac{1}{2}\phi, \tag{36-22}$$

and that

$$\phi = \frac{2\pi d}{\lambda} \sin\theta. \tag{36-23}$$

In Eq. 36-22, I_0 is the intensity of the light that arrives on the screen from one slit when the other slit is temporarily covered. We assume that the slits are so narrow in comparison to the wavelength that this single-slit intensity is essentially uniform over the region of the screen in which we wish to examine the fringes.

Equations 36-22 and 36-23, which together tell us how the intensity I of the fringe pattern varies with the angle θ in Fig. 36-10, necessarily contain information about the location of the maxima and minima. Let us see if we can extract that information to find equations about those locations.

Study of Eq. 36-22 shows that intensity maxima will occur when

$$\tfrac{1}{2}\phi = m\pi, \quad \text{for } m = 0, 1, 2, \ldots. \tag{36-24}$$

If we put this result into Eq. 36-23, we find

$$2m\pi = \frac{2\pi d}{\lambda} \sin\theta, \quad \text{for } m = 0, 1, 2, \ldots$$

or

$$d \sin\theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima)}, \tag{36-25}$$

which is exactly Eq. 36-11, the expression that we derived earlier for the locations of the maxima.

The minima in the fringe pattern occur when

$$\tfrac{1}{2}\phi = (m + \tfrac{1}{2})\pi, \quad \text{for } m = 0, 1, 2, \ldots \tag{36-26}$$

If we combine this relation with Eq. 36-23, we are led at once to

$$d \sin\theta = (m + \tfrac{1}{2})\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima)}, \tag{36-27}$$

which is just Eq. 36-16, the expression we derived earlier for the locations of the fringe minima.

Figure 36-18, which is a plot of Eq. 36-22, shows the intensity of double-slit interference patterns as a function of the phase difference ϕ between the waves at the screen. The horizontal solid line is I_0, the (uniform) intensity on the screen when one of the slits is covered up. Note in Eq. 36-22 and the graph that the intensity I varies from zero at the fringe minima to $4I_0$ at the fringe maxima.

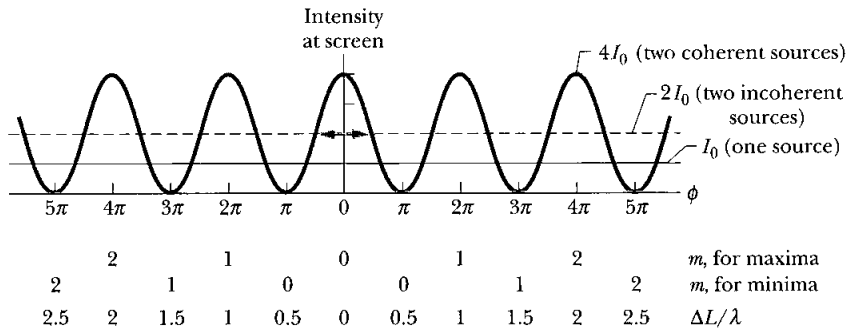


FIG. 36-18 A plot of Eq. 36-22, showing the intensity of a double-slit interference pattern as a function of the phase difference between the waves when they arrive from the two slits. I_0 is the (uniform) intensity that would appear on the screen if one slit were covered. The average intensity of the fringe pattern is $2I_0$, and the *maximum* intensity (for coherent light) is $4I_0$.

If the waves from the two sources (slits) were *incoherent,* so that no enduring phase relation existed between them, there would be no fringe pattern and the intensity would have the uniform value $2I_0$ for all points on the screen; the horizontal dashed line in Fig. 36-18 shows this uniform value.

Interference cannot create or destroy energy but merely redistributes it over the screen. Thus, the *average* intensity on the screen must be the same $2I_0$ regardless of whether the sources are coherent. This follows at once from Eq. 36-22; if we substitute $\frac{1}{2}$, the average value of the cosine-squared function, this equation reduces to $I_{avg} = 2I_0$.

Proof of Eqs. 36-22 and 36-23

We shall combine the electric field components E_1 and E_2, given by Eqs. 36-20 and 36-21, respectively, by the method of phasors as is discussed in Section 16-11, Vol. I. In Fig. 36-19*a*, the waves with components E_1 and E_2 are represented by phasors of magnitude E_0 that rotate around the origin at angular speed ω. The values of E_1 and E_2 at any time are the projections of the corresponding phasors on the vertical axis. Figure 36-19*a* shows the phasors and their projections at an arbitrary time t. Consistent with Eqs. 36-20 and 36-21, the phasor for E_1 has a rotation angle ωt and the phasor for E_2 has a rotation angle $\omega t + \phi$.

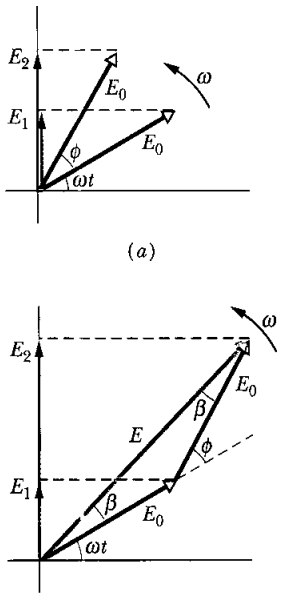


FIG. 36-19 (*a*) Phasors representing, at time t, the electric field components given by Eqs. 36-20 and 36-21. Both phasors have magnitude E_0 and rotate with angular speed ω. Their phase difference is ϕ. (*b*) Vector addition of the two phasors gives the phasor representing the resultant wave, with amplitude E and phase constant β.

To combine the field components E_1 and E_2 at any point P in Fig. 36-11, we add their phasors vectorially, as shown in Fig. 36-19*b*. The magnitude of the vector sum is the amplitude E of the resultant wave at point P, and that wave has a certain phase constant β. To find the amplitude E in Fig. 36-19*b*, we first note that the two angles marked β are equal because they are opposite equal-length sides of a triangle. From the theorem (for triangles) that an exterior angle (here ϕ, as shown in Fig. 36-19*b*) is equal to the sum of the two opposite interior angles (here that sum is $\beta + \beta$), we see that $\beta = \frac{1}{2}\phi$. Thus, we have

$$\begin{aligned} E &= 2(E_0 \cos \beta) \\ &= 2E_0 \cos \tfrac{1}{2}\phi. \end{aligned} \tag{36-28}$$

If we square each side of this relation, we obtain

$$E^2 = 4E_0^2 \cos^2 \tfrac{1}{2}\phi. \tag{36-29}$$

Now, from Eq. 33-26, we know that the intensity of an electromagnetic wave is proportional to the square of its amplitude. Therefore, the waves we are combining in Fig. 36-19*b*, whose amplitudes are E_0, each has an intensity I_0 that is proportional to E_0^2, and the resultant wave, with amplitude E, has an intensity I that is proportional to E^2. Thus,

$$\frac{I}{I_0} = \frac{E^2}{E_0^2}.$$

Substituting Eq. 36-29 into this equation and rearranging then yield

$$I = 4I_0 \cos^2 \tfrac{1}{2}\phi,$$

which is Eq. 36-22, which we set out to prove.

We still must prove Eq. 36-23, which relates the phase difference ϕ between the waves arriving at any point P on the screen of Fig. 36-11 to the angle θ that serves as a locator of that point.

The phase difference ϕ in Eq. 36-21 is associated with the path length difference S_1b in Fig. 36-11*b*. If S_1b is $\frac{1}{2}\lambda$, then ϕ is π; if S_1b is λ, then ϕ is 2π, and so on. This suggests

$$\begin{pmatrix}\text{phase}\\ \text{difference}\end{pmatrix} = \frac{2\pi}{\lambda}\begin{pmatrix}\text{path length}\\ \text{difference}\end{pmatrix} \tag{36-30}$$

The path length difference S_1b in Fig. 36-11*b* is $d\sin\theta$; so Eq. 36-30 for the phase difference between the two waves arriving at point P on the screen becomes

$$\phi = \frac{2\pi d}{\lambda}\sin\theta,$$

which is Eq. 36-23, the other equation that we set out to prove to relate ϕ to the angle θ that locates P.

Combining More Than Two Waves

In a more general case, we might want to find the resultant of more than two sinusoidally varying waves at a point. Whatever the number of waves is, our general procedure is this:

1. Construct a series of phasors representing the waves to be combined. Draw them end to end, maintaining the proper phase relations between adjacent phasors.
2. Construct the vector sum of this array. The length of this vector sum gives the amplitude of the resultant phasor. The angle between the vector sum and the first phasor is the phase of the resultant with respect to this first phasor. The projection of this vector-sum phasor on the vertical axis gives the time variation of the resultant wave.

✓**CHECKPOINT 4** Each of four pairs of light waves arrives at a certain point on a screen. The waves have the same wavelength. At the arrival point, their amplitudes and phase differences are (a) $2E_0$, $6E_0$, and π rad; (b) $3E_0$, $5E_0$, and π rad; (c) $9E_0$, $7E_0$, and 3π rad; (d) $2E_0$, $2E_0$, and 0 rad. Rank the four pairs according to the intensity of the light at the arrival point, greatest first. (*Hint:* Draw phasors.)

Sample Problem 36-5

Three light waves combine at a certain point where their electric field components are

$$E_1 = E_0\sin\omega t,$$
$$E_2 = E_0\sin(\omega t + 60°),$$
$$E_3 = E_0\sin(\omega t - 30°).$$

Find their resultant component $E(t)$ at that point.

Conceptualize/Classify: The resultant wave is

$$E(t) = E_1(t) + E_2(t) + E_3(t).$$

We can use the method of phasors to find this sum, and we are free to evaluate the phasors at any time t.

Compute: To simplify the solution, we choose $t = 0$, for which the phasors representing the three waves are shown in Fig. 36-20. We can add these three phasors either directly on a vector-capable calculator or by components. For the component approach, we first write the sum of their horizontal components as

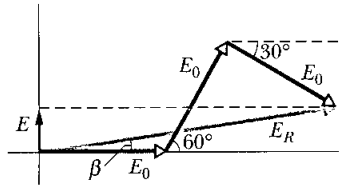


FIG. 36-20 Three phasors, representing waves with equal amplitudes E_0 and with phase constants 0°, 60°, and −30°, shown at time $t = 0$. The phasors combine to give a resultant phasor with magnitude E_R, at angle β.

$$\sum E_h = E_0\cos 0 + E_0\cos 60° + E_0\cos(-30°) = 2.37E_0.$$

The sum of their vertical components, which is the value of E at $t = 0$, is

$$\sum E_v = E_0\sin 0 + E_0\sin 60° + E_0\sin(-30°) = 0.366E_0.$$

The resultant wave $E(t)$ thus has an amplitude E_R of

$$E_R = \sqrt{(2.37E_0)^2 + (0.366E_0)^2} = 2.4E_0,$$

and a phase angle β relative to the phasor representing E_1 of

$$\beta = \tan^{-1}\left(\frac{0.366E_0}{2.37E_0}\right) = 8.8°.$$

We can now write, for the resultant wave $E(t)$,

$$\begin{aligned} E &= E_R \sin(\omega t + \beta) \\ &= 2.4E_0 \sin(\omega t + 8.8°). \end{aligned} \qquad \text{(Answer)}$$

Conclude: Be careful to interpret the angle β correctly in Fig. 36-20: It is the constant angle between E_R and the phasor representing E_1 as the four phasors rotate as a single unit around the origin. The angle between E_R and the horizontal axis in Fig. 36-20 does not remain equal to β.

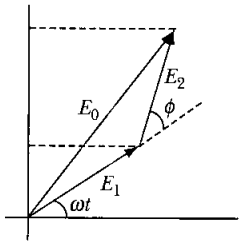


FIG. 36-21 Phasors representing, at time t, the electric field components given by Eqs. 36-20 and 36-21. Phasors have magnitude E_1 and E_2 and rotate with angular speed ω. Their phase difference is ϕ. Vector addition of the two phasors gives the phasor representing the resultant wave, with amplitude E_0.

Here we assumed that the intensity due to both the sources is the same. But in many of the cases, especially when the slits are of unequal width, the intensity due to the two silts on the screen is not the same. In such a case, we can find the resultant electric field amplitude by adding the electric field amplitudes vectorially as can be seen in Fig. 36-21.

If two waves of amplitudes E_1 and E_2, differing in phase by ϕ, superpose, the resultant amplitude A is given by,

$$E_0^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\phi.$$

Since intensity of a wave is proportional to the square of its amplitude, if two waves of intensities I_1 and I_2 superpose, the resultant intensity I is given by

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi. \qquad (36\text{-}31)$$

Constructive Interference

The resultant intensity I is maximum if $\cos\phi = +1$, that is

$$\phi = 2m\pi, \qquad (36\text{-}32)$$

where $m = 0, 1, 2, 3, \ldots$, etc. is an integer,

or

$$\frac{2\pi\Delta}{\lambda} = 2m\pi \text{ or } \Delta = m\lambda, \qquad (36\text{-}33)$$

where Δ is the path difference between the interfering waves. In this case

$$\begin{aligned} I_{max} &= I_1 + I_2 + 2\sqrt{I_1I_2}, \\ I_{min} &= I_1 + I_2 - 2\sqrt{I_1I_2}. \end{aligned} \qquad (36\text{-}34)$$

Sample Problem 36-6

Interference pattern is obtained with two coherent light sources of intensity ratio n. Show that in the interference pattern,

$$\frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{n}}{n+1}.$$

Conceptualize/Classify: Here we can apply Eq. 36-34.

Compute: Given $I_1/I_2 = n$; therefore, the amplitude ratio is

$$\frac{A_1}{A_2} = \sqrt{n}.$$

Also $I_{max} = A_1 + A_2$ and $I_{min} = A_1 + A_2$. Therefore,

$$\begin{aligned} \frac{I_{max} - I_{min}}{I_{max} + I_{min}} &= \frac{(A_1 + A_2)^2 - (A_1 - A_2)^2}{(A_1 + A_2)^2 + (A_1 - A_2)^2} \\ &= \frac{2A_1A_2}{A_1^2 + A_2^2} \\ &= \frac{2A_1/A_2}{(A_1^2/A_2^2) + 1} = \frac{2\sqrt{n}}{n+1}. \end{aligned}$$

Sample Problem 36-7 Build your skill

In Young's double-slit experiment, the upper slit is covered by a thin glass plate of refractive index 1.4 while the lower slit is covered by another glass plate having the same thickness as the first but having refractive index 1.7. Interference pattern is observed using light of wavelength 5400 Å. It is found that the point P on the screen where the central maximum ($n = 0$) fell before the glass plates were introduced now has three-fourth (3/4) of the original intensity. It is further observed that what used to be the fifth maximum earlier lies below the point P, while the sixth minimum lies above P. Calculate the thickness of the glass plate. (Absorption of light by glass plate may be neglected.)

Conceptualize/Classify: Refer to Fig. 36-22. In Young's double-slit experiment, slits S_1 and S_2 are equidistant from the source S and the slits have equal widths. Hence, light emerging from S_1 and S_2 has the same intensity, say I_0. Before the plates are introduced, the path difference (and hence the phase difference ϕ) at the central point P between the interfering waves is zero.

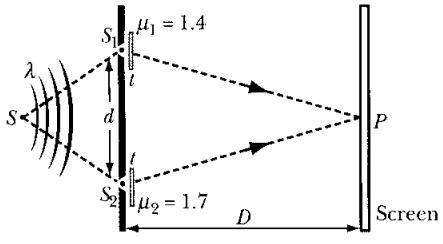


FIG. 36-22 The interference pattern when both the slits are covered by thin slabs.

Compute: The intensity at point P is

$$I = I_0 + I_0 + 2\sqrt{I_0 I_0}\cos 0° = 4I_0.$$

When the two plates are introduced, the path difference at the central point P between the interfering waves becomes

$$\Delta = (\mu_2 - 1)t - (\mu_1 - 1)t = (\mu_2 - \mu_1)t.$$

Therefore, the phase difference is given by

$$\phi = \frac{2\pi\Delta}{\lambda} = \frac{2\pi}{\lambda}(\mu_2 - \mu_1)t.$$

Hence, the intensity at point P becomes

$$I' = I_0 + I_0 + 2\sqrt{I_0 I_0}\cos\phi = 2I_0 + 2I_0\cos\phi.$$

It is given that

$$I' = \frac{3I}{4} = \frac{3}{4}\times 4I_0 = 3I_0.$$

Thus

$$3I_0 = 2I_0 + 2I_0\cos\phi.$$

Now

$$\cos\phi = \tfrac{1}{2}$$

$$\phi = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \cdots$$

$$\phi = \frac{\pi}{3}(6m \pm 1); \quad m = 0, 1, 2, 3, \ldots, \qquad (36\text{-}35)$$

where the negative value $\phi = -\pi/3$ corresponding to $m = 0$ is not admissible since ϕ is positive as $\mu_2 > \mu_1$. The upward shift of the interference pattern due to the upper plate is

$$(\Delta y)_1 = (\mu_1 - 1)\frac{tD}{d}.$$

The downward shift due to the lower plate is

$$(\Delta y)_2 = (\mu_2 - 1)\frac{tD}{d}.$$

Therefore the new downward shift is

$$\Delta y = (\Delta y)_2 - (\Delta y)_1 = (\mu_2 - \mu_1)\frac{tD}{d}$$

$$= (1.7 - 1.4)\frac{tD}{d} = \frac{0.3tD}{d}.$$

Now, the distance of the fifth maximum from P is $5\lambda D/d$, and the distance of sixth minimum from P is $[6-(1/2)]\lambda D/d = 5.5\lambda D/d$. It is given that the net shift lies between $5\lambda D/d$ and $5.5\ \lambda D/d$, that is

$$\frac{5\lambda D}{d} < \frac{0.3tD}{d} < \frac{5.5\lambda D}{d}$$

$$\frac{5\lambda}{0.3} < t < \frac{5.5\lambda}{0.3}$$

$$\frac{5\times 5.4\times 10^{-7}}{0.3}\text{m} < t < \frac{5.5\times 5.4\times 10^{-7}}{0.3}\text{m}$$

$$9.0\times 10^{-6}\text{ m} < t < 9.9\times 10^{-6}\text{ m}. \qquad (36\text{-}36)$$

We have seen that the phase difference,

$$\phi = \frac{2\pi}{\lambda}(\mu_2 - \mu_1)t$$

$$= \frac{2\pi}{\lambda}(1.7 - 1.4)t$$

$$= \frac{2\pi}{\lambda}(0.3)t. \qquad (36\text{-}37)$$

Using Eqs. 36-34 and 36-37 we have

$$\frac{\pi}{3}(6m \pm 1) = \frac{2\pi}{\lambda}(0.3t)$$

$$6m \pm 1 = \frac{6}{\lambda}(0.3)t = \frac{1.8t}{5.4\times10^{-7}}$$

$$t = 3(6m \pm 1)\times10^{-7}\ \text{m.} \qquad (36\text{-}38)$$

It is clear that Eq. 36-38 will yield a value of t which satisfies the requirement (Eq. 36-36) if $m = 5$. Hence

$$t = 3(6\times5 \pm 1)\times10^{-7}$$
$$= 3(30 \pm 1)\times10^{-7}$$
$$= 9.3\times10^{-6}\ \text{m or } 8.7\times10^{-6}\ \text{m.}$$

Conclude: The value of $t = 8.7\times10^{-6}$ m does not satisfy the requirement in Eq. 36-36. Hence, $t = 9.3\times10^{-6}$ m.

Shape of Fringes

As for a given fringe, n is constant, each fringe will be the locus of the point P for which

$$S_1P - S_2P = \text{constant.}$$

But from coordinate geometry we know that this locus is a hyperbola with S_1 and S_2 as foci, as seen in Fig. 36-23*a*. If there are two point sources, we can say that they will produce such hyperbola in all the directions. To get this directional distribution, we rotate a hyperbola about the line S_1S_2, as seen in Fig. 36-23*b*. We thus get hyperboloids of revolution, *that is, the interference fringes for two coherent point sources are hyperboloids with axis* S_1S_2. The fringes seen on a screen are sections of these hyperboloids so:

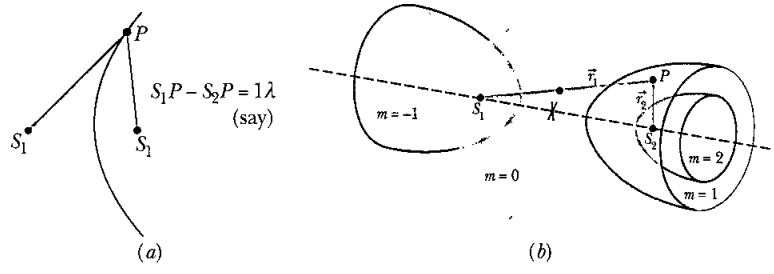


FIG. 36-23 (*a*) The locus of points in space where the path difference is constant is a hyperbola. (*b*) A combination of all such hyperbolas makes a hyperboloid in space. Please note that the figure is greatly exaggerated for the purpose of clarity. In reality, these hyperboloid are so flat that they almost look like a plane.

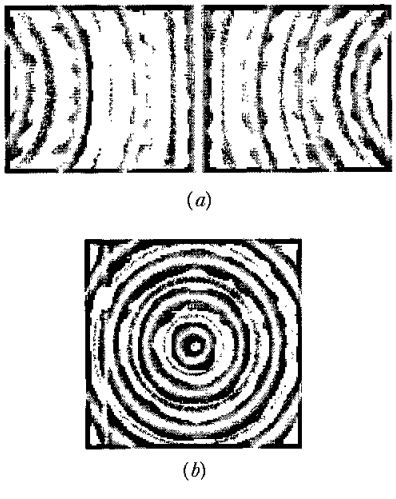


FIG. 36-24 Shape of interference pattern on a screen kept (*a*) parallel to the line S_1S_2, (*b*) perpendicular to line S_1S_2.

1. If the screen is parallel to line S_1S_2 which is usually the case, the fringes will be hyperbolic with straight-central bright fringe (Fig. 36-24*a*).
2. If the screen is perpendicular to line joining the sources, the fringes will be circular concentric circles with their centers on the point intersection of screen with line S_1S_2. In this situation central fringe will be bright if $S_1S_2 = n\lambda$ and dark if $S_1S_2 = (2n-1)\lambda/2$.

If instead of pinholes the sources are slits, we can visualize that in space fringes will be obtained on a hyperbolic plane. This hyperbolic plane intersects with the screen to produce an interference pattern. For a screen perpendicular to the slits, the shape of fringes will be hyperbolic with straight-central bright fringe (Fig. 36-25*b*) while for screen parallel to the slits, fringes will be straight lines parallel to the slits, as shown in Figs. 36-25 and *c*. For the screen parallel to the slits as well as parallel to the plane of the slits, central fringe will always be bright. For the screen parallel to the slits but perpendicular to the plane of the slits, central fringe (the fringe in the plane of the slits) will be bright if $S_1S_2 = n\lambda$ and dark if $S_1S_2 = (2n-1)\lambda/2$. For the screen perpendicular to the slits as well as the plane of the slits, central fringe will always be bright.

From all this it is clear that *shape of fringes depends on nature of sources* (pinholes or slits) *and direction of observation,* that is, the position of screen.

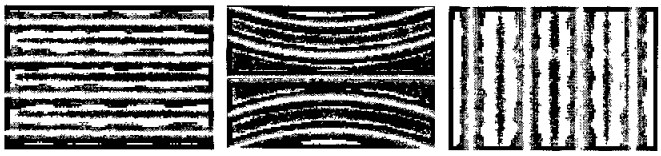


FIG. 36-25 Shape of interference pattern on a screen kept (*a*) parallel to the slits, (*b*) perpendicular to the slits, and (*c*) parallel to the slits but perpendicular the plane of the slits.

36-7 | Modifications of Young's Double-Slit Experiment

There are many other arrangements that can be considered for the formation of an interference pattern on the screen. In all these situations, note how the arrangement produces two coherent sources whose waves interfere to produce an interference pattern. Here we are going to consider a few popular examples of the same.

Sample Problem 36-8

Two point sources oscillating in the same phase are on a straight line perpendicular to a screen. The nearest source is at a distance of $D \gg \lambda$ from the screen. What shape will the interference fringes have on the screen? What is the distance on the screen from the perpendicular to the nearest bright fringe if the distance between the sources is $l = n\lambda$ (n is an integer)? Note that $\lambda \ll l$.

Conceptualize/Classify: Here we first try to see why an interference pattern will be obtained. The interference pattern is observed because as seen on the screen, the paths of the waves coming from two sources are not equal.

Compute: The intensity on the screen will be maximum when $d_2 - d_1 = k\lambda$. The locus of points on the screen reached by rays from both sources with this difference is a circle with its center at point A (Fig. 36-26). For this reason the interference fringes will have the form of concentric circle.

When $l = n\lambda$ the intensity will be maximum at point A (an interference maximum of the nth order). The nearest bright interference fringe circle of the $(n-1)$th order is at a distance from point A determined from the equation,

$$d_2 - d_1 = \sqrt{(n\lambda + D)^2 + h_{n-1}^2} - \sqrt{D^2 + h_{n-1}^2} = (n-1)\lambda.$$

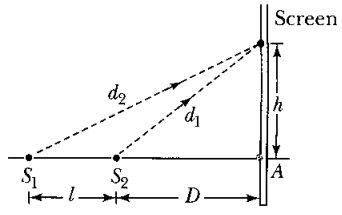


FIG. 36-26 Interference pattern when two point sources S_1 and S_2 are kept on a line perpendicular to the screen.

Bearing in mind the conditions of the problem that $h \ll D$ and $\lambda \ll l$, we get

$$h_{n-1} \cong \sqrt{\frac{2D(D+n\lambda)}{n}} = \sqrt{2D\lambda\left(\frac{D}{l}+1\right)}.$$

Conclude: One interesting aspect of this problem is that the order of maxima at any point other than the center will be less than n. Recall your mathematics classes. In any triangle, the third side is always longer than the difference of the other two sides. Since in the triangle formed by the two sources and point on the screen, the distance between two sources is $n\lambda$, the path difference at any arbitrary point should be less than that.

Sample Problem 36-9

Light from source S is incident on the Fresnel biprism shown in Fig. 36-27. The light beams refracted by the different faces of the prism partly overlap and produce an interference pattern on a screen on its section AB.

(a) Find the distance between adjacent interference fringes if the distance from the source to the prism is $a = 1$ m and that from the prism to the screen is $b = 4$ m. The angle of refraction of the prism is $\alpha = 2 \times 10^{-3}$ rad. The glass of prism has a refractive index of $n = 1.5$. The wavelength of the light is $\lambda = 6000$ Å.

(b) How many interference fringes can be observed on a screen?

Conceptualize: To find the distance d it is first necessary to calculate the distance l between the virtual source

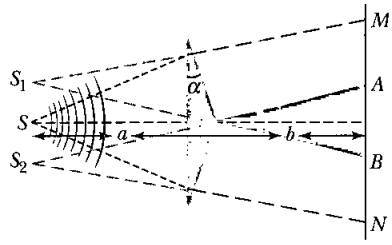


FIG. 36-27 Fresnel biprism used to form interference pattern on screen *MN*. Here the light refracted from the upper portion and the light refracted from the lower portion interfere to produce the interference pattern.

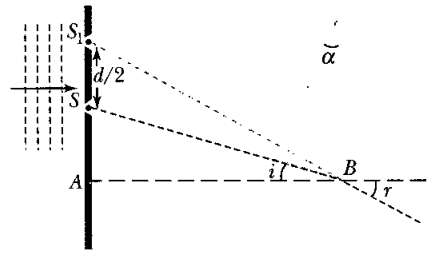


FIG. 36-28 The path of an imaginary ray incident from *S* on upper part of biprism. Since α is small, all rays undergo the same deviation. The refracted rays can be extended to meet at S_1, which acts as virtual source.

S_1 and S_2 located at the point of intersection to the continuation of the rays refracted by the prism faces.

Classify: For this purpose it will be the simplest to consider the path of the ray normally incident on the face of the prism (Fig. 36-28).

Compute: No such ray actually exists, but it can be plotted by mentally extending the upper prism downward. All the beams from point source refracted by a prism can be assumed to converge at a point. Since the refracting angle of the prism is small (the prism is thin), the virtual images S_1 and S_2 of the source can be regarded as lying at the same distance from the prism as source *S*. It can be seen from Fig. 38-28 that $i = \alpha$ and $SA = a\alpha$. According to the law of refraction, $r \cong n\alpha$. Considering the triangle AS_1B, we can write

$$\frac{d}{2} + a\alpha \cong a\alpha n.$$

Hence,

$$d = 2a\alpha(n-1).$$

We find that

$$\beta = \frac{\lambda D}{d} = \frac{\lambda(a+b)}{2a\alpha(n-1)} = 0.15 \text{ cm}.$$

(b) $N = L/\beta$, where L is the width of the interference pattern. It can be seen from Fig. 36-27 that $AB = (b/a)d$ where d is the distance between the virtual images S_1 and S_2 of the source. So we get

$$N = \frac{4ab\alpha^2(n-1)^2}{(a+b)\lambda} \cong 5.$$

Sample Problem 36-10

Two flat mirrors form an angle close to 180° (Fig. 36-29). A source of light *S* is placed at an equal distance *b* from the mirrors. Find the distance between adjacent interference fringes on screen *MN* at a distance $OA = a$, from the point of intersection of the mirrors. The wavelength of the light is equal to λ. Shield *C* does not allow the light to pass directly from the source to the screen.

Conceptualize: As you might recall, the object and image distances from the mirror are equal. This would imply that the image is located behind the mirror at an angle $\alpha/2$ behind the line *OA*. Similarly for the second mirror. The image should also be located at a distance *b* from the point of intersection of the mirrors. This can be seen in Fig. 36-30.

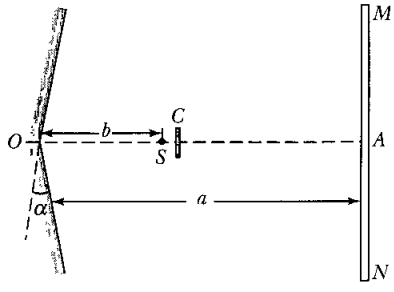


FIG. 36-29 Fresnel's mirror method to produce the interference pattern on a screen.

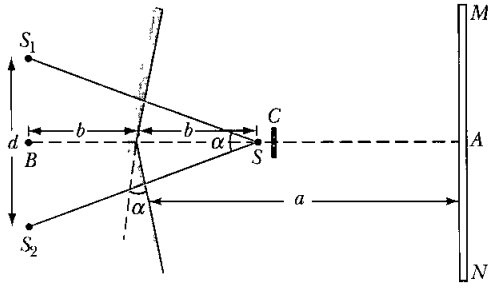


FIG. 36-30 In Fresnel's mirrors, the images of source *S* act as two coherent sources of light.

Classify: This is again a situation where the two mirrors are producing the two images. These two images act as coherent sources.

Compute: The distance between interference bands is $\beta = \lambda D/d$. In our case $D = AB \equiv a + b$, and $d = S_1S_2$ is the distance between the images S_1 and S_2 of the source in the flat mirrors (Fig. 36-30). The value of d can be found from triangle S_1SB:

$$\frac{d}{2} = 2b\frac{\alpha}{2}$$
$$d = 2b\alpha.$$

Hence,

$$\beta = \frac{\lambda(a+b)}{2b\alpha}.$$

Conclude: The light from source S, after reflection from M_1 and M_2, splits into two beams which illuminate different regions of the screen. The formation of interference fringes can be interpreted as due to two wavefronts coming from virtual sources S_1 and S_2 which are images of S in mirrors M_1 and M_2, respectively. The linear distance between S_1 and S_2 is, therefore, equivalent to separation d between the slits.

Sample Problem 36-10

In Fig. 36-31, S is a monochromatic point source emitting light of wavelength l = 500 nm. A thin lens of circular shape and of focal length 0.10 m is cut into two identical halves L_1 and L_2 by a plane passing through a diameter.

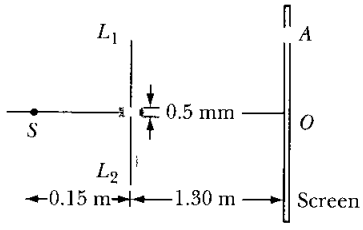


FIG. 36-31 Billet split lens method to produce interference pattern on the screen.

The two halves are placed symmetrically about the central axis SO with a gap of 0.5 mm. The distance along the axis from S to L_1 and L_2 is 0.15 m, while that from L_1 and L_2 to O is 1.30 m. The screen at O is normal to SO. If the third intensity maximum occurs at point A on the screen, find the distance OA.

Conceptualize/Classify: The two pieces L_1 and L_2 act as independent lenses. S_1 is the image of source S due to lens L_1 and S_2 that due to lens L_2. S_1 and S_2 serve as the two coherent light sources which produce interference pattern on the screen (Fig. 36-32).

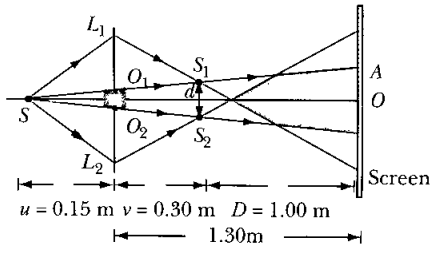


FIG. 36-32 L_1 and L_2 produce two images S_1 and S_2 which act as coherent sources of light producing interference pattern on the screen shown by yellowish color.

Compute: Given f = 0.10 m and u = −0.15 m. The image distance v is obtained by using the lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

which gives

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} = \frac{1}{0.10} + \frac{1}{-0.15} = \frac{1}{0.30}$$

or v = 0.30 m. Rays from S that pass through the optical center O_1 of lens L_1 and through O_2 of lens L_2 go through undeviated. Therefore, triangles SO_1O_2 and SS_1S_2 are similar. Hence

$$\frac{S_1S_2}{O_1O_2} = \frac{|u|+|v|}{|u|}. \qquad (36\text{-}39)$$

However O_1O_2 = 0.5 mm (given). Therefore,

$$d = S_1S_2 = \frac{(0.15+0.30)\text{m}}{(0.15)\text{m}} \times 0.5 \text{ mm} = 1.5 \text{ mm}.$$

The distance of the vertical plane containing S_1 and S_2 from the screen is

$$D = 1.30 - 0.30 = 1.00 \text{ m}.$$

The distance of the nth maximum from the central point O is given by

$$y_n = \frac{n\lambda D}{d}.$$

Since the third-order (n = 3) maximum occurs at A, we have

$$y_3 = OA = \frac{3\lambda D}{d} = \frac{3\times 500\times 10^{-9}\times 1.00}{1.5\times 10^{-3}}$$
$$= 1.0\times 10^{-3} \text{ m} = 1.0 \text{ mm}.$$

Conclude: What happens if the gap O_1O_2 is reduced? If the gap O_1O_2 is reduced, it follows from Eq. 36-39 that $S_1S_2 = d$ is also reduced as u and v remain unchanged. Hence OA increases.

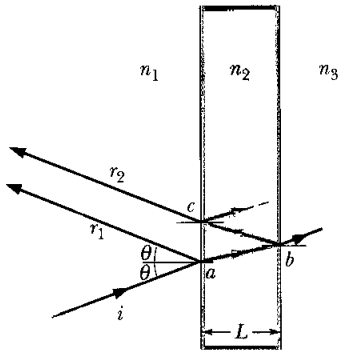


FIG. 36-33 Light waves, represented with ray i, are incident on a thin film of thickness L and index of refraction n_2. Rays r_1 and r_2 represent light waves that have been reflected by the front and back surfaces of the film, respectively. (All three rays are actually nearly perpendicular to the film.) The interference of the waves of r_1 and r_2 with each other depends on their phase difference. The index of refraction n_1 of the medium at the left can differ from the index of refraction n_3 of the medium at the right, but for now we assume that both media are air, with $n_1 = n_3 = 1.0$, which is less than n_2.

36-8 | Interference from Thin Films

The colors we see when sunlight illuminates a soap bubble or an oil slick are caused by the interference of light waves reflected from the front and back surfaces of a thin transparent film. The thickness of the soap or oil film is typically of the order of magnitude of the wavelength of the (visible) light involved. (Greater thicknesses spoil the coherence of the light needed to produce the colors due to interference.)

Figure 36-33 shows a thin transparent film of uniform thickness L and index of refraction n_2, illuminated by bright light of wavelength λ from a distant point source. For now, we assume that air lies on both sides of the film and thus that $n_1 = n_3$ in Fig. 36-33. For simplicity, we also assume that the light rays are almost perpendicular to the film ($\theta \approx 0$). We are interested in whether the film is bright or dark to an observer viewing it almost perpendicularly. (Since the film is brightly illuminated, how could it possibly be dark? You will see.)

The incident light, represented by ray i, intercepts the front (left) surface of the film at point a and undergoes both reflection and refraction there. The reflected ray r_1 is intercepted by the observer's eye. The refracted light crosses the film to point b on the back surface, where it undergoes both reflection and refraction. The light reflected at b crosses back through the film to point c, where it undergoes both reflection and refraction. The light refracted at c, represented by ray r_2, is intercepted by the observer's eye.

If the light waves of rays r_1 and r_2 are exactly in phase at the eye, they produce an interference maximum and region ac on the film is bright to the observer. If they are exactly out of phase, they produce an interference minimum and region ac is dark to the observer, *even though it is illuminated.* If there is some intermediate phase difference, there are intermediate interference and brightness.

Thus, the key to what the observer sees is the phase difference between the waves of rays r_1 and r_2. Both rays are derived from the same ray i, but the path involved in producing r_2 involves light traveling twice across the film (a to b, and then b to c), whereas the path involved in producing r_1 involves no travel through the film. Because θ is about zero, we approximate the path length difference between the waves of r_1 and r_2 as $2L$. However, to find the phase difference between the waves, we cannot just find the number of wavelengths λ that is equivalent to a path length difference of $2L$. This simple approach is impossible for two reasons: (1) the path length difference occurs in a medium other than air, and (2) reflections are involved, which can change the phase.

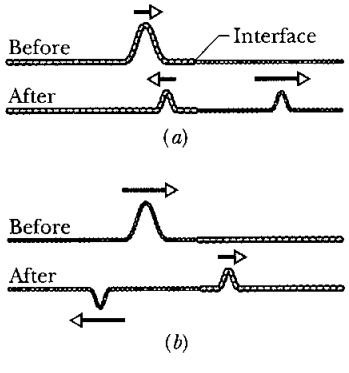


FIG. 36-34 Phase changes when a pulse is reflected at the interface between two stretched strings of different linear densities. The wave speed is greater in the lighter string. (*a*) The incident pulse is in the denser string. (*b*) The incident pulse is in the lighter string. Only here is there a phase change, and only in the reflected wave.

☞ The phase difference between two waves can change if one or both are reflected.

Before we continue our discussion of interference from thin films, we must discuss changes in phase that are caused by reflections.

Reflection Phase Shifts

Refraction at an interface never causes a phase change—but reflection can, depending on the indexes of refraction on the two sides of the interface. Figure 36-34 shows what happens when reflection causes a phase change, using as an example pulses on a denser string (along which pulse travel is relatively slow) and a lighter string (along which pulse travel is relatively fast).

When a pulse traveling relatively slowly along the denser string in Fig. 36-34*a* reaches the interface with the lighter string, the pulse is partially transmitted and partially reflected, with no change in orientation. For light, this situation corresponds to the incident wave traveling in the medium of greater index of refraction n (recall that greater n means slower speed). In that case, the wave that is reflected at the interface does not undergo a change in phase; that is, its *reflection phase shift* is zero.

When a pulse traveling more quickly along the lighter string in Fig. 36-34*b* reaches the interface with the denser string, the pulse is again partially transmitted

and partially reflected. The transmitted pulse again has the same orientation as the incident pulse, but now the reflected pulse is inverted. For a sinusoidal wave, such an inversion involves a phase change of π rad, or half a wavelength. For light, this situation corresponds to the incident wave traveling in the medium of lesser index of refraction (with greater speed). In that case, the wave that is reflected at the interface undergoes a phase shift of π rad, or half a wavelength.

We can summarize these results for light in terms of the index of refraction of the medium off which (or from which) the light reflects:



Reflection	Reflection phase shift
Off lower index	0
Off higher index	0.5 wavelength

This might be remembered as "higher means half."

Equations for Thin-Film Interference

In this chapter we have now seen three ways in which the phase difference between two waves can change:

1. by reflection
2. by the waves traveling along paths of different lengths
3. by the waves traveling through media of different indexes of refraction

When light reflects from a thin film, producing the waves of rays r_1 and r_2 shown in Fig. 36-33, all three ways are involved. Let us consider them one by one.

We first reexamine the two reflections in Fig. 36-33. At point a on the front interface, the incident wave (in air) reflects from the medium having the higher of the two indexes of refraction; so the wave of reflected ray r_1 has its phase shifted by 0.5 wavelength. At point b on the back interface, the incident wave reflects from the medium (air) having the lower of the two indexes of refraction; so the wave reflected there is not shifted in phase by the reflection, and thus neither is the portion of it that exits the film as ray r_2. We can organize this information with the first line in Table 36-1, which refers to the simplified drawing in Fig. 36-35 for a thin film in air. So far, as a result of the reflection phase shifts, the waves of r_1 and r_2 have a phase difference of 0.5 wavelength and thus are exactly out of phase.

Now we must consider the path length difference $2L$ that occurs because the wave of ray r_2 crosses the film twice. (This difference $2L$ is shown on the second line in Table 36-1.) If the waves of r_1 and r_2 are to be exactly in phase so that they produce fully constructive interference, the path length $2L$ must cause an additional phase difference of 0.5, 1.5, 2.5, ... wavelengths. Only then will the net phase difference be an integer number of wavelengths. Thus, for a bright film, we must have

$$2L = \frac{\text{odd number}}{2} \times \text{wavelength} \qquad \text{(in-phase waves).} \qquad (36\text{-}40)$$

The wavelength we need here is the wavelength λ_{n2} of the light in the medium containing path length $2L$—that is, in the medium with index of refraction n_2. Thus, we can rewrite Eq. 36-40 as

$$2L = \frac{\text{odd number}}{2} \times \lambda_{n2} \qquad \text{(in-phase waves).} \qquad (36\text{-}41)$$

If, instead, the waves are to be exactly out of phase so that there is fully destructive interference, the path length $2L$ must cause either no additional phase difference or a phase difference of 1, 2, 3, ... wavelengths. Only then will

TABLE 36-1

An Organizing Table for Thin-Film Interference in Air (Fig. 36-35)[a]

Reflection phase shifts	r_1: 0.5 wavelength	r_2: 0
Path length difference	$2L$	
Index in which path length difference occurs	n_2	
In phase[a]:	$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2}$	
Out of phase[a]:	$2L = \text{integer} \times \frac{\lambda}{n_2}$	

[a]Valid for $n_2 > n_1$ and $n_2 > n_3$.

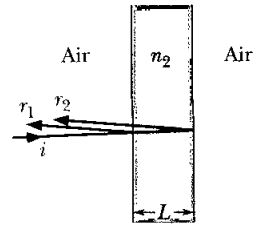


FIG. 36-35 Reflections from a thin film in air.

the net phase difference be an odd number of half-wavelengths. For a dark film, we must have

$$2L = \text{integer} \times \text{wavelength}, \tag{36-42}$$

where, again, the wavelength is the wavelength λ_{n2} in the medium containing $2L$. Thus, this time we have

$$2L = \text{integer} \times \lambda_{n2} \quad \text{(out-of-phase waves)}. \tag{36-43}$$

Now we can use Eq. 36-8 ($\lambda_n = \lambda/n$) to write the wavelength of the wave of ray r_2 inside the film as

$$\lambda_{n2} = \frac{\lambda}{n_2}, \tag{36-44}$$

where λ is the wavelength of the incident light in vacuum (and approximately also in air). Substituting Eq. 36-44 into Eq. 36-41 and replacing "odd number/2" with $(m+\frac{1}{2})$ give us

$$2L = (m+\tfrac{1}{2})\frac{\lambda}{n_2}, \quad \text{for } m = 0,1,2,\ldots \quad \text{(maxima—bright film in air)}. \tag{36-45}$$



FIG. 36-36 The reflection of light from a soapy water film spanning a vertical loop. The top portion is so thin that the light reflected there undergoes destructive interference, making that portion dark. Colored interference fringes, or bands, decorate the rest of the film but are marred by circulation of liquid within the film as the liquid is gradually pulled downward by gravitation. *(Richard Megna/Fundamental Photographs)*

Similarly, with m replacing "integer," Eq. 36-43 yields

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0,1,2,\ldots \quad \text{(minima—dark film in air)}. \tag{36-46}$$

For a given film thickness L, Eqs. 36-45 and 36-46 tell us the wavelengths of light for which the film appears bright and dark, respectively, one wavelength for each value of m. Intermediate wavelengths give intermediate brightnesses. For a given wavelength λ, Eqs. 36-45 and 36-46 tell us the thicknesses of the films that appear bright and dark in that light, respectively, one thickness for each value of m. Intermediate thicknesses give intermediate brightnesses.

Film Thickness Much Less Than λ

A special situation arises when a film is so thin that L is much less than λ, say, $L < 0.1\lambda$. Then the path length difference $2L$ can be neglected, and the phase difference between r_1 and r_2 is due *only* to reflection phase shifts. If the film of Fig. 36-35 where the reflections cause a phase difference of 0.5 wavelength, has thickness $L < 0.1\lambda$, then r_1 and r_2 are exactly out of phase, and thus the film is dark, regardless of the wavelength and intensity of the light. This special situation corresponds to $m = 0$ in Eq. 36-46. We shall count *any* thickness $L < 0.1\lambda$ as being the least thickness specified by Eq. 36-46 to make the film of Fig. 36-35 dark. (Every such thickness will correspond to $m = 0$.) The next greater thickness that will make the film dark is that corresponding to $m = 1$.

Figure 36-36 shows a vertical soap film whose thickness increases from top to bottom because gravitation has caused the film to slump. Bright white light illuminates the film. However, the top portion is so thin that it is dark. In the (somewhat thicker) middle we see fringes, or bands, whose color depends primarily on the wavelength at which reflected light undergoes fully constructive interference for a particular thickness. Toward the (thickest) bottom the fringes become progressively narrower and the colors begin to overlap and fade.

Why is Interference Observed Only in Thin Films?

Interference effects can be ignored with a thick film because it thickness is large. In this case the alternate bright and dark regions will be so close to each other that these will appear to merge into one another and interference pattern will not be visible.

Further, when we see such a film in white light, the various complementary colors will be so close to each other that these merge into one another and make the appearance of the thick film white. To understand this let us consider film of water (refractive index, $\mu = 4/3$) 1 cm thick on top of a glass surface (RI = 3/2). Assume that it is illuminated from above (Fig. 36-37). Then

$$\Delta x = \text{path differerence between 1 and 2} = 2\mu_\omega t.$$

For constructive interference in the reflected waves

$$2\mu_\omega t = n\lambda.$$

Because $t = 1$ cm is very large in comparison to the wavelength of visible light, the values of n in this equation will be large. For instance, if wavelength corresponding to orange-red light ($\lambda = 667$ nm) is to be strongly reflected then

$$n = \frac{2\mu_\omega t}{\lambda} = \frac{(2)(1.33)(1\times10^{-2})}{6.67\times10^{-7}} = 40\,000.$$

We may feel that this thick film would appear strongly orange-red; however, there will also be strong reflection of a slightly longer wavelength corresponding to $n = 39{,}999$ and of slightly shorter wavelength corresponding to $n = 40{,}001$. These wavelengths differ from each other by only one part in 40,000 or 667 nm/40,000 = 0.17 nm. This is a much smaller wavelength difference than the eye can detect. Hence, no one wavelength appears to be reinforced more than any other in the light which is reflected from a thick film. If the film is illuminated by a white light, the reflected light appears white. For thin films, the integer n will be small, the difference between adjacent strongly reflected wavelengths will be substantial, and the preferential reflection of certain wavelengths will be easily observed by the eye.

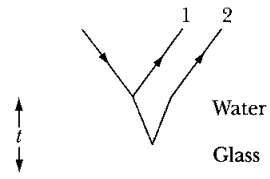


FIG. 36-37 A light wave undergoing successive reflections.

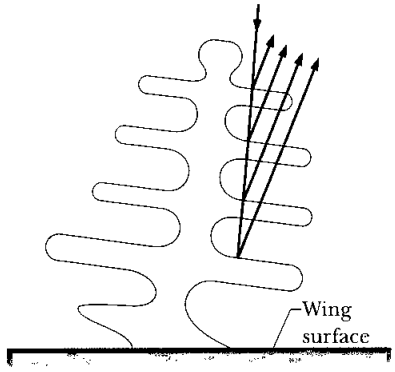


FIG. 36-38 Reflecting structure extending up from a Morpho butterfly wing. Reflections from the top surfaces of the transparent "terraces" give an interference color to the wing.

Color Shifting by Butterflies, Inks, and Paints

A surface that displays colors due to thin-film interference is said to be *iridescent* because the tints of the colors change as you change your view of the surface. The iridescence of the top surface of a *Morpho* butterfly wing is due to thin-film interference of light reflected by thin terraces of transparent cuticle-like material on the wing (Fig. 36-38). These terraces are arranged like wide, flat branches on a tree-like structure that extends perpendicular to the wing.

Suppose you look directly down on these terraces as white light shines directly down on the wing. Then the light reflected back up to you from the terraces undergoes fully constructive interference in the blue-green region of the visible spectrum. Light in the yellow and red regions, at the opposite end of the spectrum, is weaker because it undergoes only intermediate interference. Thus, the top surface of the wing looks blue-green to you.

If you intercept light that reflects from the wing in some other direction, the light has traveled along a slanted path through the terraces. Then the wavelength at which there is fully constructive interference is somewhat different from that for light reflected directly upward. Thus, if the wing moves in your view so that the angle at which you view it changes, the color at which the wing is brightest changes somewhat, producing the iridescence of the wing.

The color-shifting inks and paints used on paper currencies, cars, guitars, and other objects function in almost the same way as the color-shifting wing on a *Morpho* butterfly. Figure 36-39*a* shows a cross section of the ink layer used on some currencies. The color shifting is due to thin multilayered flakes suspended in regular ink. Figure 36-39*b* shows a cross section through one of the flakes. Light penetrating the regular ink above the flake travels through thin layers of chromium (Cr), magnesium fluoride (MgF_2), and aluminum (Al). The Cr layers function as weak mirrors, the Al layer functions as a better mirror,

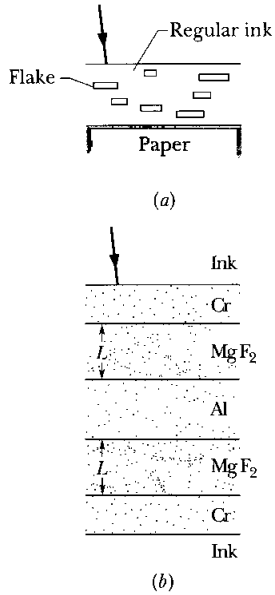


FIG. 36-39 (*a*) Color-shifting ink on a paper currency consists of multilayered thin-film flakes suspended in regular ink. (*b*) Cross section of one of the flakes. Light penetrates the five layers, reflecting from each boundary. The color that results from the interference of these reflected light waves is determined by the thickness L of the magnesium fluoride layers.

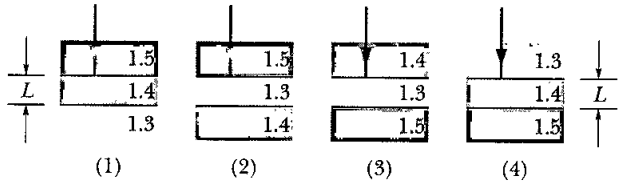
and the MgF_2 layers function like soap films. The result is that light reflected upward from each boundary between layers passes back through the regular ink and then undergoes interference at an observer's eye. Which color undergoes fully constructive interference depends on the thickness L of the MgF_2 layers. In Indian currency printed with color-shifting inks, the value of L is designed to give fully constructive interference for green or greenish yellow light when the observer looks directly down on the currency. When the observer tilts the currency and thus each flake, the light reaching the observer from the flakes undergoes constructive interference for blue light. Thus, by changing the angle of view, the observer can shift the color. Other countries use other designs of thin-film flakes to achieve different shifts in the colors on their currencies.

PROBLEM-SOLVING TACTICS

Tactic 1: *Thin-Film Equations* Some students believe that Eq. 36-45 gives the maxima and Eq. 36-46 gives the minima for *all* thin-film situations. This is not true. These relations were derived only for the situation in which $n_2 > n_1$ and $n_2 > n_3$ in Fig. 36-33 and for the situation of Fig. 36-35, in which a thin film lies in air.

The appropriate equations for other relative values of the indexes of refraction can be derived by following the reasoning of this section and constructing new versions of Table 36-1. In each case you will end up with Eqs. 36-45 and 36-46, but sometimes Eq. 36-45 will give the minima and Eq. 36-46 will give the maxima—the opposite of what we found here. Which equation gives which depends on whether the reflections at the two interfaces give the same reflection phase shift.

CHECKPOINT 5 The figure shows four situations in which light reflects perpendicularly from a thin film of thickness L, with indexes of refraction as given. (a) For which situations does reflection at the film interfaces cause a zero phase difference for the two reflected rays? (b) For which situations will the film be dark if the path length difference $2L$ causes a phase difference of 0.5 wavelength?



Sample Problem 36-12

White light, with a uniform intensity across the visible wavelength range of 400 to 690 nm, is perpendicularly incident on a water film, of index of refraction $n_2 = 1.33$ and thickness $L = 320$ nm, that is suspended in air. At what wavelength λ is the light reflected by the film brightest to an observer?

Conceptualize/Classify: The reflected light from the film is brightest at the wavelengths λ for which the reflected rays are in phase with one another. The equation relating these wavelengths λ to the given film thickness L and film index of refraction n_2 is either Eq. 36-45 or Eq. 36-46, depending on the reflection phase shifts for this particular film.

Compute: To determine which equation is needed, we should fill out an organizing table like Table 36-1. However, because there is air on both sides of the water film, the situation here is exactly like that in Fig. 36-36, and thus the table would be exactly like Table 36-1. Then from Table 36-1, we see that the reflected rays are in phase (and thus the film is brightest) when

$$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2},$$

which leads to Eq. 36-45:

$$2L = (m + \tfrac{1}{2})\frac{\lambda}{n_2}.$$

Solving for λ and substituting for L and n_2, we find

$$\lambda = \frac{2n_2L}{m+\frac{1}{2}} = \frac{(2)(1.33)(320\text{ nm})}{m+\frac{1}{2}} = \frac{851\text{ nm}}{m+\frac{1}{2}}.$$

For $m = 0$, this gives us $\lambda = 1700$ nm, which is in the infrared region. For $m = 1$, we find $\lambda = 567$ nm, which is yellow-green light, near the middle of the visible spectrum. For $m = 2$, $\lambda = 340$ nm, which is in the ultraviolet region. Thus, the wavelength at which the light seen by the observer is brightest is

$$\lambda = 567\text{ nm}. \qquad \text{(Answer)}$$

Sample Problem 36-13

In Fig. 36-40, a glass lens is coated on one side with a thin film of magnesium fluoride (MgF_2) to reduce reflection from the lens surface. The index of refraction of MgF_2 is 1.38; that of the glass is 1.50. What is the least coating thickness that eliminates (via interference) the reflections at the middle of the visible spectrum ($\lambda = 550$ nm)? Assume that the light is approximately perpendicular to the lens surface.

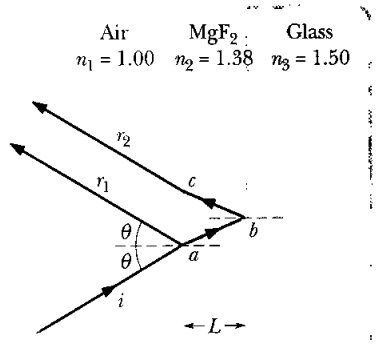


FIG. 36-40 Unwanted reflections from glass can be suppressed (at a chosen wavelength) by coating the glass with a thin transparent film of magnesium fluoride of the properly chosen thickness.

Conceptualize/Classify: Reflection is eliminated if the film thickness L is such that light waves reflected from the two film interfaces are exactly out of phase. The equation relating L to the given wavelength λ and the index of refraction n_2 of the thin film is either Eq. 36-45 or Eq. 36-46, depending on the reflection phase shifts at the interfaces.

Compute: To determine which equation is needed, we fill out an organizing table like Table 36-1. At the first interface, the incident light is in air, which has a lesser index of refraction than the MgF_2 (the thin film). Thus, we fill in 0.5 wavelength under r_1 in our organizing table (meaning that the waves of ray r_1 are shifted by 0.5λ at the first interface). At the second interface, the incident light is in the MgF_2, which has a lesser index of refraction than the glass on the other side of the interface. Thus, we fill in 0.5 wavelength under r_2 in our table.

Because both reflections cause the same phase shift, they tend to put the waves of r_1 and r_2 in phase. Since we want those waves to be *out of phase,* their path length difference $2L$ must be an odd number of half-wavelengths:

$$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2}.$$

This leads to Eq. 36-45. Solving that equation for L then gives us the film thicknesses that will eliminate reflection from the lens and coating:

$$L = (m + \tfrac{1}{2})\frac{\lambda}{2n_2}, \quad \text{for } m = 0, 1, 2, \ldots. \quad (36\text{-}47)$$

We want the least thickness for the coating—that is, the smallest value of L. Thus, we choose $m = 0$, the smallest possible value of m. Substituting it and the given data in Eq. 36-47, we obtain

$$L = \frac{\lambda}{4n_2} = \frac{550\,\text{nm}}{(4)(1.38)} = 99.6\,\text{nm}. \quad \text{(Answer)}$$

Sample Problem 36-14 Build your skill

Figure 36-41*a* shows a transparent plastic block with a thin wedge of air at the right. (The wedge thickness is exaggerated in the figure.) A broad beam of red light, with wavelength $\lambda = 632.8$ nm, is directed downward through the top of the block (at an incidence angle of 0°). Some of the light that passes into the plastic is reflected back up from the top and bottom surfaces of the wedge, which acts as a thin film (of air) with a thickness that varies uniformly and gradually from L_L at the left-hand end to L_R at the right-hand end. (The plastic layers above and below the wedge of air are too thick to act as thin films.) An observer looking down on the block sees an interference pattern consisting of six dark fringes and five bright red fringes along the wedge. What is the change in thickness ΔL ($= L_R - L_L$) along the wedge?

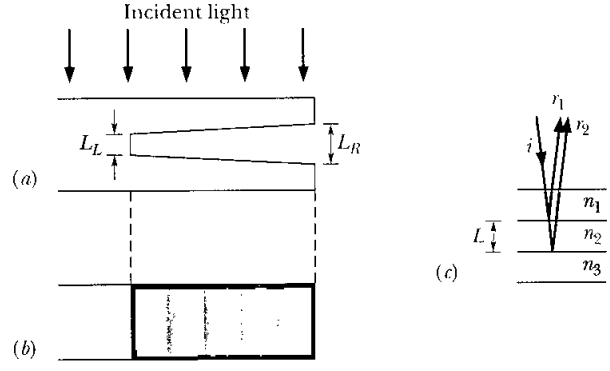


FIG. 36-41 (*a*) Red light is incident on a thin, air-filled wedge in the side of a transparent plastic block. The thickness of the wedge is L_L at the left end and L_R at the right end. (*b*) The view from above the block: an interference pattern of six dark fringes and five bright red fringes lies over the region of the wedge. (*c*) A representation of the incident ray *i*, reflected rays r_1 and r_2, and thickness L of the wedge anywhere along the length of the wedge.

Conceptualize/Classify: (1) The brightness at any point along the left–right length of the air wedge is

due to the interference of the waves reflected at the top and bottom interfaces of the wedge. (2) The variation of brightness in the pattern of bright and dark fringes is due to the variation in the thickness of the wedge. In some regions, the thickness puts the reflected waves in phase and thus produces a bright reflection (a bright red fringe). In other regions, the thickness puts the reflected waves out of phase and thus produces no reflection (a dark fringe).

Compute: Organizing the reflections: Because the observer sees more dark fringes than bright fringes, we can assume that a dark fringe is produced at both the left and right ends of the wedge. Thus, the interference pattern is that shown in Fig. 36-41*b*.

We can represent the reflection of light at the top and bottom interfaces of the wedge, at any point along its length, with Fig. 36-41*c*, in which L is the wedge thickness at that point. Let us apply this figure to the left end of the wedge, where the reflections give a dark fringe.

We know that, for a dark fringe, the waves of rays r_1 and r_2 in Fig. 36-41*c* must be out of phase. We also know that the equation relating the film thickness L to the light's wavelength λ and the film's index of refraction n_2 is either Eq. 36-45 or Eq. 36-46 depending on the reflection phase shifts. To determine which equation gives a dark fringe at the left end of the wedge, we should fill out an organizing table like Table 36-1.

At the top interface of the wedge, the incident light is in the plastic, which has a greater n than the air beneath that interface. So, we fill in 0 under r_1 in our organizing table. At the bottom interface of the wedge, the incident light is in air, which has a lesser n than the plastic beneath that interface. So we fill in 0.5 wavelength under r_2. Thus the reflections alone tend to put the waves of r_1 and r_2 out of phase.

Reflections at left end: Because the waves are out of phase at the left end of the air wedge, the path length difference $2L$ at that end of the wedge must be given by

$$2L = \text{integer} \times \frac{\lambda}{n_2},$$

which leads to Eq. 36-46:

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots. \tag{36-48}$$

Reflections at right end: Equation 36-48 holds not only for the left end of the wedge but also for any point along the wedge where a dark fringe is observed, including the right end, with a different integer value of m for each fringe. The least value of m is associated with the least thickness of the wedge where a dark fringe is observed. Progressively greater values of m are associated with progressively greater thicknesses of the wedge where a dark fringe is observed. Let m_L be the value at the left end. Then the value at the right end must be $m_L + 5$ because, from Fig. 36-41*b*, the right end is located at the fifth dark fringe from the left end.

Thickness difference: To find ΔL, we first solve Eq. 36-48 twice—once for the thickness L_L at the left end and once for the thickness L_R at the right end:

$$L_L = (m_L)\frac{\lambda}{2n_2}, \quad L_R = (m_L + 5)\frac{\lambda}{2n_2}. \tag{36-49}$$

We can now subtract L_L from L_R and substitute $n_2 = 1.00$ for the air within the wedge and $\lambda = 632.8 \times 10^{-9}$ m:

$$\Delta L = L_R - L_L = \frac{(m_L + 5)\lambda}{2n_2} - \frac{m_L\lambda}{2n_2} = \frac{5}{2}\frac{\lambda}{n_2}$$
$$= 1.58 \times 10^{-6} \text{ m}. \qquad \text{(Answer)}$$

36-9 | Diffraction and the Wave Theory of Light

We defined diffraction rather loosely as the flaring of light as it emerges from a narrow slit. More than just flaring occurs, however, because the light produces an interference pattern called a **diffraction pattern**. For example, when monochromatic light from a distant source (or a laser) passes through a narrow slit and is then intercepted by a viewing screen, the light produces on the screen a diffraction pattern like that in Fig. 36-42. This pattern consists of a broad and intense (very bright) central maximum plus a number of narrower and less intense maxima (called **secondary** or **side** maxima) to both sides. In between the maxima are minima.

Diffraction is not limited to situations when light passes through a narrow opening (such as a slit or pinhole). It also occurs when light passes an edge, such as the edges of the razor blade whose diffraction pattern is shown in Fig. 36-43. Note the lines of maxima and minima that run approximately parallel to the edges, at both the inside edges of the blade and the outside edges. As the light passes, say, the vertical edge at the left, it flares left and right and undergoes

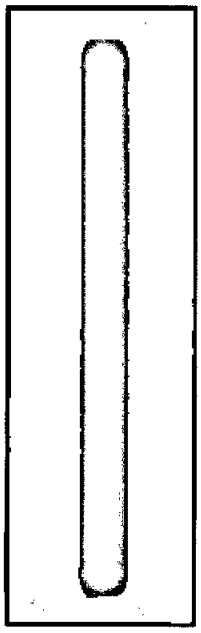


FIG. 36-42 This diffraction pattern appeared on a viewing screen when light that had passed through a narrow vertical slit reached the screen. Diffraction caused the light to flare out perpendicular to the long sides of the slit. That flaring produced an interference pattern consisting of a broad central maximum plus less intense and narrower secondary (or side) maxima, with minima between them. *(Ken Kay/Fundamental Photographs)*



FIG. 36-43 The diffraction pattern produced by a razor blade in monochromatic light. Note the lines of alternating maximum and minimum intensity. *(Ken Kay/ Fundamental Photographs)*

interference, producing the pattern along the left edge. The rightmost portion of that pattern actually lies behind the blade, within what would be the blade's shadow if geometrical optics prevailed.

You encounter a common example of diffraction when you look at a clear blue sky and see tiny specks and hairlike structures floating in your view. These *floaters*, as they are called, are produced when light passes the edges of tiny deposits in the vitreous humor, the transparent material filling most of the eyeball. What you are seeing when a floater is in your field of vision is the diffraction pattern produced on the retina by one of these deposits. If you sight through a pinhole in a piece of cardboard so as to make the light entering your eye approximately a plane wave, you can distinguish individual maxima and minima in the patterns.

Diffraction is a wave effect. That is, it occurs because light is a wave and it occurs with other types of waves as well. For example, you have probably seen diffraction in action at football games. When a cheerleader near the playing field yells up at several thousand noisy fans, the yell can hardly be heard because the sound waves diffract when they pass through the narrow opening of the cheerleader's mouth. This flaring leaves little of the waves traveling toward the fans in front of the cheerleader. To offset the diffraction, the cheerleader can yell through a megaphone. The sound waves then emerge from the much wider opening at the end of the megaphone. The flaring is thus reduced, and much more of the sound reaches the fans in front of the cheerleader.

36-10 | Diffraction by a Single Slit: Locating the Minima

Let us now examine the diffraction pattern of plane waves of light of wavelength λ that are diffracted by a single long, narrow slit of width a in an otherwise opaque screen B, as shown in cross section in Fig. 36-44. (In that figure, the slit's length extends into and out of the page, and the incoming wavefronts are parallel to screen B.) When the diffracted light reaches viewing screen C, waves from different points within the slit undergo interference and produce a diffraction pattern of bright and dark fringes (interference maxima and minima) on the screen. To locate the fringes, we shall use a procedure somewhat similar to the one we used to locate the fringes in a two-slit interference pattern. However,

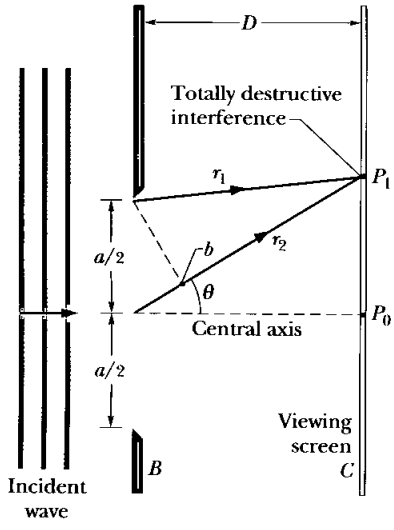


FIG. 36-44 Waves from the top points of two zones of width $a/2$ undergo fully destructive interference at point P_1 on viewing screen C.

diffraction is more mathematically challenging, and here we shall be able to find equations for only the dark fringes.

Before we do that, however, we can justify the central bright fringe seen in Fig. 36-42 by noting that the Huygens wavelets from all points in the slit travel about the same distance to reach the center of the pattern and thus are in phase there. As for the other bright fringes, we can say only that they are approximately halfway between adjacent dark fringes.

To find the dark fringes, we shall use a clever (and simplifying) strategy that involves pairing up all the rays coming through the slit and then finding what conditions cause the wavelets of the rays in each pair to cancel each other. We apply this strategy in Fig. 36-44 to locate the first dark fringe, at point P_1. First, we mentally divide the slit into two *zones* of equal widths $a/2$. Then we extend to P_1 a light ray r_1 from the top point of the top zone and a light ray r_2 from the top point of the bottom zone. A central axis is drawn from the center of the slit to screen C, and P_1 is located at an angle θ to that axis.

The wavelets of the pair of rays r_1 and r_2 are in phase within the slit because they originate from the same wavefront passing through the slit, along the width of the slit. However, to produce the first dark fringe they must be out of phase by $\lambda/2$ when they reach P_1; this phase difference is due to their path length difference, with the path traveled by the wavelet of r_2 to reach P_1 being longer than the path traveled by the wavelet of r_1. To display this path length difference, we find a point b on ray r_2 such that the path length from b to P_1 matches the path length of ray r_1. Then the path length difference between the two rays is the distance from the center of the slit to b.

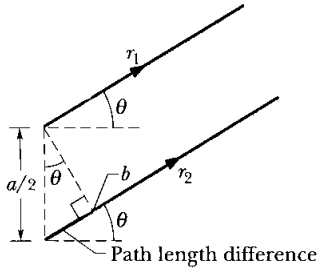


FIG. 36-45 For $D \gg a$, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis.

When viewing screen C is near screen B, as in Fig. 36-44, the diffraction pattern on C is difficult to describe mathematically. However, we can simplify the mathematics considerably if we arrange for the screen separation D to be much larger than the slit width a. Then we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis (Fig. 36-45). We can also approximate the triangle formed by point b, the top point of the slit, and the center point of the slit as being a right triangle, and one of the angles inside that triangle as being θ. The path length difference between rays r_1 and r_2 (which is still the distance from the center of the slit to point b) is then equal to $(a/2) \sin \theta$.

We can repeat this analysis for any other pair of rays originating at corresponding points in the two zones (say, at the midpoints of the zones) and extending to point P_1. Each such pair of rays has the same path length difference $(a/2) \sin \theta$. Setting this common path length difference equal to $\lambda/2$ (our condition for the first dark fringe), we have

$$\frac{a}{2} \sin \theta = \frac{\lambda}{2},$$

which gives us

$$a \sin \theta = \lambda \qquad \text{(first minimum)}. \tag{36-50}$$

Given slit width a and wavelength λ, Eq. 36-50 tells us the angle θ of the first dark fringe above and (by symmetry) below the central axis.

Note that if we begin with $a > \lambda$ and then narrow the slit while holding the wavelength constant, we increase the angle at which the first dark fringes appear; that is, the extent of the diffraction (the extent of the flaring and the width of the pattern) is *greater* for a *narrower* slit. When we have reduced the slit width to the wavelength (that is, $a = \lambda$), the angle of the first dark fringes is 90°. Since the first dark fringes mark the two edges of the central bright fringe, that bright fringe must then cover the entire viewing screen.

We find the second dark fringes above and below the central axis as we found the first dark fringes, except that we now divide the slit into *four* zones of equal widths $a/4$, as shown in Fig. 36-46*a*. We then extend rays r_1, r_2, r_3, and r_4 from the top points of the zones to point P_2, the location of the second dark fringe above

the central axis. To produce that fringe, the path length difference between r_1 and r_2, that between r_2 and r_3, and that between r_3 and r_4 must all be equal to $\lambda/2$.

For $D \gg a$, we can approximate these four rays as being parallel, at angle θ to the central axis. To display their path length differences, we extend a perpendicular line through each adjacent pair of rays, as shown in Fig. 36-46*b*, to form a series of right triangles, each of which has a path length difference as one side. We see from the top triangle that the path length difference between r_1 and r_2 is $(a/4)\sin\theta$. Similarly, from the bottom triangle, the path length difference between r_3 and r_4 is also $(a/4)\sin\theta$. In fact, the path length difference for any two rays that originate at corresponding points in two adjacent zones is $(a/4)\sin\theta$. Since in each such case the path length difference is equal to $\lambda/2$, we have

$$\frac{a}{4}\sin\theta = \frac{\lambda}{2},$$

which gives us

$$a\sin\theta = 2\lambda \qquad \text{(second minimum)}. \tag{36-51}$$

We could now continue to locate dark fringes in the diffraction pattern by splitting up the slit into more zones of equal width. We would always choose an even number of zones so that the zones (and their waves) could be paired as we have been doing. We would find that the dark fringes above and below the central axis can be located with the general equation

$$a\sin\theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \qquad \text{(minima—dark fringes)}. \tag{36-52}$$

You can remember this result in the following way. Draw a triangle like the one in Fig. 36-45, but for the full slit width a, and note that the path length difference between the top and bottom rays equals $a\sin\theta$. Thus, Eq. 36-52 says:

> In a single-slit diffraction experiment, dark fringes are produced where the path length differences ($a\sin\theta$) between the top and bottom rays are equal to $\lambda, 2\lambda, 3\lambda, \ldots$.

This may seem to be wrong because the waves of those two particular rays will be exactly in phase with each other when their path length difference is an integer number of wavelengths. However, they each will still be part of a pair of waves that are exactly out of phase with each other; thus, *each* wave will be canceled by some other wave, resulting in darkness.

Equations 36-50, 36-51, and 36-52 are derived for the case of $D \gg a$. However, they also apply if we place a converging lens between the slit and the viewing screen and then move the screen in so that it coincides with the focal plane of the lens. The lens ensures that rays which now reach any point on the screen are *exactly* parallel (rather than approximately) back at the slit.

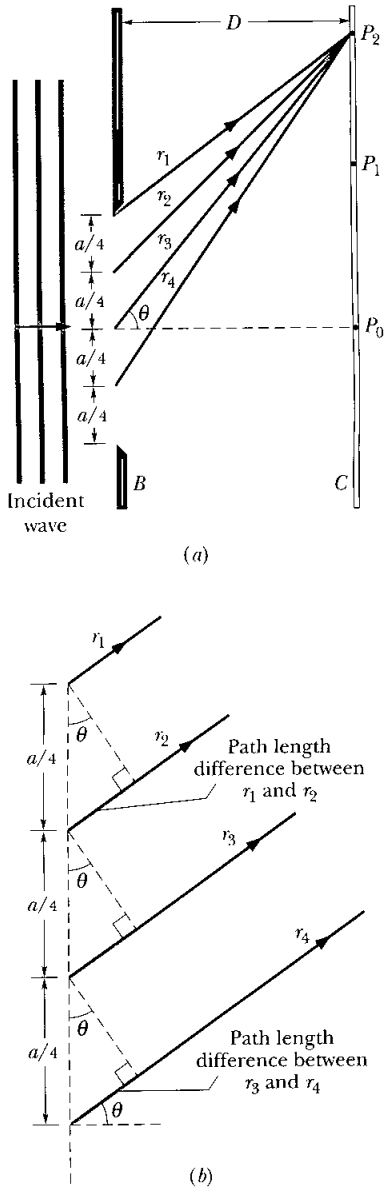


FIG. 36-46 (*a*) Waves from the top points of four zones of width $a/4$ undergo fully destructive interference at point P_2. (*b*) For $D \gg a$, we can approximate rays r_1, r_2, r_3, and r_4 as being parallel, at angle θ to the central axis.

CHECKPOINT 6 We produce a diffraction pattern on a viewing screen by means of a long narrow slit illuminated by blue light. Does the pattern expand away from the bright center (the maxima and minima shift away from the center) or contract toward it if we (a) switch to yellow light or (b) decrease the slit width?

Sample Problem 36-15

A slit of width a is illuminated by white light.

(a) For what value of a will the first minimum for red light of wavelength $\lambda = 650$ nm appear at $\theta = 15°$?

Conceptualize / Classify: Diffraction occurs separately for each wavelength in the range of wavelengths passing through the slit, with the locations of the minima for each wavelength given by Eq. 36-52 ($a\sin\theta = m\lambda$).

Compute: When we set $m = 1$ (for the first minimum) and substitute the given values of θ and λ, Eq. 36-52 yields

$$a = \frac{m\lambda}{\sin\theta} = \frac{(1)(650\text{ nm})}{\sin 15°}$$
$$= 2511\text{ nm} \approx 2.5\ \mu\text{m}. \qquad \text{(Answer)}$$

Conclude: For the incident light to flare out that much (±15° to the first minima) the slit has to be very fine indeed—in this case, a mere four times the wavelength. For comparison, note that a fine human hair may be about 100 μm in diameter.

(b) What is the wavelength λ' of the light whose first side diffraction maximum is at 15°, thus coinciding with the first minimum for the red light?

Conceptualize/Classify: The first side maximum for any wavelength is about halfway between the first and second minima for that wavelength.

Compute: Those first and second minima can be located with Eq. 36-52 by setting $m = 1$ and $m = 2$, respectively. Thus, the first side maximum can be located *approximately* by setting $m = 1.5$. Then Eq. 36-52 becomes

$$a \sin\theta = 1.5\ \lambda'.$$

Solving for λ' and substituting known data yield

$$\lambda' = \frac{a\sin\theta}{1.5} = \frac{(2511\text{ nm})(\sin 15°)}{1.5}$$
$$= 430\text{ nm}. \qquad \text{(Answer)}$$

Conclude: Light of this wavelength is violet. From the two equations we used, can you see that the first side maximum for light of wavelength 430 nm will always coincide with the first minimum for light of wavelength 650 nm, no matter what the slit width is? However, the angle θ at which this overlap occurs does depend on slit width. If the slit is relatively narrow, the angle will be relatively large, and conversely.

36-11 | Intensity in Single-Slit Diffraction, Quantitatively

Equation 36-52 tells us how to locate the minima of the single-slit diffraction pattern on screen C of Fig. 36-44 as a function of the angle θ in that figure. Here we wish to derive an expression for the intensity $I(\theta)$ of the pattern as a function of θ. We state, without proof, that the intensity is given by

$$I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad (36\text{-}53)$$

where

$$\alpha = \tfrac{1}{2}\phi = \frac{\pi a}{\lambda}\sin\theta. \qquad (36\text{-}54)$$

The symbol α is just a convenient connection between the angle θ that locates a point on the viewing screen and the light intensity $I(\theta)$ at that point. The intensity I_m is the greatest value of the intensities $I(\theta)$ in the pattern and occurs at the central maximum (where $\theta = 0$), and ϕ is the phase difference (in radians) between the top and bottom rays from the slit of width a.

Study of Eq. 36-53 shows that intensity minima will occur where

$$\alpha = m\pi, \quad \text{for } m = 1, 2, 3, \ldots. \qquad (36\text{-}55)$$

If we put this result into Eq. 36-54, we find

$$m\pi = \frac{\pi a}{\lambda}\sin\theta, \quad \text{for } m = 1, 2, 3, \ldots,$$

or

$$a\sin\theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \quad \text{(minima—dark fringes)}, \qquad (36\text{-}56)$$

which is exactly Eq. 36-53, the expression that we derived earlier for the location of the minima.

Figure 36-47 shows plots of the intensity of a single-slit diffraction pattern, calculated with Eqs. 36-53 and 36-54 for three slit widths: $a = \lambda$, $a = 5\lambda$, and $a = 10\lambda$. Note that as the slit width increases (relative to the wavelength), the width of the

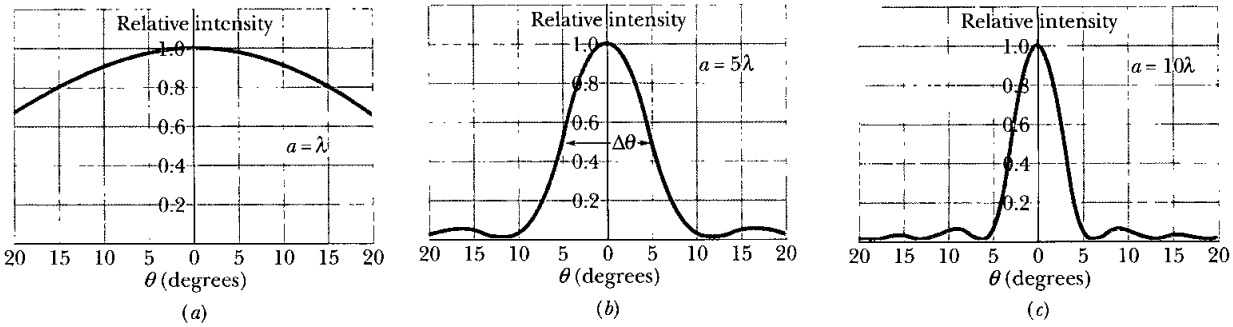


FIG. 36-47 The relative intensity in single-slit diffraction for three values of the ratio a/λ. The wider the slit is, the narrower is the central diffraction maximum.

central diffraction maximum (the central hill-like region of the graphs) decreases; that is, the light undergoes less flaring by the slit. The secondary maxima also decrease in width (and become weaker). In the limit of slit width a being much greater than wavelength λ, the secondary maxima due to the slit disappear; we then no longer have single-slit diffraction (but we still have diffraction due to the edges of the wide slit, like that produced by the edges of the razor blade in Fig. 36-43).

Sample Problem 36-16

Find the intensities of the first three secondary maxima (side maxima) in the single-slit diffraction pattern of Fig. 36-42, measured as a percentage of the intensity of the central maximum.

Conceptualize/Classify: The secondary maxima lie approximately halfway between the minima, whose angular locations are given by Eq. 36-55 ($\alpha = m\pi$). The locations of the secondary maxima are then given (approximately) by

$$a = (m+\tfrac{1}{2})\pi, \quad \text{for } m = 1,2,3,\ldots,$$

with α in radian measure. We can relate the intensity I at any point in the diffraction pattern to the intensity I_m of the central maximum via Eq. 36-53.

Compute: Substituting the approximate values of α for the secondary maxima into Eq. 36-53 to obtain the relative intensities at those maxima, we get

$$\frac{I}{I_m} = \left(\frac{\sin\alpha}{\alpha}\right)^2 = \left(\frac{\sin(m+\frac{1}{2})\pi}{(m+\frac{1}{2})\pi}\right)^2, \quad \text{for } m = 1,2,3,\ldots.$$

The first of the secondary maxima occurs for $m = 1$, and its relative intensity is

$$\frac{I_1}{I_m} = \left(\frac{\sin(1+\frac{1}{2})\pi}{(1+\frac{1}{2})\pi}\right)^2 = \left(\frac{\sin 1.5\pi}{1.5\pi}\right)^2$$

$$= 4.50\times10^{-2} \approx 4.5\%. \qquad \text{(Answer)}$$

For $m = 2$ and $m = 3$ we find that

$$\frac{I_2}{I_m} = 1.6\% \quad \text{and} \quad \frac{I_3}{I_m} = 0.83\%. \qquad \text{(Answer)}$$

Compute: Successive secondary maxima decrease rapidly in intensity. Figure 36-42 was deliberately overexposed to reveal them.

36-12 | Diffraction by a Circular Aperture

Here we consider diffraction by a circular aperture—that is, a circular opening, such as a circular lens, through which light can pass. Figure 36-48 shows the image of a distant point source of light (a star, for instance) formed on photographic film placed in the focal plane of a converging lens. This image is not a point, as geometrical optics would suggest, but a circular disk surrounded by several progressively fainter secondary rings. Comparison with Fig. 36-42 leaves little doubt that we are dealing with a diffraction phenomenon. Here, however, the aperture is a circle of diameter d rather than a rectangular slit.

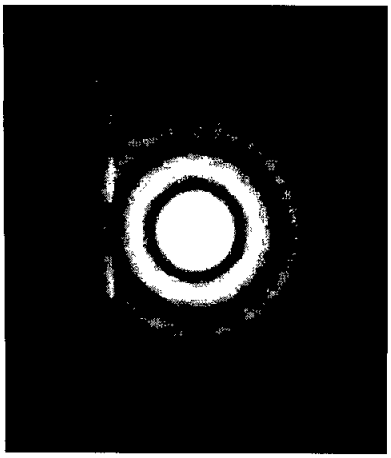


FIG. 36-48 The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum. *(From Michel Cagnet, Maurice Franzon, and Jean Thierr, Atlas of Optical Phenomena. Springer-Verlag, New York, 1962. Reproduced with permission.)*

The (complex) analysis of such patterns shows that the first minimum for the diffraction pattern of a circular aperture of diameter d is located by

$$\sin\theta = 1.22\frac{\lambda}{d} \qquad \text{(first minimum—circular aperture).} \tag{36-57}$$

The angle θ here is the angle from the central axis to any point on that (circular) minimum. Compare this with Eq. 36-50,

$$\sin\theta = \frac{\lambda}{a} \qquad \text{(first minimum—single slit),} \tag{36-58}$$

which locates the first minimum for a long narrow slit of width a. The main difference is the factor 1.22, which enters because of the circular shape of the aperture.

Resolvability

The fact that lens images are diffraction patterns is important when we wish to *resolve* (distinguish) two distant point objects whose angular separation is small. Figure 36-49 shows, in three different cases, the visual appearance and corresponding intensity pattern for two distant point objects (stars, say) with small angular separation. In Fig. 36-49*a*, the objects are not resolved because of diffraction; that is, their diffraction patterns (mainly their central maxima) overlap so much that the two objects cannot be distinguished from a single point object. In Fig. 36-49*b* the objects are barely resolved, and in Fig. 36-49*c* they are fully resolved.

In Fig. 36-49*b* the angular separation of the two point sources is such that the central maximum of the diffraction pattern of one source is centered on the first minimum of the diffraction pattern of the other, a condition called **Rayleigh's criterion** for resolvability. From Eq. 36-57, two objects that are barely resolvable by this criterion must have an angular separation θ_R of

$$\theta_R = \sin^{-1}\frac{1.22\lambda}{d}.$$

Since the angles are small, we can replace $\sin\theta_R$ with θ_R expressed in radians:

$$\theta_R = 1.22\frac{\lambda}{d} \qquad \text{(Rayleigh's criterion).} \tag{36-59}$$

Applying Rayleigh's criterion for resolvability to human vision is only an approximation because visual resolvability depends on many factors, such as

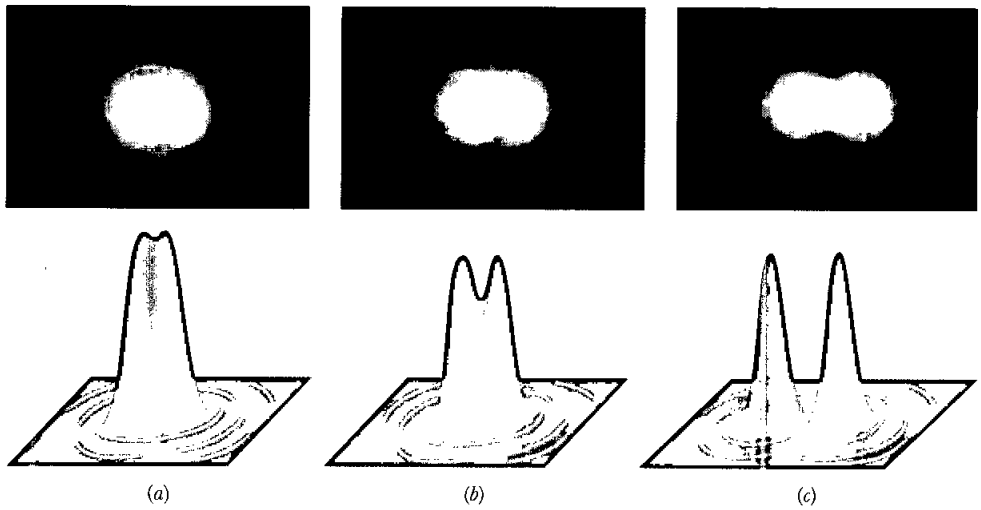


FIG. 36-49 At the top, the images of two point sources (stars) formed by a converging lens. At the bottom, representations of the image intensities. In (*a*) the angular separation of the sources is too small for them to be distinguished, in (*b*) they can be marginally distinguished, and in (*c*) they are clearly distinguished. Rayleigh's criterion is satisfied in (*b*), with the central maximum of one diffraction pattern coinciding with the first minimum of the other.

the relative brightness of the sources and their surroundings, turbulence in the air between the sources and the observer, and the functioning of the observer's visual system. Experimental results show that the least angular separation that can actually be resolved by a person is generally somewhat greater than the value given by Eq. 36-59. However, for calculations here, we shall take Eq. 36-59 as being a precise criterion: If the angular separation θ between the sources is greater than θ_R, we can visually resolve the sources; if it is less, we cannot.

When we wish to use a lens instead of our visual system to resolve objects of small angular separation, it is desirable to make the diffraction pattern as small as possible. According to Eq. 36-59, this can be done either by increasing the lens diameter or by using light of a shorter wavelength. For this reason ultraviolet light is often used with microscopes; because of its shorter wavelength, it permits finer detail to be examined than would be possible for the same microscope operated with visible light.

CHECKPOINT 7 Suppose that you can barely resolve two red dots because of diffraction by the pupil of your eye. If we increase the general illumination around you so that the pupil decreases in diameter, does the resolvability of the dots improve or diminish? Consider only diffraction. (You might experiment to check your answer.)

Sample Problem 36-17

A circular converging lens, with diameter $d = 32$ mm and focal length $f = 24$ cm, forms images of distant point objects in the focal plane of the lens. The wavelength is $\lambda = 550$ nm.

(a) Considering diffraction by the lens, what angular separation must two distant point objects have to satisfy Rayleigh's criterion?

Conceptualize/Classify: Figure 36-50 shows two distant point objects P_1 and P_2, the lens, and a viewing screen in the focal plane of the lens. It also shows, on the right, plots of light intensity I versus position on the screen for the central maxima of the images formed by the lens. Note that the angular separation θ_o of the objects equals the angular separation θ_i of the images. Thus, if the images are to satisfy Rayleigh's criterion for resolvability, the angular separations on both sides of the lens must be given by Eq. 36-59 (assuming small angles).

Compute: From Eq. 36-59, we obtain

$$\theta_0 = \theta_i = \theta_R = 1.22\frac{\lambda}{d}$$

$$= \frac{(1.22)(550 \times 10^{-9}\,\text{m})}{32 \times 10^{-3}\,\text{m}} = 2.1 \times 10^{-5}\,\text{rad}. \quad \text{(Answer)}$$

Conclude: At this angular separation, each central maximum in the two intensity curves of Fig. 36-50 is centered on the first minimum of the other curve.

(b) What is the separation Δx of the centers of the *images* in the focal plane? (That is, what is the separation of the *central* peaks in the two intensity-versus-position curves?)

Compute: From either triangle between the lens and the screen in Fig. 36-50, we see that $\tan \theta_i/2 = \Delta x/2f$. Rearranging this equation and making the approximation $\tan\theta \approx \theta$, we find

$$\Delta x = f\theta_i, \qquad (36\text{-}60)$$

where θ_i is in radian measure. Substituting known data then yields

$$\Delta x = (0.24\,\text{m})(2.1 \times 10^{-5}\,\text{rad}) = 5.0\,\mu\text{m}. \quad \text{(Answer)}$$

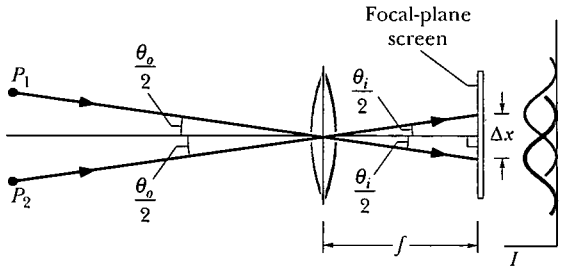


FIG. 36-50 Light from two distant point objects P_1 and P_2 passes through a converging lens and forms images on a viewing screen in the focal plane of the lens. Only one representative ray from each object is shown. The images are not points but diffraction patterns, with intensities approximately as plotted at the right. The angular separation of the objects is θ_o and that of the images is θ_i; the central maxima of the images have a separation Δx.

36-13 | Diffraction by a Double Slit

In the double-slit experiments we implicitly assumed that the slits were much narrower than the wavelength of the light illuminating them; that is, $a \ll \lambda$. For such narrow slits, the central maximum of the diffraction pattern of either slit covers the entire viewing screen. Moreover, the interference of light from the two slits produces bright fringes with approximately the same intensity (Fig. 36-18).

In practice with visible light, however, the condition $a \ll \lambda$ is often not met. For relatively wide slits, the interference of light from two slits produces bright fringes that do not all have the same intensity. That is, the intensities of the fringes produced by double-slit interference are modified by diffraction of the light passing through each slit.

As an example, the intensity plot of Fig. 36-51*a* suggests the double-slit interference pattern that would occur if the slits were infinitely narrow (and thus $a \ll \lambda$); all the bright interference fringes would have the same intensity. The intensity plot of Fig. 36-51*b* is that for diffraction by a single actual slit; the diffraction pattern has a broad central maximum and weaker secondary maxima at ±17°. The plot of Fig. 36-51*c* suggests the interference pattern for two actual slits. That plot was constructed by using the curve of Fig. 36-51*b* as an *envelope* on the intensity plot in Fig. 36-51*a*. The positions of the fringes are not changed; only the intensities are affected.

Figure 36-52*a* shows an actual pattern in which both double-slit interference and diffraction are evident. If one slit is covered, the single-slit diffraction pattern of Fig. 36-52*b* results. Note the correspondence between Figs. 36-52*a* and 36-51*c*, and between Figs. 36-52*b* and 36-51*b*. In comparing these figures, bear in mind that Fig. 36-52 has been deliberately overexposed to bring out the faint secondary maxima and that two secondary maxima (rather than one) are shown.

With diffraction effects taken into account, the intensity of a double-slit interference pattern is given by

$$I(\theta) = I_m(\cos^2 \beta)\left(\frac{\sin \alpha}{\alpha}\right)^2 \qquad \text{(double slit)}, \tag{36-61}$$

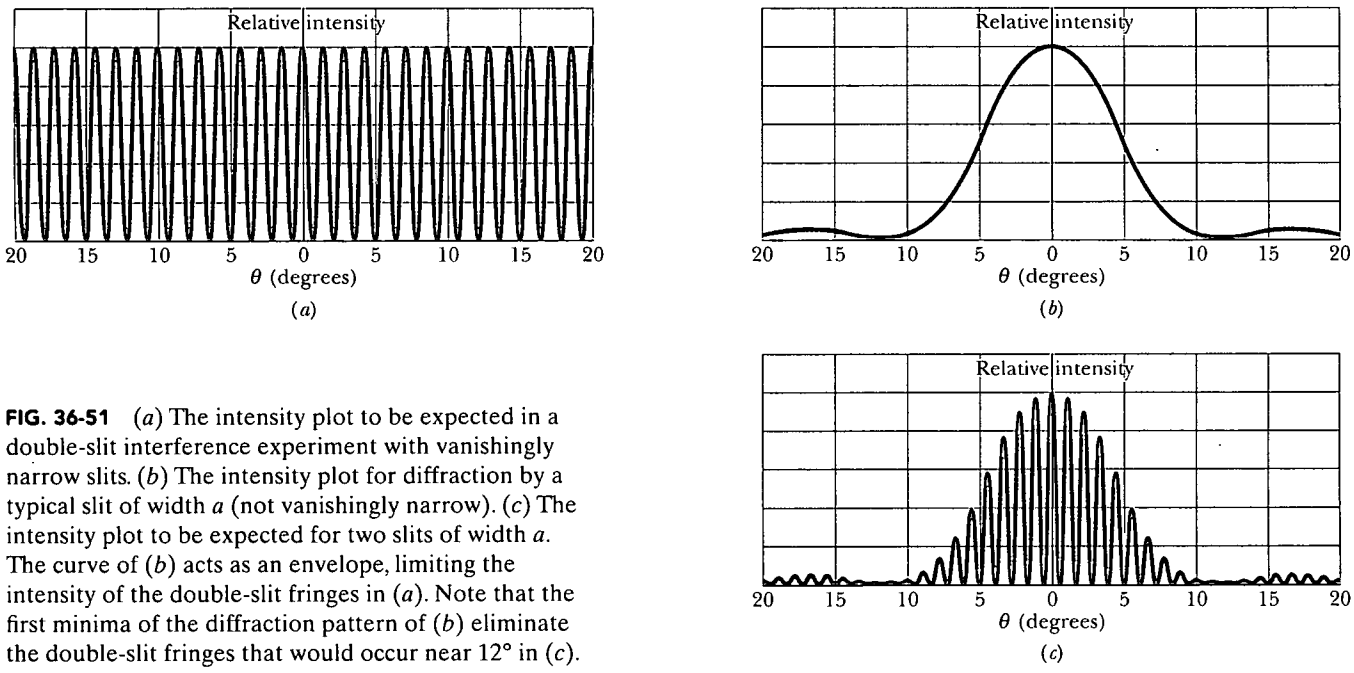


FIG. 36-51 (*a*) The intensity plot to be expected in a double-slit interference experiment with vanishingly narrow slits. (*b*) The intensity plot for diffraction by a typical slit of width a (not vanishingly narrow). (*c*) The intensity plot to be expected for two slits of width a. The curve of (*b*) acts as an envelope, limiting the intensity of the double-slit fringes in (*a*). Note that the first minima of the diffraction pattern of (*b*) eliminate the double-slit fringes that would occur near 12° in (*c*).

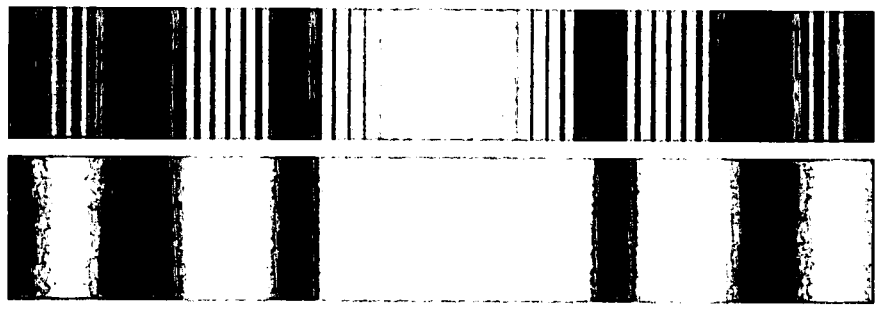


FIG. 36-52 (*a*) Interference fringes for an actual double-slit system; compare with Fig. 36-51*c*. (*b*) The diffraction pattern of a single slit; compare with Fig. 36-51*b*. *(From Michel Cagnet, Maurice Franzon, and Jean Thierr, Atlas of Optical Phenomena. Springer-Verlag, New York, 1962. Reproduced with permission.)*

in which
$$\beta = \frac{\pi d}{\lambda}\sin\theta \tag{36-62}$$

and
$$\alpha = \frac{\pi a}{\lambda}\sin\theta. \tag{36-63}$$

Here d is the distance between the centers of the slits and a is the slit width. Note carefully that the right side of Eq. 36-61 is the product of I_m and two factors. (1) The *interference factor* $\cos^2\beta$ is due to the interference between two slits with slit separation d (as given by Eqs. 36-22 and 36-23). (2) The *diffraction factor* $[(\sin\alpha)/\alpha]^2$ is due to diffraction by a single slit of width a (as given by Eqs. 36-53 and 36-54).

Let us check these factors. If we let $a \to 0$ in Eq. 36-63, for example, then $\alpha \to 0$ and $(\sin\alpha)/\alpha \to 1$. Equation 36-61 then reduces, as it must, to an equation describing the interference pattern for a pair of vanishingly narrow slits with slit separation d. Similarly, putting $d = 0$ in Eq. 36-62 is equivalent physically to causing the two slits to merge into a single slit of width a. Then Eq. 36-62 yields $\beta = 0$ and $\cos^2\beta = 1$. In this case Eq. 36-61 reduces, as it must, to an equation describing the diffraction pattern for a single slit of width a.

The double-slit pattern described by Eq. 36-61 and displayed in Fig. 36-52*a* combines interference and diffraction in an intimate way. Both are superposition effects, in that they result from the combining of waves with different phases at a given point. If the combining waves originate from a small number of elementary coherent sources—as in a double-slit experiment with $a \ll \lambda$—we call the process *interference*. If the combining waves originate in a single wavefront—as in a single-slit experiment—we call the process *diffraction*. This distinction between interference and diffraction (which is somewhat arbitrary and not always adhered to) is a convenient one, but we should not forget that both are superposition effects and usually both are present simultaneously (as in Fig. 36-52*a*).

Sample Problem 36-18 Build your skill

In a double-slit experiment, the wavelength λ of the light source is 405 nm, the slit separation d is 19.44 μm, and the slit width a is 4.050 μm. Consider the interference of the light from the two slits and also the diffraction of the light through each slit.

(a) How many bright interference fringes are within the central peak of the diffraction envelope?

Conceptualize/Classify: We first analyze the two basic mechanisms responsible for the optical pattern produced in the experiment:

1. *Single-slit diffraction:* The limits of the central peak are the first minima in the diffraction pattern due to either slit individually. (See Fig. 36-51.) The angular locations of those minima are given by Eq. 36-52 ($a \sin\theta = m\lambda$). Here let us rewrite this equation as $a \sin\theta = m_1\lambda$, with the subscript 1 referring to the one-slit diffraction. For the first minima in the diffraction pattern, we substitute $m_1 = 1$, obtaining

$$a \sin\theta = \lambda. \tag{36-64}$$

2. *Double-slit interference:* The angular locations of the bright fringes of the double-slit interference pattern are given by Eq. 36-11, which we can write as

$$d \sin \theta = m_2 \lambda, \quad \text{for } m_2 = 0, 1, 2, \ldots \qquad (36\text{-}65)$$

Here the subscript 2 refers to the double-slit interference.

Compute: We can locate the first diffraction minimum within the double-slit fringe pattern by dividing Eq. 36-65 by Eq. 36-64 and solving for m_2. By doing so and then substituting the given data, we obtain

$$m_2 = \frac{d}{a} = \frac{19.44\ \mu\text{m}}{4.050\ \mu\text{m}} = 4.8.$$

This tells us that the bright interference fringe for $m_2 = 4$ fits into the central peak of the one-slit diffraction pattern, but the fringe for $m_2 = 5$ does not fit. Within the central diffraction peak we have the central bright fringe ($m_2 = 0$), and four bright fringes (up to $m_2 = 4$) on each side of it. Thus, a total of nine bright fringes of the double-slit interference pattern are within the central peak of the diffraction envelope. The bright fringes to one side of the central bright fringe are shown in Fig. 36-53.

(b) How many bright fringes are within either of the first side peaks of the diffraction envelope?

Conceptualize/Classify: The outer limits of the first side diffraction peaks are the second diffraction minima, each of which is at the angle θ given by $a \sin \theta = m_1 \lambda$ with $m_1 = 2$:

$$a \sin \theta = 2\lambda. \qquad (36\text{-}66)$$

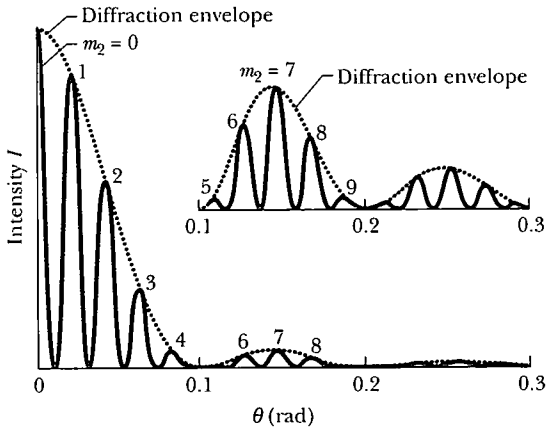


FIG. 36-53 One side of the intensity plot for a two-slit interference experiment. The inset shows (vertically expanded) the plot within the first and second side peaks of the diffraction envelope.

Compute: Dividing Eq. 36-65 by Eq. 36-66, we find

$$m_2 = \frac{2d}{a} = \frac{(2)(19.44\ \mu\text{m})}{4.050\ \mu\text{m}} = 9.6.$$

This tells us that the second diffraction minimum occurs just before the bright interference fringe for $m_2 = 10$ in Eq. 36-65. Within either first side diffraction peak we have the fringes from $m_2 = 5$ to $m_2 = 9$, for a total of five bright fringes of the double-slit interference pattern (shown in the inset of Fig. 36-53). However, if the $m_2 = 5$ bright fringe, which is almost eliminated by the first diffraction minimum, is considered too dim to count, then only four bright fringes are in the first side diffraction peak.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: In Young's double-slit experiment apparatus interference won't be produced if light directly falls on the slit.

Statement 2: Interference occurs only if the condition of coherence is achieved.

2. Statement 1: The color of a monochromatic light wave changes when it enters water.

Statement 2: The wavelength of a monochromatic light wave changes when it enters water.

3. Statement 1: Interference produces rich color in thin films only.

Statement 2: In case of thick films, the order of maxima is very high, so that neighboring wavelengths produce maxima and minima.

4. Statement 1: Interference fringes in case of double slit are hyperbola when screen is placed parallel to the slits.

Statement 2: The locus of all points in space where the path difference from two fixed points is constant is hyperbola.

MULTIPLE CHOICE QUESTIONS

sec. 36-2 Light as a Wave

1. When a ray of light of frequency 6×10^{14} Hz travels from water of refractive index 4/3 to glass of refractive index 8/5, its

(a) Frequency becomes five-sixth (5/6) of its initial value

(b) Speed becomes five-sixth (5/6) of its initial value

(c) Wavelength becomes six-fifth (6/5) of its initial value

(d) Speed becomes six-fifth (6/5) of its initial value

2. A wavefront AB passing through a system C emerges as DE (Fig. 36-54). The system C could be

(a) A slit

(b) A biprism

(c) A prism

(d) A glass slab

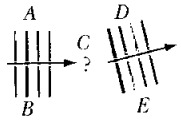


FIG. 36-54 Multiple choice question 2.

3. Mark the correct statements(s):

(a) Direction of wave propagation is along the normal to wavefront

(b) For a point source of light, the shape of wavefronts can be considered to be plane at very large distance from the source

(c) A point source of light is placed at the focus of a thin spherical lens, then the shape of the wavefront for emerged light can be plane

(d) The shape of the wavefront for the light incident on a thin spherical lens is plane, the shape of the wavefront corresponding to emergent light would always be spherical

4. The wave front of a light beam is given by the equation $x + 2y + 3z = c$, (where c is arbitary constant) then the angle made by the direction of light with the y axis is

(a) $\cos^{-1}\frac{1}{\sqrt{14}}$

(b) $\sin^{-1}\frac{2}{\sqrt{14}}$

(c) $\cos^{-1}\frac{2}{\sqrt{14}}$

(d) $\sin^{-1}\frac{3}{\sqrt{14}}$

sec. 36-4 Young's Interference Experiment

5. In Young's double-slit interference experiment a first screen with a single narrow slit is used in addition to the double-slit screen. An interference pattern is observed on the screen as shown in Fig. 36-9. What happens if the first screen is removed and light from an extended but monochromatic source, e.g., yellow light from large sodium vapor lamp, is allowed to illuminate the double-slit screen directly?

(a) The interference pattern disappears

(b) The intensity of light on the viewing screen is reduced

(c) The width of the dark fringes reduces noticeably

(d) The width of the bright fringes reduces noticeably

6. Two sources, in phase and a distance d apart, each emit a wave of wavelength λ (see Fig. 36-55). Which of the choices for the path difference $\Delta L = \Delta p = S_2p - S_1p$ $\Delta L = L_1 - L_2$ will always produce constructive interference at point P?

(a) $d\sin\theta$

(b) λ

(c) $(x/\lambda)d$

(d) $\lambda/2$

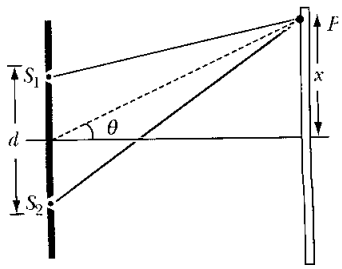


FIG. 36-55 Multiple choice question 6.

7. In a double-slit interference pattern, the first maxima for infrared light would be

(a) At the same place as the first maxima for green light

(b) Closer to the center than the first maxima for green light

(c) Farther from the center than the first maxima for green light

(d) Infrared light does not produce an interference pattern

8. The fringes produced on a screen by blue monochromatic light falling upon a double-slit arrangement were found to be too close together for measurements to be taken on them. The fringe separation could be increased by

(a) Increasing the distance between the slits and the screen

(b) Replacing the light with monochromatic red light

(c) Increasing the separation of the slits

9. Light shining through two very narrow slits produces an interference maximum at point P when the entire apparatus is in air as shown in Fig. 36-56. For the interference maximum represented, light through the bottom slit travels one wavelength further than light through the top slit before reaching point P. If the entire apparatus is immersed in water, the angle θ to the interference maximum

(a) Is unchanged

(b) Decreases because the frequency decreases

(c) Decreases because the wavelength decreases

(d) Increases because the frequency increases

(e) Increases because the wavelength increases

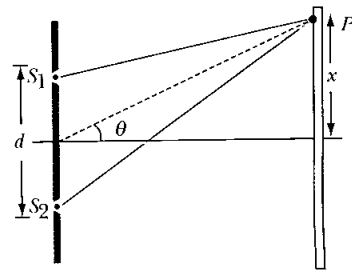


FIG. 36-56 Multiple choice question 9.

10. In a Young's double-slit experiment the slit separation is 0.5 mm and the screen is 0.5 m from the slit. For a monochromatic light of wavelength 500 nm the distance of third maxima from the second minima on the other side is

(a) 2.75 mm

(b) 2.5 mm

(c) 22.5 mm

(d) 2.25 mm

11. Two different color beams (yellow and blue) from a point source are incident on both slits of a Young's double-slit experiment setup

(a) There will be no interference pattern seen on the screen

(b) There will be interference pattern and where there is constructive interference the spot will be green

(c) There will be alternate bands of yellow, blue, and dark regions with equal separation between them

(d) There will be yellow and blue light interference patterns which are slightly separated from each other because the angles at which the maxima and minima occur for the two wavelengths are different

12. Two wavelength of light λ_1 and λ_2 are sent through Young's double-slit apparatus simultaneously. What must be true about λ_1 and λ_2 if the third-order bright fringe of λ_1 coincides with fifth-order dark fringe of λ_2

(a) $3\lambda_1 = 2\lambda_2$

(b) $2\lambda_1 = 3\lambda_2$

(c) $3\lambda_1 = 5\lambda_2$

(d) $5\lambda_1 = 3\lambda_2$

13. In a Young's double-slit experiment, if the incident light consists of two wavelengths λ_1 and λ_2, the slit separation is d, and the distance between the slit and the screen is D, the maxima due to each wavelength will coincide at a distance from the central maxima, given by

(a) $\dfrac{\lambda_1 + \lambda_2}{2Dd}$

(b) LCM of $\dfrac{\lambda_1 D}{d}$ and $\dfrac{\lambda_2 D}{d}$

(c) $(\lambda_1 - \lambda_2)\dfrac{2d}{D}$

(d) HCF of $\dfrac{\lambda_1 D}{d}$ and $\dfrac{\lambda_2 D}{d}$

14. Two plane monochromatic coherent waves produce interference pattern (of alternately bright and dark bands) on a screen. When the angle θ between the two beams is decreased the fringe width

(a) Increases

(b) Is unaffected

(c) Decreases

(d) Increases and then decreases

15. Path followed by two rays through a thin lens in air is shown in Fig. 36-57. (Assume value of μ of lens same for both rays.) Optical path followed for two rays from A to B is

(a) Greater in case 2 than 1

(b) Greater in case 1 than 2

(c) Equal in both the cases

(d) Value of μ is needed to compare

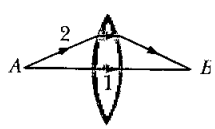


FIG. 36-57 Multiple choice question 15.

16. An interference pattern is formed on a screen by shining a planar wave on a double-slit arrangement. If we cover one slit with a glass plate, the phases of the two emerging waves will be different because the wavelength is shorter in glass than in air. If the phase difference is 180°, how is the interference pattern altered?

(a) The pattern vanishes

(b) The bright spots lie closer together

(c) The bright spots are farther apart

(d) The pattern gets shifted in such a way that center of the screen is a dark band

17. A glass slab of thickness 4 cm contains the same number of waves as 5 cm of water when both are traversed by the same monochromatic light. If the refractive index of water is 4/3, what is the refractive index of glass?

(a) 5/3

(b) 5/4

(c) 16/15

(d) 3/2

18. On introducing a thin sheet of mica (thickness 12×10^{-5} cm) in path of one of the interfering beams ($\lambda_{\text{vacuum}} = 600$ nm) in Young's double-slit experiment, the central fringe is shifted through a distance equal to the spacing between successive bright fringes. Calculate the refractive index of mica

(a) 1.33

(b) 1.4

(c) 1.5

(d) 2.5

19. A Young's double-slit experiment is conducted in water (μ_1) as shown in Fig. 36-58, and a glass plate of thickness t and refractive index μ_2 is placed in the path of S_2. Wavelength of light in water is λ. Find the magnitude of the phase difference between waves coming from S_1 and S_2 at 'O'.

(a) $\left|\left(\dfrac{\mu_2}{\mu_1} - 1\right)t\right|\dfrac{2\pi}{\lambda}$

(b) $\left|\left(\dfrac{\mu_1}{\mu_2} - 1\right)t\right|\dfrac{2\pi}{\lambda}$

(c) $|(\mu_2 - \mu_1)t|\dfrac{2\pi}{\lambda}$

(d) $|(\mu_2 - 1)t|\dfrac{2\pi}{\lambda}$

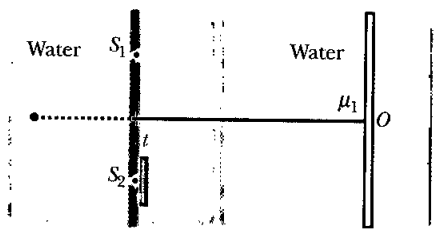


FIG. 36-58 Multiple choice question 19.

20. In a Young's double-slit experiment, green light is incident on the two slits. The interference pattern is observed on a screen. Which one of the following changes would cause the observed fringes to be more closely spaced?

(a) Reducing the separation between the slits

(b) Using blue light instead of green light

(c) Using red light instead of green light

(d) Moving the light source further away from the slits

sec. 36-5 Coherence

21. Why isn't the interference pattern like the one from Young's double-slit experiment produced from the two headlights of a car?

(a) The lights are too far apart

(b) The lights are not bright enough

(c) The lights are not rectangular slit in shape

(d) The light from one lamp is not coherent with the other

22. Coherent light is incident on two fine parallel slits S_1 and S_2 as shown Fig. 36-59. A dark fringe occurs at P when, n being an integer, the phase difference between the wave trains from S_1 and S_2 is

(a) $n\pi$ rad

(b) $(n + 1/2)\pi$ rad

(c) $(2n - 1)\pi$ rad

(d) $(2n + 1/2)\pi$ rad

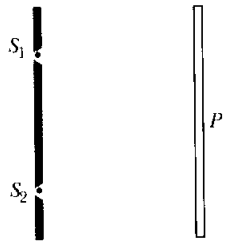


FIG. 36-59 Multiple choice question 22.

sec. 36-6 Intensity in Double-Slit Interference

23. Three coherent, equal intensity light rays arrive at a point P on a screen to produce an interference minimum of zero intensity. If any two of the rays are blocked, the intensity of the light at P is I_1. What is the intensity of the light at P if only one of the rays is blocked?

(a) 0

(b) $I_1/2$

(c) I_1

(d) $2I_1$

24. In a Young's double slit experiment, if the slits are of unequal width,

(a) Fringes will not be formed

(b) The positions of minimum intensity will not be completely dark

(c) Bright fringe will not be formed at the center of the screen

(d) Distance between two consecutive bright fringes will not be equal to the distance between two consecutive dark fringes

25. Intensities of light due to the two slits of Young's double-slit experiment are I and $4I$. How far from the central maxima will the intensity be equal to the average intensity on the screen? (β is the fringe width.)

(a) $\frac{\beta}{2}$

(b) $\frac{\beta}{3}$

(c) $\frac{\beta}{6}$

(d) $\frac{\beta}{4}$

26. In a Young's double-slit interference pattern, the intensity of the central fringe at $P = I_0$. When one slit width is reduced to half, the intensity at P will be

(a) $\frac{I_0}{2}$

(b) $\frac{I_0}{4}$

(c) $\frac{I_0}{8}(3+2\sqrt{2})$

(d) $\frac{3}{4}I_0$

27. In a Young's double-slit experiment, the central bright fringe can be identified

(a) As it has greater intensity than the other bright fringes

(b) As it is wider than the other bright fringes

(c) As it is narrower than the other bright fringes

(d) By using white light instead of monochromatic light

sec. 36-7 Modifications of Young's Double-Slit Experiment

28. As shown in Fig. 36-32 of billet lens, a point light source is placed at distance $2f$ from a lens with focus length f and a screen is placed at $4f$ from the lens. The lens is then cut at the middle into two equal portions: upper half and lower half. The upper half is moved upward by a small distance d comparable to the light wavelength and the lower half is moved downward by the same distance d. What is the light pattern on the screen?

(a) Bright and dark strips similar to the pattern seen in Young's double-slit experiment

(b) Bright and dark concentric rings

(c) Two large, bright, and partly overlapping patches

(d) Two separate bright spots

29. A Fresnel biprism is used to form the interference fringes. The distance between the source and the biprism is 20 cm and that between the biprism and the screen is 80 cm. If $\lambda = 6563$Å and the separation between the virtual sources is 3.6 mm, then the fringe width is

(a) 1.82 cm

(b) 0.182 cm

(c) 0.0182 cm

(d) 0.00182 cm

30. In a biprism experiment, the biprism is made of glass ($\mu = 1.5$). When the setup is shifted from air to the inside of a still, clear lake water ($\mu = 4/3$) then

(a) The fringe pattern gets enlarged

(b) The fringe pattern gets shrunk

(c) The fringe pattern gets shifted

(d) The fringe pattern remains unchanged

31. Two radio stations that are 250 m apart emit radio waves of wavelength 100 m. Point A is 400 m from both stations and point B is 450 m from both stations. Point C is 400 m from one station and 450 m from the other. The radio stations emit in phase. Which of the following statements is true?

(a) There will be constructive interference at A and B and destructive interference at C

(b) There will be destructive interference at A and B and constructive interference at C

(c) There will be constructive interference at A and destructive interference at B and C

(d) There will be constructive interference at B and C and destructive interference at A

sec. 36-8 Interference from Thin Films

32. A soap film of thickness t is surrounded by air. It is illuminated at near normal incidence by monochromatic light which

has wavelength λ in the film. What will be the film thickness that will produce maximum brightness of the reflected light?

(a) $(1/4)\lambda$

(b) $(1/2)\lambda$

(c) λ

(d) 2λ

33. A very thin oil film of refractive index 1.2 floats on the surface of water ($\mu = 4/3$). When its thickness is 10^{-7} m, then it appears dark when seen normally. The minimum change in thickness for which it will appear bright in normally reflected light of the same color is

(a) 10^{-7} m

(b) 2×10^{-7} m

(c) 3×10^{-7} m

(d) 5×10^{-7} m

34. White light is normally incident on a soap bubble in air, and the reflected light is viewed. As the wall thickness of the bubble approaches zero (just before it breaks) the reflected light

(a) Becomes dim for all wavelengths

(b) Becomes unusually bright for all wavelengths

(c) Becomes bright only for certain wavelengths, dependent on the index of refraction of the bubble

(d) None of the above occurs

35. Light is reflecting off a wedge-shaped thin piece of glass producing bright and dark interference fringes. If a certain location has a bright fringe, a nearby point will have a dark fringe if the thickness of the glass increases by

(a) 1/8 of a wavelength of the light in glass

(b) 1/4 of a wavelength of the light in glass

(c) 1/2 of a wavelength of the light in glass

(d) 1 wavelength of the light in glass

36. Consider two identical microscope slides in air illuminated with monochromatic light. The bottom slide is rotated (counterclockwise about the point of contact in the side view) so that the wedge angle gets a bit smaller. What happens to the fringes?

(a) They are spaced farther apart

(b) They are spaced closer together

(c) They don't change

(d) Bright and dark spots are interchanged

sec. 36-9 Diffraction and Wave Theory of Light

37. Radio waves are readily diffracted around buildings whereas light waves are negligibly diffracted around buildings. This is because radio waves

(a) Are plane polarized

(b) Have much longer wavelengths than light waves

(c) Have much shorter wavelengths than light waves

(d) Are nearly monochromatic (single frequency)

38. The shimmering or wavy lines that can often be seen near the ground on a hot day are due to

(a) Brownian movement

(b) Reflection

(c) Refraction

(d) Diffraction

39. If the Huygen's principle applies to any point anywhere in a beam path, why doesn't a laser beam without any slit spread out in all directions?

(a) Because all waves that spread interfere destructively

(b) It does spread, but the spread is so small that we normally don't notice it

(c) We can't apply the Huygen's principle anywhere but in slits and apertures

sec. 36-11 Intensity in Single-Slit Diffraction, Quantitatively

40. No fringes are seen in a single-slit diffraction pattern if

(a) The screen is far away

(b) The wavelength is less than the slit width

(c) The wavelength is greater than the slit width

(d) The distance to the screen is greater than the slit width

41. Monochromatic plane waves of light are incident normally on a single slit. Which option correctly shows the diffraction pattern observed on a distant screen?

(a) 

(b) 

(c) 

(d) 

42. A single slit with the direction to a point P on a distant screen as shown in Fig. 36-60. At P, the pattern has its second minimum (from its central maximum). If X and Y are the edge of the slit, what is the path length difference $(PX) - (PY)$?

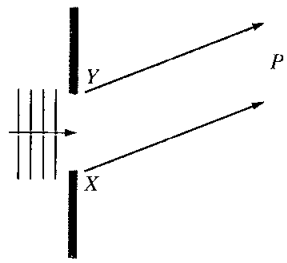


FIG. 36-60 Multiple choice question 42.

(a) $\lambda/2$

(b) λ

(c) $3\lambda/2$

(d) 2λ

43. At the first minimum adjacent to the central maximum of a single-slit diffraction pattern the phase difference between the Huygen's wavelet from the top of the slit and the wavelet from the midpoint of the slit is

(a) $\pi/8$ rad

(b) $\pi/4$ rad

(c) $\pi/2$ rad

(d) π rad

sec. 36-12 Diffraction by a Circular Aperture

44. A diffraction-limited laser of length l and aperture diameter d generates light of wavelength λ. If the beam is directed at the surface of the Moon a distance D away, the radius of the illuminated area on the Moon is approximately

(a) dD/l

(b) dD/λ

(c) $D\lambda/l$

(d) D

45. Two stars that are close together are photographed through a telescope. The black and white film is equally sensitive to all colors. Which situation would result in the most clearly separated images of the stars?

(a) Small lens, red stars

(b) Small lens, blue stars

(c) Large lens, red stars

(d) Large lens, blue stars

46. The resolving power of a telescope can be increased by

(a) Increasing the objective focal length and decreasing the eyepiece focal length

(b) Increasing the lens diameters

(c) Decreasing the lens diameters

(d) Inserting a correction lens between objective and eyepiece

sec. 36-13 Diffraction by a Double Slit

47. In a double-slit diffraction experiment, the number of interference fringes within the central diffraction maximum can be increased by

(a) Increasing the wavelength

(b) Decreasing the wavelength

(c) Decreasing the slit separation

(d) Decreasing the slit width

PROBLEMS

sec. 36-2 Light as a Wave

1. The speed of yellow light (from a sodium lamp) in a certain liquid is measured to be 1.92×10^8 m/s. What is the index of refraction of this liquid for the light?

2. The wavelength of yellow sodium light in air is 589 nm. (a) What is its frequency? (b) What is its wavelength in glass whose index of refraction is 1.52? (c) From the results of (a) and (b), find its speed in this glass.

3. In Fig. 36-61, a light wave along ray r_1 reflects once from a mirror and a light wave along ray r_2 reflects twice from that same mirror and once from a tiny mirror at distance L from the bigger mirror. (Neglect the slight tilt of the rays.) The waves have wavelength λ and are initially exactly out of phase. What are the (a) smallest, (b) second smallest, and (c) third smallest values of L/λ that result in the final waves being exactly in phase?

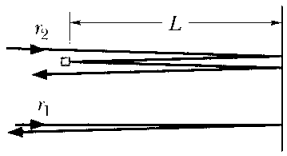


FIG. 36-61 Problems 3 and 4.

4. In Fig. 36-61, a light wave along ray r_1 reflects once from a mirror and a light wave along ray r_2 reflects twice from that same mirror and once from a tiny mirror at distance L from the bigger mirror. (Neglect the slight tilt of the rays.) The waves have wavelength 620 nm and are initially in phase. (a) What is the smallest value of L that puts the final light waves exactly out of phase? (b) With the tiny mirror initially at that value of L, how far must it be moved away from the bigger mirror to again put the final waves out of phase?

5. Suppose that the two waves in Fig. 36-6 have wavelength $\lambda = 500$ nm in air. What multiple of λ gives their phase difference when they emerge if (a) $n_1 = 1.50$, $n_2 = 1.60$, and $L = 8.50\ \mu$m; (b) $n_1 = 1.62$, $n_2 = 1.72$, and $L = 8.50\ \mu$m; and (c) $n_1 = 1.59$, $n_2 = 1.79$, and $L = 3.25\ \mu$m? (d) Suppose that in each of these three situations the waves arrive at a common point (with the same amplitude) after emerging. Rank the situations according to the brightness the waves produce at the common point.

6. In Fig. 36-62, two light rays go through different paths by reflecting from the various flat surfaces shown. The light waves have a wavelength of 420.0 nm and are initially in phase. What are the (a) smallest and (b) second smallest value of distance L that will put the waves exactly out of phase as they emerge from the region?

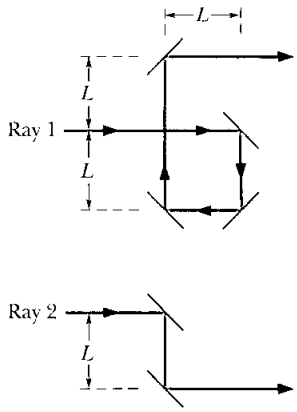


FIG. 36-62 Problems 6 and 68.

7. Two waves of light in air, of wavelength $\lambda = 600.0$ nm, are initially in phase. They then travel through plastic layers as shown in Fig. 36-63, with $L_1 = 4.00\ \mu$m, $L_2 = 3.50\ \mu$m, $n_1 = 1.40$, and $n_2 = 1.60$. (a) What multiple of λ gives their phase difference after they both have emerged from the layers? (b) If the waves later arrive at some common point with the same amplitude, is their interference fully constructive, fully destructive, intermediate but closer to fully constructive, or intermediate but closer to fully destructive?

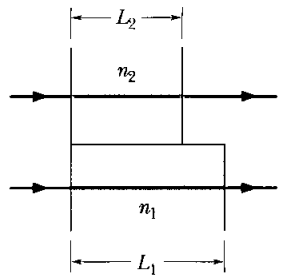


FIG. 36-63 Problem 7.

sec. 36-4 Young's Interference Experiment

8. Monochromatic green light, of wavelength 550 nm, illuminates two parallel narrow slits 7. 7. 70 μm apart. Calculate the angular deviation (θ in Fig. 36-11) of the third-order (for $m = 3$) bright fringe (a) in radians and (b) in degrees.

9. In Fig. 36-64, two radio-frequency point sources S_1 and S_2, separated by distance $d = 2.0$ m, are radiating in phase with $\lambda = 0.50$ m. A detector moves in a large circular path around

the two sources in a plane containing them. How many maxima does it detect?

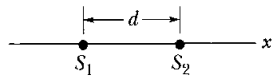


FIG. 36-64 Problems 8 and 15.

10. In a double-slit arrangement the slits are separated by a distance equal to 100 times the wavelength of the light passing through the slits. (a) What is the angular separation in radians between the central maximum and an adjacent maximum? (b) What is the distance between these maxima on a screen 50.0 cm from the slits?

11. Suppose that Young's experiment is performed with blue-green light of wavelength 500 nm. The slits are 1.20 mm apart, and the viewing screen is 5.40 m from the slits. How far apart are the bright fringes near the center of the interference pattern?

12. In Fig. 36-65, sources A and B emit long-range radio waves of wavelength 400 m, with the phase of the emission from A ahead of that from source B by 90°. The distance r_A from A to detector D is greater than the corresponding distance r_B by 100 m. What is the phase difference of the waves at D?

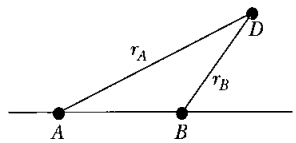


FIG. 36-65 Problem 12.

13. Sunlight is used in a double-slit interference experiment. The fourth-order maximum for a wavelength of 450 nm occurs at an angle of $\theta = 90°$. Thus, it is on the verge of being eliminated from the pattern because θ cannot exceed 90° in Eq. 36-11. (a) What range of wavelengths in the visible range (400 nm to 700 nm) are not present in the third-order maxima? To eliminate all of the visible light in the fourth-order maximum, (b) should the slit separation be increased or decreased and (c) what least change in separation is needed?

14. In Fig. 36-66, two isotropic point sources of light (S_1 and S_2) are separated by distance 2.70 μm along a y axis and emit in phase at wavelength 900 nm and at the same amplitude. A light detector is located at point P at coordinate x_P on the x axis. What is the greatest value of x_P at which the detected light is minimum due to destructive interference?

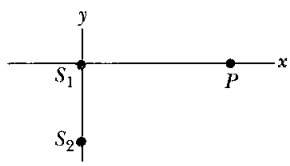


FIG. 36-66 Problems 14, and 19.

15. In Fig. 36-64, two isotropic point sources S_1 and S_2 emit identical light waves in phase at wavelength λ. The sources lie at separation d on an x axis, and a light detector is moved in a circle of large radius around the midpoint between them. It detects 30 points of zero intensity, including two on the x axis, one of them to the left of the sources and the other to the right of the sources. What is the value of d/λ?

16. In a double-slit experiment, the distance between slits is 5.0 mm and the slits are 1.0 m from the screen. Two interference patterns can be seen on the screen: one due to light of wavelength 480 nm, and the other due to light of wavelength 600 nm. What is the separation on the screen between the third-order ($m = 3$) bright fringes of the two interference patterns?

17. In Fig. 36-67, two isotropic point sources S_1 and S_2 emit light in phase at wavelength λ and at the same amplitude. The sources are separated by distance $2d = 6.00\lambda$. They lie on an axis that is parallel to an x axis, which runs along a viewing screen at distance $D = 20.0\lambda$. The origin lies on the perpendicular bisector between the sources. The figure shows two rays reaching point P on the screen, at position x_P. (a) At what value of x_P do the rays have the minimum possible phase difference? (b) What multiple of λ gives that minimum phase difference? (c) At what value of x_P do the rays have the maximum possible phase difference? What multiple of λ gives (d) that maximum phase difference and (e) the phase difference when $x_P = 6.00\lambda$? (f) When $x_P = 6.00\lambda$, is the resulting intensity at point P maximum, minimum, intermediate but closer to maximum, or intermediate but closer to minimum?

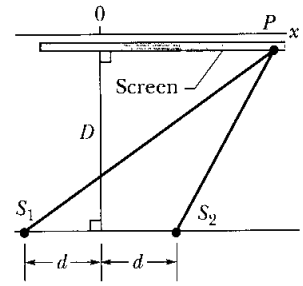


FIG. 36-67 Problem 17.

18. A thin flake of mica ($n = 1.58$) is used to cover one slit of a double-slit interference arrangement. The central point on the viewing screen is now occupied by what had been the seventh bright side fringe ($m = 7$). If $\lambda = 550$ nm, what is the thickness of the mica?

19. Figure 36-66 shows two isotropic point sources of light (S_1 and S_2) that emit in phase at wavelength 400 nm and at the same amplitude. A detection point P is shown on an x axis that extends through source S_1. The phase difference ϕ between the light arriving at point P from the two sources is to be measured as P is moved along the x axis from $x = 0$ out to $x = +\infty$. The results out to $x_s = 10 \times 10^{-7}$ m are given in Fig. 36-68. On the way out to $+\infty$, what is the greatest value of x at which the light arriving at P from S_1 is exactly out of phase with the light arriving at P from S_2?

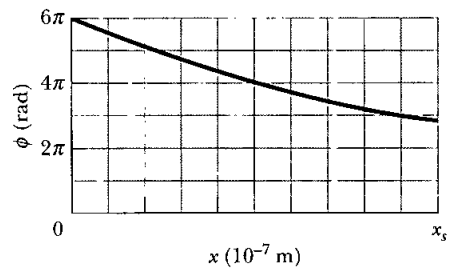


FIG. 36-68 Problem 19.

sec. 36-6 Intensity in Double-Slit Interference

20. Two waves of the same frequency have amplitudes 1.00 and 2.00. They interfere at a point where their phase difference is 60.0°. What is the resultant amplitude?

21. Three electromagnetic waves travel through a certain point P along an x axis. They are polarized parallel to a y axis, with the following variations in their amplitudes. Find their resultant at P.

$$E_1 = (10.0\ \mu\text{V/m}) \sin[(2.0 \times 10^{14}\ \text{rad/s})t]$$

$$E_2 = (5.00\ \mu\text{V/m})\ \sin[(2.0 \times 10^{14}\ \text{rad/s})t + 45.0°]$$

$$E_3 = (5.00\ \mu\text{V/m})\ \sin[(2.0 \times 10^{14}\ \text{rad/s})t - 45.0°]$$

22. In the double-slit experiment of Fig. 36-11, the electric fields of the waves arriving at point P are given by

$$E_1 = (2.00\ \mu\text{V/m})\ \sin[(1.26 \times 10^{15})t]$$

$$E_2 = (2.00\ \mu\text{V/m})\ \sin[(1.26 \times 10^{15})t + 39.6\ \text{rad}],$$

where time t is in seconds. (a) What is the amplitude of the resultant electric field at point P? (b) What is the ratio of the intensity I_P at point P to the intensity I_{cen} at the center of the interference pattern? (c) Describe where point P is in the interference pattern by giving the maximum or minimum on which it lies, or the maximum and minimum between which it lies. In a phasor diagram of the electric fields, (d) at what rate would the phasors rotate around the origin and (e) what is the angle between the phasors?

23. In the double-slit experiment of Fig. 36-11, the viewing screen is at distance $D = 4.00$ m, point P lies at distance $y = 20.5$ cm from the center of the pattern, the slit separation d is 4.50 μm, and the wavelength λ is 580 nm. (a) Determine where point P is in the interference pattern by giving the maximum or minimum on which it lies, or the maximum and minimum between which it lies. (b) What is the ratio of the intensity I_P at point P to the intensity I_{cen} at the center of the pattern?

24. The ratio of intensities due to upper and lower slits on the screen of a Young's double-slit experiment is 16:9. Find the ratio of maximum to average intensity on screen.

25. In a Young's double-slit experiment experiment, I_0 is given to be the intensity of the central bright fringe and β is the fringe width. What will be the intensity at a distance y from central bright fringe?

26. In a Young's double-slit experiment apparatus, two identical slits are separated by 1 mm and distance between slits and screen is 1 m. The wavelength of light used is 6000 Å. What is the minimum distance between two points on the screen having 75% intensity of the maximum intensity?

sec. 36-7 Modifications of Young's Double-Slit Experiment

27. Two identical monochromatic light sources A and B intensity 10^{-15} W/m^2 each produce wavelength of light 4000 Å. A glass of thickness 3 mm is placed in the path of the ray as shown in the Fig. 36-69. The glass has a variable refractive index $n = 1 + \sqrt{x}$, where x (in mm) is distance of plate from left to right. Calculate total intensity at focal point F of the lens.

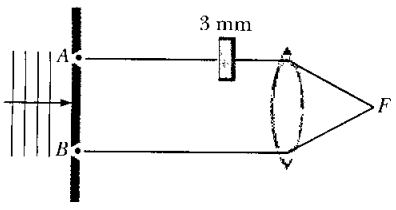


FIG. 36-69 Problem 27.

28. Two coherent point sources S_1 and S_2 are located at a distance 4λ from each other. Perpendicular to the straight line joining S_1 and S_2, there is a large screen at a distance of 1000 λ from S_1 as shown in Fig. 36-70. Find the radius of the third maxima starting from the point O.

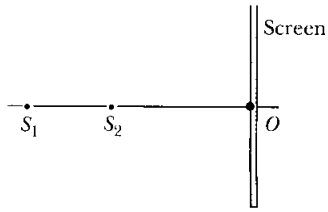


FIG. 36-70 Problem 28.

29. In Lloyd's mirror experiment, the source slit S lies in a plane 20 cm to the left from the left edge of the mirror (Fig. 36-71). The mirror is 30 cm long and a screen is placed touching the right edge of the mirror. The distance of S from plane of mirror is 2 mm and wavelength of light used = 6400 Å. Find the number of maxima formed.

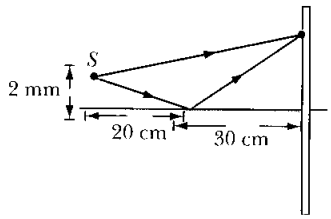


FIG. 36-71 Problem 29.

30. The distance between a slit and a biprism of acute angle 2° is 10 cm. Find (a) the fringe width and (b) the width of the entire fringe pattern when observation is made on screen at a distance of 90 cm from the biprism. The refractive index of the biprism is 1.5 and $\lambda = 5890$ Å.

sec. 36-8 Interference from Thin Films

31. The rhinestones in costume jewelry are glass with index of refraction 1.50. To make them more reflective, they are often coated with a layer of silicon monoxide of index of refraction 2.00. What is the minimum coating thickness needed to ensure that light of wavelength 560 nm and of perpendicular incidence will be reflected from the two surfaces of the coating with fully constructive interference?

32. White light is sent downward onto a horizontal thin film that is sandwiched between two materials. The indexes of refraction are 1.80 for the top material, 1.70 for the thin film, and 1.50 for the bottom material. The film thickness is 5.00×10^{-7} m. Of the visible wavelengths (400 to 700 nm) that result in fully constructive interference at an observer above the film, which is the (a) longer and (b) shorter wavelength? The materials and film are then heated so that the film thickness increases. (c) Does the light resulting in fully constructive interference shift toward longer or shorter wavelengths?

33. Light of wavelength 624 nm is incident perpendicularly on a soap film ($n = 1.33$) suspended in air. What are the (a) least and (b) second least thicknesses of the film for which the reflections from the film undergo fully constructive interference?

34. We wish to coat flat glass ($n = 1.50$) with a transparent material ($n = 1.25$) so that reflection of light at wavelength 600 nm is eliminated by interference. What minimum thickness can the coating have to do this?

35. A thin film of acetone ($n = 1.25$) coats a thick glass plate ($n = 1.50$). White light is incident normal to the film. In the

reflections, fully destructive interference occurs at 600 nm and fully constructive interference at 700 nm. Calculate the thickness of the acetone film.

36. A disabled tanker leaks kerosene ($n = 1.20$) into the Persian Gulf, creating a large slick on top of the water ($n = 1.30$). (a) If you are looking straight down from an airplane, while the Sun is overhead, at a region of the slick where its thickness is 460 nm, for which wavelength(s) of visible light is the reflection brightest because of constructive interference? (b) If you are scuba diving directly under this same region of the slick, for which wavelength(s) of visible light is the transmitted intensity strongest?

37. In Fig. 36-72, a broad beam of light of wavelength 630 nm is incident at 90° on a thin, wedge-shaped film with index of refraction 1.50. An observer intercepting the light transmitted by the film sees 10 bright and 9 dark fringes along the length of the film. By how much does the film thickness change over this length?

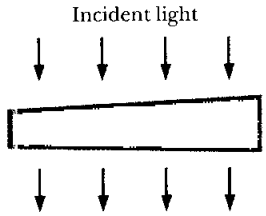


FIG. 36-72 Problem 37.

38. In Fig. 36-73, a broad beam of light of wavelength 620 nm is sent directly downward through the top plate of a pair of glass plates touching at the left end. The air between the plates acts as a thin film, and an interference pattern can be seen from above the plates. Initially, a dark fringe lies at the left end, a bright fringe lies at the right end, and nine dark fringes lie between those two end fringes. The plates are then very gradually squeezed together at a constant rate to decrease the angle between them. As a result, the fringe at the right side changes between being bright to being dark every 15.0 s. (a) At what rate is the spacing between the plates at the right end being changed? (b) By how much has the spacing there changed when both left and right ends have a dark fringe and there are five dark fringes between them?

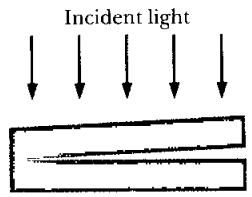


FIG. 36-73 Problems 38–40, 73.

39. In Fig. 36-73, two microscope slides touch at one end and are separated at the other end. When light of wavelength 500 nm shines vertically down on the slides, an overhead observer sees an interference pattern on the slides with the dark fringes separated by 1.2 mm. What is the angle between the slides?

40. In Fig. 36-73, a broad beam of monochromatic light is directed perpendicularly through two glass plates that are held together at one end to create a wedge of air between them. An observer intercepting light reflected from the wedge of air, which acts as a thin film, sees 4001 dark fringes along the length of the wedge. When the air between the plates is evacuated, only 4000 dark fringes are seen. Calculate to six significant figures the index of refraction of air from these data.

sec. 36-10 Diffraction by a Single Slit: Locating the Minima

41. A single slit is illuminated by light of wavelengths λ_a and λ_b, chosen so that the first diffraction minimum of the λ_a component coincides with the second minimum of the λ_b component. (a) If $\lambda_b = 350$ nm, what is λ_a? For what order number m_b (if any) does a minimum of the λ_b component coincide with the minimum of the λ_a component in the order number (b) $m_a = 2$ and (c) $m_a = 3$?

42. Monochromatic light of wavelength 441 nm is incident on a narrow slit. On a screen 2.00 m away, the distance between the second diffraction minimum and the central maximum is 1.50 cm. (a) Calculate the angle of diffraction θ of the second minimum. (b) Find the width of the slit.

43. A plane wave of wavelength 590 nm is incident on a slit with a width of $a = 0.40$ mm. A thin converging lens of focal length +70 cm is placed between the slit and a viewing screen and focuses the light on the screen. (a) How far is the screen from the lens? (b) What is the distance on the screen from the center of the diffraction pattern to the first minimum?

44. In conventional television, signals are broadcast from towers to home receivers. Even when a receiver is not in direct view of a tower because of a hill or building, it can still intercept a signal if the signal diffracts enough around the obstacle, into the obstacle's "shadow region." Previously, television signals had a wavelength of about 50 cm, but digital television signals that are transmitted from towers have a wavelength of about 10 mm. (a) Did this change in wavelength increase or decrease the diffraction of the signals into the shadow regions of obstacles? Assume that a signal passes through an opening of 5.0 m width between two adjacent buildings. What is the angular spread of the central diffraction maximum (out to the first minima) for wavelengths of (b) 50 cm and (c) 10 mm?

45. Sound waves with frequency 3000 Hz and speed 343 m/s diffract through the rectangular opening of a speaker cabinet and into a large auditorium of length $d = 100$ m. The opening, which has a horizontal width of 30.0 cm, faces a wall 100 m away (Fig. 36-74). Along that wall, how far from the central axis will a listener be at the first diffraction minimum and thus have difficulty hearing the sound? (Neglect reflections.)

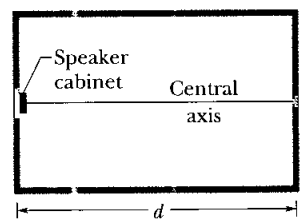


FIG. 36-74 Problem 45.

sec. 36-11 Intensity in Single-Slit Diffraction, Quantitatively

46. Monochromatic light with wavelength 538 nm is incident on a slit with width 0.025 mm. The distance from the slit to a screen is 3.5 m. Consider a point on the screen 1.1 cm from the central maximum. Calculate (a) θ for that point, (b) α, and (c) the ratio of the intensity at that point to the intensity at the central maximum.

47. A 0.10-mm-wide slit is illuminated by light of wavelength 589 nm. Consider a point P on a viewing screen on which the diffraction pattern of the slit is viewed; the point is at 30° from the central axis of the slit. What is the phase difference between the Huygens wavelets arriving at point P from the top and midpoint of the slit?

48. Figure 36-75 gives α versus the sine of the angle θ in a single-slit diffraction experiment using light of wavelength 610 nm. The vertical axis scale is set by $\alpha_s = 12$ rad. What are (a) the slit width, (b) the total number of diffraction minima in the pattern (count them on both sides of the center of the diffraction pattern), (c) the least angle for a minimum, and (d) the greatest angle for a minimum?

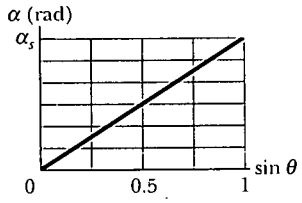


FIG. 36-75 Problem 48.

49. (a) Show that the values of α at which intensity maxima for single-slit diffraction occur can be found exactly by differentiating Eq. 36-53 with respect to α and equating the result to zero, obtaining the condition $\tan\alpha = \alpha$. To find values of α satisfying this relation, plot the curve $y = \tan\alpha$ and the straight line $y = \alpha$ and then find their intersections, or use a calculator to find an appropriate value of α by trial and error. Next, from $\alpha = (m + \frac{1}{2})\pi$, determine the values of m associated with the maxima in the single-slit pattern. (These m values are *not* integers because secondary maxima do not lie exactly halfway between minima.) What are the (b) smallest α and (c) associated m, the (d) second smallest α and (e) associated m, and the (f) third smallest α and (g) associated m?

sec. 36-12 Diffraction by a Circular Aperture

50. The telescopes on some commercial surveillance satellites can resolve objects on the ground as small as 85 cm across (see Google Earth), and the telescopes on military surveillance satellites reportedly can resolve objects as small as 10 cm across. Assume first that object resolution is determined entirely by Rayleigh's criterion and is not degraded by turbulence in the atmosphere. Also assume that the satellites are at a typical altitude of 400 km and that the wavelength of visible light is 550 nm. What would be the required diameter of the telescope aperture for (a) 85 cm resolution and (b) 10 cm resolution? (c) Now, considering that turbulence is certain to degrade resolution and that the aperture diameter of the Hubble Space Telescope is 2.4 m, what can you say about the answer to (b) and about how the military surveillance resolutions are accomplished?

51. Assume that Rayleigh's criterion gives the limit of resolution of an astronaut's eye looking down on Earth's surface from a typical space shuttle altitude of 400 km. (a) Under that idealized assumption, estimate the smallest linear width on Earth's surface that the astronaut can resolve. Take the astronaut's pupil diameter to be 5 mm and the wavelength of visible light to be 550 nm. (b) Can the astronaut resolve the Great Wall of China (Fig. 36-76), which is more than 3000 km long, 5 to 10 m thick at its base, 4 m thick at its top, and 8 m in height? (c) Would the astronaut be able to resolve any unmistakable sign of intelligent life on Earth's surface?


FIG. 36-76 Problem 51. The Great Wall of China. *(AP/Wide World Photos)*

52. The two headlights of an approaching automobile are 1.4 m apart. At what (a) angular separation and (b) maximum distance will the eye resolve them? Assume that the pupil diameter is 5.0 mm, and use a wavelength of 550 nm for the light. Also assume that diffraction effects alone limit the resolution so that Rayleigh's criterion can be applied.

53. The wall of a large room is covered with acoustic tile in which small holes are drilled 5.0 mm from center to center. How far can a person be from such a tile and still distinguish the individual holes, assuming ideal conditions, the pupil diameter of the observer's eye to be 4.0 mm, and the wavelength of the room light to be 550 nm?

54. (a) A circular diaphragm 60 cm in diameter oscillates at a frequency of 25 kHz as an underwater source of sound used for submarine detection. Far from the source, the sound intensity is distributed as the diffraction pattern of a circular hole whose diameter equals that of the diaphragm. Take the speed of sound in water to be 1450 m/s and find the angle between the normal to the diaphragm and a line from the diaphragm to the first minimum. (b) Is there such a minimum for a source having an (audible) frequency of 1.0 kHz?

sec. 36-13 Diffraction by a Double Slit

55. In a double-slit experiment, the slit separation d is 2.00 times the slit width w. How many bright interference fringes are in the central diffraction envelope?

56. Suppose that the central diffraction envelope of a double-slit diffraction pattern contains 11 bright fringes and the first diffraction minima eliminate (are coincident with) bright fringes. How many bright fringes lie between the first and second minima of the diffraction envelope?

57. Two slits of width a and separation d are illuminated by a coherent beam of light of wavelength λ. What is the linear separation of the bright interference fringes observed on a screen that is at a distance D away?

58. (a) In a double-slit experiment, what ratio of d to a causes diffraction to eliminate the fourth bright side fringe? (b) What other bright fringes are also eliminated?

ADDITIONAL PROBLEMS

59. Ocean waves moving at a speed of 4.0 m/s are approaching a beach at angle $\theta_1 = 30°$ to the normal, as shown from above in Fig. 36-77. Suppose the water depth changes abruptly at a certain distance from the beach and the wave speed there drops to 3.0 m/s. (a) Close to the beach, what is the angle θ_2 between the direction of wave motion and the normal? (Assume the same law of refraction as for light.) (b) Explain why most waves come in normal to a shore even though at large distances they approach at a variety of angles.

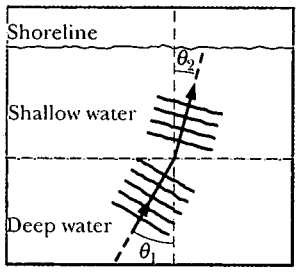


FIG. 36-77 Problem 59.

60. Figure 36-78*a* shows two light rays that are initially in phase as they travel upward through a block of plastic, with wavelength 400 nm as measured in air. Light ray r_1 exits directly into air. However, before light ray r_2 exits into air, it travels through a liquid in a hollow cylinder within the plastic. Initially the height L_{liq} of the liquid is 40.0 μm, but then the liquid begins to evaporate. Let ϕ be the phase difference between rays r_1 and r_2 once they both exit into the air. Figure 36-78*b* shows ϕ versus the liquid's height L_{liq} until the liquid disappears, with ϕ given in terms of wavelength and the horizontal scale set by $L_s = 40.00$ μm. What are (a) the index of refraction of the plastic and (b) the index of refraction of the liquid?

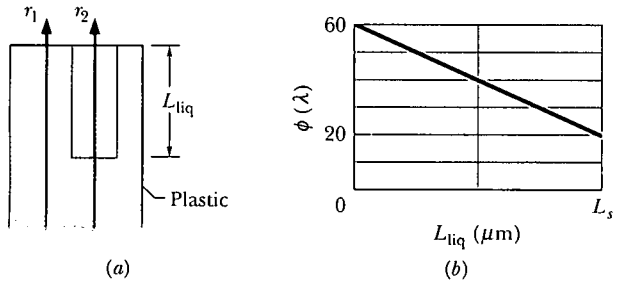


FIG. 36-78 Problem 60.

61. Two light rays, initially in phase and with a wavelength of 500 nm, go through different paths by reflecting from the various mirrors shown in Fig. 36-79. (Such a reflection does not itself produce a phase shift.) (a) What least value of distance d will put the rays exactly out of phase when they emerge from the region? (Ignore the slight tilt of the path for ray 2.) (b) Repeat the question assuming that the entire apparatus is immersed in a protein solution with an index of refraction of 1.38.

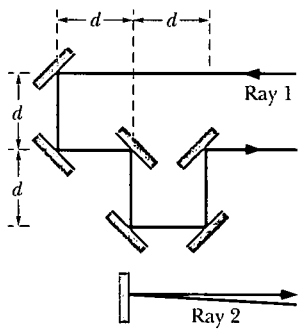


FIG. 36-79 Problem 61.

62. In Figure 36-80, two isotropic point sources S_1 and S_2 emit light in phase at wavelength λ and at the same amplitude. The sources are separated by distance $d = 6.00\lambda$ on an x axis. A viewing screen is at distance $D = 20.0\lambda$ from S_2 and parallel to the y axis. The figure shows two rays reaching point P on the screen, at height y_P. (a) At what value of y_P do the rays have the minimum possible phase difference? (b) What multiple of λ gives that minimum phase difference? (c) At what value of y_P do the rays have the maximum possible phase difference?

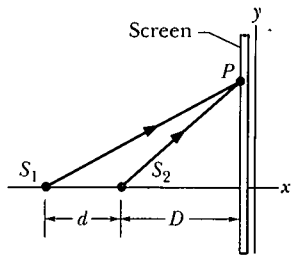


FIG. 36-80 Problem 62.

63. In Fig. 36-81, a microwave transmitter at height a above the water level of a wide lake transmits microwaves of wavelength λ toward a receiver on the opposite shore, a distance x above the water level. The microwaves reflecting from the water interfere with the microwaves arriving directly from the transmitter. Assuming that the lake width D is much greater than a and x, and that $\lambda \geq a$, find an expression that gives the values of x for which the signal at the receiver is maximum. (*Hint:* Does the reflection cause a phase change?)

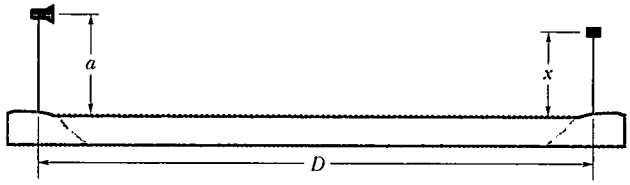


FIG. 36-81 Problem 63.

64. In Fig. 36-82, two isotropic point sources S_1 and S_2 emit light at wavelength $\lambda = 400$ nm. Source S_1 is located at $y = 640$ nm; source S_2 is located at $y = -640$ nm. At point P_1 (at $x = 720$ nm), the wave from S_2 arrives ahead of the wave from S_1 by a phase difference of 0.600π rad. (a) What multiple of λ gives the phase difference between the waves from the two sources as the waves arrive at point P_2, which is located at $y = 720$ nm. (The figure is not drawn to scale.) (b) If the waves arrive at P_2 with equal amplitudes, is the interference there fully constructive, fully destructive, intermediate but closer to fully constructive, or intermediate but closer to fully destructive?

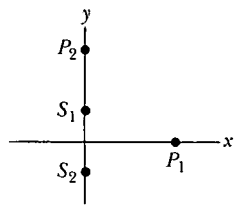


FIG. 36-82 Problem 64.

65. In Fig. 36-83*a*, the waves along rays 1 and 2 are initially in phase, with the same wavelength λ in air. Ray 2 goes through a material with length L and index of refraction n. The rays are then reflected by mirrors to a common point P on a screen. Suppose that we can vary n from $n = 1.0$ to $n = 2.5$. Suppose

also that, from $n = 1.0$ to $n = n_s = 1.5$, the intensity I of the light at point P varies with n as given in Fig. 36-83*b*. At what values of n greater than 1.4 is intensity I (a) maximum and (b) zero? (c) What multiple of λ gives the phase difference between the rays at point P when $n = 2.0$?

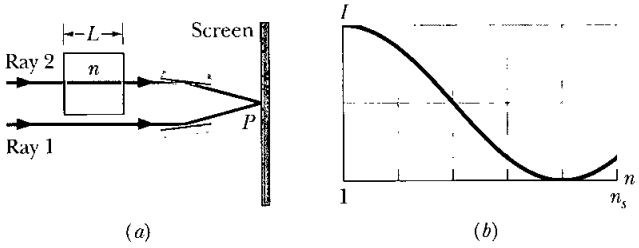


FIG. 36-83 Problems 65.

66. In a phasor diagram for the waves at any point on the viewing screen for the two-slit experiment in Fig. 36-11, the phasor of the resultant wave rotates 60.0° in 2.50×10^{-16} s. What is the wavelength of the light?

67. Light of wavelength 700.0 nm is sent along a route of length 2000 nm. The route is then filled with a medium having an index of refraction of 1.400. In degrees, by how much does the medium phase-shift the light? Give (a) the full shift and (b) the equivalent shift that has a value less than 360°.

68. In two experiments, light is to be sent along the two paths shown in Fig. 36-62 by reflecting it from the various flat surfaces shown. In the first experiment, rays 1 and 2 are initially in phase and have a wavelength of 620.0 nm. In the second experiment, rays 1 and 2 are initially in phase and have a wavelength of 496.0 nm. What least value of distance L is required such that the 620.0 nm waves emerge from the region exactly in phase but the 496.0 nm waves emerge exactly out of phase?

69. A thin film suspended in air is 0.410 μm thick and is illuminated with white light incident perpendicularly on its surface. The index of refraction of the film is 1.50. At what wavelength will visible light that is reflected from the two surfaces of the film undergo fully constructive interference?

70. If the distance between the first and tenth minima of a double-slit pattern is 18.0 mm and the slits are separated by 0.150 mm with the screen 50.0 cm from the slits, what is the wavelength of the light used?

71. Laser light of wavelength 632.8 nm passes through a double-slit arrangement at the front of a lecture room, reflects off a mirror 20.0 m away at the back of the room, and then produces an interference pattern on a screen at the front of the room. The distance between adjacent bright fringes is 10.0 cm. (a) What is the slit separation? (b) What is at the center of the pattern when the lecturer places a thin cellophane sheet over one slit, thereby increasing by 2.50 the number of wavelengths along the path that includes the cellophane?

72. One slit of a double-slit arrangement is covered by a thin glass plate with index of refraction 1.4, and the other by a thin glass plate with index of refraction 1.7. The point on the screen at which the central maximum fell before the glass plates were inserted is now occupied by what had been the $m = 5$ bright fringe. Assuming that $\lambda = 480$ nm and that the plates have the same thickness t, find t.

73. In Fig. 36-73, two glass plates are held together at one end to form a wedge of air that acts as a thin film. A broad beam of light of wavelength 480 nm is directed through the plates, perpendicular to the first plate. An observer intercepting light reflected from the plates sees on the plates an interference pattern that is due to the wedge of air. How much thicker is the wedge at the sixteenth bright fringe than it is at the sixth bright fringe, counting from where the plates touch?

74. Sodium light ($\lambda = 589$ nm) illuminates two slits separated by $d = 2.0$ mm. The slit—screen distance D is 40 mm. What percentage error is made by using Eq. 36-11 to locate the $m = 10$ bright fringe on the screen rather than using the exact path length difference?

75. When an electron moves through a medium at a speed exceeding the speed of light in that medium, the electron radiates electromagnetic energy (the *Cerenkov effect*). What minimum speed must an electron have in a liquid with index of refraction 1.54 in order to radiate?

76. The second dark band in a double-slit interference pattern is 1.2 cm from the central maximum of the pattern. The separation between the two slits is equal to 800 wavelengths of the monochromatic light incident (perpendicularly) on the slits. What is the distance between the plane of the slits and the viewing screen?

77. In a double-slit interference experiment, the slit separation is 2.00 μm, the light wavelength is 500 nm, and the separation between the slits and the screen is 4.00 m. (a) What is the angle between the center and the third side bright fringe? If we decrease the light frequency to 90.0% of its initial value, (b) does the third side bright fringe move along the screen toward or away from the pattern's center and (c) how far does it move?

78. Slits of unequal widths are used in a double-slit arrangement to produce an interference pattern on a distant screen. If only the narrower slit 1 is illuminated (the wider slit 2 is covered), the light reaching the center of the pattern has amplitude E_0 and intensity I_0. If only slit 2 is illuminated, the amplitude of the light reaching the pattern center is $2E_0$. When both slits are illuminated and a two-slit interference pattern is on the screen, what is the intensity $I(\theta)$ in the pattern as a function of angle θ? Answer in the style of Eqs. 36-22 and 36-23.

79. In Fig. 36-84, A, B, and C are three slits each of them individually producing the same intensity I_0 at P_0 when they are illuminated by parallel beam of light of wavelength λ. It is given that $B\,P_0 - A\,P_0 = \lambda/2$ Also given that $d \ll D$. Find (a) λ in terms of d and D and (b) resultant intensity at P_0.

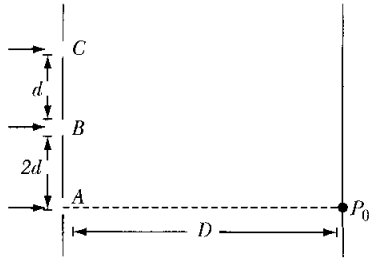


FIG. 36-84 Problem 79.

80. A mica strip and a polystyrene strip of equal thickness are fitted on the two slits of a double-slit apparatus. The thickness

of the strips is 0.50 mm and the separation between the slits is 0.12 cm. The refractive indexes of mica and polystyrenes are 1.58 and 1.55, respectively, for the light of wavelength 590 nm. The interference is observed on screen a distance 1 m away. (a) What would be the fringe width? (b) At what distance from the center will the closest maximum be located?

81. In June 1985, a laser beam was sent out from the Air Force Optical Station on Maui, Hawaii, and reflected back from the shuttle *Discovery* as it sped by 354 km overhead. The diameter of the central maximum of the beam at the shuttle position was said to be 9.1 m, and the beam wavelength was 500 nm. What is the effective diameter of the laser aperture at the Maui ground station? (*Hint:* A laser beam spreads only because of diffraction; assume a circular exit aperture.)

82. An astronaut in a space shuttle claims she can just barely resolve two point sources on Earth's surface, 160 km below. Calculate their (a) angular and (b) linear separation, assuming ideal conditions. Take $\lambda = 540$ nm and the pupil diameter of the astronaut's eye to be 5.0 mm.

83. When monochromatic light is incident on a slit 22.0 μm wide, the first diffraction minimum lies at 1.80° from the direction of the incident light. What is the wavelength?

84. In a single-slit diffraction experiment, there is a minimum of intensity for orange light ($\lambda = 600$ nm) and a minimum of intensity for blue-green light ($\lambda = 500$ nm) at the same angle of 1.00 mrad. For what minimum slit width is this possible?

85. In a two-slit interference pattern, what is the ratio of slit separation to slit width if there are 17 bright fringes within the central diffraction envelope and the diffraction minima coincide with two-slit interference maxima?

86. In two-slit interference, if the slit separation is 14 μm and the slit widths are each 2.0 μm, (a) how many two-slit maxima are in the central peak of the diffraction envelope and (b) how many are in either of the first side peak of the diffraction envelope?

87. A single-slit diffraction experiment is set up with light of wavelength 420 nm, incident perpendicularly on a slit of width 5.10 μm. The viewing screen is 3.20 m distant. On the screen, what is the distance between the center of the diffraction pattern and the second diffraction minimum?

88. If you look at something 40 m from you, what is the smallest length (perpendicular to your line of sight) that you can resolve, according to Rayleigh's criterion? Assume the pupil of your eye has a diameter of 4.00 mm, and use 500 nm as the wavelength of the light reaching you.

89. A spy satellite orbiting at 160 km above Earth's surface has a lens with a focal length of 3.6 m and can resolve objects on the ground as small as 30 cm. For example, it can easily measure the size of an aircraft's air intake port. What is the effective diameter of the lens as determined by diffraction consideration alone? Assume $\lambda = 550$ nm.

90. If you double the width of a single slit, the intensity of the central maximum of the diffraction pattern increases by a factor of 4, even though the energy passing through the slit only doubles. Explain this quantitatively.

91. Suppose that two points are separated by 2.0 cm. If they are viewed by an eye with a pupil opening of 5.0 mm, what distance from the viewer puts them at the Rayleigh limit of resolution? Assume a light wavelength of 500 nm.

92. An acoustic double-slit system (of slit separation d and slit width a) is driven by two loudspeakers as shown in Fig. 36-85. By use of a variable delay line, the phase of one of the speakers may be varied relative to the other speaker. Describe in detail what changes occur in the double-slit diffraction pattern at large distances as the phase difference between the speakers is varied from zero to 2π. Take both interference and diffraction effects into account.

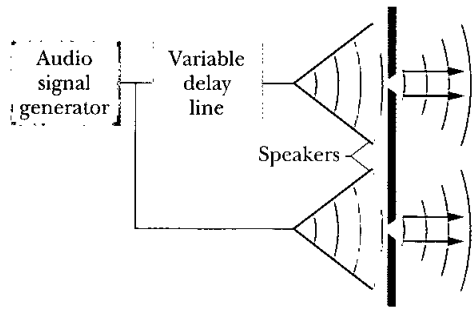


FIG. 36-85 Problem 92.

ANSWERS

Checkpoints

1. B (least n), c, a

2. (a) top; (b) bright intermediate illumination (phase difference is 2.1 wavelengths)

3. (a) 3λ, 3; (b) 2.5λ; 2.5

4. a and d tie (amplitude of resultant wave is $4E_0$), then b and c tie (amplitude of resultant wave is $2E_0$)

5. (a) 1 and 4; (b) 1 and 4

6. (a) expand; (b) expand

7. diminish

Reasoning Type Questions

1. (a)

2. (d)

3. (a)

4. (d)

Multiple Choice Questions

1. (b)
2. (c)
3. (a), (b), (c), (d)
4. (c)
5. (a)
6. (b)
7. (c)
8. (a), (b)
9. (c)
10. (d)
11. (d)
12. (b)
13. (b)
14. (a)
15. (c)
16. (d)
17. (a)
18. (c)
19. (a)
20. (b)
21. (d)
22. (c)
23. (c)
24. (b)
25. (d)
26. (c)
27. (d)
28. (a)
29. (c)
30. (a)
31. (a)
32. (a)
33. (a)
34. (a)
35. (b)
36. (a)
37. (b)
38. (c)
39. (a)
40. (c)
41. (b)
42. (d)
43. (d)
44. (d)
45. (d)
46. (b)
47. (d)

Problems

1. 1.56
2. (a) 5.09×10^{14} Hz; (b) 387.5 nm; (c) 1.97×10^{8} m/s
3. (a) 0.25; (b) 0.75; (c) 1.25
4. (a) 155 nm; (b) 310 nm
5. (a) 1.70; (b) 1.70; (c) 1.30; (d) all tie
6. (a) 52.50 nm; (b) 157.5 nm
7. (a) 0.833; (b) intermediate closer to fully constructive
8. (a) 0.214 rad; (b) 12.3°
9. 16
10. (a) 0.010 rad; (b) 5.0 mm
11. 2.25 mm
12. 0
13. (a) 600 nm, any wavelength greater than this will not be seen. Thus, 600 nm $< \theta \leq$ 700 nm are absent; (b) the slit separation d needs to be decreased; (c) 0.20 μm
14. 7.88 μm
15. 7.5
16. 72 μm
17. (a) 0; (b) 0; (c) ∞; (d) 12π; (e) 3.6π; (f) "intermediate" but closer to a minimum than to a maximum
18. 6.64 μm
19. 3.5 μm
20. 2.65
21. $(17.1\ \mu\text{V/m})\sin[(2.0 \times 10^{14}\ \text{rad/s})t]$
22. (a) 2.33 μV/m; (b) 0.338; (c) point P is between the sixth side maximum (at which $\phi = 6$ wavelengths) and the seventh minimum (at which $\phi = 6\frac{1}{2}$ wavelengths); (d) 1.26×10^{15} rad/s; (e) 2270°
23. (a) Between the central maximum (zero wavelength difference) and the first minimum; (b) 0.404
24. 49/25
25. $I_0 \cos^2(\pi y/\beta)$
26. 0.20 mm
27. 4×10^{-15} W
28. $\sqrt{3} \times 1000\lambda$
29. 38
30. (a) 0.168 mm; (b) 31.5 mm

31. 70.0 nm

32. (a) 567 nm; (b) 425 nm; (c) longer wavelength

33. (a) 0.117 μm; (b) 0.352 μm

34. 120 nm

35. 840 nm

36. (a) 552 nm; (b) 442 nm

37. 1.89 μm

38. (a) 10.3 nm/s; (b) 1.09 μm

39. 0.012°

40. $n_{air} = (1.00000)\dfrac{4001}{4000} = 1.00025$

41. (a) 700 nm; (b) 4; (c) 6

42. (a) 0.430°; (b) 0.118 mm

43. (a) 70 cm; (b) 1.0 mm

44. (a) Diffraction effects in general would decrease; (b) 11°; (c) 0.010 m

45. 41.2 m

46. (a) 0.18°; (b) 0.46 rad; (c) 0.93

47. 160°

48. (a) 2.33 μm; (b) 3.82; (c) 15.2°; (d) 51.8°

49. (b) 0; (c) –0.5; (d) 4.493 rad; (e) 0.930; (f) 7.725 rad; (g) 1.96

50. (a) 0.32 m; (b) 2.7 m; (c) the military satellites do not use Hubble Telescope-sized apertures.

51. (a) 50 m; (b) no; (c) no

52. (a) 1.34×10^{-4} rad; (b) 10 km

53. 30 m

54. (a) 6.8°; (b) since $\sin\theta$ cannot exceed 1 there is no minimum

55. 3

56. 5

57. $\dfrac{D\lambda}{d}$

58. (a) $4a$; (b) every fourth fringe is missing

Additional Problems

59. (a) 22°; (b) refraction reduces θ

60. (a) 1.6; (b) 1.4

61. (a) 50.0 nm; (b) 36.2 nm

62. (a) ∞; (b) 0; (c) At $y_P = 0$ (where the screen crosses the x axis)

63. $x = (D/2a)(m + 0.5)\lambda$, for $m = 0, 1, 2, \ldots$

64. (a) 2.90 λ; (b) intermediate, but close to fully constructive

65. (a) 1.8; (b) 2.2; (c) 1.25

66. 450 nm

67. (a) 411.4°; (b) 51.4°

68. 310.0 nm

69. 492 nm

70. 600 nm

71. (a) 0.253 mm; (b) there is now a minimum instead of a maximum

72. 8.0 μm

73. 2.4 μm

74. 0.032%

75. 1.95×10^8 m/s

76. 6.4 m

77. (a) 48.6°; (b) away; (c) 1.49 m

78. $I_0[1 + 8\cos^2(\phi/2)]$, with $\phi = (2\pi d/\lambda)\sin\theta$

79. $\dfrac{4d^2}{D}, I_0$

80. (a) 4.9×10^{-4} m; (b) 0.021 cm

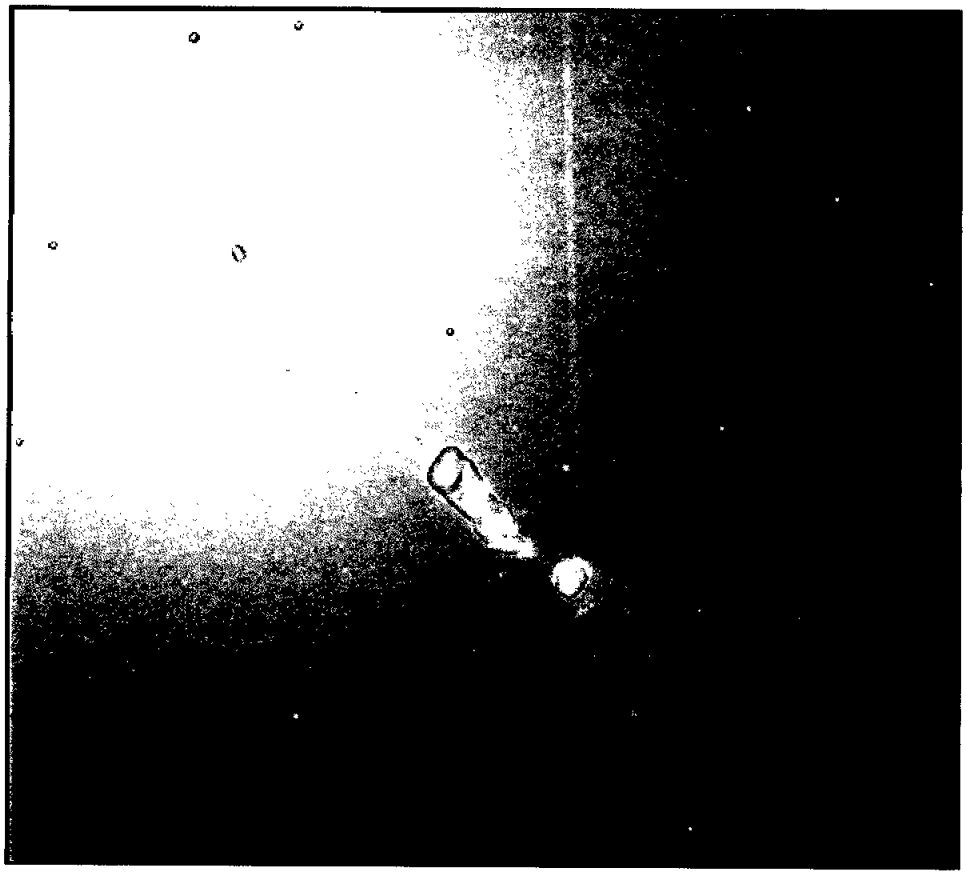
81. 4.7 cm

82. (a) 1.3×10^{-4} rad; (b) 21 m

83. 691 nm

84. 3.0 mm

Special Relativity 37



Courtesy J. A. Biretta, et al., Hubble Heritage Team (STScI/AURA), NASA

A jet 5000 light-years long streams from the center of galaxy M87 (the bright spot at the upper left end of the jet), which is 5×10^7 light-years from us. The jet is formed by electrons moving at nearly the speed of light. Something very strange must be occurring at the center of M87 in order for the electrons to be ejected from it at such high speed. Is there some type of monster there? Unfortunately, M87 is too distant for us to see any object at its center.

How can we tell what monster lies at the center of M87?

The answer is in this chapter.

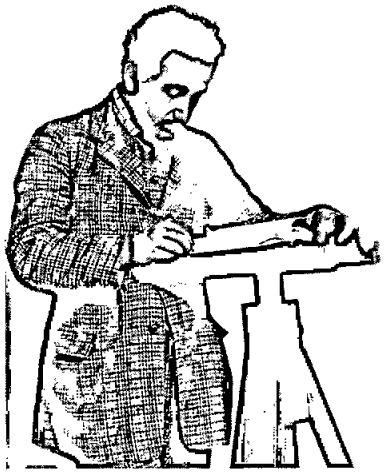


FIG. 37-1 Einstein posing for a photograph as fame began to accumulate. *(Corbis Images)*

37-1 WHAT IS PHYSICS?

One principal subject of physics is **relativity**, the field of study that measures events (things that happen): where and when they happen, and by how much any two events are separated in space and in time. In addition, relativity has to do with transforming such measurements between reference frames that move relative to each other. (Hence the name *relativity.*)

Transformations and moving reference frames, such as those we discussed in the chapter on Kinematics. (Chapter 4, Vol. I), were well understood and quite routine to physicists in 1905. Then Albert Einstein (Fig. 37-1) published his **special theory of relativity**. The adjective *special* means that the theory deals only with **inertial reference frames**, which are frames in which Newton's laws are valid. (Einstein's *general theory of relativity* treats the more challenging situation in which reference frames can undergo gravitational acceleration; in this chapter the term *relativity* implies only inertial reference frames.)

Starting with two deceivingly simple postulates, Einstein stunned the scientific world by showing that the old ideas about relativity were wrong, even though everyone was so accustomed to them that they seemed to be unquestionable common sense. This supposed common sense, however, was derived only from experience with things that move rather slowly. Einstein's relativity, which turns out to be correct for all possible speeds, predicted many effects that were, at first study, bizarre because no one had ever experienced them.

In particular, Einstein demonstrated that space and time are entangled; that is, the time between two events depends on how far apart they occur, and vice versa. Also, the entanglement is different for observers who move relative to each other. One result is that time does not pass at a fixed rate, as if it were ticked off with mechanical regularity on some master grandfather clock that controls the universe. Rather, that rate is adjustable: Relative motion can change the rate at which time passes. Prior to 1905, no one but a few daydreamers would have thought that. Now, engineers and scientists take it for granted because their experience with special relativity has reshaped their common sense. For example, any engineer involved with the Global Positioning System of the NAVSTAR satellites must routinely use relativity to determine the rate at which time passes on the satellites because that rate differs from the rate on Earth's surface.

Special relativity has the reputation of being difficult. It is not difficult mathematically, at least not here. However, it is difficult in that we must be very careful about *who* measures *what* about an event and just *how* that measurement is made—and it can be difficult because it can contradict routine experience.

37-2 | The Postulates

We now examine the two postulates of relativity, on which Einstein's theory is based:

☞ **1. The Relativity Postulate:** The laws of physics are the same for observers in all inertial reference frames. No one frame is preferred over any other.

Galileo assumed that the laws of *mechanics* were the same in all inertial reference frames. Einstein extended that idea to include *all* the laws of physics, especially those of electromagnetism and optics. This postulate does *not* say that the measured values of all physical quantities are the same for all inertial observers; most are not the same. It is the *laws of physics*, which relate these measurements to one another, that are the same.

☞ **2. The Speed of Light Postulate:** The speed of light in vacuum has the same value c in all directions and in all inertial reference frames.

We can also phrase this postulate to say that there is in nature an *ultimate speed c*, the same in all directions and in all inertial reference frames. Light happens to travel at this ultimate speed. However, no entity that carries energy or information can exceed this limit. Moreover, no particle that has mass can actually reach speed c, no matter how much or for how long that particle is accelerated. (Alas, the faster-than-light warp drive used in many science fiction stories appears to be impossible.)

Both postulates have been exhaustively tested, and no exceptions have ever been found.

The Ultimate Speed

The existence of a limit to the speed of accelerated electrons was shown in a 1964 experiment by W. Bertozzi, who accelerated electrons to various measured speeds and—by an independent method—measured their kinetic energies. He found that as the force on a very fast electron is increased, the electron's measured kinetic energy increases toward very large values but its speed does not increase appreciably (Fig. 37-2). Electrons have been accelerated in laboratories to at least 0.999 999 999 95 times the speed of light but—close though it may be—that speed is still less than the ultimate speed c.

This ultimate speed has been defined to be exactly

$$c = 299\,792\,458 \text{ m/s}. \tag{37-1}$$

Caution: So far in this book we have (appropriately) approximated c as 3.0×10^8 m/s, but in this chapter we shall often use the exact value. You might want to store the exact value in your calculator's memory (if it is not there already), to be called up when needed.

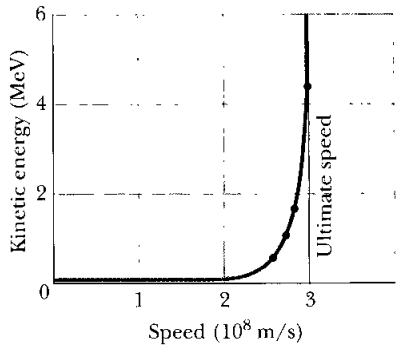


FIG. 37-2 The dots show measured values of the kinetic energy of an electron plotted against its measured speed. No matter how much energy is given to an electron (or to any other particle having mass), its speed can never equal or exceed the ultimate limiting speed c. (The plotted curve through the dots shows the predictions of Einstein's special theory of relativity.)

Testing the Speed of Light Postulate

If the speed of light is the same in all inertial reference frames, then the speed of light emitted by a source relative to, say, a laboratory should be the same as the speed of light that is emitted by a source at rest in the laboratory. This claim has been tested directly, in an experiment of high precision. The "light source" was the *neutral pion* (symbol π^0), an unstable, short-lived particle that can be produced by collisions in a particle accelerator. It decays (transforms) into two gamma rays by the process

$$\pi^0 \rightarrow \gamma + \gamma. \tag{37-2}$$

Gamma rays are part of the electromagnetic spectrum (at very high frequencies) and so obey the speed of light postulate, just as visible light does. (In this chapter we shall use the term light for any type of electromagnetic wave, visible or not.)

In 1964, physicists at CERN, the European particle-physics laboratory near Geneva, generated a beam of pions moving at a speed of $0.999\,75c$ with respect to the laboratory. The experimenters then measured the speed of the gamma rays emitted from these very rapidly moving sources. They found that the speed of the light emitted by the pions was the same as it would be if the pions were at rest in the laboratory, namely c.

37-3 | Measuring an Event

An **event** is something that happens, and every event can be assigned three space coordinates and one time coordinate. Among many possible events are (1) the turning on or off of a tiny lightbulb, (2) the collision of two particles, (3) the passage of a pulse of light through a specified point, (4) an explosion, and (5) the sweeping of the hand of a clock past a marker on the rim of the clock. A certain observer, fixed in a certain inertial reference frame, might, for example, assign to an event A the coordinates given in Table 37-1. Because space and time are entangled

TABLE 37-1
Record of Event *A*

Coordinate	Value
x	3.58 m
y	1.29 m
z	0 m
t	34.5 s

with each other in relativity, we can describe these coordinates collectively as *spacetime* coordinates. The coordinate system itself is part of the reference frame of the observer.

A given event may be recorded by any number of observers, each in a different inertial reference frame. In general, different observers will assign different spacetime coordinates to the same event. Note that an event does not "belong" to any particular inertial reference frame. An event is just something that happens, and anyone in any reference frame may detect it and assign spacetime coordinates to it.

Making such an assignment can be complicated by a practical problem. For example, suppose a balloon bursts 1 km to your right while a firecracker pops 2 km to your left, both at 9:00 A.M. However, you do not detect either event precisely at 9:00 A.M. because at that instant light from the events has not yet reached you. Because light from the firecracker pop has farther to go, it arrives at your eyes later than does light from the balloon burst, and thus the pop will seem to have occurred later than the burst. To sort out the actual times and to assign 9:00 A.M. as the happening time for both events, you must calculate the travel times of the light and then subtract these times from the arrival times.

This procedure can be very messy in more challenging situations, and we need an easier procedure that automatically eliminates any concern about the travel time from an event to an observer. To set up such a procedure, we shall construct an imaginary array of measuring rods and clocks throughout the observer's inertial frame (the array moves rigidly with the observer). This construction may seem contrived, but it spares us much confusion and calculation and allows us to find the space coordinates, the time coordinate, and the spacetime coordinates, as follows.

1. **The Space Coordinates.** We imagine the observer's coordinate system fitted with a close-packed, three-dimensional array of measuring rods, one set of rods parallel to each of the three coordinate axes. These rods provide a way to determine coordinates along the axes. Thus, if the event is, say, the turning on of a small lightbulb, the observer, in order to locate the position of the event, need only read the three space coordinates at the bulb's location.

2. **The Time Coordinate.** For the time coordinate, we imagine that every point of intersection in the array of measuring rods includes a tiny clock, which the observer can read because the clock is illuminated by the light generated by the event. Figure 37-3 suggests one plane in the "jungle gym" of clocks and measuring rods we have described.

   The array of clocks must be synchronized properly. It is not enough to assemble a set of identical clocks, set them all to the same time, and then move them to their assigned positions. We do not know, for example, whether moving the clocks will change their rates. (Actually, it will.) We must put the clocks in place and *then* synchronize them.

   If we had a method of transmitting signals at infinite speed, synchronization would be a simple matter. However, no known signal has this property. We therefore choose light (any part of the electromagnetic spectrum) to send out our synchronizing signals because, in vacuum, light travels at the greatest possible speed, the limiting speed c.

   Here is one of many ways in which an observer might synchronize an array of clocks using light signals: The observer enlists the help of a great number of temporary helpers, one for each clock. The observer then stands at a point selected as the origin and sends out a pulse of light when the origin clock reads $t = 0$. When the light pulse reaches the location of a helper, that helper sets the clock there to read $t = r/c$, where r is the distance between the helper and the origin. The clocks are then synchronized.

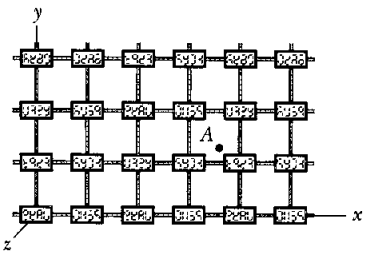


FIG. 37-3 One section of a three-dimensional array of clocks and measuring rods by which an observer can assign spacetime coordinates to an event, such as a flash of light at point A. The event's space coordinates are approximately $x = 3.6$ rod lengths, $y = 1.3$ rod lengths, and $z = 0$. The time coordinate is whatever time appears on the clock closest to A at the instant of the flash.

3. **The Spacetime Coordinates.** The observer can now assign spacetime coordinates to an event by simply recording the time on the clock nearest the event and the position as measured on the nearest measuring rods. If there are two events, the observer computes their separation in time as the difference in the times on clocks near each and their separation in space from the differences in coordinates on rods near each. We thus avoid the practical problem of calculating the travel times of the signals to the observer from the events.

37-4 | The Relativity of Simultaneity

Suppose that one observer (Sam) notes that two independent events (event Red and event Blue) occur at the same time. Suppose also that another observer (Sally), who is moving at a constant velocity $\vec{v}$ with respect to Sam, also records these same two events. Will Sally also find that they occur at the same time?

The answer is that in general she will not:

☞ If two observers are in relative motion, they will not, in general, agree as to whether two events are simultaneous. If one observer finds them to be simultaneous, the other generally will not.

We cannot say that one observer is right and the other wrong. Their observations are equally valid, and there is no reason to favor one over the other.

The realization that two contradictory statements about the same natural event can be correct is a seemingly strange outcome of Einstein's theory. However, in Chapter 17, Vol. I, we saw another way in which motion can affect measurement without balking at the contradictory results: In the Doppler effect, the frequency an observer measures for a sound wave depends on the relative motion of observer and source. Thus, two observers moving relative to each other can measure different frequencies for the same wave, and both measurements are correct.

We conclude the following:

☞ Simultaneity is not an absolute concept but rather a relative one, depending on the motion of the observer.

If the relative speed of the observers is very much less than the speed of light, then measured departures from simultaneity are so small that they are not noticeable. Such is the case for all our experiences of daily living; that is why the relativity of simultaneity is unfamiliar.

A Closer Look at Simultaneity

Let us clarify the relativity of simultaneity with an example based on the postulates of relativity, no clocks or measuring rods being directly involved. Figure 37-4 shows two long spaceships (the SS *Sally* and the SS *Sam*), which can serve as inertial reference frames for observers Sally and Sam. The two observers are stationed at the midpoints of their ships. The ships are separating along a common x axis, the relative velocity of *Sally* with respect to *Sam* being $\vec{v}$. Figure 37-4*a* shows the ships with the two observer stations momentarily aligned opposite each other.

Two large meteorites strike the ships, one setting off a red flare (event Red) and the other a blue flare (event Blue), not necessarily simultaneously. Each event leaves a permanent mark on each ship, at positions RR' and BB'.

Let us suppose that the expanding wavefronts from the two events happen to reach Sam at the same time, as Fig. 37-4*b* shows. Let us further suppose that, after the episode, Sam finds, by measuring the marks on his spaceship, that he was

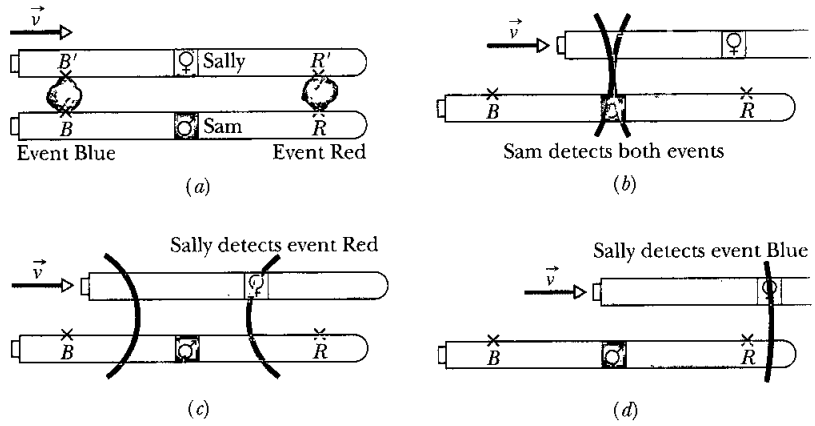


FIG. 37-4 The spaceships of Sally and Sam and the occurrences of events from Sam's view. Sally's ship moves rightward with velocity $\vec{v}$. (*a*) Event Red occurs at positions RR' and event Blue occurs at positions BB'; each event sends out a wave of light. (*b*) Sam simultaneously detects the waves from event Red and event Blue. (*c*) Sally detects the wave from event Red. (*d*) Sally detects the wave from event Blue.

indeed stationed exactly halfway between the markers B and R on his ship when the two events occurred. He will say:

Sam Light from event Red and light from event Blue reached me at the same time. From the marks on my spaceship, I find that I was standing halfway between the two sources. Therefore, event Red and event Blue were simultaneous events.

As study of Fig. 37-4 shows, Sally and the expanding wavefront from event Red are moving *toward* each other, while she and the expanding wavefront from event Blue are moving in the *same direction.* Thus, the wavefront from event Red will reach Sally *before* the wavefront from event Blue does. She will say:

Sally Light from event Red reached me before light from event Blue did. From the marks on my spaceship, I found that I too was standing halfway between the two sources. Therefore, the events were not simultaneous; event Red occurred first, followed by event Blue.

These reports do not agree. Nevertheless, *both* observers are correct.

Note carefully that there is only one wavefront expanding from the site of each event and that *this wavefront travels with the same speed c in both reference frames*, exactly as the speed of light postulate requires.

It *might* have happened that the meteorites struck the ships in such a way that the two hits appeared to Sally to be simultaneous. If that had been the case, then Sam would have declared them not to be simultaneous.

Let us study this concept more closely. What we are going to do is show that relativity of simultaneity implies that a moving observer sees that the reading in two clocks may be different, even though they may have been synchronized in their own frame. For this, let us consider two clocks positioned at the ends of a train of length L (as measured in its own frame). They are synchronized in the train frame. We will show here that the clocks are not synchronized in the observer's frame. (Observer is standing on the ground.)

Let us put a light source on the train, and position it so that the light hits the clocks at the ends of the train at the same time *in the observer's frame.* The relative speeds of the wavefront of light and the clocks are $c + v$ and $c - v$ (as viewed in the observer's frame). To make the time of travel for the wavefront equal, we need to divide the train into lengths in this ratio, in the observer's frame. So in

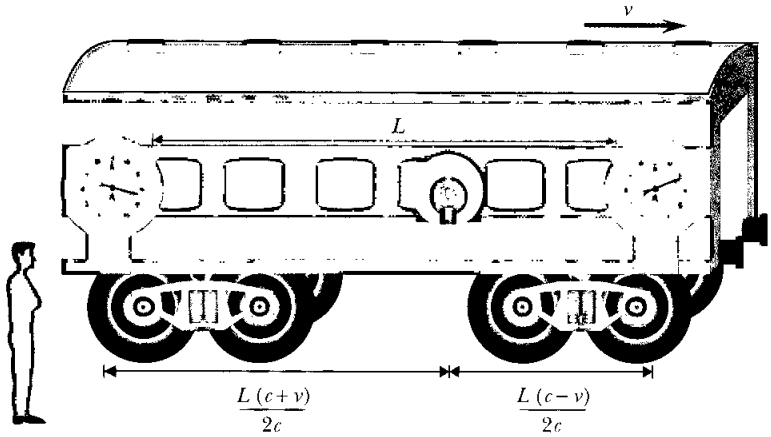


FIG. 37-5 A light source is positioned in a train so that its light hits the rear end and the front end simultaneously as seen by an observer on ground.

the train frame, one can quickly show that two numbers that are in this ratio, and that add up to L, are $L(c + v)/2c$ and $L(c - v)/2c$.

The source of light is to be positioned such that the light hits the rear end and front end simultaneously. This can happen only if the light source is closer to the front end. After all, the light has the same velocity in all the directions. The front end is running away from the wavefront and the rear end is running toward it.

The situation in the train frame therefore looks like that in Fig. 37-5. The light must travel an extra distance of $L(c + v)/2c - L(c - v)/2c = Lv/c$ to reach the rear clock. The extra time is therefore Lv/c^2. Hence, the rear clock reads Lv/c^2 more when it is hit by the backward wavefront, compared to what the front clock reads when it is hit by the forward wavefront. Note that our result does *not* say that you see the rear clock ticking at a faster rate than the front clock. They run at the same rate. The back clock is simply a fixed time ahead of the front clock, as seen by you. If we were to make a array of clocks, the clocks stationary with respect to you would look like in Fig. 37-6*a* and the clocks moving with respect to you would look like in Fig. 37-6*b*.

What is the moral of the story? The clocks that are synchronized in one frame are not necessarily synchronized in another frame. This follows from the relativity of simultaneity: What is simultaneous in the one frame is not simultaneous in other frame.

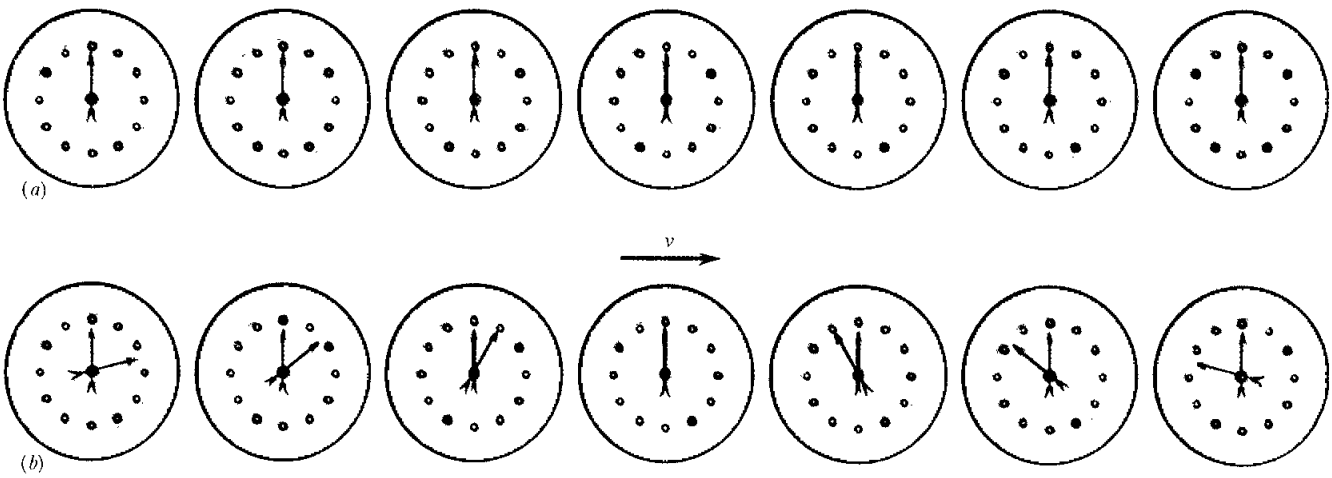


FIG. 37.6 (*a*) The time shown by the synchronized clocks which are stationary in the observer's frame. (*b*) The time shown by the clocks are moving with respect to the observer.

37-5 | The Relativity of Time

If observers who move relative to each other measure the time interval (or *temporal separation*) between two events, they generally will find different results. Why? Because the spatial separation of the events can affect the time intervals measured by the observers.

☞ The time interval between two events depends on how far apart they occur in both space and time; that is, their spatial and temporal separations are entangled.

In this section we discuss this entanglement by means of an example; however, the example is restricted in a crucial way: *To one of two observers, the two events occur at the same location.* We shall not get to more general examples until Section 37-7.

Figure 37-7*a* shows the basics of an experiment Sally conducts while she and her equipment—a light source, a mirror, and a clock—ride in a train moving with constant velocity $\vec{v}$ relative to a station. A pulse of light leaves the light source B (event 1), travels vertically upward, is reflected vertically downward by the mirror, and then is detected back at the source (event 2). Sally measures a certain time interval Δt_0 between the two events, related to the distance D from source to mirror by

$$\Delta t_0 = \frac{2D}{c} \qquad \text{(Sally)}. \tag{37-3}$$

The two events occur at the same location in Sally's reference frame, and she needs only one clock C at that location to measure the time interval. Clock C is shown twice in Fig. 37-7*a*, at the beginning and end of the interval.

Consider now how these same two events are measured by Sam, who is standing on the station platform as the train passes. Because the equipment moves with the train during the travel time of the light, Sam sees the path of the light as shown in Fig. 37-7*b*. For him, the two events occur at different places in his reference frame, and so to measure the time interval between events, Sam must use *two* synchronized clocks, C_1 and C_2, one at each event. According to Einstein's speed of light postulate, the light travels at the same speed c for Sam as for Sally. Now,

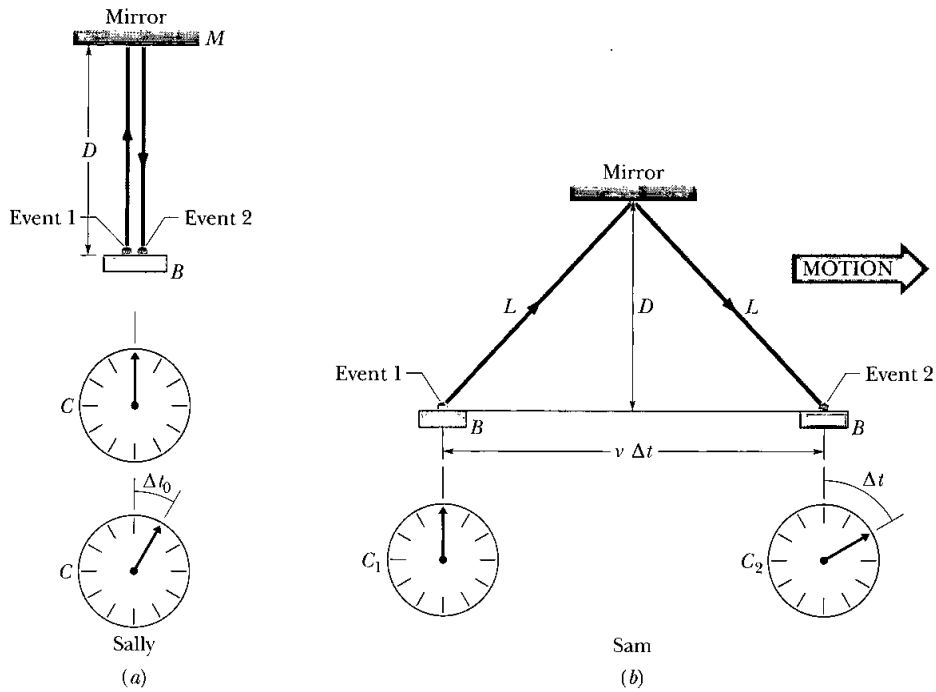


FIG. 37-7 (*a*) Sally, on the train, measures the time interval Δt_0 between events 1 and 2 using a single clock C on the train. That clock is shown twice: first for event 1 and then for event 2. (*b*) Sam, watching from the station as the events occur, requires two synchronized clocks, C_1 at event 1 and C_2 at event 2, to measure the time interval between the two events; his measured time interval is Δt.

however, the light travels distance $2L$ between events 1 and 2. The time interval measured by Sam between the two events is

$$\Delta t = \frac{2L}{c} \qquad \text{(Sam)}, \tag{37-4}$$

in which

$$L = \sqrt{(\tfrac{1}{2}v\Delta t)^2 + D^2}. \tag{37-5}$$

From Eq. 37-3, we can write this as

$$L = \sqrt{(\tfrac{1}{2}v\Delta t)^2 + (\tfrac{1}{2}c\Delta t_0)^2}. \tag{37-6}$$

If we eliminate L between Eqs. 37-4 and 37-6 and solve for Δt, we find

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}}. \tag{37-7}$$

Equation 37-7 tells us how Sam's measured interval Δt between the events compares with Sally's interval Δt_0. Because v must be less than c, the denominator in Eq. 37-7 must be less than unity. Thus, Δt must be greater than Δt_0: Sam measures a *greater* time interval between the two events than does Sally. Sam and Sally have measured the time interval between the *same* two events, but the relative motion between Sam and Sally made their measurements *different.* We conclude that relative motion can change the *rate* at which time passes between two events; the key to this effect is the fact that the speed of light is the same for both observers.

We distinguish between the measurements of Sam and Sally with the following terminology:

When two events occur at the same location in an inertial reference frame, the time interval between them, measured in that frame, is called the **proper time interval** or the **proper time**. Measurements of the same time interval from any other inertial reference frame are always greater.

Thus, Sally measures a proper time interval, and Sam measures a greater time interval. (The term *proper* is unfortunate in that it implies that any other measurement is improper or nonreal. That is just not so.) The amount by which a measured time interval is greater than the corresponding proper time interval is called **time dilation**. (To dilate is to expand or stretch; here the time interval is expanded or stretched.)

Often the dimensionless ratio v/c in Eq. 37-7 is replaced with β, called the **speed parameter**, and the dimensionless inverse square root in Eq. 37-7 is often replaced with γ, called the **Lorentz factor:**

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(v/c)^2}}. \tag{37-8}$$

With these replacements, we can rewrite Eq. 37-7 as

$$\Delta t = \gamma \Delta t_0 \qquad \text{(time dilation)}. \tag{37-9}$$

The speed parameter β is always less than unity, and, provided v is not zero, γ is always greater than unity. However, the difference between γ and 1 is not significant unless $v > 0.1c$. Thus, in general, "old relativity" works well enough for $v < 0.1c$, but we must use special relativity for greater values of v. As shown in

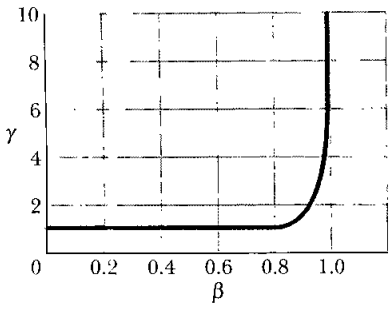


FIG. 37-8 A plot of the Lorentz factor γ as a function of the speed parameter β (= v/c).

Fig. 37-8, γ increases rapidly in magnitude as β approaches 1 (as v approaches c). Therefore, the greater the relative speed between Sally and Sam is, the greater will be the time interval measured by Sam, until at a great enough speed, the interval takes "forever."

You might wonder what Sally says about Sam's having measured a greater time interval than she did. His measurement comes as no surprise to her, because to her, he failed to synchronize his clocks C_1 and C_2 in spite of his insistence that he did. Recall that observers in relative motion generally do not agree about simultaneity. Here, Sam insists that his two clocks simultaneously read the same time when event 1 occurred. To Sally, however, Sam's clock C_2 was erroneously set ahead during the synchronization process. Thus, when Sam read the time of event 2 on it, to Sally he was reading off a time that was too large, and that is why the time interval he measured between the two events was greater than the interval she measured.

Two Tests of Time Dilation

1. **Microscopic Clocks.** Subatomic particles called *muons* are unstable; that is, when a muon is produced, it lasts for only a short time before it *decays* (transforms into particles of other types). The *lifetime* of a muon is the time interval between its production (event 1) and its decay (event 2). When muons are stationary and their lifetimes are measured with stationary clocks (say, in a laboratory), their average lifetime is 2.200 μs. This is a proper time interval because, for each muon, events 1 and 2 occur at the same location in the reference frame of the muon—namely, at the muon itself. We can represent this proper time interval with Δt_0; moreover, we can call the reference frame in which it is measured the *rest frame* of the muon.

   If, instead, the muons are moving, say, through a laboratory, then measurements of their lifetimes made with the laboratory clocks should yield a greater average lifetime (a dilated average lifetime). To check this conclusion, measurements were made of the average lifetime of muons moving with a speed of 0.9994c relative to laboratory clocks. From Eq. 37-8, with $\beta = 0.9994$, the Lorentz factor for this speed is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(0.9994)^2}} = 28.87.$$

   Equation 37-9 then yields, for the average dilated lifetime,

$$\Delta t = \gamma \Delta t_0 = (28.87)(2.200\ \mu\text{s}) = 63.51\ \mu\text{s}.$$

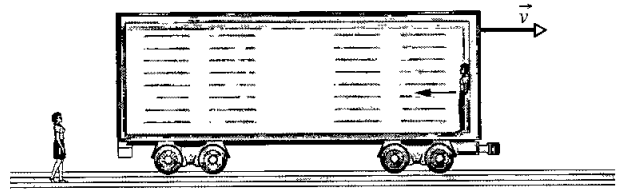
   The actual measured value matched this result within experimental error.

2. **Macroscopic Clocks.** In October 1977, Joseph Hafele and Richard Keating carried out what must have been a grueling experiment. They flew four portable atomic clocks twice around the world on commercial airlines, once in each direction. Their purpose was "to test Einstein's theory of relativity with macroscopic clocks." As we have just seen, the time dilation predictions of Einstein's theory have been confirmed on a microscopic scale, but there is great comfort in seeing a confirmation made with an actual clock. Such macroscopic measurements became possible only because of the very high precision of modern atomic clocks. Hafele and Keating verified the predictions of the theory to within 10%. (Einstein's *general* theory of relativity, which predicts that the rate at which time passes on a clock is influenced by the gravitational force on the clock, also plays a role in this experiment.)

   A few years later, physicists at the University of Maryland carried out a similar experiment with improved precision. They flew an atomic clock round and round over Chesapeake Bay for flights lasting 15 h and succeeded in checking

the time dilation prediction to better than 1%. Today, when atomic clocks are transported from one place to another for calibration or other purposes, the time dilation caused by their motion is always taken into account.

CHECKPOINT 1 Standing beside railroad tracks, we are suddenly startled by a relativistic boxcar traveling past us as shown in the figure. Inside, a well-equipped hobo fires a laser pulse from the front of the boxcar to its rear. (a) Is our measurement of the speed of the pulse greater than, less than, or the same as that measured by the hobo? (b) Is his measurement of the flight time of the pulse a proper time? (c) Are his measurement and our measurement of the flight time related by Eq. 37-9?



Sample Problem 37-1

Your starship passes Earth with a relative speed of $0.9990c$. After traveling 10.0 y (your time), you stop at lookout post LP13, turn, and then travel back to Earth with the same relative speed. The trip back takes another 10.0 y (your time). How long does the round trip take according to measurements made on Earth? (Neglect any effects due to the accelerations involved with stopping, turning, and getting back up to speed.)

Conceptualize/Classify: We begin by analyzing the outward trip:

1. This problem involves measurements made from two (inertial) reference frames, one attached to Earth and the other (your reference frame) attached to your ship.
2. The outward trip involves two events: the start of the trip at Earth and the end of the trip at LP13.
3. Your measurement of 10.0 y for the outward trip is the proper time Δt_0 between those two events, because the events occur at the same location in your reference frame—namely, on your ship.
4. The Earth-frame measurement of the time interval Δt for the outward trip must be greater than Δt_0, according to Eq. 37-9 ($\Delta t = \gamma \Delta t_0$) for time dilation.

Compute: Using Eq. 37-8 to substitute for γ in Eq. 37-9, we find

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}}$$

$$= \frac{10.0 \text{ y}}{\sqrt{1-(0.9990c/c)^2}} = (22.37)(10.0 \text{ y}) = 224 \text{ y}.$$

On the return trip, we have the same situation and the same data. Thus, the round trip requires 20 y of your time but

$$\Delta t_{\text{total}} = (2)(224 \text{ y}) = 448 \text{ y} \qquad \text{(Answer)}$$

of Earth time.

Conclude: In other words, you have aged 20 y while the Earth has aged 448 y. Although you cannot travel into the past (as far as we know), you can travel into the future of, say, Earth, by using high-speed relative motion to adjust the rate at which time passes.

Sample Problem 37-2 Build your skill

The elementary particle known as the *positive kaon* (K^+) has, on average, a lifetime of $0.1237\ \mu s$ when stationary—that is, when the lifetime is measured in the rest frame of the kaon. If a positive kaon has a speed of $0.990c$ relative to a laboratory reference frame when the kaon is produced, how far can it travel in that frame during its lifetime according to *classical physics* (which is a reasonable approximation for speeds much less than c) and according to special relativity (which is correct for all physically possible speeds)?

Conceptualize/Classify:

1. This problem involves measurements made from two (inertial) reference frames, one attached to the kaon and the other attached to the laboratory.
2. This problem also involves two events: the start of the kaon's travel (when the kaon is produced) and the end of that travel (at the end of the kaon's lifetime).

3. The distance traveled by the kaon between those two events is related to its speed v and the time interval for the travel by

$$v = \frac{\text{distance}}{\text{time interval}}. \qquad (37\text{-}10)$$

With these ideas in mind, let us solve for the distance first with classical physics and then with special relativity.

Compute: Classical physics: In classical physics we would find the same distance and time interval (in Eq. 37-10) whether we measured them from the kaon frame or from the laboratory frame. Thus, we need not be careful about the frame in which the measurements are made. To find the kaon's travel distance d_{cp} according to classical physics, we first rewrite Eq. 37-10 as

$$d_{cp} = v\,\Delta t, \qquad (37\text{-}11)$$

where Δt is the time interval between the two events in either frame. Then, substituting $0.990c$ for v and $0.1237\ \mu s$ for Δt in Eq. 37-11, we find

$$\begin{aligned} d_{cp} &= (0.990c)\Delta t \\ &= (0.990)(299\,792\,458\ \text{m/s})(0.1237\times10^{-6}\ \text{s}) \\ &= 36.7\ \text{m}. \qquad \text{(Answer)} \end{aligned}$$

This is how far the kaon would travel if classical physics were correct at speeds close to c.

Special relativity: In special relativity we must be very careful that both the distance and the time interval in Eq. 37-10 are measured in the *same* reference frame—especially when the speed is close to c, as here. Thus, to find the actual travel distance d_{sr} of the kaon *as measured from the laboratory frame* and according to special relativity, we rewrite Eq. 37-10 as

$$d_{sr} = v\,\Delta t, \qquad (37\text{-}12)$$

where Δt is the time interval between the two events *as measured from the laboratory frame.*

Before we can evaluate d_{sr} in Eq. 37-12, we must find Δt. The $0.1237\ \mu s$ time interval is a proper time because the two events occur at the same location in the kaon frame—namely, at the kaon itself. Therefore, let Δt_0 represent this proper time interval. Then we can use Eq. 37-9 ($\Delta t = \gamma\,\Delta t_0$) for time dilation to find the time interval Δt as measured from the laboratory frame. Using Eq. 37-8 to substitute for γ in Eq. 37-9 leads to

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{0.1237\times10^{-6}\ \text{s}}{\sqrt{1-(0.990c/c)^2}} = 8.769\times10^{-7}\ \text{s}.$$

This is about seven times longer than the kaon's proper lifetime. That is, the kaon's lifetime is about seven times longer in the laboratory frame than in its own frame—the kaon's lifetime is dilated. We can now evaluate Eq. 37-12 for the travel distance d_{sr} in the laboratory frame as

$$\begin{aligned} d_{sr} &= v\,\Delta t = (0.990c)\Delta t \\ &= (0.990)(299\,792\,458\ \text{m/s})(8.769\times10^{-7}\ \text{s}) \\ &= 260\ \text{m}. \qquad \text{(Answer)} \end{aligned}$$

This is about seven times d_{cp}.

Conclude: Experiments like the one outlined here, which verify special relativity, became routine in physics laboratories decades ago. The engineering design and the construction of any scientific or medical facility that employs high-speed particles must take relativity into account.

37-6 | The Relativity of Length

If you want to measure the length of a rod that is at rest with respect to you, you can—at your leisure—note the positions of its end points on a long stationary scale and subtract one reading from the other. If the rod is moving, however, you must note the positions of the end points *simultaneously* (in your reference frame) or your measurement cannot be called a length. Figure 37-9 suggests the difficulty of trying to measure the length of a moving penguin by locating its front and back at different times. Because simultaneity is relative and it enters into length measurements, length should also be a relative quantity.

Let L_0 be the length of a rod that you measure when the rod is stationary (meaning you and it are in the same reference frame, the rod's rest frame). If, instead, there is relative motion at speed v between you and the rod *along the length of the rod,* then with simultaneous measurements you obtain a length L given by

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma} \qquad \text{(length contraction)}. \qquad (37\text{-}13)$$

Because the Lorentz factor γ is always greater than unity if there is relative motion, L is less than L_0. The relative motion causes a *length contraction,* and L is called a *contracted length.* Because γ increases with speed v, the length contraction also increases with v.

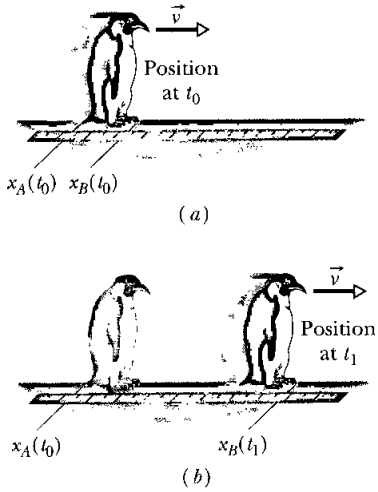


FIG. 37-9 If you want to measure the front-to-back length of a penguin while it is moving, you must mark the positions of its front and back simultaneously (in your reference frame), as in (*a*), rather than at different times, as in (*b*).

☞ The length L_0 of an object measured in the rest frame of the object is its **proper length** or **rest length.** Measurements of the length from any reference frame that is in relative motion parallel to that length are always less than the proper length.

Be careful: Length contraction occurs only along the direction of relative motion. Also, the length that is measured does not have to be that of an object like a rod or a circle. Instead, it can be the length (or distance) between two objects in the same rest frame—for example, the Sun and a nearby star (which are, at least approximately, at rest relative to each other).

Does a moving object *really* shrink? Reality is based on observations and measurements; if the results are always consistent and if no error can be determined, then what is observed and measured is real. In that sense, the object really does shrink. However, a more precise statement is that the object *is really measured* to shrink—motion affects that measurement and thus reality.

When you measure a contracted length for, say, a rod, what does an observer moving with the rod say of your measurement? To that observer, you did not locate the two ends of the rod simultaneously. (Recall that observers in motion relative to each other do not agree about simultaneity.) To the observer, you first located the rod's front end and then, slightly later, its rear end, and that is why you measured a length that is less than the proper length.

Proof of Eq. 37-13

Length contraction is a direct consequence of time dilation. Consider once more our two observers. This time, both Sally, seated on a train moving through a station, and Sam, again on the station platform, want to measure the length of the platform. Sam, using a tape measure, finds the length to be L_0, a proper length because the platform is at rest with respect to him. Sam also notes that Sally, on the train, moves through this length in a time $\Delta t = L_0/v$, where v is the speed of the train; that is,

$$L_0 = v\,\Delta t \qquad \text{(Sam)}. \tag{37-14}$$

This time interval Δt is not a proper time interval because the two events that define it (Sally passes the back of the platform and Sally passes the front of the platform) occur at two different places, and therefore Sam must use two synchronized clocks to measure the time interval Δt.

For Sally, however, the platform is moving past her. She finds that the two events measured by Sam occur *at the same place* in her reference frame. She can time them with a single stationary clock, and so the interval Δt_0 that she measures is a proper time interval. To her, the length L of the platform is given by

$$L = v\,\Delta t_0 \qquad \text{(Sally)}. \tag{37-15}$$

If we divide Eq. 37-15 by Eq. 37-14 and apply Eq. 37-9, the time dilation equation, we have

$$\frac{L}{L_0} = \frac{v\,\Delta t_0}{v\,\Delta t} = \frac{1}{\gamma},$$

or

$$L = \frac{L_0}{\gamma}, \tag{37-16}$$

which is Eq. 37-13, the length contraction equation.

Sample Problem 37-3

In Fig. 37-10, Sally (at point A) and Sam's spaceship (of proper length $L_0 = 230$ m) pass each other with constant relative speed v. Sally measures a time interval of 3.57 μs for the ship to pass her (from the passage of point B to the passage of point C). In terms of c, what is the relative speed v between Sally and the ship?

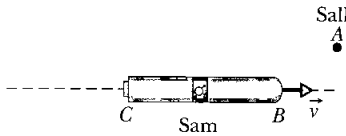


FIG. 37-10 Sally, at point A, measures the time a spaceship takes to pass her.

Conceptualize/Classify: Let's assume that speed v is near c. Then,

1. This problem involves measurements made from two (inertial) reference frames, one attached to Sally and the other attached to Sam and his spaceship.
2. This problem also involves two events: the first is the passage of point B past Sally and the second is the passage of point C past her.
3. From either reference frame, the other reference frame passes at speed v and moves a certain distance in the time interval between the two events:

$$v = \frac{\text{distance}}{\text{time interval}}. \tag{37-17}$$

Because speed v is assumed to be near the speed of light, we must be careful that the distance and the time interval in Eq. 37-17 are measured in the *same* reference frame.

Compute: We are free to use either frame for the measurements. Because we know that the time interval Δt between the two events measured from Sally's frame is 3.57 μs, let us also use the distance L between the two events measured from her frame. Equation 37-17 then becomes

$$v = \frac{L}{\Delta t}. \tag{37-18}$$

We do not know L, but we can relate it to the given L_0: The distance between the two events as measured from Sam's frame is the ship's proper length L_0. Thus, the distance L measured from Sally's frame must be less than L_0, as given by Eq. 37-13 ($L = L_0/\gamma$) for length contraction. Substituting L_0/γ for L in Eq. 37-18 and then substituting Eq. 37-8 for γ, we find

$$v = \frac{L_0/\gamma}{\Delta t} = \frac{L_0\sqrt{(1-(v/c)^2}}{\Delta t}.$$

Solving this equation for v (notice that it is on the left and also buried in the Lorentz factor) leads us to

$$\begin{aligned} v &= \frac{L_0 c}{\sqrt{(c\,\Delta t)^2 + L_0^2}} \\ &= \frac{(230\text{ m})c}{\sqrt{(299\,792\,458\text{ m/s})^2(3.57\times10^{-6}\text{ s})^2 + (230\text{ m})^2}} \\ &= 0.210c. \qquad \text{(Answer)} \end{aligned}$$

Conclude: Thus, the relative speed between Sally and the ship is 21% of the speed of light. Note that only the relative motion of Sally and Sam matters here; whether either is stationary relative to, say, a space station is irrelevant. In Fig. 37-8 we took Sally to be stationary, but we could instead have taken the ship to be stationary, with Sally moving to the left past it. Nothing would have changed in our result.

Sample Problem 37-4

Caught by surprise near a supernova, you race away from the explosion in your spaceship, hoping to outrun the high-speed material ejected toward you. Your Lorentz factor γ relative to the inertial reference frame of the local stars is 22.4.

(a) To reach a safe distance, you figure you need to cover 9.00×10^{16} m as measured in the reference frame of the local stars. How long will the flight take, as measured in that frame?

Conceptualize/Classify: Just as we did in Chapter 2, Vol. I, we can find the time required to travel a certain distance at constant speed with the definition of speed v:

$$\text{speed} = \frac{\text{distance}}{\text{time interval}}. \tag{37-19}$$

From Fig. 37-8, we see that because your Lorentz factor γ relative to the stars is 22.4 (large), your relative speed v is almost c—so close that we can approximate it as c. Then for speed $v \approx c$, we must be careful that the distance and the time interval in Eq. 37-19 are measured in the *same* reference frame.

Compute: The given distance (9.00×10^{16} m) for the length of your travel path is measured in the reference frame of the stars, and the requested time interval Δt is to be measured in that same frame. Thus, we can write

$$\begin{pmatrix}\text{time interval}\\ \text{relative to stars}\end{pmatrix} = \frac{\text{distance relative to stars}}{c},$$

Then substituting the given distance, we find that

$$\begin{pmatrix}\text{time interval}\\ \text{relative to stars}\end{pmatrix} = \frac{9.00\times10^{16}\text{ m}}{299\,792\,458\text{ m/s}}$$
$$= 3.00\times10^{8}\text{ s} = 9.51\text{ y}. \quad \text{(Answer)}$$

(b) How long does that trip take according to you (in your reference frame)?

Conceptualize/Classify:

1. We now want the time interval measured in a different reference frame—namely, yours. Thus, we need to transform the data given in the reference frame of the stars to your frame.
2. The given path length of 9.00×10^{16} m, measured in the reference frame of the stars, is a proper length L_0, because the two ends of the path are at rest in that frame. As observed from your reference frame, the stars' reference frame and those two ends of the path race past you at a relative speed of $v \approx c$.
3. You measure a contracted length L_0/γ for the path, not the proper length L_0.

Compute: We can now rewrite Eq. 37-19 as

$$\begin{pmatrix}\text{time interval}\\ \text{relative to stars}\end{pmatrix} = \frac{\text{distance relative to you}}{c} = \frac{L_0/\gamma}{c}.$$

Substituting known data, we find

$$\begin{pmatrix}\text{time interval}\\ \text{relative to stars}\end{pmatrix} = \frac{(9.00\times10^{16}\text{ m})/22.4}{299\,792\,458\text{ m/s}}$$
$$= 1.340\times10^{7}\text{ s} = 0.425\text{ y}. \quad \text{(Answer)}$$

Conclude: In part (a) we found that the flight takes 9.51 y in the reference frame of the stars. However, here we find that it takes only 0.425 y in your frame, due to the relative motion and the resulting contracted length of the path.

37-7 | The Lorentz Transformation

Figure 37-11 shows inertial reference frame S' moving with speed v relative to frame S, in the common positive direction of their horizontal axes (marked x and x'). An observer in S reports spacetime coordinates x, y, z, t for an event, and an observer in S' reports x', y', z', t' for the same event. How are these sets of numbers related?

We claim at once (although it requires proof) that the y and z coordinates, which are perpendicular to the motion, are not affected by the motion; that is, $y = y'$ and $z = z'$. Our interest then reduces to the relation between x and x' and that between t and t'.

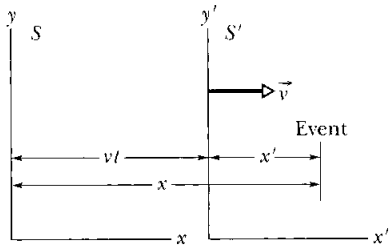


FIG. 37-11 Two intertial reference frames: frams S' has velocity $\vec{v}$ relative to frame S.

The Galilean Transformation Equations

Prior to Einstein's publication of his special theory of relativity, the four coordinates of interest were assumed to be related by the *Galilean transformation equations:*

$$x' = x - vt \qquad t' = t \qquad \text{(Galilean transformation equations; approximately valid at low speeds).} \qquad (37\text{-}20)$$

(These equations are written with the assumption that $t = t' = 0$ when the origins of S and S' coincide.) You can verify the first equation with Fig. 37-11. The second equation effectively claims that time passes at the same rate for observers in both reference frames. That would have been so obviously true to a scientist prior to Einstein that it would not even have been mentioned. When speed v is small compared to c, Eqs. 37-20 generally work well.

The Lorentz Transformation Equations

We state without proof that the correct transformation equations, which remain valid for all speeds up to the speed of light, can be derived from the postulates of

relativity. The results, called the **Lorentz transformation equations*** or sometimes (more loosely) just the Lorentz transformations, are

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx/c^2) \end{aligned} \qquad \begin{array}{l}\text{(Lorentz transformation equations;}\\ \text{valid at all physically possible speeds).}\end{array} \qquad (37\text{-}21)$$

(The equations are written with the assumption that $t = t' = 0$ when the origins of S and S' coincide.) Note that the spatial values x and the temporal values t are bound together in the first and last equations. This entanglement of space and time was a prime message of Einstein's theory, a message that was long rejected by many of his contemporaries.

It is a formal requirement of relativistic equations that they should reduce to familiar classical equations if we let c approach infinity. That is, if the speed of light were infinitely great, *all* finite speeds would be "low" and classical equations would never fail. If we let $c \to \infty$ in Eqs. 37-21, $\gamma \to 1$ and these equations reduce—as we expect—to the Galilean equations (Eqs. 37-20). You should check this.

Equations 37-21 are written in a form that is useful if we are given x and t and wish to find x' and t'. We may wish to go the other way, however. In that case we simply solve Eqs. 37-21 for x and t, obtaining

$$x = \gamma(x' + vt') \quad \text{and} \quad t = \gamma(t' + vx'/c^2). \qquad (37\text{-}22)$$

Comparison shows that, starting from either Eqs. 37-21 or Eqs. 37-22, you can find the other set by interchanging primed and unprimed quantities and reversing the sign of the relative velocity v.

Equations 37-21 and 37-22 relate the coordinates of a single event as seen by two observers. Sometimes we want to know not the coordinates of a single event but the differences between coordinates for a pair of events. That is, if we label our events 1 and 2, we may want to relate

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta t = t_2 - t_1,$$

as measured by an observer in S, and

$$\Delta x' = x_2' - x_1' \quad \text{and} \quad \Delta t' = t_2' = t_1',$$

as measured by an observer in S'.

Table 37-2 displays the Lorentz equations in difference form, suitable for analyzing pairs of events. The equations in the table were derived by simply substituting differences (such as Δx and $\Delta x'$) for the four variables in Eqs. 37-21 and 37-22.

TABLE 37-2

The Lorentz Transformation Equations for Pairs of Events

1. $\Delta x = \gamma(\Delta x' + v\,\Delta t')$	1′. $\Delta x' = \gamma(\Delta x - v\,\Delta t)$
2. $\Delta t = \gamma(\Delta t' + v\,\Delta x'/c^2)$	2′. $\Delta t' = \gamma(\Delta t - v\,\Delta x/c^2)$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

Frame S' moves at velocity v relative to frame S.

* You may wonder why we do not call these the *Einstein transformation equations* (and why not the *Einstein factor for* γ). H. A. Lorentz actually derived these equations before Einstein did, but as the great Dutch physicist graciously conceded, he did not take the further bold step of interpreting these equations as describing the true nature of space and time. It is this interpretation, first made by Einstein, that is at the heart of relativity.

Be careful: When substituting values for these differences, you must be consistent and not mix the values for the first event with those for the second event. Also, if, say, Δx is a negative quantity, you must be certain to include the minus sign in a substitution.

CHECKPOINT 2 In Fig. 37-11, frame S' has velocity $0.90c$ relative to frame S. An observer in frame S' measures two events as occurring at the following spacetime coordinates: event Yellow at (5.0 m, 20 ns) and event Green at (−2.0 m, 45 ns). An observer in frame S wants to find the temporal separation $\Delta t_{GY} = t_G - t_Y$ between the events. (a) Which equation in Table 37-2 should be used? (b) Should $+0.90c$ or $-0.90c$ be substituted for v in the parentheses on the equation's right side and in the Lorentz factor γ? What value should be substituted into the (c) first and (d) second term in the parentheses?

37-8 | Some Consequences of the Lorentz Equations

Here we use the equations of Table 37-2 to affirm some of the conclusions that we reached earlier by arguments based directly on the postulates.

Simultaneity

Consider Eq. 2 of Table 37-2,

$$\Delta t = \gamma\left(\Delta t' + \frac{v\,\Delta x'}{c^2}\right). \tag{37-23}$$

If two events occur at different places in reference frame S' of Fig. 37-11, then $\Delta x'$ in this equation is not zero. It follows that even if the events are simultaneous in S' (thus $\Delta t' = 0$), they will not be simultaneous in frame S. (This is in accord with our conclusion in Section 37-4.) The time interval between the events in S will be

$$\Delta t = \gamma\frac{v\,\Delta x'}{c^2} \qquad \text{(simultaneous events in } S'\text{)}.$$

Time Dilation

Suppose now that two events occur at the same place in S' (thus $\Delta x' = 0$) but at different times (thus $\Delta t' \neq 0$). Equation 37-23 then reduces to

$$\Delta t = \gamma\,\Delta t' \qquad \text{(events in same place in } S'\text{)}. \tag{37-24}$$

This confirms time dilation between frames S and S'. Moreover, because the two events occur at the same place in S', the time interval $\Delta t'$ between them can be measured with a single clock, located at that place. Under these conditions, the measured interval is a proper time interval, and we can label it Δt_0 as we have previously labeled proper times. Thus, with that label Eq. 37-24 becomes

$$\Delta t = \gamma\,\Delta t_0 \qquad \text{(time dilation)},$$

which is exactly Eq. 37-9, the time dilation equation.

Length Contraction

Consider Eq. 1′ of Table 37-2,

$$\Delta x' = \gamma\,(\Delta x - v\,\Delta t). \tag{37-25}$$

If a rod lies parallel to the x and x' axes of Fig. 37-9 and is at rest in reference frame S', an observer in S' can measure its length at leisure. One way to do so is by subtracting the coordinates of the end points of the rod. The value of $\Delta x'$ that

is obtained will be the proper length L_0 of the rod because the measurements are made in a frame where the rod is at rest.

Suppose the rod is moving in frame S. This means that Δx can be identified as the length L of the rod in frame S only if the coordinates of the rod's end points are measured *simultaneously*—that is, if $\Delta t = 0$. If we put $\Delta x' = L_0$, $\Delta x = L$, and $\Delta t = 0$ in Eq. 37-25, we find

$$L = \frac{L_0}{\gamma} \qquad \text{(length contraction)}, \tag{37-26}$$

which is exactly Eq. 37-13, the length contraction equation.

Sample Problem 37-5

An Earth starship has been sent to check an Earth outpost on the planet P1407, whose moon houses a battle group of the often hostile Reptulians. As the ship follows a straight-line course first past the planet and then past the moon, it detects a high-energy microwave burst at the Reptulian moon base and then, 1.10 s later, an explosion at the Earth outpost, which is 4.00×10^8 m from the Reptulian base as measured from the ship's reference frame. The Reptulians have obviously attacked the Earth outpost, and so the starship begins to prepare for a confrontation with them.

(a) The speed of the ship relative to the planet and its moon is $0.980c$. What are the distance and time interval between the burst and the explosion as measured in the planet–moon inertial frame (and thus according to the occupants of the stations)?

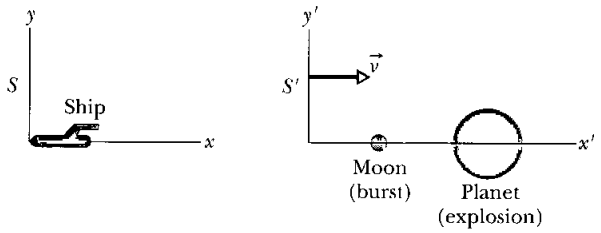


FIG. 37-12 A planet and its moon in reference frame S' move rightward with speed v relative to a starship in reference frame S.

Conceptualize/Classify:

1. This problem involves measurements made from two reference frames, the planet–moon frame and the starship frame.
2. This problem involves two events: the burst and the explosion.
3. We need to transform the given data about the time and distance between the two events as measured in the starship frame to the corresponding data as measured in the planet–moon frame.

Compute: Starship frame: Before we get to the transformation, we need to carefully choose our notation. We begin with a sketch of the situation as shown in Fig. 37-12. There, we have chosen the ship's frame S to be stationary and the planet–moon frame S' to be moving with positive velocity (rightward). (This is an arbitrary choice; we could, instead, have chosen the planet–moon frame to be stationary. Then we would redraw $\vec{v}$ in Fig. 37-12 as being attached to the S frame and indicating leftward motion; v would then be a negative quantity. The results would be the same.) Let subscripts e and b represent the explosion and burst, respectively. Then the given data, all in the unprimed (starship) reference frame, are

$$\Delta x = x_e - x_b = +\,4.00 \times 10^8 \text{ m}$$

and

$$\Delta t = t_e - t_b = +1.10 \text{ s}.$$

Here, Δx is a positive quantity because in Fig. 37-12, the coordinate x_e for the explosion is greater than the coordinate x_b for the burst; Δt is also a positive quantity because the time t_e of the explosion is greater (later) than the time t_b of the burst.

Planet–moon frame: We seek $\Delta x'$ and $\Delta t'$, which we shall get by transforming the given S-frame data to the planet–moon frame S'. Because we are considering a pair of events, we choose transformation equations from Table 37-2—namely, Eqs. 1′ and 2′:

$$\Delta x' = \gamma(\Delta x - v\,\Delta t) \tag{37-27}$$

and

$$\Delta t' = \gamma\left(\Delta t - \frac{v\,\Delta x}{c^2}\right). \tag{37-28}$$

Here, $v = +0.980c$ and the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(+0.980c/c)^2}} = 5.0252.$$

Equation 37-27 then becomes

$$\Delta x' = (5.0252)[4.00 \times 10^8 \text{ m} - (+0.980\,c)(1.10 \text{ s})]$$
$$= 3.86 \times 10^8 \text{ m}, \qquad \text{(Answer)}$$

and Eq. 37-28 becomes

$$\Delta t' = (5.0252)\left[(1.10\,\text{s}) - \frac{(+0.980c)(4.00\times10^8\,\text{m})}{c^2}\right]$$
$$= -1.04\,\text{s}. \qquad \text{(Answer)}$$

(b) What is the meaning of the minus sign in the value for $\Delta t'$?

Compute: We must be consistent with the notation we set up in part (a). Recall how we originally defined the time interval between burst and explosion: $\Delta t = t_e - t_b = +1.10$ s. To be consistent with that choice of notation, our definition of $\Delta t'$ must be $t'_e - t'_b$; thus, we have found that

$$\Delta t' = t'_e - t'_b = -1.04\,\text{s}.$$

The minus sign here tells us that $t'_b > t'_e$; that is, in the planet–moon reference frame, the burst occurred 1.04 s *after* the explosion, not 1.10 s *before* the explosion as detected in the ship frame.

(c) Did the burst cause the explosion, or vice versa?

Conceptualize/Classify: The sequence of events measured in the planet–moon reference frame is the reverse of that measured in the ship frame. In either situation, if there is a causal relationship between the two events, information must travel from the location of one event to the location of the other to cause it.

Compute: Let us check the required speed of the information. In the ship frame, this speed is

$$v_{\text{info}} = \frac{\Delta x}{\Delta t} = \frac{4.00\times10^8\,\text{m}}{1.10\,\text{s}} = 3.64\times10^8\,\text{m/s},$$

but that speed is impossible because it exceeds c. In the planet–moon frame, the speed comes out to be 3.70×10^8 m/s, also impossible. Therefore, neither event could possibly have caused the other event; that is, they are *unrelated* events. Thus, the starship should not confront the Reptulians.

37-9 | Electricity and Magnetism

The theoretical implications of the relation between electricity and magnetism led Einstein to discover special relativity. The relation between electric and magnetic field can be explained taking into account the theory of relativity.

Though the full explanation of how relativity links electricity and magnetism is mathematically complex, some aspects of it are quite easy to understand. Let us consider an example: the origin of magnetic force between two parallel currents. Note that that the electric charge is relativistically invariant which implies that a charge whose magnitude is Q in one frame of reference is also Q in all other frames.

Consider two ideal conductors (Fig. 37-13*a*) having equally spaced equal numbers of positive and negative charges at rest. As the conductors are electrically neutral, there is no force between them.

Figure 37-13 explains how two parallel current carrying wires attract each other. We assume that the current flows because of the flow of positive as well as negative charges in the opposite direction with same velocity v. In Fig. 37-13*a*, since the charges are at rest and being equal in number do not exert any net force on each other. In Fig. 37-13*b*, since the charges are moving they exert magnetic force on each other. Figure 37-13*c* explains the reason why. As there is a negative charge in wire I, the negative charges in the other wire are at rest; however, the positive charges move with a velocity of $2v$. This results in Lorentz contraction between them as well as a net attractive force, because to the electron the number of positive charges in the other wire seems more. The same phenomena from the perspective of a proton in the wire is explained in Fig. 37-13*c*. Thus the magnetic force in one frame reduces to electric force in other frame.

Figure 37-13*b* illustrates the same conductors when they carry currents i_{I} and i_{II} in the same direction. The positive and negative charges move to the right and the left, respectively, both at the same speed v as seen from the laboratory frame of reference. (Actual currents in metal consist of flows of negative electrons only, of course, but the electrically equivalent model here is easier to analyze and the results are the same.) Owing to the fact that the charges are moving, their

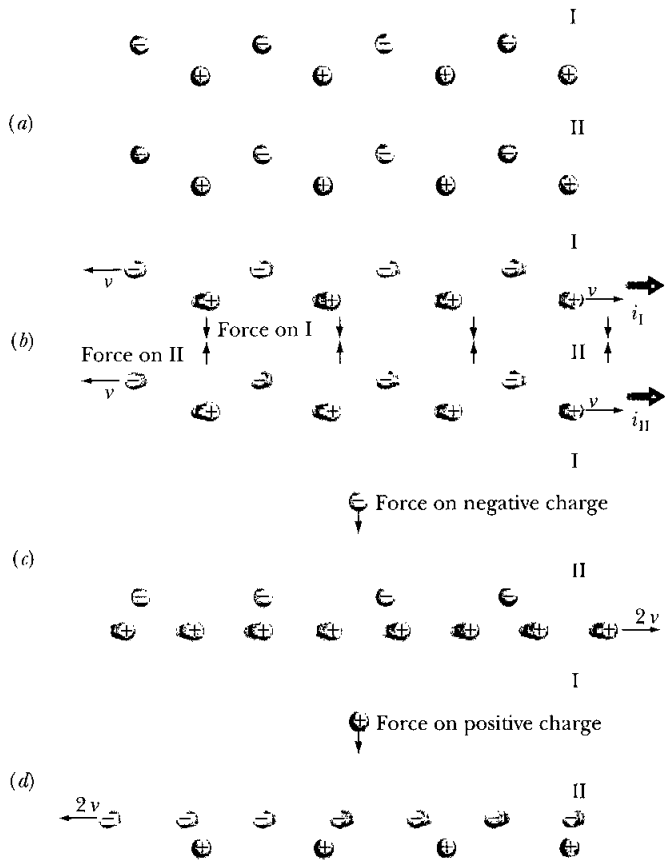


FIG. 37-13 Magnetic attraction between two parallel current-carrying wires. (a) Two neutral conductors do not exert any force on each other. (b) If the conductors carry current, the distance between the moving charges undergoes contraction as seen from lab frame of reference. (c) If the currents in both the wires are the same, the electrons have the same velocity. As seen by electrons of the first wire, the electrons of the other wire are at rest, but the protons are moving. This relative motion produces a contraction. So this electron will see protons more closely spaced, and thus feel a net attractive force. (d) Similar phenomena occurs as seen from the frame of a positive charge. The distances are exaggerated for clarity.

spacing is smaller than before by the factor $\sqrt{1-v^2/c^2}$. Since v is the same for both sets of charges, their spacings shrink by the same amounts, and both conductors remain neutral to an observer in the laboratory. However, the conductors now attract each other. Let us see why.

Consider conductor II from the frame of reference of one of the negative charges in conductor I. As can be seen in Fig. 37-13*c* because the negative charges in conductor II appear at rest in this frame, their spacing is not contracted. On the other hand, the positive charges in II have a velocity $2v$, and their spacing is accordingly contracted to a greater extent than they are in the laboratory form. Conductor II, therefore, appears to have a net positive charge, and an attractive force acts on the negative charge in conductor I.

Look at conductor II from the frame of reference of one of the positive charges in conductor I. The positive charges in conductor II are now at rest, and the negative charges there move to the left with speed $2v$. Hence, the negative charges are closer together than the positive ones, as in Fig. 37-13*d*, and the entire conductor appears negatively charged. An attractive force, therefore, acts on the positive charges in conductor I.

To each charge, the force on it is an "ordinary" electric force that arises because the charges of opposite sign in the other conductor are closer together

than the charges of the same sign, in which case the other conductor appears to have a net charge. From the laboratory frame, the situation is less straightforward. Both conductors are electrically neutral in this frame, and it is natural to explain their mutual attraction by attributing it to a special "magnetic" interaction between the currents.

Although it is convenient to think of magnetic forces as being different from electric ones, they both result from a single electromagnetic interaction that occurs between charged particles.

Clearly a current-carrying conductor that is electrically neutral in one frame of reference might not be neutral in another frame. How can this observation be reconciled with charge invariance? Answer: We must consider the entire circuit of which the conductor is a part. As the circuit must be closed for a current to run in it, for every current element in one direction that a moving observer finds to have, say a positive charge, there must be another current element in the opposite direction which the same observer finds to have, here a negative charge. Thus, the circuit as a whole appears electrically neutral to all observers.

The discussion here considered only a particular magnetic effect. All other magnetic phenomena can also be interpreted on the basis of Coulomb's law, charge invariance, and special relativity, although the analysis is usually much more complicated.

37-10 | The Relativity of Velocities

Here we wish to use the Lorentz transformation equations to compare the velocities that two observers in different inertial reference frames S and S' would measure for the same moving particle. Let S' move with velocity v relative to S.

Suppose that the particle, moving with constant velocity parallel to the x and x' axes in Fig. 37-14, sends out two signals as it moves. Each observer measures the space interval and the time interval between these two events. These four measurements are related by Eqs. 1 and 2 of Table 37-2,

$$\Delta x = \gamma(\Delta x' + v\,\Delta t')$$

and

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right).$$

If we divide the first of these equations by the second, we find

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + v\Delta x'/c^2}.$$

Dividing the numerator and denominator of the right side by $\Delta t'$, we find

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x'/\Delta t' + v}{1 + v(\Delta x'/\Delta t')/c^2}.$$

However, in the differential limit, $\Delta x/\Delta t$ is u, the velocity of the particle as measured in S, and $\Delta x'/\Delta t'$ is u', the velocity of the particle as measured in S'. Then we have, finally,

$$u = \frac{u' + v}{1 + u'v/c^2} \qquad \text{(relativistic velocity transformation)} \qquad (37\text{-}29)$$

as the relativistic velocity transformation equation. This equation reduces to the classical, or Galilean, velocity transformation equation,

$$u = u' + v \qquad \text{(classical velocity transformation)}, \qquad (37\text{-}30)$$

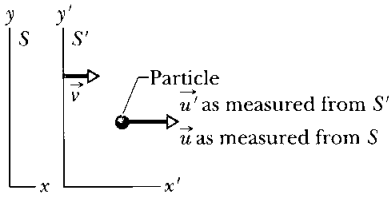


FIG. 37-14 Reference frame S' moves with velocity $\vec{v}$ relative to frame S. A particle has velocity $\vec{u}'$ relative to reference frame S' and velocity $\vec{u}$ relative to reference frame S.

when we apply the formal test of letting $c \to \infty$. In other words, Eq. 37-29 is correct for all physically possible speeds, but Eq. 37-30 is approximately correct for speeds much less than c.

37-11 | Doppler Effect for Light

In Section 17-9, Vol. I, we discussed the Doppler effect (a shift in detected frequency) for sound waves traveling in air. For such waves, the Doppler effect depends on two velocities—namely, the velocities of the source and detector with respect to the air. Air is the medium that transmits the waves.

That is not the situation with light waves, for they (and other electromagnetic waves) require no medium, being able to travel even through vacuum. The Doppler effect for light waves depends on only one velocity, the relative velocity $\vec{v}$ between source and detector, as measured from the reference frame of either. Let f_0 represent the **proper frequency** of the source—that is, the frequency that is measured by an observer in the rest frame of the source. Let f represent the frequency detected by an observer moving with velocity $\vec{v}$ relative to that rest frame. Then, when the direction of $\vec{v}$ is directly away from the source,

$$f = f_0 \sqrt{\frac{1-\beta}{1+\beta}} \qquad \text{(source and detector separating)}, \tag{37-31}$$

where $\beta = v/c$. When the direction of $\vec{v}$ is directly toward the source, we must change the signs in front of both β symbols in Eq. 37-31.

Low-Speed Doppler Effect

For low speeds ($\beta \ll 1$), Eq. 37-31 can be expanded in a power series in β and approximated as

$$f = f_0(1-\beta+\tfrac{1}{2}\beta^2) \qquad \text{(source and detector separating, } \beta \ll 1). \tag{37-32}$$

The corresponding low-speed equation for the Doppler effect with sound waves (or any waves except light waves) has the same first two terms but a different coefficient in the third term. Thus, the relativistic effect for low-speed light sources and detectors shows up only with the β^2 term.

A police radar unit employs the Doppler effect with microwaves to measure the speed v of a car. A source in the radar unit emits a microwave beam at a certain (proper) frequency f_0 along the road. A car that is moving toward the unit intercepts that beam but at a frequency that is shifted upward by the Doppler effect due to the car's motion toward the radar unit. The car reflects the beam back toward the radar unit. Because the car is moving toward the radar unit, the detector in the unit intercepts a reflected beam that is further shifted up in frequency. The unit compares that detected frequency with f_0 and computes the speed v of the car.

Astronomical Doppler Effect

In astronomical observations of stars, galaxies, and other sources of light, we can determine how fast the sources are moving, either directly away from us or directly toward us, by measuring the *Doppler shift* of the light that reaches us. If a certain star were at rest relative to us, we would detect light from it with a certain proper frequency f_0. However, if the star is moving either directly away from us or directly toward us, the light we detect has a frequency f that is shifted from f_0 by

the Doppler effect. This Doppler shift is due only to the *radial* motion of the star (its motion directly toward us or away from us), and the speed we can determine by measuring this Doppler shift is only the *radial speed* v of the star—that is, only the radial component of the star's velocity relative to us.

Suppose a star (or any other light source) moves away from us with a radial speed v that is low enough (β is small enough) for us to neglect the β^2 term in Eq. 37-32. Then we have

$$f = f_0(1 - \beta). \tag{37-33}$$

Because astronomical measurements involving light are usually done in wavelengths rather than frequencies, let us replace f with c/λ and f_0 with c/λ_0, where λ is the measured wavelength and λ_0 is the **proper wavelength** (the wavelength associated with f_0). We then have

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}(1 - \beta),$$

or

$$\lambda = \lambda_0(1 - \beta)^{-1}. \tag{37-34}$$

Because we assume β is small, we can expand $(1 - \beta)^{-1}$ in a power series. Doing so and retaining only the first power of β, we have

$$\lambda = \lambda_0(1 + \beta),$$

or

$$\beta = \frac{\lambda - \lambda_0}{\lambda_0}. \tag{37-35}$$

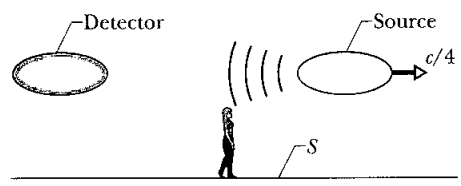
Replacing β with v/c and $\lambda - \lambda_0$ with $|\Delta\lambda|$ leads to

$$v = \frac{|\Delta\lambda|}{\lambda_0} c \qquad \text{(radial speed of light source, } v \ll c\text{)}. \tag{37-36}$$

The difference $\Delta\lambda$ is the *wavelength Doppler shift* of the light source. We enclose it with an absolute sign so that we always have a magnitude of the shift.

Equation 37-36 is an approximation that can be applied only when $v \ll c$. Under that condition, Eq. 37-36 can be applied whether the light source is moving toward or away from us. If it is moving away from us, then λ is longer than λ_0, $\Delta\lambda$ is positive, and the Doppler shift is called a *red shift*. (The term *red* does not mean the detected light is red or even visible. It merely serves as a memory device because red is at the *long* wavelength end of the visible spectrum. Thus λ is longer than λ_0.) If the light source is moving toward us, then λ is shorter than λ_0, $\Delta\lambda$ is negative, and the Doppler shift is called a *blue shift*.

CHECKPOINT 3 The figure shows a source that emits light of proper frequency f_0 while moving directly toward the right with speed $c/4$ as measured from reference frame S. The figure also shows a light detector, which measures a frequency $f > f_0$ for the emitted light. (a) Is the detector moving toward the left or the right? (b) Is the speed of the detector as measured from reference frame S more than $c/4$, less than $c/4$, or equal to $c/4$?



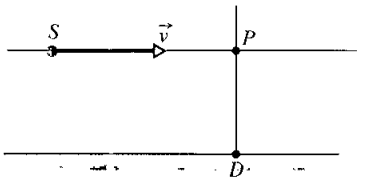


FIG. 37-15 A light source S travels with velocity $\vec{v}$ past a detector at D. The special theory of relativity predicts a transverse Doppler effect as the source passes through point P, where the direction of travel is perpendicular to the line extending through D. Classical theory predicts no such effect.

Transverse Doppler Effect

So far, we have discussed the Doppler effect, here and in the chapter on Longitudinal Waves (Chapter 17, Vol. I), only for situations in which the source and the detector move either directly toward or directly away from each other. Figure 37-15 shows a different arrangement, in which a source S moves past a detector D. When S reaches point P, the velocity of S is perpendicular to the line joining P and D, and at that instant S is moving neither toward nor away from D. If the source is emitting sound waves of frequency f_0, D detects that frequency (with no Doppler effect) when it intercepts the waves that were emitted at point P. However, if the source is emitting light waves, there is still a Doppler effect, called the **transverse Doppler effect**. In this situation, the detected frequency of the light emitted when the source is at point P is

$$f = f_0\sqrt{1-\beta^2} \qquad \text{(transverse Doppler effect).} \tag{37-37}$$

For low speeds ($\beta \ll 1$), Eq. 37-37 can be expanded in a power series in β and approximated as

$$f = f_0(1-\tfrac{1}{2}\beta^2) \qquad \text{(low speeds).} \tag{37-38}$$

Here the first term is what we would expect for sound waves, and again the relativistic effect for low-speed light sources and detectors appears with the β^2 term.

In principle, a police radar unit can determine the speed of a car even when the path of the radar beam is perpendicular (transverse) to the path of the car. However, Eq. 37-38 tells us that because β is small even for a fast car, the relativistic term $\beta^2/2$ in the transverse Doppler effect is extremely small. Thus, $f \approx f_0$ and the radar unit computes a speed of zero. For this reason, police officers always try to direct the radar beam along the car's path to get a Doppler shift that gives the car's actual speed. Any deviation from that alignment works in favor of the motorist because it reduces the measured speed.

The transverse Doppler effect is really another test of time dilation. If we rewrite Eq. 37-37 in terms of the period T of oscillation of the emitted light wave instead of the frequency, we have, because $T = 1/f$,

$$T = \frac{T_0}{\sqrt{1-\beta^2}} = \gamma T_0, \tag{37-39}$$

in which T_0 ($= 1/f_0$) is the **proper period** of the source. As comparison with Eq. 37-9 shows, Eq. 37-39 is simply the time dilation formula because a period is a time interval.

The NAVSTAR Navigation System

Each NAVSTAR satellite in the Global Positioning System (GPS) continuously signals its location by broadcasting radio signals at a set frequency controlled by precise atomic clocks. When the signal is sensed by a GPS detector on, say, a commercial aircraft, the frequency has been Doppler shifted because of the relative motion between the satellite and the detector. By detecting the signals from several NAVSTAR satellites simultaneously, the detector can determine the direction to any one of them and the direction of the satellite velocity. From the Doppler shift of the signal, the detector then determines the speed of the aircraft.

Although the relativistic contribution to the Doppler shift is extremely slight (roughly 4.5×10^{-10} of the shift is due to the relativity of this chapter), engineers must take the contribution into account to keep the NAVSTAR system accurate. They must also take into account the effects of Einstein's general theory of relativity because the gravitational acceleration at the altitude of a satellite is different from that at the altitude of a ground-based detector. Einstein's special and general theories of relativity were considered bizarre for a long time after they were published. Now they are indispensable in global positioning and long-range navigation.

Sample Problem 37-6

Figure 37-16*a* shows curves of intensity versus wavelength for light reaching us from interstellar gas on two opposite sides of galaxy M87 (Fig. 37-16*b*). One curve peaks at 499.8 nm, the other at 501.6 nm. The gas orbits the core of the galaxy at a radius $r = 100$ light-years, moving toward us on one side of the core and moving away from us on the opposite side.

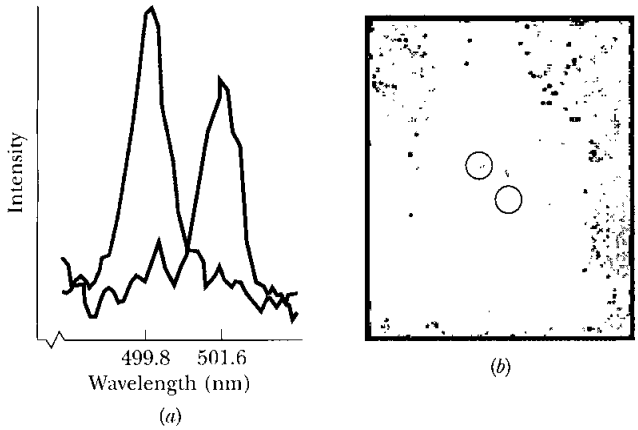


FIG. 37-16 (*a*) Plots of intensity versus wavelength for light emitted by gas on opposite sides of galaxy M87 and detected on Earth. (*b*) The central region of M87. The circles indicate the locations of the gas whose intensity is given in (*a*). The core of M87 is halfway between the circles. *(Courtesy NASA)*

(a) Which curve corresponds to the gas moving toward us? What is the speed of the gas relative to us (and relative to the galaxy's core)?

Conceptualize/Classify:

1. If the gas were not moving, the light from it would be detected at a certain wavelength.
2. The motion of the gas changes the detected wavelength via the Doppler effect, increasing the wavelength for the gas moving away from us and decreasing it for the gas moving toward us.

Thus, the curve peaking at 501.6 nm corresponds to motion away from us, and that peaking at 499.8 nm corresponds to motion toward us.

Compute: Let us assume that the increase and the decrease in wavelength due to the motion of the gas are equal in magnitude. Then the unshifted wavelength, which we shall take as the proper wavelength λ_0, must be the average of the two shifted wavelengths:

$$\lambda_0 = \frac{501.6\text{ nm} + 499.8\text{ nm}}{2} = 500.7\text{ nm}.$$

The Doppler shift $\Delta\lambda$ of the light from the gas moving away from us is then

$$\Delta\lambda = |\lambda - \lambda_0| = 501.6\text{ nm} - 500.7\text{ nm} = 0.90\text{ nm}.$$

Substituting this and $\lambda_0 = 500.7$ nm into Eq. 37-36, we find that the speed of the gas is

$$v = \frac{\Delta\lambda}{\lambda_0}c = \frac{0.90\text{ nm}}{500.7\text{ nm}}299\,792\,458\text{ m/s} = 5.39\times10^5\text{ m/s}. \quad \text{(Answer)}$$

(b) The gas orbits the core of the galaxy because the gas experiences a gravitational force due to the mass M of the core. What is that mass in multiples of the Sun's mass M_S ($= 1.99\times10^{30}$ kg)?

Conceptualize/Classify:

1. From Eq. 13-1, Vol. I, the magnitude F of the gravitational force on an orbiting gas element of mass m at orbital radius r is
$$F = \frac{GMm}{r^2}.$$
2. If the gas element orbits the galaxy core in a circle, then the element must have a centripetal acceleration of magnitude $a = v^2/r$, directed toward the core.
3. Newton's second law, written for a radial axis extending from the core to the gas element, tells us that $F = ma$.

Compute: Putting these three ideas together, we have

$$\frac{GMm}{r^2} = m\frac{v^2}{r}.$$

Solving this for M and substituting known data, we find

$$M = \frac{v^2 r}{G} = \frac{(5.39\times10^5\text{ m/s})^2(100\text{ ly})(9.46\times10^{15}\text{ m/ly})}{6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2} = 4.12\times10^{39}\text{ kg} = (2.1\times10^9)\,M_S. \quad \text{(Answer)}$$

This result tells us that a mass equivalent to two billion suns has been compacted into the core of the galaxy, strongly suggesting that a *supermassive black hole* occupies the core.

Conclude: The monster of M87: The M87 black hole is about 1000 times larger than the supermassive black hole lying at the center of our Milky Way galaxy and thus is indeed a monster. It rotates about an axis, and as material (stars and dust) fall into it, the material generates strong electromagnetic forces that shoot electrons away from the black hole, outward along the rotation axis and out of M87. The ejection is so severe that the electrons move at nearly the speed of light. The image that opens this chapter shows the jet that is angled somewhat in our direction; we cannot see the jet in the opposite direction.

37-12 | A New Look at Momentum

Suppose that a number of observers, each in a different inertial reference frame, watch an isolated collision between two particles. In classical mechanics, we have seen that—even though the observers measure different velocities for the colliding particles—they all find that the law of conservation of momentum holds. That is, they find that the total momentum of the system of particles after the collision is the same as it was before the collision.

How is this situation affected by relativity? We find that if we continue to define the momentum $\vec{p}$ of a particle as $m\vec{v}$, the product of its mass and its velocity, total momentum is *not* conserved for the observers in different inertial frames. We have two choices: (1) Give up the law of conservation of momentum or (2) see whether we can refine our definition of momentum in some new way so that the law of conservation of momentum still holds. The correct choice is the second one.

Consider a particle moving with constant speed v in the positive direction of an x axis. Classically, its momentum has magnitude

$$p = mv = m\frac{\Delta x}{\Delta t} \qquad \text{(classical momentum)}, \tag{37-40}$$

in which Δx is the distance it travels in time Δt. To find a relativistic expression for momentum, we start with the new definition

$$p = m\frac{\Delta x}{\Delta t_0}.$$

Here, as before, Δx is the distance traveled by a moving particle as viewed by an observer watching that particle. However, Δt_0 is the time required to travel that distance, measured not by the observer watching the moving particle but by an observer moving with the particle. The particle is at rest with respect to this second observer; thus that measured time is a proper time.

Using the time dilation formula, $\Delta t = \gamma \Delta t_0$ (Eq. 37-9), we can then write

$$p = m\frac{\Delta x}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\frac{\Delta t}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\gamma.$$

However, since $\Delta x/\Delta t$ is just the particle velocity v, we have

$$p = \gamma mv \qquad \text{(momentum)}. \tag{37-41}$$

Note that this differs from the classical definition of Eq. 37-40 only by the Lorentz factor γ. However, that difference is important: Unlike classical momentum, relativistic momentum approaches an infinite value as v approaches c.

We can generalize the definition of Eq. 37-41 to vector form as

$$\vec{p} = \gamma m\vec{v} \qquad \text{(momentum)}. \tag{37-42}$$

This equation gives the correct definition of momentum for all physically possible speeds. For a speed much less than c, it reduces to the classical definition of momentum ($\vec{p} = m\vec{v}$).

37-13 | A New Look at Energy

Mass Energy

The science of chemistry was initially developed with the assumption that in chemical reactions, energy and mass are conserved separately. In 1905, Einstein showed that as a consequence of his theory of special relativity, mass can be

considered to be another form of energy. Thus, the law of conservation of energy is really the law of conservation of mass–energy.

In a *chemical reaction* (a process in which atoms or molecules interact), the amount of mass that is transferred into other forms of energy (or vice versa) is such a tiny fraction of the total mass involved that there is no hope of measuring the mass change with even the best laboratory balances. Mass and energy truly *seem* to be separately conserved. However, in a *nuclear reaction* (in which nuclei or fundamental particles interact), the energy released is often about a million times greater than in a chemical reaction, and the change in mass can easily be measured. Taking mass–energy transfers into account in nuclear reactions became routine long ago.

An object's mass m and the equivalent energy E_0 are related by

$$E_0 = mc^2, \tag{37-43}$$

which, without the subscript 0, is the best-known science equation of all time. This energy that is associated with the mass of an object is called **mass energy** or **rest energy**. The second name suggests that E_0 is an energy that the object has even when it is at rest, simply because it has mass. (If you continue your study of physics beyond this book, you will see more refined discussions of the relation between mass and energy. You might even encounter disagreements about just what that relation is and means.)

Table 37-3 shows the (approximate) mass energy, or rest energy, of a few objects. The mass energy of, say, a U.S. penny is enormous; the equivalent amount of electrical energy would cost well over a million dollars. On the other hand, the entire annual U.S. electrical energy production corresponds to a mass of only a few hundred kilograms of matter (stones, burritos, or anything else).

In practice, SI units are rarely used with Eq. 37-43 because they are too large to be convenient. Masses are usually measured in atomic mass units, where

$$1\ \text{u} = 1.660\,538\,86 \times 10^{-27}\ \text{kg}, \tag{37-44}$$

and energies are usually measured in electron-volts or multiples of it, where

$$1\ \text{eV} = 1.602\,176\,462 \times 10^{-19}\ \text{J}. \tag{37-45}$$

In the units of Eqs. 37-44 and 37-45, the multiplying constant c^2 has the values

$$\begin{aligned} c^2 &= 9.314\,940\,13 \times 10^8\ \text{eV/u} = 9.314\,940\,13 \times 10^5\ \text{keV/u} \\ &= 931.494\,013\ \text{MeV/u}. \end{aligned} \tag{37-46}$$

Total Energy

Equation 37-43 gives, for any object, the mass energy E_0 that is associated with the object's mass m, regardless of whether the object is at rest or moving. If the object is moving, it has additional energy in the form of kinetic energy K. If we assume

TABLE 37-3

The Energy Equivalents of a Few Objects

Object	Mass (kg)	Energy Equivalent	
Electron	$\approx 9.11 \times 10^{-31}$	$\approx 8.19 \times 10^{-14}$ J	(≈ 511 keV)
Proton	$\approx 1.67 \times 10^{-27}$	$\approx 1.50 \times 10^{-10}$ J	(≈ 938 MeV)
Uranium atom	$\approx 3.95 \times 10^{-25}$	$\approx 3.55 \times 10^{-8}$ J	(≈ 225 GeV)
Dust particle	$\approx 1 \times 10^{-13}$	$\approx 1 \times 10^{4}$ J	(≈ 2 kcal)
U.S. penny	$\approx 3.1 \times 10^{-3}$	$\approx 2.8 \times 10^{14}$ J	(≈ 78 GW · h)

that the object's potential energy is zero, then its total energy E is the sum of its mass energy and its kinetic energy:

$$E = E_0 + K = mc^2 + K. \tag{37-47}$$

Although we shall not prove it, the total energy E can also be written as

$$E = \gamma mc^2, \tag{37-48}$$

where γ is the Lorentz factor for the object's motion.

In the previous volume and this one also, we have discussed many examples involving changes in the total energy of a particle or a system of particles. However, we did not include mass energy in the discussions because the changes in mass energy were either zero or small enough to be neglected. The law of conservation of total energy still applies when changes in mass energy are significant. Thus, regardless of what happens to the mass energy, the following statement from Section 8-8, Vol. I, is still true:

☞ The total energy E of an *isolated system* cannot change.

For example, if the total mass energy of two interacting particles in an isolated system decreases, some other type of energy in the system must increase because the total energy cannot change.

In a system undergoing a chemical or nuclear reaction, a change in the total mass energy of the system due to the reaction is often given as a Q value. The Q value for a reaction is obtained from the relation

$$\begin{pmatrix}\text{system's initial}\\ \text{total mass energy}\end{pmatrix} = \begin{pmatrix}\text{system's final}\\ \text{total mass energy}\end{pmatrix} + Q$$

or
$$E_{0i} = E_{0f} + Q. \tag{37-49}$$

Using Eq. 37-43 ($E_0 = mc^2$), we can rewrite this in terms of the initial *total* mass M_i and the final *total* mass M_f as

$$M_i c^2 = M_f c^2 + Q$$

or
$$Q = M_i c^2 - M_f c^2 = -\Delta M\, c^2, \tag{37-50}$$

where the change in mass due to the reaction is $\Delta M = M_f - M_i$.

If a reaction results in the transfer of energy from mass energy to, say, kinetic energy of the reaction products, the system's total mass energy E_0 (and total mass M) decreases and Q is positive. If, instead, a reaction requires that energy be transferred to mass energy, the system's total mass energy E_0 (and its total mass M) increases and Q is negative.

For example, suppose two hydrogen nuclei undergo a *fusion reaction* in which they join together to form a single nucleus and release two particles in the process. The total mass energy (and total mass) of the resultant single nucleus and two released particles is less than the total mass energy (and total mass) of the initial hydrogen nuclei. Thus, the Q of the fusion reaction is positive, and energy is said to be *released* (transferred from mass energy) by the reaction. This release is important to you because the fusion of hydrogen nuclei in the Sun is one part of the process that results in sunshine on Earth and makes life here possible.

Kinetic Energy

In Chapter 7, Vol. I, we defined the kinetic energy K of an object of mass m moving at speed v well below c to be

$$K = \tfrac{1}{2}mv^2. \tag{37-51}$$

However, this classical equation is only an approximation that is good enough when the speed is well below the speed of light.

Let us now find an expression for kinetic energy that is correct for *all* physically possible speeds, including speeds close to c. Solving Eq. 37-47 for K and then substituting for E from Eq. 37-48 lead to

$$\begin{aligned} K &= E - mc^2 = \gamma mc^2 - mc^2 \\ &= mc^2(\gamma - 1) \qquad \text{(kinetic energy)}, \end{aligned} \tag{37-52}$$

where $\gamma [= 1/\sqrt{1-(v/c)^2}]$ is the Lorentz factor for the object's motion.

Figure 37-17 shows plots of the kinetic energy of an electron as calculated with the correct definition (Eq. 37-52) and the classical approximation (Eq. 37-51), both as functions of v/c. Note that on the left side of the graph the two plots coincide; this is the part of the graph—at lower speeds—where we have calculated kinetic energies so far in this book. This part of the graph tells us that we have been justified in calculating kinetic energy with the classical expression of Eq. 37-51. However, on the right side of the graph—at speeds near c—the two plots differ significantly. As v/c approaches 1.0, the plot for the classical definition of kinetic energy increases only moderately while the plot for the correct definition of kinetic energy increases dramatically, approaching an infinite value as v/c approaches 1.0. Thus, when an object's speed v is near c, we *must* use Eq. 37-52 to calculate its kinetic energy.

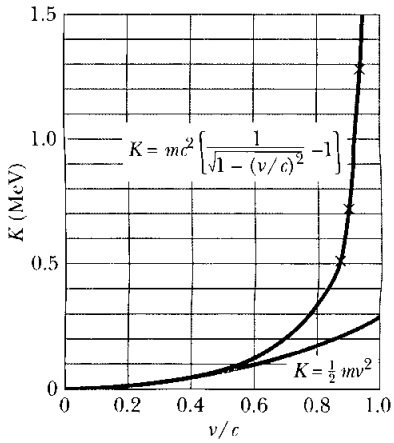


FIG. 37-17 The relativistic (Eq. 37-52) and classical (Eq. 37-51) equations for the kinetic energy of an electron, plotted as a function of v/c, where v is the speed of the electron and c is the speed of light. Note that the two curves blend together at low speeds and diverge widely at high speeds. Experimental data (at the × marks) show that at high speeds the relativistic curve agrees with experiment but the classical curve does not.

Figure 37-17 also tells us something about the work we must do on an object to increase its speed by, say, 1%. The required work W is equal to the resulting change ΔK in the object's kinetic energy. If the change is to occur on the low-speed, left side of Fig. 37-17, the required work might be modest. However, if the change is to occur on the high-speed, right side of Fig. 37-17, the required work could be enormous because the kinetic energy K increases so rapidly there with an increase in speed v. To increase an object's speed to c would require, in principle, an infinite amount of energy; thus, doing so is impossible.

The kinetic energies of electrons, protons, and other particles are often stated with the unit electron-volt or one of its multiples used as an adjective. For example, an electron with a kinetic energy of 20 MeV may be described as a 20 MeV electron.

Momentum and Kinetic Energy

In classical mechanics, the momentum p of a particle is mv and its kinetic energy K is $\frac{1}{2}mv^2$. If we eliminate v between these two expressions, we find a direct relation between momentum and kinetic energy:

$$p^2 = 2Km \qquad \text{(classical)}. \tag{37-53}$$

We can find a similar connection in relativity by eliminating v between the relativistic definition of momentum (Eq. 37-41) and the relativistic definition of kinetic energy (Eq. 37-52). Doing so leads, after some algebra, to

$$(pc)^2 = K^2 + 2Kmc^2. \tag{37-54}$$

With the aid of Eq. 37-47, we can transform Eq. 37-54 into a relation between the momentum p and the total energy E of a particle:

$$E^2 = (pc)^2 + (mc^2)^2. \tag{37-55}$$

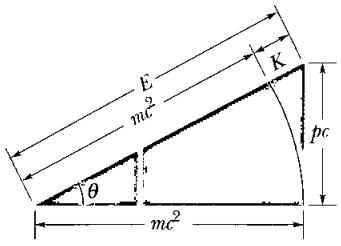


FIG. 37-18 A useful memory diagram for the relativistic relations among the total energy E, the rest energy or mass energy mc^2, the kinetic energy K, and the momentum magnitude p.

The right triangle of Fig. 37-18 can help you keep these useful relations in mind. You can also show that, in that triangle,

$$\sin\theta = \beta \text{ and } \cos\theta = 1/\gamma. \tag{37-56}$$

With Eq. 37-55 we can see that the product pc must have the same unit as energy E; thus, we can express the unit of momentum p as an energy unit divided by c, usually as MeV/c or GeV/c in fundamental particle physics.

✓**CHECKPOINT 4** Are (a) the kinetic energy and (b) the total energy of a 1 GeV electron more than, less than, or equal to those of a 1 GeV proton?

Sample Problem 37-7

(a) What is the total energy E of a 2.53 MeV electron?

Conceptualize/Classify: From Eq. 37-47, the total energy E is the sum of the electron's mass energy (or rest energy) mc^2 and its kinetic energy:

$$E = mc^2 + K. \tag{37-57}$$

Compute: The adjective "2.53 MeV" in the problem statement means that the electron's kinetic energy is 2.53 MeV. To evaluate the electron's mass energy mc^2, we substitute the electron's mass m from Appendix B, obtaining

$$mc^2 = (9.109\times10^{-31}\text{ kg})(299\,792\,458\text{ m/s})^2$$
$$= 8.187\times10^{-14}\text{ J}.$$

Then dividing this result by 1.602×10^{-13} J/MeV gives us 0.511 MeV as the electron's mass energy (confirming the value in Table 37-3). Equation 37-57 then yields

$$E = 0.511\text{ MeV} + 2.53\text{ MeV} = 3.04\text{ MeV}. \text{ (Answer)}$$

(b) What is the magnitude p of the electron's momentum, in the unit MeV/c?

Conceptualize/Classify: We can find p from the total energy E and the mass energy mc^2 via Eq. 37-55,

$$E^2 = (pc)^2 + (mc^2)^2.$$

Compute: Solving for pc gives us

$$pc = \sqrt{E^2 - (mc^2)^2}$$
$$= \sqrt{(3.04\text{ Mev})^2 - (0.511\text{ MeV})^2} = 3.00\text{ MeV}.$$

Finally, dividing both sides by c we find

$$p = 3.00\text{ MeV}/c. \text{ (Answer)}$$

Sample Problem 37-8

The most energetic proton ever detected in the cosmic rays coming to Earth from space had an astounding kinetic energy of 3.0×10^{20} eV (enough energy to warm a teaspoon of water by a few degrees).

(a) What were the proton's Lorentz factor γ and speed v (both relative to the ground-based detector)?

Conceptualize/Classify: (1) The proton's Lorentz factor γ relates its total energy E to its mass energy mc^2 via Eq. 37-48 ($E = \gamma mc^2$). (2) The proton's total energy is the sum of its mass energy mc^2 and its (given) kinetic energy K.

Compute: Putting these ideas together we have

$$\gamma = \frac{E}{mc^2} = \frac{mc^2 + K}{mc^2} = 1 + \frac{K}{mc^2}. \tag{37-58}$$

We can calculate the proton's mass energy mc^2 from its mass given in Appendix B, as we did for the electron in Sample Problem 37-7a. We find that mc^2 is 938 MeV (as listed in Table 37-3). Substituting this and the given kinetic energy into Eq. 37-58, we obtain

$$\gamma = 1 + \frac{3.0\times10^{20}\text{ eV}}{938\times10^{6}\text{ eV}}$$
$$= 3.198\times10^{11} \approx 3.2\times10^{11}. \text{ (Answer)}$$

This computed value for γ is so large that we cannot use the definition of γ (Eq. 37-8) to find v. Try it; your calculator will tell you that β is effectively equal to 1 and thus that v is effectively equal to c. Actually, v is almost c, but we want a more accurate answer, which we can obtain by first solving Eq. 37-8 for $1 - \beta$. To begin we write

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1-\beta)(1+\beta)}} \approx \frac{1}{\sqrt{2(1-\beta)}},$$

where we have used the fact that β is so close to unity that $1+\beta$ is very close to 2. The velocity we seek is contained in the $1-\beta$ term. Solving for $1-\beta$ then yields

$$1-\beta = \frac{1}{2\gamma^2} = \frac{1}{(2)(3.198\times10^{11})^2}$$
$$= 4.9\times10^{-24} \approx 5\times10^{-24}.$$

Thus, $$\beta = 1 - 5\times10^{-24}$$

and, since $v = \beta c$,

$$v \approx 0.999\ 999\ 999\ 999\ 999\ 999\ 995c. \quad \text{(Answer)}$$

(b) Suppose that the proton travels along a diameter (9.8×10^4 ly) of the Milky Way galaxy. Approximately how long does the proton take to travel that diameter as measured from the common reference frame of Earth and the Galaxy?

Compute: We just saw that this *ultrarelativistic* proton is traveling at a speed barely less than c. By the definition of light-year, light takes 1 y to travel a distance of 1 ly, and so light should take 9.8×10^4 y to travel 9.8×10^4 ly, and this proton should take almost the same time. Thus, from our Earth–Milky Way reference frame, the proton's trip takes

$$\Delta t = 9.8\times10^4 \text{ y}. \quad \text{(Answer)}$$

(c) How long does the trip take as measured in the reference frame of the proton?

Conceptualize/Classify:

1. This problem involves measurements made from two (inertial) reference frames: one is the Earth–Milky Way frame and the other is attached to the proton.
2. This problem also involves two events: the first is when the proton passes one end of the diameter along the Galaxy, and the second is when it passes the opposite end.
3. The time interval between those two events as measured in the proton's reference frame is the proper time interval Δt_0 because the events occur at the same location in that frame—namely, at the proton itself.
4. We can find the proper time interval Δt_0 from the time interval Δt measured in the Earth–Milky Way frame by using Eq. 37-9 ($\Delta t = \gamma \Delta t_0$) for time dilation.

Compute: Solving Eq. 37-9 for Δt_0 and substituting γ from (a) and Δt from (b), we find

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{9.8\times10^4 \text{ y}}{3.198\times10^{11}}$$
$$= 3.06\times10^{-7} \text{ y} = 9.7 \text{ s}. \quad \text{(Answer)}$$

Conclude: In our frame, the trip takes 98 000 y. In the proton's frame, it takes 9.7 s! As promised at the start of this chapter, relative motion can alter the rate at which time passes, and we have here an extreme example.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: True rigid objects are not possible according to theory of relativity.

Statement 2: If a moving observer observes any stationary object, it will be seen to be compressed and hence there will be elastic stress in such bodies.

2. Statement 1: If a circle were to traverse parallel to its diameter, it would appear as ellipse to a stationary observer.

Statement 2: The Lorentz contraction occurs only in the direction of motion.

3. An astronaut **A** goes away on a space trip leaving behind his twin **B** on Earth. He returns back after 40 years.

(i) Statement 1: **B** finds that **A** has aged less than 40 years.
Statement 2: **A** was moving relative to **B**.

(ii) Statement 1: **A** finds that **B** has aged less than 40 years.
Statement 2: **B** was moving relative to **A**.

4. Statement 1: An earthworm has eight hearts located at different parts of its body. The eight hearts must all beat at the same time in order to produce effective blood circulation. If an earthworm flies past us in a rocket ship traveling at 3/5th the speed of light, it front hearts will be out of synch with its rear hearts. Nevertheless, the earthworm remains alive.

Statement 2: The worm is shorter (due to length contraction).

5. Statement 1: The second postulate of relativity can be derived from the first postulate.

Statement 2: The value of speed of light can be derived from Maxwell's equations.

MULTIPLE CHOICE QUESTIONS

sec. 37-4 The Relativity of Simultaneity

1. A passenger train travels east at high speed. One passenger is located at the east side of one car, another is located in the west side of that car. In the train's frame, these two passengers glance up at the same time. In the earth's frame,

(a) They glance up simultaneously

(b) The passenger at the east side glances up first

(c) The passenger at the west side glances up first

(d) The passengers glance sideways

2. Two events occur simultaneously on the x axis of reference frame S, one at $x = 0$ and the other at $x = +1$. According to an observer moving in the positive x direction

(a) The event at $x = +1$ occurs first

(b) The event at $x = 0$ occurs first

(c) Either event might occur first, depending on the value of (a) and the observer's speed

(d) The events are simultaneous

3. Two events occur simultaneously at separated points on the y axis of reference frame S. According to an observer moving in the positive x direction

(a) The event with the greater y coordinate occurs first

(b) The event with the greater y coordinate occurs last

(c) Either event might occur first, depending on the observer's speed

(d) The events are simultaneous

4. Two events occur on the x axis separated in time by Δt and in space by Δx. A reference frame, traveling at less than the speed of light, in which the two events occur at the same time

(a) Exists no matter what the values of Δx and Δt

(b) Exists only if $\Delta x/\Delta t < c$

(c) Exists only if $\Delta x/\Delta t > c$

(d) Does not exist under any condition

sec. 37-5 The Relativity of Time

5. An astronaut aged 3 years when traveling at 99% of the speed of light to the Andromeda galaxy and back. The space officials who greeted her on her return have aged less than

(a) 3 years

(b) Less than 3 years

(c) More than 3 years

6. The spaceship Enterprise, traveling through the galaxy, sends out a smaller explorer craft that travels to a nearby planet and signals its findings back. The proper time for the trip to the planet is measured by clocks

(a) On board the Enterprise

(b) On board the explorer craft

(c) On Earth

(d) At the center of the galaxy

7. A meson when at rest decays 2 μs after it is created. If moving in the laboratory at $0.99c$, its lifetime according to laboratory clocks would be

(a) The same

(b) 0.28 s

(c) 14 μs

(d) 4.6 s

sec. 37-6 The Relativity of Length

8. A 1-meter long spear is thrown at a relativistic speed through a pipe that is 1 meter long (Fig. 37-19). Both these dimensions are measured when each is at rest. When the spear passes through the pipe, which of these statements best describes what is observed?

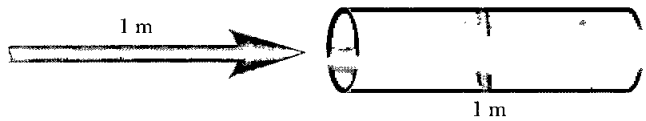


FIG. 37-19 Multiple choice question 8.

(a) The spear shrinks so that the pipe completely covers it at some point

(b) The pipe shrinks so that the spear extends from both ends at some point.

(c) Both shrink equally so that the pipe completely covers it at some point.

(d) Any of the above, depending on the motion of the observer.

9. A certain automobile is 6 m long if at rest. If it is measured to be 4/5 as long, its speed is

(a) $0.1c$

(b) $0.3c$

(c) $0.6c$

(d) $0.8c$

sec. 37-8 Some Consequences of the Lorentz Equations

10. If you were in a spaceship traveling at a speed close to the speed of light (with respect to Earth) you would notice that

(a) Your pulse rate is different than normal

(b) Some of your physical dimensions were smaller than normal

(c) Your mass is different than normal

(d) None of these effects occur

sec. 37-10 The Relativity of Velocities

11. Frame S_1 moves in the positive x direction at $0.6c$ with respect to frame S_2. A particle moves in the positive x direction at $0.4c$ as measured by an observer in S_1. The speed of the particle as measured by an observer in S_2 is

(a) $c/5$

(b) $5c/19$

(c) $8c/25$

(d) $25c/31$

sec. 37-11 Doppler Effect for Light

12. A spectral line of a certain star is observed to be "red shifted" from a wavelength of 500 nm to a wavelength of 1500 nm. Interpreting this as a Doppler effect, the speed of recession of this star is

(a) $0.33c$

(b) $0.50c$

(c) $0.71c$

(d) $0.8c$

sec. 37-13 A New Look at Energy

13. The special theory of relativity predicts that there is an upper limit to the speed of a particle. It thus follows that there is also an upper limit on the following property of a particle

(a) The linear momentum

(b) The total energy

(c) The kinetic energy

(d) More than one of these

(e) None of these

14. If the kinetic energy of a free particle is much greater than its rest energy then its kinetic energy is proportional to

(a) The magnitude of its momentum

(b) The square of the magnitude of its momentum

(c) The square root of the magnitude of its momentum

(d) The reciprocal of the magnitude of its momentum

15. If the kinetic energy of a free particle is much less than its rest energy then its kinetic energy is proportional to

(a) The magnitude of its momentum

(b) The square of the magnitude of its momentum

(c) The square root of the magnitude of its momentum

(d) The reciprocal of the magnitude of its momentum

PROBLEMS

sec. 37-5 The Relativity of Time

1. The mean lifetime of stationary muons is measured to be 2.2000 μs. The mean lifetime of high-speed muons in a burst of cosmic rays observed from Earth is measured to be 16.000 μs. To five significant figures, what is the speed parameter β of these cosmic-ray muons relative to Earth?

2. An unstable high-energy particle enters a detector and leaves a track of length 1.05 mm before it decays. Its speed relative to the detector was $0.992c$. What is its proper lifetime? That is, how long would the particle have lasted before decay had it been at rest with respect to the detector?

3. Reference frame S' is to pass reference frame S at speed v along the common direction of the x' and x axes, as in Fig. 37-11. An observer who rides along with frame S' is to count off a certain time interval on his wristwatch. The corresponding time interval Δt is to be *measured* by an observer in frame S. Figure 37-20 gives Δt versus speed parameter β for a range of values for β. The vertical axis scale is set by $\Delta t_a = 14.0$ s. What is interval Δt if $v = 0.98c$?

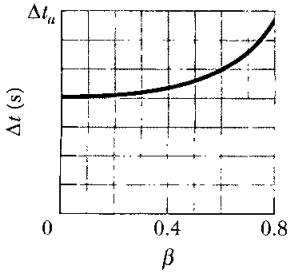


FIG. 37-20 Problem 3.

4. The premise of the *Planet of the Apes* movies and book is that hibernating astronauts travel far into Earth's future, to a time when human civilization has been replaced by an ape civilization. Considering only special relativity, determine how far into Earth's future the astronauts would travel if they slept for 120 y while traveling relative to Earth with a speed of $0.9990c$, first outward from Earth and then back again.

5. *(Come) back to the future.* Suppose that a father is 20.00 y older than his daughter. He wants to travel outward from Earth for 2.000 y and then back to Earth for another 2.000 y (both intervals as he measures them) such that he is then 20.00 y *younger* than his daughter. What constant speed parameter β (relative to Earth) is required for the trip?

sec. 37-6 The Relativity of Length

6. A meter stick in frame S' makes an angle of 30° with the x' axis. If that frame moves parallel to the x axis of frame S with speed $0.90c$ relative to frame S, what is the length of the stick as measured from S?

7. A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of $0.630c$. Its rest length is 1.70 m. What will be its measured length in frame S?

8. A spaceship of rest length 130 m races past a timing station at a speed of $0.740c$. (a) What is the length of the spaceship as measured by the timing station? (b) What time interval will the station clock record between the passage of the front and back ends of the ship?

9. A rod is to move at constant speed v along the x axis of reference frame S, with the rod's length parallel to that axis. An observer in frame S is to measure the length L of the rod. Figure 37-21 gives length L versus speed parameter β for a range of values for β. The vertical axis scale is set by $L_a = 1.00$ m. What is L if $v = 0.95c$?

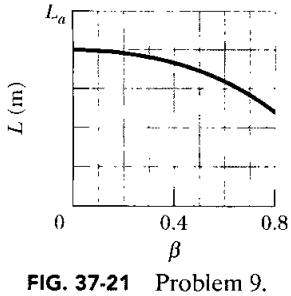


FIG. 37-21 Problem 9.

10. The length of a spaceship is measured to be exactly half its rest length. (a) To three significant figures, what is the speed parameter β of the spaceship relative to the observer's frame? (b) By what factor do the spaceship's clocks run slow relative to clocks in the observer's frame?

11. A space traveler takes off from Earth and moves at speed $0.9900c$ toward the star Vega, which is 26.00 ly distant. How much time will have elapsed by Earth clocks (a) when the traveler reaches Vega and (b) when Earth observers receive word from the traveler that she has arrived? (c) How much older will Earth observers calculate the traveler to be (measured from her frame) when she reaches Vega than she was when she started the trip?

sec. 37-8 Some Consequences of the Lorentz Equations

12. Inertial frame S' moves at a speed of $0.60c$ with respect to frame S (Fig. 37-11). Further, $x = x' = 0$ at $t = t' = 0$. Two events are recorded. In frame S, event 1 occurs at the origin at $t = 0$ and event 2 occurs on the x axis at $x = 3.0$ km at $t = 4.0\ \mu$s. According to observer S', what is the time of (a) event 1 and (b) event 2? (c) Do the two observers see the two events in the same sequence or the reverse sequence?

13. An experimenter arranges to trigger two flashbulbs simultaneously, producing a big flash located at the origin of his reference frame and a small flash at $x = 30.0$ km. An observer moving at a speed of $0.250c$ in the positive direction of x also views the flashes. (a) What is the time interval between them according to her? (b) Which flash does she say occurs first?

14. Observer S reports that an event occurred on the x axis of his reference frame at $x = 3.00 \times 10^8$ m at time $t = 2.50$ s. Observer S' and her frame are moving in the positive direction of the x axis at a speed of $0.400c$. Further, $x = x' = 0$ at $t = t' = 0$. What are the (a) spatial and (b) temporal coordinate of the event according to S'? If S' were, instead, moving in the *negative* direction of the x axis, what would be the (c) spatial and (d) temporal coordinate of the event according to S'?

15. Bullwinkle in reference frame S' passes you in reference frame S along the common direction of the x' and x axes, as in Fig. 37-11. He carries three meter sticks: meter stick 1 is parallel to the x' axis, meter stick 2 is parallel to the y' axis, and meter stick 3 is parallel to the z' axis. On his wristwatch he counts off 15.0 s, which takes 30.0 s according to you. Two events occur during his passage. According to you, event 1 occurs at $x_1 = 33.0$ m and $t_1 = 22.0$ ns, and event 2 occurs at $x_2 = 53.0$ m and $t_2 = 62.0$ ns. According to your measurements, what is the length of (a) meter stick 1, (b) meter stick 2, and (c) meter stick 3? According to Bullwinkle, what are (d) the spatial separation and (e) the temporal separation between events 1 and 2, and (f) which event occurs first?

16. A clock moves along an x axis at a speed of $0.600c$ and reads zero as it passes the origin. (a) Calculate the clock's Lorentz factor. (b) What time does the clock read as it passes $x = 180$ m?

17. As in Fig. 37-9, reference frame S' passes reference frame S with a certain velocity. Events 1 and 2 are to have a certain temporal separation $\Delta t'$ according to the S' observer. However, their spatial separation $\Delta x'$ according to that observer has not been set yet. Figure 37-22 gives their temporal separation Δt according to the S observer as a function of $\Delta x'$ for a range of $\Delta x'$ values. The vertical axis scale is set by $\Delta t_a = 6.00\ \mu$s. What is $\Delta t'$?

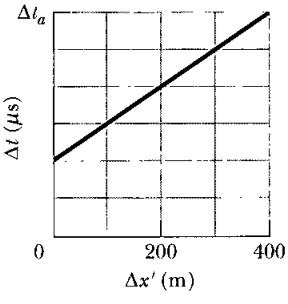
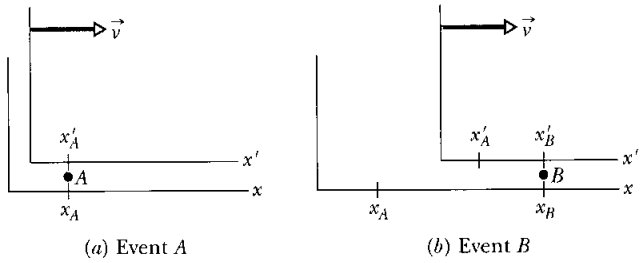


FIG. 37-22 Problem 17.

18. In Fig. 37-11, observer S detects two flashes of light. A big flash occurs at $x_1 = 1200$ m and, 5.00 μs later, a small flash occurs at $x_2 = 480$ m. As detected by observer S', the two flashes occur at a single coordinate x'. (a) What is the speed parameter of S', and (b) is S' moving in the positive or negative direction of the x axis? To S', (c) which flash occurs first and (d) what is the time interval between the flashes?

19. In Fig. 37-11, observer S detects two flashes of light. A big flash occurs at $x_1 = 1200$ m and, slightly later, a small flash occurs at $x_2 = 480$ m. The time interval between the flashes is $\Delta t = t_2 - t_1$. What is the smallest value of Δt for which observer S' will determine that the two flashes occur at the same x' coordinate?



(a) Event A (b) Event B

FIG. 37-23 Problems 20, 55, and 56.

20. For the passing reference frames in Fig. 37-23, events A and B occur at the following spacetime coordinates: according to the unprimed frame, (x_A, t'_A) and (x_B, t_B); according to the primed frame, (x'_A, t'_A) and (x'_B, t'_B). In the unprimed frame, $\Delta t = t_B - t_A = 1.00\ \mu$s and $\Delta x = x_B - x_A = 400$ m. (a) Find an expression for $\Delta x'$ in terms of the speed parameter β and the given data. Graph $\Delta x'$ versus β for two ranges of β: (b) 0 to 0.01 and (c) 0.1 to 1. (d) At what value of β is $\Delta x'$ minimum, and (e) what is that minimum?

sec. 37-10 The Relativity of Velocities

21. Galaxy A is reported to be receding from us with a speed of $0.35c$. Galaxy B, located in precisely the opposite direction, is also found to be receding from us at this same speed. What multiple of c gives the recessional speed an observer on Galaxy A would find for (a) our galaxy and (b) Galaxy B?

22. Stellar system Q_1 moves away from us at a speed of $0.800c$. Stellar system Q_2, which lies in the same direction in space but is closer to us, moves away from us at speed $0.400c$. What multiple of c gives the speed of Q_2 as measured by an observer in the reference frame of Q_1?

23. A particle moves along the x' axis of frame S' with velocity $0.40c$. Frame S' moves with velocity $0.60c$ with respect to frame S. What is the velocity of the particle with respect to frame S?

24. In Fig. 37-14, frame S' moves relative to frame S with velocity $0.62c\hat{i}$ while a particle moves parallel to the common x and x' axes. An observer attached to frame S' measures the particle's velocity to be $0.47c\hat{i}$. In terms of c, what is the particle's velocity as measured by an observer attached to frame S according to the (a) relativistic and (b) classical velocity transformation? Suppose, instead, that the S' measure of the particle's velocity is $0.47c\hat{i}$. What velocity does the observer in S now measure according to the (c) relativistic and (d) classical velocity transformation?

25. An armada of spaceships that is 1.00 ly long (in its rest frame) moves with speed $0.800c$ relative to a ground station in frame S. A messenger travels from the rear of the armada to the front with a speed of $0.950c$ relative to S. How long does the trip take as measured (a) in the messenger's rest frame, (b) in the armada's rest frame, and (c) by an observer in frame S?

26. In Fig. 37-24*a*, particle P is to move parallel to the x and x' axes of reference frames S and S', at a certain velocity relative to frame S. Frame S' is to move parallel to the x axis of frame S

at velocity v. Figure 37-24*b* gives the velocity u' of the particle relative to frame S' for a range of values for v. The vertical axis scale is set by $u'_a = 0.800c$. What value will u' have if (a) $v = 0.90c$ and (b) $v \to c$?

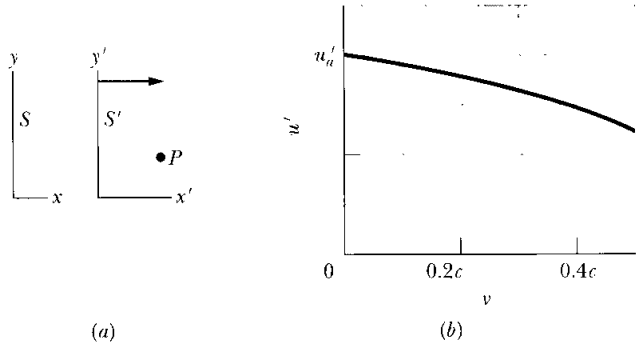


FIG. 37-24 Problem 26.

27. A spaceship whose rest length is 350 m has a speed of $0.82c$ with respect to a certain reference frame. A micrometeorite, also with a speed of $0.82c$ in this frame, passes the spaceship on an antiparallel track. How long does it take this object to pass the ship as measured on the ship?

sec. 37-11 Doppler Effect for Light

28. Certain wavelengths in the light from a galaxy in the constellation Virgo are observed to be 0.4% longer than the corresponding light from Earth sources. (a) What is the radial speed of this galaxy with respect to Earth? (b) Is the galaxy approaching or receding from Earth?

29. Assuming that Eq. 37-36 holds, find how fast you would have to go through a red light to have it appear green. Take 620 nm as the wavelength of red light and 540 nm as the wavelength of green light.

30. A spaceship, moving away from Earth at a speed of $0.900c$, reports back by transmitting at a frequency (measured in the spaceship frame) of 100 MHz. To what frequency must Earth receivers be tuned to receive the report?

31. A sodium light source moves in a horizontal circle at a constant speed of $0.100c$ while emitting light at the proper wavelength of $\lambda_0 = 589.00$ nm. Wavelength λ is measured for that light by a detector fixed at the center of the circle. What is the wavelength shift $\lambda - \lambda_0$?

32. A spaceship is moving away from Earth at speed $0.20c$. A source on the rear of the ship emits light at wavelength 450 nm according to someone on the ship. What (a) wavelength and (b) color (blue, green, yellow, or red) are detected by someone on Earth watching the ship?

sec. 37-13 A New Look at Energy

33. What is the minimum energy that is required to break a nucleus of ^{12}C (of mass 11.996 71 u) into three nuclei of ^{4}He (of mass 4.001 51 u each)?

34. How much work must be done to increase the speed of an electron (a) from $0.18c$ to $0.19c$ and (b) from $0.98c$ to $0.99c$? Note that the speed increase is $0.01c$ in both cases.

35. In the reaction $p + {}^{19}F \to \alpha + {}^{16}O$, the masses are

$$m(p) = 1.007825 \text{ u}, \qquad m(\alpha) = 4.002603 \text{ u},$$
$$m(F) = 18.998405 \text{ u}, \qquad m(O) = 15.994915 \text{ u}.$$

Calculate the Q of the reaction from these data.

36. The mass of an electron is $9.109\,381\,88 \times 10^{-31}$ kg. To six significant figures, find (a) γ and (b) β for an electron with kinetic energy $K = 100.000$ MeV.

37. How much work must be done to increase the speed of an electron from rest to (a) $0.500c$, (b) $0.990c$, and (c) $0.9990c$?

38. What must be the momentum of a particle with mass m so that the total energy of the particle is 3.00 times its rest energy?

39. The mass of an electron is $9.109\,381\,88 \times 10^{-31}$ kg. To eight significant figures, find the following for the given electron kinetic energy: (a) γ and (b) β for $K = 1.000\,000\,0$ keV, (c) γ and (d) β for $K = 1.000\,000\,0$ MeV, and (e) γ and (f) β for $K = 1.000\,000\,0$ GeV.

40. As you read this page (on paper or monitor screen), a cosmic ray proton passes along the left–right width of the page with relative speed v and a total energy of 14.24 nJ. According to your measurements, that left–right width is 21.0 cm. (a) What is the width according to the proton's reference frame? How much time did the passage take according to (b) your frame and (c) the proton's frame?

41. (a) The energy released in the explosion of 1.00 mol of TNT is 3.40 MJ. The molar mass of TNT is 0.227 kg/mol. What weight of TNT is needed for an explosive release of 1.80×10^{14} J? (b) Can you carry that weight in a backpack, or is a truck or train required? (c) Suppose that in an explosion of a fission bomb, 0.080% of the fissionable mass is converted to released energy. What weight of fissionable material is needed for an explosive release of 1.80×10^{14} J? (d) Can you carry that weight in a backpack, or is a truck or train required?

42. A certain particle of mass m has momentum of magnitude mc. What are (a) β, (b) γ, and (c) the ratio K/E_0?

43. What is β for a particle with (a) $K = 2.00E_0$ and (b) $E = 2.00E_0$?

44. Quasars are thought to be the nuclei of active galaxies in the early stages of their formation. A typical quasar radiates energy at the rate of 10^{41} W. At what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit (1 smu $= 2.0 \times 10^{30}$ kg) is the mass of our Sun.

45. (a) If m is a particle's mass, p is its momentum magnitude, and K is its kinetic energy, show that

$$m = \frac{(pc)^2 - K^2}{2Kc^2}.$$

(b) For low particle speeds, show that the right side of the equation reduces to m. (c) If a particle has $K = 55.0$ MeV when $p = 121$ MeV/c, what is the ratio m/m_e of its mass to the electron mass?

46. A 5.00-grain aspirin tablet has a mass of 320 mg. For how many kilometers would the energy equivalent of this mass power an automobile? Assume 12.75 km/L and a heat of combustion of 3.65×10^7 J/L for the gasoline used in the automobile.

47. To four significant figures, find the following when the kinetic energy is 10.00 MeV: (a) γ and (b) β for an electron ($E_0 = 0.510\,998$ MeV), (c) γ and (d) β for a proton ($E_0 = 938.272$ MeV), and (e) γ and (f) β for an α particle ($E_0 = 3727.40$ MeV).

48. In Section 29-4, we showed that a particle of charge q and mass m will move in a circle of radius $r = mv/|q|B$ when its

velocity $\vec{v}$ is perpendicular to a uniform magnetic field $\vec{B}$. We also found that the period T of the motion is independent of speed v. These two results are approximately correct if $v \ll c$. For relativistic speeds, we must use the correct equation for the radius:

$$r = \frac{p}{|q|B} = \frac{\gamma mv}{|q|B}.$$

(a) Using this equation and the definition of period ($T = 2\pi r/v$), find the correct expression for the period. (b) Is T independent of v? If a 10.0 MeV electron moves in a circular path in a uniform magnetic field of magnitude 2.20 T, what are (c) the radius according to Chapter 29, (d) the correct radius, (e) the period according to Chapter 29, and (f) the correct period?

49. The mass of a muon is 207 times the electron mass; the average lifetime of muons at rest is 2.20 μs. In a certain experiment, muons moving through a laboratory are measured to have an average lifetime of 6.90 μs. For the moving muons, what are (a) β, (b) K, and (c) p (in MeV/c)?

50. In a high-energy collision between a cosmic-ray particle and a particle near the top of Earth's atmosphere, 120 km above sea level, a pion is created. The pion has a total energy E of 1.35×10^5 MeV and is traveling vertically downward. In the pion's rest frame, the pion decays 35.0 ns after its creation. At what altitude above sea level, as measured from Earth's reference frame, does the decay occur? The rest energy of a pion is 139.6 MeV.

51. Apply the binomial theorem (Appendix E) to the last part of Eq. 37-52 for the kinetic energy of a particle. (a) Retain the first two terms of the expansion to show the kinetic energy in the form

$$K = (\text{first term}) + (\text{second term}).$$

The first term is the classical expression for kinetic energy. The second term is the first-order correction to the classical expression. Assume the particle is an electron. If its speed v is $c/20$, what is the value of (b) the classical expression and (c) the first-order correction? If the electron's speed is $0.80c$, what is the value of (d) the classical expression and (e) the first-order correction? (f) At what speed parameter β does the first-order correction become 10% or greater of the classical expression?

52. An alpha particle with kinetic energy 7.70 MeV collides with an ^{14}N nucleus at rest, and the two transform into an ^{17}O nucleus and a proton. The proton is emitted at 90° to the direction of the incident alpha particle and has a kinetic energy of 4.44 MeV. The masses of the various particles are alpha particle, 4.00260 u; ^{14}N, 14.00307 u; proton, 1.007825 u; and ^{17}O, 16.99914 u. In MeV, what are (a) the kinetic energy of the oxygen nucleus and (b) the Q of the reaction? (*Hint:* The speeds of the particles are much less than c.)

ADDITIONAL PROBLEMS

53. In Fig. 37-25*a*, particle P is to move parallel to the x and x' axes of reference frames S and S', at a certain velocity relative to frame S. Frame S' is to move parallel to the x axis of frame S at velocity v. Figure 37-25*b* gives the velocity u' of the particle relative to frame S' for a range of values for v. The vertical axis scale is set by $u'_a = -0.800c$. What value will u' have if (a) $v = 0.80c$ and (b) $v \to c$?

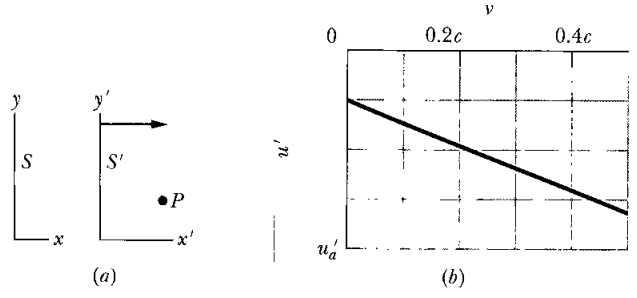


FIG. 37-25 Problem 53.

54. *Superluminal jets.* Figure 37-26*a* shows the path taken by a knot in a jet of ionized gas that has been expelled from a galaxy. The knot travels at constant velocity $\vec{v}$ at angle θ from the direction of Earth. The knot occasionally emits a burst of light, which is eventually detected on Earth. Two bursts are indicated in Fig. 37-26*a*, separated by time t as measured in a stationary frame near the bursts. The bursts are shown in Fig. 37-26*b* as if they were photographed on the same piece of film, first when light from burst 1 arrived on Earth and then later when light from burst 2 arrived. The apparent distance D_{app} traveled by the knot between the two bursts is the distance across an Earth-observer's view of the knot's path. The apparent time T_{app} between the bursts is the difference in

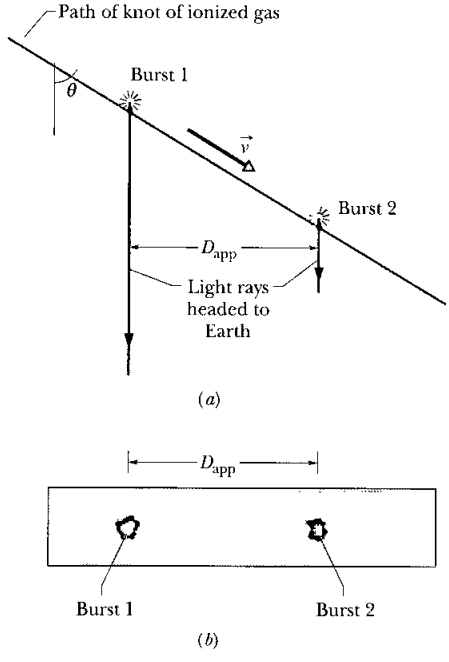


FIG. 37-26 Problem 54.

the arrival times of the light from them. The apparent speed of the knot is then $V_{app} = D_{app}/T_{app}$. In terms of v, t, and θ, what are (a) D_{app} and (b) T_{app}? (c) Evaluate V_{app} for $v = 0.980c$ and $\theta = 30.0°$. When superluminal (faster than light) jets were first observed, they seemed to defy special relativity—at least until the correct geometry (Fig. 37-26*a*) was understood.

55. *Temporal separation between two events.* Events A and B occur with the following spacetime coordinates in the reference

frames of Fig. 37-23: according to the unprimed frame, (x_A, t_A) and (x_B, t_B); according to the primed frame, (x'_A,t'_A) and (x'_B,t'_B). In the unprimed frame, $\Delta t = t_B - t_A = 1.00\ \mu s$ and $\Delta x = x_B - x_A = 240$ m. (a) Find an expression for $\Delta t'$ in terms of the speed parameter β and the given data. Graph $\Delta t'$ versus β for the following two ranges of β: (b) 0 to 0.01 and (c) 0.1 to 1. (d) At what value of β is $\Delta t'$ minimum and (e) what is that minimum? (f) Can one of these events cause the other? Explain.

56. *Spatial separation between two events.* For the passing reference frames of Fig. 37-23, events A and B occur with the following spacetime coordinates: according to the unprimed frame, (x_A, t_A) and (x_B, t_B); according to the primed frame, (x'_A,t'_A) and (x'_B,t'_B). In the unprimed frame, $\Delta t = t_B - t_A = 1.00\ \mu s$ and $\Delta x = x_B - x_A = 240$ m. (a) Find an expression for $\Delta x'$ in terms of the speed parameter β and the given data. Graph $\Delta x'$ versus β for two ranges of β: (b) 0 to 0.01 and (c) 0.1 to 1. (d) At what value of β is $\Delta x' = 0$?

57. *The car-in-the-garage problem.* Carman has just purchased the world's longest stretch limo, which has a proper length of $L_c = 30.5$ m. In Fig. 37-27, it is shown parked in front of a garage with a proper length of $L_g = 6.00$ m. The garage has a front door (shown open) and a back door (shown closed). The limo is obviously longer than the garage. Still, Garageman, who owns the garage and knows something about relativistic length contraction, makes a bet with Carman that the limo can fit in the garage with both doors closed. Carman, who dropped his physics course before reaching special relativity, says such a thing, even in principle, is impossible.

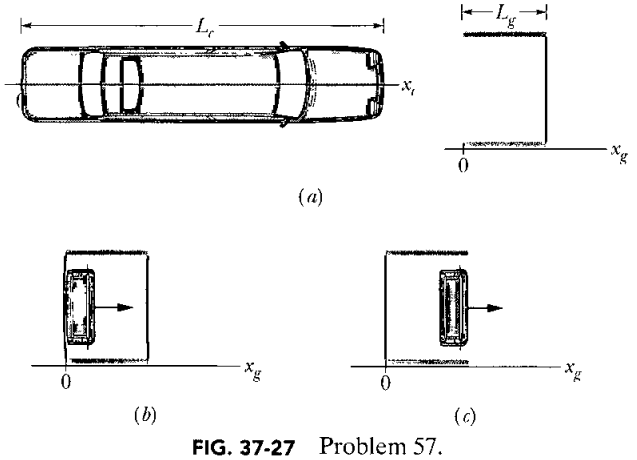


FIG. 37-27 Problem 57.

To analyze Garageman's scheme, an x_c axis is attached to the limo, with $x_c = 0$ at the rear bumper, and an x_g axis is attached to the garage, with $x_g = 0$ at the (now open) front door. Then Carman is to drive the limo directly toward the front door at a velocity of $0.9980c$ (which is, of course, both technically and financially impossible). Carman is stationary in the x_c reference frame; Garageman is stationary in the x_g reference frame.

There are two events to consider. *Event 1:* When the rear bumper clears the front door, the front door is closed. Let the time of this event be zero to both Carman and Garageman: $t_{g1} = t_{c1} = 0$. The event occurs at $x_c = x_g = 0$. Figure 37-27 shows event 1 according to the x_g reference frame. *Event 2:* When the front bumper reaches the back door, that door opens. Figure 37-27 shows event 2 according to the x_g reference frame.

According to Garageman, (a) what is the length of the limo, and what are the spacetime coordinates (b) x_{g2} and (c) t_{g2} of event 2? (d) For how long is the limo temporarily "trapped" inside the garage with both doors shut? Now consider the situation from the x_c reference frame, in which the garage comes racing past the limo at a velocity of $-0.9980c$. According to Carman, (e) what is the length of the passing garage, what are the spacetime coordinates (f) x_{c2} and (g) t_{c2} of event 2, (h) is the limo ever in the garage with both doors shut, and (i) which event occurs first? (j) Sketch events 1 and 2 as seen by Carman. (k) Are the events causally related; that is, does one of them cause the other? (l) Finally, who wins the bet?

58. Reference frame S' passes reference frame S with a certain velocity as in Fig. 37-11. Events 1 and 2 are to have a certain spatial separation $\Delta x'$ according to the S' observer. However, their temporal separation $\Delta t'$ according to that observer has not been set yet. Figure 37-28 gives their spatial separation Δx according to the S observer as a function of $\Delta t'$ for a range of $\Delta t'$ values. The vertical axis scale is set by $\Delta x_a = 10.0$ m. What is $\Delta x'$?

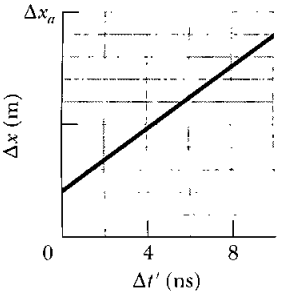


FIG. 37-28 Problem 58.

59. *Another approach to velocity transformations.* In Fig. 37-29, reference frames B and C move past reference frame A in the common direction of their x axes. Represent the x components of the velocities of one frame relative to another with a two-letter subscript. For example, v_{AB} is the x component of the velocity of A relative to B. Similarly, represent the corresponding speed parameters with two-letter subscripts. For example, β_{AB} $(= v_{AB}/c)$ is the speed parameter corresponding to v_{AB}. (a) Show that

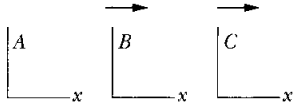


FIG. 37-29 Problems 59, 60, and 61.

$$\beta_{AC} = \frac{\beta_{AB} + \beta_{BC}}{1 + \beta_{AB}\beta_{BC}}.$$

Let M_{AB} represent the ratio $(1 - \beta_{AB})/(1 + \beta_{AB})$, and let M_{BC} and M_{AC} represent similar ratios. (b) Show that the relation

$$M_{AC} = M_{AB}M_{BC}$$

is true by deriving the equation of part (a) from it.

60. *Continuation of Problem 59.* Use the result of part (b) in Problem 59 for the motion along a single axis in the following situation. Frame A in Fig. 37-29 is attached to a particle that moves with velocity $+0.500c$ past frame B, which moves past frame C with a velocity of $+0.500c$. What are (a) M_{AC}, (b) β_{AC}, and (c) the velocity of the particle relative to frame C?

61. *Continuation of Problem 59.* Let reference frame C in Fig. 37-33 move past reference frame D (not shown). (a) Show that

$$M_{AD} = M_{AB}M_{BC}M_{CD}.$$

(b) Now put this general result to work: Three particles move parallel to a single axis on which an observer is stationed. Let plus and minus signs indicate the directions of motion along that axis. Particle A moves past particle B at $\beta_{AB} = +0.20$. Particle B moves past particle C at $\beta_{BC} = -0.40$. Particle C moves past observer D at $\beta_{CD} = +0.60$. What is the velocity of particle

A relative to observer D? (The solution technique here is *much* faster than using Eq. 37-29.)

62. The total energy of a proton passing through a laboratory apparatus is 10.611 nJ. What is its speed parameter β? Use the proton mass given in Appendix B under "Best Value," not the commonly remembered rounded number.

63. If we intercept an electron having total energy 1533 MeV that came from Vega, which is 26 ly from us, how far in light-years was the trip in the rest frame of the electron?

64. A pion is created in the higher reaches of Earth's atmosphere when an incoming high-energy cosmic-ray particle collides with an atomic nucleus. A pion so formed descends toward Earth with a speed of $0.99c$. In a reference frame in which they are at rest, pions decay with an average life of 26 ns. As measured in a frame fixed with respect to Earth, how far (on the average) will such a pion move through the atmosphere before it decays?

65. What is the momentum in MeV/c of an electron with a kinetic energy of 2.00 MeV?

66. How much work is needed to accelerate a proton from a speed of $0.9850c$ to a speed of $0.9860c$?

67. An airplane whose rest length is 40.0 m is moving at uniform velocity with respect to Earth, at a speed of 630 m/s. (a) By what fraction of its rest length is it shortened to an observer on Earth? (b) How long would it take, according to Earth clocks, for the airplane's clock to fall behind by 1.00 μs? (Use special relativity in your calculations.)

68. To circle Earth in low orbit, a satellite must have a speed of about 2.7×10^4 km/h. Suppose that two such satellites orbit Earth in opposite directions. (a) What is their relative speed as they pass, according to the classical Galilean velocity transformation equation? (b) What fractional error do you make in (a) by not using the (correct) relativistic transformation equation?

69. A particle with mass m has speed $c/2$ relative to inertial frame S. The particle collides with an identical particle at rest relative to frame S. Relative to S, what is the speed of a frame S' in which the total momentum of these particles is zero? This frame is called the *center of momentum frame.*

70. An elementary particle produced in a laboratory experiment travels 0.230 mm through the lab at a relative speed of $0.960c$ before it decays (becomes another particle). (a) What is the proper lifetime of the particle? (b) What is the distance the particle travels as measured from its rest frame?

71. What are (a) K, (b) E, and (c) p (in GeV/c) for a proton moving at speed $0.990c$? What are (d) K, (e) E, and (f) p (in MeV/c) for an electron moving at speed $0.990c$?

72. In the red shift of radiation from a distant galaxy, a certain radiation, known to have a wavelength of 434 nm when observed in the laboratory, has a wavelength of 462 nm. (a) What is the radial speed of the galaxy relative to Earth? (b) Is the galaxy approaching or receding from Earth?

73. (a) What potential difference would accelerate an electron to speed c according to classical physics? (b) With this potential difference, what speed would the electron actually attain?

74. The radius of Earth is 6370 km, and its orbital speed about the Sun is 30 km/s. Suppose Earth moves past an observer at this speed. To the observer, by how much does Earth's diameter contract along the direction of motion?

75. A spaceship at rest in a certain reference frame S is given a speed increment of $0.50c$. Relative to its new rest frame, it is then given a further $0.50c$ increment. This process is continued until its speed with respect to its original frame S exceeds $0.999c$. How many increments does this process require?

76. A Foron cruiser moving directly toward a Reptulian scout ship fires a decoy toward the scout ship. Relative to the scout ship, the speed of the decoy is $0.980c$ and the speed of the Foron cruiser is $0.900c$. What is the speed of the decoy relative to the cruiser?

77. One cosmic-ray particle approaches Earth along Earth's north–south axis with a speed of $0.80c$ toward the geographic north pole, and another approaches with a speed of $0.60c$ toward the geographic south pole (Fig. 37-30). What is the relative speed of approach of one particle with respect to the other?

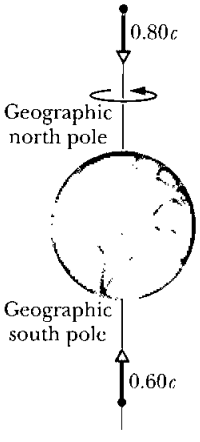


FIG. 37-30 Problem 77.

78. (a) How much energy is released in the explosion of a fission bomb containing 3.0 kg of fissionable material? Assume that 0.10% of the mass is converted to released energy. (b) What mass of TNT would have to explode to provide the same energy release? Assume that each mole of TNT liberates 3.4 MJ of energy on exploding. The molecular mass of TNT is 0.227 kg/mol. (c) For the same mass of explosive, what is the ratio of the energy released in a nuclear explosion to that released in a TNT explosion?

ANSWERS

Checkpoints

1. (a) Same (speed of light postulate); (b) no (the start and end of the flight are spatially sparated); (c) no (because his measurement is not a proper time)

2. (a) Eq. 37-2; (b) +0.90c; (c) 25 ns; (d) −7.0 m

3. (a) Right; (b) more

4. (a) Equal; (b) less

Reasoning Type Questions

1. (c)

2. (a)

3. (i) (a); (ii) (d)

4. (b)

5. (a)

Multiple Choice Questions

1. (c)
2. (a)
3. (d)
4. (c)
5. (c)
6. (b)
7. (c)
8. (d)
9. (c)
10. (d)
11. (d)
12. (d)
13. (e)
14. (a)
15. (b)

Problems

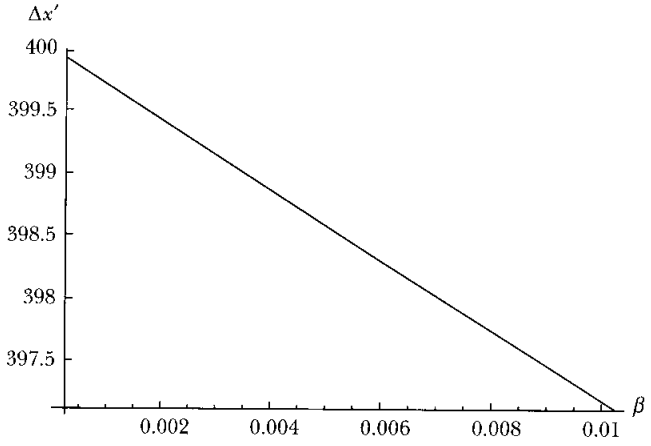
1. 0.99050
2. 0.446 ps
3. 40 s
4. $\approx 2.68 \times 10^3$ y
5. 0.9959
6. 0.63 m
7. 1.32 m
8. (a) 87.4 m; (b) 394 ns
9. 0.25 m
10. (a) 0.866; (b) 2.00
11. (a) 26.26 y; (b) 52.26 y; (c) 3.705 y
12. (a) $(x'_1, t'_1) = (0, 0)$;

    (b) $x'_2 = \dfrac{x - vt}{\sqrt{1-\beta^2}} = 2.85 \times 10^3 \text{ m}; t'_2 = \dfrac{t - \beta x/c}{\sqrt{1-\beta^2}} = -2.5 \times 10^{-6} \text{ s};$

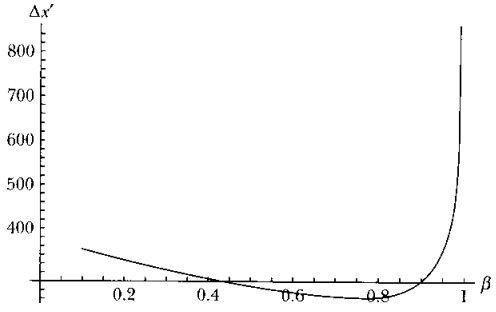
    (c) The two events in frame S occur in the order: first 1, then 2. However, in frame S' where $t'_2 < 0$, they occur in the reverse order: first 2, then 1. So the two observers see the two events in the reverse sequence.
13. (a) 25.8 μs; (b) small flash
14. (a) $2.7 \times 10^5 \text{ m} \approx 0$; (b) 2.29 s; (c) 6.54×10^8 m; (d) 3.16 s
15. 0.500 m; (b) 1.00 m; (c) 1.00 m; (d) 19.2 m; (e) –35.5 ns; (f) event 2 occurred before event 1
16. (a) 1.25; (b) 0.800 μs
17. 6.3×10^{-7} s
18. (a) 0.480; (b) negative; (c) big flash; (d) 4.39 μs
19. 2.40×10^{-6} s
20. (a) From Table 37-2, we find

    $$\Delta x' = [400 \text{ m} - (299.8 \text{ m})\beta]/\sqrt{1-\beta^2}$$

    (b) A plot of $\Delta x'$ as a function of β with $0 < \beta < 0.01$ is shown:



    (c) A plot of $\Delta x'$ as a function of β with $0.1 < \beta < 1$ is shown below:



    (d) $\beta = 0.7495 \approx 0.750$; (e) $\Delta x' \approx 265$ m
21. (a) 0.35; (b) 0.62
22. 0.588
23. $0.81c$
24. (a) $0.84c$; (b) $(1.1c)\hat{\imath}$; (c) $(0.21c)\hat{\imath}$; (d) $(0.15c)\hat{\imath}$
25. (a) 1.25 y; (b) 1.60 y; (c) 4.00 y
26. (a) $-0.36c$; (b) $-c$
27. 1.2 μs
28. (a) 1×10^6 m/s; (b) the galaxy is receding
29. $0.13c$
30. 22.9 MHz
31. + 2.97 nm
32. (a) 550 nm; (b) yellow
33. 7.28 MeV
34. (a) 1.0 keV; (b) 1.1 MeV

35. 8.12 MeV

36. (a) 196.695; (b) 0.999 987

37. (a) 79.1 keV; (b) 3.11 MeV; (c) 10.9 MeV

38. $2.83mc$

39. (a) 1.0019570; (b) 0.062469542; (c) 2.9569514; (d) 0.94107924; (e) 1957.9514; (f) 0.999 99987

40. (a) 0.222 cm; (b) 701 ps; (c) 7.40 ps

41. (a) 1.20×10^7 kg; (b) this is certainly more than can be carried in a backpack. Presumably, a train would be required; (c) 25 N; (d) this can be carried in a backpack

42. (a) 0.707; (b) 1.41; (c) 0.414

43. (a) ≈ 0.943; (b) ≈ 0.866

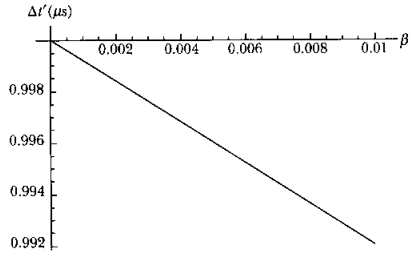
44. 18 smu/y

Additional Problems

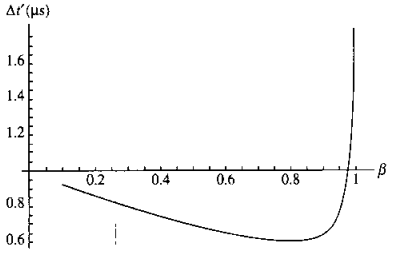
53. (a) $-0.86c$; (b) $-c$

54. (a) $vt \sin\theta$; (b) $t[1-(v/c)\cos\theta]$; (c) $3.24c$

55. (a) $\Delta t' = \gamma(1.00 - 0.800\,\beta)\ \mu$s; (b) a plot of $\Delta t'$ is shown for the range $0 < \beta < 0.01$

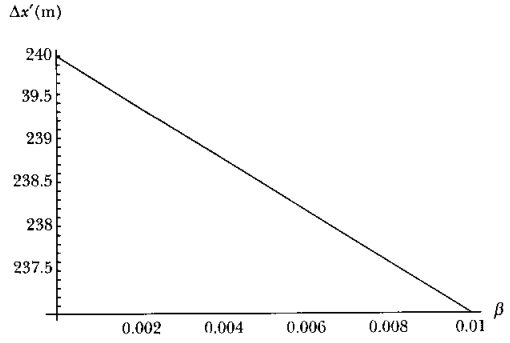


(c) A plot of $\Delta t'$ is shown for the range $0.1 < \beta < 1$



(d) $\beta = 0.801$; (e) $\Delta t' = 0.599\ \mu$s; (f) yes.

56. (a) $\Delta x' = \gamma(\Delta x - \beta c\Delta t) = \gamma[(240\text{ m}) - \beta(299.8\text{ m})]$; (b) a plot of $\Delta x'$ for $0 < \beta < 0.01$ is shown:



(c) A plot of $\Delta x'$ for $0.01 < \beta < 1$ is shown:

45. (a) $m = [(pc)^2 - K^2]/2Kc^2$; (b) $m \to 1/mv = m$; (c) 105.6 MeV/c^2

46. 1.01×10^7 km

47. (a) 20.57; (b) 0.9988; (c) ≈ 1.011; (d) ≈ 0.1448; (e) ≈ 1.003; (f) 0.07310

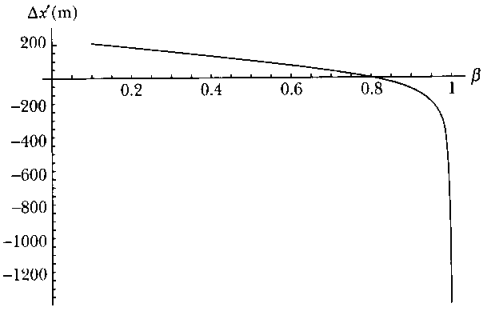
48. (a) $\gamma(2\pi m/|q|B)$; (b) no; (c) 4.85 mm; (d) 15.9 mm; (e) 16.3 ps; (f) 0.334 ns

49. (a) 0.948; (b) 226 MeV; (c) 314 MeV/c

50. 110 km

51. (a) $\frac{1}{2}mv^2 + \frac{3}{8}mv^4/c^2$; (b) 1.0×10^{-16} J; (c) 1.9×10^{-19} J; (d) 2.6×10^{-14} J; (e) 1.3×10^{-14} J; (f) ≈ 0.37

52. (a) ≈ 2.08 MeV; (b) 2.05 MeV



(d) 0.801

57. (a) 1.93 m; (b) 6.00 m; (c) 13.6 ns; (d) 13.6 ns; (e) 0.379 m; (f) 30.5 m; (g) −101 ns; (h) no; (i) 2; (k) no; (l) both

58. 0.79 m

60. (a) 1/9; (b) 0.80; (c) $0.80c$

61. $\beta = 0.44$; (b) $+0.44c$

62. 0.99990

63. 8.7×10^{-3} ly

64. 55 m

65. 2.46 MeV/c

66. 189 MeV

67. (a) 2.21×10^{-12}; (b) 5.25 d

68. (a) 5.4×10^4 km/h; (b) 6.3×10^{-10}

69. $0.27c$

70. (a) 2.24×10^{-13} s; (b) 6.44×10^{-5} m

71. (a) 5.71 GeV; (b) 6.65 GeV; (c) 6.58 GeV/c; (d) 3.11 MeV; (e) 3.62 MeV; (f) 3.59 MeV/c

72. (a) 1.87×10^7 m/s; (b) since it is shifted "toward the red" (toward longer wavelengths) then the galaxy is moving away from us (receding).

73. (a) 256 kV; (b) $0.745c$

74. 0.064 m

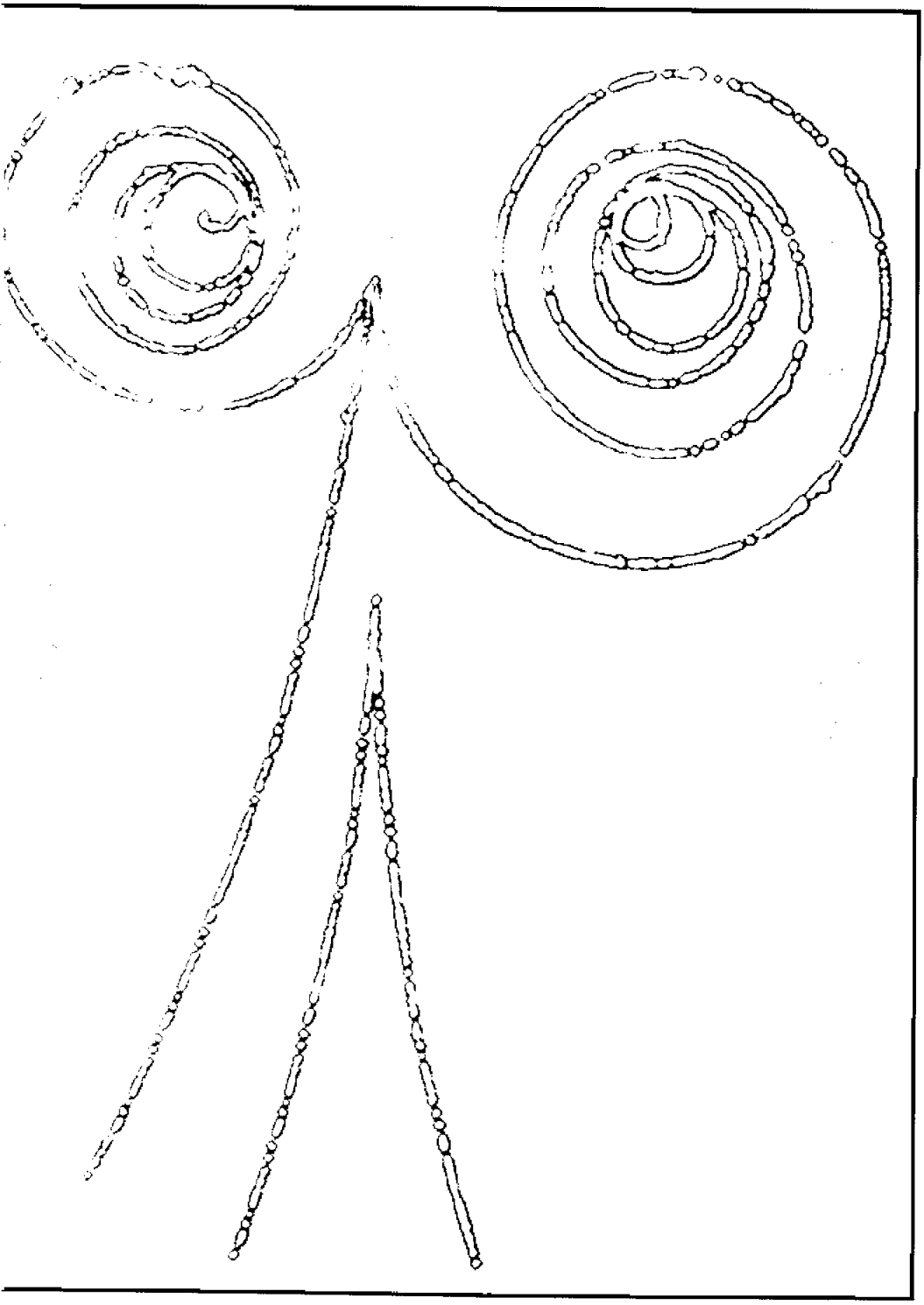
75. 7

76. $0.678c$

77. $0.95c$

78. (a) 2.7×10^{14} J; (b) 1.8×10^7 kg; (c) 6.0×10^6

Light Quanta, Matter Waves 38



Tracks of tiny vapor bubbles in this bubble-chamber image reveal where electrons (tracks color-coded green) and positrons (red) moved. A gamma ray (which left no track when it entered at the top) kicked an electron out of one of the hydrogen atoms filling the chamber and then converted to an electron–positron pair. Another gamma ray underwent another pair production farther down. These tracks (curved because of a magnetic field) clearly show that electrons and positrons are particles that move along narrow paths. Yet, those particles can also be interpreted in terms of waves.

Can a particle be a wave?

The answer is in this chapter.

38-1 WHAT IS PHYSICS?

One primary focus of physics is Einstein's theory of relativity, which took us into a world far beyond that of ordinary experience—the world of objects moving at speeds close to the speed of light. Among other surprises, Einstein's theory predicts that the rate at which a clock runs depends on how fast the clock is moving relative to the observer: the faster the motion, the slower the clock rate. This and other predictions of the theory have passed every experimental test devised thus far, and relativity theory has led us to a deeper and more satisfying view of the nature of space and time.

Now you are about to explore a second world that is outside ordinary experience—the subatomic world. You will encounter a new set of surprises that, though they may sometimes seem bizarre, have led physicists step by step to a deeper view of reality.

Quantum physics, as our new subject is called, answers such questions as: Why do the stars shine? Why do the elements exhibit the order that is so apparent in the periodic table? How do transistors and other microelectronic devices work? Why does copper conduct electricity but glass does not?

Some of the predictions of quantum physics seem strange even to the physicists and philosophers who study its foundations. Still, experiment after experiment has proved the theory correct, and many have exposed even stranger aspects of the theory. The quantum world is an amusement park full of wonderful rides that are guaranteed to shake up the commonsense world view you have developed since childhood. We begin our exploration of that quantum park with the photon.

38-2 | The Photon, the Quantum of Light

Quantum physics (which is also known as *quantum mechanics* and *quantum theory*) is largely the study of the microscopic world. In that world, many quantities are found only in certain minimum (*elementary*) amounts, or integer multiples of those elementary amounts; these quantities are then said to be *quantized.* The elementary amount that is associated with such a quantity is called the **quantum** of that quantity (*quanta* is the plural). For example, the number of students in your class are quantized because the number of students in the class can only be an integer.

In 1905, Einstein proposed that electromagnetic radiation (or simply *light*) is quantized and exists in elementary amounts (quanta) that we now call **photons.** This proposal should seem strange to you because we have just spent several chapters discussing the classical idea that light is a sinusoidal wave, with a wavelength λ, a frequency f, and a speed c such that

$$f = \frac{c}{\lambda}. \tag{38-1}$$

Furthermore, in Chapter 33 we discussed the classical light wave as being an interdependent combination of electric and magnetic fields, each oscillating at frequency f. How can this wave of oscillating fields consist of an elementary amount of something—the light quantum? What *is* a photon?

The concept of a light quantum, or a photon, turns out to be far more subtle and mysterious than Einstein imagined. Indeed, it is still very poorly understood. In this book, we shall discuss only some of the basic aspects of the photon concept, somewhat along the lines of Einstein's proposal.

According to that proposal, the quantum of a light wave of frequency f has the energy

$$E = hf \qquad \text{(photon energy)}. \tag{38-2}$$

Here h is the **Planck constant** which has the value

$$h = 6.63 \times 10^{-34}\ \text{J}\cdot\text{s} = 4.14 \times 10^{-15}\ \text{eV}\cdot\text{s}. \tag{38-3}$$

The smallest amount of energy a light wave of frequency f can have is hf, the energy of a single photon. If the wave has more energy, its total energy must be an integer multiple of hf, just as the number of students in the class has to be an integral multiple of 1. The light cannot have an energy of, say, $0.6hf$ or $75.5hf$.

Einstein further proposed that when light is absorbed or emitted by an object (matter), the absorption or emission event occurs in the atoms of the object. When light of frequency f is absorbed by an atom, the energy hf of one photon is transferred from the light to the atom. In this *absorption event,* the photon vanishes and the atom is said to absorb it. When light of frequency f is emitted by an atom, an amount of energy hf is transferred from the atom to the light. In this *emission event,* a photon suddenly appears and the atom is said to emit it. Thus, we can have *photon absorption* and *photon emission* by atoms in an object.

For an object consisting of many atoms, there can be many photon absorptions (such as with sunglasses) or photon emissions (such as with lamps). However, each absorption or emission event still involves the transfer of an amount of energy equal to that of a single photon of the light.

When we discussed the absorption or emission of light in previous chapters, our examples involved so much light that we had no need of quantum physics, and we got by with classical physics. However, in the late 20th century, technology became advanced enough that single-photon experiments could be conducted and put to practical use. Since then quantum physics has become part of standard engineering practice, especially in optical engineering.

Before going ahead with our journey of Quantum Physics, let us consider the historical development of the photon or quantum nature of light. The physicists of late 19th century and early 20th century were quite perplexed about the radiation emitted by a black body. As you have already studied in Volume 1 of this book, at a given temperature, the spectral emissive power, $dI/d\lambda$, of light emitted by a black body depends on the wavelength emitted as shown in Fig. 38-1. This could not be explained on the basis of the classical electromagnetic theory of light. Furthermore, Wein found out that if we increase the temperature of the black body, the wavelength corresponding to the most intense radiation decreases. This has been illustrated graphically in Fig. 38-1. Thus, by Wein's displacement law,

$$\lambda_m \times T = b,$$

where b is Wein's constant having value of 0.288 cm · K, λ_m is the wavelength corresponding to the most intense radiation, and T is the absolute temperature of the black body.

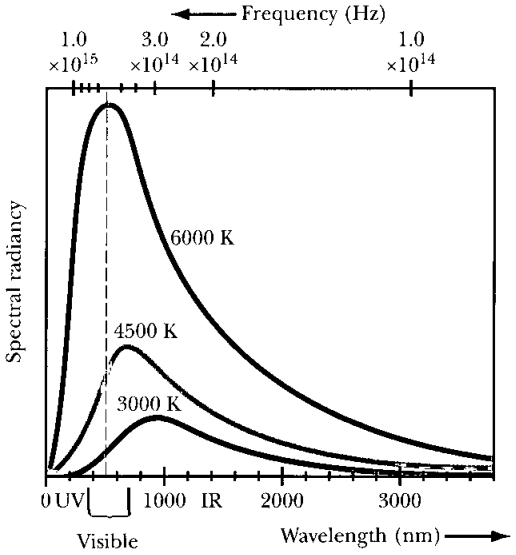


FIG. 38-1 Measured spectra of wavelengths and frequencies emitted by a blackbody at three different temperatures.

Planck's Quantum Theory

In 1901, Planck showed that a successful theory of radiation was possible by making a revolutionary assumption regarding the way in which radiation is emitted or absorbed by atoms. He supposed that energy E could only be emitted or absorbed as radiation of frequency v in integral multiples of n such that

$$E = nhv.$$

It, thus, appeared that energy must also be regarded as having an atomic nature. At first, Planck and his contemporaries found this idea of a discontinuous emission of energy very hard to accept, but it soon became the only explanation of several other phenomena in physics and was recognized as the foundation of modern atomic physics.

It is worthwhile considering how this assumption accounts for the general form of the energy distribution before proceeding to the other evidence for quantum theory. In a solid we have to suppose that the atoms are fixed relative to one another and only capable of oscillation about a mean position. The thermal energy of the body will be distributed among the atoms as kinetic and potential energy, in much the same way as we supposed in our discussion of the kinetic theory of gases. Not all atoms will have the same energy, but the mean kinetic energy is dependent upon the temperature. A few atoms will have energies greatly in excess of the mean while the energies of others will be very much smaller than the mean.

The atom will emit energy in proportion to the energy it contains. If it has less energy, the quantum of energy it emits is small. The intensity I will depend on the number of quanta (or photons) emitted per unit time per unit area and the energy of one photon:

$$I = hv \times \frac{dN}{Adt}.$$

As we proceed to higher values of v the quantum hv increases until it exceeds the mean energy of the atoms in the solid. Beyond this value fewer and fewer atoms will have sufficient energy available to radiate the necessary quantum hv. Thus, the intensity radiated related to light of a particular wavelength rises to a maximum and falls away again as we proceed to higher and higher frequencies.

Moreover, as we raise the temperature of the body, the average kinetic energy of the atoms is raised, so that this energy corresponds to a larger quantum hv and a correspondingly larger frequency. At a higher temperature there will be a greater probability of an atom acquiring sufficient energy to radiate a really large quantum of energy hv. Thus we can see in a general way how raising the temperature of a body must lead to shift of the radiation maximum towards the region of higher frequency (i.e., shorter wavelength).

CHECKPOINT 1 Rank the following radiations according to their associated photon energies, greatest first: (a) yellow light from a sodium vapor lamp, (b) a gamma ray emitted by a radioactive nucleus, (c) a radio wave emitted by the antenna of a commercial radio station, (d) a microwave beam emitted by airport traffic control radar.

Sample Problem 38-1

A sodium vapor lamp is placed at the center of a large sphere that absorbs all the light reaching it. The rate at which the lamp emits energy is 100 W; assume that the emission is entirely at a wavelength of 590 nm. At what rate are photons absorbed by the sphere?

Conceptualize: We assume that all the light emitted by the lamp reaches (and thus is absorbed by) the sphere. So, the rate R at which photons are absorbed by the sphere is equal to the rate R_{emit} at which photons are emitted by the lamp.

Classify: The light is emitted and absorbed as photons.

Compute: That rate is

$$R_{\text{emit}} = \frac{\text{rate of energy emission}}{\text{energy per emitted photon}} = \frac{P_{\text{emit}}}{E}.$$

Into this we can substitute from Eq. 38-2 ($E = hf$), Einstein's proposal about the energy E of each quantum of light (which we here call a photon in modern language):

$$R = R_{\text{emit}} = \frac{P_{\text{emit}}}{hf}.$$

Using Eq. 38-1 ($f = c/\lambda$) to substitute for f and then entering known data, we obtain

$$R = \frac{P_{\text{emit}}\lambda}{hc}$$

$$= \frac{(100\text{ W})(590\times10^{-9}\text{ m})}{(6.63\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^{8}\text{ m/s})}$$

$$= 2.97\times10^{20}\text{ photons/s.} \qquad \text{(Answer)}$$

Conclude: Do not be surprised at the large number of photons per second. Each photon has such a small energy that in order to be detected, the number of photons has to be quite large. The only exception seems to be human eye. It can detect as small as 5 photons/s.

38-3 | The Photoelectric Effect

If you direct a beam of light of short enough wavelength onto a clean metal surface, the light will cause electrons to leave that surface (the light will *eject* the electrons from the surface). This **photoelectric effect** is used in many devices, including TV cameras, camcorders, and night vision viewers. Einstein supported his photon concept by using it to explain this effect, which simply cannot be understood without quantum physics.

Let us analyze two basic photoelectric experiments, each using the apparatus of Fig. 38-2, in which light of frequency f is directed onto target T and ejects electrons from it. A potential difference V is maintained between target T and collector cup C to sweep up these electrons, said to be **photoelectrons**. This collection produces a **photoelectric current** i that is measured with meter A.

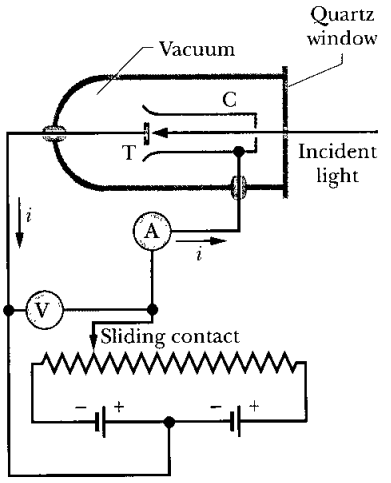


FIG. 38-2 An apparatus used to study the photoelectric effect. The incident light shines on target T, ejecting electrons, which are collected by collector cup C. The electrons move in the circuit in a direction opposite the conventional current arrows. The batteries and the variable resistor are used to produce and adjust the electric potential difference between T and C.

First Photoelectric Experiment

We adjust the potential difference V by moving the sliding contact in Fig. 38-2 so that collector C is slightly negative with respect to target T. This potential difference acts to slow down the ejected electrons. We then vary V until it reaches a certain value, called the **stopping potential** V_{stop}, at which point the reading of meter A has just dropped to zero. When $V = V_{\text{stop}}$, the most energetic ejected electrons are turned back just before reaching the collector. Then K_{max}, the kinetic energy of these most energetic electrons, is

$$K_{\text{max}} = eV_{\text{stop}}, \qquad (38\text{-}4)$$

where e is the elementary charge.

Measurements show that for light of a given frequency, K_{max} *does not depend on the intensity of the light source.* Whether the source is dazzling bright or so feeble that you can scarcely detect it (or has some intermediate brightness), the maximum kinetic energy of the ejected electrons always has the same value.

This experimental result is a puzzle for classical physics. Classically, the incident light is a sinusoidally oscillating electromagnetic wave. An electron in the target should oscillate sinusoidally due to the oscillating electric force on it from the wave's electric field. If the amplitude of the electron's oscillation is great enough, the electron should break free of the target's surface—that is, be ejected from the target. Thus, if we increase the amplitude of the wave and its oscillating electric field, the electron should get a more energetic "kick" as it is being ejected. *However, that is not what happens.* For a given frequency, intense light beams and feeble light beams give exactly the same maximum kick to ejected electrons.

The actual result follows naturally if we think in terms of photons. Now the energy that can be transferred from the incident light to an electron in the target is that of a single photon. Increasing the light intensity increases the *number* of photons in the light, but the photon energy, given by Eq. 38-2 ($E = hf$), is unchanged because the frequency is unchanged. Thus, the energy transferred to the kinetic energy of an electron is also unchanged.

Second Photoelectric Experiment

Now we vary the frequency f of the incident light and measure the associated stopping potential V_{stop}. Figure 38-3 is a plot of V_{stop} versus f. Note that the photoelectric effect does not occur if the frequency is below a certain **cutoff frequency** f_0 or, equivalently, if the wavelength is greater than the corresponding **cutoff wavelength** $\lambda_0 = c/f_0$. This is so *no matter how intense the incident light is.*

This is another puzzle for classical physics. If you view light as an electromagnetic wave, you must expect that no matter how low the frequency, electrons can always be ejected by light if you supply them with enough energy—that is, if you use a light source that is bright enough. *That is not what happens.* For light below the cutoff frequency f_0, the photoelectric effect does not occur, no matter how bright the light source.

The existence of a cutoff frequency is, however, just what we should expect if the energy is transferred via photons. The electrons within the target are held there by electric forces. (If they weren't, they would drip out of the target due to the gravitational force on them.) To just escape from the target, an electron must pick up a certain minimum energy Φ, where Φ is a property of the target material called its **work function**. If the energy hf transferred to an electron by a photon exceeds the work function of the material (if $hf > \Phi$), the electron can escape the target. If the energy transferred does not exceed the work function (that is, if $hf < \Phi$), the electron cannot escape. This is what Fig. 38-3 shows.

The Photoelectric Equation

Einstein summed up the results of such photoelectric experiments in the equation

$$hf = K_{max} + \Phi \qquad \text{(photoelectric equation).} \tag{38-5}$$

This is a statement of the conservation of energy for a single photon absorption by a target with work function Φ. Energy equal to the photon's energy hf is transferred to a single electron in the material of the target. If the electron is to escape from the target, it must pick up energy at least equal to Φ. Any additional energy ($hf - \Phi$) that the electron acquires from the photon appears as kinetic energy K of the electron. In the most favorable circumstance, the electron can escape through the surface without losing any of this kinetic energy in the process; it then appears outside the target with the maximum possible kinetic energy K_{max}. In most of the cases, the electron which gains the energy from the photon loses it by random collision within the metal lattice. So the energy of the ejected electron varies from 0 to K_{max}.

In most of the cases, the electron which gains the energy from the photon loses it by random collision within the metal lattice. So the energy of the ejected electron varies from 0 to K_{max}.

Let us rewrite Eq. 38-5 by substituting for K_{max} from Eq. 38-4 ($K_{max} = eV_{stop}$). After a little rearranging we get

$$V_{stop} = \left(\frac{h}{e}\right) f - \frac{\Phi}{e}. \tag{38-6}$$

The ratios h/e and Φ/e are constants, and so we would expect a plot of the measured stopping potential V_{stop} versus the frequency f of the light to be a straight

line, as it is in Fig. 38-3. Further, the slope of that straight line should be h/e. As a check, we measure ab and bc in Fig. 38-3 and write

$$\frac{h}{e} = \frac{ab}{bc} = \frac{2.35\text{ V} - 0.72\text{ V}}{(11.2\times10^{14} - 7.2\times10^{14})\text{ Hz}}$$
$$= 4.1\times10^{-15}\text{ V}\cdot\text{s}.$$

Multiplying this result by the elementary charge e, we find

$$h = (4.1\times10^{-15}\text{ V}\cdot\text{s})(1.6\times10^{-19}\text{ C}) = 6.6\times10^{-34}\text{ J}\cdot\text{s},$$

which agrees with values measured by many other methods.

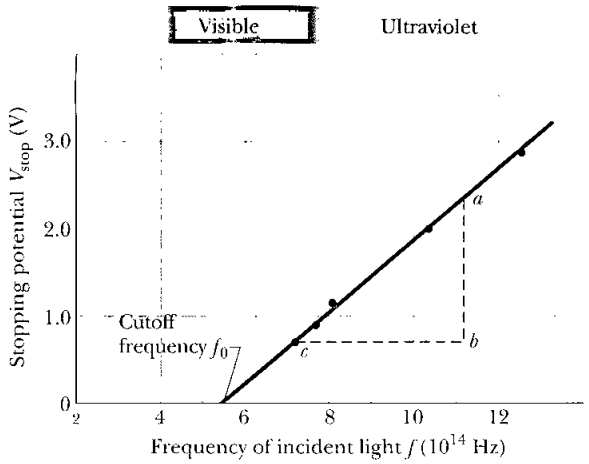
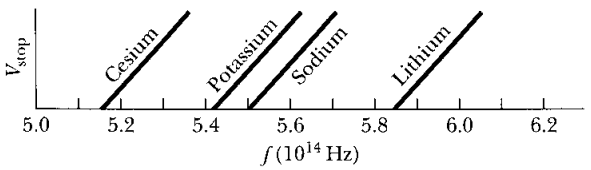


FIG. 38-3 The stopping potential V_{stop} as a function of the frequency f of the incident light for a sodium target T in the apparatus of Fig. 38-2. (Data reported by R. A. Millikan in 1916.)

CHECKPOINT 2 The figure shows data like those of Fig. 38-3 for targets of cesium, potassium, sodium, and lithium. The plots are parallel. (a) Rank the targets according to their work functions, greatest first. (b) Rank the plots according to the value of h they yield, greatest first.



Sample Problem 38-2

A potassium foil is a distance $r = 3.5$ m from an isotropic light source that emits energy at the rate $P = 1.5$ W. The work function Φ of potassium is 2.2 eV. Suppose that the energy transported by the incident light were transferred to the target foil continuously and smoothly (that is, if classical physics prevailed instead of quantum physics). How long would it take for the foil to absorb enough energy to eject an electron? Assume that the foil totally absorbs all the energy reaching it and that the to-be-ejected electron collects energy from a circular patch of the foil whose radius is 5.0×10^{-11} m, about that of a typical atom.

Conceptualize:

1. The time interval Δt required for the patch to absorb energy ΔE depends on the rate P_{abs} at which the energy is absorbed:

$$\Delta t = \frac{\Delta E}{P_{abs}}.$$

2. If the electron is to be ejected from the foil, the smallest amount of energy ΔE it must gain from the light is equal to the work function Φ of potassium. Thus,

$$\Delta t = \frac{\Phi}{P_{abs}}.$$

3. Because the patch is totally absorbing, the rate of absorption P_{abs} is equal to the rate P_{arr} at which energy arrives at the patch; that is,

$$\Delta t = \frac{\Phi}{P_{arr}}.$$

4. With Eq. 33-25 [$I = (\text{power/area})_{avg}$], we can relate the energy arrival rate P_{arr} to the intensity I of the light at the patch and the area A of the patch:

$$P_{arr} = IA.$$

Then
$$\Delta t = \frac{\Phi}{IA}.$$

5. Because the light source is isotropic, the light intensity I at distance r from the source depends on the rate P_{emit} at which energy is emitted by the source, according to Eq. 33-29:

$$I = \frac{P_{emit}}{4\pi r^2}.$$

Classify: Here we are going to use the classical theory of electromagnetic waves, not the quantum physics.

Compute: Putting these ideas together yields

$$\Delta t = \frac{4\pi r^2 \Phi}{P_{emit}\, A}.$$

The detection area A is $\pi(5.0\times10^{-11}\text{ m})^2 = 7.85\times10^{-21}\text{ m}^2$, and the work function Φ is 2.2 eV = 3.5×10^{-19} J. Substituting these and other data, we find that

$$\Delta t = \frac{4\pi(3.5\text{ m})^2(3.5\times10^{-19}\text{ J})}{(1.5\text{ W})(7.85\times10^{-21}\text{ m}^2)}$$
$$= 4580\text{ s} \approx 1.3\text{ h}. \qquad \text{(Answer)}$$

Conclude: Thus, classical physics tells us that we would have to wait more than an hour after turning on the light source for a photoelectron to be ejected. The actual waiting time is less than 10^{-9} s. Apparently, then, an electron does *not* gradually absorb energy from the light arriving at the patch containing the electron. Rather, either the electron does not absorb any energy at all or it absorbs a quantum of energy instantaneously, by absorbing a photon from the light.

Sample Problem 38-3

Find the work function Φ of sodium from Fig. 38-3.

Conceptualize/Classify: We can find the work function Φ from the cutoff frequency f_0 (which we can measure on the plot). The reasoning is this: At the cutoff frequency, the kinetic energy K_{max} in Eq. 38-5 is zero. Thus, all the energy hf that is transferred from a photon to an electron goes into the electron's escape, which requires an energy of Φ.

Compute: From that last idea, Eq. 38-5 then gives us, with $f = f_0$,

$$hf_0 = 0 + \Phi = \Phi.$$

In Fig. 38-3, the cutoff frequency f_0 is the frequency at which the plotted line intercepts the horizontal frequency axis, about 5.5×10^{14} Hz. We then have

$$\Phi = hf_0 = (6.63\times10^{-34}\text{ J}\cdot\text{s})\,(5.5\times10^{14}\text{ Hz})$$
$$= 3.6\times10^{-19}\text{ J} = 2.3\text{ eV}. \qquad \text{(Answer)}$$

Third Photoelectric Experiment

Let us now perform another experiment where we keep the frequency of the incident electromangnetic waves same, but we change the intensity of the incident radiations. If we increase the intensity of the incident radiations, it is seen that the maximum energy of the emitted photoelectrons do not change (So, the stopping potential remains the same.) But it seems that the number of photoelectrons emitted per unit time increase because the number of electrons reaching the collector per unit time increase. In Chapter 27, we noted that the current flowing through any cross section is given by

$$i = \frac{dq}{dt} = e\frac{dN}{dt}.$$

We know that the current is due to the flow of electrons only. Here e refers to the charge on the electron and N is the number of electrons emitted in a given time duration. On performing the experiment, it is seen that the current in the photoelectric tube does not follow Ohm's law. The I–V graph can be seen in Figure 38-4.

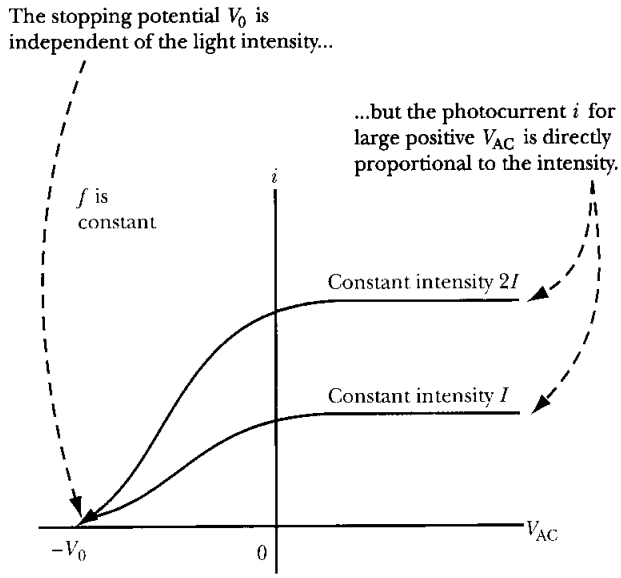


FIG. 38-4 The plot of photocurrent i for a givem monochromatic light. As the collector plate is made sufficiently positive, the current becomes constant, called saturation current. It doubles on doubling the intensity.

The phenomena can be easily understood in terms of the photon model. Of the photoelectrons emitted from the emitter plate, if the potential is sufficiently positive, all the electrons are pulled to the other side. So even an electron emitted with negligible velocity is attracted to the positively charged collector plate. This maximum possible current at a given intensity and frequency is known as *saturation current.* You might be surprised at seeing that if we increase the potential of the collector plate with respect to the emitter plate, the current does not increase beyond a certain limit. However, our experience with the electrostatics revealed that when an electron is made to undergo a higher potential drop, its kinetic energy changes by a greater factor. So the electron should arrive with a greater speed. But wait a minute! This does not necessarily mean that the number of electrons reaching the plate somehow increases. They will arrive more quickly, of course. But not in a greater number! The number of the emitted electrons will remain the same in either case, and all the emitted electrons are going to end up on the other plate. So the current remains constant.

However, if the potential of the collector plate with respect to emitter plate is maintained at zero, some of the electrons get stuck up in between, constituting what is known as *space charge.* This building of charge decreases the probability of the electron reaching the collector. So even at $V_{AC} = 0$, the current is not maximum.

If the electrons reach the collector plate, the circuit is complete and a current is registered in the ammeter connected. If the potential of the collector plate is kept sufficiently negative with respect to the emitter plate (lesser than the stopping potential), electrons are emitted from the plate but fail to reach the collector. Even the most energetic photoelectron turns back before reaching the collector plate.

It is to be noted that on doubling the intensity of incident light, the number of photons incident on the plate per unit time doubles. Naturally, this would mean that the number of electrons emitted per unit time would double. Not all the photons would be successful in emitting the photons. Only a miniscule fraction of the photons is able to emit the photoelectrons. This fraction is known as *photoelectric efficiency*. It can be measured by measuring the saturation current:

$$i = \frac{dq}{dt} = e\frac{dN}{dt},$$

$$\frac{dN}{dt} = \eta \times \frac{dN_p}{dt},$$

where η is photoelectric efficiency and dN_p/dt is number of photons incident on the emitter plate per unit time.

Note that it should not be misunderstood here that the electrons are supposed to leave perpendicular to the metal plate only. They can be emitted in any direction with any velocity subject to a maximum value of the velocity.

Sample Problem 38-4

A beam of light has three wavelengths 4144 Å, 4972 Å, and 6215 Å with a total intensity of 3.6×10^{-3} W·m^{-2} equally distributed amongst three wavelengths. The beam falls normally on an area 1.0 cm^2 of clean metallic surface of work function 2.3 eV. Assume that there is no loss of ligth by reflection and that photoelectric efficiency is 0.01% (same for all the three photons). Calculate the number of photoelectrons liberated in 2 s.

Conceptualize: Here we can first see that this is a photoelectric effect where photons of three different frequencies are ejecting photoelectrons. Although they are incident on the same area, we can safely assume that the process of liberation of photoelectrons of them will be independent of one another.

Classify: The three beams of photons can be independently treated in order to obtain the photoelectrons liberated.

Compute: But first we will have to check out which photons are capable of liberating photoelectrons.

$$E_1 = \text{Energy of 1st photon} = \frac{6.63\times10^{-34}\times3\times10^{8}}{(1.6\times10^{-19})(4144\times10^{-10})} = 3\text{ eV}.$$

$$E_2 = \text{Energy of 2nd photon} = \frac{6.63\times10^{-34}\times3\times10^{8}}{(1.6\times10^{-19})(4972\times10^{-10})} = 2.5\text{ eV}.$$

$$E_3 = \text{Energy of 3rd photon} = \frac{6.63\times10^{-34}\times3\times10^{8}}{(1.6\times10^{-19})(6216\times10^{-10})} = 2\text{ eV}.$$

One convenient point to remember is that

$$h = 4.14\times10^{-15}\text{ eV}\cdot\text{s}.$$

Also, only first two photons have energy more that 2.3 eV and only these photons are capable of ejecting photoelectrons. The intensity corresponding to each photon is thus

$$I = 1.2\times10^{-3}\text{ W/m}^2.$$

Also

$$\frac{dn_1}{dt} = \text{Number of photons of wavelength 4144 Å incident per unit time}$$

$$= \frac{IA}{E_1} = \left(\frac{1.2\times10^{3}}{3\times1.6\times10^{-19}}\right)(1\times10^{-4}) = 2.5\times10^{11}\text{/s}.$$

$$\frac{dn_2}{dt} = \text{Number of photons of wavelength 4144 Å incident per unit time}$$

$$= \frac{IA}{E_2} = \left(\frac{1.2\times10^{3}}{2.5\times1.6\times10^{-19}}\right)(1\times10^{-4}) = 3\times10^{11}\text{/s}.$$

Therefore, the number of photons emitted per second is

$$\eta \times \text{number of photons incident per second} = 0.01\times5.5\times1011/100.$$

So the number of photoelectrons emitted in 2 s = 11×10^7.

Conclude: Normally these liberated photoelectrons will leave the surface charged. In a photoelectric tube, as the target or the emitter plate is connected to the collector plate via a battery, therefore the charge does not build up on the emitter plate and hence we do not have to bother about the possibility of the electrons returning to the plate after the emission. But otherwise, the target plate accumulates a steady charge after some time. It becomes so very positively charge that its potential becomes equal to the stopping potential. So the electrons liberated will land back in no time.

38-4 | Photons Have Momentum

In 1916, Einstein extended his concept of light quanta (photons) by proposing that a quantum of light has linear momentum. For a photon with energy hf, the

magnitude of that momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda} \qquad \text{(photon momentum)}, \tag{38-7}$$

where we have substituted for f from Eq. 38-1 ($f = c/\lambda$). Thus, when a photon interacts with matter, energy *and* momentum are transferred, *as if* there were a collision between the photon and matter in the classical sense.

Proof: We have just seen that the total energy of a single photon is given by $E = hv$. Because a photon always travels at the speed of light, it is truly a relativistic particle. Thus we must use relativistic formulas for dealing with its mass, energy and momentum. The momentum of any particle of rest mass m is given by

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}},$$

as we had seen in Eq. 37-42. Since for a photon, the velocity is always equal to the velocity of light, the denominator is zero. To avoid having infinite momentum, we conclude that the photon's rest mass must be zero:

$$m_0 = 0.$$

A photon's kinetic energy is its total energy. The momentum of photon can be obtained from the relativistic formula:

$$E^2 = p^2c^2 + m_0^2c^4.$$

As in Eq. 37-55, putting $m_0 = 0$, we get

$$E = pc.$$

So

$$p = \frac{E}{c}.$$

Since $E = hv$ for a photon, its momentum is related to its wavelength by

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}. \tag{38-8}$$

Sample Problem 38-5

Suppose that a light bulb of power 100 W emits 10^{19} photons/s. Suppose that somehow all these photons are focused onto a piece of black paper and absorbed. (a) Calculate the momentum of one photon and (b) estimate the force all these photons could exert on the paper. Take the wavelength of the incident light as 500 nm.

Conceptualize/Classify: The momentum of each photon is obtained from Eq. 38-7, $p = h/\lambda$. Next, each absorbed photon's momentum changes from $p = h/\lambda$ to zero. We use Newton's second law, $F = \Delta p/\Delta t$, to get the force.

Compute: (a) Each photon has a momentum,

$$p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}\ \text{J} \cdot \text{s}}{500 \times 10^{-9}\ \text{m}} = 1.3 \times 10^{-27}\ \text{kg} \cdot \text{m/s}.$$

(b) Using Newton's second law for $N = 10^{19}$ photons whose momentum changes from h/λ to 0, we obtain:

$$F = \frac{\Delta p}{\Delta t} = \frac{(Nh/\lambda) - 0}{1\ \text{s}} = N\frac{h}{\lambda}$$

$$\approx (10^{19}\ \text{s}^{-1})(10^{-27}\ \text{kg} \cdot \text{m/s}) \approx 10^{-8}\ \text{N}.$$

Conclude: This is a very small force but we can see that a very strong light source could exert measurable force. Near sun or star the force due to photons could be considerable. That's why you see a comet tail always pointing away from the sun.

Sample Problem 38-6

A plank of mass "m" is lying on a smooth surface in situation as shown in Fig. 38-5. Assume that the plank slips over the surface and surface of plank exposed to radiation is black body. Find the force on the plank due to radiation.

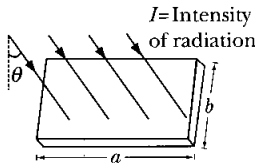


FIG. 38-5 A light beamincident on a black body at an angle θ to the normal.

Conceptualize/Classify: First of all let us find the energy received per second by surface exposed to radiation. The area of the surface is ab. But the area of cross section of the beam incident on the surface is not ab (Fig. 38-5). From the figure it is clear that the area of cross section of the beam is $ab\cos\theta$.

Compute: So the power of the incident beam is

$$P = Iab\cos\theta.$$

The number of photons incident per unit time can be found by

$$P = \frac{dn}{dt} \times \frac{hc}{\lambda}.$$

So

$$\frac{dn}{dt} = \frac{Iab\cos\theta\,\lambda}{hc}.$$

Each of the photon will be completely absorbed by the surface. This will produce an impulse on the slab in the horizontal direction alone. The impulse in the normal direction will be absorbed by the surface on which it is kept.

Radiation force experience by surface is

$$F = \frac{dn}{dt} \times p,$$

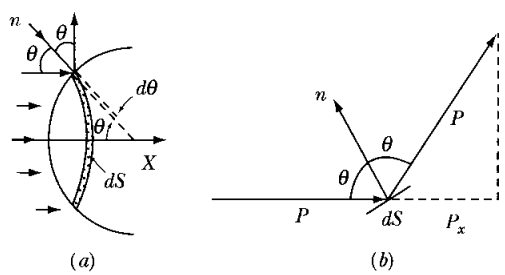
where p is the momentum of the photon incident. By de Broglie's hypothesis,

$$p = \frac{h}{\lambda},$$

$$F = \frac{Iab\cos\theta}{c}.$$

Sample Problem 38-7

A perfectly reflecting solid sphere of radius r is kept in the path of a parallel beam of light of large aperture as in Fig. 38-6. If the beam carries intensity I, find the force exerted by the beam on the sphere.



(a) (b)

FIG. 38-6 (a) A plane beam is incident on a perfectly reflecting sphere. (b) Change in momentum of a photon.

Conceptualize: The light beam is incident on the sphere as a parallel beam. However, since the normal at every point in not parallel to the normal at other points, the beam will not be reflected as a parallel beam.

Classify: For the analysis, let us consider a small element of surface at angle θ to horizontal direction. This element of the shape of ring subtends an angle $d\theta$ at the center as shown in Fig. 38-6a.

Compute: The momentum of a photon is

$$P = \frac{h}{\lambda}.$$

From symmetry, we can say that the impulse on the sphere in all direction perpendicular to incident beam direction is zero. The momentum of the photon in x direction changes as can be seen in Fig. 38-6b. Therefore

$$\Delta P_x = \frac{h}{\lambda} - \left(-\frac{h}{\lambda}\cos 2\theta\right)$$

$$= \frac{h}{\lambda}(1+\cos 2\theta)$$

$$= \frac{h}{\lambda} \times 2\cos^2\theta.$$

The number of photons incident unit time on this element is given by

$$\frac{IdA \times \lambda}{hc}.$$

Note carefully that dA is a cross-section area of incident beam and not the element of the sphere. Area of the element is given by

$$dA_e = Rd\theta \times 2\pi R\sin\theta.$$

Therefore, the area of cross-section of incident beam would be lesser by a factor of cos θ:

$$dA = 2\pi R^2 \sin\theta\, d\theta \times \cos\theta.$$

Thus,

$$dF = \frac{\text{number of photons incident}}{\text{time}} \times \Delta P_x$$
$$= \frac{I \times 2\pi R^2 \sin\theta\cos\theta\, d\theta}{hc} \times \lambda \times \frac{h}{\lambda} \times 2\cos^2\theta$$
$$= \frac{4\pi R^2 I}{c}\cos^3\theta \sin\theta\, d\theta.$$

Also

$$F = \int dF$$
$$= \frac{4\pi R^2 I}{c}\int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta$$
$$= \frac{\pi R^2 I}{c}.$$

Conclude: Interestingly, if the sphere were perfect absorber, the force exerted by light beam on the sphere would be the same.

38-5 | Light as a Probability Wave

A fundamental mystery in physics is how light can be a wave (which spreads out over a region) in classical physics but be emitted and absorbed as photons (which originate and vanish at points) in quantum physics. The double-slit experiment of Section 36-4 lies at the heart of this mystery. Let us discuss three versions of that experiment.

The Standard Version

Figure 38-7 is a sketch of the original experiment carried out by Thomas Young in 1801 (see also Fig. 36-9). Light shines on screen B, which contains two narrow parallel slits. The light waves emerging from the two slits spread out by diffraction and overlap on screen C where, by interference, they form a pattern of alternating intensity maxima and minima. In Section 36-4 we took the existence of these interference fringes as compelling evidence for the wave nature of light.

Let us place a tiny photon detector D at one point in the plane of screen C. Let the detector be a photoelectric device that clicks when it absorbs a photon. We would find that the detector produces a series of clicks, randomly spaced in time, each click signaling the transfer of energy from the light wave to the screen via a photon absorption.

If we moved the detector very slowly up or down as indicated by the black arrow in Fig. 38-7, we would find that the click rate increases and decreases, passing through alternate maxima and minima that correspond exactly to the maxima and minima of the interference fringes.

The point of this thought experiment is as follows. We cannot predict when a photon will be detected at any particular point on screen C; photons are detected at individual points at random times. We can, however, predict that the relative *probability* that a single photon will be detected at a particular point in a specified time interval is proportional to the light intensity at that point.

We know from Eq. 33-28 ($I = E_{rms}^2 / c\mu_0$) in Section 33-5 that the intensity I of a light wave at any point is proportional to the square of E_m, the amplitude of the oscillating electric field vector of the wave at that point. Thus,

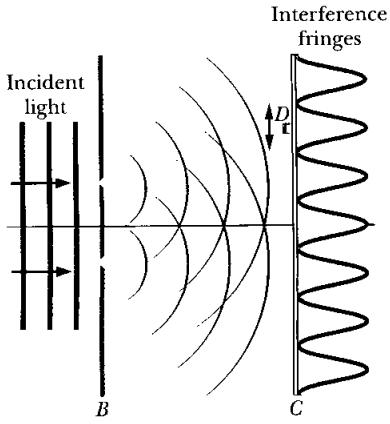


FIG. 38-7 Light is directed onto screen B, which contains two parallel slits. Light emerging from these slits spreads out by diffraction. The two diffracted waves overlap at screen C and form a pattern of interference fringes. A small photon detector D in the plane of screen C generates a sharp click for each photon that it absorbs.

☞ The probability (per unit time interval) that a photon will be detected in any small volume centered on a given point in a light wave is proportional to the square of the amplitude of the wave's electric field vector at that point.

We now have a probabilistic description of a light wave, hence another way to view light. It is not only an electromagnetic wave but also a **probability wave**. That is,

to every point in a light wave we can attach a numerical probability (per unit time interval) that a photon can be detected in any small volume centered on that point.

The Single-Photon Version

A single-photon version of the double-slit experiment was first carried out by G. I. Taylor in 1909 and has been repeated many times since. It differs from the standard version in that the light source in the Taylor experiment is so extremely feeble that it emits only one photon at a time, at random intervals. Astonishingly, interference fringes still build up on screen *C* if the experiment runs long enough (several months for Taylor's early experiment).

What explanation can we offer for the result of this single-photon double-slit experiment? Before we can even consider the result, we are compelled to ask questions like these: If the photons move through the apparatus one at a time, through which of the two slits in screen *B* does a given photon pass? How does a given photon even "know" that there is another slit present so that interference is a possibility? Can a single photon somehow pass through both slits and interfere with itself?

Bear in mind that the only thing we can know about photons is when light interacts with matter—we have no way of detecting them without an interaction with matter, such as with a detector or a screen. Thus, in the experiment of Fig. 38-7, all we can know is that photons originate at the light source and vanish at the screen. Between source and screen, we cannot know what the photon is or does. However, because an interference pattern eventually builds up on the screen, we can speculate that each photon travels from source to screen *as a wave* that fills up the space between source and screen and then vanishes in a photon absorption at some point on the screen, with a transfer of energy and momentum to the screen at that point.

We *cannot* predict where this transfer will occur (where a photon will be detected) for any given photon originating at the source. However, we *can* predict the probability that a transfer will occur at any given point on the screen. Transfers will tend to occur (and thus photons will tend to be absorbed) in the regions of the bright fringes in the interference pattern that builds up on the screen. Transfers will tend *not* to occur (and thus photons will tend *not* to be absorbed) in the regions of the dark fringes in the built-up pattern. Thus, we can say that the wave traveling from the source to the screen is a *probability wave,* which produces a pattern of "probability fringes" on the screen.

The Single-Photon, Wide-Angle Version

In the past, physicists tried to explain the single-photon double-slit experiment in terms of small packets of classical light waves that are individually sent toward the slits. They would define these small packets as photons. However, modern experiments invalidate this explanation and definition. Figure 38-8 shows the arrangement of one of these experiments, reported in 1992 by Ming Lai and Jean-Claude Diels of the University of New Mexico. Source S contains molecules that emit photons at well separated times. Mirrors M_1 and M_2 are positioned to reflect light that the source emits along two distinct paths, 1 and 2, that are separated by an angle θ, which is close to 180°. This arrangement differs from the standard two-slit experiment, in which the angle between the paths of the light reaching two slits is very small.

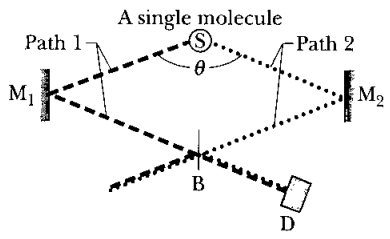


FIG. 38-8 The light from a single photon emission in source S travels over two widely separated paths and interferes with itself at detector D after being recombined by beam splitter B. (After Ming Lai and Jean-Claude Diels, *Journal of the Optical Society of America B*, **9**, 2290–2294, December 1992.)

After reflection from mirrors M_1 and M_2, the light waves traveling along paths 1 and 2 meet at beam splitter B. (A beam splitter is an optical device that transmits half the light incident upon it and reflects the other half.) On the right side of the beam splitter in Fig. 38-8, the light wave traveling along path 2 and reflected by B combines with the light wave traveling along path 1 and transmitted by B. These two waves then interfere with each other as they arrive at detector D (a *photomultiplier tube* that can detect individual photons).

The output of the detector is a randomly spaced series of electronic pulses, one for each detected photon. In the experiment, the beam splitter is moved slowly in a horizontal direction (in the reported experiment, a distance of only about 50 μm maximum), and the detector output is recorded on a chart recorder. Moving the beam splitter changes the lengths of paths 1 and 2, producing a phase shift between the light waves arriving at detector D. Interference maxima and minima appear in the detector's output signal.

This experiment is difficult to understand in traditional terms. For example, when a molecule in the source emits a single photon, does that photon travel along path 1 or path 2 in Fig. 38-8 (or along any other path)? Or can it move in both directions at once? To answer, we assume that when a molecule emits a photon, a probability wave radiates in all directions from it. The experiment samples this wave in two of those directions, chosen to be nearly opposite each other.

We see that we can interpret all three versions of the double-slit experiment if we assume that (1) light is generated in the source as photons, (2) light is absorbed in the detector as photons, and (3) light travels between source and detector as a probability wave.

38-6 | Electrons and Matter Waves

In 1924, French physicist Louis de Broglie made the following appeal to symmetry: A beam of light is a wave, but it transfers energy and momentum to matter only at points, via photons. Why can't a beam of particles have the same properties? That is, why can't we think of a moving electron—or any other particle—as a **matter wave** that transfers energy and momentum to other matter at points?

In particular, de Broglie suggested that Eq. 38-7 ($p = h/\lambda$) might apply not only to photons but also to electrons. We used that equation in Section 38-4 to assign a momentum p to a photon of light with wavelength λ. We now use it, in the form

$$\lambda = \frac{h}{p} \qquad \text{(de Broglie wavelength)}, \tag{38-9}$$

to assign a wavelength λ to a particle with momentum of magnitude p. The wavelength calculated from Eq. 38-9 is called the **de Broglie wavelength** of the moving particle. De Broglie's prediction of the existence of matter waves was first verified experimentally in 1927, by C. J. Davisson and L. H. Germer of the Bell Telephone Laboratories and by George P. Thomson of the University of Aberdeen in Scotland.

Figure 38-9 shows photographic proof of matter waves in a more recent experiment. In the experiment, an interference pattern was built up when electrons were sent, *one by one,* through a double-slit apparatus. The apparatus was like the ones we have previously used to demonstrate optical interference, except that the viewing screen was similar to a conventional television screen. When an electron hit the screen, it caused a flash of light whose position was recorded.

The first several electrons (top two photos) revealed nothing interesting and seemingly hit the screen at random points. However, after many thousands of electrons were sent through the apparatus, a pattern appeared on the screen, revealing fringes where many electrons had hit the screen and fringes where few had hit the screen. The pattern is exactly what we would expect for wave interference. Thus, *each* electron passed through the apparatus as a matter wave—the portion of the matter wave that traveled through one slit interfered with the portion that traveled through the other slit. That interference then determined the probability that the electron would materialize at a given point on the screen, hitting the screen there. Many electrons materialized in regions corresponding to bright fringes in optical interference, and few electrons materialized in regions corresponding to dark fringes.

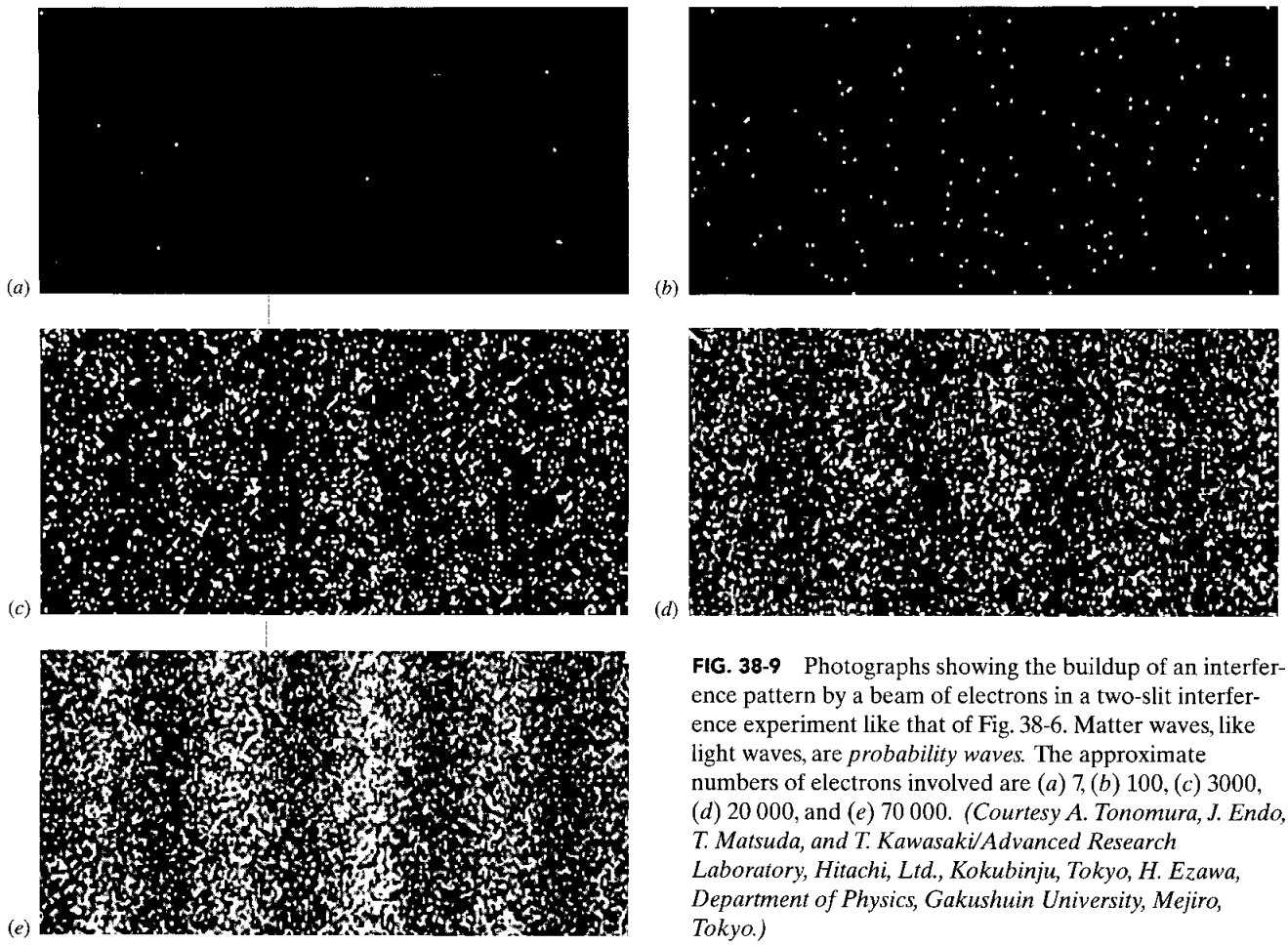
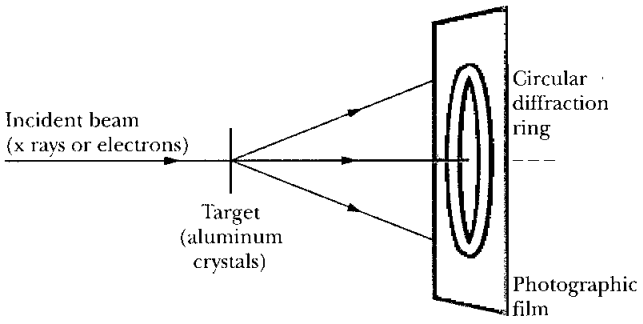


FIG. 38-9 Photographs showing the buildup of an interference pattern by a beam of electrons in a two-slit interference experiment like that of Fig. 38-6. Matter waves, like light waves, are *probability waves.* The approximate numbers of electrons involved are (*a*) 7, (*b*) 100, (*c*) 3000, (*d*) 20 000, and (*e*) 70 000. *(Courtesy A. Tonomura, J. Endo, T. Matsuda, and T. Kawasaki/Advanced Research Laboratory, Hitachi, Ltd., Kokubinju, Tokyo, H. Ezawa, Department of Physics, Gakushuin University, Mejiro, Tokyo.)*

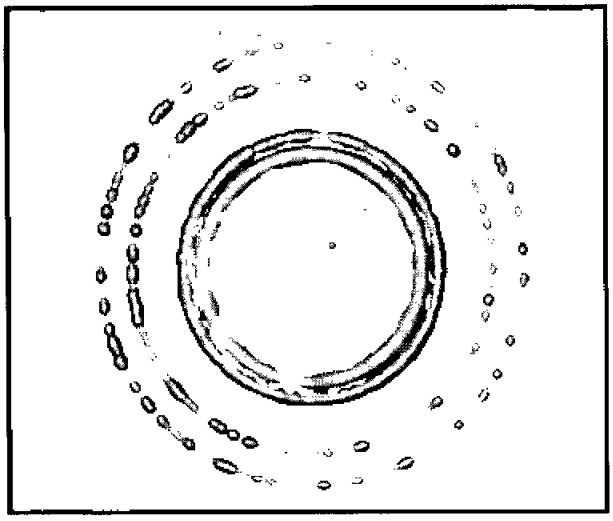
Similar interference has been demonstrated with protons, neutrons, and various atoms. In 1994, it was demonstrated with iodine molecules I_2, which are not only 500 000 times more massive than electrons but far more complex. In 1999, it was demonstrated with the even more complex *fullerenes* (or *buckyballs*) C_{60} and C_{70}. (Fullerenes are molecules of carbon atoms that are arranged in a structure resembling a soccer ball, 60 carbon atoms in C_{60} and 70 carbon atoms in C_{70}.) Apparently, such small objects as electrons, protons, atoms, and molecules travel as matter waves. However, as we consider larger and more complex objects, there must come a point at which we are no longer justified in considering the wave nature of an object. At that point, we are back in our familiar nonquantum world, with the physics of earlier chapters of this book. In short, an electron is a matter wave and can undergo interference with itself, but a cat is not a matter wave and cannot undergo interference with itself (which must be a relief to cats).

The wave nature of particles and atoms is now taken for granted in many scientific and engineering fields. For example, electron diffraction and neutron diffraction are used to study the atomic structures of solids and liquids, and electron diffraction is used to study the atomic features of surfaces on solids.

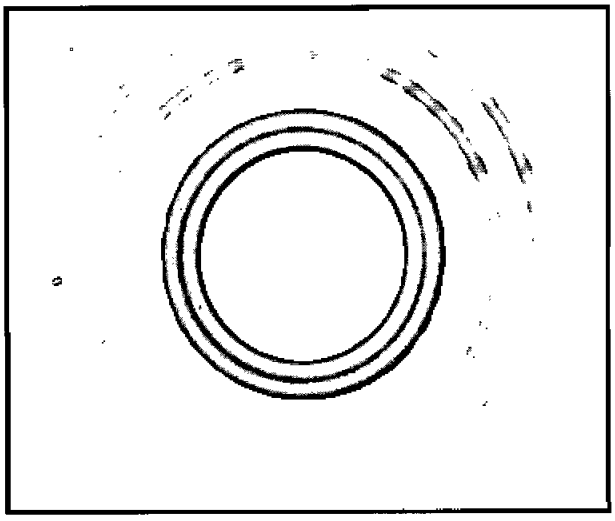
Figure 38-10*a* shows an arrangement that can be used to demonstrate the scattering of either x rays or electrons by crystals. A beam of one or the other is directed onto a target consisting of a layer of tiny aluminum crystals. The x rays have a certain wavelength λ. The electrons are given enough energy so that their de Broglie wavelength is the same wavelength λ. The scatter of x rays or electrons by the crystals produces a circular interference pattern on a photographic film. Figure 38-10*b* shows the pattern for the scatter of x rays, and Fig. 38-10*c* shows the pattern for the scatter of electrons. The patterns are the same—both x rays and electrons are waves.



(*a*)



(*b*)



(*c*)

FIG. 38-10 (*a*) An experimental arrangement used to demonstrate, by diffraction techniques, the wave-like character of the incident beam. Photographs of the diffraction patterns when the incident beam is (*b*) an x-ray beam (light wave) and (*c*) an electron beam (matter wave). Note that the two patterns are geometrically identical to each other. *(Photo (b) Cameca, Inc. Photo (c) from PSSC film "Matter Waves," courtesy Education Development Center, Newton, Massachusetts.)*

Waves and Particles

Figures 38-9 and 38-10 are convincing evidence of the *wave* nature of matter, but we have countless experiments that suggest its *particle* nature. Figure 38-11, for example, shows the tracks of particles (rather than waves) revealed in a bubble chamber. When a charged particle passes through the liquid hydrogen that fills such a chamber, the particle causes the liquid to vaporize along the particle's path. A series of bubbles thus marks the path, which is usually curved due to a magnetic field set up perpendicular to the plane of the chamber.

In Fig. 38-11, a gamma ray left no track when it entered at the top because the ray is electrically neutral. However, it collided with one of the hydrogen atoms, kicking an electron out of that atom; the curved path taken by the electron to the bottom of the photograph has been color coded green. Simultaneous with the collision, the gamma ray transformed into an electron and a positron that then moved in tight spirals (color coded green for the electron and red for the positron) as they gradually lost energy in repeated collisions with hydrogen atoms. Surely these tracks are evidence of the particle nature of the electron and positron, but is there any evidence of waves in Fig. 38-11?

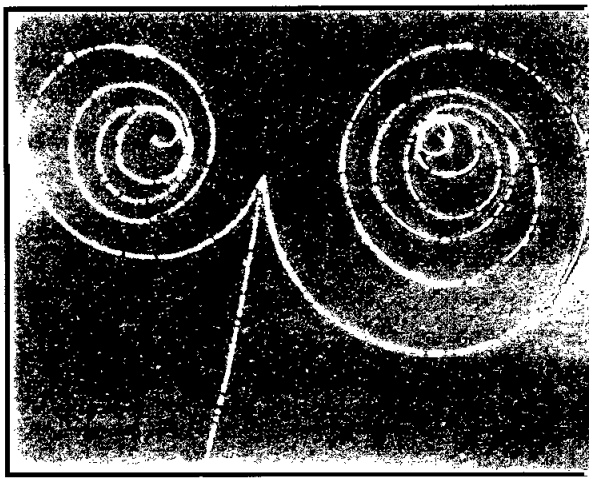


FIG. 38-11 A bubble-chamber image showing where two electrons (paths color coded green) and one positron (red) moved after a gamma ray entered the chamber. *(Lawrence Berkeley Laboratory/Science Photo Library/Photo Researchers)*

To simplify the situation, let us turn off the magnetic field so that the strings of bubbles will be straight. We can view each bubble as a detection point for the electron. Matter waves traveling between detection points such as I and F in Fig. 38-12 will explore all possible paths, a few of which are shown.

In general, for every path connecting I and F (except the straight-line path), there will be a neighboring path such that matter waves following the two paths cancel each other by interference. This is not true, however, for the straight-line

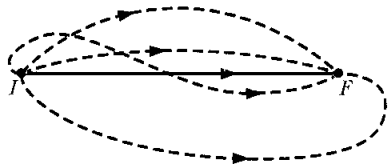


FIG. 38-12 A few of the many paths that connect two particle detection points I and F. Only matter waves that follow paths close to the straight line between these points interfere constructively. For all other paths, the waves following any pair of neighboring paths interfere destructively. Thus, a matter wave leaves a straight track.

path joining I and F; in this case, matter waves traversing all neighboring paths reinforce the wave following the direct path. You can think of the bubbles that form the track as a series of detection points at which the matter wave undergoes constructive interference.

CHECKPOINT 3 For an electron and a proton that have the same (a) kinetic energy, (b) momentum, or (c) speed, which particle has the shorter de Broglie wavelength?

Sample Problem 38-8 Optional study

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

Conceptualize/Classify: (1) We can find the electron's de Broglie wavelength λ from Eq. 38-9 ($\lambda = h/p$) if we first find the magnitude of its momentum p. (2) We find p from the given kinetic energy K of the electron. That kinetic energy is much less than the rest energy of an electron (0.511 MeV, from Table 37-3). Thus, we can get by with the classical approximations for momentum p ($= mv$) and kinetic energy K ($= \frac{1}{2}mv^2$).

Compute: In order to use the de Broglie relation, we first solve the kinetic energy equation for v and then substitute into the momentum equation, finding

$$p = \sqrt{2mK}$$
$$= \sqrt{(2)(9.11\times10^{-31}\ \text{kg})(120\ \text{eV})(1.60\times10^{-19}\ \text{J/eV})}$$
$$= 5.91\times10^{-24}\ \text{kg}\cdot\text{m/s}.$$

From Eq. 38-9 then

$$\lambda = \frac{h}{p}$$
$$= \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{5.91\times10^{-24}\ \text{kg}\cdot\text{m/s}}$$
$$= 1.12\times10^{-10}\ \text{m} = 112\ \text{pm}. \qquad \text{(Answer)}$$

Conclude: This wavelength associated with the electron is about the size of a typical atom. If we increase the electron's kinetic energy, the wavelength becomes even smaller.

What we would like to emphasize is that electron and light possess both wave-like features as well as particle-like features. But essentially, both of them are quite different from each other. For example, the momentum of both of them is given by $p = h/\lambda$. But the energy of a photon is given by hc/λ whereas the energy of an electron does not have a direct relation with the wavelength. Similarly, the velocity of a photon is given by $c = f\lambda$, whereas the velocity of an electron does not have a fixed value. Electrons have charge but photons do not have any charge.

38-7 | Schrödinger's Equation

A simple traveling wave of any kind, be it a wave on a string, a sound wave, or a light wave, is described in terms of some quantity that varies in a wave-like fashion. For light waves, for example, this quantity is $\vec{E}(x, y, z, t)$, the electric field component of the wave. Its observed value at any point depends on the location of that point and on the time at which the observation is made.

What varying quantity should we use to describe a matter wave? We should expect this quantity, which we call the **wave function** $\Psi(x, y, z, t)$, to be more

complicated than the corresponding quantity for a light wave because a matter wave, in addition to energy and momentum, transports mass and (often) electric charge. It turns out that Ψ, the uppercase Greek letter psi, usually represents a function that is complex in the mathematical sense; that is, we can always write its values in the form $a + ib$, in which a and b are real numbers and $i^2 = -1$.

In all the situations you will meet here, the space and time variables can be grouped separately and Ψ can be written in the form

$$\Psi(x, y, z, t) = \psi(x, y, z)\, e^{-i\omega t}, \tag{38-10}$$

where $\omega (= 2\pi f)$ is the angular frequency of the matter wave. Note that ψ, the lowercase Greek letter psi, represents only the space-dependent part of the complete, time-dependent wave function Ψ. We shall deal almost exclusively with ψ. Two questions arise: What is meant by the wave function? How do we find it?

What does the wave function mean? It has to do with the fact that a matter wave, like a light wave, is a probability wave. Suppose that a matter wave reaches a particle detector that is small; then the probability that a particle will be detected in a specified time interval is proportional to $|\psi|^2$, where $|\psi|$ is the absolute value of the wave function at the location of the detector. Although ψ is usually a complex quantity, $|\psi|^2$ is always both real and positive. It is, then, $|\psi|^2$, which we call the **probability density,** and not ψ, that has *physical* meaning. Speaking loosely, the meaning is this:

☞ The probability (per unit time) of detecting a particle in a small volume centered on a given point in a matter wave is proportional to the value of $|\psi|^2$ at that point.

Because ψ is usually a complex quantity, we find the square of its absolute value by multiplying ψ by ψ^*, the *complex conjugate* of ψ. (To find ψ^* we replace the imaginary number i in ψ with $-i$, wherever it occurs.)

How do we find the wave function? Sound waves and waves on strings are described by the equations of Newtonian mechanics. Light waves are described by Maxwell's equations. Matter waves are described by **Schrödinger's equation,** advanced in 1926 by Austrian physicist Erwin Schrödinger.

Many of the situations that we shall discuss involve a particle traveling in the x direction through a region in which forces acting on the particle cause it to have a potential energy $U(x)$. In this special case, Schrödinger's equation reduces to

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0 \qquad \begin{matrix}\text{(Schrödinger's equation,}\\ \text{one-dimensional motion),}\end{matrix} \tag{38-11}$$

in which E is the total mechanical energy (potential energy plus kinetic energy) of the moving particle. (We do *not* consider mass energy in this nonrelativistic equation.) We cannot derive Schrödinger's equation from more basic principles; it *is* the basic principle.

If $U(x)$ in Eq. 38-11 is zero, that equation describes a **free particle**—that is, a moving particle on which no net force acts. The particle's total energy in this case is all kinetic, and thus E in Eq. 38-11 is $\frac{1}{2}mv^2$. That equation then becomes

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left(\frac{mv^2}{2}\right)\psi = 0,$$

which we can recast as

$$\frac{d^2\psi}{dx^2} + \left(2\pi\frac{p}{h}\right)^2\psi = 0.$$

To obtain this equation, we replaced mv with the momentum p and regrouped terms.

From Eq. 38-9 ($\lambda = h/p$) we recognize p/h in the equation above as $1/\lambda$, where λ is the de Broglie wavelength of the moving particle. We further recognize $2\pi/\lambda$ as the *angular wave number* k, which we defined in Eq. 16-7, Vol. I, with these substitutions, the equation above becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad \text{(Schrödinger's equation, free particle).} \tag{38-12}$$

The most general solution of Eq. 38-12 is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{38-13}$$

in which A and B are arbitrary constants. You can show that this equation is indeed a solution of Eq. 38-12 by substituting $\psi(x)$ and its second derivative into that equation and noting that an identity results.

If we combine Eqs. 38-10 and 38-13 we find, for the time-dependent wave function Ψ of a free particle traveling in the x direction,

$$\begin{aligned}\Psi(x,t) &= \psi(x)e^{-i\omega t} = (Ae^{ikx} + Be^{-ikx})e^{-i\omega t} \\ &= Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}.\end{aligned} \tag{38-14}$$

Finding the Probability Density $|\psi|^2$

In Section 16-5, Vol. I, we saw that *any function* F of the form $F(kx \pm \omega t)$ represents a traveling wave. This applies to exponential functions like those in Eq. 38-14 as well as to the sinusoidal functions we have used to describe waves on strings. In fact, these two representations of functions are related by

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta,$$

where θ is any angle.

The first term on the right in Eq. 38-14 thus represents a wave traveling in the positive direction of x and the second term represents a wave traveling in the negative direction of x. However, we have assumed that the free particle we are considering travels only in the positive direction of x. To reduce the general solution (Eq. 38-14) to our case of interest, we choose the arbitrary constant B in Eqs. 38-14 and 38-13 to be zero. At the same time, we relabel the constant A as ψ_0. Equation 38-13 then becomes

$$\psi(x) = \psi_0 e^{ikx}. \tag{38-15}$$

To calculate the probability density, we take the square of the absolute value of $\psi(x)$. We get

$$|\psi|^2 = |\psi_0 e^{ikx}|^2 = (\psi_0^2)|e^{ikx}|^2.$$

Now, because

$$|e^{ikx}|^2 = (e^{ikx})(e^{ikx})^* = e^{ikx}e^{-ikx} = e^{ikx-ikx} = e^0 = 1,$$

we get

$$|\psi|^2 = (\psi_0^2)(1)^2 = \psi_0^2 \qquad \text{(a constant).}$$

Figure 38-13 is a plot of the probability density $|\psi|^2$ versus x for a free particle—a straight line parallel to the x axis from $-\infty$ to $+\infty$. We see that the probability density $|\psi|^2$ is the same for all values of x, which means that the particle has equal probabilities of being *anywhere* along the x axis. There is no distinguishing feature by which we can predict a most likely position for the particle. That is, all positions are equally likely.

We'll see what this means in the next section.

38-8 | Heisenberg's Uncertainty Principle

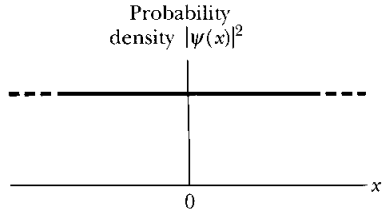


FIG. 38-13 A plot of the probability density $|\psi|^2$ for a free particle moving in the positive x direction. Since $|\psi|^2$ has the same constant value for all values of x, the particle has the same probability of detection at all points along its path.

Our inability to predict the position of a free particle, as indicated by Fig. 38-13, is our first example of **Heisenberg's uncertainty principle**, proposed in 1927 by German physicist Werner Heisenberg. It states that measured values cannot be assigned to the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision.

For the components of $\vec{r}$ and $\vec{p}$, Heisenberg's principle gives the following limits in terms of $\hbar = h/2\pi$ (called "h-bar"):

$$\begin{aligned} \Delta x \cdot \Delta p_x &\geq \hbar \\ \Delta y \cdot \Delta p_y &\geq \hbar \qquad \text{(Heisenberg's uncertainty principle).} \\ \Delta z \cdot \Delta p_z &\geq \hbar \end{aligned} \tag{38-16}$$

Here Δx and Δp_x represent the intrinsic uncertainties in the measurements of the x components of $\vec{r}$ and $\vec{p}$, with parallel meanings for the y and z terms. Even with the best measuring instruments that technology could ever provide, each product of a position uncertainty and a momentum uncertainty in Eq. 38-16 will be greater than $\hbar$; it can *never* be less.

The particle whose probability density is plotted in Fig. 38-13 is a free particle; that is, no force acts on it, and so its momentum $\vec{p}$ must be constant. We implied—without making a point of it—that we can determine $\vec{p}$ with absolute precision; in other words, we assumed that $\Delta p_x = \Delta p_y = \Delta p_z = 0$ in Eq. 38-16. That assumption then requires $\Delta x \to \infty$, $\Delta y \to \infty$, and $\Delta z \to \infty$. With such infinitely great uncertainties, the position of the particle is completely unspecified.

Do not think that the particle *really has* a sharply defined position that is, for some reason, hidden from us. If its momentum can be specified with absolute precision, the words "position of the particle" simply lose all meaning. The particle in Fig. 38-13 can be found *with equal probability* anywhere along the x axis.

Sample Problem 38-9

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

Conceptualize: We need only consider components along the x axis because we have motion only along that axis and want the uncertainty Δx in location along that axis. Since we want the minimum allowed uncertainty, we use the equality instead of the inequality in the x-axis part of Eq. 38-16, writing $\Delta x \cdot \Delta p_x = \hbar$.

Classify: The minimum uncertainty allowed by quantum theory is given by Heisenberg's uncertainty principle in Eq. 38-16.

Compute: To evaluate the uncertainty Δp_x in the momentum, we must first evaluate the momentum component p_x. Because the electron's speed v_x is much less than the speed of light c, we can evaluate p_x with the classical expression for momentum instead of using a relativistic expression. We find

$$\begin{aligned} p_x &= mv_x = (9.11\times 10^{-31}\,\text{kg})(2.05 \times 106\,\text{m/s}) \\ &= 1.87 \times 10^{-24}\,\text{kg}\cdot\text{m/s}. \end{aligned}$$

The uncertainty in the speed is given as 0.50% of the measured speed. Because p_x depends directly on speed, the uncertainty Δp_x in the momentum must be 0.50% of the momentum:

$$\begin{aligned} \Delta p_x &= (0.0050)p_x \\ &= (0.0050)(1.87\times 10^{-24}\,\text{kg}\cdot\text{m/s}) \\ &= 9.35\times 10^{-27}\,\text{kg}\cdot\text{m/s}. \end{aligned}$$

Then the uncertainty principle gives us

$$\begin{aligned} \Delta x = \frac{\hbar}{\Delta p_x} &= \frac{(6.63\times 10^{-34}\,\text{J}\cdot\text{s})/2\pi}{9.35\times 10^{-27}\,\text{kg}\cdot\text{m/s}} \\ &= 1.13\times 10^{-8}\,\text{m} \approx 11\,\text{nm}, \qquad \text{(Answer)} \end{aligned}$$

which is about 100 atomic diameters.

Conclude: Given your measurement of the electron's speed, it makes no sense to try to pin down the electron's position to any greater precision.

38-9 | Barrier Tunneling

Suppose you slide a puck over frictionless ice toward an ice-covered hill (Fig. 38-14). As the puck climbs the hill, kinetic energy K is transformed into gravitational potential energy U. If the puck reaches the top, its potential energy is U_b. Thus, the puck can pass over the top only if its initial mechanical energy $E > U_b$. Otherwise, the puck eventually stops its climb up the left side of the hill and slides back to the left. For instance, if $U_b = 20$ J and $E = 10$ J, you cannot expect the puck to pass over the hill. We say that the hill acts as a **potential energy barrier** (or, for short, a **potential barrier**) and that, in this case, the barrier has a *height* of $U_b = 20$ J.

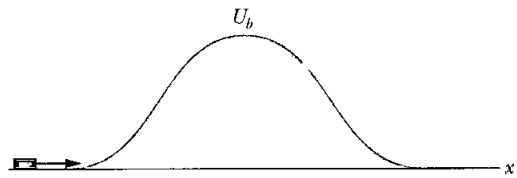


FIG. 38-14 A puck slides over frictionless ice toward a hill. The puck's gravitational potential at the top of the hill will be U_b.

Figure 38-15 shows a potential barrier for a nonrelativistic electron traveling along an idealized wire of negligible thickness. The electron, with mechanical energy E, approaches a region (the barrier) in which the electric potential V_b is negative. Because it is negatively charged, the electron will have a positive potential energy $U_b (= qV_b)$ in that region (Fig. 38-16). If $E > U_b$, we expect the electron to pass through the barrier region and come out to the right of $x = L$ in Fig. 38-15. Nothing surprising there. If $E < U_b$, we expect the electron to be unable to pass through the barrier region. Instead, it should end up traveling leftward, much as the puck would slide back down the hill in Fig. 38-14 if the puck has $E < U_b$.

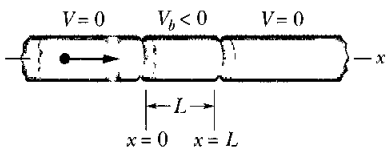


FIG. 38-15 The elements of an idealized thin wire in which an electron (the dot) approaches a negative electric potential V_b in the region $x = 0$ to $x = L$.

However, something astounding can happen to the electron when $E < U_b$. Because it is a matter wave, the electron has a finite probability of leaking (or, better, *tunneling*) through the barrier and materializing on the other side, moving rightward with energy E as though nothing (strange or otherwise) had happened in the region of $0 \le x \le L$.

The wave function $\psi(x)$ describing the electron can be found by solving Schrödinger's equation (Eq. 38-11) separately for the three regions in Fig. 38-15: (1) to the left of the barrier, (2) within the barrier, and (3) to the right of the barrier. The arbitrary constants that appear in the solutions can then be chosen so that the values of $\psi(x)$ and its derivative with respect to x join smoothly (no jumps, no kinks) at $x = 0$ and at $x = L$. Squaring the absolute value of $\psi(x)$ then yields the probability density.

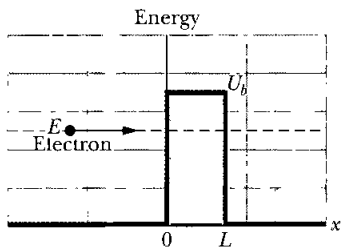


FIG. 38-16 An energy diagram containing two plots for the situation of Fig. 38-14: (1) The electron's mechanical energy E is plotted when the electron is at any coordinate $x < 0$. (2) The electron's electric potential energy U is plotted as a function of the electron's position x, *assuming* that the electron can reach any value of x. The nonzero part of the plot (the potential barrier) has height U_b and thickness L.

Figure 38-17 shows a plot of the result. The oscillating curve to the left of the barrier (for $x < 0$) is a combination of the incident matter wave and the reflected matter wave (which has a smaller amplitude than the incident wave). The oscillations occur because these two waves, traveling in opposite directions, interfere with each other, setting up a standing wave pattern.

Within the barrier (for $0 < x < L$) the probability density decreases exponentially with x. However, if L is small, the probability density is not quite zero at $x = L$.

To the right of the barrier (for $x > L$), the probability density plot describes a transmitted (through the barrier) wave with low but constant amplitude. Thus, the electron can be detected in this region but with a relatively small probability. (Compare this part of the figure with Fig. 38-13 for a free particle.)

We can assign a **transmission coefficient** T to the incident matter wave and the barrier. This coefficient gives the probability with which an approaching electron will be transmitted through the barrier—that is, that tunneling will occur.

As an example, if $T = 0.020$, then of every 1000 electrons fired at the barrier, 20 (on average) will tunnel through it and 980 will be reflected. The transmission coefficient T is approximately

$$T \approx e^{-2bL}, \tag{38-17}$$

in which

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}, \tag{38-18}$$

and e is the exponential function. Because of the exponential form of Eq. 38-17, the value of T is very sensitive to the three variables on which it depends: particle mass m, barrier thickness L, and energy difference $U_b - E$. (Because we do not include relativistic effects here, E does not include mass energy.)

Barrier tunneling finds many applications in technology, including the tunnel diode, in which a flow of electrons produced by tunneling can be rapidly turned on or off by controlling the barrier height. The 1973 Nobel Prize in physics was shared by three "tunnelers," Leo Esaki (for tunneling in semiconductors), Ivar Giaever (for tunneling in superconductors), and Brian Josephson (for the Josephson junction, a rapid quantum switching device based on tunneling). The 1986 Nobel Prize was awarded to Gerd Binnig and Heinrich Rohrer for development of the scanning tunneling microscope.

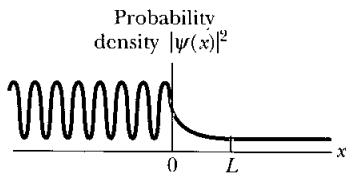


FIG. 38-17 A plot of the probability density $|\psi|^2$ of the electron matter wave for the situation of Fig. 38-16. The value of $|\psi|^2$ is nonzero to the right of the potential barrier.

CHECKPOINT 4 Is the wavelength of the transmitted wave in Fig. 38-17 larger than, smaller than, or the same as that of the incident wave?

Sample Problem 38-10

Suppose that the electron in Fig. 38-16, having a total energy E of 5.1 eV, approaches a barrier of height $U_b = 6.8$ eV and thickness $L = 750$ pm.

(a) What is the approximate probability that the electron will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

Conceptualize/Classify: The probability we seek is the transmission coefficient T as given by Eq. 38-17 ($T \approx e^{-2bL}$), where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}.$$

Compute: The numerator of the fraction under the square-root sign is

$$(8\pi^2)(9.11 \times 10^{-31}\ \text{kg})(6.8\ \text{eV} - 5.1\ \text{eV}) \times (1.60 \times 10^{-19}\ \text{J/eV}) = 1.956 \times 10^{-47}\ \text{J}\cdot\text{kg}.$$

Thus, $$b = \sqrt{\frac{1.956 \times 10^{-47}\ \text{J}\cdot\text{kg}}{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2}} = 6.67 \times 10^9\ \text{m}^{-1}.$$

The (dimensionless) quantity $2bL$ is then

$$2bL = (2)(6.67 \times 10^9\ \text{m}^{-1})(750 \times 10^{-12}\ \text{m}) = 10.0$$

and, from Eq. 38-17, the transmission coefficient is

$$T \approx e^{-2bL} = e^{-10.0} = 45 \times 10^{-6}. \qquad \text{(Answer)}$$

Conclude: Thus, of every million electrons that strike the barrier, about 45 will tunnel through it.

(b) What is the approximate probability that a proton with the same total energy of 5.1 eV will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

Reasoning: The transmission coefficient T (and thus the probability of transmission) depends on the mass of the particle. Indeed, because mass m is one of the factors in the exponent of e in the equation for T, the probability of transmission is very sensitive to the mass of the particle. This time, the mass is that of a proton (1.67×10^{-27} kg), which is significantly greater than that of the electron in (a). By substituting the proton's mass for the mass in (a) and then continuing as we did there, we find that $T \approx 10^{-186}$. Thus, although the probability that the proton will be transmitted is not exactly zero, it is barely more than zero. For even more massive particles with the same total energy of 5.1 eV, the probability of transmission is exponentially lower.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: If two photons traveling in vacuum have the same wavelength, they have the same momentum.
Statement 2: All the photons travel in vacuum with the same speed.

2. Statement 1: If two electrons have the same wavelength, they have the same kinetic energy.
Statement 2: by de Broglie hypothesis, $p = h/\lambda$

3. Statement 1: An electron in free space cannot absorb a photon completely.
Statement 2: In any process, energy as well as momentum must be conserved.

4. Statement 1: A beam of light of frequency above cut-off frequency is incident on an isolated metallic sphere. After some time, the charge on the sphere becomes constant.
Statement 2: For a photoelectron to be emitted, its energy must be greater than the work function.

5. Statement 1: By de Broglie hypothesis momentum of an electron, $p = h/\lambda$
Statement 2: Energy of an electron is given by $E = hc/\lambda$

6. Statement 1: If the frequency of the incident light in photoelectric effect experiment is doubled, keeping the intensity constant, the saturation current also doubles.
Statement 2: If the frequency of the incident light is doubled, the kinetic energy of the emitted photoelectron increases.

7. Statement 1: The tail of a comet always points away from the Sun.
Statement 2: The radiation pressure on the particles of comet tail is in the direction of motion of the photons emitted by the Sun.

MULTIPLE CHOICE QUESTIONS

sec. 38-2 The Photon, the Quantum of Light

1. Two separate segments of equal area are isolated in the energy distribution of black body radiation (Fig. 38-18). Let us assume that the number of photons emitted by the body per unit time in the segments is n_1 and n_2, and the energy of the photons are E_1 and E_2, respectively. Then

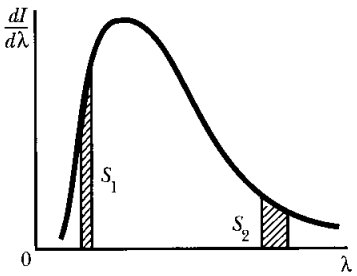


FIG. 38-18 Multiple choice question 1.

(a) $n_1 > n_2$

(b) $n_1 < n_2$

(c) $E_1 < E_2$

(d) $E_1 > E_2$

2. Let n_r and n_b be the number of photons emitted by a red bulb and a blue bulb, respectively, of equal power in a given time. Then

(a) $n_r = n_b$

(b) $n_r > n_b$

(c) $n_r < n_b$

(d) The information is insufficient to get a relation between n_r and n_b

3. Consider tuning a radio to higher frequency station. The energy of radio photons detected

(a) Is zero because radio waves do not carry energy

(b) Is smaller

(c) Is greater

(d) Is the same as before, but the radio waves are of higher frequency

sec. 38-3 The Photoelectric Effect

4. A beam of ultraviolet light is incident on the metal ball of an electroscope that is initially uncharged. Does the electroscope acquire a charge?

(a) Yes, it acquires a positive charge

(b) Yes, it acquires a negative charge

(c) No, it does not acquire a charge

(d) Its acquiring the charge depends on the work function of the metal

5. A xenon arc lamp is covered with an interference filter that only transmits light of 400 nm wavelength. When the transmitted light strikes a metal surface, a stream of electrons emerges from the metal. If the intensity of the light striking the surface is doubled, then

(a) More electrons are emitted in a given time interval

(b) The electrons that are emitted are more energetic

(c) Both of the above

(d) Neither of the above

6. A xenon arc lamp is covered with an interference filter that only transmits light of 400 nm wavelength. When the transmitted light strikes a metal surface, a stream of electrons emerges from the metal. The interference filter is then replaced with one transmitting at 300 nm and the lamp adjusted so that the

intensity of the light striking the surface is the same as it was for the 400 nm light. With the 300-nm light

(a) More electrons are emitted in a given time interval

(b) The electrons that are emitted are more energetic

(c) Both of the above

(d) Neither of the above

7. A girl is performing a photoelectric experiment but no photoelectrons are being ejected. To eject electrons, she should change the light by

(a) Decreasing the frequency

(b) Increasing the frequency

(c) Increasing the intensity

(d) Increasing the wavelength

8. In Fig. 38-19 gives the stopping voltage V versus the wavelength λ of light for three different materials. Rank the materials according to their work function, greatest first.

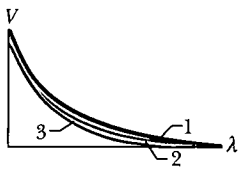


FIG. 38-19 Multiple choice question 8.

(a) $\phi_3 > \phi_2 > \phi_1$

(b) $\phi_1 > \phi_2 > \phi_3$

(c) $\phi_2 > \phi_3 > \phi_1$

(d) $\phi_3 > \phi_1 > \phi_2$

9. A non-monochromatic light is used in an experiment on photoelectric effect. The stopping potential is

(a) Related to the mean wavelength

(b) Related to the longest wavelength

(c) Related to the shortest wavelength

(d) Not related to the wavelength

10. Figure 38-20 shows the variation of photocurrent with potential of the collector plate for a photosensitive surface for three different radiations. Let I_a, I_b, and I_c be the intensities and f_a, f_b, and f_c be the frequencies for the curves a, b, and c, respectively.

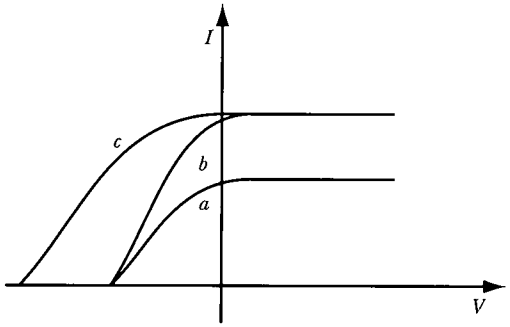


FIG. 38-20 Multiple choice question 10.

(a) $f_a = f_b$ and $I_a = I_b$

(b) $f_a = f_c$ and $I_a = I_c$

(c) $f_a = f_b$ and $I_a < I_b$

(d) $f_a = f_c$ and $I_b = I_c$

11. At the stopping potential

(a) The electrons are not at all emitted by the plate

(b) The electrons are emitted by the plate but reverse their direction of motion just before reaching the collector plate

(c) The most energetic electrons emitted reverse their direction of motion just before reaching the other plate

(d) The collector plate is negatively charged with respect to the emitter

12. In a photoelectric tube, the voltage applied is such that the photoelectrons emitted reverse in direction after traveling half the distance. If we keep the voltage same but halve the distance, then

(a) The photoelectrons reverse the direction at half of the new distance

(b) The photoelectrons are just able to reach the other electrode

(c) The photoelectrons reverse the direction at one fourth of the new distance

(d) The photoelectrons reach the other electrode with some kinetic energy

13. Which of the following are not dependent on the intensity of the incident radiation in a photoelectric experiment?

(a) Amount of photoelectric current

(b) Stopping potential to reduce the photoelectric current to zero

(c) Work function of the surface

(d) Maximum kinetic energy of photoelectrons

14. An image of the Sun is formed by a lens of focal length of 30 cm on the metal surface of a photoelectric cell and a photoelectric current i is produced. The lens forming the image is then replaced by another of the same diameter but of focal length of 15 cm. The photoelectric current in this case is

(a) $i/2$

(b) i

(c) $2i$

(d) $4i$

sec. 38-4 Photons Have Momentum

15. If a parallel beam of light having intensity I is incident normally on a perfectly reflecting surface, the force exerted on the surface, equal F. When the surface is held at an angle θ, the force is

(a) $2F \tan \theta$

(b) $F \cos \theta$

(c) $F \cos^2 \theta$

(d) $2F$

16. The difference between the photons and other material particles that we know of is that

(a) The photon is always chargeless whereas all the material particles have some charge

(b) The rest mass of photon is zero whereas all the material particles have some mass

(c) Photon cannot be accelerated by applying some force

(d) Photons have a wave associated with them but material particles cannot be associated with a wave

sec. 38-6 Electrons and Matter Waves

17. A beta particle, gamma ray, and alpha particle all have the same momentum. Which has the longest wavelength?

(a) Beta particle

(b) Gamma ray

(c) Alpha particle

(d) All the same

18. If a proton and an electron have the same kinetic energy, which has the longer de Broglie wavelength?

(a) Electron

(b) Proton

(c) Both have the same wavelength

19. The equation $E = pc$ is valid

(a) For an electron

(b) For a photon

(c) For a photon as well as for an electron

(d) Neither for an electron nor for a photon

20. A particle of mass $4m$ at rest decays into two particles of masses m and $3m$ having nonzero velocities. The ratio of the de Broglie wavelengths of the particles 1 and 2 is

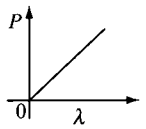
(a) 1/2

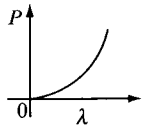
(b) 1/4

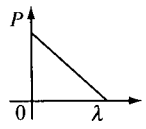
(c) 2

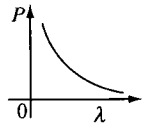
(d) 1

21. Which of the following graph represent the variation of particle momentum and associated de Broglie wavelength?

(a) 

(b) 

(c) 

(d) 

22. A beam of electrons with speed v_0 passes through double slits and then is allowed to strike a fluorescent screen. An interference pattern is observed on the screen.

(i) The speed of the electrons is increased to $2v_0$. What happens to the spacing of the interference fringes on the screen?

(a) The spacing increases

(b) The spacing decreases

(c) The spacing remains the same

(ii) The electrons are instead replaced by protons moving with speed v_0. Compared to electrons at the same speed, what happens to the spacing of the interference fringes on the screen?

(a) The spacing increases

(b) The spacing decreases

(c) The spacing remains the same

23. The frequency of matter waves is expressed as

(a) Hf

(b) E/h

(c) $1/T$

(d) None of these

sec. 38-7 Schrödinger's Equation

24. $\psi(x)$ is the wave function for a particle moving along the x axis. The probability that the particle is in the interval from $x = a$ to $x = b$ is given by

(a) $\psi(b) - \psi(a)$

(b) $|\psi(b)|/|\psi(a)|$

(c) $|\psi(b)|^2/|\psi(a)|^2$

(d) $\int_a^b \psi(x)\,dx$

(e) $\int_a^b |\psi(x)|^2\,dx$

25. The significance of $|\psi|^2 x$ is

(a) Probability

(b) Energy

(c) Probability density

(d) Energy density

(e) Wavelength

sec. 38-8 Heisenberg's Uncertainty Principle

26. A free electron in motion along the x axis has a localized wave function. The uncertainty in its momentum is decreased if

(a) The wave function is made more narrow

(b) The wave function is made less narrow

(c) The wave function remains the same but the energy of the electron is increased

(d) The wave function remains the same but the energy of the electron is decreased

(e) None of the above

27. The uncertainty in position of an electron in a certain state is 5×10^{-10} m. The uncertainty in its momentum might be

(a) 5.0×10^{-24} kg·m/s

(b) 4.0×10^{-24} kg·m/s

(c) 3.0×10^{-24} kg·m/s

(d) All of the above

(e) None of the above

sec. 38-9 Barrier Tunneling

28. The reflection coefficient R for a certain barrier tunneling problem is 0.80. The corresponding transmission coefficient T is

(a) 0.80
(b) 0.60
(c) 0.50
(d) 0.20
(e) 0

29. An electron with energy E is incident on a potential energy barrier of height E_{pot} and thickness L. The probability of tunneling increases if

(a) E decreases without any other changes
(b) E_{pot} increases without any other changes
(c) L decreases without any other changes
(d) E and E_{pot} increase by the same amount
(e) E and E_{pot} decrease by the same amount

30. An electron with energy E is incident upon a potential energy barrier of height $E_{pot} > E$ and thickness L. The transmission coefficient T

(a) Is zero
(b) Decreases exponentially with L
(c) Is proportional to $1/L$
(d) Is proportional to $1/L^2$
(e) Is nonzero and independent of L

PROBLEMS

sec. 38-2 The Photon, the Quantum of Light

1. A helium–neon laser emits red light having a wavelength of 621.0 nm and a power of 0.50 mW. How many photons are emitted each second?

2. Solar radiation fall on Earth's surface at a rate of 1400 W/m^2. Assuming that the radiation has an average wavelength of 550 nm, how many photons per square meter per second fall on the surface?

3. Find the number of photons per cubic meter in a monochromatic beam of radiation of intensity 10^{-5} W/cm^2. Take the wavelength to be 0.1 Å.

4. Find the energy of 700-nm photon.

5. How fast must an electron move to have a kinetic energy equal to the photon energy of sodium light at wavelength 590 nm?

6. A 1.00 kW radio transmitter operates at a frequency of 880 kHz. How many photons per seconds does it emit?

7. The night-adapted eye can detect as few as several photons per second say five to be definite. Assuming that the pupil of the eye is 0.6 cm in diameter, from what distance would it be possible for the naked eye to detect the 60-W light bulb (take the wavelength of the light to be 600 nm)?

sec. 38-3 The Photoelectric Effect

8. The work function of tungsten is 4.50 eV. Calculate the speed of the fastest electrons ejected from a tungsten surface when light whose photon energy is 5.80 eV shines on the surface.

9. When ultraviolet photons with a wavelength of 2.484×10^{-7} m are incident on an unknown metal surface in a vacuum, electrons with a maximum kinetic energy of 1.52 eV are emitted from the surface. What is the work function of the metal?

10. Molybdenum has a work function of 4.20 eV. (a) Find the cutoff wavelength and cutoff frequency for the photoelectric effect. (b) What is the stopping potential if the incident light has a wavelength of 180 nm?

11. Two light sources are used in a photoelectric experiment to determine the work function for a particular metal surface. When green light from a mercury lamp (λ = 546.1nm) is used, a stopping potential of 0.376 V reduces the photocurrent to zero. (a) Based on this measurement, what is the work function for this metal? (b) What stopping potential is observed when using the yellow light from a helium discharge tube (λ = 587.5 nm)?

12. A metal has work function 4.7 eV. What is the kinetic energy of a photoelectrons if radiation of wavelength 200 nm falls on the surface of the metal?

13. 1.5 mW of 400-nm light is directed on a photoelectric cell. If 0.10% of the incident photons produce photoelectrons, find the current in the cell.

14. Suppose the *photoelectric efficiency* of a cesium surface (with work function 1.80 eV) is 1.0×10^{-16}. What would be the current of electrons ejected from such a surface if it were illuminated with 621 nm light from a 2.00 mW laser and all the ejected electrons took part in the charge flow?

15. X rays with a wavelength of 62.1 pm are directed onto a gold foil and eject tightly bound electrons from the gold atoms. The ejected electrons then move in circular paths of radius r in a region of uniform magnetic field $\vec{B}$. For the fastest of the ejected electrons, the product Br is equal to 4.00×10^{-4} Tm. Find (a) the maximum kinetic energy of those electrons and (b) the work done in removing them from the gold atoms.

16. A monochromatic light source of power 5 mW emits 8×10^{15} photons per second. This light ejects photoelectrons from metal surface. The stopping potential for this setup is 2.0 V. Calculate the work function of the metal.

17. Find the (a) maximum kinetic energy of photoelectrons ejected from potassium surface, for which $\phi_0 = 2.1$ eV, by photons of wavelengths 2000 Å and 5000 Å. (b) What is the threshold frequency ν_0 and (c) the corresponding wavelength?

18. In a photoelectric experiment, the collector plate is at 2.0 V with respect to the target plate made of copper (ϕ = 4.5 eV). The target is illuminated by a source of monochromatic light of wavelength 200 nm. Find the (a) minimum and (b) maximum kinetic energy of the photoelectrons reaching the collector.

19. An isolated copper sphere of radius 5.0 cm, initially uncharged, is illuminated by ultraviolet light of wavelength 200 nm. What charge will the photoelectric effect induce on the sphere? The work function for copper is 4.70 eV.

20. Photons of wavelength λ are incident on a metal. The most energetic electrons ejected from the metal are bent into

a circular arc of radius R by a magnetic field having magnitude B. What is the work function of the metal?

21. Light described at a place by the equation $E = (100\ \text{V/m})[\sin(5 \times 10^{15}\ \text{s}^{-1})t + \sin(8 \times 10^{15}\ \text{s}^{-1})t]$ falls on a metal surface having work function 2.0 eV. Calculate the maximum kinetic energy of the photoelectrons.

sec. 38-4 Photons Have Momentum

22. What is the radiation pressure on a plane surface 1.5 m away from a 500 W lightbulb? Assume that the surface on which the pressure is exerted faces the bulb and is perfectly absorbing and that the bulb radiates uniformly in all directions.

23. Radiation from the Sun reaching Earth (just outside the atmosphere) has an intensity of 1.4 kW/m^2. (a) Assuming that Earth (and its atmosphere) behaves like a flat disk perpendicular to the Sun's rays and that all the incident energy is absorbed, calculate the force on Earth due to radiation pressure. (b) For comparison, calculate the force due to the Sun's gravitational attraction.

24. A parallel beam of monochromatic light of wavelength 500 nm is incident normally on a perfectly absorbing surface. The power through any cross section of the beam is 10 W. Find (a) the number of photons absorbed per second by the surface and (b) the force exerted by the light beam on the surface.

25. A small laser emits light at power 5.00 mW and wavelength 633 nm. The laser beam is focused (narrowed) until its diameter matches the 1266 nm diameter of a sphere placed in its path. The sphere is perfectly absorbing and has density 5.00×10^3 kg/m^3. What are (a) the beam intensity at the sphere's location, (b) the radiation pressure on the sphere, (c) the magnitude of the corresponding force, and (d) the magnitude of the acceleration that force alone would give the sphere?

26. A parallel beam of monochromatic light of wavelength 663 nm is incident on a totally reflecting plane mirror. The angle of incidence is 60° and the number of photons striking the mirror per second is 1.0×10^{19}. Calculate the force exerted by the light beam on the mirror.

27. A beam of white light is incident normally on a plane surface absorbing 50% of the light and reflecting the rest. If the incident beam carries 10 W of power, find the force exerted by it on the surface.

28. A black wall absorbs all the photons that strike it. (a) How many photons per second would have to be emitted by a laser that fires a beam of cross-sectional area 1 nm^2 perpendicularly at the wall to exert a pressure of 1 atm on the wall? The laser emits light of wavelength 600 nm. (b) What would be the power in this beam?

sec. 38-6 Electrons and Matter Waves

29. Calculate the de Broglie wavelength of (a) a 1.00 keV electron, (b) a 1.00 keV photon, and (c) a 1.00 keV neutron.

30. Green light has a wavelength of about 500 nm. Through what potential difference must an electron be accelerated to have this wavelength?

31. An electron and a photon each have a wavelength of 0.20 nm. What is the momentum (in kg m/s)of the (a) electron and (b) photon? What is the energy (in eV)of the (c) electron and (d) photon?

32. A nonrelativistic particle is moving three times as fast as an electron. The ratio of the de Broglie wavelength of the particle to that of the electron is 1.813×10^{-4}. By calculating its mass, identify the particle.

sec. 38-7 Schrödinger's Equation

33. Show that the angular wave number k for a nonrelativistic free particle of mass m can be written as

$$k = \frac{2\pi\sqrt{2mK}}{h},$$

in which K is the particle's kinetic energy.

34. The function $\psi(x)$ displayed in Eq. 38-15 describes a free particle, for which we assumed that $U(x) = 0$ in Schrödinger's equation (Eq. 38-11). Assume now that $U(x) = U_0$ = a constant in that equation. Show that Eq. 38-15 is still a solution of Schrödinger's equation, with

$$k = \frac{2\pi}{h}\sqrt{2m(E - U_0)}$$

now giving the angular wave number k of the particle.

sec. 38-8 Heisenberg's Uncertainty Principle

35. The uncertainty in the position of an electron along an x axis is given as 50 pm, which is about equal to the radius of a hydrogen atom. What is the least uncertainty in any simultaneous measurement of the momentum component p_x of this electron?

sec. 38-9 Barrier Tunneling

36. (a) Suppose a beam of 5.0 eV protons strikes a potential energy barrier of height 6.0 eV and thickness 0.70 nm, at a rate equivalent to a current of 1000 A. How long would you have to wait – on average – for one proton to be transmitted? (b) How long would you have to wait if the beam consisted of electrons rather than protons?

37. Consider a potential energy barrier like that of Fig. 38-16 but whose height U_b is 6.0 eV and whose thickness L is 0.70 nm. What is the energy of an incident electron whose transmission coefficient is 0.0010?

38. Consider the barrier-tunneling situation in Sample Problem 38-9. What percentage change in the transmission coefficient T occurs for a 1.0% change in (a) the barrier height, (b) the barrier thickness, and (c) the kinetic energy of the incident electron?

ADDITIONAL PROBLEMS

39. A satellite in Earth orbit maintains a panel of solar cells of area 2.60 m^2 perpendicular to the direction of the Sun's light rays. The intensity of the light at the panel is 1.39 kW/m^2. (a) At what rate does solar energy arrive at the panel? (b) At what rate are solar photons absorbed by the panel? Assume that the solar radiation is monochromatic, with a wavelength of 550 nm,

and that all the solar radiation striking the panel is absorbed. (c) How long would it take for a "mole of photons" to be absorbed by the panel?

40. A light detector has an absorbing area of 2.00×10^{-6} m^2 and absorbs 50% of the incident light, which is at wavelength 600 nm. The detector faces an isotropic source, 12.0 m from the source. The energy E emitted by the source versus time t is given in Fig. 38-21 (E_s = 7.2 nJ, t_s = 2.0 s). At what rate are photons absorbed by the detector?

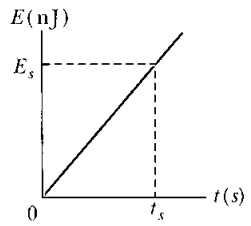


FIG. 38-21 Problem 40.

41. A point source of light is placed at the center of curvature of hemispherical surface. The radius of curvature is r and the inner surface is completely reflecting. Find the force on the hemisphere due to the light falling on it if the source emits a power W.

42. A light beam of wavelength 400 nm is incident on a metal plate of work function 2.2 eV. (a) A particular electron absorbs a photon and makes two collisions before coming out of the metal. Assuming that 10% of the extra energy of the metal is lost in each collision, find the kinetic energy of this electron as it comes out of the metal. (b) Under the same assumption, find the maximum number of collisions the electron can suffer before it is unable to come out of the metal.

43. Radiation of wavelength 180 nm ejects photoelectrons from a metal plate whose work function is 2.0 eV. If a uniform magnetic field of flux density 5.0×10^{-5} T (or Wb m^{-2}) is applied parallel to the plate, what should be the radius of the path followed by electrons ejected normally from the plate with maximum energy.

44. A perfectly absorbing, black, solid sphere with constant density and radius R hovers stationary above the Sun. This is because the gravitational attraction of the Sun is balanced by the pressure due to the Sun's light. Assume the Sun is far away that it closely approximates a point source of light. What is the distance from the center of the sun at which the sphere hovers?

45. A point source of light emits isotropically with a power of 200 W. What is the force due to the light on a totally absorbing sphere of radius 2.0 cm at a distance of 20 m from the source?

ANSWERS

Checkpoints

1. b, a, d, c

2. (a) lithium, sodium, potassium, cesium; (b) all tie

3. (a) proton; (b) same; (c) proton

4. same

Reasoning Type Questions

1. (d)

2. (a)

3. (a)

4. (b)

5. (c)

6. (d)

7. (a)

Multiple Choice Questions

1. (b), (d)

2. (b)

3. (c)

4. (d)

5. (a)

6. (b)

7. (b)

8. (a)

9. (c)

10. (c)

11. (c), (d)

12. (b)

13. (b), (c), (d)

14. (b)

15. (c)

16. (b)

17. (d)

18. (a)

19. (b)

20. (d)

21. (d)

22. (i) (b); (ii) (b)

23. (d)

24. (e)

25. (c)

26. (b)

27. (d)

28. (d)

29. (c)

30. (b)

Problems

1. 1.562×10^{15} photons/s

2. 3.87×10^{21} photons/m^{2}s

3. 1.67×10^{4} m^3

4. 1.78 eV

5. 8.59×10^{5} m/s

6. 1.72×10^{30} photons/s

7. 1.2×10^{7} m

8. 6.76×10^{5} m/s

9. 3.48 eV

10. (a) 296 nm, 1.01 PHz; (b) 2.71 V

11. (a) 1.90 eV, (b) 0.216 V

12. 1.5 eV

13. 0.48 μA

14. 9.68×10^{-20} A

15. (a) 1.4×10^{4} eV; (b) 6000 eV

16. 1.9 eV

17. (a) 4.1 eV, 0.4 eV; (b) 5.08×10^{14} Hz; (c) 5900 Å

18. (a) 2.0 eV; (b) 3.7 eV

19. 8.41 pC

20. $\dfrac{hc}{\lambda} - \dfrac{e^2B^2R^2}{2m_e}$

21. 3.27 eV

22. 5.9×10^{-8} Pa

23. (a) 6×10^{8} N; (b) 3.6×10^{22} N

24. (a) 2.5×10^{19} ; (b) 3.33×10^{-8} N

25. (a) 3.97×10^{9} W/m^2; (b) 13.2 Pa; (c) 1.67×10^{-11} N; (d) 3.14×10^{3} m/s^2

26. 1×10^{-8} N

27. 5×10^{-8} N

28. (a) 9×10^{31}/s; (b) 3×10^{-5} W

29. (a) 0.0388 nm; (b) 1.24 nm; (c) 9.06×10^{-13} m

30. 6.0 μV

31. (a) 3.3×10^{-24} kg · m/s; (b) 3.3×10^{-24} kg · m/s; (c) 38 eV; (d) 6.2 keV

32. Proton

35. 2.1×10^{-24} kg · m/s

36. (a) 3.37×10^{111} s $\approx 10^{104}$ y; (b) 2.1×10^{-19} s

37. 5.1 eV

38. (a) −0.20, a 20% decrease in the transmission coefficient; (b) −0.10, a 10% decrease in the transmission coefficient; (c) 0.15, a 15% increase in the transmission coefficient

Additional Problems

39. (a) 3.61 kW; (b) 1.00×10^{22} photon/s; (c) 60.2 s

40. 6.0 photons/s

41. $\dfrac{W}{2c}$

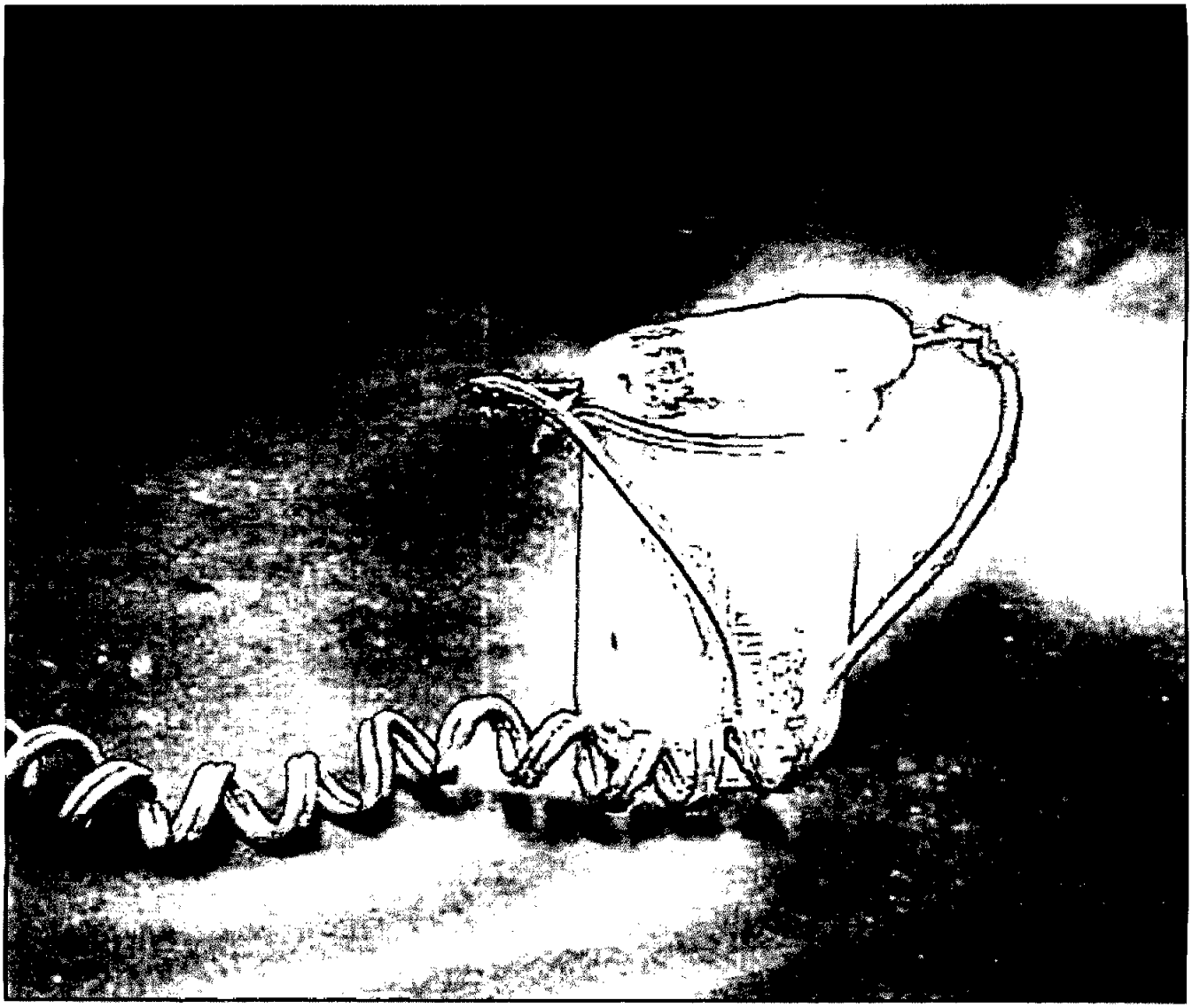
42. (a) 0.31 eV; (b) 4

43. 14.9 cm

44. The particle can hover at any distance above sun

45. 1.7×10^{-13} N

Hydrogen Atom 39



When a piece of pickle is connected to a source of alternating current, the pickle glows impressively, predominantly at one end. The amazing thing is that wherever we do the experiment, the glow is of yellow color.

Why does the pickle glow only yellow in color?

The answer is in this chapter.

39-1 WHAT IS PHYSICS?

One of the long-standing goals of physics has been to understand the nature of atoms. Early in the 20th century nobody knew how the electrons in an atom are arranged, what their motions are, how atoms emit or absorb light, or even why atoms are stable. Without this knowledge it was not possible to understand how atoms combine to form molecules or stack up to form solids.

In 1926, all these questions and many others were answered with the development of quantum physics. Its basic premise is that moving electrons, protons, and particles of any kind are best viewed as matter waves, whose motions are governed by Schrödinger's equation. Although quantum theory also applies to massive particles, there is no point in treating baseballs, automobiles, planets, and such objects with quantum theory. For such massive, slow-moving objects, Newtonian physics and quantum physics yield the same answers.

Before we can apply quantum physics to the problem of atomic structure, we need to develop some insights by applying quantum ideas in a few simpler situations. Some of these situations may seem simplistic and unreal, but they allow us to discuss the basic principles of the quantum physics of atoms without having to deal with the often overwhelming complexity of atoms. Besides, with the advances in nanotechnology, situations that were previously found only in textbooks are now being produced in laboratories and put to use in modern electronics and materials science applications.

39-2 | String Waves and Matter Waves

In Chapter 16, Vol. I, we saw that waves of two kinds can be set up on a stretched string. If the string is so long that we can take it to be infinitely long, we can set up a *traveling wave* of essentially any frequency. However, if the stretched string has only a finite length, perhaps because it is rigidly clamped at both ends, we can set up only *standing waves* on it; further, these standing waves can have only discrete frequencies. In other words, confining the wave to a finite region of space leads to *quantization* of the motion—to the existence of discrete *states* for the wave, each state with a sharply defined frequency.

This observation applies to waves of all kinds, including matter waves. For matter waves, however, it is more convenient to deal with the energy E of the associated particle than with the frequency f of the wave. In all that follows we shall focus on the matter wave associated with an electron, but the results apply to any confined matter wave.

Consider the matter wave associated with an electron moving in the positive x direction and subject to no net force—a so-called *free particle.* The energy of such an electron can have any reasonable value, just as a wave traveling along a stretched string of infinite length can have any reasonable frequency.

Consider next the matter wave associated with an atomic electron, perhaps the *valence* (least tightly bound) electron in a sodium atom. Such an electron—held within the atom by the attractive Coulomb force between it and the positively charged nucleus—is *not* a free particle. It can exist only in a set of discrete states, each having a discrete energy E. This sounds much like the discrete states and quantized frequencies that are available to a stretched string of finite length. For matter waves, then, as for all other kinds of waves, we may state a **confinement principle**:

☞ Confinement of a wave leads to quantization—that is, to the existence of discrete states with discrete energies. The wave can have only those energies.

39-3 | Energies of a Trapped Electron

One-Dimensional Traps

Here we examine the matter wave associated with a nonrelativistic electron confined to a limited region of space. We do so by analogy with standing waves on a string of finite length, stretched along an x axis and confined between rigid supports. Because the supports are rigid, the two ends of the string are nodes, or points at which the string is always at rest. There may be other nodes along the string, but these two must always be present, as Fig. 16-32 shows.

The states, or discrete standing wave patterns in which the string can oscillate, are those for which the length L of the string is equal to an integer number of half-wavelengths. That is, the string can occupy only states for which

$$L = \frac{n\lambda}{2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-1}$$

Each value of n identifies a state of the oscillating string; using the language of quantum physics, we can call the integer n a **quantum number**.

For each state of the string permitted by Eq. 39-1, the transverse displacement of the string at any position x along the string is given by

$$y_n(x) = A\sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-2}$$

in which the quantum number n identifies the oscillation pattern and A depends on the time at which you inspect the string. (Equation 39-2 is a short version of Eq. 16-55.) We see that for all values of n and for all times, there is a point of zero displacement (a node) at $x = 0$ and at $x = L$, as there must be. Figure 16-32 shows time exposures of such a stretched string for $n = 2$, 3, and 4.

Now let us turn our attention to matter waves. Our first problem is to physically confine an electron that is moving along the x axis so that it remains within a finite segment of that axis. Figure 39-1 shows a conceivable one-dimensional *electron trap.* It consists of two semi-infinitely long cylinders, each of which has an electric potential approaching $-\infty$; between them is a hollow cylinder of length L, which has an electric potential of zero. We put a single electron into this central cylinder to trap it.

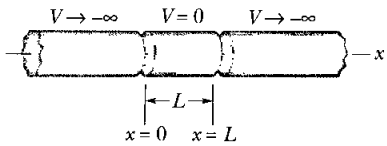


FIG. 39-1 The elements of an idealized "trap" designed to confine an electron to the central cylinder. We take the semi-infinitely long end cylinders to be at an infinitely great negative potential and the central cylinder to be at zero potential.

The trap of Fig. 39-1 is easy to analyze but is not very practical. Single electrons *can*, however, be trapped in the laboratory with traps that are more complex in design but similar in concept. At the University of Washington, for example, a single electron has been held in a trap for months on end, permitting scientists to make extremely precise measurements of its properties.

Finding the Quantized Energies

Figure 39-2 shows the potential energy of the electron as a function of its position along the x axis of the idealized trap of Fig. 39-1. When the electron is in the central cylinder, its potential energy U ($= -eV$) is zero because there the potential V is zero. If the electron could get outside this region, its potential energy would be positive and of infinite magnitude because there $V \to -\infty$. We call the potential energy pattern of Fig. 39-2 an **infinitely deep potential energy well** or, for short, an *infinite potential well.* It is a "well" because an electron placed in the central cylinder of Fig. 39-1 cannot escape from it. As the electron approaches either end of the cylinder, a force of essentially infinite magnitude reverses the electron's motion, thus trapping it. Because the electron can move along only a single axis, this trap can be called a *one-dimensional infinite potential well.*

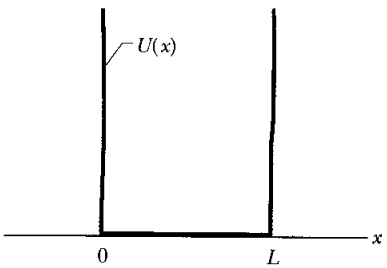


FIG. 39-2 The electric potential energy $U(x)$ of an electron confined to the central cylinder of the idealized trap of Fig. 39-1. We see that $U = 0$ for $0 < x < L$, and $U \to \infty$ for $x < 0$ and $x > L$.

Just like the standing wave in a length of stretched string, the matter wave describing the confined electron must have nodes at $x = 0$ and $x = L$. Moreover, Eq. 39-1 applies to such a matter wave if we interpret λ in that equation as the de Broglie wavelength associated with the moving electron.

The de Broglie wavelength λ is defined in Eq. 38-13 as $\lambda = h/p$, where p is the magnitude of the electron's momentum. Because the electron is nonrelativistic, this momentum magnitude p is related to the electron's kinetic energy K by $p = \sqrt{2mK}$, where m is the mass of the electron. For an electron moving within the central cylinder of Fig. 39-1, where $U = 0$, the total (mechanical) energy E is equal to the kinetic energy. Hence, we can write the de Broglie wavelength of this electron as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}. \tag{39-3}$$

If we substitute Eq. 39-3 into Eq. 39-1 and solve for the energy E, we find that E depends on n according to

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-4}$$

The positive integer n here is the quantum number of the electron's quantum state in the trap.

Equation 39-4 tells us something important: Because the electron is confined to the trap, it can have only the energies given by the equation. It *cannot* have an energy that is, say, halfway between the values for $n = 1$ and $n = 2$. Why this restriction? Because an electron is a matter wave. Were it, instead, a particle as assumed in classical physics, it could have *any* value of energy while it is confined to the trap.

Figure 39-3 is a graph showing the lowest five allowed energy values for an electron in an infinite well with $L = 100$ pm (about the size of a typical atom). The values are called *energy levels*, and they are drawn in Fig. 39-3 as levels, or steps, on a ladder, in an *energy-level diagram.* Energy is plotted vertically; nothing is plotted horizontally.

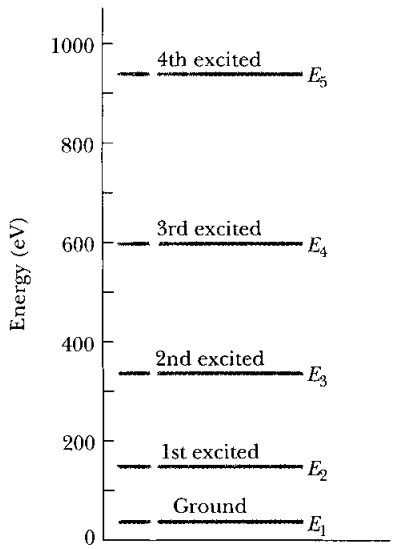


FIG. 39-3 Several of the allowed energies given by Eq. 39-4 for an electron confined to the infinite well of Fig. 39-2. Here width $L = 100$ pm. Such a plot is called an *energy-level diagram.*

The quantum state with the lowest possible energy level E_1 allowed by Eq. 39-4, with quantum number $n = 1$, is called the *ground state* of the electron. The electron tends to be in this lowest energy state. All the quantum states with greater energies (corresponding to quantum numbers $n = 2$ or greater) are called *excited states* of the electron. The state with energy level E_2, for quantum number $n = 2$, is called the *first excited state* because it is the first of the excited states as we move up the energy-level diagram. Similarly, the state with energy level E_3 is called the *second excited state.*

Energy Changes

A trapped electron tends to have the lowest allowed energy and thus to be in its ground state. It can be changed to an excited state (in which it has greater energy) only if an external source provides the additional energy that is required for the change. Let E_{low} be the initial energy of the electron and E_{high} be the greater energy in a state that is higher on its energy-level diagram. Then the amount of energy that is required for the electron's change of state is

$$\Delta E = E_{high} - E_{low}. \tag{39-5}$$

An electron that receives such energy is said to make a *quantum jump* (or *transition*), or to be *excited* from the lower-energy state to the higher-energy state. Figure 39-4*a* represents a quantum jump from the ground state (with energy

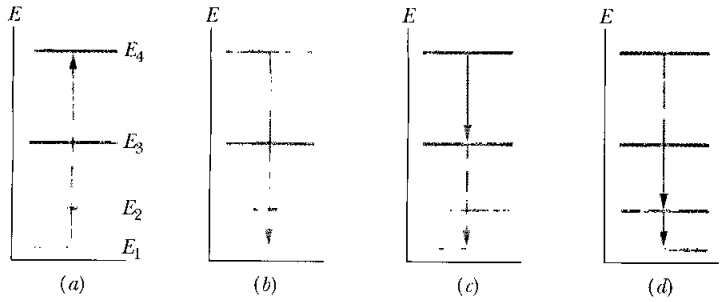


FIG. 39-4 (*a*) Excitation of a trapped electron from the energy level of its ground state to the level of its third excited state. (*b*)–(*d*) Three of four possible ways the electron can de-excite to return to the energy level of its ground state. (Which way is not shown?)

level E_1) to the third excited state (with energy level E_4). As shown, the jump *must* be from one energy level to another, but it can bypass one or more intermediate energy levels.

One way an electron can gain energy to make a quantum jump up to a greater energy level is to absorb a photon. However, this absorption and quantum jump can occur only if the following condition is met:

☞ If a confined electron is to absorb a photon, the energy hf of the photon must equal the energy difference ΔE between the initial energy level of the electron and a higher level.

Thus, excitation by the absorption of light requires that

$$hf = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{39-6}$$

When an electron reaches an excited state, it does not stay there but quickly *de-excites* by decreasing its energy. Figures 39-4*b* to *d* represent some of the possible quantum jumps down from the energy level of the third excited state. The electron can reach its ground-state level either with one direct quantum jump (Fig. 39-4*b*) or with shorter jumps via intermediate levels (Figs. 39-4*c* and *d*).

One way in which an electron can decrease its energy is by emitting a photon, but only if the following condition is met:

☞ If a confined electron emits a photon, the energy hf of that photon must equal the energy difference ΔE between the initial energy level of the electron and a lower level.

Thus, Eq. 39-6 applies to both the absorption and the emission of light by a confined electron. That is, the absorbed or emitted light can have only certain values of hf and thus only certain values of frequency f and wavelength λ.

✓**CHECKPOINT 1** Rank the following pairs of quantum states for an electron confined to an infinite well according to the energy differences between the states, greatest first: (a) $n = 3$ to $n = 1$, (b) $n = 5$ to $n = 4$, (c) $n = 4$ to $n = 3$.

Sample Problem 39-1

An electron is confined to a one-dimensional, infinitely deep potential energy well of width $L = 100$ pm.

(a) What is the smallest amount of energy the electron can have?

Conceptualize: Because the well is infinitely deep, the allowed energies are given by Eq. 39-4 ($E_n = (h^2/8mL^2)n^2$), with the quantum number n a positive integer.

Classify: Confinement of the electron (a matter wave) to the well leads to quantization of its energy.

Compute: Lowest energy level: Here, the collection of constants in front of n^2 in Eq. 39-4 is evaluated as

$$\frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{(8)(9.11\times10^{-31}\ \text{kg})(100\times10^{-12}\ \text{m})^2} = 6.031\times10^{-18}\ \text{J}. \quad (39\text{-}7)$$

The smallest amount of energy the electron can have corresponds to the lowest quantum number, which is $n = 1$ for the ground state of the electron. Thus, Eqs. 39-4 and 39-7 give us

$$E_1 = \left(\frac{h^2}{8mL^2}\right)n^2 = (6.031\times10^{-18}\ \text{J})(1^2) \approx 6.03\times10^{-18}\ \text{J} = 37.7\ \text{eV}. \quad \text{(Answer)}$$

(b) How much energy must be transferred to the electron if it is to make a quantum jump from its ground state to its second excited state?

Conceptualize/Classify: *First a caution:* Note that, from Fig. 39-3, the *second* excited state corresponds to the *third* energy level, with quantum number $n = 3$. Then if the electron is to jump from the $n = 1$ level to the $n = 3$ level, the required change in its energy is, from Eq. 39-5,

$$\Delta E_{31} = E_3 - E_1. \quad (39\text{-}8)$$

Compute: Upward jump: The energies E_3 and E_1 depend on the quantum number n, according to Eq. 39-4. Therefore, substituting that equation into Eq. 39-8 for energies E_3 and E_1 and using Eq. 39-7 lead to

$$\Delta E_{31} = \left(\frac{h^2}{8mL^2}\right)(3)^2 - \left(\frac{h^2}{8mL^2}\right)(1)^2 = \frac{h^2}{8mL^2}(3^2 - 1^2) = (6.031\times10^{-18}\ \text{J})(8) = 4.83\times10^{-17}\ \text{J} = 301\ \text{eV}. \quad \text{(Answer)}$$

(c) If the electron gains the energy for the jump from energy level E_1 to energy level E_3 by absorbing light, what light wavelength is required?

Conceptualize/Classify: (1) If light is to transfer energy to the electron, the transfer must be by photon absorption. (2) The photon's energy must equal the energy difference ΔE between the initial energy level of the electron and a higher level, according to Eq. 39-6 ($hf = \Delta E$). Otherwise, a photon *cannot* be absorbed.

Compute: Wavelength: Substituting c/λ for f, we can rewrite Eq. 39-6 as

$$\lambda = \frac{hc}{\Delta E}. \quad (39\text{-}9)$$

For the energy difference ΔE_{31} we found in (b), this equation gives us

$$\lambda = \frac{hc}{\Delta E_{31}} = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{4.83\times10^{-17}\ \text{J}} = 4.12\times10^{-9}\ \text{m}. \quad \text{(Answer)}$$

(d) Once the electron has been excited to the second excited state, what wavelengths of light can it emit by de-excitation?

Conceptualize/Classify:

1. The electron tends to de-excite, rather than remain in an excited state, until it reaches the ground state ($n = 1$).
2. If the electron is to de-excite, it must lose just enough energy to jump to a lower energy level.
3. If it is to lose energy by emitting light, then the loss of energy must be by emission of a photon.

Compute: Downward jumps: Starting in the second excited state (at the $n = 3$ level), the electron can reach the ground state ($n = 1$) by *either* making a quantum jump directly to the ground-state energy level (Fig. 39-5*a*) or by making two *separate* jumps by way of the $n = 2$ level (Figs. 39-5*b* and *c*).

The direct jump involves the same energy difference ΔE_{31} we found in (c). Then the wavelength is the same as we calculated in (c)—except now the wavelength is

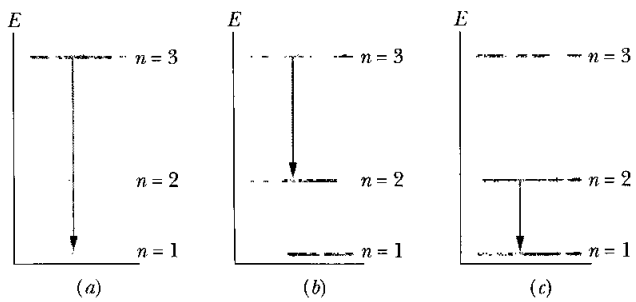


FIG. 39-5 De-excitation from the second excited state to the ground state either directly (*a*) or via the first excited state (*b*, *c*).

for light that is emitted, not absorbed. Thus, the electron can jump directly to the ground state by emitting light of wavelength

$$\lambda = 4.12 \times 10^{-9}\ \text{m}. \qquad \text{(Answer)}$$

Following the procedure of part (b), you can show that the energy differences for the jumps of Figs. 39-5*b* and *c* are

$$\Delta E_{32} = 3.016 \times 10^{-17}\ \text{J} \quad \text{and} \quad \Delta E_{21} = 1.809 \times 10^{-17}\ \text{J}.$$

From Eq. 39-9, we then find that the wavelength of the light emitted in the first of these jumps (from $n = 3$ to $n = 2$) is

$$\lambda = 6.60 \times 10^{-9}\ \text{m}, \qquad \text{(Answer)}$$

and the wavelength of the light emitted in the second of these jumps (from $n = 2$ to $n = 1$) is

$$\lambda = 1.10 \times 10^{-8}\ \text{m}. \qquad \text{(Answer)}$$

39-4 | Wave Functions of a Trapped Electron

If we solve Schrödinger's equation for an electron trapped in a one-dimensional infinite potential well of width L, we find that the wave functions for the electron are given by

$$\psi_n(x) = A \sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \qquad (39\text{-}10)$$

for $0 \le x \le L$ (the wave function is zero outside that range). We shall soon evaluate the amplitude constant A in this equation.

Note that the wave functions $\psi_n(x)$ have the same form as the displacement functions $y_n(x)$ for a standing wave on a string stretched between rigid supports (see Eq. 39-2). We can picture an electron trapped in a one-dimensional well between infinite-potential walls as being a standing matter wave.

Probability of Detection

The wave function $\psi_n(x)$ cannot be detected or directly measured in any way—we cannot simply look inside the well to see the wave the way we can see, say, a wave in a bathtub of water. All we can do is insert a probe of some kind to try to detect the electron. At the instant of detection, the electron would materialize at the point of detection, at some position along the x axis within the well.

If we repeated this detection procedure at many positions throughout the well, we would find that the probability of detecting the electron is related to the probe's position x in the well. In fact, they are related by the *probability density* $\psi_n^2(x)$. Recall from Section 38-7 that in general the probability that a particle can be detected in a specified infinitesimal volume centered on a specified point is proportional to $|\psi_n^2|$. Here, with the electron trapped in a one-dimensional well, we are concerned only with detection of the electron along the x axis. Thus, the probability density $\psi_n^2(x)$ here is a probability per unit length along the x axis. (We can omit the absolute value sign here because $\psi_n(x)$ in Eq. 39-10 is a real quantity, not a complex one.) The probability $p(x)$ that an electron can be detected at position x within the well is),

$$\begin{pmatrix}\text{probability } p(x)\\ \text{of detection in width } dx\\ \text{centered on position } x\end{pmatrix} = \begin{pmatrix}\text{probability density } \psi_n^2(x)\\ \text{at position } x\end{pmatrix}(\text{width } dx),$$

or

$$p(x) = \psi_n^2(x)dx. \qquad (39\text{-}11)$$

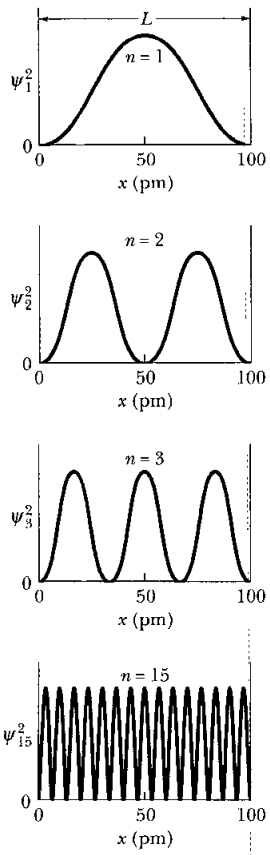


FIG. 39-6 The probability density $\psi_n^2(x)$ for four states of an electron trapped in a one-dimensional infinite well; their quantum numbers are $n = 1, 2, 3$, and 15. The electron is most likely to be found where $\psi_n^2(x)$ is greatest and least likely to be found where $\psi_n^2(x)$ is least.

From Eq. 39-10, we see that the probability density $\psi_n^2(x)$ for the trapped electron is

$$\psi_n^2(x) = A^2 \sin^2\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-12}$$

for the range $0 \le x \le L$ (the probability density is zero outside that range). Figure 39-6 shows $\psi_n^2(x)$ for $n = 1$, 2, 3, and 15 for an electron in an infinite well whose width L is 100 pm.

To find the probability that the electron can be detected in any finite section of the well—say, between point x_1 and point x_2—we must integrate $p(x)$ between those points. Thus, from Eqs. 39-11 and 39-12,

$$\begin{pmatrix}\text{probability of detection}\\ \text{between } x_1 \text{ and } x_2\end{pmatrix} = \int_{x_1}^{x_2} p(x)$$

$$= \int_{x_1}^{x_2} A^2 \sin^2\left(\frac{n\pi}{L}x\right) dx. \tag{39-13}$$

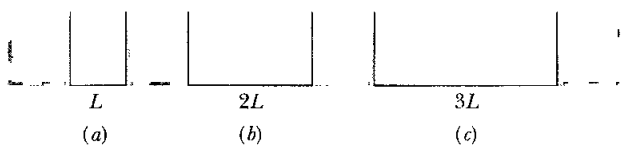
If classical physics prevailed, we would expect the trapped electron to be detectable with equal probabilities in all parts of the well. From Fig. 39-6 we see that it is not. For example, inspection of that figure or of Eq. 39-12 shows that for the state with $n = 2$, the electron is most likely to be detected near $x = 25$ pm and $x = 75$ pm. It can be detected with near-zero probability near $x = 0$, $x = 50$ pm, and $x = 100$ pm.

The case of $n = 15$ in Fig. 39-6 suggests that as n increases, the probability of detection becomes more and more uniform across the well. This result is an instance of a general principle called the **correspondence principle**:

> At large enough quantum numbers, the predictions of quantum physics merge smoothly with those of classical physics.

This principle, first advanced by Danish physicist Niels Bohr, holds for all quantum predictions.

CHECKPOINT 2 The figure shows three infinite potential wells of widths L, $2L$, and $3L$; each contains an electron in the state for which $n = 10$. Rank the wells according to (a) the number of maxima for the probability density of the electron and (b) the energy of the electron, greatest first.



Normalization

The product $\psi_n^2(x)dx$ gives the probability that an electron in an infinite well can be detected in the interval of the x axis that lies between x and $x + dx$. We know that the electron must be *somewhere* in the infinite well; so it must be true that

$$\int_{-\infty}^{+\infty} \psi_n^2(x)dx = 1 \qquad \text{(normalization equation)}, \tag{39-14}$$

because the probability 1 corresponds to certainty. Although the integral is taken over the entire x axis, only the region from $x = 0$ to $x = L$ makes any contribution to the probability. Graphically, the integral in Eq. 39-14 represents the area under each of the plots of Fig. 39-6.

In Sample Problem 39-2 we shall see that if we substitute $\psi_n^2(x)$ from Eq. 39-12 into Eq. 39-14, it is possible to assign a specific value to the amplitude constant A that appears in Eq. 39-12; namely, $A = \sqrt{2/L}$. This process of using Eq. 39-14 to evaluate the amplitude of a wave function is called **normalizing** the wave function. The process applies to *all* one-dimensional wave functions.

Zero-Point Energy

Substituting $n = 1$ in Eq. 39-4 defines the state of lowest energy for an electron in an infinite potential well, the ground state. That is the state the confined electron will occupy unless energy is supplied to it to raise it to an excited state.

The question arises: Why can't we include $n = 0$ among the possibilities listed for n in Eq. 39-4? Putting $n = 0$ in this equation would indeed yield a ground-state energy of zero. However, putting $n = 0$ in Eq. 39-12 would also yield $\psi_n^2(x) = 0$ for all x, which we can interpret only to mean that there is no electron in the well. We know that there is; so $n = 0$ is not a possible quantum number.

It is an important conclusion of quantum physics that confined systems cannot exist in states with zero energy. They must always have a certain minimum energy called the **zero-point energy**.

We can make the zero-point energy as small as we like by making the infinite well wider—that is, by increasing L in Eq. 39-4 for $n = 1$. In the limit as $L \to \infty$, the zero-point energy E_1 approaches zero. In this limit, however, with an infinitely wide well, the electron is a free particle, no longer confined in the x direction. Also, because the energy of a free particle is not quantized, that energy can have any value, including zero. Only a confined particle must have a finite zero-point energy and can never be at rest.

CHECKPOINT 3 Each of the following particles is confined to an infinite well, and all four wells have the same width: (a) an electron, (b) a proton, (c) a deuteron, and (d) an alpha particle. Rank their zero-point energies, greatest first. The particles are listed in order of increasing mass.

Sample Problem 39-2

Evaluate the amplitude constant A in Eq. 39-10 for an infinite potential well extending from $x = 0$ to $x = L$.

Conceptualize/Classify: The wave functions of Eq. 39-10 must satisfy the normalization requirement of Eq. 39-14, which states that the probability that the electron can be detected somewhere along the x axis is 1.

Compute: Substituting Eq. 39-10 into Eq. 39-14 and taking the constant A outside the integral yield

$$A^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx = 1. \qquad (39\text{-}15)$$

We have changed the limits of the integral from $-\infty$ and $+\infty$ to 0 and L because the wave function is zero outside these new limits (so there's no need to integrate out there).

We can simplify the indicated integration by changing the variable from x to the dimensionless variable y, where

$$y = \frac{n\pi}{L}x, \qquad (39\text{-}16)$$

hence

$$dx = \frac{L}{n\pi}dy.$$

When we change the variable, we must also change the integration limits (again). Equation 39-16 tells us that $y = 0$ when $x = 0$ and that $y = n\pi$ when $x = L$; thus 0 and $n\pi$ are our new limits. With all these substitutions, Eq. 39-15 becomes

$$A^2 \frac{L}{n\pi} \int_0^{n\pi} (\sin^2 y)\, dy = 1.$$

We can use integral 11 in Appendix E to evaluate the integral, obtaining the equation

$$\frac{A^2 L}{n\pi}\left[\frac{y}{2} - \frac{\sin 2y}{4}\right]_0^{n\pi} = 1.$$

Evaluating at the limits yields

$$\frac{A^2 L}{n\pi}\frac{n\pi}{2} = 1;$$

thus $$A = \sqrt{\frac{2}{L}}. \qquad \text{(Answer)} \qquad (39\text{-}17)$$

Conclude: This result tells us that the dimension for A^2, and thus for $\psi_n^2(x)$, is an inverse length. This is appropriate because the probability density of Eq. 39-12 is a probability *per unit length.*

Sample Problem 39-3

A ground-state electron is trapped in the one-dimensional infinite potential well of Fig. 39-2, with width $L = 100$ pm.

(a) What is the probability that the electron can be detected in the left one-third of the well ($x_1 = 0$ to $x_2 = L/3$)?

Conceptualize/Classify: (1) If we probe the left one-third of the well, there is no guarantee that we will detect the electron. However, we can calculate the probability of detecting it with the integral of Eq. 39-13. (2) The probability very much depends on which state the electron is in – that is, the value of quantum number n.

Compute: Because here the electron is in the ground state, we set $n = 1$ in Eq. 39-13. We also set the limits of integration as the positions $x_1 = 0$ and $x_2 = L/3$ and, from Sample Problem 39-2, set the amplitude constant A as $\sqrt{2/L}$. We then see that

$$\begin{pmatrix}\text{probability of detection}\\ \text{in left one-third}\end{pmatrix} = \int_0^{L/3} \frac{2}{L}\sin^2\left(\frac{1\pi}{L}x\right)dx.$$

We could find this probability by substituting 100×10^{-12} m for L and then using a graphing calculator or a computer math package to evaluate the integral. Instead, we shall follow the steps of Sample Problem 39-2. From Eq. 39-16, we obtain for the new integration variable y,

$$y = \frac{\pi}{L}x \quad \text{and} \quad dx = \frac{L}{\pi}dy.$$

From the first of these equations, we find the new limits of integration to be $y_1 = 0$ for $x_1 = 0$ and $y_2 = \pi/3$ for $x_2 = L/3$. We then must evaluate

$$\text{probability} = \left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_0^{\pi/3}(\sin^2 y)\,dy.$$

Using integral 11 in Appendix E, we then find

$$\text{probability} = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_0^{\pi/3} = 0.20.$$

Thus, we have

$$\begin{pmatrix}\text{probability of detection}\\ \text{in left one-third}\end{pmatrix} = 0.20. \qquad \text{(Answer)}$$

That is, if we repeatedly probe the left one-third of the well, then on average we can detect the electron with 20% of the probes.

(b) What is the probability that the electron can be detected in the middle one-third of the well (between $x_1 = L/3$ and $x_2 = 2L/3$)?

Reasoning: We now know that the probability of detection in the left one-third of the well is 0.20. By symmetry, the probability of detection in the right one-third of the well is also 0.20. Because the electron is certainly in the well, the probability of detection in the entire well is 1. Thus, the probability of detection in the middle one-third of the well is

$$\begin{pmatrix}\text{probability of detection}\\ \text{in middle one-third}\end{pmatrix} = 1 - 0.20 - 0.20$$
$$= 0.60. \qquad \text{(Answer)}$$

39-5 | The Bohr Model of the Hydrogen Atom

We now move from artificial and fictitious electron traps to natural ones—atoms. In this chapter we focus on the simplest example, a hydrogen atom. This atom consists of a single electron electrically bound to a proton, which is the only constituent of the atom's nucleus at the atom's center. Here we do not consider anything about the nucleus. Rather, we simply use the fact that the negatively charged electron is attracted by the Coulomb force to the positively charged proton. Because the proton mass is much greater than the electron mass, we assume that the proton is fixed in place and that the electron is confined to being near the proton. That is, the electron is trapped.

We have now discussed at length that confinement of an electron means that the electron's energy E is quantized and thus so is any change ΔE in its energy. In this section we want to calculate the quantized energies of the electron in a hydrogen atom. Before we apply the wave approach we used in infinite and finite potential wells, however, let's explore the hydrogen atom at the dawn of quantum physics, when physicists first discovered that atoms are quantized systems.

By the early 1900s, scientists understood that matter came in tiny pieces called atoms and that an atom of hydrogen contained positive charge $+e$ at its center and negative charge $-e$ (an electron) outside that center. However, no one understood why the electrical attraction between the electron and the positive charge did not simply cause the two to collapse together.

One clue came from the fact that a hydrogen atom cannot emit and absorb all wavelengths of visible light. Rather, it can emit and absorb only four particular wavelengths from the visible range. By guesswork, Johann Balmer devised a formula that gave those wavelengths:

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \quad \text{for } n = 3, 4, 5, \text{and } 6, \tag{39-18}$$

where R is a constant. However, neither Balmer nor anyone else knew why the visible emission and absorption by a hydrogen atom are limited to the four wavelengths given by this formula. Why doesn't a hydrogen atom emit and absorb light at *any* visible wavelength?

No one knew until 1913, when Bohr saw Balmer's equation and quickly realized that he could derive it if he made several bold (completely unjustified) assumptions: (1) The electron in a hydrogen atom orbits the nucleus in a circle much like Earth orbits the Sun (Fig. 39-7*a*). (2) The magnitude of the angular momentum $\vec{L}$ of the electron in its orbit is restricted to the values

$$L = n\hbar, \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-19}$$

where $\hbar$ (h-bar) is $h/2\pi$ and n is a quantum number.

Here, Bohr was just fitting in the empirical data, just like his predecessor Balmer. He did not know why the angular momentum was quantized at all. In fact, is this not what most of us do all the time? When we are unable to obtain an answer logically, we try to somehow manipulate the calculations to get the answer. In most of the cases, we are unable to justify this at all. Neither was Bohr able to!

However in 1924, 11 years after Bohr presented his theory, de Broglie gave an interpretation of the Bohr's rule for quantization of angular momentum. If we represent the circulating electron in terms of its de Broglie wave, then the stable states are those in which the electron's de Broglie wave joins onto itself with the same phase in each revolution; otherwise, the wave would destroy itself by destructive interference. This is presented in Fig. 39-8. In other words, the de Broglie wavelength must fit around the circumference of the orbit an integral number of times, or

$$n\lambda = 2\pi r.$$

Also

$$\lambda = \frac{h}{mv}.$$

Therefore,

$$mv_n r_n = \frac{nh}{2\pi}.$$

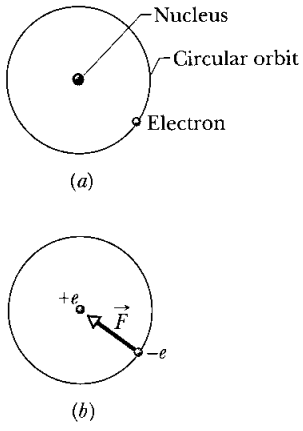


FIG. 39-7 (*a*) Circular orbit of an electron in the Bohr model of the hydrogen atom. (*b*) The Coulomb force $\vec{F}$ on the electron is directed radially inward toward the nucleus.

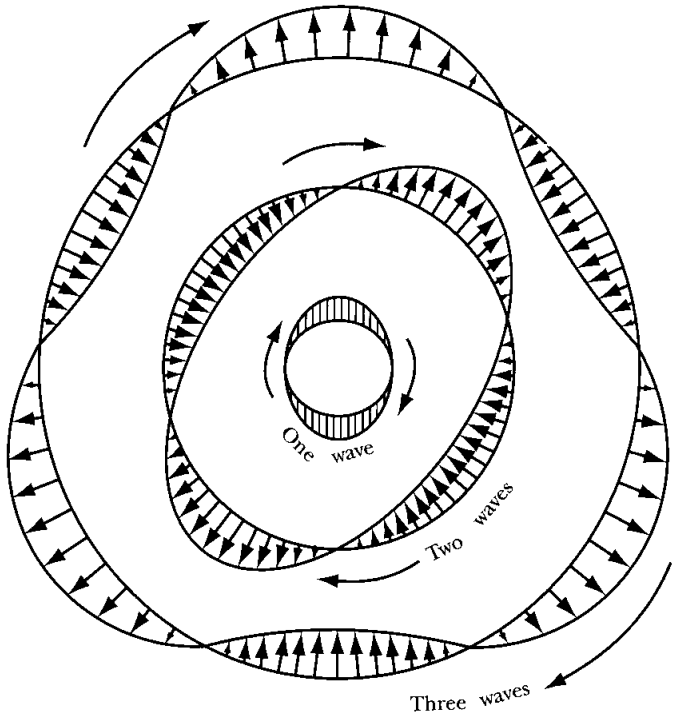


FIG. 39-8 Wave pattern for the electron of the hydrogen atom in its $n = 1, 2$, and 3 states.

The Orbital Radius Is Quantized in the Bohr Model

Let's examine the orbital motion of the electron in the Bohr model. The force holding the electron in an orbit of radius r is the Coulomb force. From Eq. 22-1, we know that the magnitude of this force is

$$F = k\frac{|q_1||q_2|}{r^2},$$

with $k = 1/4\pi\varepsilon_0$. Here q_1 is the charge $-e$ of the electron and q_2 is the charge Ze of the nucleus. For generality, let us assume that the nucleus has a charge Ze. This would make the model applicable to all hydrogen-like atoms, that is atoms having a single electron; for example, a helium atom stripped of an electron, a lithium atom stripped of 2 electrons and so on. The electron's acceleration is the centripetal acceleration, with a magnitude given by $a = v^2/r$, where v is the electron's speed. Both force $\vec{F}$ and acceleration $\vec{a}$ are radially inward (the negative direction on a radial axis), toward the nucleus (Fig. 39-17*b*). Thus, we can write Newton's second law ($F = ma$) for a radial axis as

$$-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} = m\left(-\frac{v^2}{r}\right), \tag{39-20}$$

where m is the electron mass.

We next introduce quantization by using Bohr's assumption expressed in Eq. 39-19. The magnitude ℓ of the angular momentum of a particle of mass m and speed v moving in a circle of radius r is $\ell = rmv \sin\phi$, where ϕ (the angle between $\vec{r}$ and $\vec{v}$) is 90°. Replacing L in Eq. 39-19 with $rmv \sin 90°$ gives us

$$rmv = n\hbar,$$

or

$$v = \frac{n\hbar}{rm}. \tag{39-21}$$

Substituting this equation into Eq. 39-20, replacing $\hbar$ with $h/2\pi$, and rearranging, we find

$$r = \frac{h^2\varepsilon_0}{\pi m Z e^2} n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-22}$$

If we are considering the radius of hydrogen atom, $Z = 1$, then we can rewrite this as

$$r = an^2, \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-23}$$

where

$$a = \frac{h^2\varepsilon_0}{\pi m e^2} = 5.29 \times 10^{-11} \text{ m} \approx 52.9 \text{ pm}. \tag{39-24}$$

These last three equations tell us that, in the *Bohr model of the hydrogen atom*, the electron's orbital radius r is quantized and the smallest possible orbital radius (for $n = 1$) is a, which is now called the *Bohr radius*. According to the Bohr model, the electron cannot get any closer to the nucleus than orbital radius a, and that is why the attraction between electron and nucleus does not simply collapse them together.

For hydrogen-like atoms,

$$r = \frac{an^2}{Z}.$$

Sample Problem 39-4

Find the speed of the electron in a Bohr orbit.

Conceptualize/Classify: We know that the electron speed and the radius of Bohr orbit are related by Bohr's condition of angular momentum quantization in Eq. 39-19.

Compute: We have

$$v_r = \frac{nh}{2\pi mr}.$$

By substituting the value of r from Eq. 39-22, we get

$$v = \frac{Ze^2}{2nh\varepsilon_0}. \tag{39-25}$$

Conclude: The interesting point about the speed of the electron in the orbit is that it does not depend on the mass of the electron. So, if we substitute the electron with a charged particle with the same charge as the electron but different mass, its radius will change but not the speed.

Orbital Energy Is Quantized

Let's next find the energy of the hydrogen atom according to the Bohr model. The electron has kinetic energy $K = \frac{1}{2}mv^2$, and the electron—nucleus system has electric potential energy $U = q_1q_2/4\pi\varepsilon_0 r$ (Eq. 24-5). Again, let q_1 be the electron's charge $-e$ and q_2 be the nuclear charge $+e$. Then the mechanical energy is

$$\begin{aligned} E &= K + U \\ &= \tfrac{1}{2}mv^2 + \left(-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\right). \end{aligned} \tag{39-26}$$

Solving Eq. 39-20 for mv^2 and substituting the result in Eq. 39-26 lead to

$$E = -\frac{1}{8\pi\varepsilon_0}\frac{Ze^2}{r}. \tag{39-27}$$

Next, replacing r with its equivalent from Eq. 39-22, we have

$$E_n = -\frac{mZe^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-28}$$

where the subscript n on E signals that we have now quantized the energy. Evaluating the constants in Eq. 39-28 gives us

$$E_n = -\frac{2.180 \times 10^{-18}\ Z^2\ \text{J}}{n^2} = -\frac{13.6Z^2\ \text{J}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-29}$$

This equation tells us that the energy E_n of the hydrogen atom is quantized; that is, E_n is restricted by its dependence on the quantum number n. Because the nucleus is assumed to be fixed in place and only the electron has motion, we can assign the energy values of Eq. 39-29 either to the atom as a whole or to the electron alone.

Thus, the allowed energy levels of hydrogen atom are

$$E_1 = -13.6\ \text{eV}$$
$$E_2 = -3.4\ \text{eV}$$
$$E_3 = -1.51\ \text{eV}$$
$$E_4 = -0.85\ \text{eV}$$
$$E_5 = -0.54\ \text{eV}$$

Energy Changes

The energy of a hydrogen atom (or, equivalently, of its electron) changes when the atom emits or absorbs light. As we have seen several times since Eq. 39-6, emission and absorption involve a quantum of light according to

$$hf = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{39-30}$$

Let's make three changes to Eq. 39-30. On the left side, we substitute c/λ for f. On the right side, we use Eq. 39-28 twice to replace the energy terms. Then, with a simple rearrangement, we have

$$\frac{1}{\lambda} = -\frac{mZ^2e^4}{8\varepsilon_0^2 h^3 c}\left(\frac{1}{n_{\text{high}}^2} - \frac{1}{n_{\text{low}}^2}\right). \tag{39-31}$$

We can rewrite this as

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right), \tag{39-32}$$

in which

$$R = \frac{me^4}{8\varepsilon_0^2 h^3 c} = 1.097\,373 \times 10^7\ \text{m}^{-1} \tag{39-33}$$

is now known as the *Rydberg constant.*

Compare Eq. 39-32 from the Bohr model with Eq. 39-18 from Balmer's work. In Eq. 39-32, if we replace n_{low} with 2 and then restrict n_{high} to be 3, 4, 5, and 6, we have Balmer's equation. This match was a triumph for Bohr and ushered in the quantum physics of atoms. The triumph was short-lived, however, because even though the Bohr model gives the correct emission and absorption wavelengths for the hydrogen atom, the model is *not* correct because the electron does *not* orbit the nucleus like a planet orbiting the Sun. Indeed, researchers found little success in extending the Bohr model to atoms more complicated than hydrogen. The reason for this lack of success is that an electron trapped in any atom is a matter wave confined to a potential well, and to find the resulting quantized energy values we

must apply Schrödinger's equation to the electron. Nonetheless, Bohr's theory has an artistic elegance which is a very good stepping stone for the entry of a neophyte in the world of quantum physics. So here we study this model in greater detail.

39-6 | Spectra of the Hydrogen Atom

The potential well of a hydrogen atom depends on the electrical potential energy function

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r}. \tag{39-34}$$

Because this well is three-dimensional, it is more complex than our previous one-dimensional well. Moreover, it does not have sharply defined walls. Rather, its walls vary in depth with radial distance r. Figure 39-9 is probably the best we can do in drawing the hydrogen potential well, but even that drawing takes much effort to interpret.

Energy Levels and Spectra of the Hydrogen Atom

Although we shall not do so here, we can apply Schrödinger's equation for an electron trapped in the potential well given by Eq. 39-34. In doing so, we would find that the energy values are quantized and that, amazingly, those values are given by Eq. 39-29 just as for the (incorrect) Bohr model. Thus, changes ΔE in energy due to emission or absorption of light are given by Eq. 39-30, and the wavelengths corresponding to ΔE are given by Eq. 39-32. Let's explore these results.

Figure 39-10 shows the energy levels corresponding to various values of n in Eq. 39-29. The lowest level, for $n = 1$, is the ground state of hydrogen. Higher levels correspond to excited states, just as we saw for our simpler potential traps. Note several differences, however. (1) The energy levels now have negative values rather than the positive values we previously chose in Fig. 39-3. (2) The levels now become progressively closer as we move to higher levels. (3) The energy for the greatest value of n—namely, $n = \infty$—is now $E_\infty = 0$. For any energy greater than $E_\infty = 0$, the electron and proton are not bound together (there is no hydrogen atom), and the $E > 0$ region in Fig. 39-10 is the nonquantized region.

A hydrogen atom can jump between quantized energy levels by emitting or absorbing light at the wavelengths given by Eq. 39-32. Any photon other than this will pass through the sample and will not be absorbed by the atom. Any such wavelength is often called a *line* because of the way it is detected with a spectroscope; thus, a hydrogen atom has *absorption lines* and *emission lines*. A collection of such lines, such as in those in the visible range, is called a **spectrum** of the hydrogen atom.

The lines for hydrogen are said to be grouped into *series*, according to the level at which upward jumps start and downward jumps end. For example, the emission and absorption lines for all possible jumps up from the $n = 1$ level and down

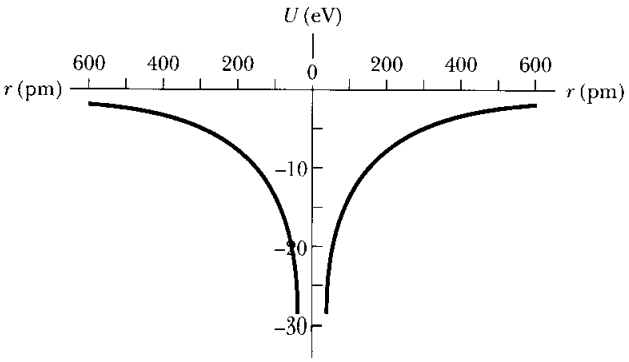


FIG. 39-9 The potential energy U of a hydrogen atom as a function of the separation r between the electron and the central proton. The plot is shown twice (on the left and on the right) to suggest the three-dimensional spherically symmetric trap in which the electron is confined.

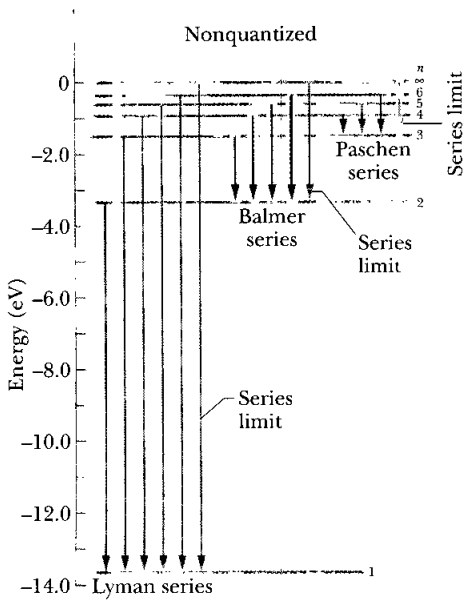


FIG. 39-10 A plot of Eq. 39-29, showing a few of the energy levels of the hydrogen atom. The transitions are grouped into series, each labeled with the name of a person.

to the $n = 1$ level are said to be in the *Lyman series*, named after the person who first studied those lines. Further, we can say that the Lyman series has a *home-base level* of $n = 1$. Similarly, the *Balmer series* has a home-base level of $n = 2$, and the *Paschen series* has a home-base level of $n = 3$.

Some of the downward quantum jumps for these three series are shown in Fig. 39-10. Four lines in the Balmer series are in the visible range and are the ones Balmer studied. They are represented in Fig. 39-10 with arrows corresponding to their colors. The shortest of those arrows represents the shortest jump in the series, from the $n = 3$ level to the $n = 2$ level. Thus, that jump involves the smallest change in the electron's energy and the smallest amount of emitted photon energy for the series. The emitted light is red. The next jump in the series, from $n = 4$ to $n = 2$, is longer, the photon energy is greater, the wavelength of the emitted light is shorter, and the light is green. The third, fourth, and fifth arrows represent longer jumps and shorter wavelengths. For the fifth jump, the emitted light is in the ultraviolet range and thus is not visible.

The *series limit* of a series is the line produced by the jump between the home-base level and the highest energy level, which is the level with quantum number $n = \infty$. Thus, the series limit is the shortest wavelength in the series. Figure 39-11 is a photograph of the Balmer emission lines taken with a spectroscope. The series limit for the series is marked with a small triangle.

The energy needed to take the electron from its ground state to an excited state is called the excitation energy of that excited state. For example, the excitation energy of the first excited state is 10.2 eV because the energy difference between the ground state and the first excited state is $13.6 - 3.4 = 10.2$ eV.

If a jump is upward into the nonquantized portion of Fig. 39-10, the electron's energy is no longer given by Eq. 39-29 because the electron is no longer trapped in the atom. That is, the hydrogen atom has been *ionized*, meaning that the electron has been removed to a distance so great that the Coulomb force on it from the nucleus is negligible. The atom can be ionized if it absorbs any wavelength greater than the series limit. The free electron then has only kinetic energy $K(=\frac{1}{2}mv^2$, assuming a nonrelativistic situation).

It is now clear that when an electron jumps from a higher energy level to a lower energy level, the radiation is emitted in the form of photons. The radiation emitted in such a transition corresponds to spectral lines in atomic spectra of hydrogen atom. In general, the number of spectral lines emitted when an electron falls from state n_2 to state n_1 is

$$\frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2}.$$

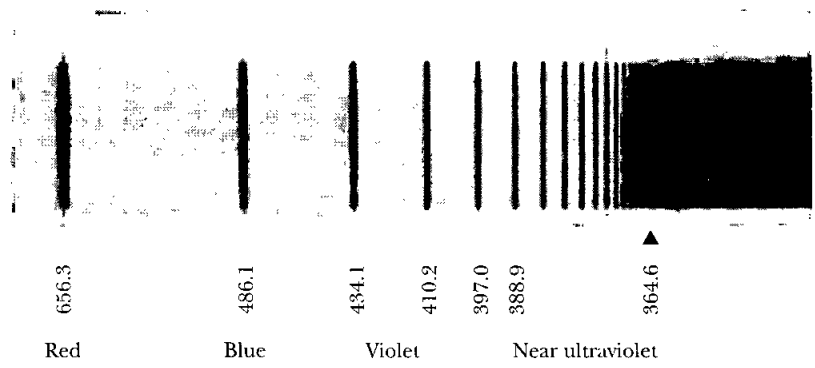


FIG. 39-11 The spectrum of emission lines for the Balmer series of the hydrogen atom (wavelengths in nanometers). Whereas Fig. 39-10 shows four transitions in this series, along with the series limit, this figure shows about a dozen lines of the series; note how they are progressively closer toward the series limit, which is marked with a triangle. *(From W. Finkelburg,* Structure of Matter, *Springer-Verlag, 1964. Reproduced with permission)*

The spectral lines are defined for hydrogen atom only and not for hydrogen-like ions.

Lyman series: It lies in ultraviolet region. When an electron jumps from a higher energy state n to ground state, $n = 1$, the series of spectral lines emitted is called Lyman series. All of them lie in near ultraviolet region.

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right). \tag{39-35}$$

Balmer series: *When an electron* jumps from a higher energy state n to state 2, the series of spectral lines emitted is called Balmer series.

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n_2^2}\right). \tag{39-36}$$

Paschen series: When an electron jumps from a higher energy state n to state 3, the series of spectral lines emitted is called Paschen series. All of them lie in near infrared region.

$$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n_2^2}\right). \tag{39-37}$$

Brackett series: State n to 4 (all of them lie in far infrared region)

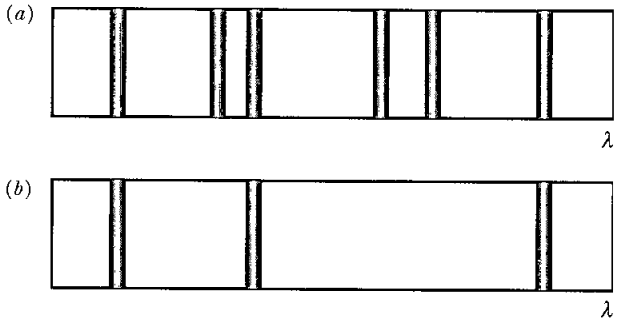
$$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n_2^2}\right). \tag{39-38}$$

Pfund series: State n to 5 (all of them lie in far–far infrared region)

$$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n_2^2}\right). \tag{39-39}$$

We have to understand that the absorption spectra are just a subset of the emission spectra. Most of the atoms at the room temperature are in ground state, so the atom absorbs photons corresponding to Lyman series only. But during emission, the atom can emit photons belonging to other series as well.

CHECKPOINT 4 In the following figure, one of the spectra is emission spectra and another is absorption spectra of the same element. Identify which is which?



Sample Problem 39-5

(a) What is the wavelength of light for the least energetic photon emitted in the Lyman series of the hydrogen atom spectrum lines?

Conceptualize: For the Lyman series, the home-base level is at n = 1 (Fig. 39-10). Thus, the transition that produces the least energetic photon is the transition from the n = 2 level to the $n = 1$ level.

Classify: For any series, the transition that produces the least energetic photon is the transition between the home-base level that defines the series and the level immediately above it.

Compute: From Eq. 39-29 the energy difference is

$$\Delta E = E_2 - E_1 = -(13.60\text{ eV})\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.20\text{ eV}.$$

Then from Eq. 39-6 ($\Delta E = hf$), with c/λ replacing f, we have

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^{8}\text{ m/s})}{(10.20\text{ eV})(1.60\times10^{-19}\text{ J/eV})}$$
$$= 1.22\times10^{-7}\text{ m} = 122\text{ nm}. \qquad \text{(Answer)}$$

Light with this wavelength is in the ultraviolet range.

(b) What is the wavelength of the series limit for the Lyman series?

Conceptualize/Classify: The series limit corresponds to a jump between the home-base level ($n = 1$ for the Lyman series) and the level at the limit $n = \infty$.

Compute: Now that we have identified the values of n for the transition, we could proceed as in (a) to find the corresponding wavelength λ. Instead, let's use a more direct procedure. From Eq. 39-32, we find

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right)$$
$$= 1.097\,373\times10^{7}\text{ m}^{-1}\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right),$$

which yields

$$\lambda = 9.11\times10^{-8}\text{ m} = 91.1\text{ nm}. \qquad \text{(Answer)}$$

Conclude: Light with this wavelength is also in the ultraviolet range.

Excitation by Collision

Besides the excitation by photons, the atom can also be excited by collision. This is shown schematically in Fig. 39-12. For example, in a discharge tube filled with hydrogen at a low pressure, the electron is accelerated under the influence of an electric field. Near the end of the tube, the electron will have the maximum kinetic energy. It can collide with a hydrogen atom there and excite the electron in ground state to a higher excited state. The state to which the electron within the atom is excited depends on the kinetic energy of the incident electron. Let us see how.

We have seen in the chapter on momentum that the kinetic energy of two particle system from laboratory frame of reference is

$$K_L = \tfrac{1}{2}M_{\text{tot}}V_{\text{cm}}^2 + \tfrac{1}{2}\mu V_{\text{rel}}^2, \qquad (39\text{-}40)$$

where μ is the reduced mass of the system given by $\mu = mM/(m+M)$; M_{tot} is the total mass of the system; V_{rel} is the velocity of the one particle relative to the other; and V_{cm} is the velocity of the center of mass. We are aware that in absence of external force, V_{cm} remains constant. The only variable part in this equation for the kinetic energy of the system is $(1/2)\mu V_{\text{rel}}^2$. This is known as the kinetic energy of the system in the center of mass frame of reference.

As we have seen in momentum conservation, for an elastic collision, the magnitude of relative velocity before and after the collision remains the same. So $(1/2)\mu V_{\text{rel}}^2$ remains the same. But for an inelastic collision, V_{rel} decreases. So $(1/2)\mu V_{\text{rel}}^2$ decreases. For a perfectly inelastic collision, after the collision, $V_{\text{rel}} = 0$. So the energy loss will be maximum in this case; it will be equal to $(1/2)\mu V_{\text{rel}}^2$. This energy is lost in the form of heat.

However, in the case of atomic collisions, the heat liberated has no meaning. Furthermore, the collision can be elastic and inelastic depending on $(1/2)\mu V_{\text{rel}}^2$. How? We have seen that the energy available to be lost is only $(1/2)\mu V_{\text{rel}}^2$. But an inelastic collision on atomic level means that the electron in the atom is

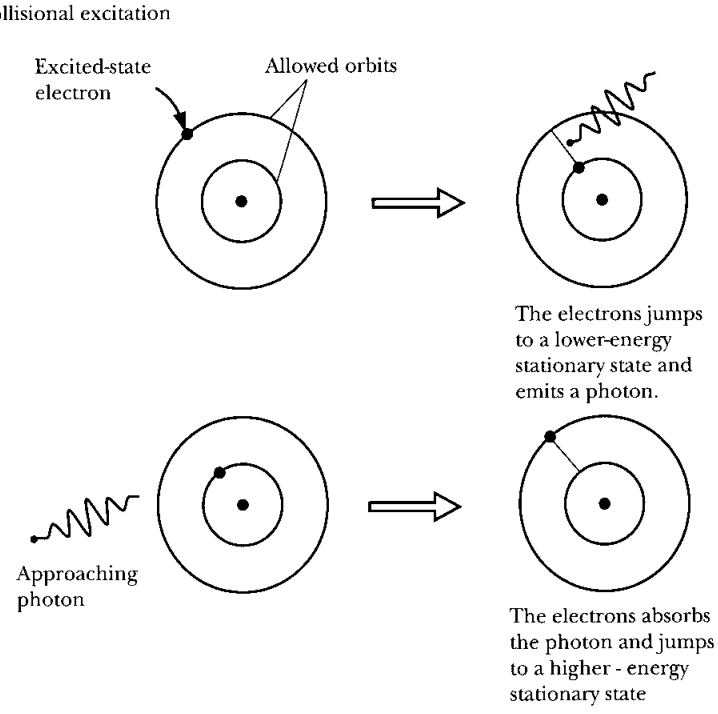


FIG. 39-12 An atom can change stationary states by emitting or absorbing a photon or by undergoing a collision.

excited to a higher state. This is the only mechanism the energy can be lost in this case. So if

$$\tfrac{1}{2}\mu V_{\text{rel}}^2 \geq \text{energy required for excitation}$$

then the collision can be inelastic; otherwise the collision will be elastic.
Let us show this by an example. Consider an electron moving with energy E. We want to find the maximum value of this energy so that it does not excite any electron in a hydrogen atom it strikes:

$$\mu = \frac{m_e m_h}{m_e + m_h} \approx m_e.$$

So the kinetic energy of the striking electron is almost entirely available for excitation. We know that to excite an electron in hydrogen atom from ground state to $n = 2$ at least requires 10.2 eV. So if we do not want the excitation to occur, then

$$E < 10.2 \text{ eV}.$$

In other words, the potential drop in the discharge tube should not be greater than 10.2 V.

Sample Problem 39-6

A neutron moving with a speed v strikes a hydrogen atom in the ground state moving toward it with the same speed. Find the minimum kinetic energy of the neutron for which inelastic (completely or partially) collision might take place. (Take $m_n = m_h$.)

Conceptualize/Classify: Here the neutron has no electronic structure, but the hydrogen atom has. So it can be excited to a higher state. As some mechanical energy will be lost from the system, we can consider the collision to be inelastic.

Compute: We have

$$\mu = \frac{m_n m_h}{m_n + m_h} = \frac{m_h}{2}.$$

For excitation, collision should be inelastic. So

$$\tfrac{1}{2}\mu V_{rel}^2 \geq 10.2 \text{ eV}$$

$$\tfrac{1}{2} m_n v^2 \geq 20.4 \text{ eV}.$$

Conclude: The interesting part is that the striking neutron needs to have a kinetic energy greater than the energy required for excitation, in fact double that energy. Why? Consider that the collision is inelastic. In that case, by the law of conservation of momentum, the two will stick together and move with the same velocity. So the entire energy cannot be lost in the form of a photon. Some energy will remain in form of the kinetic energy of the system.

Since a discharge tube can also be used to produce the emission spectra of the gas, many times the terms related to discharge tube are also used in spectroscopy. Thus, *excitation potential* of an excited state refers to the potential difference across the discharge tube, required to excite the atom from state $n = 1$ to that state. Its numerical value is equal to the excitation energy in electron volts. *Ionization potential* of an atom refers to the potential difference across the discharge tube required to ionize the atom. Its numerical value is equal to the ionization energy in electron volts. This spectrum is characteristic of the hydrogen atom. It is like a signature which can be used to identify the atom. in the case of multielectron atom, the spectra are much more complicated, and their pattern can be predicted using only quantum theory. In the chapter opener question regarding the "Pickle glow," the pickle conducts electricity due to vinegar solution (acetic acid) and sodium chloride salt used to make it. Because of the heat and sparks near the electrodes, the sodium atoms near the electrodes are in a gaseous phase. These sodium atoms get excited due to electrons flowing through the pickle. So they emit yellow light characteristic of sodium as they come down to ground state. If we use rock salt to make the pickle, the glow will be pinkish because of the presence of potassium atoms.

Sample Problem 39-7

Consider an excited hydrogen atom in state n moving with a velocity v ($v \ll c$). It emits a photon in the direction of its motion and changes its state to a lower state m. Apply momentum and energy conservation principles to calculate the frequency of emitted radiation. Compare this with the frequency emitted if the atom were at rest.

Conceptualize/Classify: Normally, we ignore the recoil of the atom that is emitting the photon. But here the atom that is emitting the photon will also change its momentum due to recoil.

Compute: Let us assume that the atom emits a photon of frequency ν and has a velocity of v_0 after the emission. By the law of conservation of energy, we have

$$\tfrac{1}{2} m v^2 + E_n = \tfrac{1}{2} m v_0^2 + E_m + h\nu. \qquad (39\text{-}41)$$

By the law of conservation of momentum, we have

$$mv = mv_0 + h\frac{\nu}{c}. \qquad (39\text{-}42)$$

In absence of recoil, the atom would have emitted a photon of frequency ν_0:

$$E_n - E_m = h\nu_0. \qquad (39\text{-}43)$$

So we have the equations as

$$\tfrac{1}{2}m(v^2 - v_0^2) = h(\nu_0 - \nu)$$

and

$$m(v - v_0) = h\frac{\nu}{c}.$$

By dividing the two equations, we get

$$\frac{v + v_0}{2c} = \frac{\nu_0 - \nu}{\nu}.$$

Since v and $v_0 \ll c$, we get

$$\nu \approx \nu_0\left(1 + \frac{v}{c}\right).$$

Conclude: If you notice, there is a striking similarity between this expression and the expression for the frequency of the wave emitted by a moving source. There we get the formula for the frequency as

$$\nu = \nu_0\left(1 - \frac{v}{c}\right)^{-1}.$$

By applying binomial approximation, we get

$$\nu \approx \nu_0\left(1 + \frac{v}{c}\right).$$

Effect of Motion of the Nucleus on the Spectra

It has been assumed in the above considerations that the electron moved around a fixed nucleus. This would only be strictly true if the mass of the nucleus were infinite. The fact that it is only some 1836 times as heavy as the electron implies that they each move around a common center of gravity, as shown in Fig. 39-13. By mass moments, we know that

$$\frac{m}{M} = \frac{A}{a}.$$

Therefore

$$a = \frac{Mr}{M+m}$$

and

$$A = \frac{mr}{M+m},$$

where A and a are the distances of the nucleus of mass M and electron of mass m, respectively, from the common center of gravity and r is the separation of the nucleus and the electron. When ω is the angular velocity of the system, the total kinetic energy is given by

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}MA^2\omega^2 + \tfrac{1}{2}ma^2\omega^2$$

$$= \tfrac{1}{2}\frac{Mm}{M+m}r^2\omega^2$$

$$= \tfrac{1}{2}\mu r^2\omega^2.$$

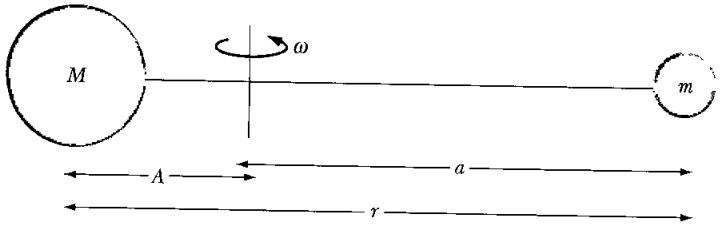


FIG. 39-13 Electron and proton moving around a common center of gravity.

It, thus, appears that to allow for the motion of the nucleus we must replace the mass m of the electron by μ, the reduced mass. The Rydberg constant then becomes

$$R = \frac{2\pi^2 \mu e^4}{ch^3(4\pi\varepsilon_0)^2} = \frac{2\pi^2 me^4}{ch^3(4\pi\varepsilon_0)^2} = \frac{1}{1+m/M} = \frac{R_\infty}{1+m/M}.$$

Isotopes carry the same nuclear charge, but have different nuclear masses. This gives rise to slightly different values for the Rydberg constant. So different isotopes of a given element would absorb or emit light of slightly different wavelengths while undergoing the same transition.

Sample Problem 39-8

A muon is unstable elementary particle whose mass is $207m_e$ and whose charge is either $+e$ or $-e$. A negative muon (μ^-) can be captured by a proton to form a muonic atom. (a) Find the distance between the muon and proton for this atom. (b) Find the ionization energy of the atom.

Conceptualize: Here, we should replace the electronic mass by the reduced mass.

Classify: Here, we can see that the mass of the nucleus and the mass of the muon are comparable. So the motion of nucleus cann ot be neglected.

Compute:
(a) Here $m = 207\ m_e$ and $M = 1836\ m_e$, so the reduced mass is

$$\mu = \frac{mM}{m+M} = \frac{(207m_e)(1836m_e)}{207m_e + 1836m_e} = 186m_e.$$

According to Eq. 39-22, the orbit radius corresponding to $n = 1$ is

$$r_1 = \frac{h^2\varepsilon_0}{\pi m_e e^2},$$

where $r_1 = a_0 = 5.29 \times 10^{-11}$ m. Hence, the distance between the muon and the proton r that corresponds to the reduced mass μ is

$$r_1' = \left(\frac{m}{\mu}\right) r_1 = \left(\frac{m_e}{186m_e}\right) a_0 = 2.85 \times 10^{-13} \text{ m}.$$

The muon is 186 times closer to the proton than an electron would be.

(b) From Eq. 39-28, we have, with $n = 1$ and $E_1 = -13.6$ eV:

$$E_1' = \left(\frac{\mu}{m}\right) E_1 = 186E_1 = -2.53 \times 10^3 \text{ eV} = -2.53 \text{ keV}.$$

The ionization energy is therefore 2.53 keV, 186 times that for an ordinary hydrogen atom.

Conclude: If we did not take the effect of reduced mass into account, the radius would have been 207 times smaller, but here we see the effect is only 186 times smaller. Similarly, we can say about the energy. But the velocity of muon in first Bohr orbit would be the same because the velocity of the electron does not depend on the electronic mass.

CHECKPOINT 5 What would be the effect of nuclear motion on the radius of the electronic orbit? Would the radius be the same as that of the predicted Bohr orbit by Eq. 39-22, more or less?

39-7 | Quantum Numbers for the Hydrogen Atom (Optional)

Although the energies of the hydrogen atom states can be described by the single quantum number n, the wave functions describing these states require three quantum numbers, corresponding to the three dimensions in which the electron can move. The three quantum numbers, along with their names and the values that they may have, are shown in Table 39-1.

Each set of quantum numbers (n, ℓ, m_ℓ) identifies the wave function of a particular quantum state. The quantum number n, called the **principal quantum number**, appears in Eq. 39-29 for the energy of the state. The **orbital quantum number** ℓ is a measure of the magnitude of the angular momentum associated with the quantum state. The **orbital magnetic quantum number** m_ℓ is related to the orientation in space of this angular momentum vector. The restrictions on the values of the quantum numbers for the hydrogen atom, as listed in Table 39-1, are not arbitrary but come out of the solution to Schrödinger's equation. Note that for the ground state ($n = 1$), the restrictions require that $\ell = 0$ and $m_\ell = 0$. That is, the hydrogen atom in its ground state has zero angular momentum, which is not predicted by Eq. 39-19 in the Bohr model.

CHECKPOINT 6 (a) A group of quantum states of the hydrogen atom has $n = 5$. How many values of ℓ are possible for states in this group? (b) A subgroup of hydrogen atom states in the $n = 5$ group has $\ell = 3$. How many values of m_ℓ are possible for states in this subgroup?

The Wave Function of the Hydrogen Atom's Ground State

The wave function for the ground state of the hydrogen atom, as obtained by solving the three-dimensional Schrödinger equation and normalizing the result, is

$$\psi(r) = \frac{1}{\sqrt{\pi}a^{3/2}} e^{-r/a} \quad \text{(ground state)}, \tag{39-44}$$

where a is the Bohr radius (Eq. 39-24). This radius is loosely taken to be the effective radius of a hydrogen atom and turns out to be a convenient unit of length for other situations involving atomic dimensions.

As with other wave functions, $\psi(r)$ in Eq. 39-44 does not have physical meaning but $\psi^2(r)$ does, being the probability density—the probability per unit volume—that the electron can be detected. Specifically, $\psi^2(r)\,dV$ is the probability that the electron can be detected in any given (infinitesimal) volume element dV located at radius r from the center of the atom:

$$\begin{pmatrix}\text{probability of detection}\\ \text{in volume } dV\\ \text{at radius } r\end{pmatrix} = \begin{pmatrix}\text{volume probability}\\ \text{density } \psi^2(r)\\ \text{at radius } r\end{pmatrix} (\text{volume } dV). \tag{39-45}$$

Because $\psi^2(r)$ here depends only on r, it makes sense to choose, as a volume element dV, the volume between two concentric spherical shells whose radii are r and $r + dr$. That is, we take the volume element dV to be

$$dV = (4\pi r^2)\,dr, \tag{39-46}$$

in which $4\pi r^2$ is the surface area of the inner shell and dr is the radial distance between the two shells. Then, combining Eqs. 39-44, 39-45, and 39-46 gives us

TABLE 39-1

Quantum Numbers for the Hydrogen Atom

Symbol	Name	Allowed Values
n	Principal quantum number	$1, 2, 3, \ldots$
l	Orbital quantum number	$0, 1, 2, \ldots, n-1$
m	Orbital magnetic quantum number	$-\ell, -(\ell-1), \ldots, +(\ell-1), +\ell$

$$\begin{pmatrix}\text{probability of detection}\\ \text{in volume } dV\\ \text{at radius } r\end{pmatrix} = \psi^2(r)dV = \frac{4}{a^3}e^{-2r/a}r^2dr. \tag{39-47}$$

Describing the probability of detecting an electron is easier if we work with a **radial probability density** $P(r)$ instead of a volume probability density $\psi^2(r)$. This $P(r)$ is a linear probability density such that

$$\begin{pmatrix}\text{radial probability}\\ \text{density } P(r)\\ \text{at radius } r\end{pmatrix}\begin{pmatrix}\text{radial}\\ \text{width } dr\end{pmatrix} = \begin{pmatrix}\text{volume probability}\\ \text{density } \psi^2(r)\\ \text{at radius } r\end{pmatrix}(\text{volume } dV)$$

or
$$P(r)\,dr = \psi^2(r)\,dV. \tag{39-48}$$

Substituting for $\psi^2(r)\,dV$ from Eq. 39-47, we obtain

$$P(r) = \frac{4}{a^3}r^2e^{-2r/a} \qquad \text{(radial probability density, hydrogen atom ground state).} \tag{39-49}$$

Figure 39-14 is a plot of Eq. 39-49. The area under the plot is unity; that is,

$$\int_0^\infty P(r)\,dr = 1. \tag{39-50}$$

This equation states that in a hydrogen atom, the electron must be *somewhere* in the space surrounding the nucleus.

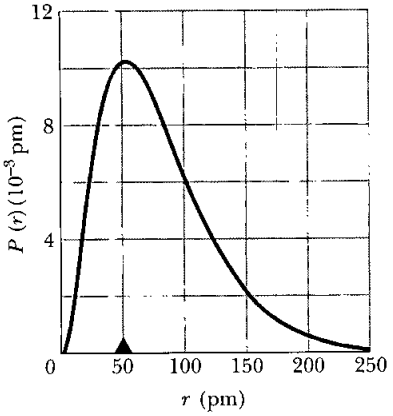


FIG. 39-14 A plot of the radial probability density $P(r)$ for the ground state of the hydrogen atom. The triangular marker is located at one Bohr radius from the origin, and the origin represents the center of the atom.

The triangular marker on the horizontal axis of Fig. 39-14 is located one Bohr radius from the origin. The graph tells us that in the ground state of the hydrogen atom, the electron is most likely to be found at about this distance from the center of the atom.

Figure 39-14 conflicts sharply with the popular view that electrons in atoms follow well-defined orbits like planets moving around the Sun. *This popular view, however familiar, is incorrect.* Figure 39-14 shows us all that we can ever know about the location of the electron in the ground state of the hydrogen atom. The appropriate question is not "When will the electron arrive at such-and-such a point?" but "What are the odds that the electron will be detected in a small volume centered on such-and-such a point?" Figure 39-15, which we call a dot plot, suggests the probabilistic nature of the wave function and provides a useful mental model of the hydrogen atom in its ground state. Think of the atom in this state as a fuzzy ball with no sharply defined boundary and no hint of orbits.

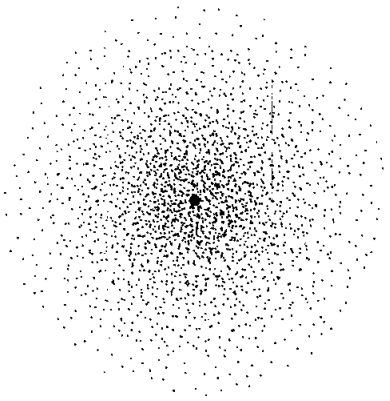


FIG. 39-15 A "dot plot" showing the volume probability density $\psi^2(r)$ – not the *radial* probability density $P(r)$ – for the ground state of the hydrogen atom. The density of dots drops exponentially with increasing distance from the nucleus, which is represented here by a red spot. Such dot plots provide a mental image of the "electron cloud" of an atom.

It is not easy for a beginner to envision subatomic particles in this probabilistic way. The difficulty is our natural impulse to regard an electron as something like a tiny jelly bean, located at certain places at certain times and following a well-defined path. Electrons and other subatomic particles simply do not behave in this way.

The energy of the ground state, found by putting $n = 1$ in Eq. 39-29, is $E_1 = -13.60$ eV. The wave function of Eq. 39-44 results if you solve Schrödinger's equation with this value of the energy. Actually, you can find a solution of Schrödinger's equation for *any* value of the energy – say, $E = -11.6$ eV or -14.3 eV. This may suggest that the energies of the hydrogen atom states are not quantized – but we know that they are.

The puzzle was solved when physicists realized that such solutions of Schrödinger's equation are not physically acceptable because they yield increasingly large values as $r \to \infty$. These "wave functions" tell us that the electron is more likely to be found very far from the nucleus rather than closer to it, which makes no sense. We get rid of these unwanted solutions by imposing what is called

a **boundary condition**, in which we agree to accept only solutions of Schrödinger's equation for which $\psi(r) \to 0$ as $r \to \infty$; that is, we agree to deal only with *confined* electrons. With this restriction, the solutions of Schrödinger's equation form a discrete set, with quantized energies given by Eq. 39-29.

Sample Problem 39-9

Show that the radial probability density for the ground state of the hydrogen atom has a maximum at $r = a$.

Conceptualize/Classify: (1) The radial probability density for a ground-state hydrogen atom is given by Eq. 39-49,

$$P(r) = \frac{4}{a^3} r^2 e^{-2r/a}.$$

(2) To find the maximum (or minimum) of any function, we must differentiate the function and set the result equal to zero.

Compute: If we differentiate $P(r)$ with respect to r, using derivative 7 of Appendix E and the chain rule for differentiating products, we get

$$\frac{dP}{dr} = \frac{4}{a^3} r^2 \left(\frac{-2}{a}\right) e^{-2r/a} + \frac{4}{a^3} 2r e^{-2r/a}$$

$$= \frac{8r}{a^3} e^{-2r/a} - \frac{8r^2}{a^4} e^{-2r/a}$$

$$= \frac{8}{a^4} r(a - r) e^{-2r/a}.$$

If we set the right side equal to zero, we obtain an equation that is true if $r = a$. In other words, dP/dr is equal to zero when $r = a$. (Note that we also have $dP/dr = 0$ at $r = 0$ and at $r = \infty$. However, these conditions correspond to a *minimum* in $P(r)$, as you can see in Fig. 39-14.)

Sample Problem 39-10

It can be shown that the probability $p(r)$ that the electron in the ground state of the hydrogen atom will be detected inside a sphere of radius r is given by

$$p(r) = 1 - e^{-2x}(1 + 2x + 2x^2),$$

in which x, a dimensionless quantity, is equal to r/a. Find r for $p(r) = 0.90$.

Conceptualize/Classify: There is no guarantee of detecting the electron at any particular radial distance r from the center of the hydrogen atom. However, with the given function, we can calculate the probability that the electron will be detected *somewhere* within a sphere of radius r.

Compute: We seek the radius of a sphere for which $p(r) = 0.90$. Substituting that value in the expression for $p(r)$, we have

$$0.90 = 1 - e^{-2x}(1 + 2x + 2x^2)$$

or

$$10\, e^{-2x}(1 + 2x + 2x^2) = 1.$$

We must find the value of x that satisfies this equality. It is not possible to solve explicitly for x, but an equation solver on a calculator yields $x = 2.66$. This means that the radius of a sphere within which the electron will be detected 90% of the time is $2.66a$. Mark this position on the horizontal axis of Fig. 39-14—is it a reasonable answer?

Hydrogen Atom States with $n = 2$

According to the requirements of Table 39-1, there are four states of the hydrogen atom with $n = 2$; their quantum numbers are listed in Table 39-2. Consider first the state with $n = 2$ and $\ell = m_\ell = 0$; its probability density is represented by the dot plot of Fig. 39-16. Note that this plot, like the plot for the ground state shown in Fig. 39-15, is spherically symmetric. That is, in a spherical coordinate system like that defined in Fig. 39-17, the probability density is a function of the radial coordinate r only and is independent of the angular coordinates θ and ϕ.

TABLE 39-2
Quantum Numbers for Hydrogen Atom States with $n = 2$

n	ℓ	m_ℓ
2	0	0
2	1	+1
2	1	0
2	1	−1

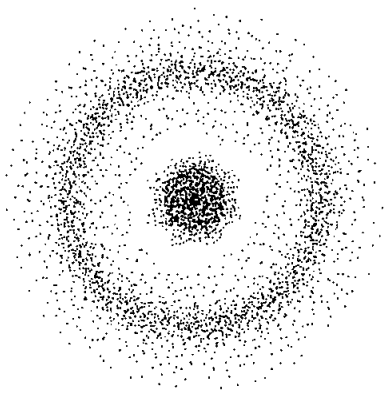


FIG. 39-16 A dot plot showing the volume probability density $\psi^2(r)$ for the hydrogen atom in the quantum state with $n = 2$ 1 $\ell = 0$, and $m_\ell = 0$. The plot has spherical symmetry about the central nucleus. The gap in the dot density pattern marks a spherical surface over which $\psi^2(r) = 0$.

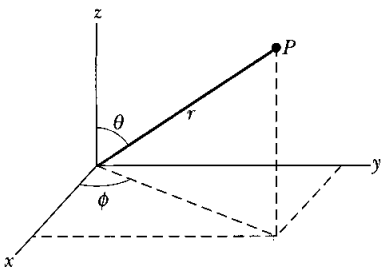


FIG. 39-17 The relationship between the coordinates x, y, and z of the rectangular coordinate system and the coordinates r, θ, and ϕ of the spherical coordinate system. The latter are more appropriate for analyzing situations involving spherical symmetry, such as the hydrogen atom.

It turns out that all quantum states with $\ell = 0$ have spherically symmetric wave functions. This is reasonable because the quantum number ℓ is a measure of the angular momentum associated with a given state. If $\ell = 0$, the angular momentum is also zero, which requires that the probability density representing the state have no preferred axis of symmetry.

Dot plots of ψ^2 for the three states with $n = 2$ and $\ell = 1$ are shown in Fig. 39-18. The probability densities for the states with $m_\ell = +1$ and $m_\ell = -1$ are identical. Although these plots are symmetric about the z axis, they are *not* spherically symmetric. That is, the probability densities for these three states are functions of both r and the angular coordinate θ.

Here is a puzzle: What is there about the hydrogen atom that establishes the axis of symmetry that is so obvious in Fig. 39-18? The answer: *absolutely nothing.*

The solution to this puzzle comes about when we realize that all three states shown in Fig. 39-18 have the same energy. Recall that the energy of a state, given by Eq. 39-29, depends only on the principal quantum number n and is independent of ℓ and m_ℓ. In fact, for an *isolated* hydrogen atom there is no way to differentiate experimentally among the three states of Fig. 39-18.

If we add the volume probability densities for the three states for which $n = 2$ and $\ell = 1$, the combined probability density turns out to be spherically symmetrical, with no unique axis. One can, then, think of the electron as spending one-third of its time in each of the three states of Fig. 39-18, and one can think of the weighted sum of the three independent wave functions as defining a spherically symmetric **subshell** specified by the quantum numbers $n = 2$, $\ell = 1$. The individual states will display their separate existence only if we place the hydrogen atom in an external electric or magnetic field. The three states of the $n = 2$, $\ell = 1$ subshell will then have different energies, and the field direction will establish the necessary symmetry axis.

The $n = 2$, $\ell = 0$ state, whose volume probability density is shown in Fig. 39-16, *also* has the same energy as each of the three states of Fig. 39-18. We can view all four states whose quantum numbers are listed in Table 39-2 as forming a spherically symmetric **shell** specified by the single quantum number n. The importance of shells and subshells will become evident in Chapter 40, where we discuss atoms having more than one electron.

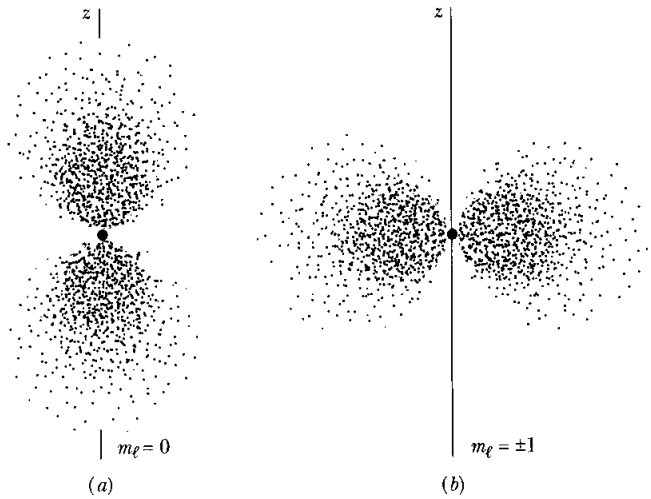


FIG. 39-18 Dot plots of the volume probability density $\psi^2(r, \theta)$ for the hydrogen atom in states with $n = 2$ and $\ell = 1$. (a) Plot for $m_\ell = 0$. (b) Plot for $m_\ell = +1$ and $n_\ell = -1$. Both plots show that the probability density is symmetric about the z axis.

To round out our picture of the hydrogen atom, we display in Fig. 39-19 a dot plot of the *radial* probability density for a hydrogen atom state with a relatively high quantum number ($n = 45$) and the highest orbital quantum number that the restrictions of Table 39-1 permit($\ell = n - 1 = 44$). The probability density forms a ring that is symmetrical about the z axis and lies very close to the xy plane. The mean radius of the ring is n^2a, where a is the Bohr radius. This mean radius is more than 2000 times the effective radius of the hydrogen atom in its ground state.

Figure 39-19 suggests the electron orbit of classical physics—it resembles the circular orbit of a planet around a star. Thus, we have another illustration of Bohr's correspondence principle—namely, that at large quantum numbers the predictions of quantum mechanics merge smoothly with those of classical physics. Imagine what a dot plot like that of Figure 39-19 would look like for *really* large values of n and ℓ—say, $n = 1000$ and $\ell = 999$.

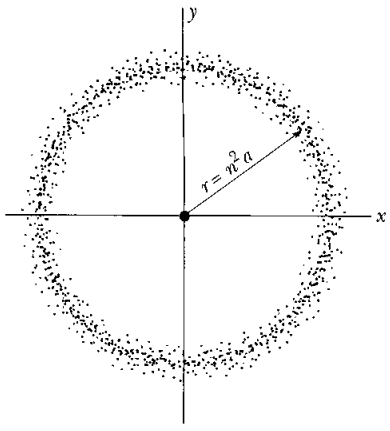


FIG. 39-19 A dot plot of the radial probability density $P(r)$ for the hydrogen atom in a quantum state with a relatively large principal quantum number—namely, $n = 45$—and angular momentum quantum number $\ell = n - 1 = 44$. The dots lie close to the xy plane, the ring of dots suggesting a classical electron orbit.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: On earth, the absorption spectra hydrogen gas at room temperature contains only Lyman series.

Statement 2: Any system tries to occupy a state of minimum potential energy.

2. Statement 1: In a cathode ray tube filled with hydrogen at high pressure, discharge is obtained when the tube is operating at potential slightly higher than 13.6 V.

Statement 2: The ionization potential of hydrogen is 13.6 V.

3. Statement 1: In Bohr's model of hydrogen atom, only those orbits radii are allowed which have circumference as $n\lambda$ (n is integer).

Statement 2: By de Broglie's hypothesis, the electron wave forms a standing wave pattern in Bohr's orbit.

4. Statement 1: In a sample of hydrogen atom, all atoms are in an excited state. Three different frequency photons are observed from the sample. Then there is a combination such that $f_1 = f_2 + f_3$.

Statement 2: The frequency of revolution of electrons is quantized.

5. Statement 1: Balmer's series is not observed in absorption experiments done on earth, but is observed in sunlight coming to earth.

Statement 2: The temperature on the earth surface is much lesser than on the Sun.

6. Statement 1: The absorption line observed in spectra of an element is never completely dark.

Statement 2: The sample used for absorption is thin, so that all photons corresponding to a transition may not be absorbed.

MULTIPLE CHOICE QUESTIONS

sec. 39-3 Energies of a Trapped Electron

1. A particle in a box has quantum states with energies $E = E_0n^2$, with $n = 1, 2, 3, 4, \ldots$ and $E_0 = 1$ eV. Which of these photons could in principle be absorbed?

(a) 1 eV

(b) 2 eV

(c) 4 eV

(d) 5 eV

(e) 6 eV

2. Which of the following transitions produce the longest wavelength photon (Fig. 39-20)?

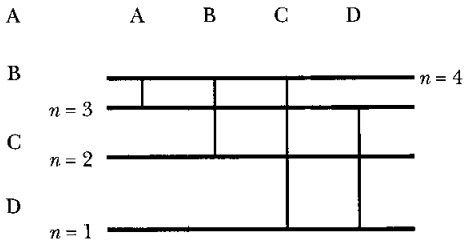


FIG. 39-20 Multiple choice question 2.

(a) $n = 4 \rightarrow 3$

(b) $n = 4 \rightarrow 2$

(c) $n = 3 \rightarrow 1$

(d) $n = 3 \rightarrow 2$

3. For a quantum particle in a box, the lowest energy quantum state has $\frac{1}{2}$ of the particle de Broglie wave fitting in the box. The next highest state corresponds to

(a) 1/4 de Broglie wavelength

(b) 1/2 de Broglie wavelength

(c) 1 de Broglie wavelength

(d) 2 de Broglie wavelength

sec. 39-4 Wave Functions of a Trapped Electron

4. An electron is in a one-dimensional trap with zero potential energy in the interior and infinite potential energy at the walls. A graph of its wave function $\psi(x)$ versus x is shown in Fig. 39-21. The value of quantum number n is

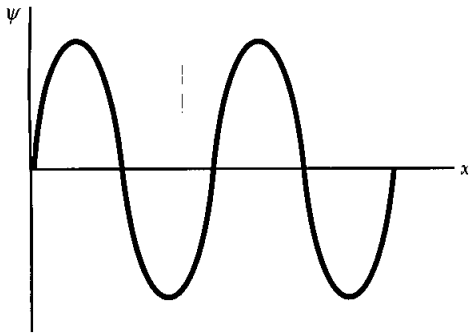


FIG. 39-21 Multiple choice question 4.

(a) 0

(b) 2

(c) 4

(d) 6

(e) 8

5. A particle is trapped in an infinite potential energy well. It is in the state with quantum number $n = 14$. How many nodes does the probability density have (counting the nodes at the ends of the well)?

(a) None

(b) 7

(c) 13

(d) 14

(e) 15

6. A particle is confined to a one-dimensional trap by infinite potential energy walls. Of the following states, designed by the quantum number n, for which one is the probability density greatest near the center of the well?

(a) $n = 2$

(b) $n = 3$

(c) $n = 4$

(d) $n = 5$

(e) $n = 6$

sec. 39-5 The Bohr Model of the Hydrogen Atom

7. In the Bohr model of hydrogen, why is the atom 9 times larger in the $n = 3$ state than in the ground state?

(a) The electrons are moving 9 times slower

(b) The de Broglie wavelength is 9 times longer

(c) There are 9 times as many wavelengths in the orbit

(d) The de Broglie wavelength is 3 times longer, and there are 3 times as many wavelengths

8. Orbital electrons do not spiral into the nucleus because of

(a) Electromagnetic forces

(b) Angular momentum conservation

(c) The large nuclear size compared to the electron's size

(d) The wave nature of the electron

9. Bohr's model cannot explain the spectrum of neutral Lithium atoms because

(a) Lithium has neutrons in its nucleus

(b) Lithium has 3 protons in its nucleus

(c) Each lithium atom has more than one electron

(d) Neutral lithium atoms have no spectral lines at all

10. Compared to hydrogen the atom of helium has

(a) More mass and is larger in size

(b) More mass and is about the same in size

(c) More mass and is smaller in size

(d) None of the above

11. If the value of Planck's constant were to increase by a factor of two, the size of a hydrogen atom would

(a) Decrease by a factor of 4

(b) No change

(c) Increase by a factor of 2

(d) Decrease by a factor of 2

(e) Increase by a factor of 4

12. In the Bohr model, if an electron moves in an orbit of greater radius

(a) Its total energy increases but its kinetic energy decreases

(b) Its total energy decreases but its kinetic energy increases

(c) Its total energy as well as its kinetic energy decreases

(d) Its total energy as well as its kinetic energy increases

13. In the Bohr hydrogen atom mode, R, V, and E represent the radius of the orbit, speed of the electron, and total energy of the electron, respectively. Which of the following quantities are proportional to the quantum number n?

(a) VR

(b) RE

(c) VE^{-1}

(d) RE^{-1}

14. When Z is doubled in an atom, which of the following statements are consistent with Bohr's theory?

(a) Energy of a state is doubled

(b) Radius of an orbit is doubled

(c) Velocity of electrons in an orbit is doubled

(d) Radius of an orbit is halved

15. Which of the following statements is (are) wrong for hydrogen atom?

(a) The rotational frequency of an electron in an orbit of radius r proportional to the cube of the radius of the orbit

(b) The linear momentum of the revolving electron is inversely proportional to the square root of the radius of the orbit

(c) The total energy of the system is inversely proportional to the radius of the orbit

(d) The magnetic moment of the electron is proportional to square of the radius of the orbit

16. If electron of the hydrogen atom is replaced by another particle of same charge but of double the mass, then

(a) Bohr radius will increase

(b) Ionization energy of the atom will be doubled

(c) Speed of the new particle in a given state will be lesser than the electron's speed in same orbit

(d) Gap between energy levels will now be doubled

17. The total energy of an electron in the excited state corresponding to $n = 3$ state is E. What is its potential energy with proper sign?

(a) $-2E$

(b) $2E$

(c) $-E$

(d) E

18. If the potential energy is taken zero at ground state of hydrogen atom in place of infinity then

(a) Total energy of atom will get changed

(b) Binding energy of hydrogen atom in ground state will remain unchanged

(c) Kinetic energy of electron in ground state will get changed

(d) Energy of photon generated because of de-excitation of atom from first excited state will remain unchanged

sec. 39-6 Spectra of the Hydrogen Atom

19. Identify the highest energy photons of the following

(a) Industrial strength microwave cookers

(b) High power AM radio transmitters that cover thousands of square miles

(c) Waves from atomic electron transitions among outer orbits

(d) Waves from atomic transitions to inner orbits

20. A "toy" atom consists of three energy levels given by a ground state with energy $E_0 = 0$, the first excited state with energy $E_1 = \varepsilon$ and the second excited state with energy $E_2 = 2\varepsilon$ where $\varepsilon > 0$. The atom is initially in the ground state. Light from a laser that emits photons with energy 1.5ε is incident on the atom. What happens

(a) Two photons are absorbed, putting one atom in a state E_1 and one atom in a state E_2

(b) A photon will always be absorbed, but half the time the atom will go into the state will energy ε and the other half into the state with energy 2ε. In this way energy will be conserved on the average

(c) The atom absorbs a photon, goes into the first excited state with energy ε, and emits a photon with energy 0.5ε to conserve energy

(d) The atom does not absorb any photons and stays in the ground state

21. When radiation with a continuous spectrum is passed through a volume of hydrogen gas whose atoms are all in the ground state, which spectral series will be present in the resulting absorption spectrum?

(a) Lyman

(b) Balmer

(c) Paschen

(d) Brackett

22. What effect would you expect the rapid random motion of the atoms of an excited gas to have on the spectral lines they produce?

(a) Wavelength will be more

(b) Wavelength will be lesser

(c) Wavelength will be same

23. Which of the following statement is correct in connection with hydrogen spectrum

(a) The longest wavelength in the Balmer series is longer than the longest wavelength in Lyman series

(b) The shortest wavelength in the Balmer series is shorter than the shortest wavelength in the Lyman series

(c) The longest wavelength in both Balmer and Lyman series is equal

(d) The longest wavelength in Balmer series is shorter than the longest wavelength in the Lyman series

24. A hydrogen atom is excited up to 9th level. The total number of possible spectral lines emitted by the hydrogen atom is

(a) 36

(b) 35

(c) 37

(d) 38

25. A hydrogen atom makes a transition from $n = 2$ to $n = 1$ and emits a photon. This photon strikes a doubly ionized lithium atom in the excited state and completely removes the orbiting electron. The least quantum number for the excited state of the ion required for the process is

(a) 2

(b) 5

(c) 3

(d) 4

26. In a sample of hydrogen-like atoms all of which are in ground state, a photon beam containing photons of various energies is passed. In absorption spectrum, five dark lines are observed. The number of bright lines in the emission spectrum will be (assume that all transitions take place)

(a) 5

(b) 10

(c) 15

(d) None of these

27. A beam of ultraviolet light of a mixture of several wavelengths passes through hydrogen gas at room temperature, in the x direction. Assume that all the photons emitted due to electron transitions inside the gas emerge in the y direction. Let A and B denote the lights emerging from the gas in the x and y directions.

(a) Some of the incident wavelengths will be absent in A

(b) Only those wavelengths will be present in B which are absent in A

(c) B will contain some visible light

(d) B will contain some infrared light

28. A free hydrogen atom in ground state is at rest. A neutron of kinetic energy "K" collides with the hydrogen atom. After collision hydrogen atom emits two photons in succession one of which has energy 2.55 eV. (Assume that the hydrogen atom and neutron has same mass.)

(a) Minimum value of "K" is 25.5 eV

(b) Minimum value of "K" is 12.75 eV

(c) The other photon has energy 10.2 eV

(d) The upper energy level is of excitation energy 12.75 eV.

29. At some place in universe an atom consists of a positron revolving round an antiproton. The ratio of the wavelength of the corresponding spectral lines from this atom and ordinary hydrogen atoms is

(a) 1

(b) $\gg 1$

(c) $\ll 1$

(d) Infinite

30. A hydrogen atom in its ground state absorbs a photon of energy 10.2 eV and gets excited. The change in angular momentum of the electrons in this process is (in kg/m^2/s) nearly

(a) 1×10^{-34}

(b) 2×10^{-34}

(c) 0.5×10^{-34}

(d) Cannot be determined

31. An electron with kinetic energy "E" eV collides with a hydrogen atom in the ground state. The collision will be elastic

(a) For all values of E

(b) For $E < 10.2$ eV

(c) For $E < 13.6$ eV

(d) Only for $E < 3.4$ eV

Sec. 39-7 Quantum Numbers for the Hydrogen Atom

32. Consider the following:

(i) The probability density for an $\ell = 0$ state

(ii) The probability density for a state with $\ell \neq 0$

(iii) The average of the probability densities for all states in an $\ell \neq 0$ subshell

Of these, which are spherically symmetric?

(a) Only (i)

(b) Only (ii)

(c) Only (i) and (ii)

(d) Only (i) and (iii)

(e) (i), (ii), and (iii)

33. If $P(r)$ is the radial probability density for a hydrogen atom then the probability that the separation of the electron and proton is between r and $r + dr$ is

(a) $P\,dr$

(b) $|P|^2\,dr$

(c) $4\pi r^2 P\,dr$

(d) $4\pi r^2 |P|\,dr$

(e) $4\pi |P|^2\,dr$

34. The quantum-mechanical probability cloud for the electron in the hydrogen atom has an average radius

(a) Quite different from the radius predicted by Bohr

(b) That agrees with the orbital radius of Bohr

35. Which of the following sets of quantum numbers is possible for an electron in a hydrogen atom?

(a) $n = 4, \quad \ell = 3, \quad m_l = -3$

(b) $n = 4, \quad \ell = 4, \quad m_l = -2$

(c) $n = 5, \quad \ell = -1, \quad m_l = 2$

(d) $n = 3, \quad \ell = 1, \quad m_l = -2$

(e) $n = 2, \quad \ell = 3, \quad m_l = -2$

PROBLEMS

sec. 39-3 Energies of a Trapped Electron

1. The ground-state energy of an electron trapped in a one-dimensional infinite potential well is 2.6 eV. What will this quantity be if the width of the potential well is doubled?

2. An electron, trapped in a one-dimensional infinite potential well 250 pm wide, is in its ground state. How much energy must it absorb if it is to jump up to the state with $n = 4$?

3. Consider an atomic nucleus to be equivalent to a one-dimensional infinite potential well with $L = 1.4 \times 10^{-14}$ m, a typical nuclear diameter. What would be the ground-state energy of an electron if it were trapped in such a potential well? (*Note:* Nuclei do not contain electrons.)

sec. 39-4 Wave Functions of a Trapped Electron

4. An electron is trapped in a one-dimensional infinite potential well that is 100 pm wide; the electron is in its ground state. What is the probability that you can detect the electron in an interval of width $\Delta x = 5.0$ pm centered at $x =$ (a) 25 pm, (b) 50 pm, and (c) 90 pm? (*Hint:* The interval Δx is so narrow that you can take the probability density to be constant within it.)

5. A particle is confined to the one-dimensional infinite potential well of Fig. 39-2. If the particle is in its ground state, what is its probability of detection between (a) $x = 0$ and $x = 0.25L$, (b) $x = 0.75L$ and $x = L$, and (c) $x = 0.25L$ and $x = 0.75L$?

6. An electron is in a certain energy state in a one-dimensional, infinite potential well from $x = 0$ to $x = L = 200$ pm. The electron's probability density is zero at $x = 0.300L$, and $x = 0.400L$; it is not zero at intermediate values of x. The electron then jumps to the next lower energy level by emitting light. What is the change in the electron's energy?

sec. 39-5 The Bohr Model of the Hydrogen Atom

7. In Bohr model of the hydrogen atom, what is the de Broglie wavelength for the electron when it is in (a) The $n = 1$ level? (b) Compare the de Broglie wavelength to the circumference $2\pi r_n$ of the orbit.

8. Find the maximum Coulomb's force that can act on the electron due to the nucleus in a hydrogen atom.

9. In the ground state of the hydrogen atom, the electron has a total energy of −13.6 eV. What are (a) its kinetic energy and (b) its potential energy if the electron is one Bohr radius from the central nucleus?

10. Using the Bohr model, (a) calculate the speed of the electrons in a hydrogen atom in $n = 1, 2$, and 3 level. and (b) calculate the orbital period in each of these levels.

11. (a) How many revolutions does an electron in the first Bohr orbit of hydrogen make each second? (b) To what current does this correspond? (c) Find the magnetic induction at the proton due to the motion of the electron.

12. Average lifetime of a hydrogen atom excited to $n = 2$ state is 10^{-8} s. Find the number of revolution made by the electron on the average before it jumps to the ground state.

13. The electron of a hydrogen atom can be replaced by a muon for a short time. The new atom is called muonium. (The muon is similar to the electron but its mass is 207 times greater than the mass of the electron, and its decay time is very short. (a) Find the radius of the muonium in its ground state. (b) How much energy is needed to ionize the muonium atom? Ignore the reduced mass effects.

sec. 39-6 Spectra of the Hydrogen Atom

14. An atom (not a hydrogen atom) absorbs a photon whose associated frequency is 6.2 × 1014 Hz. By what amount does the energy of the atom increase?

15. How much energy is required to cause an electron hydrogen to move from the $n = 1$ state to $n = 2$ state?

16. Hydrogen atoms are in an excited state with $n = 5$. In terms of the Bohr's model, (a) how many spectral lines can be radiated as these atoms return to be ground state? (b) Find the energies of the line with the longest and shortest wavelengths.

17. Calculate the wavelengths of the first two lines and of the series limit for the Balmer series of ionized helium.

18. The energy needed to detach the electron of a hydrogen-like ion in ground state is 54.4 eV. (a) What is the wavelength of the radiation emitted when the electron jumps from the first excited state to the ground state? (b) What is the radius of the first orbit for this atom?

19. Light of wavelength 121.6 nm is emitted by a hydrogen atom. What are the (a) higher quantum number and (b) lower quantum number of the transition producing this emission? (c) What is the name of the series that includes the transition?

20. A very small sample of extremely rarefied hydrogen gas consists of all of its atoms in excited state $n = 4$. If all the transitions corresponding to the final state $n = 1$ are observed, what are the minimum number of atoms in the sample?

21. Suppose that a hydrogen atom in the ground state absorbs photons of wavelength 15 nm. (a) Will the atom be ionized? (b) If so, what will be the kinetic energy of the electron when it gets far away from its atom of origin?

22. Monochromatic radiation of wavelength λ is incident on a hydrogen sample in ground state. Hydrogen atoms absorb a fraction of light and subsequently emit radiation of six different wavelengths. Find the value of λ.

23. Hydrogen atoms are initially in the $n = 4$ energy level. How many different photon energies will be emitted if the atoms go to the ground state, with all possible transition represented? Assuming that from any given excited state, all possible downward transitions are equally probable (this assumption is not true), what is the total number of photons emitted if there are 600 atoms initially in the $n = 4$ state?

24. A single electron orbits around a stationary nucleus of charge $+Ze$, where Z is a constant and e is the magnitude of electronic charge. It requires 47.2 eV to excite the electron from the second Bohr orbit to third Bohr orbit. Find (a) the value of Z; (b) the energy required to excite the electron from the third to the fourth Bohr orbit; (c) the wavelength of electromagnetic radiation required to remove the electron from the first Bohr orbit to infinity; and(d) the kinetic energy, potential energy and the angular momentum of the electron in the first Bohr orbit.

25. A hydrogen sample is prepared in a particular excited state a. Photons of energy 2.55 eV get absorbed into the sample to take some of the electrons to a further excited state B. Find the quantum number of the states A and B.

26. A certain atom has the following energy-level scheme –8, –5, –2, –1, –0.5 eV. (a) Calculate the wavelengths corresponding to all possible transitions between these levels. (b) Which one lies in the visible spectrum: the ultraviolet or the infrared?

27. An electron with mass m collides inelastically with an atom of mass M, initially at rest and excites it to an energy E above the ground state. Show that the initial energy of the electron must be at least $(1 + m/M)E$.

28. A hydrogen atom at rest is struck by a second hydrogen atom with kinetic energy K. Show that the minimum K which is required to raise one of the atoms to its first excited state is 1.5 times the ionization energy.

29. Calculate the smallest kinetic energy which an electron may have and still excite a hydrogen atom initially at rest. What minimum kinetic energy is necessary to ionize the hydrogen atom?

30. A neutron with a kinetic energy of 6.0 eV collides with a stationary hydrogen atom in its ground state. Explain why the collision must be elastic – that is, why kinetic energy must be conserved. (*Hint:* Show that the hydrogen atom cannot be excited as a result of the collision.)

31. A beam of electrons bombards a sample of hydrogen. Through what potential difference must the electron have been accelerated if the first line of the Balmer series is to be emitted?

32. A uniform magnetic field B exists in a region. An electron projected perpendicular to the field goes in a circle. Assuming Bohr's quantization rule for angular momentum, calculate (a) the smallest possible radius of the electron, (b) the radius of the nth orbit, and (c) the minimum possible speed of the electron.

sec. 39-7 Quantum Numbers for the Hydrogen Atom

33. For the hydrogen atom in its ground state, calculate (a) the probability density $\psi^2(r)$ and (b) the radial probability density $P(r)$ for $r = a$, where a is the Bohr radius.

34. (a) What is the energy E of the hydrogen-atom electron whose probability density is represented by the dot plot of Fig. 39-16? (b) What minimum energy is needed to remove this electron from the atom?

35. What is the probability that in the ground state of the hydrogen atom, the electron will be found at a radius greater than the Bohr radius?

36. Calculate the probability that the electron in the hydrogen atom, in its ground state, will be found between spherical shells whose radii are a and $2a$, where a is the Bohr radius.

ADDITIONAL PROBLEMS

37. A hydrogen atom, initially at rest in the $n = 4$ quantum state, undergoes a transition to the ground state, emitting a photon in the process. What is the speed of the recoiling hydrogen atom?

38. The general quantization of motion in circular orbits is obtained by the angular-momentum quantization condition $mvr = nh/2\pi$. Use this procedure to calculate the spectrum for circular motion in the potential $U = u_0 r$.

39. A hydrogen atom moving at a speed v absorbs a photon of wavelength 122 nm and stops. Find the value of v. Given mass of a hydrogen atom = 1.67×10^{-27} kg.

40. The electrons and its antiparticle, the positron, can form a bound system called positronium. The system is actually unstable, because the electrons and positron can annihilate each other producing two very energetic photons via the reaction $e^+ + e^- \rightarrow \gamma + \gamma$. Before the decay occurs, positronium exhibits hydrogen-like spectrum. (a) What is the difference in energy between the ground state and the first excited state? (b) What is the radius of the ground state of positronium? Note that reduced mass effect cannot be neglected here.

41. A beam of ultraviolent radiation having wavelength between 100 nm and 200 nm is incident on a sample atomic hydrogen gas. Assuming that the atoms are in ground state, which wavelengths will have low intensity in the transmitted beam?

42. Electrons are emitted from an electron gun at almost zero velocity and are accelerated by an electric field E through distance of 1.0 m. The electrons are now scattered by an atomic hydrogen sample in ground state. What should be the minimum value of E so that red light of wavelength 656.3 nm may be emitted by the hydrogen?

43. A filter transmits only radiation of wavelength greater than 440 nm. Radiation from a hydrogen-discharge tube goes through such a filter and it incident on a metal work function 2.0 eV. Find the stopping potential which can stop the photoelectrons.

44. An electron at rest is released far away from a proton, toward which it moves. (a) Show that the de-Broglie wavelength of the electron is proportional to $\sqrt{r}$, where r is the distance of the electron from the proton. (b) Find the wavelength of the electron when it is a_0 from the proton. How does this compare with the wavelength of electron in a ground-state Bohr orbit? (c) In order for the electron to be captured by the proton to form a ground state hydrogen atom, energy must be lost by the system. How much energy is lost?

45. A hydrogen-like atom (atomic number Z) is in a higher excited state of quantum number n. This excited atom can make a transition to the first excited state by successively emitting two photons of energies 10.20 eV and 17.00 eV, respectively. Alternatively, the atom from the same excited state can make a transition to the second excited state by successively emitting two photons of energies 4.25 eV and 5.95 eV, respectively. Determine the values of n and Z.

46. Consider a simple model of the helium atom in which two electrons, each with mass m move around the nucleus (charge $+2e$) in the same circular orbit. Each electron has orbital angular momentum h (that is, the orbit is the smallest-radius Bohr orbit), and the two electrons are always on opposite sides of the nucleus. Ignore the effect of spin. (a) Determine the radius of the orbit and the orbital speed of each electron. Each electron experiences an attractive force from the nucleus and a repulsive force from the other electron. (b) What is the total kinetic energy of the electron? (c) What is the potential energy of the system (the nucleus and the two electrons)? (d) In this model, how much energy is required to remove both electron to infinity? How does this compare to the experimental value of 79.0 eV?

47. A spectroscopic instrument can resolve two nearby wavelengths λ and $\lambda + \Delta\lambda$ if $\lambda/\Delta\lambda$ is small than 5×10^5. This is used to study the spectral lines of the Balmer series of hydrogen. Find the quantum number of the highest level of which the line would be just resolved by its neighbors?

48. A negative muon (an elementary particle with mass of 207 electrons and charge of one electron) essentially at rest (that is, its kinetic energy can be taken to be zero) enters a gas of hydrogen atoms. It is captured by one of the protons and ends up in ground state. The photons that were emitted while the muon descended by steps of $\Delta n = 1$ from n = infinity escaped freely. The proton and the muon form a neutral system much smaller than the original atom consisting of an electron and a proton. The electron that was originally bound to the proton is suddenly neutralized. What will be the kinetic energy of the suddenly freed electron?

ANSWERS

Checkpoints

1. b, a, c

2. (a) all tie; (b) a, b, c

3. a, b, c, d

4. (a) emission spectra; (b) absorption spectra

5. The radius would be the same

6. (a) 5; (b) 7

Reasoning Type Questions

1. (a)

2. (d)

3. (a)

4. (b)

5. (a)

6. (c)

Multiple Choice Questions

1. (d)

2. (a)

3. (c)

4. (c)

5. (e)

6. (b)

7. (d)

8. (d)

9. (c)

10. (c)

11. (e)

12. (a)

13. (a), (c)

14. (c), (d)

15. (a), (d)

16. (b), (d)

17. (b)

18. (a), (b), (d)

19. (d)

20. (d)

21. (a)

22. (a), (b), (c)

23. (a)

24. (a)

25. (d)

26. (c)

27. (a), (c), (d)

28. (a), (c), (d)

29. (a)

30. (a)

31. (b)

32. (d)

33. (a)

34. (b)

35. (a)

Problems

1. 0.65 eV

2. 90.3 eV

3. 1.9 GeV

4. (a) 0.050; (b) 0.10; (c) 0.0095

5. (a) 0.091; (b) 0.091; (c) 0.82

6. 2.86×10^{-17} J

7. (a) 3.32×10^{-10} m; (b) $\lambda = 2\pi r$

8. 8.2×10^{-8} N

9. (a) 13.6 eV; (b) −27.2 eV

10. (a) 2.18×10^{6} m/s, 1.09×10^{6} m/s, 7.27×10^{6} m/s; (b) 1.53×10^{-16} s, 1.22×10^{-15} s, 4.13×10^{-15} s

11. (a) $\frac{v}{2\pi r}$; (b) $\frac{ev}{2\pi r}$; (c) $\frac{\mu_0 e^7 \pi m^2}{8\varepsilon_0^3 h^5}$

12. 8.2×10^{6}

13. (a) 2.85×10^{-13} m; (b) 2.53 keV

14. 2.6 eV

15. 10.2 eV

16. (a) 10; (b) 4000 nm, 95 nm

17. 164 nm, 122 nm, 91 nm

18. (a) 30.4 nm (b) 26.5 pm

19. (a) 2; (b) 1; (c) Lyman series transitions

20. 4

21. Yes, 69 eV

22. 97.5 nm

23. 6, 1100

24. (a) 5; (b) 16.53 eV; (c) 36.06 Å; (d) 340 eV, − 680 eV, 1.06×10^{-34} J · s

25. A, 2; B, 4

26. (a) 2484 nm, 1242 nm, 828 nm, 414 nm, 310 nm, 276 nm, 207 nm, 177 nm, 165 nm. First 3 in infrared, 414 nm in visible, rest in UV range

29. 10.2 eV, 13.6 eV

31. 12.1 V

32. (a) $\sqrt{\frac{h}{2\pi eB}}$; (b) $\sqrt{\frac{nh}{2\pi eB}}$; (c) $\sqrt{\frac{heB}{2\pi m^2}}$

33. (a) 291 nm^{-3}; (b) 10.2 nm^{-1}

34. (a) −3.4 eV; (b) zero

35. 0.68

36. 0.439

Additional Problems

37. 4.1 m/s

38. $\lambda = (4c/3)(mh/2\pi u_0^2)^{1/3}/(n^{2/3} - m^{2/3})$

39. 3.25 m/s

40. (a) 5.1 eV; (b) 1.06×10^{-10} m

41. 122 nm, 103 nm

42. 12.1 V/m

43. 0.55 V

44. (b) $\sqrt{2}\lambda_{\text{Bohr}}$; (c) 13.6 eV

45. $n = 6, z = 3$

46. (a) $4a_0/7, 7v_0/4$; (b) 83.5 eV; (c) −166.9 eV; (d) 83.4 eV

47. 158

48. 13.6 eV

Atoms, X Rays, and Lasers 40



Yoav Levy/Phototake

Soon after lasers were invented in the 1960s, they became novel sources of light in research laboratories. Today, lasers are ubiquitous and are found in such diverse applications as voice and data transmission, surveying, welding, and grocery-store price scanning. The photograph shows a patient being prepared for LASIK (laser assisted in situ keratomileusis), in which part of the woman's cornea will be reshaped via etching by an ultraviolet laser beam in order to correct her vision. Light from a laser and light from any other source are both due to emissions by atoms.

> ***What, then, is so different about the light from a laser?***

The answer is in this chapter.

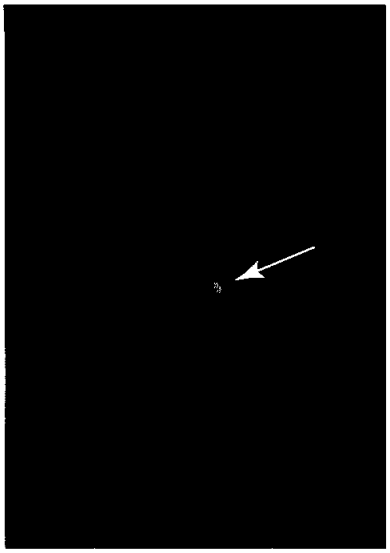


FIG. 40-1 The blue dot is a photograph of the light emitted from a single barium ion held for a long time in a trap at the University of Washington. Special techniques caused the ion to emit light over and over again as it underwent transitions between the same pair of energy levels. The dot represents the cumulative emission of many photons. *(Courtesy Warren Nagourney)*

40-1 WHAT IS PHYSICS?

In this chapter we continue with a primary goal of physics—discovering and understanding the properties of atoms. About 100 years ago, researchers struggled to find experiments that would prove the existence of atoms. Now we take their existence for granted and even have photographs (scanning tunneling microscope images) of atoms. We can drag them around on surfaces, such as to make the quantum. We can even hold an individual atom indefinitely in a trap (Fig. 40-1) so as to study its properties when it is completely isolated from other atoms.

40-2 | Some Properties of Atoms

You may think the details of atomic physics are remote from your daily life. However, consider how the following properties of atoms—so basic that we rarely think about them—affect the way we live in our world.

Atoms are stable. Essentially all the atoms that form our tangible world have existed without change for billions of years. What would the world be like if atoms continually changed into other forms, perhaps every few weeks or every few years?

Atoms combine with each other. They stick together to form stable molecules and stack up to form rigid solids. An atom is mostly empty space, but you can stand on a floor—made up of atoms—without falling through it.

These basic properties of atoms can be explained by quantum physics, as can the three less apparent properties that follow.

Atoms Are Put Together Systematically

Figure 40-2 shows an example of a repetitive property of the elements as a function of their position in the periodic table (Appendix G). The figure is a plot of the **ionization energy** of the elements; the energy required to remove the most loosely bound electron from a neutral atom is plotted as a function of the position in the periodic table of the element to which the atom belongs. The remarkable similarities in the chemical and physical properties of the elements in each vertical column of the periodic table are evidence enough that the atoms are constructed according to systematic rules.

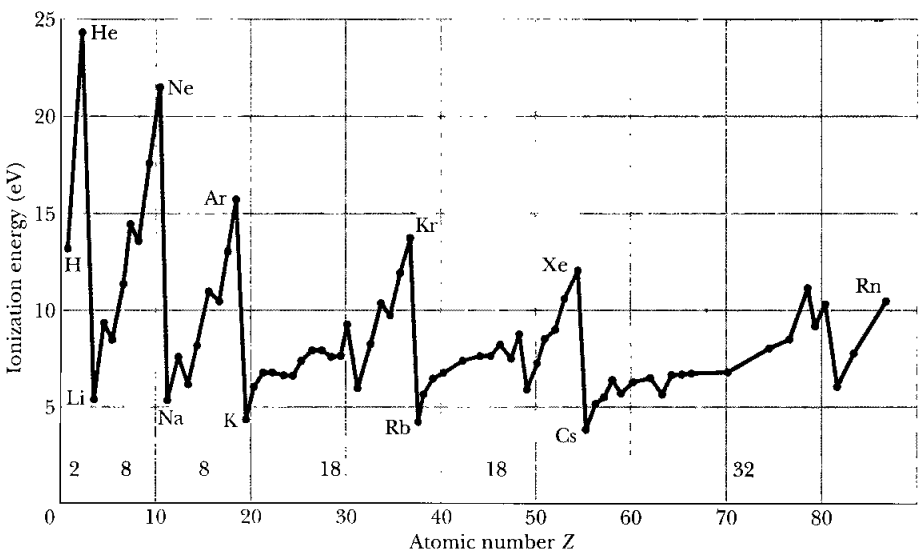


FIG. 40-2 A plot of the ionization energies of the elements as a function of atomic number, showing the periodic repetition of properties through the six complete horizontal periods of the periodic table. The number of elements in each of these periods is indicated.

The elements are arranged in the periodic table in six complete horizontal **periods** (and a seventh incomplete period): except for the first, each period starts at the left with a highly reactive alkali metal (lithium, sodium, potassium, and so on) and ends at the right with a chemically inert noble gas (neon, argon, krypton, and so on). Quantum physics accounts for the chemical properties of these elements. The numbers of elements in the six periods are

$$2, 8, 8, 18, 18, \text{and } 32.$$

Quantum physics predicts these numbers.

Atoms Emit and Absorb Light

We have already seen that atoms can exist only in discrete quantum states, each state having a certain energy. An atom can make a transition from one state to another by emitting light (to jump to a lower energy level E_{low}) or by absorbing light (to jump to a higher energy level E_{high}). As we first discussed in Section 39-3, the light is emitted or absorbed as a photon with energy

$$hf = E_{high} - E_{low}. \tag{40-1}$$

Thus, the problem of finding the frequencies of light emitted or absorbed by an atom reduces to the problem of finding the energies of the quantum states of that atom. Quantum physics allows us—in principle at least—to calculate these energies.

Atoms Have Angular Momentum and Magnetism

Figure 40-3 shows a negatively charged particle moving in a circular orbit around a fixed center. The orbiting particle has both an angular momentum $\vec{L}$ and (because its path is equivalent to a tiny current loop) a magnetic dipole moment $\vec{\mu}$. As Fig. 40-3 shows, vectors $\vec{L}$ and $\vec{\mu}$ are both perpendicular to the plane of the orbit but, because the charge is negative, they point in opposite directions.

The model of Fig. 40-3 is strictly classical and does not accurately represent an electron in an atom. In quantum physics, the rigid orbit model has been replaced by the probability density model, best visualized as a dot plot. In quantum physics, however, it is still true that in general, each quantum state of an electron in an atom involves an angular momentum $\vec{L}$ and a magnetic dipole moment $\vec{\mu}$ that have opposite directions (those vector quantities are said to be *coupled*).

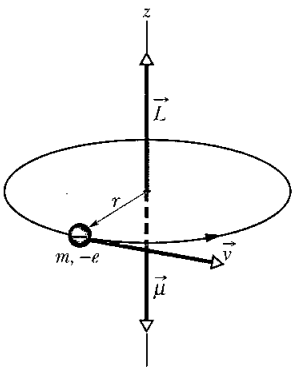


FIG. 40-3 A classical model showing a particle of mass m and charge $-e$ moving with speed v in a circle of radius r. The moving particle has an angular momentum $\vec{L}$ given by $\vec{r} \times \vec{p}$ where $\vec{p}$ is its linear momentum $m\vec{v}$. The particle's motion is equivalent to a current loop that has an associated magnetic moment $\vec{\mu}$ that is directed opposite $\vec{L}$.

The Einstein–de Haas Experiment

In 1915, well before the discovery of quantum physics, Albert Einstein and Dutch physicist W. J. de Haas carried out a clever experiment designed to show that the angular momentum and magnetic moment of individual atoms are coupled.

Einstein and de Haas suspended an iron cylinder from a thin fiber, as shown in Fig. 40-4. A solenoid was placed around the cylinder but not touching it. Initially, the magnetic dipole moments $\vec{\mu}$ of the atoms of the cylinder point in random directions, and so their external magnetic effects cancel (Fig. 40-4*a*). However, when a current is switched on in the solenoid (Fig. 40-4*b*) so that a magnetic field $\vec{B}$ is set up parallel to the long axis of the cylinder, the magnetic dipole moments of the atoms of the cylinder reorient themselves, lining up with that field. If the angular momentum $\vec{L}$ of each atom is coupled to its magnetic moment $\vec{\mu}$, then this alignment of the atomic magnetic moments must cause an alignment of the atomic angular momenta opposite the magnetic field.

No external torques initially act on the cylinder; thus, its angular momentum must remain at its initial zero value. However, when $\vec{B}$ is turned on and the atomic angular momenta line up antiparallel to $\vec{B}$, they tend to give a net angular momentum $\vec{L}_{net}$ to the cylinder as a whole (directed downward in Fig. 40-4*b*).

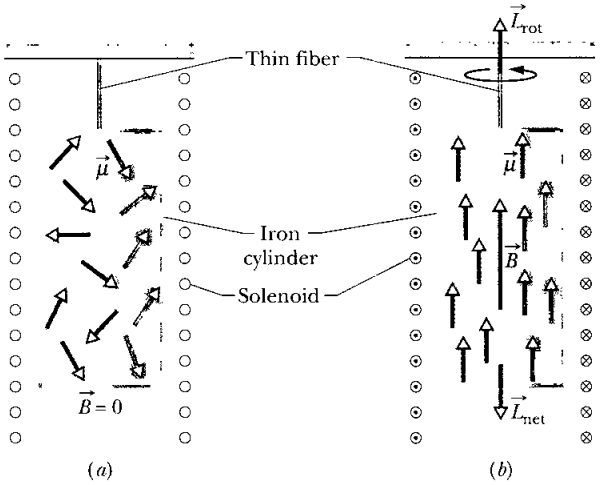


FIG. 40-4 The Einstein–de Haas experimental setup. (*a*) Initially, the magnetic field in the iron cylinder is zero and the magnetic dipole moment vectors $\vec{\mu}$ of its atoms are randomly oriented. The atomic angular momentum vectors (not shown) are directed opposite the magnetic dipole moment vectors and thus are also randomly oriented. (*b*) When a magnetic field $\vec{B}$ is set up along the cylinder's axis, the magnetic dipole moment vectors line up parallel to $\vec{B}$, which means that the angular momentum vectors line up opposite $\vec{B}$. Because the cylinder is initially isolated from external torques, its angular momentum is conserved and the cylinder as a whole must begin to rotate as shown.

To maintain zero angular momentum, the cylinder begins to rotate around its central axis to produce an angular momentum $\vec{L}_{rot}$ in the opposite direction (upward in Fig. 40-4*b*).

Were it not for the fiber, the cylinder would continue to rotate for as long as the magnetic field is present. However, the twisting of the fiber quickly produces a torque that momentarily stops the cylinder's rotation and then rotates the cylinder in the opposite direction as the twisting is undone. Thereafter, the fiber will twist and untwist as the cylinder oscillates about its initial orientation in angular simple harmonic motion.

Observation of the cylinder's rotation verified that the angular momentum and the magnetic dipole moment of an atom are coupled in opposite directions. Moreover, it dramatically demonstrated that the angular momenta associated with quantum states of atoms can result in *visible* rotation of an object of everyday size.

40-3 | Electron Spin

Whether an electron is *trapped* in an atom or is *free*, it has an intrinsic **spin angular momentum** $\vec{S}$, often called simply **spin**. (Recall that *intrinsic* means that $\vec{S}$ is a basic characteristic of an electron, like its mass and electric charge.) As we shall discuss in the next section, the magnitude of $\vec{S}$ is quantized and depends on a **spin quantum number** s, which is always $\frac{1}{2}$ for electrons (and for protons and neutrons). In addition, the component of $\vec{S}$ measured along any axis is quantized and depends on a **spin magnetic quantum number** m_s, which can have only the value $+\frac{1}{2}$ or $-\frac{1}{2}$.

It is tempting to account for electron spin by thinking of the electron as a tiny sphere spinning about an axis. However, that classical model, like the classical model of orbits, does not hold up. In quantum physics, spin angular momentum is best thought of as a measurable intrinsic property of the electron; you simply can't visualize it with a classical model.

In Section 39-6, we briefly discussed the quantum numbers generated by applying Schrödinger's equation to the electron in a hydrogen atom (Table 39-1). We can now extend the list of quantum numbers by including s and m_s, as shown in Table 40-1. This set of five quantum numbers completely specify the quantum state of an electron in a hydrogen atom or any other atom. All states with the same value of n form a **shell**. By counting the allowed values of ℓ and m_ℓ and then doubling the number to account for the two allowed values of m_s, you can verify that a shell defined by quantum number n has $2n^2$ states. All states

TABLE 40-1

Electron States for an Atom

Quantum Number	Symbol	Allowed Values	Related to
Principal	n	$1, 2, 3, \ldots$	Distance from the nucleus
Orbital	ℓ	$0, 1, 2, \ldots, (n-1)$	Orbital angular momentum
Orbital magnetic	m_ℓ	$0, \pm 1, \pm 2, \ldots, \pm \ell$	Orbital angular momentum (z component)
Spin	s	$\frac{1}{2}$	Spin angular momentum
Spin magnetic	m_s	$\pm \frac{1}{2}$	Spin angular momentum (z component)

with the same value of n and ℓ form a **subshell** and have the same energy. You can verify that a subshell defined by quantum number ℓ has $2(2\ell + 1)$states.

40-4 | Angular Momenta and Magnetic Dipole Moments

Every quantum state of an electron in an atom has an associated orbital angular momentum and a corresponding orbital magnetic dipole moment. Every electron, whether trapped in an atom or free, has a spin angular momentum and a corresponding spin magnetic dipole moment. We discuss these quantities separately first, and then in combination.

Orbital Angular Momentum and Magnetism

The magnitude L of the **orbital angular momentum** $\vec{L}$ of an electron *in an atom* is quantized; that is, it can have only certain values. These values are

$$L = \sqrt{\ell(\ell+1)}\hbar, \tag{40-2}$$

in which ℓ is the orbital quantum number and $\hbar$ is $h/2\pi$. According to Table 40-1, ℓ must be either zero or a positive integer no greater than $n - 1$. For a state with $n = 3$, for example, only $\ell = 2$, $\ell = 1$ and $\ell = 0$ are permitted.

A magnetic dipole is associated with the orbital angular momentum $\vec{L}$ of an electron in an atom. This magnetic dipole has an **orbital magnetic dipole moment** $\vec{\mu}_{\text{orb}}$, which is related to the angular momentum:

$$\vec{\mu}_{\text{orb}} = -\frac{e}{2m}\vec{L}. \tag{40-3}$$

The minus sign in this relation means that $\vec{\mu}_{\text{orb}}$ is directed opposite $\vec{L}$. Because the magnitude of $\vec{L}$ is quantized (Eq. 40-2), the magnitude of $\vec{\mu}_{\text{orb}}$ must also be quantized and given by

$$\mu_{\text{orb}} = \frac{e}{2m}\sqrt{\ell(\ell+1)}\hbar. \tag{40-4}$$

Neither $\vec{\mu}_{\text{orb}}$ nor $\vec{L}$ can be measured in any way. However, we *can* measure the components of those two vectors along a given axis. Let us imagine that the atom is located in a magnetic field $\vec{B}$; assume that a z axis extends in the direction of the field lines at the atom's location. Then we can measure the z components of $\vec{\mu}_{\text{orb}}$ and $\vec{L}$ along that axis.

The components $\mu_{\text{orb},z}$ of the orbital magnetic dipole moment are quantized and given by

$$\mu_{\text{orb},z} = -m_\ell \mu_B. \tag{40-5}$$

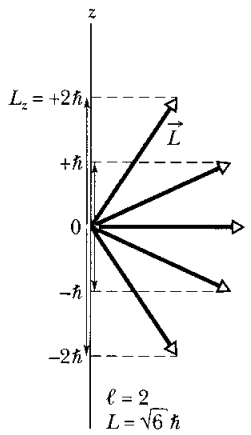


FIG. 40-5 The allowed values of L_z for an electron in a quantum state with $\ell = 2$. For every orbital angular momentum vector $\vec{L}$ in the figure, there is a vector pointing in the opposite direction, representing the magnitude and direction of the orbital magnetic dipole moment μ_{orb}.

Here m_ℓ is the orbital magnetic quantum number of Table 40-1 and μ_B is the **Bohr magneton**:

$$\mu_B = \frac{eh}{4\pi m} = \frac{e\hbar}{2m} = 9.274\times10^{-24}\ \text{J/T} \qquad \text{(Bohr magneton)}. \tag{40-6}$$

where m is the electron mass.

The components L_z of the angular momentum are also quantized, and they are given by

$$L_z = m_\ell \hbar. \tag{40-7}$$

Figure 40-5 shows the five quantized components L_z of the orbital angular momentum for an electron with $\ell = 2$, as well as the associated orientations of the angular momentum $\vec{L}$. However, *do not take the figure literally* because we cannot detect $\vec{L}$ in any way. Thus, drawing it in a figure like Fig. 40-5 is merely a visual aide. We can extend that visual aide by saying that $\vec{L}$ makes a certain angle θ with the z axis, such that

$$\cos\theta = \frac{L_z}{L}. \tag{40-8}$$

We can call θ the *semi-classical angle* between vector $\vec{L}$ and the z axis because θ is a classical measure of something that quantum theory tells us cannot be measured.

Spin Angular Momentum and Spin Magnetic Dipole Moment

The magnitude S of the spin angular momentum $\vec{S}$ of any electron, whether *free or trapped*, has the single value given by

$$\begin{aligned} S &= \sqrt{s(s+1)}\hbar \\ &= \sqrt{\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)}\hbar = 0.866\hbar, \end{aligned} \tag{40-9}$$

where $S\ (=\frac{1}{2})$ is the spin quantum number of the electron.

An electron has an intrinsic magnetic dipole that is associated with its spin angular momentum $\vec{S}$, whether the electron is confined to an atom or free. This magnetic dipole has a **spin magnetic dipole moment** $\vec{\mu}_s$, which is related to the spin angular momentum $\vec{S}$:

$$\vec{\mu}_s = -\frac{e}{m}\vec{S}. \tag{40-10}$$

The minus sign in this relation means that $\vec{\mu}_s$ is directed opposite $\vec{S}$. Because the magnitude of $\vec{S}$ is quantized (Eq. 40-9), the magnitude of $\vec{\mu}_s$ must also be quantized and given by

$$\mu_s = \frac{e}{m}\sqrt{s(s+1)}\hbar. \tag{40-11}$$

Neither $\vec{S}$ nor $\vec{\mu}_s$ can be measured in any way. However, we *can* measure their components along any given axis – call it the z axis. The components S_z of the spin angular momentum are quantized and given by

$$S_z = m_s\hbar, \tag{40-12}$$

where m_s is the spin magnetic quantum number of Table 40-1. That quantum number can have only two values: $m_s = +\frac{1}{2}$ (the electron is said to be *spin up*) and $m_s = -\frac{1}{2}$ (the electron is said to be *spin down*).

The components $\mu_{s,z}$ of the spin magnetic dipole moment are also quantized, and they are given by

$$\mu_{s,z} = -2m_s\mu_B. \tag{40-13}$$

Figure 40-6 shows the two quantized components S_z of the spin angular momentum for an electron and the associated orientations of vector $\vec{S}$. It also shows the quantized components $\mu_{s,z}$ of the spin magnetic dipole moment and the associated orientations of $\vec{\mu}_s$.

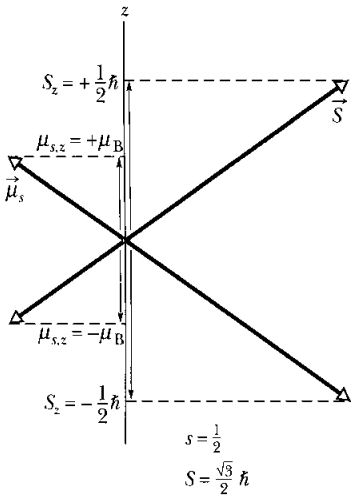


FIG. 40-6 The allowed values of S_z and μ_z for an electron.

Orbital and Spin Angular Momenta Combined

For an atom containing more than one electron, we define a total angular momentum $\vec{J}$, which is the vector sum of the angular momenta of the individual electrons—both their orbital and their spin angular momenta. Each element in the periodic table is defined by the number of protons in the nucleus of an atom of the element. This number of protons is defined as being the **atomic number** (or **charge number**) Z of the element. Because an electrically neutral atom contains equal numbers of protons and electrons, Z is also the number of electrons in the neutral atom, and we use this fact to indicate a $\vec{J}$ value for a neutral atom:

$$\vec{J} = (\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_z) + (\vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \cdots + \vec{S}_z). \qquad (40\text{-}14)$$

Similarly, the total magnetic dipole moment of a multielectron atom is the vector sum of the magnetic dipole moments (both orbital and spin) of its individual electrons. However, because of the factor 2 in Eq. 40-13, the resultant magnetic dipole moment for the atom does not have the direction of vector $-\vec{J}$; instead, it makes a certain angle with that vector. The **effective magnetic dipole moment** $\vec{\mu}_{\text{eff}}$ for the atom is the component of the vector sum of the individual magnetic dipole moments in the direction of $-\vec{J}$ (Fig. 40-7). In typical atoms the orbital angular momenta and the spin angular momenta of most of the electrons sum vectorially to zero. Then $\vec{J}$ and $\vec{\mu}_{\text{eff}}$ of those atoms are due to a relatively small number of electrons, often only a single valence electron.

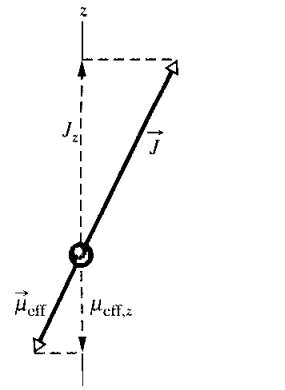


FIG. 40-7 A classical model showing the total angular momentum vector $\vec{J}$ and the effective magnetic moment vector $\vec{\mu}_{\text{eff}}$.

CHECKPOINT 1 An electron is in a quantum state for which the magnitude of the electron's orbital angular momentum $\vec{L}$ is $2\sqrt{3}\hbar$. How many projections of the electron's orbital magnetic dipole moment on a z axis are allowed?

40-5 | The Pauli Exclusion Principle

In Chapter 39 we considered a variety of electron traps, from fictional one-dimensional traps to the real three-dimensional trap of a hydrogen atom. In all those examples, we trapped only one electron. However, when we discuss traps containing two or more electrons (as we shall in the next two sections), we must consider a principle that governs any particle whose spin quantum number s is not zero or an integer. This principle applies not only to electrons but also to protons and neutrons, all of which have $s = \frac{1}{2}$. The principle is known as the **Pauli exclusion principle** after Wolfgang Pauli, who formulated it in 1925. For electrons, it states that

> No two electrons confined to the same trap can have the same set of values for their quantum numbers.

As we shall discuss in Section 40-6, this principle means that no two electrons in an atom can have the same four values for the quantum numbers n, ℓ, m_ℓ, and m_s. All electrons have the same quantum number $s = \frac{1}{2}$. Thus, any two electrons in an atom must differ in at least one of these other quantum numbers. Were this not true, atoms would collapse, and thus you and the world could not exist.

40-6 | Building the Periodic Table

The four quantum numbers n, ℓ, m_ℓ and m_s identify the quantum states of individual electrons in a multielectron atom. The wave functions for these states, however, are not the same as the wave functions for the corresponding states of the hydrogen atom because, in multielectron atoms, the potential energy associated with a given electron is determined not only by the charge and position of the atom's nucleus but also by the charges and positions of all the other electrons in the atom. Solutions of Schrödinger's equation for multielectron atoms can be carried out numerically—in principle at least—using a computer.

As we discussed in Sections 39-6 and 40-3, all states with the same values of the quantum numbers n and ℓ form a subshell. For a given value of ℓ, there are $2\ell + 1$ possible values of the magnetic quantum number m_ℓ and, for each m_ℓ, there are two possible values for the spin quantum number m_s. Thus, there are $2(2\ell + 1)$ states in a subshell. It turns out that *all states in a given subshell have the same energy,* its value being determined primarily by the value of n and to a lesser extent by the value of ℓ.

For the purpose of labeling subshells, the values of ℓ are represented by letters:

$$\begin{array}{cccccccc} \ell = 0 & 1 & 2 & 3 & 4 & 5 \ldots \\ s & p & d & f & g & h \ldots . \end{array}$$

For example, the $n = 3$, $\ell = 2$ subshell would be labeled the $3d$ subshell.

When we assign electrons to states in a multielectron atom, we must be guided by the Pauli exclusion principle of Section 40-5; that is, no two electrons in an atom can have the same set of the quantum numbers n, ℓ m_ℓ, and m_s. If this important principle did not hold, *all* the electrons in any atom could jump to the atom's lowest energy level, which would eliminate the chemistry of atoms and molecules, and thus also eliminate biochemistry and us. Let us examine the atoms of a few elements to see how the Pauli exclusion principle operates in the building up of the periodic table.

Neon

The neon atom has 10 electrons. Only two of them fit into the lowest-energy subshell, the $1s$ subshell. These two electrons both have $n = 1$, $\ell = 0$, and $m_\ell = 0$, but one has $m_s = +\frac{1}{2}$ and the other has $m_s = -\frac{1}{2}$. The $1s$ subshell contains $2[2(0) + 1] = 2$ states. Because this subshell then contains all the electrons permitted by the Pauli principle, it is said to be **closed**.

Two of the remaining eight electrons fill the next lowest energy subshell, the $2s$ subshell. The last six electrons just fill the $2p$ subshell which, with $\ell = 1$, holds $2[2(1) + 1] = 6$ states.

In a closed subshell, all allowed z projections of the orbital angular momentum vector $\vec{L}$ are present and, as you can verify from Fig. 40-5, these projections cancel for the subshell as a whole; for every positive projection there is a corresponding negative projection of the same magnitude. Similarly, the z projections of the spin angular momenta also cancel. Thus, a closed subshell has no angular momentum and no magnetic moment of any kind. Furthermore, its probability density is spherically symmetric. Then neon with its three closed subshells ($1s$, $2s$, and $2p$) has no "loosely dangling electrons" to encourage chemical interaction with other atoms. Neon, like the other **noble gases** that form the right-hand column of the periodic table, is almost chemically inert.

Sodium

Next after neon in the periodic table comes sodium, with 11 electrons. Ten of them form a closed neon-like core, which, as we have seen, has zero angular momentum. The remaining electron is largely outside this inert core, in the $3s$ subshell—the next lowest energy subshell. Because this **valence electron**

of sodium is in a state with $\ell = 0$ (that is, an s state using the lettering system above), the sodium atom's angular momentum and magnetic dipole moment must be due entirely to the spin of this single electron.

Sodium readily combines with other atoms that have a "vacancy" into which sodium's loosely bound valence electron can fit. Sodium, like the other **alkali metals** that form the left-hand column of the periodic table, is chemically active.

Chlorine

The chlorine atom, which has 17 electrons, has a closed 10-electron, neon-like core, with 7 electrons left over. Two of them fill the $3s$ subshell, leaving five to be assigned to the $3p$ subshell, which is the subshell next lowest in energy. This subshell, which has $\ell = 1$, can hold $2[2\ell(0) + 1] = 6$ electrons, and so there is a vacancy, or a "hole," in this subshell.

Chlorine is receptive to interacting with other atoms that have a valence electron that might fill this hole. Sodium chloride (NaCl), for example, is a very stable compound. Chlorine, like the other **halogens** that form column VIIA of the periodic table, is chemically active.

Iron

The arrangement of the 26 electrons of the iron atom can be represented as follows:

$$\underbrace{1s^2\ 2s^2\,2p^6\ 3s^2\,3p^6}\ 3d^6\ 4s^2.$$

The subshells are listed in numerical order and, following convention, a superscript gives the number of electrons in each subshell. From Table 40-1 we can see that an s subshell ($\ell = 0$) can hold 2 electrons, a p subshell ($\ell = 1$) can hold 6, and a d subshell ($\ell = 2$) can hold 10. Thus, iron's first 18 electrons form the five filled subshells that are marked off by the bracket, leaving 8 electrons to be accounted for. Six of the eight go into the $3d$ subshell, and the remaining two go into the $4s$ subshell.

The reason the last two electrons do not also go into the $3d$ subshell (which can hold 10 electrons) is that the $3d^6\ 4s^2$ configuration results in a lower-energy state for the atom as a whole than would the $3d^8$ configuration. An iron atom with 8 electrons (rather than 6) in the $3d$ subshell would quickly make a transition to the $3d^6\ 4s^2$ configuration, emitting electromagnetic radiation in the process. The lesson here is that except for the simplest elements, the states may not be filled in what we might think of as their "logical" sequence.

40-7 | X Rays and the Ordering of the Elements

When a solid target, such as solid copper or tungsten, is bombarded with electrons whose kinetic energies are in the kiloelectron-volt range, electromagnetic radiation called **x rays** is emitted. A device used to produce x rays is generally called an x-ray tube. Figure 40-8 shows a schematic diagram of such a device. This was originally made by Coolidge and is known as Coolidge tube.

A filament F and a metallic target are fixed in an evacuated chamber C. The filament is heated electrically and thus emits electron by thermionic emission. A constant potential difference is maintained between the filament and the target using a DC power supply so that the target is at a higher potential than the filament. The electrons emitted by the filament are, therefore, accelerated by the electric field setup between the filament and the target, and in the process x rays are emitted. These x rays are brought out of the tube through a window made up of thin mica or Mylar or some such material that does not absorb x ray appreciably. In the process, a large amount of heat is developed, and thus an arrangement

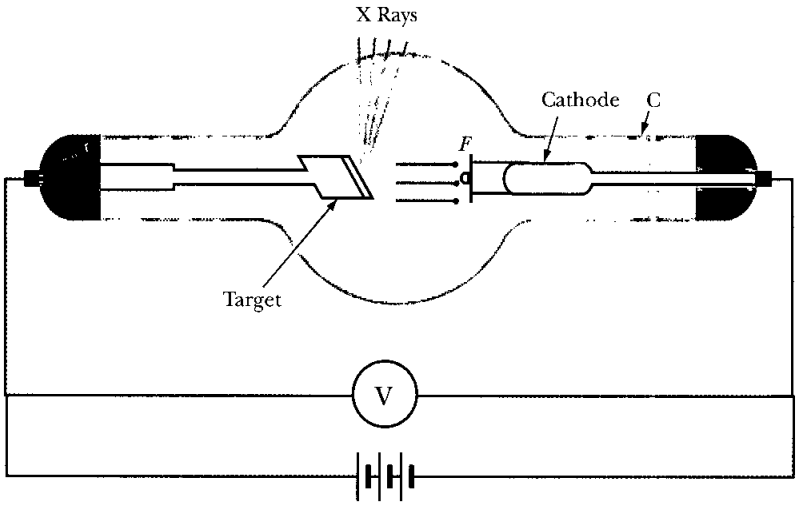


FIG. 40-8 Production of x rays by stopping fast electrons in dense materials.

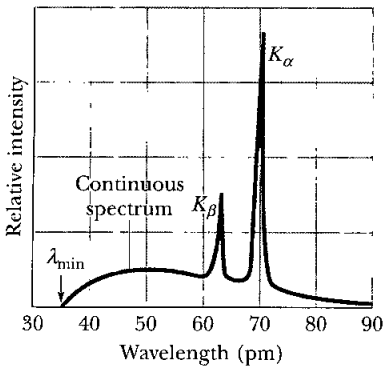


FIG. 40-9 The distribution by wavelength of the x rays produced when 35 keV electrons strike a molybdenum target. The sharp peaks and the continuous spectrum from which they rise are produced by different mechanisms.

is provided to cool down the tube by running water. The x rays are electromagnetic waves having wavelength of the order of 0.1 nm.

Our concern here is what these rays—whose medical, dental, and industrial usefulness is so well known and widespread—can teach us about the atoms that absorb or emit them. Figure 40-9 shows the wavelength spectrum of the x rays produced when a beam of 35 keV electrons falls on a molybdenum target. We see a broad, continuous spectrum of radiation on which are superimposed two peaks of sharply defined wavelengths. The continuous spectrum and the peaks arise in different ways, which we next discuss separately.

The Continuous X-Ray Spectrum

Here we examine the continuous x-ray spectrum of Fig. 40-9, ignoring for the time being the two prominent peaks that rise from it. Consider an electron of initial kinetic energy K_0 that collides (interacts) with one of the target atoms, as in Fig. 40-10. The electron may lose an amount of energy ΔK, which will appear as the energy of an x-ray photon that is radiated away from the site of the collision. (Very little energy is transferred to the recoiling atom because of the relatively large mass of the atom; here we neglect that transfer.) When the electrons are suddenly decelerated, they radiate energy known in German as Bremsstrahlung or "braking radiation."

The scattered electron in Fig. 40-10, whose energy is now less than K_0, may have a second collision with a target atom, generating a second photon, with a different photon energy. This electron-scattering process can continue until the electron is approximately stationary. All the photons generated by these collisions form part of the continuous x-ray spectrum.

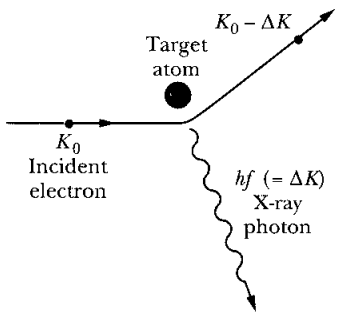


FIG. 40-10 An electron of kinetic energy K_0 passing near an atom in the target may generate an x-ray photon, the electron losing part of its energy in the process. The continuous x-ray spectrum arises in this way.

A prominent feature of that spectrum in Fig. 40-9 is the sharply defined **cutoff wavelength** λ_{min}, below which the continuous spectrum does not exist. This minimum wavelength corresponds to a collision in which an incident electron loses *all* its initial kinetic energy K_0 in a single head-on collision with a target atom. Essentially all this energy appears as the energy of a single photon, whose associated wavelength—the minimum possible x-ray wavelength—is found from

$$K_0 = hf = \frac{hc}{\lambda_{min}},$$

or

$$\lambda_{min} = \frac{hc}{K_0} \quad \text{(cutoff wavelength).} \tag{40-15}$$

The cutoff wavelength is totally independent of the target material. If we were to switch from a molybdenum target to a copper target, for example, all features of the x-ray spectrum of Fig. 40-9 would change *except* the cutoff wavelength.

CHECKPOINT 2 Does the cutoff wavelength λ_{min} of the continuous x-ray spectrum increase, decrease, or remain the same if you (a) increase the kinetic energy of the electrons that strike the x-ray target, (b) allow the electrons to strike a thin foil rather than a thick block of the target material, (c) change the target to an element of higher atomic number?

Sample Problem 40-1

A beam of 35.0 keV electrons strikes a molybdenum target, generating the x rays whose spectrum is shown in Fig. 40-9. What is the cutoff wavelength?

Conceptualize/Classify: The cutoff wavelength λ_{min} corresponds to an electron transferring (approximately) all of its energy to an x-ray photon, thus producing a photon with the greatest possible frequency and least possible wavelength.

Compute: From Eq. 40-15, we have

$$\lambda_{min} = \frac{hc}{K_0} = \frac{(4.14 \times 10^{-15}\,\text{eV}\cdot\text{s})(3.00 \times 10^{8}\,\text{m/s})}{35.0 \times 10^{3}\,\text{eV}}$$

$$= 3.55 \times 10^{-11}\,\text{m} = 35.5\,\text{pm}. \qquad \text{(Answer)}$$

The Characteristic X-Ray Spectrum

We now turn our attention to the two peaks of Fig. 40-9, labeled K_α and K_β. These (and other peaks that appear at wavelengths beyond the range displayed in Fig. 40-9) form the **characteristic x-ray spectrum** of the target material.

The peaks arise in a two-part process. (1) An energetic electron strikes an atom in the target and, while it is being scattered, the incident electron knocks out one of the atom's deep-lying (low n value) electrons. If the deep-lying electron is in the shell defined by $n = 1$ (called, for historical reasons, the K shell), there remains a vacancy, or *hole,* in this shell. Obviously, the striking electron must have an energy greater than the binding energy of the electron in the shell to knock it out. (2) An electron in one of the shells with a higher energy jumps to the K shell, filling the hole in this shell. During this jump, the atom emits a characteristic x-ray photon. If the electron that fills the K-shell vacancy jumps from the shell with $n = 2$ (called the L shell), the emitted radiation is the K_α line of Fig. 40-9; if it jumps from the shell with $n = 3$ (called the M shell), it produces the K_β line. Similarly, when the striking electron ejects an electron from the L shell, the vacancy in the L shell may be filled from an electron in the M shell. This process involves radiation of a photon that is labeled as L_α and so on. The hole left in either the L or M shell will be filled by an electron from still farther out in the atom. So you can see that a K_α line will always be accompanied by line in L series.

In studying x rays, it is more convenient to keep track of where a hole is created deep in the atom's "electron cloud" than to record the changes in the quantum state of the electrons that jump to fill that hole. Figure 40-11 does exactly that; it is an energy-level diagram for molybdenum, the element to which Fig. 40-9 refers. The baseline ($E = 0$) represents the neutral atom in its ground state. The level marked K (at $E = 20$ keV) represents the energy of the molybdenum atom with a hole in its K shell, the level marked L (at $E = 2.7$ keV) represents the atom with a hole in its L shell, and so on.

The transitions marked K_α and K_β in Fig. 40-11 are the ones that produce the two x-ray peaks in Fig. 40-9. The K_α spectral line, for example, originates when an electron from the L shell fills a hole in the K shell. To state this transition in terms of what the arrows in Fig. 40-11 show, a hole originally in the K shell moves to the L shell.

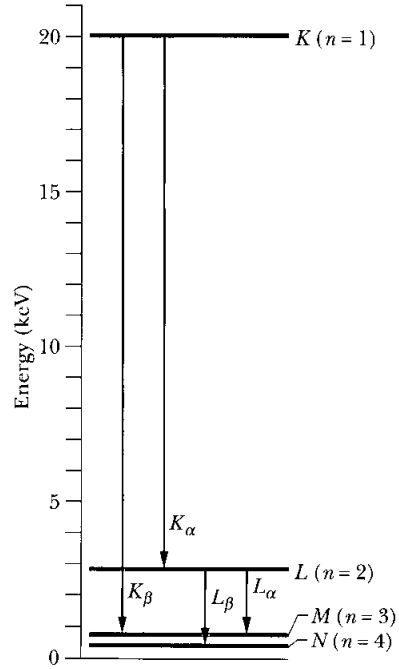


FIG. 40-11 A simplified energy-level diagram for a molybdenum atom, showing the transitions (of holes rather than electrons) that give rise to some of the characteristic x rays of that element. Each horizontal line represents the energy of the atom with a hole (a missing electron) in the shell indicated.

Ordering the Elements

In 1913, British physicist H. G. J. Moseley generated characteristic x rays for as many elements as he could find—he found 38—by using them as targets for electron bombardment in an evacuated tube of his own design. By means of a trolley manipulated by strings, Moseley was able to move the individual targets into the path of an electron beam. He measured the wavelengths of the emitted x rays by the crystal diffraction method.

Moseley then sought (and found) regularities in these spectra as he moved from element to element in the periodic table. In particular, he noted that if, for a given spectral line such as K_α, he plotted for each element the square root of the frequency f against the position of the element in the periodic table, a straight line resulted. Figure 40-12 shows a portion of his extensive data. Moseley's conclusion was this:

> *We have here a proof that there is in the atom a fundamental quantity, which increases by regular steps as we pass from one element to the next. This quantity can only be the charge on the central nucleus.*

As a result of Moseley's work, the characteristic x-ray spectrum became the universally accepted signature of an element, permitting the solution of a number of periodic table puzzles. Prior to that time (1913), the positions of elements in the table were assigned in order of atomic *mass,* although it was necessary to invert this order for several pairs of elements because of compelling chemical evidence; Moseley showed that it is the nuclear charge (that is, atomic number Z) that is the real basis for ordering the elements.

In 1913 the periodic table had several empty squares, and a surprising number of claims for new elements had been advanced. The x-ray spectrum provided a conclusive test of such claims. The lanthanide elements, often called the rare earth elements, had been sorted out only imperfectly because their similar chemical properties made sorting difficult. Once Moseley's work was reported, these elements were properly organized.

It is not hard to see why the characteristic x-ray spectrum shows such impressive regularities from element to element whereas the optical spectrum in the visible and near-visible region does not: The key to the identity of an element is the charge on its nucleus. Gold, for example, is what it is because its atoms have a nuclear charge of $+79e$ (that is, $Z = 79$). An atom with one more elementary charge on its nucleus is mercury; with one fewer, it is platinum. The K electrons, which play such a large role in the production of the x-ray spectrum, lie very close to the nucleus and are thus sensitive probes of its charge. The optical spectrum, on

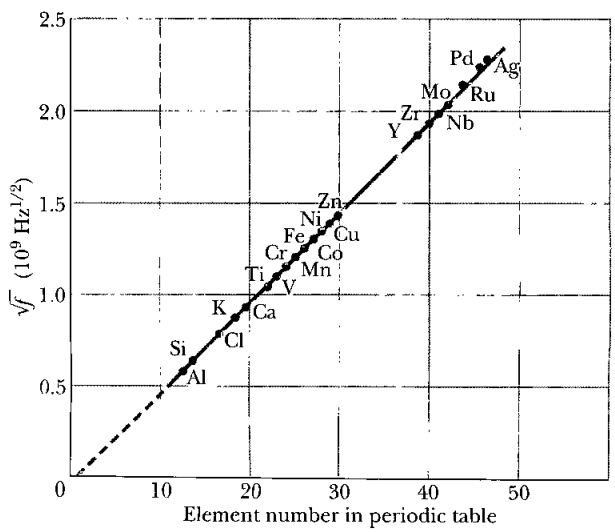


FIG. 40-12 A Moseley plot of the K_α line of the characteristic x-ray spectra of 21 elements. The frequency is calculated from the measured wavelength.

the other hand, involves transitions of the outermost electrons, which are heavily screened from the nucleus by the remaining electrons of the atom and thus are *not* sensitive probes of nuclear charge.

Accounting for the Moseley Plot

Moseley's experimental data, of which the Moseley plot of Fig. 40-12 is but a part, can be used directly to assign the elements to their proper places in the periodic table.

According to Eq. 39-33, the energy of the hydrogen atom is

$$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} = -\frac{13.60\text{ eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{40-16}$$

Consider now one of the two innermost electrons in the K shell of a multielectron atom. Because of the presence of the other K-shell electron, our electron "sees" an effective nuclear charge of approximately $(Z-1)e$, where e is the elementary charge and Z is the atomic number of the element. The factor e^4 in Eq. 40-16 is the product of e^2—the square of hydrogen's nuclear charge—and $(-e)^2$—the square of an electron's charge. For a multielectron atom, we can approximate the effective energy of the atom by replacing the factor e^4 in Eq. 40-16 with $(Z-1)^2 e^2 \times (-e)^2$, or $e^4(Z-1)^2$. That gives us

$$E_n = -\frac{(13.60\text{ eV})(Z-1)^2}{n^2}. \tag{40-17}$$

We saw that the K_α x-ray photon (of energy hf) arises when an electron makes a transition from the L shell (with $n = 2$ and energy E_2) to the K shell (with $n = 1$ and energy E_1). Thus, using Eq. 40-17, we may write the energy change as

$$\begin{aligned}\Delta E &= E_2 - E_1 \\ &= \frac{-(13.60\text{ eV})(Z-1)^2}{2^2} - \frac{-(13.60\text{ eV})(Z-1)^2}{1^2} \\ &= (10.2\text{ eV})(Z-1)^2.\end{aligned}$$

Then the frequency f of the K_α line is

$$\begin{aligned}f = \frac{\Delta E}{h} &= \frac{(10.2\text{ eV})(Z-1)^2}{(4.14\times10^{-15}\text{ eV}\cdot\text{s})} \\ &= (2.46\times10^{15}\text{ Hz})(Z-1)^2.\end{aligned} \tag{40-18}$$

Taking the square root of both sides yields

$$\sqrt{f} = CZ - C, \tag{40-19}$$

in which C is a constant ($= 4.96\times10^7\text{ Hz}^{1/2}$). Equation 40-19 is the equation of a straight line. It shows that if we plot the square root of the frequency of the K_α x-ray spectral line against the atomic number Z, we should obtain a straight line. As Fig. 40-12 shows, that is exactly what Moseley found. In general, Moseley's law is valid only for the frequency of a K_α line. If we want to find the frequency of any other line, then we can imagine that the effective nuclear charge cannot be so easily calculated because the shells other than $n = 1$ are not spherically symmetric. The general formula for any other line can be written as

$$\begin{aligned}\nu &= \frac{\Delta E}{h} = 13.60\left(\frac{1}{1^2} - \frac{1}{n^2}\right)(Z-b)^2, \\ \sqrt{\nu} &= a(Z-b),\end{aligned}$$

where a and b are constants.

Sample Problem 40-2

A cobalt target is bombarded with electrons, and the wavelengths of its characteristic x-ray spectrum are measured. There is also a second, fainter characteristic spectrum, which is due to an impurity in the cobalt. The wavelengths of the K_α lines are 178.9 pm (cobalt) and 143.5 pm (impurity), and the proton number for cobalt is $Z_{Co} = 27$. Determine the impurity using only these data.

Conceptualize/Classify: The wavelengths of the K_α lines for both the cobalt (Co) and the impurity (X) fall on a K_α Moseley plot, and Eq. 40-19 is the equation for that plot.

Compute: Substituting c/λ for f in Eq. 40-19, we obtain

$$\sqrt{\frac{c}{\lambda_{Co}}} = CZ_{Co} - C \quad \text{and} \quad \sqrt{\frac{c}{\lambda_X}} = CZ_X - C.$$

Dividing the second equation by the first neatly eliminates C, yielding

$$\sqrt{\frac{\lambda_{Co}}{\lambda_X}} = \frac{Z_X - 1}{Z_{Co} - 1}.$$

Substituting the given data yields

$$\sqrt{\frac{178.9\ \text{pm}}{143.5\ \text{pm}}} = \frac{Z_X - 1}{27 - 1}.$$

Solving for the unknown, we find that

$$Z_X = 30.0. \qquad \text{(Answer)}$$

A glance at the periodic table identifies the impurity as zinc.

Conclude: Note that with a larger value of Z than cobalt, zinc has a smaller value of the K_α line.

Auger Effect

We have seen that when a characteristic x-ray spectra is emitted, the atom loses an electron belonging to an inner shell. This gap in the electronic configuration is filled by an electron in an outer shell. This energy is emitted as an x-ray photon. This atom can also lose excitation energy by Auger effect. In this effect, an outer shell electron is ejected from the atom at the same time when another outer shell electron drops to the incomplete inner shell. This ejected electron carries the atom's excitation energy instead of the photon. Thus, it is like an internal photoelectric effect. Normally, both the process, x-ray emission and Auger effect, go on in metals. However, this electron does not emerge out whereas the characteristic x ray emerges out.

X-Ray Absorption Spectra

Just like the metals can radiate x rays when struck with fast moving electrons, they can also absorb the x rays. These x-ray absorption spectra enable the structures of the atomic levels to be found directly. In hydrogen, atom emission and absorption spectra are identical, except that an absorption spectrum is simpler because it is usually limited to transitions from the ground or lowest energy state. This is no longer true with x rays, which show absorption edges rather than absorption lines.

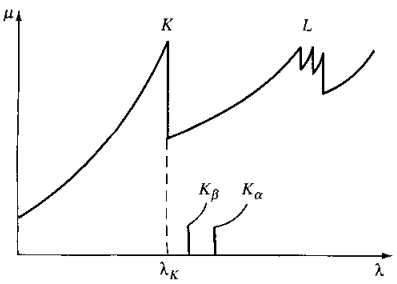


FIG. 40-13 A graph showing percentage of incident x-ray photons absorbed by a thin metal foil as a function of the wavelength of the photons.

The x-ray absorption spectrum of a metal may be examined by passing a narrow beam of x rays through the metal foil. The spectrum of the incident x rays should be continuous. Readings with and without the absorber in position at each wavelength are made and a graph of fraction of the x rays absorbed is plotted against wavelength as shown in Fig. 40-13. In general, the absorption increases with wavelength, but at certain critical wavelengths the absorption falls suddenly as shown in Fig. 40-13. This can be explained by supposing that with increasing wavelength, the energy of the x-ray photon is becoming smaller. For a wavelength lesser than that of K absorption edge, it has sufficient energy to excite an electron in the K shell. So the probability of its absorption is high. As higher shells are already occupied, the excited electron can only be completely removed from the atom.

Close examination of the wavelength associated with the K absorption edge shows that it does not coincide with any of the characteristic lines in the K series. It does, however, coincide with the limit of the K series, which corresponds to the

removal of an electron from the K shell ($n = 1$) of an atom to an infinite distance, the wavenumber of the absorbed photon being given by the formula

$$\bar{\nu} = R(Z-1)^2\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right) = R(Z-1)^2. \tag{40-20}$$

At a wavelength higher than the K absorption edge, the energy of the photon is insufficient to remove the K-shell electron. So there is a sudden drop in the absorption. Proceeding to greater wavelengths, the photon energy becomes still smaller until it is suddenly unable to excite an electron in the L shell. The fact that we observe three absorption edges for the L shell indicates that this level has three subshells with different energy levels. In the same way, it has been shown that M shell has five levels.

Hard and Soft X Rays

You have probably heard the terms "soft" and "hard" x rays. This is merely a different way of indicating the wavelength. Long-wavelength x rays are soft, that is, they have little penetrating power. These x rays will penetrate a sheet of paper, but a piece of heavy cardboard will weaken them appreciably and a human hand will stop them. Hence, they are of no use in medicine, except when a doctor wants to irradiate the skin without affecting the tissues below. There are still softer x rays that have trouble even in penetrating ordinary air; to study these, the physicist has to pump out the air from his entire apparatus so that the x rays can proceed unhindered. If you experiment with longer waves, you will find yourself at last dealing with rays that we earlier called ultraviolet. Although the properties of radiation change profoundly within this region, we must realize that these changes are gradual and that there is no difference in kind between x rays and visible light.

Moreover, if you make the wavelength shorter, the x rays get harder. Hard x rays are useful for their power of penetrating thick steel plates and castings, making internal flaws visible; they are also used in medicine when the doctor wants to irradiate some deep-seated tumor without damaging the tissues above it. In order to produce harder x rays, one has to increase the speed of the electrons that strikes the target. This is done by increasing the electric potential that drives the electrons toward the target. In the betatron, electrons can be brought up to the same speed as if accelerated by a potential of 20 million volts. The x rays that this machine produces can go through several feet of solid steel.

40-8 | Lasers and Laser Light

In the early 1960s, quantum physics made one of its many contributions to technology: the **laser**. Laser light, like the light from an ordinary lightbulb, is emitted when atoms make a transition from one quantum state to a lower one. However, in a laser, the transitions are coordinated, with these results:

1. ***Laser light is highly monochromatic.*** Light from an ordinary incandescent lightbulb is spread over a continuous range of wavelengths and is certainly not monochromatic. The radiation from a fluorescent neon sign is monochromatic, true, to about 1 part in 10^6, but the sharpness of definition of laser light can be many times greater, as much as 1 part in 10^{15}.
2. ***Laser light is highly coherent.*** Individual long waves (*wave trains*) for laser light can be several hundred kilometers long. When two separated beams that have traveled such distances over separate paths are recombined, they "remember" their common origin and are able to form a pattern of interference fringes. The corresponding *coherence length* for wave trains emitted by a lightbulb is typically less than a meter.



FIG. 40-14 A patient's head is scanned and mapped by (red) laser light in preparation for brain surgery. During the surgery, the laser-derived image of the head will be superimposed on the model of the brain shown on the monitor, to guide the surgical team into the region shown in green on the model. *(Sam Ogden/Photo Researchers)*

3. ***Laser light is highly directional.*** A laser beam spreads very little; it departs from strict parallelism only because of diffraction at the exit aperture of the laser. For example, a laser pulse used to measure the distance to the Moon generates a spot on the Moon's surface with a diameter of only a few meters. Light from an ordinary bulb can be made into an approximately parallel beam by a lens, but the beam divergence is much greater than for laser light. Each point on a lightbulb's filament forms its own separate beam, and the angular divergence of the overall composite beam is set by the size of the filament.
4. ***Laser light can be sharply focused.*** If two light beams transport the same amount of energy, the beam that can be focused to the smaller spot will have the greater intensity at that spot. For laser light, the focused spot can be so small that an intensity of 10^{17} W/cm^2 is readily obtained. An oxyacetylene flame, by contrast, has an intensity of only about 10^3 W/cm^2.

Lasers Have Many Uses

The smallest lasers, used for voice and data transmission over optical fibers, have as their active medium a semiconducting crystal about the size of a pinhead. Small as they are, such lasers can generate about 200 mW of power. The largest lasers, used for nuclear fusion research and for astronomical and military applications, fill a large building. The largest such laser can generate brief pulses of laser light with a power level, during the pulse, of about 10^{14} W. This is a few hundred times greater than the total electrical power generating capacity of the United States. To avoid a brief national power blackout during a pulse, the energy required for each pulse is stored up at a steady rate during the relatively long interpulse interval.

Among the many uses of lasers are reading bar codes, manufacturing and reading compact discs and DVDs, performing surgery of many kinds (see the opening photo of this chapter and Fig. 40-14), surveying, cutting cloth in the garment industry (several hundred layers at a time), welding auto bodies, and generating holograms.

40-9 | How Lasers Work

Because the word "laser" is an acronym for "light amplification by the stimulated emission of radiation," you should not be surprised that stimulated emission is the key to laser operation. Einstein introduced this concept in 1917. Although the world had to wait until 1960 to see an operating laser, the groundwork for its development was put in place decades earlier.

Consider an isolated atom that can exist either in its state of lowest energy (its ground state), whose energy is E_0, or in a state of higher energy (an excited state), whose energy is E_x. Here are three processes by which the atom can move from one of these states to the other:

1. ***Absorption.*** Figure 40-15*a* shows the atom initially in its ground state. If the atom is placed in an electromagnetic field that is alternating at frequency f, the atom can absorb an amount of energy hf from that field and move to the higher-energy state. From the principle of conservation of energy we have

$$hf = E_x - E_0. \tag{40-21}$$

   We call this process **absorption**.
2. ***Spontaneous emission.*** In Fig. 40-15*b* the atom is in its excited state and no external radiation is present. After a time, the atom will move *of its own accord* to its ground state, emitting a photon of energy hf in the process. We call this process **spontaneous emission**—*spontaneous* because the event was not triggered by any outside influence. The light from the filament of an ordinary lightbulb is generated in this way.

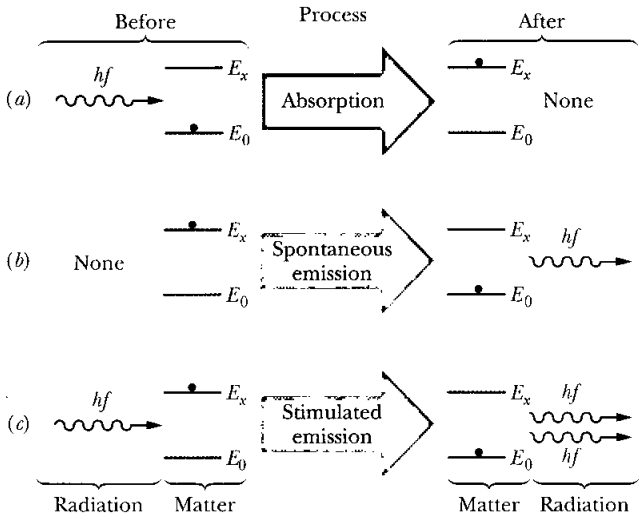


FIG. 40-15 The interaction of radiation and matter in the processes of (*a*) absorption, (*b*) spontaneous emission, and (*c*) stimulated emission. An atom (matter) is represented by the red dot; the atom is in either a lower quantum state with energy E_0 or a higher quantum state with energy E_x. In (*a*) the atom absorbs a photon of energy hf from a passing light wave. In (*b*) it emits a light wave by emitting a photon of energy hf. In (*c*) a passing light wave with photon energy hf causes the atom to emit a photon of the same energy, increasing the energy of the light wave.

Normally, the mean life of excited atoms before spontaneous emission occurs is about 10^{-8} s. However, for some excited states, this mean life is perhaps as much as 10^5 times longer. We call such long-lived states **metastable**; they play an important role in laser operation.

3. ***Stimulated emission.*** In Fig. 40-15*c* the atom is again in its excited state, but this time radiation with a frequency given by Eq. 40-21 is present. A photon of energy hf can stimulate the atom to move to its ground state, during which process the atom emits an additional photon, whose energy is also hf. We call this process **stimulated emission**—*stimulated* because the event is triggered by the external photon. The emitted photon is in every way identical to the stimulating photon. Thus, the waves associated with the photons have the same energy, phase, polarization, and direction of travel.

Figure 40-15*c* describes stimulated emission for a single atom. Suppose now that a sample contains a large number of atoms in thermal equilibrium at temperature T. Before any radiation is directed at the sample, a number N_0 of these atoms are in their ground state with energy E_0 and a number N_x are in a state of higher energy E_x. Ludwig Boltzmann showed that N_x is given in terms of N_0 by

$$N_x = N_0 e^{-(E_x - E_0)/kT}, \tag{40-22}$$

in which k is Boltzmann's constant. This equation seems reasonable. The quantity kT is the mean kinetic energy of an atom at temperature T. The higher the temperature, the more atoms—on average—will have been "bumped up" by thermal agitation (that is, by atom–atom collisions) to the higher energy state E_x. Also, because $E_x > E_0$, Eq. 40-22 requires that $N_x < N_0$; that is, there will always be fewer atoms in the excited state than in the ground state. This is what we expect if the level populations N_0 and N_x are determined only by the action of thermal agitation. Figure 40-16*a* illustrates this situation.

If we now flood the atoms of Fig. 40-16*a* with photons of energy $E_x - E_0$, photons will disappear via absorption by ground-state atoms and photons will be generated largely via stimulated emission of excited-state atoms. Einstein showed that the probabilities per atom for these two processes are identical. Thus, because there are more atoms in the ground state, the *net* effect will be the absorption of photons.

To produce laser light, we must have more photons emitted than absorbed; that is, we must have a situation in which stimulated emission dominates. The direct way to bring this about is to start with more atoms in the excited state than in the ground state, as in Fig. 40-16*b*. However, because such a **population inversion** is not consistent with thermal equilibrium, we must think up clever ways to set up and maintain one.

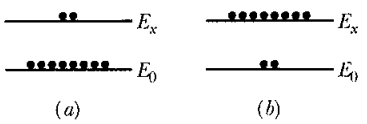


FIG. 40-16 (*a*) The equilibrium distribution of atoms between the ground state E_0 and excited state E_x accounted for by thermal agitation. (*b*) An inverted population, obtained by special methods. Such a population inversion is essential for laser action.

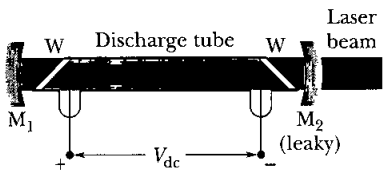


FIG. 40-17 The elements of a helium—neon gas laser. An applied potential V_{dc} sends electrons through a discharge tube containing a mixture of helium gas and neon gas. Electrons collide with helium atoms, which then collide with neon atoms, which emit light along the length of the tube. The light passes through transparent windows W and reflects back and forth through the tube from mirrors M_1 and M_2 to cause more neon atom emissions. Some of the light leaks through mirror M_2 to form the laser beam.

The Helium–Neon Gas Laser

Figure 40-17 shows a type of laser commonly found in student laboratories. It was developed in 1961 by Ali Javan and his coworkers. The glass discharge tube is filled with a 20:80 mixture of helium and neon gases, neon being the medium in which laser action occurs.

Figure 40-18 shows simplified energy-level diagrams for the two types of atoms. An electric current passed through the helium—neon gas mixture serves—through collisions between helium atoms and electrons of the current—to raise many helium atoms to state E_3, which is metastable.

The energy of helium state E_3 (20.61 eV) is very close to the energy of neon state E_2 (20.66 eV). Thus, when a metastable (E_3) helium atom and a ground-state (E_0) neon atom collide, the excitation energy of the helium atom is often transferred to the neon atom, which then moves to state E_2. In this manner, neon level E_2 in Fig. 40-18 can become more heavily populated than neon level E_1.

This population inversion is relatively easy to set up because (1) initially there are essentially no neon atoms in state E_1, (2) the metastability of helium level E_3 ensures a ready supply of neon atoms in level E_2, and (3) neon atoms in level E_1 decay rapidly (through intermediate levels not shown) to the neon ground state E_0.

Suppose now that a single photon is spontaneously emitted as a neon atom transfers from state E_2 to state E_1. Such a photon can trigger a stimulated emission event, which, in turn, can trigger other stimulated emission events. Through such a chain reaction, a coherent beam of laser light, moving parallel to the tube axis, can build up rapidly. This light, of wavelength 632.8 nm (red), moves through the discharge tube many times by successive reflections from mirrors M_1 and M_2 shown in Fig. 40-18, accumulating additional stimulated emission photons with each passage. M_1 is totally reflecting, but M_2 is slightly "leaky" so that a small fraction of the laser light escapes to form a useful external beam.

CHECKPOINT 3 The wavelength of light from laser *A* (a helium—neon gas laser) is 632.8 nm; that from laser *B* (a carbon dioxide gas laser) is 10.6 μm; that from laser *C* (a gallium arsenide semiconductor laser) is 840 nm. Rank these lasers according to the energy interval between the two quantum states responsible for laser action, greatest first.

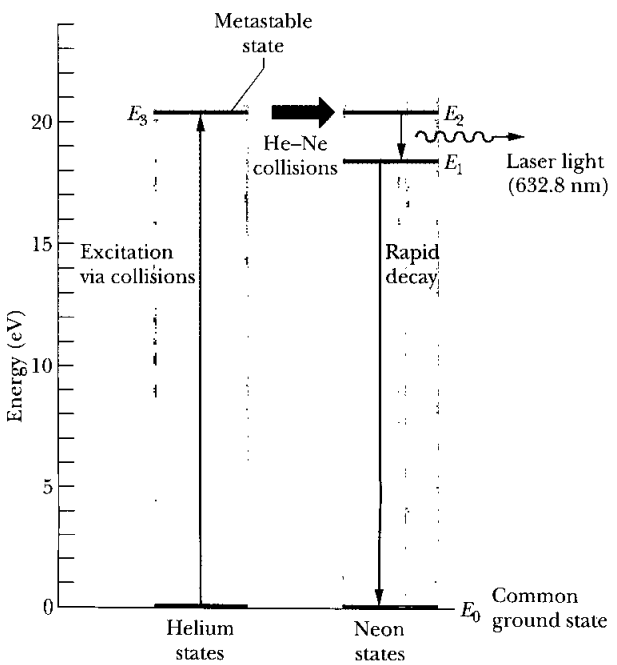


FIG. 40-18 Five essential energy levels for helium and neon atoms in a helium—neon gas laser. Laser action occurs between levels E_2 and E_1 of neon when more atoms are at the E_2 level than at the E_1 level.

Sample Problem 40-3

In the helium–neon laser of Fig. 40-17, laser action occurs between two excited states of the neon atom. However, in many lasers, laser action (*lasing*) occurs between the ground state and an excited state, as suggested in Fig. 40-16*b*.

(a) Consider such a laser that emits at wavelength $\lambda = 550$ nm. If a population inversion is not generated, what is the ratio of the population of atoms in state E_x to the population in the ground state E_0, with the atoms at room temperature?

Conceptualize/Classify: (1) The naturally occurring population ratio N_x/N_0 of the two states is due to thermal agitation of the gas atoms, according to Eq. 40-22, which we can write as

$$N_x / N_0 = e^{-(E_x - E_0)/kT}. \qquad (40\text{-}23)$$

To find N_x/N_0 with Eq. 40-23, we need to find the energy separation $E_x - E_0$ between the two states. (2) We can obtain $E_x - E_0$ from the given wavelength of 550 nm for the lasing between those two states.

Compute: The lasing wavelength gives us

$$E_x - E_0 = hf = \frac{hc}{\lambda}$$
$$= \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^{8}\ \text{m/s})}{(550\times10^{-9}\ \text{m})(1.60\times10^{-19}\ \text{J/eV})}$$
$$= 2.26\ \text{eV}.$$

To solve Eq. 40-23, we also need the mean energy of thermal agitation kT for an atom at room temperature (assumed to be 300 K), which is

$$kT = (8.62\times10^{-5}\ \text{eV/K})(300\ \text{K}) = 0.0259\ \text{eV},$$

in which k is Boltzmann's constant.

Substituting the last two results into Eq. 40-23 gives us the population ratio at room temperature:

$$N_x / N_0 = e^{-(2.26\ \text{eV})/(0.0259\ \text{eV})}$$
$$\approx 1.3\times10^{-38}. \qquad \text{(Answer)}$$

Conclude: This is an extremely small number. It is not unreasonable, however. Atoms with a mean thermal agitation energy of only 0.0259 eV will not often impart an energy of 2.26 eV to another atom in a collision.

(b) For the conditions of (a), at what temperature would the ratio N_x/N_0 be 1/2?

Compute: Now we want the temperature T such that thermal agitation has bumped enough neon atoms up to the higher-energy state to give $N_x/N_0 = 1/2$. Substituting that ratio into Eq. 40-23, taking the natural logarithm of both sides, and solving for T yield

$$T = \frac{E_x - E_0}{k(\ln 2)} = \frac{2.26\,\text{eV}}{(8.62\times10^{-5}\,\text{eV/K})(\ln 2)}$$
$$= 38\,000\ \text{K}. \qquad \text{(Answer)}$$

Conclude: This is much hotter than the surface of the Sun. It is clear that if we are to invert the populations of these two levels, some specific mechanism for bringing this about is needed—that is, we must "pump" the atoms. No temperature, however high, will naturally generate a population inversion by thermal agitation.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: If we use two different targets in Coolidge tube keeping other factors same, we get the same continuous x-ray spectra.

Statement 2: If we use two different targets in Coolidge tube keeping other factors same, we get same characteristic x-ray spectra.

2. Statement 1: X-ray photons have energy of the order of thousands of electrons volts.

Statement 2: The accelerating potential used in x-ray tube is of the order of thousand of volts.

3. Statement 1: If we use a Coolidge tube to produce soft x rays, it may not contain characteristic x-ray spectra.

Statement 2: Characteristic x rays are produced only when target is a hard metal.

4. Statement 1: It is possible to have an x-ray spectrum, where λ_{min} coincides with a peak of characteristic x ray.

Statement 2: The mechanism of production of characteristic x ray and continuous x ray is completely different.

5. Statement 1: A beam of visible light and a beam of monochromatic x ray have the same wavelength in vacuum.

Statement 2: A beam of visible light and a beam of monochromatic x ray have the same velocity in vacuum.

MULTIPLE CHOICE QUESTIONS

sec. 40-3 Electron Spin

1. The number of values of the orbital quantum number associated with the principal quantum number $n = 3$ is

(a) 1

(b) 2

(c) 3

(d) 4

2. The number of states in a shell with principal quantum number $n = 3$ is

(a) 3

(b) 9

(c) 15

(d) 18

(e) 25

3. Of the following states, $5s$, $3p$, $4f$, $5p$, $4g$, $3d$, and $2p$, the one which is NOT allowed is

(a) $3p$

(b) $4f$

(c) $3d$

(d) $4g$

(e) $2p$

sec. 40-4 Angular Momenta and Magnetic Dipole Moments

4. An electron is in a quantum state for which the magnitude of the orbital angular momentum is $6\sqrt{2}h/2\pi$. How many allowed values of the z component of the angular momentum are there?

(a) 4

(b) 5

(c) 7

(d) 8

5. An electron in an atom is in a state with $l = 1$. The minimum angle (in degrees) between $\vec{L}$ and the z axis is

(a) 0

(b) 18.0

(c) 45

(d) 36.7

6. The spin angular momentum of an electron is equal to

(a) $\sqrt{3}$

(b) $h\sqrt{3}/4\pi$

(c) $h/2$

(d) $\pm h/2$

sec. 40-7 X Rays and the Ordering of the Elements

7. The x-ray beam coming from an x-ray tube

(a) Is monochromatic

(b) Has all wavelengths smaller than a certain maximum wavelength

(c) Has all wavelengths greater than a certain minimum wavelength

(d) Has all wavelengths lying between a minimum and a maximum wavelength

8. The minimum wavelength of x ray that can be produced in a Coolidge tube depends on

(a) The metal used as the target

(b) The intensity of the electron beam striking the target

(c) The current flowing through the filament

(d) The potential difference between the cathode and the anode

9. If the potential difference applied to the tube is doubled and the separation between the filament and the target is also doubled, the cutoff wavelength

(a) Will remain unchanged

(b) Will be doubled

(c) Will be halved

(d) Will become four times the original

10. If the current in the circuit for heating the filament is increased, the cutoff wavelength

(a) Will increase

(b) Will decrease

(c) Will remain unchanged

(d) Will change

11. If a potential difference of 20 000 V is applied across an x-ray tube, the cutoff wavelength will be

(a) 6.21×10^{-10} m

(b) 6.21×10^{-11} m

(c) 6.21×10^{-12} m

(d) 3.1×10^{-11} m

12. The potential difference applied to an x-ray tube is 5 kV and the current through it is 3.2 mA. Then the number of electrons striking the target per second is

(a) 2×10^{16}

(b) 5×10^{6}

(c) 1×10^{17}

(d) 4×10^{15}

13. In an x-ray tube, the target should be made up of material having

(a) High atomic number and low thermal conductivity

(b) High atomic number and high thermal conductivity

(c) Low atomic number and low thermal conductivity

(d) Low atomic number and high thermal conductivity

14. X ray can be made harder by

(a) increasing the potential difference between cathode and target

(b) decreasing the potential difference between cathode and target

(c) increasing the current through filament

(d) decreasing the current through filament

15. Which of the following wavelength falls in an x-ray region?

(a) 10000 Å

(b) 1000 Å

(c) 1 Å

(d) 10^{-2} Å

16. An x-ray tube operates at an accelerating potential of 20 kV. Which of the following wavelengths will be absent in the continuous spectrum of x ray?

(a) 12 pm

(b) 45 pm

(c) 65 pm

(d) 95 pm

17. The energy of a photon of characteristic x ray from a Coolidge tube comes from

(a) The kinetic energy of the striking electron

(b) The kinetic energy of the free electrons of the target

(c) The kinetic energy of the ions of the target

(d) An atomic transition in the target

18. Figure 40-19 shows the intensity–wavelength relations of x rays coming from two different Coolidge tubes. The solid curve represents the relation for tube A in which the potential difference between the target and the filament is V_A and the atomic number of the target material is Z_A. These quantities are V_B and Z_B for the other tube. Then,

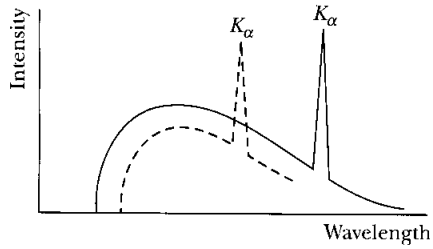


FIG. 40-19 Multiple choice question 18.

(a) $V_A > V_B, Z_A > Z_B$

(b) $V_A > V_B, Z_A < Z_B$

(c) $V_A < V_B, Z_A > Z_B$

(d) $V_A < V_B, Z_A < Z_B$

19. In a characteristic x-ray spectra (Fig. 40-20), L shell of some element is superimposed on continuous x-ray spectra. Here

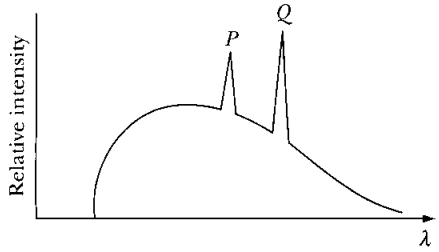


FIG. 40-20 Multiple choice question 19.

(a) P can represent L_α line

(b) Q represents L_α line

(c) Q and P represent L_α and L_β lines, respectively

(d) Q and P represent L_β and L_α lines, respectively

20. The K_α x-ray emission line of tungsten occurs at λ = 0.021 nm. The energy difference between K and L levels in this atom is about

(a) 0.51 MeV

(b) 1.2 MeV

(c) 59 keV

(d) 13.6 eV

21. Electrons with energy 80 keV are incident on the tungsten target of an x-ray tube. K-shell electrons of tungsten have 72.5 keV energy. X rays emitted by the tube contain

(a) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of 0.155 Å

(b) A continuous x-ray spectrum (Bremsstrahlung) with all wavelengths

(c) The characteristic x-ray spectrum of tungsten

(d) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of 0.155 Å and the characteristic x-ray spectrum of tungsten

22. Mosley's law for characteristic x rays is given by $\sqrt{f} = a(Z - b)$

Choose the correct statement(s).

(a) Both a and b depend on the target material

(b) Both a and b are independent of the target material

(c) Both a and b depend on the energy of the electron beam

(d) Both a and b depend on the nature of the lines (i.e., K, L, M, ...)

23. The wavelength of the K_α line for an element of atomic number 57 is α. What is the wavelength of the K_α line for the element of atomic number 29?

(a) α

(b) 2α

(c) 4α

(d) 8α

24. If the frequencies of K_α, K_β, and L_α x rays for a material $\nu_{K_\alpha}, \nu_{K_\beta}, \nu_{L_\alpha}$, respectively, then

(a) $\nu_{K_\alpha} = \nu_{K_\beta} + \nu_{L_\alpha}$

(b) $\nu_{L_\alpha} = \nu_{K_\alpha} + \nu_{K_\beta}$

(c) $\nu_{K_\beta} = \nu_{K_\alpha} + \nu_{L_\alpha}$

(d) None of these

25. The K, L, and M energy levels of platinum lie roughly at 78, 12, and 3 KeV, respectively. The ratio of wavelength of K_α line to that of K_β line in x-ray spectrum is

(a) $\frac{22}{3}$

(b) $\frac{3}{22}$

(c) $\frac{22}{25}$

(d) $\frac{25}{22}$

26. The potential difference applied to an x-ray tube is V. The ratio of de Broglie's wavelength of electron to the minimum wavelength of x ray is directly proportional to

(a) V

(b) $V^{1/2}$

(c) $\frac{1}{\sqrt{V}}$

(d) Independent of V

27. The energy of a tungsten atom with a vacancy in L shell is 11.3 keV. Wavelength of K_α photon for tungsten is 21.3 pm. If a potential difference of 62 kV is applied across the x-rays tube following characteristic x rays will be produced.

(a) K, L series

(b) Only K_α and L series

(c) Only L series

(d) None

sec. 40-9 How Lasers Work

28. Which (if any) of the following are essential for laser action to occur between two energy levels of an atom?

(a) There are more atoms in the upper level than in the lower

(b) The upper level is metastable

(c) The lower level is metastable

(d) The lower level is the ground state of the atom

29. Photons in a laser beam have the same energy, wavelength, polarization direction, and phase because

(a) Each is produced in an emission that is stimulated by another

(b) All come from the same atom

(c) The lasing material has only two quantum states

(d) All photons are alike, no matter what their source

(e) None of the above

30. Photons in a laser beam are produced by

(a) Transitions from a metastable state

(b) Transitions to a metastable state

(c) Transitions from a state that decays rapidly

(d) Pumping

31. A metastable state is important for the generation of a laser beam because it assures that

(a) Spontaneous emission does not occur more often than stimulated emission

(b) More photons are emitted than are absorbed

(c) Photons do not collide with each other

(d) Photons do not make upward transitions

32. Which of the lettered transitions on the energy level shown in Fig. 40-21 would be the best candidate for light amplification by stimulated emission?

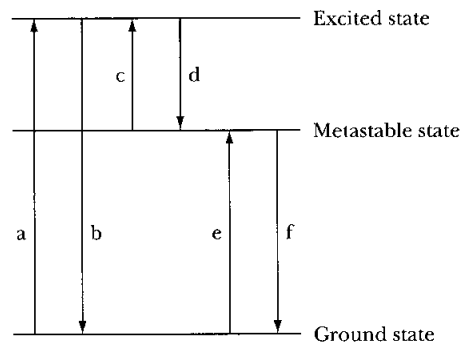


FIG. 40-21 Multiple choice question 32.

(a) a (c) c (e) e

(b) b (d) d (f) f

PROBLEMS

sec. 40-3 Electron Spin

1. An electron in a mercury atom is in the $3d$ subshell. Which of the following m_l values are possible for it: (a) −3, (b) −1, (c) 0, (d) 1, (e) 2?

2. An atom of uranium has closed $6p$ and $7s$ subshells. Which subshell has the greater number of electrons?

sec. 40-4 Angular Momenta and Magnetic Dipole Moments

3. An electron in a multielectron atom has $m_\ell = +4$. For this electron, what are: (a) the value of ℓ, (b) the smallest possible value of n, and (c) the number of possible values of m_s?

4. In the subshell $\ell = 3$, (a) what is the greatest (most positive) m_l value, (b) how many states are available with the greatest m_l value, and (c) what is the total number of states available in the subshell?

5. How many electron states are in these subshells: (a) $n = 4$, $\ell = 3$; (b) $n = 3$, $\ell = 1$; (c) $n = 4$, $\ell = 1$; (d) $n = 2$, $\ell = 0$?

6. (a) What is the magnitude of the orbital angular momentum in a state with ? (b) What is the magnitude of its largest projection on an imposed z axis?

7. An electron in a hydrogen atom is in a state with $l = 5$. What is the minimum possible value of the semiclassical angle between $\vec{L}$ and L_z?

8. If orbital angular momentum $\vec{L}$ is measured along, say, a z axis to obtain a value for L_z, show that

$$(L_x^2 + L_y^2)^{1/2} = [l(l+1) - m_l^2]^{1/2}\hbar$$

is the most that can be said about the other two components of the orbital angular momentum.

9. An electron is in a state with $n = 3$. What are (a) the number of possible values of ℓ, (b) the number of possible values of m_ℓ, (c) the number of possible values of m_s, (d) the number of states in the $n = 3$ shell, and (e) the number of subshells in the $n = 3$ shell?

10. An electron is in a state with $l = 3$. (a) What multiple of $\hbar$ gives the magnitude of $\vec{L}$? (b) What multiple of μ_B gives the magnitude of $\vec{\mu}$? (c) What is the largest possible value of m_ℓ, (d) what multiple of $\hbar$ gives the corresponding value of L_z, and (e) what multiple of μ_B gives the corresponding value of $\mu_{orb,z}$? (f) What is the value of the semiclassical angle θ between the directions of L_z and $\vec{L}$? What is the value of angle θ for (g) the second largest possible value of m_ℓ and (h) the smallest (i.e., most negative) possible value of m_ℓ?

sec. 40-6 Building the Periodic Table

11. A recently named element is darmstadtium (Ds), which has 110 electrons. Assume that you can put the 110 electrons into the atomic shells one by one and can neglect any electron–electron interaction. With the atom in ground state, what is the spectroscopic notation for the quantum number l for the last electron?

12. For a helium atom in its ground state, what are quantum numbers (n, ℓ, m_ℓ, and m_s) for the (a) spin-up electron and (b) spin-down electron?

13. Consider the elements selenium (Z = 34), bromine ($Z = 35$), and krypton ($Z = 36$). In their part of the periodic table, the subshells of the electronic states are filled in the sequence

$$1s\ 2s\ 2p\ 3s\ 3p\ 3d\ 4s\ 4p \ldots$$

What are (a) the highest occupied subshell for selenium and (b) the number of electrons in it, (c) the highest occupied subshell for bromine and (d) the number of electrons in it, and (e) the highest occupied subshell for krypton and (f) the number of electrons in it?

14. Suppose two electrons in an atom have quantum numbers $n = 2$ and $\ell = 1$. (a) How many states are possible for those two electrons? (Keep in mind that the electrons are indistinguishable.) (b) If the Pauli exclusion principle did not apply to the electrons, how many states would be possible?

15. Two of the three electrons in a lithium atom have quantum numbers (n, ℓ, m_ℓ, m_s) of (1, 0, 0, + 1/2) and (1, 0, 0, −1/2). What quantum numbers are possible for the third electron if the atom is (a) in the ground state and (b) in the first excited state?

16. Show that the number of states with the same quantum number n is $2n^2$.

sec. 40-7 X Rays and the Ordering of the Elements

17. Through what minimum potential difference must an electron in an x-ray tube be accelerated so that it can produce x rays with a wavelength of 0.100 nm?

18. An x-ray tube operates at 20 kV. A particular electron loses 5% of its kinetic energy to emit an x-ray photon at the first collision. Find the wavelength corresponding to this photon.

19. The electric current in an x-ray tube (from the target to the filament) operating at 40 kV is 10 mA. Assume that on an average, 1% of the total kinetic energy of the electrons hitting the target are converted into x rays. (a) What is the total power emitted as x rays? (b) How much heat is produced in the target every second?

20. What are the initial and final shells for an M_β line in an x-ray spectrum?

21. When electrons bombard a molybdenum target, they produce both continuous and characteristic x rays. The kinetic energy of the incident electrons is 35.0 keV. If the accelerating potential is increased to 50.0 keV, (a) what is the value of λ_{min}, and (b) do the wavelengths of the K_α and K_β lines increase, decrease, or remain the same?

22. Show that a moving electron cannot spontaneously change into an x-ray photon in free space. A third body (atom or nucleus) must be present. Why is it needed? (*Hint:* Examine the conservation of energy and momentum.)

23. The wavelength of the K_α line from iron is 193 pm. What is the energy difference between the two states of the iron atom that give rise to this transition?

24. A tungsten (Z = 74) target is bombarded by electrons in an x-ray tube. The K, L, and M energy levels for tungsten (compare Fig. 40-11) have the energies 69.5, 11.3, and 2.30 keV, respectively. (a) What is the minimum value of the accelerating potential that will permit the production of the characteristic K_α and K_β lines of tungsten? (b) For this same accelerating potential, what is λ_{min}? What are the (c) K_α and (d) K_β wavelengths?

25. Figure 40-22 shows the energy levels of a mercury atom. Electrons with kinetic energies of 7.0 eV are introduced into mercury vapor. (a) Convert the energy levels to the system that sets the ground state at 0 eV. (b) What is the kinetic energy of an electron after an elastic collision with a mercury atom? (c) What are the possible kinetic energies of an electron after an inelastic collision with a mercury atom? (d) What are the energies of the photons coming from the mercury vapor?

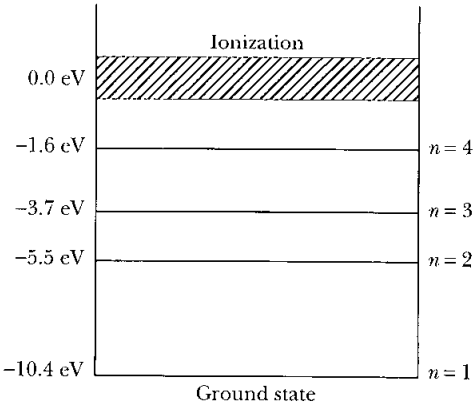


FIG. 40-22 Problem 26.

26. Which element has a K_α x-ray line whose wavelength is 0.180 nm?

27. K_α x rays of molybdenum has wavelength 71 pm. If the energy of a molybdenum atom with a K electron knocked out is 23.32 keV, what will be the energy of this atom when an L electron is knocked out?

28. The K_α x rays of aluminum (Z = 13) and zinc (Z = 30) have wavelengths 887 pm and 146 pm, respectively. Use Moseley's law $\sqrt{\nu} = a(Z - b)$ to find the wavelength of the K_α x ray of iron ($Z = 26$).

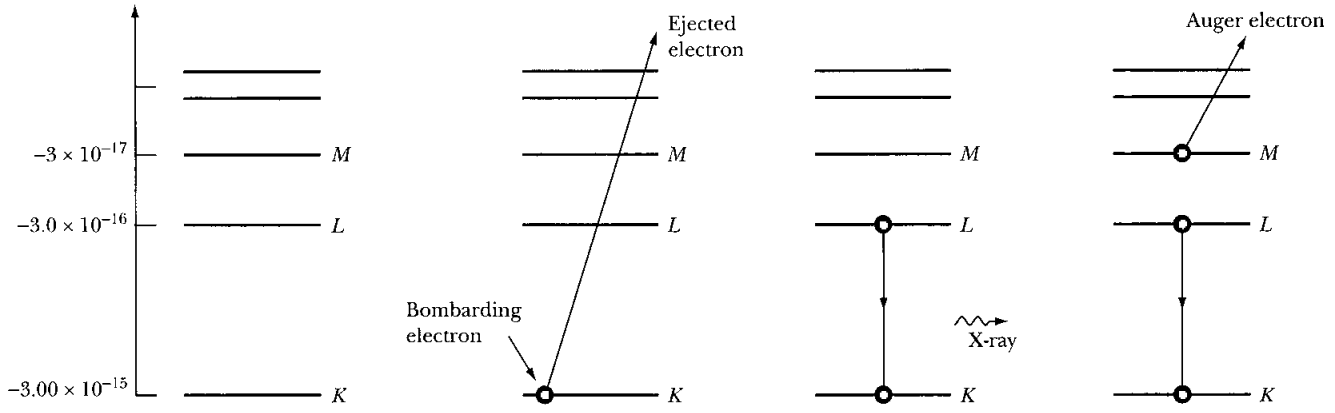


FIG. 40-23 Problem 30.

29. (a) Find the energy in electron volts required to strip a calcium atom ($Z = 20$) of its last electron, assuming the other 19 have been removed. How does this compare with the energy required to excite the K_α x-ray lines of Ca, which is about 3.7 keV? (b) Why there is a difference?

30. An imaginary atom whose energy levels are shown here undergoes an Auger transition. This implies that the x-ray photon emitted during transition from L to K state is absorbed by another electron in M shell and it comes out of the atom. Calculate the initial energy of the Auger electron as in Fig. 40-23.

31. A material whose K-absorption edge is 0.2 Å is irradiated by x rays of wavelength 0.15 Å. Find the maximum energy of the photoelectrons that are emitted from the K shell.

sec. 40-9 How Lasers Work

32. A pulsed laser emits light at a wavelength of 694.4 nm. The pulse duration is 12 ps, and the energy per pulse is 0.150 J. (a) What is the length of the pulse? (b) How many photons are emitted in each pulse?

33. A laser emits at 424 nm in a single pulse that lasts 0.500 μs. The power of the pulse is 2.80 MW. If we assume that the atoms contributing to the pulse underwent stimulated emission only once during the 0.500 μs, how many atoms contributed?

34. A hypothetical atom has energy levels uniformly separated by 1.2 eV. At a temperature of 2000 K, what is the ratio of the number of atoms in the 13th excited state to the number in the 11th excited state?

35. A hypothetical atom has only two atomic energy levels, separated by 3.2 eV. Suppose that at a certain altitude in the atmosphere of a star there are $6.1 \times 10^{13}/\text{cm}^3$ of these atoms in the higher-energy state and $2.5 \times 10^{15}/\text{cm}^3$ in the lower energy state. What is the temperature of the star's atmosphere at that altitude?

36. Assume that lasers are available whose wavelengths can be precisely "tuned" to anywhere in the visible range—that is, in the range 450 nm $< \lambda <$ 650 nm. If every television channel occupies a bandwidth of 10 MHz, how many channels can be accommodated within this wavelength range?

37. A population inversion for two energy levels is often described by assigning a negative Kelvin temperature to the system. What negative temperature would describe a system in which the population of the upper energy level exceeds that of the lower level by 10% and the energy difference between the two levels is 2.26 eV?

38. The active volume of a laser constructed of the semiconductor GaAlAs is only 200 μm^3 (smaller than a grain of sand), and yet the laser can continuously deliver 5.0 mW of power at a wavelength of 0.80 μm. At what rate does it generate photons?

39. A high-powered laser beam ($\lambda = 600$ nm) with a beam diameter of 12 cm is aimed at the Moon, 3.8×10^5 km distant. The beam spreads only because of diffraction. The angular location of the edge of the central diffraction disk is given by

$$\sin\theta = \frac{1.22\lambda}{d},$$

where d is the diameter of the beam aperture. What is the diameter of the central diffraction disk on the Moon's surface?

40. Figure 40-24 shows the energy levels of two types of atoms. Atoms A are in one tube, and atoms B are in another tube. The energies (relative to a ground-state energy of zero) are indicated; the average lifetime of atoms in each level is also indicated. All the atoms are initially pumped to levels higher than the levels shown in the figure. The atoms then drop down through the levels, and many become "stuck" on certain levels, leading to population inversion and lasing. The light emitted by A illuminates B and can cause stimulated emission of B. What is the energy per photon of that stimulated emission of B?

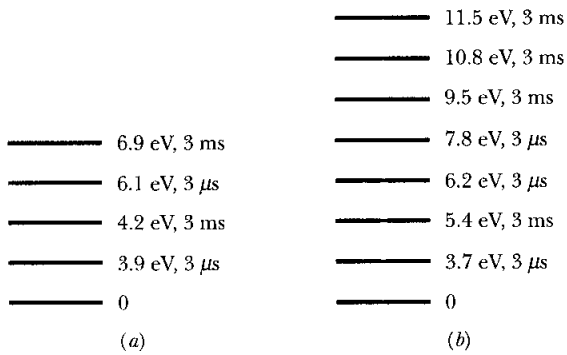


FIG. 40-24 Problem 40.

41. Ruby lases at a wavelength of 694 nm. A certain ruby crystal has 4.00×10^{19} Cr ions (which are the atoms that lase).

The lasing transition is between the first excited state and the ground state, and the output is a light pulse lasting 2.00 μs. As the pulse begins, 60.0% of the Cr ions are in the first excited state and the rest are in the ground state. What is the average power emitted during the pulse? (*Hint:* Don't just ignore the ground-state ions.)

42. The active medium in a particular laser that generates laser light at a wavelength of 694 nm is 6.00 cm long and 1.00 cm in diameter. (a) Treat the medium as an optical resonance cavity analogous to a closed organ pipe. How many standing-wave nodes are there along the laser axis? (b) By what amount Δf would the beam frequency have to shift to increase this number by one? (c) Show that Δf is just the inverse of the travel time of laser light for one round trip back and forth along the laser axis. (d) What is the corresponding fractional frequency shift $\Delta f/f$? The appropriate index of refraction of the lasing medium (a ruby crystal) is 1.75.

43. The mirrors in the laser of Fig. 40-17, which are separated by 8.0 cm, form an optical cavity in which standing waves of laser light can be set up. Each standing wave has an integral number n of half wavelengths in the 8.0 cm length, where n is large and the waves differ slightly in wavelength. Near $\lambda = 533$ nm, how far apart in wavelength are the standing waves?

44. A hypothetical atom has two energy levels, with a transition wavelength between them of 580 nm. In a particular sample at 300 K, 4.0×10^{20} such atoms are in the state of lower energy. (a) How many atoms are in the upper state, assuming conditions of thermal equilibrium? (b) Suppose, instead, that 3.0×10^{20} of these atoms are "pumped" into the upper state by an external process, with 1.0×10^{20} atoms remaining in the lower state. What is the maximum energy that could be released by the atoms in a single laser pulse if each atom jumps once between those two states (either via absorption or via stimulated emission)? region of the electromagnetic spectrum was the emission?

ADDITIONAL PROBLEMS

45. A molybdenum ($Z = 42$) target is bombarded with 35.0 keV electrons and the x-ray spectrum of Fig. 40-9 results. The K_β and K_α wavelengths are 63.0 and 71.0 pm, respectively. What photon energy corresponds to the (a) K_β and (b) K_α radiation? The two radiations are to be filtered through one of the substances in the following table such that the substance absorbs the K_β line more strongly than the K_α line. A substance will absorb radiation x_1 more strongly than it absorbs radiation x_2 if a photon of x_1 has enough energy to eject a K electron from an atom of the substance but a photon of x_2 does not. The table gives the ionization energy of the K electron in molybdenum and four other substances. Which substance in the table will serve (c) best and (d) second best as the filter?

	Zr	Nb	Mo	Tc	Ru
Z	40	40	42	43	44
E_K(keV)	18.00	18.99	20.00	21.04	22.12

46. Suppose that the electron had no spin and that the Pauli's exclusion principle still held. Which, if any, of the present noble gases would remain in that category?

47. Show that if the 63 electrons in an atom of europium were assigned to shells according to the "logical" sequence of quantum numbers, this element would be chemically similar to sodium.

48. Knowing that the minimum x-ray wavelength produced by 40.0 keV electrons striking a target is 31.1 pm, determine the Planck constant h.

49. An electron in a multielectron atom is known to have the quantum number $l = 3$. What are its possible n, m_l, and m_s quantum numbers?

50. The K_α x-ray line for any element arises because of a transition between the K shell ($n = 1$) and the L shell ($n = 2$). Fig. 40-9 shows this line (for a molybdenum target) occurring at a single wavelength. With higher resolution, however, the line splits into several wavelength components because the L shell does not have a unique energy. (a) How many components does the K_α line have? (b) Similarly, how many components does the K_β line have?

ANSWERS

Checkpoints

1. 7

2. (a) decrease; (b)–(c) remains the same

3. A, C, B

Reasoning Type Questions

1. (c)

2. (a)

3. (b)

4. (d)

5. (d)

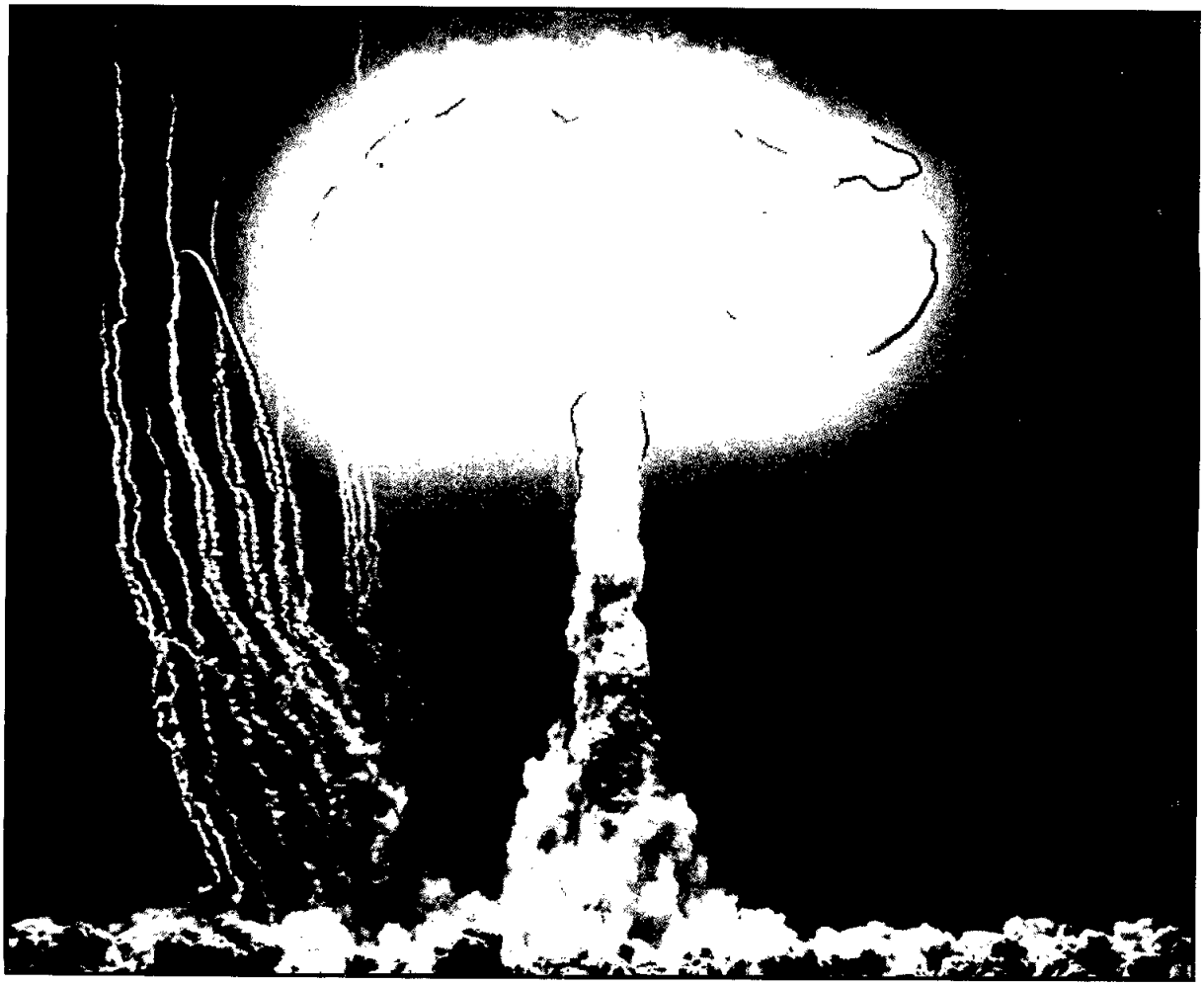
Multiple Choice Questions

1. (c)
2. (d)
3. (d)
4. (d)
5. (c)
6. (b)
7. (c)
8. (d)
9. (c)
10. (c)
11. (b)
12. (a)
13. (b)
14. (a)
15. (c)
16. (a), (b)
17. (d)
18. (b)
19. (c)
20. (c)
21. (a), (c), (d)
22. (d)
23. (c)
24. (c)
25. (d)
26. (b)
27. (c)
28. (a), (b)
29. (a)
30. (a)
31. (a)
32. (f)

Problems

1. 2, –1, 0, and 1
2. 6p
3. (a) 4; (b) 5; (c) 2
4. (a) 3; (b) 2; (c) 14
5. (a) 14; (b) 6; (c) 6; (d) 2
6. (a) 3.65×10^{-34} J·s; (b) 3.16×10^{-34} J·s
7. 24.1°
9. (a) 0, 1, and 2; (b) 5; (c) $\pm\frac{1}{2}$; (d) 18; (e) there are three subshells in the $n = 3$ shell
10. (a) $\sqrt{12} \approx 3.46$; (b) $\sqrt{12} \approx 3.46$; (c) 3; (d) 3; (e) –3; (f) 30.0°; (g) 54.7°; (h) 150°
11. g
12. (a) 1,0,0,+1/2; (b) 1,0,0,–1/2
13. (a) 4p; (b) 4; (c) 4p; (d) 5; (e) 4p; (f) 6
14. (a) 15; (b) 21
15. (a) [2, 0, 0, +1/2] and [2, 0, 0, –1/2]; (b) [2,1,1, +1/2], [2,1,1, –1/2], [2, 1, 0, + 1/2], [2, 1, 0, – 1/2], [2, 1, – 1, + 1/2] and [2, 1, –1, –1/2]
17. 12.4 kV
18. 1.24 nm
19. (a) 4 W, (b) 396 J
20. $n = 5$ to $n = 3$
21. (a) 0.025 nm; (b) same
23. 6.42 keV
24. (a) 69.5 kV; (b) 17.8 pm; (c) 21.3 pm; (d) 18.5 pm
25. (a) 0 eV, 4.9 eV, 6.7 eV, 8.8 eV and so on; (b) 7 eV; (c) 2.12 eV, 0.3 eV; (d) 6.7 eV, 4.9 eV, 1.8 eV
26. Cobalt, atomic no. 27
27. 5.82 keV
28. 198 pm
29. (a) 5.4 keV; (b) because other electrons change the potential
30. 2.67×10^{-15} J
31. 3.31×10^{-15} J
32. (a) 3.60 mm; (b) 5.24×10^{17}
33. 3.0×10^{18}
34. 9.0×10^{-7}
35. 1.0×10^{4} K
36. 2×10^{7}
37. -2.75×10^{5} K
38. 2.0×10^{16} s^{-1}
39. 4.7 km
40. 3.0 eV
41. 1.1×10^{6} W
42. (a) 3.03×10^{5}; (b) 1.43 GHz; (c) The speed of light in the crystal is c/n and the round-trip distance is $2L$, so the round-trip travel time is $2nL/c$. This is the same as the reciprocal of the change in frequency; (d) 3.31×10^{-6}
43. 1.8 pm
44. (a) 0; (b) 68 J
45. (a) ≈ 20 ke V; (b) ≈ 18 ke V; (c) Zr is the best choice; (d) Nb is the second best choice
46. Argon
48. 6.63×10^{-34} J·s
49. The principal quantum number n must be greater than 3. The magnetic quantum number m_ℓ can have any of the values – 3, – 2, – 1, 0, +1, +2, or +3. The spin quantum number can have either of the values –1/2 or +1/2
50. (a) 3; (b) 5

The Nucleus 45



Courtesy U.S. Department of Energy

This image has transfixed the world since World War II. When Robert Oppenheimer, the head of the scientific team that developed the atomic bomb, witnessed the first atomic explosion, he quoted from Bhagavad Gita: "... The great destroyer of worlds, and I have come here to destroy all people."

What is the physics behind this image that has so horrified the world?

The answer is in this chapter.

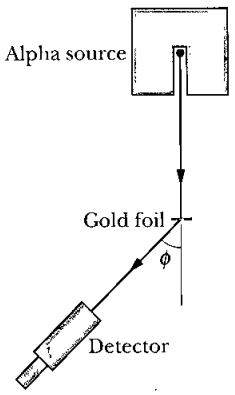


FIG. 41-1 An arrangement (top view) used in Rutherford's laboratory in 1911–1913 to study the scattering of α particles by thin metal foils. The detector can be rotated to various values of the scattering angle ϕ. The alpha source was radon gas, a decay product of radium. With this simple "tabletop" apparatus, the atomic nucleus was discovered.

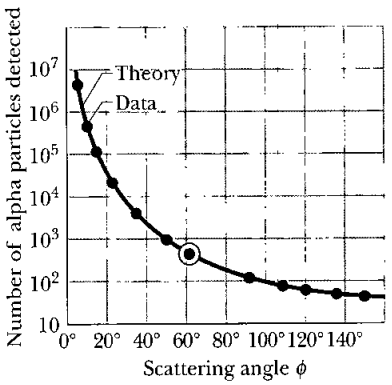


FIG. 41-2 The dots are alpha-particle scattering data for a gold foil, obtained by Geiger and Marsden using the apparatus of Fig. 41-1. The solid curve is the theoretical prediction, based on the assumption that the atom has a small, massive, positively charged nucleus. Note that the vertical scale is logarithmic, covering six orders of magnitude. The data have been adjusted to fit the theoretical curve at the experimental point that is enclosed in a circle.

41-1 WHAT IS PHYSICS?

Thus far in our exploration of the quantum amusement park, we have examined electrons trapped within various potential wells, including atoms, but we have largely neglected what lies at the center of an atom—the nucleus. For the last 90 years, a principal goal of physics has been to work out the quantum physics of nuclei, and, for almost as long, a principal goal of some types of engineering has been to apply that quantum physics with applications ranging from radiation therapy in the war on cancer to detectors of radon gas in basements.

Before we get to such applications and the quantum physics of nuclei, let's first discuss how physicists discovered that an atom has a nucleus. As obvious as that fact is today, it initially came as an incredible surprise.

41-2 | Discovering the Nucleus

In the first years of the 20th century, not much was known about the structure of atoms beyond the fact that they contain electrons. The electron had been discovered (by J. J. Thomson) in 1897, and its mass was unknown in those early days. Thus, it was not possible even to say how many negatively charged electrons a given atom contained. Scientists reasoned that because atoms were electrically neutral, they must also contain some positive charge, but nobody knew what form this compensating positive charge took.

In 1911 Ernest Rutherford proposed that the positive charge of the atom is densely concentrated at the center of the atom, forming its **nucleus**, and that, furthermore, the nucleus is responsible for most of the mass of the atom. Rutherford's proposal was no mere conjecture but was based firmly on the results of an experiment suggested by him and carried out by his collaborators, Hans Geiger (of Geiger counter fame) and Ernest Marsden, a 20-year-old student who had not yet earned his bachelor's degree.

In Rutherford's day it was known that certain elements, called **radioactive**, transform into other elements spontaneously, emitting particles in the process. One such element is radon, which emits alpha (α) particles that have an energy of about 5.5 MeV. We now know that these particles are helium nuclei.

Rutherford's idea was to direct energetic alpha particles at a thin target foil and measure the extent to which they were deflected as they passed through the foil. Alpha particles, which are about 7300 times more massive than electrons, have a charge of $+2e$.

Figure 41-1 shows the experimental arrangement of Geiger and Marsden. Their alpha source was a thin-walled glass tube of radon gas. The experiment involves counting the number of alpha particles that are deflected through various scattering angles ϕ.

Figure 41-2 shows their results. Note especially that the vertical scale is logarithmic. We see that most of the particles are scattered through rather small angles, but—and this was the big surprise—a very small fraction of them are scattered through very large angles, approaching 180°. In Rutherford's words: "It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you had fired a 15-inch shell at a piece of tissue paper and it [the shell] came back and hit you."

Why was Rutherford so surprised? At the time of these experiments, most physicists believed in the so-called plum pudding model of the atom, which had been advanced by J. J. Thomson. In this view the positive charge of the atom was thought to be spread out through the entire volume of the atom. The electrons (the "plums") were thought to vibrate about fixed points within this sphere of positive charge (the "pudding").

The maximum deflecting force that could act on an alpha particle as it passed through such a large positive sphere of charge would be far too small to deflect the alpha particle by even as much as 1°. (The expected deflection has been compared to what you would observe if you fired a bullet through a sack of snowballs.) The electrons in the atom would also have very little effect on the massive, energetic alpha particle.

Rutherford saw that, to deflect the alpha particle backward, there must be a large force; this force could be provided if the positive charge, instead of being spread throughout the atom, were concentrated tightly at its center. Then the incoming alpha particle could get very close to the positive charge without penetrating it; such a close encounter would result in a large deflecting force.

Figure 41-3 shows possible paths taken by typical alpha particles as they pass through the atoms of the target foil. As we see, most are either undeflected or only slightly deflected, but a few (those whose incoming paths pass, by chance, very close to a nucleus) are deflected through large angles. From an analysis of the data, Rutherford concluded that the radius of the nucleus must be smaller than the radius of an atom by a factor of about 10^4. In other words, the atom is mostly empty space.

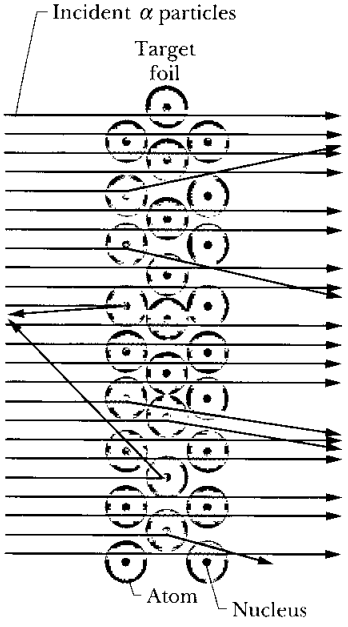


FIG. 41-3 The angle through which an incident alpha particle is scattered depends on how close the particle's path lies to an atomic nucleus. Large deflections result only from very close encounters.

Sample Problem 41-1

A 5.30 MeV alpha particle happens, by chance, to be headed directly toward the nucleus of an atom of gold, which contains 79 protons. How close does the alpha particle get to the center of the nucleus before coming momentarily to rest and reversing its motion? Neglect the recoil of the relatively massive nucleus.

Conceptualize: Throughout this process, the total mechanical energy E of the system of alpha particle and gold nucleus is conserved.

Classify: In particular, the system's initial mechanical energy E_i, before the particle and nucleus interact, is equal to its mechanical energy E_f when the alpha particle momentarily stops. The initial energy E_i is just the kinetic energy K_α of the incoming alpha particle. The final energy E_f is just the electric potential energy U of the system (the kinetic energy is then zero). We can find U with Eq. 24-5 ($U = q_1q_2/4\pi\varepsilon_0 r$).

Compute: Let d be the center-to-center distance between the alpha particle and the gold nucleus when the alpha particle is at its stopping point. Then we can write the conservation of energy $E_i = E_f$ as

$$K_\alpha = \frac{1}{4\pi\varepsilon_0}\frac{q_\alpha q_{Au}}{d},$$

in which q_α (= $2e$) is the charge of the alpha particle (2 protons) and q_{Au} (= $79e$) is the charge of the gold nucleus (79 protons).

Substituting for the charges and solving for d yield

$$d = \frac{(2e)(79e)}{4\pi\varepsilon_0 K_\alpha}$$

$$= \frac{(2\times 79)(1.60\times 10^{-19}\ \text{C})^2}{4\pi\varepsilon_0 (5.30\ \text{MeV})(1.60\times 10^{-13}\ \text{J/MeV})}$$

$$= 4.29\times 10^{-14}\ \text{m}. \qquad \text{(Answer)}$$

Conclude: This is a small distance by atomic standards but not by nuclear standards. It is, in fact, considerably larger than the sum of the radii of the gold nucleus and the alpha particle. Thus, this alpha particle reverses its motion without ever actually "touching" the gold nucleus.

41-3 | Some Nuclear Properties

When we are interested primarily in their properties as specific nuclear species (rather than as parts of atoms), we call these particles **nuclides**.

Some Nuclear Terminology

Nuclei are made up of protons and neutrons. The number of protons in a nucleus (called the **atomic number** or **proton number** of the nucleus) is represented by the symbol Z; the number of neutrons (the **neutron number**) is represented by the symbol N. The total number of neutrons and protons in a nucleus is called its **mass number** A; thus

$$A = Z + N. \tag{41-1}$$

Neutrons and protons, when considered collectively, are called **nucleons**.

We represent nuclides with symbols such as those displayed here. Consider ^{197}Au, for example. The superscript 197 is the mass number A. The chemical symbol Au tells us that this element is gold, whose atomic number is 79. From Eq. 41-1, the neutron number of this nuclide is 197 – 79, or 118.

Nuclides with the same atomic number Z but different neutron numbers N are called **isotopes** of one another. The element gold has 32 isotopes, ranging from ^{173}Au to ^{204}Au. Only one of them (^{197}Au) is stable; the remaining 31 are radioactive. Such **radionuclides** undergo **decay** (or **disintegration**) by emitting a particle and thereby transforming to a different nuclide. During such a nuclear reaction, the total number of nucleons before and after the reaction remains the same. Also the total charge of the system remains conserved by the law of conservation of charge. For example, when ^{198}Au undergoes beta decay, the nuclear reaction is

$$^{198}_{79}\text{Au} \rightarrow {}^{198}_{80}\text{Hg} + {}^{0}_{-1}\text{e} + \bar{\nu}.$$

Here on left-hand side, we can see that the nucleus contains 79 protons and 119 neutrons. On decay, the number of protons increases to 80, but the number of neutrons decreases to 118. The charge is conserved because an electron is emitted during the process. The $\bar{\nu}$ (antineutrino) is an uncharged particle.

Organizing the Nuclides

The neutral atoms of all isotopes of an element (all with the same Z) have the same number of electrons and the same chemical properties, and they fit into the same box in the periodic table of the elements. The *nuclear* properties of the isotopes of a given element, however, are very different from one isotope to another. Thus, the periodic table is of limited use to the nuclear physicist, the nuclear chemist, or the nuclear engineer.

We organize the nuclides on a **nuclidic chart** like that in Fig. 41-4, in which a nuclide is represented by plotting its proton number against its neutron number. The stable nuclides in this figure are represented by the green, the radionuclides by the beige. As you can see, the radionuclides tend to lie on either side of—and at the upper end of—a well-defined band of stable nuclides. Note too that light stable nuclides tend to lie close to the line $N = Z$, which means that they have about the same numbers of neutrons and protons. Heavier nuclides, however, tend to have many more neutrons than protons. As an example, we saw that ^{197}Au has 118 neutrons and only 79 protons, a *neutron excess* of 39.

Nuclidic charts are available as wall charts, in which each small box on the chart is filled with data about the nuclide it represents. Figure 41-5 shows a section of such a chart, centered on ^{197}Au. Relative abundances (usually, as found on Earth) are shown for stable nuclides, and half-lives (a measure of decay rate) are shown for radionuclides. The sloping line points out a line of **isobars**—nuclides of the same mass number, $A = 198$ in this case.

CHECKPOINT 1 What is the slope of the line in Fig. 41-5?

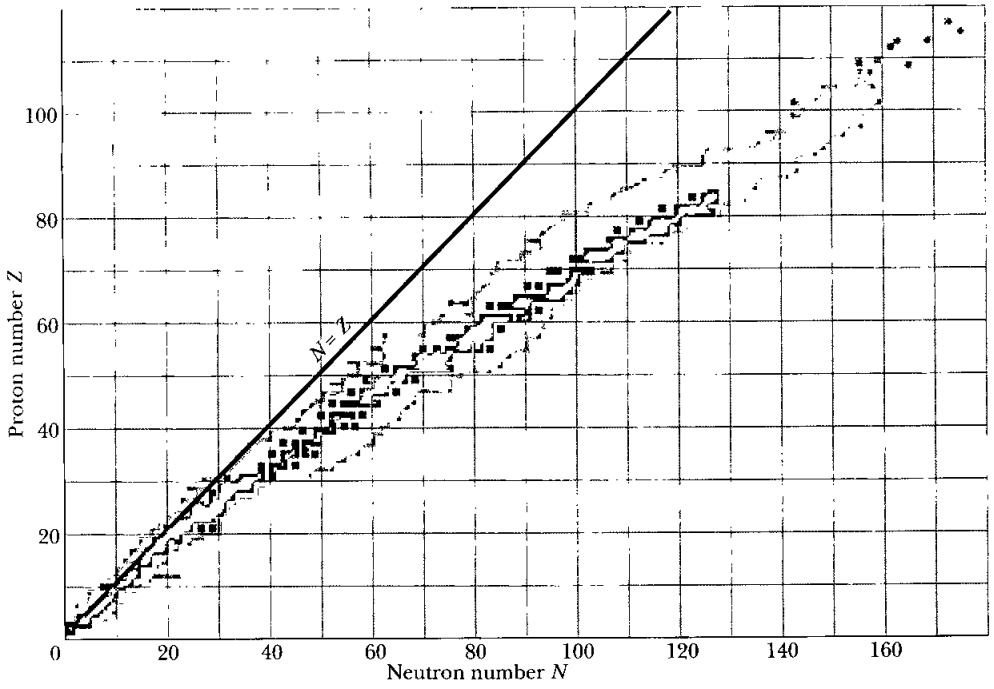


FIG. 41-4 A plot of the known nuclides. The green shading identifies the band of stable nuclides, the beige shading the radionuclides. Low-mass, stable nuclides have essentially equal numbers of neutrons and protons, but more massive nuclides have an increasing excess of neutrons. The figure shows that there are no stable nuclides with $Z > 83$ (bismuth).

As of early 2007, nuclides with atomic numbers as high as $Z = 118$ ($A = 294$) had been found in laboratory experiments (no elements with Z greater than 92 occur naturally). Although large nuclides generally should be highly unstable and last only a very brief time, certain supermassive nuclides are relatively stable, with fairly long lifetimes. These stable supermassive nuclides and other predicted ones form an *island of stability* at high values of Z and N on a nuclidic chart like Fig. 41-4.

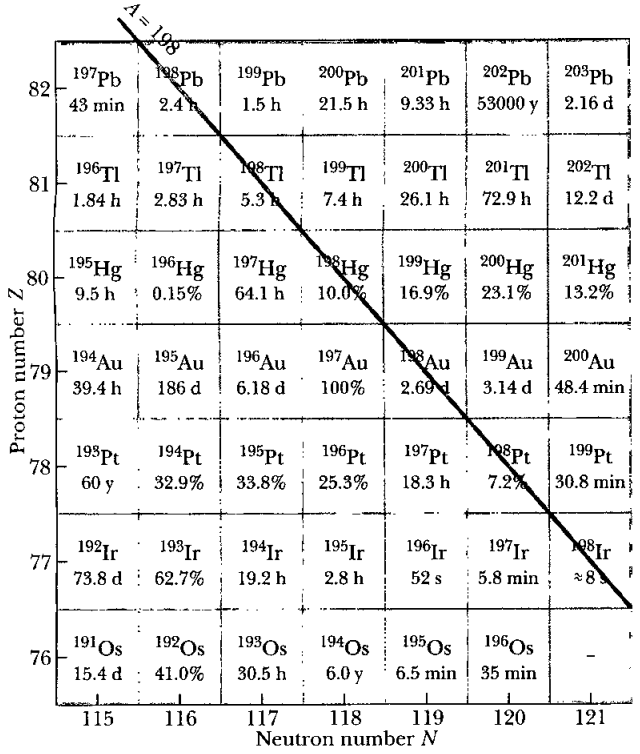


FIG. 41-5 An enlarged and detailed section of the nuclidic chart of Fig. 42-4, centered on ^{197}Au. Green squares represent stable nuclides, for which relative isotopic abundances are given. Beige squares represent radionuclides, for which half-lives are given. Isobaric lines of constant mass number A slope as shown by the example line for $A = 198$.

CHECKPOINT 2 Based on Fig. 41-4, which of the following nuclides do you conclude are not likely to be detected: ^{52}Fe ($Z = 26$), ^{90}As ($Z = 33$), ^{158}Nd ($Z = 60$), ^{175}Lu ($Z = 71$), ^{208}Pb ($Z = 82$)?

Nuclear Radii

A convenient unit for measuring distances on the scale of nuclei is the *femtometer* ($= 10^{-15}$ m). This unit is often called the *fermi;* the two names share the same abbreviation. Thus,

$$1 \text{ femtometer} = 1 \text{ fermi} = 1 \text{ fm} = 10^{-15} \text{ m}. \tag{41-2}$$

We can learn about the size and structure of nuclei by bombarding them with a beam of high-energy electrons and observing how the nuclei deflect the incident electrons. The electrons must be energetic enough (at least 200 MeV) to have de Broglie wavelengths that are smaller than the nuclear structures they are to probe.

The nucleus, like the atom, is not a solid object with a well-defined surface. Furthermore, although most nuclides are spherical, some are notably ellipsoidal. Nevertheless, electron-scattering experiments (as well as experiments of other kinds) allow us to assign to each nuclide an effective radius given by

$$r = r_0 A^{1/3}, \tag{41-3}$$

in which A is the mass number and $r_0 \approx 1.2$ fm. We see that the volume of a nucleus, which is proportional to r^3, is directly proportional to the mass number A and is independent of the separate values of Z and N. This means that the density of the nucleus is the same for every nuclide. Thus, we can picture the nucleus as made up of nucleons that are incompressible. It is like a ball of snow that children make. As more snow is added to the ball, the snow ball gets bigger, not denser. The shape of the nucleus will be the same as shown in Fig. 41-6.

In fact, we can use Eq. 41-3 to estimate the density of nuclear matter. Consider a nucleus with mass number A. Its mass within 1% is A atomic mass units. Thus,

$$\begin{aligned}
\rho_{\text{nuc}} &\approx \frac{A \text{ u}}{(4/3)\pi r^3} \\
&= \frac{A \text{ u}}{(4/3)\pi r_0^3 A} \\
&= \frac{1 \text{ u}}{(4/3)\pi r_0^3} \\
&= \frac{1.66 \times 10^{-27} \text{ kg}}{(4/3)\pi (1.2 \times 10^{-15} \text{ m})^3} \\
&= 2.3 \times 10^{17} \text{ kg/m}^3.
\end{aligned} \tag{41-4}$$

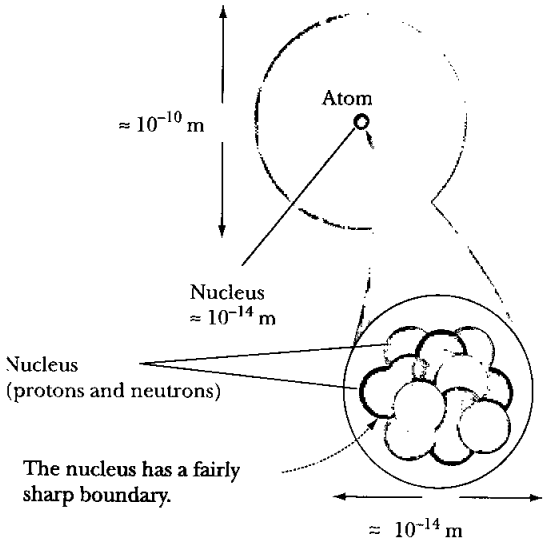


FIG. 41-6 The nucleus is a tiny sphere within an atom.

Atomic Masses

Atomic masses are now measured to great precision, but usually nuclear masses are not directly measurable because stripping off all the electrons from an atom is difficult. As we briefly discussed in Section 37-13, atomic masses are often reported in *atomic mass units,* a system in which the atomic mass of neutral ^{12}C is defined to be exactly 12 u.

Precise atomic masses are available in tables on the Web and are usually provided in homework problems. However, sometimes we need only an approximation of the mass of either a nucleus alone or a neutral atom. The mass number A of a nuclide gives such an approximate mass in atomic mass units. For example, the approximate mass of both the nucleus and the neutral atom for ^{197}Au is 197 u, which is close to the actual atomic mass of 196.966 552 u.

As we saw in Section 37-13,

$$1\text{ u} = 1.660\,538\,86 \times 10^{-27}\text{ kg}. \tag{41-5}$$

We also saw that if the total mass of the participants in a nuclear reaction changes by an amount Δm, there is an energy release or absorption given by Eq. 37-50 ($Q = -\Delta m\, c^2$). As we shall now see, nuclear energies are often reported in multiples of 1 MeV. Thus, a convenient conversion between mass units and energy units is provided by Eq. 37-46:

$$c^2 = 931.494\,013\text{ MeV/u}. \tag{41-6}$$

Scientists and engineers working with atomic masses often prefer to report the mass of an atom by means of the atom's *mass excess* Δ, defined as

$$\Delta = M - A \text{ (mass excess)}, \tag{41-7}$$

where M is the actual mass of the atom in atomic mass units and A is the mass number for that atom's nucleus.

Nuclear Binding Energies

The mass M of a nucleus is *less* than the total mass Σm of its individual protons and neutrons. That means that the mass energy Mc^2 of a nucleus is *less* than the total mass energy $\Sigma(mc^2)$ of its individual protons and neutrons. The difference between these two energies is called the **binding energy** of the nucleus:

$$\Delta E_{be} = \Sigma(mc^2) - Mc^2 \text{ (binding energy)}. \tag{41-8}$$

Caution: Binding energy is not an energy that resides in the nucleus. Rather, it is a *difference* in mass energy between a nucleus and its individual nucleons: If we were able to separate a nucleus into its nucleons, we would have to transfer a total energy equal to ΔE_{be} to those particles during the separating process. Although we cannot actually tear apart a nucleus in this way, the nuclear binding energy is still a convenient measure of how well a nucleus is held together.

A better measure is the **binding energy per nucleon** ΔE_{ben}, which is the ratio of the binding energy ΔE_{be} of a nucleus to the number A of nucleons in that nucleus:

$$\Delta E_{ben} = \frac{\Delta E_{be}}{A} \text{ (binding energy per nucleon)}. \tag{41-9}$$

We can think of the binding energy per nucleon as the average energy needed to separate a nucleus into its individual nucleons.

Figure 41-7 is a plot of the binding energy per nucleon ΔE_{ben} versus mass number A for a large number of nuclei. Those high on the plot are very tightly bound; that is, we would have to supply a great amount of energy per nucleon to break apart one of those nuclei. The nuclei that are lower on the plot, at the left and right sides, are less tightly bound, and less energy per nucleon would be required to break them apart.

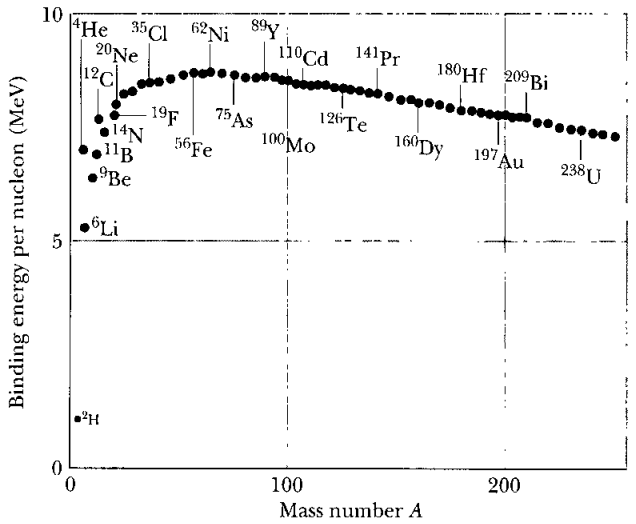


FIG. 41-7 The binding energy per nucleon for some representative nuclides. The nickel nuclide ^{62}Ni has the highest binding energy per nucleon (about 8.794 60 MeV/nucleon) of any known stable nuclide. Note that the alpha particle (^{4}He) has a higher binding energy per nucleon than its neighbors in the periodic table and thus is also particularly stable.

These simple statements about Fig. 41-7 have profound consequences. The nucleons in a nucleus on the right side of the plot would be more tightly bound if that nucleus were to split into two nuclei that lie near the top of the plot. Such a process, called **fission**, occurs naturally with large (high mass number A) nuclei such as uranium, which can undergo fission spontaneously (that is, without an external cause or source of energy). The process can also occur in nuclear weapons, in which many uranium or plutonium nuclei are made to fission all at once, to create an explosion.

The nucleons in any pair of nuclei on the left side of the plot would be more tightly bound if the pair were to combine to form a single nucleus that lies near the top of the plot. Such a process, called **fusion**, occurs naturally in stars. Were this not true, the Sun would not shine and thus life could not exist on Earth.

Nuclear Energy Levels

The energy of nuclei, like that of atoms, is quantized. That is, nuclei can exist only in discrete quantum states, each with a well-defined energy. Figure 41-8 shows some of these energy levels for ^{28}Al, a typical low-mass nuclide. Note that the energy scale is in millions of electron-volts, rather than the electron-volts used for atoms. When a nucleus makes a transition from one level to a level of lower energy, the emitted photon is typically in the gamma-ray region of the electromagnetic spectrum.

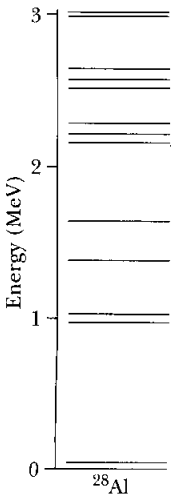


FIG. 41-8 Energy levels for the nuclide ^{28}Al, deduced from nuclear reaction experiments.

Nuclear Spin and Magnetism

Many nuclides have an intrinsic *nuclear angular momentum,* or spin, and an associated intrinsic *nuclear magnetic moment.* Although nuclear angular momenta are roughly of the same magnitude as the angular momenta of atomic electrons, nuclear magnetic moments are much smaller than typical atomic magnetic moments.

The Nuclear Force

The force that controls the motions of atomic electrons is the familiar electromagnetic force. To bind the nucleus together, however, there must be a strong attractive nuclear force of a totally different kind, strong enough to overcome the repulsive force between the (positively charged) nuclear protons and to bind both

protons and neutrons into the tiny nuclear volume. The nuclear force must also be of short range because its influence does not extend very far beyond the nuclear "surface." Its range is of the order of 2 fm.

The present view is that the nuclear force that binds neutrons and protons in the nucleus is not a fundamental force of nature but is a secondary, or "spillover," effect of the **strong force** that binds quarks together to form neutrons and protons. In much the same way, the attractive force between certain neutral molecules is a spillover effect of the Coulomb electric force that acts within each molecule to bind it together. This strong force is independent of the charge. This means that the strong force of proton–proton, neutron–neutron, proton–neutron interactions is the same, apart from the additional repulsive Coulomb force for the proton–proton interaction. It is customary to talk of the potential energy when we talk of nuclear forces. Here, the potential energy of interaction of a proton and a neutron is shown in Fig. 41-9*a*.

But when we see the potential energy of interaction between a proton and another proton (Fig. 41-9*b*), the graph looks a little different. This includes the effect of Coulomb repulsion as well. We can also say from the graph that the strong nuclear force is independent of the charge as the potential well has the same shape for both the figures.

For a very short distance, less than 0.4 fm, the nuclear force becomes repulsive. If it were not so, then all the nuclei would collapse to zero size.

This strong nuclear force does not affect the electrons. So the electrons can be used as a projectile to probe the inner details of the nucleus. On the basis of these experiments, we have been able to conclude that its magnitude depends on the relative spin orientation of the nucleons. This will help us to understand Fig. 41-4. The green points refer to stable nuclides (about 270). In heavy stable nuclei, $N > Z$. As the number of protons increases, the strength of the Coulomb force increases which tends to break apart the nucleus. As a result, Mother Nature adds more neutrons to keep the nucleus stable because the neutrons experience only attractive force. At $Z = 83$ and beyond, the addition of more neutrons cannot compensate the repulsive force between the protons. So none of the nuclides with $Z > 83$ are stable.

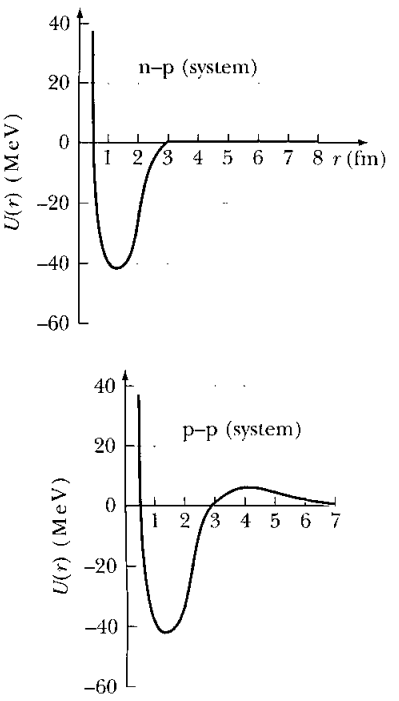


FIG. 41-9 (*a*) Potential energy versus separation for neutron–proton system. (*b*) Potential energy versus separation for proton–proton system. The difference is two curves is due to the Coulomb repulsion between protons.

Sample Problem 41-2

What is the binding energy per nucleon for ^{120}Sn?

Conceptualize/Classify:

1. We can find the binding energy per nucleon ΔE_{ben} if we first find the binding energy ΔE_{be} and then divide by the number of nucleons A in the nucleus, according to Eq. 41-9 ($\Delta E_{ben} = \Delta E_{be}/A$).
2. We can find ΔE_{be} by finding the difference between the mass energy Mc^2 of the nucleus and the total mass energy $\Sigma(mc^2)$ of the individual nucleons that make up the nucleus, according to Eq. 41-8 ($\Delta E_{be} = \Sigma(mc^2) - Mc^2$).

Compute: From Appendix F, we see that a ^{120}Sn nucleus consists of 50 protons ($Z = 50$) and 70 neutrons ($N = A - Z = 120 - 50 = 70$). Thus, we need to imagine a ^{120}Sn nucleus being separated into its 50 protons and 70 neutrons,

$$\left(^{120}\text{Sn nucleus}\right) \rightarrow 50\begin{pmatrix}\text{separate}\\ \text{protons}\end{pmatrix} + 70\begin{pmatrix}\text{separate}\\ \text{neutrons}\end{pmatrix}, \quad (41\text{-}10)$$

and then compute the resulting change in mass energy.

For that computation, we need the masses of a ^{120}Sn nucleus, a proton, and a neutron. However, because the mass of a neutral atom (nucleus *plus* electrons) is much easier to measure than the mass of a bare nucleus, calculations of binding energies are traditionally done with atomic masses. Thus, let's modify Eq. 41-10 so that it has a neutral ^{120}Sn atom on the left side. To do that, we include 50 electrons on the left side (to match the 50 protons in the ^{120}Sn nucleus). We must also add 50 electrons on the right side to balance Eq. 41-10. Those 50 electrons can be combined with the 50 protons, to form 50 neutral hydrogen atoms. We then have

$$\left(^{120}\text{Sn atom}\right) \rightarrow 50\begin{pmatrix}\text{separate}\\ \text{H atoms}\end{pmatrix} + 70\begin{pmatrix}\text{separate}\\ \text{neutrons}\end{pmatrix}. \quad (41\text{-}11)$$

In Appendix F the mass M_{Sn} of a ^{120}Sn atom is 119.902 197 u and the mass m_H of a hydrogen atom is 1.007 825 u; the mass m_n of a neutron is 1.008 665 u. Thus, Eq. 41-8 yields

$$\begin{aligned}\Delta E_{be} &= \sum(mc^2) - Mc^2 \\ &= 50(m_H c^2) + 70(m_n c^2) - M_{sn}c^2 \\ &= 50(1.007\,825\text{ u})c^2 + 70(1.008\,665\text{ u})c^2 \\ &\quad -(119.902\,197\text{ u})c^2 \\ &= (1.095\,603\text{ u})c^2 \\ &= (1.095\,603\text{ u})(931.494\,013\text{ MeV/u}) \\ &= 1020.5\text{ MeV},\end{aligned}$$

where Eq. 41-6 ($c^2 = 931.494\,013$ MeV/u) provides an easy unit conversion. Note that using atomic masses instead of nuclear masses does not affect the result because the mass of the 50 electrons in the ^{120}Sn atom subtracts out from the mass of the electrons in the 50 hydrogen atoms.

Now Eq. 41-9 gives us the binding energy per nucleon as

$$\begin{aligned}\Delta E_{ben} &= \frac{\Delta E_{be}}{A} = \frac{1020.5\text{ MeV}}{120} \\ &= 8.50\text{ MeV/nucleon.} \qquad \text{(Answer)}\end{aligned}$$

41-4 | Radioactive Decay

As Fig. 41-4 shows, most of the nuclides that have been identified are radioactive. A radioactive nuclide spontaneously emits a particle, transforming itself in the process into a different nuclide, occupying a different square on the nuclidic chart.

Radioactive decay provided the first evidence that the laws that govern the subatomic world are statistical. Consider, for example, a 1 mg sample of uranium metal. It contains 2.5×10^{18} atoms of the very long-lived radionuclide ^{238}U. The nuclei of these particular atoms have existed without decaying since they were created—well before the formation of our solar system. During any given second, only about 12 of the nuclei in our sample will happen to decay by emitting an alpha particle, transforming themselves into nuclei of ^{234}Th. However,

There is absolutely no way to predict whether any given nucleus in a radioactive sample will be among the small number of nuclei that decay during the next second. All have the same chance.

Although we cannot predict which nuclei in a sample will decay, we can say that if a sample contains N radioactive nuclei, then the rate ($= -dN/dt$) at which nuclei will decay is proportional to N:

$$-\frac{dN}{dt} = \lambda N, \tag{41-12}$$

in which λ, the **disintegration constant** (or **decay constant**) has a characteristic value for every radionuclide. Its SI unit is the inverse second (s^{-1}).

To find N as a function of time t, we first rearrange Eq. 41-12 as

$$\frac{dN}{N} = -\lambda\, dt, \tag{41-13}$$

and then integrate both sides, obtaining

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_{t_0}^{t} dt,$$

or

$$\ln N - \ln N_0 = -\lambda(t - t_0). \tag{41-14}$$

Here N_0 is the number of radioactive nuclei in the sample at some arbitrary initial time t_0. Setting $t_0 = 0$ and rearranging Eq. 41-14 give us

$$\ln \frac{N}{N_0} = -\lambda t. \tag{41-15}$$

Taking the exponential of both sides (the exponential function is the antifunction of the natural logarithm) leads to

$$\frac{N}{N_0} = e^{-\lambda t}$$

or
$$N = N_0 e^{-\lambda t} \text{ (radioactive decay)}, \tag{41-16}$$

in which N_0 is the number of radioactive nuclei in the sample at $t = 0$ and N is the number remaining at any subsequent time t. Note that lightbulbs (for one example) follow no such exponential decay law. If we life-test 1000 bulbs, we expect that they will all "decay" (that is, burn out) at more or less the same time. The decay of radionuclides follows quite a different law.

We are often more interested in the decay rate R ($= -dN/dt$) than in N itself. Differentiating Eq. 41-16, we find

$$R = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t}$$

or
$$R = R_0 e^{-\lambda t} \text{ (radioactive decay)}, \tag{41-17}$$

an alternative form of the law of radioactive decay (Eq. 41-16). Here R_0 is the decay rate at time $t = 0$ and R is the rate at any subsequent time t. We can now rewrite Eq. 41-12 in terms of the decay rate R of the sample as

$$R = \lambda N, \tag{41-18}$$

where R and the number of radioactive nuclei N that have not yet undergone decay must be evaluated at the same instant.

The total decay rate R of a sample of one or more radionuclides is called the **activity** of that sample. The SI unit for activity is the **becquerel**, named for Henri Becquerel, the discoverer of radioactivity:

$$1 \text{ becquerel} = 1 \text{ Bq} = 1 \text{ decay per second}.$$

An older unit, the **curie**, is still in common use:

$$1 \text{ curie} = 1 \text{ Ci} = 3.7 \times 10^{10} \text{ Bq}.$$

Here is an example using these units: "The activity of spent reactor fuel rod #5658 on January 15, 2004, was 3.5×10^{15} Bq ($= 9.5 \times 10^4$ Ci)." Thus, on that day 3.5×10^{15} radioactive nuclei in the rod decayed each second. The identities of the radionuclides in the fuel rod, their disintegration constants λ, and the types of radiation they emit have no bearing on this measure of activity.

There are two common time measures of how long any given type of radionuclides lasts. One measure is the **half-life** $T_{1/2}$ of a radionuclide, which is the time at which both N and R have been reduced to one-half their initial values. The other measure is the **mean life** τ, which is the time at which both N and R have been reduced to e^{-1} of their initial values.

To relate $T_{1/2}$ to the disintegration constant λ, we put $R = \frac{1}{2}R_0$ in Eq. 41-17 and substitute $T_{1/2}$ for t. We obtain

$$\tfrac{1}{2}R_0 = R_0 e^{-\lambda T_{1/2}}.$$

Taking the natural logarithm of both sides and solving for $T_{1/2}$, we find

$$T_{1/2} = \frac{\ln 2}{\lambda}.$$

Similarly, to relate τ to λ, we put $R = e^{-1}R_0$ in Eq. 41-17, substitute τ for t, and solve for τ, finding

$$\tau = \frac{1}{\lambda}.$$

We summarize these results with the following:

$$T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2. \tag{41-19}$$

CHECKPOINT 3 The nuclide ^{131}I is radioactive, with a half-life of 8.04 days. At noon on January 1, the activity of a certain sample is 600 Bq. Using the concept of half-life, without written calculation, determine whether the activity at noon on January 24 will be a little less than 200 Bq, a little more than 200 Bq, a little less than 75 Bq, or a little more than 75 Bq.

41-5 | Probability of Decay

As the decay process is governed by statistics, it is appropriate to talk of probability of a decay or survival of a nucleus over here. Thus, if we take a sample of a radioactive element, we can say that its probability of decay in time dt is

$$\frac{dN}{N} = -\lambda dt.$$

Over a large interval of time say t, the probability of survival will be

$$\frac{N}{N_0} = e^{-\lambda t}.$$

Thus, the probability of the decay of a given nuclei will be

$$\frac{N_0 - N}{N_0} = 1 - e^{-\lambda t}.$$

Sample Problem 41-3

Suppose that we have a large sample of radioactive nuclei. Out of that sample, we pick up four nuclei. What is the probability that exactly two nuclei would have decayed after one half-life?

Conceptualize/Classify: Here, we should remember the statement that after one half-life, half of the nuclei decay is just an estimate. In reality, the decay of each nuclei is an independent process and we have to consider the probability that half of the nuclei decay during that time period and half of the nuclei survive. *Note the word exactly.*

Compute: Out of the four nuclei available, we can choose any two nuclei that are to decay in

$$^4_2C = \frac{4!}{2! \times 2!} = 6 \text{ ways.}$$

The probability of survival of each nuclei in one half-life is

$$e^{-(\ln 2/T)\times T} = \tfrac{1}{2}.$$

Similarly, the probability of decay of each nuclei in one half-life is also

$$1 - e^{-\lambda t} = \tfrac{1}{2}.$$

So the probability required would be

$$^4_2C(e^{-\lambda t})^2 \times (1 - e^{-\lambda t})^2 = \tfrac{3}{8}.$$

Conclude: To examine it more deeply, we can say that in any time interval t, there can be two outcomes. Either the nuclei can decay or it can survive. So the sum of the probability of survival and the probability of decay should be 1. In general, if n nuclei are being considered,

$$(P_s + P_d)^n = 1,$$

where P_s represents the probability of survival and P_d represents the probability of decay.

The different terms in this binomial expression represent the probability of different outcomes. Thus, the probability that all the nuclei will decay and none would survive will be

$$^n_0C(P_s)^0(P_d)^n,$$

and so on.

Sample Problem 41-4

The table that follows shows some measurements of the decay rate of a sample of ^{128}I, a radionuclide often used medically as a tracer to measure the rate at which iodine is absorbed by the thyroid gland.

Time (min)	R (counts/s)	Time (min)	R (counts/s)
4	392.2	132	10.9
36	161.4	164	4.56
68	65.5	196	1.86
100	26.8	218	1.00

Find the disintegration constant λ and the half-life $T_{1/2}$ for this radionuclide.

Conceptualize/Classify: The disintegration constant λ determines the exponential rate at which the decay rate R decreases with time t (as indicated by Eq. 41-17, $R = R_0 e^{-\lambda t}$). Therefore, we should be able to determine λ by plotting the measurements of R against the measurement times t. However, obtaining λ from a plot of R versus t is difficult because R decreases exponentially with t, according to Eq. 41-17. A neat solution is to transform Eq. 41-17 into a linear function of t, so that we can easily find λ. To do so, we take the natural logarithms of both sides of Eq. 41-17.

Compute: We obtain

$$\ln R = \ln(R_0 e^{-\lambda t}) = \ln R_0 + \ln(e^{-\lambda t}) = \ln R_0 - \lambda t. \quad (41\text{-}20)$$

Because Eq. 41-20 is of the form $y = b + mx$, with b and m constants, it is a linear equation giving the quantity $\ln R$ as a function of t. Thus, if we plot $\ln R$ (instead of R) versus t, we should get a straight line. Further, the slope of the line should be equal to $-\lambda$.

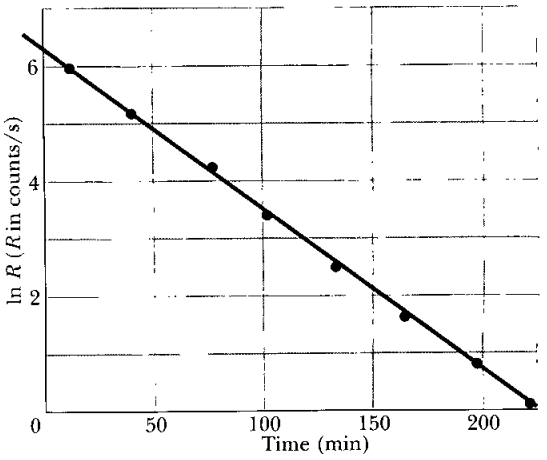


FIG. 41-10 A semilogarithmic plot of the decay of a sample of ^{128}I, based on the data in the table.

Figure 41-10 shows a plot of $\ln R$ versus time t for the given measurements. The slope of the straight line that fits through the plotted points is

$$\text{slope} = \frac{0 - 6.2}{225\ \text{min} - 0} = -0.0276\ \text{min}^{-1}.$$

Thus, $$-\lambda = -0.0276\ \text{min}^{-1}$$

or $$\lambda = 0.0276\ \text{min}^{-1} \approx 1.7\ \text{h}^{-1}. \quad \text{(Answer)}$$

The time for the decay rate R to decrease by 1/2 is related to the disintegration constant λ via Eq. 41-19 ($T_{1/2} = (\ln 2)/\lambda$). From that equation, we find

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{\ln 2}{0.0276\ \text{min}^{-1}} \approx 25\,\text{min} \quad \text{(Answer)}$$

Sample Problem 41-5

A 2.71 g sample of KCl from the chemistry stockroom is found to be radioactive, and it is decaying at a constant rate of 4490 Bq. The decays are traced to the element potassium and in particular to the isotope ^{40}K, which constitutes 1.17% of normal potassium. Calculate the half-life of this nuclide.

Conceptualize/Classify:

1. Because the activity R of the sample is apparently constant, we cannot find the half-life $T_{1/2}$ by plotting $\ln R$ versus time t as we did in Sample Problem 42-4. (We would just get a horizontal plot.) However, we can use the following ideas.
2. We can relate the half-life $T_{1/2}$ to the disintegration constant λ via Eq. 41-19 ($T_{1/2} = (\ln 2)/\lambda$).
3. We can then relate λ to the given activity R of 4490 Bq by means of Eq. 41-18 ($R = \lambda N$), where N is the number of ^{40}K nuclei (and thus atoms) in the sample.

Compute: Combining Eqs. 41-19 and 41-18 yields

$$T_{1/2} = \frac{N \ln 2}{R}. \quad (41\text{-}21)$$

We know that N in this equation is 1.17% of the total number N_K of potassium atoms in the sample. We also know that N_K must equal the number N_{KCl} of molecules in the sample. We can obtain N_{KCl} from the molar mass M_{KCl} of KCl (the mass of one mole of KCl) and the given mass M_{sam} of the sample by combining $n = N/N_A$ and $n = M_{sam}/M$ to write

$$N_{KCl} = \begin{pmatrix}\text{number of moles}\\ \text{in sample}\end{pmatrix} N_A = \frac{M_{sam}}{M_{KCl}} N_A, \qquad (41\text{-}22)$$

where N_A is Avogadro's number ($6.02 \times 10^{23}\ \text{mol}^{-1}$). From Appendix F, we see that the molar mass of potassium is 39.102 g/mol and the molar mass of chlorine is 35.453 g/mol; thus, the molar mass of KCl is 74.555 g/mol. Equation 41-22 then gives us

$$N_{KCl} = \frac{(2.71\ \text{g})(6.02 \times 10^{23}\ \text{mol}^{-1})}{74.555\ \text{g/mol}} = 2.188 \times 10^{22}$$

as the number of KCl molecules in the sample. Thus, the total number N_K of potassium atoms is also 2.188×10^{22}, and the number of ^{40}K in the sample must be

$$N = 0.0117 N_K = (0.0117)(2.188 \times 10^{22}) = 2.560 \times 10^{20}.$$

Substituting this value for N and the given activity of 4490 Bq ($= 4490\ \text{s}^{-1}$) for R into Eq. 41-21 leads to

$$T_{1/2} = \frac{(2.560 \times 10^{20}) \ln 2}{4490\ \text{s}^{-1}} = 3.95 \times 10^{16}\ \text{s} = 1.25 \times 10^{9}\ \text{y}. \qquad \text{(Answer)}$$

Conclude: This half-life of ^{40}K turns out to have the same order of magnitude as the age of the universe. Thus, the activity of ^{40}K in the stockroom sample decreases *very* slowly, too slowly for us to detect during a few days of observation or even an entire lifetime. A portion of the potassium in our bodies consists of this radioisotope, which means that we are all slightly radioactive.

41-6 | Alpha Decay

When a nucleus undergoes **alpha decay**, it transforms to a different nuclide by emitting an alpha particle (a helium nucleus, ^{4}He). For example, when uranium ^{238}U undergoes alpha decay, it transforms to thorium ^{234}Th:

$$^{238}\text{U} \rightarrow {}^{234}\text{Th} + {}^{4}\text{He}. \qquad (41\text{-}23)$$

This alpha decay of ^{238}U can occur spontaneously (without an external source of energy) because the total mass of the decay products ^{234}Th and ^{4}He is less than the mass of the original ^{238}U. Thus, the total mass energy of the decay products is less than the mass energy of the original nuclide. As defined by Eq. 37-50 ($Q = -\Delta M c^2$), in such a process the difference between the initial mass energy and the total final mass energy is called the Q of the process.

For a nuclear decay, we say that the difference in mass energy is the decay's *disintegration energy* Q. The Q for the decay in Eq. 41-23 is 4.25 MeV—that amount of energy is said to be released by the alpha decay of ^{238}U, with the energy transferred from mass energy to the kinetic energy of the two products.

The half-life of ^{238}U for this decay process is 4.5×10^9 y. Why so long? If ^{238}U can decay in this way, why doesn't every ^{238}U nuclide in a sample of ^{238}U atoms simply decay at once? To answer the questions, we must examine the process of alpha decay.

We choose a model in which the alpha particle is imagined to exist (already formed) inside the nucleus before it escapes from the nucleus. Figure 41-11 shows

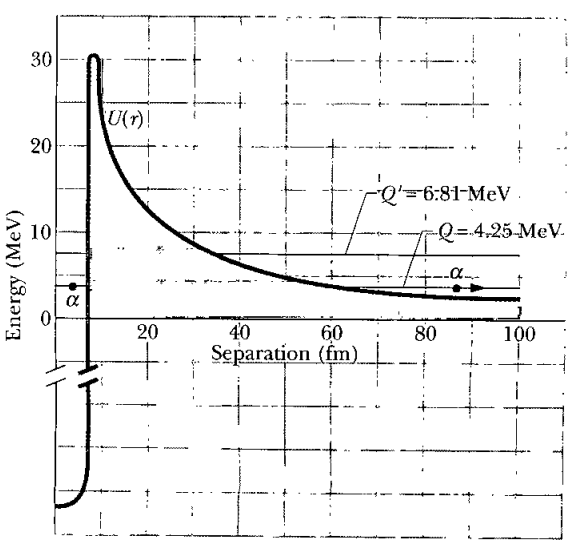


FIG. 41-11 A potential energy function for the emission of an alpha particle by ^{238}U. The horizontal black line marked $Q = 4.25$ MeV shows the disintegration energy for the process. The thick gray portion of this line represents separations r that are classically forbidden to the alpha particle. The alpha particle is represented by a dot, both inside this potential energy barrier (at the left) and outside it (at the right), after the particle has tunneled through. The horizontal black line marked $Q' = 6.81$ MeV shows the disintegration energy for the alpha decay of ^{228}U. (Both isotopes have the same potential energy function because they have the same nuclear charge.)

TABLE 41-1

Two Alpha Emitters Compared

Radionuclide	Q	Half-Life
^{238}U	4.25 MeV	4.5×10^9 y
^{228}U	6.81 MeV	9.1 min

the approximate potential energy $U(r)$ of the system consisting of the alpha particle and the residual ^{234}Th nucleus, as a function of their separation r. This energy is a combination of (1) the potential energy associated with the (attractive) strong nuclear force that acts in the nuclear interior and (2) a Coulomb potential associated with the (repulsive) electric force that acts between the two particles before and after the decay has occurred.

The horizontal black line marked $Q = 4.25$ MeV shows the disintegration energy for the process. If we assume that this represents the total energy of the alpha particle during the decay process, then the part of the $U(r)$ curve above this line constitutes a potential energy barrier like that in Fig. 38-11. This barrier cannot be surmounted. If the alpha particle were able to be at some separation r within the barrier, its potential energy U would exceed its total energy E. This would mean, classically, that its kinetic energy K (which equals $E - U$) would be negative, an impossible situation.

We can see now why the alpha particle is not immediately emitted from the ^{238}U nucleus. That nucleus is surrounded by an impressive potential barrier, occupying—if you think of it in three dimensions—the volume lying between two spherical shells (of radii about 8 and 60 fm). This argument is so convincing that we now change our last question and ask: Since the particle seems permanently trapped inside the nucleus by the barrier, how can the ^{238}U nucleus *ever* emit an alpha particle? The answer is that there is a finite probability that a particle can tunnel through an energy barrier that is classically insurmountable. In fact, alpha decay occurs as a result of barrier tunneling.

The very long half-life of ^{238}U tells us that the barrier is apparently not very "leaky." The alpha particle, presumed to be rattling back and forth within the nucleus, must arrive at the inner surface of the barrier about 10^{38} times before it succeeds in tunneling through the barrier. This is about 10^{21} times per second for about 4×10^9 years (the age of Earth)! We, of course, are waiting on the outside, able to count only the alpha particles that *do* manage to escape.

We can test this explanation of alpha decay by examining other alpha emitters. For an extreme contrast, consider the alpha decay of another uranium isotope, ^{228}U, which has a disintegration energy Q' of 6.81 MeV, about 60% higher than that of ^{238}U. (The value of Q' is also shown as a horizontal black line in Fig. 41-11.) The transmission coefficient of a barrier is very sensitive to small changes in the total energy of the particle seeking to penetrate it. Thus, we expect alpha decay to occur more readily for this nuclide than for ^{238}U. Indeed it does. As Table 41-1 shows, its half-life is only 9.1 min! An increase in Q by a factor of only 1.6 produces a decrease in half-life (that is, in the effectiveness of the barrier) by a factor of 3×10^{14}. This is sensitivity indeed.

Sample Problem 41-6

We are given the following atomic masses:

^{238}U 238.050 79 u ^{4}He 4.002 60 u

^{234}Th 234.043 63 u ^{1}H 1.007 83 u

^{237}Pa 237.051 21 u

Here Pa is the symbol for the element protactinium ($Z = 91$).

(a) Calculate the energy released during the alpha decay of ^{238}U. The decay process is

$$^{238}U \rightarrow {}^{234}TH + {}^{4}He.$$

Note, incidentally, how nuclear charge is conserved in this equation: The atomic numbers of thorium (90) and helium (2) add up to the atomic number of

uranium (92). The number of nucleons is also conserved: 238 = 234 + 4.

Conceptualize/Classify: The energy released in the decay is the disintegration energy Q, which we can calculate from the change in mass ΔM due to the ^{238}U decay.

Compute: To do this, we use Eq. 37-50,

$$Q = M_i c^2 - M_f c^2, \tag{41-24}$$

where the initial mass M_i is that of ^{238}U and the final mass M_f is the sum of the ^{234}Th and ^{4}He masses. As in Sample Problem 41-2, we must do this calculation for neutral atoms—that is, with atomic masses. Using the atomic masses given in the problem statement, Eq. 41-24 becomes

$$\begin{aligned} Q &= (238.050\ 79\ \text{u})c^2 - (234.043\ 63\ \text{u} + 4.002\ 60\ \text{u})c^2 \\ &= (0.004\ 56\ \text{u})c^2 = (0.004\ 56\ \text{u})(931.494\ 013\ \text{MeV/u}) \\ &= 4.25\ \text{MeV}. \qquad \text{(Answer)} \end{aligned}$$

Note that using atomic masses instead of nuclear masses does not affect the result because the total mass of the electrons in the products subtracts out from the mass of the nucleons + electrons in the original ^{238}U.

(b) Show that ^{238}U cannot spontaneously emit a proton; that is, protons do not leak out of the nucleus in spite of the proton–proton repulsion within the nucleus.

Compute: If this happened, the decay process would be

$$^{238}\text{U} \rightarrow {}^{237}\text{Pa} + {}^{1}\text{H}.$$

(You should verify that both nuclear charge and the number of nucleons are conserved in this process.) Using the same Key Idea as in part (a) and proceeding as we did there, we would find that the mass of the two decay products (= 237.051 21 u + 1.007 83 u) would *exceed* the mass of ^{238}U by $\Delta m = 0.008\ 25$ u, with disintegration energy $Q = -7.68$ MeV. The minus sign indicates that we must *add* 7.68 MeV to a ^{238}U nucleus before it will emit a proton; it will certainly not do so spontaneously.

41-7 | Beta Decay

A nucleus that decays spontaneously by emitting an electron or a positron (a positively charged particle with the mass of an electron) is said to undergo **beta decay**. Like alpha decay, this is a spontaneous process, with a definite disintegration energy and half-life. Again like alpha decay, beta decay is a statistical process, governed by Eqs. 41-16 and 41-17. In *beta-minus* (β^-) decay, an electron is emitted by a nucleus, as in the decay

$$^{32}\text{P} \rightarrow {}^{32}\text{S} + \text{e}^- + \bar{\nu} \quad (T_{1/2} = 14.3\ \text{d}). \tag{41-25}$$

In *beta-plus* (β^+) decay, a positron is emitted by a nucleus, as in the decay

$$^{64}\text{Cu} \rightarrow {}^{64}\text{Ni} + \text{e}^+ + \nu \quad (T_{1/2} = 12.7\ \text{h}). \tag{41-26}$$

The symbol ν represents a **neutrino**, a neutral particle which has a very small mass or zero mass (it is not sure as to what its mass is), that is emitted from the nucleus along with the electron or positron during the decay process. During beta-minus decay, the particle that is emitted is called antineutrino. Neutrinos interact only very weakly with matter and—for that reason—are so extremely difficult to detect that their presence long went unnoticed. A process which competes with beta-plus decay is electron capture or K-shell capture. It occurs when a parent nucleus captures one of the electron belonging to K-shell. One example is capture of an electron by copper, as in the decay

$$^{64}_{29}\text{Cu} + {}^{0}_{-1}e \rightarrow {}^{64}_{28}\text{Ni} + \nu.$$

K-shell capture is accompanied by the emission of K-series x rays thereafter.

Both charge and nucleon number are conserved in the above three processes. In the decay of Eq. 41-25, for example, we can write for charge conservation

$$(+15e) = (+16e) + (-e) + (0),$$

because ^{32}P has 15 protons, ^{32}S has 16 protons, and the neutrino ν has zero charge. Similarly, for nucleon conservation, we can write

$$(32) = (32) + (0) + (0),$$

because ^{32}P and ^{32}S each have 32 nucleons and neither the electron nor the neutrino is a nucleon.

It may seem surprising that nuclei can emit electrons, positrons, and neutrinos, since we have said that nuclei are made up of neutrons and protons only. However, we saw earlier that atoms emit photons, and we certainly do not say that atoms "contain" photons. We say that the photons are created during the emission process.

It is the same with the electrons, positrons, and neutrinos emitted from nuclei during beta decay. They are created during the emission process. For beta-minus decay, a neutron transforms into a proton within the nucleus according to

$$n \rightarrow p + e^- + \bar{\nu}. \qquad (41\text{-}27)$$

For beta-plus decay, a proton transforms into a neutron via

$$p \rightarrow n + e^+ + \nu. \qquad (41\text{-}28)$$

Both of these beta-decay processes provide evidence that—as was pointed out—neutrons and protons are not truly fundamental particles. These processes show why the mass number A of a nuclide undergoing beta decay does not change; one of its constituent nucleons simply changes its character according to Eq. 41-27 or 41-28.

In both alpha decay and beta decay, the same amount of energy is released in every individual decay of a particular radionuclide. In the alpha decay of a particular radionuclide, every emitted alpha particle has the same sharply defined kinetic energy. However, in the beta-minus decay of Eq. 41-27 with electron emission, the disintegration energy Q is shared—in varying proportions—between the emitted electron and neutrino. Sometimes the electron gets nearly all the energy, sometimes the neutrino does. In every case, however, the sum of the electron's energy and the neutrino's energy gives the same value Q. A similar sharing of energy, with a sum equal to Q, occurs in beta-plus decay (Eq. 41-28).

Thus, in beta decay the energy of the emitted electrons or positrons may range from zero up to a certain maximum K_{max}. Figure 41-12 shows the distribution of positron energies for the beta decay of ^{64}Cu (see Eq. 41-26). The maximum positron energy K_{max} must equal the disintegration energy Q because the neutrino has approximately zero energy when the positron has K_{max}:

$$Q = K_{max}. \qquad (42\text{-}29)$$

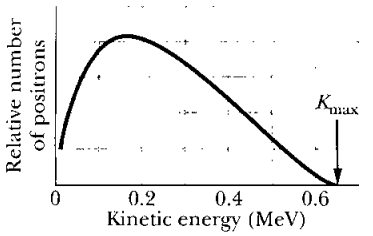


FIG. 41-12 The distribution of the kinetic energies of positrons emitted in the beta decay of ^{64}Cu. The maximum kinetic energy of the distribution (K_{max}) is 0.653 MeV. In all ^{64}Cu decay events, this energy is shared between the positron and the neutrino, in varying proportions. The *most probable* energy for an emitted positron is about 0.15 MeV.

The Neutrino

Wolfgang Pauli first suggested the existence of neutrinos in 1930. His neutrino hypothesis not only permitted an understanding of the energy distribution of electrons or positrons in beta decay but also solved another early beta-decay puzzle involving "missing" angular momentum.

The neutrino is a truly elusive particle; the mean free path of an energetic neutrino in water has been calculated as no less than several thousand light-years. At the same time, neutrinos left over from the big bang that presumably marked the creation of the universe are the most abundant particles of physics. Billions of them pass through our bodies every second, leaving no trace.

In spite of their elusive character, neutrinos have been detected in the laboratory. This was first done in 1953 by F. Reines and C. L. Cowan, using neutrinos generated in a high-power nuclear reactor. (In 1995, Reines received a Nobel Prize for this work.) In spite of the difficulties of detection, experimental neutrino physics is now a well-developed branch of experimental physics, with avid practitioners at laboratories throughout the world.

The Sun emits neutrinos copiously from the nuclear furnace at its core, and at night these messengers from the center of the Sun come up at us from below, Earth being almost totally transparent to them. In February 1987, light from an exploding star in the Large Magellanic Cloud (a nearby galaxy) reached Earth after having traveled for 170 000 years. Enormous numbers of neutrinos were generated in this explosion, and about 10 of them were picked up by a sensitive neutrino detector in Japan; Fig. 41-13 shows a record of their passage.

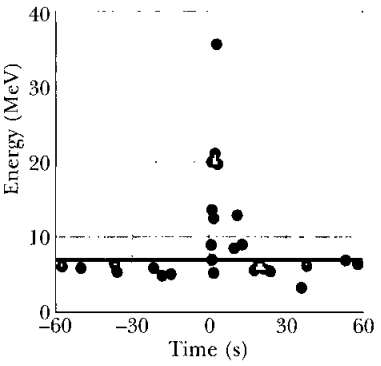


FIG. 41-13 A burst of neutrinos from the supernova SN 1987A, which occurred at (relative) time 0, stands out from the usual *background* of neutrinos. (For neutrinos, 10 is a "burst.") The particles were detected by an elaborate detector housed deep in a mine in Japan. The supernova was visible only in the Southern Hemisphere; so the neutrinos had to penetrate Earth (a trifling barrier for them) to reach the detector.

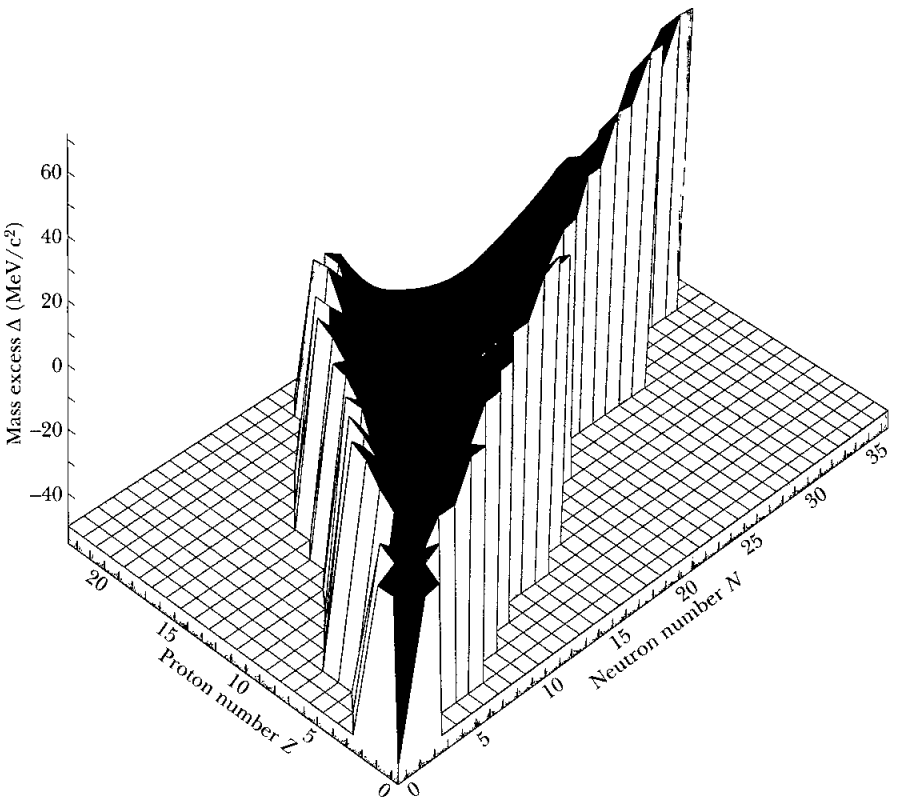


FIG. 41-14 A portion of the valley of the nuclides, showing only the nuclides of low mass. Deuterium, tritium, and helium lie at the near end of the plot, with helium at the high point. The valley stretches away from us, with the plot stopping at about $Z = 22$ and $N = 35$. Nuclides with large values of A, which would be plotted much beyond the valley, can decay into the valley by repeated alpha emissions and by fission (splitting of a nuclide).

Radioactivity and the Nuclidic Chart

We can increase the amount of information obtainable from the nuclidic chart of Fig. 41-44 by including a third axis showing the mass excess Δ expressed in the unit MeV/c^2. The inclusion of such an axis gives Fig. 41-14, which reveals the degree of nuclear stability of the nuclides. For the low-mass nuclides, we find a "valley of the nuclides," with the stability band of Fig. 41-4 running along its bottom. Nuclides on the proton-rich side of the valley decay into it by emitting positrons, and those on the neutron-rich side do so by emitting electrons.

CHECKPOINT 4 ^{238}U decays to ^{234}Th by the emission of an alpha particle. There follows a chain of further radioactive decays, either by alpha decay or by beta decay. Eventually a stable nuclide is reached and, after that, no further radioactive decay is possible. Which of the following stable nuclides is the end product of the ^{238}U radioactive decay chain: ^{206}Pb, ^{207}Pb, ^{208}Pb, or ^{209}Pb? (*Hint:* You can decide by considering the changes in mass number A for the two types of decay.)

Sample Problem 41-7

Calculate the disintegration energy Q for the beta decay of ^{32}P, as described by Eq. 41-25. The needed atomic masses are 31.973 91 u for ^{32}P and 31.972 07 u for ^{32}S.

Conceptualize/Classify: The disintegration energy Q for the beta decay is the amount by which the mass energy is changed by the decay.

Compute: Q is given by Eq. 37-50 ($Q = -\Delta M c^2$). However, we must be careful to distinguish between nuclear masses (which we do not know) and atomic masses (which we do know). Let the boldface symbols $\mathbf{m}_P$ and $\mathbf{m}_S$ represent the nuclear masses of ^{32}P and ^{32}S, and let the italic symbols m_P and m_S represent their atomic masses. Then we can write the change in mass for the decay of Eq. 41-25 as

$$\Delta m = (\mathbf{m}_S + m_e) - \mathbf{m}_P,$$

in which m_e is the mass of the electron. If we add and subtract $15m_e$ on the right side of this equation, we obtain

$$\Delta m = (\mathbf{m}_S + 16m_e) - (\mathbf{m}_P + 15m_e).$$

The quantities in parentheses are the atomic masses of ^{32}S and ^{32}P; so

$$\Delta m = m_S - m_P.$$

We thus see that if we subtract only the atomic masses, the mass of the emitted electron is automatically taken into account. (This procedure will not work for positron emission.)

The disintegration energy for the ^{32}P decay is then

$$Q = -\Delta m\, c^2$$
$$= -(31.972\,07\ \text{u} - 31.973\,91\ \text{u})(931.494\,013\ \text{MeV/u})$$
$$= 1.71\ \text{MeV}. \qquad \text{(Answer)}$$

Experimentally, this calculated quantity proves to be equal to K_{max}, the maximum energy the emitted electrons can have. Although 1.71 MeV is released every time a ^{32}P nucleus decays, in essentially every case the electron carries away less energy than this. The neutrino gets all the rest, carrying it stealthily out of the laboratory.

41-8 | Gamma Decay

As we have seen, the energy of internal motion of a nucleus is quantized. A typical nucleus has a set of allowed energy levels, including a ground state (state of lowest energy) and several excited states. Because of the great strength of nuclear interactions, excitation energies of nuclei are typically of the order of 1 MeV. In ordinary physical and chemical transformations, the nucleus always remains in its ground state. When a nucleus is placed in an excited state, either by bombardment with high-energy particles or by a radioactive transformation, it can decay to the ground state by emission of one or more photons called gamma rays, with typical energies of 10 keV–5 MeV. This process is called gamma (γ) decay. For example, alpha particles emitted from ^{226}Ra have two possible kinetic energies, either 4.784 or 4.602 MeV. Including the recoil energy of the resulting ^{222}Rn nucleus, these correspond to a total released energy of 4.871 or 4.685 MeV, respectively. When an alpha particle with the smaller energy is emitted, the ^{222}Rn nucleus is left in an excited state. It then decays to its ground state by emitting a gamma-ray photon with energy, 4.871 – 4.685 MeV = 0.186 MeV. A photon with this energy is observed during this decay (Fig. 41-15).

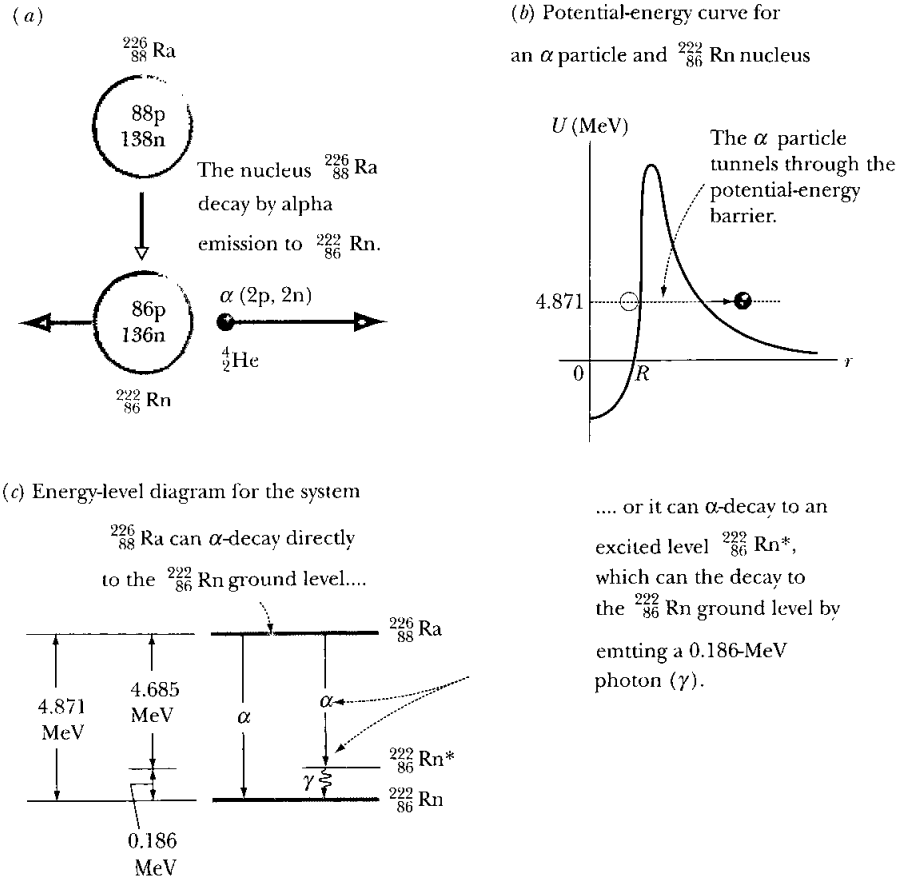


FIG. 41-15 Alpha decay of the unstable radium nuclide $^{226}_{88}$Ra.

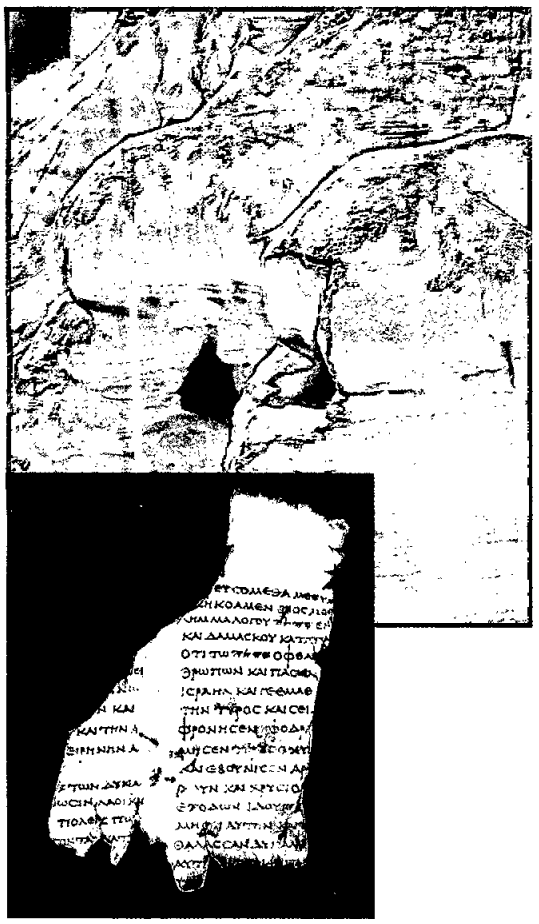


FIG. 41-16 A fragment of the Dead Sea scrolls and the caves from which the scrolls were recovered. *Top photo: (George Rockwin/Bruce Coleman, Inc.) Insert: (R. Perry/Corbis Sygma)*

41-9 | Radioactive Dating

If you know the half-life of a given radionuclide, you can in principle use the decay of that radionuclide as a clock to measure time intervals. The decay of very long-lived nuclides, for example, can be used to measure the age of rocks – that is, the time that has elapsed since they were formed. Such measurements for rocks from Earth yield a maximum age of about 4.5×10^9 y for these bodies.

The radionuclide ^{40}K, for example, decays to ^{40}Ar, a stable isotope of the noble gas argon. The half-life for this decay is 1.25×10^9 y. A measurement of the ratio of ^{40}K to ^{40}Ar, as found in the rock in question, can be used to calculate the age of that rock. Other long-lived decays, such as that of ^{235}U to ^{207}Pb (involving a number of intermediate stages of unstable nuclei), can be used to verify this calculation.

For measuring shorter time intervals, in the range of historical interest, radiocarbon dating has proved invaluable. The radionuclide ^{14}C (with $T_{1/2} = 5730$ y) is produced at a constant rate in the upper atmosphere as atmospheric nitrogen is bombarded by cosmic rays. This radiocarbon mixes with the carbon that is normally present in the atmosphere (as CO_2) so that there is about one atom of ^{14}C for every 10^{13} atoms of ordinary stable ^{12}C. Through biological activity such as photosynthesis and breathing, the atoms of atmospheric carbon trade places randomly, one atom at a time, with the atoms of carbon in every living thing, including broccoli, mushrooms, penguins, and humans. Eventually an exchange equilibrium is reached at which the carbon atoms of every living thing contain a fixed small fraction of the radioactive nuclide ^{14}C.

This equilibrium persists as long as the organism is alive. When the organism dies, the exchange with the atmosphere stops and the amount of radiocarbon trapped in the organism, since it is no longer being replenished, dwindles away with a half-life of 5730 y. By measuring the amount of radiocarbon per gram of organic matter, it is possible to measure the time that has elapsed since the organism died. Charcoal from ancient campfires, the Dead Sea scrolls (Fig. 41-16), and many prehistoric artifacts have been dated in this way. The age of the scrolls was determined by radiocarbon dating a sample of the cloth used to plug the jars in which the scrolls were sealed.

Sample Problem 41-8

Mass spectrometric analysis of potassium and argon atoms in a Moon rock sample shows that the ratio of the number of (stable) ^{40}Ar atoms present to the number of (radioactive) ^{40}K atoms is 10.3. Assume that all the argon atoms were produced by the decay of potassium atoms, with a half-life of 1.25×10^9 y. (We neglect here several complications about the decay of potassium atoms.) How old is the rock?

Conceptualize/Classify: (1) If N_0 potassium atoms were present at the time the rock was formed by solidification from a molten form, the number of potassium atoms remaining at the time of analysis is, from Eq. 41-16,

$$N_K = N_0 e^{-\lambda t}, \tag{41-30}$$

in which t is the age of the rock. (2) For every potassium atom that decays, an argon atom is produced. Thus, the number of argon atoms present at the time of the analysis is

$$N_{Ar} = N_0 - N_K. \tag{41-31}$$

Compute: We cannot measure N_0; so let's eliminate it from Eqs. 41-30 and 41-31. We find, after some algebra, that

$$\lambda t = \ln\left(1 + \frac{N_{Ar}}{N_K}\right), \tag{41-32}$$

in which N_{Ar}/N_K *can* be measured. Solving for t and using Eq. 41-19 to replace λ with $(\ln 2)/T_{1/2}$ yield

$$t = \frac{T_{1/2}\ \ln(1 + N_{Ar}/N_K)}{\ln 2}$$
$$= \frac{(1.25 \times 10^9\ \text{y})[\ln(1 + 10.3)]}{\ln 2}$$
$$= 4.37 \times 10^9\ \text{y}. \qquad \text{(Answer)}$$

Conclude: Lesser ages may be found for other lunar or terrestrial rock samples, but no substantially greater ones. Thus, the oldest rocks were formed soon after the solar system formed, and the solar system must be about 4 billion years old.

41-10 | Nuclear Models (Optional)

Nuclei are more complicated than atoms. For atoms, the basic force law (Coulomb's law) is simple in form and there is a natural force center, the nucleus. For nuclei, the force law is complicated and cannot, in fact, be written down explicitly in full detail. Furthermore, the nucleus—a jumble of protons and neutrons—has no natural force center to simplify the calculations.

In the absence of a comprehensive nuclear *theory*, we turn to the construction of nuclear *models*. A nuclear model is simply a way of looking at the nucleus that gives a physical insight into as wide a range of its properties as possible. The usefulness of a model is tested by its ability to provide predictions that can be verified experimentally in the laboratory.

Two models of the nucleus have proved useful. Although based on assumptions that seem flatly to exclude each other, each accounts very well for a selected group of nuclear properties. After describing them separately, we shall see how these two models may be combined to form a single coherent picture of the atomic nucleus.

The Collective Model

In the *collective model,* formulated by Niels Bohr, the nucleons, moving around within the nucleus at random, are imagined to interact strongly with each other, like the molecules in a drop of liquid. A given nucleon collides frequently with other nucleons in the nuclear interior, its mean free path as it moves about being substantially less than the nuclear radius.

The collective model permits us to correlate many facts about nuclear masses and binding energies; it is useful (as you will see later) in explaining nuclear fission. It is also useful for understanding a large class of nuclear reactions.

Consider, for example, a generalized nuclear reaction of the form

$$X + a \rightarrow C \rightarrow Y + b. \qquad (41\text{-}33)$$

We imagine that projectile a enters target nucleus X, forming a **compound nucleus** C and conveying to it a certain amount of excitation energy. The projectile, perhaps a neutron, is at once caught up by the random motions that characterize the nuclear interior. It quickly loses its identity—so to speak—and the excitation energy it carried into the nucleus is quickly shared with all the other nucleons in C.

The quasi-stable state represented by C in Eq. 41-33 may have a mean life of 10^{-16} s before it decays to Y and b. By nuclear standards, this is a very long time, being about one million times longer than the time required for a nucleon with a few million electron-volts of energy to travel across a nucleus.

The central feature of this compound-nucleus concept is that the formation of the compound nucleus and its eventual decay are totally independent events. At the time of its decay, the compound nucleus has "forgotten" how it was formed. Hence, its mode of decay is not influenced by its mode of formation. As an example, Fig. 41-17 shows three possible ways in which the compound nucleus ^{20}Ne might be formed and three in which it might decay. Any of the three formation modes can lead to any of the three decay modes.

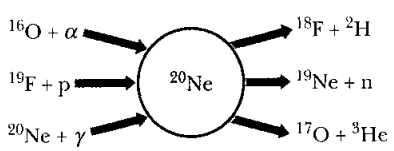


FIG. 41-17 The formation modes and the decay modes of the compound nucleus ^{20}Ne.

The Independent Particle Model

In the collective model, we assume that the nucleons move around at random and bump into one another frequently. The *independent particle model*, however, is based on just the opposite assumption—namely, that each nucleon remains in a well-defined quantum state within the nucleus and makes hardly any collisions at all! The nucleus, unlike the atom, has no fixed center of charge; we assume in this model that each nucleon moves in a potential well that is determined by the smeared-out (time-averaged) motions of all the other nucleons.

A nucleon in a nucleus, like an electron in an atom, has a set of quantum numbers that defines its state of motion. Also, nucleons obey the Pauli exclusion principle, just as electrons do; that is, no two nucleons in a nucleus may occupy the same quantum state at the same time. In this regard, the neutrons and the protons are treated separately, each particle type with its own set of quantum states.

The fact that nucleons obey the Pauli exclusion principle helps us to understand the relative stability of nucleon states. If two nucleons within the nucleus are to collide, the energy of each of them after the collision must correspond to the energy of an *unoccupied* state. If no such state is available, the collision simply cannot occur. Thus, any given nucleon experiencing repeated "frustrated collision opportunities" will maintain its state of motion long enough to give meaning to the statement that it exists in a quantum state with a well-defined energy.

In the atomic realm, the repetitions of physical and chemical properties that we find in the periodic table are associated with a property of atomic electrons—namely, they arrange themselves in shells that have a special stability when fully occupied. We can take the atomic numbers of the noble gases,

$$2, 10, 18, 36, 54, 86, \ldots,$$

as *magic electron numbers* that mark the completion (or closure) of such shells.

Nuclei also show such closed-shell effects, associated with certain **magic nucleon numbers**:

$$2, 8, 20, 28, 50, 82, 126, \ldots.$$

Any nuclide whose proton number Z or neutron number N has one of these values turns out to have a special stability that may be made apparent in a variety of ways.

Examples of "magic" nuclides are ^{18}O ($Z = 8$), ^{40}Ca ($Z = 20, N = 20$), ^{92}Mo ($N = 50$), and ^{208}Pb ($Z = 82, N = 126$). Both ^{40}Ca and ^{208}Pb are said to be "doubly magic" because they contain both filled shells of protons *and* filled shells of neutrons.

The magic number 2 shows up in the exceptional stability of the alpha particle (^4He), which, with $Z = N = 2$, is doubly magic. For example, on the binding energy curve of Fig. 41-7, the binding energy per nucleon for this nuclide stands well above those of its periodic-table neighbors hydrogen, lithium, and beryllium. The neutrons and protons making up the alpha particle are so tightly bound to one another, in fact, that it is impossible to add another proton or neutron to it; there is no stable nuclide with $A = 5$.

The central idea of a closed shell is that a single particle outside a closed shell can be relatively easily removed, but considerably more energy must be expended to remove a particle from the shell itself. The sodium atom, for example, has one (valence) electron outside a closed electron shell. Only about 5 eV is required to strip the valence electron away from a sodium atom; however, to remove a *second* electron (which must be plucked out of a closed shell) requires 22 eV. As a nuclear case, consider ^{121}Sb ($Z = 51$), which contains a single proton outside a closed shell of 50 protons. To remove this lone proton requires 5.8 MeV; to remove a *second* proton, however, requires an energy of 11 MeV. There is much

additional experimental evidence that the nucleons in a nucleus form closed shells and that these shells exhibit stable properties.

We have seen that quantum theory can account beautifully for the magic electron numbers—that is, for the populations of the subshells into which atomic electrons are grouped. It turns out that, under certain assumptions, quantum theory can account equally well for the magic nucleon numbers! The 1963 Nobel Prize in physics was, in fact, awarded to Maria Mayer and Hans Jensen "for their discoveries concerning nuclear shell structure."

A Combined Model

Consider a nucleus in which a small number of neutrons (or protons) exist outside a core of closed shells that contains magic numbers of neutrons or protons. The outside nucleons occupy quantized states in a potential well established by the central core, thus preserving the central feature of the independent-particle model. These outside nucleons also interact with the core, deforming it and setting up "tidal wave" motions of rotation or vibration within it. These collective motions of the core preserve the central feature of the collective model. Such a model of nuclear structure thus succeeds in combining the seemingly irreconcilable points of view of the collective and independent-particle models. It has been remarkably successful in explaining observed nuclear properties.

Sample Problem 41-9

Consider the neutron capture reaction

$$^{109}\text{Ag} + \text{n} \rightarrow {}^{110}\text{Ag} \rightarrow {}^{110}\text{Ag} + \gamma, \qquad (41\text{-}34)$$

in which a compound nucleus (^{110}Ag) is formed. Figure 41-18 shows the relative rate at which such events take place, plotted against the energy of the incoming neutron. Find the mean lifetime of this compound nucleus by using the uncertainty principle in the form

$$\Delta E \cdot \Delta t \approx \hbar. \qquad (41\text{-}35)$$

Here ΔE is a measure of the uncertainty with which the energy of a state can be defined. The quantity Δt is a measure of the time available to measure this energy.

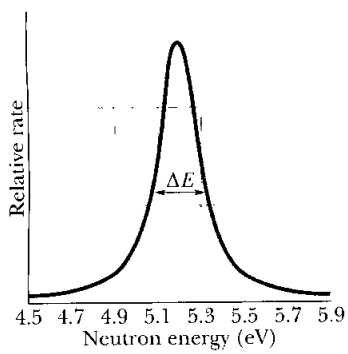


FIG. 41-18 A plot of the relative number of reaction events of the type described by Eq. 41-34 as a function of the energy of the incident neutron. The half-width ΔE of the resonance peak is about 0.20 eV.

In fact, here Δt is just t_{avg}, the average life of the compound nucleus before it decays to its ground state.

Conceptualize/Classify: We see that the relative reaction rate peaks sharply at a neutron energy of about 5.2 eV. This suggests that we are dealing with a single excited energy level of the compound nucleus ^{110}Ag. When the available energy (of the incoming neutron) just matches the energy of this level above the ^{110}Ag ground state, we have "resonance" and the reaction of Eq. 41-34 really "goes."

However, the resonance peak is not infinitely sharp but has an approximate half-width (ΔE in the figure) of about 0.20 eV. We can account for this resonance-peak width by saying that the excited level is not sharply defined in energy but has an energy uncertainty ΔE of about 0.20 eV.

Compute: Substituting that uncertainty into Eq. 41-35 gives us

$$\Delta t = t_{avg} \approx \frac{\hbar}{\Delta E} \approx \frac{(4.14\times10^{-15}\ \text{eV}\cdot\text{s})/2\pi}{0.20\ \text{eV}}$$
$$\approx 3\times10^{-15}\ \text{s}. \qquad \text{(Answer)}$$

Conclude: This is several hundred times greater than the time a 0.20 eV neutron takes to cross the diameter of a ^{109}Ag nucleus. Therefore, the neutron is spending this time of 3×10^{-15} s *as part of* the nucleus.

41-11 | Nuclear Fission: The Basic Process

In 1932 English physicist James Chadwick discovered the neutron. A few years later Enrico Fermi in Rome found that when various elements are bombarded by neutrons, new radioactive elements are produced. Fermi had predicted that the neutron, being uncharged, would be a useful nuclear projectile; unlike the proton or the alpha particle, it experiences no repulsive Coulomb force when it nears a nuclear surface. Even *thermal neutrons*, which are slowly moving neutrons in thermal equilibrium with the surrounding matter at room temperature, with a kinetic energy of only about 0.04 eV, are useful projectiles in nuclear studies.

In the late 1930s, physicist Lise Meitner and chemists Otto Hahn and Fritz Strassmann bombarded solutions of uranium salts with such thermal neutrons. They found that after the bombardment a number of new radionuclides were present. In 1939 one of the radionuclides produced in this way was positively identified, by repeated tests, as barium. But how, Hahn and Strassmann wondered, could this middle-mass element ($Z = 56$) be produced by bombarding uranium ($Z = 92$) with neutrons?

The puzzle was solved within a few weeks by Meitner and her nephew Otto Frisch. They suggested the mechanism by which a uranium nucleus, having absorbed a thermal neutron, could split, with the release of energy, into two roughly equal parts, one of which might well be barium. Frisch named the process **fission**.

A Closer Look at Fission

Figure 41-19 shows the distribution by mass number of the fragments produced when ^{235}U is bombarded with thermal neutrons. The most probable mass numbers, occurring in about 7% of the events, are centered around $A \approx 95$ and $A \approx 140$. Curiously, the "double-peaked" character of Fig. 41-19 is still not understood.

In a typical ^{235}U fission event, a ^{235}U nucleus absorbs a thermal neutron, producing a compound nucleus ^{236}U in a highly excited state. It is *this* nucleus that actually undergoes fission, splitting into two fragments. These fragments—between them—rapidly emit two neutrons, leaving (in a typical case) ^{140}Xe ($Z = 54$) and ^{94}Sr ($Z = 38$) as fission fragments. Thus, the stepwise fission equation for this event is

$$^{235}U + n \rightarrow {}^{236}U \rightarrow {}^{140}Xe + {}^{94}Sr + 2n. \qquad (41\text{-}36)$$

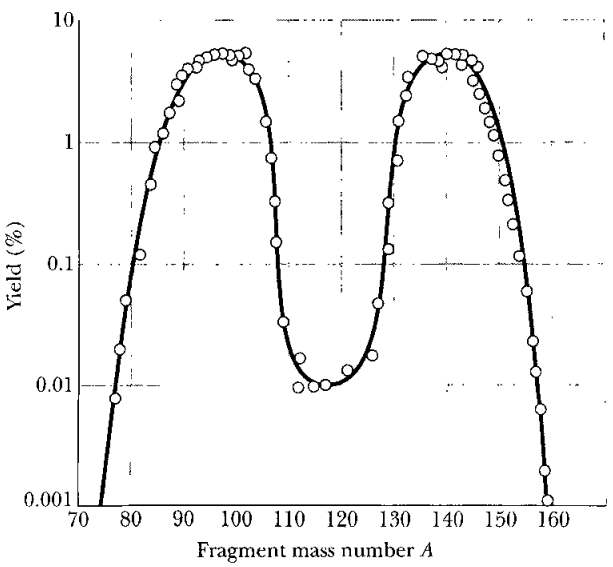


FIG. 41-19 The distribution by mass number of the fragments that are found when many fission events of ^{235}U are examined. Note that the vertical scale is logarithmic.

Note that during the formation and fission of the compound nucleus, there is conservation of the number of protons and of the number of neutrons involved in the process (and thus conservation of their total number and the net charge).

In Eq. 41-36, the fragments ^{140}Xe and ^{94}Sr are both highly unstable, undergoing beta decay (with the conversion of a neutron to a proton and the emission of an electron and a neutrino) until each reaches a stable end product. For xenon, the decay chain is

	^{140}Xe $\rightarrow$	^{140}Cs $\rightarrow$	^{140}Ba $\rightarrow$	^{140}La $\rightarrow$	^{140}Ce
$T_{1/2}$	14 s	64 s	13 d	40 h	Stable
Z	54	55	56	57	58

(41-37)

For strontium, it is

	^{94}Sr $\rightarrow$	^{94}Y $\rightarrow$	^{94}Zr
$T_{1/2}$	75 s	19 min	Stable
Z	38	39	40

(41-38)

As we should expect from Section 41-7, the mass numbers (140 and 94) of the fragments remain unchanged during these beta-decay processes and the atomic numbers (initially 54 and 38) increase by unity at each step.

Inspection of the stability band on the nuclidic chart of Fig. 41-4 shows why the fission fragments are unstable. The nuclide ^{236}U, which is the fissioning nucleus in the reaction of Eq. 41-36, has 92 protons and 236 – 92, or 144, neutrons, for a neutron/proton ratio of about 1.6. The primary fragments formed immediately after the fission reaction have about this same neutron/proton ratio. However, stable nuclides in the middle-mass region have smaller neutron/proton ratios, in the range of 1.3 to 1.4. The primary fragments are thus *neutron rich* (they have too many neutrons) and will eject a few neutrons, two in the case of the reaction of Eq. 41-36. The fragments that remain are still too neutron rich to be stable. Beta decay offers a mechanism for getting rid of the excess neutrons—namely, by changing them into protons within the nucleus.

We can estimate the energy released by the fission of a high-mass nuclide by examining the total binding energy per nucleon ΔE_{ben} before and after the fission. The idea is that fission can occur because the total mass energy will decrease; that is, ΔE_{ben} will *increase* so that the products of the fission are *more* tightly bound. Thus, the energy Q released by the fission is

$$Q = \begin{pmatrix}\text{total final}\\ \text{binding energy}\end{pmatrix} - \begin{pmatrix}\text{initial}\\ \text{binding energy}\end{pmatrix}. \tag{41-39}$$

For our estimate, let us assume that fission transforms an initial high-mass nucleus to two middle-mass nuclei with the same number of nucleons. Then we have

$$Q = \begin{pmatrix}\text{final}\\ \Delta E_{ben}\end{pmatrix}\begin{pmatrix}\text{final number}\\ \text{of nucleons}\end{pmatrix} - \begin{pmatrix}\text{initial}\\ \Delta E_{ben}\end{pmatrix}\begin{pmatrix}\text{initial number}\\ \text{of nucleons}\end{pmatrix}. \tag{41-40}$$

From Fig. 41-7, we see that for a high-mass nuclide ($A \approx 240$), the binding energy per nucleon is about 7.6 MeV/nucleon. For middle-mass nuclides ($A \approx 120$), it is about 8.5 MeV/nucleon. Thus, the energy released by fission of a high-mass nuclide to two middle-mass nuclides is

$$Q = \left(8.5\frac{\text{MeV}}{\text{nucleon}}\right)(2\text{ nuclei})\left(120\frac{\text{nucleons}}{\text{nucleus}}\right) - \left(7.6\frac{\text{MeV}}{\text{nucleon}}\right)(240\text{ nucleons}) \approx 200\text{ MeV}. \tag{41-41}$$

CHECKPOINT 5 A generic fission event is

$$^{235}\mathrm{U} + \mathrm{n} \rightarrow X + Y + 2\mathrm{n}$$

Which of the following pairs *cannot* represent X and Y: (a) ^{141}Xe and ^{93}Sr; (b) ^{139}Cs and ^{95}Rb; (c) ^{156}Nd and ^{79}Ge; (d) ^{121}In and ^{113}Ru?

Sample Problem 41-10

Find the disintegration energy Q for the fission event of Eq. 41-36, taking into account the decay of the fission fragments as displayed in Eqs. 41-37 and 41-38. Some needed atomic and particle masses are

^{235}U 235.0439 u ^{140}Ce 139.9054 u

n 1.008 66 u ^{94}Zr 93.9063 u

Conceptualize/Classify: (1) The disintegration energy Q is the energy transferred from mass energy to kinetic energy of the decay products. (2) $Q = -\Delta m\, c^2$, where Δm is the change in mass.

Compute: Because we are to include the decay of the fission fragments, we combine Eqs. 41-36, 41-37, and 41-38 to write the overall transformation as

$$^{235}\mathrm{U} \rightarrow {}^{140}\mathrm{Ce} + {}^{94}\mathrm{Zr} + \mathrm{n}. \qquad (41\text{-}42)$$

Only the single neutron appears here because the initiating neutron on the left side of Eq. 41-36 cancels one of the two neutrons on the right of that equation. The mass difference for the reaction of Eq. 41-41 is

$$\begin{aligned}\Delta m &= (139.9054\ \mathrm{u} + 93.9063\ u + 1.008\ 66\ \mathrm{u}) \\ &\quad - (235.0439\ \mathrm{u}) \\ &= -0.223\ 54\ \mathrm{u}.\end{aligned}$$

and the corresponding disintegration energy is

$$\begin{aligned}Q &= -\Delta m c^2 = -(-0.223\ 54\ \mathrm{u})(931.494\ 013\ \mathrm{MeV/u}) \\ &= 208\ \mathrm{MeV}, \qquad \text{(Answer)}\end{aligned}$$

which is in good agreement with our estimate of Eq. 41-41.

Conclude: If the fission event takes place in a bulk solid, most of this disintegration energy, which first goes into kinetic energy of the decay products, appears eventually as an increase in the internal energy of that body, revealing itself as a rise in temperature. Five or six percent or so of the disintegration energy, however, is associated with neutrinos that are emitted during the beta decay of the primary fission fragments. This energy is carried out of the system and is lost.

41-12 | Thermonuclear Fusion: The Basic Process

The binding energy curve of Fig. 41-7 shows that energy can be released if two light nuclei combine to form a single larger nucleus, a process called nuclear **fusion**. That process is hindered by the Coulomb repulsion that acts to prevent the two positively charged particles from getting close enough to be within range of their attractive nuclear forces and thus "fusing." The height of this *Coulomb barrier* depends on the charges and the radii of the two interacting nuclei. We show in Sample Problem 43-4 that, for two protons ($Z = 1$), the barrier height is 400 keV. For more highly charged particles, of course, the barrier is correspondingly higher.

To generate useful amounts of energy, nuclear fusion must occur in bulk matter. The best hope for bringing this about is to raise the temperature of the material until the particles have enough energy—due to their thermal motions alone—to penetrate the Coulomb barrier. We call this process **thermonuclear fusion**.

In thermonuclear studies, temperatures are reported in terms of the kinetic energy K of interacting particles via the relation

$$K = 3kT/2 \qquad (41\text{-}43)$$

in which K is the average kinetic energy of the interacting particles, k is the Boltzmann constant, and the temperature T is in kelvins. Thus, rather than saying, "The temperature at the center of the Sun is 1.5×10^7 K," it is more common to say, "The temperature at the center of the Sun is 2 keV."

Room temperature corresponds to $K \approx 0.03$ eV; a particle with only this amount of energy could not hope to overcome a barrier as high as, say, 400 keV. Even at the center of the Sun, the outlook for thermonuclear fusion does not seem promising at first glance. Yet we know that thermonuclear fusion not only occurs in the core of the Sun but is the dominant feature of that body and of all other stars.

The puzzle is solved when we realize two facts: (1) The energy calculated with Eq. 41-43 is that of the particles with the *root mean* speed; there is a long tail of particles with much higher speeds and, correspondingly, much higher energies. (2) The barrier heights that we have calculated represent the *peaks* of the barriers. Barrier tunneling can occur at energies considerably below those peaks, as we saw with alpha decay in Section 41-6.

CHECKPOINT 6 Which of these potential fusion reactions will *not* result in the net release of energy: (a) ^{6}Li + ^{6}Li, (b) ^{4}He + ^{4}He, (c) ^{12}C + ^{12}C, (d) ^{20}Ne + ^{20}Ne, (e) ^{35}Cl + ^{35}Cl, and (f) ^{14}N + ^{35}Cl? (*Hint:* Consult the curve of Fig. 41-7.)

Sample Problem 41-11

Assume a proton is a sphere of radius $R \approx 1$ fm. Two protons are fired at each other with the same kinetic energy K.

(a) What must K be if the particles are brought to rest by their mutual Coulomb repulsion when they are just "touching" each other? We can take this value of K as a representative measure of the height of the Coulomb barrier.

Conceptualize/Classify: The mechanical energy E of the two-proton system is conserved as the protons move toward each other and momentarily stop. In particular, the initial mechanical energy E_i is equal to the mechanical energy E_f when they stop. The initial energy E_i consists only of the total kinetic energy $2K$ of the two protons. When the protons stop, energy E_f consists only of the electric potential energy U of the system, as given by Eq. 24-5 ($U = q_1q_2/4\pi\varepsilon_0 r$).

Compute: Here the distance r between the protons when they stop is their center-to-center distance $2R$, and their charges q_1 and q_2 are both e. Then we can write the conservation of energy $E_i = E_f$ as

$$2K = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2R}.$$

This yields, with known values,

$$K = \frac{e^2}{16\pi\varepsilon_0 R}$$
$$= \frac{(1.60\times10^{-19}\text{ C})^2}{(16\pi)(8.85\times10^{-12}\text{ F/m})(1\times10^{-15}\text{ m})}$$
$$= 5.75\times10^{-14}\text{ J} = 360\text{ keV} \approx 400\text{ keV}. \quad \text{(Answer)}$$

(b) At what temperature would a proton in a gas of protons have the average kinetic energy calculated in (a) and thus have energy equal to the height of the Coulomb barrier?

Conceptualize/Classify: If we treat the proton gas as an ideal gas, then from Eq. 20-16, Vol. I, the average energy of the protons is $K_{avg} = \frac{3}{2}kT$, where k is the Boltzmann constant.

Compute: Solving that equation for T and using the result of (a) yield

$$T = \frac{2K_{avg}}{3k} = \frac{(2)(5.75\times10^{-14}\text{ J})}{(3)(1.38\times10^{-23}\text{ J/K})}$$
$$\approx 3\times10^9\text{ K}. \quad \text{(Answer)}$$

Conclude: The temperature of the core of the Sun is only about 1.5×10^7 K; thus fusion in the Sun's core must involve protons whose energies are *far* above the average energy.

41-13 | Thermonuclear Fusion in the Sun and Other Stars

The Sun radiates energy at the rate of 3.9×10^{26} W and has been doing so for several billion years. Where does all this energy come from? Chemical burning is ruled out; if the Sun had been made of coal and oxygen—in the right proportions

for combustion—it would have lasted for only about 1000 y. Another possibility is that the Sun is slowly shrinking, under the action of its own gravitational forces. By transferring gravitational potential energy to thermal energy, the Sun might maintain its temperature and continue to radiate. Calculation shows, however, that this mechanism also fails; it produces a solar lifetime that is too short by a factor of at least 500. That leaves only thermonuclear fusion. The Sun, as you will see, burns not coal but hydrogen, and in a nuclear furnace, not an atomic or chemical one.

The fusion reaction in the Sun is a multistep process in which hydrogen is burned to form helium, hydrogen being the "fuel" and helium the "ashes." Figure 43-10 shows the **proton–proton** (p-p) **cycle** by which this occurs.

The p-p cycle starts with the collision of two protons ($^1H + {}^1H$) to form a deuteron (^2H), with the simultaneous creation of a positron (e^+) and a neutrino (ν). The positron very quickly encounters a free electron (e^-) in the Sun, and both particles annihilate (see Section 22-5), their mass energy appearing as two gamma-ray photons (γ).

A pair of such events is shown in the top row of Fig. 41-20. These events are actually extremely rare. In fact, only once in about 10^{26} proton–proton collisions is a deuteron formed; in the vast majority of cases, the two protons simply rebound elastically from each other. It is the slowness of this "bottleneck" process that regulates the rate of energy production and keeps the Sun from exploding. In spite of this slowness, there are so very many protons in the huge and dense volume of the Sun's core that deuterium is produced in just this way at the rate of 10^{12} kg/s.

Once a deuteron has been produced, it quickly collides with another proton and forms a ^3He nucleus, as the middle row of Fig. 41-20 shows. Two such ^3He nuclei may eventually (within 10^5 y; there is plenty of time) find each other, forming an alpha particle (^4He) and two protons, as the bottom row in the figure shows.

Overall, we see from Fig. 41-20 that the p-p cycle amounts to the combination of four protons and two electrons to form an alpha particle, two neutrinos, and six gamma-ray photons. That is,

$$4\,{}^1H + 2e^- \rightarrow {}^4He + 2\nu + 6\gamma. \tag{41-44}$$

Let us now add two electrons to each side of Eq. 41-44, obtaining

$$(4\,{}^1H + 4e^-) \rightarrow ({}^4He + 2e^-) + 2\nu + 6\gamma. \tag{41-45}$$

The quantities in the two sets of parentheses then represent *atoms* (not bare nuclei) of hydrogen and of helium. That allows us to compute the energy release in the overall reaction of Eq. 41-44 (and Eq. 41-45) as

$$\begin{aligned} Q &= \Delta m\, c^2 \\ &= -[4.002\,603\text{ u} - (4)(1.007\,825\text{ u})][931.5\text{ MeV/u}] \\ &= 26.7\text{ MeV}, \end{aligned}$$

in which 4.002 603 u is the mass of a helium atom and 1.007 825 u is the mass of a hydrogen atom. Neutrinos have a negligibly small mass, and gamma-ray photons have no mass; thus, they do not enter into the calculation of the disintegration energy.

This same value of Q follows (as it must) from adding up the Q values for the separate steps of the proton–proton cycle in Fig. 41-20. Thus,

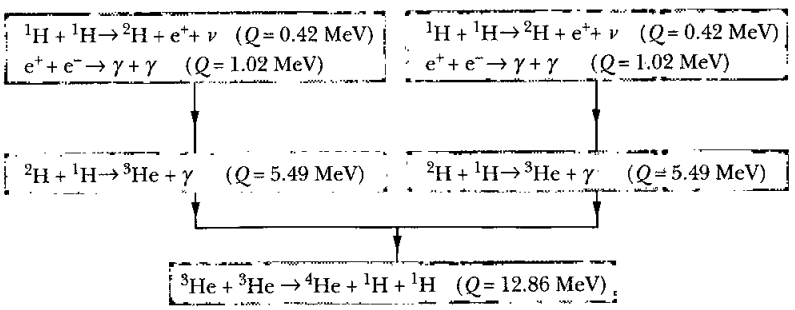


FIG. 41-20 The proton–proton mechanism that accounts for energy production in the Sun. In this process, protons fuse to form an alpha particle (^4He), with a net energy release of 26.7 MeV for each event.

$$Q = (2)(0.42\ \text{MeV}) + (2)(1.02\ \text{MeV}) + (2)(5.49\ \text{MeV}) + 12.86\ \text{MeV}$$
$$= 26.7\ \text{MeV}.$$

About 0.5 MeV of this energy is carried out of the Sun by the two neutrinos indicated in Eqs. 41-44 and 41-45; the rest (= 26.2 MeV) is deposited in the core of the Sun as thermal energy. That thermal energy is then gradually transported to the Sun's surface, where it is radiated away from the Sun as electromagnetic waves, including visible light. Note that in fission as well as fusion, there should be at least two products, else the law of conservation of momentum as well as of energy will not hold true.

REASONING TYPE QUESTIONS

In the following set of questions, a statement 1 is given and a corresponding statement 2 is given just below it. Mark the correct answer as:

(a) If both Statement 1 and Statement 2 are true and Statement 2 is the correct explanation of Statement 1.

(b) If both Statement 1 and Statement 2 are true but Statement 2 is not the correct explanation of Statement 1.

(c) If Statement 1 is true but Statement 2 is false.

(d) If Statement 1 is false but Statement 2 is true.

1. Statement 1: Two protons in the same nucleus do not fly apart under the influence of electrostatic repulsion.

Statement 2: Coulomb's law is valid only for large distances.

2. Statement 1: The atomic mass number of an element is always greater than its atomic number.

Statement 2: $A = N + Z$.

3. Statement 1: Mass of all the nuclides is greater than A amu, where A is their respective mass number.

Statement 2: Mass of both proton and neutron is greater than 1 amu.

4. Statement 1: The greatest mass per nucleon is for hydrogen nuclei.

Statement 2: The binding energy of hydrogen nuclei is zero.

5. Statement 1: On earth, at present, the amount of stable nuclides is estimated to be more than the amount of unstable nuclides.

Statement 2: The half-life of stable nuclides is infinite.

6. Statement 1: In any radioactive sample, after one half-life exactly half of the nuclei would have decayed.

Statement 2: According to the law of radioactivity, $N = N_0 e^{-\lambda t}$.

7. Statement 1: An electron capture is indicated by the characteristic x-ray spectra emitted after the reaction.

Statement 2: After an electron capture, the nucleus remains in an excited state and subsequently radiates the excess energy in form of a photon.

8. Statement 1: Nuclear fusion occurs only at very high temperatures.

Statement 2: The average kinetic energies of the nuclei undergoing fusion should be sufficient to overcome Coulomb barrier.

9. Statement 1: When nuclides lighter than ^{62}Ni split into two fragments, energy is released.

Statement 2: For every nuclei except hydrogen, the mass of the nucleus is less than the sum of the masses of the constituent nucleons.

10. Statement 1: Neither fission nor fusion of ^{62}Ni would yield energy.

Statement 2: The binding energy per nucleon of ^{62}Ni is maximum.

11. Statement 1: When ^{235}U undergoes fission, energy is released.

Statement 2: When a heavy nucleus undergoes fission, the number of nucleons after the decay is less than the number of nucleons before the decay.

MULTIPLE CHOICE QUESTIONS

sec. 41-2 Discovering the Nucleus

1. In Rutherford's famous gold foil scattering experiment, he found that most alpha particles would pass through the foil undeflected. Which of the following nuclear properties can be inferred from this observation?

(a) The nucleus must have a positive charge

(b) Most of the mass of an atom is in the nucleus

(c) The nucleus contains both protons and neutrons

(d) The diameter of the nucleus is small compared to the diameter of the atom

sec. 41-3 Some Nuclear Properties

2. The electron, neutron, and proton, listed in descending order of mass are

(a) Electron, neutron, proton

(b) Electron, proton, neutron

(c) Proton, neutron, electron

(d) Neutron, proton, electron

3. Which of the following is a wrong description of binding energy of a nucleus?

(a) It is the energy required to break a nucleus into its constituent nucleons

(b) It is the energy made available when free nucleons combine to form a nucleus

(c) It is proportional to the sum of the rest mass energies of its nucleons minus the rest mass energy of the nucleus

(d) It is the sum of the kinetic energy of all the nucleons in the nucleus

4. In the options given below, let E denote the mass energy of a nucleus and $E(n)$ that of a neutron. The correct option is

(a) $E(^{236}_{92}\mathrm{U}) > E(^{137}_{53}\mathrm{I}) + E(^{97}_{39}\mathrm{Y}) + 2E(n)$

(b) $E(^{236}_{92}\mathrm{U}) < E(^{137}_{53}\mathrm{I}) + E(^{97}_{39}\mathrm{Y}) + 2E(n)$

(c) $E(^{236}_{92}\mathrm{U}) < E(^{140}_{56}\mathrm{Ba}) + E(^{94}_{36}\mathrm{Kr}) + 2E(n)$

(d) $E(^{236}_{92}\mathrm{U}) > E(^{140}_{56}\mathrm{Ba}) + E(^{94}_{36}\mathrm{Kr}) + 2E(n)$

5. The binding energies of the atom of elements A and B are E_a and E_b, respectively. Three atoms of the element B fuse to give one atom of element A. This fusion process is accompanied by release of energy e. Then E_a and E_b are related to each other as

(a) $E_a + e = 3E_b$

(b) $E_a = 3E_b$

(c) $E_a - e = 3E_b$

(d) $E_a + 3E_b + e = 0$

6. How does the rest energy of a nucleus compare to the sum of the rest energies of the individual nucleons when they are apart from the nucleus?

(a) It is greater than the sum

(b) It is less than the sum

(c) It is equal to the sum

(d) It is in the ratio of 1:1.25.

7. The binding energies of the nuclei of $^{4}_{2}\mathrm{He}$, $^{7}_{3}\mathrm{Li}$, $^{12}_{6}\mathrm{C}$, and $^{14}_{7}\mathrm{N}$ are 28, 52, 90, and 98 MeV, respectively. Which of these are most stable?

(a) $^{4}_{2}\mathrm{He}$

(b) $^{7}_{3}\mathrm{Li}$

(c) $^{12}_{6}\mathrm{C}$

(d) $^{14}_{7}\mathrm{N}$

8. Let m_p be the mass of proton, m_n the mass of a neutron, M_1 the mass of a $^{20}_{10}\mathrm{Ne}$ nucleus and M_2 the mass of a $^{40}_{20}\mathrm{Ca}$ nucleus. Then

(a) $M_2 = 2M_1$

(b) $M_2 > 2M_1$

(c) $M_2 < 2M_1$

(d) $M_1 < 10(M_n + m_p)$

9. In the nuclear reaction described by the equation below, an alpha particle reacts with a nucleus to form another nucleus and an additional particle "X." $^{4}_{2}\mathrm{He} + ^{12}_{6}\mathrm{C} \rightarrow ^{13}_{7}\mathrm{N} + X$. This additional particle, "X," is a

(a) Positron

(b) Neutron

(c) Proton

(d) Tritium nucleus

10. Nucleus A is converted into C through the following reactions

$$A \rightarrow B + \alpha \text{ and } B \rightarrow C + 2e^{-}$$

(a) A and B are isotopes

(b) A and C are isobars

(c) A and B are isobars

(d) A and C are isotopes

11. Which of the following are evidences for the existence of neutrons in an atom?

(a) The ratio of the charge to the mass of atom's nucleus is smaller than that of a hydrogen nucleus

(b) An atom is always electrically neutral

(c) Isotopes are present

(d) Some atoms are radioactive in nature

12. The instability of the nucleus can be due to various causes. An unstable nucleus emits radiations if possible to transform into less unstable state. Then the cause and the result can be

(a) A nucleus of excess nucleons is α active

(b) An excited nucleus of excess protons is β^- active

(c) An excited nucleus of excess protons is β^+ active

(d) A nucleus of excess neutrons is β^- active

sec. 41-4 Radioactive Decay

13. In one average life (assuming the decay to proceed according to radioactive law)

(a) Half the active nuclei decay

(b) Less than half the active nuclei decay

(c) More than half the active nuclei decay

(d) All the nuclei decay

14. In total, 90% of the active nuclei present in a radioactive sample are found to remain undecayed after 1 day. From the start, the percentage of undecayed nuclei left after 2 days will be

(a) 85%

(b) 81%

(c) 80%

(d) 79%

15. A fraction f_1 of a radioactive sample decays in one mean life, and a fraction f_2 decays in one half-life. Then

(a) $f_1 > f_2$

(b) $f_1 < f_2$

(c) $f_1 = f_2$

(d) $f_1 \neq f_2$

16. Let T be the mean life of a radioactive sample. 75% of the active nuclei present in the sample initially will decay in time

(a) $2T$

(b) $0.5T(\ln 2)$

(c) $4T$

(d) $2(\ln 2)T$

17. A freshly prepared radioactive source of half-life 2 h emits radiation of intensity which is 64 times the permissible safe level. The minimum time after which it would be possible to work safely with this source is

(a) 6 h

(b) 12 h

(c) 24 h

(d) 128 h

18. The half-life of ^{131}I is 8 days. Given a sample of ^{131}I at time $t = 0$, we can assert that

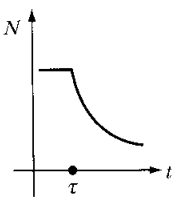
(a) No nucleus will decay before $t = 4$ days

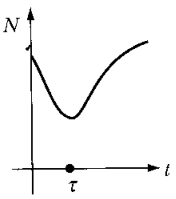
(b) No nucleus will decay $t = 8$ days

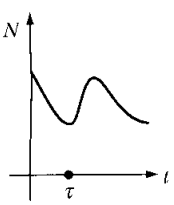
(c) All nuclei will decay before $t = 16$ days

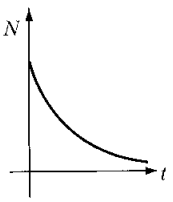
(d) A given nucleus may decay at any time after $t = 0$

19. A radioactive sample consists of two distinct species having equal number of atoms initially. The mean life of one species is τ and that of the other is 5τ. The decay products in both cases are stable. A plot is made of the total number of radioactive nuclei as a function of time. Which of the following figure best represent the form of this plot

(a) 

(b) 

(c) 

(d) 

20. Half-lives of two radioactive substances A and B are 20 min and 40 min, respectively. Initially, the samples had equal number of nuclei. After 80 min, the ratio of decayed numbers of A and B nuclei is

(a) 1:16

(b) 4:1

(c) 1:4

(d) 5:4

21. The ratio of molecular mass of two radioactive substances is 3/2 and the ratio of their decay constant is 4/3. Then the ratio of their initial activity per mole will be

(a) 2

(b) 8/9

(c) 4/3

(d) 9/8

22. An unstable nucleus decays in three different modes, each mode having a different half-life. T_1, T_2, T_3 ($T_1 \gg T_2 \gg T_3$). The overall half-life of a sample will be given by

(a) $T \approx T_1$

(b) $T \approx T_3$

(c) $T \approx (T_1 + T_2 + T_3)/3$

(d) $T \approx T_2$

23. The decay constant of a radioactive material represents the fraction of nuclei of a particular isotope that

(a) Will eventually decay if you wait long enough

(b) Will decay per unit time

(c) Are present in an element containing a mixture of isotopes

(d) Are present in a chemical compound containing elements of different atomic numbers

24. Samples of two different isotopes, X and Y, both contain the same number of radioactive atoms. Sample X has a half-life twice than that of Y. How do their activities be compared?

(a) X has a smaller activity than Y

(b) X has a greater activity than Y

(c) Activities of X and Y are equal

(d) Need to know number of neutrons in nucleus

25. A particular isotope has a half-life of 10 min. What does that statement imply about a sample of this type of nucleus?

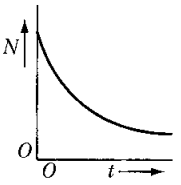
(a) All of the nuclei will decay in 10 min

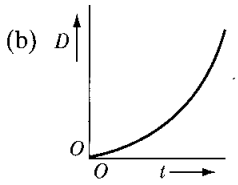
(b) All of the nuclei will decay in 20 min

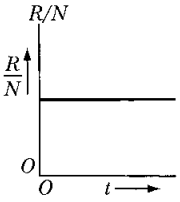
(c) Every time you start with 10 nuclei at the beginning of a 10-min interval, 5 nuclei will be left at the end of the interval.

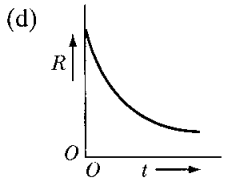
(d) The expected number of nuclei to decay in 10 min will be one-half the number at the beginning of the 10 min.

26. A large population of radioactive nucleus starts disintegrating at $t = 0$. At time t, if N = number of parent nuclei present, D = the number of daughter nuclei present, and R = rate at which the daughter nuclei are produced, then the correct representation will be

(a) 

(b) 

(c) 

(d) 

27. A radioactive substance is dissolved in a liquid and the solution is heated. The activity of the solution

(a) Is smaller than that of element

(b) Is greater than that of element

(c) Is equal to that of element

(d) Will be smaller or greater depending upon whether the solution is weak or concentrated.

sec. 41-6 Alpha Decay

28. Nuclear reactions can occur spontaneously when

(a) Q is negative

(b) Q is positive

(c) Q is negative, positive, or imaginary.

(d) Q is either positive or negative, but not when Q is imaginary

29. A nucleus with mass number 220 initially at rest emits an α particle. If the Q value of the reaction is 5.5 MeV, identify the kinetic energy of the α particle

(a) 4.4 MeV

(b) 5.4 MeV

(c) 5.6 MeV

(d) 6.5 MeV

sec. 41-7 Beta Decay

30. In β decay, the Q-value of the process is E. Then,

(a) Kinetic energy of a β-particle cannot exceed E

(b) Kinetic energy of anti-neutrino emitted lies between zero and E

(c) N/Z ratio of the nucleus is altered

(d) Mass number (A) of the nucleus is altered

31. During a negative beta decay,

(a) An atomic electron is ejected

(b) An electron which is already present within the nucleus is ejected

(c) A neutron in the nucleus decays emitting an electron

(d) A part of the excitation energy of the nucleus is converted into an electron

32. Consider the following nuclear reactions and select the correct statements from the option given below

Reaction I: $n \rightarrow p + e^- + \nu$ Reaction II: $p \rightarrow n + e^+ + \nu$

(a) Free neutron is unstable, therefore reaction I is possible

(b) Free proton is stable, therefore reaction II is not possible

(c) Inside a nucleus, both decays (reactions I and II) are possible

(d) Inside a nucleus, reaction I is not possible but reaction II is possible

33. If ${}_{92}U^{238}$ changes to ${}_{85}Al^{210}$ by a series of α and β decays, the number of α and β decays undergone is

(a) 7 and 5

(b) 7 and 7

(c) 5 and 7

(d) 7 and 9

34. Ten grams of ${}^{57}Co$ is kept in an open container beta decays with a half-life of 270 days. The weight of the material inside the container after 540 days will be very nearly

(a) 10 g

(b) 5 g

(c) 2.5 g

(d) 1.25 g

35. Which one of the following combinations of radioactive decay results in the formation of an isotope of the original nucleus?

(a) One alpha and four beta

(b) One alpha and two beta

(c) One alpha and one beta

(d) Two alpha and one beta

(e) Four alpha and one beta

sec. 41-8 Gamma Decay

36. What is the essential distinction between x rays and γ rays

(a) γ rays have shorter wavelength than x rays

(b) γ rays are extraterrestrial, x rays are man-made

(c) γ rays have less penetrating power than x rays

(d) γ rays have greater ionizing powers than x rays

(e) γ rays originate from within an atomic nucleus, x rays from outside an atomic nucleus

37. If a beam consisting of α, β, and γ radiations is passed through an electric field perpendicular to the beam, the deflections suffered by the components, in decreasing order, are

(a) α, β, γ

(b) α, γ, β

(c) β, α, γ

(d) β, γ, α

38. A γ-ray photon is emitted

(a) After ionization of an atom

(b) Due to conversion of a neutron into a proton in the nucleus

(c) After de-excitation of a nucleus

(d) Due to conversion of a proton into a neutron in the nucleus

39. When the nucleus of an electrically neutral atom undergoes a radioactive decay process, it will remain neutral after the decay if the process is

(a) α decay

(b) β decay

(c) γ decay

(d) K-capture

sec. 41-9 Radioactive Dating

40. Which of the following is an assumption in Carbon dating process?

(a) The ratio of ^{14}C to ^{12}C is constant in living organisms

(b) The half-life of a nuclide remains constant over the years

(c) ^{14}C stops being absorbed when an organism dies

(d) Fluctuation of ^{14}C and ^{12}C ratios occurs in well-known periodic intervals

41. An archaeologist extracts a sample of carbon from an ancient ax handle and finds that it emits an average of 10 beta emissions per minute. She finds that the same mass of carbon from a living tree emits 40 beta per minute. Knowing that the half-life of carbon-14 is 5730 years, she concludes that the age of the ax handle is about

(a) 2865 years

(b) 5730 years

(c) 11 460 years

(d) 17 190 years

sec. 41-11 Nuclear Fission: The Basic Process

42. When a U-235 nucleus absorbs a neutron and undergoes nuclear fission, about 200 MeV of energy is released. But in what form? Interestingly, most of this energy initially appears in the form of

(a) γ rays

(b) Kinetic energy of emitted neutrons

(c) Kinetic energy of the fission fragments

(d) Heat

sec. 41-12 Thermonuclear Fusion: The Basic Process

43 A star initially has 10^{40} deuterons. It produces energy via the processes

(i) $_1H^2 + {}_1H^2 \rightarrow {}_1H^3 + p$ and (ii) $_1H^2 + {}_1H^3 \rightarrow {}_2He^4 + n$

If the average power radiated by the star is 10^{16} W, the deuteron supply of the star is exhausted in a time of the order of

(a) 10^6 s

(b) 10^8 s

(c) 10^{12} s

(d) 10^{16} s

44. Both nuclear fission and nuclear fusion release enormous energy. When a uranium nucleus fissions, the released energy is mainly kinetic energy of the repelling fragments. When a pair of hydrogen isotopes fuses, the energy initially released is in the form of

(a) Gamma radiation

(b) Kinetic energy of recoiling particles

(c) Potential energy of the helium nucleus that is formed

(d) Heat

45. Which of the following is a correct statement

(a) In fission process, two fragments are necessary but in fusion two products are not necessary

(b) In fission process, two fragments are not necessary but in fusion two products are necessary

(c) In fission process as well as fusion, two products are necessary

(d) In fission process or fusion, two products are not necessary

46. Which of the following statements are correct?

(a) Nuclei with small mass number and atomic number are more likely to undergo fusion than fission

(b) Nuclei having a small binding energy per nucleon are more likely to undergo fusion than fission.

(c) Nuclei having large binding energy per nucleon are more likely to undergo fusion than fission.

(d) Nuclei with large atomic number are more likely to undergo fusion than fission.

47. The total binding energy of an α particle ($_2He^4$) is 24.4 MeV whereas the total binding energy of a deutron ($_1H^2$) is merely 2.2 MeV. When two deutrons are made to combine

(a) An α particle will be obtained

(b) An amount of 22.2 MeV energy will be released

(c) An amount of 20.0 MeV energy will be released

(d) An amount of 22.2 MeV energy will be consumed

48. Which of the following(s) is(are) correct?

(a) The rest mass of a stable nucleus is less than the sum of the rest masses of its separated nucleons

(b) The rest mass of a stable nucleus is greater than the sum of the rest masses of its separated nucleons

(c) In nuclear fusion, energy is released by fusion of two nuclei of medium mass (approximately 100 amu)

(d) In nuclear fission, energy is released by fragmentation of a very heavy nucleus

PROBLEMS

sec. 41-2 Discovering the Nucleus

1. A $^{7}Li_{3}$ nucleus with a kinetic energy of 3.00 MeV is sent toward a $^{232}Th_{90}$ nucleus. What is the least center-to-center separation between the two nuclei, assuming that the (more massive) ^{232}Th nucleus does not move?

2. When an alpha particle collides elastically with a nucleus, the nucleus recoils. Suppose a 5.00 MeV alpha particle has a head-on elastic collision with a gold nucleus that is initially at rest. What is the kinetic energy of (a) the recoiling nucleus and (b) the rebounding alpha particle?

sec. 41-3 Some Nuclear Properties

3. Complete these nuclear reactions

(a) $^{6}_{3}\mathrm{Li} + ? \rightarrow {}^{7}_{4}\mathrm{Be} + {}^{1}_{0}\mathrm{n}$

(b) $^{35}_{17}\mathrm{Cl} + ? \rightarrow {}^{35}_{16}\mathrm{S} + {}^{4}_{2}\mathrm{He}$

(c) $^{9}_{4}\mathrm{Cl} + {}^{4}_{2}\mathrm{He} \rightarrow 3\ {}^{4}_{2}\mathrm{He} + ?$

4. Find the radius of a nucleus of $^{1}_{1}$H.

5. A star ending its life with a mass of two times the mass of the Sun is expected to collapse, combining its protons and electrons to form a neutron star. Such a star could be thought of as a gigantic atomic nucleus. If $M_{star} = 2 \times 1.99 \times 10^{30}$ kg collapsed into neutrons ($m_n = 1.67 \times 10^{-27}$ kg), what would be its radius?

6. Suppose that we could use the energy released when 1 g of antimatter annihilates 1 g of matter to lift a mass 1 km from the Earth's surface. How much mass could we lift?

7. What is the mass excess Δm of ^{1}H (whose actual mass is 1.007 825 u) in (a) atomic mass units and (b) MeV/c^2? What is the mass excess Δ_n of a neutron (actual mass is 1.008 665 u) in (c) atomic mass units and (d) MeV/c^2? What is the mass excess Δ_{120} of ^{120}Sn (actual mass is 119.902 197 u) in (e) atomic mass units and (f) MeV/c^2?

8. The nuclide ^{14}C contains (a) how many protons and (b) how many neutrons?

9. The strong neutron excess (defined as $N - Z$) of highmass nuclei is illustrated by noting that most high-mass nuclides could never fission into two stable nuclei without neutrons being left over. For example, consider the spontaneous fission of a ^{235}U nucleus into two stable *daughter nuclei* with atomic numbers 39 and 53. From Appendix F, determine the name of the (a) first and (b) second daughter nucleus. From Fig. 41-4, approximately how many neutrons are in the (c) first and (d) second? (e) Approximately how many neutrons are left over?

10. A penny has a mass of 3.0 g. Calculate the energy that would be required to separate all the neutrons and protons in this coin from one another. For simplicity, assume that the penny is made entirely of ^{63}Cu atoms (of mass 62.929 60 u). The masses of the proton-plus-electron and the neutron are 1.007 83 u and 1.008 66 u, respectively.

11. An α particle (^{4}He nucleus) is to be taken apart in the following steps. Give the energy (work) required for each step: (a) remove a proton, (b) remove a neutron, and (c) separate the remaining proton and neutron. For an α particle, what are (d) the total binding energy and (e) the binding energy per nucleon? (f) Does either match an answer to (a), (b), or (c)? Here are some atomic masses and the neutron mass.

^{4}He	4.002 60 u	^{2}H	2.014 10 u
^{3}H	3.016 05 u	^{1}H	1.007 83 u
n	1.008 67 u		

12. What is the binding energy per nucleon of ^{262}Bh? The mass of the atom is 262.1231 u.

13. (a) Show that the total binding energy E_{be} of a given nuclide is

$$E_{be} = Z\Delta_H + N\Delta_n - \Delta,$$

where Δ_H is the mass excess of ^{1}H, Δ_n is the mass excess of a neutron, and Δ is the mass excess of the given nuclide. (b) Using this method, calculate the binding energy per nucleon for ^{197}Au. The needed mass excesses, rounded to three significant figures, are $\Delta_H = +7.29$ MeV, $\Delta_n = 8.07$ MeV, and $\Delta_{197} = -31.2$ MeV. Note the economy of calculation that results when mass excesses are used in place of the actual masses.

14. Because the neutron has no charge, its mass must be found in some way other than by using a mass spectrometer. When a neutron and a proton meet (assume both to be almost stationary), they combine and form a deuteron, emitting a gamma ray whose energy is 2.2233 MeV. The masses of the proton and the deuteron are 1.007 276 467 u and 2.013 553 212 u, respectively. Find the mass of the neutron from these data.

15. When the nuclear reaction is endothermic, the disintegration energy Q is negative. For the reaction to proceed, the incoming particle must have minimum energy called the threshold energy E_{th}. Some fraction of the energy of the incident particle is transferred to the compound nucleus to conserve momentum. Therefore, E_{th} must be greater than Q. Show that

$$E_{th} = -Q\left(1 + \frac{M_a}{M_x}\right).$$

16. The most common isotope of uranium has atomic mass 238.050783 amu. Calculate (a) the mass defect, (b) the binding energy in MeV, and (c) binding energy per nucleon.

17. The binding energy of the ^{14}N nucleus is 16.76 pJ and the ^{14}C nucleus is 16.86 pJ. Which nucleus is the decay product of the other?

sec. 41-4 Radioactive Decay

18. Consider an initially pure 3.35 g sample of ^{67}Ga, an isotope that has a half-life of 78 h. (a) What is its initial decay rate? (b) What is its decay rate 78 h later?

19. The half-life of a particular radioactive isotope is 6.5 h. If there are initially 48×10^{19} atoms of this isotope, how many remain at the end of 26 h?

20. A radioactive isotope of mercury, ^{197}Hg, decays to gold, ^{197}Au, with a disintegration constant of 0.0108 h^{-1}. (a) Calculate the half-life of the ^{197}Hg. What fraction of a sample will remain at the end of (b) three half-lives?

21. The plutonium isotope ^{239}Pu is produced as a by-product in nuclear reactors and hence is accumulating in our environment. It is radioactive, decaying with a half-life of 2.41×10^4 y. (a) How many nuclei of Pu constitute a chemically lethal dose of 2.00 mg? (b) What is the decay rate of this amount?

22. In 1992, Swiss police arrested two men who were attempting to smuggle osmium out of Eastern Europe for a clandestine sale. However, by error, the smugglers had picked up ^{137}Cs. Reportedly, each smuggler was carrying a 1.0 g sample of ^{137}Cs *in a pocket*! In (a) bequerels and (b) curies, what was the activity of each sample? The isotope ^{137}Cs has a halflife of 30.2 y. (The activities of radioisotopes commonly used in hospitals range up to a few millicuries.)

23. The radionuclide ^{64}Cu has a half-life of 12.7 h. If a sample contains 5.50 g of initially pure ^{64}Cu at $t = 0$, how much of it will decay between $t = 14.0$ h and $t = 16.0$ h?

24. A source contains two phosphorus radionuclides, ^{32}P ($T_{1/2}$ = 14.3 d) and ^{33}P ($T_{1/2}$ = 25.3 d). Initially, 10.0% of the decays come from ^{33}P. How long must one wait until 90.0% do so?

25. A sample of radioactive material contains 1.00×10^{15} atoms and has an activity of 6.00×10^{11} Bq. What is its half-life?

26. A certain radionuclide is being manufactured in a cyclotron at a constant rate R. It is also decaying with disintegration constant λ. Assume that the production process has been going on for a time that is much longer than the half-life of the radionuclide. (a) Show that the number of radioactive nuclei present after such time remains constant and is given by $N = R/\lambda$. (b) Now show that this result holds no matter how many radioactive nuclei were present initially. The nuclide is said to be in *secular equilibrium* with its source; in this state its decay rate is just equal to its production rate.

27. Find the probability that a particular nucleus of will undergo beta decay in any 1.00 s period. The half-life of is 37.2 min.

28. A particular nucleus in a large population of identical radioactive nuclei survived five half-lives of that isotope. What is the probability that this nucleus will survive the next half-life also?

29. What is the mass of $^{198}_{79}$Au in a radioactive sample of activity 1 Ci given that half-life of decay is 2.7 days?

30. A freshly prepared sample of a certain radioactive isotope has an activity of 10.0 mCi. After 4 h, its activity is 8.00 mCi. (a) Find the decay constant and half-life. (b) How many atoms of the isotope were contained in the freshly prepared sample? (c) What is the sample's activity 30 h after it is prepared?

31. Consider a radioactive sample. Determine the ratio of the number of atoms decaying during the first half of its half-life to the number of atoms decaying during the second half of its half-life.

32. At 1:00 P.M., a sample of a particular radioactive isotope has an activity 8.00 mCi. At 7:00 P.M., the activity of the sample has decreased to 2.00 mCi. At what time will the activity of the sample be 1.00 mCi?

33. Radioactive ^{24}Na, which has a half-life of 3 h, is flown from Chambal River outside Kota to a Hospital in Delhi. It takes 15 h for the transportation. If the activity must be 10 mCi when it is used in the hospital, what is the activity value, in mCi, it needs to have when it leaves Chambal River?

34. Consider that ^{235}U, which has a nuclear half-life of 0.7×10^9 years, decays through a series of stages, all of short life to ^{207}Pb. What is now the ratio of ^{207}Pb nuclei to ^{235}U in a rock which solidified 2.1×10^9 years ago, assuming that all the ^{207}Pb came from ^{235}U in the rock?

35. The half-life of ^{40}K is 1.30×10^9 years. A sample of 1.00 g of pure KCl gives 160 counts/s. Calculate the relative abundance of ^{40}K (fraction of ^{40}K present) in natural potassium.

36. The half-life of a substance A is 1 h. The half-life of another substance B is 10 years. A mixture of the two substances has an activity of 1000 Bq. After 1 h, its activity drops to 900 Bq. What will be its activity after 20 years?

37. A radioactive nucleus can decay by two different processes. The half-life for the first processes is t_1 and that for the second process is t_2. Show that the effective half-life of the nucleus is given by

$$\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}.$$

38. A nucleus decays in two channels with probabilities of 0.25 and 0.75, respectively. Its total half-life is 4 h. What are the half-lives of the two channels?

39. ^{57}Co decays to ^{57}Fe by β^+ emission. The resulting ^{57}Fe is in its excited state and comes to the ground state by emitting γ rays. The half-life of β^+ decay is 270 days and that of the γ emission is 10^{-8} s. A sample of ^{57}Co gives 5.0×10^9 γ rays per second. How much time will elapse before the emission rate of γ rays drops to 2.5×10^9 per second.

40. As a health physicalist, you are being consulted about a spill in radiochemistry laboratory. The isotope spilled was 500 μCi ^{131}Ba, which has a half-life of 12 days (a) What mass of ^{131}Ba was spilled? (b) Your recommendation is clear to the laboratory until the radiation level is down to 1.00 μCi. How long will the laboratory be closed?

sec. 41-6 Alpha Decay

41. The alpha decay of the ^{238}U is followed by two beta decay events. What are the proton and the nucleon numbers of the great grandchild of the ^{238}U?

42. How much energy is released when a ^{238}U nucleus decays by emitting (a) an alpha particle and (b) a sequence of neutron, proton, neutron, proton? (c) Convince yourself both by reasoned argument and by direct calculation that the difference between these two numbers is just the total binding energy of the alpha particle. (d) Find that binding energy. Some needed atomic and particle masses are

^{238}U 238.050 79 u	^{234}Th 234.043 63 u
^{237}U 237.048 73 u	^{4}He 4.002 60 u

^{236}Pa 236.048 91 u ^{1}H 1.007 83 u
^{235}Pa 235.045 44 u n 1.008 66 u

43. Under certain rare circumstances, a nucleus can decay by emitting a particle more massive than an alpha particle. Consider the decays

$$^{223}\text{Ra} \rightarrow {}^{209}\text{Pb} + {}^{14}\text{C} \text{ and } {}^{223}\text{Ra} \rightarrow {}^{219}\text{Rn} + {}^{4}\text{He}$$

Calculate the Q value for the (a) first and (b) second decay and determine that both are energetically possible. The needed atomic masses are

^{223}Ra 223.018 50 u ^{14}C 14.003 24 u
^{209}Pb 208.981 07 u ^{4}He 4.002 60 u
^{219}Rn 219.009 48 u

44. A ^{238}U nucleus emits a 4.196 MeV alpha particle. Calculate the disintegration energy Q for this process, taking the recoil energy of the residual ^{234}Th nucleus into account.

sec. 41-7 Beta Decay

45. If one starts with $_{83}$Np^{237}, what nuclide remains after six alpha and three β decays?

46. (a) Is the decay $n \rightarrow p + \beta^- + \bar{\nu}_e$ energetically possible? If not, explain why not. If so, calculate the total energy released. (b) Is the decay $p \rightarrow n + \beta^+ + \nu_e$ energetically possible? If not, explain why not. If so, calculate the total energy released.

47. The cesium isotope ^{137}Cs is present in the fallout from aboveground detonations of nuclear bombs. Because it decays with a slow (30.2 y) half-life into ^{137}Ba, releasing considerable energy in the process, it is of environmental concern. The atomic masses of the Cs and Ba are 136.9071 and 136.9058 u, respectively; calculate the total energy released in such a decay.

48. Some radionuclides decay by capturing one of their own atomic electrons, a K-shell electron, say. An example is

$$^{49}\text{V} + e^- \rightarrow {}^{49}\text{Ti} + \nu, \quad T_{1/2} = 331\text{d}.$$

Show that the disintegration energy Q for this process is given by

$$Q = (m_V - m_{Ti})c^2 - E_K$$

where m_V and m_{Ti} are the atomic masses of ^{49}V and ^{49}Ti, respectively, and E_K is the binding energy of the vanadium K-shell electron. (*Hint:* Put $\mathbf{m}_V$ and $\mathbf{m}_{Ti}$ as the corresponding nuclear masses and proceed as in Sample Problem 41-8.)

49. The radionuclide ^{11}C decays according to

$$^{11}\text{C} \rightarrow {}^{11}\text{B} + e^+ + \nu, \quad T_{1/2} = 20.3 \text{ min}.$$

The maximum energy of the emitted positrons is 0.960 MeV. (a) Show that the disintegration energy Q for this process is given by

$$Q = (m_C - m_B - 2m_e)c^2,$$

where m_C and m_B are the atomic masses of ^{11}C and ^{11}B, respectively, and m_e is the mass of a positron. (b) Given the mass values m_C = 11.011 424 u, m_B = 11.009 305 u, and m_e = 0.000 548 6 u, calculate Q and compare it with the maximum energy of the emitted positron given above. (*Hint:* Let $\mathbf{m}_C$ and $\mathbf{m}_B$ be the nuclear masses and proceed as in Sample Problem 41-8 for beta decay. Note that beta-plus decay is an exception to the general rule that if atomic masses are used in nuclear decay calculations, the mass of the emitted electron is automatically taken care of.)

sec. 41-8 Gamma Decay

50. Calculate the maximum possible energy of the γ rays emitted by an excited B^{11} nucleus formed by the capture of a slow neutron by a B^{10} nucleus. Take atomic mass of B^{10} = 10.012938 amu; B^{11} = 11.009305 amu.

sec. 41-9 Nuclear Models (Optional)

51. In the following list of nuclides, identify (a) those with filled nucleon shells, (b) those with one nucleon outside a filled shell, and (c) those with one vacancy in an otherwise filled shell: ^{13}C, ^{18}O, ^{40}K, ^{49}Ti, ^{60}Ni, ^{91}Zr, ^{92}Mo, ^{121}Sb, ^{143}Nd, ^{144}Sm, ^{205}Tl, and ^{207}Pb.

52. An intermediate nucleus in a particular nuclear reaction decays within 10^{-22} s of its formation. (a) What is the uncertainty ΔE in our knowledge of this intermediate state? (b) Can this state be called a compound nucleus? (See Sample Problem 41-9.)

53. A typical kinetic energy for a nucleon in a middle-mass nucleus may be taken as 5.00 MeV. To what effective nuclear temperature does this correspond, based on the assumptions of the collective model of nuclear structure?

54. Consider the three formation processes shown for the compound nucleus ^{20}Ne in Fig. 41-17. Here are some of the masses:

^{20}Ne 19.992 44 u α 4.002 60 u
^{19}F 18.998 40 u p 1.007 83 u
^{16}O 15.994 91 u

What energy must (a) the alpha particle, (b) the proton, and (c) the γ-ray photon have to provide 25.0 MeV of excitation energy to the compound nucleus?

55. The Sun has a mass of 2.0×10^{30} kg and radiates energy at the rate of 3.9×10^{26} W. (a) At what rate does the Sun transfer its mass to other forms of energy? (b) What fraction of its original mass has the Sun lost in this way since it began to burn hydrogen, about 4.5×10^9 y ago?

56. Roughly 0.0150% of the mass of ordinary water is due to "heavy water," in which one of the two hydrogens in an H_2O molecule is replaced with deuterium, ^{2}H. How much average fusion power could be obtained if we "burned" all the ^{2}H in 1.00 liter of water in 1.00 day by somehow causing the deuterium to fuse via the reaction $^2\text{H} + {}^2\text{H} \rightarrow {}^3\text{He} + \text{n}$?

57. Find the Q values of the reaction $\text{He}^4(\alpha, \text{p})\ \text{Li}^7$. If a beam of α particles is incident on a stationary helium target, calculate the minimum kinetic energy required of the incident α particles to excite this reaction. (Take atomic mass of He^4 = 4.002603 amu and Li^7 = 7.016004 amu.)

ADDITIONAL PROBLEMS

58. In a certain rock, the ratio of lead atoms to uranium atoms is 0.300. Assume that uranium has a half-life of 4.47×10^9 years and that the rock had no lead atoms when it was formed. How old is the rock?

59. Figure 41-21 shows part of the decay scheme of ^{237}Np on a plot of mass number A versus proton number Z; five lines that represent either alpha decay or beta-minus decay connect dots that represent isotopes. What is the isotope at the end of the five decays (as marked with a question mark in Fig. 41-21)?

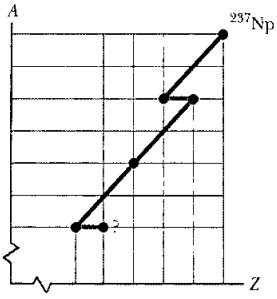


FIG. 41-21 Problem 59.

60. Radioactive element AA can decay to either element BB or element CC. The decay depends on chance, but the ratio of the resulting number of BB atoms to the resulting number of CC atoms is always 2/1. The decay has a half-life of 8.00 days. We start with a sample of pure AA. How long must we wait until the number of CC atoms is 1.50 times the number of AA atoms?

61. How many years are needed to reduce the activity of ^{14}C to 0.020 of its original activity? The half-life of ^{14}C is 5730 y.

62. A certain stable nuclide, after absorbing a neutron, emits an electron, and the new nuclide splits spontaneously into two alpha particles. Identify the nuclide.

63. The effective Q for the proton–proton cycle of Fig. 41-20 is 26.2 MeV. (a) Express this as energy per kilogram of hydrogen consumed. (b) The power of the Sun is 3.9×10^{26} W. If its energy derives from the proton–proton cycle, at what rate is it losing hydrogen? (c) At what rate is it losing mass? (d) Account for the difference in the results for (b) and (c). (e) The mass of the Sun is 2.0×10^{30} kg. If it loses mass at the constant rate calculated in (c) how long will it take to lose 0.10% of its mass?

64. (a) Find the radius of the $^{12}_{6}$C nucleus and (b) the force of repulsion between a proton at the surface of a $^{12}_{6}$C nucleus and the remaining five protons. (c) How much work (in MeV) has to be done to overcome this electrostatic repulsion to put the last proton into the nucleus?

65. The energy of the alpha particle emitted by ^{210}Po ($T_{1/2}$ = 138 days) is 5.30 MeV. (a) What mass of ^{210}Po is needed to power a thermoelectric cell of 1.00-W output if the efficiency of energy conversion is 8.00%? (b) What would the power output be after 1.00 years?

66. A body contains 120 g of stable potassium ^{39}K. 3.2×10^4 Bq of ^{42}K (half-life of 12 h) was injected into the body. After 24 h, a sample of serum was found to contain 200 mg of stable potassium. (a) Calculate the activity of ^{42}K in the sample. (b) What will be the activity of ^{42}K in another 200 mg sample taken 12 h later? Neglect biological excretion, but take nuclear decay into account.

67. $^{197}_{80}$Hg decays to $^{197}_{79}$Au through electron capture. (a) What other particle or particles are emitted in the decay? (b) Assume that the electron is captured from the K shell. Use Mosely's law $\sqrt{\nu} = \alpha(Z - b)$ ($\alpha = 4.95 \times 10^7$ s$^{-1/2}$ and $b = 1$) to find the wavelength of the K_α x ray emitted following the electron capture.

68. In a slow reaction, heat is being evolved at a rate about 10 mW in a liquid. If the heat were being generated by the decay of ^{32}P, a radioactive isotope of phosphorus that has half-life of 14 days emits only beta particles with a mean energy of 700 KeV. Estimate the number of ^{32}P atoms in the liquid.

69. A polonium nucleus ($^{210}_{84}$Po) emits an alpha particlee ($^{4}_{2}$He) and is converted into a lead nucleus ($^{206}_{82}$Pb). This reaction is used for producing electric power in a space mission. ^{210}Po has half-life of 138.6 days. Assuming an efficiency of 10%, (a) how much ^{210}Po is required to produce 1.2×10^7 J of electrical energy per day at the end of 693 days. (b) Also find the activity of the mass at that time. (Given, masses of $^{210}_{84}$Po nuclei = 209.98264 amu, $^{206}_{82}$Pb = 205.97440 amu, $^{4}_{2}$He = 4.00260 amu.)

ANSWERS

Checkpoints

1. –1

2. ^{90}As and ^{158}Nd

3. a little more than 75 Bq (elapsed time is a little less than three half-lives)

4. ^{206}Pb

5. c and d

6. (a) no; (b) yes; (c) no

Reasoning Type Questions

1. (c)

2. (d)

3. (d)

4. (a)

5. (a)

6. (d)

7. (b)

8. (a)

9. (d)

10. (a)

11. (c)

Multiple Choice Questions

1. (d)
2. (d)
3. (d)
4. (a), (d)
5. (c)
6. (b)
7. (c)
8. (c), (d)
9. (d)
10. (d)
11. (a), (c)
12. (a), (c), (d)
13. (c)
14. (b)
15. (a), (d)
16. (d)
17. (b)
18. (d)
19. (d)
20. (d)
21. (c)
22. (b)
23. (b)
24. (a)
25. (d)
26. (a), (c), (d)
27. (c)
28. (b)
29. (b)
30. (a), (b), (c)
31. (c)
32. (a), (b), (c)
33. (b)
34. (a)
35. (b)
36. (a), (b), (c)
37. (c)
38. (c)
39. (c), (d)
40. (a), (b), (c)
41. (c)
42. (c)
43. (c)
44. (a), (b)
45. (c)
46. (a)
47. (a), (c)
48. (a), (d)

Problems

1. 1.3×10^{-13} m or 130 fm
2. (a) 0.390 MeV; (b) 4.61 MeV
3. (a) ${}_1^2\mathrm{H}$; (b) ${}_1^4\mathrm{H}$; (c) ${}_0^1\mathrm{n}$
4. 1.2 fm
5. 1.6×10^4 m
6. 1.8×10^{10} kg
7. (a) 0.007825 u; (b) +7.290 MeV/c^2; (c) 0.008665 u; (d) +8.071 MeV/c^2; (e) – 0.09780 u; (f) – 91.10 MeV/c^2
8. (a) 6 protons; (b) 8 neutrons
9. (a) yttrium; (b) iodine; (c) 50 neutrons; (d) 74 neutrons; (e) 19
10. 1.6×10^{25} MeV
11. (a) 19.8 MeV; (b) 6.26 MeV; (c) 2.23 MeV; (d) 28.3 MeV; (e) 7.07 MeV; (f) No, the answers do not match
12. 7.31 MeV/nucleon
13. (b) 7.92 MeV
14. 1.0087 u
16. (a) 1.93 u; (b) 1.80×10^3; (c) 7.56 MeV/nucleon
17. ^{14}C is the decay product of ^{14}N
18. (a) 0.030 g/h; (b) 0.015 g/h
19. 3.0×10^{19}
20. (a) 64.2 h; (b) $0.125N_0$
21. (a) 5.04×10^{18} nuclei; (b) 4.60×10^6 Bq
22. (a) 3.2×10^{12} Bq; (b) 86 Ci
23. 0.265 g
24. 209 d
25. 1160 s
27. 3.1×10^{-4}
28. 1/2
29. 4×10^{-6} g
30. (a) 1.55×10^{-5}/s; 12.4 h; (b) 2.39×10^{13} atoms; (c) 1.87 mCi
31. 0.58
32. 10 pm
33. 320 mCi
34. 7
35. 0.12%
36. 200 Bq
38. 16 h and 16/3 h

39. 270 days

40. (a) 6 ng; (b) 108 days

41. 92, 234

42. (a) 4.25 MeV; (b) –24.1 MeV; (c) 28.3 MeV

43. (a) 31.8 MeV; (b) 5.98 MeV

44. 4.269 MeV

45. $^{213}_{74}W$

46. (a) Yes, 783 keV; (b) no

47. 1.21 MeV

49. (b) 0.961 MeV

50. 11.44 MeV

51. (a) ^{18}O, ^{60}Ni, ^{92}Mo, ^{144}Sm, and ^{207}Pb each have a filled shell for either the protons or the neutrons; (b) ^{40}K, ^{91}Zr, ^{121}Sb, and ^{143}Nd; (c) ^{13}C, ^{40}K, ^{49}Ti, ^{205}Tl, and ^{207}Pb

52. 6.6 MeV; (b) no

53. 3.87×10^{10} K

54. (a) 25.4 MeV; (b) 12.8 MeV; (c) 25.0 MeV

55. (a) 4.3×10^{9} kg/s; (b) 3.1×10^{-4}

56. 14.4 kW

57. 17.34 MeV

Additional Problems

58. 1.69×10^{9} y

59. $^{225}_{89}Ac$

60. 19.7 d

61. 3.2×10^{4} y

62. $^{7}_{3}Li$

63. (a) 6.3×10^{14} J/kg; (b) 6.2×10^{11} kg/s; (c) 4.3×10^{9} kg/s; (e) 1.5×10^{10} y

64. (a) 2.75 fm; (b) 152 N; (c) 2.62 MeV

65. (a) 0.0884 g; (b) 0.16 W

66. (a) 13.33 Bq; 6.67 Bq

67. (a) neutrino; (b) 20 pm

68. 1.55×10^{17}

69. (a) 320 g; (b) 4.57×10^{21} reaction per day

The International System of Units (SI)*

TABLE 1

The SI Base Units

Quantity	Name	Symbol	Definition
length	meter	m	". . . the length of the path traveled by light *in vacuum* in 1/299,792,458 of a second." (1983)
mass	kilogram	kg	". . . this prototype [a certain platinum–iridium cylinder] shall henceforth be considered to be the unit of mass." (1889)
time	second	s	". . . the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom." (1967)
electric current	ampere	A	". . . that constant current which, if maintained in two straight parallel conductors of *infinite* length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length." (1946)
thermodynamic temperature	kelvin	K	". . . the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." (1967)
amount of substance	mole	mol	". . . the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12." (1971)
luminous intensity	candela	cd	". . . the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." (1979)

*Adapted from "The International System of Units (SI)," National Bureau of Standards Special Publication 330, 1972 edition. The definitions above were adopted by the General Conference of Weights and Measures, an international body, on the dates shown. In this book we do not use the candela.

TABLE 2

Some SI Derived Units

Quantity	Name of Unit	Symbol	
area	square meter	m^2	
volume	cubic meter	m^3	
frequency	hertz	Hz	s^{-1}
mass density (density)	kilogram per cubic meter	kg/m^3	
speed, velocity	meter per second	m/s	
angular velocity	radian per second	rad/s	
acceleration	meter per second per second	m/s^2	
angular acceleration	radian per second per second	rad/s^2	
force	newton	N	$kg \cdot m/s^2$
pressure	pascal	Pa	N/m^2
work, energy, quantity of heat	joule	J	$N \cdot m$
power	watt	W	J/s
quantity of electric charge	coulomb	C	$A \cdot s$
potential difference, electromotive force	volt	V	W/A
electric field strength	volt per meter (or newton per coulomb)	V/m	N/C
electric resistance	ohm	Ω	V/A
capacitance	farad	F	$A \cdot s/V$
magnetic flux	weber	Wb	$V \cdot s$
inductance	henry	H	$V \cdot s/A$
magnetic flux density	tesla	T	Wb/m^2
magnetic field strength	ampere per meter	A/m	
entropy	joule per kelvin	J/K	
specific heat	joule per kilogram kelvin	$J/(kg \cdot K)$	
thermal conductivity	watt per meter kelvin	$W/(m \cdot K)$	
radiant intensity	watt per steradian	W/sr	

TABLE 3

The SI Supplementary Units

Quantity	Name of Unit	Symbol
plane angle	radian	rad
solid angle	steradian	sr

Some Fundamental Constants of Physics*

Constant	Symbol	Computational Value	Best (1998) Value	
			Value[a]	Uncertainty[b]
Speed of light in a vacuum	c	3.00×10^{8} m/s	2.997 924 58	exact
Elementary charge	e	1.60×10^{-19} C	1.602 176 462	0.039
Gravitational constant	G	6.67×10^{-11} m^3/s$^2 \cdot$ kg	6.673	1500
Universal gas constant	R	8.31 J/mol $\cdot$ K	8.314 472	1.7
Avogadro constant	N_A	6.02×10^{23} mol^{-1}	6.022 141 99	0.079
Boltzmann constant	k	1.38×10^{-23} J/K	1.380 650 3	1.7
Stefan–Boltzmann constant	σ	5.67×10^{-8} W/m$^2 \cdot$ K^4	5.670 400	7.0
Molar volume of ideal gas at STP[d]	V_m	2.27×10^{-2} m^3/mol	2.271 098 1	1.7
Permittivity constant	ϵ_0	8.85×10^{-12} F/m	8.854 187 817 62	exact
Permeability constant	μ_0	1.26×10^{-6} H/m	1.256 637 061 43	exact
Planck constant	h	6.63×10^{-34} J $\cdot$ s	6.626 068 76	0.078
Electron mass[c]	m_e	9.11×10^{-31} kg	9.109 381 88	0.079
		5.49×10^{-4} u	5.485 799 110	0.0021
Proton mass[c]	m_p	1.67×10^{-27} kg	1.672 621 58	0.079
		1.0073 u	1.007 276 466 88	1.3×10^{-4}
Ratio of proton mass to electron mass	m_p/m_e	1840	1836.152 667 5	0.0021
Electron charge-to-mass ratio	e/m_e	1.76×10^{11} C/kg	1.758 820 174	0.040
Neutron mass[c]	m_n	1.68×10^{-27} kg	1.674 927 16	0.079
		1.0087 u	1.008 664 915 78	5.4×10^{-4}
Hydrogen atom mass[c]	$m_{^1\mathrm{H}}$	1.0078 u	1.007 825 031 6	0.0005
Deuterium atom mass[c]	$m_{^2\mathrm{H}}$	2.0141 u	2.014 101 777 9	0.0005
Helium atom mass[c]	$m_{^4\mathrm{He}}$	4.0026 u	4.002 603 2	0.067
Muon mass	m_μ	1.88×10^{-28} kg	1.883 531 09	0.084
Electron magnetic moment	μ_e	9.28×10^{-24} J/T	9.284 763 62	0.040
Proton magnetic moment	μ_p	1.41×10^{-26} J/T	1.410 606 663	0.041
Bohr magneton	μ_B	9.27×10^{-24} J/T	9.274 008 99	0.040
Nuclear magneton	μ_N	5.05×10^{-27} J/T	5.050 783 17	0.040
Bohr radius	a	5.29×10^{-11} m	5.291 772 083	0.0037
Rydberg constant	R	1.10×10^{7} m^{-1}	1.097 373 156 854 8	7.6×10^{-6}
Electron Compton wavelength	λ_C	2.43×10^{-12} m	2.426 310 215	0.0073

[a]Values given in this column should be given the same unit and power of 10 as the computational value.

[b]Parts per million.

[c]Masses given in u are in unified atomic mass units, where 1 u = $1.660\,538\,86 \times 10^{-27}$ kg.

[d]STP means standard temperature and pressure: 0°C and 1.0 atm (0.1 MPa).

*The values in this table were selected from the 1998 CODATA recommended values (www.physics.nist.gov).

C Some Astronomical Data

Some Distances from Earth

To the Moon*	3.82×10^8 m	To the center of our galaxy	2.2×10^{20} m
To the Sun*	1.50×10^{11} m	To the Andromeda Galaxy	2.1×10^{22} m
To the nearest star (Proxima Centauri)	4.04×10^{16} m	To the edge of the observable universe	$\sim 10^{26}$ m

*Mean distance.

The Sun, Earth, and the Moon

Property	Unit	Sun	Earth	Moon
Mass	kg	1.99×10^{30}	5.98×10^{24}	7.36×10^{22}
Mean radius	m	6.96×10^8	6.37×10^6	1.74×10^6
Mean density	kg/m^3	1410	5520	3340
Free-fall acceleration at the surface	m/s^2	274	9.81	1.67
Escape velocity	km/s	618	11.2	2.38
Period of rotation[a]	—	37 d at poles[b] 26 d at equator[b]	23 h 56 min	27.3 d
Radiation power[c]	W	3.90×10^{26}		

[a]Measured with respect to the distant stars.

[b]The Sun, a ball of gas, does not rotate as a rigid body.

[c]Just outside Earth's atmosphere solar energy is received, assuming normal incidence, at the rate of 1340 W/m^2.

Some Properties of the Planets

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Mean distance from Sun, 10^6 km	57.9	108	150	228	778	1430	2870	4500	5900
Period of revolution, y	0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248
Period of rotation,[a] d	58.7	−243[b]	0.997	1.03	0.409	0.426	−0.451[b]	0.658	6.39
Orbital speed, km/s	47.9	35.0	29.8	24.1	13.1	9.64	6.81	5.43	4.74
Inclination of axis to orbit	<28°	≈3°	23.4°	25.0°	3.08°	26.7°	97.9°	29.6°	57.5°
Inclination of orbit to Earth's orbit	7.00°	3.39°		1.85°	1.30°	2.49°	0.77°	1.77°	17.2°
Eccentricity of orbit	0.206	0.0068	0.0167	0.0934	0.0485	0.0556	0.0472	0.0086	0.250
Equatorial diameter, km	4880	12 100	12 800	6790	143 000	120 000	51 800	49 500	2300
Mass (Earth = 1)	0.0558	0.815	1.000	0.107	318	95.1	14.5	17.2	0.002
Density (water = 1)	5.60	5.20	5.52	3.95	1.31	0.704	1.21	1.67	2.03
Surface value of g,[c] m/s^2	3.78	8.60	9.78	3.72	22.9	9.05	7.77	11.0	0.5
Escape velocity,[c] km/s	4.3	10.3	11.2	5.0	59.5	35.6	21.2	23.6	1.1
Known satellites	0	0	1	2	60 + ring	31 + rings	21 + rings	11 + rings	1

[a]Measured with respect to the distant stars.

[b]Venus and Uranus rotate opposite their orbital motion.

[c]Gravitational acceleration measured at the planet's equator.

Conversion Factors D

Conversion factors may be read directly from these tables. For example, 1 degree = 2.778×10^{-3} revolutions, so $16.7° = 16.7 \times 2.778 \times 10^{-3}$ rev. The SI units are fully capitalized. Adapted in part from G. Shortley and D. Williams, *Elements of Physics,* 1971, Prentice-Hall, Englewood Cliffs, NJ.

Plane Angle

°	′	″	RADIAN	rev
1 degree = 1	60	3600	1.745×10^{-2}	2.778×10^{-3}
1 minute = 1.667×10^{-2}	1	60	2.909×10^{-4}	4.630×10^{-5}
1 second = 2.778×10^{-4}	1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 RADIAN = 57.30	3438	2.063×10^{5}	1	0.1592
1 revolution = 360	2.16×10^{4}	1.296×10^{6}	6.283	1

Solid Angle

1 sphere = 4π steradians = 12.57 steradians

Length

cm	METER	km	in.	ft	mi
1 centimeter = 1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 METER = 100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer = 10^{5}	1000	1	3.937×10^{4}	3281	0.6214
1 inch = 2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot = 30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile = 1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

1 angström = 10^{-10} m
1 nautical mile = 1852 m = 1.151 miles = 6076 ft
1 fermi = 10^{-15} m
1 light-year = 9.461×10^{12} km
1 parsec = 3.084×10^{13} km
1 fathom = 6 ft
1 Bohr radius = 5.292×10^{-11} m
1 yard = 3 ft
1 rod = 16.5 ft
1 mil = 10^{-3} in.
1 nm = 10^{-9} m

Area

METER^2	cm^2	ft^2	in.^{2}
1 SQUARE METER = 1	10^{4}	10.76	1550
1 square centimeter = 10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot = 9.290×10^{-2}	929.0	1	144
1 square inch = 6.452×10^{-4}	6.452	6.944×10^{-3}	1

1 square mile = 2.788×10^{7} ft^2 = 640 acres
1 barn = 10^{-28} m^2
1 acre = 43 560 ft^2
1 hectare = 10^{4} m^2 = 2.471 acres

Volume

METER^3	cm^3	L	ft^3	in.^{3}
1 CUBIC METER = 1	10^6	1000	35.31	6.102×10^4
1 cubic centimeter = 10^{-6}	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter = 1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot = 2.832×10^{-2}	2.832×10^4	28.32	1	1728
1 cubic inch = 1.639×10^{-5}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. pints = 128 U.S. fluid ounces = 231 in.^{3}

1 British imperial gallon = 277.4 in.^{3} = 1.201 U.S. fluid gallons

Mass

Quantities in the colored areas are not mass units but are often used as such. For example, when we write 1 kg "=" 2.205 lb, this means that a kilogram is a *mass* that *weighs* 2.205 pounds at a location where g has the standard value of 9.80665 m/s^2.

g	KILOGRAM	slug	u	oz	lb	ton
1 gram = 1	0.001	6.852×10^{-5}	6.022×10^{23}	3.527×10^{-2}	2.205×10^{-3}	1.102×10^{-6}
1 KILOGRAM = 1000	1	6.852×10^{-2}	6.022×10^{26}	35.27	2.205	1.102×10^{-3}
1 slug = 1.459×10^4	14.59	1	8.786×10^{27}	514.8	32.17	1.609×10^{-2}
1 atomic mass unit = 1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1	5.857×10^{-26}	3.662×10^{-27}	1.830×10^{-30}
1 ounce = 28.35	2.835×10^{-2}	1.943×10^{-3}	1.718×10^{25}	1	6.250×10^{-2}	3.125×10^{-5}
1 pound = 453.6	0.4536	3.108×10^{-2}	2.732×10^{26}	16	1	0.0005
1 ton = 9.072×10^5	907.2	62.16	5.463×10^{29}	3.2×10^4	2000	1

1 metric ton = 1000 kg

Density

Quantities in the colored areas are weight densities and, as such, are dimensionally different from mass densities. See the note for the mass table.

slug/ft^3	KILOGRAM/ METER^3	g/cm^3	lb/ft^3	lb/in.^{3}
1 slug per foot^3 = 1	515.4	0.5154	32.17	1.862×10^{-2}
1 KILOGRAM per METER^3 = 1.940×10^{-3}	1	0.001	6.243×10^{-2}	3.613×10^{-5}
1 gram per centimeter^3 = 1.940	1000	1	62.43	3.613×10^{-2}
1 pound per foot^3 = 3.108×10^{-2}	16.02	16.02×10^{-2}	1	5.787×10^{-4}
1 pound per inch^3 = 53.71	2.768×10^4	27.68	1728	1

Time

y	d	h	min	SECOND
1 year = 1	365.25	8.766×10^3	5.259×10^5	3.156×10^7
1 day = 2.738×10^{-3}	1	24	1440	8.640×10^4
1 hour = 1.141×10^{-4}	4.167×10^{-2}	1	60	3600
1 minute = 1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 SECOND = 3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

Speed

	ft/s	km/h	METER/SECOND	mi/h	cm/s
1 foot per second =	1	1.097	0.3048	0.6818	30.48
1 kilometer per hour =	0.9113	1	0.2778	0.6214	27.78
1 METER per SECOND =	3.281	3.6	1	2.237	100
1 mile per hour =	1.467	1.609	0.4470	1	44.70
1 centimeter per second =	3.281×10^{-2}	3.6×10^{-2}	0.01	2.237×10^{-2}	1

1 knot = 1 nautical mi/h = 1.688 ft/s 1 mi/min = 88.00 ft/s = 60.00 mi/h

Force

Force units in the colored areas are now little used. To clarify: 1 gram-force (= 1 gf) is the force of gravity that would act on an object whose mass is 1 gram at a location where g has the standard value of 9.80665 m/s^2.

	dyne	NEWTON	lb	pdl	gf	kgf
1 dyne =	1	10^{-5}	2.248×10^{-6}	7.233×10^{-5}	1.020×10^{-3}	1.020×10^{-6}
1 NEWTON =	10^5	1	0.2248	7.233	102.0	0.1020
1 pound =	4.448×10^5	4.448	1	32.17	453.6	0.4536
1 poundal =	1.383×10^4	0.1383	3.108×10^{-2}	1	14.10	1.410×10^2
1 gram-force =	980.7	9.807×10^{-3}	2.205×10^{-3}	7.093×10^{-2}	1	0.001
1 kilogram-force =	9.807×10^5	9.807	2.205	70.93	1000	1

1 ton = 2000 lb

Pressure

	atm	dyne/cm^2	inch of water	cm Hg	PASCAL	lb/in.^{2}	lb/ft^2
1 atmosphere =	1	1.013×10^6	406.8	76	1.013×10^5	14.70	2116
1 dyne per centimeter^2 =	9.869×10^{-7}	1	4.015×10^{-4}	7.501×10^{-5}	0.1	1.405×10^{-5}	2.089×10^{-3}
1 inch of water[a] at 4°C =	2.458×10^{-3}	2491	1	0.1868	249.1	3.613×10^{-2}	5.202
1 centimeter of mercury[a] at 0°C =	1.316×10^{-2}	1.333×10^4	5.353	1	1333	0.1934	27.85
1 PASCAL =	9.869×10^{-6}	10	4.015×10^{-3}	7.501×10^{-4}	1	1.450×10^{-4}	2.089×10^{-2}
1 pound per inch^2 =	6.805×10^{-2}	6.895×10^4	27.68	5.171	6.895×10^3	1	144
1 pound per foot^2 =	4.725×10^{-4}	478.8	0.1922	3.591×10^{-2}	47.88	6.944×10^{-3}	1

[a]Where the acceleration of gravity has the standard value of 9.80665 m/s^2.

1 bar = 10^6 dyne/cm^2 = 0.1 MPa 1 millibar = 10^3 dyne/cm^2 = 10^2 Pa 1 torr = 1 mm Hg

Energy, Work, Heat

Quantities in the colored areas are not energy units but are included for convenience. They arise from the relativistic mass–energy equivalence formula $E = mc^2$ and represent the energy released if a kilogram or unified atomic mass unit (u) is completely converted to energy (bottom two rows) or the mass that would be completely converted to one unit of energy (rightmost two columns).

	Btu	erg	ft · lb	hp · h	JOULE	cal	kW · h	eV	MeV	kg	u
1 British thermal unit =	1	1.055×10^{10}	777.9	3.929×10^{-4}	1055	252.0	2.930×10^{-4}	6.585×10^{21}	6.585×10^{15}	1.174×10^{-14}	7.070×10^{12}
1 erg =	9.481×10^{-11}	1	7.376×10^{-8}	3.725×10^{-14}	10^{-7}	2.389×10^{-8}	2.778×10^{-14}	6.242×10^{11}	6.242×10^{5}	1.113×10^{-24}	670.2
1 foot-pound =	1.285×10^{-3}	1.356×10^{7}	1	5.051×10^{-7}	1.356	0.3238	3.766×10^{-7}	8.464×10^{18}	8.464×10^{12}	1.509×10^{-17}	9.037×10^{9}
1 horsepower-hour =	2545	2.685×10^{13}	1.980×10^{6}	1	2.685×10^{6}	6.413×10^{5}	0.7457	1.676×10^{25}	1.676×10^{19}	2.988×10^{-11}	1.799×10^{16}
1 JOULE =	9.481×10^{-4}	10^{7}	0.7376	3.725×10^{-7}	1	0.2389	2.778×10^{-7}	6.242×10^{18}	6.242×10^{12}	1.113×10^{-17}	6.702×10^{9}
1 calorie =	3.968×10^{-3}	4.1868×10^{7}	3.088	1.560×10^{-6}	4.1868	1	1.163×10^{-6}	2.613×10^{19}	2.613×10^{13}	4.660×10^{-17}	2.806×10^{10}
1 kilowatt-hour =	3413	3.600×10^{13}	2.655×10^{6}	1.341	3.600×10^{6}	8.600×10^{5}	1	2.247×10^{25}	2.247×10^{19}	4.007×10^{-11}	2.413×10^{16}
1 electron-volt =	1.519×10^{-22}	1.602×10^{-12}	1.182×10^{-19}	5.967×10^{-26}	1.602×10^{-19}	3.827×10^{-20}	4.450×10^{-26}	1	10^{-6}	1.783×10^{-36}	1.074×10^{-9}
1 million electron-volts =	1.519×10^{-16}	1.602×10^{-6}	1.182×10^{-13}	5.967×10^{-20}	1.602×10^{-13}	3.827×10^{-14}	4.450×10^{-20}	10^{-6}	1	1.783×10^{-30}	1.074×10^{-3}
1 kilogram =	8.521×10^{13}	8.987×10^{23}	6.629×10^{16}	3.348×10^{10}	8.987×10^{16}	2.146×10^{16}	2.497×10^{10}	5.610×10^{35}	5.610×10^{29}	1	6.022×10^{26}
1 unified atomic mass unit =	1.415×10^{-13}	1.492×10^{-3}	1.101×10^{-10}	5.559×10^{-17}	1.492×10^{-10}	3.564×10^{-11}	4.146×10^{-17}	9.320×10^{8}	932.0	1.661×10^{-27}	1

Power

	Btu/h	ft · lb/s	hp	cal/s	kW	WATT
1 British thermal unit per hour =	1	0.2161	3.929×10^{-4}	6.998×10^{-2}	2.930×10^{-4}	0.2930
1 foot-pound per second =	4.628	1	1.818×10^{-3}	0.3239	1.356×10^{-3}	1.356
1 horsepower =	2545	550	1	178.1	0.7457	745.7
1 calorie per second =	14.29	3.088	5.615×10^{-3}	1	4.186×10^{-3}	4.186
1 kilowatt =	3413	737.6	1.341	238.9	1	1000
1 WATT =	3.413	0.7376	1.341×10^{-3}	0.2389	0.001	1

Magnetic Field

	gauss	TESLA	milligauss
1 gauss =	1	10^{-4}	1000
1 TESLA =	10^{4}	1	10^{7}
1 milligauss =	0.001	10^{-7}	1

1 tesla = 1 weber/meter^2

Magnetic Flux

	maxwell	WEBER
1 maxwell =	1	10^{-8}
1 WEBER =	10^{8}	1

Mathematical Formulas

Geometry

Circle of radius r: circumference $= 2\pi r$; area $= \pi r^2$.

Sphere of radius r: area $= 4\pi r^2$; volume $= \frac{4}{3}\pi r^3$.

Right circular cylinder of radius r and height h:
area $= 2\pi r^2 + 2\pi rh$; volume $= \pi r^2 h$.

Triangle of base a and altitude h: area $= \frac{1}{2}ah$.

Quadratic Formula

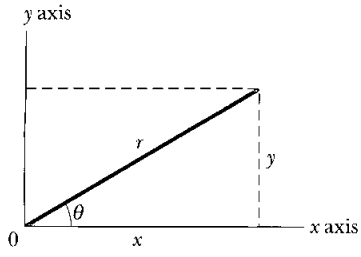
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Trigonometric Functions of Angle θ

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r}$$

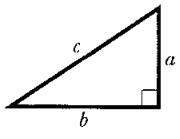
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\sec\theta = \frac{r}{x} \quad \csc\theta = \frac{r}{y}$$



Pythagorean Theorem

In this right triangle,
$a^2 + b^2 = c^2$



Triangles

Angles are A, B, C

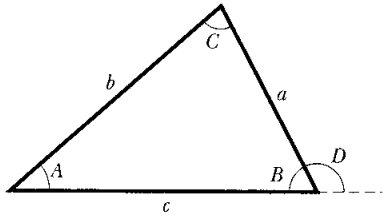
Opposite sides are a, b, c

Angles $A + B + C = 180°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$c^2 = a^2 + b^2 - 2ab\cos C$

Exterior angle $D = A + C$



Mathematical Signs and Symbols

$=$ equals

$\approx$ equals approximately

$\sim$ is the order of magnitude of

$\neq$ is not equal to

$\equiv$ is identical to, is defined as

$>$ is greater than ($\gg$ is much greater than)

$<$ is less than ($\ll$ is much less than)

$\geq$ is greater than or equal to (or, is no less than)

$\leq$ is less than or equal to (or, is no more than)

$\pm$ plus or minus

$\propto$ is proportional to

Σ the sum of

x_{avg} the average value of x

Trigonometric Identities

$\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$

$\sin\theta/\cos\theta = \tan\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta - \tan^2\theta = 1$

$\csc^2\theta - \cot^2\theta = 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$

$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$

$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$

Binomial Theorem

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots \qquad (x^2 < 1)$$

Exponential Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Logarithmic Expansion

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots \qquad (|x| < 1)$$

Trigonometric Expansions (θ in radians)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Cramer's Rule

Two simultaneous equations in unknowns x and y,

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2,$$

have the solutions

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Products of Vectors

Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ be unit vectors in the x, y, and z directions. Then

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1, \quad \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0,$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0,$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

Any vector $\vec{a}$ with components a_x, a_y, and a_z along the x, y, and z axes can be written as

$$\vec{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}.$$

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be arbitrary vectors with magnitudes a, b, and c. Then

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

$$(s\vec{a}) \times \vec{b} = \vec{a} \times (s\vec{b}) = s(\vec{a} \times \vec{b}) \qquad (s = \text{a scalar}).$$

Let θ be the smaller of the two angles between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = a_xb_x + a_yb_y + a_zb_z = ab\cos\theta$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{\mathbf{i}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= (a_yb_z - b_ya_z)\hat{\mathbf{i}} + (a_zb_x - b_za_x)\hat{\mathbf{j}} + (a_xb_y - b_xa_y)\hat{\mathbf{k}}$$

$$|\vec{a} \times \vec{b}| = ab\sin\theta$$

$$\vec{a}\cdot(\vec{b} \times \vec{c}) = \vec{b}\cdot(\vec{c} \times \vec{a}) = \vec{c}\cdot(\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Derivatives and Integrals

In what follows, the letters u and v stand for any functions of x, and a and m are constants. To each of the indefinite integrals should be added an arbitrary constant of integration. The *Handbook of Chemistry and Physics* (CRC Press Inc.) gives a more extensive tabulation.

1. $\dfrac{dx}{dx} = 1$
2. $\dfrac{d}{dx}(au) = a\dfrac{du}{dx}$
3. $\dfrac{d}{dx}(u + v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$
4. $\dfrac{d}{dx}x^m = mx^{m-1}$
5. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
6. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
7. $\dfrac{d}{dx}e^x = e^x$
8. $\dfrac{d}{dx}\sin x = \cos x$
9. $\dfrac{d}{dx}\cos x = -\sin x$
10. $\dfrac{d}{dx}\tan x = \sec^2 x$
11. $\dfrac{d}{dx}\cot x = -\csc^2 x$
12. $\dfrac{d}{dx}\sec x = \tan x \sec x$
13. $\dfrac{d}{dx}\csc x = -\cot x \csc x$
14. $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$
15. $\dfrac{d}{dx}\sin u = \cos u\dfrac{du}{dx}$
16. $\dfrac{d}{dx}\cos u = -\sin u\dfrac{du}{dx}$

1. $\displaystyle\int dx = x$
2. $\displaystyle\int au\,dx = a\int u\,dx$
3. $\displaystyle\int (u + v)\,dx = \int u\,dx + \int v\,dx$
4. $\displaystyle\int x^m\,dx = \frac{x^{m+1}}{m+1}\quad (m \neq -1)$
5. $\displaystyle\int \frac{dx}{x} = \ln|x|$
6. $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
7. $\displaystyle\int e^x\,dx = e^x$
8. $\displaystyle\int \sin x\,dx = -\cos x$
9. $\displaystyle\int \cos x\,dx = \sin x$
10. $\displaystyle\int \tan x\,dx = \ln|\sec x|$
11. $\displaystyle\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x$
12. $\displaystyle\int e^{-ax}\,dx = -\frac{1}{a}e^{-ax}$
13. $\displaystyle\int xe^{-ax}\,dx = -\frac{1}{a^2}(ax + 1)\,e^{-ax}$
14. $\displaystyle\int x^2e^{-ax}\,dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2)e^{-ax}$
15. $\displaystyle\int_0^\infty x^ne^{-ax}\,dx = \frac{n!}{a^{n+1}}$
16. $\displaystyle\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$
17. $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$
18. $\displaystyle\int \frac{x\,dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{(x^2 + a^2)^{1/2}}$
19. $\displaystyle\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2(x^2 + a^2)^{1/2}}$
20. $\displaystyle\int_0^\infty x^{2n+1}\,e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}\quad (a > 0)$
21. $\displaystyle\int \frac{x\,dx}{x + d} = x - d\ln(x + d)$

F Properties of the Elements

All physical properties are for a pressure of 1 atm unless otherwise specified.

Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g · °C) at 25°C
Actinium	Ac	89	(227)	10.06	1323	(3473)	0.092
Aluminum	Al	13	26.9815	2.699	660	2450	0.900
Americium	Am	95	(243)	13.67	1541	—	—
Antimony	Sb	51	121.75	6.691	630.5	1380	0.205
Argon	Ar	18	39.948	1.6626×10^{-3}	−189.4	−185.8	0.523
Arsenic	As	33	74.9216	5.78	817 (28 atm)	613	0.331
Astatine	At	85	(210)	—	(302)	—	—
Barium	Ba	56	137.34	3.594	729	1640	0.205
Berkelium	Bk	97	(247)	14.79	—	—	—
Beryllium	Be	4	9.0122	1.848	1287	2770	1.83
Bismuth	Bi	83	208.980	9.747	271.37	1560	0.122
Bohrium	Bh	107	262.12	—	—	—	—
Boron	B	5	10.811	2.34	2030	—	1.11
Bromine	Br	35	79.909	3.12 (liquid)	−7.2	58	0.293
Cadmium	Cd	48	112.40	8.65	321.03	765	0.226
Calcium	Ca	20	40.08	1.55	838	1440	0.624
Californium	Cf	98	(251)	—	—	—	—
Carbon	C	6	12.01115	2.26	3727	4830	0.691
Cerium	Ce	58	140.12	6.768	804	3470	0.188
Cesium	Cs	55	132.905	1.873	28.40	690	0.243
Chlorine	Cl	17	35.453	3.214×10^{-3} (0°C)	−101	−34.7	0.486
Chromium	Cr	24	51.996	7.19	1857	2665	0.448
Cobalt	Co	27	58.9332	8.85	1495	2900	0.423
Copper	Cu	29	63.54	8.96	1083.40	2595	0.385
Curium	Cm	96	(247)	13.3	—	—	—
Darmstadtium	Ds	110	(271)	—	—	—	—
Dubnium	Db	105	262.114	—	—	—	—
Dysprosium	Dy	66	162.50	8.55	1409	2330	0.172
Einsteinium	Es	99	(254)	—	—	—	—
Erbium	Er	68	167.26	9.15	1522	2630	0.167
Europium	Eu	63	151.96	5.243	817	1490	0.163
Fermium	Fm	100	(237)	—	—	—	—
Fluorine	F	9	18.9984	1.696×10^{-3} (0°C)	−219.6	−188.2	0.753
Francium	Fr	87	(223)	—	(27)	—	—
Gadolinium	Gd	64	157.25	7.90	1312	2730	0.234
Gallium	Ga	31	69.72	5.907	29.75	2237	0.377
Germanium	Ge	32	72.59	5.323	937.25	2830	0.322
Gold	Au	79	196.967	19.32	1064.43	2970	0.131

Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g · °C) at 25°C
Hafnium	Hf	72	178.49	13.31	2227	5400	0.144
Hassium	Hs	108	(265)	—	—	—	—
Helium	He	2	4.0026	0.1664×10^{-3}	−269.7	−268.9	5.23
Holmium	Ho	67	164.930	8.79	1470	2330	0.165
Hydrogen	H	1	1.00797	0.08375×10^{-3}	−259.19	−252.7	14.4
Indium	In	49	114.82	7.31	156.634	2000	0.233
Iodine	I	53	126.9044	4.93	113.7	183	0.218
Iridium	Ir	77	192.2	22.5	2447	(5300)	0.130
Iron	Fe	26	55.847	7.874	1536.5	3000	0.447
Krypton	Kr	36	83.80	3.488×10^{-3}	−157.37	−152	0.247
Lanthanum	La	57	138.91	6.189	920	3470	0.195
Lawrencium	Lr	103	(257)	—	—	—	—
Lead	Pb	82	207.19	11.35	327.45	1725	0.129
Lithium	Li	3	6.939	0.534	180.55	1300	3.58
Lutetium	Lu	71	174.97	9.849	1663	1930	0.155
Magnesium	Mg	12	24.312	1.738	650	1107	1.03
Manganese	Mn	25	54.9380	7.44	1244	2150	0.481
Meitnerium	Mt	109	(266)	—	—	—	—
Mendelevium	Md	101	(256)	—	—	—	—
Mercury	Hg	80	200.59	13.55	−38.87	357	0.138
Molybdenum	Mo	42	95.94	10.22	2617	5560	0.251
Neodymium	Nd	60	144.24	7.007	1016	3180	0.188
Neon	Ne	10	20.183	0.8387×10^{-3}	−248.597	−246.0	1.03
Neptunium	Np	93	(237)	20.25	637	—	1.26
Nickel	Ni	28	58.71	8.902	1453	2730	0.444
Niobium	Nb	41	92.906	8.57	2468	4927	0.264
Nitrogen	N	7	14.0067	1.1649×10^{-3}	−210	−195.8	1.03
Nobelium	No	102	(255)	—	—	—	—
Osmium	Os	76	190.2	22.59	3027	5500	0.130
Oxygen	O	8	15.9994	1.3318×10^{-3}	−218.80	−183.0	0.913
Palladium	Pd	46	106.4	12.02	1552	3980	0.243
Phosphorus	P	15	30.9738	1.83	44.25	280	0.741
Platinum	Pt	78	195.09	21.45	1769	4530	0.134
Plutonium	Pu	94	(244)	19.8	640	3235	0.130
Polonium	Po	84	(210)	9.32	254	—	—
Potassium	K	19	39.102	0.862	63.20	760	0.758
Praseodymium	Pr	59	140.907	6.773	931	3020	0.197
Promethium	Pm	61	(145)	7.22	(1027)	—	—
Protactinium	Pa	91	(231)	15.37 (estimated)	(1230)	—	—
Radium	Ra	88	(226)	5.0	700	—	—
Radon	Rn	86	(222)	9.96×10^{-3} (0°C)	(−71)	−61.8	0.092
Rhenium	Re	75	186.2	21.02	3180	5900	0.134
Rhodium	Rh	45	102.905	12.41	1963	4500	0.243
Rubidium	Rb	37	85.47	1.532	39.49	688	0.364
Ruthenium	Ru	44	101.107	12.37	2250	4900	0.239
Rutherfordium	Rf	104	261.11	—	—	—	—

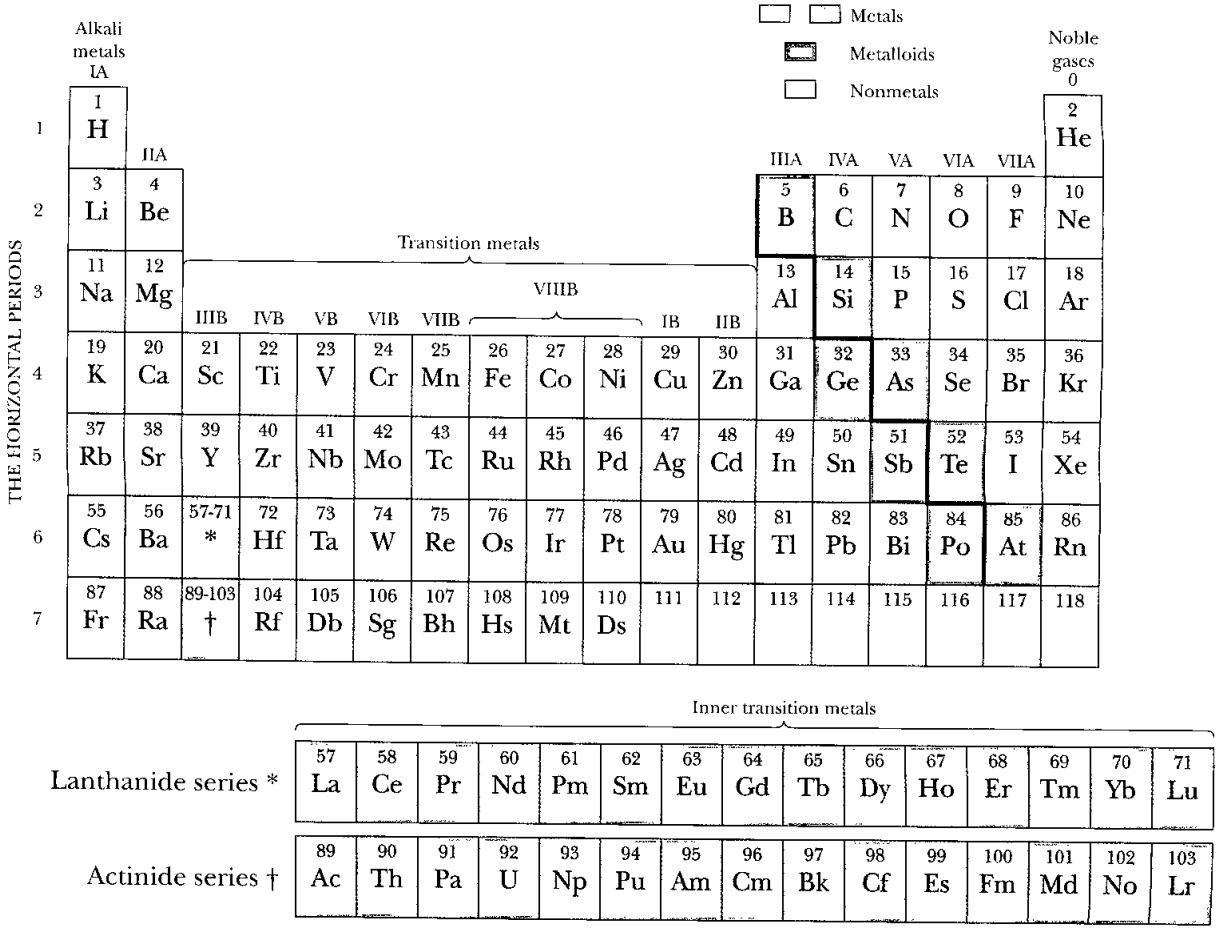
Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g · °C) at 25°C
Samarium	Sm	62	150.35	7.52	1072	1630	0.197
Scandium	Sc	21	44.956	2.99	1539	2730	0.569
Seaborgium	Sg	106	263.118	—	—	—	—
Selenium	Se	34	78.96	4.79	221	685	0.318
Silicon	Si	14	28.086	2.33	1412	2680	0.712
Silver	Ag	47	107.870	10.49	960.8	2210	0.234
Sodium	Na	11	22.9898	0.9712	97.85	892	1.23
Strontium	Sr	38	87.62	2.54	768	1380	0.737
Sulfur	S	16	32.064	2.07	119.0	444.6	0.707
Tantalum	Ta	73	180.948	16.6	3014	5425	0.138
Technetium	Tc	43	(99)	11.46	2200	—	0.209
Tellurium	Te	52	127.60	6.24	449.5	990	0.201
Terbium	Tb	65	158.924	8.229	1357	2530	0.180
Thallium	Tl	81	204.37	11.85	304	1457	0.130
Thorium	Th	90	(232)	11.72	1755	(3850)	0.117
Thulium	Tm	69	168.934	9.32	1545	1720	0.159
Tin	Sn	50	118.69	7.2984	231.868	2270	0.226
Titanium	Ti	22	47.90	4.54	1670	3260	0.523
Tungsten	W	74	183.85	19.3	3380	5930	0.134
Unnamed	Uuu	111	(272)	—	—	—	—
Unnamed	Uub	112	(285)	—	—	—	—
Unnamed	Uut	113	—	—	—	—	—
Unnamed	Unq	114	(289)	—	—	—	—
Unnamed	Uup	115	—	—	—	—	—
Unnamed	Uuh	116	—	—	—	—	—
Unnamed	Uus	117	—	—	—	—	—
Unnamed	Uuo	118	(293)	—	—	—	—
Uranium	U	92	(238)	18.95	1132	3818	0.117
Vanadium	V	23	50.942	6.11	1902	3400	0.490
Xenon	Xe	54	131.30	5.495×10^{-3}	−111.79	−108	0.159
Ytterbium	Yb	70	173.04	6.965	824	1530	0.155
Yttrium	Y	39	88.905	4.469	1526	3030	0.297
Zinc	Zn	30	65.37	7.133	419.58	906	0.389
Zirconium	Zr	40	91.22	6.506	1852	3580	0.276

The values in parentheses in the column of molar masses are the mass numbers of the longest-lived isotopes of those elements that are radioactive. Melting points and boiling points in parentheses are uncertain.

The data for gases are valid only when these are in their usual molecular state, such as H_2, He, O_2, Ne, etc. The specific heats of the gases are the values at constant pressure.

Source: Adapted from J. Emsley, *The Elements,* 3rd ed., 1998, Clarendon Press, Oxford. See also www.webelements.com for the latest values and newest elements.

Periodic Table of the Elements



Elements 111, 112, 114, and 116 have been discovered but, as of 2003, have not yet been named. Evidence for the discovery of elements 113 and 115 has been reported. See www.webelements.com for the latest information and newest elements.

IIT-JEE PHYSICS PAPER 2010

Paper I

SECTION – I (Single Correct Choice Type)

This section contains **8 multiple choice questions.** Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

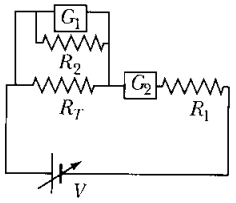
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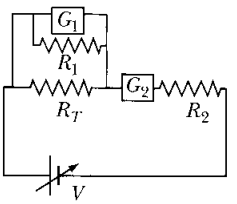
1. Incandescent bulbs are designed by keeping in mind that the resistance of their filament increases with the increase in temperature. If at room temperature, 100 W, 60 W and 40 W bulbs have filament resistances R_{100}, R_{60}, and R_{40}, respectively, the relation between these resistances is

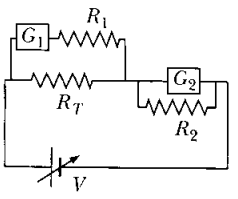
(A) $\frac{1}{R_{100}} = \frac{1}{R_{40}} + \frac{1}{R_{60}}$ (B) $R_{100} = R_{40} + R_{60}$

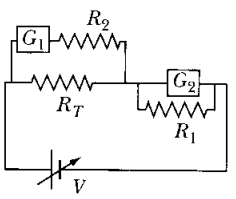
(C) $R_{100} > R_{60} > R_{40}$ (D) $\frac{1}{R_{100}} > \frac{1}{R_{60}} > \frac{1}{R_{40}}$

2. To verify Ohm's law, a student is provided with a test resistor R_T, a high resistance R_1, a small resistance R_2, two identical galvanometers G_1 and G_2, and a variable voltage source V. The correct circuit to carry out the experiment is

(A)


(B)


(C)


(D)


3. An AC voltage source of variable angular frequency ω and fixed amplitude V_0 is connected in series with a capacitance C and an electric bulb of resistance R (inductance zero). When ω is increased

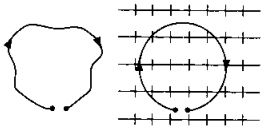
(A) the bulb glows dimmer

(B) the bulb glows brighter

(C) total impedance of the circuit is unchanged

(D) total impedance of the circuit increases

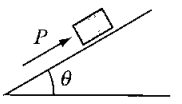
4. A thin flexible wire of length L is connected to two adjacent fixed points and carries a current I in the clockwise direction, as shown in the figure. When the system is put in a uniform magnetic field of strength B going into the plane of the paper, the wire takes the shape of a circle. The tension in the wire is

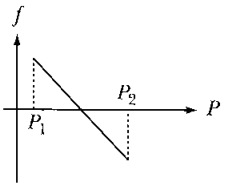


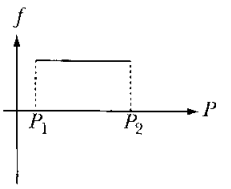
(A) IBL (B) $\frac{IBL}{\pi}$

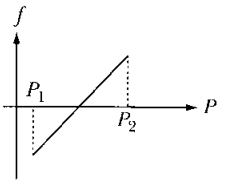
(C) $\frac{IBL}{2\pi}$ (D) $\frac{IBL}{4\pi}$

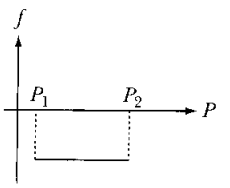
5. A block of mass m is placed on an inclined plane of angle θ. The coefficient of friction between the block and the plane is μ and $\tan\theta > \mu$. The block is held stationary by applying a force P parallel to the plane. The direction of force pointing up the plane is taken to be positive. As P is varied from $P_1 = mg(\sin\theta - \mu\cos\theta)$ to $P_2 = (\sin\theta + \mu\cos\theta)$, the frictional force f versus P graph will look like



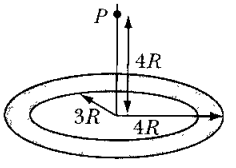
(A)


(B)


(C)


(D)


6. A thin uniform annular disc (see figure) of mass M has outer radius $4R$ and inner radius $3R$. The work required to take a unit mass from point P on its axis to infinity is



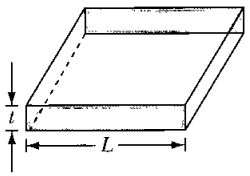
(A) $\frac{2GM}{7R}(4\sqrt{2} - 5)$

(B) $-\frac{2GM}{7R}(4\sqrt{2} - 5)$

(C) $\frac{GM}{4R}$

(D) $\frac{2GM}{5R}(\sqrt{2} - 1)$

7. Consider a thin square sheet of side L and thickness t, made of a material of resistivity ρ. The resistance between two opposite faces, shown by the shaded areas in the figure is



(A) directly proportional to L

(B) directly proportional to t

(C) independent of L

(D) independent of t

8. A real gas behaves like an ideal gas if its

(A) pressure and temperature are both high

(B) pressure and temperature are both low

(C) pressure is high and temperature is low

(D) pressure is low and temperature is high

SECTION – II (Multiple Correct Choice Type)

This section contains **5 multiple choice questions.** Each question has four choices (A), (B), (C) and (D) out of which **ONE OR More** may be correct.

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9. A point of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverses its direction and moves with a speed of 2 m/s. Which of the following statement(s) is (are) correct for the system of these two masses?

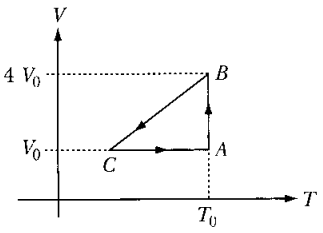
(A) Total momentum of the system is 3 kg m/s

(B) Momentum of 5 kg mass after collision is 4 kg m/s

(C) Kinetic energy of the center of mass is 0.75 J

(D) Total kinetic energy of the system is 4 J

10. One mole of an ideal gas in initial state A undergoes a cyclic process $ABCA$, as shown in the figure. Its pressure at A is P_0. Choose the correct option(s) from the following



(A) Internal energies at A and B are the same

(B) Work done by the gas in process AB is $P_0V_0 \ln 4$

(C) Pressure at C is $P_0/4$

(D) Temperature at C is $T_0/4$

11. A student uses a simple pendulum of exactly 1 m length to determine g, the acceleration due to gravity. He uses a stop watch with the least count of 1 s for this and records 40 s for 20 oscillations. For this observation, which of the following statement(s) is (are) true?

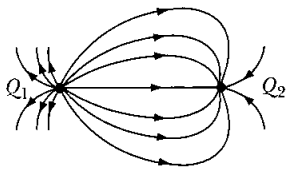
(A) Error ΔT in measuring T, the time period, is 0.05 s

(B) Error ΔT in measuring T, the time period, is 1 s

(C) Percentage error in the determination of g is 5%

(D) Percentage error in the determination of g is 2.5%

12. A few electric field lines for a system of two chares Q_1 and Q_2 fixed at two different points on the x-axis are shown in the figure. These lines suggest that



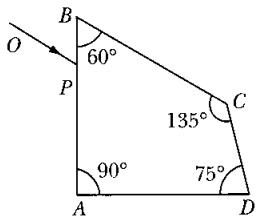
(A) $|Q_1| > |Q_2|$

(B) $|Q_1| < |Q_2|$

(C) at a finite distance to the left of Q_1 the electric field is zero

(D) at a finite distance to the right of Q_2 the electric field is zero

13. A ray OP of monochromatic light is incident on the face AB of prism $ABCD$ near vertex B at an incident angle of 60° (see figure). If the refractive index of the material of the prism is $\sqrt{3}$, which of the following is (are) correct?



(A) The ray gets totally internally reflected at face CD

(B) The ray comes out through face AD

(C) The angle between the incident ray and the emergent ray is 90°

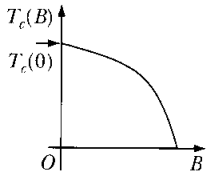
(D) The angle between the incident ray and the emergent ray is 120°

SECTION – III (Paragraph Type)

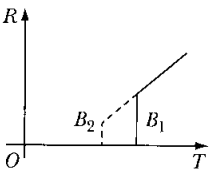
This section contains **2 paragraphs.** Based upon the first paragraph **2 multiple choice questions** and based upon the second paragraph **3 multiple choice questions** have to be answered. Each of these questions has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

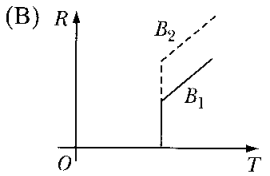
Paragraph for Questions 14 to 15

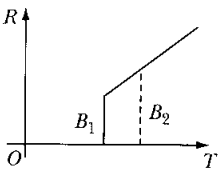
Electrical resistance of certain materials, known as superconductors, changes abruptly from a nonzero value to zero as their temperature is lowered below a critical temperature $T_c(0)$. An interesting property of superconductors is that their critical temperature becomes smaller than $T_c(0)$ if they are placed in a magnetic field, i.e., the critical temperature $T_c(B)$ is a function of the magnetic field strength B. The dependence of $T_c(B)$ on B is shown in the adjacent figure.

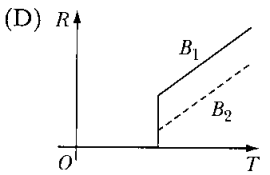


14. In the graph below, the resistance R of a superconductor is shown as a friction of its temperature T for two different magnetic fields B_1 (solid line) and B_2 (dashed line). If B_2 is larger than B_1 which of the following graphs shows the correct variation of R with T in these fields?

(A) 

(B) 

(C) 

(D) 

15. A superconductor has $T_c(0) = 100\,\text{K}$. When a magnetic field of 7.5 T is applied, its T_c decreases to 75 K. For this material one can definitely say that when

(A) $B = 5\,\text{T}$, $T_c(B) = 80\,\text{K}$

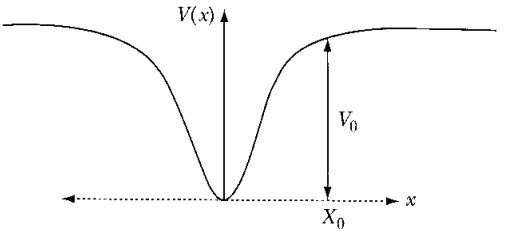
(B) $B = 5\,\text{T}$, $75\,\text{K} < T_c(B) < 100\,\text{K}$

(C) $B = 10\,\text{T}$, $75\,\text{K} < T_c < 100\,\text{K}$

(D) $B = 10\,\text{T}$, $T_c = 70\,\text{K}$

Paragraph for Questions 16 to 18

When a particle of mass m moves on the x-axis in a potential of the form $V(x) = kx^2$ it performs simple harmonic motion. The corresponding time period is proportional to $\sqrt{\frac{m}{k}}$, as can be seen easily using dimensional analysis. However, the motion of a particle can be periodic even when its potential energy increases on both sides of $x = 0$ in a way different from $k x^2$ and its total energy is such that the particle does not escape to infinity. Consider a particle of mass m moving on the x-axis. Its potential energy is $V(x) = \alpha x^4 (\alpha > 0)$ for $|x|$ near the origin and becomes a constant equal to V_0 for $|x| \geq X_0$ (see figure).



16. If the total energy of the particle is E, it will perform periodic motion only if

(A) $E < 0$ (B) $E > 0$

(C) $V_0 > E > 0$ (D) $E > V_0$

17. For periodic motion of small amplitude A, the time period T of this particle is proportional to

(A) $A\sqrt{\frac{m}{\alpha}}$ (B) $\frac{1}{A}\sqrt{\frac{m}{\alpha}}$

(C) $A\sqrt{\frac{\alpha}{m}}$ (D) $A\sqrt{\frac{\alpha}{m}}$

18. The acceleration of this particle for $|x| > X_0$ is

(A) proportional to V_0

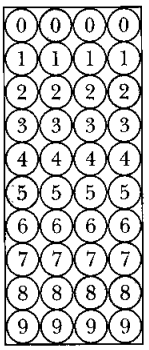
(B) proportional to $\frac{V_0}{mX_0}$

(C) proportional to $\sqrt{\frac{V_0}{mX_0}}$

(D) zero

SECTION – IV (Integer Type)

This section contains **TEN** questions. The answer to each question is a **single-digit integer** ranging from 0 to 9. The correct digit below the question number in the **ORS** is to be bubbled.



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19. Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11}g$, where g is the gravitational acceleration on the surface of the Earth. The average mass density of the planet is $\frac{2}{3}$ times that of the Earth. If the escape speed on the surface of the earth is taken to be $11\,\text{km s}^{-1}$, the escape speed on the surface of the planet in km s^{-1} will be _____.

20. A piece of ice (heat capacity = $2100\,\text{J kg}^{-1}\,^\circ\text{C}^{-1}$ and latent heat = $3.36\times10^5\,\text{J kg}^{-1}$) of mass m grams is at –5°C at atmospheric pressure. It is given 420 J of heat so that the ice starts melting. Finally when the ice–water mixture is in equilibrium, it is found that 1 g of ice has melted. Assuming there is no other heat exchange in the process, the value of m is _____.

21. A stationary source is emitting sound at a fixed frequency f_0, which is reflected by two cars approaching the source. The difference between the frequencies of sound reflected from the cars is 1.2% of f_0. What is the difference in the speeds of the cars (in km per hour) to the nearest integer? The cars are moving at constant speeds much smaller than the speed of sound which is $330\,\text{ms}^{-1}$.

22. The focal length of a thin biconvex lens is 20 cm. When an object is moved from a distance of 25 cm in front of it to 50 cm, the magnification of its image changes from m_{25} to m_{50}. The ratio m_{25}/m_{50} is _____.

23. An α-particle and a proton are accelerated from rest by a potential difference of 100 V. After this, their de Broglie wavelengths are λ_α and λ_p, respectively. The ratio λ_p/λ_α, to the nearest integer, is _____.

24. When two identical batteries of internal resistance $1\,\Omega$ each are connected in series across a resistor R, the rate of heat produced in R is J_1. When the same batteries are connected in parallel across R, the rate is J_2. If $J_1 = 2.25J_2$ then the value of R in Ω is _____.

25. Two spherical bodies A (radius 6 cm) and B (radius 18 cm) are at temperatures T_1 and T_2, respectively. The maximum intensity in the emission spectrum of A is at 500 nm and in that of B is at 1500 nm. Considering them to be black bodies, what will be the ratio of the rate of total energy radiated by A to that of B?

26. When two progressive waves $y_1 = 4\sin(2x - 6t)$ and $y_2 = 3\sin(2x - 6t - \pi/2)$ are superimposed, the amplitude of the resultant wave is _____.

27. A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its cross-sectional area is $4.9\times10^{-7}\,\text{m}^2$. If the mass is pulled a little in the vertically downward direction and released, it performs simple harmonic motion of angular frequency $140\,\text{rad s}^{-1}$. If the Young's modulus of the material of the wire is $n\times10^9\,\text{Nm}^{-2}$, the value of n is _____.

28. A binary star consists of two stars A (mass $2.2M_s$) and B (mass $11M_s$), where M_s is the mass of the sun. They are separated by distance d and are rotating about their centre of mass, which is stationary. The ratio of the total angular momentum of the binary star to the angular momentum of star B about the centre of mass is _____.

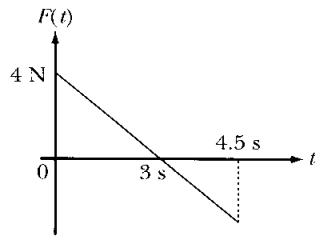
Paper II

SECTION – I (Single Correct Choice Type)

This section contains **6 multiple choice questions.** Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

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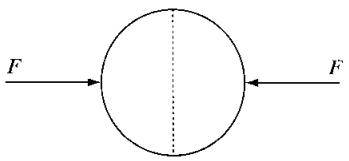
29. A block of mass 2 kg is free to move along the x-axis. It is at rest and from $t = 0$ onwards it is subjected to a time-dependent force $F(t)$ in the x direction. The force $F(t)$ varies with t as shown in the figure. The kinetic energy of the block after 4.5 s is



(A) 4.50 J (B) 7.50 J

(C) 5.06 J (D) 14.06 J

30. A uniformly charged thin spherical shell of radius R carries uniform surface charge density of σ per unit area. It is made of two hemispherical shells, held together by pressing them with force F (see figure). F is proportional to



(A) $\frac{1}{\varepsilon_0}\sigma^2 R^2$ (B) $\frac{1}{\varepsilon_0}\sigma^2 R$

(C) $\frac{1}{\varepsilon_0}\frac{\sigma^2}{R}$ (D) $\frac{1}{\varepsilon_0}\frac{\sigma^2}{R^2}$

31. A tiny spherical oil drop carrying a net charge q is balanced in still air with a vertical uniform electric field of strength $81\pi/7 \times 10^5$ Vn. When the field is switched off, the drop is observed to fall with terminal velocity 2×10^{-3} m s^{-1}. Given $g = 9.8$ m s^{-2}, viscosity of the air $= 1.8 \times 10^{-5}$ Ns m^{-2} and the density of oil $= 900$ kg m^{-3}, the magnitude of q is

(A) 1.6×10^{-19} C (B) 3.2×10^{-19} C

(C) 4.8×10^{-19} C (D) 8.0×10^{-19} C

32. A Vernier caliper has 1 mm marks on the main scale. It has 20 equal divisions on the Vernier scale which match with 16 main scale divisions. For this Vernier caliper, the least count is

(A) 0.02 mm (B) 0.05 mm

(C) 0.1 mm (D) 0.2 mm

33. A biconvex lens of focal length 15 cm is in front of a plane mirror. The distance between the lens and the mirror is 10 cm. A small object is kept at a distance of 30 cm from the lens. The final image is

(A) virtual and at a distance of 16 cm from the mirror

(B) real and at a distance of 16 cm from the mirror

(C) virtual and at a distance of 20 cm from the mirror

(D) real and at a distance of 20 cm from the mirror

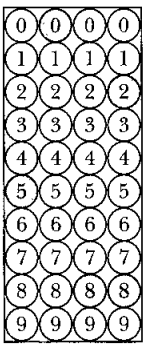
34. A hollow pipe of length 0.8 m is closed at one end. At its open end a 0.5 m long uniform string is vibrating in its second harmonic and it resonates with the fundamental frequency of the pipe. If the tension in the wire is 50 N and the speed of sound is 320 ms^{-1}, the mass of the string is

(A) 5 g (B) 10 g

(C) 20 g (D) 40 g

SECTION – II (Integer Type)

This section contains **5 questions.** The answer to each question is a **single-digit integer** ranging from 0 to 9. The correct digit below the question number in the **ORS** is to be bubbled.

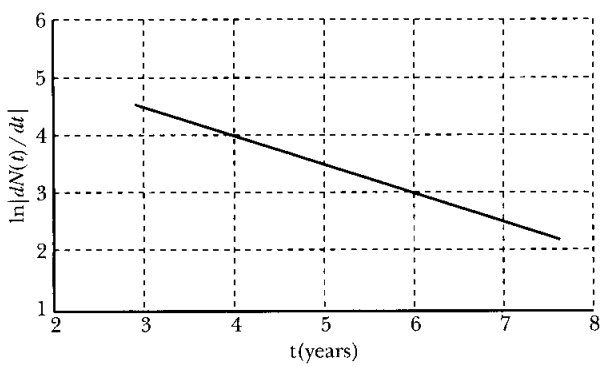


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35. A large glass slab ($\mu = 5/3$) of thickness 8 cm is placed over a point source of light on a plane surface. It is seen that light emerges out of the top surface of the slab from a circular area of radius R cm. What is the value of R?

36. Image of an object approaching a convex mirror of radius of curvature 20 m along its optical axis is observed to move from 25/3 m to 50/7 m in 30 s. What is the speed of the object in km per hour?

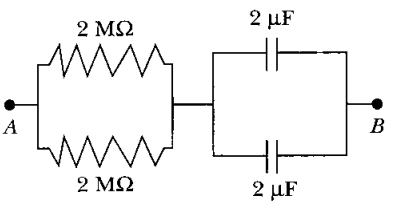
37. To determine the half life of a radioactive element, a student plots a graph of $\ln \left| dN(t)/dt \right|$ versus t. Here $dN(t)/dt$ is the rate of radioactive decay at time t. If the number of radioactive nuclei of this element decreases by a factor of p after 4.16 years, the value of p is _____.



38. A diatomic ideal gas is compressed adiabatically to1/32 of its initial volume. If the initial temperature of the gas is T_i (in Kelvin) and the final temperature is aT_i, the value of a is _____.

39. At time $t = 0$, a battery of 10 V is connected across points A and B in the given circuit. If the capacitors have no charge initially, at what time (in seconds) does the voltage across them become 4 V?

[Take ln 5 = 1.6, ln 3 = 1.1.]



SECTION – III (Paragraph Type)

This section contains **2 paragraphs.** Based upon each of the paragraphs **3 multiple choice questions** have to be answered. Each of these questions has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Paragraphs for Questions 40 to 42

When liquid medicine of density ρ is to be put in the eye, it is done with the help of a dropper. As the bulb on the top of the dropper is pressed, a drop forms at the opening of the dropper. We wish to estimate the size of the drop. We first assume that the drop formed at the opening is spherical because that requires a minimum increase in its surface energy. To determine the size, we calculate the net vertical force due to the surface tension T when the radius of the drop is R. When the force becomes *smaller than the weight* of the drop, the drop gets detached from the dropper.

40. If the radius of the opening of the dropper is r, the vertical force due to the surface tension on the drop of radius R (assuming $r \ll R$) is

(A) $2\pi rT$ (B) $2\pi RT$

(C) $\frac{2\pi r^2 T}{R}$ (D) $\frac{2\pi R^2 T}{r}$

41. If $r = 5 \times 10^{-4}$ m, $\rho = 10^3$ kg m^{-3}, $g = 10$ m/s^2, $T = 0.11$ Nm^{-1}, the radius of the drop when it detaches from the dropper is approximately

(A) 1.4×10^{-3} m (B) 3.3×10^{-3} m

(C) 2.0×10^{-3} m (D) 4.1×10^{-3} m

42. After the drop detaches, its surface energy is

(A) 1.4×10^{-6} J (B) 2.7×10^{-6} J

(C) 5.4×10^{-6} J (D) 8.1×10^{-6} J

Paragraph for Questions 43 to 45

The key feature of Bohr's theory of spectrum of hydrogen atom is the quantization of angular momentum when an electron is revolving around a proton. We will extend this to a general rotational motion to find quantized rotational energy of a diatomic molecule assuming it to be rigid. The rule to be applied is Bohr's quantization condition.

43. A diatomic molecule has moment of inertia I. By Bohr's quantization condition its rotational energy in the n^{th} level ($n = 0$ is not allowed) is

(A) $\frac{1}{n^2}\left(\frac{h^2}{8\pi^2 I}\right)$ (B) $\frac{1}{n}\left(\frac{h^2}{8\pi^2 I}\right)$

(C) $n\left(\frac{h^2}{8\pi^2 I}\right)$ (D) $n^2\left(\frac{h^2}{8\pi^2 I}\right)$

44. It is found that the excitation frequency from ground to the first excited state of rotation for the CO molecule is close to $4/\pi \times 10^{11}$ Hz. Then the moment of inertia of CO molecule about its centre of mass is close to (take $h = 2\pi \times 10^{-34}$ Js)

(A) 2.76×10^{-46} kg m^2 (B) 1.87×10^{-46} kg m^2

(C) 4.67×10^{-47} kg m^2 (D) 1.17×10^{-47} kg m^2

45. In a CO molecule, the distance between C (mass = 12 a.m.u.) and O (mass = 16 a.m.u.), where 1 a.m.u. = $5/3 \times 10^{-27}$ kg, is close to

(A) 2.4×10^{-10} m (B) 1.9×10^{-10} m

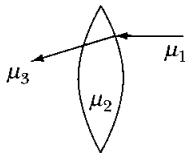
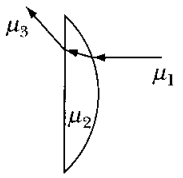
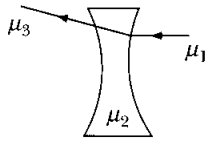
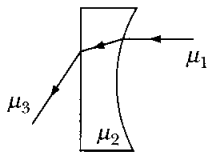
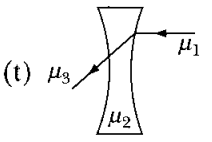
(C) 1.3×10^{-10} m (D) 4.4×10^{-11} m

SECTION – IV (Matrix Type)

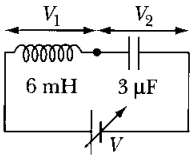
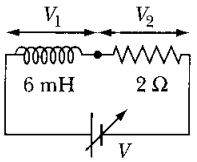
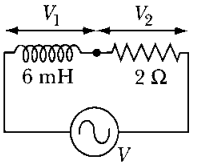
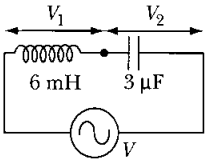
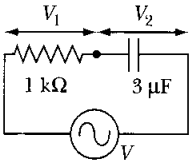
This section contains **2 questions.** Each question has **four statements** (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.** For example, if for a given question, statement B matches with the statements given in q and r, then for that particular question, against statement B, darken the bubbles corresponding to q and r in the **ORS.**

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

46. Two transparent media of refractive indices μ_1 and μ_3 have a solid lens shaped transparent material of refractive index μ_2 between them as shown in figures in Column II. A ray traversing these media is also shown in the figures. In Column I different relationships between μ_1, μ_2 and μ_3 are given. Match them to the ray diagram shown in Column II.

Column I	Column II
(A) $\mu_1 < \mu_2$	(p) 
(B) $\mu_1 > \mu_2$	(q) 
(C) $\mu_2 = \mu_3$	(r) 
(D) $\mu_2 > \mu_3$	(s) 
	(t) 

47. You are given many resistances, capacitors and inductors. These are connected to a variable DC voltage source (the first two circuits) or an AC voltage source of 50 Hz frequency (the next three circuits) in different ways as shown in Column II. When a current I (steady state for DC or rms for AC) flows through the circuit, the corresponding voltages V_1 and V_2 (indicated in circuits) are related as shown in Column I. Match the two

Column I	Column II
(A) $I \neq 0$, V_1 is proportional to I	(p) 
(B) $I \neq 0$, $V_2 > V_1$	(q) 
(C) $V_1 = 0$, $V_2 = V$	(r) 
(D) $I \neq 0$, V_2 is proportional to I	(s) 
	(t) 

ANSWERS

Paper I

1. (D)	**8.** (D)	**15.** (B)	**22.** (6)
2. (C)	**9.** (A, C)	**16.** (C)	**23.** (3)
3. (B)	**10.** (A, B)	**17.** (B)	**24.** (4)
4. (C)	**11.** (A, C)	**18.** (D)	**25.** (9)
5. (A)	**12.** (A, D)	**19.** (3)	**26.** (5)
6. (A)	**13.** (A, B, C)	**20.** (8)	**27.** (4)
7. (C)	**14.** (A)	**21.** (7)	**28.** (6)

Paper II

29. (C)

30. (A)

31. (D)

32. (D)

33. (B)

34. (B)

35. (6)

36. (3)

37. (8)

38. (4)

39. (2)

40. (C)

41. (A)

42. (B)

43. (D)

44. (B)

45. (C)

46. (A) → (p, r)
(B) → (q, s, t)
(C) → (p, r, t)
(D) → (q, s)

47. (A) → (r, s, t)
(B) → (q, r, s, t)
(C) → (p, q)
(D) → (q, r, s, t)